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From quantum fluctuations in the Early Universe to classical structures on cosmological scales

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Contents

Chapter 1

Introduction

Inflation theory is the standard theory in cosmology describing the primordial Universe. It was introduced to explain some shortcomings of the Hot Big Bang model, such as the horizon problem and the flatness problem [1], but in the modern view the most important property of inflation theory is that it could have generated the seeds for the Large Scale Structures (LSS) we observe nowadays, from galaxies to CMB anisotropies [2]. Inflation is an early phase of accelerated expansion taking place before the standard radiation domination epoch, (presumably) at energies scales much higher than the ones we can probe at Earth laboratories. Despite the fact that the nature of the fields driving inflation, their number and their interactions are still not known precisely, there is a wide group of models which are compatible with the experimental observations. During the inflationary epoch, quantum fluctuations of the primordial vacuum generates small deviations from perfect homogeneity and isotropy, which are stretched by the accelerated expansion into the LSS we observe today. Assuming an initial Bunch-Davies vacuum state for the fluctuations, it was shown that they evolve into a two-squeezed quantum state which, as we will see, discloses both a classical and a quantum interpretation [3].

This prediction of Inflation theory is widely accepted by the cosmologists' community, however, a key question remains: how did these fluctuations transition from a quantum state to the classical realm we see in our Universe? This problem is referred to as "quantum-to-classical" transition and the main mechanism which was proposed to explain the classicalization of the primordial quantum fluctuations in our Universe is the quantum decoherence. This phenomenon is well known in Physics, especially in quantum optics, and it was observed in laboratory [4]. Quantum decoherence relies on the fact that the Early Universe is not a closed system, but can be considered as an open quantum system (OQS) interacting with an environment. During the last fifteen years this OQS approach has been applied to cosmology as a possible solution to explain the quantum-to-classical transition, keeping attention to the fact that some assumptions it relies on are broken by the expansion of the Universe. Within this technique the Early Universe is partitioned into a system and an environment which respectively consist in the degrees of freedom which are observable today and the ones which are not. It should be clear that the choice of what is the system and what is the environment is crucial, because if it changes, it redefines the observables we can access, the entanglement and the quantum information properties we can test. For this reason, choosing a well justified bipartition is fundamental. In literature there are two main choices for this bipartition: a system field interacting with an external field, e.g. [5], and a system made of Super-Hubble scale modes (observable today) self-interacting with an environment made of the shorter wavelengths (unobservable today), e.g. [6, 7]. In this work we will consider a two-field model [8], with the system made of the scalar curvature perturbation (which is directly related to the inflaton quantum fluctuations) and an environment made by a second external field that plays the role of the so called "isocurvature" perturbation. In the OQS setting, the state of the Universe is represented by the density matrix, whose evolution is described by the Liouville equation. This equation is usually too difficult to be solved, then some approximations are needed, giving raise to a plethora of so-called master equations, which will be presented in the following.

The Thesis is organized as follows. In chapter 2 we will provide a short review of the inflation mechanism, starting with the reasons that led to the introduction of this new theory, then focusing on the raising of quantum fluctuations. In chapter 3 we will summarize the outcome of a two-squeezed quantum state for the inflationary perturbations in case of no decoherence (that as we will see has been described both as a highly classical and a highly quantum state), and then we will introduce the OQS formalism, by deriving the master equation that describes the quantum state evolution of the inflationary perturbations. In chapter 4 we will present our two-field model for the Early Universe [8], where we consider a linear interaction between the curvature, also called adiabatic, (system) and the isocurvature, also called entropic, (environment) sectors showing that for such a simple model, we observe an initial period of decoherence, followed by a recoherence, where the quantum information is transferred from the environment to the system. The phenomenon of recoherence was not much studied in literature (it was found for the first time in [8]) and it is peculiar of the model we considered. In chapter 5 we will generalize this model by introducing a non-linear self-interaction in the environmental degrees of freedom and we will see how the decoherence and recoherence of the previous model are modified. The latter generalization constitute the main original part of this Thesis work. Finally we will draw our conclusions, along with some future prospects and present some appendixes making explicit some computations.

Chapter 2

Inflation

2.1 Hot Big Bang model

During the 20s of the last century the astronomer Edwin Hubble observed that the astrophysical objects outside our galaxy were receding from us, independently on the direction one is looking at. This discovery allowed him to formulate the famous Hubble law, which claims that (for sufficiently low velocities) there exists a linear relation between the receding velocity of the galaxies and their distance [9]. Before this observation the Universe was thought to be static (Einstein himself put into play the cosmological constant to guarantee a static Universe), but this simple relation, together with the Copernican principle (i.e. we are not privileged observers in the Universe), led to the discovery of the Universe expansion. In order to explain this expansion two theories were formulated: the steady-state theory and the Hot Big Bang theory. These two theories competed until 1964, when the discovery of the Cosmic Microwave Background (CMB) [10] secured the Big Bang theory as the best theory of the origin and evolution of the Universe (together with the evidences of the Big Bang Nucleosynthesis). In the following years many other predictions of the Big Bang theory were confirmed by the observations, such as the abundance of light elements, the fact that the age of the Universe is comparable to direct age measurements of object within the Universe and that given the irregularities in the CMB, it provides an explanation for the subsequent formation of structure, through gravitational collapse.
However, in spite of all its successes, the Big Bang theory is not a flawless model, but it has some shortcomings which were solved only in the 80s with the introduction of the inflation theory by Starobinskij [11] and Guth [1].

2.1.1 Big Bang cosmology

The Hot Big Bang model is based on the cosmological principle which claims that the Universe is homogeneous and isotropic on large scales and that it can be described by the so-called FLRW (Friedmann-Lemaître-Robertson-Walker) metric [12]:

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\Omega^2\right], \tag{2.1}$$

where t is the cosmic time, $a(t)$ is the scale factor which measures the physical size of the Universe, $\mathrm{d}\Omega$ is the infinitesimal solid angle and k is the curvature that can be either positive (closed Universe), negative (open Universe) or null (flat Universe). We will call "comoving" observer an observer which sees the cosmic fluid at rest. The physical coordinates are related to the comoving ones through the relation:

$$\text{Physical distance} = a(t) \times \text{Comoving distance}. \tag{2.2}$$

If we interpret the Universe as a perfect fluid, the expansion will be characterized by its pressure P and its energy density ρ, which are in general connected through an equation of state:

$$P = w\rho, \tag{2.3}$$

with $w = 0$ for non-relativistic matter, $w = 1/3$ for radiation, $w = -1$ for the cosmological constant. In general relativity (GR) the Einstein equations relate the geometry of the spacetime to the matter distributions within it:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2.4}$$

where $G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, G is the gravitational constant and $T_{\mu\nu}$ is the energy-momentum tensor. This equation is obtained by varying with respect to the metric the following action:

$$S = S_{HE} + S_m = \int \mathrm{d}^4x \sqrt{-g} \frac{R}{16\pi G} + \int \mathrm{d}^4x \sqrt{-g} \mathcal{L}_m, \tag{2.5}$$

where S_{HE} is the Einstein Hilbert action, and S_m describes the matter content of the Universe. In particular from the variation $\delta S_{HE}/\delta g_{\mu\nu}$ one finds the LHS of (2.4) while the energy-momentum tensor on the RHS is defined as:

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g_{\mu\nu}}. \tag{2.6}$$

If we focus on the global properties and treat the Universe as a perfect fluid, we obtain the following expression for the energy-momentum tensor:

$$T_{\mu\nu} = u_\mu u_\nu (\rho + P) + P g_{\mu\nu}, \tag{2.7}$$

where u_μ is the four-velocity of the fluid in its reference frame.
If we replace (2.1) and (2.7) in (2.4) we obtain the Friedmann equations [12]:

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho, \tag{2.8}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P), \tag{2.9}$$

$$\dot{\rho} + 3H(\rho + P) = 0, \tag{2.10}$$

where overdots represents derivative with respect to cosmic time and $H = \dot{a}/a$ is the Hubble parameter. Only two of these three equations are independent. When $k = 0$ Friedmann equations can be easily solved leading to the following solutions:

$$\begin{array}{llll} \text{Matter domination} & P = 0 & \Longrightarrow \quad \rho \propto a^{-3} & a(t) \propto t^{2/3} \\ \text{Radiation domination} & P = 1/3 & \Longrightarrow \quad \rho \propto a^{-4} & a(t) \propto t^{1/2}. \end{array} \tag{2.11}$$

A more interesting solution, which we will recall when we will talk about inflation is the one obtained assuming the cosmological constant domination, i.e. $w = -1$:

$$P = -\rho \quad \Longrightarrow \quad a(t) \propto e^{Ht}. \tag{2.12}$$

Density parameter and horizons

The energy density of a flat Universe ($k = 0$) is called critical density:

$$\rho_c(t) = \frac{3 M_{Pl}^2 H^2}{8\pi}. \tag{2.13}$$

Then the density parameter is:

$$\Omega(t) \equiv \frac{\rho}{\rho_c}. \tag{2.14}$$

The expanding Universe has two characteristic scales: the Hubble length H^{-1} giving the characteristic timescale of evolution of $a(t)$ and the curvature scale $a|k|^{-1/2}$ giving the distance up to which space can be taken as having a flat geometry. From the Friedmann equation we obtain the following relation:

$$\sqrt{|\Omega - 1|} = \frac{H^{-1}}{a|k|^{-1/2}}. \tag{2.15}$$

Since light speed is finite, it has travelled only a finite distance from the beginning of the Universe t^*, so we can define the *cosmological horizon* as the radius of the region of causal connection centered on us:

$$d_H(t) = a(t) \int_0^t \frac{dt'}{a(t')}. \tag{2.16}$$

An object farther than this radius is causally disconnected from us.
Another useful definition is that of *comoving Hubble radius*:

$$r_H(t) = \frac{1}{\dot{a}(t)} = \frac{1}{a(t)H(t)}. \tag{2.17}$$

This is the distance travelled by light in the Hubble time H^{-1}.
Another important quantity is the *redshift*, defined as:

$$1 + z = \frac{a(t_0)}{a(t_{\text{emission}})}. \tag{2.18}$$

2.1.2 Shortcomings of the Big Bang model

In this section we will give a short review of the original motivations that gave rise to the inflationary model. We remind that today these problems are marginal and the main focus is on the ability of the inflation to predict the formation of large scale structure (LSS).

The flatness problem

As we have seen we can write the density parameter as:

$$|\Omega - 1| = \frac{|k|}{a^2 H^2}. \tag{2.19}$$

During the standard Big Bang evolution Ω moves away from 1, since:

$$\begin{aligned} \text{Matter domination}: &\quad |\Omega - 1| \propto t^{2/3}, \\ \text{Radiation domination}: &\quad |\Omega - 1| \propto t. \end{aligned} \tag{2.20}$$

So if Ω is exactly equal to 1 it will remains zero, but if Ω is even slightly different from 1, it gets farther and farther from flatness. Nowadays Ω is measured to be very close to 1, so it must have been even much closer in the past. In particular, assuming radiation domination up to the present days we obtain:

$$\begin{aligned} \text{Nucleosynthesis } (t \sim 1sec): &\quad |\Omega - 1| < \mathcal{O}(10^{-16}) \\ \text{Electro-weak scale } (t \sim 10^{-11} sec): &\quad |\Omega - 1| < \mathcal{O}(10^{-27}). \end{aligned} \tag{2.21}$$

These results give rise to a *fine-tuning problem*: if we want to explain the smallness of Ω today we need to fine tune the initial value of Ω up to the 27^{th} decimal figure, and this appear quite unnatural and difficult to accept. The situation is even more dramatic at the Planck scale where $|\Omega - 1| < \mathcal{O}(10^{-60})$.

The horizon problem

Looking the CMB spectrum we see that photons emitted in opposite directions in the sky appear to be at almost the same temperature. The simplest explanation is that the Universe has indeed reached a state of thermal equilibrium, through interactions between different regions. Unfortunately this explanation cannot be used in the context of the Big Bang model. Indeed there was no time for those regions to interact before the photon decoupling, because of the finite size of the horizon:

$$\int_{t^*}^{t_{dec}} \frac{\mathrm{d}t}{a(t)} \ll \int_{t_{dec}}^{t_0} \frac{\mathrm{d}t}{a(t)}. \tag{2.22}$$

This relation claims that the distance photons could have travelled before decoupling is much smaller than the present horizon distance. In fact, in the Big Bang theory, regions separated by more than 2 degrees should be causally disconnected.

Unwanted relics

Particle physics predict that in the Early Universe, a variety of massive particles, dubbed as "unwanted relics", should have formed (e.g. magnetic monopoles, domain walls etc.). If they were created in the Early Universe they should contribute today to the density parameter, leading to $\Omega \gg 1$ in sharp contrast with the observations.

2.1.3 The inflationary solution

Inflation theory was proposed for the first time by Aleksej Starobinskij [11] and Alan Guth [1] in the early 80s as a possible solution to the shortcomings of the Hot Big Bang model. The inflationary cosmology is not a replacement of the Hot Big Bang model, but rather an add-on that occurs at very early times without affecting any of its successes.
Going back to the flatness problem:

$$|\Omega - 1| = \frac{|k|}{a^2 H^2}. \tag{2.23}$$

The problem arises because aH always decreases, causing Ω to depart from 1. In order to solve the problem we need a mechanism that lead Ω to approach 1. This is possible if we consider an early epoch characterized by an accelerated expansion, which can be expressed using different conditions:

$$\ddot{a} > 0 \quad \Longleftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{H^{-1}}{a} < 0 \quad \Longleftrightarrow \quad P < -\frac{1}{3}\rho. \tag{2.24}$$

The second condition has a clear geometrical interpretation: since H^{-1}/a is the comoving Hubble length, the condition for inflation is that the comoving Hubble length is decreasing with time. In comoving coordinates this means that the observable Universe becomes smaller during inflation because the characteristic scale occupies a smaller and smaller region as inflation proceeds. From the last condition we understand that radiation and non-relativistic matter are not sufficient to obtain inflation, while the cosmological constant would do the job. In figure (2.1) it is shown the Hubble radius as a function of time in case of inflation.
Since we want to recover the Hot Big Bang behaviour and all its successful predictions, we cannot permit inflation to go on forever and it has to stop at some point. Now let us show briefly how the inflation theory solves the previous shortcomings.

Solution to the shortcomings

The flatness problem is easily solved in the inflationary scenario because looking at the density parameter (2.19) we see that if aH increases, Ω is pushed towards one, regardless the initial conditions (in this sense we call inflation an attractor solution). This is the case in the inflation theory where the scale factor increases like (2.12) and the Hubble parameter is almost constant (at least in the case $w = -1$).

The horizon problem is solved if we show that the following condition holds:

$$\int_{t^*}^{t_{dec}} \frac{\mathrm{d}t}{a(t)} \gg \int_{t_{dec}}^{t_0} \frac{\mathrm{d}t}{a(t)}, \tag{2.25}$$

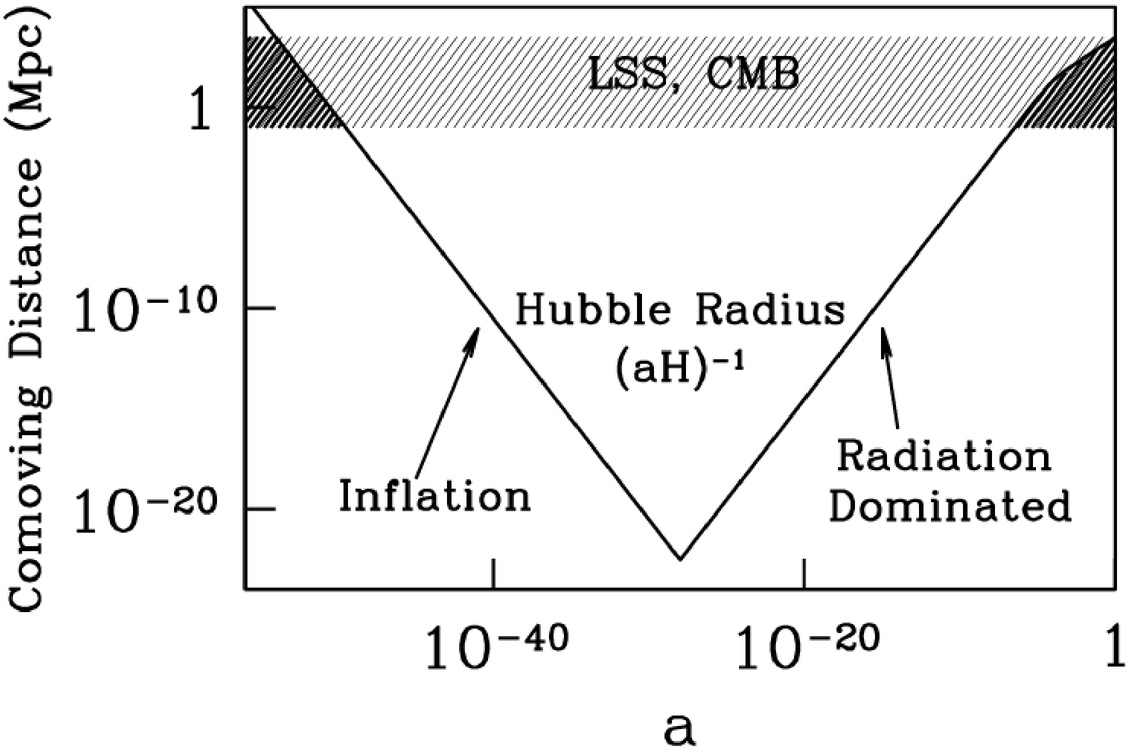


Figure 2.1: Comoving Hubble radius as a function of time. The scales that are inside the horizons are called sub-Hubble scales, the ones which have crossed the horizon are called super-Hubble scales. (The figure was taken by [13]).

so that light can travel much further before decoupling than it can afterwards. To understand the validity of this relation, remember that inflation corresponds to a decreasing of the comoving Hubble wavelength (2.24), which is a good measure of how far light can travel in the Universe. This is telling us that the region of the Universe we can see after inflation is much smaller than the region which would have been visible before inflation started (if inflation lasts enough time). Hence before inflation our observable Universe was well within the causal horizon, allowing to reach the thermal equilibrium we observe in the CMB spectrum.

The unwanted relics problem is solved by inflation because the energy density during inflation falls off more slowly (as a^{-2} or slower) then the relic particle density, so that the latter becomes rapidly negligible. This solution can only work if, after inflation, the energy density of the Universe can be turned into conventional matter without recreating the unwanted relics. This can be achieved by ensuring that during the conversion, known as reheating, the temperature never gets hot enough again to allow their thermal recreation.

Inflation duration

It should be clear that the previous shortcomings are solved by inflation only if inflation lasts enough, a short period of expansion would not be sufficient. In this sense we can introduce a useful quantity called the number of e-folds, defined as:

$$N = \int_{t_i}^{t_f} H(t)\mathrm{d}t = \ln\left(\frac{a_f}{a_i}\right), \tag{2.26}$$

where "i" and "f" refers to the beginning and the end of inflation, respectively. So this number immediately tells us how much the Universe has inflated at the end of inflation with respect to its beginning and the name comes from the fact that $\frac{a_f}{a_i} = e^N$.
Let us see what is the minimum number of e-folds inflation should last to solve the horizon problem. The minimal requirement is that the Hubble radius today is smaller than the Hubble radius at the

beginning of inflation:

$$r_H(t_0) \leq r_H(t_i) \quad \Longrightarrow \quad \frac{1}{a_0 H_0} \leq \frac{1}{a_i H_i}. \tag{2.27}$$

Multiplying both sides for a_f:

$$\frac{a_f}{a_0} H_0^{-1} \leq \frac{a_f}{a_i} H_i^{-1} \quad \Longrightarrow \quad \underbrace{\frac{a_f}{a_i}}_{e^N} \geq \frac{a_f}{a_0} \frac{H_i}{H_0}. \tag{2.28}$$

Using the fact that as the Universe expands the temperature decreases as the inverse of the scale factor, $a(t) \propto T^{-1}$:

$$e^N \geq \frac{T_0}{H_0} \frac{H_i}{T_f} \quad \Longrightarrow \quad N \geq \ln\left(\frac{T_0}{H_0}\right) + \ln\left(\frac{H_i}{T_f}\right). \tag{2.29}$$

We know that $T_0 \approx 2.7K \approx 10^{-13} GeV$ and $H_0 \approx 10^{-42} GeV$, so

$$\ln\left(\frac{T_0}{H_0}\right) \approx 67. \tag{2.30}$$

The ratio H_i/T_f is model dependent, but we can estimate it assuming a quasi-De Sitter phase with H constant:

$$H \approx const. \quad \Longrightarrow \quad H^2(t_i) = H^2(t_f). \tag{2.31}$$

Since at the end of inflation there is a radiation epoch we can impose:

$$H^2(t_f) = \frac{8}{3}\pi G \rho_\gamma|_{T_f} = \frac{8}{3}\pi G \frac{\pi^2}{30} g_* T^4|_{T_f} = \frac{8}{3}\frac{\pi^3}{30} g_* \frac{T_f^4}{M_{Pl}^2}, \tag{2.32}$$

$$\Longrightarrow H(t_i) = H(t_f) \sim \frac{T_f^2}{M_{Pl}}. \tag{2.33}$$

Then, replacing this result in (2.29) we get:

$$N \geq \ln\left(\frac{T_0}{H_0}\right) + \ln\left(\frac{T_f}{M_{Pl}}\right). \tag{2.34}$$

Depending on the inflationary model we have $10^{-5} \leq T_f/M_{Pl} \leq 1$ so that $\ln(T_f/M_{Pl}) = [-11, 0]$, so finally:

$$N \geq 67 + [-11, 0] = [56, 67]. \tag{2.35}$$

Here we can better understand the utility of the number of e-folds: we can say that in order to solve the Horizon problem the Universe had to be expanded by a factor of $e^{60} \sim 10^{26}$. One can show that this minimal requirement is the same necessary to solve the other shortcomings of the Big Bang model.

2.1.4 The inflaton field

As we have seen, in order to obtain inflation we need a particle with the unusual property of a negative pressure. The simplest choice is to consider a scalar field, which is called the inflaton. It is useful to split this field as the sum of a classical background field depending only on time plus a fluctuations field depending also on the spatial coordinates:

$$\phi(\mathbf{x}, t) = \phi_0(t) + \delta\phi(\mathbf{x}, t). \tag{2.36}$$

This splitting makes sense if the fluctuations are much smaller than the background field, so that we can treat them in perturbation theory. For the moment we will focus on the background field (we will neglect the index 0).

The background dynamics

Comparing the energy-momentum tensor of the inflaton with that of a perfect fluid it can be shown that the energy density and the pressure of a homogeneous scalar field $\phi \equiv \phi_0(t)$ satisfy the following equations [2]:

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{2.37}$$

$$P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi), \tag{2.38}$$

where $V(\phi)$ is the inflaton potential and different choices of this potential correspond to different inflationary models. The equation of state in this case becomes:

$$w_\phi = \frac{P_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}. \tag{2.39}$$

Thus if the potential energy dominates over the kinetic term, $V(\phi) \gg \frac{1}{2}\dot{\phi}^2$, we get $w_\phi \sim -1 < -\frac{1}{3}$ as we required to have an accelerated expansion. The condition $V(\phi) \gg \frac{1}{2}\dot{\phi}^2$ is satisfied if the potential is sufficiently flat. This regime is called slow-roll regime and it is the standard scenario to realize inflation.

Assuming a spatially flat Universe, $k = 0$ (reasonably since during inflation the Universe is pushed to be flat) and replacing (2.37) and (2.38) in the Friedmann equations (2.8) and (2.10) we obtain:

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{\partial V}{\partial \phi}, \tag{2.40}$$

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right). \tag{2.41}$$

The standard strategy to solve this equation is the so-called slow-roll approximation (SRA): it consists in the first slow-roll condition we already introduced:

$$1^{st} \text{ slow-roll condition:} \quad \dot{\phi}^2 \ll V(\phi), \tag{2.42}$$

which is realized if the inflaton potential is sufficiently flat. Moreover, given the flatness of the potential, we expect that also the higher order derivatives are negligible. So we can impose:

$$2^{nd} \text{ slow-roll condition:} \quad \ddot{\phi} \ll 3H\dot{\phi}. \tag{2.43}$$

Then our evolution equations in the slow-roll approximations read:

$$\begin{aligned} H^2 &\approx \frac{8\pi G}{3}V(\phi), \\ 3H\dot{\phi} &\approx -\frac{\partial V}{\partial \phi}. \end{aligned} \tag{2.44}$$

In order to quantify the slow-roll regime dynamics we can introduce the so-called *slow-roll parameters*:

$$\varepsilon = -\frac{\dot{H}}{H^2} \qquad \eta = -\frac{\ddot{\phi}}{H\dot{\phi}}. \tag{2.45}$$

We can easily see that a necessary condition for inflation to happen is:

$$\ddot{a} = (aH)^{\cdot} = \dot{a}H + a\dot{H} = aH^2 + a\dot{H} = aH^2\left(1 + \frac{\dot{H}}{H^2}\right) = aH^2(1-\varepsilon) > 0. \tag{2.46}$$

So inflation happens only if:

$$\ddot{a} > 0 \qquad \Longleftrightarrow \qquad \varepsilon < 1. \tag{2.47}$$

Moreover, the slow-roll parameters can be expressed in terms of the inflaton potential as:

$$\varepsilon = \frac{3}{2}\frac{\dot{\phi}^2}{V(\phi)} = \frac{1}{16\pi G}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \qquad \eta = \frac{V''}{3H^2} - \frac{\dot{H}}{H^2}\frac{V'}{3H\dot{\phi}} \approx \eta_V - \varepsilon, \tag{2.48}$$

where we defined $\eta_V = V''/(3H^2)$. This implies that in order to satisfy both the slow-roll conditions we need to require:

$$\varepsilon, \eta \ll 1. \tag{2.49}$$

This condition guarantees that inflation happens and that it lasts enough to solve the shortcomings of the Big Bang model.

2.2 Quantum fluctuations of the inflaton field

Up to this moment we focused on the evolution of ϕ_0, now we want to study the evolution of the quantum fluctuation $\delta\phi(\mathbf{x}, t)$, which is the quantity we are mostly interested in. First of all we need to understand how it is possible to generate the cosmological perturbations on large cosmological scales (such us the anisotropies we observe in the CMB spectrum), starting from the small-scale microscopic quantum fluctuations $\delta\phi$ of the inflaton field.
Let us consider the evolution equations for the background field and for the fluctuations:

$$\begin{aligned} \ddot{\phi}_0 + 3H\dot{\phi}_0 &= -V'(\phi), \\ \ddot{\delta\phi} + 3H\dot{\delta\phi} - \frac{\nabla^2\delta\phi}{a^2} &= -V''(\phi)\delta\phi. \end{aligned} \tag{2.50}$$

If we derive with respect to time the first one we get:

$$\begin{aligned} \left(\dot{\phi}_0\right)^{\cdot\cdot} + 3H\left(\dot{\phi}_0\right)^{\cdot} &= -V''(\phi)\dot{\phi}_0, \\ \ddot{\delta\phi} + 3H\dot{\delta\phi} - \frac{\nabla^2\delta\phi}{a^2} &= -V''(\phi)\delta\phi. \end{aligned} \tag{2.51}$$

We can see that $\dot{\phi}_0$ and $\delta\phi$ have the same equation of motion apart from the laplacian term. As we will show soon, we are most interested on super-horizon scale, characterized by $k \ll aH$, so going to Fourier space one can immediately see that the laplacian term becomes negligible. This approach consists in performing a coarse-graining of the system, since we are not interested in the microscopic behaviour we take an average over small-scale regions. Since $\dot{\phi}_0$ and $\delta\phi$ have the same equation of motion on super-Hubble scale, it turns out that the two solutions are not independent and one obtains [14]:

$$\delta\phi(\mathbf{x}, t) = -\delta t(\mathbf{x})\dot{\phi}_0(t). \tag{2.52}$$

Thus the scalar field is related to its background evolution through:

$$\phi(\mathbf{x}, t) = \phi_0(t - \delta t(\mathbf{x})). \tag{2.53}$$

The last equation tells us that, on large scales, the scalar field goes locally through the same expansion history of the background field, but at a different time due to quantum fluctuations.

2.2.1 Solution to the equation of motion

In this section we want to solve the equation:

$$\ddot{\delta\phi} + 3H\dot{\delta\phi} - \frac{\nabla^2\delta\phi}{a^2} = -\frac{\partial^2 V(\phi)}{\partial\phi\partial\phi}\delta\phi. \tag{2.54}$$

The Fourier transformation of the inflaton fluctuations is:

$$\delta\phi(\mathbf{x}, t) = \frac{1}{(2\pi)^3}\int \mathrm{d}^3\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}\delta\phi(\mathbf{k}, t). \tag{2.55}$$

Notice that we are considering 3-dim Fourier space because our Universe is invariant under translations and rotations, but not under time translation (because it is expanding). Since the evolution equation is linear different modes evolve independently and we can focus on a single mode. The equation of motion for a given Fourier component reads:

$$\delta\ddot{\phi}_k + 3H\delta\dot{\phi}_k - \frac{k^2\delta\phi_k}{a^2} = -\frac{\partial^2 V(\phi)}{\partial\phi_k\partial\phi_k}\delta\phi_k. \tag{2.56}$$

Now we promote $\delta\phi$ to a quantum operator, rescale it and pass to conformal time ($\mathrm{d}\tau = \mathrm{d}t/a$):

$$\delta\phi(\mathbf{x},\tau) = \frac{\delta\hat{\phi}(\mathbf{x},\tau)}{a(\tau)}, \tag{2.57}$$

from now on dots represent derivative with respect to cosmic time, while ' represent derivative with respect to conformal time. We can expand the field operator in terms of the creation and annihilation operators as:

$$\delta\hat{\phi}(\mathbf{x},\tau) = \frac{1}{(2\pi)^3}\int \mathrm{d}^3\mathbf{k}\left[u_k(\tau)\hat{a}_k e^{-i\mathbf{k}\cdot\mathbf{x}} + u_k^*(\tau)\hat{a}_k^\dagger e^{i\mathbf{k}\cdot\mathbf{x}}\right], \tag{2.58}$$

where $\hat{a}_k$ and $\hat{a}_k^\dagger$ are the annihilation and creation operators of the free vacuum since we are not considering any interaction. In order to have canonical commutation relations,

$$\left[\hat{a}_k, \hat{a}_{k'}\right] = \left[\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger\right] = 0 \qquad \left[\hat{a}_k, \hat{a}_{k'}^\dagger\right] = \delta^3(\mathbf{k}-\mathbf{k}'), \tag{2.59}$$

we need to require the following normalization condition for the mode functions:

$$u_k^*(\tau)u_k'(\tau) - u_k(\tau)u_k^{*'}(\tau) = -i. \tag{2.60}$$

In flat spacetime:

$$u_k(t) = \frac{1}{\sqrt{2\omega_k}}e^{-i\omega_k t} \qquad \text{with} \quad \omega_k = \sqrt{k^2+m^2} \qquad \text{Plane wave.} \tag{2.61}$$

In a curved spacetime u_k is not necessarily a plane wave; there is indeed an ambiguity in the definition of the vacuum state. Nevertheless, if we consider a small patch of the spacetime, $\lambda \ll H^{-1}$, we can use the equivalence principle and consider a flat metric, obtaining a plane wave expression:

$$u_k(\tau) \approx \frac{e^{-ik\tau}}{\sqrt{2k}} \qquad \text{for} \quad k \gg aH. \tag{2.62}$$

This choice is called *Bunch-Davies vacuum* choice.
More generally, in a curved spacetime, starting from the equation of motion of the inflaton field, one can show that $u_k(\tau)$ satisfies the following evolution equation:

$$u_k''(\tau) + \left(k^2 - \frac{a''}{a} + V''a^2\right)u_k(\tau) = 0. \tag{2.63}$$

De Sitter solution

If we consider a massless scalar field, $m_\phi^2 = V''(\phi)$=0 in pure De Sitter regime ($a(t) \propto e^{Ht}$, and H =const.) the equation simplifies to:

$$u_k''(\tau) + \left(k^2 - \frac{a''}{a}\right)u_k(\tau) = 0, \tag{2.64}$$

and using the definition of conformal time we get:

$$\mathrm{d}\tau = \frac{\mathrm{d}t}{a} \propto \frac{\mathrm{d}t}{e^{Ht}} \quad \Longrightarrow \quad \tau \propto -\frac{1}{H}e^{-Ht} = -\frac{1}{aH} < 0 \tag{2.65}$$

so conventionally, during inflation, τ is negative ($\tau \in]-\infty, 0[$) and:

$$a(\tau) = -\frac{1}{\tau H} \quad \Longrightarrow \quad \frac{a''}{a} = \left(\frac{1}{\tau^2 H}\right)'(-\tau H) = \frac{2}{\tau^2} = 2a^2H^2 = \frac{2}{r_H^2}. \tag{2.66}$$

Now we can solve equation (2.64) in two different regimes:

1. **Sub-horizon regime:** $\lambda \ll \mathbf{H^{-1}} \quad \longleftrightarrow \quad \mathbf{k \gg aH}$
   In this limit the equation of motion reduces to a harmonic oscillator equation:
$$u_k'' + k^2 u_k = 0 \quad \Longrightarrow \quad u_k(\tau) = \frac{1}{\sqrt{2k}} e^{-ik\tau}, \tag{2.67}$$
   in agreement with the Bunch-Davies argument we have considered previously. In terms of the field:
$$\delta\phi_k = \frac{u_k}{a} = \frac{1}{a}\frac{1}{\sqrt{2k}} e^{-ik\tau}, \tag{2.68}$$
   we notice that it has a decreasing amplitude $|\delta\phi| = 1/(a\sqrt{2k})$, so small scale fluctuations are very suppressed during inflation.

2. **Super-horizon regime:** $\lambda \gg H^{-1} \quad \longleftrightarrow \quad k \ll aH$
   In this limit the equation of motion reduces to:
$$u_k'' - \frac{a''}{a} u_k = 0, \tag{2.69}$$
   whose general solution is:
$$u_k(\tau) = B(k)a(\tau) + A(k)a^{-2}(\tau), \tag{2.70}$$
   where A and B are integration constant on τ which depend on k. In terms of the field:
$$\delta\phi_k = \frac{u_k}{a} = B(k) + A(k)a^{-3}(\tau) \approx B(k), \tag{2.71}$$
   where we neglected the decaying term which gets washed away by inflation. So we see that Super-horizon fluctuations survive during inflation because they are constant.

At horizon crossing we can match the two behaviours to get:
$$|\delta\phi_k| = |B(k)| = \frac{1}{a\sqrt{2k}}|_{k=aH} = \frac{H}{\sqrt{2k^3}}. \tag{2.72}$$
In general if we make an average of the quantum fluctuations on sub-horizon scales, in a macroscopic time interval we get $\langle\delta\phi\rangle = 0$ because positive fluctuations compensate negative fluctuations (in other words particles are continuously created and annihilated). However, the background is exponentially inflating so their physical wavelengths grow exponentially,
$$\lambda \propto a(t) \propto e^{Ht}, \tag{2.73}$$
until they become greater than the horizon H^{-1}. Once crossed the horizon these fluctuations get frozen: they do not vanish if averaged on a macroscopic time interval. In other words we have realized a state with a net number of particles. This is called *gravitational amplification mechanism*, which the gravitational analogue of the creation of couples e^+e^- in a strong electromagnetic field.

Quasi-De Sitter solution

Now we want to solve equation (2.63) in the massless case and in the quasi-De Sitter regime, which differs from De Sitter because H is not constant, but $\varepsilon = -\frac{\dot{H}}{H^2} \ll 1$. In this regime, using an expansion at linear order in ε we obtain the following equation:
$$u_k''(\tau) + \left[k^2 - \frac{\nu^2 - \frac{1}{4}}{\tau^2}\right] u_k(\tau) = 0 \qquad \text{with} \quad \nu^2 \equiv \frac{9}{4} + 3\varepsilon. \tag{2.74}$$
In this form it is equivalent to a Bessel equation, whose solutions are of the form:
$$u_k(\tau) = \sqrt{-\tau}\Big[c_1(k)H_\nu^{(1)}(-k\tau) + c_2(k)H_\nu^{(2)}(-k\tau)\Big] \qquad \text{with} \quad H_\nu^{(2)} = (H_\nu^{(1)})^*. \tag{2.75}$$

where $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are the Henkel functions of the first and the second type. Imposing the previous solution on sub-horizon scale we find that $c_1(k) = \frac{\sqrt{\pi}}{2} e^{i\left(\nu+\frac{1}{2}\right)\frac{\pi}{2}}$ and $c_2(k) = 0$, so on small scales we get:

$$u_k(\tau) = \sqrt{-\tau}\frac{\sqrt{\pi}}{2} e^{i\left(\nu+\frac{1}{2}\right)\frac{\pi}{2}} \frac{e^{-ik\tau}}{-k\tau}. \tag{2.76}$$

On super-horizon scale instead we find:

$$u_k(\tau) = \frac{1}{\sqrt{2k}} e^{i\left(\nu-\frac{1}{2}\right)\frac{\pi}{2}} 2^{\nu-\frac{3}{2}} \frac{\Gamma(\nu)}{\Gamma(3/2)} (-k\tau)^{\frac{1}{2}-\nu}, \tag{2.77}$$

in terms of the field:

$$|\delta\phi_k| = \frac{H}{\sqrt{2k^3}} \left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu}, \tag{2.78}$$

where the De Sitter solution is recovered with $\varepsilon = 0$, implying $\nu = 3/2$.
If we consider a field with a small mass we get the same solution with a different scale dependence:

$$|\delta\phi_k| = \frac{H}{\sqrt{2k^3}} \left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu} \qquad \text{with} \quad \frac{3}{2} - \nu = \eta_V - \varepsilon, \tag{2.79}$$

where $\eta_V = m_\phi^2/(3H^2)$ (we consider only light field because a heavy field would violate the slow-roll condition $\eta_V \ll 1$).

Notice that in our computation there is an inconsistency: we have perturbed the scalar field without perturbing the spacetime metric. Indeed if we consider Einstein equations, assuming the inflaton domination during inflation,

$$G_{\mu\nu} = 8\pi G T^\phi_{\mu\nu}, \tag{2.80}$$

perturbations in the inflaton lead to perturbations in the energy-momentum tensor, which through Einstein equations cause perturbations in the Einstein tensor and so in the metric:

$$\delta T^\phi_{\mu\nu} \longrightarrow \delta G_{\mu\nu} \longrightarrow \delta g_{\mu\nu}. \tag{2.81}$$

In order to take into account both the metric and the inflaton perturbations we can define a gauge invariant perturbation, called the *Mukhanov-Sasaki variable*:

$$v_\phi = \delta\phi + \frac{\dot{\phi}}{H}\hat{\varphi} \qquad \text{where} \quad \hat{\varphi} = \varphi + \frac{1}{6}\boldsymbol{\nabla}^2\chi^{\parallel}, \tag{2.82}$$

where $\hat{\varphi}$ is related to the scalar perturbations of the spatial part of the metric δg_{ij}:

$$\begin{aligned} g_{ij} &= a^2(\tau)[(1-2\varphi)\delta_{ij} + \chi_{ij}] \\ \chi_{ij} &= D_{ij}\chi^{\parallel} + \partial_i\chi_j^{\perp} + \partial_j\chi_i^{\perp} + \chi_{ij}^T \\ D_{ij} &= \partial_i\partial_j - \frac{1}{3}\delta_{ij}\boldsymbol{\nabla}^2. \end{aligned} \tag{2.83}$$

For a good review of cosmological perturbation theory see [15, 14].
One can show that the operator $\hat{v}_\phi = v_\phi/a$ satisfies a Bessel equation,

$$\hat{v}_\phi''(\tau) + \left[k^2 - \frac{\nu^2 - \frac{1}{4}}{\tau^2}\right]\hat{v}_\phi(\tau) = 0 \qquad \text{with} \quad \nu^2 = \frac{9}{4} + 9\varepsilon - 3\eta_V, \tag{2.84}$$

and the amplitude is found to be:

$$|v_\phi| \approx \frac{H}{\sqrt{2k^3}} \left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu} \qquad \text{with} \quad \frac{3}{2} - \nu \approx \eta_V - 3\varepsilon, \tag{2.85}$$

where we have approximated $\nu \approx \frac{3}{2} + 3\varepsilon - \eta_V$.

Another important predictions of the inflation theory is the production of primordial gravitational waves (GWs) [2]. They are produced from the quantum vacuum fluctuations of the metric itself with a similar mechanism to the one we already described, since the two polarization degrees of freedom of the GWs evolve like two minimally coupled, massless scalar fields.

2.2.2 From quantum fluctuations to primordial energy density perturbations

Since we do not observe directly the inflaton fluctuations, we need to connect them to the primordial perturbations we observe today (e.g. CMB temperature anisotropies). In order to do that a very useful quantity can be introduced: the *curvature perturbation on uniform energy density hypersurfaces*, which will be simply called from now on curvature perturbation

$$\zeta(\mathbf{x}, t) = -\hat{\varphi} - H\frac{\delta\rho}{\dot{\rho}}. \tag{2.86}$$

In the spatially flat gauge $\hat{\varphi} = 0$, so:

$$\zeta(\mathbf{x}, t) = -H\frac{\delta\rho}{\dot{\rho}}. \tag{2.87}$$

The great advantage of ζ is that it is constant, provided that the pressure perturbation is negligible (see [2]). This turns out to be the case on scales well outside the horizon regardless of the matter content of the Universe. Moreover, this definition is completely general, so it holds throughout the evolution of the Universe, also when other fields are dominant. This allows us to consider for example a scale k which crosses out the horizon during inflation (at the moment $t_H^{(1)}(k)$) and reenters the horizon during the radiation epoch (at the time $t_H^{(2)}(k)$) and to compare the inflaton fluctuations with radiation energy density perturbations:

$$\zeta\Big|_{t_H^{(1)}(k)} = \zeta\Big|_{t_H^{(2)}(k)} \qquad \Longrightarrow \qquad -H\frac{\delta\phi}{\dot{\phi}}\Big|_{t_H^{(1)}(k)} = -H\frac{\delta\rho_\gamma}{\dot{\rho}_\gamma}\Big|_{t_H^{(2)}(k)}, \tag{2.88}$$

but since in radiation epoch $\dot{\rho}_\gamma = -4H\rho_\gamma$:

$$-H\frac{\delta\phi}{\dot{\phi}}\Big|_{t_H^{(1)}(k)} = \frac{1}{4}\frac{\delta\rho_\gamma}{\rho_\gamma}\Big|_{t_H^{(2)}(k)}, \tag{2.89}$$

and using the fact that $\rho_\gamma \propto T^4$ we get $\delta\rho/\rho = 4\delta T/T$ we finally obtain:

$$\zeta_\phi \sim \frac{\delta T}{T} \sim 10^{-5}, \tag{2.90}$$

so we found a direct link between the inflaton fluctuations and the CMB temperature anisotropies we observe today.
Notice that ζ_ϕ depends on the inflation model we consider, since ρ is proportional to the inflaton potential; so comparing the predictions of different inflationary models with the observed CMB anisotropies, we can rule out some of these models.
Moreover, notice that the crossing time $t_H(k)$ depends on k, because depending on the scale considered it will cross the horizon after or earlier.

2.3 Two-field inflation

In this section we want to briefly introduce two-field inflation and the concept of adiabatic and entropic perturbations [16] which will be useful because in the model we will consider we will have the interaction between an adiabatic and an entropic sector. Two-field inflation was studied in several works, e.g. [17, 18]. An interesting aspect is that these models are the simplest where non-gaussianities appear [19]. In the simplest models of inflation driven by a single scalar field, the quantum fluctuations of the inflaton produce a primordial adiabatic spectrum whose amplitude can be characterized by the comoving curvature perturbation ζ, which remains constant on super-Hubble scales until the perturbation comes back within the Hubble scale long after inflation has ended. As soon as one considers a model with more than one scalar field, one has to take into account also non-adiabatic fluctuations. These new fluctuations can affect the curvature perturbation (which is adiabatic) and can also generate isocurvature (or entropy) perturbations. First of all we need to underline the difference between adiabatic and entropic fluctuations: the former correspond to a local

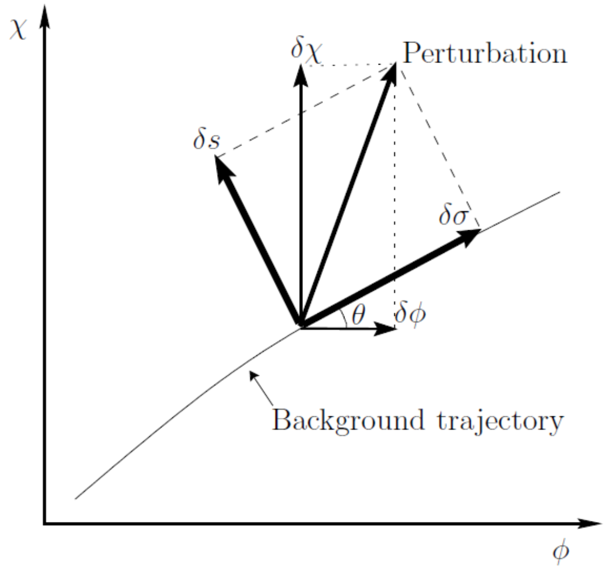


Figure 2.2: An illustration of the decomposition of an arbitrary perturbation into an adiabatic ($\delta\sigma$) and entropy (δs) component. The angle of the tangent to the background trajectory is denoted by θ. The usual perturbation decomposition, along the ϕ and χ axes, is also shown. (The figure was taken by [16]).

perturbation in the total energy density of the Universe, while the latter correspond to perturbations between the energy densities of the different components of the Universe (for this reason they appear only if we consider at least two fields).

Let us consider a model with two interacting scalar fields, ϕ and χ. Given a generic perturbation we can always decompose it in an adiabatic and an entropic fluctuation, by a simple rotation in field space. We can define the adiabatic field as:

$$\dot{\sigma} = (\cos\theta)\dot{\phi} + (\sin\theta)\dot{\chi}, \tag{2.91}$$

where:

$$\cos\theta = \frac{\dot{\phi}}{\sqrt{\dot{\phi}^2 + \dot{\chi}^2}} \qquad \sin\theta = \frac{\dot{\chi}}{\sqrt{\dot{\phi}^2 + \dot{\chi}^2}}. \tag{2.92}$$

The perturbation of the adiabatic field $\delta\sigma$ represents the component of the total perturbation along the trajectory of the background field. Instead fluctuations orthogonal to the background trajectory represent the "non-adiabatic" fluctuations (see figure 2.2):

$$\delta s = (\cos\theta)\delta\chi - (\sin\theta)\delta\phi. \tag{2.93}$$

As it is explained in [16] the entropy and adiabatic perturbations satisfy the following equation of motion:

$$\ddot{\delta s} + 3H\dot{\delta s} + \left(\frac{k^2}{a^2} + V_{ss} + 3\dot{\theta}^2\right)\delta s = \frac{\dot{\theta}}{\dot{\sigma}}\frac{k^2}{2\pi G a^2}\Psi, \tag{2.94}$$

$$\ddot{Q}_\sigma + 3H\dot{Q}_\sigma + \left[\frac{k^2}{a^2} + V_{\sigma\sigma} - \dot{\theta}^2 - \frac{8\pi G}{a^3}\left(\frac{a^3\dot{\sigma}^2}{H}\right)^{\cdot}\right]Q_\sigma = 2\left(\dot{\theta}\delta s\right)^{\cdot} - 2\left(\frac{V_\sigma}{\dot{\sigma}} + \frac{\dot{H}}{H}\right)\dot{\theta}\delta s, \tag{2.95}$$

where V_σ, V_{ss} and $V_{\sigma\sigma}$ are related to the potential term, Ψ is the curvature perturbation in the zero-shear gauge, Q_σ is the Mukhanov-Sasaki variable for the adiabatic perturbation. The interesting aspect we want to focus on is that the evolution equation for the entropy perturbation (2.94) is a homogeneous equation decoupled from the adiabatic field [16], while if the background trajectory is curved, i.e. $\dot{\theta} \neq 0$, the entropy perturbation acts as an additional source term in the equation of motion for the adiabatic field perturbation (2.95). This means that we can have significant change in the large-scale comoving curvature perturbation if there is a non-negligible entropy perturbation, and the background trajectory in field space is curved.

Chapter 3

Open Quantum System approach

In the previous chapter we have seen how we can trace back the origin of the primordial energy density fluctuations we observe today to the quantum fluctuations of the inflaton field generated during inflation. However, inflation theory does not answer an important question: the fluctuations we observe in our Universe are classical, so how did the quantum fluctuations of the inflaton become classical? There is still no clear answer to this question, and this problem was dubbed the "quantum-to-classical transition". In the last years this topic has generated a great debate, but the main question remains: can we prove the quantum origin of the classical fluctuations we observe today? One of the main obstruction to this observation is the phenomenon of quantum decoherence that could have cancelled any quantum signature in the present Universe. This phenomenon is widely studied in the OQS framework (a good introduction of Open Quantum Systems is found in [20]) and our goal in this chapter is to show how to deal with the quantum decoherence during inflation within this framework.

First of all we are going to show that, in the free field case (without interaction, and so without the possibility of decoherence), primordial quantum fluctuations are placed in a very peculiar quantum state, called *two-mode squeezed state* [21]. We will understand why it has been proposed to interpret these states within a classical stochastic theory [22] and why they are actually authentic quantum states. Then we will introduce the concept of quantum decoherence applied to cosmology and the OQS techniques to deal with it, along with the master equations that will be used in the next chapters.

