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Geometric invariant theory for spaces of hypersurfaces

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Introduction

Invariant theory is a classical subject in both algebra and geometry.
Especially after the introduction of projective coordinates at the beginning of the nineteenth-century, mathematicians became interested in properties of plane curves that were invariant after linear change of coordinates. As time went on, they realized that such a property can be formulated as the invariance under an action of a group, usually SL_n or GL_n. Since the geometric objects were defined by polynomial equations, the problem turned out to be the following one: given a "nice" action of a group G on the polynomial ring $S = k[x_1, ..., x_n]$, find the elements of S that are invariant under G. The subalgebra of invariant elements is denoted S^G, and they noted that in some cases S^G was finitely generated.
At the end of the century, one of the main problems was to find the invariants of the natural action of $SL_2(\mathbb{C})$ on $\mathbb{C}[x_0, x_1]_d$, the so called "Problem of invariants of binary forms of degree d". A great contribution was given by David Hilbert, who proved that the ring of invariants is finitely generated in many cases, such as the one of binary forms. This led to the formulation of Hilbert's 14th problem, asking whether the ring of invariants is always finitely generated: this was solved by Nagata in 1959, with a negative answer. However, Nagata himself proved that the ring of invariants is always finitely generated if we make a stronger assumption on the group G, that is, if G is geometrically reductive.
Geometric invariant theory (GIT) is a method for constructing quotients for the actions of algebraic groups on varieties and it is frequently used in the context of *moduli*. The starting point is Nagata's result, so that GIT is concerned with the action of geometrically reductive groups. A moduli problem is essentially a classification problem: we would like to classify geometric objects up to some notion of equivalence. A key example is given by the classification of degree d hypersurfaces in $\mathbb{P}^n$ up to projective equivalence, which is the core of this thesis.
Given an action of an algebraic G on a variety X, the set of orbits cannot be structured as a variety in general, because the action typically has non-

closed orbits. However, we can relax our hypotesis of having an orbit space, in order to get a quotient with better geometrical properties: this leads to the definition of good quotient. Geometric invariant theory, as developed by Mumford in [13], shows that for a linear action of a geometrically reductive group on a projective variety X, there always exists an open subset U of the variety and a good quotient for the action of G on U. Moreover, the quotient is itself projective.
The thesis is divided into three chapters.

In Chapter 1 we give the definition and the main examples of algebraic groups and discuss their actions on varieties. We present the notion of geometrically reductive groups, which is of great importance in the context of GIT. Finally, we give the statement and the proof of Nagata's Theorem.

In Chapter 2 we first show how to construct GIT quotients in the affine case, and we give some examples of explicit constructions. Then we focus our attention on the projective case: this leads to the definitions of stable and semi-stable points of a projective variety under a linear action of a geometrically reductive group. Finally, we present Hilbert-Mumford criterion, a very useful tool in order to determine (semi-)stability of a point under a given action. We give a proof of the criterion in the particular case where the group is SL_n.

The third chapter is the central part of the thesis. Here we study the natural action of SL_{n+1} on the space of projective hypersurfaces of degree d in $\mathbb{P}^n$, which we denote by $\mathrm{Hyp}_d(n)$. This is precisely the moduli problem of the classification of projective hypersurfaces up to projective equivalence.
First, we emphasize the role of smooth hypersurfaces, then we study in detail some particular cases for small values of d and n using the Hilbert-Mumford criterion. We start with the case of quadric hypersufaces in $\mathbb{P}^n$ and the one of binary forms of degree d. Great emphasis is given to the case of plane cubics, where we give a complete description of the stable and semi-stable locus.
Then we deal with stability of plane quartics and cubic surfaces, from which we can deduce that a breakdown in stability is always due to "bad" singularities of the hypersurface.

The main references for this thesis are [11] and [15] for the results in the first two chapters, while in the third chapter we mainly refer to [5], [12] and [13]. Finally, for the last example where we deal with cubic surfaces, the main references are [3] and [6].

Notations and conventions

Throughout the thesis we fix an algebraically closed field k of any characteristic, unless otherwise specified: for instance, in chapter 3 we will assume char $k \neq 2$ when we study projective quadrics. By a variety, we mean a separated prevariety over the field k, i.e. a ringed space $(X, \mathcal{O}_X)$ where X is irreducible and $\mathcal{O}_X$ is a sheaf of k-valued functions, which admits an open cover by finitely many affine varieties, such that the diagonal Δ is closed in $X \times X$. In particular, we always assume that varieties are irreducible.
If X is an affine variety, we will denote by $A(X)$ (instead of $\mathcal{O}_X(X)$) its ring of regular functions.

Chapter 1

Actions of algebraic groups

In this chapter we will introduce the notion of algebraic group and discuss the actions of algebraic groups on varieties, which will allow us to construct quotients for these actions. Finally, we will define geometrically reductive groups and give a proof of Nagata's Theorem, which asserts that under a rational action of a geometrically reductive group on a finitely generated k-algebra, the subalgebra of invariants is always finitely generated.
The main references here are [11] (Sections 3.1 and 3.2) and [15] (Sections 3.1 and 3.2)

1.1 Algebraic groups

Definition 1.1.1. An *algebraic group* is a group G together with a structure of a variety such that the maps

$$\begin{gathered} G \times G \to G \\ (g, g') \longmapsto gg' \\ G \to G \\ g \longmapsto g^{-1} \end{gathered}$$

are morphisms of varieties. A homomorphism of algebraic groups is a map which is both a group homomorphism and a morphism of varieties.

Definition 1.1.2. An action of an algebraic group G on a variety X is a morphism

$$G \times X \to X$$

such that $g(g'x) = (gg')x$ and $1x = x$ for any $g, g' \in G$ and for any $x \in X$.

Remark. Just a matter of notation: if x is any point in the variety X, we denote by $O(x)$ the orbit of the point x, where $O(x) = \{gx : g \in G\}$.
Moreover, we denote by G_x the stabiliser of x.
$G_x = \{g \in G : gx = x\}$ is a closed subgroup of G, since $G_x = \sigma_x^{-1}(x)$, where σ_x is the following morphism

$$\sigma_x : G \to X$$
$$\sigma_x(g) = gx$$

We recall that a point x (subset W) of X is invariant under G if $gx = x$ ($gW = W$) for all $g \in G$.

Definition 1.1.3. If G is an algebraic group acting on the varieties X and Y, then we say that a morphism $\phi : X \to Y$ is a *G-morphism* if $\phi(gx) = g\phi(x)$ for all $g \in G$, $x \in X$.
In the particular case when G acts trivially on Y, the morphism ϕ is said to be *G-invariant*. Equivalently, we can say that a morphism is G-invariant if and only if it is constant on orbits.

Examples.

- $GL(n)$ is the standard example of an algebraic group.
  Indeed, $GL(n) = \{(x_{i,j}, t) \in k^{n^2+1} : det(x_{i,j})t - 1 = 0\}$.
  Hence, $GL(n) = V(f)$ where $f = det(x_{i,j})t - 1$ and it is an affine variety.
  Moreover, the group operation $GL(n) \times GL(n) \to GL(n)$ and the inverse map $GL(n) \to GL(n)$ are polynomial maps.
  In fact, if $(X, Y) \in GL(n) \times GL(n)$ then $(XY)_{i,j} = \Sigma_k x_{ik}\, y_{kj}$ and this is a polynomial. The case of the inverse is analogous, just recall the formula for the inverse of a matrix using the cofactor matrix.
  Hence these maps are morphisms of affine varieties, as desired.
  The group $GL(1) = k^*$ is usually denoted by $\mathbb{G}_m$.

- $SL(n)$ is of course a closed subgroup of $GL(n)$, hence it is an algebraic group.
  In general, we say that an algebraic group isomorphic to a closed subgroup of $GL(n)$ for some n is a *linear algebraic group*.

- The additive group k is an algebraic group, since the underlying variety is the affine line and the operations of sum and opposite are clearly polynomial maps. This group is usually denoted by $\mathbb{G}_a$.
  It is a linear algebraic group, since we have the following embedding in $GL(2)$
  $$a \longmapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

Definition 1.1.4. Let G be an algebraic group acting on a variety X. The pair (Y, ϕ) is a *categorical quotient* for the action of G on X if Y is a variety and $\phi : X \to Y$ is a G-invariant morphism which is universal, that is, any G-invariant morphism $f : X \to Z$ factors uniquely through ϕ, i.e. there exists a unique morphism $h : Y \to Z$ such that $f = h \circ \phi$, where Z is any variety.
Moreover, if the preimage of each point $y \in Y$ is a single orbit, we say that (Y, ϕ) is an *orbit space*.

Remark. $GL(n)$ acts in a natural way on k^n. Hence, if we are given an homomorphism of algebraic groups $\rho : G \to GL(n)$ we obtain an action of G on k^n given by $gv = \rho(g)v$ for any $v \in k^n$. The homomorphism ρ is called a *rational representation* of G, and the corresponding action on k^n a *linear action.*

Assume that G is an algebraic group acting on a variety X. Then we get an induced action of G on $A(X)$, where $A(X)$ is the ring of regular functions on X, in the following way: if $f \in A(X)$ we define f^g by

$$f^g(x) = f(gx)$$

It is immediate to see that the properties of an action are satisfied, hence in particular $f \longmapsto f^g$ is a k-algebra automorphism for any $g \in G$.

Lemma 1.1.1. *Let G be an algebraic group acting on a variety X, let W be a finite-dimensional vector subspace of $A(X)$(let us recall that $A(X)$ is a k-algebra, hence in particular a vector space over k). Then*

a) if W is invariant, the action of G on W is given by a rational representation,

b) W is contained in a finite-dimensional invariant subspace of $A(X)$.

Proof. a) Let $f_1, ..., f_n$ be a basis of W. Since W is invariant, $f_i^g \in W$. Hence we can write in a unique way

$$f_i^g = \sum_{i=1}^{n} \rho_{ij}(g) f_j \qquad (*)$$

where $\rho_{ij}(g) \in k$. The map $g \longmapsto (\rho_{ij}(g))$ defines a group homomorphism $\rho : G \to GL(n)$.
Moreover, the action of G on W is given by

$$(\sum_{i=1}^{n} \lambda_i f_i)^g = \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \rho_{ij}(g) f_j$$

We have to show that ρ is a morphism. Since the f_i are linearly independent, there exist $x_1, ..., x_n \in X$ such that $\det(f_j(x_k)) \neq 0$. Let $A = (a_{jk}) := (f_j(x_k))$. Then by the equation $(*)$ we get

$$(\rho_{i1}(g), ..., \rho_{in}(g)) = (f_i(gx_1), ..., f_i(gx_n))A^{-1}.$$

This implies that for any i, j the function ρ_{ij} is regular on G.

b) Again, let $f_1, ..., f_n$ be a basis of W. Let W' be the subspace of $A(X)$ generated by the f_i^g, for any $i = 1, ..., n$ and for any $g \in G$. Since $W \subseteq W'$ and W' is invariant by construction, it is enough to show that W' is finite dimensional.
Let us define $F_i \in A(G \times X)$ by $F_i(g, x) = f_i(gx)$.
Since $A(G \times X) = A(G) \otimes_k A(X)$, we can write each F_i as a finite sum

$$F_i = \sum_{j=1}^{k} G_{ij} \otimes H_{ij}$$

for suitable $G_{ij} \in A(G), H_{ij} \in A(X)$. Let W'' be the subspace of $A(X)$ generated by the H_{ij}. Then W'' is finite-dimensional.
Since

$$f_i^g(x) = F_i(g, x) = \sum_{j=1}^{k} G_{ij}(g)H_{ij}(x)$$

we have $f_i^g \in W''$ for any i and g. Hence $W' \subset W''$ and so W' is finite-dimensional.

$\square$

The previous lemma may suggest the following definition.

Definition 1.1.5. Let G be an algebraic group and R a k-algebra. A *rational action* of G on R is a map

$$\begin{aligned} R \times G &\to R \\ (f, g) &\longmapsto f^g \end{aligned}$$

such that

1) $f^{gg'} = (f^g)^{g'}$ and $f^1 = f$ for all $f \in R, g, g' \in G$

2) the map $f \longmapsto f^g$ is a k-algebra automorphism of R for all $g \in G$

3) every element of R is contained in a finite-dimensional invariant subspace on which G acts by a rational representation.

Remark. Given a rational action of G on a finitely generated k-algebra R, it makes sense to ask if the subalgebra

$$R^G = \{f \in R : f^g = f \ \forall g \in G\}$$

is finitely generated too. The answer is negative in general as shown by Nagata (see [14]), but it is true if we make a stronger assumption on the group G, as we will see soon.
This question also has a geometrical meaning: assume that X is an affine variety and that (Y, ϕ) is a categorical quotient for the action of G on X. By definition of a categorical quotient, a morphism $X \to k$ factors uniquely trough ϕ if and only if it is constant on orbits. This is equivalent to require that

$$\phi^* : A(Y) \to A(X)$$

is an isomorphism between $A(Y)$ and $A(X)^G$. Thus, if Y is affine, then $A(X)^G$ must be finitely generated.

By Lemma 1.1.1, given an action of an algebraic group on k^n we have an induced rational action of G on $k[x_1, ..., x_n]$. In this sense, we can speak of *invariant polynomials*.

Definition 1.1.6. A linear algebraic group G is *geometrically reductive* (resp. *linearly reductive*) if, for every linear action of G on k^n and every invariant point v of k^n, $v \neq 0$, there exists an invariant homogeneous polynomial f of degree ≥ 1(resp. $= 1$) such that $f(v) \neq 0$.

There is another definition of reductivity, which is given in algebraic terms.

Definition 1.1.7. A linear algebraic group G is *unipotent* if every non-trivial linear representation $\rho : G \to GL_n$ has a non zero G-invariant point.
A linear algebraic group is *reductive* if the unipotent radical of G (which is the maximal connected unipotent normal subgroup of G) is trivial.

Let us summarise the main results relating these three notions of reductivity in the following theorem, whose proof is beyond the scope of this thesis.

Theorem 1.1.1. *(Weyl, Nagata, Mumford, Haboush)*

i) Every linearly reductive group is geometrically reductive.

ii) In characteristic zero, every reductive group is linearly reductive.

iii) A linear algebraic group is reductive if and only if it is geometrically reductive.

In particular, all three notions coincide in characteristic 0.

Remark. This general results are due to the work of different mathematicians: part *ii*) was proved by Weyl, Nagata proved that every geometrically reductive group is reductive and the converse was first conjectured by Mumford and finally proved by Haboush.
By part *iii*), from now on we will always deal with geometrically reductive groups.

Let us start with a geometrical property.

Lemma 1.1.2. *Let G be a geometrically reductive group acting on an affine variety X. Let W_1, W_2 be disjoint closed invariant subsets of X.
Then there exists $f \in A(X)^G$ such that $f(W_1) = 0$ and $f(W_2) = 1$*

Proof. Let $h \in A(X)$ be such that $h(W_1) = 0$ and $h(W_2) = 1$.
By Lemma 1.1.1, the subspace of $A(X)$ generated by h^g for any $g \in G$ is invariant and finite dimensional. Let $h_1, ..., h_n$ be a basis of this subspace. Then

$$h_i^g = \Sigma a_{ij}(g)h_j$$

where the map $g \longmapsto (a_{ij}(g))$ is a rational representation for every $i, j = 1...n$. Hence we get a linear action of G on k^n. Let us define the following morphism

$$\psi : X \to k^n,\ \psi(x) = (h_1(x), ..., h_n(x))$$

Since

$$\begin{array}{rcl} \psi(gx) & = & (h_1(gx), ..., h_n(gx)) = \\ (h_1^g(x), ..., h_n^g(x)) & = & (\Sigma a_{1j}(g)h_j(x), ..., \Sigma a_{nj}(g)h_j(x)) = g\psi(x) \end{array}$$

ψ is a G-morphism. Moreover, since W_1 and W_2 are invariant, we get $\psi(W_1) = (0, ..., 0)$ and $\psi(W_2)$ is just a single point $v \neq (0, ..., 0)$.
By assumption, the group G is geometrically reductive, hence there exists $f' \in k[X_1, ..., X_n]^G$ such that $f'(v) \neq 0$ and $f'(0) = 0$ (since f' is homogeneous). Up to multiply f' by $1/f'(v)$, we may assume $f'(v) = 1$.
Let us define $f = f' \circ \psi$.
For any $g \in G$ and $x \in X$,

$$f^g(x) = f(gx) = f'(\psi(gx)) = f'(g\psi(x)) = f'(\psi(x)) = f(x)$$

where the third and the fourth equality hold since ψ is a G-morphism and f' is invariant.
Hence $f \in A(X)^G$. Moreover, $f(W_1) = f'(0) = 0$ and $f(W_2) = f'(v) = 1$ as desired. □

Remark. It can be shown that $GL(n), SL(n)$ and $PGL(n)$ are all geometrically reductive groups. These are the geometrically reductive groups that most often appear in practice.
However, even "nice" groups such as $\mathbb{G}_a$ are not geometrically reductive.
In fact, let us consider the rational representation of $\mathbb{G}_a$

$$a \longmapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

and the corresponding linear action on k^2.
We notice that the point $v = (1, 0)$ is invariant under this action.
Consider $p = (x, y) \in k^2$. Then $O(p) = \{(x + ay, y) : a \in k\}$.
Hence in this case the invariant polynomials are the ones which are constant on the lines parallel to the x-axis, so that they must be of the form $f(y)$.
This implies that any homogeneous invariant polynomial of positive degree must vanish at v, hence $\mathbb{G}_a$ is not geometrically reductive.

1.2 Nagata's Theorem

Geometrically reductive groups will play a fundamental role for our purpose, mainly due to the following result.

Theorem 1.2.1. *(Nagata) Let G be a geometrically reductive group and let R be a finitely generated k-algebra on which G acts rationally.*
Then R^G is a finitely generated k-algebra.

In order to give the proof of this theorem, we need some preliminary results.
The first step is to state and prove two lemmas, both due to Nagata.

Lemma 1.2.1. *Let G be a geometrically reductive group acting rationally on a finitely generated k-algebra R. Let J be an invariant ideal of R.*
If $f \in (R/J)^G$, then $f^t \in R^G/J \cap R^G$ for some positive integer t.

