DIPARTIMENTO DI INGEGNERIA INDUSTRIALE

Università degli Studi di Padova – Dipartimento di Ingegneria Industriale

Corso di Laurea in Ingegneria Aerospaziale

Relazione per la prova finale «Manifold dynamics in the restricted three body problem and applications to orbital mechanics»

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Padova, 16/03/2022

INTRODUCTION

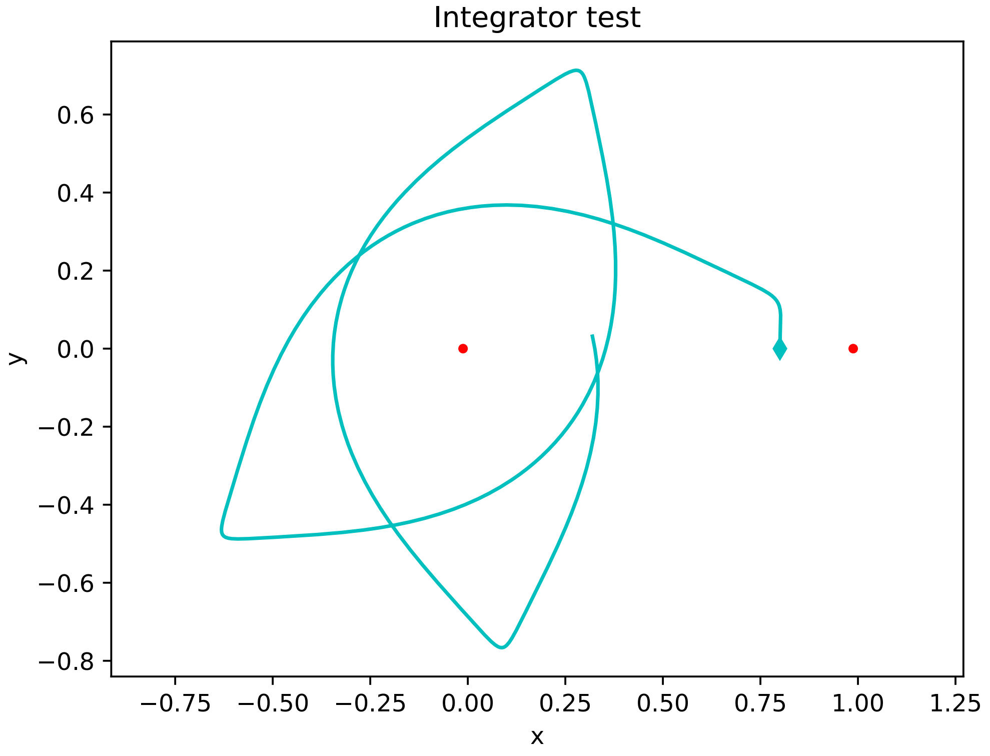
- Simple analysis of the CR3BP
- Collinear Lagrange points
- Linearization of the Hamilton equations
- Lyapunov orbit
- Unstable and stable manifolds of Lyapunov orbit
- Heteroclinic intersection
- Heteroclinic transfer

