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MEAN-VARIANCE-LIQUIDITY OPTIMIZATION: AN EMPIRICAL INVESTIGATION IN THE EUROPEAN MARKET

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Firma dello studente

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Introduction

Since its early development, Modern Portfolio Theory (or MPT) has represented a building block of portfolio management and, more in general, of finance. Markowitz and his pioneering work opened the door to a more quantitative approach to asset allocation, showing the extreme benefits of diversification and the great easiness wherewith it was possible to improve the portfolio performances. While the idea of diversifying risks was already known, Markowitz showed how it was theoretically possible to combine the asset to reduce the overall risk, without necessarily sacrificing the returns. He demonstrated that, for a given universe of securities, it was possible to create a set of dominating portfolios, such that it was impossible to improve the return furthermore without increasing the risk. Therefore, investors have just to choose across such set of superior portfolios depending on their risk-return preferences. When plotted in a return-standard deviation space, this set of portfolios results in a branch of a parabola, commonly called efficient frontier.

Despite the easiness and attractiveness of Markowitz's ideas, his model presents several drawbacks when applied to the real world. Being based on several strict assumptions, optimal portfolios often result in asset allocations which are not feasible, go against real world limitations or that would require large cost to set up. An example is the large changes into the portfolio composition due to the high sensibility of the optimization process to slight changes in the inputs (returns and variance-covariance matrix). Moreover, several studies proved that these so-called efficient portfolios are often outperformed by more naïve approaches, such as an equally weighted portfolio. Among the main fallacies, the model does not take into account the market capitalization and the relative level of liquidity of the assets that the investors are required to invest in. This represent a huge limitation, especially for large institutional investors. Even if they were feasible, the establishment of large positions on low-liquid assets would have a huge market impact and therefore resulting on worse execution price and lower portfolio return than what predicted by the theoretical model.

The aim of this work is to try to introduce a new parameter, namely the asset liquidity, into the optimization problem. The resulting efficient frontier will be now a 3-dimensional surface, graphing the trade-off between portfolio return, risk and liquidity. It will be investigated whether the introduction of liquidity characteristics results in more feasible portfolio from several points of view, and how gross and net returns respond to the liquidity parameter. The data set used to conduct this analysis includes prices, turnover volumes, Bid-Ask spread, market capitalization and outstanding shares of the constituents of the EuroStoxx600, from 1999 to 2019, with observations recorded daily. The dissertation is then organized as it follows:

Chapter 1 presents a review of Markowitz's mean-variance original framework and further developments, highlighting advantages and fallacies of the optimization problem and the various results.

Chapter 2 gives a description of the concept of liquidity, its multiple definitions and sources and the role that it plays in driving the investment decisions. Moreover, the most common liquidity measures used by the literature and in real world applications will be described as, some of them, will be later implemented in the empirical analysis.

Chapter 3 describes the optimizations models used in the empirical investigation. The liquidity will be proxied by two very popular measures: The Bid-Ask spread and the Amihud measure of illiquidity. The first represents the easiest and most direct approximation of transaction costs, the second is a rough proxy of the market impact. Liquidity will be implemented as an additional constraint in the optimization process, so that each portfolio satisfies a target level of liquidity. Firstly, this constraint will be introduced in a static contest. Hence, we will attempt at creating a Mean-Variance-Liquidity surface in opposition to the more classical mean-variance frontier. We will analyze how portfolio allocation, returns and performance measures vary at the different level of liquidity imposed. Subsequently the liquidity will be introduced also from an active perspective. With a monthly rebalancing, we will analyze four portfolios with a minimum level of illiquidity to understand whether liquidity can play a role in determining the success of an investment strategy.

Chapter 4 provides the data used and reports the MATLAB results of the analysis and optimizations described in part 3.

1. Literature Review: Markowitz's optimization problem

1.1. Before Modern Portfolio Theory

Before the groundbreaking work of Harry Markowitz and the subsequent developments of the modern portfolio theory, asset allocation and portfolio strategies were mainly based on the concepts elaborated by John Burr Williams in his book in 1938, "The Theory of Investment Value". This book, which can be considered as one of the pioneers of the current MPT, introduced notions which have been used, later on, by Markowitz, Modigliani-Miller and Fama. In his work, John Burr Williams made a significant contribution to the field of fundamental analysis, developing the concept of discount cash flow and, in particular, the dividend-based valuation. According to his theories, investors should allocate wealth in those stocks which are undervalued, hoping that the price will, sooner or later, correct to its fundamental value. Therefore, before the 50's, Williams' theory determined the way portfolio managers were investing: they were screening across hundreds of stocks in order to find potentially undervalued investment opportunities. However, as Markowitz pointed out in one of his works, if the value of a stock is the present value of its future dividends, as Williams asserted, but investors care only about the expected value of a security, and consequently the expected value of their portfolio, they should put all their money in the security delivering the highest expected return. However, this makes no sense, since it is well known that the allocation of all your eggs into one basket is not wise choice. Moreover, Williams' investment strategies were little or no concerned about the risk associated with the securities and how the risks of different securities interact with each other. It should be pointed out that Markowitz did not introduce the concept of diversification, which was already well known, even before the 50's. In fact, there are evidences, despite being quite rudimental, of the concept of diversification even during the 16th century, when Shakespeare wrote "My ventures are not in one bottom trusted, nor to one place; nor is my whole estate upon the fortune of this present year; therefore, my merchandise makes me not sad. Act I, Scene 1" in the famous Merchant of Venice (Markowitz, 1999). However, what was lacking at that time, was an adequate theory of investment that could cover the effects of diversification when risks are correlated and analyze risk-return trade-off on the portfolio as a whole (Markowitz, 1999).

1.2. Mean Variance Framework

Markowitz's work in 1952 opened the doors to what it is currently called Modern Portfolio Theory or mean-variance analysis, since it is based on the expected returns (mean) and the standard deviation (variance) of the different portfolios. His groundbreaking work has shown the positive benefits of diversification in portfolio construction. He emphasized that the overall quality of the portfolio can be different from the quality of the individual securities constituting the portfolio itself. In fact, by choosing securities which are not positively correlated (i.e. do not move together in the same direction), it is possible to reduce the overall risk of the portfolio, still maintaining an adequate level of return. In his original work, Markowitz stated that all investors share two preferences: they all prefer higher returns to lower and they all dislike uncertainty. These assumptions ensure that investors are rational and risk averse. Rational investors like more than less, as this maximize the utility obtained, and they like more what is certain than what is uncertain, as the estimation and forecast of future utility becomes easier. These represent few of the several assumptions postulated by Markowitz. Although being quite restricting and hardly to be met in reality, they work quite well for academic purposes as it is possible to obtain a close solution to the optimization problem.

Considering a portfolio of *n* securities, in which investors can invest a fraction ω_i, $i=1, 2, ..., n$, of the available funds into *i-th* the security. The returns on individual securities are assumed to be jointly randomly distributed and represented by the variable R_i, with the expected value being equal to $E(R_i)=r_i$. Letting $\omega = (\omega_1, \omega_2, ..., \omega_n)$ and $r = (r1, r2, ..., r_n)$ be respectively the vectors of securities weights and expected returns, it is possible to define the expected portfolio return as:

$$E(R_p) = E\left(\sum_{i=1}^{n} \omega_i R_i\right) = \sum_{i=1}^{n} \omega_i E(R_i) = \sum_{i=1}^{n} \omega_i r_i = \omega' r \quad (1)$$

Hence, the resulting portfolio return is a simple linear combination of the security specific return and its weight. Concerning the portfolio risk, there is no unique and universal variable to effectively describe this concept. Investors have used different dimensions to express and measure risk, but Markowitz quantified the risk using two well-known statistical measures: the variance and, its square root, the standard deviation. Computations are generally easier when dealing with the variance. Therefore, it is convenient to set up the problem in terms of the variance and then just compute the square root to recover the standard deviation (Fabozzi, 2011). The variance can be expressed as it follows:

$$V_p = \sum_{i=1}^{n} \sum_{j=1}^{n} \omega_i \, \omega_j \sigma_{ij}$$

Where $\sigma_{ij} = E[(R_i - r_i) - (R_j - r_j)]$ represents the covariance between the return of the *i-th* and *j-th* asset. In matrix notation it takes this form:

$$\sigma_p^2 = V_p = \omega' \Omega \omega$$

Where Ω represents the variance-covariance matrix. As we can see from the first equation, portfolio return is just a weighted linear combination of securities returns while the variance of the portfolio depends not only on the variance of the single securities constituting the portfolio, but also on the way these securities move (the correlation), and the amount invested in each. In fact, "other things being equal, the more returns on individual securities tend to move up and down together, the less do variations in individual securities cancel out each other and hence the greater is the variability of return on the portfolio" (Markowitz, 1959). Recalling Williams theories, investors should choose the portfolio which guarantees the highest expected return. No or little attention was paid to the concept of risk. It was claimed that by the law of large number, by investing in a sufficient number of securities, it was possible to eliminate the portfolio risk. However, as Markowitz mentioned in his work, the presumption that the law of large numbers applies to a portfolio of securities cannot be accepted because returns are too intercorrelated. Thus, diversification cannot entirely eliminate the variance (Markowitz, 1952). Markowitz had the brilliant insight that, despite correlation cannot completely eliminate the risk, it might reduce it, without impairing portfolio returns. In fact, correlation and the resulting diversification do play a key role, since is possible to obtain portfolios which dominate single securities. This means that is possible to combine securities in a portfolio whose either return is higher that the single security or the risk is lower (the risk is not completely eliminated as stated by Williams). Any combination of securities that results in a portfolio is called feasible portfolio. The universe of all these feasible solutions is called feasible set of portfolios. Across these, there are some which cannot be dominated in terms of risk and return, the efficient portfolios. A portfolio is said to be efficient when, for a given level of risk, no other combination of securities can generate a higher level of return without increasing the risk. In the same way, is also efficient the portfolio which delivers a given level of return with a lower level of risk (lower variance). The collection of all these portfolios creates a line called efficient frontier. All the feasible portfolios that lie below this line are not efficient enough since for same level of risk, the return can be increased, or risk can be decreased for the same level of reward. The efficient frontier clearly shows the trade-off between risk and reward: as we move to left, the return increases but also does the risk. Being a mere combination of securities' characteristics, the efficient frontier is the same for all the investors, regardless their risk preferences.

There are two optimization problems behind the formulations of the main principle of the mean-variance analysis: investors want to either maximize the return for a given level or risk or minimize the risk for a given level of return. Both the structures deliver the same results, as they represent the same optimization problem tackled from two different points of view. The problem, in matrix notation, is the following:

$$min_w \, \omega' \Omega \, \omega$$
$$subject\ to \begin{cases} \omega' r = \mu_p \\ \omega' e = 1 \end{cases}$$

Where *e* is a column vector of ones, size *n* by 1. This problem aims at minimizing the variance of the portfolio under two constraints: first, the overall return of the portfolio is equal to the target return μ_p, which depends on the investor's risk preferences; the second is a full budget constraint which ensures that 100% of the funds available for investing are allocated in the portfolio. Very often this minimization process is presented with in a similar way, by multiplying the variance by 1/2. This is just a convenience with makes mathematical computations a little easier. However, from a practical point of view, the two processes are equivalent and lead to the same results. Equivalently, the other optimization problem aims at maximizing the portfolio return, keeping the variance at the target level.

$$max_w \, \omega' r$$
$$subject\ to \begin{cases} \omega' \Omega \omega = \sigma_p^2 \\ \omega' e = 1 \end{cases}$$

Where the first constraint defines the target variance σ_p^2 to be achieved. It can be noticed that Markowitz's original optimization process is completely boundless. Except for the full budget constraint, no other restrictions are imposed, so that theoretically any position can be taken on the single assets. As described later, this is far from real world applications and further assumptions are required in order to make the optimization feasible. Both the optimization problems can be solved using the Lagrangian multipliers approach. For the purpose of this work, we will focus on the minimization process. To obtain the minimum, the Lagrangian function has to be obtain as it follows:

$$L = \omega'\Omega\omega - \lambda_1(\omega' r - \mu_p) - \lambda_2(\omega' e - 1)$$

Then, partial derivatives with respect to the vector of weights and the Lagrangian multipliers have to be taken and set equal to zero.

$$\begin{cases} \frac{\partial L}{\partial \omega} = 2\omega\Omega - \lambda_1 r - \lambda_2 e = 0 \\ \frac{\partial L}{\partial \lambda_1} = -\omega' r + \mu_p = 0 \\ \frac{\partial L}{\partial \lambda_2} = -\omega' e + 1 = 0 \end{cases}$$

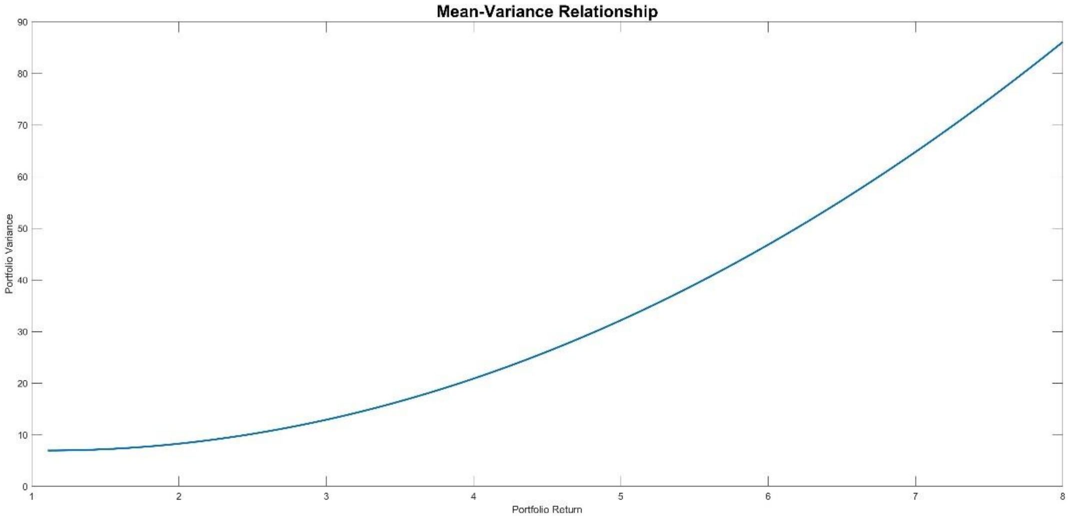
By solving the system for the ω, λ_1 and λ_2, and replacing the values of the Lagrangian multipliers into ω, it is possible to derive the optimal weights equation, ω^*:

$$\omega^* = \frac{(\gamma\Omega^{-1}r - \beta\Omega^{-1}e)\mu_p + (\alpha\Omega^{-1}e - \beta\Omega^{-1}r)}{\alpha\gamma - \beta^2}$$

Where $\alpha = r'\Omega^{-1}r$, $\beta = r'\Omega^{-1}e$ and $\gamma = e'\Omega^{-1}e$. The vector ω^* represents the optimal weight for each asset in the portfolio, satisfying both the return and the full budget constraint. The efficient frontier is then derived by varying the target return in the constraint and keep solving the optimization problem for the different levels. Also, the equation describing the efficient frontier can be easily retrieved by substituting the weights' equation back into the minimization problem.

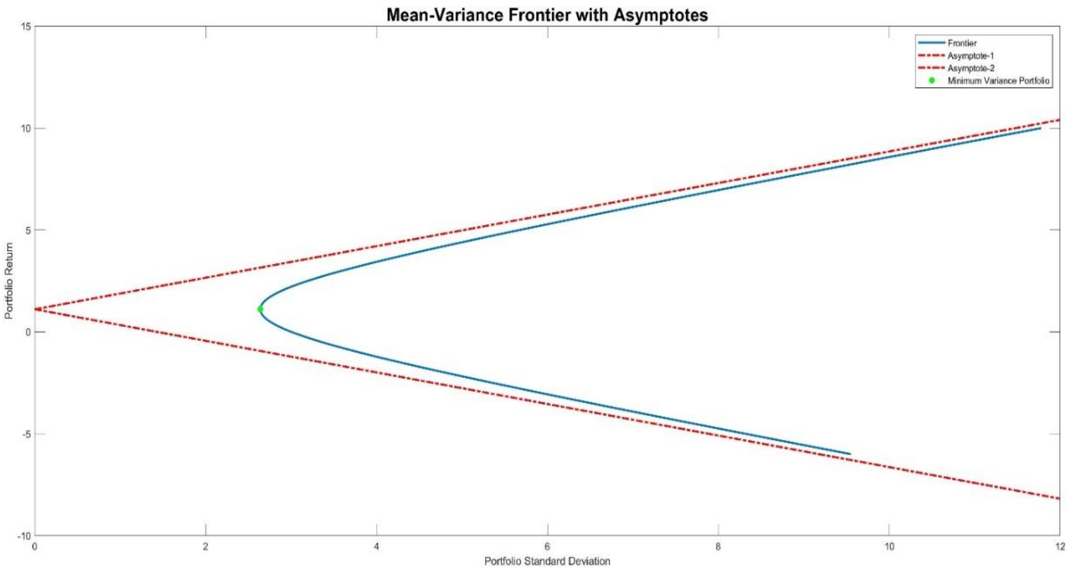
$$\omega'\Omega\omega = \sigma_p^2 = \frac{\gamma{\mu_p}^2 - 2\beta\mu_p + \alpha}{\alpha\gamma - \beta^2}$$

This expression represents the analytic equation of the efficient frontier, relating the portfolio variance with the expected return. As it can be noticed, the variance is a quadratic function of the portfolio expected return. Therefore, the frontier has a parabolic shape in a mean-variance space, where the portfolio variance is represented on the y-axis and the expected return on the x-axis.



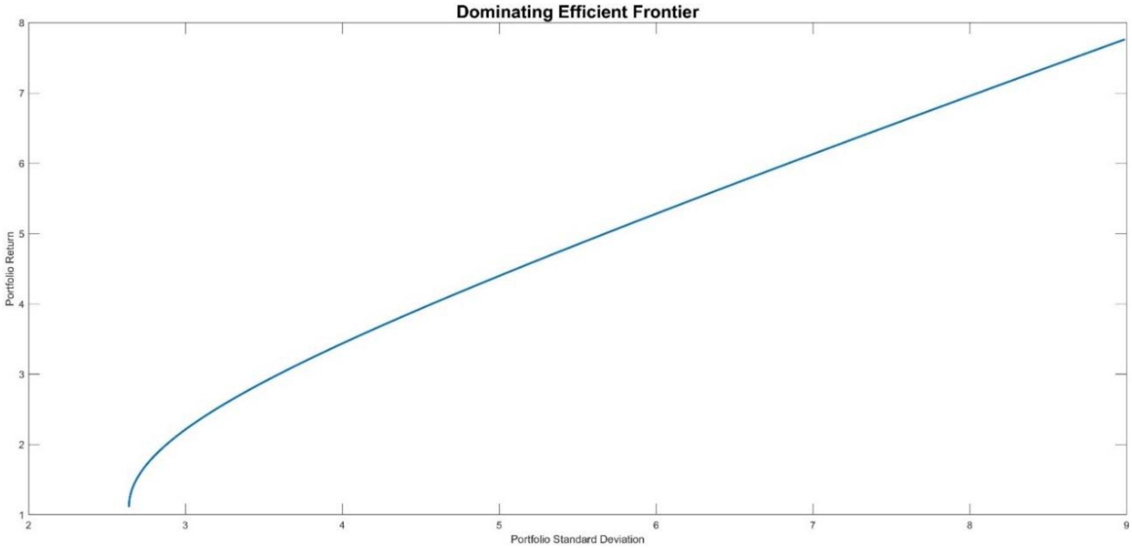
(Figure 1. Mean-Variance parabola, computed for the EuroStoxx 50, time window: 2009-2019)

However, given such shape, the understating of the existence of a dominating solution in the space is not immediate. To this end, it is much more convenient and easier to represent the frontier in a mean-standard deviation plane, with returns on the y-axis and standard deviation on the x-axis, given also that the two variables have the same unit of measurement. In fact, being the standard deviation a direct transformation of the variance, the optimization problem could be solved directly for the standard deviation and the results would be identical. In this scenario, the frontier is represented by a horizontal parabola, with the vertex located on $\left(\frac{1}{\sqrt{\gamma}}, \frac{\beta}{\gamma}\right)$ and the two asymptotes following the equation $\mu_p = \frac{\beta}{\gamma} \pm \sigma_p \sqrt{\left(\frac{\alpha\gamma - \beta^2}{\gamma}\right)}$.



(Figure 2. Full mean-standard deviation frontier with the green dot representing the vertex and the red lines the asymptotes, computed for the EuroStoxx 50, time window: 2009-2019)

The vertex represents one of the portfolios of major interest, the Global Minimum Variance. Such portfolio is the one characterized by the lowest possible volatility. As the figure shows, no matter what the reduction in the expected return is, it is not possible to reduce the risk of the portfolio below the Minimum Variance level. The Global Minimum Variance is also important because it represents the threshold between the dominating efficient frontier, the part of the curve above the vertex, and the so-called inefficient frontier, the part below it. The part below is called inefficient because, despite having even lower returns than the Global Minimum Variance, they have a larger standard deviation and hence higher risk. Since these portfolios are exactly on the same standard deviation level of a portfolio on the upper curve, they are dominated by those as they have same risk but lower return.



(Figure 3. Dominating Mean-Standard deviation Efficient Frontier, computed for the EuroStoxx 50, time window: 2009-2019)

In building the efficient frontier, the correlation between the securities represents a key driver. Recalling that the correlation moves in the interval [-1,1], these benefits can be easily demonstrated considering an example with just two securities. When the correlation is perfectly positive (equals to 1), there is no space for diversification benefits. The efficient frontier is a simple line going through the two assets; thus, the expected return of the portfolio is linearly increasing in the risk. The opposite extreme, correlation equals -1, represents the best possible scenario as it is theoretically possible to combine the two securities in a portfolio with essentially zero volatility. When correlation moves in between the two extremes, the efficient frontier moves between these two cases. Of course, this is a clear extremization of the reality, but it is still important to explain why negative correlation and potential for diversification are important drivers in building the efficient frontier.

One of the most attractive features of this framework is that, provided the assumptions are satisfied, the only inputs necessary are returns, variance and correlation. Despite this, the estimation of such parameters is not easy, and tend to produce several drawbacks in the asset allocation. Markowitz himself did not specify how to estimate those parameters, whether investors should use historical values or not to compute the expected returns and the variance-covariance matrix. What he did say was that investors should use “relevant beliefs about future performances” without specifying how to get those (Markowitz, 1952). Later, however, he stated that average past returns, and past covariance could actually be assumed as a good proxy for future return and risk measures. This poses several problems both from the results validity and appropriateness of the measures point of view. However, as much as it might be an oversimplification, for academic purpose, the use of historical data makes the computations much easier. That is why, historical values of returns and variance will be used to compute the mean-variance analysis and any further development in this dissertation.

1.3. Relevant portfolios: Global Minimum Variance and Maximum Trade-off

As mentioned, the optimal portfolio for an investor strongly depends on his risk aversion characteristics. More risk averse individuals will prefer safer portfolios, with less risk attached, while more risk lover investors will prefer portfolios with a stronger risk component. However, across all the efficient portfolios, there are two which have desirable and interesting characteristics: the Global Minimum Variance and the Maximum Trade-off (also known as Max Sharpe) portfolio. These are, respectively, the portfolios with the lowest possible risk (assuming the variance/standard deviation as proxy for the risk) and the portfolio with the highest Sharpe ratio, hence the highest risk-return trade-off. These portfolios will be later used in the empirical analysis. The Sharpe measure is indeed defined as the ratio between the portfolio excess return over the risk-free r_f rate divided by the standard deviation of the portfolio itself.

$$\frac{\mu_p - r_f}{\sigma_p}$$

It essentially measures the reward over the risk-free rate per unit or risk taken. Ideally investors are looking for portfolios with positive and high Sharpe ratio. Negative values suggest that the portfolio is underperforming the risk-free security, despite the higher level of risk taken. In this case, investors would be better off by switching their investments to the risk-free asset. Given their characteristics, these portfolios are easily recognizable along the efficient frontier. The popularity of the Global Minimum Variance has risen quite a bit in the literature in the past years. This popularity comes from the fact that the Global Minimum Variance has some desirable features, both from the statistical and portfolio composition point of view. First, stock returns are difficult to estimate (Merton 1980). Estimates might significantly differ from the real values. This estimation errors would affect the optimization process and ultimately result in a suboptimal portfolio composition, which might lead to poor performances (Kempf, Memmel 2005). However, the Global Minimum Variance is the result of variance minimization problem and thus it is not affected by errors in the estimation of returns. In fact, variance-covariance matrix can generally be estimated much more precisely than returns, therefore improving the optimization results (Kempf, Memmel 2005). Moreover, there exist more and better methods to cope with the uncertainty in the variance-covariance matrix than with the estimation of returns. Second, it has been often proved that low volatility stocks do not perform much worse than those with much higher volatility (Coqueret 2015). Therefore, the Global Minimum Variance is the result of a portfolio optimization problem, where the only 'binding' constraint is the budget one (no target return constraint):

$$min_{\omega}\ \ \omega'\Omega\omega$$
$$subject\ to\ \omega'e = 1$$

The Lagrangian reduces to:

$$L = \omega'\Omega\omega - \ \lambda_1(\omega'e - 1)$$

With the following partial first derivatives:

$$\begin{cases} \dfrac{\partial L}{\partial \omega} = 2\omega\Omega - \lambda_1 e = 0 \\ \dfrac{\partial L}{\partial \lambda_1} = -\omega'e + 1 = 0 \end{cases}$$

Solving for λ_1 and the substituting back into the weights' formula, we obtain:

$$\omega_{GMV} = \frac{\Omega^{-1}e}{e'\Omega^{-1}e} \quad r_{GMV} = \frac{r'\Omega^{-1}e}{e'\Omega^{-1}e} = \frac{\beta}{\gamma} \quad SD_{GMV} = \frac{1}{\sqrt{\gamma}}$$

As anticipated, the Global Minimum Variance is located on the vertex of the efficient frontier parabola with coordinates $\left(\frac{1}{\sqrt{\gamma}}, \frac{\beta}{\gamma}\right)$. Moreover, the weight combination is a mere function of the variance-covariance matrix, thus reducing the risk of mis-allocation due to errors in the estimation of the returns. On the other hand, the Maximum Trade-off portfolio is the result of a more complex optimization problem, which aims at maximizing the Sharpe ratio, still under the budget constraint.

$$max_{\omega} \ \frac{\omega' r}{\sqrt{\omega' \Omega \omega}}$$
$$subject\ to\ \omega' e = 1$$

Again, solving the first derivatives for ω and for $\lambda_1$and substituting back it into the weights' formula, we obtain the following quantities:

$$\omega_{TAN} = \frac{\Omega^{-1} r}{e' \Omega^{-1} r} \quad r_{TAN} = \frac{r' \Omega^{-1} r}{e' \Omega^{-1} r} = \frac{\alpha}{\beta} \quad SD_{TAN} = \frac{\sqrt{\alpha}}{|\beta|}$$

The Maximum Trade-off has coordinates $\left(\frac{\sqrt{\alpha}}{|\beta|}, \frac{\alpha}{\beta}\right)$. Differently from the Global Minimum Variance, the weights are a function of both the variance-covariance matrix and the returns. Therefore, the estimation of the returns plays a key role in determining the portfolio composition. The Maximum Trade-off portfolio acquires particular importance when a risk-free security is added to the analysis. In this scenario, according to the one fund theorem later described, the Maximum Trade-off, also referred as the tangency portfolio, represents the optimal risky portfolio in which every individual should invest in, regardless the risk preferences. Then, the investor can control the overall risk and adapt it to his preferences by taking a position on the risk-free security. A deeper discussion is presented in the next paragraph. However, it is important to point out that, when the risk-free security is introduced, the resulting Maximum Trade-off portfolio will be different from the one estimated from the efficient frontier with only risky securities, as now the portfolio's Sharpe ratio to maximize is defined with respect to the risk-free rate.

In alternative to the Global Minimum Variance and the Maximum Trade-off, there is another portfolio which, despite not necessarily lying on the efficient frontier, is of particular interest for the investors: the equally weighted portfolio. This portfolio is often referred as a naïve portfolio because it is based on a naïve diversification rule. Going against what introduced by Markowitz, the equally weighted portfolio invests evenly across all the *N* securities available, ignoring the data and any optimization process. Despite its simplicity, a naïve diversification approach is widely used as benchmark for several reasons. First, it does not involve any estimation error as moments do not need to be computed in order to find the optimal asset allocation. Moreover, many studies show a lack of diversification in the portfolio holdings. An equal weighted approach ensures sufficient diversification, avoiding large concentrations in the same small group of assets. This, in turn, might improve the portfolio liquidity and performances by partially reducing the market impact of Markowitz's portfolios. Second, even after decades of new models and new methods to estimate parameters and to cope with the estimation errors, many investors keep using a naïve approach to allocate wealth across assets, especially when looking at passive investment strategies (DeMiguel, Garlappi, Uppal 2009). Moreover, many studies proved that the equally weighted portfolios perform not much worse, or even outperform, more advanced strategies, especially when the number of securities involved is high, since, even if naively, there is more potential for diversification while the increasing number of parameters to be estimated for the more advanced models also increases the potential for estimation errors. Similarly, naïve approaches work best when the data available are limited, as the parameters estimation becomes less precise (DeMiguel, Garlappi, Uppal 2009).

1.4. Efficient frontier with risk-free asset

Markowitz's biggest achievement is, without any doubt, the quantification of the importance of diversification in the portfolio construction. Moreover, the introduction of the MPT, provided the foundations for several other developments. Particularly remarkable has been the work done by Tobin (1958), Sharpe and Lintner in analyzing the impact of introduction of a risk-free asset in the analysis. They showed that adding a risk-free component generates efficient portfolios which are superior to those available to investors without it (Rachev, Stoyanov, Fabozzi 2008). Moreover, they found out that the efficient portfolios are a combination of a specific risky portfolio, called market (or tangent) portfolio, common to all the investors, and the risk-free securities. Investors are still assumed to optimize their holdings as stated by Markowitz, but now they are also allowed to borrow or lend as much as they want at a risk-free rete (the assumption of unlimited borrowing has been widely proved to be unrealistic). Lending corresponds to having a long position, while shorting the risk-free security corresponds to borrowing. This assumption has important effect on the budget constraint, as the weights of the risky portfolio no longer have to sum to one, as investor are allowed to short the risk-free security and thus increase their long position on the risky assets above 100%. However, it is still assumed that the positions on the two assets equal the unity, so that all the funds are still invested (such assumption is implicitly incorporated in the target return constraint). Therefore, investors allocate their funds between a portfolio of (risky) securities and the risk-free asset. The expected return of this combined portfolio is a simple linear combination of the returns,

$$ER = \omega'\mu + \omega_{rf} r_f$$

With ω_{rf} representing the fraction invested in the risk-free asset (it is a scalar value since it is assumed that there is only one risk-free asset). The variance (and standard deviation), on the other hand, has the same structure as before, since the risk-free asset has zero variance.

$$V_p = \omega'\Omega\omega$$

The optimization problem takes the following form

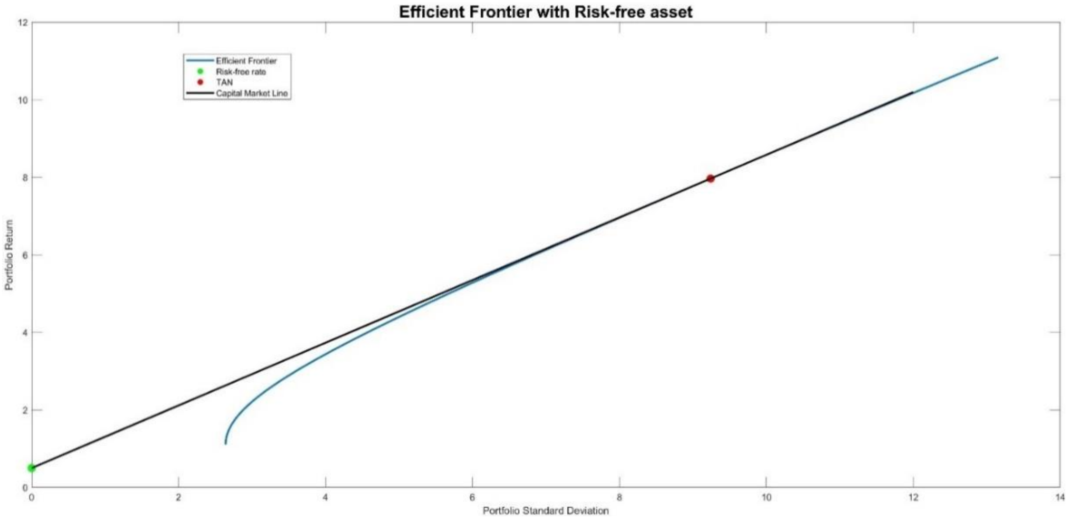
$$min_w \, \omega'\Omega\,\omega$$

$$subject\ to\ \omega' r + (1 - \omega' e) r_f = \mu_p$$

And the resulting equation of the efficient, expressed in a return-standard deviation space, is:

$$\mu_p = r_f \pm \sigma_p \sqrt{H}$$

With H being equal to $H = (r - er_f)'\Omega^{-1}(r - er_f)$. In this case the efficient frontier is no longer a parabola in the mean-standard deviation space, but it is a straight line, which intercepts the y-axis at the risk-free rate value. In the extreme case where $r_f = \beta/\gamma$, the efficient line would equate the asymptotes previously calculated for the risk-only frontier. However, if that was the case, the Global Minimum Variance would have the same expected return of the riskless rate, which is clearly impossible (Hubbert 2007). Therefore, the only feasible case is when $r_f < \beta/\gamma$. Since such frontier represents all the possible combinations of the risk-free security and a risky portfolio, it is possible to create an infinite number of lines which differ only in the slope's coefficient (Vasicek, McQuown 1972). It can be proved that the slope of this new efficient frontier coincides with the Sharpe ratio of the risky portfolio selected. Since all investors are utility maximizer, they would like to maximize the risk-return trade-off, hence, they would like to invest in a portfolio which lies on the highest feasible line. Given this assumption, the best efficient line is the one going through the tangency portfolio, since it has the highest Sharpe ratio. Thus, the efficient frontier with risk-free asset can be thought as any combination of the tangency portfolio and the riskless asset. This assumption is commonly called one fund theorem. The tangency portfolio is often referred as market portfolio since, under certain assumptions, it consists of the entire universe of available assets. Indeed, if all the investors share the same views about asset moments, they will all invest a fraction in the tangent portfolio and borrow or lend the risk-free security, hence demanding the same risky portfolio. Then, assuming that markets are in equilibrium so that supply equals demand, every investor's holdings will be made up of a part of the same portfolio. It follows that the tangency portfolio comprises all the shares outstanding of all the common stocks in the market. This portfolio is thus called "market portfolio" and includes all the outstanding shares, proportionally to their market capitalization (Vasicek, McQuown 1972). The resulting frontier is commonly known Capital Market Line (CML) and it can be represented as it follows:



(Figure 4. Capital Market Line, computed for the EuroStoxx 50, time window: 2009-2019)

It is easy to notice that all portfolios that lie on the risky efficient frontier (Markowitz's original frontier) are now inferior solutions to the portfolios on the CML, in the sense that they result in lower expected returns for the same amount of risk (Vasicek, McQuown 1972). This theorem has important implication as, differently from the world without risk-free asset, all investor should now pick the same portfolio regardless their risk preferences. Risk aversion does not determine the composition of the risky portfolio anymore, but it affects the composition of the overall efficient portfolio, resulting from the combination of the risk-free asset and the market portfolio. More risk averse individual will position themselves to the right of the optimal portfolio along the CML, taking a long, but smaller than the unity, position on the tangency portfolio and investing mainly in the risk-free asset. On the other hand, very risk lover might opt to invest everything in the tangency portfolio or even short the risk-free asset (borrow) in order to take a long position exceeding the unity. The crucial conclusion thereby suggested is that every investor must resolve what risk level he is willing to assume but need not to select particular stocks nor be concerned with how to combine them into a portfolio (Vasicek, McQuown 1972). Theoretically, if all the model's assumptions are satisfied, he could simply invest in all the stocks in the market and borrow/invest in the risk-free asset and automatically obtain the efficient portfolio.