Proof. If $f = 0$ the result is trivial, so we may assume $f \neq 0$. Let $h \in R$ be an element whose image in R/J is f. We claim that there exist $h_0 \in R^G$ and a suitable integer t such that $h_0 - h^t \in J$.
Since G acts rationally on R, the subspace M generated by $h^g (g \in G)$ is finite dimensional. Let us consider $N = M \cap J$. By assumption $f \neq 0$, hence $h \notin J$ and so $h \notin N$, but $h^g - h \in N$ for any $g \in G$, since f is invariant.
M is generated by h^g and we can write $h^g = (h^g - h) + h \in N \oplus < h >$, it follows that we have dim M=dim $N + 1$.
Hence we can write any element of M in a unique way as a sum

$$ah + h' \qquad (a \in k, h' \in N)$$

Therefore, we can define a linear map $l : M \to k$ by

$$l(ah + h') = a$$

We notice that l is G-invariant since

$$(ah + h')^g = ah + a(h^g - h) + h'^g \in ah + N$$

where we used that N is G-invariant, since so are M and J. Hence we have $l^g(ah + h') = l((ah + h')^g) = a$.
Let M^* be the dual space of M. We choose a basis $h_2, ..., h_r$ of N, then $h, h_2, ..., h_r$ is a basis of M, and we may identify M^* with k^r by the dual basis. Under this identification, $l = (1, 0, ..., 0)$ and it is invariant, as we noticed before.
By assumption G is geometrically reductive, so there exists an invariant homogeneous polynomial $F \in k[x_1, ..., x_r]$ of degree $t \geq 1$ such that $F(l) \neq 0$. This condition implies that the coefficient of x_1^t in F is non-zero; up to scalar multiplication we may assume that this coefficient is 1.
Let us now consider the k-algebra homomorphism

$$\alpha : k[x_1, ..., x_r] \to R$$
$$x_1 \longmapsto h, \; x_i \longmapsto h_i \; i = 2, ..., r$$

Since the action on k^r is induced by the action on M, it follows that this homomorphism commutes with the action of G. In particular, since F is invariant, $h_0 = \alpha(F) \in R^G$.
Finally, since the coefficient of x_1^t in F is 1, the polynomial h_0 belongs to $h^t + S$ where S is the ideal of R generated by $h_2, ..., h_r$.
Then $h_0 - h^t \in S \subseteq J$, as desired. □

Lemma 1.2.2. *Let G be a geometrically reductive group acting rationally on a finitely generated k-algebra R.*
If $f_1, ..., f_s \in R^G$ and $f \in (\sum_{i=1}^{n} f_i R) \cap R^G$, then $f^t \in \Sigma f_i R^G$ for some $t \in \mathbb{N}$.

Proof. The proof will be by induction on s. For $s = 1$, let $f \in f_1R \cap R^G$; then $f = f_1f'$ and $(f_1f')^g = f_1^g f'^g = f_1f'^g$ since $f_1 \in R^G$. Hence

$$f_1(f'^g - f') = 0 \qquad (*)$$

Let us consider the ideal $J = \{h \in R : f_1h = 0\}$, the annihilator of f_1: it is invariant under G because f_1 is invariant, moreover the image of f' on R/J is invariant by $(*)$. Hence we can apply Lemma 1.2.1, obtaining $f'' \in R^G$ and $t \in \mathbb{N}$ such that the image of f'' and f'^t in R/J coincide, which is equivalent to $f'' - f'^t \in J$.
It follows that $f_1(f'' - f'^t) = 0$, which implies

$$f^t = f_1^t f'^t = f_1f'' \in f_1R^G.$$

Now assume $s > 1$. In order to simplify notation, let $\overline{R} = R/f_1R$ and $\overline{f}$ be the image of f in $\overline{R}$ for any $f \in R$.
If $f \in (\sum_{i=1}^{s} f_iR) \cap R^G$, then $\overline{f} \in (\sum_{i=2}^{s} \overline{f_i}\ \overline{R}) \cap \overline{R}^G$, so we can apply the inductive hypothesis to get a positive integer t such that

$$\overline{f}^t \in \sum_{i=2}^{r} \overline{f_i}\ \overline{R}^G.$$

So $f^t = \sum_{i=1}^{s} f_ih_i$ with $h_i \in R, \overline{h_2}, ..., \overline{h_s} \in \overline{R}^G$.
The ideal $J = f_1R$ is invariant and $\overline{h_s} \in \overline{R/J}^G$. Hence we can apply Lemma 1.2.1 to get a positive integer u and $h'_s \in R^G$ such that $\overline{h'_s} = \overline{h_s}^u$. It follows that

$$f^{tu} - f_s^u h'_s \in (\sum_{i=1}^{s-1} f_iR) \cap R^G.$$

We can apply again the inductive hypothesis and we get a positive integer v such that

$$(f^{tu} - f_s^u h'_s)^v \in \sum_{i=1}^{s-1} f_iR^G.$$

This implies $f^{tuv} \in \sum_{i=1}^{s} f_iR^G$, as required. □

Now we state a proposition containing some results of commutative algebra and field theory, which will be needed in the proof of the theorem of Nagata. Since they are not directly linked to invariant theory we won't give a full proof of them, for part (4) we refer to [8]. For this proposition, we mainly refer to well-known commutative algebra texts, such as [2] and [7].

Proposition 1.2.1. *1) Let $R = \oplus_{i\geq 0} R_i$ be a graded k-algebra, where $R_0 = k$. Then R is a finitely generated k-algebra if and only if $R_+ = \oplus_{i\geq 1} R_i$ is a finitely generated ideal of R.*

2) Let R be a finitely generated k-algebra, integral over a subalgebra S. Then S is a finitely generated k-algebra.

3) Let R be a k-algebra, integral over a subalgebra S. Assume that R is a domain, and that its field of fractions L is a finitely generated extension of k. If S is a finitely generated k-algebra, then R is a finite S-module, and so it is finitely generated as a k-algebra.

4) Let L' be a finitely generated extension of k, and L a subfield of L' containing k. Then L is a finitely generated extension of k.

Proof. 1) $\Rightarrow$) Clear, since R is Noetherian by Hilbert's basis theorem.
$\Leftarrow$) Coversely, assume R_+ is generated by $f_1, ..., f_n$ where the f_j are homogeneous of positive degree. We prove that for any i $R_i \subseteq k[f_1, ..., f_n]$, by induction on $i \in \mathbb{N}$.
If $i = 0$ there is nothing to prove, since $R_0 = k$.
Assume $f \in R_{i+1}$; by assumption we can write $f = h_1 f_1 + ...h_n f_n$ for suitable $h_j \in R$, where the h_j are homogeneous of degree $\leq i$. By the induction hypothesis $h_j \in k[f_1, ..., f_n]$ and so $f \in k[f_1, ..., f_n]$ too.

2) Let $f_1, ..., f_n$ generate R. Since R is integral over S, each f_i satisfies an equation of the form

$$f_i^{r_i} + a_{i1} f_i^{r_i - 1} + ... + a_{ir_i} = 0 \qquad (*)$$

with $a_{ij} \in S$, where $i = 1, ..., n$ and $j = 1, ..., r_i$. Let S_0 be the subalgebra of S generated by the a_{ij}. By construction S_0 is finitely generated, hence Noetherian.
By equation $(*)$, we notice that R is generated as an S_0-module by the products

$$f_1^{s_1} f_2^{s_2} ... f_n^{s_n} \ (0 \leq s_i \leq r_i - 1)$$

and so R is a finitely generated S_0-module. Since S_0 is a Noetherian ring, this implies that R is a Noetherian S_0-module, so that S is a finitely generated S_0-module. Hence S is a finitely generated k-algebra, as desired.

3) Let L' be the field of fractions of S. Then L' is a finitely generated extension of k and L is an algebraic extension of L'. By assumption L is a finitely generated extension of k, hence L is indeed a finite extension of L', being algebraic and finitely generated. So by [8] [Corollary 5.49] the integral closure S' of S in L is a finite S-module. But S is Noetherian since it is a finitely generated k-algebra, and $R \subseteq S'$, hence R is also a finite S-module.

4) See [8] [Lemma 2.25].

□

Now we are able to give the proof of Theorem 1.2.1.
Since R is finitely generated and G acts rationally on R, there exist linearly independent elements $f_1, ..., f_n \in R$ that generate R and such that the subspace of R generated by them is invariant under G. Moreover, the action of G on this subspace is given by

$$f_i^g = \sum_{j=1}^{n} a_{ij}(g) f_j$$

where $g \longmapsto (a_{ij}(g))$ is a rational representation.
Let $S = k[x_1, ..., x_n]$. We notice that there exists a unique rational action of G on S such that

$$x_i^g = \sum_{j=1}^{n} a_{ij}(g) x_j$$

for $i = 1, ..., n$ and $g \in G$.
Now consider the k-algebra homomorphism $S \to R$ sending x_i to f_i. Clearly it commutes with the actions of G on S and R, so it suffices to prove the following result.

Theorem 1.2.2. *Let G be a geometrically reductive group acting rationally on $S = k[x_1, ..., x_n]$ in such a way that it preserves the degree of any homogeneous element, let Q be an ideal in S which is invariant under G. Then we have an induced action of G on $R := S/Q$ and R^G is a finitely generated k-algebra.*

Proof. The proof will be by contradiction. First, we assume that there exists a homogeneous ideal Q of S such that R^G is not finitely generated.
Since S is Noetherian, we may assume that Q is maximal among such ideals.
Notice that since Q is homogeneous, R is a graded algebra.
Moreover, by the maximality of Q, if J is any non-zero homogeneous ideal of R, then $(R/J)^G$ is finitely generated.
By Lemma 1.2.1 $(R/J)^G$ is integral over $(R^G/J \cap R^G)$ so we can apply 2) of Proposition 1.2.1 and we have that

$$(R^G/J \cap R^G) \text{ is finitely generated.} \qquad (A)$$

Moreover, by the proof of 2) of Proposition 1.2.1, we get that

$$(R/J)^G \text{ is a finite } (R^G/J \cap R^G)\text{-module.} \qquad (B)$$

Clearly $(R^G)_+ \neq 0$ since R^G is not finitely generated, so let f be a non-zero homogeneous element of R^G of positive degree. If f is not a zero-divisor, we claim that we have

$$fR \cap R^G = fR^G$$

In fact, let $a \in fR \cap R^G$. Then $a = fh$ and $a^g = a$ for any $g \in G$. This implies $fh = (fh)^g = f^g h^g = fh^g$ for all $g \in G$, so that $f(h^g - h) = 0$ and $h^g = h$. Hence $h \in R^G$ and $a \in fR^G$, as desired.
We get the equality since the other inclusion is trivial as we have $f \in R^G$.
By (A) for $J = (f)$, the k-algebra R^G/fR^G is finitely generated, hence by part 1) of Proposition 1.2.1 $(R^G/fR^G)_+ = R^G_+/fR^G$ is a finitely generated ideal of R^G/fR^G. This implies that R^G_+ is a finitely generated ideal of R^G, so that R^G is a finitely generated k-algebra by part (1) again, which is a contradiction.
Otherwise, if f is a zero-divisor, let us consider the annihilator of f, the ideal $I = \{h \in R : fh = 0\}$. Since f is homogeneous, I is homogeneous too; hence by (A) the algebras $R^G/fR \cap R^G$ and $R^G/I \cap R^G$ are both finitely generated. Hence there exists a finitely generated subalgebra R_1 of R^G such that the natural homomorphisms from R_1 to these algebras are both surjective.
By (B) the k-algebra $(R/I)^G$ is a finite $(R^G/I \cap R^G)$-module; let $c_1, ..., c_r \in R$ be the lifts of the generators of this module.
Let $g \in G$. We have

$$(fc_i)^g = f^g c_i^g = fc_i^g = fc_i$$

where the last equality holds because $c_i^g - c_i \in I$. It follows that $fc_i \in R^G$ for any $i = 1, ..., r$, hence $R_1[fc_1, ..., fc_r] \subseteq R^G$.
We claim that actually $R^G = R_1[fc_1, ..., fc_r]$, so that R^G is finitely generated and we get the required contradiction.
Indeed, let $h \in R^G$. Since the natural map from R_1 to $R^G/fR \cap R^G$ is surjective, we get the existence of $h' \in R_1$ such that $h - h' = fb$ for some $b \in R$. Hence $fb \in R^G$ and we get

$$0 = (fb)^g - fb = f(b^g - b)$$

which implies $b^g - b \in I$, so that the image of b in R/I belongs to $(R/I)^G$. Recalling that $(R/I)^G$ is generated as an $R^G/I \cap R^G$-module by the c_i, there exist $f_1, ..., f_r \in R^G$ such that

$$b - \sum_{i=1}^{r} f_i c_i \in I.$$

Since the $f_i \in R^G$ and the natural map from R_1 to $R^G/I \cap R^G$ is surjective, we get the existence of $f_i' \in R_1$ such that $f_i - f_i' \in I$ for any i. This implies

$$b - \sum_{i=1}^{r} f_i' c_i \in I.$$

Hence we have

$$h = h' + fb = h' + \sum f f_i' c_i \in R_1[fc_1, ..., fc_r].$$

Now we consider the general case, where Q is a maximal invariant ideal in S such that R^G is not finitely generated.
If R^G contains a zero-divisor, we get a contradiction exactly as in the homogeneous case. Hence we may assume that R^G is a domain.
Now Q is not homogeneous, hence $R = S/Q$ is not a graded k-algebra, so the proof before doesn't work anymore and we have to find another way.
First, we notice that by the homogeneous case S^G is finitely generated. moreover by Lemma 1.2.1 the k-algebra $R^G = (S/Q)^G$ is integral over $S^G/Q \cap S^G$. It is therefore sufficient to show that the field of fractions L of R^G is a finitely generated extension of k, by part 3) of Proposition 1.2.1.
Let T be the set of non-zero divisors of R, let $T^{-1}R$ denote the corresponding ring of fractions. Clearly $R \subseteq T^{-1}R$ since the natural map from R into the localization is injective. Moreover, if $\mathbf{m}$ is a proper ideal of $T^{-1}R$, then $\mathbf{m} \cap R^G = 0$ since R^G is a domain. In particular, if $\mathbf{m}$ is a maximal ideal $(T^{-1}R)/\mathbf{m}$ is a field and L is a subfield, up to isomorphism.
By part 4) of Proposition 1.2.1, it suffices to show that $(T^{-1}R)/\mathbf{m}$ is a finitely generated extension of k. This is clear, since $(T^{-1}R)/\mathbf{m}$ is the field of fractions of the finitely generated k-algebra $R/\mathbf{m} \cap R$.

□

Chapter 2

Construction of quotients

Let us consider the action of an algebraic group G on a variety X. One might hope that the orbit set X/G endowed with the quotient topology could be made into a variety in such a way that the natural projection is a morphism. Unfortunately, this is not the case, even if we consider very simple actions.
For instance, assume that $\mathbb{G}_m$ acts on $\mathbb{A}^1$ by multiplication. Then there are exactly 2 orbits, namely the origin and $\mathbb{A}^1 \setminus \{0\}$. However, the origin belongs to the closure of the set of non-zero elements, so that X/G consists of 2 points and one of them is in the closure of the other. Hence it cannot be structured as a variety, since any finite algebraic set has the discrete topology.
However, we could ask if we can relax the idea of having an orbit space, in order to get quotients with better geometrical properties, such as categorical quotients.
In this chapter, we will use Nagata's Theorem in order to construct quotients in both the affine and projective case. As we will see, the situation in the affine case is quite straightforward, since there always exists a global quotient which is itself affine. Unlike the affine case, the projective one will require more attention, and the result which we will state are not global in general.
Finally, we will state the Hilbert-Mumford criterion, which will play a central role in the last chapter.
We will follow the exposition of [11] (Chapter 4) and of [15] (Sections 3.3, 3.4, 4.1, 4.2).

2.1 Affine quotients

Let us start with the simpler case, where a geometrically reductive group acts on an affine variety. The main result is the following theorem.

Theorem 2.1.1. *Let G be a geometrically reductive group which acts linearly on an affine variety X. Then there exists an affine variety Y and a morphism $\phi : X \to Y$ such that*

i) *ϕ is G-invariant*

ii) *ϕ is surjective*

iii) *if U is open in Y, then*

$$\phi^* : A(U) \to A(\phi^{-1}(U))$$

is an isomorphism of $A(U)$ onto $A(\phi^{-1}(U))^G$

iv) *if W is a closed invariant subset of X, then $\phi(W)$ is closed*

v) *if W_1, W_2 are disjoint closed invariant subsets of X, then $\phi(W_1) \cap \phi(W_2) = \emptyset$*

Proof. By the theorem of Nagata, $A(X)^G$ is a finitely generated k-algebra. Since it is a subalgebra of $A(X)$, it is reduced. Hence there exists an affine variety Y whose coordinate ring is $A(X)^G$. Let $\phi : X \to Y$ be the morphism induced by the natural inclusion $A(Y) = A(X)^G \subset A(X)$.

i) By contradiction, let us assume there exists $x \in X, g \in G$ such that $\phi(gx) \neq \phi(x)$. Let $f \in A(Y)$ such that

$$f(\phi(gx)) \neq f(\phi(x))$$

but this implies that $\phi^* f(gx) \neq \phi^* f(x)$, contradicting the fact that $\phi^* f \in A(X)^G$.

ii) Let $y \in Y$, let $f_1, ..., f_r$ generate the maximal ideal in $A(X)^G$ corresponding to the point y.
First, we notice that the f_i generate a proper ideal in $A(X)$. By contradiction, if $\sum_{i=1}^{n} f_i A(X) = A(X)$, then by Lemma 1.2.2
$1 \in \sum_{i=1}^{n} f_i A(X)^G$, contradicting the fact that $\sum_{i=1}^{n} f_i A(X)^G$ is a maximal

ideal.
Hence there exists a maximal ideal of $A(X)$ containing $\sum_{i=1}^{n} f_i A(X)$.
Let x be the point of X corresponding to this maximal ideal. Then $x \in V(\sum_{i=1}^{n} f_i A(X))$ so that $f_i(x) = 0$ for any $i = 1, ..., r$. Since the f_i are elements of $A(Y)$, it follows that

$$f_i(\phi(x)) = \phi^*(f_i(x)) = f_i(x) = 0$$

which implies $\phi(x) = y$.

iii) It suffices to prove the claim for distinguished open subsets in Y, so we may assume that $U = Y_f$ for some $f \in A(Y)$. In this case, $\phi^{-1}(U) = X_f$. Hence we have to prove that $(A(X)^G)_f = (A(X)_f)^G$ for any $f \in A(X)^G$.
Let $h/f^n \in (A(X)_f)^G$. Then, for any $g \in G$, $(h/f^n)^g = h/f^n$. Let $x \in X$.

$$(h/f^n)(gx) = h(gx)/f^n(gx) = h(gx)/f^n(x) \text{ since } f \in A(X)^G$$

This must be equal to $h(x)/f^n(x)$. But this implies that

$$f^n(x)(h(gx) - h(x)) = 0 \text{ in } k.$$

Then necessarily $h(gx) = h(x)$ and $h \in A(X)^G$.
Hence we get $(A(X)_f)^G \subseteq (A(X)^G)_f$. Since the other inclusion is trivial, we get $(A(X)^G)_f = (A(X)_f)^G$ as desired.

iv) Follows immediately by (v). If W is a closed invariant subset of X and $y \in \overline{\phi(W)} \setminus \phi(W)$, we can apply (v) to $W_1 = W$ and $W_2 = \phi^{-1}(y)$ to get $y \notin \overline{\phi(W)}$, which is a contradiction.
Notice that W_2 is indeed an invariant subset of X, since ϕ is G-invariant by (i).

v) Let W_1 and W_2 be disjoint closed invariant subsets of X. By Lemma 1.1.2 there exists $f \in A(X)^G$ such that $f(W_1) = 0$ and $f(W_2) = 1$. Since $A(X)^G = A(Y)$, we may consider f as an element of $A(Y)$ in order to get $f(\phi(W_1)) = 0$, $f(\phi(W_2)) = 1$.
This implies that $\overline{\phi(W_1)} \cap \overline{\phi(W_2)} = \emptyset$, since f is constant on both $\phi(W_1)$ and $\phi(W_2)$, so it is constant on the respective closures.