3.1 Two-mode squeezed states

The linear action for the curvature perturbation is given by [23]:

$$S_\zeta^{(2)} = \int \mathrm{d}\eta \mathrm{d}^3\mathbf{x} a^2 \varepsilon M_{Pl}^2 \left[\zeta'^2 - (\partial_i \zeta)^2\right], \tag{3.1}$$

we can introduce the "Mukhanov-Sasaki" variable $v(\eta, \mathbf{x}) \equiv z(\eta)\zeta(\eta, \mathbf{x})$ where $z(\eta) = a(\eta)\sqrt{2\varepsilon} M_{Pl}$. In terms of this variable the previous action reads:

$$S_v^{(2)} = \frac{1}{2} \int \mathrm{d}\eta \mathrm{d}^3\mathbf{x} \left[(v')^2 - (\partial_i v)^2 + \frac{z''}{z} v^2\right]. \tag{3.2}$$

From the action we can derive the Hamiltonian of the system, which reads:

$$H = \frac{1}{2} \int \mathrm{d}^3\mathbf{x} \left[p^2 + (\partial_i v)^2 + 2\frac{z'}{z} vp\right], \tag{3.3}$$

where p is the conjugate momentum associated to the v variable. Now we promote the fields to quantum operators and Fourier transform them as:

$$\begin{aligned} \hat{v} &= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}} \hat{v}_k e^{i\mathbf{k}\cdot\mathbf{x}}, \\ \hat{p} &= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}} \hat{p}_k e^{i\mathbf{k}\cdot\mathbf{x}}, \end{aligned} \tag{3.4}$$

then the two-mode Hamiltonian operator for a given $\mathbf{k}$ is given by:

$$\hat{\mathcal{H}}_k = \hat{p}_{-k}\hat{p}_k + k^2\hat{v}_{-k}\hat{v}_k + \frac{z'}{z}(\hat{p}_{-k}\hat{v}_k + \hat{v}_{-k}\hat{p}_k). \tag{3.5}$$

Decomposing the operators in terms of the mode functions as:

$$\begin{aligned} \hat{v}_k &= \frac{1}{\sqrt{2k}}\left(\hat{a}_k + \hat{a}^\dagger_{-k}\right), \\ \hat{p}_k &= -i\sqrt{\frac{k}{2}}\left(\hat{a}_k - \hat{a}^\dagger_{-k}\right). \end{aligned} \tag{3.6}$$

the two mode Hamiltonian can be rewritten as:

$$\hat{\mathcal{H}}_k = F_k\left(\hat{a}^\dagger_k\hat{a}_k + \hat{a}^\dagger_{-k}\hat{a}_{-k} + 1\right) + iR_k\big(e^{-2i\Theta_k}\hat{a}_k\hat{a}_{-k} - \text{h.c.}\big), \tag{3.7}$$

where we have defined [24]:

$$\begin{aligned} F_k &= \frac{k}{2}, \\ R_k &= \left[\left(\frac{k}{2}\right)^2 + \left(\frac{z'}{z}\right)^2\right]^{1/2}, \\ \Theta_k &= -\frac{\pi}{2} + \frac{1}{2}\arctan\left(\frac{k}{2}\frac{z}{z'}\right), \end{aligned} \tag{3.8}$$

where F_k is the frequency of the harmonic free evolution piece of the Hamiltonian, while R_k is the parametric amplification of the squeezing part of the Hamiltonian which is due to the curvature of the background that yields a time-dependent mass to the field, $m^2_{eff} = -z''/z$. The evolution operator generated by this Hamiltonian can be factorized in three parts:

$$\hat{\mathcal{U}}(\eta, \eta_0) = \hat{\mathcal{R}}(\varphi_k)\hat{\mathcal{Z}}(r_k)\hat{\mathcal{R}}(\theta_k), \tag{3.9}$$

where $\hat{\mathcal{R}}$ is the *phase-shift*:

$$\hat{\mathcal{R}}(\varphi_k) = \exp\left(i\varphi_k\left[\hat{a}^\dagger_k\hat{a}_k + \hat{a}^\dagger_{-k}\hat{a}_{-k} + 1\right]\right), \tag{3.10}$$

and $\hat{\mathcal{Z}}$ is the *squeezer*:

$$\hat{\mathcal{Z}}(r_k) = \exp\left[r_k\left(\hat{a}^\dagger_k\hat{a}^\dagger_{-k} - \hat{a}_k\hat{a}_{-k}\right)\right], \tag{3.11}$$

and $(\varphi_k, r_k, \theta_k)$ are called *squeezing parameters* and characterize the dynamics of the state (their explicit expression can be found in [24]).
Now we want to apply this evolution operator to a primordial vacuum. Let us start by the Bunch-Davies vacuum:

$$|0(\eta_0)\rangle = \prod_{\mathbf{k}\in\mathbb{R}^{3+}} |0_k, 0_{-k}\rangle\,, \tag{3.12}$$

acting with the evolution operator defined above we get the following quantum state [24]:

$$\prod_{\mathbf{k}\in\mathbb{R}^{3+}} |2MSS_k\rangle = \prod_{\mathbf{k}\in\mathbb{R}^{3+}}\left[\frac{e^{i\varphi_k}}{\cosh r_k}\sum_{n=0}^{\infty}(-1)^n e^{2in\theta_k}\tanh^n r_k\,|n_k, n_{-k}\rangle\right], \tag{3.13}$$

where

$$|n_k, n_{-k}\rangle = \frac{1}{n!}(\hat{a}^\dagger_k)^n\frac{1}{n!}(\hat{a}^\dagger_{-k})^n\,|0_k, 0_{-k}\rangle\,. \tag{3.14}$$

This class of quantum states are called *two-mode squeezed states*, due to the entanglement between the mode k and the mode $-k$. The squeezer, $\hat{\mathcal{Z}}(r_k)$, is the responsible for the generation of infinite pairs of pairs of entangled quanta, due to the operator $\hat{a}^\dagger_k\hat{a}^\dagger_{-k}$. It can be shown that during inflation, r_k grows like the number of efolds and that θ_k gets aligned to the value of $\pi/4$ [3]. So the inflationary dynamics generates a two-mode squeezed state with a squeezing parameter of at least $r_k \sim 50 - 60$. This value is way bigger than the typical values of the squeezed quantum states created in laboratory. The discovery of the squeezed states as states describing the primordial fluctuations was first interpreted as a way to justify the "classicalization" of the original quantum fluctuations [22]. To understand better this statement we can consider the quantum state in the phase-space representation where it is described by the so-called Wigner function, defined for a two-mode squeezed state as:

$$W_k[v_k, p_k] = \int \mathrm{d}x e^{-ip_k x} \left\langle v_k + \frac{x}{2} \middle| 2MSS_k \right\rangle \left\langle 2MSS_k \middle| v_k - \frac{x}{2} \right\rangle, \tag{3.15}$$

with v_k the eigenstate of the operator $\hat{v}_k$. In case of a gaussian state, the Wigner function reduces to:

$$W_k[v_k, p_k] = \frac{1}{(2\pi)^2 \sqrt{\det \mathbf{Cov}}} \exp\left[-\frac{1}{2} \begin{pmatrix} v_k & p_k \end{pmatrix} \mathbf{Cov}^{-1} \begin{pmatrix} v_k \\ p_k \end{pmatrix} \right], \tag{3.16}$$

where the covariance matrix is defined in terms of the power spectra as:

$$\mathbf{Cov} = \begin{pmatrix} \langle 2MSS_k | \hat{v}_k \hat{v}_{-k} | 2MSS_k \rangle & \frac{1}{2} \langle 2MSS_k | \{\hat{v}_k, \hat{p}_{-k}\} | 2MSS_k \rangle \\ \frac{1}{2} \langle 2MSS_k | \{\hat{v}_k, \hat{p}_{-k}\} | 2MSS_k \rangle & \langle 2MSS_k | \hat{p}_k \hat{p}_{-k} | 2MSS_k \rangle . \end{pmatrix} \tag{3.17}$$

Inserting expression (3.13) we can compute the Wigner function as a function of the squeezing parameters. Representing this function in the large-squeezing limit we obtain an elongated ellipse. This mechanism has been proposed [25] to interpret the system as a classical stochastic system. Indeed, in this case, the position variable v_k is free to take any value, but given the value of v_k the momentum variable, p_k, is almost fixed. In this sense the system behaves as if it follows an infinite number of classical trajectories with a definite probability on each of them, defining essentially a classical stochastic system. This interpretation must be confronted with the fact that a squeezed state is still a pure and coherent quantum state and this apparent classicalization is only due to the peculiar organization of the uncertainty principle:

$$\Delta \hat{v}_k \Delta \hat{p}_k = \frac{1}{2}, \tag{3.18}$$

where for a squeezed state $\Delta \hat{v}_k \sim e^r$ and $\Delta \hat{p}_k \sim e^{-r}$, so the uncertainty over the momentum gets rapidly suppressed. This is a consequence of the existence of a decaying mode in inflation, recast in the language of squeezed quantum states.

3.2 Dissipation and Decoherence

The FLRW metric, describing the early Universe, is less symmetric than the Minkoswski one because it is not invariant under time reversal. This implies that energy gets diluted with the expansion, a phenomenon resembling dissipation, so we need some techniques to take into account this contribution. At the same time, as we said, it is believed that the early Universe underwent some form of quantum decoherence during the inflationary period, which causes the primordial quantum fluctuations to become classical. OQS techniques allow us to describe both dissipation and decoherence due to the interaction of a system with an environment. In particular the former describes energy exchanges, while the latter describes the generation of entanglement. These effects cannot be described by a unitary evolution, which only could describe the renormalization of the energy levels due to the interaction with the environment, but we need also to consider non-unitary contributions. Moreover, OQS techniques, are widely studied in the context of non-perturbative QFT, due to their ability to resum late time effects. The idea behind non-perturbative resummation relies on deriving perturbatively an object, such as the generator of the dynamical map and then solving the dynamics non-perturbatively, without any further expansion [3].
Here we want to introduce the techniques which are typical of the OQS framework.

3.3 System-environment bipartition

The first thing to do when dealing with an open quantum system is to create a separation between what we will consider as the "system" and what we will consider as the "environment" [3].

This bipartition is fundamental and it will affect the results we will find. For this reason it is very important to choose a reasonable, well justified bipartition. The idea is that the system is made of all the degrees of freedom we have access to, while the environment is made of the unobservable degrees of freedom about which we do not have (and we do not need to have) much information. A typical example of system is provided by the curvature perturbation, whose imprints are observed today in the CMB. An example of environment is given by the sub-Hubble modes which crossed the horizon later than the scales we can observe with the CMB spectrum or a heavy field.

In order to make the discussion more quantitative, let us introduce two fields, ζ and $\mathcal{F}$, with the former representing an observable while the latter an unobservable degree of freedom. The action can be split as:

$$S[\zeta, \mathcal{F}] = S_\zeta[\zeta] + S_{\mathcal{F}}[\mathcal{F}] + S_{int}[\zeta; \mathcal{F}]. \tag{3.19}$$

Now we are interested in the evolution of the system and we want to see how we can track down the effect of the environment in this evolution, giving up a detailed description of the environment. The basic idea, widely used also in the QFT context, is to integrate out, sum over or coarse-grain the degrees of freedom we are not interested in. Let us see how to implement this procedure.

Consider a pure quantum state described by the ket $|\Psi\rangle$. The density matrix $\hat{\rho} = |\Psi\rangle \langle\Psi|$ gives a representation of the quantum state in the Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_\varepsilon$, with $\mathcal{H}_S$ and $\mathcal{H}_\varepsilon$ the Hilbert space of the system and the environment respectively. Since we are only interested in the system degrees of freedom, we define the *reduced density matrix* as the partial trace of the full density matrix over the environmental degrees of freedom:

$$\hat{\rho}_{red} \equiv \mathrm{Tr}_\varepsilon\, \hat{\rho} = \sum_\alpha \langle\alpha|\, \hat{\rho}\, |\alpha\rangle\,, \tag{3.20}$$

where $\{|\alpha\rangle\}$ represents a generic basis of the Hilbert space of the environment, $\mathcal{H}_\varepsilon$. This quantity describes the state of the system, once taken into account the effect of the environment. For this reason our goal is to derive an evolution equation for $\hat{\rho}_{red}(t)$. In the context of OQS these equations are called *master equations.*

3.4 Master equations

The exact evolution of the reduced density matrix is given by the Liouville equation [26]:

$$\frac{\mathrm{d}\hat{\rho}_{red}}{\mathrm{d}t} = \mathcal{L}_S[\hat{\rho}_{red}], \tag{3.21}$$

where $\mathcal{L}_S$ is the Liouvillian, which describes both the unitary and non-unitary effects. This is an exact equation and starting from this, using various kinds of approximation we will see soon, we can obtain different master equations, valid in different contexts.

The first approximation we can consider is the so-called *Born approximation*, consistent in assuming a weak coupling between the system and the environment. This is a reasonable approximation because if the coupling were strong, we would have a strong mixing between system and environment and it would not make sense to distinguish a system and an environment at all. Consider a Hamiltonian of the form:

$$\hat{H} = \hat{H}_S + \hat{H}_\varepsilon + g\hat{H}_{int}, \tag{3.22}$$

where $\hat{H}_S$ and $\hat{H}_\varepsilon$ are respectively the free Hamiltonian of the system and the environment, $\hat{H}_{int}$ is the interaction Hamiltonian and g is the coupling constant. Using the Born approximation we want to expand the dynamics in powers of the coupling constant. We will work in the interaction picture where quantum states evolve with the interaction Hamiltonian, $\hat{H}_{int}$ while operators evolve with the free Hamiltonian, $\hat{H}_0 = \hat{H}_S + \hat{H}_\varepsilon$. From now on we will use an overall tilde to characterize quantities in the interaction picture and an overall hat to characterize quantities in the Schrödinger and Heisenberg picture. In the interaction picture the Liouville equation becomes:

$$\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}t} = -ig\Big[\tilde{H}_{int}(t), \tilde{\rho}(t)\Big]. \tag{3.23}$$

Formally we can integrate this equation obtaining:

$$\tilde{\rho}(t) = \tilde{\rho}(t_0) - ig \int_{t_0}^{t} \mathrm{d}t' \Big[\tilde{H}_{int}(t'), \tilde{\rho}(t')\Big]. \tag{3.24}$$

Putting (3.24) into (3.23) and tracing over the environmental degrees of freedom we obtain the following evolution equation for the reduced density matrix:

$$\frac{\mathrm{d}\tilde{\rho}_{red}}{\mathrm{d}t} = -g^2 \int_{t_0}^{t} \mathrm{d}t' \,\mathrm{Tr}_\varepsilon \left[\tilde{H}_{int}(t), \left[\tilde{H}_{int}(t'), \tilde{\rho}(t')\right]\right]. \tag{3.25}$$

Under this form this equation is still exact and difficult to solve as the Liouville equation (3.23).

3.4.1 Perturbative master equation

Notice that the derivative of the reduced density matrix is proportional to g^2, thus at lowest order in g, $\tilde{\rho}_{red}(t)$ is constant and for an initial state we can factorize it as $\hat{\rho}(t_0) = \hat{\rho}_S^{(0)} \otimes \hat{\rho}_\varepsilon^{(0)}$ (we used the hat because at initial time the three pictures coincide) and then at a generic time:

$$\tilde{\rho}_{red}(t') = \hat{\rho}_S^{(0)} \otimes \hat{\rho}_\varepsilon^{(0)} + \mathcal{O}(g), \tag{3.26}$$

then at order $\mathcal{O}(g^2)$ the master equation reads:

$$\frac{\mathrm{d}\tilde{\rho}_{red}}{\mathrm{d}t} = -g^2 \int_{t_0}^{t} \mathrm{d}t' \,\mathrm{Tr}_\varepsilon \left[\tilde{H}_{int}(t), \left[\tilde{H}_{int}(t'), \hat{\rho}_S^{(0)} \otimes \hat{\rho}_\varepsilon^{(0)}\right]\right] + \mathcal{O}(g^3). \tag{3.27}$$

In order to make the physics more explicit let us rewrite this equation using the following decomposition for the interaction Hamiltonian:

$$\hat{H}_{int}(t) = \int \mathrm{d}^3\mathbf{x} \hat{J}_S(t,\mathbf{x}) \otimes \hat{J}_\varepsilon(t,\mathbf{x}), \tag{3.28}$$

then:

$$\begin{aligned}
\frac{\mathrm{d}\tilde{\rho}_{red}}{\mathrm{d}t} &= -g^2 \int_{t_0}^{t} dt' \,\mathrm{Tr}_\varepsilon \Big[\int d^3\boldsymbol{x} \hat{J}_S(x) \otimes \hat{J}_\varepsilon(x), \Big[\int d^3\boldsymbol{y} \hat{J}_S(y) \otimes \hat{J}_\varepsilon(y), \hat{\rho}_S^{(0)} \otimes \hat{\rho}_\varepsilon^{(0)}\Big]\Big] \\
&= -g^2 \int_{t_0}^{t} dt' \int d^3\boldsymbol{x} \int d^3\boldsymbol{y} \,\mathrm{Tr}_\varepsilon \Big[\hat{J}_S(x) \otimes \hat{J}_\varepsilon(x), \hat{J}_S(y)\hat{\rho}_S^{(0)} \hat{J}_\varepsilon(y)\hat{\rho}_\varepsilon^{(0)} - \hat{\rho}_S^{(0)} \hat{J}_S(y)\hat{\rho}_\varepsilon^{(0)} \hat{J}_\varepsilon(y)\Big] \\
&= -g^2 \int_{t_0}^{t} dt' \int d^3\boldsymbol{x} \int d^3\boldsymbol{y} \,\mathrm{Tr}_\varepsilon \Big[\hat{J}_S(x)\hat{J}_S(y)\hat{\rho}_S^{(0)} \hat{J}_\varepsilon(x)\hat{J}_\varepsilon(y)\hat{\rho}_\varepsilon^{(0)} - \hat{J}_S(x)\hat{\rho}_S^{(0)} \hat{J}_S(y)\hat{J}_\varepsilon(x)\hat{\rho}_\varepsilon^{(0)} \hat{J}_\varepsilon(y) \\
&- \hat{J}_S(y)\hat{\rho}_S^{(0)} \hat{J}_S(x)\hat{J}_\varepsilon(y)\hat{\rho}_\varepsilon^{(0)} \hat{J}_\varepsilon(x) + \hat{\rho}_S^{(0)} \hat{J}_S(y)\hat{J}_S(x)\hat{\rho}_\varepsilon^{(0)} \hat{J}_\varepsilon(y)\hat{J}_\varepsilon(x)\Big] \\
&= -g^2 \int_{t_0}^{t} dt' \int d^3\boldsymbol{x} \int d^3\boldsymbol{y} \Big\{\Big[\hat{J}_S(x)\hat{J}_S(y)\hat{\rho}_S^{(0)} - \hat{J}_S(y)\hat{\rho}_S^{(0)} \hat{J}_S(x)\Big] \,\mathrm{Tr}_\varepsilon \Big[\hat{J}_\varepsilon(x)\hat{J}_\varepsilon(y)\hat{\rho}_\varepsilon^{(0)}\Big] \\
&- \Big[\hat{J}_S(x)\hat{\rho}_S^{(0)} \hat{J}_S(y) - \hat{\rho}_S^{(0)} \hat{J}_S(y)\hat{J}_S(x)\Big] \,\mathrm{Tr}_\varepsilon \Big[\hat{J}_\varepsilon(x) Je(y)\hat{\rho}_\varepsilon^{(0)}\Big]\Big\} \\
&= -g^2 \int_{t_0}^{t} dt' \int d^3\boldsymbol{x} \int d^3\boldsymbol{y} \Big\{\Big[\hat{J}_S(x)\hat{J}_S(y)\hat{\rho}_S^{(0)} - \hat{J}_S(y)\hat{\rho}_S^{(0)} \hat{J}_S(x)\Big]\mathcal{K}^{>}(x,y) \\
&- \Big[\hat{J}_S(x)\hat{\rho}_S^{(0)} \hat{J}_S(y) - \hat{\rho}_S^{(0)} \hat{J}_S(y)\hat{J}_S(x)\Big]\Big[\mathcal{K}^{>}(x,y)\Big]^*\Big\},
\end{aligned} \tag{3.29}$$

where we have defined:

$$\mathcal{K}^{>}(x,y) \equiv \mathrm{Tr}_\varepsilon \left[\hat{J}_\varepsilon(x)\hat{J}_\varepsilon(y)\hat{\rho}_\varepsilon^{(0)}\right] = \langle \hat{J}_\varepsilon(x)\hat{J}_\varepsilon(y)\rangle_0, \tag{3.30}$$

where $\mathcal{K}^{>}(x,y)$ is called *memory kernel* and it takes into account the effect of the environment on the system, indeed it has the form of a two-point correlation function of the environment, depending on its properties. Let us now try to explain why this object is called memory kernel. As we can see it depends on two different times, let us call them t and t'. If the time over which the environment evolves is much shorter than that of the system, the correlation function will rapidly decay as $|t-t'|$ and the integral appearing in (3.29) is dominated by the most recent contributions, i.e. those close to the upper bound t. In this limit we can say that the dynamics "looses memory" of the past history and it is only determined by the instantaneous interaction. This limit is called *Markovian limit* and it will be more deeply analyzed in the following sections.

3.4.2 The exact Nakajima-Zwanzig master equation

Now we want to derive the so-called *Nakajima-Zwanzig* master equation [3] which is nothing else that a recast of the exact Liouville equation, but it will be more useful to implement some approximation schemes. Let us start from the Liouville equation:

$$\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}t} = -ig\left[\tilde{H}_{int}(t), \tilde{\rho}(t)\right] \equiv g\mathcal{L}(t)\tilde{\rho}(t). \tag{3.31}$$

Now we introduce the following projection super-operator which acts on the full density matrix:

$$\mathcal{P}\tilde{\rho} \equiv \mathrm{Tr}_{\varepsilon}(\tilde{\rho}) \otimes \hat{\rho}_{\varepsilon}^{(0)}, \tag{3.32}$$

and its complementary

$$\mathcal{Q} = \mathbb{1} - \mathcal{P}. \tag{3.33}$$

$\mathcal{P}$ and $\mathcal{Q}$ are projectors, i.e. $\mathcal{P}^2 = \mathcal{P}$, and one can show that $\mathcal{P}\tilde{\rho}$ contains the same information that are contained in $\tilde{\rho}_{red}$. Acting with $\mathcal{P}$ and $\mathcal{Q}$ on the Liouville equation we obtain:

$$\begin{aligned} \frac{\partial}{\partial t}\mathcal{P}\tilde{\rho}(t) &= g\mathcal{P}\mathcal{L}(t)\tilde{\rho}(t), \\ \frac{\partial}{\partial t}\mathcal{Q}\tilde{\rho}(t) &= g\mathcal{Q}\mathcal{L}(t)\tilde{\rho}(t), \end{aligned} \tag{3.34}$$

where we could commute the time derivative with the projectors because $\hat{\rho}_{\varepsilon}^{(0)}$ is time-independent. Inserting the identity $\mathbb{1} = \mathcal{P} + \mathcal{Q}$

$$\frac{\partial}{\partial t}\mathcal{P}\tilde{\rho}(t) = g\mathcal{P}\mathcal{L}(t)\mathcal{P}\tilde{\rho}(t) + g\mathcal{P}\mathcal{L}(t)\mathcal{Q}\tilde{\rho}(t), \tag{3.35}$$

$$\frac{\partial}{\partial t}\mathcal{Q}\tilde{\rho}(t) = g\mathcal{Q}\mathcal{L}(t)\mathcal{P}\tilde{\rho}(t) + g\mathcal{Q}\mathcal{L}(t)\mathcal{Q}\tilde{\rho}(t). \tag{3.36}$$

The last equation is a first order linear differential equation which can be formally integrated as:

$$\mathcal{Q}\tilde{\rho}(t) = \mathcal{G}_{\mathcal{Q}}(t,t_0)\mathcal{Q}\tilde{\rho}(t_0) + g\int_{t_0}^{t} dt'\mathcal{G}_{\mathcal{Q}}(t,t')\mathcal{Q}\mathcal{L}(t')\mathcal{P}\tilde{\rho}(t'), \tag{3.37}$$

where $\mathcal{G}_{\mathcal{Q}}(t,t_0)$ is the propagator defined as:

$$\mathcal{G}_{\mathcal{Q}}(t,t_0) \equiv \mathcal{T}\exp\left[g\int_{t'}^{t} \mathrm{d}t''\mathcal{Q}\mathcal{L}(t'')\right]. \tag{3.38}$$

Inserting equation (3.37) in equation (3.35) we obtain:

$$\frac{\partial}{\partial t}\mathcal{P}\tilde{\rho}(t) = g\mathcal{P}\mathcal{L}(t)\mathcal{P}\tilde{\rho}(t) + g\mathcal{P}\mathcal{L}(t)\mathcal{G}_{\mathcal{Q}}(t,t_0)\mathcal{Q}\tilde{\rho}(t_0) + g^2\int_{t_0}^{t} dt'\mathcal{P}\mathcal{L}(t)\mathcal{G}_{\mathcal{Q}}(t,t'). \tag{3.39}$$

This is the so-called *Nakajima-Zwanzig* (NZ) equation. It is an exact equation for the reduced density matrix evolution. We can make some assumptions to simplify this equation:

- No correlations system-environment in the initial state:

$$\begin{aligned} &\tilde{\rho}(t_0) = \mathrm{Tr}_\varepsilon(\tilde{\rho}^{(0)}) \otimes \mathrm{Tr}_S(\tilde{\rho}^{(0)}) = \hat{\rho}_S^{(0)} \otimes \hat{\rho}_\varepsilon^{(0)}, \\ &\Rightarrow \mathcal{Q}\tilde{\rho}(t_0) = (\mathbb{1} - \mathcal{P})\tilde{\rho}(t_0) = \tilde{\rho}(t_0) - \underbrace{\underbrace{\mathrm{Tr}_\varepsilon \tilde{\rho}^{(0)}}_{\hat{\rho}_S^{(0)}} \otimes \hat{\rho}_\varepsilon^{(0)}}_{\tilde{\rho}(t_0)} = 0. \end{aligned} \tag{3.40}$$

- Interaction Hamiltonian vanishes in the reference state:

$$\langle \hat{H}_{int} \rangle_0 = 0 \qquad \text{always possible after a field redefinition.} \quad \Rightarrow \mathcal{P}\mathcal{L}(t)\mathcal{P} = 0. \tag{3.41}$$

Then the NZ equation reduces to:

$$\frac{\partial}{\partial t}\mathcal{P}\tilde{\rho}(t) = g^2 \int_{t_0}^{t} dt' \mathcal{K}(t,t')\mathcal{P}\tilde{\rho}(t'). \tag{3.42}$$

Where the memory kernel is defined as:

$$\mathcal{K}(t,t') = \mathcal{P}\mathcal{L}(t)\mathcal{G}_{\mathcal{Q}}(t,t')\mathcal{Q}\mathcal{L}(t')\mathcal{P}. \tag{3.43}$$

This master equation is as difficult to solve as the complete Liouville equation, but it allows approximation schemes to be designed.

3.4.3 Perturbative Nakajima-Zwanzig equation (NZ$_n$)

We have already said that the bipartition system/environment only makes sense if the two sectors are weakly coupled, this provides a natural small parameter, the coupling constant, with respect to we can perturbatively expand the NZ equation. This is the *Born approximation*. There are several ways to perform this approximation, the simplest one is to expand the memory kernel in powers of the coupling constant. In particular we can expand $\mathcal{Q} = \mathbb{1} + \dots$ and $\mathcal{G}_{\mathcal{Q}}(t,t') = 1 + \dots$ and at order $\mathcal{O}(g^2)$ the NZ equation becomes:

$$\frac{\partial}{\partial t}\mathcal{P}\tilde{\rho}(t) = g^2 \int_{t_0}^{t} dt' \mathcal{P}\mathcal{L}(t)\mathcal{L}(t')\mathcal{P}\tilde{\rho}(t') \qquad (NZ_2 \text{ equation}). \tag{3.44}$$

In terms of the interaction Hamiltonian (making explicit $\mathcal{P}$ and $\mathcal{L}(t)$):

$$\begin{aligned} \frac{\partial}{\partial t}\Big(\mathrm{Tr}_\varepsilon(\tilde{\rho}(t)) \otimes \hat{\rho}_\varepsilon^{(0)}\Big) &= g^2 \int_{t_0}^{t} dt' \Big\{(-i)(-i)\mathcal{P}\Big[\tilde{H}_{int}(t), \Big[\tilde{H}_{int}(t'), \hat{\rho}_{red}(t') \otimes \hat{\rho}_\varepsilon^{(0)}\Big]\Big]\Big\} \\ &= -g^2 \int_{t_0}^{t} dt' \,\mathrm{Tr}_\varepsilon \Big[\tilde{H}_{int}(t), \Big[\tilde{H}_{int}(t'), \hat{\rho}_{red}(t') \otimes \hat{\rho}_\varepsilon^{(0)}\Big]\Big] \otimes \hat{\rho}_\varepsilon^{(0)} \\ \Rightarrow \frac{\mathrm{d}\tilde{\rho}_{red}(t)}{\mathrm{dt}} &= -g^2 \int_{t_0}^{t} dt' \,\mathrm{Tr}_\varepsilon \Big[\tilde{H}_{int}(t), \Big[\tilde{H}_{int}(t'), \hat{\rho}_{red}(t') \otimes \hat{\rho}_\varepsilon^{(0)}\Big]\Big]. \end{aligned} \tag{3.45}$$

Comparing this equation with (3.27) we see that the only difference is that in (3.27) it appears the initial state of the system while in (3.45) it appears $\tilde{\rho}_{red}(t')$. Due to this t' dependence this equation is still very hard to solve, even numerically. For this reason we need to exploit another assumption.

3.4.4 Time-Convolutionless master equation (TCL$_2$)

The Time-Convolutionless method allows us to remove the convolution in (3.45) by noticing that the non-local in time contributions are also organised in powers of the coupling constant g, which therefore must be neglected if we focus on a given order. The TCL$_2$ is indeed given by:

$$\frac{\mathrm{d}\tilde{\rho}_{red}(t)}{\mathrm{dt}} = -g^2 \int_{t_0}^{t} dt' \,\mathrm{Tr}_\varepsilon \Big[\tilde{H}_{int}(t), \Big[\tilde{H}_{int}(t'), \hat{\rho}_{red}(t) \otimes \hat{\rho}_\varepsilon^{(0)}\Big]\Big]. \tag{3.46}$$

At a given order, let us say $\mathcal{O}(g^2)$, NZ_2 and TCL_2 approximate the exact dynamics at the same accuracy. Let us stress that despite being written in a time-local form, the TCL_2 remains a non-Markovian master equation. Indeed the definition of Markovian map is the following [27]: given a dynamical map acting on the density matrix

$$\tilde{\rho}(t) \quad\rightarrow\quad \tilde{\rho}(t') = \mathcal{V}_{t\rightarrow t'}\tilde{\rho}(t), \tag{3.47}$$

it is said to be Markovian if it satisfies the semi-group property:

$$\mathcal{V}_{t\rightarrow t'} = \mathcal{V}_{t''\rightarrow t'}\mathcal{V}_{t\rightarrow t''}. \tag{3.48}$$

In order for this property to be satisfied the environment must be much larger than the system, such that it is not affected by the system dynamics. In this case the environment is called *bath*. The assumption of Markovianity is related to the concept of *irreversibility*, since the information can only flow from the system to the environment, but we never have a backreaction. Vice versa a non-markovian master equation can account for memory effects and information backflow as we will explain later. Notice that a Markovian master equation is necessarily local in time, but the opposite is not generally true and this is the case of the TCL_2 master equation. The lesson to learn from this discussion is that Markovianity and time-locality are two different concepts and only the former implies the latter, the vice versa does not hold in general.

3.4.5 Developing the TCL_2 equation

Now we want to focus on the TCL_2 master equation.

$$\frac{\mathrm{d}\tilde{\rho}_{red}(t)}{\mathrm{d}t} = -g^2\int_{t_0}^{t} dt'\,\mathrm{Tr}_{\varepsilon}\left[\tilde{H}_{int}(t),\left[\tilde{H}_{int}(t'),\hat{\rho}_{red}(t)\otimes\hat{\rho}_{\varepsilon}^{(0)}\right]\right]. \tag{3.49}$$

Using again:

$$\tilde{H}_{int}(t) = \int d^3\boldsymbol{x}\tilde{J}_S(t,\boldsymbol{x})\otimes\tilde{J}_{\varepsilon}(t,\boldsymbol{x}). \tag{3.50}$$

We get again the previous equation with $\tilde{\rho}_{red}$ instead of $\hat{\rho}_S^{(0)}$:

$$\begin{aligned}\frac{\mathrm{d}\tilde{\rho}_{red}}{\mathrm{d}t} =& -g^2\int_{t_0}^{t} dt'\int d^3\boldsymbol{x}\int d^3\boldsymbol{y}\Big\{\Big[\tilde{J}_S(x)\tilde{J}_S(y)\tilde{\rho}_{red} - \tilde{J}_S(y)\tilde{\rho}_{red}\tilde{J}_S(x)\Big]\mathcal{K}^{>}(x,y)\\ &-\Big[\tilde{J}_S(x)\tilde{\rho}_{red}\tilde{J}_S(y) - \tilde{\rho}_{red}\tilde{J}_S(y)\tilde{J}_S(x)\Big]\Big[\mathcal{K}^{>}(x,y)\Big]^*\Big\}.\end{aligned} \tag{3.51}$$

For simplicity we consider localised currents and bilinear interactions, which means:

$$\tilde{J}_S(x) = \tilde{J}_S(t)\delta^3(\boldsymbol{x}-\boldsymbol{x}_0) \qquad\qquad \tilde{H}_{int}(t) = \tilde{\boldsymbol{z}}_{\zeta}^T(t)\boldsymbol{V}(t)\tilde{\boldsymbol{z}}_{\mathcal{F}}(t), \tag{3.52}$$

where $\boldsymbol{V}(t)$ is an arbitrary 2×2 matrix containing the linear coupling between the two fields and $\tilde{\boldsymbol{z}}_{\alpha} = (\tilde{\alpha},\tilde{p}_{\alpha})^T$ for $\alpha = \zeta,\mathcal{F}$.
Now we can use the fact that in the interaction picture the operators evolve with the free Hamiltonian, so:

$$\tilde{\boldsymbol{z}}_{\zeta}(t') = \bar{\mathcal{T}}\exp\left[i\int_t^{t'}\tilde{H}_0(t'')dt''\right]\tilde{\boldsymbol{z}}_{\zeta}(t)\mathcal{T}\exp\left[-i\int_t^{t'}\tilde{H}_0(t'')dt''\right], \tag{3.53}$$

but the linearity of the dynamics simplifies the computation as:

$$\tilde{\boldsymbol{z}}_{\zeta}(t') = \boldsymbol{G}^{\mathcal{S}}(t',t)\tilde{\boldsymbol{z}}_{\zeta}(t), \tag{3.54}$$

where $\boldsymbol{G}^{\mathcal{S}}(t',t)$ is the Green's matrix of the unperturbed system. Then the TCL$_2$ becomes (here the coupling constant g is put inside $\boldsymbol{V}(t)$):

$$
\begin{aligned}
\frac{\mathrm{d}\tilde{\rho}_{red}}{\mathrm{d}t} &= -\int_{t_0}^{t} \mathrm{Tr}_{\varepsilon}\Big[\tilde{H}_{int}(t)\tilde{H}_{int}(t')\tilde{\rho}_{red}(t)\otimes\hat{\rho}_{\varepsilon}^{(0)} - \tilde{H}_{int}(t)\tilde{\rho}_{red}(t)\otimes\hat{\rho}_{\varepsilon}^{(0)}\tilde{H}_{int}(t') \\
&- \tilde{H}_{int}(t')\tilde{\rho}_{red}(t)\otimes\hat{\rho}_{\varepsilon}^{(0)}\tilde{H}_{int}(t) + \tilde{\rho}_{red}(t)\otimes\hat{\rho}_{\varepsilon}^{(0)}\tilde{H}_{int}(t')\tilde{H}_{int}(t)\Big] \\
&= -\int_{t_0}^{t} dt'\,\mathrm{Tr}_{\varepsilon}\Big[\tilde{z}_{\zeta,i}(t)V_{ij}(t)\tilde{z}_{\mathcal{F},j}(t)\tilde{z}_{\zeta,k}(t')V_{kl}(t)\tilde{z}_{\mathcal{F},l}(t')\tilde{\rho}_{red}(t)\otimes\hat{\rho}_{\varepsilon}^{(0)} \\
&- \tilde{z}_{\zeta,i}(t)V_{ij}(t)\tilde{z}_{\mathcal{F},j}(t)\tilde{\rho}_{red}(t)\otimes\hat{\rho}_{\varepsilon}^{(0)}\tilde{z}_{\zeta,k}(t')V_{kl}(t')\tilde{z}_{\mathcal{F},l}(t') \\
&- \tilde{z}_{\zeta,k}(t')V_{kl}(t')\tilde{z}_{\mathcal{F},l}(t')\tilde{\rho}_{red}(t)\otimes\hat{\rho}_{\varepsilon}^{(0)}\tilde{z}_{\zeta,i}(t)V_{ij}(t)\tilde{z}_{\mathcal{F},j}(t) \\
&+ \tilde{\rho}_{red}(t)\otimes\hat{\rho}_{\varepsilon}^{(0)}\tilde{z}_{\zeta,k}(t')V_{kl}(t')\tilde{z}_{\mathcal{F},l}(t')\tilde{z}_{\zeta,i}(t)V_{ij}(t)\tilde{z}_{\mathcal{F},j}(t)\Big] \\
&= -\int_{t_0}^{t} dt'\Big\{\Big[\tilde{z}_{\zeta,i}(t)V_{ij}(t)\tilde{z}_{\zeta,k}(t')V_{kl}(t')\tilde{\rho}_{red}(t) - \tilde{z}_{\zeta,k}(t')V_{kl}(t')\tilde{\rho}_{red}(t)\tilde{z}_{\zeta,i}(t)V_{ij}(t)\Big]\times \\
&\times \mathrm{Tr}_{\varepsilon}\Big[\tilde{z}_{\mathcal{F},j}(t)\tilde{z}_{\mathcal{F},l}(t')\hat{\rho}_{\varepsilon}^{(0)}\Big] \\
&- \Big[\tilde{z}_{\zeta,i}(t)V_{ij}(t)\tilde{\rho}_{red}(t)\tilde{z}_{\zeta,k}(t')V_{kl}(t') - \tilde{\rho}_{red}(t)\tilde{z}_{\zeta,k}(t')V_{kl}(t')\tilde{z}_{\zeta,i}(t)V_{ij}(t)\Big]\,\mathrm{Tr}_{\varepsilon}\Big[\tilde{z}_{\mathcal{F},l}(t')\tilde{z}_{\mathcal{F},j}(t)\hat{\rho}_{\varepsilon}^{(0)}\Big]\Big\} \\
&= -\int_{t_0}^{t} dt'\Big\{\Big[\tilde{z}_{\zeta,i}(t)V_{ij}(t)G^{(S)}_{kn}(t',t)\tilde{z}_{\zeta,n}(t)V_{kl}(t')\tilde{\rho}_{red}(t) \\
&- G^{(S)}_{kn}(t',t)\tilde{z}_{\zeta,n}(t)V_{kl}(t')\tilde{\rho}_{red}(t)\tilde{z}_{\zeta,i}(t)V_{ij}(t)\Big]\mathcal{K}^{>}_{jl}(t,t') \\
&- \Big[\tilde{z}_{\zeta,i}(t)V_{ij}(t)\tilde{\rho}_{red}(t)G^{(S)}_{kn}(t',t)\tilde{z}_{\zeta,n}(t)V_{kl}(t') - \tilde{\rho}_{red}(t)G^{(S)}_{kn}(t',t)\tilde{z}_{\zeta,n}(t)V_{kl}(t')\tilde{z}_{\zeta,i}(t)V_{ij}(t)\Big]\times \\
&\times \Big[\mathcal{K}^{>}_{jl}(t,t')\Big]^{*}\Big\} \\
&= -\int_{t_0}^{t} dt'\Big\{\Big[\tilde{z}_{\zeta,i}(t)\tilde{z}_{\zeta,n}(t)\tilde{\rho}_{red}(t) - \tilde{z}_{\zeta,n}(t)\tilde{\rho}_{red}(t)\tilde{z}_{\zeta,i}(t)\Big]\times V_{ij}(t)\mathcal{K}^{>}_{jl}(t,t')V_{kl}(t')G^{(S)}_{kn}(t',t) \\
&- \Big[\tilde{z}_{\zeta,i}(t)\tilde{\rho}_{red}(t)\tilde{z}_{\zeta,n}(t) - \tilde{\rho}_{red}(t)\tilde{z}_{\zeta,n}(t)\tilde{z}_{\zeta,i}(t)\Big]\times V_{ij}(t)(\mathcal{K}^{>}_{jl})^{*}V_{kl}(t')G^{(S)}_{kn}(t',t)\Big\} \\
&= -\int_{t_0}^{t} dt'\Big\{\Big[\tilde{z}_{\zeta,i}(t)\tilde{z}_{\zeta,j}(t)\tilde{\rho}_{red}(t) - \tilde{z}_{\zeta,j}(t)\tilde{\rho}_{red}(t)\tilde{z}_{\zeta,i}(t)\Big]D^{>}_{ij}(t,t') \\
&- \Big[\tilde{z}_{\zeta,i}(t)\tilde{\rho}_{red}(t)\tilde{z}_{\zeta,j}(t) - \tilde{\rho}_{red}(t)\tilde{z}_{\zeta,j}(t)\tilde{z}_{\zeta,i}(t)\Big]D^{>*}_{ij}(t,t')\Big\},
\end{aligned}
\tag{3.55}
$$

where we have defined:

$$\mathcal{K}^{>}(t,t') \equiv \mathrm{Tr}_{\varepsilon}\Big[\tilde{\boldsymbol{z}}^{T}_{\mathcal{F}}(t)\tilde{\boldsymbol{z}}_{\mathcal{F}}(t')\hat{\rho}_{\varepsilon}^{(0)}\Big], \tag{3.56}$$

$$\boldsymbol{D}^{>}(t,t') \equiv \boldsymbol{V}(t)\boldsymbol{\mathcal{K}}^{>}(t,t')\boldsymbol{V}^{T}(t')\boldsymbol{G}^{(S)}(t',t). \tag{3.57}$$

Now it is easy to express the previous equation in Schrödinger picture, using:

$$\tilde{\boldsymbol{z}}(t) = \hat{U}_0^{\dagger}(t)\hat{\boldsymbol{z}}\hat{U}_0(t), \tag{3.58}$$

$$\tilde{\rho}(t) = \hat{U}_0^{\dagger}(t)\hat{\rho}(t)\hat{U}_0(t), \tag{3.59}$$

so the LHS becomes:

$$\frac{\mathrm{d}\tilde{\rho}_{red}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\big(\hat{U}_0^\dagger(t)\hat{\rho}_{red}(t)\hat{U}_0(t)\big) = \frac{\mathrm{d}\hat{U}_0^\dagger}{\mathrm{d}t}\hat{\rho}_{red}\hat{U}_0 + \hat{U}_0^\dagger\hat{\rho}_{red}\frac{\mathrm{d}\hat{U}_0}{\mathrm{d}t} + \hat{U}_0^\dagger\frac{\mathrm{d}\hat{\rho}_{red}}{\mathrm{d}t}\hat{U}_0. \tag{3.60}$$

Using the equation of motion of the evolution operator:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{U}_0(t) = -i\hat{H}_0\hat{U}_0(t), \tag{3.61}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{U}_0^\dagger(t) = i\hat{U}_0^\dagger(t)\hat{H}_0. \tag{3.62}$$

Since $\hat{H}_0 = \hat{H}_S + \hat{H}_\varepsilon$ is applied on $\hat{\rho}_{red}$, it reduces to $\hat{H}_S$

$$\frac{\mathrm{d}\tilde{\rho}_{red}}{\mathrm{d}t} = i\hat{U}_0^\dagger\hat{H}_S\hat{\rho}_{red}\hat{U}_0 - i\hat{U}_0^\dagger\hat{\rho}_{red}\hat{H}_S\hat{U}_0 + \hat{U}_0^\dagger\frac{\mathrm{d}\hat{\rho}_{red}}{\mathrm{d}t}\hat{U}_0 = i\hat{U}_0^\dagger\Big[\hat{H}_S, \hat{\rho}_{red}(t)\Big]\hat{U}_0 + \hat{U}_0^\dagger\frac{\mathrm{d}\hat{\rho}_{red}}{\mathrm{d}t}\hat{U}_0, \tag{3.63}$$

while in the RHS (we take only one term because for the others is the same):

$$\begin{aligned}\tilde{z}_{\zeta,i}(t)\tilde{z}_{\zeta,j}(t)\tilde{\rho}_{red}(t) &= \hat{U}_0^\dagger(t)\hat{z}_{\zeta,i}\hat{U}_0(t)\hat{U}_0^\dagger(t)\hat{z}_{\zeta,j}\hat{U}_0(t)\hat{U}_0^\dagger(t)\hat{\rho}_{red}(t)\hat{U}_0(t)\\ &= \hat{U}_0^\dagger(t)\hat{z}_{\zeta,i}\hat{z}_{\zeta,j}\hat{\rho}_{red}(t)\hat{U}_0(t).\end{aligned} \tag{3.64}$$

So both in the LHS and the RHS we can cancel the $\hat{U}_0^\dagger$ at the beginning and the $\hat{U}_0$ at the end to obtain:

$$\begin{aligned}\frac{\mathrm{d}\hat{\rho}_{red}}{\mathrm{d}t} = -i\Big[\hat{H}_S(t), \hat{\rho}_{red}(t)\Big] - \int_{t_0}^{t} dt' \Bigg\{ &\Big[\hat{z}_{\zeta,i}\hat{z}_{\zeta,j}\hat{\rho}_{red}(t) - \hat{z}_{\zeta,j}\hat{\rho}_{red}(t)\hat{z}_{\zeta,i}\Big] D_{ij}^{>}(t,t')\\ &- \Big[\hat{z}_{\zeta,i}\hat{\rho}_{red}(t)\hat{z}_{\zeta,j} - \hat{\rho}_{red}(t)\hat{z}_{\zeta,j}\hat{z}_{\zeta,i}\Big] D_{ij}^{>*}(t,t')\Bigg\}.\end{aligned} \tag{3.65}$$

Now we can use the following decomposition:

$$\boldsymbol{D}^{>}(t,t') \equiv \boldsymbol{D}^{Re}(t,t') + i\boldsymbol{D}^{Im}(t,t'), \tag{3.66}$$

$$A_{ij} \equiv A_{(ij)} + A_{12}\omega_{ij} \qquad \text{with} \quad \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3.67}$$

$$\Rightarrow D_{ij}^{>}(t,t') = D_{(ij)}^{Re}(t,t') + D_{12}^{Re}(t,t')\omega_{ij} + iD_{(ij)}^{Im}(t,t') + iD_{12}^{Im}(t,t')\omega_{ij}. \tag{3.68}$$

Now let us define:

$$\Delta_{ij}(t) = 2\int_{t_0}^{t} dt' D_{(ij)}^{Im}(t,t'), \tag{3.69}$$

$$D_{ij}(t) = 2\int_{t_0}^{t} dt' D_{(ij)}^{Re}(t,t'), \tag{3.70}$$

$$\Rightarrow \int_{t_0}^{t} dt' D_{ij}^{>}(t,t') = \frac{1}{2}D_{ij}(t) + \int_{t_0}^{t} D_{12}^{Re}(t,t')\omega_{ij} + \frac{i}{2}\Delta_{ij}(t) + \frac{i}{2}\Delta_{12}(t)\omega_{ij}. \tag{3.71}$$

So our equation becomes:

$$\begin{aligned}
\frac{\mathrm{d}\hat{\rho}_{red}}{\mathrm{d}t} &= -i\Big[\hat{H}_S(t), \hat{\rho}_{red}(t)\Big] \\
&- \Bigg\{ \Big[\hat{z}_{\zeta,i}\hat{z}_{\zeta,j}\hat{\rho}_{red}(t) - \hat{z}_{\zeta,j}\hat{\rho}_{red}(t)\hat{z}_{\zeta,i}\Big]\Big[\frac{1}{2}D_{ij}(t) + \int_{t_0}^{t} D_{12}^{Re}(t,t')\omega_{ij} + \frac{i}{2}\Delta_{ij}(t) + \frac{i}{2}\Delta_{12}(t)\omega_{ij}\Big] \\
&+ \Big[\hat{\rho}_{red}(t)\hat{z}_{\zeta,j}\hat{z}_{\zeta,i} - \hat{z}_{\zeta,i}\hat{\rho}_{red}(t)\hat{z}_{\zeta,j}\Big]\Big[\frac{1}{2}D_{ij}(t) + \int_{t_0}^{t} D_{12}^{Re}(t,t')\omega_{ij} - \frac{i}{2}\Delta_{ij}(t) - \frac{i}{2}\Delta_{12}(t)\omega_{ij}\Big]\Bigg\} \\
&= -i\Big[\hat{H}_S(t), \hat{\rho}_{red}(t)\Big] - \Bigg\{\frac{1}{2}\hat{z}_{\zeta,i}D_{ij}(t)\hat{z}_{\zeta,j}\hat{\rho}_{red}(t) - \frac{1}{2}\hat{z}_{\zeta,j}\hat{\rho}_{red}(t)D_{ij}(t)\hat{z}_{\zeta,i} + \frac{i}{2}\hat{z}_{\zeta,i}\Delta_{ij}(t)\hat{z}_{\zeta,j}\hat{\rho}_{red}(t) \\
&- \frac{i}{2}\hat{z}_{\zeta,j}\hat{\rho}_{red}(t)\Delta_{ij}(t)\hat{z}_{\zeta,i} + \frac{i}{2}\Delta_{12}(t)\hat{z}_{\zeta,i}\omega_{ij}\hat{z}_{\zeta,j}\hat{\rho}_{red}(t) - \frac{i}{2}\Delta_{12}(t)\hat{z}_{\zeta,j}\hat{\rho}_{red}(t)\omega_{ij}\hat{z}_{\zeta,i} \\
&+ \frac{1}{2}\hat{\rho}_{red}(t)\hat{z}_{\zeta,j}D_{ij}(t)\hat{z}_{\zeta,i} - \frac{1}{2}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t)D_{ij}(t)\hat{z}_{\zeta,j} - \frac{i}{2}\hat{\rho}_{red}(t)\hat{z}_{\zeta,j}\Delta_{ij}(t)\hat{z}_{\zeta,i} + \frac{i}{2}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t)\Delta_{ij}(t)\hat{z}_{\zeta,j} \\
&- \frac{i}{2}\Delta_{12}(t)\hat{\rho}_{red}\hat{z}_{\zeta,j}\omega_{ij}\hat{z}_{\zeta,i} + \frac{i}{2}\Delta_{12}(t)\hat{z}_{\zeta,i}\hat{\rho}_{red}(t)\omega_{ij}\hat{z}_{\zeta,j}\Bigg\},
\end{aligned} \tag{3.72}$$

where we noticed that calling $A_{ij} \equiv \int_{t_0}^{t} D_{12}^{Re}(t,t')\omega_{ij}$ (antisymmetric)

$$\begin{aligned}
&A_{ij}\Big[\hat{z}_{\zeta,i}\hat{z}_{\zeta,j}\hat{\rho}_{red}(t) - \hat{\rho}_{red}(t)\hat{z}_{\zeta,j}\hat{z}_{\zeta,i} - \hat{z}_{\zeta,j}\hat{\rho}_{red}(t)\hat{z}_{\zeta,i} - \hat{z}_{\zeta,i}\hat{\rho}_{red}(t)\hat{z}_{\zeta,j}\Big] \\
&\quad = A_{ij}\Big\{\underbrace{\hat{z}_{\zeta,i}\hat{z}_{\zeta,j}}_{\text{symmetric under } i \leftrightarrow j}, \hat{\rho}_{red}(t)\Big\} - A_{ij}\Big[\underbrace{\underbrace{\hat{z}_{\zeta,j}\hat{\rho}_{red}(t)\hat{z}_{\zeta,i}}_{T_{ji}} + \underbrace{\hat{z}_{\zeta,i}\hat{\rho}_{red}(t)\hat{z}_{\zeta,j}}_{T_{ij}}}_{2T_{(ij)}}\Big] = 0,
\end{aligned} \tag{3.73}$$

because the product of a symmetric and an antisymmetric matrix is zero.
Then, since Δ_{ij} is symmetric, we can freely exchange $i \leftrightarrow j$:

$$\frac{i}{2}\Big[\hat{z}_{\zeta,i}\hat{\rho}_{red}(t)\Delta_{ij}(t)\hat{z}_{\zeta,j} - \hat{z}_{\zeta,j}\hat{\rho}_{red}(t)\Delta_{ij}(t)\hat{z}_{\zeta,i}\Big] = 0. \tag{3.74}$$

So, using the symmetry and antisymmetry of Δ_{ij} and D_{ij} respectively, our equation becomes:

$$\begin{aligned}
\frac{\mathrm{d}\hat{\rho}_{red}}{\mathrm{d}t} &= -i\Big[\hat{H}_S(t), \hat{\rho}_{red}(t)\Big] - \frac{i}{2}\Big[\hat{z}_{\zeta,i}\Delta_{ij}(t)\hat{z}_{\zeta,j}, \hat{\rho}_{red}(t)\Big] + D_{ij}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t)\hat{z}_{\zeta,j} \\
&- i\Delta_{12}(t)\omega_{ij}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t)\hat{z}_{\zeta,j} - \frac{1}{2}D_{ij}\Big\{\hat{z}_{\zeta,i}\hat{z}_{\zeta,j}, \hat{\rho}_{red}(t)\Big\} + \frac{i}{2}\Delta_{12}(t)\omega_{ij}\Big\{\hat{z}_{\zeta,i}\hat{z}_{\zeta,j}, \hat{\rho}_{red}(t)\Big\}.
\end{aligned} \tag{3.75}$$

Now if we define:

$$\hat{H}_{LS}(t) \equiv \frac{1}{2}\hat{z}_{\zeta,i}\Delta_{ij}\hat{z}_{\zeta,j}, \tag{3.76}$$

$$\gamma_{ij}(t) \equiv D_{ij}(t) - i\Delta_{12}(t)\omega_{ij}. \tag{3.77}$$

We finally get:

$$\frac{\mathrm{d}\hat{\rho}_{red}}{\mathrm{d}t} = \underbrace{-i\Big[\hat{H}_S(t) + \hat{H}_{LS}(t), \hat{\rho}_{red}(t)\Big]}_{\text{UNITARY CONTRIBUTION}} + \underbrace{\gamma_{ij}(t)\Big(\hat{z}_{\zeta,i}\hat{\rho}_{red}(t)\hat{z}_{\zeta,j} - \frac{1}{2}\big\{\hat{z}_{\zeta,j}\hat{z}_{\zeta,i}, \hat{\rho}_{red}(t)\big\}\Big)}_{\text{NON-UNITARY CONTRIBUTION}}. \tag{3.78}$$

We see that the interaction with an environment has two main effects: the first one is the appearance of $\hat{H}_{LS}(t)$ which takes into account the renormalization of the energy levels due to the interaction with the environment; the second one is the appearance of a non-unitary contribution, absent in the free theory, that describes the effects of dissipation and decoherence induced by the environment. In particular dissipation is due to the imaginary part of Δ_{12}, whereas decoherence is described by the real part of D_{ij}.