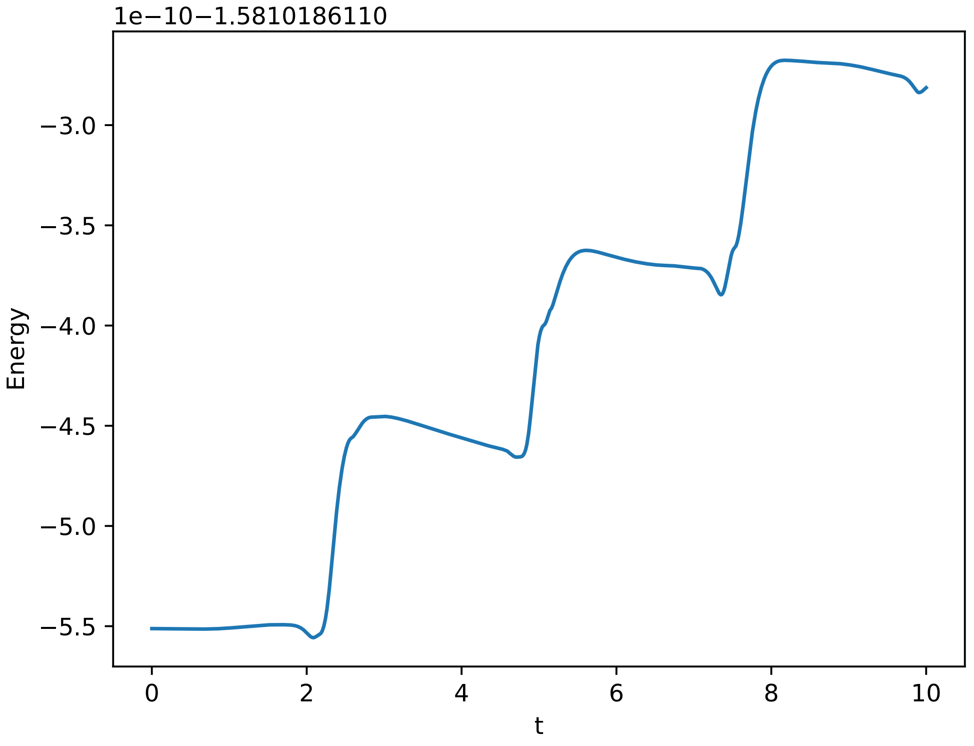
1. The CR3BP

- The Circular Restricted Three Body Problem, with Hamiltonian

$$H = \frac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) + p_x y - p_y x - \frac{1-\mu}{\sqrt{(x+\mu)^2 + y^2 + z^2}} - \frac{\mu}{\sqrt{(x+\mu-1)^2 + y^2 + z^2}}$$