□

Corollary 2.1.1. *Let $x_1, x_2 \in X$ and let Y and $\phi : X \to Y$ be as above. Then*
$\phi(x_1) = \phi(x_2) \Leftrightarrow \overline{O(x_1)} \cap \overline{O(x_2)} \neq \emptyset$

Proof. $\Longrightarrow$) Take $W_i = \overline{O(x_i)}$. By contradiction, assume $W_1 \cap W_2 = \emptyset$. Then, by part v) of Theorem 2.1.1

$$\overline{\phi(W_1)} \cap \overline{\phi(W_2)} = \emptyset.$$

But $\phi(W_i) = \phi(x_i)$ since ϕ is G-invariant, hence we get $\phi(x_1) \neq \phi(x_2)$, contradicting the assumption.

$\Longleftarrow$) This is clear, since ϕ is constant on orbits and so it is constant on their closures. □

Now, we are going to prove that the variety Y constructed in Theorem 2.1.1 is a categorical quotient for the action of G on X. Actually, an even stronger result is true. In order to state this result, we need the following lemma.

Lemma 2.1.1. *Let X, G, Y be as in Theorem 2.1.1, and let U be an open subset of Y. If W_1, W_2 are closed disjoint invariant subsets of $\phi^{-1}(U)$, then $\phi(W_1) \cap \phi(W_2) = \emptyset$.*

Proof. Let $\overline{W_i}$ be the closure of W_i in X. Assume that there exists $y \in \phi(W_1) \cap \phi(W_2)$.
Let us consider the closed subsets $\phi^{-1}(y) \cap \overline{W_1}$ and $\overline{W_2}$ whose images under ϕ both contain the point y. Hence, by part (v) again we get

$$\phi^{-1}(y) \cap \overline{W_1} \cap \overline{W_2} \neq \emptyset$$

However, by assumption W_1 and W_2 are closed in $\phi^{-1}(U)$. Hence

$$\phi^{-1}(U) \cap \overline{W_1} \cap \overline{W_2} = W_1 \cap W_2 = \emptyset$$

and this leads to a contradiction, since clearly

$$\phi^{-1}(y) \cap \overline{W_1} \cap \overline{W_2} \subset \phi^{-1}(U) \cap \overline{W_1} \cap \overline{W_2}.$$

□

Corollary 2.1.2. *Let X, G, Y be as in Theorem 2.1.1. Let U be an open subset of Y.*
Then $(U, \phi_{|_U})$ is a categorical quotient of $\phi^{-1}(U)$ by G.

Proof. Let Z be any variety, and $\psi : \phi^{-1}(U) \to Z$ be a G-invariant morphism. We have to show that there exists a unique morphism $\chi : U \to Z$ such that $\chi \circ \phi = \psi$. Uniqueness immediately follows from the fact that ϕ is surjective, so we have only to prove existence.
We first prove that there exists a map χ such that $\chi \circ \phi = \psi$. The claim is that $\psi(\phi^{-1}(y))$ consists of one point for any $y \in U$. So, let $z_1, z_2 \in \psi(\phi^{-1}(y))$, $z_1 \neq z_2$. If we take $W_i = \psi^{-1}(z_i)$, we get two disjoint closed invariant subsets of $\phi^{-1}(U)$ such that $\phi(W_1) \cap \phi(W_2) \neq \emptyset$, contradicting lemma 2.1.1. Notice that to conclude that the W_i are invariant we used the fact that ψ is G-invariant.
Hence we can define $\chi : U \to Z$ in the following way:

$$\chi(y) := \psi(\phi^{-1}(y))$$

This is well defined and clearly satisfies $\chi \circ \phi = \psi$.
Let V be an open subset of Z. Since ϕ is surjective by part *ii*) of Thereom 2.1.1, we get

$$\chi^{-1}(V) = Y \setminus \phi[X \setminus \phi^{-1}(\chi^{-1}(V)] = Y \setminus \phi[X \setminus \psi^{-1}(V)]$$

The set $\psi^{-1}(V)$ is open since ψ is a morphism, so that $X \setminus \psi^{-1}(V)$ is closed in X. Moreover, it is invariant since ψ is G-invariant. Applying (iv) we get that $\phi(X \setminus \psi^{-1}(V))$ is closed, hence $\chi^{-1}(V)$ is open.
Therefore, in order to prove that χ is a morphism, it is enough to prove that $\chi \mid_{\chi^{-1}(V)}$ is a morphism when V is an affine open subset of Z. Since ψ is G-invariant, the image of

$$\psi^* : A(V) \to A(\psi^{-1}(V))$$

is contained in $A(\psi^{-1}(V))^G$. By part (iii) of Theorem 2.1.1

$$\phi^* : A(\chi^{-1}(V)) \to A(\phi^{-1}(\chi^{-1}(V))^G = A(\psi^{-1}(V))^G$$

is an isomorphism, so we can consider its inverse $(\phi^*)^{-1}$. Hence we get a k-algebra homomorphism

$$(\phi^*)^{-1} \circ \psi^* : A(V) \to A(\chi^{-1}(V))$$

which determines a morphism $\chi' : \chi^{-1}(V) \to V$ since V is affine. By construction of χ', we have

$$\chi' \circ (\phi \mid_{\psi^{-1}(V)}) = \psi \mid_{\psi^{-1}(V)}$$

Since $\chi \circ \phi = \psi$, we also have

$$(\chi \mid_{\chi^{-1}(V)}) \circ (\phi \mid_{\psi^{-1}(V)}) = \psi \mid_{\psi^{-1}(V)}$$

By the surjectivity of ϕ, this implies $\chi \mid_{\chi^{-1}(V)} = \chi'$, so that $\chi \mid_{\chi^{-1}(V)}$ is a morphism. $\square$

Corollary 2.1.3. *Assume we are in the same setting as Theorem 2.1.1. Moreover, let U be an open subset of Y such that the action of G on $\phi^{-1}(U)$ is closed. Then U is an orbit space for this action.*

Proof. We have already shown that U is a categorical quotient.
Let $x_1, x_2 \in \phi^{-1}(U)$ be points belonging to different orbits. We have to show $\phi(x_1) \neq \phi(x_2)$.
By Corollary 2.1.1, it suffices to show that $O(x_1) \cap O(x_2) = \emptyset$.
This follows by Lemma 2.1.1, by simply taking $W_i = O(x_i)$, where this choice makes sense since the action of G on $\phi^{-1}(U)$ is closed.

□

Examples. *Construction of affine quotients*

- Let $\mathbb{G}_m$ acts on $\mathbb{A}^n$ by scalar multiplication: $t(x_1, ..., x_n) = (tx_1, ..., tx_n)$. There are two types of orbits: punctured lines through the origin and the origin itself. We notice that in this case the origin is the only closed orbit, moreover, it belongs to the closure of every orbit.
  It follows that the invariant polynomials are indeed constant, since they must be constant on orbit closures, hence $A(\mathbb{A}^n)^{\mathbb{G}_m} = k$. By Corollary 2.1.2, the affine variety corresponding to k, which is a single point, is a categorical quotient for this action.
  However, this quotient is not an orbit space, since the action is not closed. Even worse, by Corollary 2.1.1 all the orbits are identified in the categorical quotient, since their closures contain the origin.

- Let $\mathbb{G}_m$ acts on $\mathbb{A}^2$ by $t(x, y) = (tx, t^{-1}y)$. There are four types of orbits:

  i) conics of the form $\{(x, y) : xy = \alpha\}$ for any $\alpha \in k^*$,
  ii) the punctured x-axis $\{(x, 0) : x \neq 0\}$,
  iii) the punctured y-axis $\{(0, y) : y \neq 0\}$,
  iv) the origin.

  Again, in order to construct a categorical quotient for the action, we have to determine the ring of invariants.
  Let us consider the following k-algebra homomorphism.

$$f : k[z] \to k[x, y]$$
$$z \longmapsto xy$$

We claim that this is an isomorphism onto the ring of $\mathbb{G}_m$- invariant functions.
First, it is clear from the form of the orbits that $k[xy] \subseteq A(\mathbb{A}^2)^{\mathbb{G}_m}$.
Conversely, assume that $\sum_{i,j} a_{ij}x^iy^j$ is invariant. Since the action of $\mathbb{G}_m$ on $k[x,y]$ is given by

$$t(\sum_{i,j} a_{ij}x^iy^j) = \sum_{i,j} t^{i-j}a_{ij}x^iy^j$$

it follows that $\sum_{i,j} t^{i-j}a_{ij}x^iy^j = \sum_{i,j} a_{ij}x^iy^j$ for any $t \in \mathbb{G}_m$.
This is true only if $a_{ij} = 0$ whenever $i \neq j$.
This is equivalent to say that the given polynomial belongs to $k[xy]$, as desired.
Since we have shown that the ring of invariants is $k[z]$, we are able to conclude that the pair $(\mathbb{A}^1, \phi)$ is a categorical quotient, where
$\phi : \mathbb{A}^2 \to \mathbb{A}^1$ is the morphism sending $(x, y) \longmapsto xy$, induced by the inclusion $k[xy] \subseteq k[x, y]$.

- Let $\mathbb{G}_m$ acts on the affine variety $\mathbb{A}^1 \setminus \{0\}$ by multiplication. In this simple case, we have only one orbit which is the whole $\mathbb{A}^1 \setminus \{0\}$. Then any $\mathbb{G}_m$-invariant morphism $\mathbb{A}^1 \setminus \{0\} \to Z$ must be constant, hence we have a single point as a categorical quotient.
  In this case the categorical quotient is an orbit space, since the action is trivially closed.

2.2 Projective quotients

In this section we are going to construct quotients for the actions of geometrically reductive groups on projective varieties. The idea is the following one: we would like to cover our projective variety X by affine open subsets invariant under the action of the group and glue the respective affine quotients. However, in general it is not possible to cover all of X in such a way, so we will actually construct quotients for a suitable open invariant subset of X.
We start by giving the following general definitions.

Definition 2.2.1. Let G be an algebraic group acting on a variety X.
A *good quotient* of X by G is a pair (Y, ϕ) where Y is a variety and $\phi : X \to Y$ is an affine morphism, satisfying the properties $(i) - (v)$ of Theorem 2.1.1.
A *geometric quotient* is a good quotient which is also an orbit space.

Proposition 2.2.1. *The concepts of good and geometric quotients are local with respect to Y, that is,*

1) *if (Y, ϕ) is a good (geometric) quotient of X by G and U is open in Y, then $(U, \phi_{|U})$ is a good (geometric) quotient of $\phi^{-1}(U)$ by G;*

2) *if $\phi : X \to Y$ is a morphism and $\{U_i\}$ is an open covering of Y such that for any i the pair $(U_i, \phi_{|U_i})$ is a good (geometric) quotient of $\phi^{-1}(U_i)$ by G, then (Y, ϕ) is a good (resp. geometric) quotient of X by G.*

Proof. 1) Parts $(i) - (iii)$ are clear, so we have only to show (v), since we recall that (iv) follows by (v). But (v) is just Lemma 2.1.1.

2) Parts (i) and (ii) are clear.
For (iii), let U be an open subset of Y. Notice that $\{U \cap U_i\}$ is an open covering of U.
For any i, let us consider the following commutative diagram

$$\begin{array}{ccc} A(U) & \xrightarrow{\phi^*(U)} & A(\phi^{-1}(U))^G \\ \downarrow & & \downarrow \\ A(U \cap U_i) & \xrightarrow{\phi^*(U \cap U_i)} & A(\phi^{-1}(U \cap U_i))^G \end{array}$$

where the vertical maps are the restrictions. If $f \in A(\phi^{-1}(U))^G$, then $f_{|U \cap U_i} \in A(\phi^{-1}(U \cap U_i))^G$, and since $\phi^*(U \cap U_i)$ is an isomorphism by assumption, we get $h_i \in A(U \cap U_i)$ whose image is $f_{|U \cap U_i}$.
Since the h_i are compatible on the restrictions, i.e $h_{i|U \cap U_i \cap U_j} = h_{j|U \cap U_i \cap U_j}$, and the regular functions form a sheaf, we get the existence of a unique element $h \in A(U)$ such that $h_{|U \cap U_i} = h_i$. By the commutativity of the diagram, $\phi^*(U)(h) = f$ and it is an isomorphism, as desired.

For (v), let W_1, W_2 be disjoint closed invariant subsets of X and by contradiction assume there exists $y \in \phi(W_1) \cap \phi(W_2)$. Hence there exist $w_1 \in W_1$ and $w_2 \in W_2$, such that $\phi(w_1) = \phi(w_2) = y$. Moreover, since the U_i cover Y, there exists an index j such that $y \in U_j$. Let us consider $W_1' = W_1 \cap \phi^{-1}(U_j)$ and $W_2' = W_2 \cap \phi^{-1}(U_j)$: since $w_i \in W_i'$ for $i = 1, 2$ and ϕ is G-invariant, it follows that they are non-empty disjoint closed invariant subsets of $\phi^{-1}(U_j)$. But $y \in \phi(W_1') \cap \phi(W_2')$, contradicting that $(U_j, \phi_{|U_j})$ is a good quotient of $\phi^{-1}(U_j)$ by G.

□

Proposition 2.2.2. *Let (Y, ϕ) be a good quotient of X by G. Then*

1) *(Y, ϕ) is a categorical quotient of X by G;*

2) *$\phi(x_1) = \phi(x_2) \Leftrightarrow \overline{O(x_1)} \cap \overline{O(x_2)} \neq \emptyset$;*

3) *if the action of G on X is closed, then (Y, ϕ) is a geometric quotient.*

Proof. The result follows from the proofs of Corollaries 2.1.1, 2.1.2, 2.1.3, since there we never used that our varieties were affine, but only the fact that we had a good quotient. □

Now, let us assume that X is a projective variety. In order to consider the action of a geometrically reductive group G on X, we need to talk about invariant polynomials. However, in the context of projective varieties, the action of G on X does not induce an action on $k[x_0, ..., x_n]$ or any quotient of this ring, since polynomials are not functions on $\mathbb{P}^n$. We therefore make the following definition.

Definition 2.2.2. A *linearisation* of an action of an algebraic group G on a projective variety X is a linear action of G on k^{n+1} which induces the given action on X.
A *linear action* of G on X is an action of G together with a linearisation.

Clearly, a linear action of G on X induces an action on $k[x_0, ..., x_n]$. Hence the following definition makes sense.

Definition 2.2.3. Let X be a projective variety in $\mathbb{P}^n$. For any linear action of a geometrically reductive group G on X, a point $x \in X$ is called

- *semi-stable* if there exists an invariant homogeneous polynomial f of positive degree such that $f(x) \neq 0$;

- *stable* if dim $O(x) =$ dim G and there exists an invariant homogeneous polynomial f of positive degree such that $f(x) \neq 0$ and the action of G on X_f is closed.

Remark. We denote by X^{SS}(resp. X_0^S) the set of semi-stable (resp. stable) points of X. These sets do not depend only on the action of G on X, but also on the embedding of X in $\mathbb{P}^n$ and the chosen linearisation of the action. Notice that a point is not semi-stable if all non-constant invariant homogeneous polynomial vanish at the point. The set of non-semi-stable points is called the *nullcone*, denoted by N, and it is closed, since $N = V(S(X)_+^G)$ where $S(X)$ is the homogeneous coordinate ring of X.

Our goal now is to show that X_0^S and X^{SS} are open in X. In order to do this, we need some general results.

Proposition 2.2.3. *Let $f : X \to Y$ be a surjective morphism of varieties, let $n = dim\ X$ and $m = dim\ Y$. Then the sets $Y_k := \{y \in Y : dim\ f^{-1}(y) \geq k\}$ are closed in Y for all integers k.*

Proof. The proof will be by induction on m. If $m = 0$ the result is trivial, so we may assume $m > 0$.
By the Theorem on the dimension of a fibre, $Y_{n-m} = Y$ and there exists a dense open subset U of Y such that dim $f^{-1}(y) = n - m$ for any $y \in U$.
Let Y' be the complement of U. Then $Y_k \subseteq Y'$ if $k > n - m$ and Y' is a proper closed subset of Y. Let Z_i denote the irreducible components of Y', then dim $Z_i < m$, so if we consider the restriction $f_i : f^{-1}(Z_i) \to Z_i$, we can conclude that Y_k is closed in Z_i by the inductive hypothesis. Hence Y_k is closed in Y, as desired.

□

Lemma 2.2.1. *Let G be an algebraic group acting on a variety X.*

i) For any $x \in X$, dim $O(x)$ = dim G − dim G_x;

ii) For any integer n, the set $\{x \in X : dim\ O(x) \geq n\}$ is open, that is, dim $O(x)$ is a lower semi-continuous function of x.

Proof. i) Observe that $O(x)$ is the image of the morphism

$$\sigma_x : G \to X$$
$$\sigma_x(g) = gx$$

and the fibres of σ_x are cosets of G_x, so that they all have dimension equal to dim G_x. By a well-known result in dimension theory, it follows that dim G = dim $O(x)$ + dim G_x.

ii) By *i)* it suffices to show that dim G_x is an upper semi-continuous function of x. Let us consider the following morphism

$$\phi : G \times X \to X \times X$$
$$(g, x) \longmapsto (gx, x).$$

By the previous proposition, it follows that the function

$$G \times X \to \mathbb{N}$$
$$P \longmapsto \text{dim } \phi^{-1}(\phi(P))$$

is upper semi-continuous.
By restricting to $\{(1,x) : x \in X\}$, which is clearly isomorphic to X, it follows that the set $\{x \in X\colon \dim \phi^{-1}(x,x) \geq n\}$ is closed in X.
We are done, since the fibre $\phi^{-1}(x,x)$ is $G_x \times \{x\} \simeq G_x$.

□

Lemma 2.2.2. *The sets X^{SS} and X_0^S are open in X.*

Proof. For X^{SS} it is clear from the remark after Definition 2.2.3, since it is the complement of the nullcone.
For X_0^S, we first observe that the set $L = \{x \in X : \dim O(x) = \dim G\}$ is open in X by part *ii)* of the previous Lemma.
Let $X^0 := \bigcup X_f$ where the union is taken over all invariant polynomials f such that the action of G on X_f is closed.
It follows that $X_0^S = L \cap X^0$ is open in X^0, which is open in X, thus the stable set is open. □

We are now ready to state the main result of this section.