1.5. Optimal Portfolio and Utility Maximization

In portfolio theory, as in many other situations in economics, individuals have to make choices in a world dominated by uncertainty: in this context, they are asked to choose across a set of portfolios, each with its own risk-return profile, and, thus, different level of utility. Since rational individuals prefer more to less, they always make decisions in order to maximize the resulting expected utility. Therefore, portfolio choice should also be concerned about maximizing the utility. If the mean-variance dominating portfolio individuated by Markowitz are truly superior choices compared to the non-efficient ones, they should also produce a higher level of utility than all the non-dominating portfolios, otherwise investors might opt for those portfolios that do not lie on the efficient frontier. In the standard portfolio analysis, the primary concerns for investors are risk and returns. Thus, utility function mainly express individual's preferences towards these parameters, which represents only the first two moments of the distribution of returns. In particular, the slope of the function is positive to reflect the fact that, to maintain the same level of utility, a risk-averse investor has to be compensated with higher returns in order to accept higher risks (Fabozzi 2012). Moreover, the more risk averse an individual is (i.e. the larger is the risk aversion coefficient), the steeper is the utility curve, since he must be extremely rewarded in order to accept an even small increment in the risk. As mentioned above, the optimal portfolio for an investor depends on his risk preferences, embedded in his utility function. Therefore, the problem reduces to the estimation of investor's utility function. The literature behind the estimation of the indifference curves is very wide, but also very debated. While the existence of a function that can perfectly explain and quantify individuals' preferences would be extremely useful, in reality this is not always the case. In fact, there are many evidences from both traditional and behavioral studies, that not only it is difficult to determine the utility function, but also individuals are not fully rational as Markowitz predicted and the utility function type, as well the risk preferences, might change depending on the circumstances and time (Fabozzi, Pachamanova 2010). An example is the fact that households' portfolio tends to be home-biased, with extreme low diversification across foreign stocks. It seems that foreign stocks are perceived risker than similar domestic stocks. Clearly, this behavior reflects non-rational beliefs. French and Poterba (1991) have shown that investors in the USA and Japan allocate more than 90% of their overall investment in domestic stock, resulting in a much lower diversification than what standard portfolio models suggest. Moreover, the assumption that utility is expressed just by the first two moments of the distribution of returns is a departure from the expected utility theory. According to it, when facing many uncertain scenarios, an individual always chooses the prospect yielding the highest level of expected utility. In that case, the choice depends on the entire probability distribution function of the return and not merely on the expected mean and the standard deviations (Biswas 1997). However, assuming that investors are fully rational and utility maximizer makes the analysis easier and allows for a solution to the optimization problem. Then, under certain assumptions, the mean-variance utility coincides with the expected utility theory. Let still be r_p the random portfolio return, W the final wealth, $W = W_0(1 + \mu_p)$, and W_0 the initial wealth which can be normalized to the unity so that the only argument of the utility function is the return. Because of the difficulties in estimating the correct utility, practitioners often work with a mean-variance approximation of the chosen utility function (Fabozzi, Pachamanova 2010). In fact, by applying a Taylor series expansion around μ_p, it is possible to show that a mean-variance framework is reconcilable with the expected utility theory, under specific assumptions about investor's preferences and returns distribution. Despite being just an approximation, many studies showed that a two moments expansion works as a useful proxy to the expected utility when it comes to selecting portfolios of common stocks and including higher moments do not always improve the results (Hlawitschka 1994). A common assumption about the utility shape is that it is quadratic. A quadratic utility function has several mathematical advantages, such as that all the derivatives above the third power are zero. Moreover, Levy and Markowitz (1979) showed that, when dealing with mutual funds, the expected utility function can be very well approximated by a function of the mean and the variance of the portfolio returns. They found out that a mean-variance approximation performs particularly well when returns range from -30% to 60%. Moreover, they demonstrated that the mean variance frontier included the portfolio maximizing the true utility function (Cremers, Kriztman 2003). Therefore, the quadratic approximation can take the following form

$$U_{quad}(r_p) = \omega' r - \frac{A_{quad}}{2} (\omega' r)^2$$

Where $A_{quad} > 0$ represents the investor's attitude towards risk and it is generally different from the one described by Pratt (1964). However, the presupposition of a quadratic function results in some unrealistic assumptions about the investor's behavior. First, a quadratic utility is characterized by increasing absolute risk aversion, which implies a reduction in the nominal amount invested in risky assets as the wealth increases. Second, it exhibits a positive marginal utility function only up to a certain level of wealth, after which it starts declining (Ingersoll 1987). Alternatively, similar conclusions can be derived assuming a multivariate normal distribution of the return, as investors can infer the entire distribution of returns from its mean and variance and, therefore, higher moments become irrelevant (Cremers, Kriztman 2003). When returns follow a normal distribution, a mean-variance approach makes sense regardless the shape of investor's utility. A very common choice when dealing with normally distributed returns is the negative exponential utility function o CARA (Constant Absolute Risk Aversion), which is easier to optimize than some of the other utility functions.

$$U_{exp}(W) = -e^{-A_{exp}W}$$

With A_{exp} representing the Arrow-Pratt constant coefficient of risk aversion. Then, the maximization of such utility is equivalent to the so-called mean-variance utility function expressed as (Ingersoll 1987)

$$\omega' r - \frac{A_{exp}}{2}\omega'\Omega\omega$$

In both cases, the resulting expected utilities will depend merely on the first two moments of the returns. Hereof, the connection with the mean-variance framework. Therefore, besides the methods previously shown, there is another way to derive the efficient frontier, which consist on the direct maximization of a mean-variance utility function (A_{exp} is replace with A to simplify the expression)

$$max_{\omega}\ \ \omega' r - \frac{A}{2}\omega'\Omega\omega$$

$$subeject\ to\ \omega' e = 1$$

Recalling that the maximization of a function equals the minimization of its negative form, we can express the Lagrangian in the following form:

$$L = -\omega' r + \frac{A}{2}\omega'\Omega\omega - \lambda_1(\omega' e - 1)$$

The solution to this minimization problem can be easily obtain by replicating the steps followed for the standard optimization problem. The resulting solution is:

$$\omega_U = \frac{1}{A}\Omega^{-1}\left(r - \frac{\beta - A}{\gamma}e\right)$$

Or, alternatively,

$$\omega_U = \frac{\beta}{A}\omega_{TAN} - \left(\frac{\beta - A}{A}\omega_{GMV}\right)$$

The last representation is particularly interesting because it shows that the optimal portfolio, and in general any optimal portfolio for a given risk parameter A, is a linear combination of the Global Minimum Variance and Maximum Trade-off portfolio. When the investor is infinitely risk lover, so that $A \rightarrow 0$, the solution is given by an extreme long position in the Maximum Trade-off and a short on the Global Minimum Variance. On the other hand, when $A \rightarrow \infty$ the investor is infinitely risk averse, and the solution simply converges to the Global Minimum Variance portfolio, with zero holdings in the Maximum Trade-off. Therefore, by varying the risk parameter and solving the optimization it is possible to derive the efficient frontier, which is exactly equivalent to the one derived in the previous sections. It is easy to understand the attractiveness of the mean-variance approximation. It provides a direct way to compute the portfolio maximizing the expected utility, instead of having to compute the utility for each possible portfolio's composition along the frontier (Cremers, Kriztman 2003). Despite being quite straightforward, this procedure is not exempt from criticisms. Even if all the assumptions about the utility form and the distributions of returns are correct, there still would be several pitfalls. The main drawback concerns the estimation of the risk aversion parameters. In fact, despite the large literature behind, nowadays there is no procedure which produces a unique result. There have been many attempts to directly elicit the risk aversion preferences, using both qualitative and quantitative procedure. Qualitative procedures exploit surveys and questionnaires, which are also widely applied in psychology. The main shortcoming with these kinds of procedures is that they do not directly quantify a risk aversion parameter but simply allow to determine whether people are more or less risk averse. Results tend also to be biased, as people might not fully understand the questions. Moreover, and this probably represents the biggest problem, people do not distinguish between risk aversion and risk perception (which means that probabilities are not held constant). Therefore, some people might get classified as extremely risk averse only because they attach a much bigger probability to a risky event that what others do. Quantitative approaches, on the other hand, can provide an estimate of the risk aversion parameter. These approaches are based on simulated games or scenarios where people are asked how they would behave depending on the conditions. Then, by using those results and making some assumptions about the utility function, it is possible to infer the risk aversion parameter, provided that individuals are utility maximizer. The main advantage, compare to qualitative methods, is that now probabilities are given (and thus held constant across individuals), therefore there is no error caused by different risk perceptions. However, also these approaches have some drawbacks. Many evidences show that people involved in these simulations tend to underestimate their willingness to pay, hence the resulting risk aversion is overstated. Moreover, the answers tend to change in accordance to the size of the game. Once that the efficient frontier has been constructed, the next step is to determine the optimal portfolio. Since the higher is the indifference curve, the higher is the final utility, rational investors would always like to be on the highest possible indifference curve. Therefore, the resulting optimal portfolio, will be determined by the tangency of the highest utility curve with the efficient frontier, or with the CML line in case the risk-free asset is included in the analysis.

Concluding, while Bernoulli and Von Neumann and Morgenstern's work suggests that investors wish to rationally maximize their utility, practitioners have found that for most investors, the utility function is an impractical device for selecting portfolios. In their experience, they find that investors do not fully understand the concept of utility and are generally unable to provide the information required to determine their function analytically. This also explain why the literature is mainly focused in dealing with simpler approximation (Guerard 2010).

1.6. Drawback and Fallacies of Mean Variance Framework

The mean-variance framework, and the portfolio selection methods arising from it, have become standard investment tools. Aside from their theoretical appeal, the easiness of the practical implementation was surely among the crucial determinants of their great popularity. The constant development of technology allows to run optimization algorithms for the computations of the minimum variance and the market portfolio quite fast also on personal computers. Moreover, it is also relatively easy to estimate returns, variances and covariances, at least using historical data (Morawski 2008). However, despite such attractiveness, the mean-variance analysis is subject to several limitations and fallacies. Michaud (1989) pointed out that the traditional procedure often leads to financially irrelevant or false "optimal" portfolios and, sometimes, even a naïve approach, as an equally weighted portfolio, might turn out to be a superior solution.

The first shortcoming concerns the estimation of the risk-return parameters. In fact, while using historical data simplifies a lot the process, the results obtained have little meaning for future portfolio allocations and performance. Therefore, a forward-looking estimate is required. However, this complicates the analysts' work. In fact, as Michaud (1989) and Black and Litterman (1992) pointed out, Markowitz' allocation process tends to predilect those securities with high expected returns and negative correlations compared to those with low return and positive correlation. However, these securities are also those more prone to estimation errors (Michaud 2001). Therefore, the optimization procedure ends up maximizing these errors. This problem might be partially alleviated by focusing only on the minimization of the risk. However, even in this case, the estimation procedure is not so easier. If, for example, a portfolio is composed by 50 securities, this requires the estimation of 50 expected returns and 1225 covariances (the number of covariance is $N(N-1)/2$). While the estimation of 50 returns might still be reasonable, it is clear that 1225 covariances are not feasible to be estimated without incurring in significant errors. In fact, Markowitz realized that some kind of model for the covariance structure was needed for the practical application of normative analysis to large portfolios (Morawski 2008). Moreover, Markowitz introduced the variance as a proxy for risk. However, the variance is more a measure of uncertainty and, while it is straightforward that there is a positive relationship between risk and uncertainty, the variance does not really capture the concept of risk. In fact, the variance treats all the deviations from the mean as the same, either they are positive or negative. However, risks are not symmetrical, and investors are generally more concerned with the negative ones (Rachev, Stoyanov, Fabozzi 2008). This is especially the case of hedge funds, whose potential downsides are extremely large, and the variance cannot really capture the full extent of the risk taken (Fung, Hsieh 1999). Markowitz suggested to use the downside semi-standard deviation as a proxy for risk, which is the deviation of the returns falling below a given threshold. This approach also has the advantage of being tailored to the specific objectives and risk profile of investors with different levels of target return. Another measure of dispersion, the Mean Absolute Deviation (MAD), might be used. Differently from the standard deviation, the MAD measures the absolute distance from the mean, rather that the dispersion squared. The MAD tends to be more robust to outliers, those observations falling in the tails of the distribution (Morawski 2008). Alternatively, "safety first risk measures" can be used. These risk measures generally value the probability that portfolio returns are lower than a given target. The most famous measure is the Value at Risk (commonly abbreviate to VaR). However, even the VaR has its own limitations, as it does not consider the concentration of returns in the tails beyond the threshold. To overcome these limitations, conditional Value at Risk (CVaR) has been introduces. CVaR, also called expected shortfall or expected tail loss, measures the expected value of portfolio returns, when returns fall beyond the VaR threshold. In this sense, the CVaR represent a much more coherent risk measure (Morawski 2008).

Another main drawback, highly linked with estimation error, is the fact that the mean-variance model tends to be quite unstable. As the inputs slightly change, the resulting portfolio composition might be completely different. Since trading costs are directly proportional to the size and number of trades, such instability would translate in very high transaction costs, thus undermining the portfolio performances (Kourtis 2015). Moreover, Markowitz's original optimization process, does not include any constraint at asset level, so that the optimization can determine positions of any size. The optimal portfolio can result in extreme large positions, which in reality might be unfeasible. This is particularly true for many institutional investors which have investment restrictions and might not be allowed to short securities. That is why, in order to obtain more realistic and stable results, robust estimators should be included in the analysis, as they result less sensitive to outliers (Morawski 2008). Such solutions include the Black-Litterman Model, which allows investors to make assumptions about future returns (so-called views) and express how reliable they think these views are. The views, weighted by the assigned probability, are then combined with equilibrium values to determine the expected returns and standard deviations. Michaud, on the other hand, developed a model to reduce the impact of estimation errors by computing mean and covariance matrix from a sample of simulated returns. Another drawback is that the model does not allow to express the level of uncertainty for the estimates of the parameters. This is a quite relevant issue especially during period of high volatility and uncertainty, such as during financial crisis. Indeed, it is reasonable to assume that estimates of parameters during such periods are likely to have an extra component of uncertainty compared to period of steady growth, where forecasting is much easier. Since these periods are quite different, Chow created a model where observations are distinguished in inlier and outlier, with respect to a threshold parameter, and portfolio allocation is calculated in both scenarios. Then, the resulting optimal portfolio allocation is a weighted average of the inlier and outlier portfolio, where the weights are the probability of being in a situation of steady growth (inlier observations) or during a financial crisis (outlier observations).

However, the main fallacy linked to the purpose of this dissertation is that the mean-variance model does not take into account information about the market capitalization and liquidity of the stocks during optimization process. While this is likely to be irrelevant for the small, private investor given the size of his trades and portfolio, it might significantly affect the large institutional investors' decisions. As mentioned, sometimes the portfolio allocation resulting from the optimization requires very large long and/or short positions, which may be very costly or even impossible to establish. However, problems arise also with small position, when these have to be taken on illiquid assets (i.e. Small or Mid-Cap stocks or asset with low liquidity such as real estate). In fact, generally, the more capitalized is a company, the more shares are usually available for trading and the more liquid it can be. Such stocks would allow investors to establish quite large positions without significantly impact the execution price. On the other hand, for less liquid asset, it might be difficult even to take a small position without affecting the market, and thus negatively impacting the portfolio performances. Therefore, stocks' liquidity can play a key role in affecting portfolio feasibility and returns. There have been attempts to include a "liquidity dimension" into the analysis. A popular approach consists in incorporating a measure of transaction costs into the model. The idea behind is to penalize the portfolio turnover, which means to penalize the changes in the vector composition respect to an initial portfolio, by setting a given parameter k which should reflect the extent of the transaction costs (Kourtis 2015). The model would still aim at minimizing the variance, but under the following constraint:

$$\omega' r - k\,|\omega - \widetilde{\omega}| = \zeta$$

Where $\widetilde{\omega}$ represent the initial portfolio composition. However, given the nonlinear form of transaction costs, such problem tends not to have a closed form solution and computations might be hard and inefficient, especially when the number of assets is large (Kourtis 2015). Given the large dimension of the universe and time window used in the empirical analysis, this approach has been considered unpracticable. Alternatively, Kourtis (2015) proposed a model to improve the stability, penalizing the portfolio changes. He suggested to include a such penalty directly into the minimization function, so that investors face a trade-off between efficiency and stability, which can be controlled through the stability parameter c.

$$min_w\, \omega'\Omega\, \omega + c(\omega - \widetilde{\omega})'\Omega(\omega - \widetilde{\omega})$$

The larger is the parameter, the more the stability will be considered important, the closer will be the optimal portfolio to $\widetilde{\omega}$ and thus the larger will be the deviation to the Markowitz's equivalent portfolio (Kourtis 2015). The optimal portfolio will be just a combination of a portfolio along the efficient frontier and the initial portfolio $\widetilde{\omega}$.

2. Liquidity

In the literature, it is difficult to find a unique definition of what liquidity is. This is because there are different types and dimensions of liquidity. There is an asset specific type of liquidity, there is a market liquidity and there is also a concept of liquidity at a corporate level, when looking at the company's solvency and ability to pay. While the latter is of little relevance for the purpose of this analysis, the first two are of greater interest. Most researchers and investors can ideally identify liquidity characteristics when analyzing a security or the market. However, it is a bit more difficult to try to uniquely define it. As it is explained after, over time, a lot of different definitions and measures have been developed, each of them describing a given aspect of the liquidity. However, given that no unique solution currently exists, from a practical point of view, investor have a sort of "discretionary" approach in defining what is liquid. The only thing sure is that liquidity is used on a daily basis as investment criterion by many investment funds and banks, and, at the end of the year, liquidity can represent a significant variable in determining the sign of the trading P&L. First, it is important to understand why liquidity is so important. The easiest answer is that liquidity determines the market value of a product, not only when it comes to stocks or financial assets, but basically for any product in the world. Given everything else equal, the easier it is to find, the less you are willing to pay. To get a little bit more technical, the degree of liquidity affects the asset's value, as more marketable product tends to have a higher value than more illiquid. This is because investors value the ability and the speed at which is possible to convert their holdings into cash if necessary. Therefore, having low liquid assets poses extra risk challenges as they might end up holding those securities more than what they would ideally want, and thus increasing their exposure to market fluctuations (Dyl, Jiang 2008). However, as mentioned, there are different measures of liquidity, and none of them can ultimately determine whether an asset is liquid or not, but at least it is possible to get an idea of what is more or less liquid than others. Probably, the most widely recognized proxy of liquidity when working with stocks is the Bid-Ask spread, which has been extensively used in empirical studies over the years.

These risks, associated with unforeseen impossibility to liquidate a position when required, have also stimulated the literature to try to find whether this extra risk component (extra in the sense that does not directly depend on prices changes) is actually rewarded somehow. Indeed, many researches, including Amihud and Haim Mendelson (1986) and Amihud (2002), suggested that part of stock excess return is due to illiquidity premium: they proved a positive relationship between stock return and different measures of illiquidity (mainly Bid-Ask spread and a function of the stock dollar volume), suggesting that small firm stocks should be better compensate in terms of returns as result of their market "thinness". If this is true, there would be repercussion on the portfolio composition. An early liquidation of a portfolio with a heavy exposure on Small-Cap might significantly reduce the portfolio performances. Thus, investors with a short-term investment horizon should focus on more liquid securities, while long-term investors can attenuate the consequences of trading illiquid asset and therefore capture the potential premium return. It is clear how liquidity can affect not only portfolio characteristics but also its profitability. Many measures of liquidity are linked with costs estimates, as more liquid markets and securities are assumed to be cheaper than more illiquid. While for the household, those cents spent in transaction cost might be irrelevant, for large traders they might determine between who wins and who loses. Moreover, as mentioned, Markowitz's optimization does not take into account stocks' features. Hence, the portfolio composition might load aggressively on those low-liquid securities, given the potential higher returns they offer, resulting in unfeasible portfolio or significant differences in the theoretical and realized returns. Furthermore, Ohler (1990) conducted a study questioning financial advisors about the investment characteristics that are considered as the most important by their clients. Aside from risk and return, which unsurprisingly were ranked as the most important, liquidity classified in the top 3 desirable characteristics, showing the importance of such component also for the clients. Therefore, it seems reasonable to analyzed whether it makes sense to introduce a measure of liquidity in the mean-variance framework and what are the potential implications and results. However, first it is necessary to try to define what asset liquidity and market liquidity are and their consequences on portfolio allocation. Subsequently, we will try to understand where liquidity come from, which might be the sources of different degree of liquidity across similar assets. Finally, we will analyze the most famous measures of liquidity developed by the literature in order to try to give a numerical representation to this variable.

2.1. What is Liquidity?

When dealing with stocks, or pretty much any other security traded in financial markets, one of the most well recognized definition of liquidity is the ease to buy or sell an asset quickly, in large quantities, without substantially affecting the execution price (basically the ease at which is possible to trade an asset at its fair market value). According to this definition, shares in large blue-chip stocks like General Motors or General Electric are more liquid, because they are continuously traded during the day and therefore the stock price is unlikely to move dramatically following few trading orders (Morawski 2008.). This definition captures more the idea of market liquidity than the liquidity of the single asset itself which can be thought as the ease with which it can be converted into cash. Thus, cash and equivalents are generally considered as the most liquid assets available in the market, since they are or can be immediately converted into cash, hence reducing basically to zero the liquidity risk. Stocks tend to be pretty liquid (even if with differences depending on their specific characteristics and the market organization and conditions) as, nowadays, can be easily traded and converted in cash. Among the most illiquid assets there is real estate, for which the liquidation process might take days or weeks and generally involve very high transaction costs. Since the purpose of this work is to show how liquidity affects portfolio allocation and, ultimately, the performances, we will mainly focus on the market definition. Despite being quite easy, that definition is very powerful and captures several dimensions of the market structure, including time (how much does it take to liquidate a given position?), cost (at which prices will the trade be executed?) and quantity (how large is the position that can be liquidated?) (Morawski 2008). These quantities are also referred as immediacy (time), size or depth (quantity) and width or breadth (cost) and they represent the three main dimensions of market liquidity. Market breadth is defining as the cost of doing a trade at a given size. It is primarily associated with the Bid-Ask spread of the security traded, especially when the trade size is small and the market impact null or almost null. Market depth generally refers to the size of the market, interpreted either as the market players willing to trade with you, or the number of units that can be traded at a given price. The idea is that the more participants there are, the easier should be to find someone on the other side of the market to trade with. Thus, larger and more organized markets (i.e. NYSE) are considered deeper than smaller markets. However, such identification might actually be misleading. As pointed out by Persaud (2002), the size itself is not a good indicator of liquidity if it is not followed by a sufficient diversification among the traders. A large market where everyone wants to trade on the same side cannot be considered more liquid than a small one where it is always possible to find someone on the other side willing to trade with you. This balance in the market is ensured by the presence of noise trader, people trading for exogenous reasons other than profiting from under/overvalued securities. Indeed, if all the market players share the same information and trade only with respect to under/overvalued securities, everyone would buy or sell at the same time, making the trades impossible to be executed, thus making the market extremely illiquid (Morawski 2008). Breadth and depth are two sides of the same coin, as in both cases investors try to execute the trade with the minimum price impact. Immediacy refers to the time necessary to execute a trade of a given size at a given price. The speed of the trade firstly depends of the type of order submitted. Market orders are generally executed immediately. When submitting a market order, investors are willing to trade at the prevailing market conditions. This means that the order is likely to be fast, as not particular conditions are demanded, but also likely to be executed at inferior prices. On the other hand, with standing limit order, investors can expect to obtain better prices on average, but more time is required. However, also the speed greatly depends on the overall market liquidity, as in case of low-deep market, it might take time even to execute a market order. There exists a clear trade-off between these quantities. When traders are willing to search longer, they can expect to find better prices for a given size or better sizes for a given price, but this comes at the cost of sacrificing immediacy. On the other hand, if they want to trade larger sizes, they should expect lower prices, as the trade is likely to occur at a discount price, and longer searching time. Thus, it is not possible to fast trading large quantities at the market price. One dimension must be sacrificed in order to maximize the other two.

2.2. Origins of liquidity

Now that there is a clearer idea of what liquidity is, it is useful to try to understand where it comes from, what are the factors that determine the level of liquidity of an asset. As defined before, an asset is liquid provided that it can be traded quickly and without discount. This definition implies two main things: first the possibility to find a counterparty, otherwise the liquidation is impossible, second, the transaction should have no or little impact on the execution price. If selling an asset is impossible, then the liquidity is zero by definition (Morawski 2008). This is probably the worst-case scenario that all the investors fear: the will to liquidate a position before the market moves, but the impossibility to do so. Therefore, it is clear that what determines the liquidity are those factors ultimately affecting the duration and execution of a trade. The factors can be clustered in three main categories: transaction costs, trading organization and infrastructures and different asset valuations (Morawski 2008). The role played by the first group of variables is clear. Whether those costs are more or less explicit, they lower the liquidation value thereby reducing the trading frequency. The second group regards the characteristics and infrastructures of the market which will ultimately affect not only the easiness and ability to find a counterparty, but also the duration of the transactions. Finally, also different beliefs regarding the assets will affect the liquidity, as different valuation estimates will result in longer and more difficult transactions.

2.2.1. Transaction costs

When discussing liquidity, transaction costs are often one of the concepts introduced at the beginning. They represent the measurable component of liquidity and thus researchers try to estimate them and establish a relationship with the liquidity. As mentioned earlier in this chapter, transaction costs take a variety of forms, going from pure explicit costs, such as commissions paid to brokers, to less explicit, like the spread, to pure implicit costs of difficult estimation, such as the price impact and the opportunity costs. Commissions represent the compensation paid to the brokers for a variety of services provided to the costumers, including the search of a viable counterparty. This service is quite valuable, especially for small, private investors, who lack of the necessary expertise or in those markets where the search for a counterparty would be extremely costly if done on your own. In turn, this provides significant liquidity-related benefits for the investors: first brokers have better knowledge of the market and the actors playing in it. They can find a better counterparty faster and more cheaply that what investors could do on their own, thus increasing the liquidation speed and decreasing the liquidity risk. Moreover, they are better negotiators, and this can significantly improve the resulting execution price, thus decreasing again the liquidity risks. However, commissions are generally of less relevance compared to other transaction costs for two main reasons: they are smaller and known upfront, hence their impact on the uncertainty of the execution price is often quite small.

The second major type of costs include the spread and price impact. These two measures are grouped together since, similarly to commission, they impact negatively the execution price by lowering the revenues obtain or increasing the price paid, but differently from commission they are not know upfront and their size depends of multiple factors, including order size, type of market, market's conditions. Spread and market impact affect liquidity in two different ways. Spread is a pure cost: it forces the investor to buy higher and sell lower than what is the current fair market value (this is because the spread represents the dealer's profit, thus the current fair market price is assumed to be in between Bid and Ask prices). It has a very similar effect to commissions. Higher spread reduces the willingness to trade and the trading frequency, leading to less market activity and thus lower market depth (Morawski 2008). Market impact, on the other hand, affects liquidity through a different channel. As mentioned, market impact is the result of trading huge quantities in a market which is unable to absorb such sizes without significantly impact the execution price. In these situations, market players are afraid that those large trades might be driven by an informational advantage and thus require some price concessions in order to take that risk and be the counterparty. It is clear that market impact affects the liquidation price. Very often, investors hide their large trades by splitting them in multiple smaller trades, hoping that their real position is not discovered and so the resulting execution price will be better than what they would have gotten by executing the entire trade at once. However, this comes at the cost of increasing the liquidation time and thus being subject to unexpected market fluctuations (Morawski 2008). What is really interesting is that several studies show that the "indirect" costs of trading share the same sources of liquidity itself, like Keim and Madhavan (1998) demonstrated that many determinants of transaction costs (i.e. stock-specific characteristics, returns volatility, market infrastructures, type and features of trading orders) are also factors affecting the liquidity itself. This is why, transaction costs and liquidity can be used as interchangeable words to describe the same concept.

The last type of transaction costs groups the opportunity costs. They represent the cost of missed opportunities: it includes all the losses and gains that could have been avoided or obtained if, instead of waiting and postponing, the transaction would have been completed earlier. As rational investors want to maximize their utility, they are looking to maximize their liquidation price. Assuming that the time of liquidation is not random but pre-determined, investors might decide to postpone their transaction, hoping to get more favorable terms (Morawski 2008). This search, however, results in extending the liquidation period and increasing the liquidity risk by being exposing to market fluctuations. Of course, market fluctuations might turn out to be positive for the investor, but anyway they represent an additional component of uncertainty. Concluding, transactions costs, whether explicit or implicit, seem to affect liquidity in all its dimensions.

2.2.2. Type of Market

Market infrastructures have been already mentioned multiple times has a factor affecting liquidity. This is not surprising. The type of market, infrastructures and organization determine the way buyers and seller are brought together, as well as the execution timing and price. Thus, different markets result in different liquidation duration and price, therefore affecting the liquidity overall. There are several different types of markets. The simplest ones are the direct search market, as they require individuals to find a counterparty on their own. These markets are quite common for real estate and commodities, but also for those financial products of difficult standardization (i.e. credit default swaps). Liquidity is severally impaired by the market characteristics. Liquidation process can be extremely long as there is absolutely no intermediation which can help to find a willing counterparty to trade with. Moreover, also the valuation of these assets becomes more difficult, especially if they lack of comparables with an active market. This might lead to very different valuations, resulting in even longer liquidation process and potentially lower liquidation proceeds. It is a market where bargain power can significantly affect the outcome of the transaction. This also means that it is possible to execute operations at terms that would be impossible to obtain on more regulated markets. A similar type of market is the brokered market. It is a market where investors still have to find each other, but this function is delegated to brokers. As mentioned before, brokers are extremely useful because they possess a set of skills which can significantly increase the liquidity in the market. Not only they take care of bringing their clients together, they also provide a set of corollary services, such as order management. This is particularly useful for price-sensitive clients that can benefit from a significant reduction in the price impact, hence decreasing the transaction costs and improving the liquidation prices. Moreover, brokers are experienced negotiators. They know better the market and the counterparties, thus improving the quality and the speed of the transactions, resulting in larger trading frequency and subsequently greater liquidity in the market. A third form of market is dealer market. Dealers play a similar role to brokers, but instead of trading on their clients' behalf, the trade for their own account. This means that instead of looking for a buyer for their clients that are selling, dealers will directly buy from the clients. They act as buyers for the sellers and sellers for the buyers. In order to stay in business, they must profit from this activity. This is achieved by offering to buy at the Bid price and sell at a higher Ask price. This difference is indeed the spread and compensate the dealer for offering an extreme valuable service: immediacy. Dealers trade whenever their clients want to trade, providing thus liquidity in exchange of a compensation that takes the form of the spread. However, the spread does not ensure that dealers will profit from buying and selling a given securities, as they might have to buy a greater quantity than the one, they end up selling later. If the market moves when some of those stocks are still in the dealer's storage, he might need to reduce his Ask price thus losing money. Dealers market are very liquid because, assuming an investor accepts the transaction price (Bid or Ask), he can trade immediately at that price, reducing the liquidation time close to zero (of course it then depends on the size traded). However, the most common form of where stocks, and the majority of financial instruments are traded, are auction-market. They main characteristics is that these markets are centralized and organized. Buyers and sellers do not even know who the counterparty of their trades is. They simply place their orders, in terms of price and quantity, and the exchange will take care of matching them with someone on the other side of the market. There are different auction-markets depending on how orders are executed, whether it is on a continuous basis or at specific point in time. Execution price is then determined in accordance with the pricing rules, which also depend on the type of market. Overall, these markets are fairly liquid. Especially in the continuous type, as soon as an order is available on the other side of the market, it will be matched and executed. Moreover, since investors are trading directly with other investors, without the intermediation of dealers, transaction costs are lower, thus potentially increasing the trading frequency and liquidity.