3.4.6 Transport equations

Now that we have derived a master equation, which describes the dynamics of the quantum state, we want to use it to derive an evolution equation for the observables we are interested in. This kind of equations are commonly called *transport equations*. Starting from the expression of the correlator in the Schrödinger picture [3] (where operators do not evolve)

$$\langle\hat{O}(t)\rangle = \mathrm{Tr}_S\left[\hat{O}\hat{\rho}_{red}(t)\right], \tag{3.79}$$

and differentiate it with respect to time:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{O}(t)\rangle = \mathrm{Tr}_S\left[\hat{O}\frac{\mathrm{d}\hat{\rho}_{red}(t)}{\mathrm{d}t}\right]. \tag{3.80}$$

Now injecting the master equation we have derived in this expression we obtain (let us consider term by term):

- First term:

$$\begin{aligned}
&-i\,\mathrm{Tr}_S\left(\hat{O}\Big[\underbrace{\hat{H}_S(t)}_{\frac{1}{2}\hat{z}_{\zeta,i}\hat{H}_{0ij}\hat{z}_{\zeta,j}} + \underbrace{\hat{H}_{LS}(t)}_{\frac{1}{2}\hat{z}_{\zeta,i}\Delta_{ij}\hat{z}_{\zeta,j}},\hat{\rho}_{red}(t)\Big]\right) = -\frac{i}{2}\mathrm{Tr}_S\left(\hat{O}\big(\hat{H}_0+\Delta\big)_{ij}\left[\hat{z}_{\zeta,i}\hat{z}_{\zeta,j},\hat{\rho}_{red}(t)\right]\right)\\
&= -\frac{i}{2}\big(\hat{H}_0+\Delta\big)_{ij}\,\mathrm{Tr}_S\left(\hat{O}\hat{z}_{\zeta,i}\hat{z}_{\zeta,j}\hat{\rho}_{red}(t)-\hat{O}\hat{\rho}_{red}(t)\hat{z}_{\zeta,i}\hat{z}_{\zeta,j}\right)\\
&= -\frac{i}{2}\big(\hat{H}_0+\Delta\big)_{ij}\,\mathrm{Tr}_S\left(\hat{O}\hat{z}_{\zeta,i}\hat{z}_{\zeta,j}\hat{\rho}_{red}(t)-\hat{z}_{\zeta,i}\hat{z}_{\zeta,j}\hat{O}\hat{\rho}_{red}(t)\right)\\
&= \frac{i}{2}\big(\hat{H}_0+\Delta\big)_{ij}\,\mathrm{Tr}_S\left(\left[\hat{z}_{\zeta,i}\hat{z}_{\zeta,j},\hat{O}\right]\hat{\rho}_{red}(t)\right)\\
&= \frac{i}{2}\big(\hat{H}_0+\Delta\big)_{ij}\Big\langle\left[\hat{z}_{\zeta,i}\hat{z}_{\zeta,j},\hat{O}\right]\Big\rangle.
\end{aligned} \tag{3.81}$$

- Second term:

$$\begin{aligned}
&\mathrm{Tr}_S\left(\hat{O}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t)\hat{z}_{\zeta,j}-\frac{1}{2}\hat{O}\Big\{\hat{z}_{\zeta,j}\hat{z}_{\zeta,i},\hat{\rho}_{red}(t)\Big\}\right) = \mathrm{Tr}_S\Big(\hat{O}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t)\hat{z}_{\zeta,j}-\frac{1}{2}\hat{O}\hat{z}_{\zeta,j}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t)\\
&-\frac{1}{2}\hat{O}\hat{\rho}_{red}(t)\hat{z}_{\zeta,j}\hat{z}_{\zeta,i}\Big) = \mathrm{Tr}_S\Big(\frac{1}{2}\hat{O}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t)\hat{z}_{\zeta,j}\frac{1}{2}\hat{O}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t)\hat{z}_{\zeta,j}-\frac{1}{2}\hat{O}\hat{z}_{\zeta,j}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t)\\
&-\frac{1}{2}\hat{O}\hat{\rho}_{red}(t)\hat{z}_{\zeta,j}\hat{z}_{\zeta,i}\Big) = \frac{1}{2}\mathrm{Tr}_S\Big(\hat{z}_{\zeta,j}\hat{O}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t)+\hat{z}_{\zeta,j}\hat{O}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t)-\hat{O}\hat{z}_{\zeta,j}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t)\\
&-\hat{z}_{\zeta,j}\hat{z}_{\zeta,i}\hat{O}\hat{\rho}_{red}(t)\Big) = \frac{1}{2}\mathrm{Tr}_S\left(\hat{z}_{\zeta,j}\left[\hat{O},\hat{z}_{\zeta,i}\right]\hat{\rho}_{red}(t)\right)+\frac{1}{2}\mathrm{Tr}_S\left(\left[\hat{z}_{\zeta,j},\hat{O}\right]\hat{z}_{\zeta,i}\hat{\rho}_{red}(t)\right)\\
&= \frac{1}{2}\left\langle\hat{z}_{\zeta,j}\left[\hat{O},\hat{z}_{\zeta,i}\right]\right\rangle+\frac{1}{2}\left\langle\left[\hat{z}_{\zeta,j},\hat{O}\right],\hat{z}_{\zeta,i}\right\rangle.
\end{aligned} \tag{3.82}$$

So putting together these results, we get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\langle\hat{O}(t)\big\rangle = \frac{i}{2}\left[\hat{H}_0(t)+\Delta(t)\right]_{ij}\Big\langle\left[\hat{z}_{\zeta,i}\hat{z}_{\zeta,j},\hat{O}\right]\Big\rangle+\gamma_{ij}(t)\left[\frac{1}{2}\Big\langle\left[\hat{z}_{\zeta,j},\hat{O}\right]\hat{z}_{\zeta,i}\Big\rangle+\frac{1}{2}\Big\langle\hat{z}_{\zeta,j}\left[\hat{O},\hat{z}_{\zeta,i}\right]\Big\rangle\right]. \tag{3.83}$$

If we consider as observable the covariance matrix:

$$\Sigma_{ab}(t) = \frac{1}{2}\mathrm{Tr}_S\left[\Big\{\hat{z}_{\zeta,a},\hat{z}_{\zeta,b}\Big\}\hat{\rho}_{red}(t)\right], \tag{3.84}$$

and using the canonical commutation relations $\left[\hat{z}_{\zeta,l},\hat{z}_{\zeta,a}\right]=i\omega_{la}$, leading to:

$$\begin{aligned}
\left[\hat{z}_{\zeta,l},\frac{1}{2}\Big\{\hat{z}_{\zeta,a},\hat{z}_{\zeta,b}\Big\}\right] &= \frac{1}{2}\left[\hat{z}_{\zeta,l},\hat{z}_{\zeta,a}\hat{z}_{\zeta,b}\right]+\frac{1}{2}\left[\hat{z}_{\zeta,l},\hat{z}_{\zeta,b}\hat{z}_{\zeta,a}\right]\\
&= \frac{1}{2}\Big\{\hat{z}_{\zeta,a}\left[\hat{z}_{\zeta,l},\hat{z}_{\zeta,b}\right]+\left[\hat{z}_{\zeta,l},\hat{z}_{\zeta,a}\right]\hat{z}_{\zeta,b}+\hat{z}_{\zeta,b}\left[\hat{z}_{\zeta,l},\hat{z}_{\zeta,a}\right]+\left[\hat{z}_{\zeta,l},\hat{z}_{\zeta,b}\right]\hat{z}_{\zeta,a}\Big\}\\
&= i\omega_{lb}\hat{z}_{\zeta,a}+i\omega_{la}\hat{z}_{\zeta,b}.
\end{aligned} \tag{3.85}$$

So differentiating the covariance matrix and using the master equation:

$$\frac{\mathrm{d}\Sigma}{\mathrm{d}t} = \frac{1}{2}\,\mathrm{Tr}_S\left[\left\{\hat{z}_{\zeta,a}, \hat{z}_{\zeta,b}\right\}\frac{\mathrm{d}\hat{\rho}_{red}(t)}{\mathrm{d}t}\right]. \tag{3.86}$$

Consider again term by term:

- First term (the complete computation is shown in appendix A.1):

$$-\frac{i}{2}\,\mathrm{Tr}_S\left[\left\{\hat{z}_{\zeta,a}, \hat{z}_{\zeta,b}\right\}\left[\hat{H}_S(t) + \hat{H}_{LS}(t), \hat{\rho}_{red}(t)\right]\right] = \;= \boldsymbol{\omega}(\hat{\boldsymbol{H}}_0 + \boldsymbol{\Delta})\boldsymbol{\Sigma} - \boldsymbol{\Sigma}(\hat{\boldsymbol{H}}_0 + \boldsymbol{\Delta})\boldsymbol{\omega}. \tag{3.87}$$

- Second term

$$\begin{aligned}
&\frac{1}{2}\,\mathrm{Tr}_S\left[\left\{\hat{z}_{\zeta,a}, \hat{z}_{\zeta,b}\right\}D_{ij}\left(\hat{z}_{\zeta,i}\hat{\rho}_{red}(t)\hat{z}_{\zeta,j} - \frac{1}{2}\left\{\hat{z}_{\zeta,j}\hat{z}_{\zeta,i}, \hat{\rho}_{red}(t)\right\}\right)\right]\\
&= \frac{1}{2}D_{ij}\,\mathrm{Tr}_S\left[\hat{z}_{\zeta,a}\hat{z}_{\zeta,b}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t)\hat{z}_{\zeta,j} - \frac{1}{2}\hat{z}_{\zeta,a}\hat{z}_{\zeta,b}\hat{z}_{\zeta,j}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t) - \frac{1}{2}\hat{z}_{\zeta,a}\hat{z}_{\zeta,b}\hat{\rho}_{red}(t)\hat{z}_{\zeta,j}\hat{z}_{\zeta,i} + (a \leftrightarrow b)\right]\\
&= \frac{1}{2}D_{ij}\,\mathrm{Tr}_S\left[\hat{z}_{\zeta,j}\hat{z}_{\zeta,a}\hat{z}_{\zeta,b}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t) - \frac{1}{2}\hat{z}_{\zeta,a}\hat{z}_{\zeta,b}\hat{z}_{\zeta,j}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t) - \frac{1}{2}\hat{z}_{\zeta,j}\hat{z}_{\zeta,i}\hat{z}_{\zeta,a}\hat{z}_{\zeta,b}\hat{\rho}_{red}(t)\right.\\
&\left. + \hat{z}_{\zeta,j}\hat{z}_{\zeta,b}\hat{z}_{\zeta,a}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t) - \frac{1}{2}\hat{z}_{\zeta,b}\hat{z}_{\zeta,a}\hat{z}_{\zeta,j}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t) - \frac{1}{2}\hat{z}_{\zeta,j}\hat{z}_{\zeta,i}\hat{z}_{\zeta,b}\hat{z}_{\zeta,a}\hat{\rho}_{red}(t)\right]\\
&= \frac{1}{2}D_{ij}\,\mathrm{Tr}_S\left[\hat{z}_{\zeta,j}\left\{\hat{z}_{\zeta,a}, \hat{z}_{\zeta,b}\right\}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t) - \frac{1}{2}\left\{\hat{z}_{\zeta,a}, \hat{z}_{\zeta,b}\right\}\hat{z}_{\zeta,j}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t) - \frac{1}{2}\hat{z}_{\zeta,j}\hat{z}_{\zeta,i}\left\{\hat{z}_{\zeta,a}, \hat{z}_{\zeta,b}\right\}\hat{\rho}_{red}(t)\right]\\
&= \frac{1}{4}D_{ij}\,\mathrm{Tr}_S\left[\hat{z}_{\zeta,j}\left\{\hat{z}_{\zeta,a}, \hat{z}_{\zeta,b}\right\}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t) + \hat{z}_{\zeta,j}\left\{\hat{z}_{\zeta,a}, \hat{z}_{\zeta,b}\right\}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t)\right.\\
&\left. - \left\{\hat{z}_{\zeta,a}, \hat{z}_{\zeta,b}\right\}\hat{z}_{\zeta,j}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t) - \hat{z}_{\zeta,j}\hat{z}_{\zeta,i}\left\{\hat{z}_{\zeta,a}, \hat{z}_{\zeta,b}\right\}\hat{\rho}_{red}(t)\right]\\
&= \frac{1}{2}D_{ij}\,\mathrm{Tr}_S\left[\left[\hat{z}_{\zeta,j}, \frac{1}{2}\left\{\hat{z}_{\zeta,a}, \hat{z}_{\zeta,b}\right\}\right]\hat{z}_{\zeta,i}\hat{\rho}_{red}(t) + \hat{z}_{\zeta,j}\underbrace{\left[\frac{1}{2}\left\{\hat{z}_{\zeta,a}, \hat{z}_{\zeta,b}\right\}, \hat{z}_{\zeta,i}\right]}_{-\left[\hat{z}_{\zeta,i}, \frac{1}{2}\left\{\hat{z}_{\zeta,a}, \hat{z}_{\zeta,b}\right\}\right]}\hat{\rho}_{red}(t)\right]\\
&= \frac{1}{2}D_{ij}\,\mathrm{Tr}_S\left[i\omega_{ja}\hat{z}_{\zeta,b}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t) + i\omega_{jb}\hat{z}_{\zeta,a}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t) - i\underbrace{\omega_{ia}\hat{z}_{\zeta,j}\hat{z}_{\zeta,b}}_{\omega_{ja}\hat{z}_{\zeta,i}\hat{z}_{\zeta,b}}\hat{\rho}_{red}(t) - i\underbrace{\omega_{ib}\hat{z}_{\zeta,j}\hat{z}_{\zeta,a}}_{\omega_{jb}\hat{z}_{\zeta,i}\hat{z}_{\zeta,a}}\hat{\rho}_{red}(t)\right]\\
&= \frac{i}{2}D_{ij}\,\mathrm{Tr}_S\left[\omega_{ja}\left[\hat{z}_{\zeta,b}, \hat{z}_{\zeta,i}\right]\hat{\rho}_{red}(t) + \omega_{jb}\left[\hat{z}_{\zeta,a}, \hat{z}_{\zeta,i}\right]\right]\\
&= -\frac{1}{2}D_{ij}\,\mathrm{Tr}_S\left[\omega_{ja}\omega_{bi}\hat{\rho}_{red}(t) + \omega_{jb}\omega_{ai}\hat{\rho}_{red}(t)\right]\\
&= -\frac{1}{2}D_{ij}\left[\omega_{ja}\omega_{bi} + \omega_{jb}\omega_{ai}\right]\underbrace{\mathrm{Tr}_S\left[\hat{\rho}_{red}(t)\right]}_{1}\\
&= -\frac{1}{2}D_{ij}\left[\underbrace{\omega_{ia}\omega_{bj}}_{\omega_{ai}\omega_{jb}} + \omega_{jb}\omega_{ai}\right] = -\omega_{ai}D_{ij}\omega_{jb} = -\boldsymbol{\omega}\boldsymbol{D}\boldsymbol{\omega}.
\end{aligned} \tag{3.88}$$

- Third term: for the last term we can follow the same procedure using the antisymmetry of ω_{ij} instead of the symmetry of D_{ij}

$$
\begin{aligned}
&- \frac{i}{2}\Delta_{12}\omega_{ij}\,\mathrm{Tr}_S\left[i\omega_{ja}\hat{z}_{\zeta,b}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t) + i\omega_{jb}\hat{z}_{\zeta,a}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t) - i\omega_{ia}\hat{z}_{\zeta,j}\hat{z}_{\zeta,b}\hat{\rho}_{red}(t) - i\omega_{ib}\hat{z}_{\zeta,j}\hat{z}_{\zeta,a}\hat{\rho}_{red}(t)\right] \\
&= -\frac{i}{2}\Delta_{12}\omega_{ij}\,\mathrm{Tr}_S\left[i\omega_{ja}\hat{z}_{\zeta,b}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t) + i\omega_{jb}\hat{z}_{\zeta,a}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t) + i\omega_{ja}\hat{z}_{\zeta,i}\hat{z}_{\zeta,b}\hat{\rho}_{red}(t) + i\omega_{jb}\hat{z}_{\zeta,i}\hat{z}_{\zeta,a}\hat{\rho}_{red}(t)\right] \\
&= \frac{1}{2}\Delta_{12}\omega_{ij}\,\mathrm{Tr}_S\left[\omega_{ja}\Big\{\hat{z}_{\zeta,i},\hat{z}_{\zeta,b}\Big\}\hat{\rho}_{red}(t) + \omega_{jb}\Big\{\hat{z}_{\zeta,i},\hat{z}_{\zeta,a}\Big\}\hat{\rho}_{red}(t)\right] \\
&= \Delta_{12}\,\mathrm{Tr}_S\left[\underbrace{\omega_{ij}\omega_{ja}}_{\mathbb{1}}\underbrace{\frac{1}{2}\Big\{\hat{z}_{\zeta,i},\hat{z}_{\zeta,b}\Big\}\hat{\rho}_{red}(t)}_{\Sigma} + \underbrace{\omega_{ij}\omega_{jb}}_{\mathbb{1}}\underbrace{\frac{1}{2}\Big\{\hat{z}_{\zeta,i},\hat{z}_{\zeta,a}\Big\}\hat{\rho}_{red}(t)}_{\Sigma}\right] \\
&= 2\Delta_{12}\boldsymbol{\Sigma}.
\end{aligned}
\tag{3.89}
$$

Finally, putting together all the terms, we get the transport equation for the covariance matrix:

$$
\frac{\mathrm{d}\Sigma}{\mathrm{d}t} = \underbrace{\boldsymbol{\omega}(\boldsymbol{H}_0+\boldsymbol{\Delta})\boldsymbol{\Sigma} - \boldsymbol{\Sigma}(\boldsymbol{H}_0+\boldsymbol{\Delta})\boldsymbol{\omega}}_{\text{UNITARY EVOLUTION}} - \underbrace{\boldsymbol{\omega}\boldsymbol{D}\boldsymbol{\omega}}_{\text{DECOHERENCE}} + \underbrace{2\Delta_{12}\boldsymbol{\Sigma}}_{\text{DISSIPATION}} . \tag{3.90}
$$

As in the TCL_2 master equation we have a unitary term, giving the renormalization of the energy levels, a term proportional to Δ_{12} taking into account the dissipation process and a term proportional to $\mathbf{D}$ characterizing the decoherence. In case of gaussian systems, the covariance matrix contains all the information about the quantum state, for example it allows us to access to the curvature perturbation power spectrum, simply through $\Sigma_{11} = \langle\hat{\zeta}^2\rangle$.

Beyond the gaussian case, non-linearities prevent the set of coupled equations to close [3]. This is a general problem, related to the non-integrability of non-linear system in absence of symmetries. Just to sketch the problem, in a non-gaussian system the dynamical equation of the n-point function requires the knowledge of the $n+1$-functions, indeed:

$$
\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{\zeta}^n\rangle = F\Big(\langle\hat{\zeta}^n\rangle\Big) + G\Big(\langle\hat{\zeta}^{n+1}\rangle\Big), \tag{3.91}
$$

where F and G are functions which depend on the details of the dynamics, so we need to know the correlators at all order to fully specify the quantum state, then the covariance matrix, which includes only the two-point functions, is not enough. In order for the system of differential equation to close, one must perform an approximation scheme to express $G\Big(\langle\hat{\zeta}^{n+1}\rangle\Big)$ in terms of the lower order statistics.

3.4.7 Lindblad equation

In this section we want to derive the most used master equation: the Lindblad equation. We will highlight the assumptions it is based on, stressing the differences with the non-Markovian master equation derived so far. We will follow closely the derivation done in [28].
Consider again the full density matrix ρ in interaction picture and its exact evolution equation:

$$
\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}t} = -ig\Big[\tilde{H}_{int}, \tilde{\rho(t)}\Big], \tag{3.92}
$$

which can be formally integrated as:

$$
\tilde{\rho}(t+\Delta t) = \tilde{\rho}(t) - ig\int_t^{t+\Delta t}\mathrm{d}t'\Big[\tilde{H}_{int}(t'), \tilde{\rho}(t')\Big]. \tag{3.93}
$$

Using the Born approximation we can iteratively expand the density matrix on the RHS, obtaining at order $\mathcal{O}(g^2)$:

$$
\begin{aligned}
\tilde{\rho}(t+\Delta t) - \tilde{\rho}(t) = &-ig\int_t^{t+\Delta t}\mathrm{d}t'\Big[\tilde{H}_{int}(t'), \tilde{\rho}(t)\Big] \\
&- g^2\int_t^{t+\Delta t}\mathrm{d}t'\int_t^{t'}\mathrm{d}t''\Big[\tilde{H}_{int}(t'), \Big[\tilde{H}_{int}(t''), \tilde{\rho}(t'')\Big]\Big] + \mathcal{O}(g^3).
\end{aligned}
\tag{3.94}
$$

From now on we will drop the $\mathcal{O}(g^3)$, assuming we are working at order $\mathcal{O}(g^2)$. Tracing out the environment degrees of freedom we obtain the reduced density matrix of the system:

$$\tilde{\rho}_{red}(t) \equiv \mathrm{Tr}_{\varepsilon}[\tilde{\rho}(t)]. \tag{3.95}$$

Tracing out equation (3.94) we obtain:

$$\begin{aligned} \tilde{\rho}_{red}(t+\Delta t) - \tilde{\rho}_{red}(t) = &-ig \int_t^{t+\Delta t} \mathrm{d}t' \, \mathrm{Tr}_{\varepsilon} \left[\tilde{H}_{int}(t'), \tilde{\rho}(t) \right] \\ &- g^2 \int_t^{t+\Delta t} \mathrm{d}t' \int_t^{t'} \mathrm{d}t'' \, \mathrm{Tr}_{\varepsilon} \left[\tilde{H}_{int}(t'), \left[\tilde{H}_{int}(t''), \tilde{\rho}(t'') \right] \right]. \end{aligned} \tag{3.96}$$

Analogously to (3.95) we can define a reduced density matrix for the environment, as:

$$\tilde{\rho}_{\varepsilon}(t) \equiv \mathrm{Tr}_{S}[\tilde{\rho}(t)]. \tag{3.97}$$

It is important to stress that in general $\tilde{\rho}(t) \neq \tilde{\rho}_{red}(t) \otimes \tilde{\rho}_{\varepsilon}(t)$, but instead

$$\tilde{\rho}(t) = \tilde{\rho}_{red}(t) \otimes \tilde{\rho}_{\varepsilon}(t) + g^p \tilde{\rho}_{correl}(t), \tag{3.98}$$

where $\tilde{\rho}_{correl}$ describe the correlation between system and environment, with g appearing because if we start with $\tilde{\rho}_{correl}(t_0) = 0$ it will get different from zero only if the interaction is switched on. p is an unknown integer. Replacing (3.98) in (3.96) we obtain:

$$\begin{aligned} \tilde{\rho}_{red}(t+\Delta t) - \tilde{\rho}_{red}(t) = &-ig \int_t^{t+\Delta t} \mathrm{d}t' \, \mathrm{Tr}_{\varepsilon} \left[\tilde{H}_{int}(t'), \tilde{\rho}_{red}(t) \otimes \tilde{\rho}_{\varepsilon}(t) \right] \\ &- ig^{p+1} \int_t^{t+\Delta t} \mathrm{d}t' \, \mathrm{Tr}_{\varepsilon}[\tilde{H}_{int}(t'), \tilde{\rho}_{correl}(t)] \\ &- g^2 \int_t^{t+\Delta t} \mathrm{d}t' \int_t^{t'} \mathrm{d}t'' \, \mathrm{Tr}_{\varepsilon} \left[\tilde{H}_{int}(t'), \left[\tilde{H}_{int}(t''), \tilde{\rho}_{red}(t'') \otimes \tilde{\rho}_{\varepsilon}(t'') \right] \right] \\ &- g^{p+2} \int_t^{t+\Delta t} \mathrm{d}t' \int_t^{t'} \mathrm{d}t'' \, \mathrm{Tr}_{\varepsilon} \left[\tilde{H}_{int}(t'), \left[\tilde{H}_{int}(t''), \tilde{\rho}_{correl}(t'') \right] \right]. \end{aligned} \tag{3.99}$$

In order to understand if some of these terms can be neglected, we need to specify the interaction Hamiltonian. Assume it can be written as:

$$\tilde{H}_{int}(t) = \tilde{A}(t) \otimes \tilde{R}(t), \tag{3.100}$$

with $\tilde{A}$ acting on the system and $\tilde{R}$ acting on the environment Hilbert space. Consider the first term of (3.99):

$$\begin{aligned} &\mathrm{Tr}_{\varepsilon} \left[\tilde{H}_{int}(t'), \tilde{\rho}_{red}(t) \otimes \tilde{\rho}_{\varepsilon}(t) \right] = \mathrm{Tr}_{\varepsilon} \left[\tilde{A}(t') \otimes \tilde{R}(t'), \tilde{\rho}_{red}(t) \otimes \tilde{\rho}_{\varepsilon}(t) \right] \\ &= \tilde{A}(t') \tilde{\rho}_{red}(t) \, \mathrm{tr}_{\varepsilon} \left[\tilde{R}(t') \tilde{\rho}_{\varepsilon}(t) \right] - \tilde{\rho}_{red}(t) \tilde{A}(t') \, \mathrm{Tr}_{\varepsilon} \left[\tilde{\rho}_{\varepsilon}(t) \tilde{R}(t') \right] \\ &= \mathrm{Tr}_{\varepsilon} \left[\tilde{R}(t') \tilde{\rho}_{\varepsilon}(t) \right] \left[\tilde{A}(t'), \tilde{\rho}_{red}(t) \right]. \end{aligned} \tag{3.101}$$

Here we want to implement the approximations holding behind the Lindblad equation:

1. The system is small compared to the environment and it has no influence on the latter. So we can assume:

$$\tilde{\rho}_{\varepsilon}(t) \simeq \tilde{\rho}_{\varepsilon}^{(0)} \equiv \tilde{\rho}_{\varepsilon}, \tag{3.102}$$

   constant in interaction picture.

2. The environment is in a stationary state, i.e. H_{ε} does not depend explicitly on time, so $[\tilde{\rho}_{\varepsilon}, H_{\varepsilon}] = 0$. This implies that the evolution operator for the environment can be written as $U_{\varepsilon} = e^{-iH_{\varepsilon}(t)}$, then $[\tilde{\rho}_{\varepsilon}, U_{\varepsilon}] = 0$. This also implies that $\rho_{\varepsilon}(t) = e^{-iH_{\varepsilon}(t)} \tilde{\rho}_{\varepsilon} e^{iH_{\varepsilon}(t)}$ is time independent, so $\rho_{\varepsilon} = \tilde{\rho}_{\varepsilon}$. As a consequence $[\rho_{\varepsilon}, H_{\varepsilon}] = 0$ and the environment density operator can be decomposed as:

$$\tilde{\rho}_{\varepsilon} = \sum_n p_n \left| n \right\rangle \left\langle n \right|, \tag{3.103}$$

   with $|n\rangle$ eigenvectors of H_{ε} with eigenvalue E_n and p_n real constant coefficients.

3. The mean value of $R(t)$ vanishes:

$$\langle R \rangle = \mathrm{Tr}_\varepsilon(R\tilde{\rho}_\varepsilon) = 0, \tag{3.104}$$

then using the ciclicity of the trace and the fact that $\tilde{\rho}_\varepsilon$ commutes with U_ε we get:

$$\mathrm{Tr}_\varepsilon(\tilde{R}\tilde{\rho}_\varepsilon) = \mathrm{Tr}_\varepsilon\left(U_\varepsilon^\dagger R U_\varepsilon \tilde{\rho}_\varepsilon\right) = \mathrm{Tr}_\varepsilon\left(U_\varepsilon^\dagger R \tilde{\rho}_\varepsilon U_\varepsilon\right) = \mathrm{Tr}_\varepsilon\left(U_\varepsilon U_\varepsilon^\dagger R \tilde{\rho}_\varepsilon\right) = \mathrm{Tr}_\varepsilon(R\tilde{\rho}_\varepsilon) = 0. \tag{3.105}$$

This implies that (3.101) vanishes.

Now we are able to determine the value of the integer p in equation (3.99). At leading order in g the LHS of equation (3.99), $\tilde{\rho}_{red}(t+\Delta t) - \tilde{\rho}_{red}(t)$ must be proportional to $g^p \tilde{\rho}_{correl}$ since in absence of the interaction term $\tilde{\rho}_{red}$ does not evolve in interaction picture. So the LHS is of order p, while on the RHS we have terms of order 2, $p+1$ and $p+2$. The only possibility that allows us to recognize a dominant term on the RHS is that $p=2$, so that the terms of order $p+1$ and $p+2$ are subdominant and can be neglected. Then finally:

$$\tilde{\rho}_{red}(t+\Delta t) - \tilde{\rho}_{red}(t) = -g^2 \int_t^{t+\Delta t} \mathrm{d}t' \int_t^{t'} \mathrm{d}t'' \, \mathrm{Tr}_\varepsilon \left[\tilde{H}_{int}(t'), \left[\tilde{H}_{int}(t''), \tilde{\rho}_{red}(t'') \otimes \tilde{\rho}_\varepsilon(t'')\right]\right]. \tag{3.106}$$

This holds at leading order so we need a fourth assumption, i.e. the interaction modifies the dynamics of the system in the perturbative regime only. Let us now evaluate the remaining term:

$$\begin{aligned}
&\mathrm{Tr}_\varepsilon \left[\tilde{H}_{int}(t'), \left[\tilde{H}_{int}(t''), \tilde{\rho}_{red}(t'') \otimes \tilde{\rho}_\varepsilon(t'')\right]\right] = \mathrm{Tr}_\varepsilon \left[\tilde{A}(t') \otimes \tilde{R}(t'), \left[\tilde{A}(t'') \otimes \tilde{R}(t''), \tilde{\rho}_{red}(t'') \otimes \tilde{\rho}_\varepsilon(t'')\right]\right] \\
&= \tilde{A}(t')\tilde{A}(t'')\tilde{\rho}_{red}(t'')\,\mathrm{Tr}_\varepsilon\left[\tilde{R}(t')\tilde{R}(t'')\tilde{\rho}_\varepsilon\right] - \tilde{A}(t'')\tilde{\rho}_{red}(t'')\tilde{A}(t')\,\mathrm{Tr}_\varepsilon\left[\tilde{R}(t'')\tilde{\rho}_\varepsilon\tilde{R}(t')\right] \\
&- \tilde{A}(t')\tilde{\rho}_{red}(t'')\tilde{A}(t'')\,\mathrm{Tr}_\varepsilon\left[\tilde{R}(t')\tilde{\rho}_\varepsilon\tilde{R}(t'')\right] + \tilde{\rho}_{red}(t'')\tilde{A}(t'')\tilde{A}(t')\,\mathrm{Tr}_\varepsilon\left[\tilde{\rho}_\varepsilon\tilde{R}(t'')\tilde{R}(t')\right] \\
&= \tilde{A}(t')\tilde{A}(t'')\tilde{\rho}_{red}(t'')\,\mathrm{Tr}_\varepsilon\left[\tilde{\rho}_\varepsilon\tilde{R}(t')\tilde{R}(t'')\right] - \tilde{A}(t'')\tilde{\rho}_{red}(t'')\tilde{A}(t')\,\mathrm{Tr}_\varepsilon\left[\tilde{\rho}_\varepsilon\tilde{R}(t')\tilde{R}(t'')\right] \\
&- \tilde{A}(t')\tilde{\rho}_{red}(t'')\tilde{A}(t'')\,\mathrm{Tr}_\varepsilon\left[\tilde{\rho}_\varepsilon\tilde{R}(t'')\tilde{R}(t')\right] + \tilde{\rho}_{red}(t'')\tilde{A}(t'')\tilde{A}(t')\,\mathrm{Tr}_\varepsilon\left[\tilde{\rho}_\varepsilon\tilde{R}(t'')\tilde{R}(t')\right] \\
&= \left[\tilde{A}(t')\tilde{A}(t'')\tilde{\rho}_{red}(t'') - \tilde{A}(t'')\tilde{\rho}_{red}(t'')\tilde{A}(t')\right] C_R(t'-t'') \\
&- \left[\tilde{A}(t')\tilde{\rho}_{red}(t'')\tilde{A}(t'') - \tilde{\rho}_{red}(t'')\tilde{A}(t'')\tilde{A}(t')\right] C_R(t''-t'),
\end{aligned} \tag{3.107}$$

where we defined:

$$C_R(t,t') \equiv \mathrm{Tr}_\varepsilon\left[\tilde{\rho}_\varepsilon\tilde{R}(t)\tilde{R}(t')\right], \tag{3.108}$$

where due to the fact that the environment is a stationary state, one can show that $C_R(t,t')$ is a function of $\tau \equiv t-t'$ only:

$$\begin{aligned}
C_R(t,t') &= \mathrm{Tr}_\varepsilon\left[\tilde{\rho}_\varepsilon e^{iH_\varepsilon t}\tilde{R}(0)e^{-iH_\varepsilon t}e^{iH_\varepsilon t'}\tilde{R}(0)e^{-iH_\varepsilon t'}\right] \\
&= \mathrm{Tr}_\varepsilon\left[\tilde{\rho}_\varepsilon e^{iH_\varepsilon t}e^{-iH_\varepsilon t'}e^{iH_\varepsilon t'}\tilde{R}(0)e^{-iH_\varepsilon \tau}\tilde{R}(0)e^{-iH_\varepsilon t'}\right] \\
&= \mathrm{Tr}_\varepsilon\left[\tilde{\rho}_\varepsilon e^{iH_\varepsilon \tau}e^{iH_\varepsilon t'}\tilde{R}(0)e^{-iH_\varepsilon \tau}\tilde{R}(0)e^{-iH_\varepsilon t'}\right] \\
&= \mathrm{Tr}_\varepsilon\left[\tilde{\rho}_\varepsilon e^{iH_\varepsilon t'}e^{iH_\varepsilon \tau}\tilde{R}(0)e^{-iH_\varepsilon \tau}\tilde{R}(0)e^{-iH_\varepsilon t'}\right] \\
&= \mathrm{Tr}_\varepsilon\left[e^{iH_\varepsilon t'}\tilde{\rho}_\varepsilon\tilde{R}(\tau)\tilde{R}(0)e^{-iH_\varepsilon t'}\right] \\
&= \mathrm{Tr}_\varepsilon\left[\tilde{\rho}_\varepsilon\tilde{R}(\tau)\tilde{R}(0)\right] = C_R(\tau).
\end{aligned} \tag{3.109}$$

Recalling equation (3.103):

$$\begin{aligned}
C_R(\tau) &= \mathrm{Tr}_\varepsilon\left[\tilde{\rho}_\varepsilon \tilde{R}(\tau)\tilde{R}(0)\right] = \sum_m \langle m| \left[\sum_n p_n |n\rangle \langle n| \tilde{R}(\tau)\tilde{R}(0)\right] |m\rangle \\
&= \sum_{m,n} p_n \underbrace{\langle m|n\rangle}_{\delta_{mn}} \langle n| \tilde{R}(\tau)\tilde{R}(0) |m\rangle = \sum_n p_n \langle n| \tilde{R}(\tau)\tilde{R}(0) |n\rangle \\
&= \sum_n p_n \langle n| e^{iH_\varepsilon \tau} \tilde{R}(0) e^{-iH_\varepsilon \tau} \tilde{R}(0) |n\rangle \\
&= \sum_{n,m,p,q} p_n \underbrace{\langle n| e^{iH_\varepsilon \tau} |m\rangle}_{e^{iE_n\tau}} \langle m| \tilde{R}(0) |p\rangle \underbrace{\langle p| e^{-iH_\varepsilon \tau} |q\rangle}_{e^{-iE_p\tau}} \langle q| \tilde{R}(0) |n\rangle \\
&= \sum_{n,p} p_n e^{i(E_n - E_p)\tau} \langle n| \tilde{R}(0) |p\rangle \langle p| \tilde{R}(0) |n\rangle \\
&= \sum_{n,p} p_n e^{i(E_n - E_p)\tau} \left| \langle n| \tilde{R}(0) |p\rangle \right|^2 .
\end{aligned} \tag{3.110}$$

In particular $C_R(-\tau) = C_R^*(\tau)$. We see that $C_R(\tau)$ is a sum of exponential oscillating with frequencies $E_n - E_p$. In the limit where the environment is large and has an almost continuous spectrum of energy levels, the oscillating exponentials sum in a destructive way, rapidly driving $C_R(\tau)$ to zero with a characteristic time t_c, such that $C_R(\tau) \simeq C_R(0)e^{-|\tau|/t_c}$. Using our result, equation (3.106) becomes:

$$\begin{aligned}
&\int_t^{t+\Delta t} \mathrm{d}t' \int_t^{t'} \mathrm{d}t'' \,\mathrm{Tr}_\varepsilon \left[\tilde{H}_{int}(t'), \left[\tilde{H}_{int}(t''), \tilde{\rho}_{red}(t'') \otimes \tilde{\rho}_\varepsilon(t'')\right]\right] = \\
&= \int_t^{t+\Delta t} \mathrm{d}t' \int_t^{t'} \mathrm{d}t'' \Big\{ \left[\tilde{A}(t')\tilde{A}(t'')\tilde{\rho}_{red}(t'') - \tilde{A}(t'')\tilde{\rho}_{red}(t'')\tilde{A}(t')\right] C_R(t' - t'') \\
&- \left[\tilde{A}(t')\tilde{\rho}_{red}(t'')\tilde{A}(t'') - \tilde{\rho}_{red}(t'')\tilde{A}(t'')\tilde{A}(t')\right] C_R(t'' - t') \Big\}.
\end{aligned} \tag{3.111}$$

We can reparametrize the integration domain in terms of t' and $\tau = t' - t''$ as:

$$\int_t^{t+\Delta t} \mathrm{d}t' \int_t^{t'} \mathrm{d}t'' = \int_0^{\Delta t} \mathrm{d}\tau \int_{t+\tau}^{t+\Delta t} \mathrm{d}t'. \tag{3.112}$$

It can be shown [28] that the extended integration domain

$$\int_0^\infty \mathrm{d}\tau \int_t^{t+\Delta t} \mathrm{d}t', \tag{3.113}$$

is a good approximation of the previous one in the limit $t_c \ll \Delta t$, so when the environment dynamics evolve in a much shorter time than the system dynamics. This is the fifth and last assumption in the derivation of the Lindblad equation. Then we obtain:

$$\begin{aligned}
\tilde{\rho}_{red}(t + \Delta t) - \tilde{\rho}_{red}(t) \simeq -g^2 \int_0^\infty \mathrm{d}\tau \int_t^{t+\Delta t} \mathrm{d}t' \Big\{ &\left[\tilde{A}(t')\tilde{A}(t' - \tau)\tilde{\rho}_{red}(t' - \tau) - \tilde{A}(t' - \tau)\tilde{\rho}_{red}(t' - \tau)\tilde{A}(t')\right] \\
\times C_R(\tau) - &\left[\tilde{A}(t')\tilde{\rho}_{red}(t' - \tau)\tilde{A}(t' - \tau) - \tilde{\rho}_{red}(t' - \tau)\tilde{A}(t' - \tau)\tilde{A}(t')\right] C_R(-\tau) \Big\}.
\end{aligned} \tag{3.114}$$

The time derivative of $\tilde{\rho}_{red}$ is obtained by dividing both sides by Δt. If Δt is much smaller than the typical time scale of A we can approximate $\tilde{A}(t') \simeq \tilde{A}(t)$ and $\tilde{A}(t' - \tau) \simeq \tilde{A}(t - \tau)$. Given the fifth assumption, $t_c \ll \Delta t$ this means that A should vary on scales much larger than the autocorrelation time of the environment. Furthermore, the variation of $\tilde{\rho}_{red}$ between time t and $t + \Delta t$ is of order g^2. Since the RHS is already of order g^2, we can evaluate $\tilde{\rho}_{red}$ on the RHS as $\tilde{\rho}_{red}(t' - \tau) \simeq \tilde{\rho}_{red}(t)$. In this way the integral over t' becomes trivial and we obtain:

$$\begin{aligned}
\frac{\Delta \tilde{\rho}_{red}}{\Delta t} = -g^2 \int_0^\infty \mathrm{d}\tau \Big\{ &\left[\tilde{A}(t)\tilde{A}(t - \tau)\tilde{\rho}_{red}(t) - \tilde{A}(t - \tau)\tilde{\rho}_{red}(t)\tilde{A}(t)\right] C_R(\tau) \\
- &\left[\tilde{A}(t)\tilde{\rho}_{red}(t)\tilde{A}(t - \tau) - \tilde{\rho}_{red}(t)\tilde{A}(t - \tau)\tilde{A}(t)\right] C_R(-\tau) \Big\}.
\end{aligned} \tag{3.115}$$

We see that the RHS depends only on t, consequence of the Markovian nature of this equation. We can write this equation in a compact way by defining:

$$\begin{aligned} L_1(t) &\equiv g^2 \int_0^\infty \mathrm{d}\tau C_R(\tau)\tilde{A}(t-\tau) \\ L_2(t) &\equiv g^2 \int_0^\infty \mathrm{d}\tau C_R(-\tau)\tilde{A}(t-\tau) = g^2 \int_0^\infty \mathrm{d}\tau C_R^*(\tau)\tilde{A}(t-\tau) = L_1^\dagger(t), \end{aligned} \tag{3.116}$$

where we assumed $\tilde{A}$ hermitian. Then the master equation becomes:

$$\frac{\Delta\tilde{\rho}_{red}}{\Delta t} = -\tilde{A}(t)L_1(t)\tilde{\rho}_{red}(t) + L_1(t)\tilde{\rho}_{red}(t)\tilde{A}(t) + \tilde{A}(t)\tilde{\rho}_{red}(t)L_2(t) - \tilde{\rho}_{red}(t)L_2(t)\tilde{A}(t). \tag{3.117}$$

We can further simplify the operators L_1 and L_2 exploiting the condition $t_c \ll \Delta t$ and remembering that $C_R(\tau) = C_R(0)e^{-|\tau|/t_c}$ goes rapidly to zero, the integrals in the definition of L_1 and L_2 are dominated by the bottom of the integration interval. As we have said, in this short time $\tilde{A}$ does not evolve significantly and we can evaluate $\tilde{A}(t-\tau) \simeq \tilde{A}(t)$, so:

$$L_1(t) = g^2 \int_0^\infty \mathrm{d}\tau C_R(\tau)\tilde{A}(t-\tau) \simeq g^2 \int_0^\infty \mathrm{d}\tau C_R(0)e^{-\tau/t_c}\tilde{A}(t) = g^2 C_R(0)t_c\tilde{A}(t) = L_2(t). \tag{3.118}$$

Then our master equation (3.117) gets rewritten as:

$$\frac{\mathrm{d}\tilde{\rho}_{red}}{\mathrm{d}t} = -g^2 C_R(0)t_c\Big[\tilde{A}, \Big[\tilde{A}, \tilde{\rho}_{red}\Big]\Big]. \tag{3.119}$$

Going back to the standard picture (and using $i\frac{\mathrm{d}U_s}{\mathrm{d}t} = H_s U_s$):

$$\begin{aligned} \frac{\mathrm{d}\tilde{\rho}_{red}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(U_s^\dagger \rho_{red} U_s\Big) = -g^2 C_R(0)t_c\Big[U_s^\dagger A U_s, \Big[U_s^\dagger A U_s, U_s^\dagger \rho_{red} U_s\Big]\Big] \\ &\implies \frac{\mathrm{d}U_s^\dagger}{\mathrm{d}t}\rho_{red}U_s + U_s^\dagger \frac{\mathrm{d}\rho_{red}}{\mathrm{d}t}U_s + U_s^\dagger \rho_{red}\frac{\mathrm{d}U_s}{\mathrm{d}t} = -g^2 C_R(0)t_c U_s^\dagger[A,[A,\rho_{red}]]U_s \\ &\implies iU_s^\dagger H_s \rho_{red} U_s + U_s^\dagger \frac{\mathrm{d}\rho_{red}}{\mathrm{d}t}U_s - iU_s^\dagger \rho_{red} H_s U_s = -g^2 C_R(0)t_c U_s^\dagger[A,[A,\rho_{red}]]U_s \\ &\implies \boxed{\frac{\mathrm{d}\rho_{red}}{\mathrm{d}t} = i[\rho_{red}, H_s] - g^2 C_R(0)t_c[A,[A,\rho_{red}]]}. \end{aligned} \tag{3.120}$$

This is the standard form of the *Lindblad equation*. It can be generalized for an interaction Hamiltonian of the type:

$$H_{int} = \sum_i A_i(t) \otimes R_i(t), \tag{3.121}$$

with the environment correlators given by:

$$C_{R,ij}(t,t') \equiv \mathrm{Tr}_\varepsilon\left[\tilde{\rho}_\varepsilon \tilde{R}_i(t)\tilde{R}_j(t')\right], \tag{3.122}$$

and repeating all the previous steps we obtain:

$$\frac{\mathrm{d}\rho_s}{\mathrm{d}t} = i[\rho_s, H_s] - g^2 \sum_{i,j} C_{R,ij}(0)t_{c,ij}[A_i,[A_j\rho_s]], \tag{3.123}$$

where we assumed $C_{R,ij} = C_{R,ji}$ and $t_{c,ij}$ small for every i,j with respect to the typical time scale of the system.
If we consider continuous indexes:

$$H_{int} = \int \mathrm{d}^3\mathbf{x} A(t,\mathbf{x}) \otimes R(t,\mathbf{x}), \tag{3.124}$$

and we get:

$$\frac{\mathrm{d}\rho_s}{\mathrm{d}t} = -i[H_s,\rho_s] - \frac{\gamma}{2}\int \mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{y} C_R(\mathbf{x},\mathbf{y})[A(\mathbf{x}),[A(\mathbf{y})\rho_s]], \tag{3.125}$$

where $C_R(\mathbf{x},\mathbf{y})$ denotes $C_{R,ij}(0)$ and $\gamma \equiv 2g^2 t_c$.