Theorem 2.2.1. *Let X be a projective variety in $\mathbb{P}^n$. Then, for any linear action of a geometrically reductive group G on X,*

i) there exists a good quotient (Y, ϕ) of X^{SS} by G, and Y is projective;

ii) there exists an open subset Y^S of Y such that $\phi^{-1}(Y^S) = X_0^S$ and $(Y^S, \phi_{|X_0^S})$ is a geometric quotient of X_0^S;

iii) if $x_1, x_2 \in X^{SS}$,

$$\phi(x_1) = \phi(x_2) \Leftrightarrow \overline{O(x_1)} \cap \overline{O(x_2)} \cap X^{SS} \neq \emptyset.$$

Proof. i) Let $\tilde{X}$ be the affine cone over X, so that $A(\tilde{X}) = k[x_0, ..., x_n]/I(X)$.
By the theorem of Nagata, $A(\tilde{X})^G$ is a finitely generated k-algebra. Moreover, since the action of G on k^{n+1} is linear, the induced action on $A(\tilde{X})$ preserves the degree of any homogeneous element. Hence $A(\tilde{X})^G$ is a k-algebra graded by degree, generated by the homogeneous elements $f_1, ..., f_r$.
Let Y be the projective variety whose homogeneous coordinate ring is $A(\tilde{X})^G$.
If the f_i are not of the same degree, say f_i has degree d_i, let $d := d_1 \cdots d_r$ and let us consider

$$(A(\tilde{X})^G)^{(d)} = \oplus_{l \geq 0} A(\tilde{X})^G_{dl},$$

which is finitely generated as a k-algebra by $(A(\tilde{X})^G)_1^{(d)}$, and the corresponding projective variety Y'.
Notice that the varieties Y and Y' are naturally isomorphic, simply by taking the degree d-Veronese embedding.
Hence we have that the inclusion $A(\tilde{X})^G \subset A(\tilde{X})$ induces a well-defined rational map of projective varieties

$$\begin{aligned} \phi : X &\dashrightarrow Y \\ P &\longmapsto [f_1(P), ..., f_r(P)] \end{aligned}$$

whose indeterminacy locus is given by the points of X for which any invariant homogeneous polynomial of positive degree vanish at them, which is precisely the nullcone. Hence we get a morphism

$$\phi : X^{SS} \to Y.$$

For $f \in A(\tilde{X})^G$, the affine open subsets Y_f cover Y and by construction of ϕ we have $\phi^{-1}(Y_f) = X_f$. Moreover, notice that the X_f cover X^{SS} by definition.
Let $\tilde{X_f}$ and $\tilde{Y_f}$ denote the affine cones over X_f and Y_f respectively. We recall that if R is a graded k-algebra, R_0 denotes the subring of R given by elements of degree 0. Then we have

$$A(Y_f) = A(\tilde{Y_f})_0 = (A(\tilde{X})^G)_f)_0 = ((A(\tilde{X})_f)_0)^G = (A(\tilde{X}_f)_0)^G = A(X_f)^G$$

and so by Theorem 2.1.1 the pair $(Y_f, \phi \mid_{X_f})$ is a good quotient of X_f by G.
Finally, we apply part (2) of Proposition 2.2.1 to get that (Y, ϕ) is a good quotient of X^{SS} by G.

ii) Put $Y^S = \phi(X_0^S)$ and define Y^0 to be the union of those Y_f for which the action of G on X_f is closed.
Clearly $X_0^S \subseteq \phi^{-1}(Y^0)$ and so $Y^S \subseteq Y^0$. Let $X^0 = \phi^{-1}(Y^0)$; it follows from part 1) of Proposition 2.2.1 that $(Y^0, \phi \mid_{X^0})$ is a good quotient of X^0, but it is indeed a geometric quotient since the action of G on X^0 is closed.
We claim that we have $X_0^S = \phi^{-1}(Y^S)$.
Indeed, if $x \in \phi^{-1}(Y^S)$, then $\phi(x) \in Y^S$ so that $\phi(x) = \phi(x')$ for a suitable $x' \in X_0^S$. This implies that $O(x) \cap O(x') \neq \emptyset$ since $(Y^0, \phi \mid_{X^0})$ is a geometric quotient, hence $x = gx' \in X_0^S$, since the stable set is

invariant.
Moreover, again using that $(Y^0, \phi\mid_{X^0})$ is a geometric quotient and the fact that the stable set is invariant, we get that

$$Y^0 \setminus Y^S = \phi(X^0 \setminus X_0^S).$$

Notice that $X^0 \setminus X_0^S$ is a closed invariant subset of X^0, hence by property (iv) of a geometric quotient, $\phi(X^0 \setminus X_0^S)$ is closed in Y^0, so that Y^S is open in Y^0 and so it is open in Y.
Finally, we apply part 1) of Proposition 2.2.1 once again, to get that $(Y^S, \phi\mid_{X_0^S})$ is a geometric quotient of X_0^S, as desired.

iii) follows immediately from Proposition 2.2.2.

□

Remark. The quotient Y is usually denoted by $X//G$, in order to remind that it is not necessarily an orbit space.

Example. Consider the linear action of $\mathbb{G}_m$ on $X = \mathbb{P}^n$ by

$$t[x_0, x_1, ..., x_n] = [t^{-1}x_0, tx_1, ..., tx_n].$$

In order to construct a quotient, we have to determine the ring of invariants $k[x_0, ..., x_n]^{\mathbb{G}_m}$. Clearly, the polynomials $x_0x_1, x_0x_2, ..., x_0x_n$ are invariants.
We claim that these polynomials generate the ring of invariants.
If $f \in k[x_0, ..., x_n]$, $f = \sum_{\underline{m}=(m_0,...,m_n)} a(\underline{m})x_0^{m_0}x_1^{m_1}\cdots x_n^{m_n}$, then

$$t.f = \sum_{\underline{m}=(m_0,...,m_n)} a(\underline{m})t^{m_1+...+m_n-m_0}x_0^{m_0}x_1^{m_1}\cdots x_n^{m_n}.$$

Hence f is invariant if and only if $a(\underline{m}) = 0$ for all $\underline{m}$ such that $m_0 \neq \sum_{i=1}^{n} m_i$.
Notice that, if m satisfies $m_0 = \sum_{i=1}^{n} m_i$, then

$$x_0^{m_0}x_1^{m_1}...x_n^{m_n} = (x_0x_1)^{m_1}...(x_0x_n)^{m_n},$$

that is, if f is invariant then $f \in k[x_0x_1, ..., x_0x_n]$.
It follows that the ring of invariants is $k[x_0x_1, ..., x_0x_n]$, which is isomorphic to $k[y_0, ..., y_{n-1}]$. Hence in this case the quotient Y is $\mathbb{P}^{n-1}$.
Moreover, since we know the generators of the ring of invariants, we have an explicit expression of the rational map

$$\begin{aligned} \phi : \mathbb{P}^n &\dashrightarrow \mathbb{P}^{n-1} \\ [x_0, ..., x_n] &\longmapsto [x_0x_1, ..., x_0x_n]. \end{aligned}$$

From the expression we immediately get that the nullcone is the projective variety defined by the homogeneous ideal $(x_0x_1, ..., x_0x_n)$. It follows that

$$X^{SS} = \bigcup_{i=1}^{n} X_{x_0x_i} = \{[x_0, ..., x_n] \in \mathbb{P}^n : x_0 \neq 0 \text{ and } (x_1, ..., x_n) \neq 0\}$$

and $X^{SS} \simeq \mathbb{A}^n \setminus \{0\}$, where we are identifying X_{x_0} with $\mathbb{A}^n$ in the natural way.
Therefore we have that

$$\phi : \mathbb{A}^n \setminus \{0\} \to \mathbb{P}^{n-1}$$

is a good quotient for the action on X^{SS}. Moreover, the action of $\mathbb{G}_m$ on $\mathbb{A}^n \setminus \{0\}$ is closed, since the orbits are exactly punctured lines through the origin, hence $(\mathbb{P}^{n-1}, \phi)$ is indeed a geometric quotient of X^{SS}.

In general, the computation of the ring of invariants is difficult, so we would like to have other criterion to determine the (semi-)stability of a point. We will need the following general result. For the proof we refer to [4].

Lemma 2.2.3. *Let G be an algebraic group acting linearly on a variety X. For any $x \in X$, $O(x)$ is an open subset of $\overline{O(x)}$, hence the boundary $\overline{O(x)} \setminus O(x)$ is a union of orbits of G, all of which have dimension strictly less than dim $O(x)$.*
In particular, the orbits having minimum dimension are closed.

Proof. See [4] (1.8). □

Proposition 2.2.4. *Let G be a geometrically reductive group acting linearly on the projective variety X. A point x is stable if and only if it is semistable, $O(x)$ is closed in X^{SS} and dim $O(x)$ =dim G.*

Proof. $\Rightarrow$) Assume x is stable and let $x' \in \overline{O(x)} \cap X^{SS}$, then $\phi(x) = \phi(x')$ and so $x' \in \phi^{-1}(\phi(x)) \subset \phi^{-1}(Y^S) = X_0^S$. Since the action of G on X_0^S is closed, $x' \in O(x)$ and hence $O(x)$ is closed in X^{SS}.
$\Leftarrow$) Conversely, since x is semistable, there exists an invariant homogeneous polynomial f of positive degree such that $x \in X_f$. Since $O(x)$ is closed in X^{SS} by assumption, it is also closed in the affine open subset X_f.
By Lemma 2.2.1 the set $Z := \{z \in X_f : dim\ O(z) < dim\ G\ \}$ is closed in X_f. Hence Z and $O(x)$ are closed disjoint invariant subsets of the affine variety X_f. By Lemma 1.1.2 there exists $h \in A(X_f)^G$ such that $h(Z) = 0$, $h(O(x)) = 1$.
Since $A(X_f)$ is a quotient of $(k[x_0, ..., x_n]_f)_0$, by Lemma 1.2.1 there exist positive integers t, r and a homogeneous invariant polynomial h' such that $h^t = h'/f^r$.

Clearly $x \in X_{fh'}$ and $X_{fh'}$ is disjoint from Z, hence dim $O(y)$ =dim G for any $y \in X_{fh'}$.
It now follows from Lemma 2.2.3 that the action of G on $X_{fh'}$ is closed, since all the orbits have the same dimension, hence x is stable. □

We are now ready to give a topological criterion for stability. From now on we will always assume that G is a geometrically reductive group acting linearly on a projective variety X.

Proposition 2.2.5. *Let $x \in X$ and let $\tilde{x} \in k^{n+1}$ be a non-zero point lying over x. Then:*

i) x is semi-stable if and only if $0 \notin \overline{O(\tilde{x})}$;

ii) x is stable if and only if dim $G_{\tilde{x}} = 0$ and $O(\tilde{x})$ is closed in $\tilde{X}$.

Proof. i) If x is semi-stable, there exists a G-invariant homogeneous polynomial f such that $f(x) \neq 0$. We may view f as an invariant polynomial on $\tilde{X}$ with $f(\tilde{x}) \neq 0$. It follows that $f(\overline{O(\tilde{x})}) \neq 0$ and so $0 \notin \overline{O(\tilde{x})}$.

Conversely, if $0 \notin \overline{O(\tilde{x})}$, by Lemma 1.1.2 there exists an invariant polynomial f such that $f(0) = 0$ and $f(\overline{O(\tilde{x})}) = 1$.
We may take f to be homogeneous: in fact, if we decompose $f = f_0 + ... + f_r$ into the homogeneous components, then each f_i must be G-invariant since the action is linear, hence at least one of them does not vanish on $\tilde{x}$. It follows that x is semi-stable, as desired.

ii) If x is stable, then dim $G_x = 0$ and there exists f homogeneous invariant polynomial such that $x \in X_f$ and $O(x)$ is closed in X_f.
First, we notice that $G_{\tilde{x}} \subset G_x$, hence the stabiliser of $\tilde{x}$ is also zero dimensional. As before, we view f as a function on $\tilde{X}$ and we consider the closed subset

$$Z := \{z \in \tilde{X} : f(z) = f(\tilde{x})\}$$

of $\tilde{X}$. Hence it suffices to show that $O(\tilde{x})$ is closed in Z. The natural projection $\tilde{X} \setminus \{0\} \to X$ restricts to a finite morphism $\pi : Z \to X_f$. The preimage $\pi^{-1}(O(x))$ is closed and G-invariant, and it is the union of a finite number of orbits as π is finite. Moreover, since π is finite, every such orbit has dimension equal to dim G, and so it is closed in the preimage by Lemma 2.2.3. In particular, we get that $O(\tilde{x})$ is closed in Z.

Conversely, assume that dim $G_{\tilde{x}} = 0$ and $O(\tilde{x})$ is closed in $\tilde{X}$. Then, since the action is linear, $0 \notin O(\tilde{x}) = \overline{O(\tilde{x})}$, and thus x is semi-stable by part i). Hence there exists a non-constant homogeneous invariant polynomial f such that $x \in X_f$. As before, we consider the finite morphism $\pi : Z \to X_f$.
Since $\pi(O(\tilde{x})) = O(x)$ and π is finite (hence closed), we have that $O(x)$ is closed in X_f. Moreover, dim $O(x) =$ dim G and hence x has zero dimensional stabiliser. Since $O(x)$ is closed in X_f for every f such that $x \in X_f$, it follows that $O(x)$ is closed in $X^{SS} = \cup_f X_f$. Hence x is stable by Proposition 2.2.4.

□

2.3 Hilbert-Mumford criterion

Now we are going to give a numerical criterion which can be used to determine (semi)-stability of a point.

Definition 2.3.1. A 1-parameter subgroup ($1-$PS) of G is a non-trivial group homomorphism $\lambda : \mathbb{G}_m \to G$.

For a 1-PS λ, let $\lambda_x : \mathbb{G}_m \to X$ to be the morphism given by $\lambda_x(t) = \lambda(t)x$. There is a natural embedding of $\mathbb{G}_m$ into $\mathbb{P}^1$, namely,

$$t \longmapsto [1 : t].$$

Next, since X is a projective variety, the morphism λ_x extends uniquely to $\tilde{\lambda_x} : \mathbb{P}^1 \to X$, and we use a particular notation for this extension:

$$\lim_{t\to 0} \lambda(t)x := \tilde{\lambda_x}([1, 0])$$
$$\lim_{t\to\infty} \lambda(t)x := \tilde{\lambda_x}([0, 1])$$

Now, let $\tilde{x} \in \tilde{X}$ be a non-zero lift of x. Then we consider the morphism $\lambda_{\tilde{x}} : \mathbb{G}_m \to \tilde{X}$ given by $t \longmapsto \lambda(t)\tilde{x}$. However, $\tilde{X}$ is not projective, hence this morphism may or may not extend to $\mathbb{P}^1$. If it does, as before we denote the limits by

$$\lim_{t\to 0} \lambda(t)\tilde{x}$$
$$\lim_{t\to\infty} \lambda(t)\tilde{x}.$$

Since $\mathbb{P}^1$ is complete, $\lambda_{\tilde{x}}(\mathbb{P}^1)$ is closed in $\tilde{X}$, hence $\overline{\lambda_x(\mathbb{G}_m)} \subseteq \lambda_{\tilde{x}}(\mathbb{P}^1)$, so that any point in the boundary $\overline{\lambda_x(\mathbb{G}_m)} \setminus \lambda_x(\mathbb{G}_m)$ must be one of the two limit points.
Since the action of $\mathbb{G}_m$ on k^{n+1} is linear, it is diagonalisable; therefore there exists a basis $e_0, ..., e_n$ of k^{n+1} such that for any i

$$\lambda(t)e_i = t^{r_i}e_i$$

where $r_i \in \mathbb{Z}$. With respect to this basis, $\tilde{x} = \sum_{i=0}^{n} \tilde{x}_i e_i$ and therefore

$$\lambda(t)\tilde{x} = \sum_{i=0}^{n} t^{r_i}\tilde{x}_i e_i.$$

We let the λ-weights of x to be the integers r_i such that $x_i \neq 0$. We notice that these integers do not depend on the choice of the lift $\tilde{x}$. In fact, by taking $a\tilde{x}$ for any $a \in k^*$ we get

$$\lambda(t)a\tilde{x} = \sum_{i=0}^{n} t^{r_i}a\tilde{x}_i e_i$$

and $ax_i \neq 0 \Leftrightarrow x_i \neq 0$.

Definition 2.3.2. The Hilbert-Mumford weight of x at λ is

$$\mu(x,\lambda) := \max\{-r_i : x_i \neq 0\}$$

Remark. The Hilbert-Mumford weight $\mu(x,\lambda)$ is the unique integer μ such that $\lim_{t\to 0} t^{\mu}\lambda(t)\tilde{x}$ exists and it is non-zero.
In fact, let $\tilde{x}$ be a non-zero lift and assume as before that we have chosen a basis such that the action of $\lambda(t)$ is given by

$$\lambda(t)\tilde{x} = \lambda(t)(x_0,...,x_n) = (t^{r_0}x_0,...,t^{r_n}x_n).$$

By definition of $\mu(x,\lambda)$, we get $\mu(x,\lambda) + r_i \geq 0$ for any i such that $x_i \neq 0$, with equality for at least one index i. It follows that

$$\tilde{y} := \lim_{t\to 0} t^{\mu(x,\lambda)}\lambda(t)\tilde{x} = (y_0,...,y_n)$$

where $y_i = x_i$ if $r_i = -\mu(x,\lambda)$ and 0 otherwise. Hence $\tilde{y}$ exists and it is non-zero.

Lemma 2.3.1. *Let λ be a 1-parameter subgroup and $x \in X$. The Hilbert-Mumford weight has the following properties:*

i) $\mu(x,\lambda) < 0 \Leftrightarrow \lim_{t\to 0}\lambda(t)\tilde{x} = 0$*;*

ii) $\mu(x,\lambda) = 0 \Leftrightarrow \lim_{t\to 0}\lambda(t)\tilde{x}$ *exists and it is non-zero;*

iii) $\mu(x,\lambda) > 0 \Leftrightarrow \lim_{t\to 0}\lambda(t)\tilde{x}$ *does not exist.*

Proof. i) $\mu(x,\lambda) < 0$ if and only if $r_i > 0$ for any i.
Since $\lambda(t)\tilde{x} = (t^{r_0}x_0,...,t^{r_n}x_n)$, this is equivalent to $\lim_{t\to 0}\lambda(t)\tilde{x} = 0$.

ii) This is just the previous remark.

iii) $\mu(x, \lambda) > 0$ if and only if there exists $r_i < 0$ such that $x_i \neq 0$, which is equivalent to say that $\lim_{t\to 0} \lambda(t)\tilde{x}$ does not exist.