It is now clear that the market organization can substantially affect liquidity, primarily through its effect on liquidation time and price. These represent the two dimensions highly affected by the market's characteristics. Liquidation duration is the entire process that goes from the search of a potential counterparty to the moment when the trade is finalized. The search depends mainly on two factors: the asset's characteristics and trading frequency. Simpler assets, with active market and comparables have easier valuation process, hence facilitating the negotiations between parties. On the other hand, higher is the trading frequency, easier is to find someone on the other side of the market (Morawski 2008). Investors in direct search market are clearly penalized from this point of view. They bear the entire cost of search. This is partially alleviated in brokered market, where this responsibility is shifted to the broker. From the pure counterparty search, these two are definitely the most illiquid markets. Moreover, in these markets often non-standardized goods are traded, making also the valuation process a little bit trickier and increasing even more the liquidity risk. The others two markets are quite liquid from this perspective. They basically involve little or no search at all, thus significantly increasing the execution time. Moreover, better and more improved markets for search of a counterparty tend to attract more investors. This leads to higher chances of finding a counterparty, higher trading frequency and hence greater liquidity. The increasing liquidity and decreasing costs can attract further investors, exponentializing the results and improving the liquidity beyond the simple reduction of search costs (Morawski 2008). Regarding the liquidation price, each type of market has it benefits. Auctions allow to trade at the current market value, thus guarantee a market price liquidation. However, direct and brokered markets enable investors to choose their own transaction terms, hence allowing for potential higher liquidation values (Morawski 2008).

Concluding, market organization plays a key role in determining the level of liquidity in the market. More organized infrastructures can improve liquidity by mainly reducing the search time and costs. Dealer markets reduce the liquidation time to zero in exchange of the payment of the spread while auction market, on continuous basis, constantly attempt to match buy orders with sell orders so that, a soon as they are matched, they can be executed. However, there is little room for improvement in terms of liquidation price. On the other hand, markets that require the investors to directly or indirectly find their counterparty are associated with higher transaction costs, but given that the trade is conducted privately, investor can obtain more favorable liquidation terms, thus decreasing the liquidity risk associated with lower execution price.

2.2.3. Diverging Valuations

The last factor that has to be taken into account is the asset valuation. The role played has already been partially explained in the previous paragraphs. Valuation directly affects the liquidity through its impact on the liquidation time and costs. Whenever there is some discretionary power in deciding the transaction value, the overall duration increases as it becomes more difficult to find a partner agreeing to your own personal valuation. Why then these different valuations? Where do they come from? There are several possible explanations. Among the main factors affecting different valuations there are different information set and different expectations (Morawski 2008). Clearly, having different information concerning the asset examined will produce different estimates. This will result in either a longer search for a new potential partner or a review of the information currently used to make the valuation. Either way, this is translated in longer liquidation periods. Investors also have different expectation about the assets. This is translated in different estimates of future revenues and/or different perception of the underlying risk. In both cases, again, the resulting valuations are likely to be different. There is also another possible explanation that is called divergence of tastes, which can be defined as the difference in utility attached to the same object (Morawski 2008). While this generally the case of non-financial instruments (i.e. arts, consumption goods and so on), there are some empirical studies supporting the theory that this phenomenon is also happening in the financial world. Behavioral finance studies show that many investors are home biased in their investment strategies: they tend to invest more than what they should do in companies and stocks that are geographically closer to them. Similarly, just consider the ESG topic that is growing fast in these past years. Investors sensitive to this topic are likely to attach more value to companies that are actively involved in ESG operations that investors who are indifferent, thus resulting in different valuations for the same stock.

Concluding, also valuation plays a key role in determining the overall liquidity level. Assets that require longer valuation and negotiation process are likely to have higher liquidity cost due to higher risks. It should be pointed out that, precisely because negotiation power can play a key role in the outcome, the liquidation price can turn out to be greater than what originally forecasted. However, given that this cannot be known upfront, the overall effect is still to increase the illiquidity level due to a greater uncertainty surrounding the timing and outcome of liquidation.

2.3. Liquidity measures

Since investors are assumed be rational, given two securities with identical risk and return characteristics, they should prefer the cheapest and most liquid one. Thus, in order to include the liquidity into the decision process, a quantitative measure is required. While a broad definition of market and asset liquidity has been provided, the literature has failed to provide a unique empirical measure so far. As mentioned, the definition of market liquidity tackles several dimensions which are not that easy to quantify. The market width, representing the cost of doing a trade at a given size, is the less difficult to quantify, since it is composed by some observable elements. Commissions, being the most explicit ones, can be easily quantified as they are directly paid to the broker in exchange of the trading services provided, but also represent the smallest and less relevant component. The indirect components (spread, market impact and opportunity costs), while opaquer, often account for the largest share of the overall cost. The importance of transaction costs greatly depends on the type of trader and portfolio management. For a more passive trader with a buy hold strategy, market impact might be small or irrelevant especially if the investment horizon is long. For an aggressive investor with an active portfolio management and frequent rebalancing, market impact and, more generally, the overall transaction costs, play an important role in determining the net performances. Overall, transaction costs are sort of penalties that are paid whenever a trade is executed, either you trade frequently or not. Thus, they create a trade-off between the will of rebalancing the portfolio to keep it align as much as possible to the investment goals, and the will of minimizing the cost of trading to increase the performances. It becomes clear why transaction costs, being strictly related to the concept of liquidity, have to be included into the analysis. The focus of this work will be mainly on estimates of Bid-Ask spread, market impact and other liquidity-related measured for which the estimation requires data that can be easily retrieve for all the securities.

There are some common measures that have been widely used for research purposes, but they all measure different aspects of the broad definition of liquidity. A first popular group of measures attempts to quantify the market depth, this includes: quantity available for trading at the given quotes, the trading volume and, in particular, the turnover (Morawski 2008). More complex measures have been developed by combing these quantities with other stock-related variables. The trading volume is probably the simplest proxy of liquidity: it computes the overall number of shares traded within a given time window (i.e. minutes, days, months, years) for a given stock. Data providers often report values on a daily basis. However, by focusing on single day trading volume, the liquidity picture of a stock is likely to be biased, as it could reflect (high or low) unusual trading activity. That is why, a more popular liquidity measure based on trading volume consists in averaging the daily traded volume over a period T as it follows:

$$ADV = \sum_{1}^{T} DV_i$$

Where DV_i is the trading volume (i.e. the number of shares traded) on the day i for a given security. Even if from a retrospective point of view, this measure allows to better understand the level of trading activity, smoothing the impact of unusual market activity. The Average Daily Volume is often averaged out over 20 days, 1, 3 or 6 months. It is a widely used measure of liquidity in practice. Increasing or decreasing level of ADV, might signal a shifting market interest for the security, therefore signaling a bearish or bullish momentum depending on the price's direction. Moreover, volumes are also useful to confirm price movements. During strong price upward or downward trends, volume should also rise. If it is not the case, there may not be enough interest to support the price shift and price might revert back. Despite its wide application, it also suffers from some shortcomings: trading similar amounts for different stocks might correspond to trade significant differences in the overall float available in the market. Indeed, a higher volume might also be due to a large availability in the market. Therefore, companies with less stocks would appear to be less liquid on average compared to those with more outstanding stocks. Similarly, securities with the same trading volume might appear as equally liquid, while it might not be the case, especially when comparing Large with Mid and Small-Cap stocks.

Such limitations can be easily overcome by relating the traded shares with the total available to be traded in the market. This ratio is defined as the ADV over a given period T and the number of outstanding shares per period on the same time window.

$$Turnover = \frac{ADV}{\#\ outstanding\ shares}$$

This measure allows to correct for the biased belief that a stock is liquid simply because the trading volume is high, while that trading volume actually accounts for a very small part of the outstanding tradeable shares. However, even this measure is not flawless. First, similarly to the Bid-Ask spread, such measure does not take into account the size of the trade. Second, there is no clear relationship why larger companies, generally assumed to be more liquid, should have larger turnover. Indeed, many evidences suggest the opposite. Large companies generally have a huge number of outstanding and high price share. Thus, even if the volume appears to be quite large, it might just reflect a small portion of the outstanding shares. On the other hand, small companies have more affordable shares, which can attract more investors, thus in turn boosting the volume traded. This is also the reason companies sometimes decide to split their shares to keep them more affordable and thus more liquid. Finally, the total number of outstanding shares does not fully reflect the real number of tradeable shares. To get a better picture, this measure is sometimes corrected by multiplying the number of outstanding shares by the floating percentage. Such measure is later used on the empirical analysis but, unfortunately, the floating percent could not be obtained for all the securities and thus the first "raw" version of the ratio will be used. Overall, this group of measures has wide real-world applications when evaluating investment decisions as they are understandable, cost and time efficient and data can be obtained directly from the exchanges or from data providers. Very often, portfolio managers screen securities by imposing a minimum level of ADV or turnover ratio. This represent a first "raw" implementation of liquidity concept in their portfolio optimization.

The second group of estimates are related to transaction costs. One of the earliest and most used measure is the Bid-Ask spread. It is the first approximation of the market breadth, measured as the distance between the Bid (price for immediate purchase from the market dealer) and Ask price (price for immediate sell from the market dealer). It represents the cost of a round trip for the investors, the cost of buying and selling at the same time, the same security, from the same dealer. Such trade results in a loss for the investor as the Bid price is always equal or higher than the Ask price. This is because the Bid price represents the cost to buy immediately the security, thus a premium has to be paid to someone who is willing to offer this "immediacy" service. Similarly, if the investor wants to sell immediately, he has to offer at discount in order to compensate the buyer for the service. Thus, it is easy to understand why Bid-Ask spread is often referred as a proxy of liquidity: it broadly measures the cost for being able to execute your trade immediately, to transform your illiquid asset into a liquid one and vice versa. From the market dealer point of view the spread represents the profits for offering liquidity, in the form of immediacy, maintaining an inventory sufficiently large to cover the different buy and sell order size. The absolute difference, despite providing an initial measure of liquidity, is quite problematic for one main reason: high price stocks are more likely to have also high absolute spread, thus making any comparison between stocks meaningless. Moreover, it is an imperfect measure of the cost too, as it represents the quoted spread from the market dealers, but trade can also happen at prices in between these boundaries. That is why the realized effective spread is a better measure. To make the measure meaningful, the difference is often divided by the mid-price, estimated as the average of the Bid and Ask price. Hence, potential liquidity measures based on the concept of spread include:

Percentage quoted spread = $(P_A - P_B) / (P_A + P_B)$
Log spread = $\ln (P_A / P_B)$
Effective spread = $| 2Pt - (P_A + P_B) |$

(Gabrielsen, Marzo, Zagaglia 2011). For the purpose of this work, we will mainly focus on the percentage quoted spread. Ideally the Bid-Ask spread should be lower for more liquid securities. The more liquid as stocks is, the easier is to execute a trade, thus lowering the liquidity risk and cost for immediacy. However, the spread presents also several drawbacks. First, it can only be computed for dealer-type market, but this represent no problem for the purpose of this work. Second, the spread does not take into account the size of the trade. It measures the round-trip cost of a relatively small trade which has no market impact. Therefore, large institutional traders who trade significant quantities at a time, the Bid-Ask spread alone pictures just a partial representation of the upcoming liquidity costs. Another indirect measure of spread was postulated by Roll in 1984. Roll's idea consists in using a model to infer the effective spread based on the time series properties of observed market prices and returns, focusing on the negative autocorrelation produced by Bid-Ask bounce (Gabrielsen, Marzo, Zagaglia 2011). This measure should be positive correlated with the actual Bid-Ask spread. His model assumes that prices follow a random walk and the observed closing price *P* on day *t* is equal to the stock's true value plus or minus half of the effective spread. In these circumstances the autocovariance of returns will be negative. However, empirical evidences show that in real markets it often appears to be positive. In such cases the Roll spread is generally set to zero in that given month (Będowska-Sójka, 2017). Therefore, the Roll covariance spread estimator would take the following form:

$$Roll \begin{cases} 2 * \sqrt{- cov\ (\Delta P_t, \Delta P_{t-1})} & When\ cov\ (\Delta P_t, \Delta P_{t-1}) < 0 \\ 0 & When\ cov\ (\Delta P_t, \Delta P_{t-1})\ \geq 0 \end{cases}$$

The implication of this measure is that the higher the negative covariance is, the more illiquid should also be the stock. The main shortcoming of this measure is that it does not take into account information asymmetries. To better explain it, Roll's measure gives an estimation of the pure processing cost, thus assuming that does not change in response to trades. This condition, however, would ideally hold only if there were no informed traders in the market and the quotes did not adjust to compensate for changes in inventory positions. Indeed, Huang and Stoll's work in 1996 shows that when estimating the Roll implied spread in the NYSE, the resulting value is actually much lower than the effective half-spread (Gabrielsen, Marzo, Zagaglia 2011).

What Roll's measure is missing is the role played by information advantages when executing a trade. This topic is particularly sensitive for investors executing large trades. Indeed, processing large trades might signal that the investor possesses some sort of private informational advantage and thus it is unlikely that he will be able to execute the entire order at the quoted prices (Bid or Ask) without impacting the execution price. This effect of trading size on the execution price is commonly known as market or price impact and represent a "shadow" component of trading, as it cannot be estimated upfront and it is unknow until the trade is completed. To incentivize others to trade with them, investors offer to sell at discount and buy at premium with respect to the fair value, causing the price to move. While price impact might be neglectable for the small investors, it constitutes a big, maybe the biggest, share of costs for traders moving large quantities. They would end up paying much more that the half of the quoted spread per transaction. The market or price impact is highly correlated with the liquidity risk. When the trading size is small, the chances of finding a counterparty are higher, therefore reducing the risk of overholding the security (holding the asset more than what desired). However, as the size increases it gets more difficult to find willing parties to trade immediately therefore increasing the risk of being subject to additional prices shifts due to the impossibility of liquidating the position (at least without significantly impacting the trade execution price). This is precisely the trade-off mentioned before: the investor can delay his liquidation thus reducing the market impact, but this will ultimately pose new risks for the remaining portion of the portfolio that has not been liquidated yet (Stange, Kaserer 2009). Differently from the spread, the price impact cannot be directly observed and thus estimated. It is also hard to measure because it is the cost of trading many shares relative to the cost of trading one share, and you cannot run a controlled experiment and trade both many shares and one share under identical conditions (Grinold, Kahn 1999). It highly depends on the market conditions and characteristics. One of the first factors affecting the price impact is probably the size of the market itself. In small, illiquid markets, even small trades might cause price to diverge while, in larger and more complex markets, big trades are conducted on a daily basis. The type of the market also affects the execution price: trading the same security in a regulated exchange or OTC can result in significantly different trading price. Thus, there have been many attempts in proxying the price impact resulting from a given trade. A first, raw, measure of the market impact has been introduced by Kyle (1985) and it is called Kyle's lambda. It is derived by regressing the stock price on the average trade size in the following way:

$$P_t = \mu + \lambda V_t$$

The resulting λ coefficient of the OLS regression will be the approximated estimate of the market impact. For very short period, this can be approximated as the ratio between the price change and a measure of the volume traded, often the turnover.

$$\lambda = \frac{|\Delta P|}{Turnover}$$

According to this formulation, a high liquid stock will experience a smaller price change than a more illiquid, given the same level of turnover over a period. A second measure of market impact has been proposed by Grinold and Kahn in 1994. This is known as sigma-root-liquidity model and attempts to estimate the price change as a function of the spread, stock daily volatility σ_d, size of the trade to be conducted (Q), the average traded volume (V_d) and a constant factor c.

$$\Delta P = Spread + \sigma_d * c * \sqrt{\frac{Q}{V_d}}$$

Thus, the cost of conducting a trade is a constant function of the spread and depends proportionally on the daily volatility and the size of the trade compared to the average activity in the market during a day window. It represents a raw measure, as it relates the market impact only to the size of the trade, without taking into account the rate of trade as well as the stock market capitalization. However, it represents a good and complete approximation of the trading costs, by taking into account both a spread and market impact component, but it requires knowing upfront the trade size.

There is another group of estimates, called liquidity ratios, that attempts to combine turnover and/or other measures of trading volume and returns. A first example, also known as Amivest liquidity ratio, is defined as:

$$LR1 = \frac{\sum_{i=1}^{N} p_i * q_i}{|r_t|} = \frac{V_t}{|r_t|}$$

with r_t being the stock returns, p_i the stock price for the *i-th* security and q_i is the number of shares traded, V_t is the dollar trading volume during that day. This measure compares the traded volume to the absolute price change during a certain period. The higher the volume, the easier the price movement can be absorbed (Rico von Wyss 2004). Amihud (2002) proposed a similar measure, which became quite popular in the literature. Instead of measuring the degree of liquidity, Amihud proposed a measure of illiquidity by reverting the parameters in the Amivest ratio. He defined the illiquidity parameter, *ILLIQ*, as the average ratio between the stock return to its dollar trading volume in any given period (*T* trading days during the selected window), where the dollar trading volume is the cumulative number of shares traded during the day times the trading price:

$$ILLIQ = \frac{1}{T}\sum_{t=1}^{T}\frac{|r_t|}{V_t},$$

Amihud measure has been proved to be a good proxy for the theoretical price impact coefficient lambda discovered by Kyle. The disadvantage of this illiquidity measure is that this is not well suited for comparison between different markets and securities (Corwin, Schultz 2012). However, given the focus of the analysis on European incorporated stocks, this should not represent much of a problem. It should also be pointed out that despite its attractiveness, the Amihud measure of illiquidity as little meaning as a stand-alone variable. Most of the time, this measure is used to make liquidity comparisons between the different asset analyzed.

Ranaldo (2000) proposes another version of a liquidity ratio:

$$LR = \frac{V_t}{(N_e - N_o) * r_t}$$

where the traded volume V_t is corrected for the free float of the firm. The term ($N_e - N_o$) denotes the difference between total number of shares outstanding and the number of shares owned by the firm (Rico von Wyss, 2004). The main shortcoming of these measures is that they are all sensitive to the stock price.

Brunner (1996) overcame this problem by dividing the stock return by the number of trades N_t.

$$LR2 = \frac{\sum_{t=1}^{T} |r|}{N_t},$$

Similarly, to Amihud measure, a higher ratio shows lower liquidity. If the number of trades for certain time space is zero, this liquidity ratio would converge to infinite. The great advantage of these measures is that they can be easily computed, even with a spreadsheet, as they simply require daily values for stock returns, shares outstanding and trading volume.

The last measure that is worthy to mention is derived from option theory. As previously describe, liquidity can be defined as the ability to sell or buy whenever needed, without significantly impact the trading value. Longstaff associated this ability to sell at the chosen time to a put option and, thus, the value of liquidity can be potentially estimated using option pricing theory (Dyl, Jiang 2008). He basically determined the illiquidity as the opportunity cost and the associated loss in the asset value due to the impossibility to sell it a fashionable way. Longstaff based is model on a frictionless world, where an investor with perfect timing sells the risky asset and invest the proceeds into a riskless security to maximize the portfolio value. If then the investor is unable to sell the asset for a given time *T*, the optimal trade is no longer possible and thus the portfolio value declines. Longstaff's model measures the portfolio loss due to the impossibility to trade to trade the asset during the illiquid period T as follows:

$$Discount = \left(2 + \frac{\sigma^2 T}{2}\right) [N(d)] + \sqrt{\frac{\sigma^2 T}{2\pi}} * \exp\left(\frac{-\sigma^2 T}{8}\right) \text{-1}$$

Where σ represents the annualized daily return standard deviation, *T* the estimated illiquidity period for the selected security and *N(d)* is probability that a standardized, normally distributed, random variable is greater or less than *d*, with *d* being $\sqrt{\frac{\sigma^2 T}{2}}$. This model calculates the potential discount for the absence of liquidity that can occur in a market with rational investors with daily returns volatility being σ and the period of illiquidity is *T* days. Concluding, the model shows that the cost of illiquidity per unit of time is a function of the volatility, and the total percentage discount depends on both the cost per unit of time and the length of this "illiquidity period" where the stocks cannot be traded (Dyl, Jiang 2008). As many other measures, however, it has some shortcomings. First, as many other economics models, are based on a frictionless world, with perfectly rational investors. Moreover, it might have little meaning from a practical point of view as it would require knowing upfront the illiquidity period *T*, which is unlikely to be the case, and it also requires significant computations.

All these measures are computed on a stock level, thus refer to a single asset. However, investors managed several different positions at a time. Therefore, it might be interesting to take a look at the overall portfolio's liquidity level. Indeed, when looking at the portfolio as a whole, the picture changes, as some sort of diversification can be achieved to reduce the liquidity risk. While portfolio managers are not willing to bet everything on a high illiquid stock, they might accept to invest in more illiquid securities, knowing that in case of emergencies, they can cover that position or convert in cash part of the remaining portfolio. Hence, instead of focusing on a single asset, it may be more interesting to focus on the aggregate level of liquidity, as we focus on the aggregate level of risk and return. However, here the literature is not particularly extended, and finding a solution is a little bit tricky. Indeed, there is no clear explanation on how to combine liquidity measure across different assets. Surely, the easiest way would be to linearly aggregate them as it is done with the portfolio return. However, as returns correlate, also liquidity can correlate. Different asset classes might have different aggregate level of liquidity. In period of financial recession, safer investments attract more capital, thus liquidity might shift from the stock market to the sovereign bond market, making the liquidation of multi-asset portfolio easier.

Concluding, there are many different measures of liquidity, depending on which aspect of liquidity there is a focus on. Some of them are simpler, based on the trades that are really happening on the market (volume, spread). Others are more complex, try to give an explanation and estimation to why the resulting trading costs sometimes diverge so much from the quoted spread. It is thus clear that the absence of a unique definition of liquidity makes also difficult to establish a unique measure thereby giving the investor the arbitrary power to determine what is liquid and what is not.

3. Mean – Variance – Liquidity Optimization

As mentioned, liquidity has been hardly treated in the mean-variance framework. However, portfolio managers attempting at maximizing the performances of their portfolios are somewhat limited by what they need or want to do and the ease at which they can do it. Therefore, they are sensitive to the three main dimensions of liquidity: price, timing and quantity. If they cannot invest in perfectly liquid assets, they need to give up one or more dimensions, and this will impose costs on their portfolios, which will ultimately affect their investment decisions (Hodrick, Moulton 2009). But the introduction of liquidity parameters into the optimization poses some challenges, mostly deriving by the lack of a unique definition and measure capable of summarizing the liquidity in all its dimensions. Hence, the first question is clearly which liquidity measures should be chosen? Should we use only explicit transaction cost measures, which can be determined and estimated with higher accuracy? This is often the best proxy for the liquidity chosen in the literature. However, this solution would ignore the hidden costs from trading large quantity. Estimating market impact is quite difficult and often requires knowing the size you want to trade, which is not the case of this research paper. Therefore, the very first step is the choice of the liquidity parameters to use, and it is already challenging. Secondly, how should they be combined into the optimization problem? One possible solution would be to model them as a linear function of the portfolio weights, in the same way portfolio returns are calculated. This, however, introduces additional questions. The mean-variance optimization is a universal model, in the sense that it can be applied to any portfolio, regardless its size. Thus, the portfolio weights resulting from the optimization process are in relative terms (percentages), not in absolute (real units). In order to know how much an investor would really need to buy or sell, it is first required knowing the size of the overall investment. As mentioned before, the price impact depends on the quantity trade, which cannot be really captured by portfolio weights expressed in percentages.

This chapter will describe the methodology implemented in the empirical analysis focusing on two main parts: a static analysis, where liquidity is included into the optimization process in order to replicate Markowitz's frontier in a 3-dimensional space. Thus, the frontier shapeshifts from a simple line to a surface, representing a 3-way trade-off between volatility, return and liquidity. The second part is focused on a more active approach. The attempt is to investigate whether portfolios with different required minimum level of liquidity would perform differently over time and whether there is an illiquidity premium to profit from. Data implemented and results are then presented in Chapter 4.

3.1. Static Analysis: Mean-Variance-Liquidity Frontier

There are very few research papers that investigate this topic. One of the most famous is a paper published by some professors from the MIT called: *It's 11 pm—do you know where your liquidity is? The mean-variance-liquidity frontier*. In their paper, they try to build this 3-dimensional surface using three different approaches: a liquidity filtered surface, a liquidity constrained surface and a surface derived from a direct maximization of the utility function, including now a liquidity parameter. They applied all these approaches to a basket of 50 US stocks randomly selected. For the purpose of this work, it has been decided to implement a similar liquidity constrained approach. This seems the most reasonable approach. Indeed, from a practical point of view, liquidity is often implemented as a sort of filter, by focusing on securities with an historical minimum level of trading volume (ADV). From a static point of view (buy and hold portfolio), this approach is totally feasible. However, since this analysis is also developed from an active management perspective with monthly rebalancing, introducing such a filter could cause some issues. In particular, since this approach would exclude from the portfolio selection all those stocks with a level of liquidity below the one required, the turnover can be extremely high, as investors would be forced to completely liquidate their position on those securities, even if the security's level of liquidity is extremely close but still smaller than the threshold. Moreover, as the trading volume of a security drops below the minimum level, the investor would face unexpected liquidation precisely when liquidity is draining up, thus increasing the liquidity risk and costs. Thus, from an active management perspective, with frequent rebalancing, implementing this strategy can results in huge transaction costs and significant potential losses due to unexpected early liquidations. Such criteria are, however, often used in passive management strategies, as they require low effort, they are cost and time efficient and the low frequency of the rebalancing ensure no extreme transaction costs. On the other hand, implementing a direct utility optimization seems rather complicated. Literature shows that is already quite difficult to determine the individuals risk aversion, with empirical results being far from what theorical models predict. Introducing the liquidity in the direct optimization would require determining the risk aversion to illiquidity, for which no estimation method currently exists. Concluding, imposing liquidity as additional constraint in the optimization process seems the most feasible and appropriate approach for the purpose of this analysis.

Initially we will focus on a portfolio where short selling is not allowed. There are several reasons behind this choice. From a practical point of view, first, the liquidity constraint is easier to implement when all weights are positive. This is because liquidity parameter can be linearly combined as returns are. With negative weights, the model gets more complicated, as short sold securities would decrease the overall portfolio liquidity. This could still be easily solved by taking the absolute value of the weights or by standardizing them. Anyway, this requires the introduction of a nonlinear constraint in the optimization process, which would no longer guarantee the convexity of the problem, and thus the existence of a unique global minimum solution. Despite this shortcoming, a portfolio with short selling will be analyzed, as short selling is universally allowed, and many investors pursue long-short portfolios. However, despite being allowed, short selling is often difficult, especially for private investors as well as institutional investors. Indeed, many of the latter face regulatory constraint preventing them from short selling securities (clear example is pension funds). Since it is unlikely that transaction costs are going to heavily affect the decision of the small portfolio manager, it makes sense to first consider a case closer to real world applications. Moreover, with the recent huge migration from active to passive, less and less investment portfolios are including short selling positions. For the reasons here announced, it seems more correct to first focus on a portfolio with these characteristics.

Based on these criteria, we developed a 3-dimensional surface, which shows risk (portfolio standard deviation) on the x-axis, portfolio return on the y-axis and portfolio illiquidity on the z-axis. It was opted to focus on illiquidity rather than liquidity, because many parameters directly estimate the illiquidity (transaction costs) first as described in the previous chapter (i.e. Spread, Amihud, pure price impact measures). This surface was developed in the same way Markowitz originally developed the efficient frontier. By varying the target illiquidity and return parameter, we calculated the portfolio with the minimum variance satisfying those criteria. Therefore the “Mean-Variance-Liquidity Surface” was obtained by solving the following optimization problem:

$$min_w \, \omega' \Omega \, \omega$$

$$subject\ to \begin{cases} \omega' r = \mu_p \\ \omega' e = 1 \\ \omega_i \geq 0 \\ \omega' l = L_p \end{cases}$$

Where l represents the asset’s specific liquidity level and L_p represent the target portfolio liquidity. The asset’s specific level of liquidity has been determined by combining the security specific mid-price Bid-Ask spread and the Amihud measure of illiquidity. This choice is motivated by the fact that, first, both the measures can be easily obtained and calculated. Especially for Small-Cap securities the data are fragmented and hard to retrieve. Thus, the need to use measures that do not rely on too many parameters and are also of common use. Moreover, combining them, it allows to take into account for both explicit and implicit costs (at least partially). Indeed, the spread is a well know liquidity measure that portfolio managers try to account for when optimizing their portfolio, and Amihud measure of illiquidity, despite not having a real meaning when taken as a stand-alone measure, is useful to compare securities with different level of liquidity. The greater this combined liquidity parameter is, the lower is the liquidity of the stock. Then, each portfolio on the surface will satisfy a target level of return and liquidity, by minimizing the risk.

Subsequently, the same analysis is computed on a portfolio where short selling is allowed. It seems interesting to also analyze this case since short selling is a common practice in today's finance. The original Markowitz's optimization problem was completely boundless. However, such approach seems extremely inappropriate due to real world limitations. As it is shown later, without any bound, portfolios might take positions exceeding the unit, which is clearly not feasible. However, from the purpose of a pure static analysis, it has been opted to first introduce loose bounds, as the time window is extremely long and an eventual buy and hold portfolio over that time period would be at least more feasible than in the case with periodic rebalancing. As mentioned, allowing short positions however introduces some issues from the optimization point of view. The process now requires nonlinear constraint to handle the negative weights. Without that, a simple linear combination would make no sense as negative positions in high liquid securities would actually decrease the overall portfolio liquidity. To overcome this problem, liquidity parameters have been linearly combined with the absolute value of the weights. This approach guarantees that taking a long or a short position on an equally liquid security, would also equally impact on the overall portfolio liquidity. This is of course a rough approximation, as short positions are generally more difficult to establish, therefore the level of liquidity would not probably be the same. The optimization problem then takes the following form:

$$min_w \, \omega' \Omega \, \omega$$

$$subject\ to \begin{cases} \omega' r = \mu_p \\ \omega' e = 1 \\ abs(\omega)' l = L_p \\ \omega_i \geq -1 \ and \ \omega_i \leq +1 \end{cases}$$

All the parameters have been calculated in the same way as before. For each optimization method the 3-dimensional Mean-Variance-Liquidity surface was plotted, alongside with each efficient frontier at the different liquidity threshold L_p. Particular attention is paid to two specific portfolios with peculiar characteristics: the Global Minimum Variance (here and in after called GMV) and the Maximum Sharpe (here and in after call TAN). For each GMV and TAN, the resulting portfolio composition has been investigated, with a focus on the allocation across different market capitalization classes (Large, Mid and Small) and how these evolve in portfolios with different level of liquidity. Finally, the gross and net return as well as Sharpe ratio's evolution was analyzed for the GMV and TAN at the different levels of liquidity. The net returns have been calculated as the difference between the average gross return and the average Bid-Ask mid-price spread over the same time window, as calculated in chapter 4.