- Check energy conservation to test quality of integration

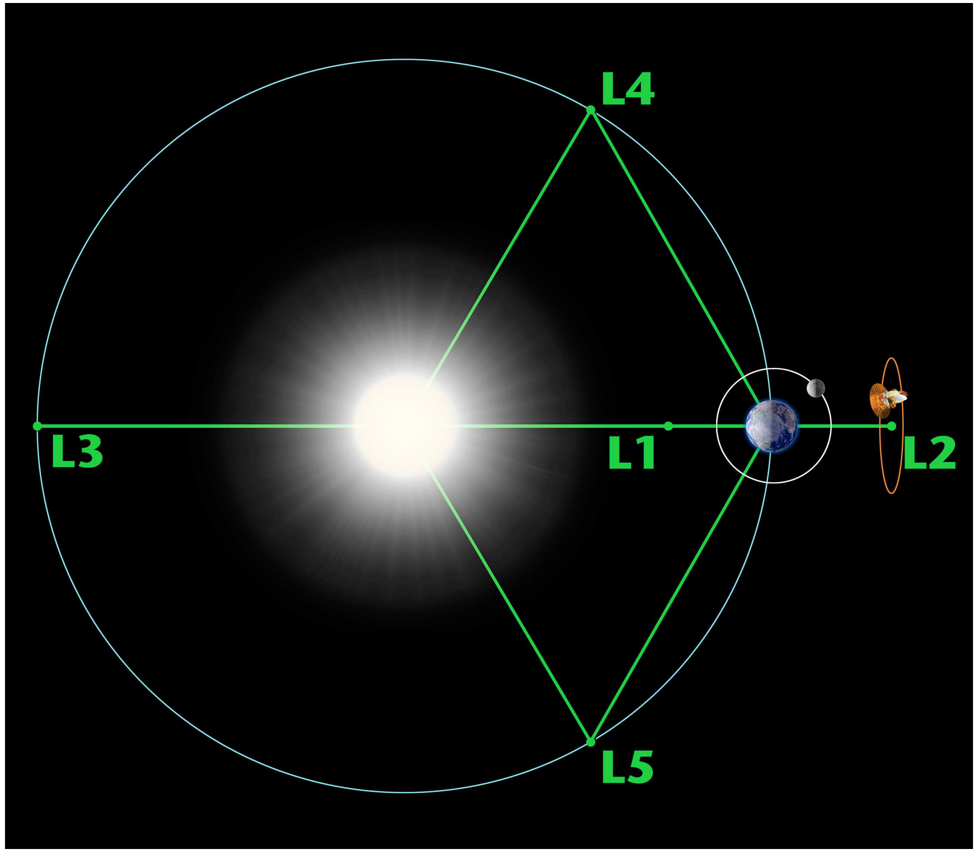




2. LAGRANGE POINTS AND STABILITY

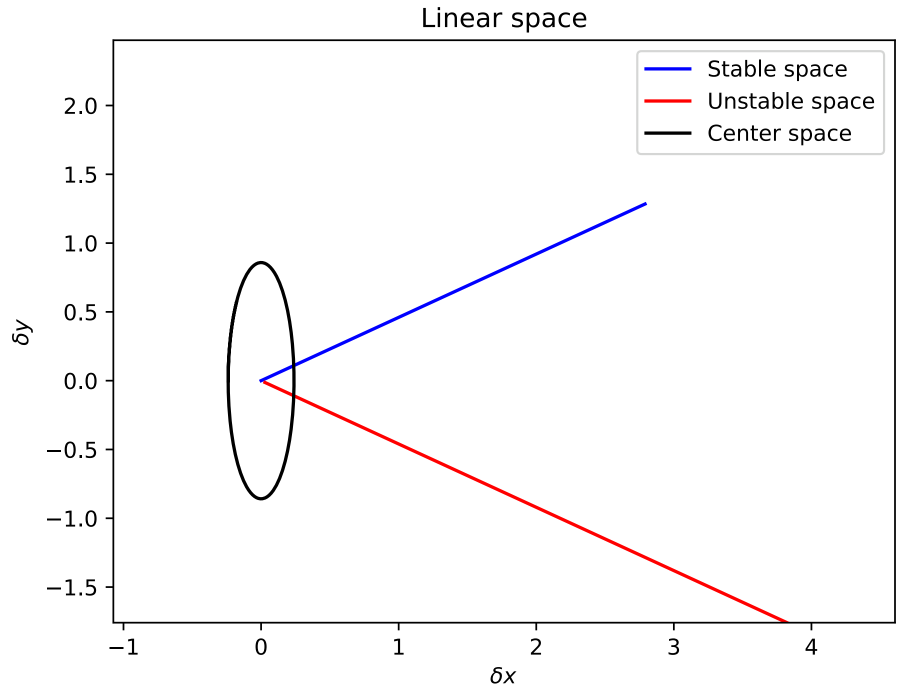
- Focus on collinear Lagrange points L_1, L_2, L_3
- Equilibrium point in phase space:
  $\underline{c} = (x_c, y_c, p_{x,c}, p_{y,c}) = (x_c, 0, 0, x_c)$
- Phase state vector:
  $\underline{r} = (x, y, p_x, p_y)$
- Can be expressed as $\underline{r} = \underline{c} + \underline{\delta}$
  $x = x_c + \delta x$
  $y = y_c + \delta y$
  $p_x = p_{x,c} + \delta p_x$
  $p_y = p_{y,c} + \delta p_y$
- If $f(\underline{r})$ are the Hamilton equations, the first order Taylor expansion is:
  $\dot{\underline{r}} = f(\underline{c}) + Df(\underline{c})(\underline{r} - \underline{c}) + \dots$
  $\dot{\underline{\delta}} = Df(\underline{c})\,\underline{\delta}$
- Where $Df(\underline{r})$ is the Jacobian:

$$\mathbf{J}_{\underline{r}} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ \frac{\partial \dot{p}_x}{\partial x} & \frac{\partial \dot{p}_x}{\partial y} & 0 & 1 \\ \frac{\partial \dot{p}_y}{\partial x} & \frac{\partial \dot{p}_y}{\partial y} & -1 & 0 \end{bmatrix}$$