$\square$

We notice that $\lim_{t\to\infty} \lambda(t)\tilde{x} = \lim_{t\to 0} \lambda^{-1}(t)\tilde{x}$, where $\lambda^{-1}(t) = \lambda(t)^{-1}$. In particular, from the previous lemma we get:

i) $\mu(x, \lambda^{-1}) < 0 \Leftrightarrow \lim_{t\to\infty} \lambda(t)\tilde{x} = 0$;

ii) $\mu(x, \lambda^{-1}) = 0 \Leftrightarrow \lim_{t\to\infty} \lambda(t)\tilde{x}$ exists and it is non-zero;

iii) $\mu(x, \lambda^{-1}) > 0 \Leftrightarrow \lim_{t\to\infty} \lambda(t)\tilde{x}$ does not exist.

Following the discussion above and the topological criterion (Proposition 2.2.5) we get the following result for (semi-)stability with respect to the action of $\lambda(\mathbb{G}_m)$ on X.

Proposition 2.3.1. *Let G be a geometrically reductive group acting linearly on the projective variety X and let $x \in X$. Assume that λ is a 1-parameter subgroup of G. Then:*

i) x is semi-stable for the action of $\lambda(\mathbb{G}_m)$ if and only if $\mu(x, \lambda) \geq 0$ and $\mu(x, \lambda^{-1}) \geq 0$;

ii) x is stable for the action of $\lambda(\mathbb{G}_m)$ if and only if $\mu(x, \lambda) > 0$ and $\mu(x, \lambda^{-1}) > 0$;

Proof. i) By the topological criterion, x is semi-stable if and only if $0 \notin \overline{\lambda_{\tilde{x}}(\mathbb{G}_m)}$. As we already noticed, any point in the boundary $\overline{\lambda_{\tilde{x}}(\mathbb{G}_m)} \setminus \lambda_{\tilde{x}}(\mathbb{G}_m)$ is either

$$\lim_{t\to 0} \lambda(t)\tilde{x} \text{ or } \lim_{t\to\infty} \lambda(t)\tilde{x} = \lim_{t\to 0} \lambda^{-1}(t)\tilde{x}.$$

Hence $0 \notin \overline{\lambda_{\tilde{x}}(\mathbb{G}_m)}$ if and only if the limits do not exist or they exist and they are non-zero. By Lemma 2.3.1 this is equivalent to $\mu(x, \lambda) \geq 0$ and $\mu(x, \lambda^{-1}) \geq 0$.

ii) By the topological criterion, x is stable if and only if $\dim \lambda(\mathbb{G}_m)_{\tilde{x}} = 0$ and $\lambda_{\tilde{x}}(\mathbb{G}_m)$ is closed. The orbit is closed if and only if the boundary is empty, that is, if and only if both limits

$$\lim_{t\to 0} \lambda(t)\tilde{x} \text{ or } \lim_{t\to\infty} \lambda(t)\tilde{x} = \lim_{t\to 0} \lambda^{-1}(t)\tilde{x}$$

do not exist, which by Lemma 2.3.1 is equivalent to $\mu(x, \lambda) > 0$ and $\mu(x, \lambda^{-1}) > 0$.
Moreover, if these inequalities hold, then $\lambda(\mathbb{G}_m)$ cannot stabilise $\tilde{x}$ (otherwise the limits would both exist) and so $\lambda(\mathbb{G}_m)_{\tilde{x}}$ is a proper closed subset of $\lambda(\mathbb{G}_m)$. Hence, we must have dim $\lambda(\mathbb{G}_m)_{\tilde{x}} = 0$.

□

Remark. If x is (semi-)stable for G, then it is (semi-)stable for all subgroups H of G, as every G-invariant function is clearly H-invariant.
Hence, by Proposition 2.3.1 we get

$$x \text{ is semi-stable} \Longrightarrow \mu(x, \lambda) \geq 0 \ \forall \text{ 1-parameter subgroup } \lambda \text{ of } G;$$
$$x \text{ is stable} \Longrightarrow \mu(x, \lambda) > 0 \ \forall \text{ 1-parameter subgroup } \lambda \text{ of } G.$$

Theorem 2.3.1 (Hilbert-Mumford Criterion)**.** *Let G be a geometrically reductive group acting on a projective variety X, let $x \in X$. Then*

1. *x is semi-stable $\Leftrightarrow \mu(x, \lambda) \geq 0 \ \forall$ 1-parameter subgroup λ of G;*

2. *x is stable $\Leftrightarrow \mu(x, \lambda) > 0 \ \forall$ 1-parameter subgroup λ of G.*

This is equivalent to say that if x is not stable, then there exists a 1-parameter subgroup λ such that $\mu(x, \lambda) \leq 0$ and analogously if x is not semi-stable. These statements are at least plausible if G has enough 1-parameter subgroups; in fact this is the case if G is geometrically reductive.
We will give the proof in the particular case where $G = SL(n)$. In order to achieve this goal, we need some preliminary results.
First we recall a general definition:

Definition 2.3.3. A morphism of varieties $f : X \to Y$ is called proper if, for every variety Z, the morphism

$$f \times 1_Z : X \times Z \to Y \times Z$$

is closed.

It is well-known that a finite morphism is proper.
Notice that a variety X is complete if and only if the morphism $X \to \{pt\}$ is proper. In particular, this implies that the preimage of a complete variety under a proper morphism is again complete.
In the proof of the criterion we will use the following result.

Lemma 2.3.2. *Let G be a linear algebraic group(not necessarily geometrically reductive) acting on arbitrary variety X. Then the morphism*

$$\sigma_x : G \to X$$
$$\sigma_x(g) = gx$$

is proper if and only if $O(x)$ is closed in X and G_x is finite.

Proof. $\Rightarrow$) If σ_x is proper, then $O(x)$ is clearly closed, being the image of σ_x. Moreover, as we already noticed $G_x = \sigma_x^{-1}(x)$ is both complete and affine, so it must be finite.
$\Leftarrow$) Conversely, assume $O(x)$ is closed and G_x is finite. In order to prove that σ_x is proper, it suffices to show that the induced morphism $\sigma'_x : G \to O(x)$ is finite.
By assumption, σ'_x has finite fibres, but being finite is a much stronger property.
Actually, it suffices to find a non-empty open subset U in $O(x)$ such that the restriction $\sigma'_x \mid_{\sigma'^{-1}_x(U)}$ is finite; since in this case we can use the action of G to cover $O(x)$ by similar open subsets.
Hence the proof follows from this more general fact. □

Proposition 2.3.2. *Let $f : X \to Y$ be a dominant morphism of varieties such that f has finite fibres. Then there exists a non-empty open subset U in Y such that*

$$f \mid_{f^{-1}(U)}: f^{-1}(U) \to U$$

is finite.

Proof. Let V, V' be non empty open affine subsets of X, Y respectively, such that $f(V) \subset V'$ and consider the induced homomorphism

$$f^* : A(V') \to A(V).$$

Since f is dominant, f^* is injective. Moreover f has finite fibres, so we have $\dim V = \dim V'$. It follows that every element g of $A(V)$ satisfies an equation of the form

$$g^r + f^*(a_1)g^{r-1} + ... + f^*(a_r) = 0$$

for some $a_1, ..., a_r$ belonging to the quotient field of $A(V')$. Notice that $f^*(a_i)$ is well defined since f^* is injective.
We can apply this to a finite set of generators of $A(V)$. Let h be an element of $A(V')$ such that $ha_i \in A(V')$ for every a_i of any of the finite number of equations (for instance, we can just take the product of all the denominators of a_i). It follows that $A(V)_{f^*(h)}$ is integral over $A(V')_h$, which is equivalent to say that the restriction

$$f : V_{h \circ f} \to V'_h$$

is finite. Since $X \setminus (V_{h\circ f})$ is a proper closed subset of X, it follows that

$$\dim X \setminus (V_{h\circ f}) < \dim X = \dim Y;$$

the desired result follows by taking $U = V'_h \cap (Y \setminus \overline{f(X \setminus V_{h\circ f})})$. □

Now we can go back to our situation of a geometrically reductive group G acting linearly on a projective variety X.

Corollary 2.3.1. *Let $x \in X$ and $\tilde{x}$ be a non-zero point lying over x. Then x is stable if and only if the morphism*

$$\begin{gathered} \sigma_{\tilde{x}} : G \to k^{n+1} \\ \sigma_{\tilde{x}}(g) = g\tilde{x} \end{gathered}$$

is proper.

Proof. This result follows immediately from the previous lemma and the topological criterion for stability. □

In the proof of the Hilbert-Mumford criterion we will use the valuative criterion for properness, that we now state in a form which is suitable for our purpose.
Let us denote by R the ring of formal power series $k[[T]]$ and let $K = k((T))$ be the field of fractions of R. For any variety X, we can consider R-valued and K-valued points of X. In particular, if X is affine, we can just consider generators of $I(X)$ and then take common zeroes of these polynomials in R^n and K^n (this makes sense since $k \subset R$). We denote these sets of points by X_R and X_K respectively.
Notice that there is a natural map $X_R \to X$ obtained by substituting 0 for the indeterminate T.

Theorem 2.3.2 (Valuative criterion for properness)**.** *Let $f : X \to Y$ be a morphism of varieties. Then*

i) if $y \in \overline{f(X)}$ there exists $\overline{x} \in X_K$ such that $f_K(\overline{x}) \in Y_R$, $(f_K(\overline{x}))_{T=0} = y$;

ii) if f is not proper, there exists $\overline{x} \in X_K$ such that $f_K(\overline{x}) \in Y_R$ and $x \notin X_R$.

The geometric idea behind this result is the following one: a morphism $f : X \to Y$ is proper if given any smooth curve C in X and any point $p \in C$, any morphism $C \setminus \{p\} \to X$ can be extended uniquely to a morphism $C \to X$ in such a way that the following diagram commutes:

$$\begin{array}{ccc} C \setminus \{p\} & \longrightarrow & X \\ \downarrow & \overset{\exists!}{\nearrow} & \downarrow f \\ C & \longrightarrow & Y \end{array}$$

Then our version can be obtained by localising the construction above. Full details, in a more general setting, can be found in [9] [Chapter 2, Theorem 7.3.8].

Lemma 2.3.3. *Let R and K be as in the valuative criterion and let M be any $n \times n$-matrix with entries in K. Then there exist $A, B \in SL(n, R)$ such that $M = ADB$ where $D = diag(a_1, ..., a_n)$ and a_i divides a_{i+1} for any i.*

Proof. Clearly we may assume $M \neq 0$ and let m_{ij} be the entry with minimum valuation. After multiplying on left and right by permutation matrices, we may assume that this entry is exactly m_{11}. We can write

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ y_2 & 1 & 0 & 0 & \cdots & 0 \\ y_3 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & \\ y_n & 0 & \cdots & & 0 & 1 \end{pmatrix} M \begin{pmatrix} 1 & z_2 & z_3 & z_4 & \cdots & z_n \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & \\ 0 & \cdots & & & 0 & 1 \end{pmatrix} = \begin{pmatrix} m_{11} & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & * & \cdots & * \\ 0 & * & \cdots & * \end{pmatrix}$$

where $y_i = -m_{i1}/m_{11}$ and $z_j = -m_{1j}/m_{11}$.
Notice that by the way we choose m_{11}, both y_i and z_j belongs to R. Thus both of the matrices on the left hand side belong to $SL(n, R)$.
Now we just repeat the same procedure on the $(n-1)\times(n-1)$ submatrix on the right hand side, until we obtain a diagonal matrix $D = diag(m_1, ..., m_n)$ where for any i the valuation of m_i is at most the valuation of m_{i+1}, as desired. □

Now we are ready to give the proof of the Hilbert-Mumford criterion.

Proof. (Theorem 2.3.1, $G = SL(m)$) Assume that x is not stable. Then, by Corollary 2.3.1 $\sigma_{\tilde{x}}$ is not proper, so we can apply part $ii)$ of the valuative criterion for properness to get $\overline{g} \in SL(m, K)$ such that $\overline{g}\tilde{x} \in R^{n+1}$ and $\overline{g} \notin SL(m, R)$.
By Lemma 2.3.3 we get

$$\overline{g} = \overline{g_1}\overline{h}\overline{g_2},$$

where $\overline{g_1}, \overline{g_2} \in SL(m, R)$ and $\overline{h} = diag(a_0, ..., a_{m-1})$ where $a_i \in K$. Since R is a DVR with maximal ideal (T), then for any i we have $a_i = u_i T^{r_i}$, where u_i is a unit of R and r_i is a suitable integer. Let us consider the matrix $D = diag(u_0^{-1}, ..., u_{m-1}^{-1})$. Notice that $D \in SL(m, R)$ since det $\overline{h} = 1$, moreover

$$\overline{g} = \overline{g_1} D^{-1}(D\overline{h})\overline{g_2}$$

and we may assume $\overline{h} = diag(T^{r_0}, ..., T^{r_{m-1}})$ up to multiplying by D and D^{-1} as above.
Since det $\overline{h} = 1$, then it follows that $r_0 + ... + r_{m-1} = 0$. Moreover, not all $r_i = 0$, since otherwise $\overline{g}$ would belong to $SL(m, R)$.
Let g_2 be the element of $SL(m)$ obtained by substituting $T = 0$ in $\overline{g_2}$, and define a 1-parameter subgroup of $SL(m)$ in the following way:

$$\lambda(t) = g_2^{-1} diag(t^{r_0}, ..., t^{r_{m-1}})g_2.$$

Let $e_0, ..., e_n$ be a basis of k^{n+1} diagonalising λ; so that there exist $l_0, ..., l_n \in \mathbb{Z}$ such that $\lambda(t)e_i = t^{l_i}e_i$.
So $\tilde{x} = \sum_{i=0}^{n} \tilde{x}_i e_i$ and we have to prove that $\mu(x, \lambda) \leq 0$, that is, if $\tilde{x}_i \neq 0$ then $l_i \geq 0$.
Notice that we may regard e_i as a basis of K^{n+1}; by definition of λ then we get

$$g_2^{-1}\overline{h}g_2 e_i = T^{l_i}e_i.$$

Now we have

$$g_2^{-1}\overline{g_1}^{-1}\overline{g}\tilde{x} = g_2^{-1}\overline{g_1}^{-1}(\overline{g_1}\overline{h}\overline{g_2})\tilde{x} = (g_2^{-1}\overline{h}g_2)g_2^{-1}\overline{g_2}\tilde{x},$$

and so

$$(g_2^{-1}\overline{g_1}^{-1}\overline{g}\tilde{x})_i = T^{l_i}(g_2^{-1}\overline{g_2}\tilde{x})_i.$$

Finally we get

$$(g_2^{-1}\overline{g_2}\tilde{x})_i = T^{-l_i}(g_2^{-1}\overline{g_1}^{-1}\overline{g}\tilde{x})_i \in T^{-l_i}R. \qquad (*)$$

Notice that in order to conclude that $(g_2^{-1}\overline{g_1}^{-1}\overline{g}\tilde{x})_i \in R$, we used the assumption $\overline{g}\tilde{x} \in R^{n+1}$.
The left hand side of $(*)$ belongs to R and has constant term $\tilde{x}_i$ by definition of g_2.
Now assume that $\tilde{x}_i \neq 0$. Then by $(*)$ it necessarily follows $l_i \geq 0$, as desired.

Now assume x is not semi-stable. Then we argue exactly as before to get equation $(*)$. Moreover, by the topological criterion x is not semi-stable if and only if $0 \in \overline{O(x)}$, so by part $i)$ of the valuative criterion for properness we may further assume that $(\overline{g}\tilde{x})_{T=0} = 0$.
Hence in this case the right hand side of $(*)$ belongs to $T^{-l_i+1}R$, so we deduce that if $\tilde{x}_i \neq 0$ then $l_i > 0$, which is equivalent to $\mu(x, \lambda) < 0$. $\square$

Now we give a simple application of the Hilbert-Mumford criterion.

Example. We consider again the action of $\mathbb{G}_m$ on $X = \mathbb{P}^n$ given by $t[x_0, ..., x_n] = [t^{-1}x_0, tx_1, ..., tx_n]$. Since $G = \mathbb{G}_m$, by Proposition 2.3.1 it suffices to compute $\mu(x, \lambda)$ and $\mu(x, \lambda^{-1})$ where λ is just the identity of $\mathbb{G}_m$.
Assume $\tilde{x} = (x_0, ..., x_n)$ lies over $x \in \mathbb{P}^n$. Then

$$\lim_{t \mapsto 0} \lambda(t)\tilde{x} = (t^{-1}x_0, tx_1, ..., tx_n)$$

exists if and only if $x_0 = 0$. In such a case, $\mu(x, \lambda) = -1$ and otherwise $\mu(x, \lambda) = 1$. Similarly

$$\lim_{t \mapsto 0} \lambda^{-1}(t)\tilde{x} = (tx_0, t^{-1}x_1, ..., t^{-1}x_n)$$

exists if and only if $x_1 = ... = x_n = 0$. In this case, $\mu(x, \lambda^{-1}) = -1$ and otherwise $\mu(x, \lambda^{-1}) = 1$.
Therefore, we get that a point $x \in \mathbb{P}^n$ is stable if and only if it is semi-stable and we have

$$X^{SS} = X_0^S = \{[x_0, ..., x_n] : x_0 \neq 0 \text{ and } (x_1, ..., x_n) \neq (0, ..., 0)\},$$

which agrees with what we have already computed using the invariants.

Chapter 3

Projective hypersurfaces

In this chapter we will consider the problem of classifying projective hypersurfaces of a fixed degree d up to a linear change of coordinates.
More precisely, let $G = SL_{n+1}$ acts on $\mathbb{A}^{n+1}$ in the natural way. This action induces an action on the subspace $k[x_0, ..., x_n]_d$ of homogeneous polynomials of degree d for any positive integer d. This space has dimension

$$N = \binom{n+d}{d}$$

and thus it can be identified with $\mathbb{A}^N$.
A non-zero homogeneous polynomial F of degree d defines a projective hypersurface in $\mathbb{P}^n$ given by $V(F)$. If F is irreducible, then this will be a variety (irreducible).
Since any non-zero scalar multiple of a homogeneous polynomial define the same hypersurface, we consider the following projective space:

$$\mathrm{Hyp}_d(n) := \mathbb{P}(k[x_0, ..., x_n]_d) \simeq \mathbb{P}^{N-1}.$$

A point of this space is called a *hypersurface* of degree d.
Clearly, the action of SL_{n+1} on $k[x_0, ..., x_n]_d$ induces a linear action of SL_{n+1} on $\mathrm{Hyp}_d(n)$ given by

$$(g \cdot F)(p) = F(g^{-1} \cdot p) \text{ for any } g \in SL_{n+1}, F \in \mathrm{Hyp}_d(n).$$

Now we are going to study this action, and we will try to describe the semistable and stable points.
For this chapter, we mainly refer to [5] (Chapter 10), [11] (Chapter 7) and [13] (Section 4.2) .