3.2. Active Analysis: Rolling analysis with minimum liquidity requirements

From an active approach point of view, this research tries to investigate whether imposing minimum level of liquidity when rebalancing the portfolio affects the portfolio profile, in terms of risk, return, turnover and weights allocation. Similar to what done in the static analysis, the focus will be on two different optimization approaches, one where short selling is not allowed, and one where it is, but upper and lower bounds at a security level are introduced. In both case a monthly rebalancing approach has been applied. At each rebalancing date, historical values are used to determine the next month portfolio composition. The active approach will be focused exclusively on the GMV and TAN. This choice is motivated by the fact that, first, these two portfolios are the most peculiar across the ones along the frontier. Since this research is generalized to all the type of inventors and there is no explicit risk aversion preference, it makes sense to pick the portfolios with peculiar characteristics. Secondly, these rolling optimizations require a lot of computational power, therefore focusing just on those allows to keep the code lighter and faster. The focus will be on four different portfolios, each of one needs to satisfy a target level of liquidity. Therefore, at the beginning of each month, the following optimization processes are run:

GMV

$$\min_w \omega' \Omega \, \omega$$

$$subject\ to \begin{cases} \omega' r = \mu_p \\ \omega' e = 1 \\ \omega_i \geq 0 \\ \omega' l \geq ILL_i \end{cases}$$

TAN

$$\max_w \frac{\omega' r}{\sqrt{\omega' \Omega \omega}}$$

$$subject\ to \begin{cases} \omega' r = \mu_p \\ \omega' e = 1 \\ \omega_i \geq 0 \\ \omega' l \geq ILL_i \end{cases}$$

Where ILL_i represents the asset's specific liquidity measure. ILL_i is the *i-th* liquidity threshold parameter as defined in chapter 4. From an active perspective it seems more reasonable to introduce an inequality constraint rather an equality for the liquidity as in the static analysis. The main reason explaining this choice is that, as shown in chapter 4, less liquid securities tend also to have larger standard deviation. Therefore, if the inequality sign was the other way around, the portfolios resulting from the rebalancing would almost be the same. Hence, the introduction of a minimum level of illiquidity to satisfy, while minimizing the illiquidity. Similar approach in case for the portfolio without short selling constraint. As mentioned before, such portfolio was bounded at an asset level so that no single position can exceed the upper and lower bound. However, differently from the static analysis, it was opted to tighten these bounds to 50%. This is because the active approach involves now a quite frequent rebalancing, and previous bounds are likely to lead to an enormous turnover. It should be pointed out that 50% still represent a quite high level compared to real world applications, and it would probably require quite strong risk lover features to be achieved. However, since originally Markowitz's optimization was completely boundless and there is no investor's profile model leading this analysis, the choice would be anyway arbitrary. Thus, the optimization problem solved at each rebalancing date is the following:

GMV

$$\min_w \omega' \Omega \, \omega$$

$$subject\ to \begin{cases} \omega' r = \mu_p \\ \omega' e = 1 \\ \omega' l \geq ILL_i \\ \omega_i \geq -0.5 \ and \ \omega_i \leq +0.5 \end{cases}$$

TAN

$$\max_w \frac{\omega' r}{\sqrt{\omega' \Omega \omega}}$$

$$subject\ to \begin{cases} \omega' r = \mu_p \\ \omega' e = 1 \\ \omega' l \geq ILL_i \\ \omega_i \geq -0.5 \ and \ \omega_i \leq +0.5 \end{cases}$$

ILL_i is calculated in the same way as in the previous case but is based on the unconstrained portfolio characteristics. For each optimization method and level of liquidity the gross and net returns are calculated. These are then used to value the portfolio performances based on a series of risk-reward measures. These measures include the followings: Sharpe, Sortino and Treynor ratio, Value-at-Risk, Expected Shortfall, Calmar and Sterling ratio based on the drawdown sequence and the Farinelli-Tibiletti ratio. The Sharpe ratio is the traditional risk-reward measure. It shows the amount of return per unit of risk taken, where the risk is proxied by the portfolio returns volatility. It takes the form previously shown in the optimization process. Sharpe ratio, despite is wide application, is often criticized because based on the total volatility as measure of risk. However, the volatility in positive returns is of smaller concern than the volatility of negative return, which could represent a better proxy for the risk. The Sortino ratio overcome this bias, by relating returns with unit of downside volatility, defined as the standard deviation of only the negative returns. Sortino takes the following form:

$$Sortino = \frac{R_p}{\sqrt{var(R_p < 0)}}$$

Another risk-reward widely used is the Treynor ratio. Differently from the previous two, Treynor does not relate the returns to the volatility, but rather to the systematic risk of the portfolio expressed by the portfolio beta β_p. This risk measure is defined as the sensitiveness of portfolio returns to movements in the market and Treynor is the following ratio.

$$Treynor = \frac{R_p}{\beta_p}$$

The portfolio beta will be calculated proxying the market with the referring benchmark from which the universe used in this analysis is extrapolated. It represents an appropriate choice since the benchmark is the EuroStoxx 600 basically approximate the entire European float-market capitalization. The next measures involve more complex definition of risk that go beyond the simple portfolio returns volatility. The first is the Value-at-Risk ratio, which is a risk-reward measure derived from the concept of Value-at-Risk (VaR). The VaR estimates, given a certain level of probability, what is the expected loss for a particular investment. Given a portfolio, a time horizon, and probability q, VaR(q) can be defined informally as the maximum possible loss that can occur during that time window and after we exclude all worse outcomes whose probability is lower than q (Fabozzi 2011).

$$VaR\ (q) = \frac{R_p}{quantile(R_p, q)}$$

In the following analysis the threshold probability q has been set to 5% (often used in real world applications). Thus, the VaR calculates the maximum possible loss occurring with a probability of 5% based on the portfolio returns distribution. The greater is the potential loss, the lower will be the ratio and thus worse will be the portfolio. The next measure is based on the Expected Shortfall. The Expected Shortfall, also known as Conditional Value at Risk (CVaR), is highly correlated with the VaR. Given again a probability *q*, instead of calculating the possible loss in the investment, it calculates the expected portfolio return in the worst *q*% cases. It represents a sort of conservative way to estimate portfolio returns by focusing only on "negative" scenarios. The Expected Shortfall is considered a more useful risk measure than VaR because it is more coherent and gives a better idea of risk. It is calculated for a given quantile-level *q* and measures the expected value of portfolio returns, given that the VaR at level *q* has been exceeded (Fabozzi 2011). This calculation is simplified as it follows:

$$CVaR\ (q) = \frac{R_p}{mean(R_p < quantile(R_p, q))}$$

Safer investments like Large-Cap stocks rarely exceed VaR by a significant amount. On the other hand, more volatile asset classes, like Small-Cap or stocks from emerging markets can have CVaRs much larger than their VaRs. Ideally, investors are looking for small CVaRs. However, very often, the investments with the most upside potential have large CVaRs (Fabozzi 2011).

The next two measures are based on the drawdown sequence and are the Calmar and Sterling ratio. First, it is necessary to introduce the concept of Drawdown. The Drawdown is a measure that focuses on the losses and their recovery in a recursive way. It basically refers to the portfolio loss from the previous peak before the portfolio recovers to a new peak. Therefore, it can be analyzed in two different dimensions: magnitude, how large was the loss before the recovery, and time, how long before the total recovery (Fabozzi 2011). This means that a drawdown is official recorded only when the portfolio goes above the previous peak. The drawdown became really popular as investors tend to focus just on gain and loss with respect to their purchase prices, without taking into account the drop from a peak arising after the acquisition. However, also that represents a sort of loss (more like missing profits) on the investment thereby worthy of being taken into account. This also means that the drawdown might not be equal to the loss if the security was sold when it plummeted. Investors are, of course, interested in finding securities with the lowest possible drawdown, as it means that, historically, the "negative" price shifts were quite small. Based on the maximum drawdown (the largest movement from a peak to a low point before its recovery) is the Calmar Ratio.

$$Calmar = \frac{R_p}{\max(DD)}$$

However, this ratio might be biased by a single large drawdown, while the average or the following ones are actually quite small. The Sterling ratio partially solves this bias by taking the ratio of the returns over the average of the *k*-largest drawdown sequences, which has been set to five in the empirical analysis. Sterling then takes the following form:

$$Calmar = \frac{R_p}{\text{mean}(Max_5(DD))}$$

The last measure implemented is the Farinelli-Tibiletti ratio. This ratio divides positive and negative volatility, as well as, big return shifts from small return shifts with respect to a predetermined threshold. The ratio takes the following form:

$$FT = \frac{E[((R_p - r_t)^p)^{1/p}}{E[((R_p - r_t)^q)^{1/q}}$$

Where r_t represents the threshold chosen and $p \geq 0$ and $q \geq 0$. For the purpose of our analysis the threshold has been set naively to zero. It is clear that the smaller *p* and *q* are, greater is a given power, greater will be the weight attached to that particular tail of distribution (i.e. smaller *p*, greater is the weight attached to extreme positive returns, thereby supporting riskier investments) (Rachev, Stoyanov, Fabozzi 2008).

4. Mean-Variance-Liquidity Empirical Analysis on the EuroStoxx 600 constituents from 1999 to 2019

In the following chapter the results of the empirical analysis will be reported and examined. The entire analysis has been conducted using the software MATLAB® and the data provider Eikon Thomson Reuters®. Table values have been rounded at the fifth digit after the decimal point.

4.1. Starting Universe and Data Used

The empirical study is based on an initial universe composed by all the current constituents of the EuroStoxx 600 index (as of May 2019). This index is provided by the STOXX® Ltd and aims at replicating the performances of the European Union economy. It has a fixed number of components, comprising Large, Mid and Small-Cap stocks, representing approximately 90% of the free-float market capitalization of the European stock market. It includes stocks from all the major European countries, such as UK (which accounts alone for roughly 27% of the index), France, Germany and Switzerland (each accounting for roughly 15% of the index), as well as Austria, Italia, Sweden, Spain and many more (STOXX® Index Methodology Guide, 2020). Such index has been specifically chosen for its composition since, differently from the EuroStoxx 50, it also includes many Small and Mid-Cap stocks, which allows to better investigate the potential effects of liquidity in the portfolio construction. For each constituent, the closing prices have been downloaded starting from 1989, with a daily frequency, using the data provider Reuters. Since the time history starts before the introduction of the common currency euro, each stock has its original currency. This includes stocks in the following currencies: Norwegian Krona, Swedish Krona, Danish Krona Polish Zloty and Swiss Franco and UK pound. For the sake of simplicity, it has been assumed a perfect hedge situation, so that prices can be easily converted to Euro denominated values by adjusting the daily prices for the daily exchange rate. Unfortunately, Reuters does not own exchange rate data before the 31/12/1998, thus the constituents' time-series have been resized to start from the first date when all the exchange rates are available. Furthermore, for the same constituents, it was downloaded also the Bid and Ask prices, the security market value, the turnover by volume and the number of shares outstanding. For portfolio performances analysis also the EuroStoxx600 levels have been downloaded during the same time window. Given the enormous amount of data, and in order to avoid an excessive portfolio rebalancing in the subsequent analysis, data has been converted on a monthly basis. Monthly returns have been calculated has the ratio between previous month last business day closing price and the current month last business day closing price. Shares outstanding, turnover and market values data, on the other hand, have been simply averaged out on the same time interval. Bid and Ask prices have been used to calculate the Bid-Ask spread percentage, with respect to the security mid-price, has describe in Chapter 2. Such measure was then averaged in the same way the others were. The turnover volume has also been converted to the turnover volume percentage as described in Chapter 2, by taking the ratio between the average turnover volume on a given month and the average number of outstanding shares over the same month. Unfortunately, Reuters provides the common shares outstanding based on annual reports, therefore the number for outstanding shares are hardly changing during the year. In addition to turnover volume percentage and Bid-Ask spread, the liquidity of a security has also been proxied by the daily Amihud illiquidity measure described in the previous chapters. Such measure has been chosen for several reasons: first, it can be easily calculated with the data Reuters provides. It requires nothing more than turnover values and prices, thus allowing to obtain enough data for such a big universe. Second, as mentioned in the previous chapters, it approximates the market impact for a given securities, which cannot be fully captured by turnover and spread.

To better investigate the liquidity characteristics and its possible impacts on the mean-variance framework, securities have been classified in Large, Mid and Small-Cap, following the EuroStoxx 600 index guideline classification, but simplifying it by removing the buffers conditions and rounding the thresholds. Specifically, securities have been classified as it follows (based on their average monthly Market Value):

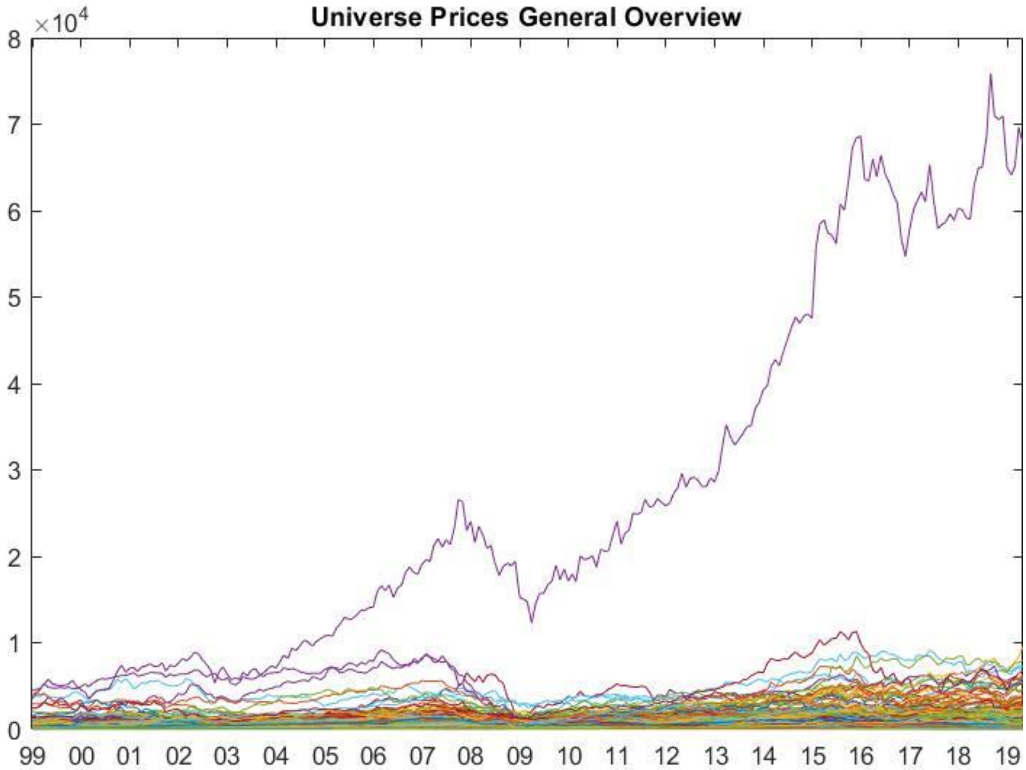
- Large-Cap account for 70% of the index;
- Mid-Cap account for 20% of the index;
- Small-Cap account for the remaining 10% of the index.

Such classification has been kept constant for entire analysis. The estimation methodology chosen for the static analysis is a combination of sample and exponentially moving mean. Specifically, the sample mean has been taken for the first 10 years (120 observations) and, after that, a weight of 2.00% has been assigned to each new monthly observation. This approach has the advantage of allowing us to track the evolution of the company over time, giving more weights to more recent events. Moreover, since the variables have been converted to a monthly basis, applying a pure exponentially smoothing approach would result in misrepresenting the data, as there would be a huge focus only on the very recent past. Unfortunately, for some securities there are less than ten years of data available, either due to lack of information by Reuters or because they got listed and became eligible for the index later. Thus, for those securities, parameters would be calculated using a pure smoothing approach. This also explain the choice of a very small smoothing factor. However, it should be pointed out that those securities, due to their limited data availability, are unlikely to be eligible for the final universe, therefore the results should not be biased. For these reasons, the proposed approached is considered to be the best solution for the analysis.

Given the huge amount of data involved in a universe of 600 securities, for a time spawn of over 20 years, it was opted to reduce the initial universe to a more appropriate one, consisting in 100 securities. However, in order to keep the final universe as close as possible to the original composition, some adjustments have been made. In particular, we aimed at keeping the same Large, Mid, Small-Cap ratio existing in the EuroStoxx 600. Concluding, the analysis that follows will be carried out on a portfolio of 100 securities, 10 Small-Cap, 20 Mid-Cap and 70 Large-Cap. To avoid possible bias due to a driven stock specific selection towards more or less liquid securities, a random simulation of 1000 portfolios has been run, and one of the resulting compositions has been chosen completely randomly.

4.2. Initial Descriptive analysis

The initial universe price evolution over the 20 years’ time window is the following (the “outlier” stock is CHOCOLADEFABRIKEN LINDT, whose prices in is currently above 80,000.00 Euro).



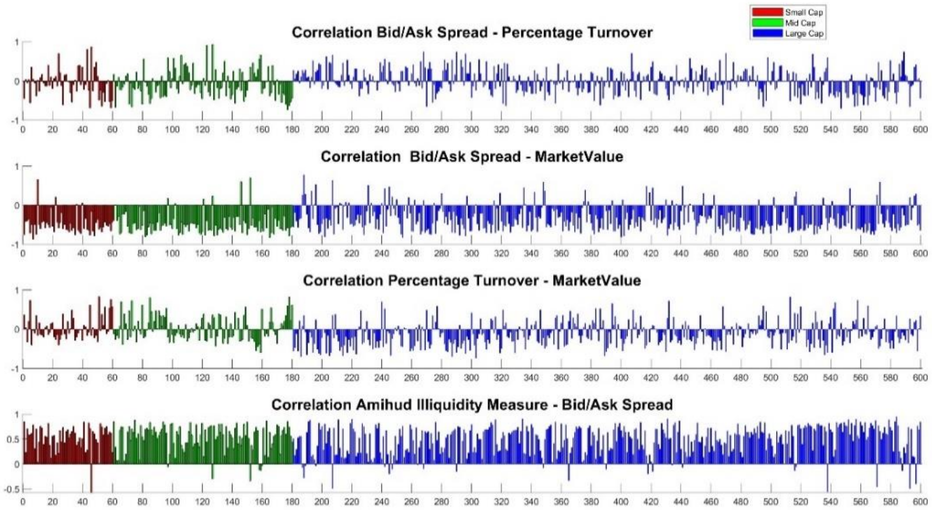
(Figure 5. Price evolution of the EuroStoxx 600 constituents, time window 1999-2019)

In order to make the results more understandable, all the stocks have been organized according to their Market Cap classification, so that the first presented are always Small-Cap, then there are Mid and Large-Cap. Initially, it is provided a brief descriptive analysis summarizing returns, standard deviation, minimum and maximum return, skewness, kurtosis and spread. The tables are reported in the Appendix at the end of the chapter. Results are already quite interesting. Over the past 20 years, it seems that Market Cap classes have had specific characteristics. In particular, it can be noticed that Small-Cap, which on average have a larger spread, are associated with higher average returns but also higher standard deviation. On the other hand, Large-Cap stocks tend to have much lower spread, less risk but also lower average returns. This seems to initially suggest some sort of illiquidity premium across the stocks (at least in the long run), as more illiquidity stocks appear to guarantee larger returns to compensate for both the higher volatility and liquidity risk. The following table shows the described situation, with values representing the sample mean of the variables over the entire time window.

BucketSize	Return	St_Dev	Sharpe_Ratio	Spread
'Small Cap'	1.57466	10.35091	0.155	0.67257
'Mid Cap'	1.15616	9.38921	0.12846	0.50957
'Large Cap'	0.87734	8.75771	0.10524	0.31559

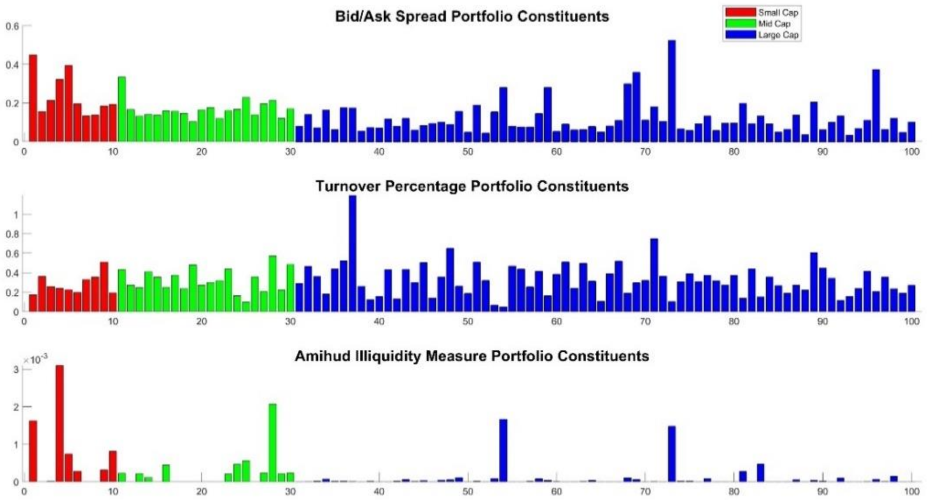
(Figure 6. Sample average descriptive statistics of the universe)

Further, the initial universe's liquidity has been investigated using some of the measures describe previously, including spread, turnover, market value and Amihud measure of illiquidity. The results are somewhat expected. First, it can be noticed that the spread, on average, tends to decline as the Market Cap increases. Such relationship has been established and widely proven in several studies. Far more interesting is the Amihud measure of illiquidity. Even if with some exceptions, it seems that such measure tends to be higher in low capitalized compared to high capitalized securities. This seems to indicate that not only low capitalized stocks are more "costly" due to a higher spread, but also, they tend to have a lower trading volume, which in turns tends to increase the expensiveness of such stocks even more following the impact of potential trades. Finally, the last measure presented is the turnover which, quite surprisingly, is larger for Small and Mid-Cap than for Large-Cap stocks. This might be due to Small-Cap stocks having less shares outstanding (almost 10 time less), thus the impact of similar trades would result in larger turnover for Small-Cap. Another possible explanation is that Small-Cap stocks tend to have less history than Large-Cap. Therefore, larger value would be attributed to those Small-Cap compared to the Large-Cap, which have been through more economics cycles. Indeed, by taking sample mean values, Large-Cap stocks appear to have larger turnover than Mid and Small-Cap. Moreover, if we look at the dollar value of the turnover, this is much larger for Large-Cap stock compared to Small-Cap. To further investigate into the liquidity characteristics, correlations among the previously described variables have been analyzed. Firstly, as mentioned above, the correlation between spread and turnover does not have a specific direction, as sometimes stocks with higher spread tend to have also higher turnover. This association seems particularly more frequent among Mid and Large than Small-Cap stocks, for which, on the other hand, the majority shows negative correlations. This is might be due to the fact that Small-Cap stocks tend to be much costlier, in several different ways, than the other stocks. Therefore, a further increase in the spread would significantly impact the trades people are willing to undertake. On the other hand, for Mid and Large-Cap stocks the market impact might be smaller and thus, even if there is an increase in the spread, it will not affect much the turnover. Furthermore, the correlations between spread and market value and between spread and Amihud illiquidity show a precise direction. In particular, the correlation between spread and Market Cap appears to be strongly negative for all the stock classes, confirming once more that the higher the Market Cap is, the lower spread tend to be. Even more interesting for the purpose of this analysis is the correlation between spread and Amihud illiquidity measure. This correlation seems to be positive and very strong among all the stocks classes. This suggest that stocks with higher spread are not only costlier in terms of pure explicit transaction costs, but, apparently, they tend to be associated with also higher average market impact. For these reasons, the liquidity constraints set in the following analysis will be a combination of the mid-price spread percentage and Amihud measure of illiquidity. The aim is to try to take into account explicit transaction costs, proxied by the spread, and more implicit transaction costs, proxied by the Amihud measure of illiquidity. As mentioned earlier, these results are based on the entire initial universe (EuroStoxx600).



(Figure 7. Correlation across the different liquidity measures, divided by Market Cap class)

The randomly chosen final universe, however, shares the same similarities, therefore ensuring that the results obtained are not entirely biased and can be generalized to the entire population. It can be noticed that when taking securities with sufficient data, for all the stock classes, Large-Cap appears to have a larger turnover than Small and Mid-Cap, even if slightly.



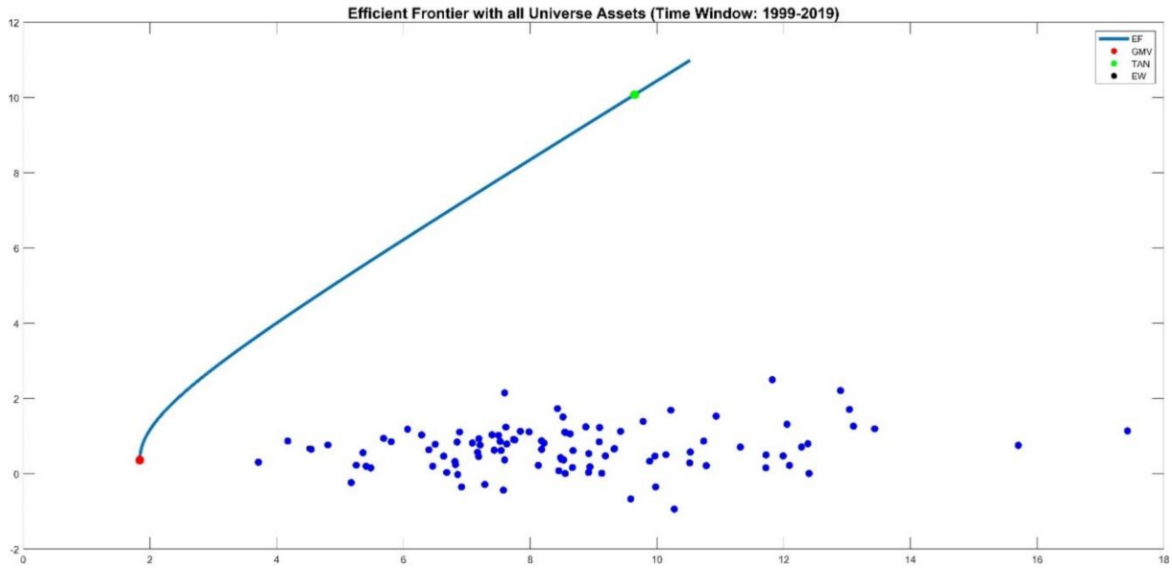
(Figure 8. Liquidity characteristics of the assets constituting resulting portfolio universe)

BucketSize	Spread	Turnover	Amihud_Illiquidity
'Small Cap'	0.23794	0.28314	0.00069
'Mid Cap'	0.16662	0.32419	0.00025
'Large Cap'	0.11942	0.32857	8e-05

(Figure 9. Average liquidity characteristics per Market Cap class)

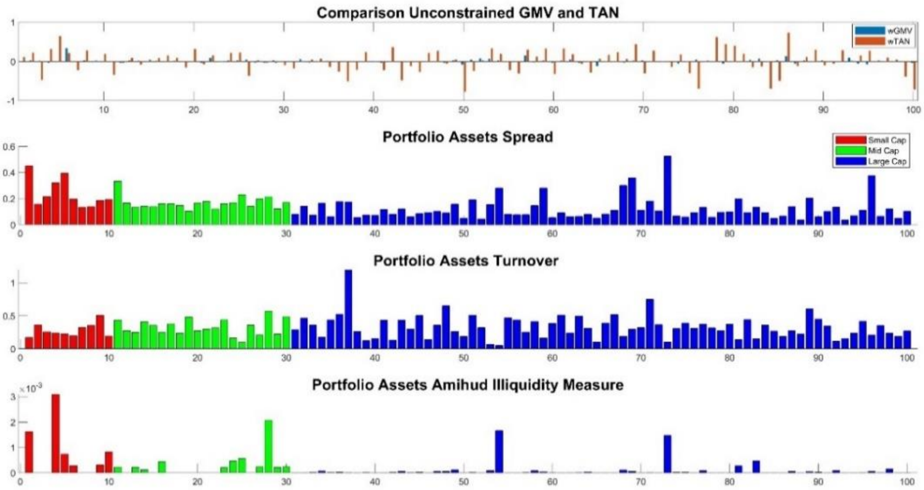
4.3. Mean-Variance Portfolio

The first step is the investigation of the mean-variance framework. To do so we run several optimizations using MATLAB and, providing as input the mean returns and variance, calculated using the previously described methodology, of the final universe for the entire time window (1999-2019). Initially, no further constraint has been set so that both long and short positions are allowed. As it can evinced by the following picture, Markowitz's idea was correct, as by combining the securities in a specific way is possible to obtain better risk-return profiles than by investing in the single securities.



(Figure 10. Efficient Frontier based on the final universe composition)

In particular, over the past 20 years, the GMV would have outperformed or at least performed as good as roughly 30% of the final universe of stocks, but with a lot less risk. However, even with a naïve approach such as equal weighting (black dot on the above graph), the portfolio would have outperformed many securities and performed almost as good as the GMV, highlighting another Markowitz drawback and calling some questions about the usefulness of all these computations. As mentioned in chapter 3, much of the subsequent analysis is focused on the GMV and TAN. The following chart shows the resulting composition for both the portfolios.



(Figure 11. Comparison of GMV and TAN Unconstrained and assets' relative liquidity measures)

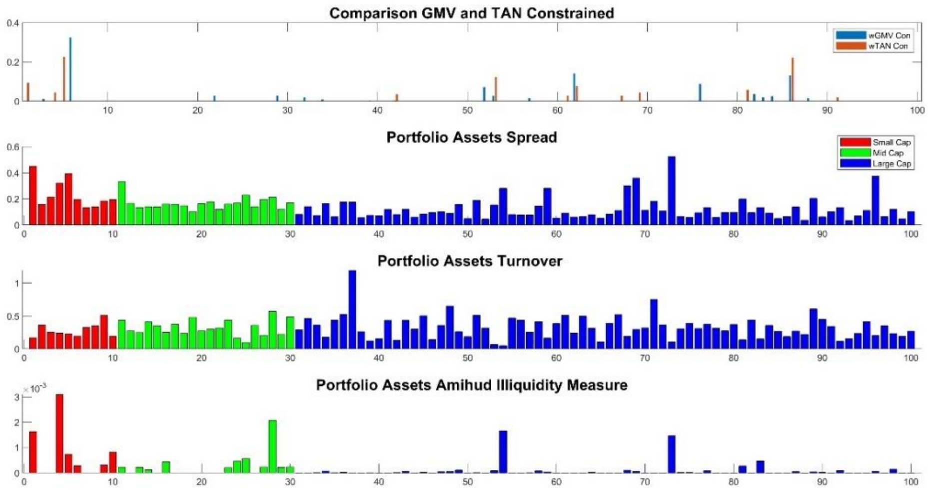
As it can be noticed, this chart shows some of the mean-variance framework's drawbacks previously discussed. According to the optimization results, the investors are required to take several large long and short positions in single assets. While this phenomenon is partially alleviated in the GMV (even if for security n° 6, COFINIMMO the optimal weight is around 35%), the TAN shows multiple extreme positions, with weights ranging from -70% to +70%. Furthermore, some of those extreme positions are on assets which appear to have higher illiquidity costs than the others. In particular, the TAN requires to allocate a significant amount of funds across some of the Small and Mid-Cap stocks with the highest Bid-Ask spread and Amihud measure of illiquidity. Overall, both TAN and GMV would reflect the starting universe, with similar proportions invested in Small, Mid and Large-Cap.

BucketSize	GMV_Unconstrained	TAN_Unconstrained
'Small Cap'	0.159	0.12082
'Mid Cap'	0.14776	0.11586
'Large Cap'	0.69324	0.76332

(Figure 12. Weights distribution across the different Market Cap classes for an Unconstrained portfolio)

Quite surprisingly is that the GMV invests slightly more in the Small-Cap class than what the TAN does. Concluding, this first results already show that while from a mathematical point of view there is a set of unique solutions dominating the other, not necessarily such solutions can be implemented, due to real world limitations. Even if it was feasible to replicate such portfolios, it might get extremely costly, with a significant impact on the final performances.

The same analysis has been conducted including a no-short selling constraint so that short positions are not allowed. As expected, the efficient frontier is lower than in the case of short selling, as less combinations are now available. Interesting is the GMV and TAN composition compared to the first, boundless, optimizations. Including the no-short selling constraint limits the level of diversification as expected, but it does not completely solve the problems listed before. The GMV's level of diversification is extremely reduce as it is now investing in 21 securities, with 4 securities accounting for 2/3 of the overall portfolio. Moreover, almost 1/3 of the funds is allocated to a Small-Cap stock with a significant degree of illiquidity, as it has the 3rd largest spread and the 7th largest Amihud illiquidity measure. Regarding the TAN, it also has quite significant concentration in few, low liquid securities, even across Large-Cap. Furthermore, introducing the no-short selling constraint extremized the Market Cap class allocation, with both GMV and TAN investing roughly the same proportion in Small and Large-Cap (35% Small 60-65% Large), with a minimum investment or no investment at all in Mid-Cap stocks.



(Figure 13. Comparison of GMV and TAN Constrained and assets' relative liquidity measures)

BucketSize	GMV_Constrained	TAN_Constrained
'Small Cap'	0.33649	0.36212
'Mid Cap'	0.05676	0
'Large Cap'	0.60674	0.63788

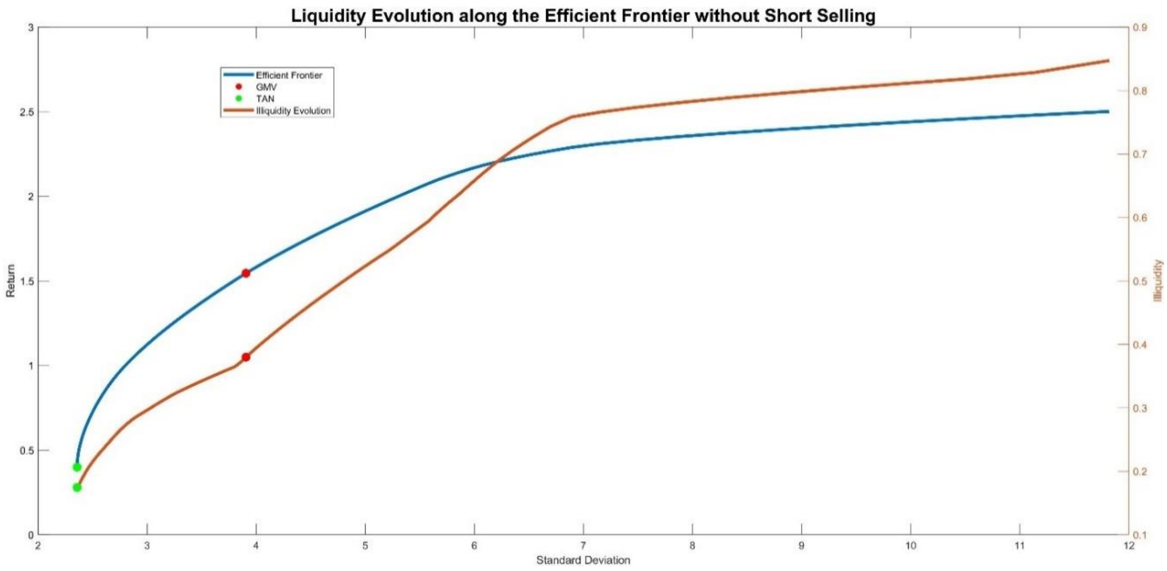
(Figure 14. Weights distribution across the different Market Cap classes for a Constrained portfolio)

4.4. Mean-Variance-Liquidity Optimization

The following paragraph will attempt to show the implications of the introduction of a liquidity constraint in the mean-variance optimization framework. The paragraph includes two sub-section, one for each optimization problem.