3.5 Tracking quantum decoherence

OQS techniques allow us to to keep trace of the quantum information properties of the system we are considering, and this is fundamental to describe the quantum decoherence of the system. There are several criteria to assess the quantum nature of the system, as explained in [29] and here we will review some of them. First of all we need to understand what is the physical process behind quantum decoherence: the loss of coherence of a quantum system consists in the transfer of correlations initially contained within the system into its surrounding environment. To keep track of the evolution of the coherence of the system we can consider different quantities. One of these is the *entropy of entanglement*, S_{ent}, defined as the Von Neumann entropy of the system once the environment has been traced over:

$$S_{ent} \equiv -\operatorname{Tr}_S\left(\hat{\rho}_{red} \ln \hat{\rho}_{red}\right). \tag{3.126}$$

This is a measure of the degree of quantum entanglement between two subsystems costituing a composite quantum system (in our case system and environment). If it is non-vanishing the system and environment are entangled.
Another quantity, which will be the one we will use in the following is the *purity*:

$$\gamma \equiv \operatorname{Tr}_S\left(\hat{\rho}_{red}^2\right). \tag{3.127}$$

The purity defines a measure on quantum states, giving information on how much a state is mixed, with $\gamma = 1$ corresponding to a pure state and $\gamma = 0$ corresponding to a maximally mixed state. Analogously the *linear entropy* is defined as $S_{lin} = 1 - \gamma$. In the case of a gaussian state with a single degree of freedom, specifying the purity is enough to characterize all the others quantumness measures [29], indeed they are related through:

$$S_{ent} = \frac{1-\gamma}{2\gamma} \ln\left(\frac{1+\gamma}{1-\gamma}\right) - \ln\left(\frac{2\gamma}{1+\gamma}\right). \tag{3.128}$$

If the system is made of more than one degree of freedom this formula and the one-to-one correspondence between entanglement and linear entropy does not hold.

Once the master equation is determined it provides the evolution of $\hat{\rho}_{red}$ but in general this is not enough to reach a close system of equations and one needs to rely on further assumptions in order to obtain some results. However, in the case of a gaussian state, there is a simple relation linking the purity with the observables of the system, which are fully contained in the covariance matrix $\mathbf{\Sigma}$, that is:

$$\gamma(t) = \frac{1}{4 \det \mathbf{\Sigma}}. \tag{3.129}$$

Hence, the problem of accessing the quantum information properties of the system reduces to the one of solving the transport equations. One can then assess decoherence by keeping track of the transition of the purity from 1 to 0.
Decoherence is thought to erase all the quantum information which was present in the Early Universe, but some works have tried to find an observational imprint of this quantum origin in the Universe we observe today, e.g. [30].

3.5.1 Decoherence with Lindblad equation

In this section we want to find an evolution equation for the purity, using the Lindblad equation. We will present this method [28] for a generic interaction and in the following we will use it in our model with non-linear interaction. Here we will use the Heisenberg picture. In the linear case the following factorization of the reduced density matrix holds:

$$\hat{\rho}_{red}(\eta) = \prod_{\mathbf{k}\in\mathbb{R}^{3+}} \prod_{s=R,I} \hat{\rho}^s_{\mathbf{k}}(\eta), \tag{3.130}$$

but in the non-linear case this is no longer true. Anyway we can define the following effective density matrix defined on the Fourier space characterized by s and $\mathbf{k}$, obtained by tracing out all the other degrees of freedom:

$$\hat{\rho}^s_{\mathbf{k}} \equiv \mathrm{Tr}_{\left\{\substack{s' \\ \mathbf{k}' \neq \mathbf{k}}\right\}, \left\{\substack{\bar{s} \\ \mathbf{k}}\right\}} (\hat{\rho}_{red}), \tag{3.131}$$

with $\bar{s} = R$ if $s = I$ and vice versa. The purity is defined as:

$$\gamma = \mathrm{Tr}_{\left\{\substack{s \\ \mathbf{k}}\right\}} \left(\hat{\rho}^{s2}_{\mathbf{k}}\right). \tag{3.132}$$

Using the linearity of the trace operator we can find an evolution equation for the purity as:

$$\frac{\mathrm{d}}{\mathrm{d}\eta} \mathrm{Tr}_{\left\{\substack{s \\ \mathbf{k}}\right\}} \left(\hat{\rho}^{s2}_{\mathbf{k}}\right) = 2\, \mathrm{Tr}_{\left\{\substack{s \\ \mathbf{k}}\right\}} \left[\hat{\rho}^s_{\mathbf{k}}\, \mathrm{Tr}_{\left\{\substack{s' \\ \mathbf{k}' \neq \mathbf{k}}\right\}, \left\{\substack{\bar{s} \\ \mathbf{k}}\right\}} \frac{\mathrm{d}\hat{\rho}_{red}}{\mathrm{d}\eta}\right]. \tag{3.133}$$

Now consider the Lindblad equation with $\hat{A} = \hat{v}^2$ with $\hat{v}$ the Mukhanov-Sasaki variable:

$$\frac{\mathrm{d}\hat{\rho}_{red}}{\mathrm{d}\eta} = -i\left[\hat{H}_v, \hat{\rho}_{red}\right] - \frac{\gamma}{2} \int \mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{y} C_R(\mathbf{x}-\mathbf{y}) \left[\hat{v}^2(\mathbf{x}), \left[\hat{v}^2(\mathbf{y}), \hat{\rho}_{red}\right]\right]. \tag{3.134}$$

It is difficult to incorporate the Hamiltonian term in the following calculation and since it vanishes in the linear case we assume it is vanishing also in this case. Using the Fourier transform:

$$\hat{v}(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3\mathbf{k} \hat{v}_{\mathbf{k}}(\eta) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{3.135}$$

$$C_R(\mathbf{x}-\mathbf{y}) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3\mathbf{k} C_R(\mathbf{k}) e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}, \tag{3.136}$$

then the Lindblad equation in Fourier space becomes:

$$\frac{\mathrm{d}\hat{\rho}_{red}}{\mathrm{d}\eta} = -\frac{\gamma}{2}\frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3\mathbf{k}_1 \mathrm{d}^3\mathbf{k}_2 \mathrm{d}^3\mathbf{k}_3 C_R(\mathbf{k}_1) [\hat{v}_{\mathbf{k}_2}\hat{v}_{-\mathbf{k}_1-\mathbf{k}_2}, [\hat{v}_{\mathbf{k}_3}\hat{v}_{\mathbf{k}_1-\mathbf{k}_3}, \hat{\rho}_{red}]]. \tag{3.137}$$

Since the RHS is of order γ, at leading order in γ the trace can be evaluated in the free theory where we can use the factorization:

$$\hat{\rho}_{red}(\eta) = \prod_{\mathbf{k}\in\mathbb{R}^{3+}} \prod_{s=R,I} \hat{\rho}^s_{\mathbf{k}}(\eta). \tag{3.138}$$

If $\mathbf{k}$ is different from all the other combinations of momenta, we can remove $\hat{\rho}^s_{\mathbf{k}}$ from the trace in (3.133) and we get a full trace that vanishes. Also in the case in which $\mathbf{k}$ is equal to only one wavenumber we get a vanishing result [28], so we need to consider the cases where $\mathbf{k}$ is equal at least to two wavenumbers.

1. **k equal to two wavenumbers:**
   Consider the case $\mathbf{k} = \mathbf{k}_2 = -\mathbf{k}_3$ and $\mathbf{k}_1 \neq 2\mathbf{k}$ and take the second trace inside equation (3.133):

$$\begin{aligned}
\mathrm{Tr}_{\left\{\substack{s' \\ \mathbf{k}' \neq \mathbf{k}}\right\}, \left\{\substack{\bar{s} \\ \mathbf{k}}\right\}} \frac{\mathrm{d}\hat{\rho}_{red}}{\mathrm{d}\eta} &= -\frac{\gamma}{2}\frac{1}{(2\pi)^{3/2}} \int d^3\mathbf{k_1} \mathrm{d}^3\mathbf{k_2} \mathrm{d}^3\mathbf{k_3} C_R(\mathbf{k}_1) \times \\
&\times \mathrm{Tr}_{\left\{\substack{s' \\ \mathbf{k}' \neq \mathbf{k}}\right\}, \left\{\substack{\bar{s} \\ \mathbf{k}}\right\}} \left(\left[\hat{v}_{\mathbf{k}_2}\hat{v}_{-\mathbf{k}_1-\mathbf{k}_2}, \left[\hat{v}_{\mathbf{k}_3}\hat{v}_{\mathbf{k}_1-\mathbf{k}_3}, \prod_{\mathbf{k}}\prod_{\mathbf{s}} \hat{\rho}^s_{\mathbf{k}}\right]\right]\right) \\
&= -\frac{\gamma}{2}\frac{1}{(2\pi)^{3/2}} \int d^3\mathbf{k_1} \mathrm{d}^3\mathbf{k_2} \mathrm{d}^3\mathbf{k_3} C_R(\mathbf{k}_1) \times \\
&\times \mathrm{Tr}_{\left\{\substack{s' \\ \mathbf{k}' \neq \mathbf{k}}\right\}, \left\{\substack{\bar{s} \\ \mathbf{k}}\right\}} \left(\left[\hat{v}_{\mathbf{k}}\hat{v}_{-\mathbf{k}-\mathbf{k}_1}, \left[\hat{v}_{-\mathbf{k}}\hat{v}_{\mathbf{k}+\mathbf{k}_1}, \prod_{\mathbf{k}}\prod_{\mathbf{s}} \hat{\rho}^s_{\mathbf{k}}\right]\right]\right).
\end{aligned} \tag{3.139}$$

Here the only $\mathbf{k'} \neq \mathbf{k}$ is $\mathbf{k}+\mathbf{k}_1$, so we have $\prod_{\mathbf{k}}\prod_{\mathbf{s}}\hat{\rho}^s_{\mathbf{k}} = \hat{\rho}^s_{\mathbf{k}}\hat{\rho}^{\bar{s}}_{\mathbf{k}}\hat{\rho}^s_{\mathbf{k}+\mathbf{k}_1}\hat{\rho}^{\bar{s}}_{\mathbf{k}+\mathbf{k}_1}$. Then:

$$\begin{aligned}
&\mathrm{Tr}_{\left\{\substack{s' \\ \mathbf{k'}\neq\mathbf{k}}\right\},\left\{\substack{\bar{s} \\ \mathbf{k}}\right\}}\left(\left[\hat{v}_{\mathbf{k}}\hat{v}_{-\mathbf{k}-\mathbf{k}_1},\left[\hat{v}_{-\mathbf{k}}\hat{v}_{\mathbf{k}+\mathbf{k}_1},\hat{\rho}_{red}\right]\right]\right) = \\
&= \mathrm{Tr}_{\left\{\substack{s' \\ \mathbf{k'}\neq\mathbf{k}}\right\},\left\{\substack{\bar{s} \\ \mathbf{k}}\right\}}\Big(\hat{v}_{\mathbf{k}}\hat{v}_{-\mathbf{k}_1-\mathbf{k}}\hat{v}_{-\mathbf{k}}\hat{v}_{\mathbf{k}+\mathbf{k}_1}\hat{\rho}_{red} - \hat{v}_{\mathbf{k}}\hat{v}_{-\mathbf{k}_1-\mathbf{k}}\hat{\rho}_{red}\hat{v}_{-\mathbf{k}}\hat{v}_{\mathbf{k}+\mathbf{k}_1} \\
&- \hat{v}_{-\mathbf{k}}\hat{v}_{\mathbf{k}_1+\mathbf{k}}\hat{\rho}_{red}\hat{v}_{\mathbf{k}}\hat{v}_{-\mathbf{k}-\mathbf{k}_1} + \hat{\rho}_{red}\hat{v}_{-\mathbf{k}}\hat{v}_{\mathbf{k}_1+\mathbf{k}}\hat{v}_{\mathbf{k}}\hat{v}_{-\mathbf{k}-\mathbf{k}_1}\Big) \\
&= \mathrm{Tr}_{\left\{\substack{\bar{s} \\ \mathbf{k}}\right\}}\left[\hat{v}_{\mathbf{k}}\hat{v}_{-\mathbf{k}}\hat{\rho}^s_{\mathbf{k}}\hat{\rho}^{\bar{s}}_{\mathbf{k}}\right]\mathrm{Tr}_{\left\{\substack{s,\bar{s} \\ \mathbf{k}+\mathbf{k}_1}\right\}}\left[\hat{v}_{-\mathbf{k}-\mathbf{k}_1}\hat{v}_{\mathbf{k}+\mathbf{k}_1}\hat{\rho}^s_{\mathbf{k}+\mathbf{k}_1}\hat{\rho}^{\bar{s}}_{\mathbf{k}+\mathbf{k}_1}\right] \\
&- \mathrm{Tr}_{\left\{\substack{\bar{s} \\ \mathbf{k}}\right\}}\left[\hat{v}_{\mathbf{k}}\hat{\rho}^s_{\mathbf{k}}\hat{\rho}^{\bar{s}}_{\mathbf{k}}\hat{v}_{-\mathbf{k}}\right]\mathrm{Tr}_{\left\{\substack{s,\bar{s} \\ \mathbf{k}+\mathbf{k}_1}\right\}}\left[\hat{v}_{-\mathbf{k}-\mathbf{k}_1}\hat{\rho}^s_{\mathbf{k}+\mathbf{k}_1}\hat{\rho}^{\bar{s}}_{\mathbf{k}+\mathbf{k}_1}\hat{v}_{\mathbf{k}+\mathbf{k}_1}\right] \\
&- \mathrm{Tr}_{\left\{\substack{\bar{s} \\ \mathbf{k}}\right\}}\left[\hat{v}_{-\mathbf{k}}\hat{\rho}^s_{\mathbf{k}}\hat{\rho}^{\bar{s}}_{\mathbf{k}}\hat{v}_{\mathbf{k}}\right]\mathrm{Tr}_{\left\{\substack{s,\bar{s} \\ \mathbf{k}+\mathbf{k}_1}\right\}}\left[\hat{v}_{\mathbf{k}+\mathbf{k}_1}\hat{\rho}^s_{\mathbf{k}+\mathbf{k}_1}\hat{\rho}^{\bar{s}}_{\mathbf{k}+\mathbf{k}_1}\hat{v}_{-\mathbf{k}-\mathbf{k}_1}\right] \\
&+ \mathrm{Tr}_{\left\{\substack{\bar{s} \\ \mathbf{k}}\right\}}\left[\hat{\rho}^s_{\mathbf{k}}\hat{\rho}^{\bar{s}}_{\mathbf{k}}\hat{v}_{-\mathbf{k}}\hat{v}_{\mathbf{k}}\right]\mathrm{Tr}_{\left\{\substack{s,\bar{s} \\ \mathbf{k}+\mathbf{k}_1}\right\}}\left[\hat{\rho}^s_{\mathbf{k}+\mathbf{k}_1}\hat{\rho}^{\bar{s}}_{\mathbf{k}+\mathbf{k}_1}\hat{v}_{\mathbf{k}+\mathbf{k}_1}\hat{v}_{-\mathbf{k}-\mathbf{k}_1}\right].
\end{aligned} \tag{3.140}$$

Now using the usual decomposition $\hat{v}_{\mathbf{k}} = \frac{\hat{v}^R_{\mathbf{k}}+i\hat{v}^I_{\mathbf{k}}}{\sqrt{2}}$ with $\hat{v}^R_{-\mathbf{k}} = \hat{v}^R_{\mathbf{k}}$ and $\hat{v}^I_{-\mathbf{k}} = -\hat{v}^I_{\mathbf{k}}$:

$$\begin{aligned}
&= \frac{1}{4}\mathrm{Tr}_{\left\{\substack{\bar{s} \\ \mathbf{k}}\right\}}\left[(\hat{v}^R_{\mathbf{k}}\hat{v}^R_{\mathbf{k}} + \hat{v}^I_{\mathbf{k}}\hat{v}^I_{\mathbf{k}})\hat{\rho}^s_{\mathbf{k}}\hat{\rho}^{\bar{s}}_{\mathbf{k}}\right]\mathrm{Tr}_{\left\{\substack{s,\bar{s} \\ \mathbf{k}+\mathbf{k}_1}\right\}}\left[(\hat{v}^R_{\mathbf{k}+\mathbf{k}_1}\hat{v}^R_{\mathbf{k}+\mathbf{k}_1} + \hat{v}^I_{\mathbf{k}+\mathbf{k}_1}\hat{v}^I_{\mathbf{k}+\mathbf{k}_1})\hat{\rho}^s_{\mathbf{k}+\mathbf{k}_1}\hat{\rho}^{\bar{s}}_{\mathbf{k}+\mathbf{k}_1}\right] \\
&- \frac{1}{2}\mathrm{Tr}_{\left\{\substack{\bar{s} \\ \mathbf{k}}\right\}}\left[\hat{v}^R_{\mathbf{k}}\hat{\rho}^s_{\mathbf{k}}\hat{\rho}^{\bar{s}}_{\mathbf{k}}\hat{v}^R_{\mathbf{k}} + \hat{v}^I_{\mathbf{k}}\hat{\rho}^s_{\mathbf{k}}\hat{\rho}^{\bar{s}}_{\mathbf{k}}\hat{v}^I_{\mathbf{k}}\right]\mathrm{Tr}_{\left\{\substack{s,\bar{s} \\ \mathbf{k}+\mathbf{k}_1}\right\}}\left[\hat{v}^R_{\mathbf{k}+\mathbf{k}_1}\hat{\rho}^s_{\mathbf{k}+\mathbf{k}_1}\hat{\rho}^{\bar{s}}_{\mathbf{k}+\mathbf{k}_1}\hat{v}^R_{\mathbf{k}+\mathbf{k}_1} + \hat{v}^I_{\mathbf{k}+\mathbf{k}_1}\hat{\rho}^s_{\mathbf{k}+\mathbf{k}_1}\hat{\rho}^{\bar{s}}_{\mathbf{k}+\mathbf{k}_1}\hat{v}^I_{\mathbf{k}+\mathbf{k}_1}\right] \\
&+ \frac{1}{4}\mathrm{Tr}_{\left\{\substack{\bar{s} \\ \mathbf{k}}\right\}}\left[\hat{\rho}^s_{\mathbf{k}}\hat{\rho}^{\bar{s}}_{\mathbf{k}}\hat{v}^R_{\mathbf{k}}\hat{v}^R_{\mathbf{k}} + \hat{\rho}^s_{\mathbf{k}}\hat{\rho}^{\bar{s}}_{\mathbf{k}}\hat{v}^I_{\mathbf{k}}\hat{v}^I_{\mathbf{k}}\right]\mathrm{Tr}_{\left\{\substack{s,\bar{s} \\ \mathbf{k}+\mathbf{k}_1}\right\}}\left[\hat{\rho}^s_{\mathbf{k}+\mathbf{k}_1}\hat{\rho}^{\bar{s}}_{\mathbf{k}+\mathbf{k}_1}\hat{v}^R_{\mathbf{k}+\mathbf{k}_1}\hat{v}^R_{\mathbf{k}+\mathbf{k}_1} + \hat{\rho}^s_{\mathbf{k}+\mathbf{k}_1}\hat{\rho}^{\bar{s}}_{\mathbf{k}+\mathbf{k}_1}\hat{v}^I_{\mathbf{k}+\mathbf{k}_1}\hat{v}^I_{\mathbf{k}+\mathbf{k}_1}\right].
\end{aligned} \tag{3.141}$$

Writing all in terms of $s = R$ and $\bar{s} = I$:

$$\begin{aligned}
&= \frac{1}{4}\mathrm{Tr}_{\left\{\substack{\bar{s} \\ \mathbf{k}}\right\}}\left[\hat{v}^s_{\mathbf{k}}\hat{v}^s_{\mathbf{k}}\hat{\rho}^s_{\mathbf{k}}\hat{\rho}^{\bar{s}}_{\mathbf{k}} + \hat{v}^{\bar{s}}_{\mathbf{k}}\hat{v}^{\bar{s}}_{\mathbf{k}}\hat{\rho}^s_{\mathbf{k}}\hat{\rho}^{\bar{s}}_{\mathbf{k}}\right]\underbrace{\mathrm{Tr}_{\left\{\substack{s,\bar{s} \\ \mathbf{k}+\mathbf{k}_1}\right\}}\left[\hat{v}^s_{\mathbf{k}+\mathbf{k}_1}\hat{v}^s_{\mathbf{k}+\mathbf{k}_1}\hat{\rho}^s_{\mathbf{k}+\mathbf{k}_1}\hat{\rho}^{\bar{s}}_{\mathbf{k}+\mathbf{k}_1} + \hat{v}^{\bar{s}}_{\mathbf{k}+\mathbf{k}_1}\hat{v}^{\bar{s}}_{\mathbf{k}+\mathbf{k}_1}\hat{\rho}^s_{\mathbf{k}+\mathbf{k}_1}\hat{\rho}^{\bar{s}}_{\mathbf{k}+\mathbf{k}_1}\right]}_{1} \\
&- \frac{1}{2}\mathrm{Tr}_{\left\{\substack{\bar{s} \\ \mathbf{k}}\right\}}\left[\hat{v}^s_{\mathbf{k}}\hat{\rho}^s_{\mathbf{k}}\hat{\rho}^{\bar{s}}_{\mathbf{k}}\hat{v}^s_{\mathbf{k}} + \hat{v}^{\bar{s}}_{\mathbf{k}}\hat{\rho}^s_{\mathbf{k}}\hat{\rho}^{\bar{s}}_{\mathbf{k}}\hat{v}^{\bar{s}}_{\mathbf{k}}\right]\underbrace{\mathrm{Tr}_{\left\{\substack{s,\bar{s} \\ \mathbf{k}+\mathbf{k}_1}\right\}}\left[\hat{v}^s_{\mathbf{k}+\mathbf{k}_1}\hat{\rho}^s_{\mathbf{k}+\mathbf{k}_1}\hat{\rho}^{\bar{s}}_{\mathbf{k}+\mathbf{k}_1}\hat{v}^s_{\mathbf{k}+\mathbf{k}_1} + \hat{v}^{\bar{s}}_{\mathbf{k}+\mathbf{k}_1}\hat{\rho}^s_{\mathbf{k}+\mathbf{k}_1}\hat{\rho}^{\bar{s}}_{\mathbf{k}+\mathbf{k}_1}\hat{v}^{\bar{s}}_{\mathbf{k}+\mathbf{k}_1}\right]}_{2} \\
&+ \frac{1}{4}\mathrm{Tr}_{\left\{\substack{\bar{s} \\ \mathbf{k}}\right\}}\left[\hat{\rho}^s_{\mathbf{k}}\hat{\rho}^{\bar{s}}_{\mathbf{k}}\hat{v}^s_{\mathbf{k}}\hat{v}^s_{\mathbf{k}} + \hat{\rho}^s_{\mathbf{k}}\hat{\rho}^{\bar{s}}_{\mathbf{k}}\hat{v}^{\bar{s}}_{\mathbf{k}}\hat{v}^{\bar{s}}_{\mathbf{k}}\right]\underbrace{\mathrm{Tr}_{\left\{\substack{s,\bar{s} \\ \mathbf{k}+\mathbf{k}_1}\right\}}\left[\hat{\rho}^s_{\mathbf{k}+\mathbf{k}_1}\hat{\rho}^{\bar{s}}_{\mathbf{k}+\mathbf{k}_1}\hat{v}^s_{\mathbf{k}+\mathbf{k}_1}\hat{v}^s_{\mathbf{k}+\mathbf{k}_1} + \hat{\rho}^s_{\mathbf{k}+\mathbf{k}_1}\hat{\rho}^{\bar{s}}_{\mathbf{k}+\mathbf{k}_1}\hat{v}^{\bar{s}}_{\mathbf{k}+\mathbf{k}_1}\hat{v}^{\bar{s}}_{\mathbf{k}+\mathbf{k}_1}\right]}_{3}.
\end{aligned} \tag{3.142}$$

Now focusing on the second traces and using $\mathrm{Tr}_{\left\{\substack{s \\ \mathbf{k}+\mathbf{k}_1}\right\}}\hat{\rho}^s_{\mathbf{k}+\mathbf{k}_1} = \mathrm{Tr}_{\left\{\substack{\bar{s} \\ \mathbf{k}+\mathbf{k}_1}\right\}}\hat{\rho}^{\bar{s}}_{\mathbf{k}+\mathbf{k}_1} = 1$ we get:

$$\begin{aligned}
1) &= \mathrm{Tr}_{\left\{\substack{s \\ \mathbf{k}+\mathbf{k}_1}\right\}}\left[\hat{v}^s_{\mathbf{k}+\mathbf{k}_1}\hat{v}^s_{\mathbf{k}+\mathbf{k}_1}\hat{\rho}^s_{\mathbf{k}+\mathbf{k}_1}\right] + \mathrm{Tr}_{\left\{\substack{\bar{s} \\ \mathbf{k}+\mathbf{k}_1}\right\}}\left[\hat{v}^{\bar{s}}_{\mathbf{k}+\mathbf{k}_1}\hat{v}^{\bar{s}}_{\mathbf{k}+\mathbf{k}_1}\hat{\rho}^{\bar{s}}_{\mathbf{k}+\mathbf{k}_1}\right] \\
&= 2\,\mathrm{Tr}_{\left\{\substack{s \\ \mathbf{k}+\mathbf{k}_1}\right\}}\left[\hat{v}^s_{\mathbf{k}+\mathbf{k}_1}\hat{v}^s_{\mathbf{k}+\mathbf{k}_1}\hat{\rho}^s_{\mathbf{k}+\mathbf{k}_1}\right] = 2P_{vv}(\mathbf{k}+\mathbf{k}_1), \\
2) &= \mathrm{Tr}_{\left\{\substack{s \\ \mathbf{k}+\mathbf{k}_1}\right\}}\left[\hat{v}^s_{\mathbf{k}+\mathbf{k}_1}\hat{\rho}^s_{\mathbf{k}+\mathbf{k}_1}\hat{v}^s_{\mathbf{k}+\mathbf{k}_1}\right] + \mathrm{Tr}_{\left\{\substack{\bar{s} \\ \mathbf{k}+\mathbf{k}_1}\right\}}\left[\hat{v}^{\bar{s}}_{\mathbf{k}+\mathbf{k}_1}\hat{\rho}^{\bar{s}}_{\mathbf{k}+\mathbf{k}_1}\hat{v}^{\bar{s}}_{\mathbf{k}+\mathbf{k}_1}\right] \\
&= 2\,\mathrm{Tr}_{\left\{\substack{s \\ \mathbf{k}+\mathbf{k}_1}\right\}}\left[\hat{v}^s_{\mathbf{k}+\mathbf{k}_1}\hat{\rho}^s_{\mathbf{k}+\mathbf{k}_1}\hat{v}^s_{\mathbf{k}+\mathbf{k}_1}\right] = 2P_{vv}(\mathbf{k}+\mathbf{k}_1), \\
3) &= \mathrm{Tr}_{\left\{\substack{s \\ \mathbf{k}+\mathbf{k}_1}\right\}}\left[\hat{\rho}^s_{\mathbf{k}+\mathbf{k}_1}\hat{v}^s_{\mathbf{k}+\mathbf{k}_1}\hat{v}^s_{\mathbf{k}+\mathbf{k}_1}\right] + \mathrm{Tr}_{\left\{\substack{\bar{s} \\ \mathbf{k}+\mathbf{k}_1}\right\}}\left[\hat{\rho}^{\bar{s}}_{\mathbf{k}+\mathbf{k}_1}\hat{v}^{\bar{s}}_{\mathbf{k}+\mathbf{k}_1}\hat{v}^{\bar{s}}_{\mathbf{k}+\mathbf{k}_1}\right] \\
&= 2\,\mathrm{Tr}_{\left\{\substack{s \\ \mathbf{k}+\mathbf{k}_1}\right\}}\left[\hat{\rho}^s_{\mathbf{k}+\mathbf{k}_1}\hat{v}^s_{\mathbf{k}+\mathbf{k}_1}\hat{v}^s_{\mathbf{k}+\mathbf{k}_1}\right] = 2P_{vv}(\mathbf{k}+\mathbf{k}_1),
\end{aligned} \tag{3.143}$$

where we recognized the power spectrum.

Then we obtain:

$$
\begin{aligned}
&\mathrm{Tr}_{\left\{\substack{s'\\ \mathbf{k}'\neq\mathbf{k}}\right\},\left\{\substack{\bar{s}\\ \mathbf{k}}\right\}}\left(\left[\hat{v}_{\mathbf{k}}\hat{v}_{-\mathbf{k}-\mathbf{k}_1},\left[\hat{v}_{-\mathbf{k}}\hat{v}_{\mathbf{k}+\mathbf{k}_1},\hat{\rho}_{red}\right]\right]\right)=\\
&=\frac{1}{2}\left[\hat{v}^s_{\mathbf{k}}\hat{v}^s_{\mathbf{k}}\hat{\rho}^s_{\mathbf{k}}+\hat{\rho}^s_{\mathbf{k}}\,\mathrm{Tr}_{\left\{\substack{\bar{s}\\ \mathbf{k}}\right\}}\left(\hat{v}^{\bar{s}}_{\mathbf{k}}\hat{v}^{\bar{s}}_{\mathbf{k}}\hat{\rho}^{\bar{s}}_{\mathbf{k}}\right)\right]P_{vv}(\mathbf{k}+\mathbf{k}_1)\\
&-\left[\hat{v}^s_{\mathbf{k}}\hat{\rho}^s_{\mathbf{k}}\hat{v}^s_{\mathbf{k}}+\hat{\rho}^s_{\mathbf{k}}\,\mathrm{Tr}_{\left\{\substack{\bar{s}\\ \mathbf{k}}\right\}}\left(\hat{v}^{\bar{s}}_{\mathbf{k}}\hat{\rho}^{\bar{s}}_{\mathbf{k}}\hat{v}^{\bar{s}}_{\mathbf{k}}\right)\right]P_{vv}(\mathbf{k}+\mathbf{k}_1)\\
&+\frac{1}{2}\left[\hat{\rho}^s_{\mathbf{k}}\hat{v}^s_{\mathbf{k}}\hat{v}^s_{\mathbf{k}}+\hat{\rho}^s_{\mathbf{k}}\,\mathrm{Tr}_{\left\{\substack{\bar{s}\\ \mathbf{k}}\right\}}\left(\hat{\rho}^{\bar{s}}_{\mathbf{k}}\hat{v}^{\bar{s}}_{\mathbf{k}}\hat{v}^{\bar{s}}_{\mathbf{k}}\right)\right]P_{vv}(\mathbf{k}+\mathbf{k}_1)\\
&=\frac{1}{2}[\hat{v}^s_{\mathbf{k}},[\hat{v}^s_{\mathbf{k}},\hat{\rho}^s_{\mathbf{k}}]]P_{vv}(\mathbf{k}+\mathbf{k}_1).
\end{aligned}
\tag{3.144}
$$

The same result is obtained with $\mathbf{k}=\mathbf{k}_2=\mathbf{k}_3-\mathbf{k}_1$, or $\mathbf{k}=-\mathbf{k}_1-\mathbf{k}_2=-\mathbf{k}_3$, or $\mathbf{k}=-\mathbf{k}_1-\mathbf{k}_2=-\mathbf{k}_1+\mathbf{k}_3$. The same result, with the only difference that the power spectrum is evaluated in $\mathbf{k}-\mathbf{k}_1$ instead of $\mathbf{k}+\mathbf{k}_1$, is obtained in the following cases:

- $\mathbf{k}=-\mathbf{k}_2=\mathbf{k}_3$,
- $\mathbf{k}=-\mathbf{k}_2=\mathbf{k}_1-\mathbf{k}_3$,
- $\mathbf{k}=\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3$,
- $\mathbf{k}=\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_1-\mathbf{k}_3$.

2. **k equal to three wavenumbers**:
in this case we obtain a vanishing result.

3. **k equal to four wavenumbers**: in this case we obtain a non-vanishing result, but this contribution is suppressed with respect to the case k equal to two wavenumbers, because here we get a finite result only for a single configuration of $\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3$, while the other case leaves one wavenumber free.

Putting together our results we obtain:

$$
\begin{aligned}
&\mathrm{Tr}_{\left\{\substack{s'\\ \mathbf{k}'\neq\mathbf{k}}\right\},\left\{\substack{\bar{s}\\ \mathbf{k}}\right\}}\left([\hat{v}_{\mathbf{k}_2}\hat{v}_{-\mathbf{k}_1-\mathbf{k}_2},[\hat{v}_{\mathbf{k}_3}\hat{v}_{\mathbf{k}_1-\mathbf{k}_3},\hat{\rho}_{red}]]\right)=\\
&=\frac{1}{2}[\hat{v}^s_{\mathbf{k}},[\hat{v}^s_{\mathbf{k}},\hat{\rho}^s_{\mathbf{k}}]]\times\Big\{P_{vv}(\mathbf{k}+\mathbf{k}_1)\Big[\delta^3(\mathbf{k}_2-\mathbf{k})\delta^3(\mathbf{k}_3+\mathbf{k})+\delta^3(\mathbf{k}_2-\mathbf{k})\delta^3(\mathbf{k}_3-\mathbf{k}_1-\mathbf{k})\\
&+\delta^3(\mathbf{k}_3+\mathbf{k})\delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k})+\delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k})\delta^3(\mathbf{k}_3-\mathbf{k}_1-\mathbf{k})\Big]\\
&+P_{vv}(\mathbf{k}-\mathbf{k}_1)\Big[\delta^3(\mathbf{k}_2+\mathbf{k})\delta^3(\mathbf{k}_3-\mathbf{k})+\delta^3(\mathbf{k}_2+\mathbf{k})\delta^3(\mathbf{k}_1-\mathbf{k}_3-\mathbf{k})\\
&+\delta^3(\mathbf{k}_3-\mathbf{k})\delta^3(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})+\delta^3(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\delta^3(\mathbf{k}_1-\mathbf{k}_3-\mathbf{k})\Big]\Big\}.
\end{aligned}
\tag{3.145}
$$

Plugging back this result in equation (3.139):

$$
\begin{aligned}
\mathrm{Tr}_{\left\{\substack{s'\\ \mathbf{k}'\neq\mathbf{k}}\right\},\left\{\substack{\bar{s}\\ \mathbf{k}}\right\}}\frac{\mathrm{d}\hat{\rho}_{red}}{\mathrm{d}\eta}&=-\frac{\gamma}{4}\frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{k_1}\mathrm{d}^3\mathbf{k_2}\mathrm{d}^3\mathbf{k_3}C_R(\mathbf{k}_1)[\hat{v}^s_{\mathbf{k}},[\hat{v}^s_{\mathbf{k}},\hat{\rho}^s_{\mathbf{k}}]]\times\\
&\times\Big\{P_{vv}(\mathbf{k}+\mathbf{k}_1)\Big[\delta^3(\mathbf{k}_2-\mathbf{k})\delta^3(\mathbf{k}_3+\mathbf{k})+\delta^3(\mathbf{k}_2-\mathbf{k})\delta^3(\mathbf{k}_3-\mathbf{k}_1-\mathbf{k})\\
&+\delta^3(\mathbf{k}_3+\mathbf{k})\delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k})+\delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k})\delta^3(\mathbf{k}_3-\mathbf{k}_1-\mathbf{k})\Big]\\
&+P_{vv}(\mathbf{k}-\mathbf{k}_1)\Big[\delta^3(\mathbf{k}_2+\mathbf{k})\delta^3(\mathbf{k}_3-\mathbf{k})+\delta^3(\mathbf{k}_2+\mathbf{k})\delta^3(\mathbf{k}_1-\mathbf{k}_3-\mathbf{k})\\
&+\delta^3(\mathbf{k}_3-\mathbf{k})\delta^3(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})+\delta^3(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\delta^3(\mathbf{k}_1-\mathbf{k}_3-\mathbf{k})\Big]\Big\}\\
&=-\gamma\frac{1}{(2\pi)^{3/2}}C_R(\mathbf{k}_1)[\hat{v}^s_{\mathbf{k}},[\hat{v}^s_{\mathbf{k}},\hat{\rho}^s_{\mathbf{k}}]]\Big\{P_{vv}(\mathbf{k}+\mathbf{k}_1)+P_{vv}(\mathbf{k}-\mathbf{k}_1)\Big\}.
\end{aligned}
\tag{3.146}
$$

Using $P_{vv}(\mathbf{k}+\mathbf{k}_1) = P_{vv}(\mathbf{k}-\mathbf{k}_1) = P_{vv}(|\mathbf{k}+\mathbf{k}_1|)$:

$$\begin{aligned} \mathrm{Tr}_{\left\{\substack{s' \\ \mathbf{k}'\neq\mathbf{k}}\right\},\left\{\substack{\bar{s} \\ \mathbf{k}}\right\}} \frac{\mathrm{d}\hat{\rho}_{red}}{\mathrm{d}\eta} &= -2\gamma\frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3\mathbf{k}_1 C_R(\mathbf{k}_1)[\hat{v}^s_{\mathbf{k}},[\hat{v}^s_{\mathbf{k}},\hat{\rho}^s_{\mathbf{k}}]]P_{vv}(|\mathbf{k}+\mathbf{k}_1|) \\ &= -\frac{1}{4}[\hat{v}^s_{\mathbf{k}},[\hat{v}^s_{\mathbf{k}},\hat{\rho}^s_{\mathbf{k}}]]S_2(\mathbf{k},\eta), \end{aligned} \tag{3.147}$$

where we defined the source function:

$$S_2(\mathbf{k},\eta) = \frac{8\gamma}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} \mathrm{d}^3\mathbf{k}' C_R(\mathbf{k}')P_{vv}(|\mathbf{k}'+\mathbf{k}|). \tag{3.148}$$

Inserting this result in the equation for the purity:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\eta}\,\mathrm{Tr}_{\left\{\substack{s \\ \mathbf{k}}\right\}}\left(\hat{\rho}^{s2}_{\mathbf{k}}\right) &= -\frac{1}{2}\,\mathrm{Tr}_{\left\{\substack{s \\ \mathbf{k}}\right\}}\left\{\hat{\rho}^s_{\mathbf{k}}[\hat{v}^s_{\mathbf{k}},[\hat{v}^s_{\mathbf{k}},\hat{\rho}^s_{\mathbf{k}}]]S_2(\mathbf{k},\eta)\right\} \\ &= -\frac{1}{2}S_2(\mathbf{k},\eta)P_{vv}(\mathbf{k},\eta), \end{aligned} \tag{3.149}$$

where we used the relation $\mathrm{Tr}\left(\hat{\rho}^s_{\mathbf{k}}[\hat{v}^s_{\mathbf{k}},[\hat{v}^s_{\mathbf{k}},\hat{\rho}^s_{\mathbf{k}}]]\right) = 2P_{vv}(\mathbf{k})$ which was shown in [28]. This method will be used in the following chapters to compute the purity evolution in the case of a non-linear interaction.

Chapter 4

Linear model

As we have said, observing a signature of the original quantum nature of the present fluctuations is a hard task, mainly because of the phenomenon of quantum decoherence which is believed to erase quantum signature, leaving a classical Universe. In this chapter we want to analyze the quantum decoherence process in an extension of the simplest single-field slow-roll inflation. We will follow closely [8]. Contrary to the common expectation, we will show that, after an initial period of decoherence, the phenomenon of *recoherence* takes place, with the system showing a large level of self-coherence in the final state.

Since inflation happened at very high-energy it is reasonable to consider a heavy field interacting with the inflaton. From a bottom-up EFT approach the dynamics of the fluctuations in the adiabatic direction ζ and the entropic direction $\mathcal{F}$ is given by the lagrangian density [31]:

$$\mathcal{L} = a^2\varepsilon M_{Pl}^2\zeta'^2 - a^2\varepsilon M_{Pl}^2(\partial_i\zeta)^2 + \frac{1}{2}a^2\mathcal{F}'^2 - \frac{1}{2}a^2(\partial_i\mathcal{F})^2 - \frac{1}{2}m^2a^4\mathcal{F}^2 - \rho a^3\sqrt{2\varepsilon}M_{Pl}\zeta'\mathcal{F}, \tag{4.1}$$

where η is the conformal time, $a(\eta)$ the scale factor, $(\partial_i\zeta)^2 \equiv \delta_{ij}\partial_i\zeta\partial_j\zeta$, ε is the first slow-roll parameter and M_{Pl} is the Planck mass. The curvature perturbation, ζ, is observed directly in the CMB, while $\mathcal{F}$ describes an unobservable heavy field. ρ is the coupling constant of the interaction term $\zeta'\mathcal{F}$, and it represents the turn rate in the field space between adiabatic and entropic directions.
We define the following gauge-invariant variables, called Mukhanov-Sasaki variables:

$$\begin{aligned} v_\zeta(\eta,\boldsymbol{x}) = -a(\eta)\sqrt{2\varepsilon}M_{Pl}\zeta(\eta,\boldsymbol{x} \quad &\rightarrow \quad \zeta(\eta,\boldsymbol{x}) = -\frac{v_\zeta(\eta,\boldsymbol{x})}{a(\eta)\sqrt{2\varepsilon}M_{Pl}}, \\ v_\mathcal{F}(\eta,\boldsymbol{x} = a(\eta)\mathcal{F}(\eta,\boldsymbol{x} \quad &\rightarrow \quad \mathcal{F}(\eta,\boldsymbol{x}) = \frac{v_\mathcal{F}(\eta,\boldsymbol{x})}{a(\eta)}, \end{aligned} \tag{4.2}$$

then the lagrangian density gets rewritten as:

$$\begin{aligned} \mathcal{L} = &\frac{1}{2}v_\zeta'^2 + \frac{1}{2}\Big(\frac{a'}{a}\Big)^2 v_\zeta^2 - \frac{a'}{a}v_\zeta' v_\zeta - \frac{1}{2}(\partial_i v_\zeta)^2 + \frac{1}{2}v_\mathcal{F}'^2 + \frac{1}{2}\Big(\frac{a'}{a}\Big)^2 v_\mathcal{F}^2 - \frac{a'}{a}v_\mathcal{F}' v_\mathcal{F} - \frac{1}{2}(\partial_i v_\mathcal{F})^2 \\ &- \frac{1}{2}m^2a^2v_\mathcal{F}^2 + \rho a v_\mathcal{F} v_\zeta' - \rho a' v_\zeta v_\mathcal{F}. \end{aligned} \tag{4.3}$$

From this we can derive the conjugated momenta:

$$\begin{aligned} p_\zeta = \frac{\partial\mathcal{L}}{\partial v_\zeta'} = v_\zeta' - \frac{a'}{a}v_\zeta + \rho a v_\mathcal{F} \quad &\rightarrow \quad v_\zeta' = p_\zeta + \frac{a'}{a}v_\zeta - \rho a v_\mathcal{F}, \\ p_\mathcal{F} = \frac{\partial\mathcal{L}}{\partial v_\mathcal{F}'} = v_\mathcal{F}' - \frac{a'}{a}v_\mathcal{F} \quad &\rightarrow \quad v_\mathcal{F}' = p_\mathcal{F} + \frac{a'}{a}v_\mathcal{F}, \end{aligned} \tag{4.4}$$

and finally the Hamiltonian density:

$$\begin{aligned}\mathcal{H}(v_\zeta, v_\mathcal{F}, p_\zeta, p_\mathcal{F}, \eta) &= p_\zeta v'_\zeta + p_\mathcal{F} v'_\mathcal{F} - \mathcal{L}(v_\zeta, v_\mathcal{F}, v'_\zeta(v_\zeta, p_\zeta, v_\mathcal{F}, p_\mathcal{F}), v'_\mathcal{F}(v_\zeta, p_\zeta, v_\mathcal{F}, p_\mathcal{F})) \\ &= \frac{1}{2}p_\zeta^2 + \frac{1}{2}p_\mathcal{F}^2 + \frac{1}{2}(\partial_i v_\zeta)^2 + \frac{1}{2}(\partial_i v_\mathcal{F})^2 + \frac{a'}{a}v_\zeta p_\zeta - \rho a v_\mathcal{F} p_\zeta \\ &+ \frac{a'}{a}v_\mathcal{F} p_\mathcal{F} + \frac{1}{2}\rho^2 a^2 v_\mathcal{F}^2 + \frac{1}{2}m^2 a^2 v_\mathcal{F}^2.\end{aligned} \tag{4.5}$$

Using the Fourier decomposition for the Mukhanov-Sasaki variable and its conjugated momentum

$$v_\alpha(\eta, \boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3\boldsymbol{k} v_\alpha(\eta, \boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{4.6}$$

the Hamiltonian becomes:

$$\begin{aligned} H &= \frac{1}{(2\pi)^3} \int d^3\boldsymbol{x} \int d^3\boldsymbol{k} \int d^3\boldsymbol{k'} \Bigg[\frac{1}{2}p_\zeta(\boldsymbol{k})p_\zeta(\boldsymbol{k'})e^{i(\boldsymbol{k}+\boldsymbol{k'})\cdot\boldsymbol{x}} + \frac{1}{2}p_\mathcal{F}(\boldsymbol{k})p_\mathcal{F}(\boldsymbol{k'})e^{i(\boldsymbol{k}+\boldsymbol{k'})\cdot\boldsymbol{x}} \\ &+ \frac{1}{2}\Big(\partial_i v_\zeta(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\Big)\Big(\partial_i v_\zeta(\boldsymbol{k'})e^{i\boldsymbol{k'}\cdot\boldsymbol{x}}\Big) + \frac{1}{2}\Big(\partial_i v_\mathcal{F}(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\Big)\Big(\partial_i v_\mathcal{F}(\boldsymbol{k'})e^{i\boldsymbol{k'}\cdot\boldsymbol{x}}\Big) \\ &+ \frac{a'}{a}\Big(v_\zeta(\boldsymbol{k})p_\zeta(\boldsymbol{k'})e^{i(\boldsymbol{k}+\boldsymbol{k'})\cdot\boldsymbol{x}}\Big) + \frac{a'}{a}\Big(v_\mathcal{F}(\boldsymbol{k})p_\mathcal{F}(\boldsymbol{k'})e^{i(\boldsymbol{k}+\boldsymbol{k'})\cdot\boldsymbol{x}}\Big) - \rho a v_\mathcal{F}(\boldsymbol{k})p_\zeta(\boldsymbol{k'})e^{i(\boldsymbol{k}+\boldsymbol{k'})\cdot\boldsymbol{x}} \\ &+ \frac{1}{2}\rho^2 a^2 v_\mathcal{F}(\boldsymbol{k})v_\mathcal{F}(\boldsymbol{k'})e^{i(\boldsymbol{k}+\boldsymbol{k'})\cdot\boldsymbol{x}} + \frac{1}{2}m^2 a^2 v_\mathcal{F}(\boldsymbol{k})v_\mathcal{F}(\boldsymbol{k'})e^{i(\boldsymbol{k}+\boldsymbol{k'})\cdot\boldsymbol{x}}\Bigg] \\ &= \int d^3\boldsymbol{k}\Bigg[\frac{1}{2}p_\zeta(\boldsymbol{k})p_\zeta(-\boldsymbol{k}) + \frac{1}{2}p_\mathcal{F}(\boldsymbol{k})p_\mathcal{F}(-\boldsymbol{k}) + \frac{1}{2}k^2 v_\zeta(\boldsymbol{k})v_\zeta(-\boldsymbol{k}) \\ &+ \frac{1}{2}(k^2 + (\rho^2 + m^2)a^2)v_\mathcal{F}(\boldsymbol{k})v_\mathcal{F}(-\boldsymbol{k}) + \frac{a'}{a}v_\zeta(\boldsymbol{k})p_\zeta(-\boldsymbol{k}) + \frac{a'}{a}v_\mathcal{F}(\boldsymbol{k})p_\mathcal{F}(-\boldsymbol{k}) - \rho a v_\mathcal{F}(\boldsymbol{k})p_\zeta(-\boldsymbol{k})\Bigg], \end{aligned} \tag{4.7}$$

which can be schematically rewritten as:

$$H = \int_{\mathcal{R}^{3+}} d^3\boldsymbol{k} \boldsymbol{z}_k^\dagger \boldsymbol{\mathcal{H}}(\eta)\boldsymbol{z}_{-k} \qquad \text{with} \qquad \boldsymbol{z} \equiv (v_\zeta, p_\zeta, v_\mathcal{F}, p_\mathcal{F})^T, \tag{4.8}$$

where

$$\boldsymbol{\mathcal{H}}(\eta) = \begin{pmatrix} \boldsymbol{\mathcal{H}}^{(S)} & \boldsymbol{V} \\ \boldsymbol{V}^T & \boldsymbol{\mathcal{H}}^{(\varepsilon)} \end{pmatrix}, \tag{4.9}$$

with

$$\boldsymbol{\mathcal{H}}^{(S)}(\eta) = \begin{pmatrix} k^2 & \frac{a'}{a} \\ \frac{a'}{a} & 1 \end{pmatrix} \qquad \boldsymbol{\mathcal{H}}^{(\varepsilon)}(\eta) = \begin{pmatrix} k^2 + (m^2+\rho^2)a^2 & \frac{a'}{a} \\ \frac{a'}{a} & 1 \end{pmatrix} \qquad \boldsymbol{V}(\eta) = \begin{pmatrix} 0 & 0 \\ -\rho a & 0 \end{pmatrix}, \tag{4.10}$$

where the integral is performed over $\mathbb{R}^{3+}$ because of the condition $\mathbf{z}^*(\eta, \mathbf{k}) = \mathbf{z}(\eta, -\mathbf{k})$ due to the reality of the original fields, ζ and $\mathcal{F}$.
Following the canonical quantization prescription we can promote the fields to quantum operators and split them in real and imaginary part in order to work with Hermitian operators:

$$\hat{\mathbf{z}} = \frac{1}{\sqrt{2}}\big(\hat{z}^R + i\hat{z}^I\big). \tag{4.11}$$

In this basis the Hamiltonian takes the form:

$$\hat{H}(\eta) = \frac{1}{2}\sum_{s=R,I}\int_{\mathbb{R}^{3+}} \mathrm{d}^3\mathbf{k}(\hat{\mathbf{z}}^s)^T \mathbf{H}(\eta)\hat{\mathbf{z}}^s. \tag{4.12}$$

Being separable there is no mixing between different modes, nor between R and I sectors, so we can focus on a given mode $\mathbf{k}$ and a given s-sector.

Since in this case the interaction is linear we can both study the decoherence in an exact way and with the OQS approach we have introduced in the previous chapters.