3.1 Smooth Hypersurfaces

Definition 3.1.1.
- Let $p \in \mathbb{P}^n$ and $F \in k[x_0, ..., x_n]_d$. The point p is *singular* for the hypersurface $V(F)$ if

$$F(p) = 0 \text{ and } \frac{\partial F}{\partial x_i}(p) = 0 \text{ for any } i = 0, ..., n.$$

- The hypersurface is *smooth* or *non-singular* if it has no singular points.

Remark. By the Euler formula we get

$$\sum_{i=0}^{n} x_i \frac{\partial F}{\partial x_i} = dF.$$

So if we assume that $char(k)$ is coprime to d, we see that p is singular if and only if $\frac{\partial F}{\partial x_i}(p) = 0$ for any i.

Let Δ be the resultant of the polynomials $\frac{\partial F}{\partial x_i}$. As usual, we will call Δ the discriminant of F. Then Δ is a homogeneous polynomial in the coefficients of F, and $\Delta(F) = 0$ if and only if the polynomials $\frac{\partial F}{\partial x_i}$ have a root in common, that is, if and only if $V(F)$ is singular.
Moreover, since the property of being singular is independent from the choice of coordinates, it follows that Δ is SL_{n+1}-invariant.
Hence we obtained the following result.

Theorem 3.1.1. *Let us assume $char(k)$ and d are coprime. Every smooth hypersuface is a semi-stable point of* $\mathrm{Hyp}_d(n)$.

Proof. Let F be the polynomial corresponding to the hypersurface. From the previous observation, Δ is an invariant homogeneous polynomial and $\Delta(F) \neq 0$, since the hypersurface is smooth. □

The projective automorphism group of a hypersurface is the subgroup of PGL_{n+1} which leaves this hypersurface invariant.
It is known that for $d \geq 3$ the projective automorphism group of any smooth hypersurface of degree d is finite; this is a classical but non-trivial result. For a proof, see [12] Theorem 5.23.
Hence we get the following stronger result.

Theorem 3.1.2. *Let $d \geq 3$. Any smooth hypersurface is a stable point of* $\mathrm{Hyp}_d(n)$.

Example. We assume $\mathrm{char}(k) \neq 2$. Let us consider $\mathrm{Hyp}_2(n)$, i.e the space of quadric hypersurfaces in $\mathbb{P}^n$. The space $k[x_0, ..., x_n]_2$ is the space of quadratic forms

$$F = \sum_{0 \leq i,j \leq n} a_{ij} x_i x_j,$$

where $a_{ij} = a_{ji}$, or equivalently, the space of symmetric $(n+1) \times (n+1)$ matrices, identifying F with the matrix $A = (a_{ij})$. In this particular case, the discriminant Δ corresponds to the determinant of the matrix A, thus a quadric is smooth if and only if the rank of the associated matrix is $n+1$. By Theorem 3.1.1, we get that a smooth quadric is semi-stable.
From the theory of quadratic forms, every quadric whose associated matrix has rank $r+1$ is projectively equivalent to

$$x_0^2 + ... + x_r^2,$$

which implies that there are exactly n orbits, each one determined by the rank.
Notice that we are considering the action of SL_{n+1}, hence we are allowed to consider only linear transformations of determinant 1, but since we are considering polynomials up to a non-zero scalar multiple, the result is the same.
Let us consider the non-degenerate form

$$F = x_0^2 + ... + x_n^2;$$

and let us look to its stabilizer. If $g \in SL_{n+1}$, then $F(x) = x^T x$ and we have

$$gF(x) = F(g^{-1}x) = x^T (g^{-1})^T g^{-1} x.$$

Hence g stabilizes F if and only if

$$(g^{-1})^T g^{-1} = \mathbb{I}_{n+1}.$$

It follows that the stabilizer is the special orthogonal group SO_{n+1}, which is positive-dimensional. This implies that a smooth quadric cannot be stable.
Moreover, we claim that the only semi-stable points are the smooth quadrics.
Let h be an invariant homogeneous polynomial of degree s in the variables a_{ij}, and let F be a non singular quadratic form. Then there exists a $\mu \in k$ such that the homogeneous polynomial

$$h^{n+1} - \mu\Delta^s \text{ takes the value 0 at } F.$$

Since this polynomial is invariant and we have a unique orbit containing all the non singular quadratic forms, we have $(h^n - \mu\Delta^s)(O(F)) = 0$. We notice that this orbit is open, since it is exactly $\mathrm{Hyp}_2(n)_\Delta$, so we must have $h^n - \mu\Delta^s \equiv 0$ because the orbit is dense.
Finally, since Δ is irreducible, we get $h = \lambda\Delta^r$ for suitable $\lambda \in k$ and integer r.
In particular, for points outside $\mathrm{Hyp}_2(n)_\Delta$ every invariant is zero, hence these points are not semi-stable.
Thus we obtained that a quadric is never stable and it is semi-stable if and only if it is smooth.
Hence $\mathrm{Hyp}_2(n)//SL_{n+1}$ is a single point: this geometrically means that every non-degenerate quadratic form is equivalent to $x_0^2 + ... + x_n^2$, as already observed.

3.2 Hilbert-Mumford criterion for hypersurfaces

In order to determine the (semi)-stable locus for the action of SL_{n+1} on $\mathrm{Hyp}_d(n)$ we can use the Hilbert-Mumford criterion.

Remark 3.2.1. It follows from the definition of the Hilbert-Mumford weight that

$$\mu(gx, \lambda) = \mu(x, g^{-1}\lambda g) \text{ for any } g \in G.$$

This allows us to replace λ by a suitable conjugate in calculations. This is particularly convenient in our case where $G = SL_{n+1}$: in fact, since any 1-parameter subgroup is diagonalisable, it is conjugate to one of the form $\lambda(t) =$diag$(t^{r_0}, ..., t^{r_n})$ where the r_i are integers such that $\sum\limits_{i=0}^{n} r_i = 0$ and $r_0 \geq r_1 \geq ... \geq r_n$. Then the action of λ is diagonal with respect to the standard basis of $k[x_0, ..., x_n]_d$ given by monomials. Furthemore, given $I = (i_0, ..., i_n)$, the weight of the monomial

$$x_I = x_0^{i_0} x_1^{i_1} ... x_n^{i_n}$$

is precisely $-\sum\limits_{j=0}^{n} r_j i_j$, where the negative sign arises since we act by the inverse of $\lambda(t)$.
Thus, given $F = \sum a_I x_I \in \mathrm{Hyp}_d(n)$, we get that

$$\mu(F, \lambda) = \max\{\sum_{j=0}^{n} r_j i_j : I = (i_0, ..., i_n) \text{ and } a_I \neq 0\}.$$

For general values of d and n it is very difficult to give a complete description of the semi-stable locus. However, we will see that for certain small values, such a description can be given.

Example. Let us start with the easiest case, when $d = 1$. In this case, $\mathrm{Hyp}_1(n)$, the space of hyperplanes in $\mathbb{P}^n$, is isomorphic to $\mathbb{P}^n$.
We claim that $(\mathbb{P}^n)^{SS} = \emptyset$. Since the action of SL_{n+1} on $\mathbb{P}^n$ is transitive, i.e there is only one orbit, it suffices to show that the point $x = [1, 0, ..., 0]$ is not semi-stable. Let us consider the $1-$parameter subgroup $\lambda(t) = (t, t^{-1}, 1, ..., 1)$: it is clear that $\mu(x, \lambda) = -1 < 0$, thus x is not semi-stable.

We already considered the case $d = 2$. It is worth noting that the cases $d = 1, 2$ are "special", in the sense that their behaviour is known for any n.

3.3 Binary forms of degree d

Now we are going to study the case of *binary forms* of degree d.

Definition 3.3.1. A binary form of degree d is a degree d homogeneous polynomial in two variables, so it is just an element of $k[x_0, x_1]_d$.

The set of zeroes of a binary form F is given by d points counted with multiplicity. Of course, this is strictly related to the (ir)reducibility of the polynomial: for istance, the reducible form $F(x_0, x_1) = x_0^d$ has the point $[0, 1]$ of multiplicity d.
Going back to our notation, we are going to study the action of SL_2 on $\mathrm{Hyp}_d(1)$.

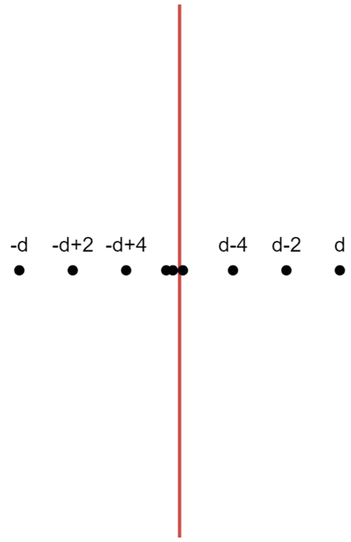
We may assume $d \geq 3$. From the remark, any 1-PS of SL_2 is conjugate to

$$\lambda(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

Assume $\tilde{F} = \sum_{i=0}^{d} a_i x_0^{d-i} x_1^i \in k[x_0, x_1]_d \setminus \{0\}$ lies over $F \in \mathrm{Hyp}_d(1)$: then we have

$$\lambda(t)\tilde{F} = \sum_{i=0}^{d} t^{2i-d} a_i x_0^{d-i} x_1^i.$$

We notice that x_0^d has weight $-d$, $x_0^{d-1}x_1$ has weight $-d+2$ and so on. The following picture shows the weight set:



According to the picture, we notice the symmetry of the weight set with respect to the monomial $x_0^{d/2}x_1^{d/2}$, which has weight 0.
Let us compute the Hilbert-Mumford weight: we have

$$\mu(F,\lambda) = \max\{-(2i-d) : a_i \neq 0\} = \max\{d-2i : a_i \neq 0\} = d - 2i_0,$$

where i_0 is the smallest index for which $a_i \neq 0$.
Hence we get

1. $\mu(F,\lambda) \geq 0$ if and only if $i_0 \leq d/2$, which is equivalent to say that the point $[1,0]$ occurs as a root with multiplicity at most $d/2$;

2. $\mu(F,\lambda) > 0$ if and only if $i_0 < d/2$, which is equivalent to say that the point $[1,0]$ occurs as a root with multiplicity strictly less than $d/2$.

From this we easily deduce the following result.

Theorem 3.3.1. $\mathrm{Hyp}_d(1)^{SS}$*(resp.* $\mathrm{Hyp}_d(1)_0^S$) *is equal to the set of hypersurfaces with roots of multiplicity less than or equal to (resp. strictly less than) $d/2$.*

Proof. We will prove the statement for semi-stability, the one for stability is exactly the same.
($\Rightarrow$) By contradiction, assume F is semi-stable and has a root $[p_0,p_1]$ of multiplicity $> d/2$. Since the action of SL_2 on $\mathbb{P}^1$ is transitive, there exists g such that $g[p_0,p_1] = [1,0]$. Then $F' = gF$ has the point $[1,0]$ has a root of multiplicity $> d/2$, since

$$F'([1,0]) = gF([1,0]) = F(g^{-1}[1,0]) = F([p_0, p_1]) = 0.$$

From the observation above, this implies that $\mu(F', \lambda) \leq 0$. But

$$\mu(F', \lambda) = \mu(gF, \lambda) = \mu(F, g^{-1}\lambda g),$$

hence F is not semi-stable by the Hilbert-Mumford criterion.

($\Leftarrow$) Conversely, assume F has no roots of multiplicity $> d/2$ and F is not semi-stable. Then there exists a 1-PS λ' such that $\mu(F, \lambda') < 0$. Let $g \in SL_2$ be such that $g^{-1}\lambda' g = \lambda$. Then $\mu(gF, \lambda) = \mu(F, \lambda') < 0$ and hence gF has the point $[1,0]$ as root of multiplicity $> d/2$, but then F has the point $g^{-1}[1,0]$ as a root of multiplicity $> d/2$, contradicting the assumption. $\square$

Corollary 3.3.1. *Assume d is odd. Then $\mathrm{Hyp}_d(1)^{SS} = \mathrm{Hyp}_d(1)^S_0$ and $\mathrm{Hyp}_d(1)//SL_2$ is a geometric quotient of the space of stable binary forms of degree d.*

Now assume d is even and let $F \in \mathrm{Hyp}_d(1)^{SS} \setminus \mathrm{Hyp}_d(1)^S_0$. This precisely means that F has a root of multiplicity $d/2$ and no roots of multiplicity greater than $d/2$. Let us consider the fibre of the morphism

$$\phi : \mathrm{Hyp}_d(1)^{SS} \to \mathrm{Hyp}_d(1)//SL_2$$

containing F. Since the quotient is a good quotient, this fibre contains a unique closed orbit. Assume that F belongs to this orbit; then its stabilizer must be positive dimensional, otherwise F would be stable. Since any element stabilizing F also stabilizes its set of roots and any subset of $\mathbb{P}^1$ consisting of at least 3 points is only stabilized by the identity, it follows that F must have exactly 2 roots. Since one has multiplicity $d/2$, the only possibility is that the other one is of multiplicity $d/2$ too.
This tells us that

$$(\mathrm{Hyp}_d(1)//SL_2) \setminus (\mathrm{Hyp}_d(1)^S/SL_2) = \{p_0\}$$

where the single point p_0 represents the orbit of the polynomial $F_0 = x_0^{d/2} x_1^{d/2}$. Here of course by $\mathrm{Hyp}_d(1)^S/SL_2$ we mean the geometric quotient of the stable locus.
Summarizing, we get that in the quotient all the semi-stable but not stable orbits are identified with the unique closed one, namely the one of binary forms having exactly two roots, each one of multiplicity $d/2$.

Let us consider some special cases for small values of d.
If $d = 3$, then the semi-stable and the stable locus coincide. This locus consists of forms with 3 distinct roots. Given any three points $p_1, p_2, p_3 \in \mathbb{P}^1$,

there is a unique automorphism of $\mathbb{P}^1$ mapping these points to any other three distinct points q_1, q_2, q_3. This implies that there is only orbit, hence in this case the quotient $\mathrm{Hyp}_3(1)//SL_2$ is a single point. This agrees with the fact that the ring of invariants is $k[\Delta]$, as one can show.

If $d = 4$, then the stable locus is the set of forms with 4 distinct roots, while the semi-stable locus is the set of forms with at most double roots. A binary quartic is given by

$$a_0x_0^4 + a_1x_0^3x_1 + a_2x_0^2x_1^2 + a_3x_0x_1^3 + a_4x_1^4.$$

Lemma 3.3.1. *The ring of invariants $S(\mathrm{Hyp}_4(1))^{SL_2}$ is generated by the two following invariants:*

$$I = a_0a_4 - 4a_1a_3 + 3a_2^2,$$
$$J = a_0a_2a_4 + 2a_1a_2a_3 - a_0a_3^2 - a_1^2a_4 - a_2^3.$$

Proof. The discriminant Δ is a homogeneous polynomial of degree 6 and we have precisely $\Delta = I^3 - 27J^2$.
Given any 4 distinct ordered points in $\mathbb{P}^1$, there exists a unique automorphism of $\mathbb{P}^1$ mapping these points to $(0, 1, \infty, \lambda)$ for some $\lambda \neq 0, 1$. The number λ is called the *cross ratio* of the four points in the given order. However, in our case the 4 roots of a stable binary form do not have a natural ordering. Let us consider the following action of the symmetric group S_4 on $k\backslash\{0,1\}$: given λ, permute $0, 1, \infty, \lambda$ according to $\alpha \in S_4$, then apply the linear transformation mapping the first three back to $0, 1, \infty$ and let $\alpha \cdot \lambda$ be the image of the fourth. The orbit of λ consists of

$$\lambda,\ 1-\lambda,\ 1/\lambda,\ (\lambda-1)/\lambda,\ \lambda/(\lambda-1),\ 1/(1-\lambda).$$

The number

$$\rho = \Big(\frac{(2\lambda-1)(\lambda-2)(\lambda+1)}{\lambda(\lambda-1)}\Big)^2$$

is symmetric in the six different values of λ, hence we have a well-defined morphism

$$\rho : \mathrm{Hyp}_4(1)_0^S \to \mathbb{A}^1$$
$$\lambda \longmapsto \rho(\lambda)$$

which is SL_2-invariant.
We notice that λ corresponds to the polynomial

$$x_0x_1(x_0 - x_1)(x_0 - \lambda x_1) = x_0^3x_1 - (1+\lambda)x_0^2x_1^2 + \lambda x_0x_1^3.$$

whose roots are precisely $[1,0],[0,1],[1,1],[\lambda,1]$. Thus we can express the coefficents $a_0,...,a_4$ in terms of λ through the following k-algebra homomorphism

$$\begin{aligned} k[a_0,a_1,a_2,a_3,a_4] &\to k[\lambda] \\ a_0 &\longmapsto 0 \\ a_1 &\longmapsto 1 \\ a_2 &\longmapsto -(1+\lambda) \\ a_3 &\longmapsto \lambda \\ a_4 &\longmapsto 0. \end{aligned}$$

Moreover, it can be shown that $\rho = 3^6 J^2/\Delta$.
For each value of $\rho \in \mathbb{A}^1 \setminus \{0,-27\}$, there are six possible choices for λ, which corresponds to a unique stable orbit. For the value 0 there are three possible choices for λ, namely $-1,2,1/2$, which again corresponds to a unique stable orbit. Finally, for the value -27, there are two possible values for λ, namely $-\omega,-\omega^2$, where $\omega^3=1$ and $\omega \neq 1$, and these points correspond again to a unique orbit.
Hence the morphism ρ is SL_2-invariant and separates the orbits, so $\mathbb{A}^1$ is indeed the geometric quotient of the action on the stable locus and the morphism ρ is the quotient morphism.
We are now ready to complete our proof. If f is any homogeneous invariant polynomial in the coefficients $a_0,...,a_4$, then its zero set in $\mathbb{P}^4$ consists of a finite union of closures of 3-dimensional orbits. These are easy to classify: for each $\alpha \in k$ there exists a single stable orbit for which $J^2/\Delta = \alpha$, from what we have observed before. We will denote the closure of these orbits by X_α. Moreover, there is another one 3-dimensional orbit, the one consisting of quartics with a unique double root, whose closure we denote by X_∞. Each X_α is irreducible, so it is the zero locus of an irreducible homogeneous polynomial f_α. Moreover, we may assume f is irreducible, so that f actually coincides with some f_α.
By checking the respective zero sets, we see that we can take f_α to be:

$$\begin{aligned} f\infty &= \Delta, \\ f_0 &= J, \\ f_{-1/27} &= I, \\ f_\alpha &= J^2 - \alpha\Delta \text{ for } \alpha \neq \infty, 0, -1/27. \end{aligned}$$