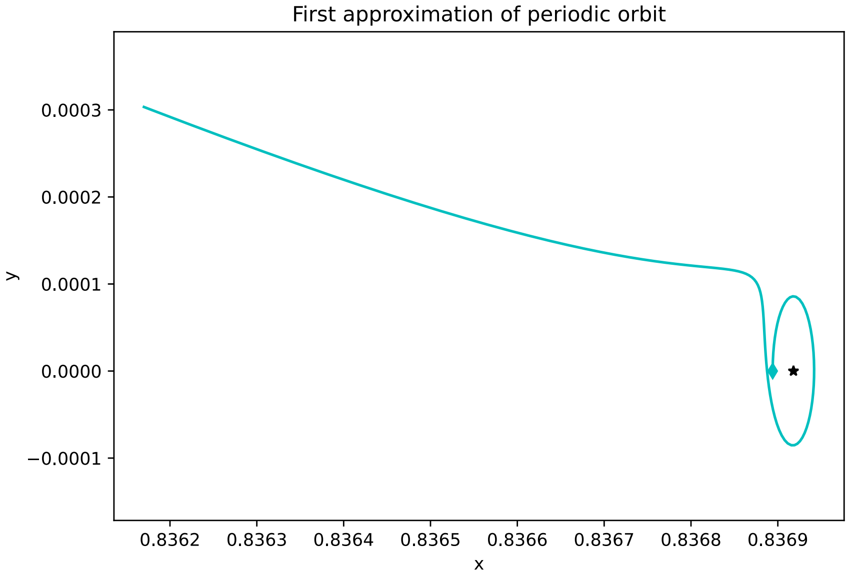
2. Lagrange points and stability

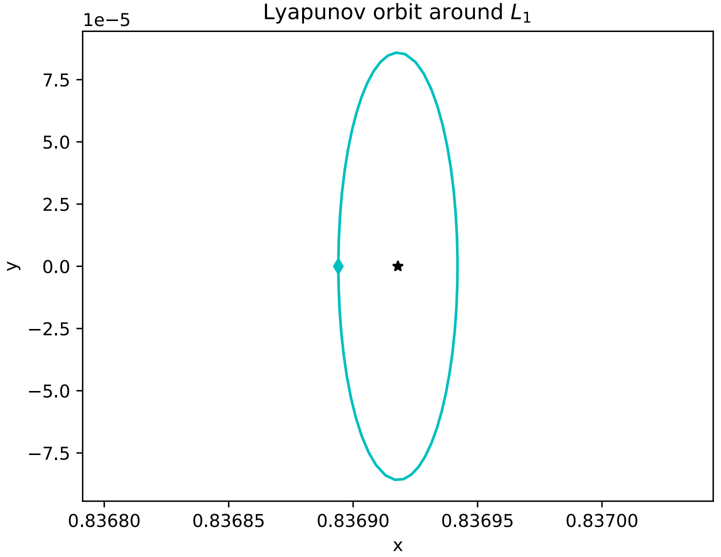
- The eigenvalues of the Jacobian give the linear stability of $\underline{c}$:
  $\lambda_1 = \lambda \qquad \lambda_2 = -\lambda \qquad \lambda_3 = i\nu \qquad \lambda_4 = -i\nu$
  therefore $\underline{c}$ is a partially hyperbolic equilibrium point
- The eigenvectors of the Jacobian give the unstable, stable and center spaces, respectively
  $\mathbf{E}^U = \langle \mathrm{v}_1 \rangle$
  $\mathbf{E}^S = \langle \mathrm{v}_2 \rangle$
  $\mathbf{E}^C = \langle \mathrm{v}_3, \mathrm{v}_4 \rangle$
- The local manifold theorem states that, for a system with partially hyperbolic equilibrium, the manifolds exist and are tangent in the equilibrium point to their respective linear spaces
- The stable, unstable and center manifolds are defined as the starting conditions in the phase space D that:
  $$W^S = \{\underline{r} \in D \backslash \underline{c} : \lim_{t \to +\infty} \phi(t, \underline{r}) = \underline{c}\}$$
  $$W^U = \{\underline{r} \in D \backslash \underline{c} : \lim_{t \to -\infty} \phi(t, \underline{r}) = \underline{c}\}$$
  $$W^C = \{\underline{r} \in D \backslash \underline{c} : \|\phi(t, \underline{r}) - \underline{c}\| < \epsilon\} \qquad \epsilon > 0, t \in \tau$$
- The family of horizontal Lyapunov orbits foliate the center manifold Wc



3. LYAPUNOV ORBIT

- Using a starting condition on the center space gives a trajectory that does not make a full orbit, the nonlinear terms dominate the dynamics
- We define the section of dimension 3:
  $\Sigma = \{\underline{r} : y = 0 \quad \dot{y} > 0\}$
- And the section map:
  $\Sigma \rightarrow \Sigma \quad (x, p_x, p_y) \mapsto (x', p_x', p_y') = \psi(x, p_x, p_y)$
- The Lyapunov orbit is a fixed point for the map $\psi(x, p_x, p_y)$
- The Lyapunov orbit is obtained by finding the root of
  $F(p_y) = x' - x_0 = \psi_x(x_0, 0, p_y) - x_0$
  using Newton's method