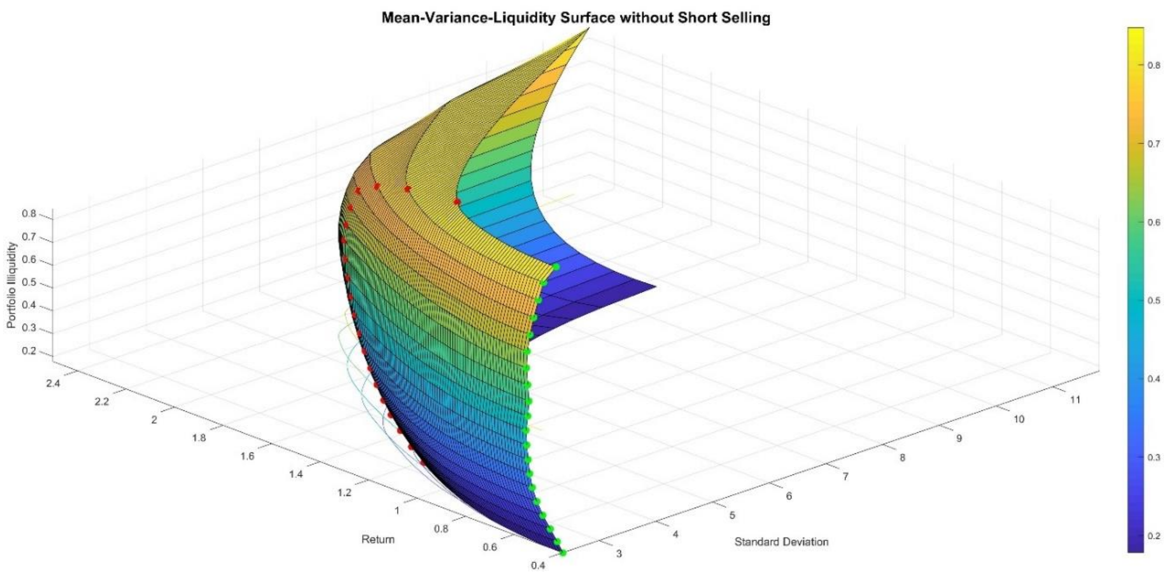
4.4.1. Portfolio with no-Short Selling

The first results provided concern the introduction of the liquidity constraint in a portfolio where short selling is not allowed. As previously described, the liquidity parameter has been computed as the linear combination of the spread and Amihud illiquidity measure. These values have been subsequently standardized so that they would range from 0 to 1, making easier the set-up and interpretation of the constraints and result. Initially, it makes sense to analyze how portfolio's liquidity behaves alongside the efficient frontier. Therefore, we calculated what is the portfolio liquidity for the set of dominant portfolios constituting the efficient frontier. There seems to be an inverse relationship between risk, return and liquidity. The higher risk and returns are, the lower also tend to be the portfolio liquidity, up until the most illiquid portfolio at the end of the efficient frontier, where the entire portfolio allocation is on a single, very high illiquid security.



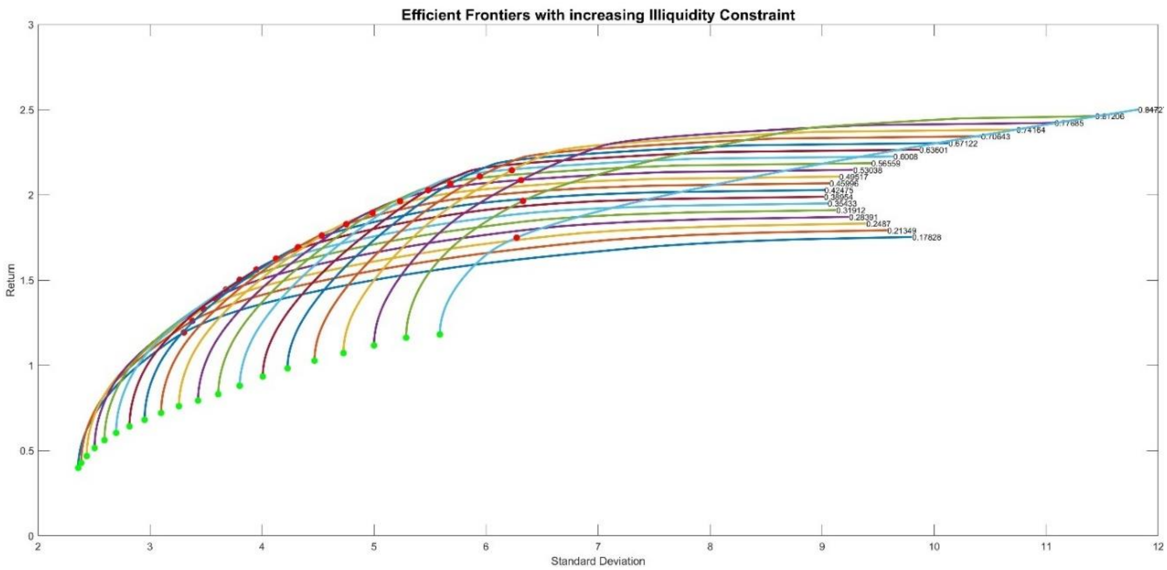
(Figure 15. Liquidity evolution along the Constrained Efficient Frontier)

To build the new Mean-Variance-Liquidity surface, several optimization iterations have been run. Given that there is no unique way to determine what is liquid and what is not, as well as, there are no particular portfolio liquidity references in the literature, the target liquidity level to be matched at each optimization iteration has been chosen by looking at the percentiles of the liquidity alongside the frontier previously shown. From the 5th up the 100th percentile, twenty different frontiers have been calculated. Thus, along the same frontier, there are portfolios with different risk-return profile, but same level of liquidity. The resulting surface is the following:



(Figure 16. Mean-Variance-Liquidity surface for a portfolio without short selling)

Returns and standard deviation are on the y-axis and x-axis as usual, while on the z-axis there is the level of portfolio illiquidity. TANs and GMVs have also been plotted on the surface and they are represented, respectively, by the red and green dots. It is interesting to notice the surface initially protrudes outside and then goes back in. The picture becomes clearer if looking at the single frontiers plotted in the standard 2-dimensional space. When the level of target liquidity is extremely high (very low illiquidity constraint), the portfolio combinations achievable are somewhat limited by the availability of securities and the positive weights constraint. However, as the liquidity constraint gets looser, the portfolio can achieve better risk-return combinations, up to the point where the level of illiquidity required is so strict, that the frontiers start moving to the right (worse solutions). Particularly interesting are those points where frontiers overlap: those represent portfolios with the same risk-return portfolio but, by lying on different frontiers, they also have different level of liquidity. This suggest that there is some sort of superiority between these portfolios: given everything else equal, a rational investor should always choose the portfolio that represents the cheapest solution, as it would maximize his final utility. Overall, this surface shows that in both case where liquidity is set very high or very low, the risk-return profiles are worse than in case of mid-range values. However, at that point, high liquid portfolios represent a superior choice, both in terms of costs and portfolio composition, compared to same risk-return profile portfolios but with higher degree of illiquidity.

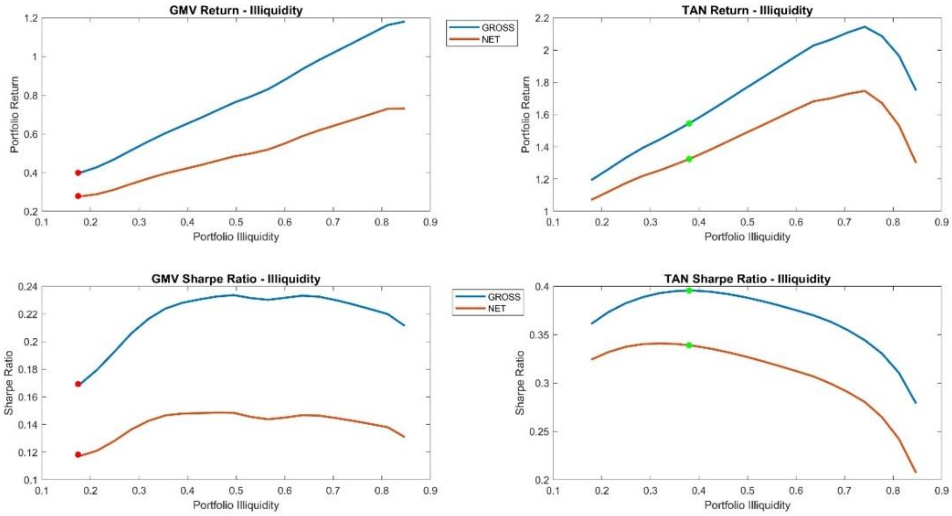


(Figure 17. All the Mean-Variance-Liquidity frontiers, with GMV and TANs, plotted in a return-standard deviation space)

As for the surface previously presented, also here TANs and GMVs have been plotted on their respective frontier calculated at each liquidity iteration. At the end of each frontier is also reported the target level of illiquidity. By taking a closer look the graph, it can be noticed that at a level of gross return of 1.5912% and standard deviation of 5.90%, there is an intersection between two different frontiers, meaning two different portfolios with same risk-return profile. However, these two portfolios lie on the two most extreme frontiers, the most liquid and the most illiquid. On the most liquid, such portfolio is towards the end, therefore the diversification is quite reduced even if the overall liquidity is very high. Indeed, despite that, both the portfolios are basically investing in the same number of securities. However, in terms of net returns there is a significant difference as, by only taking into account the spread, the return of the portfolio on the most liquid is 1.4690%, thus with just a marginal reduction from the gross, while for the portfolio on the most illiquid the net return is 1.1414%, almost 30% less than its equivalent. Concluding, given that the level of diversification is fairly similar, and assuming that investors preferences are the same and there are no other constraints, a rational investor should always choose the portfolio lying on the most liquid frontier.

In terms of diversification, as expected, a higher degree of illiquidity produces a skewed allocation towards the most illiquid assets and therefore a significant reduction in the diversification, as weights are constrained above zero. This can be noticed especially in the TAN, where some high illiquid Large-Cap stocks which had no or small weights at lower level of illiquidity, get increasing allocations. However, until the 60^{th} percentile GMV is investing in at least 10 assets while the TAN in at least 8. These numbers might seem small, but they are not far away from Markowitz's starting point of 21 assets for the GMV and only 12 for the TAN. In terms of Market Cap allocation, there are some common trends across GMV and TAN: the reduction on Mid-Cap investments as the liquidity decreases as well as the increasing allocation towards Small-Cap stocks

Subsequently, the evolution of GMV and TAN gross and net returns, as well as the Sharpe ratio, at the different levels of liquidity has been analyzed. The results are the following:

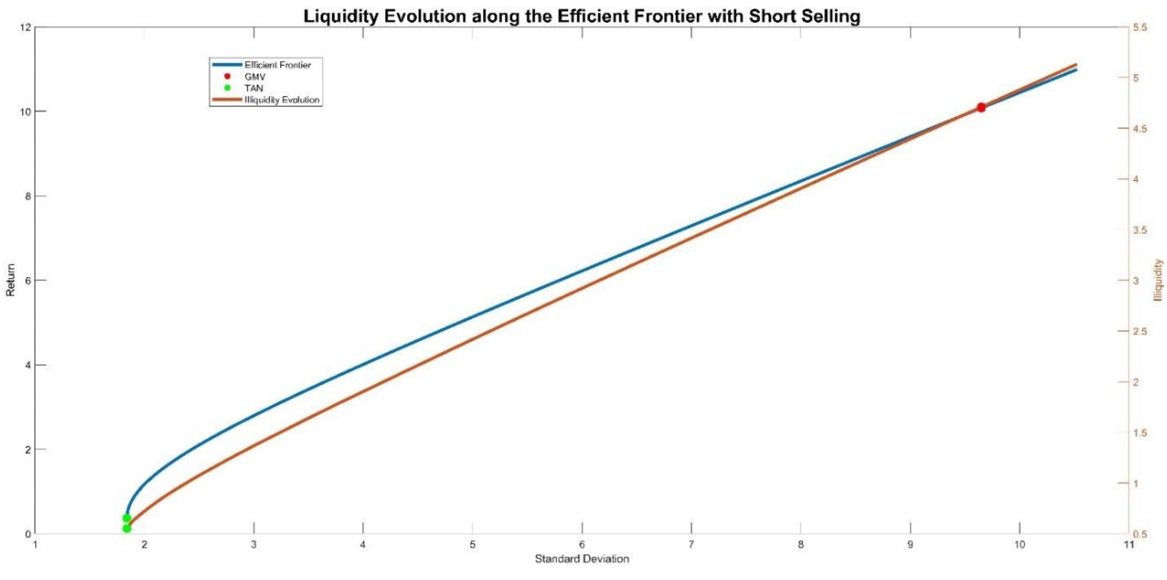


(Figure 18. Return and Sharpe ratio evolution across different level of illiquidity for a portfolio without short selling)

Markowitz's TAN and GMV are represented by the green and red dots respectively. As it can be noticed, GMV returns tend to increase quite straightforwardly as the illiquidity increases. TAN has a similar behavior, up to a level of illiquidity of 0.75 after which returns start decreasing again. Such portfolio represents a superior choice compared to the followings as it has less risk (lower Standard deviation), higher returns and lower trading costs. This seems to indicate that, while lower liquidity brings higher returns (and, of course, higher risks), there is an "efficient point" after which stretching the degree of illiquidity even more would be counterproductive from all the points of view. Concerning the Sharpe ratio, results get even more interesting. For both the GMV and TAN, the relationship resembles a reverse U-shape curve. This, again, points to an optimality point, a given level of liquidity maximizing the risk return profile of the portfolio. It is also interesting to notice that the level of illiquidity of Markowitz's TAN is in between the two level of illiquidity maximizing the gross Sharpe ratio. Therefore, it seems that Markowitz optimization is automatically selecting a portfolio among the top performing. Complete different story for the GMV, whose level of illiquidity maximizing the gross Sharpe ratio is three times bigger. However, considering the net amount flowing into the investors' pockets, the optimality is on a lower level of illiquidity as expected. Indeed, for a net perspective, the level of illiquidity maximizing the TAN Sharpe Ratio is smaller than Markowitz's value. This confirms that is possible to improve Markowitz optimization by considering the liquidity as an extra parameter, as better performing and cheaper solutions can be achieved. On the other hand, it can be noticed how the imposition of a liquidity constraint is highly penalizing the GMV net returns, as the net Sharpe ratio is almost halved. Indeed, the introduction of higher and higher illiquidity constraint seems to increase the risk and illiquidity enormously but with only a marginal impact on returns. Therefore, in period of high illiquidity, it might make more sense to try to maximize the liquidity of the portfolio, and imposing constraint on the maximum level of risk acceptable.

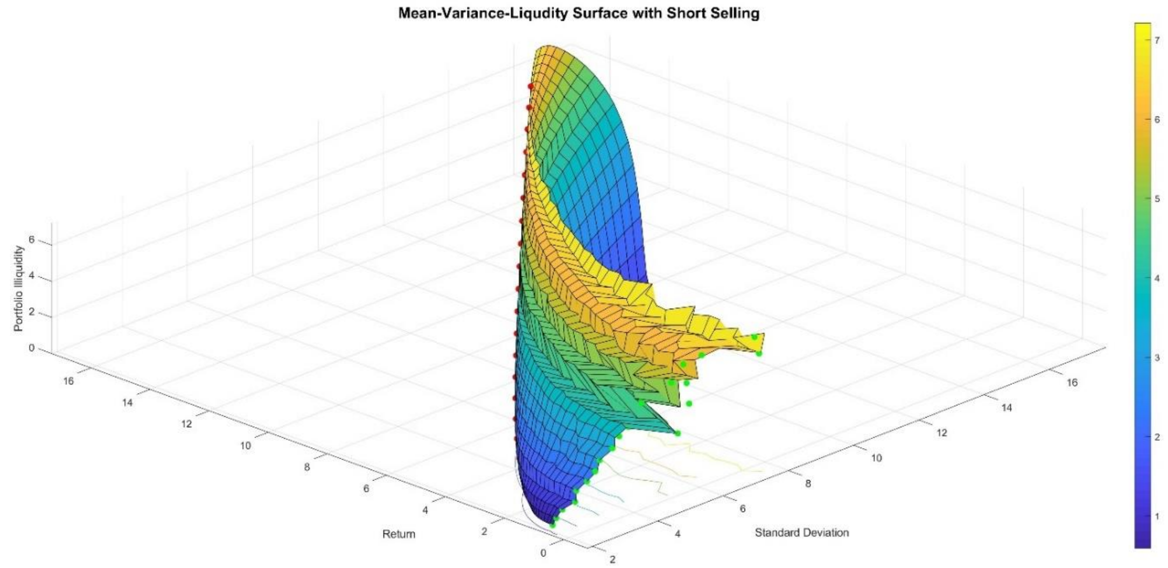
4.4.2. Portfolio with Short Selling

The same analysis has been performed on a portfolio where also short selling is allowed. Again, the liquidity target levels have been calculated using the same methodology as before but applied to the liquidity calculated along the boundless Markowitz's efficient frontier. As previously described in chapter 3, these optimizations require the use of a nonlinear constraints in order to account for the liquidity of short-sold securities. Given the nonlinearity of the constraint, the problem might not be convex and have no unique solution, especially when the constraints get tighter and tighter. To deal with this issue and reduce the margin for errors, we used the *GlobalSearch* script in MATLAB. For each optimization, the function simulates several starting points and the returns the global minimum solution, if available. The advantage of having short selling is that it allows to stretch the limits of the liquidity threshold, as higher degree of illiquidity can now be achieved. Indeed, the illiquidity constraint thresholds for the 20 frontiers have been set from the 5th percentile to 1.5x the 100th percentile of the illiquidity along the original Markowitz frontier. The resulting liquidity frontier is the following:



(Figure 19. Liquidity evolution along the Unconstrained Efficient Frontier)

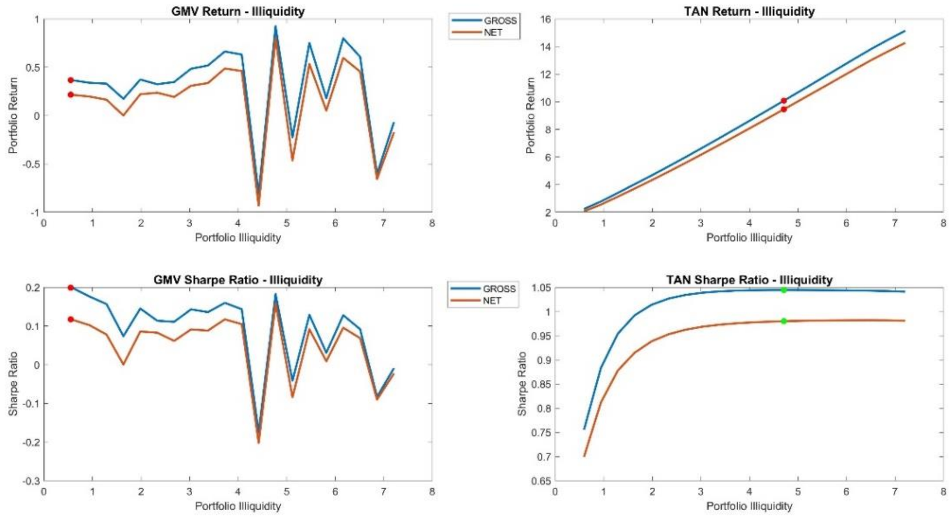
As for the constrained case, illiquidity appears to increase with the risk-return profile, following almost a straight line. As it can be notice, the overall level of illiquidity of the TAN is extremely large, being more than five times larger than the most illiquid security. Based on these values, Mean-Variance-Liquidity surface shown below is derived.



(Figure 20. Mean-Variance-Liquidity surface for a portfolio with short selling)

Again, returns, standard deviation and portfolio illiquidity are, respectively, on the y-axis, x-axis and z-axis, while green and red dots represent the GMVs and TANs. As for the no-short selling constraint, it can be notice that the surface's evolution is similar. Efficient solutions can be found, as for the same risk-return profile, there are portfolios with different degree of liquidity and thus different costs. The major difference with the constrained case is that by allowing short selling, the combination of risk-return-illiquidity that can be achieved are much higher. The TAN on the most liquid frontier in the unconstrained case generates a return higher than the top performing TAN in the constrained case, with 1/3 of the volatility and a degree of illiquidity 30% lower. Indeed, for the same percentile of illiquidity, there is a significant increase on the Sharpe ratio, which is more than doubled compared to the constrained case. Moreover, as expected, allowing short selling also had beneficial effects in terms of diversification as both GMV and TAN invest in all the 100 assets, regardless the level of liquidity. Also, the allocation across the Market Cap classes is improved. Weights are more equally distributed across the classes. As the liquidity decreases there is still a skewed allocation towards Small and Mid-Cap (TAN almost doubles its exposure to Small-Cap from 7 to 12% and GMV increases its exposure to Mid-Cap by 30%, even if there is a significant reduction to Small-Cap) but the overall allocation almost reflects the initial universe composition.

Particularly interesting is the GMV, for which imposing different liquidity constraints results in different portfolio composition rather than just a skewed version of the most liquid GMV. Regarding the returns, unfortunately, results are less straightforward than for the constrained case.

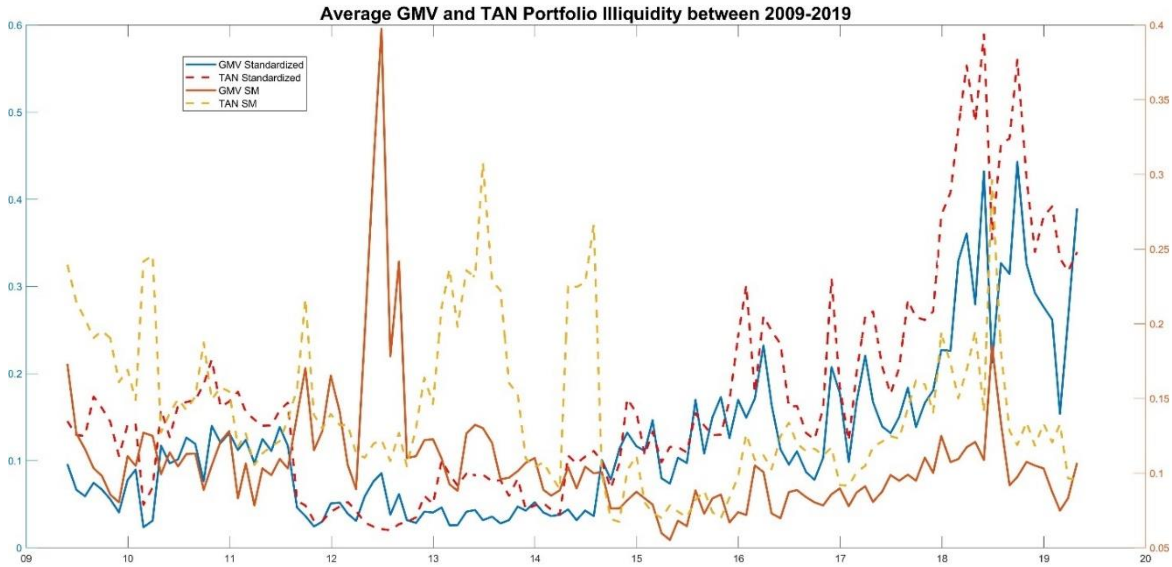


(Figure 21. Return and Sharpe Ratio evolution across different level of illiquidity for a portfolio with short selling)

Indeed, both GMV returns and Sharpe ratio do not show a clear relationship with the liquidity, as for low levels of illiquidity the Sharpe ratio is even higher than for larger levels, even if the absolute return is smaller. This seems to suggest that while increasing liquidity is beneficial for absolute returns, the risk increases proportionally more, resulting in a portfolio with lower Sharpe ratio and higher costs. However, results seem highly dependent on the level of liquidity chosen. Concerning TAN things get interesting. At first it can be noticed that the return-illiquidity relationship is almost a line, suggesting an almost linear relationship between the variables. Moreover, the Sharpe ratio curve, after an initial increase, gets almost flat, which means that as the illiquidity increases not only the absolute returns increase, but also does the risk in a proportional way such that the ratio is basically constant. Therefore, it seems that stretching the illiquidity level above the 50^{th} percentile is pointless as no better risk-return profiles can be achieved for the TAN.

4.5. Rolling Approach

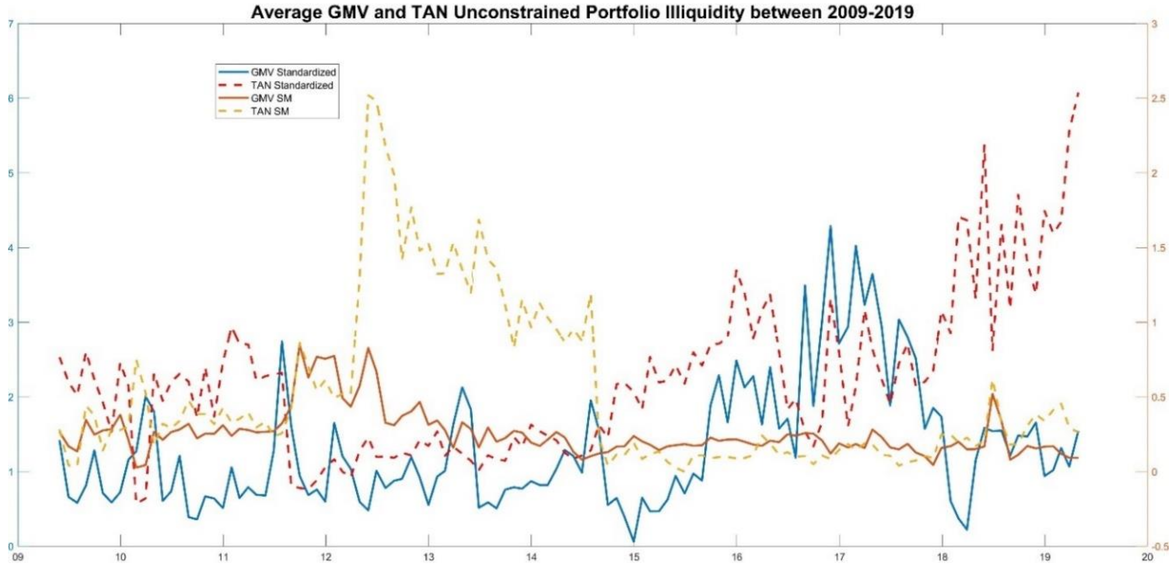
To deepen the analysis, we investigate the effect of liquidity from a dynamic point of view. We use the same three "cases" described above: standard Markowitz, liquidity constrained without short selling and liquidity constrained with short selling and apply a monthly rebalancing approach. To do so, first we computed new estimates for returns, variance and illiquidity measure. In particular, we used an exponentially weighted moving average approach, where the initial value is the sample mean for roughly the first 10 year (124 monthly observations so that there are exactly 10 years left in the rolling analysis). Thus, at the end of each month, returns, variance and illiquidity are calculated as the combination of the new monthly observation (at time *t*) and the previous 124 observations. Moreover, also the smoothing factor is calculated in a dynamic way. Starting from a value of 90%, assigned to the previous 124 observations, the smoothing factor is adjusted depending on the current average assets' volatility compared to the previous period average volatility. In particular, the factor is adjusted inversely to the change in the volatility levels, so that if the universe, as aggregate, appears to be more volatile in this period compared to the previous month, less weight will be given to the new observation. In this way, we try to account for random spikes in the volatility and period of greater uncertainty. The analysis will be again focused on the GMV and TAN, for which, at each iteration, portfolio composition, gross and net return, volatility and portfolio illiquidity will be calculated. Moreover, also the level of illiquidity alongside the frontier will be calculated at each iteration, since the liquidity constraints will be based on those levels. When introducing the liquidity constrained, we will be focused on four different levels: 10^{th}, 25^{th}, 50^{th}, 75^{th} percentile based on the current level of liquidity along Markowitz's frontier in the period. Therefore, at each iteration, those target percentiles will be calculated based on the current liquidity levels along the frontier. Moreover, percentiles have been chosen to ensure that there are relevant differences across the portfolios but, at the same time, portfolios are enough diversified (diversification which could not be achieved by imposing higher constraint thresholds). Overall, this solution seems the most reasonable because, especially over such a long-time spawn, the liquidity across assets and portfolios is likely to change. Hence, it makes more sense to define a target level in relative terms based on the current level of liquidity in the market, rather than just taking absolute values based on the entire time series. As before, the illiquidity parameter has been standardized from 0 to 1 at each iteration. The purpose of this analysis is to investigate whether portfolios with more or less stringent liquidity characteristics can perform better than standard Markowitz and whether there is a superior solution across the different liquidity constrained portfolios. An inequality constraint has been added to the portfolio, thus introducing a minimum level of illiquidity that each portfolio has to achieve, but without imposing any upper constraint on that. The optimization process takes the form described in chapter 3. Firstly, we analyze the evolution of the liquidity parameter over time for both GMV and TAN.



(Figure 22. Illiquidity evolution over time window 2009-19 for Markowitz GMV and TAN Constrained with monthly rebalancing)

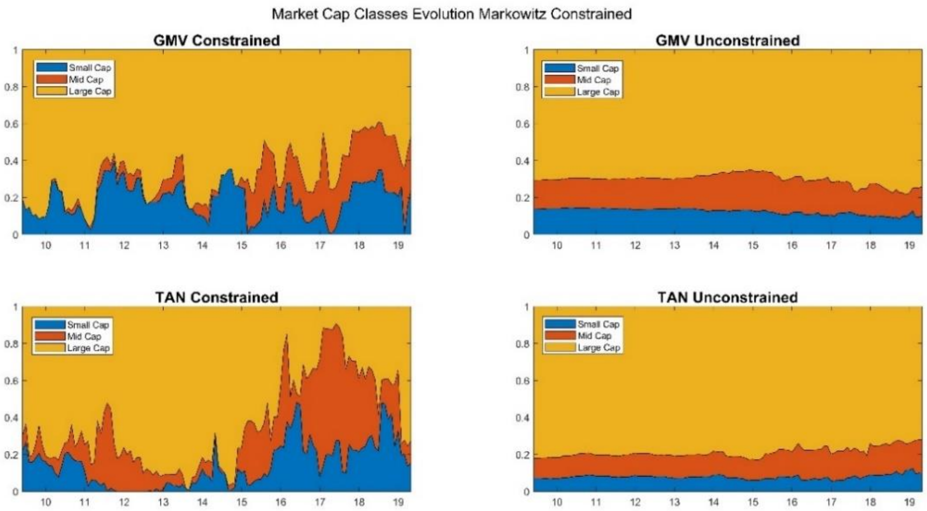
It is interesting to notice that while in absolute values the portfolio illiquidity seems to decrease over time, when analyzing using the standardize values, it actually increases. This suggest that even if portfolios are less expensive nowadays, with apparently lower transaction costs compared to 10 years ago, they also seem to invest in more illiquid securities now than what they used to do in the past.

Moving to the unconstrained case, the patterns are similar to the constrained case. There is an upward trend of increasing allocation to relative more illiquid stocks for the TAN, even if the overall level of spread and Amihud measure of illiquidity is much lower than in the triennial 2012-2015 where, in contrast, the standardized illiquidity was not much large. The GMV on other hand, does not seems to predilect more or less liquid securities, as the ratio in gross terms stays almost constant over time. As in the static analysis, allowing short position widely increases the level of illiquidity for both the TAN and GMV, with the TAN peaking at values ten times larger than in the constrained case.



(Figure 23. Illiquidity evolution over time window 2009-19 for Markowitz GMV and TAN Unconstrained with monthly rebalancing)

Subsequently, we analyzed the GMV and TAN allocation across the different Market Cap classes. As it can be noticed, overtime, GMV seems to give more weight to Small-Cap compared to the TAN, especially during the triennium 2011-2015, where TAN is allocating almost everything across Mid and Large-Cap. On the other hand, in the most recent years, there has been a significant decline in the allocation to Large-Cap for both TAN and GMV, which is consistent with the previous picture, showing an upward trend in the level of illiquidity starting from 2015.

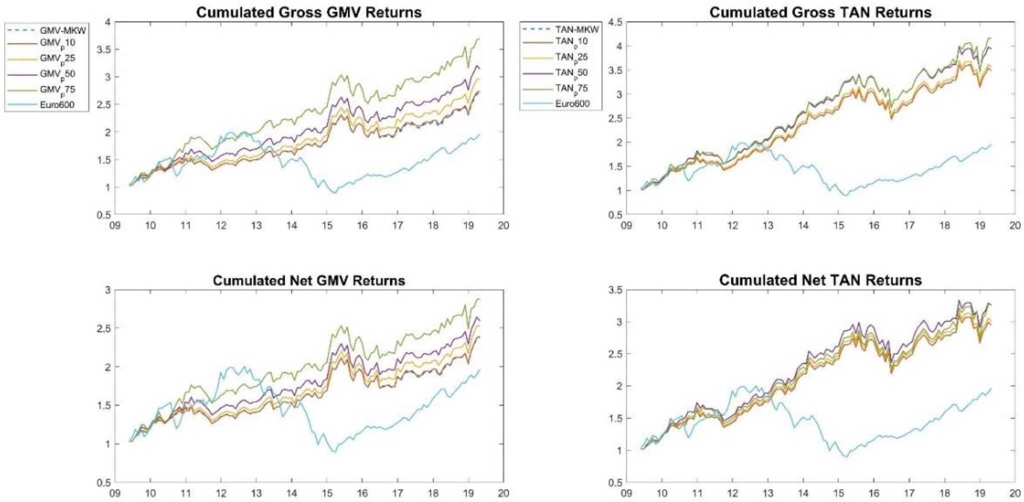


(Figure 24. Markowitz GMV and TAN allocation across the different Market Cap classes)

Significantly different is the trend for the unconstrained portfolio, whose Market Cap class allocation is almost constant over time and extremely similar to the initial universe composition.