4.1 Exact approach

Consider the covariance matrix of the full setup (system + environment) in the Heisenberg picture:

$$\Sigma_{ij}^{(S+\varepsilon)}(\eta) \equiv \frac{1}{2}\,\mathrm{Tr}\left\{\left[\hat{z}_i(\eta)\hat{z}_j(\eta) + \hat{z}_j(\eta)\hat{z}_i(\eta)\right]\hat{\rho}_0\right\}. \tag{4.13}$$

As we have already said, due to the linearity of the dynamics, the state remains gaussian and the covariance matrix contains all the information about the quantum state. Now if we differentiate it with respect to time, using the Heisenberg equation to evaluate $\mathrm{d}\hat{\mathbf{z}}/\mathrm{d}\eta$, with the Hamiltonian (4.12) and using for $\hat{\rho}_0$ the Bunch-Davies vacuum, we obtain the following transport equation:

$$\frac{\mathrm{d}\boldsymbol{\Sigma}^{(S+\varepsilon)}}{\mathrm{d}\eta} = \boldsymbol{\Omega}\boldsymbol{H}\boldsymbol{\Sigma}^{(S+\varepsilon)} - \boldsymbol{\Sigma}^{(S+\varepsilon)}\boldsymbol{H}\boldsymbol{\Omega}, \tag{4.14}$$

where Ω is a four-by-four matrix defined as:

$$\boldsymbol{\Omega} = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} \qquad \text{with} \quad \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.15}$$

From this equation is easy to derive a transport equation for $\det\boldsymbol{\Sigma}$ with $\boldsymbol{\Sigma}$ the system's covariance:

$$\frac{\mathrm{d}\det\boldsymbol{\Sigma}}{\mathrm{d}\eta} = \Sigma_{11}^{(S+\varepsilon)}\frac{\mathrm{d}\Sigma_{22}^{(S+\varepsilon)}}{\mathrm{d}\eta} + \Sigma_{22}^{(S+\varepsilon)}\frac{\mathrm{d}\Sigma_{11}^{(S+\varepsilon)}}{\mathrm{d}\eta} - 2\Sigma_{12}^{(S+\varepsilon)}\frac{\mathrm{d}\Sigma_{12}^{(S+\varepsilon)}}{\mathrm{d}\eta}. \tag{4.16}$$

Making equation (4.14) and (4.16) explicit we get a system of eleven coupled ordinary differential equations, that is:

$$\begin{aligned}
\frac{\mathrm{d}\Sigma_{11}^{(S+\varepsilon)}}{\mathrm{d}\eta} &= 2\Big(\frac{a'}{a}\Sigma_{11}^{(S+\varepsilon)} + \Sigma_{12}^{(S+\varepsilon)} - \rho a\Sigma_{13}^{(S+\varepsilon)}\Big)\\
\frac{\mathrm{d}\Sigma_{12}^{(S+\varepsilon)}}{\mathrm{d}\eta} &= \Sigma_{22}^{(S+\varepsilon)} - \rho a\Sigma_{23}^{(S+\varepsilon)} - k^2\Sigma_{11}^{(S+\varepsilon)}\\
\frac{\mathrm{d}\Sigma_{13}^{(S+\varepsilon)}}{\mathrm{d}\eta} &= 2\frac{a'}{a}\Sigma_{13}^{(S+\varepsilon)} - \rho a\Sigma_{33}^{(S+\varepsilon)} + \Sigma_{23}^{(S+\varepsilon)} + \Sigma_{14}^{(S+\varepsilon)}\\
\frac{\mathrm{d}\Sigma_{14}^{(S+\varepsilon)}}{\mathrm{d}\eta} &= \rho a\big(\Sigma_{12}^{(S+\varepsilon)} - \Sigma_{34}^{(S+\varepsilon)}\big) + \Sigma_{24}^{(S+\varepsilon)} - B\Sigma_{13}^{(S+\varepsilon)}\\
\frac{\mathrm{d}\Sigma_{22}^{(S+\varepsilon)}}{\mathrm{d}\eta} &= -2\Big(k^2\Sigma_{12}^{(S+\varepsilon)} + \frac{a'}{a}\Sigma_{22}^{(S+\varepsilon)}\Big)\\
\frac{\mathrm{d}\Sigma_{23}^{(S+\varepsilon)}}{\mathrm{d}\eta} &= \Sigma_{24}^{(S+\varepsilon)} - k^2\Sigma_{13}^{(S+\varepsilon)}\\
\frac{\mathrm{d}\Sigma_{24}^{(S+\varepsilon)}}{\mathrm{d}\eta} &= -k^2\Sigma_{14}^{(S+\varepsilon)} - 2\frac{a'}{a}\Sigma_{24}^{(S+\varepsilon)} + \rho a\Sigma_{22}^{(S+\varepsilon)} - B\Sigma_{23}^{(S+\varepsilon)}\\
\frac{\mathrm{d}\Sigma_{33}^{(S+\varepsilon)}}{\mathrm{d}\eta} &= 2\Big(\frac{a'}{a}\Sigma_{33}^{(S+\varepsilon)} + \Sigma_{34}^{(S+\varepsilon)}\Big)\\
\frac{\mathrm{d}\Sigma_{34}^{(S+\varepsilon)}}{\mathrm{d}\eta} &= \Sigma_{44}^{(S+\varepsilon)} + \rho a\Sigma_{23}^{(S+\varepsilon)} - B\Sigma_{33}^{(S+\varepsilon)}\\
\frac{\mathrm{d}\Sigma_{44}^{(S+\varepsilon)}}{\mathrm{d}\eta} &= 2\Big(\rho a\Sigma_{24}^{(S+\varepsilon)} - B\Sigma_{34}^{(S+\varepsilon)} - \frac{a'}{a}\Sigma_{44}^{(S+\varepsilon)}\Big)\\
\frac{\mathrm{d}\det\Sigma}{\mathrm{d}\eta} &= \Sigma_{11}^{(S+\varepsilon)}\frac{\mathrm{d}\Sigma_{22}^{(S+\varepsilon)}}{\mathrm{d}\eta} + \Sigma_{22}^{(S+\varepsilon)}\frac{\mathrm{d}\Sigma_{11}^{(S+\varepsilon)}}{\mathrm{d}\eta} - 2\Sigma_{12}^{(S+\varepsilon)}\frac{\mathrm{d}\Sigma_{12}^{(S+\varepsilon)}}{\mathrm{d}\eta},
\end{aligned} \tag{4.17}$$

where we have defined:

$$B = k^2 + (m^2 + \rho^2)a^2 \tag{4.18}$$

We want to express Σ as a function of $\ln\left(\frac{a}{a_*}\right)$ with $k = a_* H$ where a_* is the scale factor at Hubble crossing. We can use:

$$\eta = -\frac{1}{a(\eta)H} \qquad \Rightarrow \qquad \mathrm{d}\eta = \frac{1}{a^2 H}\mathrm{d}a \qquad \text{but} \qquad \mathrm{d}\left(\ln\frac{a}{a_*}\right) = \frac{a_*}{a}\frac{\mathrm{d}a}{a_*} = \frac{\mathrm{d}a}{a}, \tag{4.19}$$

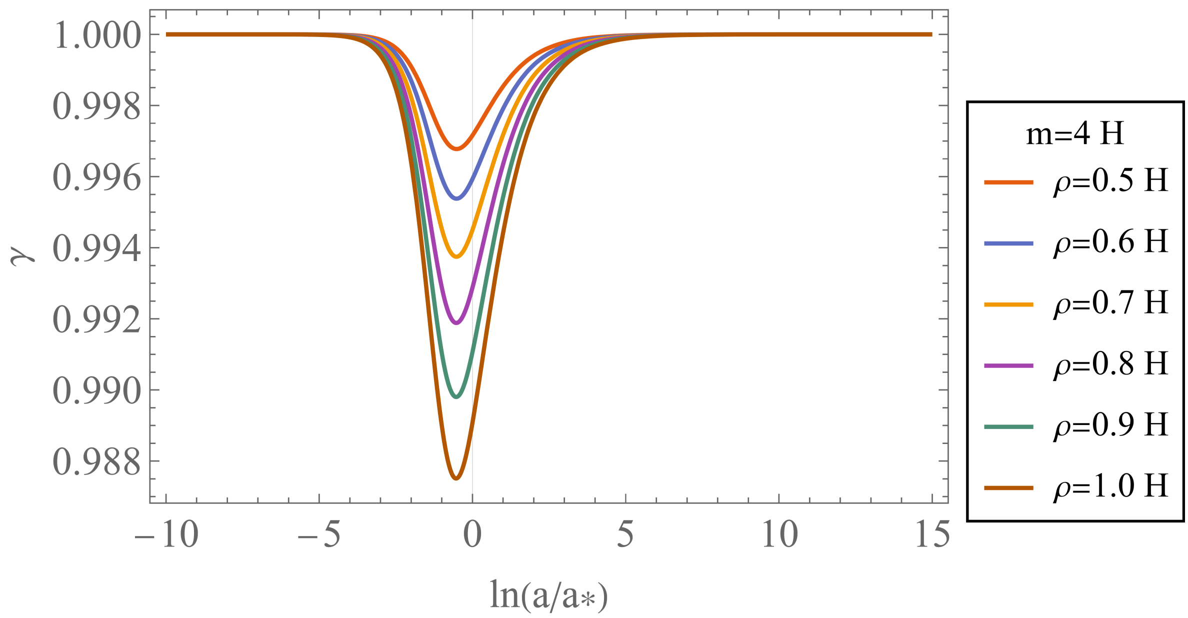
$$\Longrightarrow \frac{\mathrm{d}\Sigma}{\mathrm{d}\eta} = a^2 H \frac{\mathrm{d}\Sigma}{\mathrm{d}a} = aH\frac{\mathrm{d}\Sigma}{\mathrm{d}\ln\left(\frac{a}{a_*}\right)} = \frac{a'}{a}\frac{\mathrm{d}\Sigma}{\mathrm{d}\ln\frac{a}{a_*}}. \tag{4.20}$$

Now we can rewrite our equations as a function of $\frac{a}{a_*}$:

$$\begin{aligned}
\frac{\mathrm{d}\Sigma_{11}^{(S+\varepsilon)}}{\mathrm{d}\ln\left(\frac{a}{a_*}\right)} &= 2\Big(\Sigma_{11}^{(S+\varepsilon)} + \frac{a_*}{a}\frac{1}{k}\Sigma_{12}^{(S+\varepsilon)} - \frac{\rho}{H}\Sigma_{13}^{(S+\varepsilon)}\Big),\\
\frac{\mathrm{d}\Sigma_{12}^{(S+\varepsilon)}}{\mathrm{d}\ln\left(\frac{a}{a_*}\right)} &= \frac{a_*}{a}\frac{1}{k}\Sigma_{22}^{(S+\varepsilon)} - \frac{\rho}{H}\Sigma_{23}^{(S+\varepsilon)} - \frac{a_*}{a}k\Sigma_{11}^{(S+\varepsilon)},\\
\frac{\mathrm{d}\Sigma_{13}^{(S+\varepsilon)}}{\mathrm{d}\ln\left(\frac{a}{a_*}\right)} &= 2\Sigma_{13}^{(S+\varepsilon)} - \frac{\rho}{H}\Sigma_{33}^{(S+\varepsilon)} + \frac{a_*}{a}\frac{1}{k}\Big(\Sigma_{23}^{(S+\varepsilon)} + \Sigma_{14}^{(S+\varepsilon)}\Big),\\
\frac{\mathrm{d}\Sigma_{14}^{(S+\varepsilon)}}{\mathrm{d}\ln\left(\frac{a}{a_*}\right)} &= \frac{\rho}{H}\Big(\Sigma_{12}^{(S+\varepsilon)} - \Sigma_{34}^{(S+\varepsilon)}\Big) + \frac{a_*}{a}\frac{1}{k}\Sigma_{24}^{(S+\varepsilon)} - \Big(\frac{a_*}{a}k + \frac{a}{a_*}k\Big(\frac{m^2}{H^2} + \frac{\rho^2}{H^2}\Big)\Big)\Sigma_{13}^{(S+\varepsilon)},\\
\frac{\mathrm{d}\Sigma_{22}^{(S+\varepsilon)}}{\mathrm{d}\ln\left(\frac{a}{a_*}\right)} &= -2\Big(\frac{a_*}{a}k\Sigma_{12}^{(S+\varepsilon)} + \Sigma_{22}^{(S+\varepsilon)}\Big),\\
\frac{\mathrm{d}\Sigma_{23}^{(S+\varepsilon)}}{\mathrm{d}\ln\left(\frac{a}{a_*}\right)} &= \frac{a_*}{a}\frac{1}{k}\Sigma_{24}^{(S+\varepsilon)} - \frac{a_*}{a}k\Sigma_{13}^{(S+\varepsilon)},\\
\frac{\mathrm{d}\Sigma_{24}^{(S+\varepsilon)}}{\mathrm{d}\ln\left(\frac{a}{a_*}\right)} &= -\frac{a_*}{a}k\Sigma_{14}^{(S+\varepsilon)} - 2\Sigma_{24}^{(S+\varepsilon)} + \frac{\rho}{H}\Sigma_{22}^{(S+\varepsilon)} - \Big(\frac{a_*}{a}k + \frac{a}{a_*}k\Big(\frac{m^2}{H^2} + \frac{\rho^2}{H^2}\Big)\Big)\Sigma_{23}^{(S+\varepsilon)},\\
\frac{\mathrm{d}\Sigma_{33}^{(S+\varepsilon)}}{\mathrm{d}\ln\left(\frac{a}{a_*}\right)} &= 2\Big(\Sigma_{33}^{(S+\varepsilon)} + \frac{a_*}{a}\frac{1}{k}\Sigma_{34}^{(S+\varepsilon)}\Big),\\
\frac{\mathrm{d}\Sigma_{34}^{(S+\varepsilon)}}{\mathrm{d}\ln\left(\frac{a}{a_*}\right)} &= \frac{a_*}{a}\frac{1}{k}\Sigma_{44}^{(S+\varepsilon)} + \frac{\rho}{H}\Sigma_{23}^{(S+\varepsilon)} - \Big(\frac{a_*}{a}k + \frac{a}{a_*}k\Big(\frac{m^2}{H^2} + \frac{\rho^2}{H^2}\Big)\Big)\Sigma_{33}^{(S+\varepsilon)},\\
\frac{\mathrm{d}\Sigma_{44}^{(S+\varepsilon)}}{\mathrm{d}\ln\left(\frac{a}{a_*}\right)} &= 2\Big(\frac{\rho}{H}\Sigma_{24}^{(S+\varepsilon)} - \Big(\frac{a_*}{a}k + \frac{a}{a_*}k\Big(\frac{m^2}{H^2} + \frac{\rho^2}{H^2}\Big)\Big)\Sigma_{34}^{(S+\varepsilon)} - \Sigma_{44}^{(S+\varepsilon)}\Big),\\
\frac{\mathrm{d}\det\Sigma}{\mathrm{d}\ln\left(\frac{a}{a_*}\right)} &= \Sigma_{11}^{(S+\varepsilon)}\frac{\mathrm{d}\Sigma_{22}^{(S+\varepsilon)}}{\mathrm{d}\ln\left(\frac{a}{a_*}\right)} + \Sigma_{22}^{(S+\varepsilon)}\frac{\mathrm{d}\Sigma_{11}^{(S+\varepsilon)}}{\mathrm{d}\ln\left(\frac{a}{a_*}\right)} - 2\Sigma_{12}^{(S+\varepsilon)}\frac{\mathrm{d}\Sigma_{12}^{(S+\varepsilon)}}{\mathrm{d}\ln\left(\frac{a}{a_*}\right)}.
\end{aligned} \tag{4.21}$$

Notice that we have eleven equations for ten unknowns, indeed the equation for the determinant is redundant, but computationally it is more efficient to use all the eleven equations. Now we can integrate this set of coupled equations from $\ln(a/a_*) = -15$ to $\ln(a/a^*) = 15$ using as initial conditions those of the Bunch-Davies vacuum, where $\Sigma_{11}^{(S+\varepsilon)} = \Sigma_{33}^{(S+\varepsilon)} = 1/(2k)$ and $\Sigma_{22}^{(S+\varepsilon)} = \Sigma_{44}^{(S+\varepsilon)} = k/2$, with all the other correlation functions vanishing. As we have seen in section 3.5, for a gaussian state the covariance matrix not only contains all the observables of the system, but it fully specifies the quantum state of the system, i.e. $\hat{\rho}_{red}$, and this allows us to reproduce the transition from a pure state to a statistically mixed state. This is possible through the purity parameter, $\gamma = \mathrm{Tr}_S\left(\hat{\rho}_{red}^2\right)$, which for a gaussian system is simply related to the covariance matrix through $\gamma = 1/(4\det\mathbf{\Sigma})$. So once solved the previous system we can reproduce the purity as a function of time. The result is shown in figure, where we choose as a scale $k = 1$.

In the top graph of figure 4.1 we fixed the mass and see how the decoherence changes for some values of the coupling constant, while in the bottom graph we fixed the coupling constant and considered various values of the mass. In both cases we observe an initial decoherence, followed by an



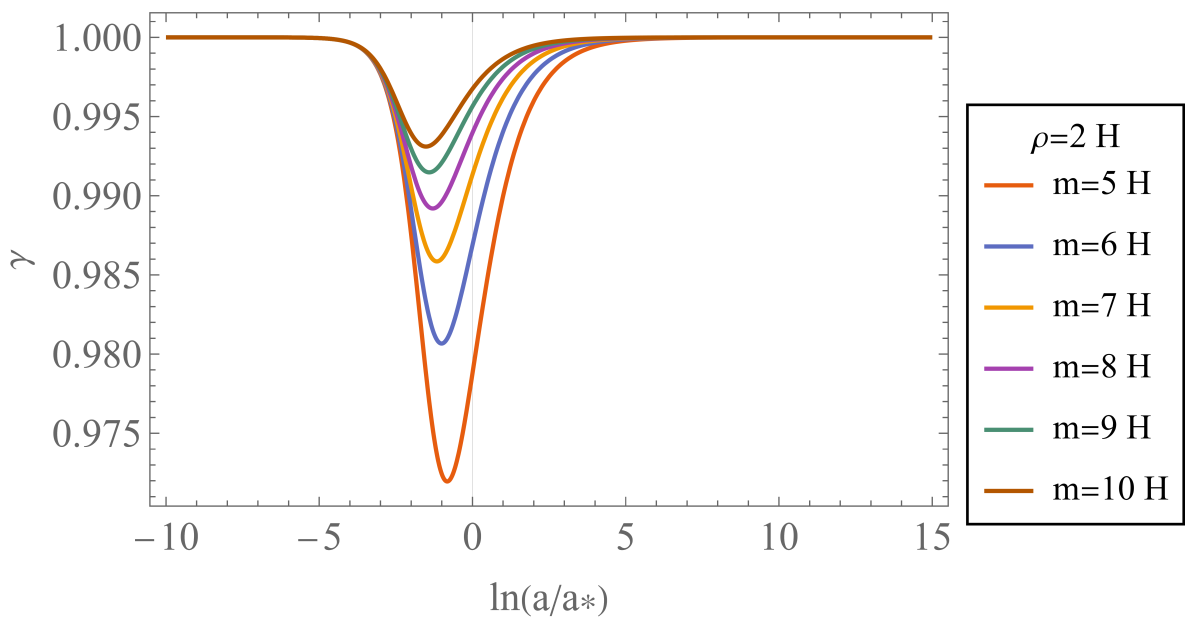


Figure 4.1: Purity as a function of the number of efolds since Hubble crossing of the reference scale $k = a_* H = 1$ for a fixed mass m and few values of the coupling constant ρ in the top figure and for a fixed value of ρ and few values of m in the bottom figure. We observe an initial period of decoherence followed by a period of recoherence where the purity saturates to 1 due to the effective turn off of the interaction explained in the main text. We see that the turning point happens on sub-Hubble scale in both figures, but in the upper figure it is exactly the same for all the values of the coupling constant, while in the bottom figure it depends on the mass, turning earlier for heavier masses. This is in agreement with the effective field theory (EFT) idea that a heavier environment leaves less imprint on the system.

increasing of the purity parameter which at late time shows large levels of self-coherence. This behaviour is called *recoherence*. We see that the turning point for the purity happens before that the scale of interest crosses the horizon, on sub-Hubble scale. [1] We also see that the decoherence is more effective for smaller masses and stronger coupling constant, as we expect from an effective field theory (EFT) intuition that heavier masses leave a smaller imprint on light degrees of freedom.

This recoherence can be interpreted as if the information that was transferred from the system to the environment during the period of decoherence, is flowing back from the environment to the system. This aspect could appear in contrast with the previous literature on decoherence, e.g. [28, 7], where due to the characteristics of the environment the information could not backflow to the system as in this case. Actually there is no contradiction, because of the small size of the environment in our model, where due to the linear coupling the system is coupled with a single Fourier mode of the environment; in this sense we cannot speak of the environment as a thermal bath as it is usually considered. In this case the bipartition between the system and the environment is based on the fact that the environment, even if it has the same size of the system, is unobservable, because the field is very massive. When the environment has many more degrees of freedom than the environment, the leakage of quantum information from the system to the environment causes the decoherence and the information backflow from the environment to the system is statistically suppressed given the enormous difference in the number of degrees of freedom. In our model instead, the quantum information is continuously exchanged between the system and the environment, given that they have the same dimension. This continuous exchange can be better understood if we consider the same model in a static spacetime [8]: in that case the purity appears to oscillate at frequencies $2\omega_s$, $2\omega_\varepsilon$, $\omega_S + \omega_\varepsilon$ and $\omega_S - \omega_\varepsilon$ where $\omega_S \equiv k$ and $\omega_\varepsilon \equiv \sqrt{k^2 + m^2}$; if the coupling is turned off the purity freezes at the time of the quench: this is exactly what happens in the case of a De Sitter spacetime where the the expansion of the Universe causes the coupling to be turned off. To better understand this aspect we can consider three regimes:

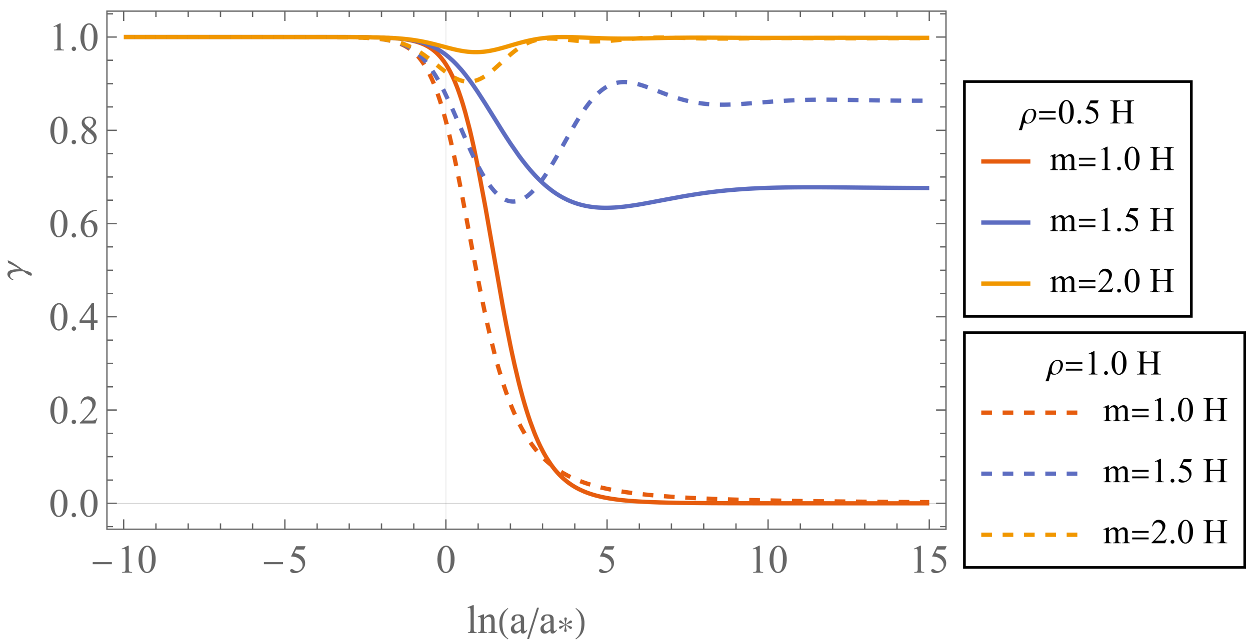


Figure 4.2: Same as figure 4.1, for lighter masses. We observe three different behaviours: $m > 3H/2$ recoherence, $m \simeq 3H/2$ purity freezing and $m < 3H/2$ decoherence

- $k \gg am$: in this case the mode functions of both fields oscillate at the same frequency k/a, in

[1] In the paper we are referring to, i.e. [8], it appears that the turning point is shifted on super-horizon scale. This is an incongruence with what is said in the text, where indeed they say that the turning point is reached in the sub-Hubble regime. To clarify this doubt, I contacted the author of the paper, Dr. Thomas Colas, to whom I am very grateful, who explain to me that reproducing the same graphs, he obtained the same result of mine, so probably there was some problem with the graphs in his paper.

their vacuum state;

- $am \gg k \gg aH$: in this regime the two frequencies differ, with the system oscillating at frequency k/a, while the environment oscillating at frequency m, so the purity oscillates as in the flat space;
- $k \ll aH$: in this regime we can observe three different behaviours, which are represented in figure 4.2:
  1. $m > \frac{3}{2}H$: entropic fluctuations are heavy, so they oscillate and quickly decay. The same happens for the curvature perturbation, since $\zeta' \propto 1/a^2$ on super-Hubble scales. Then the coupling between the adiabatic and entropic fluctuations is effectively turned off and the purity freezes to one.
  2. $m < \frac{3}{2}H$: in this case, $\mathcal{F}$ acquires a growing mode that keeps the interaction term, $\zeta'\mathcal{F}$ alive despite the decay of $\zeta'\mathcal{F}$, leading to a complete decoherence.
  3. $m \simeq \frac{3}{2}H$: in this case the purity freezes to a value different from 0 and 1, strongly dependent on the value of the coupling constant.

In all the cases we described, the system is driven towards a mixed state by the dynamical generation of entangled pairs of quanta between ζ and $\mathcal{F}$, which explains why decoherence happens even if the environment is made of a single degree of freedom. It is interesting to highlight that even if we are considering a heavy environment, varying the value of m we obtain three different behaviours: recoherence, purity freezing and decoherence.

4.2 Master equation approach

As we have seen in the previous section the model being linear can be solved exactly, but this is not possible in case of more complex interaction. For this reason we want also apply the OQS approach, using the master equation we derived previously.

Given the peculiar nature of the environment in our model, we cannot use a Markovian master equation, such as the Lindblad equation, that relies on the assumption that the environment is much larger than the system, which implies a form of irreversibility in the information flow. Clearly this is not the case in our model, and we need a non-Markovian master equation, like the TCL_2 we have derived in the previous chapter, which we report here:

$$\frac{\mathrm{d}\hat{\rho}_{red}}{\mathrm{d}\eta} = -i\Big[\hat{H}_S(\eta) + \hat{H}_{LS}(\eta), \hat{\rho}_{red}(\eta)\Big] + \gamma_{ij}(\eta)\Big(\hat{z}_{\zeta,i}\hat{\rho}_{red}(\eta)\hat{z}_{\zeta,j} - \frac{1}{2}\big\{\hat{z}_{\zeta,j}\hat{z}_{\zeta,i}, \hat{\rho}_{red}(\eta)\big\}\Big). \tag{4.22}$$

From this equation, we derived a transport equation for the covariance matrix, which being the evolution gaussian contains all the information about the quantum state and the observables we are interested in:

$$\frac{\mathrm{d}\Sigma}{\mathrm{d}\eta} = \boldsymbol{\omega}\big(\boldsymbol{H}_0 + \boldsymbol{\Delta}\big)\boldsymbol{\Sigma} - \boldsymbol{\Sigma}\big(\boldsymbol{H}_0 + \boldsymbol{\Delta}\big)\boldsymbol{\omega} - \boldsymbol{\omega}\boldsymbol{D}\boldsymbol{\omega} + 2\Delta_{12}\boldsymbol{\Sigma}. \tag{4.23}$$

Then we can also obtain a transport equation for the determinant of the system's covariance matrix, using the property $\det\boldsymbol{\Sigma} = e^{\ln\det\boldsymbol{\Sigma}} = e^{\mathrm{Tr}\ln\boldsymbol{\Sigma}}$:

$$\begin{aligned}\frac{\mathrm{d}\det\boldsymbol{\Sigma}}{\mathrm{d}\eta} &= \det\boldsymbol{\Sigma}\,\mathrm{Tr}\left(\boldsymbol{\Sigma}^{-1}\frac{\mathrm{d}\boldsymbol{\Sigma}}{\mathrm{d}\eta}\right) = \det\boldsymbol{\Sigma}\,\mathrm{Tr}\left[\boldsymbol{\Sigma}^{-1}\Big(\boldsymbol{\omega}(\boldsymbol{H}^{(S)} + \boldsymbol{\Delta})\boldsymbol{\Sigma} - \boldsymbol{\Sigma}(\boldsymbol{H}^{(S)} + \boldsymbol{\Delta})\boldsymbol{\omega} - \boldsymbol{\omega}\boldsymbol{D}\boldsymbol{\omega} + 2\Delta_{12}\boldsymbol{\Sigma}\Big)\right]\\ &= \det\Sigma\,\mathrm{Tr}\left[\boldsymbol{\omega}(\boldsymbol{H}^{(S)} + \boldsymbol{\Delta}) - (\boldsymbol{H}^{(S)} + \boldsymbol{\Delta})\boldsymbol{\omega} - \boldsymbol{\Sigma}^{-1}\boldsymbol{\omega}\boldsymbol{D}\boldsymbol{\omega} + 2\Delta_{12}\mathbb{1}\right]\\ &= 4\Delta_{12}\det\boldsymbol{\Sigma} - \det\boldsymbol{\Sigma}\,\mathrm{Tr}\left[\boldsymbol{\Sigma}^{-1}\boldsymbol{\omega}\boldsymbol{D}\boldsymbol{\omega}\right],\end{aligned} \tag{4.24}$$

where:

$$
\begin{aligned}
\boldsymbol{\Sigma}^{-1} &= \frac{1}{\det \boldsymbol{\Sigma}} \begin{pmatrix} \Sigma_{22} & -\Sigma_{12} \\ -\Sigma_{12} & \Sigma_{11} \end{pmatrix} \qquad \Rightarrow \\
\Rightarrow \qquad \boldsymbol{\Sigma}^{-1}\boldsymbol{\omega}\boldsymbol{D}\boldsymbol{\omega} &= \begin{pmatrix} \Sigma_{22} & -\Sigma_{12} \\ -\Sigma_{12} & \Sigma_{11} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} D_{11} & D_{12} \\ D_{12} & D_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= \frac{1}{\det \boldsymbol{\Sigma}} \begin{pmatrix} \Sigma_{12} & \Sigma_{22} \\ -\Sigma_{11} & -\Sigma_{12} \end{pmatrix} \begin{pmatrix} -D_{12} & D_{11} \\ -D_{22} & D_{12} \end{pmatrix} \\
&= \frac{1}{\det \boldsymbol{\Sigma}} \begin{pmatrix} -D_{12}\Sigma_{12} - D_{22}\Sigma_{22} & D_{11}\Sigma_{12} + D_{12}\Sigma_{12} \\ D_{12}\Sigma_{11} + D_{22}\Sigma_{12} & -D_{11}\Sigma_{11} - D_{12}\Sigma_{12} \end{pmatrix} \\
\Rightarrow \qquad \mathrm{Tr}\left[\boldsymbol{\Sigma}^{-1}\boldsymbol{\omega}\boldsymbol{D}\boldsymbol{\omega}\right] &= -\,\mathrm{Tr}\,[\boldsymbol{\Sigma}\boldsymbol{D}].
\end{aligned} \tag{4.25}
$$

So finally we get:

$$
\frac{\mathrm{d} \det \boldsymbol{\Sigma}}{\mathrm{d}\eta} = 4\Delta_{12} \det \boldsymbol{\Sigma} + \det \boldsymbol{\Sigma}\, \mathrm{Tr}\,[\boldsymbol{\Sigma}\mathbf{D}]. \tag{4.26}
$$

Now we want to compute the various pieces of this equation. First of all we need to compute the mode functions of the system and the environment.

4.2.1 Mode functions

In the interaction picture, operators evolve according to the free Hamiltonian $\hat{H}_0(\eta) = \hat{H}_0^S(\eta) \otimes \hat{H}_0^\varepsilon(\eta)$ where

$$
\hat{H}_0^S(\eta) = \frac{1}{2} \begin{pmatrix} \hat{v}_\zeta & \hat{p}_\zeta \end{pmatrix} \begin{pmatrix} k^2 & \frac{a'}{a} \\ \frac{a'}{a} & 1 \end{pmatrix} \begin{pmatrix} \hat{v}_\zeta \\ \hat{p}_\zeta \end{pmatrix} = \frac{1}{2}\left[\hat{p}_\zeta\hat{p}_\zeta + k^2\hat{v}_\zeta\hat{v}_\zeta + \frac{a'}{a}\{\hat{v}_\zeta, \hat{p}_\zeta\}\right], \tag{4.27}
$$

$$
\hat{H}_0^\varepsilon(\eta) = \frac{1}{2} \begin{pmatrix} \hat{v}_\mathcal{F} & \hat{p}_\mathcal{F} \end{pmatrix} \begin{pmatrix} k^2 + m^2a^2 & \frac{a'}{a} \\ \frac{a'}{a} & 1 \end{pmatrix} \begin{pmatrix} \hat{v}_\mathcal{F} \\ \hat{p}_\mathcal{F} \end{pmatrix} = \frac{1}{2}\left[\hat{p}_\mathcal{F}\hat{p}_\mathcal{F} + \left(k^2 + m^2a^2\right)\hat{v}_\mathcal{F}\hat{v}_\mathcal{F} + \frac{a'}{a}\{\hat{v}_\mathcal{F}, \hat{p}_\mathcal{F}\}\right]. \tag{4.28}
$$

The quantum states and the density matrix evolve according to the interaction Hamiltonian:

$$
\hat{H}_{int}(\eta) = \begin{pmatrix} \hat{v}_\zeta & \hat{p}_\zeta \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -\rho a & 0 \end{pmatrix} \begin{pmatrix} \hat{v}_\mathcal{F} \\ \hat{p}_\mathcal{F} \end{pmatrix} = -\rho a \hat{p}_\zeta \hat{v}_\mathcal{F}. \tag{4.29}
$$

In this picture, the field operators admit the following mode-functions decomposition:

$$
\tilde{v}_\alpha(\eta) = v_\alpha(\eta)\hat{a}_\alpha + v_\alpha^*(\eta)\hat{a}_\alpha^\dagger, \tag{4.30}
$$

where $\hat{a}_\alpha$ and $\hat{a}_\alpha^\dagger$ are the creation and annihilation operators of the uncoupled fields. Now using Heisenberg equation:

$$
\begin{aligned}
\frac{\mathrm{d}\hat{v}_\zeta}{\mathrm{d}\eta} &= i\left[\hat{H}_0^S(\eta), \hat{v}_\zeta\right] = \frac{i}{2}\left[\hat{p}_\zeta\hat{p}_\zeta + k^2\hat{v}_\zeta\hat{v}_\zeta + \frac{a'}{a}\{\hat{v}_\zeta, \hat{p}_\zeta\}, \hat{v}_\zeta\right] \\
&= \frac{i}{2}[\hat{p}_\zeta\hat{p}_\zeta, \hat{v}_\zeta] + \frac{i}{2}\frac{a'}{a}[\hat{v}_\zeta\hat{p}_\zeta, \hat{v}_\zeta] + \frac{i}{2}\frac{a'}{a}[\hat{p}_\zeta\hat{v}_\zeta, \hat{v}_\zeta] \\
&= \frac{i}{2}\hat{p}_\zeta \underbrace{[\hat{p}_\zeta, \hat{v}_\zeta]}_{-i} + \frac{i}{2}\underbrace{[\hat{p}_\zeta, \hat{v}_\zeta]}_{-i}\hat{p}_\zeta + \frac{i}{2}\frac{a'}{a}\hat{v}_\zeta \underbrace{[\hat{p}_\zeta, \hat{v}_\zeta]}_{-i} + \frac{i}{2}\frac{a'}{a}\underbrace{[\hat{p}_\zeta, \hat{v}_\zeta]}_{-i}\hat{v}_\zeta \\
&= \hat{p}_\zeta + \frac{a'}{a}\hat{v}_\zeta,
\end{aligned} \tag{4.31}
$$

$$
\begin{aligned}
\frac{\mathrm{d}\hat{p}_\zeta}{\mathrm{d}\eta} &= i\left[\hat{H}_0^S(\eta), \hat{p}_\zeta\right] = \frac{i}{2}\left[\hat{p}_\zeta\hat{p}_\zeta + k^2\hat{v}_\zeta\hat{v}_\zeta + \frac{a'}{a}\{\hat{v}_\zeta, \hat{p}_\zeta\}, \hat{p}_\zeta\right] \\
&= \frac{i}{2}k^2\hat{v}_\zeta \underbrace{[\hat{v}_\zeta, \hat{p}_\zeta]}_{i} + \frac{i}{2}k^2 \underbrace{[\hat{v}_\zeta, \hat{p}_\zeta]}_{i}\hat{v}_\zeta + \frac{i}{2}\frac{a'}{a}\underbrace{[\hat{v}_\zeta, \hat{p}_\zeta]}_{i}\hat{p}_\zeta + \frac{i}{2}\frac{a'}{a}\hat{p}_\zeta \underbrace{[\hat{v}_\zeta, \hat{p}_\zeta]}_{i} \\
&= -k^2\hat{v}_\zeta - \frac{a'}{a}\hat{p}_\zeta.
\end{aligned} \tag{4.32}
$$

So we have:

$$\begin{aligned}\frac{\mathrm{d}\hat{v}_\zeta}{\mathrm{d}\eta} &= v'_\zeta(\eta)\hat{a}_\zeta + v'^*_\zeta(\eta)\hat{a}^\dagger_\zeta = \left(p_\zeta(\eta) + \frac{a'}{a}v_\zeta(\eta)\right)\hat{a}_\zeta + \left(p^*_\zeta(\eta) + \frac{a'}{a}v^*_\zeta(\eta)\right)\hat{a}^\dagger_\zeta \\ &\Rightarrow v'_\zeta(\eta) = p_\zeta(\eta) + \frac{a'}{a}v_\zeta(\eta), \\ \frac{\mathrm{d}\hat{p}_\zeta}{\mathrm{d}\eta} &= p'_\zeta(\eta)\hat{a}_\zeta + p'^*_\zeta(\eta)\hat{a}^\dagger_\zeta = \left(-k^2v_\zeta(\eta) - \frac{a'}{a}p_\zeta(\eta)\right)\hat{a}_\zeta + \left(-k^2v^*_\zeta(\eta) - \frac{a'}{a}p^*_\zeta(\eta)\right)\hat{a}^\dagger_\zeta \\ &\Rightarrow p'_\zeta(\eta) = -k^2v_\zeta(\eta) - \frac{a'}{a}p_\zeta(\eta).\end{aligned} \tag{4.33}$$

So taking the second derivative:

$$\begin{aligned}v''_\zeta(\eta) &= p'_\zeta(\eta) + \frac{a''}{a}v_\zeta(\eta) - \frac{a'^2}{a^2}v_\zeta(\eta) + \frac{a'}{a}v'_\zeta(\eta) \\ &= -k^2v_\zeta(\eta) - \frac{a'}{a}p_\zeta(\eta) + \frac{a''}{a}v_\zeta(\eta) - \frac{a'^2}{a^2}v_\zeta(\eta) + \frac{a'}{a}p_\zeta(\eta) + \frac{a'^2}{a^2}v_\zeta(\eta) \\ &= -k^2v_\zeta(\eta) + \frac{a''}{a}v_\zeta(\eta) = -\left(k^2 - \frac{2}{\eta^2}v_\zeta(\eta)\right).\end{aligned} \tag{4.34}$$

where we used the relation

$$\frac{2}{\eta^2} = \frac{a''}{a}(1 + \mathcal{O}(\varepsilon)), \tag{4.35}$$

valid at lowest order in slow-roll parameters.
So we finally get:

$$v''_\zeta(\eta) + \left(k^2 - \frac{2}{\eta^2}\right)v_\zeta(\eta) = 0 \qquad \Rightarrow \qquad \boxed{v''_\zeta(\eta) + \left(k^2 - \frac{\nu^2 - \frac{1}{4}}{\eta^2}\right)v_\zeta(\eta) = 0}, \tag{4.36}$$

with $\nu = 3/2$.
Let us repeat the same procedure for the environment mode functions:

$$\begin{aligned}\frac{\mathrm{d}\hat{v}_\mathcal{F}}{\mathrm{d}\eta} &= i\left[\hat{H}^\varepsilon_0(\eta), \hat{v}_\mathcal{F}\right] = \frac{i}{2}\left[\hat{p}_\mathcal{F}\hat{p}_\mathcal{F} + \left(k^2 + m^2a^2\right)\hat{v}_\mathcal{F}\hat{v}_\mathcal{F} + \frac{a'}{a}\{\hat{v}_\mathcal{F}, \hat{p}_\mathcal{F}\}, \hat{v}_\mathcal{F}\right] \\ &= \frac{i}{2}\hat{p}_\mathcal{F}\underbrace{[\hat{p}_\mathcal{F}, \hat{v}_\mathcal{F}]}_{-i} + \frac{i}{2}\underbrace{[\hat{p}_\mathcal{F}, \hat{v}_\mathcal{F}]}_{-i}\hat{p}_\mathcal{F} + \frac{i}{2}\frac{a'}{a}\hat{v}_\mathcal{F}\underbrace{[\hat{p}_\mathcal{F}, \hat{v}_\mathcal{F}]}_{-i} + \frac{i}{2}\frac{a'}{a}\underbrace{[\hat{p}_\mathcal{F}, \hat{v}_\mathcal{F}]}_{-i}\hat{v}_\mathcal{F} \\ &= \hat{p}_\mathcal{F} + \frac{a'}{a}\hat{v}_\mathcal{F}, \\ \frac{\mathrm{d}\hat{p}_\mathcal{F}}{\mathrm{d}\eta} &= i\left[\hat{H}^\varepsilon_0(\eta), \hat{v}_\mathcal{F}\right] = \frac{i}{2}\left[\hat{p}_\mathcal{F}\hat{p}_\mathcal{F} + \left(k^2 + m^2a^2\right)\hat{v}_\mathcal{F}\hat{v}_\mathcal{F} + \frac{a'}{a}\{\hat{v}_\mathcal{F}, \hat{p}_\mathcal{F}\}, \hat{p}_\mathcal{F}\right] \\ &= \frac{i}{2}\left(k^2 + m^2a^2\right)\left(\hat{v}_\mathcal{F}\underbrace{[\hat{v}_\mathcal{F}, \hat{p}_\mathcal{F}]}_{i} + \underbrace{[\hat{v}_\mathcal{F}, \hat{p}_\mathcal{F}]}_{i}\hat{v}_\mathcal{F}\right) + \frac{i}{2}\frac{a'}{a}\left(\hat{p}_\mathcal{F}\underbrace{[\hat{v}_\mathcal{F}, \hat{p}_\mathcal{F}]}_{i} + \underbrace{[\hat{v}_\mathcal{F}, \hat{p}_\mathcal{F}]}_{i}\hat{p}_\mathcal{F}\right) \\ &= -\left(k^2 + m^2a^2\right)\hat{v}_\mathcal{F} - \frac{a'}{a}\hat{p}_\mathcal{F} \\ &\Rightarrow v'_\mathcal{F}(\eta) = p_\mathcal{F}(\eta) + \frac{a'}{a}v_\mathcal{F}(\eta) \\ &\Rightarrow p'_\mathcal{F}(\eta) = -\left(k^2 + m^2a^2\right)v_\mathcal{F}(\eta) - \frac{a'}{a}p_\mathcal{F}(\eta).\end{aligned} \tag{4.37}$$

Again, doing the second derivative:

$$\begin{aligned}
v''_{\mathcal{F}}(\eta) &= p'_{\mathcal{F}}(\eta) + \frac{a''}{a}v_{\mathcal{F}} - \frac{a'^2}{a^2}v_{\mathcal{F}} + \frac{a'}{a}v'_{\mathcal{F}} \\
&= -\big(k^2 + m^2a^2\big)v_{\mathcal{F}}(\eta) - \frac{a'}{a}p_{\mathcal{F}}(\eta) + \frac{a''}{a}v_{\mathcal{F}} - \frac{a'^2}{a^2}v_{\mathcal{F}} + \frac{a'}{a}p_{\mathcal{F}} + \frac{a'^2}{a^2}v_{\mathcal{F}} \\
&= -\big(k^2 + m^2a^2\big)v_{\mathcal{F}}(\eta) + \underbrace{\frac{a''}{a}}_{2/\eta^2}v_{\mathcal{F}} = -\left(k^2 - \frac{2}{\eta^2} + m^2a^2\right) \\
&= -\left(k^2 - \frac{2 - m^2a^2\eta^2}{\eta^2}\right) = -\left(k^2 - \frac{2 - \frac{m^2a^2}{a^2H^2}}{\eta^2}\right) \\
&= -\left(k^2 - \frac{\frac{9}{4} - \frac{1}{4} - \frac{m^2}{H^2}}{\eta^2}\right) = -\left(k^2 - \frac{\nu_{\mathcal{F}}^2 - \frac{1}{4}}{\eta^2}\right),
\end{aligned} \tag{4.38}$$

where we used $\eta = -\frac{1}{aH}$ and $\nu_{\mathcal{F}} \equiv \frac{3}{2}\sqrt{1 - \frac{4}{9}\frac{m^2}{H^2}} \equiv i\mu_{\mathcal{F}}$ assuming $m^2 > \frac{9}{4}H^2$.
So finally:

$$\boxed{v''_{\mathcal{F}}(\eta) + \left(k^2 - \frac{\nu_{\mathcal{F}}^2 - \frac{1}{4}}{\eta^2}\right)v_{\mathcal{F}}(\eta) = 0.} \tag{4.39}$$

Now we want to solve the mode functions equations by normalising the mode functions to the Bunch-Davies vacuum in the asymptotic, sub-Hubble past.
Let us start with the first equation:

$$v''_{\zeta}(\eta) + \left(k^2 - \frac{\nu^2 - \frac{1}{4}}{\eta^2}\right)v_{\zeta} = 0 \quad \text{with} \quad \nu = \frac{3}{2}. \tag{4.40}$$

This is equivalent to a Bessel equation which has solution of the kind:

$$v_{\zeta}(\eta) = \sqrt{-\eta}\Big[c_1(k)H_{\nu}^{(1)}(-k\eta) + c_2(k)H_{\nu}^{(2)}(-k\eta)\Big] \qquad H_{\nu}^{(2)} = \Big(H_{\nu}^{(1)}\Big)^*, \tag{4.41}$$

where $H_{\nu}^{(1)}$ and $H_{\nu}^{(2)}$ are the Hankel function of the first and second type, respectively.
To determine the constants we require that on sub-Hubble scale

$$v_{\zeta}(\eta) \sim \frac{e^{-ik\eta}}{\sqrt{2k}} \qquad \text{for } k \gg aH, \tag{4.42}$$

and

$$H_{\nu}^{(1)}(-k\eta) \xrightarrow{-k\eta\gg 1} \sqrt{-\frac{2}{\pi k\eta}}e^{i\left(-k\eta - \frac{\pi}{2}\nu - \frac{\pi}{4}\right)} \simeq \frac{e^{-ik\eta}}{\sqrt{-k\eta}}\sqrt{\frac{2}{\pi}}. \tag{4.43}$$

Since $H_{\nu}^{(2)} \sim e^{ik\eta}$ we can put $c_2(k) = 0$ and comparing the two expressions we get $c_1(k) = \frac{\sqrt{\pi}}{2}$, then

$$\Rightarrow v_{\zeta}(\eta) = \frac{\sqrt{\pi}}{2}\sqrt{\frac{-k\eta}{k}}H_{3/2}^{(1)}(-k\eta) = -\frac{1}{2}\sqrt{\frac{\pi z}{k}}H_{3/2}^{(1)}(z) = \left(1 + \frac{i}{z}\right)\frac{e^{iz}}{\sqrt{2k}}, \tag{4.44}$$

with $z \equiv -k\eta$.
The same discussion can be done for the second equation but this time $\nu = i\mu_{\mathcal{F}}$, so the general solution is:

$$v_{\mathcal{F}}(\eta) = \sqrt{-\eta}\Big[c_1(k)H_{\nu}^{(1)}(-k\eta) + c_2(k)H_{\nu}^{(2)}(-k\eta)\Big]. \tag{4.45}$$

Imposing that on sub-Hubble scale

$$v_{\mathcal{F}}(\eta) \sim \frac{e^{-ik\eta}}{\sqrt{2k}} \qquad \text{for } k \gg aH, \tag{4.46}$$

and using

$$H_{\nu}^{(1)}(-k\eta) \xrightarrow{-k\eta\gg 1} \sqrt{-\frac{2}{\pi k\eta}} e^{i\left(-k\eta-\frac{\pi}{2}\nu-\frac{\pi}{4}\right)} \qquad \text{for } k \gg aH \tag{4.47}$$

$$\Rightarrow c_1(k) = \frac{\sqrt{\pi}}{2} e^{i\frac{\pi}{2}\left(\nu+\frac{1}{2}\right)}. \tag{4.48}$$

So finally:

$$v_{\mathcal{F}}(\eta) = \frac{1}{2}\sqrt{\frac{\pi z}{k}} e^{-\frac{\pi}{2}\mu_{\mathcal{F}}+i\frac{\pi}{4}} H_{i\mu_{\mathcal{F}}}^{(1)}(-k\eta). \tag{4.49}$$