4. CALCULATING THE TUBE MANIFOLDS

- We can find the energy of the Lyapunov orbit, E
- We define the Poincaré section of dimension 2:
  $\Pi = \{\underline{r} : y = 0 \quad \dot{y} > 0, \quad H = E\}$
- And we define the Poincaré map:
  $\Pi \rightarrow \Pi \qquad \underline{s} \mapsto \underline{s}' = \underline{g}(\underline{s})$
  where $\underline{s} = (x, p_x)$
- The fixed point $\underline{p}$ of the Lyapunov orbit is again a fixed point for the Poincaré map
- We can do a first order Taylor expansion around the fixed point:
  $\underline{s}' = \underline{p} + M(\underline{s} - \underline{p}) + \dots$
  $\underline{\eta}' = M\underline{\eta}$
  where $\underline{\eta} = \underline{s} - \underline{p}$ and M is the monodromy matrix:
  $$M = \begin{bmatrix} \frac{\partial g_x}{\partial x} & \frac{\partial g_x}{\partial p_x} \\ \frac{\partial g_{p_x}}{\partial x} & \frac{\partial g_{p_x}}{\partial p_x} \end{bmatrix}$$

4. CALCULATING THE TUBE MANIFOLDS

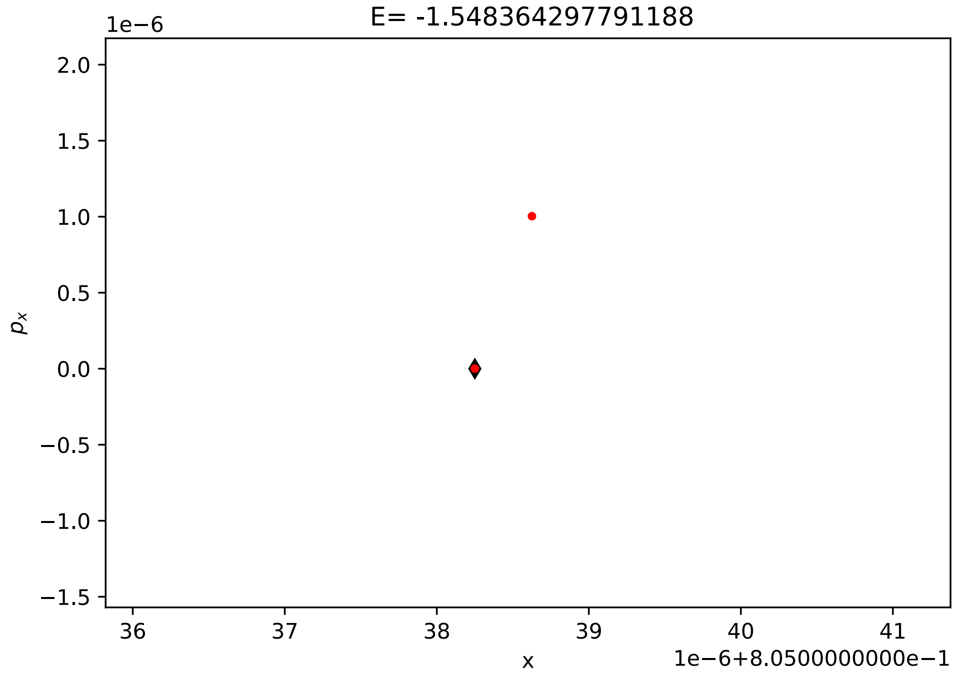
- The partial derivatives for the monodromy matrix are calculated through finite differences of first order:

$$\frac{\partial g_x}{\partial x} \approx \frac{g_x(x_0 + h,0) - g_x(x_0 - h,0)}{2h}$$

$$\frac{\partial g_x}{\partial p_x} \approx \frac{g_x(x_0, h) - g_x(x_0, -h)}{2h}$$

$$\frac{\partial g_{p_x}}{\partial x} \approx \frac{g_{p_x}(x_0 + h,0) - g_{p_x}(x_0 - h,0)}{2h}$$

$$\frac{\partial g_{p_x}}{\partial p_x} \approx \frac{g_{p_x}(x_0, h) - g_{p_x}(x_0, -h)}{2h}$$



- Check that the monodromy matrix is symplectic
  $det(M) = 1$

- The eigenvalues of the monodromy matrix give the linear stability of the fixed point
  $|\rho_1| > 1 \qquad |\rho_2| < 1$