However, these results are just at an aggregate level. As it is shown later, even if the class allocation is fairly constant, the level of turnover is significantly high, even higher than for the constrained case.

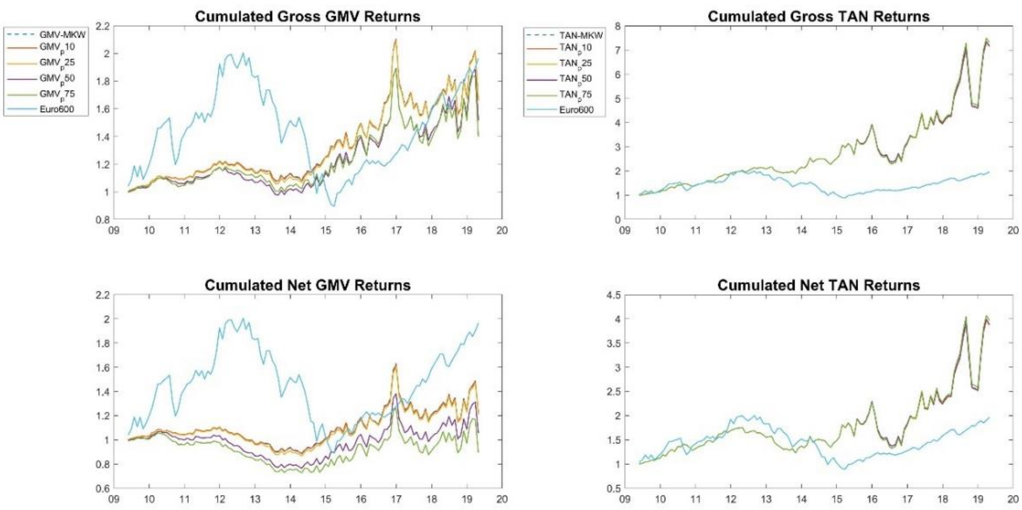
In this part results will be mainly focused on portfolio characteristics concerning gross and net returns, turnover level and portfolio performances parameters, which will be then used to determine which portfolio would have outperformed the others. Firstly, cumulated returns are analyzed. The following picture shows the comparison between the portfolios for the different levels of liquidity, plotted against the relative benchmark, the EuroStoxx 600. Unfortunately, due to the limited availability of data, it was not possible to compute a net version of the benchmark, so in both gross and net case, portfolios return will be compared to the gross cumulated return of the EuroStoxx 600.



(Figure 25. Constrained portfolio Gross and Net cumulated returns at the different illiquidity levels)

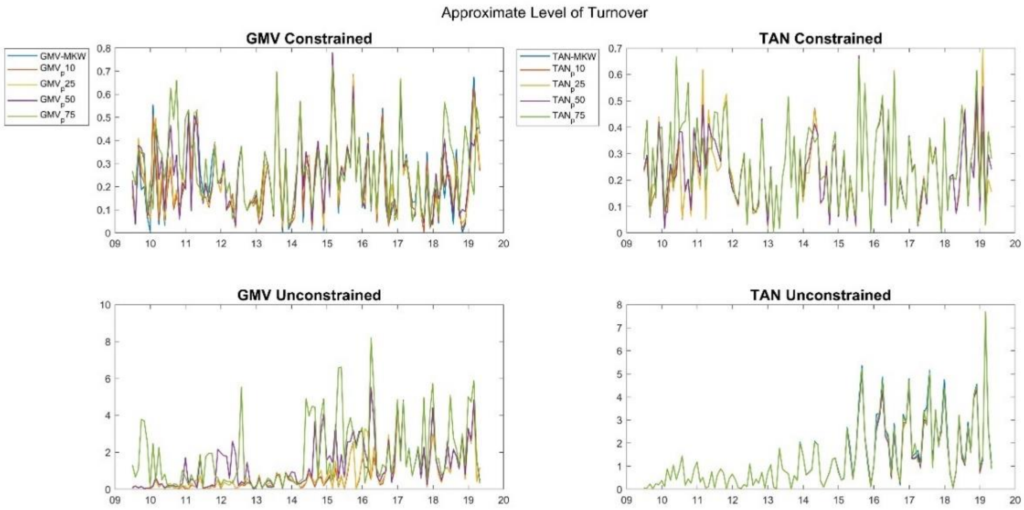
As expected, the higher the degree of illiquidity is, the higher is also the potential cumulated return of the portfolio. It can be noticed that there are only four TAN time-series in the chart. This is because also Markowitz portfolio satisfies the threshold at the 10^{th} percentile, therefore the resulting portfolio almost overlaps Markowitz's (even for the GMV the two time-series almost overlaps completely, but in this case the differences are slightly bigger). All the portfolios show a premium compared to the relative benchmark, even when moving to net returns. However, even if imposing higher illiquidity constraint seems to guarantee higher returns, it also comes at an average higher cost. It can be noticed how the most illiquid TAN has, on average, net returns 30% lower than the Markowitz's net. Even worse is the GMV case, where the average difference between gross and net returns is more than twice than the Markowitz's. However, these results were partially expected since the GMV was almost more liquid than the TAN, therefore the imposition of very high illiquidity constraint resulted in very "extreme" portfolios, with the final allocation widely changed from Markowitz's and heavily biased towards more illiquid securities which had no or very small weight in Markowitz's. It can also be noticed that, in terms on net returns, the TAN at the 50th percentile performed as good as the 75th percentile, despite a visible difference in gross performances. This again suggests that controlling the level of liquidity in the portfolio is beneficial. Too liquid portfolio does not produce enough gross returns, but with too low liquidity the premium is eaten by the transaction costs.

Moving to the unconstrained case, the results are completely different than in the constrained case. GMV and TAN are showing extreme values. It can be noticed how, in terms of gross returns, they stand at the opposites: the first is highly outperformed by the EuroStoxx 600 until the 2015, where it sharply increases up until reaching the same level of the benchmark. The TAN, on the other hand, is always outperforming the benchmark, increasing the overall value of the portfolio by 8 times in 10 years. This is enormous considering that the cumulative return of the S&P over the same period is barely 185%. As we move to net returns, performance plummet for the GMV, with the Markowitz's barely breaking even, and the most illiquid is even losing money. TAN still shows remarkable performances, but not too far apart from the constrained case.



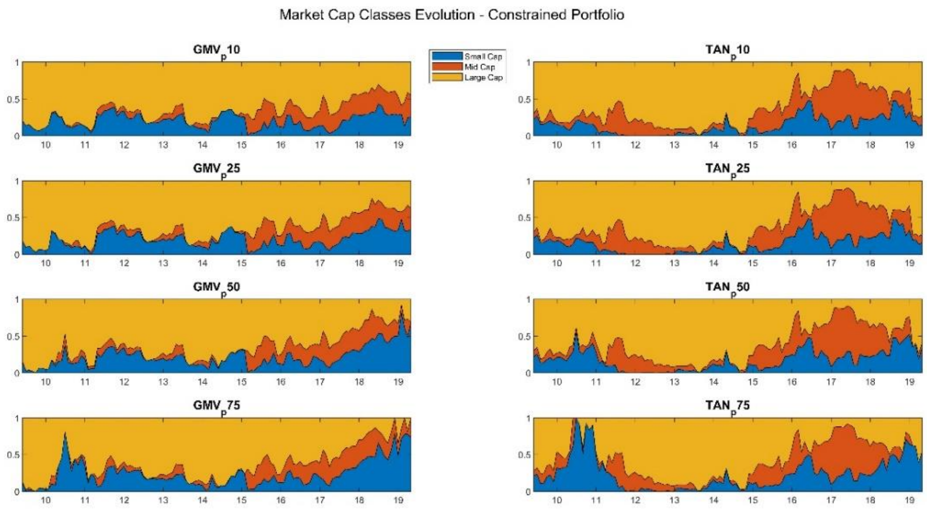
(Figure 26. Unconstrained portfolio Gross and Net cumulated returns at the different illiquidity levels)

It might thus make sense to investigate the weight allocation and turnover to determine the causes of these results. In terms of weight evolution and turnover, the overall composition of the portfolios over time is similar in term of asset allocation, with even the most illiquid guarantying a sufficient level of diversification. If looking at the evolution, there are quite significant spikes around 2011-12 and 14-15 which are probably due to the high turmoil in the market caused by the debt financial crisis and the political and economic instability that characterized the 2014-15. This can be noticed even better in the following chart, providing the approximate level of turnover at each rebalancing date, calculated roughly as absolute distance between portfolio weights at time *t-1* and time *t*. Overall, the level of turnover is not far away from Markowitz's, but they are still quite high. Despite an average of 20-25% (which is still high considering a monthly rebalancing), there are spikes around 60-80%. Changing such huge portions of the portfolio can be extremely costly, eroding all the returns. While for the constrained case the levels are still reasonable, the unconstrained portfolios show unrealistic level of turnover for both the GMV and TAN. This problem regards not only high level of illiquidity, but even Markowitz's portfolios, showing how his boundless approach, from an active perspective, is not feasible in the real world. This also explain the poor performance of the GMV. The gross return increases slightly but steadily, thanks to investment in low risky/volatile securities. However, considering a monthly rebalancing without turnover constraint, such research for low risk security leads to an extreme portfolio rebalancing and thus the complete erosion of the profits (just consider that the average standard deviation for the GMV constrained is 10 times bigger than for the unconstrained, which is almost 0). This situation is even worse for higher illiquidity levels, as there is no premium, gross performances are the same of the other level of liquidity and the final net return is -12%. Similar situation for then TAN, where the huge turnover erodes the returns, bringing them down almost to the same level of the constrained case. It should be pointed out that the level of turnover is also highly correlated with the bounds chosen. The higher is the position allowed on a single asset, the higher is also the chance that the portfolio will experience a higher level of turnover. Indeed, it can be easily shown that with stricter bounds the turnover would significantly decline. The introduction of a simple no-short selling constraint however seems to partially alleviate the problem. The TAN is still performing well, and the GMV, with less low-risk optimal allocations available, is "forced" to invest in more risky securities, which in return, boost significantly the gross and net performances.



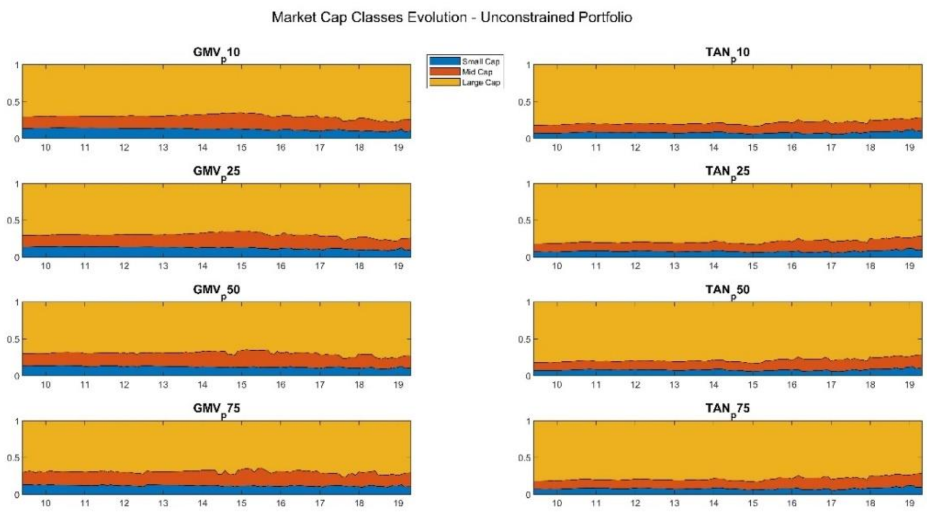
(Figure 27. Approximate turnover evolution for all the portfolios, time window: 2009-2019)

This picture highlights one of the Markowitz's drawback initially mentioned. It can be noticed how, even with only a monthly frequency, the optimization problem is quite unstable, with significant changes in the resulting allocation as the inputs change. In terms of diversification, the constrained TAN and GMV diversify in a sufficient amount of assets, with GMV investing in a minimum of 10 securities in at least 90% of the time-window and TAN investing in a minimum of 8 securities in 80% of the time-window. These number might seem small, but they are closely aligned with Markowitz's, especially when the illiquidity is low. Even the allocation across Market Cap classes is quite similar. In analyzing the constrained cases, it turns out that even across high illiquid portfolios there is an average significant investment in Mid-Cap stocks that it could not be found before in the static analysis, where the average allocation was below 5% for both GMV and TAN. Overall the trend is similar across the different level of illiquidity. It is quite interesting to notice that, despite the significant differences in the required level of illiquidity, the GMV Market Cap class allocation is not so different until 2017, after which there is a significant increasing in the Small-Cap allocation, with peaks at 80-85%. This suggests that in the past few years there might have been an increase in the risk associated with Small-Cap stocks, which was not so predominant earlier in time. TANs, on the other hand, show a consistent increase in the Small-Cap allocation as illiquidity increases, even if between 2012 and 2015, almost 75% of the portfolio is invested in Large-Cap, which not only appear to be more liquid, but also to deliver the best risk-reward profile.



(Figure 28. Market Cap allocation across different level of illiquidity for the Constrained portfolios, time window: 2009-2019)

The unconstrained cases show a very similar pattern to Markowitz's as well. In particular, it can be noticed how the allocation is almost constant over time. When the optimization is boundless (even if not completely) and thus all the assets are available for investing, GMV and TAN seem to allocate similarly across classes and time, with a slighter higher allocation in Small and Mid-Cap stocks for the GMV. This phenomenon is also quite interesting considering the high level of turnover previously shown. Hence, while the optimizations seem to require a significant asset-specific level of turnover from month to month, from a market cap class point of view, the resulting "group" turnover is much smaller, with a smoother class weight allocation.



(Figure 29. Market Cap allocation across different level of illiquidity for the Unconstrained portfolios, time window: 2009-2019)

The last step of the active analysis is a performance comparison of the portfolios. Portfolios have been evaluated using the risk-reward measures described in chapter 3. Based on these measures, each portfolio got ranked and the results are here reported. The first tables presented concern the constrained case. In analyzing the risk reward measures, some Treynor ratios turns out to have negative values. Unfortunately, Treynor has little meaning when the parameter is negative. It should be specified that such negative values are not due to bad returns, but rather a negative portfolio beta. Indeed, the "worst performing", according to Treynor Ratio, is the TAN_p50, which however classifies first according to several other parameters. Therefore, Treynor ratio will be presented but excluded from the computation to determine the final rank.

Strategy	CI	Sh	So	Tr	VaR	ES	Cal	Ste	FT
"GMV-MKW"	67	0.25884	0.36882	82.9143	0.17001	0.11719	0.09209	0.1123	0.86107
"GMV_p10"	60	0.26386	0.38635	73.48374	0.17305	0.12146	0.09299	0.1166	0.87795
"GMV_p25"	51	0.28246	0.42233	81.89108	0.18671	0.13405	0.0997	0.12905	0.91971
"GMV_p50"	32	0.29106	0.46008	159.12231	0.21978	0.14536	0.10929	0.13672	0.95886
"GMV_p75"	19	0.30158	0.46696	576.7739	0.22123	0.15049	0.10782	0.14293	0.96966
"TAN-MKW"	46	0.30576	0.4249	-112.96469	0.19763	0.1474	0.08917	0.14016	0.92828
"TAN_p10"	41	0.30576	0.4249	-112.96794	0.19763	0.1474	0.08917	0.14016	0.92828
"TAN_p25"	32	0.30959	0.43149	-156.48321	0.20083	0.14979	0.09062	0.14243	0.93631
"TAN_p50"	11	0.32584	0.4764	-233.62098	0.23154	0.16404	0.09751	0.15633	0.99105
"TAN_p75"	26	0.29719	0.44611	62.0076	0.23663	0.15166	0.09498	0.1423	0.95361

(Figure 30. Table summarizing Gross performance measures and the final rank for Constrained portfolios)

As mentioned above, the top performing portfolio is the TAN_p50. The best four portfolios are indeed the two most illiquid per type. This confirms once more that, in terms of gross returns and "gross" risk-return profile, less liquid portfolio can outperform more liquid, without sacrificing diversification as shown before. There are few interesting things that should be pointed out: first we see that the Sharpe ratio is almost always increasing in the illiquidity measure, similarly to what we described in the static analysis. In particular for the TAN, it increases up to the 50^{th} percentile and then starts declining again. The same pattern is also shown by Sortino, even if the increments from one illiquidity level to the other are much bigger. This suggests that the overall volatility changes less that what the downside volatility does. However, it is quite interesting to notice that, while the downside volatility is always larger than the upside for high liquid portfolio, the trend gets reverted for high illiquid portfolios (GMV p_75, TAN p_50 and TAN p_75) where the volatility of negative returns is actually smaller than the volatility of positive returns. Moreover, it is also interesting to highlight that for both GMV and TAN the 50^{th} liquidity percentile minimizes the downside volatility. Similarly, also the VaR and the Expected Shortfall are minimized at the 50^{th} percentile. The Maximum Drawdown is actually quite similar across the portfolios, with the exception of the 75^{th}

percentile, where there is a significant increment in the potential maximum loss. Overall, all the risk measures seem to follow the same pattern, and be minimized at the 50th percentile, regardless the object of the optimization.

The same performance analysis has been computed using net returns for the performance measures. Again, Treynor has been excluded due to the beta still being negative as it is calculated as the correlation between portfolios and benchmark gross returns.

Strategy	CI	Sh	So	Tr	VaR	ES	Cal	Ste	FT
"GMV-MKW"	66	0.2278	0.32248	73.90955	0.14685	0.10186	0.08039	0.09769	0.81129
"GMV_p10"	59	0.23061	0.33416	64.9862	0.14816	0.10472	0.08061	0.10064	0.82329
"GMV_p25"	51	0.24439	0.36364	66.2406	0.15823	0.11405	0.08551	0.10987	0.85396
"GMV_p50"	29	0.24497	0.387	105.90628	0.17869	0.11967	0.09112	0.11281	0.87564
"GMV_p75"	31	0.24834	0.37884	208.07764	0.17266	0.1206	0.0863	0.11508	0.86909
"TAN-MKW"	42	0.26678	0.36949	-116.13083	0.16887	0.1263	0.07697	0.12019	0.86143
"TAN_p10"	37	0.26678	0.36949	-116.13484	0.16887	0.1263	0.07697	0.12019	0.86143
"TAN_p25"	29	0.27055	0.37582	-168.75653	0.17185	0.12852	0.07833	0.12231	0.86886
"TAN_p50"	10	0.28328	0.41404	-243.62147	0.19562	0.13972	0.08375	0.13314	0.915
"TAN_p75"	31	0.25094	0.37674	50.99539	0.19066	0.12525	0.07854	0.11753	0.87256

(Figure 31. Table summarizing Net performance measures and the final rank for Constrained portfolios)

In this case risk measures have also been calculated on the net returns, to highlight potential liquidity effects. Again, the top performing portfolio is the TAN at the 50th percentile, followed by the GMV at the same liquidity level and the GMV and TAN at the highest illiquidity level. Thus, high illiquidity seems to reward sufficiently the investors in terms of net performances compared to the risk taken. Markowitz’s classify among the last, suggesting that a keeping a high level of liquidity is not necessarily beneficial, if not adequate rewarded (as mentioned before Markowitz portfolio has a level of liquidity very close to the 10th percentile). Risk measures follow the exact same pattern as in the case of gross returns, with the liquidity at the 50th percentile minimizing the risks. Concluding, it is interesting that the top performing portfolios are those with intermediate level of liquidity. This behavior could be explained by the fact that the search of liquidity might be as penalizing as the search for illiquidity. By focusing only on very liquid securities, profit opportunity from low-liquid, riskier securities are missed, thus resulting in less performing portfolio, even in net terms. It should always be remembered that the performance analysis is conducted with respect to only spread as transaction cost. For more illiquid stocks market impact plays a much bigger role than the quoted mid-price spread. Therefore, this picture might actually change when including the market impact. However, as already repeated, this would require knowing the size to trade, which is not the case of this research work.

Moving to the unconstrained case, we replicated the same analysis. Even with short selling allowed, some portfolios appear to have a negative beta and hence Treynor will be left out from the following analysis. Given the very poor performances of the GMV at each liquidity level, we are expecting it to be outperformed by the TAN. Indeed, the GMV is vastly outperformed by the TAN, with all the performance measures being twice larger or more for the TAN. The top performing portfolios are the TANs with intermediate level of illiquidity (50^{th} and 25^{th}). Thus, in both the constrained and unconstrained case it seems that keeping the liquidity of the portfolio not to high, might indeed be beneficial, resulting in better portfolios for an active management perspective.

Strategy	CI	Sh	So	Tr	VaR	ES	Cal	Ste	FT
"GMV-MKW"	47	0.11592	0.13681	-12.06519	0.07781	0.04658	0.03041	0.04316	0.56068
"GMV_p10"	49	0.11572	0.13685	-11.91452	0.07807	0.04653	0.0303	0.04311	0.5616
"GMV_p25"	54	0.11464	0.13679	-11.3164	0.07733	0.04652	0.02994	0.04315	0.56428
"GMV_p50"	60	0.09534	0.12085	-6.49501	0.07393	0.04033	0.02432	0.03718	0.56823
"GMV_p75"	70	0.07877	0.10012	-7.87467	0.05463	0.0335	0.01855	0.03113	0.54787
"TAN-MKW"	32	0.25107	0.39519	85.4151	0.18641	0.12932	0.08937	0.1215	0.89352
"TAN_p10"	20	0.25504	0.4004	103.78009	0.1827	0.13112	0.0933	0.12374	0.89993
"TAN_p25"	18	0.25489	0.40142	100.06549	0.18244	0.13132	0.09329	0.12402	0.9011
"TAN_p50"	11	0.25527	0.40212	104.87962	0.18443	0.13127	0.0934	0.12376	0.90191
"TAN_p75"	24	0.25502	0.39829	111.71877	0.188	0.13014	0.09196	0.12251	0.89749

(Figure 32. Table summarizing Gross performance measures and the final rank for Unconstrained portfolios)

GMV returns are highly impaired by the illiquidity, as most of the performance measures are halved compared to the gross ones. If we take a closer look at those measures indeed, it can be noticed that the Shape ratio for the TAN is actually not that different compared to the constrained case (roughly 20% less), while the difference is much bigger for the GMV as anticipated. When looking at Sortino, it is interesting to notice that for the GMV the downside volatility is always larger than the upside, while this trend is reverted for the TAN. Indeed, TANs' Sortino are four times larger than the corresponding GMVs, while the Sharpe ratio is less than three times larger. Concerning the other risk measures, they tend to be larger than their constrained counterparty. For example, the VaR, while being slightly larger for the GMV, it is almost twice larger for the TAN unconstrained compared to the constrained case. Similar pattern is shown by both the CVaR and the Drawdown sequence. Overall, differently from the constrained case, there does not seem to be a relationship between risk and liquidity level.

From the net results perspective, conclusions are unchanged. This is somewhat expected, given that the levels of illiquidity for all the TANs was very similar. Apparently, by not setting up an upper bound for the level of illiquidity, all the TAN unconstrained maximize the Sharpe ratio for almost the same level of illiquidity, which is extremely large (well above the unity, which means above the single security with the highest level of illiquidity). Similar conclusions can be made for the GMV, whose research for the minimum volatility, in a constrained liquidity environment, results in a very large degree of illiquidity, regardless the percentile chosen for the lower bound.

Strategy	CI	Sh	So	Tr	VaR	ES	Cal	Ste	FT
"GMV-MKW"	45	0.06031	0.07492	-6.24804	0.03954	0.02398	0.01574	0.02225	0.50309
"GMV_p10"	50	0.05979	0.07442	-6.15195	0.0394	0.02379	0.01558	0.02207	0.50373
"GMV_p25"	55	0.05744	0.07214	-5.77042	0.03786	0.02307	0.01493	0.02143	0.50528
"GMV_p50"	60	0.03403	0.04537	-2.39371	0.02579	0.01428	0.00866	0.01318	0.50876
"GMV_p75"	70	0.01082	0.01472	-1.09914	0.00733	0.00456	0.00254	0.00425	0.48659
"TAN-MKW"	32	0.18341	0.2988	30.30402	0.13564	0.09422	0.0648	0.08856	0.79012
"TAN_p10"	21	0.18576	0.30219	32.93148	0.13269	0.09527	0.06742	0.08992	0.79368
"TAN_p25"	19	0.18552	0.30262	32.35806	0.1324	0.09534	0.06736	0.09006	0.79435
"TAN_p50"	11	0.18598	0.30358	33.09247	0.13399	0.09541	0.06751	0.08996	0.79535
"TAN_p75"	22	0.18671	0.30212	34.35109	0.13725	0.0951	0.06683	0.08954	0.79302

(Figure 33. Table summarizing Net performance measures and the final rank for Unconstrained portfolios)

Even if we compared all the portfolios constrained and unconstrained, the rank would not change, with the constrained version always outperforming the unconstrained counterparty and TAN and GMV at the 50th percentile still being the top performer in gross and net terms. Concluding, from an active perspective, while undoubtedly there are benefits from keeping high level of liquidity in the portfolio, especially in terms of diversification and low transaction costs, it seems that such liquidity is not fully rewarded also in terms of lower risk and higher return. Indeed, by accepting a higher level of portfolio illiquidity, returns increase more than proportionally therefore allowing for better performing portfolios. Moreover, this analysis shows how Markowitz optimization process represents an unfeasible solution from the active management perspective, as it involves extreme positions shifts from month to month, resulting in high transaction costs and worse performances. By simply adding a no-short selling constraint, much more feasible solutions can be achieved, reducing significantly the level of turnover and thus ultimately boosting the overall performances.

Conclusions

Markowitz's model revolutionized the way portfolio allocation was done, re-designing the process from a more quantitative point of view, where risks and rewards were, at the same time, minimized and maximized. However, as many theoretical models, it works mostly in theory. Even if all the assumptions on which it is based are overlooked (all the economics models are based on strict and implausible assumptions), the boundless optimization originally proposed would still be of little relevance for real world applications. Indeed, it has been shown that, first, stocks are not all the same. Markowitz paid no attention to the stocks' characteristics, but as demonstrated, stocks with different level of capitalization, are associated with different risk-reward-liquidity profiles. If such characteristics are not taken into account, there is no certainty that the resulting optimization process will be feasible. Indeed, from a buy-hold portfolio perspective, it is possible to derive portfolios with the same risk-return profile, but different levels of liquidity. Based on pure risk-reward preferences, investors should be indifferent between them. However, as they prefer more to less, they should be more willing to invest in the most liquid, and thus less costly, portfolio. On the other hand, from an active management perspective, results are quite surprising. By focusing on four portfolios with minimum level of illiquidity at each rebalancing date, the top performing portfolios, both in terms of gross and net performances, were those with an intermediate level of illiquidity, thus not to illiquid, but also not to liquid. This could be due to the fact that focusing only on very liquid securities prevents investors from exploiting better opportunities arising from lower liquid but more rewarding stocks. Therefore, a more balanced portfolio, with a mix of liquid and illiquid securities might actually perform better than portfolios with extreme (positive or negative) exposure to the liquidity. This work also shows that the boundless approach is unfeasible. While the unconstrained GMV was vastly outperformed by the relative benchmark, the TAN resulted in a final value 8 times larger than the starting one. However, the turnover was so high than the net performances where even negative for the GMV, while for the TAN they were not so different from the constrained case. On the other hand, the constrained case shows a more feasible behavior, with limited turnover and downside risk, without however renouncing to good performances. This suggest that, introducing a simple constraint such as limiting the weights to be positive can significantly limit the drawbacks of Markovitz's boundless optimization. Furthermore, by introducing a liquidity constraint, it is actually possible to improve the performances of the optimization, thereby capturing some premium coming from the introduction of the liquidity parameter in the model.

Appendix

DESCRIPTIVE TABLE

Small-Cap summary of the main statistics:

	Mean_SC	Median_SC	StDev_SC	Min_SC	Max_SC	MidSpread_SC	Skew_SC	Kurt_SC	Sharpe_SC
TRITAX BIG BOX	0.69	1.10	3.92	-12.52	10.34	0.24	-0.62	3.96	0.18
CINEWORLD GR	1.16	1.87	7.80	-29.44	18.33	0.73	-0.44	3.73	0.15
ALTEN	2.14	0.93	15.37	-53.79	127.78	0.51	2.47	23.08	0.14
AMBU	2.77	1.93	10.29	-35.89	37.20	2.04	0.51	4.64	0.27
ARGE	5.71	3.66	16.97	-18.83	110.08	1.31	4.10	26.35	0.34
ASM INTERNATI	2.31	1.35	16.03	-45.18	105.70	0.26	1.66	12.33	0.14
BALFOUR BEA	0.99	0.54	10.05	-28.82	54.31	0.65	0.64	6.49	0.10
BEAZL	1.08	1.09	6.82	-20.45	27.64	0.98	0.09	4.20	0.16
BUCHER INDUST	1.36	1.63	8.18	-35.27	26.03	0.54	-0.52	5.34	0.17
ELECTROCO	0.54	-0.31	9.40	-21.56	31.29	0.43	0.36	3.29	0.06
GVC HOLDI	1.22	1.26	14.65	-56.35	54.39	3.14	0.04	5.67	0.08
HEXPOL	2.85	2.01	12.78	-38.89	64.43	0.60	0.90	8.09	0.22
IMCD GRO	2.15	2.62	5.52	-10.59	20.85	0.40	0.48	4.10	0.39
INTERPUMP GR	1.28	1.49	7.57	-26.44	25.11	0.65	-0.19	4.17	0.17
ROYAL UNIB	1.79	1.91	12.29	-48.30	88.85	0.91	1.89	22.23	0.15
SIMCO	1.86	1.54	10.97	-37.12	40.11	0.74	-0.01	4.28	0.17
SSP GRO	1.89	2.14	6.59	-18.42	15.15	0.15	-0.58	3.86	0.29
UNITE GR	1.66	1.39	12.58	-52.40	85.00	1.10	1.45	14.88	0.13
VISCOF	0.81	1.14	7.07	-22.21	27.69	0.56	0.17	4.48	0.11
WDP	0.90	0.64	4.71	-19.82	15.07	0.82	-0.40	4.78	0.19
AAK	1.42	1.58	8.41	-22.60	29.57	0.42	0.02	4.07	0.17
ALTRAN TECHNOLO	1.12	0.70	15.70	-68.99	83.89	0.30	0.17	7.94	0.07
AVEVA GR	2.18	1.60	10.62	-36.77	40.78	1.12	0.16	5.32	0.20
BRITV	0.97	1.41	7.64	-22.84	25.66	0.20	-0.16	4.34	0.13
CD PROJECT	3.69	1.57	20.21	-41.49	138.11	0.60	2.44	15.34	0.18
CLOSE BROTHERS G	0.73	0.75	8.70	-25.25	31.22	0.75	0.03	4.46	0.08
COFINIM	0.08	0.35	3.65	-20.73	12.80	0.35	-0.97	7.54	0.02
DECHRA PHARMACEUT	1.61	1.67	8.63	-48.79	29.03	1.34	-0.70	8.50	0.19
HOMESER	1.47	1.48	8.98	-44.22	37.32	1.21	-0.46	6.79	0.16
JUPITER FUND MANAG	0.96	0.77	7.89	-23.91	23.09	0.14	-0.30	3.75	0.12
KINDRED GROUP	2.14	1.57	11.14	-21.75	61.16	0.45	1.04	7.14	0.19
KONECRAN	1.07	0.85	10.15	-28.97	40.11	0.65	0.08	3.81	0.11
ROCKWOOL INTERNATIO	1.31	0.41	9.51	-25.40	39.73	0.60	0.53	5.13	0.14
SOPRA STERIA G	1.31	0.93	13.71	-61.41	84.64	0.61	0.78	10.86	0.10
TGS-NOPEC GEO	2.17	1.51	12.74	-39.02	67.16	0.45	0.45	5.84	0.17
TOMRA SYST	1.17	1.05	11.14	-29.93	49.56	0.52	0.15	4.35	0.11
VALM	2.41	2.67	6.76	-16.54	17.33	0.17	-0.16	2.94	0.36
VAT GRO	2.67	2.54	8.44	-16.36	18.31	0.13	-0.06	2.59	0.32
VICTR	1.36	1.74	8.41	-24.81	29.91	1.04	-0.07	3.43	0.16
WH SMI	1.00	0.72	8.58	-30.24	41.72	0.50	0.56	5.82	0.12
WIENERBER	0.57	0.65	10.36	-35.86	51.68	0.32	0.28	6.40	0.05
BOLSAS Y MERCADOS ESP	0.20	-0.31	7.42	-21.29	22.34	0.37	-0.09	3.46	0.03
CEMBRA MONEY BANK	1.03	1.03	5.07	-8.17	17.91	0.15	0.96	4.74	0.20
DOMETIC GR	1.09	1.82	7.83	-18.10	17.89	0.20	-0.15	3.08	0.14
FLSMIDTH AND	0.95	1.17	10.62	-35.73	30.64	0.52	0.02	3.47	0.09
GALENICA SA	0.66	1.42	5.24	-13.41	9.96	0.12	-0.61	3.41	0.13
GREENE K	0.67	0.96	7.55	-29.67	32.05	0.44	-0.10	4.67	0.09
LOOMIS	1.93	0.93	8.03	-16.06	34.15	0.35	0.88	5.10	0.24
TAKEAWAY	4.37	3.63	11.52	-18.21	28.91	0.59	0.13	2.56	0.38
TECAN '	1.28	1.79	10.01	-28.21	45.73	0.58	0.07	4.82	0.13
UDG HEALTHCARE P	1.09	0.84	7.77	-33.23	43.76	2.59	0.54	7.63	0.14
AAREAL BANK (	1.15	0.72	12.38	-50.86	58.10	0.31	0.32	6.62	0.09
AURUBIS (X	0.88	1.76	7.79	-25.13	19.85	0.46	-0.34	3.19	0.11
BECHTLE (X	1.29	1.15	10.11	-29.09	40.16	0.59	0.05	4.33	0.13
EVOTEC (X	1.81	-0.93	20.79	-44.20	171.23	0.65	3.02	23.17	0.09
GERRESHEIMER (	0.70	0.85	7.81	-28.04	32.34	0.26	0.33	5.99	0.09
MORPHOSYS (X	3.05	0.03	27.42	-43.90	288.46	0.55	6.48	59.88	0.11
NEMETSCHEK (	2.29	1.64	17.64	-51.71	103.33	1.46	1.66	12.42	0.13
SILTRONIC (X	3.11	-0.51	15.20	-25.63	39.36	0.43	0.13	2.33	0.20
TAG IMMOBILIEN (	0.39	0.76	9.98	-46.68	40.78	1.09	0.36	7.14	0.04