From these mode functions we can compute the conjugate momenta:

$$\begin{aligned}
p_\zeta &= v_\zeta' - \frac{a'}{a}v_\zeta = -i\sqrt{\frac{k}{2}}e^{iz}, \\
p_{\mathcal{F}} &= v_{\mathcal{F}}' - \frac{a'}{a}v_{\mathcal{F}} \\
&= \frac{1}{2}\sqrt{-\pi}\frac{1}{2\sqrt{\eta}}e^{-\frac{\pi}{2}\mu_{\mathcal{F}}+i\frac{\pi}{4}}H_{i\mu_{\mathcal{F}}}^{(1)}(-k\eta) + \frac{1}{2}\sqrt{-k\eta}e^{-\frac{\pi}{2}\mu_{\mathcal{F}}+i\frac{\pi}{4}}\frac{\mathrm{d}H_{i\mu_{\mathcal{F}}(-k\eta)}}{\mathrm{d}\eta} \\
&+ \frac{1}{2\eta}\sqrt{-\pi\eta}e^{-\frac{\pi}{2}\mu_{\mathcal{F}}+i\frac{\pi}{4}}H_{i\mu_{\mathcal{F}}}^{(1)}(-k\eta),
\end{aligned} \tag{4.50}$$

using the property of the Hankel function:

$$\frac{\mathrm{d}H_n^{(1)}(z)}{\mathrm{d}z} = \frac{nH_n^{(1)}(z)}{z} - H_{n+1}^{(1)}(z) \quad \Rightarrow \quad \frac{\mathrm{d}H_{i\mu_{\mathcal{F}}}^{(1)}(-k\eta)}{\mathrm{d}\eta} = -kn\frac{H_{i\mu_{\mathcal{F}}}^{(1)}(-k\eta)}{-k\eta} + kH_{i\mu_{\mathcal{F}}+1}^{(1)}(-k\eta), \tag{4.51}$$

we get:

$$\begin{aligned}
p_{\mathcal{F}} &= \frac{1}{2}e^{-\frac{\pi}{2}\mu_{\mathcal{F}}+i\frac{\pi}{4}}\left[\frac{1}{2}\sqrt{-\frac{\pi}{\eta}}H_{i\mu_{\mathcal{F}}}^{(1)}(-k\eta) + \sqrt{-\frac{\pi}{\eta}}H_{i\mu_{\mathcal{F}}}^{(1)}(-k\eta)\right. \\
&\left. + \sqrt{-\frac{\pi}{\eta}}\left(i\mu_{\mathcal{F}}H_{i\mu_{\mathcal{F}}}^{(1)}(-k\eta) + k\eta H_{i\mu_{\mathcal{F}}+1}^{(1)}(-k\eta)\right)\right] \\
&= \frac{1}{2}\sqrt{\frac{k\pi}{z}}e^{-\frac{\pi}{2}\mu_{\mathcal{F}}+i\frac{\pi}{4}}\left[\left(i\mu_{\mathcal{F}}+\frac{3}{2}\right)H_{i\mu_{\mathcal{F}}}^{(1)}(z) - zH_{i\mu_{\mathcal{F}}+1}^{(1)}(z)\right].
\end{aligned} \tag{4.52}$$

So finally:

$$\begin{aligned}
p_\zeta &= -i\sqrt{\frac{k}{2}}e^{iz}, \\
p_{\mathcal{F}} &= \frac{1}{2}\sqrt{\frac{k\pi}{z}}e^{-\frac{\pi}{2}\mu_{\mathcal{F}}+i\frac{\pi}{4}}\left[\left(i\mu_{\mathcal{F}}+\frac{3}{2}\right)H_{i\mu_{\mathcal{F}}}^{(1)}(z) - zH_{i\mu_{\mathcal{F}}+1}^{(1)}(z)\right].
\end{aligned} \tag{4.53}$$

4.2.2 Master equation coefficients

Let us go back to the master equation we derived previously. First of all we want to derive the memory kernel:

$$\mathcal{K}^{>} \equiv \mathrm{Tr}\left[\tilde{z}_{\mathcal{F}}^{T}(\eta)\tilde{z}_{\mathcal{F}}(\eta')\hat{\rho}_{\varepsilon}^{(0)}\right] = \mathrm{Tr}\left[\tilde{z}_{\mathcal{F}}(\eta')\hat{\rho}_{\varepsilon}^{(0)}\tilde{z}_{\mathcal{F}}(\eta)\right]. \tag{4.54}$$

Now using as initial state the vacuum state (i.e. the Bunch-Davies one, annihilated by $\hat{a}_{\mathcal{F}}$), $\hat{\rho}_{\varepsilon}^{(0)} = |0\rangle\langle 0|$ and using the decomposition for the mode functions:

$$\tilde{v}_{\mathcal{F}}(\eta) = v_{\mathcal{F}}(\eta)\hat{a}_{\mathcal{F}} + v_{\mathcal{F}}^{*}(\eta)\hat{a}_{\mathcal{F}}^{\dagger}, \tag{4.55}$$

$$
\begin{aligned}
\mathcal{K}_{11}^{>}(\eta,\eta') &= \mathrm{Tr}\left[\tilde{v}_{\mathcal{F}}(\eta')\left|0\right\rangle\left\langle 0\right|\tilde{v}_{\mathcal{F}}(\eta)\right] \\
&= \mathrm{Tr}\left[\left(v_{\mathcal{F}}(\eta')\hat{a}_{\mathcal{F}} + v^*_{\mathcal{F}}(\eta')\hat{a}^\dagger_{\mathcal{F}}\right)\left|0\right\rangle\left\langle 0\right|\left(v_{\mathcal{F}}(\eta)\hat{a}_{\mathcal{F}} + v^*_{\mathcal{F}}(\eta')\hat{a}^\dagger_{\mathcal{F}}\right)\right] \\
&= \mathrm{Tr}\left[v^*_{\mathcal{F}}(\eta')\hat{a}^\dagger_{\mathcal{F}}\left|0\right\rangle\left\langle 0\right|v_{\mathcal{F}}(\eta)\hat{a}_{\mathcal{F}}\right] = v_{\mathcal{F}}(\eta)v^*_{\mathcal{F}}(\eta') \qquad \underbrace{\mathrm{Tr}\left[\left|1\right\rangle\left\langle 1\right|\right]}_{\sum_{n=0}^{\infty}\langle n|1\rangle\langle 1|n\rangle=\langle 1|1\rangle\langle 1|1\rangle=1} \\
&= v_{\mathcal{F}}(\eta)v^*_{\mathcal{F}}(\eta'), \\
\mathcal{K}_{12}^{>}(\eta,\eta') &= \mathrm{Tr}\left[\tilde{v}_{\mathcal{F}}(\eta')\left|0\right\rangle\left\langle 0\right|\tilde{p}_{\mathcal{F}}(\eta)\right] \\
&= \mathrm{Tr}\left[\left(v_{\mathcal{F}}(\eta')\hat{a}_{\mathcal{F}} + v^*_{\mathcal{F}}(\eta')\hat{a}^\dagger_{\mathcal{F}}\right)\left|0\right\rangle\left\langle 0\right|\left(p_{\mathcal{F}}(\eta)\hat{a}_{\mathcal{F}} + p^*_{\mathcal{F}}(\eta)\hat{a}^\dagger_{\mathcal{F}}\right)\right] \\
&= p_{\mathcal{F}}(\eta)v^*_{\mathcal{F}}(\eta'), \\
\mathcal{K}_{21}^{>}(\eta,\eta') &= \mathrm{Tr}\left[\tilde{p}_{\mathcal{F}}(\eta')\left|0\right\rangle\left\langle 0\right|\tilde{v}_{\mathcal{F}}(\eta)\right] \\
&= \mathrm{Tr}\left[\left(p_{\mathcal{F}}(\eta')\hat{a}_{\mathcal{F}} + p^*_{\mathcal{F}}(\eta')\hat{a}^\dagger_{\mathcal{F}}\right)\left|0\right\rangle\left\langle 0\right|\left(v_{\mathcal{F}}(\eta)\hat{a}_{\mathcal{F}} + v^*_{\mathcal{F}}(\eta)\hat{a}^\dagger_{\mathcal{F}}\right)\right] \\
&= v_{\mathcal{F}}(\eta)p^*_{\mathcal{F}}(\eta'), \\
\mathcal{K}_{22}^{>}(\eta,\eta') &= \mathrm{Tr}\left[\tilde{p}_{\mathcal{F}}(\eta')\left|0\right\rangle\left\langle 0\right|\tilde{p}_{\mathcal{F}}(\eta)\right] \\
&= \mathrm{Tr}\left[\left(p_{\mathcal{F}}(\eta')\hat{a}_{\mathcal{F}} + p^*_{\mathcal{F}}(\eta')\hat{a}^\dagger_{\mathcal{F}}\right)\left|0\right\rangle\left\langle 0\right|\left(p_{\mathcal{F}}(\eta)\hat{a}_{\mathcal{F}} + p^*_{\mathcal{F}}(\eta)\hat{a}^\dagger_{\mathcal{F}}\right)\right] \\
&= p_{\mathcal{F}}(\eta)p^*_{\mathcal{F}}(\eta').
\end{aligned}
\tag{4.56}
$$

So we finally have:

$$
\mathcal{K}^{>}(\eta,\eta') = \begin{pmatrix} v_{\mathcal{F}}(\eta)v^*_{\mathcal{F}}(\eta') & p_{\mathcal{F}}(\eta)v^*_{\mathcal{F}}(\eta') \\ v_{\mathcal{F}}(\eta)p^*_{\mathcal{F}}(\eta') & p_{\mathcal{F}}(\eta)p^*_{\mathcal{F}}(\eta'). \end{pmatrix} \tag{4.57}
$$

In an analogous way we find the Green's matrix of the system, $\mathbf{G}^{(S)}(\eta',\eta) = \mathrm{Tr}\left\{\left[\tilde{z}^T_\zeta(\eta'),\tilde{z}_\zeta(\eta)\right]\hat{\rho}_S\right\}$, with $\hat{\rho}_S$ the initial state of the system:

$$
\mathbf{G}^{(S)}(\eta',\eta) = 2\begin{pmatrix} -\mathrm{Im}\left[p_\zeta(\eta)v^*_\zeta(\eta')\right] & \mathrm{Im}\left[v_\zeta(\eta)v^*_\zeta(\eta')\right] \\ -\mathrm{Im}\left[p_\zeta(\eta)p^*_\zeta(\eta')\right] & \mathrm{Im}\left[v_\zeta(\eta)p^*_\zeta(\eta')\right]. \end{pmatrix} \tag{4.58}
$$

Now we want to compute the master equation coefficients which are defined as:

$$
\begin{aligned}
\Delta_{ij}(\eta) &= 2\int_{\eta_0}^{\eta} d\eta' D^{\mathrm{Im}}_{(ij)}(\eta,\eta'), \\
D_{ij}(\eta) &= 2\int_{\eta_0}^{\eta} d\eta' D^{\mathrm{Re}}_{(ij)}(\eta,\eta'),
\end{aligned}
\tag{4.59}
$$

where the memory kernel is defined as:

$$
\mathbf{D}^{>}(\eta,\eta') = \mathbf{V}(\eta)\mathcal{K}^{>}(\eta,\eta')\mathbf{V}^T(\eta')\mathbf{G}^{(S)}(\eta',\eta), \tag{4.60}
$$

and it can be decomposed as:

$$
\mathbf{D}^{>}(\eta,\eta') = \mathbf{D}^{\mathrm{Re}}(\eta,\eta') + i\mathbf{D}^{\mathrm{Im}}(\eta,\eta'), \tag{4.61}
$$

leading to:

$$
\begin{aligned}
\mathbf{D}^{>}(\eta,\eta') &= \begin{pmatrix} 0 & 0 \\ -\rho a(\eta) & 0 \end{pmatrix}\begin{pmatrix} v_{\mathcal{F}}(\eta)v^*_{\mathcal{F}}(\eta') & p_{\mathcal{F}}(\eta)v^*_{\mathcal{F}}(\eta') \\ v_{\mathcal{F}}(\eta)p^*_{\mathcal{F}}(\eta') & p_{\mathcal{F}}(\eta)p^*_{\mathcal{F}}(\eta') \end{pmatrix}\begin{pmatrix} 0 & -\rho a(\eta') \\ 0 & 0 \end{pmatrix}\mathbf{G}^{(S)}(\eta',\eta) \\
&= \begin{pmatrix} 0 & 0 \\ -\rho a(\eta)v_{\mathcal{F}}(\eta)v^*_{\mathcal{F}}(\eta') & -\rho a(\eta)p_{\mathcal{F}}(\eta)v^*_{\mathcal{F}}(\eta') \end{pmatrix}\begin{pmatrix} 0 & -\rho a(\eta') \\ 0 & 0 \end{pmatrix}\mathbf{G}^{(S)}(\eta',\eta) \\
&= 2\begin{pmatrix} 0 & 0 \\ 0 & \rho^2 a(\eta)a(\eta')v_{\mathcal{F}}(\eta)v^*_{\mathcal{F}}(\eta') \end{pmatrix}\begin{pmatrix} -\mathrm{Im}\left[p_\zeta(\eta)v^*_\zeta(\eta')\right] & \mathrm{Im}\left[v_\zeta(\eta)v^*_\zeta(\eta')\right] \\ -\mathrm{Im}\left[p_\zeta(\eta)p^*_\zeta(\eta')\right] & \mathrm{Im}\left[v_\zeta(\eta)p^*_\zeta(\eta')\right] \end{pmatrix} \\
&= 2\begin{pmatrix} 0 & 0 \\ -\rho^2 a(\eta)a(\eta')v_{\mathcal{F}}(\eta)v^*_{\mathcal{F}}(\eta')\,\mathrm{Im}\left[p_\zeta(\eta)p^*_\zeta(\eta')\right] & \rho^2 a(\eta)a(\eta')v_{\mathcal{F}}(\eta)v^*_{\mathcal{F}}(\eta')\,\mathrm{Im}\left[v_\zeta(\eta)p^*_\zeta(\eta')\right] \end{pmatrix}.
\end{aligned}
\tag{4.62}
$$

So from the definition we see that the master equation coefficients are the following:

$$\begin{aligned}
&\Delta_{11}(\eta) = 0,\\
&\Delta_{12}(\eta) = \Delta_{21}(\eta) = -2\rho^2 a(\eta) \int_{\eta_0}^{\eta} d\eta' a(\eta') \operatorname{Im}\left[p_\zeta(\eta) p_\zeta^*(\eta')\right] \operatorname{Im}\left[v_{\mathcal{F}}(\eta) v_{\mathcal{F}}^*(\eta')\right],\\
&\Delta_{22}(\eta) = 4\rho^2 a(\eta) \int_{\eta_0}^{\eta} d\eta' a(\eta') \operatorname{Im}\left[v_\zeta(\eta) p_\zeta^*(\eta')\right] \operatorname{Im}\left[v_{\mathcal{F}}(\eta) v_{\mathcal{F}}^*(\eta')\right],\\
&D_{11}(\eta) = 0,\\
&D_{12}(\eta) = D_{21}(\eta) = -2\rho^2 a(\eta) \int_{\eta_0}^{\eta} d\eta' a(\eta') \operatorname{Im}\left[p_\zeta(\eta) p_\zeta^*(\eta')\right] \operatorname{Re}\left[v_{\mathcal{F}}(\eta) v_{\mathcal{F}}^*(\eta')\right],\\
&D_{22}(\eta) = 4\rho^2 a(\eta) \int_{\eta_0}^{\eta} d\eta' a(\eta') \operatorname{Im}\left[v_\zeta(\eta) p_\zeta^*(\eta')\right] \operatorname{Re}\left[v_{\mathcal{F}}(\eta) v_{\mathcal{F}}^*(\eta')\right],
\end{aligned} \tag{4.63}$$

By simple manipulations, using $\operatorname{Im} z = \frac{z-z^*}{2i}$ and $\operatorname{Re} z = \frac{z+z^*}{2}$, we can rewrite:

$$\begin{aligned}
\Delta_{12}(\eta) &= -\frac{2\rho^2}{4i^2} a(\eta) \int_{\eta_0}^{\eta} d\eta' a(\eta') \left[\left(p_\zeta(\eta) p_\zeta^*(\eta') - p_\zeta^*(\eta) p_\zeta(\eta')\right)\left(v_{\mathcal{F}}(\eta) v_{\mathcal{F}}^*(\eta') - v_{\mathcal{F}}^*(\eta) v_{\mathcal{F}}(\eta')\right)\right]\\
&= \frac{\rho^2}{2} a(\eta) \int_{\eta_0}^{\eta} d\eta' a(\eta') \Big[p_\zeta(\eta) v_{\mathcal{F}}(\eta) p_\zeta^*(\eta') v_{\mathcal{F}}^*(\eta') - p_\zeta(\eta) v_{\mathcal{F}}^*(\eta) p_\zeta^*(\eta') v_{\mathcal{F}}(\eta')\\
&- p_\zeta^*(\eta) v_{\mathcal{F}}(\eta) p_\zeta(\eta') v_{\mathcal{F}}^*(\eta') + p_\zeta^*(\eta) v_{\mathcal{F}}^*(\eta) p_\zeta(\eta') v_{\mathcal{F}}(\eta')\Big]\\
&= \frac{\rho^2}{2} a(\eta) \operatorname{Re}\Big[p_\zeta(\eta) v_{\mathcal{F}}(\eta) \int_{\eta_0}^{\eta} d\eta' a(\eta') p_\zeta^*(\eta') v_{\mathcal{F}}^*(\eta')\\
&- p_\zeta(\eta) v_{\mathcal{F}}^*(\eta) \int_{\eta_0}^{\eta} d\eta' a(\eta') p_\zeta^*(\eta') v_{\mathcal{F}}(\eta') + \text{h.c.}\Big]\\
&= \rho^2 a(\eta) \operatorname{Re}\Big[p_\zeta(\eta) v_{\mathcal{F}}(\eta) \int_{\eta_0}^{\eta} d\eta' a(\eta') p_\zeta^*(\eta') v_{\mathcal{F}}^*(\eta')\\
&- p_\zeta(\eta) v_{\mathcal{F}}^*(\eta) \int_{\eta_0}^{\eta} d\eta' a(\eta') p_\zeta^*(\eta') v_{\mathcal{F}}(\eta')\Big].
\end{aligned} \tag{4.64}$$

Now using $z = -k\eta \quad \Rightarrow \quad \eta = -\frac{z}{k} = -\frac{1}{aH} \quad d\eta = -\frac{dz}{k} \quad \Rightarrow \quad a(\eta) = -\frac{1}{\eta H} = \frac{k}{zH}$:

$$\begin{aligned}
\Delta_{12}(z) &= -\frac{\rho^2}{H^2}\frac{k}{z} \operatorname{Re}\left[p_\zeta(\eta) v_{\mathcal{F}}(\eta) \int_{z_0}^{z} \frac{dz'}{z'} p_\zeta^*(\eta') v_{\mathcal{F}}^*(\eta') - p_\zeta(\eta) v_{\mathcal{F}}^*(\eta) \int_{z_0}^{z} \frac{dz'}{z'} p_\zeta^*(\eta') v_{\mathcal{F}}(\eta')\right],\\
\Delta_{22}(\eta) &= 2\frac{\rho^2}{H^2}\frac{k}{z} \operatorname{Re}\left[v_\zeta(z) v_{\mathcal{F}}(z) \int_{z_0}^{z} \frac{dz'}{z'} p_\zeta^*(z') v_{\mathcal{F}}^*(z') - v_\zeta(z) v_{\mathcal{F}}^*(z) \int_{z_0}^{z} \frac{dz'}{z'} p_\zeta^*(z') v_{\mathcal{F}}(z')\right],\\
D_{12}(\eta) &= \frac{\rho^2}{H}\frac{k}{z} \operatorname{Im}\left[p_\zeta(z) v_{\mathcal{F}}(z) \int_{z_0}^{z} \frac{dz'}{z'} p_\zeta^*(z') v_{\mathcal{F}}^*(z') + p_\zeta(z) v_{\mathcal{F}}^*(z) \int_{z_0}^{z} \frac{dz'}{z'} p_\zeta^*(z') v_{\mathcal{F}}(z')\right],\\
D_{22}(\eta) &= -2\frac{\rho^2}{H}\frac{k}{z} \operatorname{Im}\left[v_\zeta(z) v_{\mathcal{F}}(z) \int_{z_0}^{z} \frac{dz'}{z'} p_\zeta^*(z') v_{\mathcal{F}}^*(z') + v_\zeta(z) v_{\mathcal{F}}^*(z) \int_{z_0}^{z} \frac{dz'}{z'} p_\zeta^*(z') v_{\mathcal{F}}(z')\right]
\end{aligned} \tag{4.65}$$

where the explicit computation is shown in appendix A.2.
To obtain analytical expressions for the master equation coefficients, we have to compute two integrals. The first one is:

$$I_1(z, z_0) = \int_{z_0}^{z} \frac{dz'}{z'} p_\zeta^*(z') v_{\mathcal{F}}^*(z'). \tag{4.66}$$

Inserting the mode functions:

$$\begin{aligned}
I_1(z, z_0) &= \int_{z_0}^{z} \frac{dz'}{z'} \underbrace{\left(\frac{1}{2}\sqrt{k\pi z'} H_{1/2}^{(1)*}(z')\right)}_{i\sqrt{\frac{k}{2}} e^{-iz'}} \left(\frac{1}{2}\sqrt{\frac{\pi z'}{k}} e^{-\frac{\pi}{2}\mu_{\mathcal{F}} - i\frac{\pi}{4}} H_{i\mu_{\mathcal{F}}}^{(1)*}(z')\right)\\
&= \frac{i}{2}\sqrt{\frac{\pi}{2}} e^{-\frac{\pi}{2}\mu_{\mathcal{F}} - i\frac{\pi}{4}} \int_{z_0}^{z} \frac{dz'}{\sqrt{z'}} e^{-iz'} H_{-i\mu_{\mathcal{F}}}^{(2)}(z'),
\end{aligned} \tag{4.67}$$

where we used the property of the Hankel function $H_\alpha^{(1)*} = H_{-\alpha}^{(2)}$. Working with Mathematica, the last integral gives:

$$\begin{aligned}
&\int_{z_0}^{z} \frac{dz'}{\sqrt{z'}} e^{-iz'} H^{(2)}_{-i\mu_\mathcal{F}}(z') = 2i2^{-i\mu_\mathcal{F}} z_0^{\frac{1}{2}-i\mu_\mathcal{F}} \times \\
&\times \Bigg[z_0^{2i\mu_\mathcal{F}} \frac{1}{\sin(\pi i\mu_\mathcal{F})} \frac{{}_2F_2\big(\frac{1}{2}+i\mu_\mathcal{F},\frac{1}{2}+i\mu_\mathcal{F};\frac{3}{2}+i\mu_\mathcal{F},1+2i\mu_\mathcal{F}\big)(-2iz_0)}{(1+2i\mu_\mathcal{F})\Gamma(1+i\mu_\mathcal{F})} + \\
&+ 2^{2i\mu_\mathcal{F}}(\cot(\pi i\mu_\mathcal{F})-i) \frac{{}_2F_2\big(\frac{1}{2}-i\mu_\mathcal{F},\frac{1}{2}-i\mu_\mathcal{F};1-2i\mu_\mathcal{F},\frac{3}{2}-i\mu_\mathcal{F}\big)(-2iz_0)}{(2i\mu_\mathcal{F}-1)\Gamma(1-i\mu_\mathcal{F})}\Bigg] - \\
&- 2i2^{-i\mu_\mathcal{F}} z^{\frac{1}{2}-i\mu_\mathcal{F}} \Bigg[z^{2i\mu_\mathcal{F}} \frac{1}{\sin(\pi i\mu_\mathcal{F})} \frac{{}_2F_2(\frac{1}{2}+i\mu_\mathcal{F},\frac{1}{2}+i\mu_\mathcal{F};\frac{3}{2}+i\mu_\mathcal{F},1+2i\mu_\mathcal{F})(-2iz)}{(2i\mu_\mathcal{F}+1)\Gamma(1+i\mu_\mathcal{F})} + \\
&+ 2^{2i\mu_\mathcal{F}}(\cot(\pi i\mu_\mathcal{F})-i) \frac{{}_2F_2(\frac{1}{2}-i\mu_\mathcal{F},\frac{1}{2}-i\mu_\mathcal{F};1-2i\mu_\mathcal{F},\frac{3}{2}-i\mu_\mathcal{F})(-2iz)}{(2i\mu_\mathcal{F}-1)\Gamma(1-i\mu_\mathcal{F})}\Bigg],
\end{aligned} \tag{4.68}$$

then using the following trigonometric properties:

$$\begin{aligned}
\sin(\pi i\mu_\mathcal{F}) &= -i\sinh(-\pi\mu_\mathcal{F}) = i\sinh(\pi\mu_\mathcal{F}), \\
\cot x &= \frac{\cos x}{\sin x} = \frac{\cosh(ix)}{-isinh(ix)} = i\coth(ix).
\end{aligned} \tag{4.69}$$

So we get:

$$\begin{aligned}
&= 2^{1-i\mu_\mathcal{F}} \Bigg\{ z^{\frac{1}{2}-i\mu_\mathcal{F}} \Bigg[-\frac{z^{2i\mu_\mathcal{F}}}{\sinh(\pi\mu_\mathcal{F})} \frac{{}_2F_2(\frac{1}{2}+i\mu_\mathcal{F},\frac{1}{2}+i\mu_\mathcal{F};\frac{3}{2}+i\mu_\mathcal{F},1+2i\mu_\mathcal{F})(-2iz)}{(2i\mu_\mathcal{F}+1)\Gamma(1+i\mu_\mathcal{F})} - \\
&- 2^{2i\mu_\mathcal{F}}(\coth(\pi\mu_\mathcal{F})+1) \frac{{}_2F_2(\frac{1}{2}-i\mu_\mathcal{F},\frac{1}{2}-i\mu_\mathcal{F};\frac{3}{2}-i\mu_\mathcal{F},1-2i\mu_\mathcal{F})(-2iz)}{(2i\mu_\mathcal{F}-1)\Gamma(1-i\mu_\mathcal{F})}\Bigg] - (z\to z_0)\Bigg\} \\
&= 2\Bigg\{ \sqrt{z}\Bigg[\left(\frac{z}{2}\right)^{i\mu_\mathcal{F}} \frac{-1}{\sinh(\pi\mu_\mathcal{F})} \frac{1}{\Gamma(1+i\mu_\mathcal{F})} \frac{1}{1+2i\mu_\mathcal{F}} {}_2F_2(\frac{1}{2}+i\mu_\mathcal{F},\frac{1}{2}+i\mu_\mathcal{F};\frac{3}{2}+i\mu_\mathcal{F},1+2i\mu_\mathcal{F})(-2iz) + \\
&+ \left(\frac{z}{2}\right)^{-i\mu_\mathcal{F}} \frac{1+\coth(\pi\mu_\mathcal{F})}{\Gamma(1-i\mu_\mathcal{F})} \frac{1}{1-2i\mu_\mathcal{F}} {}_2F_2(\frac{1}{2}-i\mu_\mathcal{F},\frac{1}{2}-i\mu_\mathcal{F};\frac{3}{2}-i\mu_\mathcal{F},1-2i\mu_\mathcal{F})(-2iz)\Bigg] - (z\to z_0)\Bigg\}.
\end{aligned} \tag{4.70}$$

Now defining the following functions:

$$\begin{aligned}
\gamma_{\mu_\mathcal{F}}(z) &\equiv \frac{1+\coth(\pi\mu_\mathcal{F})}{\Gamma(1+i\mu_\mathcal{F})}\left(\frac{z}{2}\right)^{i\mu_\mathcal{F}}, \\
\delta_{\mu_\mathcal{F}}(z) &\equiv \frac{-1}{\sinh(\pi\mu_\mathcal{F})}\frac{1}{\Gamma(1-i\mu_\mathcal{F})}\left(\frac{z}{2}\right)^{-i\mu_\mathcal{F}}, \\
g_{\mu_\mathcal{F}}(z) &\equiv \frac{1}{1-2i\mu_\mathcal{F}} {}_2F_2\left(\frac{1}{2}-i\mu_\mathcal{F},\frac{1}{2}-i\mu_\mathcal{F};\frac{3}{2}-i\mu_\mathcal{F},1-2i\mu_\mathcal{F}\right).
\end{aligned} \tag{4.71}$$

So that

$$\int_{z_0}^{z} \frac{dz'}{\sqrt{z'}} e^{-iz'} H^{(2)}_{-i\mu_\mathcal{F}}(z') = 2\sqrt{z}\big[\gamma^*_{\mu_\mathcal{F}}(z) g_{\mu_\mathcal{F}}(z) + \delta^*_{\mu_\mathcal{F}}(z) g_{-\mu_\mathcal{F}}(z)\big] - (z\to z_0). \tag{4.72}$$

Then:

$$I_1(z,z_0) = F_{I_1}(z) - F_{I_1}(z_0), \tag{4.73}$$

with

$$F_{I_1}(z) = i\sqrt{\frac{\pi}{2}} e^{-\frac{\pi}{2}\mu_\mathcal{F} - i\frac{\pi}{4}} \sqrt{z}\big[\gamma^*_{\mu_\mathcal{F}}(z) g_{\mu_\mathcal{F}}(z) + \delta^*_{\mu_\mathcal{F}}(z) g_{-\mu_\mathcal{F}}(z)\big], \tag{4.74}$$

where $g^*_{\mu_\mathcal{F}}(z) = g_{-\mu_\mathcal{F}}(-z)$.
The second integral to compute is:

$$I_2(z,z_0) = \int_{z_0}^{z} \frac{dz'}{z'} p^*_\zeta(z') v_\mathcal{F}(z'). \tag{4.75}$$

Inserting the mode functions we get:

$$I_2(z,z_0) = \frac{i}{2}\sqrt{\frac{\pi}{2}} e^{-\frac{\pi}{2}\mu_\mathcal{F} + i\frac{\pi}{4}} \int_{z_0}^{z} \frac{dz'}{\sqrt{z'}} e^{-iz'} H^{(1)}_{i\mu_\mathcal{F}}(z'). \tag{4.76}$$

Again using Mathematica we get:

$$\int_{z_0}^{z} \frac{dz'}{\sqrt{z'}} e^{-iz'} H^{(1)}_{i\mu_\mathcal{F}}(z') = 2\sqrt{z}[\gamma_{\mu_\mathcal{F}}(z) g_{-\mu_\mathcal{F}}(z) + \delta_{\mu_\mathcal{F}}(z) g_{\mu_\mathcal{F}}(z)] - (z \to z_0). \tag{4.77}$$

So we can write the second integral as:

$$I_2(z,z_0) = F_{I_2}(z) - F_{I_2}(z_0), \tag{4.78}$$

with:

$$F_{I_2}(z) = i\sqrt{\frac{\pi}{2}} e^{-\frac{\pi}{2}\mu_\mathcal{F} + i\frac{\pi}{4}} \sqrt{z}[\gamma_{\mu_\mathcal{F}}(z) g_{-\mu_\mathcal{F}}(z) + \delta_{\mu_\mathcal{F}}(z) g_{\mu_\mathcal{F}}(z)]. \tag{4.79}$$

Notice that the master equation coefficients are expressed as an integral between t_0 and t, so they can be formally written as:

$$D_{11} = F_{D_{11}}(\eta,\eta) - F_{D_{11}}(\eta,\eta_0), \tag{4.80}$$

where $F_{D_{11}}(\eta,\cdot)$ is the primitive of the integrand appearing in the definition. The second term, the one depending on the initial time η_0 is dubbed "spurious" for several reasons [32].

First, the spurious terms involve a dependence on the initial time η_0 and if the environment memory kernel (4.54) is dominated by contributions around $\eta' = \eta$, then this contributions get suppressed with respect to the non-spurious one. This is essentially a Markovian limit, but we are considering a non-Markovian dynamics, so this argument is not enough to justify the erase of the spurious terms in our model.

Second, and more important point in our case, in the exact solution we derived in the previous section, there is no initial time dependence and indeed one can show [32] that the spurious terms cancel out at all orders in perturbation theory. This is consistent with the fact that, at leading order, TCL coincides with the exact theory, which contains no spurious terms.

Third, it has been proved [32] that if we solve the transport equation for the covariance matrix using also the spurious terms, the result quickly blows up, while removing them provides a remarkably well-behaved result. For all these reasons we will remove by hand the spurious terms in the following.

Thus finally our master equation coefficients are:

$$\begin{aligned}
\Delta_{12}(z) &= -\frac{\rho^2}{H^2}\frac{k}{z}\,\mathrm{Re}\,[p_\zeta(z) v_\mathcal{F}(z) F_{I_1}(z) - p_\zeta(z) v^*_\mathcal{F}(z) F_{I_2}(z)], \\
\Delta_{22}(z) &= 2\frac{\rho^2}{H^2}\frac{k}{z}\,\mathrm{Re}\,[v_\zeta(z) v_\mathcal{F}(z) F_{I_1}(z) - v_\zeta(z) v^*_\mathcal{F}(z) F_{I_2}(z)], \\
D_{12}(z) &= \frac{\rho^2}{H^2}\frac{k}{z}\,\mathrm{Im}\,[p_\zeta(z) v_\mathcal{F}(z) F_{I_1}(z) + p_\zeta(z) v^*_\mathcal{F}(z) F_{I_2}(z)], \\
D_{22}(z) &= -2\frac{\rho^2}{H^2}\frac{k}{z}\,\mathrm{Im}\,[v_\zeta(z) v_\mathcal{F}(z) F_{I_1}(z) + v_\zeta(z) v^*_\mathcal{F}(z) F_{I_2}(z)].
\end{aligned} \tag{4.81}$$

4.2.3 Perturbative limit

Now we want to compute the previous transport equation in the perturbative limit, keeping only contributions up to order $\mathcal{O}(\rho^2)$. This amounts to replace $\mathbf{\Sigma}$ by its free-theory counterpart $\mathbf{\Sigma}^{(0)}$, when multiplied by $\mathbf{\Delta}$ or $\boldsymbol{D}$ (that are already at order $\mathcal{O}(\rho^2)$), so we get the following equation:

$$\frac{\mathrm{d}\mathbf{\Sigma}^{(2)}}{\mathrm{d}\eta} = \boldsymbol{\omega}\boldsymbol{H}^{(S)}\mathbf{\Sigma}^{(2)} - \mathbf{\Sigma}^{(2)}\boldsymbol{H}^{(S)}\omega + \boldsymbol{\omega}\mathbf{\Delta}\mathbf{\Sigma}^{(0)} - \mathbf{\Sigma}^{(0)}\mathbf{\Delta}\boldsymbol{\omega} - \boldsymbol{\omega}\boldsymbol{D}\boldsymbol{\omega} + 2\Delta_{12}\mathbf{\Sigma}^{(0)}, \tag{4.82}$$

$$\frac{\mathrm{d}\det\mathbf{\Sigma}^{(2)}}{\mathrm{d}\eta} = \mathrm{Tr}\left(\mathbf{\Sigma}^{(0)}\boldsymbol{D}\right) + \Delta_{12}, \tag{4.83}$$

where we used $\det\mathbf{\Sigma}^{(0)} = \frac{1}{4}$.

Explicitly we have:

$$\begin{aligned}
\frac{\mathrm{d}\Sigma_{11}^{(2)}}{\mathrm{d}\eta} &= 2\left(\frac{a'}{a}\Sigma_{11}^{(2)} + \Sigma_{12}^{(2)} + 2\Delta_{12}\Sigma_{11}^{(0)} + \Delta_{22}\Sigma_{12}^{(0)}\right) + D_{22},\\
\frac{\mathrm{d}\Sigma_{12}^{(2)}}{\mathrm{d}\eta} &= \Sigma_{22}^{(2)} - k^2\Sigma_{11}^{(2)} + \Delta_{22}\Sigma_{22}^{(0)} - D_{12} + 2\Delta_{12}\Sigma_{12}^{(0)},\\
\frac{\mathrm{d}\Sigma_{22}^{(2)}}{\mathrm{d}\eta} &= -2\left(k^2\Sigma_{12}^{(2)} + \frac{a'}{a}\Sigma_{22}^{(2)}\right),
\end{aligned} \tag{4.84}$$

and for the $\det\mathbf{\Sigma}$:

$$\begin{aligned}
\frac{\mathrm{d}\det\mathbf{\Sigma}^{(2)}}{\mathrm{d}\eta} &= \mathrm{Tr}\left[\begin{pmatrix}\Sigma_{11}^{(0)} & \Sigma_{12}^{(0)}\\ \Sigma_{12}^{(0)} & \Sigma_{22}^{(0)}\end{pmatrix}\begin{pmatrix}0 & D_{12}\\ D_{12} & D_{22}\end{pmatrix}\right] + \Delta_{12} = \mathrm{Tr}\begin{pmatrix}D_1 2\Sigma_{12}^{(0)} & D_{12}\Sigma_{11}^{(0)} + D_{22}\Sigma_{12}^{(0)}\\ D_{12}\Sigma_{22}^{(0)} & D_{12}\Sigma_{12}^{(0)} + D_{22}\Sigma_{22}^{(0)}\end{pmatrix} + \Delta_{12}\\
&= 2D_{12}\Sigma_{12}^{(0)} + D_{22}\Sigma_{22}^{(0)} + \Delta_{12}.
\end{aligned} \tag{4.85}$$

So in terms of $\ln\left(\frac{a}{a_*}\right) \equiv t$

$$\begin{aligned}
\frac{\mathrm{d}\Sigma_{11}^{(2)}}{\mathrm{d}t} &= 2\Sigma_{11}^{(2)} + \frac{e^{-t}}{k}\Big[2\Big(\Sigma_{12}^{(2)} + 2\Delta_{12}\Sigma_{11}^{(0)} + \Delta_{22}\Sigma_{12}^{(0)}\Big) + D_{22}\Big],\\
\frac{\mathrm{d}\Sigma_{12}^{(2)}}{\mathrm{d}t} &= \frac{e^{-t}}{k}\Big[\Sigma_{22}^{(2)} - k^2\Sigma_{11}^{(2)} + \Delta_{22}\Sigma_{22}^{(2)} - D_{12} + 2\Delta_{12}\Sigma_{12}^{(0)}\Big],\\
\frac{\mathrm{d}\Sigma_{22}^{(2)}}{\mathrm{d}t} &= -2\Sigma_{22}^{(2)} - 2ke^{-t}\Sigma_{12}^{(2)},\\
\frac{\mathrm{d}\det\mathbf{\Sigma}^{(2)}}{\mathrm{d}t} &= \frac{e^{-t}}{k}\Big[2D_{12}\Sigma_{12}^{(0)} + D_{22}\Sigma_{22}^{(0)} + \Delta_{12}\Big].
\end{aligned} \tag{4.86}$$

In figure 4.3 we can see the result obtained using the master equation and its perturbative limit, compared with the exact result found previously. We see an excellent agreement, in particular the turning point is well reproduced. Notice that the full master equation solution does not improve significantly the perturbative result. This is what we expected, since the main non-perturbative effect is the resummation at late time, but in this case we know that at late time the interaction is quenched off, so there is no effect to be resummed and the perturbative limit gives almost the same prediction of the complete master equation.

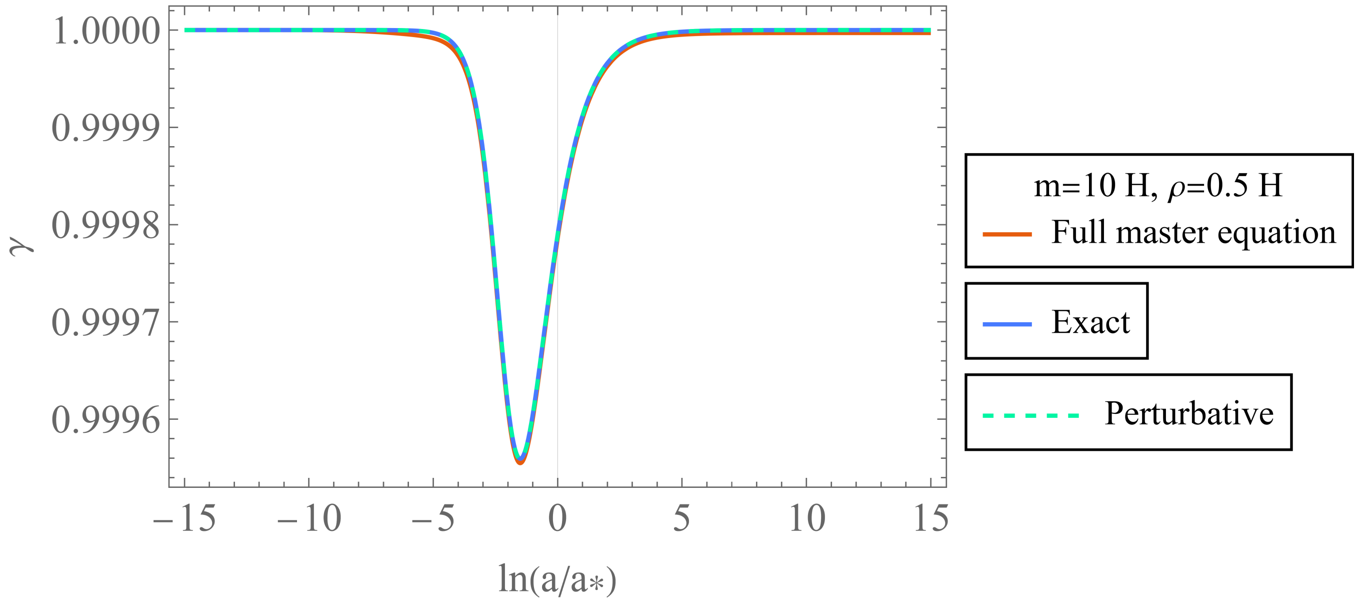


Figure 4.3: Same as figure 4.1 and 4.2, with the exact result compared with the full master equation result and the perturbative limit of the master equation. In this case we use as a scale $k = 0.2$. The agreement is excellent and get even better decreasing ρ or increasing m.

Chapter 5

Non-linear model

In this chapter we will present our original contribution to the quantum-to-classical transition problem. In the previous part we considered a model describing a system made of the curvature perturbations which are measurable in the CMB and in the LSS and an environment made of a heavy scalar field. In that case the interaction was linear, leading to a gaussian dynamics that could be solved both exactly and with a master equation approach and we observed that both methods give a good result. In this chapter we want to extend the previous model by adding a non-linear term in the environment sector and we want to study how the purity is changed by this interaction. This addition leads to a more realistic model than the previous one and will lead to some form of non-gaussianities [33]. Our expectation would be that this new term "increases" the interaction between system and environment, spoiling somehow the recoherence phenomenon and amplifying the decoherence period. Given the non-linear interaction this model is not solvable exactly and we will rely only on the master equation approach.

The new interaction term is given by

$$\mathcal{L}_{int} = \mu \mathcal{F}^3 \qquad [\mu] = E \qquad \mathcal{F} = \frac{v_{\mathcal{F}}}{a}. \tag{5.1}$$

Then the total lagrangian reads:

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}v_\zeta'^2 + \frac{1}{2}\left(\frac{a'}{a}\right)^2 v_\zeta^2 - \frac{a'}{a}v_\zeta' v_\zeta - \frac{1}{2}(\partial_i v_\zeta)^2 + \frac{1}{2}v_{\mathcal{F}}'^2 + \frac{1}{2}\left(\frac{a'}{a}\right)^2 v_{\mathcal{F}}^2 - \frac{a'}{a}v_{\mathcal{F}}' v_{\mathcal{F}} - \frac{1}{2}(\partial_i v_{\mathcal{F}})^2 \\ &- \frac{1}{2}m^2 a^2 v_{\mathcal{F}}^2 + \rho a v_{\mathcal{F}} v_\zeta' - \rho a' v_\zeta v_{\mathcal{F}} + \frac{\mu}{a^3} v_{\mathcal{F}}^3.\end{aligned} \tag{5.2}$$

The conjugated momenta are the same of the linear case:

$$\begin{aligned} p_\zeta &= \frac{\partial \mathcal{L}}{\partial v_\zeta'} = v_\zeta' - \frac{a'}{a}v_\zeta + \rho a v_{\mathcal{F}} \quad \rightarrow \quad v_\zeta' = p_\zeta + \frac{a'}{a}v_\zeta - \rho a v_{\mathcal{F}}, \\ p_{\mathcal{F}} &= \frac{\partial \mathcal{L}}{\partial v_{\mathcal{F}}'} = v_{\mathcal{F}}' - \frac{a'}{a}v_{\mathcal{F}} \quad \rightarrow \quad v_{\mathcal{F}}' = p_{\mathcal{F}} + \frac{a'}{a}v_{\mathcal{F}}, \end{aligned} \tag{5.3}$$

and the Hamiltonian density gets modified as:

$$\begin{aligned}\mathcal{H}(v_\zeta, v_{\mathcal{F}}, p_\zeta, p_{\mathcal{F}}, \eta) = &\frac{1}{2}p_\zeta^2 + \frac{1}{2}p_{\mathcal{F}}^2 + \frac{1}{2}(\partial_i v_\zeta)^2 + \frac{1}{2}(\partial_i v_{\mathcal{F}})^2 + \frac{a'}{a}v_\zeta p_\zeta - \rho a v_{\mathcal{F}} p_\zeta \\ &+ \frac{a'}{a}v_{\mathcal{F}} p_{\mathcal{F}} + \frac{1}{2}\rho^2 a^2 v_{\mathcal{F}}^2 + \frac{1}{2}m^2 a^2 v_{\mathcal{F}}^2 - \frac{\mu}{a^3}v_{\mathcal{F}}^3.\end{aligned} \tag{5.4}$$

Using the Fourier decomposition for the Mukhanov-Sasaki variable and its conjugate momentum

$$v_\alpha(\eta, \mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3\mathbf{k} v_\alpha(\eta, \mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{5.5}$$

The only term different from the linear case is the last one, which after integrating over $\mathbf{x}$ (to get the Hamiltonian) and decomposing in Fourier space becomes:

$$
\begin{aligned}
&-\frac{\mu}{a^3}\frac{1}{(2\pi)^{9/2}}\int d^3\mathbf{x}\int d^3\mathbf{k}d^3\mathbf{k}'d^3\mathbf{k}''v_{\mathcal{F}}(\mathbf{k})v_{\mathcal{F}}(\mathbf{k}')v_{\mathcal{F}}(\mathbf{k}'')e^{i(\mathbf{k}+\mathbf{k}'\mathbf{k}'')\cdot\mathbf{x}} \\
&= -\frac{\mu}{a^3}\frac{1}{(2\pi)^{9/2}}\int d^3\mathbf{k}d^3\mathbf{k}'d^3\mathbf{k}''v_{\mathcal{F}}(\mathbf{k})v_{\mathcal{F}}(\mathbf{k}')v_{\mathcal{F}}(\mathbf{k}'')\delta^3(\mathbf{k}+\mathbf{k}'+\mathbf{k}'') \\
&= -\frac{\mu}{a^3}\frac{1}{(2\pi)^{9/2}}\int d^3\mathbf{k}d^3\mathbf{k}'v_{\mathcal{F}}(\mathbf{k})v_{\mathcal{F}}(\mathbf{k}')v_{\mathcal{F}}(-\mathbf{k}-\mathbf{k}').
\end{aligned}
\tag{5.6}
$$

So the final Hamiltonian (multiplied by 2 and integrated over $\mathbb{R}^{3+}$ due to the reality condition) is:

$$
\begin{aligned}
H = 2\int_{\mathbb{R}^{3+}} d^3\mathbf{k}\Bigg[&\frac{1}{2}p_\zeta(\mathbf{k})p_\zeta(-\mathbf{k})+\frac{1}{2}p_{\mathcal{F}}(\mathbf{k})p_{\mathcal{F}}(-\mathbf{k})+\frac{1}{2}k^2v_\zeta(\mathbf{k})v_\zeta(-\mathbf{k}) \\
&+\frac{1}{2}(k^2+(\rho^2+m^2)a^2)v_{\mathcal{F}}(\mathbf{k})v_{\mathcal{F}}(-\mathbf{k})+\frac{a'}{a}v_\zeta(\mathbf{k})p_\zeta(-\mathbf{k})+\frac{a'}{a}v_{\mathcal{F}}(\mathbf{k})p_{\mathcal{F}}(-\mathbf{k})-\rho a v_{\mathcal{F}}(\mathbf{k})p_\zeta(-\mathbf{k}) \\
&-\frac{\mu}{a^3}\frac{1}{(2\pi)^{9/2}}\int d^3\mathbf{k}d^3\mathbf{k}'v_{\mathcal{F}}(\mathbf{k})v_{\mathcal{F}}(\mathbf{k}')v_{\mathcal{F}}(-\mathbf{k}-\mathbf{k}')\Bigg].
\end{aligned}
\tag{5.7}
$$

Now we promote the field variables to quantum operators and in order to work with Hermitian operators we split the field in real and imaginary parts:

$$
\tilde{\chi}_\alpha = \frac{1}{\sqrt{2}}(\tilde{\chi}_\alpha^R + i\tilde{\chi}_\alpha^I),
\tag{5.8}
$$

where $\chi = v, p$ and $\alpha = \zeta, \mathcal{F}$. Then using $\chi(-\mathbf{k}) = \chi^*(\mathbf{k})$ the linear part gives:

$$
\tilde{H}_{\text{linear}}(t) = \frac{1}{2}\sum_{s=R,I}\int_{\mathbb{R}^{3+}} d^3\mathbf{k}(\tilde{z}^s)^T\mathbf{H}(t)\tilde{z}^s,
\tag{5.9}
$$

where we used the same notation as equation (4.12). The non-linear part gives:

$$
\begin{aligned}
\tilde{H}_{\text{non-linear}}(t) = &-\frac{\mu}{\sqrt{2}a^3}\frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{k}d^3\mathbf{k}'\left(\tilde{v}^R_{\mathcal{F}}(\mathbf{k},t)+i\tilde{v}^I_{\mathcal{F}}(\mathbf{k},t)\right)\left(\tilde{v}^R_{\mathcal{F}}(\mathbf{k}',t)+i\tilde{v}^I_{\mathcal{F}}(\mathbf{k}',t)\right) \\
&\left(\tilde{v}^R_{\mathcal{F}}(\mathbf{k}+\mathbf{k}',t)-i\tilde{v}^I_{\mathcal{F}}(\mathbf{k}+\mathbf{k}',t)\right) \\
&= -\frac{\mu}{\sqrt{2}a^3}\frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{k}d^3\mathbf{k}'\Big[\tilde{v}^R_{\mathcal{F}}(\mathbf{k},t)\tilde{v}^R_{\mathcal{F}}(\mathbf{k}',t)\tilde{v}^R_{\mathcal{F}}(\mathbf{k}+\mathbf{k}',t)-i\tilde{v}^R_{\mathcal{F}}(\mathbf{k},t)\tilde{v}^R_{\mathcal{F}}(\mathbf{k}',t)\tilde{v}^I_{\mathcal{F}}(\mathbf{k}+\mathbf{k}',t)+ \\
&+i\tilde{v}^R_{\mathcal{F}}(\mathbf{k},t)\tilde{v}^I_{\mathcal{F}}(\mathbf{k}',t)\tilde{v}^R_{\mathcal{F}}(\mathbf{k}+\mathbf{k}',t)+\tilde{v}^R_{\mathcal{F}}(\mathbf{k},t)\tilde{v}^I_{\mathcal{F}}(\mathbf{k}',t)\tilde{v}^I_{\mathcal{F}}(\mathbf{k}+\mathbf{k}',t)+i\tilde{v}^I_{\mathcal{F}}(\mathbf{k},t)\tilde{v}^R_{\mathcal{F}}(\mathbf{k}',t)\tilde{v}^R_{\mathcal{F}}(\mathbf{k}+\mathbf{k}',t)+ \\
&+\tilde{v}^I_{\mathcal{F}}(\mathbf{k},t)\tilde{v}^R_{\mathcal{F}}(\mathbf{k}',t)\tilde{v}^I_{\mathcal{F}}(\mathbf{k}+\mathbf{k}',t)-\tilde{v}^I_{\mathcal{F}}(\mathbf{k},t)\tilde{v}^I_{\mathcal{F}}(\mathbf{k}',t)\tilde{v}^R_{\mathcal{F}}(\mathbf{k}+\mathbf{k}',t)+i\tilde{v}^I_{\mathcal{F}}(\mathbf{k},t)\tilde{v}^I_{\mathcal{F}}(\mathbf{k}',t)\tilde{v}^I_{\mathcal{F}}(\mathbf{k}+\mathbf{k}',t)\Big] \\
&= -\frac{\mu}{\sqrt{2}a^3}\frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{k}d^3\mathbf{k}'\sum_{s,q,\bar{q}=R,I}C_{sq\bar{q}}\tilde{v}^s_{\mathcal{F}}(\mathbf{k},t)\tilde{v}^q_{\mathcal{F}}(\mathbf{k}',t)\tilde{v}^{\bar{q}}_{\mathcal{F}}(\mathbf{k}+\mathbf{k}',t),
\end{aligned}
\tag{5.10}
$$

where we defined:

$$
C_{RRR}=1, \qquad C_{RRI}=-i, \qquad C_{RIR}=i, \qquad C_{RII}=1,
\tag{5.11}
$$

$$
C_{IRR}=i, \qquad C_{IRI}=1, \qquad C_{IIR}=-1, \qquad C_{III}=i.
\tag{5.12}
$$