- The respective eigenvectors generate the unstable and stable spaces of the linearized map

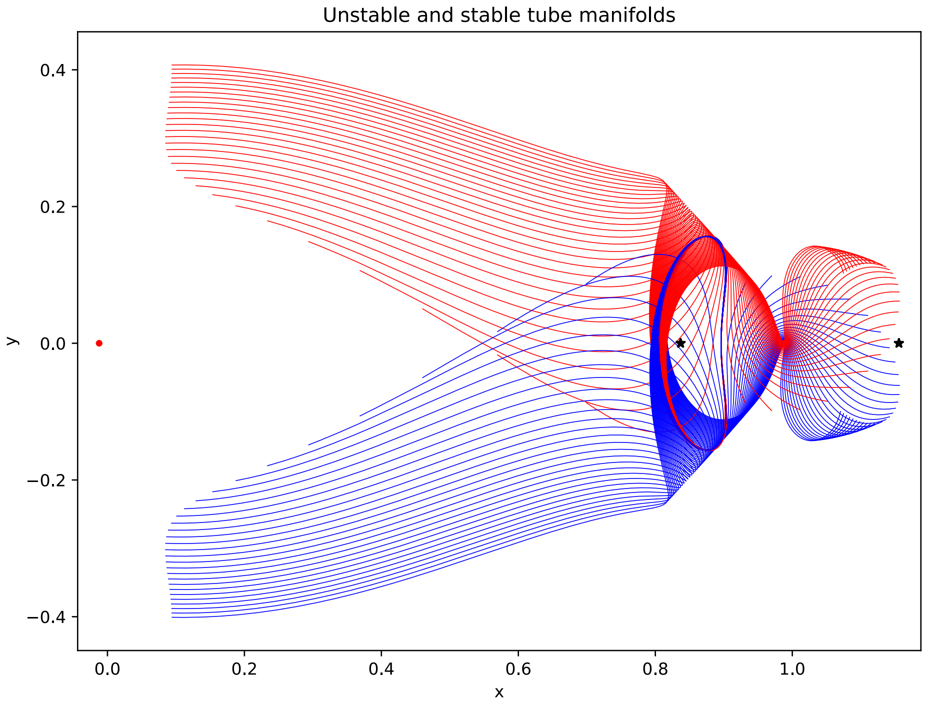
4. Calculating the tube manifolds

- **Definition**: the unstable and stable manifolds of $\underline{p}$ are the sets:

$$W_{\underline{p}}^{U} = \{x, p_x \in \Sigma \backslash \underline{p} : \lim_{k\to+\infty} \underline{g}^k = \underline{p}\}$$

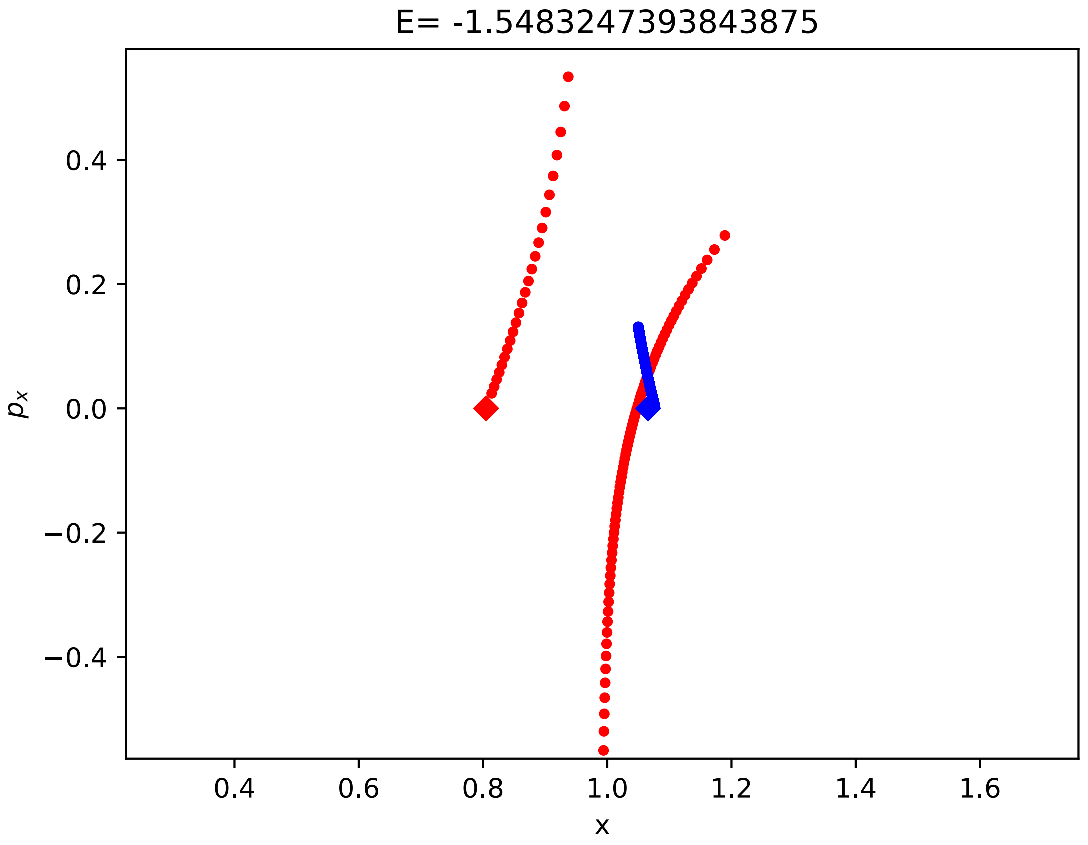
$$W_{\underline{p}}^{S} = \{x, p_x \in \Sigma \backslash \underline{p} : \lim_{k\to-\infty} \underline{g}^k = \underline{p}\}$$