Mid-Cap summary:

	Mean_MC	Median_MC	StDev_MC	Min_MC	Max_MC	MidSpread_MC	Skew_MC	Kurt_MC	Sharpe_MC
WORLDLI	2.38	2.29	7.29	-15.68	18.26	0.38	0.16	2.79	0.33
AIR FRANCE-	0.65	-0.28	12.52	-46.64	62.33	0.19	0.43	5.63	0.05
BANCO B	-0.51	-0.99	11.99	-35.03	45.38	0.41	0.05	4.08	-0.04
BANCO COMR.PORTUGUE	-1.15	-0.89	10.74	-40.51	32.07	0.41	0.08	4.36	-0.11
FINECOBANK	2.27	2.45	8.17	-19.70	18.15	0.15	-0.22	3.04	0.28
NMC HEAL	3.52	3.25	9.63	-17.97	23.85	0.46	-0.13	2.49	0.37
UNIONE DI BANCHE IT	-0.17	0.22	10.73	-30.61	36.43	0.34	0.01	3.51	-0.02
VERBU	0.82	0.85	7.98	-19.89	22.14	0.35	-0.05	3.11	0.10
ASR NEDERL	2.30	2.98	6.34	-9.24	14.52	0.25	0.02	2.14	0.36
HALMA	1.25	0.70	6.68	-23.35	24.86	0.86	-0.03	3.71	0.19
HELVETIA HOLDI	0.77	0.86	7.62	-34.45	32.04	0.48	-0.40	7.06	0.10
HISCOX	1.18	0.72	8.30	-42.37	30.84	0.83	-0.38	7.63	0.14
ICA GRUP	1.11	0.67	8.28	-19.87	38.09	0.36	0.59	5.00	0.13
KINGSPAN GR	1.70	2.27	10.47	-29.69	42.55	0.84	0.12	4.24	0.16
ORPEA	1.51	1.34	6.50	-25.00	19.55	0.34	-0.27	4.03	0.23
SPIRAX-SARCO E	1.29	1.85	6.76	-28.62	20.06	0.69	-0.55	4.61	0.19
SWEDISH ORPHAN BIOV	1.35	-0.61	11.66	-28.34	45.68	0.68	0.86	5.10	0.12
A2A	0.29	0.55	9.33	-21.57	47.41	0.45	0.98	7.60	0.03
AALBER	1.20	1.56	9.31	-23.96	43.60	0.33	0.38	4.60	0.13
AMS	2.28	2.91	14.40	-40.67	56.72	0.76	0.03	4.33	0.16
CELLNEX TELE	1.58	1.87	6.69	-11.63	22.21	0.10	0.62	3.64	0.24
COBH	0.57	0.41	7.99	-28.07	30.96	0.46	0.13	4.22	0.07
ELEKTA	2.13	1.81	10.30	-28.87	46.79	0.36	0.42	4.74	0.21
EURAZ	0.81	1.23	8.71	-30.00	54.49	0.39	0.48	8.82	0.09
GALAPAG	2.37	0.56	12.89	-32.31	70.23	0.81	1.15	6.88	0.18
GLANB	1.20	1.25	9.24	-38.67	40.68	1.33	-0.06	5.39	0.13
GN STORE N	1.60	1.44	12.13	-43.95	37.92	0.38	-0.35	4.44	0.13
JUST E	2.54	3.01	11.27	-26.92	29.93	0.13	-0.03	3.14	0.22
JYSKE BA	0.97	0.91	8.11	-33.72	26.66	0.39	-0.12	5.06	0.12
LOGITECH	2.10	0.77	12.50	-31.52	66.94	0.37	0.77	6.04	0.17
MAN GRO	0.85	0.94	9.95	-39.38	26.69	0.29	-0.46	4.67	0.09
METSO	1.10	1.16	10.46	-39.31	31.69	0.35	-0.28	3.93	0.10
OCADO GR	3.38	0.18	17.47	-37.37	66.78	0.17	0.86	5.08	0.19
PHOENIX GROUP	0.51	0.90	6.71	-21.21	23.65	0.51	0.14	4.30	0.08
REMY COINTR	1.17	0.95	8.50	-34.09	46.14	0.34	0.41	8.59	0.14
SALM	1.97	1.92	9.65	-23.18	33.75	1.78	-0.04	3.34	0.20
SECURITA	0.61	0.60	8.02	-26.01	29.90	0.25	0.04	4.43	0.08
SMITH (D	1.15	1.28	10.27	-37.89	61.05	0.92	0.42	7.93	0.11
TATE & L	0.59	0.69	7.96	-31.01	29.09	0.46	-0.24	5.03	0.07
WOOD GROUP (J	0.66	0.84	9.31	-28.74	25.89	0.41	-0.13	3.38	0.07
AUTO TRADER G	1.62	1.78	7.94	-17.24	19.82	0.09	-0.14	3.16	0.20
BBA AVIAT	0.32	0.49	8.85	-29.60	33.08	0.44	-0.23	4.21	0.04
BELLW	1.36	0.79	9.45	-36.08	33.86	0.50	0.04	4.55	0.14
BOSKALIS WESTMIN	1.25	0.81	9.32	-35.83	32.74	0.32	0.08	4.46	0.13
CASTELL	1.08	1.13	6.77	-17.91	27.08	0.35	0.23	3.92	0.16
CYBG	0.41	-0.02	9.76	-23.72	27.93	0.10	-0.20	4.37	0.04
DERWENT LON	0.98	1.47	7.24	-30.25	30.61	0.52	-0.21	5.54	0.14
DIASOR	1.65	1.00	6.96	-15.22	19.39	0.50	0.21	2.87	0.24
ELIS	0.46	0.49	6.89	-15.90	17.44	0.26	0.18	3.48	0.07
EURONE	2.38	1.00	7.57	-20.79	17.77	0.29	0.01	3.77	0.31
FABE	1.45	1.64	8.09	-26.92	33.47	0.37	0.03	5.45	0.18
GEORG FISC	1.03	0.91	9.84	-30.80	45.09	0.44	0.15	5.04	0.10
HERA	0.72	0.61	6.67	-16.09	23.49	0.49	-0.04	3.40	0.11
HIKMA PHARMACEUT	1.45	1.13	9.92	-20.98	41.62	0.26	0.44	4.08	0.15
HUHTAMA	0.88	0.83	7.79	-20.82	43.16	0.48	0.88	7.32	0.11
INGENICO GR	1.81	1.12	15.98	-38.60	160.74	0.36	4.24	42.82	0.11
INMOBILIARIA COL	0.04	0.00	13.41	-44.74	59.29	1.41	0.36	6.52	0.00
INTERMEDIATE CAPITA	1.23	1.65	10.34	-34.59	67.82	0.46	0.79	11.86	0.12
INVEST	0.72	0.50	8.87	-23.17	37.52	0.40	0.24	4.15	0.08
KESKO	0.84	0.92	7.95	-25.71	28.25	0.36	-0.13	3.90	0.11
LAGARDERE GR	0.45	0.31	8.75	-32.95	47.41	0.16	0.47	6.58	0.05
MEGGI	1.06	0.67	8.91	-36.03	45.58	0.49	0.15	6.14	0.12
MERLIN PROPERTIES	0.90	1.38	4.82	-9.63	13.00	0.23	-0.04	2.72	0.19
NIBE INDUSTRI	2.04	2.32	8.31	-33.65	33.04	0.57	0.04	4.80	0.25
NOKIAN RENK	1.53	1.14	10.43	-39.17	36.65	0.56	-0.21	4.59	0.15
PENNON GR	0.39	0.19	6.31	-33.78	25.16	0.45	-0.50	6.82	0.06
QUILT	0.76	0.75	7.95	-13.38	12.24	0.12	-0.40	2.24	0.10
REXEL	0.33	-0.29	10.36	-24.70	44.40	0.26	0.47	4.51	0.03
RIGHTMO	2.01	2.20	9.48	-24.27	34.80	0.31	0.14	4.74	0.21
ROTO	1.06	0.86	7.49	-22.08	21.56	0.81	-0.09	3.45	0.14
RUBIS	1.10	1.13	5.36	-13.66	15.20	0.45	0.08	3.02	0.20

SAAB	0.93	0.77	9.04	-34.87	46.25	0.41	0.04	6.82	0.10
SBM OFFSH	0.72	0.39	9.41	-38.44	27.72	0.25	-0.15	3.93	0.08
SCHIBSTE	1.50	0.72	12.18	-37.81	80.25	0.48	1.65	12.81	0.12
SIGNI	0.79	1.84	7.49	-17.25	15.85	0.24	-0.26	2.66	0.11
SOFI	0.70	0.72	5.18	-20.86	12.93	0.59	-0.64	4.66	0.13
SPECTR	1.35	1.34	9.49	-24.75	51.58	0.68	0.49	5.70	0.14
STOREBRA	0.84	0.70	11.43	-55.21	34.78	0.41	-0.64	5.57	0.07
SUBSEA	1.44	2.35	14.62	-54.03	62.22	0.94	-0.07	5.23	0.10
TRAVIS PERK	1.11	0.63	10.95	-44.57	65.10	0.41	1.06	10.66	0.10
WILLIAM H	0.21	-0.22	8.76	-23.82	33.55	0.22	0.09	4.08	0.02
ACKERMANS & VAN H	0.80	0.53	6.51	-29.30	24.21	0.53	-0.07	5.63	0.12
B&M EUROPEAN VAL	0.89	0.48	8.71	-24.51	18.50	0.17	-0.60	3.46	0.10
BB BIOTEC	1.32	0.99	8.66	-21.82	36.04	0.46	0.62	4.91	0.15
BTG	1.53	0.38	15.72	-53.82	69.86	1.35	0.95	6.53	0.10
DAILY MAIL	0.28	0.34	8.87	-23.38	46.52	0.48	0.53	5.75	0.03
DKSH HOLD	0.59	1.19	6.60	-21.02	14.71	0.14	-0.81	4.31	0.09
FASTIGHETS BALD	0.85	0.89	14.27	-61.90	70.76	3.29	-0.10	7.27	0.06
FLUGHAFEN ZU	0.93	1.06	9.14	-32.95	36.54	0.91	-0.04	5.68	0.10
GETINGE	1.04	0.95	8.58	-24.45	24.51	0.30	0.08	3.40	0.12
GREAT PORTLAND ES	0.75	1.10	6.48	-23.22	30.46	0.54	-0.11	4.86	0.12
HAYS	0.23	0.77	9.72	-42.33	34.57	0.38	-0.30	4.83	0.02
HOWDEN JOINERY	1.88	1.48	13.76	-30.40	96.35	0.73	1.73	12.64	0.14
HUSQVARN	0.57	0.56	8.64	-17.33	26.00	0.20	0.16	2.97	0.07
IG GROUP HOLD	1.23	1.22	9.58	-41.53	30.18	0.25	-0.69	6.92	0.13
IMI	0.87	0.96	8.24	-26.02	36.81	0.50	0.10	4.72	0.11
INCHCA	1.47	1.87	11.58	-57.70	62.68	0.70	-0.19	9.78	0.13
INMARS	0.52	0.37	8.91	-22.56	45.64	0.16	0.98	7.91	0.06
ITALG	1.97	3.34	6.85	-16.88	13.54	0.13	-0.75	3.49	0.29
LPP	3.08	1.54	12.54	-41.21	52.27	1.28	0.50	4.95	0.25
LUNDBERGFORETAG	1.21	1.03	5.89	-17.60	20.70	0.41	-0.01	3.77	0.21
OC OERLIKON CORPOR	0.92	1.39	13.94	-56.64	81.25	0.35	0.48	9.17	0.07
ORION	0.79	1.19	7.75	-31.57	27.59	0.37	-0.05	4.50	0.10
POLYMETAL INTERNAT	0.41	-0.50	10.51	-34.50	39.52	0.12	0.26	4.84	0.04
PSP SWISS PROPER	0.80	0.52	4.49	-18.01	28.16	0.29	0.90	9.82	0.18
RPC GRO	1.12	0.69	8.66	-21.47	35.69	1.08	0.47	4.46	0.13
SUNRISE COMMUNICA	-0.11	-0.07	6.88	-19.02	14.67	0.10	-0.32	3.19	-0.02
TRELLEBOR	1.45	0.25	13.64	-52.34	139.99	0.30	4.10	46.61	0.11
POLYMETAL INTERNAT	0.41	-0.50	10.51	-34.50	39.52	0.12	0.26	4.84	0.04
PSP SWISS PROPER	0.80	0.52	4.49	-18.01	28.16	0.29	0.90	9.82	0.18
RPC GRO	1.12	0.69	8.66	-21.47	35.69	1.08	0.47	4.46	0.13
SUNRISE COMMUNICA	-0.11	-0.07	6.88	-19.02	14.67	0.10	-0.32	3.19	-0.02
TRELLEBOR	1.45	0.25	13.64	-52.34	139.99	0.30	4.10	46.61	0.11
1&1 DRILLI	1.84	1.66	16.74	-57.18	82.81	1.06	0.93	7.02	0.11
AGGRE	1.03	1.56	9.67	-39.69	32.23	0.69	-0.20	4.57	0.11
SPIE	0.22	0.81	7.84	-19.08	14.76	0.30	-0.18	2.54	0.03
CARL ZEISS MEDITEC	1.11	0.72	11.20	-56.15	43.75	0.71	-0.22	7.24	0.10
FREENET (X	1.11	1.02	16.44	-43.22	67.28	0.46	0.87	6.31	0.07
GRENKE N (X	1.79	1.53	11.34	-39.56	75.21	0.83	1.05	10.96	0.16
HELLA GMBH &	1.37	-0.02	8.90	-15.26	23.62	0.25	0.26	2.82	0.15
RHEINMETALL (	1.25	1.24	11.16	-36.53	59.80	0.66	0.52	6.47	0.11
SCOUT24 (X	1.15	0.43	6.03	-10.31	13.48	0.36	0.06	2.51	0.19
AROUNDTOWN (	1.48	1.00	4.59	-9.28	12.06	1.36	0.10	2.58	0.32
FUCHS PETROLUB PF.	1.70	1.63	7.93	-29.84	34.88	0.64	0.06	5.03	0.21
SARTORIUS PREF.	2.51	1.97	10.52	-47.57	41.49	1.13	0.10	5.74	0.24

Large-Cap summary:

	Mean_LC	Median_LC	StDev_LC	Min_LC	Max_LC	MidSpread_LC	Skew_LC	Kurt_LC	Sharpe_LC
NESTLE '	0.71	0.53	4.09	-13.15	13.01	0.10	-0.09	3.35	0.17
NOVARTIS	0.40	0.06	4.58	-13.42	13.05	0.09	0.07	3.13	0.09
HSBC HOLDI	0.27	0.40	6.52	-20.81	26.01	0.09	0.30	4.73	0.04
ROCHE HOLD	0.49	0.38	5.36	-18.17	15.11	0.09	-0.07	3.28	0.09
TOTAL	0.49	0.68	5.36	-14.29	20.11	0.07	0.20	3.54	0.09
LVMH	1.38	1.32	8.21	-34.71	38.02	0.10	0.21	5.75	0.17
ROYAL DUTCH SHE	0.30	0.44	5.96	-17.38	23.12	0.07	0.06	3.91	0.05
BP	0.25	0.14	6.92	-33.52	25.29	0.10	0.01	5.72	0.04
3I GRO	0.85	0.76	9.71	-43.50	35.77	0.28	-0.25	5.29	0.09
L'ORE	0.73	0.76	5.72	-16.76	22.82	0.11	0.05	4.80	0.13
ANHEUSER-BUSCH I	0.89	1.46	7.36	-34.49	28.63	0.17	-0.52	6.01	0.12
SANO	0.50	0.67	5.88	-17.75	23.35	0.10	0.00	3.87	0.08
ASTRAZEN	0.49	0.09	7.01	-21.24	25.37	0.11	0.32	3.68	0.07
UNILEVER DUTCH C	0.47	0.39	5.43	-15.37	19.97	0.09	0.14	4.70	0.09
INDIT	1.13	1.05	6.34	-20.58	22.01	0.26	-0.31	4.39	0.18
GLAXOSMITHKL	-0.05	-0.49	5.28	-18.10	20.22	0.11	0.21	3.91	-0.01
DIAG	0.67	0.94	4.71	-13.46	13.61	0.14	-0.29	3.07	0.14
NOVO NORDISK	1.52	1.08	7.24	-28.73	28.64	0.21	-0.20	5.35	0.21
BRITISH AMERICAN TO	0.86	1.23	6.79	-21.91	34.74	0.19	0.16	5.78	0.13
AIRB	1.36	1.50	10.07	-37.04	32.68	0.11	-0.13	4.15	0.14
BANCO SANTAN	0.33	1.04	8.52	-24.08	40.08	0.17	0.08	4.91	0.04
EQUIN	0.65	0.52	7.27	-20.49	22.37	0.23	0.06	3.29	0.09
ASML HOLD	1.97	1.27	12.22	-40.36	49.92	0.11	0.37	5.47	0.16
BNP PARI	0.50	1.40	8.44	-30.66	29.34	0.08	-0.36	4.43	0.06
CHRISTIAN D	1.60	1.71	8.28	-31.20	37.82	0.26	-0.01	5.59	0.19
UNILEVER (	0.54	0.82	5.49	-17.79	19.07	0.17	-0.11	4.52	0.10
VODAFONE GR	0.04	0.14	7.27	-22.56	25.16	0.13	-0.01	3.81	0.01
RIO TIN	1.27	0.66	9.94	-46.72	28.54	0.16	-0.38	4.99	0.13
ENI	0.29	0.27	5.52	-16.33	20.95	0.40	0.15	3.63	0.05
ENEL	0.08	0.07	5.89	-17.41	15.32	0.50	-0.17	3.39	0.01
HERMES IN	1.69	1.84	8.09	-22.49	30.39	0.37	0.14	3.73	0.21
KERI	0.96	0.21	9.46	-25.90	34.45	0.13	0.16	4.49	0.10
LLOYDS BANKING G	-0.32	-0.45	10.44	-35.88	64.10	0.14	0.65	9.36	-0.03
RECKITT BENCKISER	0.95	0.94	5.96	-22.77	30.97	0.22	-0.09	6.57	0.16
AXA	0.44	0.77	10.10	-39.93	51.76	0.08	0.09	7.76	0.04
UBS GRO	0.18	0.09	8.67	-24.76	49.10	0.10	0.62	7.19	0.02
HEINEK	0.59	0.53	5.53	-17.77	15.80	0.14	-0.15	3.51	0.11
ING GRO	0.42	0.69	11.21	-51.54	70.75	0.09	0.30	11.52	0.04
AIR LIQU	0.64	1.02	4.73	-13.06	13.99	0.11	-0.13	3.37	0.14
ESSILORLUXOT	0.91	0.63	5.31	-18.23	18.16	0.21	-0.03	3.71	0.17
DANO	0.51	1.02	5.14	-20.74	16.62	0.09	-0.37	3.94	0.10
GLENCO	0.15	-0.39	11.93	-38.97	45.69	0.05	0.31	5.32	0.01
PRUDENTI	0.59	1.87	8.65	-37.68	33.63	0.18	-0.41	5.54	0.07
IBERDRO	0.70	0.57	6.43	-21.15	21.81	0.21	-0.07	4.13	0.11
SAFR	1.61	1.62	11.11	-41.16	82.06	0.26	1.44	14.74	0.14
TELEFONI	0.20	0.37	7.56	-24.98	34.87	0.17	0.35	5.33	0.03
VINCI	1.13	1.14	6.56	-19.47	23.40	0.17	0.02	3.44	0.17
BHP GRO	1.50	1.57	9.16	-24.93	43.41	0.20	0.30	4.06	0.16
EDF	-0.12	-0.96	8.75	-19.44	27.18	0.08	0.48	3.52	-0.01
INTESA SANPA	0.22	0.61	9.89	-31.73	27.99	0.51	-0.37	3.72	0.02
ORAN	0.12	-0.86	11.92	-54.60	66.86	0.07	1.01	12.79	0.01
ZURICH INSURANCE	0.15	0.57	8.26	-45.46	31.50	0.11	-0.75	7.86	0.02
BARCLA	0.29	-0.51	11.56	-44.98	97.02	0.13	2.44	24.18	0.03
ABB LTD	1.05	0.72	14.09	-59.72	149.99	0.20	4.33	54.49	0.07
BBV.ARGENTA	0.11	0.19	8.59	-24.48	37.70	0.18	0.39	5.20	0.01
SCHNEIDER ELEC	0.72	0.93	7.34	-34.43	20.49	0.13	-0.41	4.73	0.10
CREDIT AGRIC	0.39	0.28	10.17	-28.95	34.74	0.10	0.05	3.84	0.04
PERNOD-RIC	1.09	1.04	6.08	-17.81	21.51	0.17	0.27	4.07	0.18
RELX	0.64	0.57	6.22	-19.49	27.48	0.25	0.48	5.21	0.10
ROYAL BANK OF SCT	-0.25	-0.11	12.00	-62.14	76.73	0.17	0.11	13.11	-0.02
NATIONAL G	0.33	0.67	5.32	-17.78	18.65	0.19	-0.24	3.68	0.06
PHILIPS ELTN.KONINK	0.85	0.52	8.86	-28.23	27.74	0.10	-0.02	4.13	0.10
RICHEMON	1.37	1.68	8.13	-22.89	26.24	0.19	-0.16	3.80	0.17
UNICRED	-0.42	0.03	10.42	-31.50	50.48	0.50	0.29	5.88	-0.04
BT GRO	-0.10	0.53	8.83	-28.35	24.51	0.16	-0.25	3.71	-0.01
COMPASS GR	0.58	0.92	6.10	-30.16	14.83	0.20	-0.91	5.90	0.10
NOKIA	0.34	-0.20	12.39	-34.92	66.03	0.12	0.53	6.27	0.03
SOCIETE GENE	0.51	0.50	10.35	-32.75	33.13	0.11	-0.07	3.87	0.05
ANGLO AMERI	1.19	0.92	11.63	-35.19	69.34	0.19	1.03	8.84	0.10
CREDIT SUISSE G	0.05	0.27	8.88	-23.99	34.33	0.11	0.22	4.25	0.01

ENGIE	-0.23	-0.51	6.39	-15.90	17.40	0.07	0.19	2.65	-0.04
KBC GRO	0.71	0.97	11.99	-48.86	74.63	0.24	0.86	12.31	0.06
SWEDBANK	0.54	0.49	9.90	-31.31	72.73	0.21	1.34	15.20	0.05
DNB	1.15	0.83	8.92	-34.42	41.99	0.27	-0.02	6.02	0.13
FIAT CHRYSLER AU	0.92	0.98	11.74	-37.26	47.94	0.38	0.34	4.64	0.08
KONINKLIJKE AHOLD DE	0.42	0.37	9.22	-69.85	55.61	0.14	-0.94	21.37	0.05
TESCO	0.31	0.26	6.89	-20.23	23.75	0.17	0.16	3.91	0.04
VIVEN	-0.07	-0.02	8.09	-34.88	32.18	0.07	-0.32	5.15	-0.01
DASSAULT SYST	1.32	1.02	10.21	-52.77	58.66	0.19	0.87	13.05	0.13
IMPERIAL BRA	0.80	0.86	6.44	-20.86	24.72	0.26	0.24	3.81	0.12
STANDARD CHART	0.42	-0.11	9.01	-24.39	30.67	0.20	0.15	3.94	0.05
SWISS	0.31	0.30	8.79	-52.79	47.46	0.11	-0.34	13.01	0.04
ABN AMRO GR	0.35	0.51	6.65	-18.63	14.14	0.16	-0.46	3.48	0.05
AMADEUS IT G	1.82	1.78	5.65	-13.71	13.31	0.55	-0.41	3.00	0.32
REPSOL Y	0.42	0.90	7.01	-29.00	19.12	0.21	-0.56	4.24	0.06
SKANDINAVISKA ENSKILDA B	0.70	1.09	8.89	-39.06	37.90	0.24	-0.10	5.82	0.08
DANSKE B	0.55	1.00	8.76	-36.74	42.29	0.26	0.09	7.64	0.06
ERICSSON	0.74	1.06	14.91	-52.65	121.76	0.23	1.97	20.96	0.05
HENNES & MAURI	0.56	-0.03	8.11	-23.80	39.31	0.15	0.76	6.58	0.07
LAFARGEHOL	0.43	0.92	7.92	-22.57	44.08	0.55	0.53	6.48	0.05
ORST	2.36	2.14	5.23	-10.88	12.80	0.08	-0.24	3.16	0.45
SVENSKA HANDELSBAN	0.57	0.44	6.48	-24.91	27.42	0.20	0.27	4.96	0.09
TELEN	0.98	0.62	8.17	-45.94	25.90	0.24	-0.67	7.89	0.12
ARCELORMIT	1.29	0.08	15.64	-43.00	66.66	1.11	0.51	4.96	0.08
ASSICURAZIONI GEN	0.05	-0.31	7.57	-26.87	29.76	0.40	-0.07	4.75	0.01
WFD UNIBAIL RODAMCO STAPLE	0.77	1.48	5.99	-21.16	17.50	0.24	-0.25	3.36	0.13
CAIXABA	0.16	0.18	9.21	-25.22	26.91	0.34	0.26	3.84	0.02
ENDE	0.61	0.47	7.15	-24.38	24.29	0.38	0.09	4.26	0.08
NATURGY ENE	0.26	0.46	6.89	-24.09	19.86	0.33	-0.09	3.55	0.04
SWISSCOM	0.23	-0.03	5.19	-14.25	19.57	0.13	0.43	4.78	0.04
AKZO NOB	0.57	0.85	7.68	-19.76	24.64	0.13	0.02	3.66	0.07
BAE SYST	0.22	0.24	7.79	-36.71	19.60	0.21	-0.54	5.07	0.03
HEINEKEN HOL	0.66	0.43	5.43	-16.50	18.20	0.37	-0.03	3.68	0.12
KONE '	1.87	2.14	6.65	-14.62	33.91	0.46	0.44	4.66	0.28
LONZA GR	0.98	1.09	7.06	-28.99	15.96	0.16	-0.99	5.55	0.14
RENAU	0.86	0.38	11.59	-46.45	58.67	0.11	0.27	6.27	0.07
ROLLS-ROYCE HOLD	1.02	0.81	9.04	-39.73	32.64	0.25	-0.14	5.21	0.11
THAL	0.68	1.08	6.89	-29.37	22.27	0.18	-0.39	4.44	0.10
VOLVO	1.02	0.84	9.30	-35.87	26.77	0.17	-0.11	3.90	0.11
AENA S	1.54	1.49	5.63	-10.27	13.69	0.08	0.31	2.36	0.27
ASSOCIATED BRIT.	0.75	0.95	6.68	-16.67	20.28	0.35	0.14	3.39	0.11
ERSTE GROUP	1.05	1.49	10.47	-39.81	66.20	0.22	0.48	10.01	0.10
EXPERI	0.91	1.54	5.91	-17.81	13.27	0.12	-0.48	3.06	0.15
GIVAUDAN	1.03	1.07	4.94	-14.52	25.14	0.14	0.13	5.55	0.21
SAMPO '	0.97	1.06	6.66	-19.30	27.97	0.29	0.12	4.38	0.15
ATLAS COPC	1.53	1.61	8.65	-26.68	30.36	0.18	0.08	3.94	0.18
AVIVA	-0.01	0.24	8.89	-28.02	51.12	0.19	0.64	8.35	-0.00
NESTE	1.43	0.12	9.75	-16.80	34.43	0.13	0.60	3.47	0.15
ASSA ABLO	1.14	0.79	8.25	-21.18	34.50	0.26	0.62	4.88	0.14
FORT	0.88	1.11	6.78	-18.53	23.33	0.28	-0.17	3.26	0.13
KERRY GROUP	1.05	0.86	5.69	-20.15	16.53	0.49	-0.13	4.00	0.18
LEGAL & GENE	0.45	0.37	8.35	-34.90	40.44	0.29	-0.06	6.77	0.05
PEUGE	0.77	1.01	10.57	-31.62	51.78	0.12	0.20	4.96	0.07
SAINT GOB	0.49	0.60	8.56	-32.94	38.49	0.10	-0.14	5.88	0.06
SANDV	1.08	0.84	8.46	-31.87	26.35	0.18	-0.03	3.83	0.13
TELIA COMP	-0.14	-0.11	7.30	-22.16	29.14	0.21	0.10	5.13	-0.02
ATLANT	1.01	0.76	7.50	-29.25	45.49	0.41	0.67	9.08	0.14
COLOPLAS	1.40	0.93	6.14	-17.67	23.60	0.41	0.22	3.62	0.23
FERGUS	0.90	1.31	9.62	-51.08	31.98	0.34	-0.66	6.73	0.09
INVESTOR	0.84	1.19	6.69	-24.41	18.16	0.25	-0.51	4.04	0.13
LONDON STOCK EX.	1.80	1.51	9.60	-35.02	49.89	0.37	0.65	7.92	0.19
MICHEL	0.86	0.99	8.37	-22.14	39.62	0.13	0.49	5.04	0.10
NATIX	0.79	0.82	10.83	-39.57	74.01	0.32	1.12	11.97	0.07
SIKA	1.70	2.61	6.90	-23.08	24.33	0.46	-0.33	4.40	0.25
WOLTERS KLU	0.38	0.72	7.00	-35.15	22.38	0.17	-0.77	5.93	0.05
A P MOLLER MAER	0.75	0.71	9.13	-25.85	34.54	0.35	0.36	4.22	0.08
ADP	1.05	1.57	6.16	-20.09	21.11	0.19	-0.26	4.38	0.17
ADYEN	5.59	4.86	16.52	-19.97	36.34	0.45	0.01	2.81	0.34
CAPGEMI	0.62	0.83	11.79	-47.04	53.27	0.12	0.09	6.06	0.05
DSM KONINKLI	1.11	1.34	7.38	-34.91	19.18	0.14	-0.68	5.36	0.15
KUEHNE UND NAGEL INTERNA	1.44	1.34	6.98	-29.30	30.43	0.58	0.29	6.46	0.21
PARTNERS GROUP HO	1.93	2.07	7.74	-27.08	46.34	0.24	0.66	11.15	0.25