So we can write the interaction Hamiltonian as:

$$
\begin{aligned}
\tilde{H}_{int}(t) = &-\int d^3\mathbf{k}\rho a(t)\sum_s \tilde{v}^s_{\mathcal{F}}(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t)- \\
&-\frac{\mu}{\sqrt{2}a^3(t)}\frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{k}d^3\mathbf{k}'\sum_{s,q,\bar{q}}C_{sq\bar{q}}\tilde{v}^s_{\mathcal{F}}(\mathbf{k},t)\tilde{v}^q_{\mathcal{F}}(\mathbf{k}',t)\tilde{v}^{\bar{q}}_{\mathcal{F}}(\mathbf{k}+\mathbf{k}',t),
\end{aligned}
\tag{5.13}
$$

$$\tilde{H}_{int}(t') = -\int d^3\mathbf{p}\rho a(t')\sum_l \tilde{v}^l_{\mathcal{F}}(\mathbf{p},t')\tilde{p}^l_\zeta(\mathbf{p},t') - \\ - \frac{\mu}{\sqrt{2}a^3(t')}\frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{p}d^3\mathbf{p}'\sum_{\bar{l},n,\bar{n}} C_{\bar{l}n\bar{n}}\tilde{v}^{\bar{l}}_{\mathcal{F}}(\mathbf{p},t')\hat{v}^n_{\mathcal{F}}(\mathbf{p}',t')\tilde{v}^{\bar{n}}_{\mathcal{F}}(\mathbf{p}+\mathbf{p}',t'). \tag{5.14}$$

So, working in perturbation theory, neglecting $\mathcal{O}(\mu^2)$ contributions we get:

$$\begin{aligned}\tilde{H}_{int}(t)\tilde{H}_{int}(t') &= \int d^3\mathbf{k}d^3\mathbf{p}\rho^2 a(t)a(t')\sum_{s,l}\tilde{v}^s_{\mathcal{F}}(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{v}^l_{\mathcal{F}}(\mathbf{p},t')\tilde{p}^l_\zeta(\mathbf{p},t') + \\ &+ \int d^3\mathbf{k}d^3\mathbf{p}\Bigg[\frac{\rho\mu a(t)}{\sqrt{2}a^3(t')}\frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{p}'\sum_{s,\bar{l},n,\bar{n}} C_{\bar{l}n\bar{n}}\tilde{v}^s_{\mathcal{F}}(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{v}^{\bar{l}}_{\mathcal{F}}(\mathbf{p},t')\tilde{v}^n_{\mathcal{F}}(\mathbf{p}',t')\tilde{v}^{\bar{n}}_{\mathcal{F}}(\mathbf{p}+\mathbf{p}',t') + \\ &+ \frac{\rho\mu a(t')}{\sqrt{2}a^3(t)}\frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{k}'\sum_{l,\bar{s},q,\bar{q}} C_{\bar{s}q\bar{q}}\tilde{v}^{\bar{s}}_{\mathcal{F}}(\mathbf{k},t)\tilde{v}^q_{\mathcal{F}}(\mathbf{k}',t)\tilde{v}^{\bar{q}}_{\mathcal{F}}(\mathbf{k}+\mathbf{k}',t)\tilde{v}^l_{\mathcal{F}}(\mathbf{p},t')\tilde{p}^l_\zeta(\mathbf{p},t')\Bigg],\end{aligned} \tag{5.15}$$

$$\begin{aligned}\tilde{H}_{int}(t')\tilde{H}_{int}(t) &= \int d^3\mathbf{k}d^3\mathbf{p}\rho^2 a(t)a(t')\sum_{s,l}\hat{v}^l_{\mathcal{F}}(\mathbf{p},t')\tilde{p}^l_\zeta(\mathbf{p},t')\tilde{v}^s_{\mathcal{F}}(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t) + \\ &+ \int d^3\mathbf{k}d^3\mathbf{p}\Bigg[\frac{\rho\mu a(t')}{\sqrt{2}a^3(t)}\frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{k}'\sum_{l,\bar{s},q,\bar{q}} C_{\bar{s}q\bar{q}}\tilde{v}^l_{\mathcal{F}}(\mathbf{p},t')\tilde{p}^l_\zeta(\mathbf{p},t')\tilde{v}^{\bar{s}}_{\mathcal{F}}(\mathbf{k},t)\tilde{v}^q_{\mathcal{F}}(\mathbf{k}',t)\tilde{v}^{\bar{q}}_{\mathcal{F}}(\mathbf{k}+\mathbf{k}',t) + \\ &+ \frac{\rho\mu a(t)}{\sqrt{2}a^3(t')}\frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{p}'\sum_{s,\bar{l},n,\bar{n}} C_{\bar{l}n\bar{n}}\tilde{v}^{\bar{l}}_{\mathcal{F}}(\mathbf{p},t')\tilde{v}^n_{\mathcal{F}}(\mathbf{p}',t')\tilde{v}^{\bar{n}}_{\mathcal{F}}(\mathbf{p}+\mathbf{p}',t')\tilde{v}^s_{\mathcal{F}}(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t)\Bigg].\end{aligned} \tag{5.16}$$

Inserting these results in the TCL$_2$ master equation (the explicit computation is shown in appendix A.3):

$$\begin{aligned}\frac{\mathrm{d}\tilde{\rho}_{red}(t)}{\mathrm{d}t} &= -\int_{t_0}^t dt'\,\mathrm{Tr}_\varepsilon\left[\tilde{H}_{int}(t),\left[\tilde{H}_{int}(t'),\tilde{\rho}_{red}(t)\otimes\hat{\rho}^{(0)}_\varepsilon\right]\right] \\ &= -\int_{t_0}^t dt'\Bigg\{\int d^3\mathbf{k}d^3\mathbf{p}\rho^2 a(t)a(t')\sum_{s,l}\Bigg(\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{p}^l_\zeta(\mathbf{p},t')\tilde{\rho}_{red}(t)\,\mathrm{Tr}_\varepsilon\left[\tilde{v}^s_{\mathcal{F}}(\mathbf{k},t)\tilde{v}^l_{\mathcal{F}}(\mathbf{p},t')\hat{\rho}^{(0)}_\varepsilon\right] \\ &- \tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\tilde{p}^l_\zeta(\mathbf{p},t')\,\mathrm{Tr}_\varepsilon\left[\tilde{v}^s_{\mathcal{F}}(\mathbf{k},t)\hat{\rho}^{(0)}_\varepsilon\tilde{v}^l_{\mathcal{F}}(\mathbf{p},t')\right] - \tilde{p}^l_\zeta(\mathbf{p},t')\tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t)\,\mathrm{Tr}_\varepsilon\left[\tilde{v}^l_{\mathcal{F}}(\mathbf{p},t')\hat{\rho}^{(0)}_\varepsilon\tilde{v}^s_{\mathcal{F}}(\mathbf{k},t)\right] \\ &+ \tilde{\rho}_{red}(t)\tilde{p}^l_\zeta(\mathbf{p},t')\tilde{p}^s_\zeta(\mathbf{k},t)\,\mathrm{Tr}_\varepsilon\left[\hat{\rho}^{(0)}_\varepsilon\tilde{v}^l_{\mathcal{F}}(\mathbf{p},t')\tilde{v}^s_{\mathcal{F}}(\mathbf{k},t)\right]\Bigg) \\ &+ \int d^3\mathbf{k}d^3\mathbf{p}\Bigg[\frac{\rho\mu a(t)}{\sqrt{2}a^3(t')}\frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{p}'\sum_{s,\bar{l},n,\bar{n}} C_{\bar{l}n\bar{n}}\Bigg(\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\,\mathrm{Tr}_\varepsilon\left[\tilde{v}^s_{\mathcal{F}}(\mathbf{k},t)\tilde{v}^{\bar{l}}_{\mathcal{F}}(\mathbf{p},t')\tilde{v}^n_{\mathcal{F}}(\mathbf{p}',t')\right] \\ &\times\tilde{v}^{\bar{n}}_{\mathcal{F}}(\mathbf{p}+\mathbf{p}',t')\hat{\rho}^{(0)}_\varepsilon - \tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\,\mathrm{Tr}_\varepsilon\left[\tilde{v}^s_{\mathcal{F}}(\mathbf{k},t)\hat{\rho}^{(0)}_\varepsilon\tilde{v}^{\bar{l}}_{\mathcal{F}}(\mathbf{p},t')\tilde{v}^n_{\mathcal{F}}(\mathbf{p}',t')\tilde{v}^{\bar{n}}_{\mathcal{F}}(\mathbf{p}+\mathbf{p}',t')\right] \\ &- \tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t)\,\mathrm{Tr}_\varepsilon\left[\tilde{v}^{\bar{l}}_{\mathcal{F}}(\mathbf{p},t')\tilde{v}^n_{\mathcal{F}}(\mathbf{p}',t')\tilde{v}^{\bar{n}}_{\mathcal{F}}(\mathbf{p}+\mathbf{p}',t')\hat{\rho}^{(0)}_\varepsilon\tilde{v}^s_{\mathcal{F}}(\mathbf{k},t)\right] \\ &+ \tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t)\,\mathrm{Tr}_\varepsilon\left[\hat{\rho}^{(0)}_\varepsilon\tilde{v}^{\bar{l}}_{\mathcal{F}}(\mathbf{p},t')\tilde{v}^n_{\mathcal{F}}(\mathbf{p}',t')\tilde{v}^{\bar{n}}_{\mathcal{F}}(\mathbf{p}+\mathbf{p}',t')\tilde{v}^s_{\mathcal{F}}(\mathbf{k},t)\right]\Bigg)\end{aligned} \tag{5.17}$$

$$
+ \frac{\rho\mu a(t')}{\sqrt{2}a^3(t)}\frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{k}' \sum_{l,\bar{s},q,\bar{q}} C_{\bar{s}q\bar{q}}\Bigg(\tilde{p}^l_\zeta(\mathbf{p},t')\tilde{\rho}_{red}(t)\,\mathrm{Tr}_\varepsilon\left[\tilde{v}^{\bar{s}}_{\mathcal{F}}(\mathbf{k},t)\tilde{v}^{q}_{\mathcal{F}}(\mathbf{k}',t)\tilde{v}^{\bar{q}}_{\mathcal{F}}(\mathbf{k}+\mathbf{k}',t)\tilde{v}^{l}_{\mathcal{F}}(\mathbf{p},t')\hat{\rho}^{(0)}_\varepsilon\right]
$$

$$
- \tilde{\rho}_{red}(t)\tilde{p}^l_\zeta(\mathbf{p},t')\,\mathrm{Tr}_\varepsilon\left[\tilde{v}^{\bar{s}}_{\mathcal{F}}(\mathbf{k},t)\tilde{v}^{q}_{\mathcal{F}}(\mathbf{k}',t)\tilde{v}^{\bar{q}}_{\mathcal{F}}(\mathbf{k}+\mathbf{k}',t)\hat{\rho}^{(0)}_\varepsilon\tilde{v}^{l}_{\mathcal{F}}(\mathbf{p},t')\right]
$$

$$
- \tilde{p}^l_\zeta(\mathbf{p},t')\tilde{\rho}_{red}(t)\,\mathrm{Tr}_\varepsilon\left[\tilde{v}^{l}_{\mathcal{F}}(\mathbf{p},t')\hat{\rho}^{(0)}_\varepsilon\tilde{v}^{\bar{s}}_{\mathcal{F}}(\mathbf{k},t)\tilde{v}^{q}_{\mathcal{F}}(\mathbf{k}',t)\tilde{v}^{\bar{q}}_{\mathcal{F}}(\mathbf{k}+\mathbf{k}',t)\right]
$$

$$
+ \tilde{\rho}_{red}(t)\tilde{p}^l_\zeta(\mathbf{p},t')\,\mathrm{Tr}_\varepsilon\left[\hat{\rho}^{(0)}_\varepsilon\tilde{v}^{l}_{\mathcal{F}}(\mathbf{p},t')\tilde{v}^{\bar{s}}_{\mathcal{F}}(\mathbf{k},t)\tilde{v}^{q}_{\mathcal{F}}(\mathbf{k}',t)\tilde{v}^{\bar{q}}_{\mathcal{F}}(\mathbf{k}+\mathbf{k}',t)\right]\Bigg)\Bigg]\Bigg\}. \tag{5.18}
$$

Using the cyclicity of the trace the term inside the last round brackets vanishes and we finally get:

$$
\frac{\mathrm{d}\tilde{\rho}_{red}(t)}{\mathrm{d}t} = -\int_{t_0}^{t} dt'\Bigg\{\int d^3\mathbf{k}d^3\mathbf{p}\rho^2 a(t)a(t')\sum_{s,l}\Bigg(\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{p}^l_\zeta(\mathbf{p},t')\tilde{\rho}_{red}(t)\,\mathrm{Tr}_\varepsilon\left[\tilde{v}^{s}_{\mathcal{F}}(\mathbf{k},t)\tilde{v}^{l}_{\mathcal{F}}(\mathbf{p},t')\hat{\rho}^{(0)}_\varepsilon\right]
$$

$$
- \tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\tilde{p}^l_\zeta(\mathbf{p},t')\,\mathrm{Tr}_\varepsilon\left[\tilde{v}^{l}_{\mathcal{F}}(\mathbf{p},t')\tilde{v}^{s}_{\mathcal{F}}(\mathbf{k},t)\hat{\rho}^{(0)}_\varepsilon\right]
$$

$$
- \tilde{p}^l_\zeta(\mathbf{p},t')\tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t)\,\mathrm{Tr}_\varepsilon\left[\tilde{v}^{s}_{\mathcal{F}}(\mathbf{k},t)\tilde{v}^{l}_{\mathcal{F}}(\mathbf{p},t')\hat{\rho}^{(0)}_\varepsilon\right]
$$

$$
+ \hat{\rho}_{red}(t)\tilde{p}^l_\zeta(\mathbf{p},t')\tilde{p}^s_\zeta(\mathbf{k},t)\,\mathrm{Tr}_\varepsilon\left[\tilde{v}^{l}_{\mathcal{F}}(\mathbf{p},t')\tilde{v}^{s}_{\mathcal{F}}(\mathbf{k},t)\hat{\rho}^{(0)}_\varepsilon\right]\Bigg)
$$

$$
+ \frac{\rho\mu a(t)}{\sqrt{2}a^3(t')}\frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{k}d^3\mathbf{p}\int d^3\mathbf{p}'\sum_{s,\bar{l},n,\bar{n}} C_{\bar{l}n\bar{n}}
$$

$$
\Bigg(\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\,\mathrm{Tr}_\varepsilon\left[\tilde{v}^{s}_{\mathcal{F}}(\mathbf{k},t)\tilde{v}^{\bar{l}}_{\mathcal{F}}(\mathbf{p},t')\tilde{v}^{n}_{\mathcal{F}}(\mathbf{p}',t')\right]\tilde{v}^{\bar{n}}_{\mathcal{F}}(\mathbf{p}+\mathbf{p}',t')\hat{\rho}^{(0)}_\varepsilon
$$

$$
- \tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\,\mathrm{Tr}_\varepsilon\left[\tilde{v}^{\bar{l}}_{\mathcal{F}}(\mathbf{p},t')\tilde{v}^{n}_{\mathcal{F}}(\mathbf{p}',t')\tilde{v}^{\bar{n}}_{\mathcal{F}}(\mathbf{p}+\mathbf{p}',t')\tilde{v}^{s}_{\mathcal{F}}(\mathbf{k},t)\hat{\rho}^{(0)}_\varepsilon\right]
$$

$$
- \tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t)\,\mathrm{Tr}_\varepsilon\left[\tilde{v}^{s}_{\mathcal{F}}(\mathbf{k},t)\tilde{v}^{\bar{l}}_{\mathcal{F}}(\mathbf{p},t')\tilde{v}^{n}_{\mathcal{F}}(\mathbf{p}',t')\tilde{v}^{\bar{n}}_{\mathcal{F}}(\mathbf{p}+\mathbf{p}',t')\hat{\rho}^{(0)}_\varepsilon\right]
$$

$$
+ \tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t)\,\mathrm{Tr}_\varepsilon\left[\tilde{v}^{\bar{l}}_{\mathcal{F}}(\mathbf{p},t')\tilde{v}^{n}_{\mathcal{F}}(\mathbf{p}',t')\tilde{v}^{\bar{n}}_{\mathcal{F}}(\mathbf{p}+\mathbf{p}',t')\tilde{v}^{s}_{\mathcal{F}}(\mathbf{k},t)\hat{\rho}^{(0)}_\varepsilon\right]\Bigg)\Bigg\}
$$

$$
= -\int_{t_0}^{t} dt'\Bigg\{\int d^3\mathbf{k}d^3\mathbf{p}\rho^2 a(t)a(t')\sum_{s,l}
$$

$$
\Bigg[\Big(\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{p}^l_\zeta(\mathbf{p},t')\tilde{\rho}_{red}(t) - \tilde{p}^l_\zeta(\mathbf{p},t')\tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t)\Big)\,\mathrm{Tr}_\varepsilon\left[\tilde{v}^{s}_{\mathcal{F}}(\mathbf{k},t)\tilde{v}^{l}_{\mathcal{F}}(\mathbf{p},t')\hat{\rho}^{(0)}_\varepsilon\right]
$$

$$
- \Big(\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\tilde{p}^l_\zeta(\mathbf{p},t') - \hat{\rho}_{red}(t)\tilde{p}^l_\zeta(\mathbf{p},t')\tilde{p}^s_\zeta(\mathbf{k},t)\Big)\,\mathrm{Tr}_\varepsilon\left[\tilde{v}^{l}_{\mathcal{F}}(\mathbf{p},t')\tilde{v}^{s}_{\mathcal{F}}(\mathbf{k},t)\hat{\rho}^{(0)}_\varepsilon\right]\Bigg]
$$

$$
+ \frac{\rho\mu a(t)}{\sqrt{2}a^3(t')}\frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{k}d^3\mathbf{p}\int d^3\mathbf{p}'\sum_{s,\bar{l},n,\bar{n}} C_{\bar{l}n\bar{n}}\big[\tilde{p}^s_\zeta(\mathbf{k},t),\tilde{\rho}_{red}(t)\big]
$$

$$
\Bigg(\mathrm{Tr}_\varepsilon\left[\tilde{v}^{s}_{\mathcal{F}}(\mathbf{k},t)\tilde{v}^{\bar{l}}_{\mathcal{F}}(\mathbf{p},t')\tilde{v}^{n}_{\mathcal{F}}(\mathbf{p}',t')\tilde{v}^{\bar{n}}_{\mathcal{F}}(\mathbf{p}+\mathbf{p}',t')\hat{\rho}^{(0)}_\varepsilon\right] - \mathrm{Tr}_\varepsilon\left[\tilde{v}^{\bar{l}}_{\mathcal{F}}(\mathbf{p},t')\tilde{v}^{n}_{\mathcal{F}}(\mathbf{p}',t')\tilde{v}^{\bar{n}}_{\mathcal{F}}(\mathbf{p}+\mathbf{p}',t')\tilde{v}^{s}_{\mathcal{F}}(\mathbf{k},t)\hat{\rho}^{(0)}_\varepsilon\right]\Bigg). \tag{5.19}
$$

In order to compact our equation let us define the following quantities:

$$
\begin{aligned}
D_{kp}^{sl}(t,t') &\equiv \mathrm{Tr}_\varepsilon\left[\tilde{v}_{\mathcal{F}}^{s}(\mathbf{k},t)\tilde{v}_{\mathcal{F}}^{l}(\mathbf{p},t')\hat{\rho}_{\varepsilon}^{(0)}\right],\\
D_{pk}^{ls}(t',t) &\equiv \mathrm{Tr}_\varepsilon\left[\tilde{v}_{\mathcal{F}}^{l}(\mathbf{p},t')\tilde{v}_{\mathcal{F}}^{s}(\mathbf{k},t)\hat{\rho}_{\varepsilon}^{(0)}\right],\\
D_{kpp'p+p'}^{s\bar{l}n\bar{n}}(t,t',t',t') &\equiv \mathrm{Tr}_\varepsilon\left[\tilde{v}_{\mathcal{F}}^{s}(\mathbf{k},t)\tilde{v}_{\mathcal{F}}^{\bar{l}}(\mathbf{p},t')\tilde{v}_{\mathcal{F}}^{n}(\mathbf{p}',t')\tilde{v}_{\mathcal{F}}^{\bar{n}}(\mathbf{p}+\mathbf{p}',t')\hat{\rho}_{\varepsilon}^{(0)}\right],\\
D_{pp'p+p'k}^{\bar{l}n\bar{n}s}(t',t',t',t) &\equiv \mathrm{Tr}_\varepsilon\left[\tilde{v}_{\mathcal{F}}^{\bar{l}}(\mathbf{p},t')\tilde{v}_{\mathcal{F}}^{n}(\mathbf{p}',t')\tilde{v}_{\mathcal{F}}^{\bar{n}}(\mathbf{p}+\mathbf{p}',t')\tilde{v}_{\mathcal{F}}^{s}(\mathbf{k},t)\hat{\rho}_{\varepsilon}^{(0)}\right].
\end{aligned}
\tag{5.20}
$$

These quantities are 2-point and 4-point functions of the environment.
Then the master equation gets rewritten as:

$$
\begin{aligned}
\frac{\mathrm{d}\tilde{\rho}_{red}(t)}{\mathrm{d}t} = &-\int_{t_0}^{t}dt'\Bigg\{\int d^3\mathbf{k}d^3\mathbf{p}\rho^2 a(t)a(t')\sum_{s,l}\\
&\Bigg[\Big(\tilde{p}_{\zeta}^{s}(\mathbf{k},t)\tilde{p}_{\zeta}^{l}(\mathbf{p},t')\tilde{\rho}_{red}(t)-\tilde{p}_{\zeta}^{l}(\mathbf{p},t')\tilde{\rho}_{red}(t)\tilde{p}_{\zeta}^{s}(\mathbf{k},t)\Big)D_{kp}^{sl}(t,t')\\
&-\Big(\tilde{p}_{\zeta}^{s}(\mathbf{k},t)\tilde{\rho}_{red}(t)\tilde{p}_{\zeta}^{l}(\mathbf{p},t')-\hat{\rho}_{red}(t)\tilde{p}_{\zeta}^{l}(\mathbf{p},t')\tilde{p}_{\zeta}^{s}(\mathbf{k},t)\Big)D_{pk}^{ls}(t',t)\Bigg]\\
&+\frac{\rho\mu a(t)}{\sqrt{2}a^3(t')}\frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{k}d^3\mathbf{p}\int d^3\mathbf{p}'\sum_{s,\bar{l},n,\bar{n}}C_{\bar{l}n\bar{n}}\left[\tilde{p}_{\zeta}^{s}(\mathbf{k},t),\tilde{\rho}_{red}(t)\right]\\
&\Bigg(D_{kpp'p+p'}^{s\bar{l}n\bar{n}}(t,t',t',t')-D_{pp'p+p'k}^{\bar{l}n\bar{n}s}(t',t',t',t)\Bigg).
\end{aligned}
\tag{5.21}
$$

Now using the decomposition in creation and annihilation operators, $\tilde{v}_{\mathcal{F}}(\mathbf{k},t)=v_{\mathcal{F}}(\mathbf{k},t)\hat{a}_{\mathcal{F}}(\mathbf{k})+v_{\mathcal{F}}^{*}(\mathbf{k},t)\hat{a}_{\mathcal{F}}^{\dagger}(\mathbf{k})$ (in our case we are working separately with real and imaginary part, so $v_{\mathcal{F}}^{s*}=v_{\mathcal{F}}^{s}$) and using the Bunch-Davies vacuum for the environment (annihilated by $\hat{a}_{\mathcal{F}}$) we can compute the quantities we just defined:

$$
\begin{aligned}
D_{kp}^{sl}(t,t') &= \mathrm{Tr}_\varepsilon\left[\tilde{v}_{\mathcal{F}}^{s}(\mathbf{k},t)\tilde{v}_{\mathcal{F}}^{l}(\mathbf{p},t')\hat{\rho}_{\varepsilon}^{(0)}\right]=\mathrm{Tr}_\varepsilon\left[\tilde{v}_{\mathcal{F}}^{l}(\mathbf{p},t')\hat{\rho}_{\varepsilon}^{(0)}\tilde{v}_{\mathcal{F}}^{s}(\mathbf{k},t)\right]\\
&= \mathrm{Tr}_\varepsilon\left[v_{\mathcal{F}}^{l}(\mathbf{p},t')\Big(\hat{a}_{\mathcal{F}}^{l}(\mathbf{p})+\hat{a}_{\mathcal{F}}^{l\dagger}(\mathbf{p})\Big)\left|0\right\rangle\left\langle 0\right|v_{\mathcal{F}}^{s}(\mathbf{k},t)\Big(\hat{a}_{\mathcal{F}}^{s}(\mathbf{k})+\hat{a}_{\mathcal{F}}^{s\dagger}(\mathbf{k})\Big)\right]\\
&= v_{\mathcal{F}}^{l}(\mathbf{p},t')v_{\mathcal{F}}^{s}(\mathbf{k},t)\delta_{ls}\delta^3(\mathbf{k}-\mathbf{p})\underbrace{\mathrm{Tr}_\varepsilon\left(\left|1\right\rangle\left\langle 1\right|\right)}_{1}\\
&= v_{\mathcal{F}}^{l}(\mathbf{p},t')v_{\mathcal{F}}^{s}(\mathbf{k},t)\delta_{ls}\delta^3(\mathbf{k}-\mathbf{p}),\\
D_{pk}^{ls}(t',t) &= \mathrm{Tr}_\varepsilon\left[\tilde{v}_{\mathcal{F}}^{s}(\mathbf{k},t)\hat{\rho}_{\varepsilon}^{(0)}\tilde{v}_{\mathcal{F}}^{l}(\mathbf{p},t')\right]=v_{\mathcal{F}}^{s}(\mathbf{k},t)v_{\mathcal{F}}^{l}(\mathbf{p},t')\delta_{ls}\delta^3(\mathbf{k}-\mathbf{p}),
\end{aligned}
\tag{5.22}
$$

$$
\begin{aligned}
&D_{kpp'p+p'}^{s\bar{l}n\bar{n}}(t,t',t',t') = \mathrm{Tr}_\varepsilon\left[\tilde{v}_{\mathcal{F}}^{n}(\mathbf{p}',t')\tilde{v}_{\mathcal{F}}^{\bar{n}}(\mathbf{p}+\mathbf{p}',t')\hat{\rho}_{\varepsilon}^{(0)}\tilde{v}_{\mathcal{F}}^{s}(\mathbf{k},t)\tilde{v}_{\mathcal{F}}^{\bar{l}}(\mathbf{p},t')\right]\\
&= \mathrm{Tr}_\varepsilon\Big[v_{\mathcal{F}}^{n}(\mathbf{p}',t')(\hat{a}_{\mathcal{F}}^{n}(\mathbf{p}')+\hat{a}_{\mathcal{F}}^{n\dagger}(\mathbf{p}'))v_{\mathcal{F}}^{\bar{n}}(\mathbf{p}+\mathbf{p}',t')(\hat{a}_{\mathcal{F}}^{\bar{n}}(\mathbf{p}+\mathbf{p}')+\hat{a}_{\mathcal{F}}^{\bar{n}\dagger}(\mathbf{p}+\mathbf{p}'))\\
&\left|0\right\rangle\left\langle 0\right|v_{\mathcal{F}}^{s}(\mathbf{k},t)(\hat{a}_{\mathcal{F}}^{s}(\mathbf{k})+\hat{a}_{\mathcal{F}}^{s\dagger}(\mathbf{k}))v_{\mathcal{F}}^{\bar{l}}(\mathbf{p},t')(\hat{a}_{\mathcal{F}}^{\bar{l}}(\mathbf{p})+\hat{a}_{\mathcal{F}}^{\bar{l}\dagger}(\mathbf{p}))\Big]\\
&= \mathrm{Tr}_\varepsilon\left[v_{\mathcal{F}}^{n}(\mathbf{p}',t')v_{\mathcal{F}}^{\bar{n}}(\mathbf{p}+\mathbf{p}',t')\hat{a}_{\mathcal{F}}^{n}(\mathbf{p}')\hat{a}_{\mathcal{F}}^{\bar{n}\dagger}(\mathbf{p}+\mathbf{p}')\left|0\right\rangle\left\langle 0\right|v_{\mathcal{F}}^{s}(\mathbf{k},t)v_{\mathcal{F}}^{\bar{l}}(\mathbf{p},t')\hat{a}_{\mathcal{F}}^{s}(\mathbf{k})\hat{a}_{\mathcal{F}}^{\bar{l}\dagger}(\mathbf{p})\right]\\
&+\mathrm{Tr}_\varepsilon\left[v_{\mathcal{F}}^{n}(\mathbf{p}',t')v_{\mathcal{F}}^{\bar{n}}(\mathbf{p}+\mathbf{p}',t')\hat{a}_{\mathcal{F}}^{n\dagger}(\mathbf{p}')\hat{a}_{\mathcal{F}}^{\bar{n}\dagger}(\mathbf{p}+\mathbf{p}')\left|0\right\rangle\left\langle 0\right|v_{\mathcal{F}}^{s}(\mathbf{k},t)v_{\mathcal{F}}^{\bar{l}}(\mathbf{p},t')\hat{a}_{\mathcal{F}}^{s}(\mathbf{k})\hat{a}_{\mathcal{F}}^{\bar{l}}(\mathbf{p})\right].
\end{aligned}
\tag{5.23}
$$

In the first trace the only possibility is that $\hat{a}_{\mathcal{F}}^{\bar{n}\dagger}(\mathbf{p}+\mathbf{p}')$ and $\hat{a}_{\mathcal{F}}^{n}(\mathbf{p}')$ respectively create and annihilate the same particle, while in the second trace we can have a 2-particle state or two different 1-particle state. For this reason we need to distinguish different cases:

$$D^{s\bar{l}n\bar{n}}_{kpp'p+p'}(t,t',t',t') = v^n_{\mathcal{F}}(\mathbf{p}',t')v^{\bar{n}}_{\mathcal{F}}(\mathbf{p}{+}\mathbf{p}',t')v^s_{\mathcal{F}}(\mathbf{k},t)v^{\bar{l}}_{\mathcal{F}}(\mathbf{p},t') \begin{cases} \delta^{n\bar{n}}\delta^{s\bar{l}}\delta^3(\mathbf{p})\delta^3(\mathbf{k}-\mathbf{p}) + \delta^{n\bar{n}}\delta^{s\bar{l}}\delta^{sn}\delta^3(\mathbf{p})\delta^3(\mathbf{k}-\mathbf{p})\delta^3(\mathbf{k}-\mathbf{p}') \\ \delta^{n\bar{n}}\delta^{s\bar{l}}\delta^3(\mathbf{k}-\mathbf{p}) + \delta^{s\bar{n}}\delta^{\bar{l}n}\delta^3(\mathbf{k}-\mathbf{p}-\mathbf{p}')\delta^3(\mathbf{p}-\mathbf{p}') \\ \delta^{n\bar{n}}\delta^{s\bar{l}}\delta^3(\mathbf{k}-\mathbf{p}) + \delta^{\bar{n}\bar{l}}\delta^{sn}\delta^3(\mathbf{k}-\mathbf{p}')\delta^3(\mathbf{p}') \end{cases} \tag{5.24}$$

where we used $\mathrm{Tr}_\varepsilon\left(|1,1\rangle\langle 1,1|\right) = \mathrm{Tr}_\varepsilon\left(|2\rangle\langle 2|\right) = 1$.
Analogously:

$$D^{\bar{l}n\bar{n}s}_{pp'p+p'k}(t',t',t',t) = v^n_{\mathcal{F}}(\mathbf{p}',t')v^{\bar{n}}_{\mathcal{F}}(\mathbf{p}{+}\mathbf{p}',t')v^s_{\mathcal{F}}(\mathbf{k},t)v^{\bar{l}}_{\mathcal{F}}(\mathbf{p},t') \begin{cases} \delta^{s\bar{n}}\delta^{n\bar{l}}\delta^3(\mathbf{k}-\mathbf{p}-\mathbf{p}')\delta^3(\mathbf{p}-\mathbf{p}') + \delta^{s\bar{n}}\delta^{n\bar{l}}\delta^{sn}\delta^3(\mathbf{k}-\mathbf{p}-\mathbf{p}')\delta^3(\mathbf{p}-\mathbf{p}')\delta^3(\mathbf{k}-\mathbf{p}') \\ \delta^{s\bar{n}}\delta^{n\bar{l}}\delta^3(\mathbf{k}-\mathbf{p}-\mathbf{p}')\delta^3(\mathbf{p}-\mathbf{p}') + \delta^{sn}\delta^{\bar{n}\bar{l}}\delta^3(\mathbf{k}-\mathbf{p}')\delta^3(\mathbf{p}') \\ \delta^{s\bar{n}}\delta^{n\bar{l}}\delta^3(\mathbf{k}-\mathbf{p}-\mathbf{p}')\delta^3(\mathbf{p}-\mathbf{p}') + \delta^{s\bar{l}}\delta^{n\bar{n}}\delta^3(\mathbf{k}-\mathbf{p})\delta^3(\mathbf{p}) \end{cases} \tag{5.25}$$

Now, let us focus the first combination of delta, as we will see the result is the same for the other combinations, and replacing in the master equation we get:

$$\begin{aligned}
\frac{\mathrm{d}\tilde{\rho}_{red}(t)}{\mathrm{d}t} &= -\int_{t_0}^{t} dt' \Bigg\{ \int d^3\mathbf{k}d^3\mathbf{p}\rho^2 a(t)a(t') \sum_{s,l} \Big[\Big(\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{p}^l_\zeta(\mathbf{p},t')\tilde{\rho}_{red}(t) - \tilde{p}^l_\zeta(\mathbf{p},t')\tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t) - \tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\tilde{p}^l_\zeta(\mathbf{p},t') \\
&\quad - \hat{\rho}_{red}(t)\tilde{p}^l_\zeta(\mathbf{p},t')\tilde{p}^s_\zeta(\mathbf{k},t)\Big) v^s_{\mathcal{F}}(\mathbf{k},t)v^l_{\mathcal{F}}(\mathbf{p},t')\delta_{ls}\delta^3(\mathbf{k}-\mathbf{p})\Big] + \\
&\quad + \frac{\rho\mu a(t)}{\sqrt{2}a^3(t')}\frac{1}{(2\pi)^{3/2}} \int d^3\mathbf{k}d^3\mathbf{p} \int d^3\mathbf{p}' \sum_{s,\bar{l},n,\bar{n}} C_{\bar{l}n\bar{n}}\left[\tilde{p}^s_\zeta(\mathbf{k},t),\tilde{\rho}_{red}(t)\right]\times \\
&\quad \times v^n_{\mathcal{F}}(\mathbf{p}',t')v^{\bar{n}}_{\mathcal{F}}(\mathbf{p}+\mathbf{p}',t')v^s_{\mathcal{F}}(\mathbf{k},t)v^{\bar{l}}_{\mathcal{F}}(\mathbf{p},t')\Big(\delta^{n\bar{n}}\delta^{s\bar{l}}\delta^3(\mathbf{p})\delta^3(\mathbf{k}-\mathbf{p}) + \delta^{n\bar{n}}\delta^{s\bar{l}}\delta^{sn}\delta^3(\mathbf{p})\delta^3(\mathbf{k}-\mathbf{p})\delta^3(\mathbf{k}-\mathbf{p}') \\
&\quad - \delta^{s\bar{n}}\delta^{n\bar{l}}\delta^3(\mathbf{k}-\mathbf{p}-\mathbf{p}')\delta^3(\mathbf{p}-\mathbf{p}') - \delta^{s\bar{n}}\delta^{n\bar{l}}\delta^{sn}\delta^3(\mathbf{k}-\mathbf{p}-\mathbf{p}')\delta^3(\mathbf{p}-\mathbf{p}')\delta^3(\mathbf{k}-\mathbf{p}')\Big) \\
&= -\int_{t_0}^{t} dt' \Bigg\{ \int d^3\mathbf{k}\rho^2 a(t)a(t') \sum_{s} \Big[\Big(\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t')\tilde{\rho}_{red}(t) - \tilde{p}^s_\zeta(\mathbf{k},t')\tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t) - \tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t') \\
&\quad - \hat{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t')\tilde{p}^s_\zeta(\mathbf{k},t)\Big) v^s_{\mathcal{F}}(\mathbf{k},t)v^s_{\mathcal{F}}(\mathbf{k},t')\Big] \\
&\quad + \frac{\rho\mu a(t)}{\sqrt{2}a^3(t')}\frac{1}{(2\pi)^{3/2}} \Bigg[\int d^3\mathbf{p}' \sum_{s,n} C_{snn}\left[\tilde{p}^s_\zeta(0,t),\tilde{\rho}_{red}(t)\right] v^n_{\mathcal{F}}(\mathbf{p}',t')v^n_{\mathcal{F}}(\mathbf{p}',t')v^s_{\mathcal{F}}(0,t)v^s_{\mathcal{F}}(0,t') \\
&\quad + \sum_{s} C_{sss}\left[\tilde{p}^s_\zeta(0,t),\tilde{p}^s_\zeta(0,t),\tilde{\rho}_{red}(t)\right]\left[v^s_{\mathcal{F}}(0,t')\right]^3 v^s_{\mathcal{F}}(0,t) \\
&\quad - \int d^3\mathbf{p} \sum_{s,n} C_{nns}\left[\tilde{p}^s_\zeta(2\mathbf{p},t),\tilde{\rho}_{red}(t)\right] v^s_{\mathcal{F}}(2\mathbf{p},t')v^s_{\mathcal{F}}(2\mathbf{p},t)v^n_{\mathcal{F}}(\mathbf{p},t')v^n_{\mathcal{F}}(\mathbf{p},t') \\
&\quad - \sum_{s} C_{sss}\left[\tilde{p}^s_\zeta(0,t),\tilde{p}^s_\zeta(0,t),\tilde{\rho}_{red}(t)\right]\left[v^s_{\mathcal{F}}(0,t')\right]^3 v^s_{\mathcal{F}}(0,t)\Bigg]\Bigg\}.
\end{aligned} \tag{5.26}$$

Renaming indexes and momenta we get:

$$\begin{aligned}
\frac{\mathrm{d}\tilde{\rho}_{red}(t)}{\mathrm{d}t} &= -\int_{t_0}^{t} dt' \Bigg\{ \int d^3\mathbf{k}\rho^2 a(t)a(t') \sum_{s} \Big[\Big(\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t')\tilde{\rho}_{red}(t) - \tilde{p}^s_\zeta(\mathbf{k},t')\tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t) \\
&\quad - \tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t') - \hat{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t')\tilde{p}^s_\zeta(\mathbf{k},t)\Big) v^s_{\mathcal{F}}(\mathbf{k},t)v^s_{\mathcal{F}}(\mathbf{k},t')\Big] \\
&\quad + \frac{\rho\mu a(t)}{\sqrt{2}a^3(t')}\frac{1}{(2\pi)^{3/2}} \int d^3\mathbf{k} \sum_{s,n} \Big[C_{snn}\left[\tilde{p}^s_\zeta(0,t),\tilde{\rho}_{red}(t)\right] v^n_{\mathcal{F}}(\mathbf{k},t')v^n_{\mathcal{F}}(\mathbf{k},t')v^s_{\mathcal{F}}(0,t)v^s_{\mathcal{F}}(0,t') \\
&\quad - C_{nns}\left[\tilde{p}^s_\zeta(2\mathbf{k},t),\tilde{\rho}_{red}(t)\right] v^s_{\mathcal{F}}(2\mathbf{k},t')v^s_{\mathcal{F}}(2\mathbf{k},t)v^n_{\mathcal{F}}(\mathbf{k},t')v^n_{\mathcal{F}}(\mathbf{k},t')\Big]\Bigg\}.
\end{aligned} \tag{5.27}$$

Now we want to express the operators computed in t' in terms of t using:

$$\tilde{z}_\zeta(t') = \mathbf{G}^{(S)}(t',t)\tilde{z}_\zeta(t), \tag{5.28}$$

$$\begin{pmatrix} \tilde{v}_\zeta(t') \\ \tilde{p}_\zeta(t') \end{pmatrix} = \begin{pmatrix} G^{(S)}_{11}(t',t) & G^{(S)}_{12}(t',t) \\ G^{(S)}_{12}(t',t) & G^{(S)}_{22}(t',t) \end{pmatrix} \begin{pmatrix} \tilde{v}_\zeta(t) \\ \tilde{p}_\zeta(t) \end{pmatrix}, \tag{5.29}$$

$$\tilde{v}_\zeta(t') = G^{(S)}_{11}(t',t)\tilde{v}_\zeta(t) + G^{(S)}_{12}(t',t)\tilde{p}_\zeta(t), \tag{5.30}$$

$$\tilde{p}_\zeta(t') = G^{(S)}_{12}(t',t)\tilde{v}_\zeta(t) + G^{(S)}_{22}(t',t)\tilde{p}_\zeta(t). \tag{5.31}$$

So the first term of the master equation gets modified as:

$$\begin{aligned}
&\Big[\tilde{p}^s_\zeta(\mathbf{k},t)\Big(G^{(S)}_{12}(t',t)\tilde{v}^s_\zeta(\mathbf{k},t) + G^{(S)}_{22}(t',t)\tilde{p}^s_\zeta(\mathbf{k},t)\Big)\tilde{\rho}_{red}(t) \\
&- \Big(G^{(S)}_{12}(t',t)\tilde{v}^s_\zeta(\mathbf{k},t) + G^{(S)}_{22}(t',t)\tilde{p}^s_\zeta(\mathbf{k},t)\Big)\tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t) \\
&- \tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\Big(G^{(S)}_{12}(t',t)\tilde{v}^s_\zeta(\mathbf{k},t) + G^{(S)}_{22}(t',t)\tilde{p}^s_\zeta(\mathbf{k},t)\Big) \\
&+ \tilde{\rho}_{red}(t)\Big(G^{(S)}_{12}(t',t)\tilde{v}^s_\zeta(\mathbf{k},t) + G^{(S)}_{22}(t',t)\tilde{p}^s_\zeta(\mathbf{k},t)\Big)\tilde{p}^s_\zeta(\mathbf{k},t)\Big] v^s_{\mathcal{F}}(\mathbf{k},t)v^s_{\mathcal{F}}(\mathbf{k},t') \\
&= \Big[G^{(S)}_{12}(t',t)\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{v}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t) + G^{(S)}_{22}(t',t)\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t) \\
&- G^{(S)}_{12}(t',t)\tilde{v}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t) - G^{(S)}_{22}(t',t)\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t) \\
&- G^{(S)}_{12}(t',t)\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\tilde{v}^s_\zeta(\mathbf{k},t) + G^{(S)}_{22}(t',t)\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t) \\
&- G^{(S)}_{12}(t',t)\tilde{\rho}_{red}(t)\tilde{v}^s_\zeta(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t) - G^{(S)}_{22}(t',t)\tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t)\Big] v^s_{\mathcal{F}}(\mathbf{k},t)v^s_{\mathcal{F}}(\mathbf{k},t') \\
&= \Big\{\Big[\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{v}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t) - \tilde{v}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t)\Big]G^{(S)}_{12}(t',t) \\
&+ \Big[\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t) - \tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t)\Big]G^{(S)}_{22}(t',t) \\
&- \Big[\tilde{\rho}_{red}(t)\tilde{v}^s_\zeta(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t) - \tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\tilde{v}^s_\zeta(\mathbf{k},t)\Big]G^{(S)}_{12}(t',t) \\
&- \Big[\tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t) - \tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t)\Big]G^{(S)}_{22}(t',t)\Big\} v^s_{\mathcal{F}}(\mathbf{k},t')v^s_{\mathcal{F}}(\mathbf{k},t).
\end{aligned} \tag{5.32}$$

So our master equation becomes:

$$\begin{aligned}
\frac{\mathrm{d}\tilde{\rho}_{red}(t)}{\mathrm{d}t} &= -\int_{t_0}^{t} dt' \Bigg\{ \int d^3\mathbf{k}\rho^2 a(t)a(t') \sum_s \\
&\Bigg[\Big[\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{v}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t) - \tilde{v}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t)\Big]G^{(S)}_{12}(t',t) \\
&+ \Big[\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t) - \tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t)\Big]G^{(S)}_{22}(t',t) \\
&- \Big[\tilde{\rho}_{red}(t)\tilde{v}^s_\zeta(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t) - \tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\tilde{v}^s_\zeta(\mathbf{k},t)\Big]G^{(S)}_{12}(t',t) \\
&- \Big[\tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t) - \tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t)\Big]G^{(S)}_{22}(t',t)\Bigg] v^s_{\mathcal{F}}(\mathbf{k},t')v^s_{\mathcal{F}}(\mathbf{k},t) \\
&+ \frac{\rho\mu a(t)}{\sqrt{2}a^3(t')}\frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{k}\sum_{s,n}\Big[C_{snn}\big[\tilde{p}^s_\zeta(0,t),\tilde{\rho}_{red}(t)\big]v^n_{\mathcal{F}}(\mathbf{k},t')v^n_{\mathcal{F}}(\mathbf{k},t')v^s_{\mathcal{F}}(0,t)v^s_{\mathcal{F}}(0,t') \\
&- C_{nns}\big[\tilde{p}^s_\zeta(2\mathbf{k},t),\tilde{\rho}_{red}(t)\big]v^s_{\mathcal{F}}(2\mathbf{k},t')v^s_{\mathcal{F}}(2\mathbf{k},t)v^n_{\mathcal{F}}(\mathbf{k},t')v^n_{\mathcal{F}}(\mathbf{k},t')\Big]\Bigg\}.
\end{aligned} \tag{5.33}$$

Then:

$$\begin{aligned}
\frac{\mathrm{d}\tilde{\rho}_{red}(t)}{\mathrm{d}t} &= -\int d^3\mathbf{k}\Bigg\{\rho^2 a(t)\sum_s \Big[\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{v}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t) - \tilde{v}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t) \\
&- \tilde{\rho}_{red}(t)\tilde{v}^s_\zeta(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t) + \tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\tilde{v}^s_\zeta(\mathbf{k},t)\Big] v^s_{\mathcal{F}}(\mathbf{k},t)\int_{t_0}^t dt' a(t') G^{(S)}_{12}(t',t) v^s_{\mathcal{F}}(\mathbf{k},t') \\
&+ \Big[\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t) - \tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t) \\
&- \tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t) + \tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t)\Big] v^s_{\mathcal{F}}(\mathbf{k},t)\int_{t_0}^t dt' a(t') G^{(S)}_{22}(t',t) v^s_{\mathcal{F}}(\mathbf{k},t') \\
&+ \frac{\rho\mu a(t)}{\sqrt{2}a^3(t')}\frac{1}{(2\pi)^{3/2}}\sum_{s,n}\Bigg[C_{nns}\big[\tilde{p}^s_\zeta(2\mathbf{k},t),\tilde{\rho}_{red}(t)\big]\bigg(-v^s_{\mathcal{F}}(2\mathbf{k},t)\int_{t_0}^t dt' v^s_{\mathcal{F}}(2\mathbf{k},t') v^n_{\mathcal{F}}(\mathbf{k},t') v^n_{\mathcal{F}}(\mathbf{k},t')\bigg) \\
&+ C_{snn}\big[\tilde{p}^s_\zeta(0,t),\tilde{\rho}_{red}(t)\big]\bigg(v^s_{\mathcal{F}}(0,t)\int_{t_0}^t dt' v^s_{\mathcal{F}}(0,t') v^n_{\mathcal{F}}(\mathbf{k},t') v^n_{\mathcal{F}}(\mathbf{k},t')\bigg)\Bigg]\Bigg\} \\
&= -\int d^3\mathbf{k}\Bigg\{\rho^2 a(t)\sum_s \Big[\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{v}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t) - \tilde{v}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t) \\
&- \tilde{\rho}_{red}(t)\tilde{v}^s_\zeta(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t) + \tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\tilde{v}^s_\zeta(\mathbf{k},t)\Big] v^s_{\mathcal{F}}(\mathbf{k},t)\int_{t_0}^t dt' a(t') G^{(S)}_{12}(t',t) v^s_{\mathcal{F}}(\mathbf{k},t') \\
&+ \Big[\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t) - \tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\Big] v^s_{\mathcal{F}}(\mathbf{k},t)\int_{t_0}^t dt' a(t') G^{(S)}_{22}(t',t) v^s_{\mathcal{F}}(\mathbf{k},t') \\
&+ \frac{\rho\mu a(t)}{\sqrt{2}a^3(t')}\frac{1}{(2\pi)^{3/2}}\sum_{s,n}\Bigg[C_{nns}\big[\tilde{p}^s_\zeta(2\mathbf{k},t),\tilde{\rho}_{red}(t)\big]\bigg(-v^s_{\mathcal{F}}(2\mathbf{k},t)\int_{t_0}^t dt' v^s_{\mathcal{F}}(2\mathbf{k},t') v^n_{\mathcal{F}}(\mathbf{k},t') v^n_{\mathcal{F}}(\mathbf{k},t')\bigg) \\
&+ C_{snn}\big[\tilde{p}^s_\zeta(0,t),\tilde{\rho}_{red}(t)\big]\bigg(v^s_{\mathcal{F}}(0,t)\int_{t_0}^t dt' v^s_{\mathcal{F}}(0,t') v^n_{\mathcal{F}}(\mathbf{k},t') v^n_{\mathcal{F}}(\mathbf{k},t')\bigg)\Bigg]\Bigg\}.
\end{aligned} \tag{5.34}$$