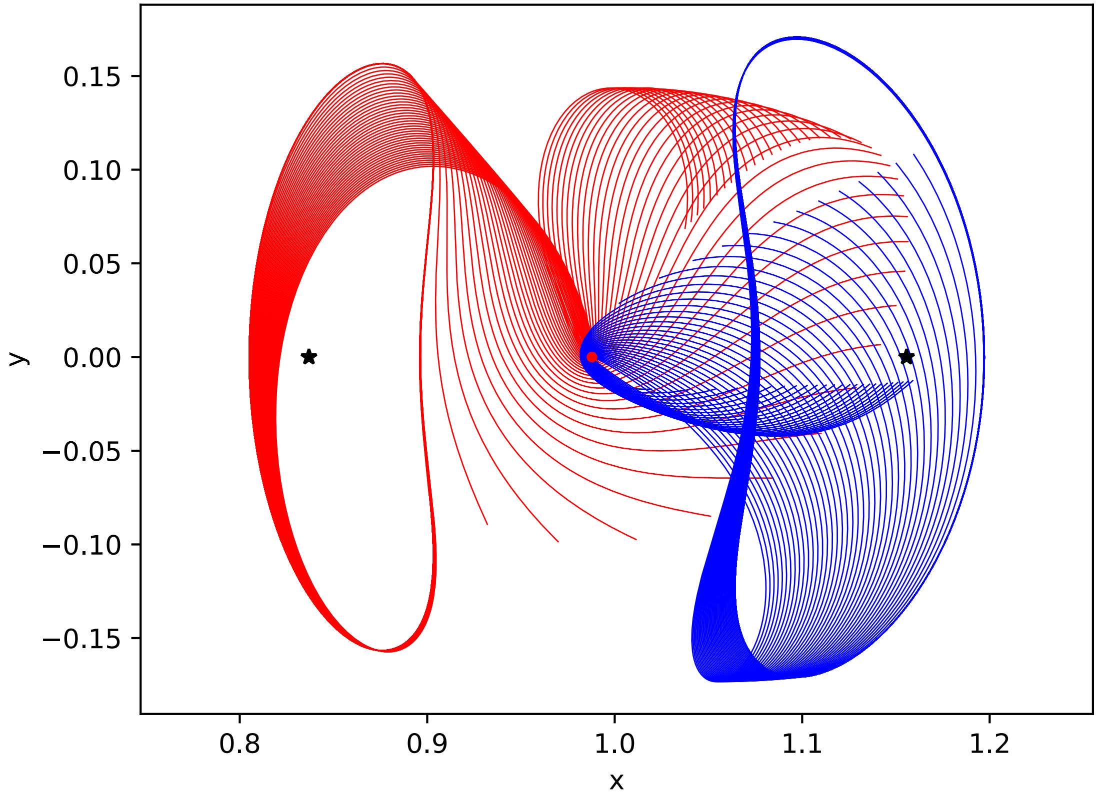
- **Proposition**: the sets $W_{\underline{p}}^{U}$ $W_{\underline{p}}^{S}$ are invariant under the action of the map $\underline{g}(\underline{s})$
- **Theorem** (Local manifold theorem): The manifolds $W_{\underline{p}}^{U}$ $W_{\underline{p}}^{S}$ exist, and are tangent to the respective invariant manifolds of the fixed point of the linearized map $\underline{\eta}' = M\underline{\eta}$
- To visualise the stable tube manifold:
  - Pick various points from the manifold in the map
  - Integrate backward in time in the flow
- To visualise the unstable tube manifold:
  - Pick various points from the manifold in the map
  - Integrate forward in time in the flow