It follows that each $f_\alpha \in k[I,J]$, as desired. □

Hence $\mathrm{Hyp}_4(1)//SL_2$ is isomorphic to $\mathbb{P}^1$ and the quotient map is given by

$$\begin{aligned} \mathrm{Hyp}_4(1)^{SS} &\to \mathbb{P}^1 \\ [a_0,...,a_4] &\longmapsto [I^3-27J^2, J^2]. \end{aligned}$$

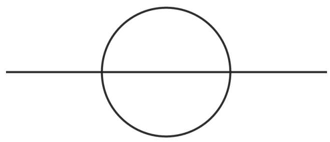
3.4 Plane cubics

Now we will study degree 3 hypersurfaces in $\mathbb{P}^2$, i.e plane cubic curves. Every homogeneous polynomial of degree 3 in x_0, x_1, x_2 is of the form

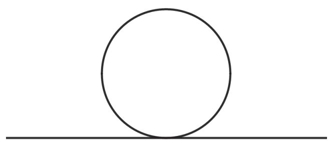
$$F(x_0, x_1, x_2) = \sum_{i=0}^{3} \sum_{j=0}^{3-i} a_{ij} x_0^{3-i-j} x_1^i x_2^j, \text{ for suitable } a_{ij} \in k.$$

First, we want to describe all plane cubics up to projective equivalence; that is, we will give a complete description of the quotient for the action of SL_3 on $\mathrm{Hyp}_3(2)$.
Let us start with the classification of reducible curves. They are of the following types:

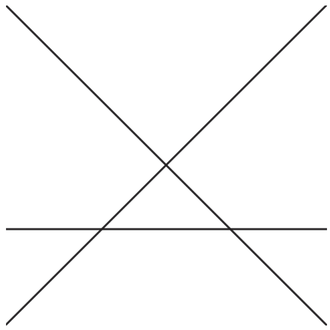
1) the union of an irreducible conic and a line intersecting it at two distinct points;



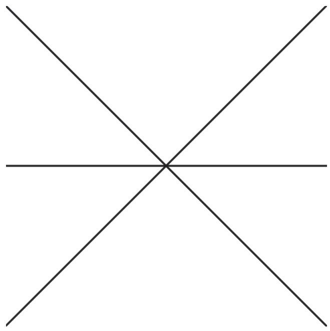
2) the union of an irreducible conic and a line tangent to it;



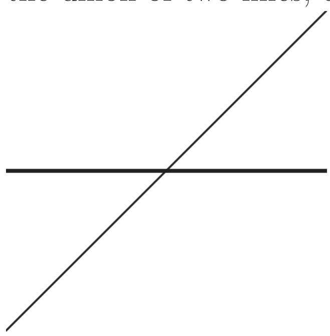
3) the union of three non-concurrent lines;



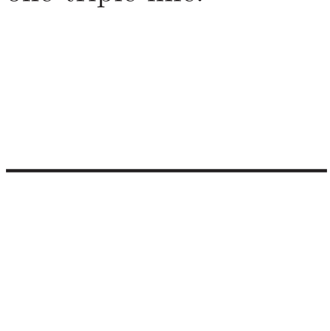
4) the union of three concurrent lines;



5) the union of two lines, one of them double;



6) one triple line.



From the theory of quadratic forms we know that every non-singular conic in $\mathbb{P}^2$ is projectively equivalent to the conic $C : x_0x_2 + x_1^2 = 0$. Moreover, since the projective automorphism group of C acts transitively on the set of lines tangent to it and on the set of lines intersecting it in two distinct points, we get that every reducible curve of type 1) and 2) is projectively equivalent to the curve:
1) $(x_0x_2 + x_1^2)x_1 = 0$;

2) $(x_0x_2 + x_1^2)x_0 = 0$
respectively.

Since the group of projective automorphisms acts transitively on the set of 3 lines, any reducible cubic of type $3) - 6)$ is projectively equivalent to the curve
3) $x_0x_1x_2 = 0$;

4) $x_0x_1(x_0 + x_1) = 0$;

5) $x_0^2(x_0 + x_1) = 0$;

6) $x_0^3 = 0$
respectively.
Let us stop for a while our classification, in order to recall some properties of singular points and tangent lines, which will be fundamental for our purpose.

Definition 3.4.1. A singular point p of a cubic defined by $F(x_0, x_1, x_2) = 0$ is called a double point if at least one second order partial derivatives of F at p is non-zero.
It is called a triple point if all second order partial derivatives of F vanish at p.

Let $p = [p_0, p_1, p_2]$ be a point of the curve $C : F(x_0, x_1, x_2) = 0$.

- If p is non-singular, then the tangent line to C at p is given by the equation

$$\frac{\partial F}{x_0}(p)x_0 + \frac{\partial F}{x_1}(p)x_1 + \frac{\partial F}{x_2}(p)x_2 = 0.$$

- If p is a double point, then the tangent cone to C at p is given by the following homogeneous polynomial of degree 2

$$x^T \begin{pmatrix} \frac{\partial F}{x_0^2}(p) & \frac{\partial F}{x_0x_1}(p) & \frac{\partial F}{x_0x_2}(p) \\ \frac{\partial F}{x_0x_1}(p) & \frac{\partial F}{x_1^2}(p) & \frac{\partial F}{x_1x_2}(p) \\ \frac{\partial F}{x_0x_2}(p) & \frac{\partial F}{x_1x_2}(p) & \frac{\partial F}{x_2^2}(p) \end{pmatrix} x = 0$$

where $x = (x_0 - p_0, x_1 - p_1, x_2 - p_2)$.

Notice that the 3×3 matrix in the equation has not full rank, since p is a double point. Hence the polynomial factorises into a product of two linear polynomials, which may be distinct or not. According to this observation, we have two different types of double points:

1. A *node* is a double point with two distinct lines in the tangent cone.
2. A *cusp* is a double point with a single line of multiplicity 2 in the tangent cone.

Lemma 3.4.1. *Let C be the plane cubic defined by the polynomial*

$$F(x_0, x_1, x_2) = \sum_{i=0}^{3} \sum_{j=0}^{3-i} a_{ij} x_0^{3-i-j} x_1^i x_2^j.$$

Then for the point $p = [1, 0, 0]$ we have:

i) $p \in C$ if and only if $a_{00} = 0$;

ii) p is a singular point of C if and only if $a_{00} = a_{10} = a_{01} = 0$;

iii) p is a triple point of C if and only if $a_{00} = a_{10} = a_{01} = a_{11} = a_{20} = a_{02} = 0$;

iv) if p is a double point of C, then its limit tangent lines are defined by

$$a_{20}x_1^2 + a_{11}x_1x_2 + a_{02}x_2^2 = 0.$$

Proof. $i)$ is immediate. For parts $ii)$ and $iii)$ the proof is just a computation of the derivatives (first and second order) of F at p, while part $iv)$ follows immediately by the formula for the tangent lines at a double point that we recalled before. □

Let us continue our classification of cubics. Now assume that the curve is irreducible. We have two possibilities: either the cubic is smooth or singular. First, let us consider the case of a smooth curve.
We can choose a system of coordinates such that $[0, 0, 1]$ is an inflection point with tangent line $x_0 = 0$. Then we can write the equation as follows:

$$x_2^2x_0 + x_2L_2(x_0, x_1) + L_3(x_0, x_1) = 0,$$

where L_2 is a form of degree 2 and L_3 is a form of degree 3. By assumption, the line $x_0 = 0$ intersects the cubic in the single point $[0, 0, 1]$ with multiplicity 3, this implies that the coefficient of x_1^2 in L_2 must be zero. Thus in affine coordinates $x = x_1/x_0$ and $y = x_2/x_0$ the equation has the form

$$y^2 + axy + by + dx^3 + ex^2 + fx + g = 0.$$

Clearly $d \neq 0$, so after scaling we may assume $d = 1$.
Assume $\mathrm{char}(k) \neq 2$. Replacing y with $y + a/2x + b/2$, we may assume $a = b = 0$. If $\mathrm{char}(k) \neq 3$, after a linear change of variables $x \longrightarrow x + e/3$, we may assume $e = 0$.
Thus, we obtain the *Weierstrass equation* of a non-singular plane cubic:

$$\begin{array}{cc} y^2 + x^3 + ax + b = 0, & char(k) \neq 2, 3 \\ y^2 + axy + by + x^3 + cx + d = 0, & char(k) = 2 \\ y^2 + x^3 + ax^2 + bx + c = 0, & char(k) = 3. \end{array}$$

The condition that the curve is non-singular is expressed by $\Delta \neq 0$, where Δ is the discriminant defined by

$$\begin{array}{cc} \Delta = 4a^3 + 27b^2, & char(k) \neq 2, 3 \\ \Delta = a^3b^3 + b^4 + a^4(abc + c^3 + a^2d), & char(k) = 2 \\ \Delta = b^3 + (b^2 - ac)a^2, & char(k) = 3. \end{array}$$

Now let us assume that the cubic is singular. We may choose the point $[0, 0, 1]$ to be the singular point.
Then the equation cannot does not involve the monomials $x_2^3, x_0x_2^2$ and $x_1x_2^2$, since otherwise $[0, 0, 1]$ would not belong to the curve or it would be a non singular point. Hence the equation has the form

$$x_2L_2(x_0, x_1) + L_3(x_0, x_1) = 0.$$

By a linear transformation of x_0, x_1 we can reduce L_2 to be one of the two forms:

1. $L_2 = x_0^2$;
2. $L_2 = x_0x_1$.

Let us consider the first case, which corresponds to a cuspidal cubic,as we will se soon. The equation is

$$x_2x_0^2 + ax_0^3 + bx_0^2x_1 + cx_0x_1^2 + dx_1^3 = 0.$$

Replacing x_2 with $x_2 + ax_0 + bx_1$ we may assume $a = b = 0$; moreover since the curve is irreducible then $d \neq 0$, so we may further assume $d = 1$ after scaling. Finally, the coefficient c may be either 0 or 1.
Let $\mathrm{char}(k) \neq 3$. From a direct computation we get that the Hessian of the curve is

$$cx_0^3 + 3x_0^2 x_1 = 0,$$

hence we have a unique inflection point(the intersection of the curve and the Hessian) given by $[1, -c/3, -2c^3/27]$. If we change the coordinates in such a way that the inflection point becomes $[1, 0, 0]$ with tangent $x_2 = 0$, we finally get that the equation becomes

$$x_2 x_0^2 + x_1^3 = 0.$$

Given in this form, it is immediate to see that $[0, 0, 1]$ is a cusp, with double tangent line $x_0 = 0$.
If instead $\mathrm{char}(k) = 3$, the Hessian is just $cx_0^3 = 0$; hence there are 2 orbits of cuspidal cubics given by $c = 0$ or $c = 1$, represented by the equations

$$x_0^2 x_2 + x_1^3 = 0 \text{ and } x_0^2 x_2 + x_0 x_1^2 + x_1^3 = 0.$$

In the first case the Hessian is identically zero, hence all non singular points of the curve are inflection points. The second curve does not have non singular inflection points.

Let us now consider the case when the quadratic form L_2 reduces to $x_0 x_1 = 0$, which corresponds to the case of nodal cubics.
The equation is of the form

$$x_0 x_1 x_2 + ax_0^3 + bx_0^2 x_1 + cx_0 x_1^2 + dx_1^3 = 0.$$

By replacing x_2 with $x_2 - bx_0 - cx_1$ we may assume $b = c = 0$. Now, since the curve is irreducible we must have $a, d \neq 0$, after scaling we may assume that they are both 1. Hence we get the equation

$$x_0 x_1 x_2 + x_0^3 + x_1^3 = 0,$$

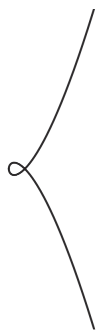
which shows that $[0, 0, 1]$ is indeed a node with tangent lines $x_0 = 0$ and $x_1 = 0$.
Summarizing, we get the following list of irreducible plane cubics(up to projective equivalence), which completes our classification:

$\mathrm{char}(k) \neq 2, 3$:
7) non singular cubic

$$x_0 x_2^2 + x_1^3 + ax_0^2 x_1 + bx_0^3 = 0, \qquad 4a^3 + 27b^2 \neq 0;$$

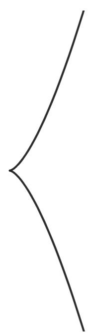
8) nodal cubic

$$x_0x_1x_2 + x_0^3 + x_1^3 = 0;$$



9) cuspidal cubic

$$x_0x_2^2 + x_1^3 = 0.$$



$\mathrm{char}(k) = 3$:

7) non-singular cubic

$$x_0x_2^2 + x_1^3 + ax_0x_1^2 + bx_0^2x_1 + cx_0^3 = 0, \qquad b^3 + (b^2 - ac)a^2 \neq 0;$$

8) nodal cubic

$$x_0x_1x_2 + x_0^3 + x_1^3 = 0;$$

9) cuspidal cubic

$$x_0^2x_2 + x_1^3 = 0 \qquad \text{or} \qquad x_0^2x_2 + x_0x_1^2 + x_1^3 = 0$$

$\mathrm{char}(k) = 2$:

7) non singular cubic

$$x_0x_2^2+x_1^3+ax_0x_1x_2+bx_0^2x_2+cx_0^2x_1+dx_0^3 = 0, \qquad a^3b^3+b^4+a^4(abc+c^3+a^2d) \neq 0;$$

8) nodal cubic

$$x_0x_1x_2 + x_0^3 + x_1^3 = 0;$$

9) cuspidal cubic

$$x_0x_2^2 + x_1^3 = 0.$$

This classification does not tell us anything about which ones are stable or semi-stable. In order to have a complete description of the (semi)stable locus, we will use the Hilbert-Mumford criterion.

Theorem 3.4.1. *A plane cubic curve C is stable if and only if it is non-singular, while it is semi-stable if and only if it has no triple point and no double point with a unique limit tangent (i.e. it has no cusps).*

Proof. Let C be defined by the vanishing of the polynomial

$$F(x_0, x_1, x_2) = \sum_{i=0}^{3}\sum_{j=0}^{3-i} a_{ij}x_0^{3-i-j}x_1^i x_2^j.$$

We have already observed that any 1-PS of SL_3 is conjugate to $\lambda(t)$ =diag(t_0^r, t_1^r, t_2^r) where $r_0 \geq r_1 \geq r_2$ and $r_0 + r_1 + r_2 = 0$.
Moreover, we have already computed the Hilbert-Mumford weight for general hypersurfaces (see Remark 3.2.1): in this particular case we get

$$\mu(F, \lambda) = \max\{(3 - i - j)r_0 + ir_1 + jr_2 : a_{ij} \neq 0\}.$$

Finally, since $\mu(gx, \lambda) = \mu(x, g^{-1}\lambda' g)$ for any 1-PS λ', we will only consider the diagonal parameter subgroup λ.

Let us first consider the case of stability. Assume that C has a singular point p, by a change of coordinates $p = [1, 0, 0]$. Hence by Lemma 3.4.1 we have $a_{00} = a_{10} = a_{01} = 0$. For $\lambda(t)$ =diag(t^2, t^{-1}, t^{-1}) we have $\mu(F, \lambda) \leq 0$ since $i + j \geq 2$. Hence F is not stable.
Conversely, assume F is not stable. Then $\mu(F, \lambda) \leq 0$; this implies $a_{00} = a_{10} = 0$ since the monomials x_0^3 and $x_0^2x_1$ have weights $-3r_0$ and $-(2r_0 + r_1)$ respectively, which are strictly negative. If we also have $a_{01} = 0$, then $[1, 0, 0]$ is a singular point by Lemma 3.4.1. If $a_{01} \neq 0$, then

$$0 \geq \mu(F, \lambda) \geq 2r_0 + r_2.$$

Since $r_2 = -(r_0 + r_1)$, we have $r_0 - r_1 \leq 0$. But $r_0 \geq r_1$ by assumption, hence we must have $r_1 = r_0$ and $r_2 = -r_0$. For these values of r_i, we get

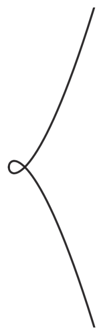
$$\mu(F, \lambda) = \max\{(3 - 3j)r_0 : a_{ij} = 0\} \leq 0.$$

Since $r_0 > 0$, we must have $a_{i0} = 0$ for any i, which is equivalent to say that x_2 divides F. It follows that $F = x_2 f'$ for some f' of degree 2; hence C is singular at every point for which $x_2 = f' = 0$.
Let us now consider the case of semi-stability. If F is not semi-stable, then

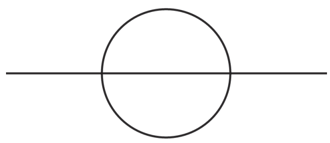
$\mu(F,\lambda) < 0$, that is, all weights of F must be positive. Since $r_0 \geq r_1 \geq r_2$, the monomials with non-positive weights are $x_0^3, x_0^2x_1, x_0x_1^2, x_0^2x_2, x_0x_1x_2$. This implies that $a_{00} = a_{10} = a_{20} = a_{01} = a_{11} = 0$. We conclude from Lemma 3.4.1 that $[1,0,0]$ is a singular point for C, which is a triple point if $a_{02} = 0$ or a cusp if $a_{02} \neq 0$.
Conversely, let us assume that C has a triple point or a double point with a unique tangent. Again, we assume that this point is $[1,0,0]$ and it follows from Lemma 3.4.1 that $a_{00} = a_{10} = a_{20} = a_{01} = a_{11} = 0$. Let us consider the 1-PS $\lambda(t)$ =diag(t^3, t^{-1}, t^{-2}): a direct computation gives $\mu(F,\lambda) < 0$, thus F is not semi-stable. □

From the classification of cubics, it is immediate to see that the singular points of the following cubics are nodes. Hence we conclude that there are three semi-stable but not stable orbits:

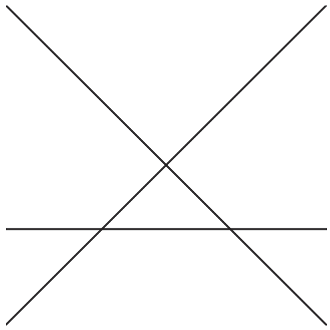
1) nodal irreducible cubics;



2) the union of an irreducible conic and a line not tangent to it;

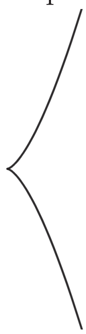


3) the union of three non-concurrent lines.