SGS '	1.31	0.84	7.76	-28.70	36.17	0.57	0.65	6.68	0.17
INTL.CONS.AIRL	1.17	2.48	8.89	-36.41	21.05	0.07	-1.12	5.98	0.13
LEGRA	0.85	1.39	6.03	-17.83	16.62	0.14	-0.20	3.12	0.14
OMV	1.11	0.70	8.66	-32.98	26.29	0.22	-0.05	3.70	0.13
RYANAIR HOLD	1.53	1.08	9.01	-26.10	28.94	0.55	0.29	3.62	0.17
SMITH & NEP	1.01	0.94	7.22	-22.35	34.33	0.35	0.39	5.11	0.14
SODE	0.59	1.41	7.13	-34.21	26.08	0.17	-0.65	5.64	0.08
UCB	0.42	0.07	7.74	-29.83	26.38	0.25	-0.02	4.60	0.05
VESTAS WINDSYS	2.37	2.33	15.24	-47.02	73.04	0.31	0.36	5.20	0.16
ACCOR	0.49	0.76	8.07	-27.61	22.91	0.12	-0.15	3.52	0.06
BOUYGU	0.68	0.47	9.21	-23.80	38.37	0.12	0.74	5.61	0.07
CARLSBER	0.78	1.48	7.93	-42.10	25.79	0.32	-0.67	6.66	0.10
CARREFO	-0.15	-0.49	7.21	-20.21	19.84	0.08	0.05	2.88	-0.02
CNH INDUSTR	0.27	0.49	6.93	-17.26	13.56	0.10	-0.07	2.53	0.04
CNP ASSURAN	0.75	0.73	7.18	-20.74	26.08	0.32	-0.05	3.58	0.10
EXOR O	2.06	1.94	8.41	-28.89	27.40	0.72	-0.06	4.48	0.25
FERROVI	1.02	0.98	8.45	-25.50	44.67	0.35	0.71	7.44	0.12
GEBERIT	1.43	1.94	6.62	-19.03	21.30	0.28	-0.13	3.74	0.22
HEXAGON	2.00	1.82	10.31	-36.27	60.04	0.45	0.50	7.38	0.19
KPN K	0.04	0.00	11.69	-43.17	75.23	0.16	1.21	13.16	0.00
NN GRO	1.26	1.25	6.04	-17.39	13.16	0.14	-0.30	3.37	0.21
SSE	0.26	0.69	5.04	-18.76	13.87	0.27	-0.34	3.44	0.05
SWISS LIFE HOL	0.76	1.16	11.45	-41.84	65.24	0.16	0.67	10.67	0.07
TENAR	1.56	1.31	10.52	-41.96	36.60	0.34	-0.09	4.87	0.15
UPM-KYMM	0.72	0.70	9.29	-24.80	57.01	0.23	0.81	8.07	0.08
AEGON	-0.28	0.03	11.12	-48.82	44.10	0.12	-0.18	6.09	-0.03
AIB GRO	-0.94	-0.46	19.53	-68.44	150.00	1.62	1.93	18.28	-0.05
CHRISTIAN HANSEN HO	1.95	1.62	5.42	-10.72	14.83	0.12	0.01	2.79	0.36
DSV '	2.05	1.16	9.17	-26.30	56.32	0.51	0.97	8.64	0.22
PUBLICIS GRO	0.90	0.66	8.86	-29.12	37.39	0.20	0.35	5.19	0.10
RAIFFEISEN BANK	0.49	-0.60	12.24	-50.97	41.53	0.23	0.00	4.88	0.04
RED ELECTR	1.22	0.87	8.09	-14.57	87.83	0.43	5.11	56.51	0.15
SNAM	0.60	1.02	4.28	-16.46	10.59	0.31	-0.59	4.02	0.14
SOLV	0.49	0.89	7.15	-22.75	23.66	0.22	-0.15	4.13	0.07
TELECOM ITA	-0.51	-0.31	9.33	-33.01	35.61	0.43	0.06	4.06	-0.05
WPP	0.71	0.85	9.00	-27.29	41.22	0.24	0.38	5.51	0.08
ACS ACTIV.CONSTR.Y	0.96	1.29	7.59	-27.83	32.72	0.42	0.07	5.32	0.13
BANK	-1.90	-1.20	21.30	-78.00	98.85	0.62	0.83	11.08	-0.09
BOLLO	1.32	0.60	7.03	-15.97	28.41	0.64	0.71	4.15	0.19
BUNZL	1.01	1.19	5.60	-20.22	15.99	0.32	-0.17	3.26	0.18
BURBERRY GR	1.37	1.60	8.89	-29.20	49.26	0.24	0.36	7.93	0.15
CENTRI	0.12	0.06	6.34	-19.01	18.65	0.28	0.04	3.44	0.02
CHOCOLADEFABRIKEN LINDT & SP	1.27	1.51	5.53	-21.29	18.54	0.61	-0.14	4.52	0.23
EDP ENERGIAS DE POR	0.17	-0.32	6.24	-22.22	29.48	0.29	0.30	4.96	0.03
GALP ENERGIA	1.03	1.51	8.98	-38.27	33.06	0.18	-0.31	5.58	0.11
GBL N	0.50	0.58	5.53	-21.93	21.86	0.30	-0.10	5.24	0.09
HARGREAVES LANS	1.95	2.78	9.29	-24.97	24.73	0.24	-0.22	2.99	0.21
IPSEN	1.22	1.55	7.62	-27.66	18.26	0.26	-0.59	4.03	0.16
MORRISON(WM)SPM	0.31	-0.04	6.49	-26.77	16.33	0.41	-0.15	4.08	0.05
MOWI	1.76	1.03	21.61	-77.63	123.08	1.88	0.98	9.55	0.08
NOVOZYME	1.28	1.60	6.81	-33.83	17.22	0.25	-0.76	5.65	0.19
PKO BA	0.76	1.11	9.59	-34.14	40.79	0.18	0.01	5.66	0.08
PLKNC.NAFTOWY O	1.10	0.87	9.59	-22.70	27.21	0.28	0.13	2.78	0.11
SAINSBUR	-0.12	0.46	7.30	-24.16	29.32	0.23	-0.11	4.30	-0.02
SIEMENS GAMESA RENEWABLE	1.09	1.81	11.77	-46.92	48.96	0.40	-0.03	4.91	0.09
VEOLIA ENVI	0.23	1.34	9.09	-33.45	35.38	0.13	-0.16	5.48	0.03
AKER	2.12	2.40	14.73	-49.74	80.40	3.71	1.41	10.82	0.14
ALST	0.45	-0.23	14.10	-51.88	87.91	0.26	0.70	10.67	0.03
AMUNDI (	1.11	3.08	8.00	-18.53	17.05	0.23	-0.42	2.61	0.14
ANTOFAGA	1.82	1.97	10.71	-33.86	50.19	0.58	0.29	4.73	0.17
BANCO DE SABA	-0.07	-0.61	8.92	-25.96	48.39	0.36	0.83	6.62	-0.01
BANKINTER	0.72	0.28	9.68	-26.57	40.19	0.33	0.58	5.06	0.07
CRODA INTERNATI	1.46	1.54	7.04	-17.08	26.06	0.65	0.16	3.71	0.21
DASSAULT AVIA	1.09	1.03	7.09	-18.03	30.39	1.26	0.53	5.00	0.15
DAVIDE CAMPARI M	1.33	1.65	5.96	-20.22	21.17	0.42	-0.23	3.76	0.22
EMS-CHEMIE	0.91	0.40	5.92	-28.29	20.21	0.34	-0.10	5.67	0.15
GRIFOLS ORD	1.86	1.47	8.60	-23.53	26.76	0.41	0.06	3.04	0.22
ICTL.HTLS.	1.45	1.81	7.30	-23.16	26.21	0.24	-0.14	4.28	0.20
INTERTEK GR	1.31	1.26	6.60	-15.61	19.43	0.24	-0.03	3.08	0.20
MEDIOBANCA BC	0.39	0.47	9.64	-26.04	37.40	0.39	0.33	4.16	0.04
MELROSE INDUST	1.80	1.94	9.82	-37.73	58.63	0.79	0.96	9.90	0.18
NORSK HY	0.78	0.77	9.06	-35.93	26.18	0.17	-0.17	4.02	0.09

COLRU	0.75	0.64	5.05	-16.00	14.94	0.34	-0.21	4.17	0.15
DCC	1.44	0.97	8.26	-17.92	51.61	1.11	1.74	12.18	0.17
EASYJ	1.05	1.61	11.01	-34.80	30.61	0.36	-0.17	3.36	0.10
EDENR	1.36	0.88	7.44	-22.80	24.43	0.17	-0.02	3.89	0.18
EIFFA	1.38	1.05	9.79	-31.31	43.93	0.33	0.50	5.72	0.14
ILIAD	1.17	1.08	8.76	-18.83	35.59	0.21	0.29	3.90	0.13
ITV	0.39	0.20	11.50	-36.92	74.46	0.37	1.12	10.11	0.03
JOHNSON MATT	1.08	0.93	7.63	-30.68	29.56	0.42	0.23	4.81	0.14
LUNDIN PETRO	3.03	1.54	12.33	-36.35	52.59	0.32	0.82	4.99	0.25
MAPF	0.60	-0.20	8.36	-28.26	31.52	0.44	0.25	4.45	0.07
MARKS & SPENCER	0.11	-0.48	8.34	-22.87	32.08	0.18	0.15	3.84	0.01
PERSIMM	1.64	1.21	10.61	-36.64	36.71	0.40	-0.20	4.47	0.16
PROXIM	0.12	0.07	5.17	-15.71	13.92	0.16	-0.01	2.99	0.02
PZU GRO	0.40	-0.36	7.56	-17.07	23.31	0.21	0.23	2.97	0.05
RECORDATI INDUA.CH	1.78	1.14	8.16	-31.29	41.43	0.58	0.66	6.64	0.22
RSA INSURANCE G	-0.09	0.26	9.76	-29.29	58.26	0.27	0.52	8.96	-0.01
SAGE GRO	1.03	0.92	10.26	-26.67	77.30	0.39	1.87	15.57	0.10
SANTANDER BANK P	1.62	1.90	10.47	-26.61	54.46	0.54	0.63	6.11	0.15
SCHRODE	0.85	0.54	8.63	-23.50	28.95	0.48	0.03	3.35	0.10
SCOR	-0.04	0.94	9.63	-63.65	39.94	0.34	-1.25	13.38	-0.00
SEB	1.17	0.63	8.26	-30.88	39.82	0.40	0.39	5.59	0.14
SEGRO	0.37	0.60	7.43	-33.03	26.58	0.36	-0.80	7.33	0.05
SMITHS GR	0.35	0.42	6.41	-22.65	14.18	0.32	-0.55	3.71	0.05
STANDARD LIFE ABE	0.22	1.25	7.67	-34.06	19.19	0.12	-0.80	5.27	0.03
STORA ENS	0.66	0.55	10.26	-31.37	63.30	0.27	0.81	8.30	0.06
SUEZ	-0.07	0.11	6.77	-18.10	13.57	0.13	-0.34	2.73	-0.01
TELE2	0.68	-0.07	10.36	-33.94	53.64	0.20	0.94	8.11	0.07
TELEPERFORMA	1.59	1.73	10.16	-27.94	49.27	0.39	0.43	5.72	0.16
TRYG	1.01	1.12	5.66	-17.90	17.50	0.19	-0.06	3.44	0.18
UBISOFT ENTERTAINMENT	2.27	0.79	16.29	-47.81	87.28	0.36	1.11	7.69	0.14
VALEO	0.75	0.85	11.01	-36.05	42.80	0.18	0.27	4.79	0.07
ADECCO GR	0.55	0.60	9.28	-30.55	34.92	0.17	0.13	4.40	0.06
ADMIRAL GR	1.32	1.71	7.42	-24.27	21.72	0.18	-0.25	3.80	0.18
BALOISE HOLD	0.51	0.92	7.37	-28.28	22.25	0.21	-0.62	4.83	0.07
BARRATT DEVELOPM	1.53	0.73	14.05	-68.43	64.65	0.42	0.22	8.72	0.11
BERKELEY GROUP	1.53	1.13	8.78	-29.10	33.26	0.51	-0.04	4.40	0.17
RANDST	0.71	0.03	11.64	-30.86	48.31	0.27	0.28	4.14	0.06
SONOVA	1.61	1.67	9.33	-34.52	30.63	0.41	-0.37	5.22	0.17
TERNA RETE ELETTRIC	0.73	1.22	4.49	-20.08	10.14	0.24	-0.76	4.72	0.16
THE SWATCH G	1.01	1.40	8.17	-23.66	33.94	0.18	0.11	3.90	0.12
UMICO	1.34	2.02	7.74	-35.33	23.44	0.27	-0.63	5.05	0.17
YARA INTERNATI	1.65	1.00	10.72	-42.41	28.19	0.17	-0.32	4.38	0.15
AGEAS (EX-FOR	0.15	1.35	11.82	-79.30	66.85	0.21	-0.64	17.08	0.01
ALFA LAV	1.39	1.19	8.17	-23.37	23.21	0.26	-0.16	3.50	0.17
ATOS	0.59	0.55	11.16	-41.42	47.35	0.24	0.15	6.25	0.05
BANK OF IRELAND	0.38	-0.80	20.23	-70.31	169.44	0.72	3.04	27.60	0.02
BARRY CALLEB	1.20	1.16	7.15	-30.84	23.71	0.56	-0.26	5.30	0.17
BRITISH L	0.35	0.82	6.89	-24.51	23.68	0.31	-0.17	3.69	0.05
CARNIV	0.89	1.53	8.79	-38.75	47.11	0.22	0.10	7.34	0.10
COCA-COLA	0.86	1.40	6.43	-16.93	14.30	0.11	-0.42	3.15	0.13
GECINA R	0.76	0.73	8.49	-35.49	44.32	0.34	0.50	9.83	0.09
GENM	1.96	0.79	15.43	-71.17	57.06	0.55	0.15	5.84	0.13
INFOR	0.98	1.13	9.76	-43.69	35.36	0.95	-0.39	6.58	0.10
JULIUS BAER GR	0.74	0.58	7.30	-13.96	19.56	0.07	0.38	2.93	0.10
KLEPIERRE R	0.74	1.27	6.93	-34.55	30.68	0.32	-0.11	7.37	0.11
MICRO FOCUS I	2.16	2.78	11.88	-51.70	31.42	0.61	-0.97	6.67	0.18
MONCL	1.62	1.83	7.86	-17.25	21.17	0.12	0.15	2.75	0.21
NEXT	1.31	1.05	8.75	-23.62	33.39	0.29	0.16	4.37	0.15
PEARS	0.15	0.18	8.16	-24.85	35.97	0.25	0.16	5.94	0.02
PIRELLI	-0.03	0.82	6.08	-10.15	13.42	0.14	0.13	2.73	-0.00
POLISH OIL AND	0.57	-0.70	8.18	-21.71	22.60	0.40	0.28	2.97	0.07
POSTE ITALI	1.12	2.16	6.42	-16.75	11.10	0.13	-0.80	3.28	0.17
RENTOKIL INI	0.29	0.77	9.21	-34.52	53.05	0.30	0.10	8.41	0.03
STRAUMANN HOL	2.27	1.47	9.87	-32.33	42.37	0.45	0.31	5.04	0.23
TEMENOS	2.33	2.63	15.77	-55.96	85.32	0.81	0.85	10.30	0.15
WARTSI	1.35	0.57	9.77	-32.93	58.85	0.37	0.72	8.00	0.14
WHITBRE	0.92	0.63	7.63	-23.70	23.63	0.26	-0.04	3.49	0.12
ARKE	1.30	0.64	10.25	-30.76	47.39	0.15	0.33	5.17	0.13
ASHTEAD GR	2.62	1.80	17.34	-77.98	103.92	1.19	0.97	10.96	0.15
BIOMERIE	1.31	1.20	6.27	-17.85	20.00	0.31	-0.02	3.72	0.21
BUREAU VERI	0.78	0.65	6.11	-22.34	12.78	0.19	-0.49	3.99	0.13
CASINO GUICHA	-0.06	-0.53	7.45	-21.63	32.63	0.17	0.32	4.64	-0.01

SAIP	0.63	0.29	10.91	-39.69	29.44	0.40	-0.51	4.51	0.06
SMURFIT KAPPA G	1.30	1.10	14.97	-53.64	80.23	0.47	1.09	9.81	0.09
ST.JAMES'S PLACE	0.97	1.50	9.93	-29.46	37.36	1.24	0.10	4.42	0.10
SVENSKA CELLULOSA AKTIEBOLAG	1.07	1.30	7.12	-22.16	29.17	0.17	0.31	4.38	0.15
TELENET GROUP HO	1.15	1.39	6.48	-24.75	21.90	0.23	-0.26	4.34	0.18
UNITED UTILITIES	0.13	0.32	5.36	-17.78	16.12	0.27	-0.11	3.70	0.02
ALCON (SWX) ORD	NaN	NaN	NaN	NaN	NaN	0.04	NaN	NaN	NaN
BIC	0.41	0.46	6.43	-23.55	26.01	0.30	-0.13	4.59	0.06
CAPI	0.52	0.58	10.32	-53.87	61.35	0.36	0.32	11.83	0.05
EUTELSAT COMMUNICA	0.43	1.07	6.96	-33.94	17.62	0.18	-0.99	6.25	0.06
ICADE RE	0.78	0.43	7.91	-23.72	42.44	3.17	0.46	6.69	0.10
IMER	0.63	0.68	7.33	-20.99	25.15	0.35	0.14	3.67	0.09
INDUSTRIVARD	0.84	1.09	7.88	-25.52	29.54	0.40	-0.20	4.08	0.11
KGHM	1.69	0.82	13.01	-41.32	38.12	0.18	0.19	3.65	0.13
KINGFISH	0.01	-0.18	8.30	-28.99	28.77	0.22	0.07	4.32	0.00
KINNEVIK	1.27	0.79	11.54	-34.97	70.16	0.52	1.13	9.18	0.11
PANDO	1.29	1.42	14.40	-68.70	39.15	0.13	-0.65	8.01	0.09
SWISS PRIME	0.64	0.49	4.25	-11.93	26.59	0.45	0.96	8.82	0.15
VOESTALP	1.16	1.48	9.81	-41.31	48.73	0.27	0.06	6.49	0.12
WEIR GRO	1.28	1.62	9.75	-42.92	29.45	0.69	-0.55	4.73	0.13
WEND	1.04	1.88	10.28	-34.56	40.93	0.39	-0.48	5.72	0.10
ADIDAS (X	1.27	1.12	8.18	-26.98	21.46	0.17	-0.28	3.73	0.16
ANDRI	1.64	2.09	8.01	-28.67	28.30	0.32	-0.33	4.52	0.20
BAYER (X	0.57	1.03	8.06	-27.98	30.90	0.12	-0.37	4.50	0.07
BMW (XE	0.82	0.58	8.55	-26.29	26.33	0.14	0.01	3.63	0.10
CONVATEC GR	-1.27	-0.67	9.93	-29.99	18.03	0.12	-1.13	5.23	-0.13
G4S	0.59	0.88	8.30	-26.43	31.04	0.46	0.04	4.70	0.07
PARGESA	0.60	0.83	6.34	-26.65	23.57	0.54	-0.26	5.42	0.09
PUMA (XE	2.04	0.74	9.50	-31.44	41.99	0.47	0.59	5.03	0.21
TULLOW O	1.36	0.76	12.16	-33.25	45.00	1.03	0.36	4.08	0.11
BASF (XE	0.88	0.53	7.31	-22.76	25.36	0.11	0.05	3.98	0.12
BEIERSDORF (	0.85	0.71	6.04	-17.28	23.33	0.33	0.29	3.78	0.14
DEUTSCHE BANK (	-0.09	-0.34	10.64	-40.57	45.81	0.09	0.13	4.92	-0.01
INFINEON TECHS.	1.03	0.53	17.72	-52.34	131.75	0.16	2.14	18.22	0.06
SIEMENS (X	0.88	0.97	9.25	-29.34	40.56	0.09	0.17	5.41	0.09
SIEMENS (XET) HEALTH	1.09	2.78	4.46	-5.80	7.75	0.14	-0.26	1.58	0.24
CLARIA	0.36	0.56	10.65	-47.56	47.64	0.21	-0.20	6.02	0.03
COVIV	0.96	0.78	6.79	-29.47	28.13	1.02	-0.08	5.15	0.14
DIRECT LINE IN.G	0.73	0.43	5.37	-15.45	10.81	0.09	-0.36	2.91	0.14
ELECTROLU	0.91	0.09	9.76	-23.08	46.69	0.21	0.77	4.77	0.09
ELISA	0.80	0.49	9.72	-43.15	47.75	0.33	0.39	7.36	0.08
ENAG	0.85	0.77	5.82	-16.38	23.99	0.32	0.17	3.99	0.15
EPIROC	0.80	2.21	10.91	-19.28	14.38	0.18	-0.53	2.31	0.07
EVRAZ	1.79	1.40	16.11	-30.30	59.99	0.14	0.47	3.81	0.11
GJENSIDIGE FORSI	1.02	0.51	6.17	-13.54	23.19	0.15	0.51	4.08	0.17
H LUNDBE	1.22	0.64	9.97	-31.56	35.22	0.36	0.30	4.18	0.12
HAMMERS	0.34	0.80	7.58	-30.76	28.82	0.40	-0.19	5.13	0.05
JCDECA	0.69	0.78	8.90	-35.71	27.23	0.39	-0.12	4.44	0.08
KONINKLIJKE V	0.87	0.67	7.24	-23.79	18.63	0.29	-0.13	3.64	0.12
LAND SECURITIES	0.26	0.15	6.76	-20.57	33.26	0.22	0.07	5.07	0.04
LEONAR	0.42	0.62	10.71	-35.77	44.34	0.44	-0.15	4.73	0.04
MERLIN ENTERTAIN	0.22	0.96	6.36	-18.93	15.00	0.08	-0.54	3.34	0.03
MONDI	1.12	0.81	9.90	-32.77	27.50	0.14	-0.17	4.26	0.11
ORKLA	0.76	0.68	7.57	-34.56	30.19	0.27	-0.34	5.59	0.10
SARTORIUS STEDIM BI	2.53	2.63	10.40	-40.28	45.80	1.06	-0.16	4.95	0.24
SEVERN TR	0.36	0.48	5.98	-20.89	18.77	0.30	-0.42	4.22	0.06
SKANSKA	0.73	0.23	8.26	-25.53	28.52	0.28	0.34	4.06	0.09
SKF B	1.20	0.71	8.32	-20.55	28.83	0.19	0.36	3.42	0.14
SWEDISH MA	1.27	1.50	5.84	-23.41	18.40	0.29	-0.25	4.22	0.22
TAYLOR WIM	1.13	1.71	14.24	-71.52	124.66	0.50	1.88	27.89	0.08
VIFOR PHA	1.61	1.05	7.57	-23.99	31.44	0.54	0.13	4.44	0.21
BANK POLSKA KASA O	0.83	0.59	9.75	-32.18	53.34	0.33	0.65	6.85	0.09
BOLID	1.23	0.89	14.98	-36.43	75.83	0.32	1.02	7.50	0.08
DEMA	1.45	1.26	8.76	-29.16	38.01	0.40	0.15	4.91	0.17
DUFRY '	1.09	1.71	12.42	-51.02	77.37	0.42	0.81	13.14	0.09
EUROFINS SCIENT	2.62	1.19	14.14	-35.10	105.26	0.70	2.02	15.49	0.19
FAUREC	0.75	-0.06	12.31	-47.09	54.40	0.42	0.54	5.34	0.06
GETLI	0.23	1.36	10.80	-53.33	62.66	0.33	0.26	14.33	0.02
ISS	0.38	0.81	5.80	-14.80	11.79	0.11	-0.25	2.83	0.07
JERONIMO MAR	0.70	1.20	8.43	-33.15	20.00	0.34	-0.40	4.02	0.08
PRYSMI	0.39	-0.37	9.03	-34.09	27.22	0.40	-0.43	4.77	0.04
ROYAL MA	-0.87	-0.09	8.33	-24.42	18.04	0.07	-0.19	3.25	-0.10

ADIDAS (X	1.27	1.12	8.18	-26.98	21.46	0.17	-0.28	3.73
ANDRI	1.64	2.09	8.01	-28.67	28.30	0.32	-0.33	4.52
BAYER (X	0.57	1.03	8.06	-27.98	30.90	0.12	-0.37	4.50
BMW (XE	0.82	0.58	8.55	-26.29	26.33	0.14	0.01	3.63
CONVATEC GR	-1.27	-0.67	9.93	-29.99	18.03	0.12	-1.13	5.23
G4S	0.59	0.88	8.30	-26.43	31.04	0.46	0.04	4.70
PARGESA	0.60	0.83	6.34	-26.65	23.57	0.54	-0.26	5.42
PUMA (XE	2.04	0.74	9.50	-31.44	41.99	0.47	0.59	5.03
TULLOW O	1.36	0.76	12.16	-33.25	45.00	1.03	0.36	4.08
BASF (XE	0.88	0.53	7.31	-22.76	25.36	0.11	0.05	3.98
BEIERSDORF (	0.85	0.71	6.04	-17.28	23.33	0.33	0.29	3.78
DEUTSCHE BANK (	-0.09	-0.34	10.64	-40.57	45.81	0.09	0.13	4.92
INFINEON TECHS.	1.03	0.53	17.72	-52.34	131.75	0.16	2.14	18.22
SIEMENS (X	0.88	0.97	9.25	-29.34	40.56	0.09	0.17	5.41
SIEMENS (XET) HEALTH	1.09	2.78	4.46	-5.80	7.75	0.14	-0.26	1.58
THYSSENKRUPP (	0.47	0.98	10.57	-38.53	29.92	0.17	-0.17	3.50
WIRECARD (X	2.31	2.36	16.91	-62.51	133.49	1.76	1.66	19.47
ALLIANZ (X	0.35	0.86	9.55	-39.74	56.54	0.09	0.13	9.58
AXEL SPRINGER (	0.64	0.00	7.67	-29.48	34.56	2.50	0.44	5.74
BOSS (HUGO) (	1.30	1.80	10.41	-41.86	35.05	1.26	-0.16	5.00
BRENNTAG (X	1.07	1.11	6.09	-13.15	14.74	0.18	0.01	2.54
COMMERZBANK (	-0.34	-0.38	12.99	-48.23	44.08	0.15	0.06	5.30
CONTINENTAL (	1.39	1.47	10.75	-51.87	55.97	0.26	-0.36	8.50
COVESTRO (X	1.71	2.29	8.83	-18.24	24.08	0.19	-0.13	2.98
DAIMLER (X	0.31	-0.29	9.55	-31.82	42.27	0.09	0.20	4.69
DELIVERY HERO (	2.27	2.64	10.13	-15.17	27.61	0.31	0.25	3.31
DEUTSCHE BOERSE	1.15	1.02	7.78	-22.27	24.74	0.18	-0.09	3.85
DEUTSCHE LUFTHANSA	0.52	0.45	9.44	-39.40	22.64	0.17	-0.31	3.68
DEUTSCHE POST (	0.49	1.26	8.13	-41.47	30.52	0.15	-0.69	6.72
DEUTSCHE TELEKOM	0.13	0.17	8.84	-33.40	43.23	0.10	0.50	7.46
DEUTSCHE WOHNEN (XET) B	0.95	0.63	13.20	-36.80	108.60	0.35	3.22	30.32
E ON N (X	0.08	0.93	7.17	-24.03	25.01	0.12	-0.29	4.09
EVONIK INDUSTRIES	-0.11	0.24	5.61	-13.32	10.98	0.20	-0.13	2.57
FRAPORT (X	0.73	0.78	8.19	-39.71	28.10	0.34	-0.54	6.95
FRESENIUS (X	1.12	0.83	8.98	-34.37	66.65	0.60	1.18	15.28
FRESENIUS MED.CARE	0.85	1.02	8.06	-24.51	44.98	0.20	0.92	9.76
SKF B	1.20	0.71	8.32	-20.55	28.83	0.19	0.36	3.42
SWEDISH MA	1.27	1.50	5.84	-23.41	18.40	0.29	-0.25	4.22
TAYLOR WIM	1.13	1.71	14.24	-71.52	124.66	0.50	1.88	27.89
VIFOR PHA	1.61	1.05	7.57	-23.99	31.44	0.54	0.13	4.44
BANK POLSKA KASA O	0.83	0.59	9.75	-32.18	53.34	0.33	0.65	6.85
BOLID	1.23	0.89	14.98	-36.43	75.83	0.32	1.02	7.50
DEMA	1.45	1.26	8.76	-29.16	38.01	0.40	0.15	4.91
DUFRY '	1.09	1.71	12.42	-51.02	77.37	0.42	0.81	13.14
EUROFINS SCIENT	2.62	1.19	14.14	-35.10	105.26	0.70	2.02	15.49
FAUREC	0.75	-0.06	12.31	-47.09	54.40	0.42	0.54	5.34
GETLI	0.23	1.36	10.80	-53.33	62.66	0.33	0.26	14.33
ISS	0.38	0.81	5.80	-14.80	11.79	0.11	-0.25	2.83
JERONIMO MAR	0.70	1.20	8.43	-33.15	20.00	0.34	-0.40	4.02
PRYSMI	0.39	-0.37	9.03	-34.09	27.22	0.40	-0.43	4.77
ROYAL MA	-0.87	-0.09	8.33	-24.42	18.04	0.07	-0.19	3.25
SAIP	0.63	0.29	10.91	-39.69	29.44	0.40	-0.51	4.51
SMURFIT KAPPA G	1.30	1.10	14.97	-53.64	80.23	0.47	1.09	9.81
ST.JAMES'S PLACE	0.97	1.50	9.93	-29.46	37.36	1.24	0.10	4.42
SVENSKA CELLULOSA AKTIEBOLAG	1.07	1.30	7.12	-22.16	29.17	0.17	0.31	4.38
TELENET GROUP HO	1.15	1.39	6.48	-24.75	21.90	0.23	-0.26	4.34
UNITED UTILITIES	0.13	0.32	5.36	-17.78	16.12	0.27	-0.11	3.70
ALCON (SWX) ORD	NaN	NaN	NaN	NaN	NaN	0.04	NaN	NaN
BIC	0.41	0.46	6.43	-23.55	26.01	0.30	-0.13	4.59
CAPI	0.52	0.58	10.32	-53.87	61.35	0.36	0.32	11.83
EUTELSAT COMMUNICA	0.43	1.07	6.96	-33.94	17.62	0.18	-0.99	6.25
ICADE RE	0.78	0.43	7.91	-23.72	42.44	3.17	0.46	6.69
IMER	0.63	0.68	7.33	-20.99	25.15	0.35	0.14	3.67
INDUSTRIVARD	0.84	1.09	7.88	-25.52	29.54	0.40	-0.20	4.08
KGHM	1.69	0.82	13.01	-41.32	38.12	0.18	0.19	3.65
KINGFISH	0.01	-0.18	8.30	-28.99	28.77	0.22	0.07	4.32
KINNEVIK	1.27	0.79	11.54	-34.97	70.16	0.52	1.13	9.18
PANDO	1.29	1.42	14.40	-68.70	39.15	0.13	-0.65	8.01
SWISS PRIME	0.64	0.49	4.25	-11.93	26.59	0.45	0.96	8.82
VOESTALP	1.16	1.48	9.81	-41.31	48.73	0.27	0.06	6.49
WEIR GRO	1.28	1.62	9.75	-42.92	29.45	0.69	-0.55	4.73
WEND	1.04	1.88	10.28	-34.56	40.93	0.39	-0.48	5.72

GEA GROUP (X	0.80	0.04	10.62	-36.47	64.19	0.34	0.63	8.95
HANNOVER RUECK (	0.90	1.41	7.65	-30.82	37.49	0.31	0.12	7.07
HEIDELBERGCEMENT	0.69	0.71	10.17	-35.85	38.86	0.36	0.07	4.41
HOCHTIEF (X	1.28	0.73	11.83	-41.98	38.09	0.49	-0.08	4.15
INNOGY (X	0.62	1.38	5.99	-15.86	17.25	0.17	-0.04	4.69
K + S (X	1.26	1.40	9.74	-41.26	30.40	0.28	-0.52	5.72
KION GROUP (	1.67	0.24	8.13	-15.14	31.02	0.25	0.60	4.06
KNORR BREMSE (	3.32	2.73	5.49	-4.69	9.78	0.34	-0.10	1.88
LANXESS (X	1.25	0.57	10.06	-38.33	28.46	0.17	-0.33	4.72
LEG IMMOBILIEN (	1.28	0.99	5.36	-10.27	12.49	0.19	-0.02	2.26
MERCK KGAA (	0.94	0.60	7.55	-27.38	24.63	0.31	0.01	3.82
METRO (X	-0.22	0.00	8.91	-16.50	27.41	0.21	1.17	5.79
MTU AERO ENGINES (XET)	1.63	1.53	8.48	-25.41	44.56	0.20	0.50	6.78
MUENCHENER RUCK.	0.43	0.43	8.73	-42.54	70.48	0.12	1.35	22.46
OSRAM LICHT (	0.61	1.09	10.38	-30.67	24.79	0.19	-0.50	3.84
PORSCHE AML.HLDG. (XET)	1.14	0.97	11.08	-38.67	57.35	0.47	0.61	7.64
PROSIEBENSAT 1 (XET)	1.57	1.08	17.87	-57.08	105.91	0.41	1.83	12.33
QIAGEN (X	1.35	0.44	11.94	-41.39	72.33	0.30	1.11	9.34
RWE (XE	0.08	-0.53	8.60	-29.31	24.68	0.14	-0.17	3.86
SAP (XE	1.11	0.91	10.85	-40.98	67.89	0.15	1.12	11.76
SYMRISE (X	1.27	2.02	7.21	-26.85	22.22	0.19	-0.61	4.67
TELEFONICA DTL. (XET)	-0.42	-0.83	6.12	-12.67	12.21	0.13	0.21	2.18
UNIPER SE (X	3.22	3.63	7.38	-11.13	20.27	0.23	0.43	2.84
UNITED INTERNET	2.35	2.06	16.66	-40.07	131.29	0.48	2.45	19.76
VONOVIA (X	1.52	1.05	5.30	-8.92	12.25	0.15	0.16	2.22
ZALANDO (X	1.88	1.37	11.94	-26.06	43.08	0.22	0.84	5.42
NORDEA BANK (	0.54	0.63	8.30	-25.47	52.01	0.23	0.82	8.81
CRH (DU	0.68	0.79	7.82	-23.46	22.50	0.32	-0.12	3.40
LINDE (X	1.38	2.27	5.00	-7.19	7.83	0.08	-0.44	2.10
TUI (LO	-0.19	1.39	9.06	-29.35	16.35	0.10	-0.95	4.45
FERRARI (M	3.40	4.48	7.84	-12.73	25.14	0.09	0.18	3.42
OLD MUTUAL LIMITED	0.52	0.01	8.91	-34.67	36.29	0.32	0.04	4.82
PADDY POWER BETFAIR	1.87	2.39	8.41	-25.04	26.91	0.80	-0.29	3.84
RTL GROUP (X	0.18	0.37	7.06	-30.98	21.70	1.00	-0.23	5.42
SES FDR (P	0.70	0.77	6.76	-24.58	18.25	0.25	-0.54	4.52
STMICROELECTRONICS	0.93	0.10	12.52	-34.34	54.86	0.34	0.53	4.95
TECHNIPFMC (	-0.86	0.28	7.97	-13.82	15.23	0.13	0.18	2.45
ESSITY	0.61	0.12	5.86	-6.73	12.25	0.08	0.73	2.65
HENKEL PREF. (	0.73	0.98	6.35	-20.50	22.42	0.20	-0.29	3.99
SCHINDLER	1.29	1.03	6.09	-22.07	21.88	0.39	-0.23	4.27
VOLKSWAGEN PREF.	1.18	0.83	10.99	-44.91	35.75	0.31	-0.52	4.72

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