5. HETEROCLINIC TRANSFER

- **Heteroclinic intersection**: transverse intersection between the unstable and stable manifolds of two different hyperbolic (or partially hyperbolic) fixed points

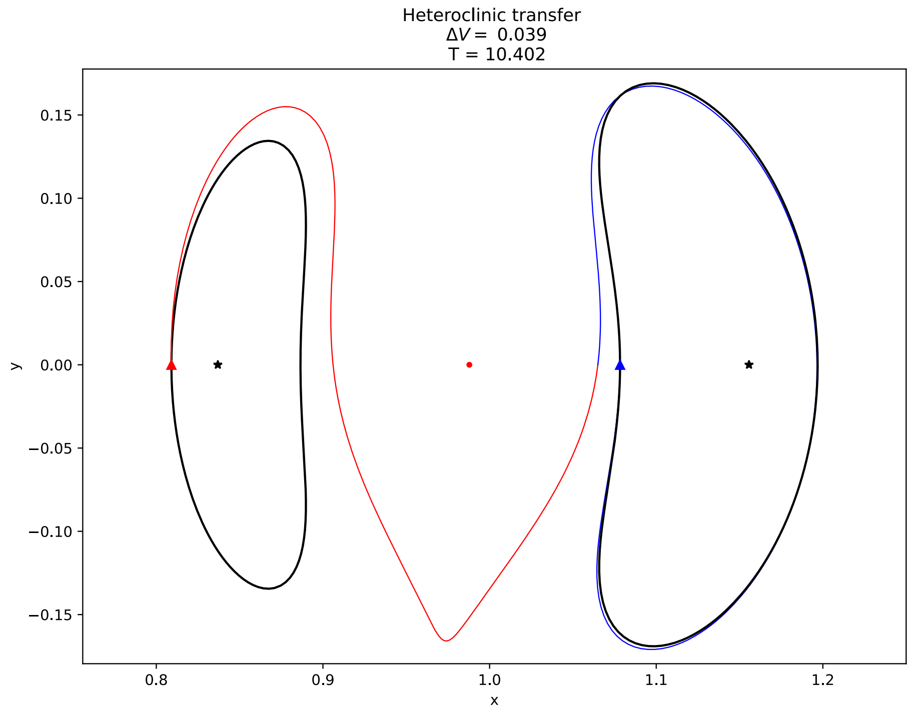




5. HETEROCLINIC TRANSFER

- From the intersection integrate backward in time to find the Lyapunov orbit around L_1
- Calculate the ΔV for orbit departure
- Transfer
- Arrive in proximity of Lyapunov orbit around L_2
- Possible ΔV for orbit insertion

- **Example**:
    - Earth - Moon system, $\mu = 0.01215$
    - $\Delta V = 0.039$
    - $T = 10.402$



CONCLUSIONS

- Lyapunov theorem → Existence of families of closed orbits around collinear L points
- Local manifold theorem → Existence of tube manifolds of horizontal Lyapunov orbit
- Heteroclinic intersections → Low cost transfer trajectories
- Numerical computation:
  - Newton's method applied to the section map Σ in order to compute the periodic orbits
  - Linearization of the Poincaré map Π and use of the local manifold theorem to identify initial conditions on the sets $W_{\underline{p}}^{U}$ $W_{\underline{p}}^{S}$
  - Numerical propagation of the initial conditions
- Applications:
  - Great distances covered with minimum fuel expenditure
  - Orbits with useful properties (space observatories, communication satellites etc.)
  - Long travel times, trade-off between speed and fuel consumption

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