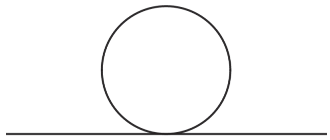


Finally, we list the non semi-stable orbits:

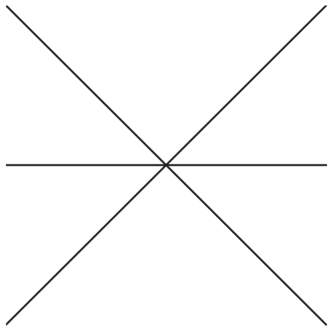
1) irreducible cuspidal cubics (two orbits if $\mathrm{char}(k) = 3$);



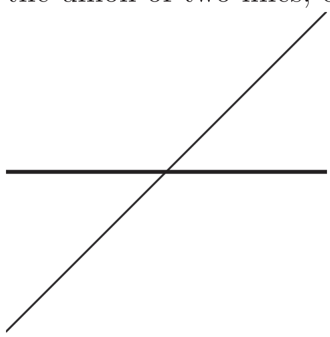
2) the union of an irreducible conic and its tangent line;



3) the union of three concurrent lines;



4) the union of two lines, one of them double;



5) a triple line.

Again, this follows from the classification up to equivalence: cubics of the form $1), 2)$ have a cusp, while the ones of the form $3), 4), 5)$ have a triple point.
Let us consider the three semistable orbits: they are given by equations

1) $x_0x_1x_2 + x_0^3 + x_1^3 = 0,$

2) $x_0x_1x_2 + x_1^3 = 0,$

3) $x_0x_1x_2 = 0.$

From the equations we see that $\lambda(t) = diag(t, 1, t^{-1})$ stabilizes the second curve, while $\lambda(t) = diag(t^2, t^{-1}, t^{-1})$ stabilizes the third curve. Hence these two orbits have positive dimensional stabilizer, so they are of dimension ≤ 7, because dim $SL_3 = 8$. More precisely, it can be shown that their dimension are 7 and 6, respectively. Let us consider the fibre of the projection

$$\mathrm{Hyp}_3(2)^{SS} \to \mathrm{Hyp}_3(2)//SL_3$$

containing any semi-stable but not stable points. The unique closed orbit contained in this fibre must be the one corresponding to three non-concurrent lines, since it is the one of minimal dimension. Moreover, from Lemma 2.2.3 we get that this orbit is contained in the closure of the one of dimension 7. Finally, it can be shown that the orbit of nodal irreducible cubics has dimension 8, thus its closure must contain both the other strictly semi-stable orbits. In particular, these orbits correspond to just one point in the quotient. The geometric quotient of the stable locus classifies smooth cubics up to isomorphism. From the theory of elliptic curves, it is known that two such cubics are isomorphic if and only if they have the same j-invariant, where

$$j = 1728\frac{4a^3}{4a^3 + 27b^2}$$

and we are referring to the Weierstrass equation

$$y^2 + x^3 + ax + b = 0.$$

Thus we get that the geometric quotient $\mathrm{Hyp}_3(2)/SL_3$ is isomorphic to $\mathbb{A}^1$, because any element of k occurs as the j-invariant of some elliptic curve. (See [10] *IV* Theorem 4.1).
Finally, the good quotient $\mathrm{Hyp}_3(2)//SL_3$ is just the compactification of the geometric quotient, where we add the single point corresponding to the strictly semi-stable orbits. Thus this quotient is isomorphic to $\mathbb{P}^1$.

3.5 Plane quartics

Let us now consider $\mathrm{Hyp}_4(2)$, the space of plane quartics. Every quartic is defined by an homogeneous polynomial of degree 4 of the following form

$$F(x_0, x_1, x_2) = \sum_{i=0}^{4}\sum_{j=0}^{4-i} a_{ij}x_0^{4-i-j}x_1^i x_2^j, \text{ for suitable } a_{ij} \in k.$$

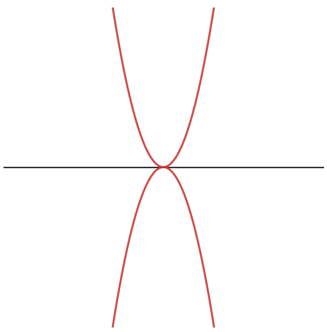
As usual, let λ be a 1-PS defined by $\lambda(t) = diag(t^{r_0}, t^{r_1}, t^{r_2})$ where $r_0 + r_1 + r_2 = 0$ and $r_0 \geq r_1 \geq r_2$.
From Remark 3.2.1 it follows that

$$\mu(F,\lambda) = \max\,\{(4-i-j)r_0 + ir_1 + jr_2 : a_{ij} \neq 0\}.$$

In order to state the result about stability of a quartic, we need to recall a definition.

Definition 3.5.1. Let C be any curve defined by the vanishing locus of the polynomial F, let $p \in C$.
We say that p is a *tacnode* if p is a double point with a unique tangent line, such that the intersection multiplicity between the curve and this line at p is at least 4.

Example. Let us consider the curve given by $y^2 - x^4 = 0$ in $\mathbb{A}^2$. This curve has a tacnode in the origin, with the x-axis as tangent line. Geometrically, this means that two branches of the curve have the same tangent line at the origin, as we can see from the following picture.



Remark. Let C be a plane quartic defined by the polynomial

$$F(x_0,x_1,x_2) = \sum_{i=0}^{4}\sum_{j=0}^{4-i} a_{ij}x_0^{4-i-j}x_1^i x_2^j.$$

We notice that the results in Lemma 3.4.1 are still true, in exactly the same form.

Theorem 3.5.1. *A plane quartic C is stable if and only if C has no triple points and no tacnodes.*

Proof. $\Leftarrow$) Let F be the polynomial defining the curve C. By contradiction, assume C is not stable. From the Hilbert-Mumford criterion, there exists a 1-PS λ such that $\mu(F,\lambda) \leq 0$. As usual, up to a suitable conjugate, we assume that $\lambda(t) = diag(t^{r_0}, t^{r_1}, t^{r_2})$ with $r_0 \geq r_1 \geq r_2$. Thus we get

$$\mu(F,\lambda) = \max\,\{(4-i-j)r_0 + ir_1 + jr_2 : a_{ij} \neq 0\} \leq 0.$$

We notice that this inequality implies

$$a_{00} = a_{10} = a_{01} = a_{11} = a_{20} = 0.$$

Moreover, if $a_{30} \neq 0$ and $a_{02} \neq 0$, then we must have

$$0 \geq r_0 + 3r_1 + 2r_0 + 2r_2 = r_0 + r_1,$$

which implies $r_2 \geq 0$, contradicting our assumption on the r_i. Now, if $a_{02} = 0$, we get that C has a triple point at $[1, 0, 0]$ from the previous remark. Otherwise, if $a_{02} \neq 0$, the double point $[1, 0, 0]$ has the unique tangent line $x_2 = 0$. Since we must have $a_{30} = 0$, the intersection multiplicity of the point is at least 4, thus it is a tacnode.

$\Rightarrow$) Conversely, by contradiction let us assume that C has a triple point; without loss of generality we may assume that this point is $[1, 0, 0]$. Thus we must have $a_{00} = a_{10} = a_{01} = a_{11} = a_{20} = a_{02} = 0$.
Let us consider the 1-PS defined by $\lambda(t) = diag(t^2, t^{-1}, t^{-1})$. Hence we get

$$\mu(F, \lambda) = \max \{8 - 3i - 3j : a_{ij} \neq 0\},$$

which implies $\mu(F, \lambda) \leq -1$ because either $i \geq 3$ or $i \geq 2$ and $j \geq 1$ (and viceversa, exchanging the roles of i and j). Here we remark that $\mu(F, \lambda) < 0$, hence we actually proved that a quartic with a triple point is not semi-stable.
Now, let us assume that C has a tacnode at $[1, 0, 0]$, with tangent line $x_2 = 0$. It follows that $a_{00} = a_{10} = a_{01} = a_{11} = a_{20} = a_{30} = 0$.
If we consider $\lambda(t) = diag(t, 1, t^{-1})$, then we get

$$\mu(F, \lambda) = \max \{4 - i - 2j : a_{ij} \neq 0\}.$$

From a direct computation, we easily get $\mu(F, \lambda) \leq 0$ and C is not stable. $\square$

3.6 Cubic surfaces

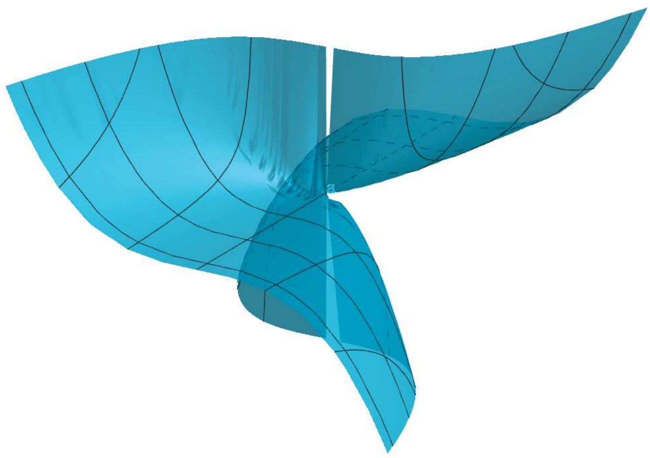
Let us now consider $\mathrm{Hyp}_3(3)$, the space of cubic surfaces in $\mathbb{P}^3$. We are going to study (semi)-stability of cubic surfaces under the action of SL_4 using the Hilbert-Mumford criterion. For this section, we refer to [3] and [1].
In this case, we will consider 1-PS of the form

$$\lambda(t) = diag(t^{r_0}, t^{r_1}, t^{r_2}, t^{r_3}),\ r_0 + ... + r_3 = 0 \text{ and } r_0 \leq r_1 \leq r_2 \leq r_3 \qquad (*)$$

We notice that here we reversed the inequalities between the r_i, because we are considering the inverse of our "typical" 1-PS. This choice is due to simplicity, since it allows us to avoid negative signs in the action.
Let us recall that an ordinary double point of a surface is a double point whose singularity is locally anatically equivalent to that one of the origin in $x^2 + y^2 + z^2 = 0$, i.e. a singular point whose tangent cone is given by an

irreducible degree 2 polynomial.
An ordinary cusp is a double point whose singularity is locally analitically equivalent to the origin in $x^2+y^2+z^3=0$, i.e. a singular point whose tangent cone consists of two distinct planes and the intersection multiplicity of the surface and the planes is exactly 3.
The main result is the following one.

Theorem 3.6.1. *A cubic surface is stable if and only if it is smooth or it has only ordinary double points.*
A cubic surface is semi-stable if and only if its singular points are ordinary cusps (and perhaps some ordinary double points).
Moreover, there is only one closed orbit of semi-stable not stable cubic surfaces, namely that of $x_0^3 = x_1x_2x_3$.



Graphic of the surface $x_0^3 = x_1x_2x_3$ in a neighborhood of the point $[0,0,0,1]$

Proof. The proof will be divided into the following steps:

a) Cubic surfaces with only ordinary double points or cusps are semi-stable.

b) Cubic surfaces with only ordinary double points are stable.

c) Cubic surfaces with a singularity which is worse than an ordinary cusp are not semi-stable. This is the case of double points whose tangent cone consists of a unique plane of multiplicity 2, not ordinary cusps (i.e cusps where the intersection multiplicity at the point is of higher order) or triple points.

d) Cubic surfaces with at least an ordinary cusp are semi-stable but not stable; they all contain in their orbit closure the orbit of the surface $x_0^3 = x_1x_2x_3$.

a) Let S be a cubic surface defined by $F(x_0, x_1, x_2, x_3) = 0$. Assume that S is not semi-stable: then we have $\mu(F, \lambda) < 0$ for the 1-PS of the form $(*)$. More precisely, if

$$F = \sum_{I=(i_0,i_1,i_2,i_3)} a_I x^I,\ i_0 + i_1 + i_2 + i_3 = 0 \text{ and } x = (x_0, x_1, x_2, x_3),$$

we have

$$\lambda(t)F = \sum_{I=(i_0,i_1,i_2,i_3)} t^{rI} a_I x^I \quad rI = r_0 i_0 + ... + r_3 i_3.$$

Since $\mu(F, \lambda) < 0$ all the weights must be positive. Thus we get

$$r_0 i_0 + ... + r_3 i_3 > 0 \text{ whenever } a_I \neq 0. \quad (A)$$

If S is reducible, then S is either the union of three planes (counted with multiplicities) or a union of a plane and an irreducible quadric. In the first case, it is clear that such a surface has at least a triple point, which is the intersection of the planes. Let us consider the second case: up to a change of coordinates we may assume that the equation is of the form

$$x_0 L_2(x_0, x_1, x_2, x_3),$$

where L_2 is irreducible. Since L_2 is a non-degenerate quadratic form, another change of coordinates leads to the form

$$x_0(x_0^2 + x_1^2 + x_2^2 + x_3^2).$$

It is clear that $[0, 1, i, 0]$ is a double point, which is a cusp since its tangent cone is $x_0(x_1 + ix_2) = 0$. Since the plane $x_0 = 0$ is contained in the surface, the point is not an ordinary cusp.
Thus we may assume S to be irreducible.
In this case, then $r_2 > 0$: otherwise every monomial appearing in F would be divisible by x_3 from inequality (A). Then F cannot contain a monomial divisible by x_0^2 or x_0x_1 because their weights are $2r_0 + r_i$ and $r_0 + r_1 + r_i$ respectively, and we have

$$2r_0 + r_i \leq r_0 + r_1 + r_i \leq r_0 + r_1 + r_3 = -r_2 < 0.$$

Hence the equation will be of the form

$$x_0 L_2(x_2, x_3) + L_3(x_1, x_2, x_3).$$

It follows that S has a double point at $[1, 0, 0, 0]$. Moreover, the tangent cone L_2 has rank ≤ 2. If this rank is stricly less than 2, we would have a unique plane of multiplicity 2. Otherwise, if the rank is exactly 2, we may assume $L_2(x_2, x_3) = x_2 x_3$ and the point is a cusp: moreover we must have $r_0 + r_2 + r_3 > 0$ from inequality (A), because the first monomial is $x_0 x_2 x_3$. However, this implies $r_1 < 0$, thus the monomial x_1^3 cannot appear in the equation of F. Hence the point $[1, 0, 0, 0]$ cannot be an ordinary cusp, because the intersection multiplicity of the surface and both the planes $x_2 = 0$ and $x_3 = 0$ is of higher order.

b) Now assume S is not stable. In this case, the inequality (A) becomes $r_0 i_0 + ... + r_3 i_3 \geq 0$. If $r_2 > 0$, we proceed in the same way as before to get that S cannot have an ordinary double point at $[1, 0, 0, 0]$. However, we may now have $r_2 = 0$. If $r_0 < r_1$, the only change is that F can contain the monomial $x_0 x_1 x_3$, but we see from (A) that it cannot contain $x_0 x_2^2$. Thus the tangent cone has rank ≤ 2 and we get again that $[1, 0, 0, 0]$ cannot be an ordinary double point. Finally, if $r_0 = r_1$, we have $\lambda(t) = diag(t^{-1}, t^{-1}, 1, t^2)$ and we can check that S has at least one cusp.
Thus we get that cubic surfaces with at most ordinary double points are stable, as desired.

c) Let us assume that $[1, 0, 0, 0]$ is a singular point which is worse than an ordinary cusp. For an appropriate choice of coordinates, the equation of F will be of the form

$$x_0 L_2(x_1, x_2, x_3) + L_3(x_1, x_2, x_3).$$

By assumption, the tangent cone $L_2(x_1, x_2, x_3) = 0$ has rank ≤ 2. If this rank is 1, then we may assume $L_2 = a x_3^2$ for some $a \in k^*$. Let us consider the 1-PS defined by $\lambda(t) = diag(t^{-5}, t, t, t^3)$: we get that S is not semi-stable by (A).
If L_2 has rank 2, we may assume $L_2 = x_2 x_3$. Let us now consider $\lambda(t) = diag(t^{-2}, 1, t, t)$.
We have

$$\lim_{t\longrightarrow 0}\lambda(t)F = x_0x_2x_3 + cx_1^3,$$

where c is the coefficient of x_1^3 in L_3. Since the point $[1,0,0,0]$ is not an ordinary cusp, we must have $c = 0$. Hence the limit is the union of three planes, which is not semi-stable. To see this, it suffices to consider the 1-PS $\lambda(t) = diag(t^{-1}, t^{-1}, t, t)$.

d) Assume S has an ordinary cusp at $[1,0,0,0]$, which means $c \neq 0$ in the notation of part c). We have

$$\lim_{t\longrightarrow 0}\lambda(t)F := F_0 = x_0x_2x_3 + cx_1^3.$$

Let S_0 be the surface corresponding to the polynomial F_0: this surface has three ordinary cusps, which are $[1,0,0,0], [0,0,0,1]$ and $[0,0,1,0]$ (and no other singularities), hence it is semi-stable. Since every orbit closure of a cuspidal surface contains S_0, it follows that this orbit is closed.
Finally, S_0 is not stable because its stabilizer contains the matrices of the form $diag(a,1,b,c)$, where $a,b,c \in k^*$ and $abc = 1$. Thus the stabilizer is positive-dimensional.

□

The algebra of invariants $S := A(\mathrm{Hyp}_3(3))^{SL_4}$ was computed by G. Salmon and A.Clebsch ([16]). It is generated by the invariants $I_8, I_{16}, I_{24}, I_{32}$, I_{40}, I_{100}, where the subscript indicates the degree. The invariant I_{32} is the discriminant.
We have a basic relation between the invariants: the square of I_{100} is expressed as a polynomial in the first five invariants, i.e.

$$I_{100}^2 = F(I_8, I_{16}, I_{24}, I_{32}, I_{40}).$$

Hence we have the following isomorphism of graded k-algebras

$$\begin{gathered} S \simeq k[t_0,t_1,t_2,t_3,t_4,t_5]/(t_5^2 - F(t_0,t_1,t_2,t_3,t_4) \\ I_8 \longmapsto t_0 \\ I_{16} \longmapsto t_1 \\ I_{24} \longmapsto t_2 \\ I_{32} \longmapsto t_3 \\ I_{40} \longmapsto t_4 \\ I_{100} \longmapsto t_5, \end{gathered}$$

where $k[t_0,t_1,t_2,t_3,t_4,t_5]$ is graded by setting

$$\deg t_0 = 8, \deg t_1 = 16, \deg t_2 = 24, \deg t_3 = 32, \deg t_4 = 40, \deg t_5 = 50,$$

so that F is a weighted homogeneous polynomial. We notice that the subalgebra of S generated by elements of even degree is just $k[t_0, t_1, t_2, t_3, t_4]$ because we have $t_5^2 = F(t_0, t_1, t_2, t_3, t_4)$. It follows that $\mathrm{Hyp}_3(3)$ is the projective variety associated to the algebra of invariants $k[t_0, t_1, t_2, t_3, t_4]$, which is the weighted projective space $\mathbb{P}(8, 16, 24, 32, 40) \simeq \mathbb{P}(1, 2, 3, 4, 5)$, a projective variety of dimension 4.

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