Now if we define:

$$\begin{aligned}
D^s_1(\mathbf{k},t) &= v^s_{\mathcal{F}}(\mathbf{k},t)\int_{t_0}^t dt' a(t') G^{(S)}_{12}(t',t) v^s_{\mathcal{F}}(\mathbf{k},t'), \\
D^s_2(\mathbf{k},t) &= v^s_{\mathcal{F}}(\mathbf{k},t)\int_{t_0}^t dt' a(t') G^{(S)}_{22}(t',t) v^s_{\mathcal{F}}(\mathbf{k},t'), \\
D^{sn}_3(\mathbf{k},t) &= v^s_{\mathcal{F}}(2\mathbf{k},t)\int_{t_0}^t dt' \frac{1}{a^3(t')} v^s_{\mathcal{F}}(2\mathbf{k},t') v^n_{\mathcal{F}}(\mathbf{k},t') v^n_{\mathcal{F}}(\mathbf{k},t'), \\
D^{sn}_4(\mathbf{k},t) &= v^s_{\mathcal{F}}(0,t)\int_{t_0}^t dt' \frac{1}{a^3(t')} v^s_{\mathcal{F}}(0,t') v^n_{\mathcal{F}}(\mathbf{k},t') v^n_{\mathcal{F}}(\mathbf{k},t'),
\end{aligned} \tag{5.35}$$

we finally get:

$$\begin{aligned}
\frac{\mathrm{d}\tilde{\rho}_{red}(t)}{\mathrm{d}t} = -\int d^3\mathbf{k}\Bigg\{&\rho^2 a(t)\sum_s \Big[\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{v}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t) - \tilde{v}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t) \\
&- \tilde{\rho}_{red}(t)\tilde{v}^s_\zeta(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t) + \tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\tilde{v}^s_\zeta(\mathbf{k},t)\Big] D^s_1(\mathbf{k},t) \\
&+ \Big[\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t) - \tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t)\Big] D^s_2(\mathbf{k},t) \\
&+ \frac{\rho\mu a(t)}{\sqrt{2}a^3(t')}\frac{1}{(2\pi)^{3/2}}\sum_{s,n}\Bigg[C_{snn}\big[\tilde{p}^s_\zeta(0,t),\tilde{\rho}_{red}(t)\big] D^{sn}_4(\mathbf{k},t) - C_{nns}\big[\tilde{p}^s_\zeta(2\mathbf{k},t),\tilde{\rho}_{red}(t)\big] D^{sn}_3(\mathbf{k},t)\Bigg]\Bigg\}.
\end{aligned} \tag{5.36}$$

5.1 Linear part

Now following the reasoning of [28] we define the following effective density matrix:

$$\hat{\rho}^s_{\mathbf{k}} = \mathrm{Tr}_{\left\{\substack{s'\\ \mathbf{k}'\neq\mathbf{k}}\right\},\left\{\substack{\bar{s}\\ \mathbf{k}}\right\}}\hat{\rho}_{red}(t) \qquad \Rightarrow \qquad \gamma = \mathrm{Tr}_{\left\{\substack{s\\ \mathbf{k}}\right\}}\left(\hat{\rho}^{s2}_{\mathbf{k}}\right), \tag{5.37}$$

and using the linearity of the trace operator we get:

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}_{\left\{\substack{s\\ \mathbf{k}}\right\}}\left(\hat{\rho}^{s2}_{\mathbf{k}}\right) = 2\,\mathrm{Tr}_{\left\{\substack{s\\ \mathbf{k}}\right\}}\left[\hat{\rho}^s_{\mathbf{k}}\,\mathrm{Tr}_{\left\{\substack{s'\\ \mathbf{k}'\neq\mathbf{k}}\right\},\left\{\substack{\bar{s}\\ \mathbf{k}}\right\}}\left(\frac{\mathrm{d}\hat{\rho}_{red}(t)}{\mathrm{d}t}\right)\right]. \tag{5.38}$$

Let us focus on the last trace and replace the linear term of the master equation:

$$\begin{aligned}
\mathrm{Tr}_{\left\{\substack{s'\\ \mathbf{k}'\neq\mathbf{k}}\right\},\left\{\substack{\bar{s}\\ \mathbf{k}}\right\}}\left(\frac{\mathrm{d}\hat{\rho}_{red}(t)}{\mathrm{d}t}\right) = -\rho^2 a(t)\int d^3\mathbf{k_1}\sum_r \mathrm{Tr}_{\left\{\substack{s'\\ \mathbf{k}'\neq\mathbf{k}}\right\},\left\{\substack{\bar{s}\\ \mathbf{k}}\right\}}\Big\{\Big[\tilde{p}^r_\zeta(\mathbf{k_1},t)\tilde{v}^r_\zeta(\mathbf{k_1},t)\tilde{\rho}_{red}(t) - \tilde{v}^r_\zeta(\mathbf{k_1},t)\tilde{\rho}_{red}(t)\tilde{p}^r_\zeta(\mathbf{k_1},t)\\
+ \tilde{\rho}_{red}(t)\tilde{v}^r_\zeta(\mathbf{k_1},t)\tilde{p}^r_\zeta(\mathbf{k_1},t) - \tilde{p}^r_\zeta(\mathbf{k_1},t)\tilde{\rho}_{red}(t)\tilde{v}^r_\zeta(\mathbf{k_1},t)\Big]D^s_1(\mathbf{k_1},t)\\
+ \Big[\tilde{p}^r_\zeta(\mathbf{k_1},t)\tilde{p}^r_\zeta(\mathbf{k_1},t)\tilde{\rho}_{red}(t) + \tilde{\rho}_{red}(t)\tilde{p}^r_\zeta(\mathbf{k_1},t)\tilde{p}^r_\zeta(\mathbf{k_1},t)\tilde{\rho}_{red}(t) - 2\tilde{p}^r_\zeta(\mathbf{k_1},t)\tilde{\rho}_{red}(t)\tilde{p}^r_\zeta(\mathbf{k}_1,t)\Big]D^r_2(\mathbf{k_1},t)\Big\}.
\end{aligned} \tag{5.39}$$

At leading order in ρ^2 we can factorize $\tilde{\rho}_{red}(t) = \prod_{\mathbf{k}}\prod_s \tilde{\rho}^s_{\mathbf{k}}$. If $\mathbf{k}\neq\mathbf{k}_1$ we can remove $\tilde{\rho}^s_{\mathbf{k}}$ from (5.38), getting a full trace instead of a partial trace and we get zero. So the only possibility is that $\mathbf{k}=\mathbf{k}_1$ and we can factorize $\tilde{\rho}_{red}(t) = \tilde{\rho}^s_{\mathbf{k}}\tilde{\rho}^{\bar{s}}_{\mathbf{k}}$, so we obtain:

$$\begin{aligned}
\mathrm{Tr}_{\left\{\substack{s'\\ \mathbf{k}'\neq\mathbf{k}}\right\},\left\{\substack{\bar{s}\\ \mathbf{k}}\right\}}\left(\frac{\mathrm{d}\hat{\rho}_{red}(t)}{\mathrm{d}t}\right) = -\rho^2 a(t)\int d^3\mathbf{k}\delta^3(\mathbf{k}-\mathbf{k}_1)\,\mathrm{Tr}_{\left\{\substack{\bar{s}\\ \mathbf{k}}\right\}}\Big[&\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{v}^s_\zeta(\mathbf{k},t)\tilde{\rho}^s_{\mathbf{k}}\tilde{\rho}^{\bar{s}}_{\mathbf{k}} - \tilde{v}^s_\zeta(\mathbf{k},t)\tilde{\rho}^s_{\mathbf{k}}\tilde{\rho}^{\bar{s}}_{\mathbf{k}}\tilde{p}^s_\zeta(\mathbf{k},t)\\
&+ \tilde{\rho}^s_{\mathbf{k}}\tilde{\rho}^{\bar{s}}_{\mathbf{k}}\tilde{v}^s_\zeta(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t) - \tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}^s_{\mathbf{k}}\tilde{\rho}^{\bar{s}}_{\mathbf{k}}\tilde{v}^s_\zeta(\mathbf{k},t)\Big]D^s_1(\mathbf{k},t)\\
&+ \Big[\tilde{p}^{\bar{s}}_\zeta(\mathbf{k},t)\tilde{v}^{\bar{s}}_\zeta(\mathbf{k},t)\tilde{\rho}^s_{\mathbf{k}}\tilde{\rho}^{\bar{s}}_{\mathbf{k}} - \tilde{v}^{\bar{s}}_\zeta(\mathbf{k},t)\tilde{\rho}^s_{\mathbf{k}}\tilde{\rho}^{\bar{s}}_{\mathbf{k}}\tilde{p}^{\bar{s}}_\zeta(\mathbf{k},t)\\
&+ \tilde{\rho}^s_{\mathbf{k}}\tilde{\rho}^{\bar{s}}_{\mathbf{k}}\tilde{v}^{\bar{s}}_\zeta(\mathbf{k},t)\tilde{p}^{\bar{s}}_\zeta(\mathbf{k},t) - \tilde{p}^{\bar{s}}_\zeta(\mathbf{k},t)\tilde{\rho}^s_{\mathbf{k}}\tilde{\rho}^{\bar{s}}_{\mathbf{k}}\tilde{v}^{\bar{s}}_\zeta(\mathbf{k},t)\Big]D^{\bar{s}}_1(\mathbf{k},t)\\
&+ \Big[\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}^s_{\mathbf{k}}\tilde{\rho}^{\bar{s}}_{\mathbf{k}} + \tilde{\rho}^s_{\mathbf{k}}\tilde{\rho}^{\bar{s}}_{\mathbf{k}}\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}^s_{\mathbf{k}}\tilde{\rho}^{\bar{s}}_{\mathbf{k}} - 2\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}^s_{\mathbf{k}}\tilde{\rho}^{\bar{s}}_{\mathbf{k}}\tilde{p}^s_\zeta(\mathbf{k},t)\Big]D^s_2(\mathbf{k},t)\\
&+ \Big[\tilde{p}^{\bar{s}}_\zeta(\mathbf{k},t)\tilde{p}^{\bar{s}}_\zeta(\mathbf{k},t)\tilde{\rho}^s_{\mathbf{k}}\tilde{\rho}^{\bar{s}}_{\mathbf{k}} + \tilde{\rho}^s_{\mathbf{k}}\tilde{\rho}^{\bar{s}}_{\mathbf{k}}\tilde{p}^{\bar{s}}_\zeta(\mathbf{k},t)\tilde{p}^{\bar{s}}_\zeta(\mathbf{k},t)\tilde{\rho}^s_{\mathbf{k}}\tilde{\rho}^{\bar{s}}_{\mathbf{k}} - 2\tilde{p}^{\bar{s}}_\zeta(\mathbf{k},t)\tilde{\rho}^s_{\mathbf{k}}\tilde{\rho}^{\bar{s}}_{\mathbf{k}}\tilde{p}^{\bar{s}}_\zeta(\mathbf{k},t)\Big]D^{\bar{s}}_2(\mathbf{k},t).
\end{aligned} \tag{5.40}$$

Using $\mathrm{Tr}_{\left\{\substack{\bar{s}\\ \mathbf{k}}\right\}}\left(\tilde{\rho}^s_{\mathbf{k}}\right) = 1$ we obtain:

$$\begin{aligned}
&\mathrm{Tr}_{\left\{\substack{s'\\ \mathbf{k}'\neq\mathbf{k}}\right\},\left\{\substack{\bar{s}\\ \mathbf{k}}\right\}}\left(\frac{\mathrm{d}\hat{\rho}_{red}(t)}{\mathrm{d}t}\right) = -\rho^2 a(t)\Big[\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{v}^s_\zeta(\mathbf{k},t)\tilde{\rho}^s_{\mathbf{k}} - \tilde{v}^s_\zeta(\mathbf{k},t)\tilde{\rho}^s_{\mathbf{k}}\tilde{p}^s_\zeta(\mathbf{k},t)\\
&+ \tilde{\rho}^s_{\mathbf{k}}\tilde{v}^s_\zeta(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t) - \tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}^s_{\mathbf{k}}\tilde{v}^s_\zeta(\mathbf{k},t)\Big]D^s_1(\mathbf{k},t)\\
&+ \Big[\tilde{\rho}^s_{\mathbf{k}}\,\mathrm{Tr}_{\left\{\substack{\bar{s}\\ \mathbf{k}}\right\}}\left(\tilde{p}^{\bar{s}}_\zeta(\mathbf{k},t)\tilde{v}^{\bar{s}}_\zeta(\mathbf{k},t)\tilde{\rho}^{\bar{s}}_{\mathbf{k}}\right) - \tilde{\rho}^s_{\mathbf{k}}\,\mathrm{Tr}_{\left\{\substack{\bar{s}\\ \mathbf{k}}\right\}}\left(\tilde{v}^{\bar{s}}_\zeta(\mathbf{k},t)\tilde{\rho}^{\bar{s}}_{\mathbf{k}}\tilde{p}^{\bar{s}}_\zeta(\mathbf{k},t)\right)\\
&+ \tilde{\rho}^s_{\mathbf{k}}\,\mathrm{Tr}_{\left\{\substack{\bar{s}\\ \mathbf{k}}\right\}}\left(\tilde{\rho}^{\bar{s}}_{\mathbf{k}}\tilde{v}^{\bar{s}}_\zeta(\mathbf{k},t)\tilde{p}^{\bar{s}}_\zeta(\mathbf{k},t)\right) - \tilde{\rho}^s_{\mathbf{k}}\,\mathrm{Tr}_{\left\{\substack{\bar{s}\\ \mathbf{k}}\right\}}\left(\tilde{p}^{\bar{s}}_\zeta(\mathbf{k},t)\tilde{\rho}^{\bar{s}}_{\mathbf{k}}\tilde{v}^{\bar{s}}_\zeta(\mathbf{k},t)\right)\Big]D^{\bar{s}}_1(\mathbf{k},t)\\
&+ \Big[\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}^s_{\mathbf{k}} + \tilde{\rho}^s_{\mathbf{k}}\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}^s_{\mathbf{k}} - 2\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}^s_{\mathbf{k}}\tilde{p}^s_\zeta(\mathbf{k},t)\Big]D^s_2(\mathbf{k},t)\\
&+ \Big[\tilde{\rho}^s_{\mathbf{k}}\,\mathrm{Tr}_{\left\{\substack{\bar{s}\\ \mathbf{k}}\right\}}\left(\tilde{p}^{\bar{s}}_\zeta(\mathbf{k},t)\tilde{p}^{\bar{s}}_\zeta(\mathbf{k},t)\tilde{\rho}^{\bar{s}}_{\mathbf{k}}\right) + \tilde{\rho}^s_{\mathbf{k}}\,\mathrm{Tr}_{\left\{\substack{\bar{s}\\ \mathbf{k}}\right\}}\left(\tilde{\rho}^{\bar{s}}_{\mathbf{k}}\tilde{p}^{\bar{s}}_\zeta(\mathbf{k},t)\tilde{p}^{\bar{s}}_\zeta(\mathbf{k},t)\right) - 2\tilde{\rho}^s_{\mathbf{k}}\,\mathrm{Tr}_{\left\{\substack{\bar{s}\\ \mathbf{k}}\right\}}\left(\tilde{p}^{\bar{s}}_\zeta(\mathbf{k},t)\tilde{\rho}^{\bar{s}}_{\mathbf{k}}\tilde{p}^{\bar{s}}_\zeta(\mathbf{k},t)\right)\Big]D^{\bar{s}}_2(\mathbf{k},t)\\
&= -\rho^2 a(t)\Big\{\big[\tilde{p}^s_\zeta(\mathbf{k},t)\left[\tilde{v}^s_\zeta(\mathbf{k},t),\tilde{\rho}^s_{\mathbf{k}}\right] - \left[\tilde{v}^s_\zeta(\mathbf{k},t),\tilde{\rho}^s_{\mathbf{k}}\right]\tilde{p}^s_\zeta(\mathbf{k},t)\big]D^s_1(\mathbf{k},t)\\
&+ \big[\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}^s_{\mathbf{k}} + \tilde{\rho}^s_{\mathbf{k}}\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t) - \tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}^s_{\mathbf{k}}\tilde{p}^s_\zeta(\mathbf{k},t) - \tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}^s_{\mathbf{k}}\tilde{p}^s_\zeta(\mathbf{k},t)\big]D^s_2(\mathbf{k},t)\Big\}\\
&= -\rho^2 a(t)\Big\{\big[\tilde{p}^s_\zeta(\mathbf{k},t),\left[\tilde{v}^s_\zeta(\mathbf{k},t),\tilde{\rho}^s_{\mathbf{k}}\right]\big]D^s_1(\mathbf{k},t) + \big[\tilde{p}^s_\zeta(\mathbf{k},t),\left[\tilde{p}^s_\zeta(\mathbf{k},t),\tilde{\rho}^s_{\mathbf{k}}\right]\big]D^s_2(\mathbf{k},t)\Big\}.
\end{aligned} \tag{5.41}$$

Putting this result in the equation for the purity (5.38):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Tr}_{\left\{{s \atop \mathbf{k}}\right\}}\left(\hat{\rho}_{\mathbf{k}}^{s2}\right) &= -2\rho^2 a(t)\,\mathrm{Tr}_{\left\{{s \atop \mathbf{k}}\right\}}\Big\{\Big[\tilde{\rho}_{\mathbf{k}}^s\tilde{p}_\zeta^s(\mathbf{k},t)\tilde{v}_\zeta^s(\mathbf{k},t)\tilde{\rho}_{\mathbf{k}}^s - \tilde{\rho}_{\mathbf{k}}^s\tilde{v}_\zeta^s(\mathbf{k},t)\tilde{\rho}_{\mathbf{k}}^s\tilde{p}_\zeta^s(\mathbf{k},t)\\
&+ \tilde{\rho}_{\mathbf{k}}^s\tilde{\rho}_{\mathbf{k}}^s\tilde{v}_\zeta^s(\mathbf{k},t)\tilde{p}_\zeta^s(\mathbf{k},t) - \tilde{\rho}_{\mathbf{k}}^s\tilde{p}_\zeta^s(\mathbf{k},t)\tilde{\rho}_{\mathbf{k}}^s\tilde{v}_\zeta^s(\mathbf{k},t)\Big]D_1^s(\mathbf{k},t)\\
&+ \Big[\tilde{\rho}_{\mathbf{k}}^s\tilde{p}_\zeta^s(\mathbf{k},t)\tilde{p}_\zeta^s(\mathbf{k},t)\tilde{\rho}_{\mathbf{k}}^s + \tilde{\rho}_{\mathbf{k}}^s\tilde{\rho}_{\mathbf{k}}^s\tilde{p}_\zeta^s(\mathbf{k},t)\tilde{p}_\zeta^s(\mathbf{k},t)\tilde{\rho}_{\mathbf{k}}^s - 2\tilde{\rho}_{\mathbf{k}}^s\tilde{p}_\zeta^s(\mathbf{k},t)\tilde{\rho}_{\mathbf{k}}^s\tilde{p}_\zeta^s(\mathbf{k},t)\Big]D_2^s(\mathbf{k},t)\Big\}\\
&= -2\rho^2 a(t)\,\mathrm{Tr}_{\left\{{s \atop \mathbf{k}}\right\}}\Big\{\Big[\tilde{\rho}_{\mathbf{k}}^s\tilde{\rho}_{\mathbf{k}}^s\tilde{p}_\zeta^s(\mathbf{k},t)\tilde{v}_\zeta^s(\mathbf{k},t) - \tilde{\rho}_{\mathbf{k}}^s\tilde{v}_\zeta^s(\mathbf{k},t)\tilde{\rho}_{\mathbf{k}}^s\tilde{p}_\zeta^s(\mathbf{k},t)\\
&+ \tilde{\rho}_{\mathbf{k}}^s\tilde{\rho}_{\mathbf{k}}^s\tilde{v}_\zeta^s(\mathbf{k},t)\tilde{p}_\zeta^s(\mathbf{k},t) - \tilde{\rho}_{\mathbf{k}}^s\tilde{p}_\zeta^s(\mathbf{k},t)\tilde{\rho}_{\mathbf{k}}^s\tilde{v}_\zeta^s(\mathbf{k},t)\Big]D_1^s(\mathbf{k},t)\\
&+ 2\big[\tilde{p}_\zeta^s(\mathbf{k},t)\tilde{p}_\zeta^s(\mathbf{k},t)\tilde{\rho}_{\mathbf{k}}^s\tilde{\rho}_{\mathbf{k}}^s - \tilde{p}_\zeta^s(\mathbf{k},t)\tilde{\rho}_{\mathbf{k}}^s\tilde{p}_\zeta^s(\mathbf{k},t)\tilde{\rho}_{\mathbf{k}}^s\big]D_2^s(\mathbf{k},t)\Big\}\\
&= -2\rho^2 a(t)\,\mathrm{Tr}_{\left\{{s \atop \mathbf{k}}\right\}}\Big\{\Big[\tilde{\rho}_{\mathbf{k}}^s\big[\tilde{\rho}_{\mathbf{k}}^s,\tilde{v}_\zeta^s(\mathbf{k},t)\big]\tilde{p}_\zeta^s(\mathbf{k},t) + \tilde{\rho}_{\mathbf{k}}^s\big[\tilde{\rho}_{\mathbf{k}}^s,\tilde{p}_\zeta^s(\mathbf{k},t)\big]\tilde{v}_\zeta^s(\mathbf{k},t)\Big]D_1^s(\mathbf{k},t)\\
&+ 2\tilde{\rho}_{\mathbf{k}}^s\big[\tilde{\rho}_{\mathbf{k}}^s,\tilde{p}_\zeta^s(\mathbf{k},t)\big]\tilde{p}_\zeta^s(\mathbf{k},t)D_2^s(\mathbf{k},t)\Big\}.\end{aligned} \tag{5.42}$$

Then using the expression that was shown in [28]:

$$\mathrm{Tr}\left(\tilde{\rho}_{\mathbf{k}}^s\big[\tilde{\rho}_{\mathbf{k}}^s,\tilde{v}_\zeta^s(\mathbf{k},t)\big]\tilde{v}_\zeta^s(\mathbf{k},t)\right) = P_{vv}(k), \tag{5.43}$$

and generalizing it as

$$\mathrm{Tr}\left(\tilde{\rho}_{\mathbf{k}}^s\big[\tilde{\rho}_{\mathbf{k}}^s,\tilde{\chi}_\zeta^s(\mathbf{k},t)\big]\tilde{\chi}_\zeta^s(\mathbf{k},t)\right) = P_{\chi\chi}(k), \tag{5.44}$$

with $\tilde{\chi}_\zeta^s = \tilde{v}_\zeta^s, \tilde{p}_\zeta^s$. Finally we find:

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Tr}_{\left\{{s \atop \mathbf{k}}\right\}}\left(\hat{\rho}_{\mathbf{k}}^{s2}\right) = -2\rho^2 a(t)[(P_{vp}+P_{pv})D_1^s(\mathbf{k},t) + 2P_{pp}D_2^s(\mathbf{k},t)], \tag{5.45}$$

which can be integrated to find:

$$\gamma = -2\rho^2\int_{-\infty}^{z}\frac{k}{z'H}[(P_{vp}+P_{pv})D_1^s(\mathbf{k},t) + 2P_{pp}D_2^s(\mathbf{k},t)]. \tag{5.46}$$

This is the evolution of the purity considering only the linear interaction term $\zeta'\mathcal{F}$. It would be interesting to compute this integral and compare the resulting purity evolution with figure (4.1) and (4.2), in order to verify the goodness of our result, which was computed with a different method with respect to the one used in [8]. We are trying to perform this computation but in this moment we have some numerical issues.

5.2 Non-linear part

Let us now replace the non-linear part of the master equation in the trace within the equation for the purity (5.38):

$$
\begin{aligned}
\mathrm{Tr}_{\left\{\substack{s' \\ \mathbf{k}'\neq\mathbf{k}}\right\},\left\{\substack{\bar{s} \\ \mathbf{k}}\right\}}\left(\frac{\mathrm{d}\tilde{\rho}_{red}}{\mathrm{d}t}\right) &= -\frac{\rho\mu a(t)}{\sqrt{2}(2\pi)^{3/2}}\int \mathrm{d}^3\mathbf{k_1}\sum_{r,n}\Big\{C_{rnn}\left[\tilde{p}^r_\zeta(0,t),\tilde{\rho}_{red}(t)\right]D_4^{rn}(\mathbf{k_1},t) \\
&- C_{nnr}\left[\tilde{p}^r_\zeta(2\mathbf{k_1},t),\tilde{\rho}_{red}(t)\right]D_3^{rn}(\mathbf{k_1},t)\Big\} \\
&= -\frac{\rho\mu a(t)}{\sqrt{2}(2\pi)^{3/2}}\int \mathrm{d}^3\mathbf{k_1}\sum_{r}\mathrm{Tr}_{\left\{\substack{s' \\ \mathbf{k}'\neq\mathbf{k}}\right\},\left\{\substack{\bar{s} \\ \mathbf{k}}\right\}}\Big\{\big(\tilde{p}^r_\zeta(0,t)\tilde{\rho}_{red}(t)-\tilde{\rho}_{red}(t)\tilde{p}^r_\zeta(0,t)\big)\times \\
&\sum_n C_{rnn}D_4^{rn}(\mathbf{k_1},t)-\big(\tilde{p}^r_\zeta(2\mathbf{k_1},t)\tilde{\rho}_{red}(t)-\tilde{\rho}_{red}(t)\tilde{p}^r_\zeta(2\mathbf{k_1},t)\big)\sum_n C_{nnr}D_3^{rn}(\mathbf{k_1},t)\Big\} \\
&= -\frac{\rho\mu a(t)}{\sqrt{2}(2\pi)^{3/2}}\int \mathrm{d}^3\mathbf{k_1}\,\mathrm{Tr}_{\left\{\substack{s' \\ \mathbf{k}'\neq\mathbf{k}}\right\},\left\{\substack{\bar{s} \\ \mathbf{k}}\right\}}\Big\{\big(\tilde{p}^r_\zeta(0,t)\tilde{\rho}_{red}(t)-\tilde{\rho}_{red}(t)\tilde{p}^r_\zeta(0,t)\big)C_{rnn}D_4^{rn}(\mathbf{k_1},t) \\
&+\big(\tilde{p}^r_\zeta(0,t)\tilde{\rho}_{red}(t)-\tilde{\rho}_{red}(t)\tilde{p}^r_\zeta(0,t)\big)C_{r\bar{n}\bar{n}}D_4^{r\bar{n}}(\mathbf{k_1},t) \\
&+\big(\tilde{p}^{\bar{r}}_\zeta(0,t)\tilde{\rho}_{red}(t)-\tilde{\rho}_{red}(t)\tilde{p}^{\bar{r}}_\zeta(0,t)\big)C_{\bar{r}nn}D_4^{\bar{r}n}(\mathbf{k_1},t) \\
&+\big(\tilde{p}^{\bar{r}}_\zeta(0,t)\tilde{\rho}_{red}(t)-\tilde{\rho}_{red}(t)\tilde{p}^{\bar{r}}_\zeta(0,t)\big)C_{\bar{r}\bar{n}\bar{n}}D_4^{\bar{r}\bar{n}}(\mathbf{k_1},t) \\
&-\big(\tilde{p}^r_\zeta(2\mathbf{k_1},t)\tilde{\rho}_{red}(t)-\tilde{\rho}_{red}(t)\tilde{p}^r_\zeta(2\mathbf{k_1},t)\big)C_{nnr}D_3^{rn}(\mathbf{k_1},t) \\
&-\big(\tilde{p}^r_\zeta(2\mathbf{k_1},t)\tilde{\rho}_{red}(t)-\tilde{\rho}_{red}(t)\tilde{p}^r_\zeta(2\mathbf{k_1},t)\big)C_{\bar{n}\bar{n}r}D_3^{r\bar{n}}(\mathbf{k_1},t) \\
&-\big(\tilde{p}^{\bar{r}}_\zeta(2\mathbf{k_1},t)\tilde{\rho}_{red}(t)-\tilde{\rho}_{red}(t)\tilde{p}^{\bar{r}}_\zeta(2\mathbf{k_1},t)\big)C_{nn\bar{r}}D_3^{\bar{r}n}(\mathbf{k_1},t) \\
&-\big(\tilde{p}^{\bar{r}}_\zeta(2\mathbf{k_1},t)\tilde{\rho}_{red}(t)-\tilde{\rho}_{red}(t)\tilde{p}^{\bar{r}}_\zeta(2\mathbf{k_1},t)\big)C_{\bar{n}\bar{n}\bar{r}}D_3^{\bar{r}\bar{n}}(\mathbf{k_1},t)\Big\}.
\end{aligned}
\tag{5.47}
$$

Now following the reasoning of [28], if $\mathbf{k}\neq 0,\mathbf{k_1},\mathbf{k_2}$ we get a full trace instead of a partial trace, so we get zero. Consider now the possible cases:

1. $\mathbf{k}=0\neq\mathbf{k_1}\neq 2\mathbf{k_1}$:
   so $\mathbf{k}'=\mathbf{k_1},\mathbf{k_2}$ with $\mathbf{k_2}\equiv 2\mathbf{k_1}$. At linear order in ρ we can factorize:

$$
\tilde{\rho}_{red}(t)=\tilde{\rho}^r_0\tilde{\rho}^{\bar{r}}_0\tilde{\rho}^r_{\mathbf{k_1}}\tilde{\rho}^{\bar{r}}_{\mathbf{k_1}}\tilde{\rho}^r_{\mathbf{k_2}}\tilde{\rho}^{\bar{r}}_{\mathbf{k_2}}.
\tag{5.48}
$$

   Then:

$$
\begin{aligned}
\mathrm{Tr}_{\left\{\substack{s' \\ \mathbf{k}'\neq\mathbf{k}}\right\},\left\{\substack{\bar{s} \\ \mathbf{k}}\right\}}\left(\frac{\mathrm{d}\tilde{\rho}_{red}}{\mathrm{d}t}\right) &= -\frac{\rho\mu a(t)}{\sqrt{2}(2\pi)^{3/2}}\int \mathrm{d}^3\mathbf{k_1}\delta^3(\mathbf{k})\,\mathrm{Tr}_{\left\{\substack{s,\,\bar{s} \\ \mathbf{k_1}}\right\},\left\{\substack{s,\,\bar{s} \\ \mathbf{k_2}}\right\},\left\{\substack{\bar{s} \\ 0}\right\}}\times \\
&\times\Big\{\big(\tilde{p}^s_\zeta(0,t)\tilde{\rho}^s_0\tilde{\rho}^{\bar{s}}_0\tilde{\rho}^s_{\mathbf{k_1}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_1}}\tilde{\rho}^s_{\mathbf{k_2}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_2}}-\tilde{\rho}^s_0\tilde{\rho}^{\bar{s}}_0\tilde{\rho}^s_{\mathbf{k_1}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_1}}\tilde{\rho}^s_{\mathbf{k_2}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_2}}\tilde{p}^s_\zeta(0,t)\big)C_{snn}D_4^{sn}(\mathbf{k_1},t) \\
&+\big(\tilde{p}^s_\zeta(0,t)\tilde{\rho}^s_0\tilde{\rho}^{\bar{s}}_0\tilde{\rho}^s_{\mathbf{k_1}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_1}}\tilde{\rho}^s_{\mathbf{k_2}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_2}}-\tilde{\rho}^s_0\tilde{\rho}^{\bar{s}}_0\tilde{\rho}^s_{\mathbf{k_1}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_1}}\tilde{\rho}^s_{\mathbf{k_2}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_2}}\tilde{p}^s_\zeta(0,t)\big)C_{s\bar{n}\bar{n}}D_4^{s\bar{n}}(\mathbf{k_1},t) \\
&+\big(\tilde{p}^{\bar{s}}_\zeta(0,t)\tilde{\rho}^s_0\tilde{\rho}^{\bar{s}}_0\tilde{\rho}^s_{\mathbf{k_1}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_1}}\tilde{\rho}^s_{\mathbf{k_2}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_2}}-\tilde{\rho}^s_0\tilde{\rho}^{\bar{s}}_0\tilde{\rho}^s_{\mathbf{k_1}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_1}}\tilde{\rho}^s_{\mathbf{k_2}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_2}}\tilde{p}^{\bar{s}}_\zeta(0,t)\big)C_{\bar{s}nn}D_4^{\bar{s}n}(\mathbf{k_1},t) \\
&+\big(\tilde{p}^{\bar{s}}_\zeta(0,t)\tilde{\rho}^s_0\tilde{\rho}^{\bar{s}}_0\tilde{\rho}^s_{\mathbf{k_1}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_1}}\tilde{\rho}^s_{\mathbf{k_2}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_2}}-\tilde{\rho}^s_0\tilde{\rho}^{\bar{s}}_0\tilde{\rho}^s_{\mathbf{k_1}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_1}}\tilde{\rho}^s_{\mathbf{k_2}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_2}}\tilde{p}^{\bar{s}}_\zeta(0,t)\big)C_{\bar{s}\bar{n}\bar{n}}D_4^{\bar{s}\bar{n}}(\mathbf{k_1},t) \\
&-\big(\tilde{p}^s_\zeta(\mathbf{k_2},t)\tilde{\rho}^s_0\tilde{\rho}^{\bar{s}}_0\tilde{\rho}^s_{\mathbf{k_1}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_1}}\tilde{\rho}^s_{\mathbf{k_2}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_2}}-\tilde{\rho}^s_0\tilde{\rho}^{\bar{s}}_0\tilde{\rho}^s_{\mathbf{k_1}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_1}}\tilde{\rho}^s_{\mathbf{k_2}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_2}}\tilde{p}^s_\zeta(\mathbf{k_2},t)\big)C_{nns}D_3^{sn}(\mathbf{k_1},t) \\
&-\big(\tilde{p}^s_\zeta(\mathbf{k_2},t)\tilde{\rho}^s_0\tilde{\rho}^{\bar{s}}_0\tilde{\rho}^s_{\mathbf{k_1}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_1}}\tilde{\rho}^s_{\mathbf{k_2}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_2}}-\tilde{\rho}^s_0\tilde{\rho}^{\bar{s}}_0\tilde{\rho}^s_{\mathbf{k_1}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_1}}\tilde{\rho}^s_{\mathbf{k_2}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_2}}\tilde{p}^s_\zeta(\mathbf{k_2},t)\big)C_{\bar{n}\bar{n}s}D_3^{s\bar{n}}(\mathbf{k_1},t) \\
&-\big(\tilde{p}^{\bar{s}}_\zeta(\mathbf{k_2},t)\tilde{\rho}^s_0\tilde{\rho}^{\bar{s}}_0\tilde{\rho}^s_{\mathbf{k_1}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_1}}\tilde{\rho}^s_{\mathbf{k_2}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_2}}-\tilde{\rho}^s_0\tilde{\rho}^{\bar{s}}_0\tilde{\rho}^s_{\mathbf{k_1}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_1}}\tilde{\rho}^s_{\mathbf{k_2}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_2}}\tilde{p}^{\bar{s}}_\zeta(\mathbf{k_2},t)\big)C_{nn\bar{s}}D_3^{\bar{s}n}(\mathbf{k_1},t) \\
&-\big(\tilde{p}^{\bar{s}}_\zeta(\mathbf{k_2},t)\tilde{\rho}^s_0\tilde{\rho}^{\bar{s}}_0\tilde{\rho}^s_{\mathbf{k_1}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_1}}\tilde{\rho}^s_{\mathbf{k_2}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_2}}-\tilde{\rho}^s_0\tilde{\rho}^{\bar{s}}_0\tilde{\rho}^s_{\mathbf{k_1}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_1}}\tilde{\rho}^s_{\mathbf{k_2}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_2}}\tilde{p}^{\bar{s}}_\zeta(\mathbf{k_2},t)\big)C_{\bar{n}\bar{n}\bar{s}}D_3^{\bar{s}\bar{n}}(\mathbf{k_1},t)\Big\}.
\end{aligned}
\tag{5.49}
$$

Acting with the trace:

$$
\begin{aligned}
\mathrm{Tr}_{\left\{\substack{s'\\ \mathbf{k}'\neq\mathbf{k}}\right\},\left\{\substack{\bar{s}\\ \mathbf{k}}\right\}}\left(\frac{\mathrm{d}\tilde{\rho}_{red}}{\mathrm{d}t}\right) &= -\frac{\rho\mu a(t)}{\sqrt{2}(2\pi)^{3/2}}\int \mathrm{d}^3\mathbf{k_1}\delta^3(\mathbf{k})\Big\{\big(\tilde{p}^s_\zeta(0,t)\tilde{\rho}^s_0-\tilde{\rho}^s_0\tilde{p}^s_\zeta(0,t)\big)CsnnD^{sn}_4(\mathbf{k_1},t)\\
&+\big(\tilde{p}^s_\zeta(0,t)\tilde{\rho}^s_0-\tilde{\rho}^s_0\tilde{p}^s_\zeta(0,t)\big)Cs\bar{n}\bar{n}D^{s\bar{n}}_4(\mathbf{k_1},t)\\
&+\underbrace{\left(\tilde{\rho}^s_0\,\mathrm{Tr}_{\left\{\substack{\bar{s}\\ 0}\right\}}\big(\tilde{p}^{\bar{s}}_\zeta(0,t)\tilde{\rho}^{\bar{s}}_0\big)-\tilde{\rho}^s_0\,\mathrm{Tr}_{\left\{\substack{\bar{s}\\ 0}\right\}}\big(\tilde{\rho}^{\bar{s}}_0\tilde{p}^{\bar{s}}_\zeta(0,t)\big)\right)}_{0}\\
&+\underbrace{\ldots}_{0}\Big\}\\
&= -\frac{\rho\mu a(t)}{\sqrt{2}(2\pi)^{3/2}}\int \mathrm{d}^3\mathbf{k_1}\delta^3(\mathbf{k})\Big\{\big[\tilde{p}^s_\zeta(0,t),\tilde{\rho}^s_0\big]\big(C_{snn}D^{sn}_4(\mathbf{k_1},t)-C_{s\bar{n}\bar{n}}D^{s\bar{n}}_4(\mathbf{k_1},t)\big)\Big\},
\end{aligned}
\tag{5.50}
$$

Putting this result inside the equation for the purity we obtain:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}_{\left\{\substack{s\\ 0}\right\}}\left(\tilde{\rho}^{s2}_0\right) = -\frac{2\rho\mu a(t)}{\sqrt{2}(2\pi)^{3/2}}\int \mathrm{d}^3\mathbf{k_1}\delta^3(\mathbf{k})\,\mathrm{Tr}_{\left\{\substack{s\\ 0}\right\}}\Big[\underbrace{\big(\tilde{\rho}^s_0\tilde{p}^s_\zeta(0,t)\tilde{\rho}^s_0-\tilde{\rho}^s_0\tilde{\rho}^s_0\tilde{p}^s_\zeta(0,t)\big)}_{0}\times\\
\times\big(C_{snn}D^{sn}_4(\mathbf{k_1},t)-C_{s\bar{n}\bar{n}}D^{s\bar{n}}_4(\mathbf{k_1},t)\big)\Big]=0,
\end{aligned}
\tag{5.51}
$$

where we get zero because of the ciclicity of the trace and the dots represent other contributions similar to the last one that give zero as well.

2. $\mathbf{k}=\mathbf{k_1}\neq 0\neq\mathbf{k_2}$
   so $\mathbf{k}'=0,\mathbf{k_2}$ and at linear order in ρ we can factorize the reduced density matrix as in (5.48). Then:

$$
\begin{aligned}
\mathrm{Tr}_{\left\{\substack{s'\\ \mathbf{k}'\neq\mathbf{k}}\right\},\left\{\substack{\bar{s}\\ \mathbf{k}}\right\}}\left(\frac{\mathrm{d}\tilde{\rho}_{red}}{\mathrm{d}t}\right) &= -\frac{\rho\mu a(t)}{\sqrt{2}(2\pi)^{3/2}}\int \mathrm{d}^3\mathbf{k_1}\delta^3(\mathbf{k}-\mathbf{k_1})\,\mathrm{Tr}_{\left\{\substack{s,\,\bar{s}\\ 0}\right\},\left\{\substack{s,\,\bar{s}\\ \mathbf{k_2}}\right\},\left\{\substack{\bar{s}\\ \mathbf{k}}\right\}}\times\\
&\times\Big\{\big(\tilde{p}^s_\zeta(0,t)\tilde{\rho}^s_0\tilde{\rho}^{\bar{s}}_0\tilde{\rho}^s_{\mathbf{k}}\tilde{\rho}^{\bar{s}}_{\mathbf{k}}\tilde{\rho}^s_{\mathbf{k_2}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_2}}-\tilde{\rho}^s_0\tilde{\rho}^{\bar{s}}_0\tilde{\rho}^s_{\mathbf{k}}\tilde{\rho}^{\bar{s}}_{\mathbf{k}}\tilde{\rho}^s_{\mathbf{k_2}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_2}}\tilde{p}^s_\zeta(0,t)\big)C_{snn}D^{sn}_4(\mathbf{k},t)\\
&+\big(\tilde{p}^s_\zeta(0,t)\tilde{\rho}^s_0\tilde{\rho}^{\bar{s}}_0\tilde{\rho}^s_{\mathbf{k}}\tilde{\rho}^{\bar{s}}_{\mathbf{k}}\tilde{\rho}^s_{\mathbf{k_2}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_2}}-\tilde{\rho}^s_0\tilde{\rho}^{\bar{s}}_0\tilde{\rho}^s_{\mathbf{k}}\tilde{\rho}^{\bar{s}}_{\mathbf{k}}\tilde{\rho}^s_{\mathbf{k_2}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_2}}\tilde{p}^s_\zeta(0,t)\big)C_{s\bar{n}\bar{n}}D^{s\bar{n}}_4(\mathbf{k},t)\\
&+\big(\tilde{p}^{\bar{s}}_\zeta(0,t)\tilde{\rho}^s_0\tilde{\rho}^{\bar{s}}_0\tilde{\rho}^s_{\mathbf{k}}\tilde{\rho}^{\bar{s}}_{\mathbf{k}}\tilde{\rho}^s_{\mathbf{k_2}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_2}}-\tilde{\rho}^s_0\tilde{\rho}^{\bar{s}}_0\tilde{\rho}^s_{\mathbf{k}}\tilde{\rho}^{\bar{s}}_{\mathbf{k}}\tilde{\rho}^s_{\mathbf{k_2}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_2}}\tilde{p}^{\bar{s}}_\zeta(0,t)\big)C_{\bar{s}nn}D^{\bar{s}n}_4(\mathbf{k},t)\\
&+\big(\tilde{p}^{\bar{s}}_\zeta(0,t)\tilde{\rho}^s_0\tilde{\rho}^{\bar{s}}_0\tilde{\rho}^s_{\mathbf{k}}\tilde{\rho}^{\bar{s}}_{\mathbf{k}}\tilde{\rho}^s_{\mathbf{k_2}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_2}}-\tilde{\rho}^s_0\tilde{\rho}^{\bar{s}}_0\tilde{\rho}^s_{\mathbf{k}}\tilde{\rho}^{\bar{s}}_{\mathbf{k}}\tilde{\rho}^s_{\mathbf{k_2}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_2}}\tilde{p}^{\bar{s}}_\zeta(0,t)\big)C_{\bar{s}\bar{n}\bar{n}}D^{\bar{s}\bar{n}}_4(\mathbf{k},t)\\
&-\big(\tilde{p}^s_\zeta(\mathbf{k_2},t)\tilde{\rho}^s_0\tilde{\rho}^{\bar{s}}_0\tilde{\rho}^s_{\mathbf{k}}\tilde{\rho}^{\bar{s}}_{\mathbf{k}}\tilde{\rho}^s_{\mathbf{k_2}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_2}}-\tilde{\rho}^s_0\tilde{\rho}^{\bar{s}}_0\tilde{\rho}^s_{\mathbf{k}}\tilde{\rho}^{\bar{s}}_{\mathbf{k}}\tilde{\rho}^s_{\mathbf{k_2}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_2}}\tilde{p}^s_\zeta(\mathbf{k_2},t)\big)C_{nns}D^{sn}_3(\mathbf{k},t)\\
&-\big(\tilde{p}^s_\zeta(\mathbf{k_2},t)\tilde{\rho}^s_0\tilde{\rho}^{\bar{s}}_0\tilde{\rho}^s_{\mathbf{k}}\tilde{\rho}^{\bar{s}}_{\mathbf{k}}\tilde{\rho}^s_{\mathbf{k_2}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_2}}-\tilde{\rho}^s_0\tilde{\rho}^{\bar{s}}_0\tilde{\rho}^s_{\mathbf{k}}\tilde{\rho}^{\bar{s}}_{\mathbf{k}}\tilde{\rho}^s_{\mathbf{k_2}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_2}}\tilde{p}^s_\zeta(\mathbf{k_2},t)\big)C_{\bar{n}\bar{n}s}D^{s\bar{n}}_3(\mathbf{k},t)\\
&-\big(\tilde{p}^{\bar{s}}_\zeta(\mathbf{k_2},t)\tilde{\rho}^s_0\tilde{\rho}^{\bar{s}}_0\tilde{\rho}^s_{\mathbf{k}}\tilde{\rho}^{\bar{s}}_{\mathbf{k}}\tilde{\rho}^s_{\mathbf{k_2}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_2}}-\tilde{\rho}^s_0\tilde{\rho}^{\bar{s}}_0\tilde{\rho}^s_{\mathbf{k}}\tilde{\rho}^{\bar{s}}_{\mathbf{k}}\tilde{\rho}^s_{\mathbf{k_2}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_2}}\tilde{p}^{\bar{s}}_\zeta(\mathbf{k_2},t)\big)C_{nn\bar{s}}D^{\bar{s}n}_3(\mathbf{k},t)\\
&-\big(\tilde{p}^{\bar{s}}_\zeta(\mathbf{k_2},t)\tilde{\rho}^s_0\tilde{\rho}^{\bar{s}}_0\tilde{\rho}^s_{\mathbf{k}}\tilde{\rho}^{\bar{s}}_{\mathbf{k}}\tilde{\rho}^s_{\mathbf{k_2}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_2}}-\tilde{\rho}^s_0\tilde{\rho}^{\bar{s}}_0\tilde{\rho}^s_{\mathbf{k}}\tilde{\rho}^{\bar{s}}_{\mathbf{k}}\tilde{\rho}^s_{\mathbf{k_2}}\tilde{\rho}^{\bar{s}}_{\mathbf{k_2}}\tilde{p}^{\bar{s}}_\zeta(\mathbf{k_2},t)\big)C_{\bar{n}\bar{n}\bar{s}}D^{\bar{s}\bar{n}}_3(\mathbf{k},t)\Big\}\\
&= -\frac{\rho\mu a(t)}{\sqrt{2}(2\pi)^{3/2}}\Big\{\underbrace{\left(\tilde{\rho}^s_{\mathbf{k}}\,\mathrm{Tr}_{\left\{\substack{s\\ 0}\right\}}\big(\tilde{p}^s_\zeta(0,t)\tilde{\rho}^s_0\big)-\tilde{\rho}^s_{\mathbf{k}}\,\mathrm{Tr}_{\left\{\substack{s\\ 0}\right\}}\big(\tilde{\rho}^s_0\tilde{p}^s_\zeta(0,t)\big)\right)}_{0}+\underbrace{\ldots}_{0}\Big\},
\end{aligned}
\tag{5.52}
$$

where using the ciclicity of the trace the term we wrote down disappears and the dots represent other contributions similar to the previous one which go to zero.

3. $\mathbf{k}=\mathbf{k_2}\neq 0\neq\mathbf{k_1}$
   The computation is analogous to the case 1 and gives zero as well.

4. $\mathbf{k}=\mathbf{k_1}=\mathbf{k_2}=0$
   In this case the only trace that survives is the trace $\mathrm{Tr}_{\left\{\substack{\bar{s}\\ 0}\right\}}$ and the reduced density matrix can

be factorized as:

$$\tilde{\rho}_{red}(t) = \tilde{\rho}_0^s \tilde{\rho}_0^{\bar{s}}. \tag{5.53}$$

Then:

$$\begin{aligned}
\mathrm{Tr}_{\left\{ {s' \atop \mathbf{k}' \neq \mathbf{k}} \right\}, \left\{ {\bar{s} \atop \mathbf{k}} \right\}} \left(\frac{\mathrm{d}\tilde{\rho}_{red}}{\mathrm{d}t} \right) &= -\frac{\rho\mu a(t)}{\sqrt{2}(2\pi)^{3/2}} \mathrm{Tr}_{\left\{ {\bar{s} \atop 0} \right\}} \times \\
&\times \Big\{ \big(\tilde{p}_\zeta^s(0,t)\tilde{\rho}_0^s\tilde{\rho}_0^{\bar{s}} - \tilde{\rho}_0^s\tilde{\rho}_0^{\bar{s}}\tilde{p}_\zeta^s(0,t)\big) C_{snn} D_4^{sn}(0,t) \\
&+ \big(\tilde{p}_\zeta^s(0,t)\tilde{\rho}_0^s\tilde{\rho}_0^{\bar{s}} - \tilde{\rho}_0^s\tilde{\rho}_0^{\bar{s}}\tilde{p}_\zeta^s(0,t)\big) C_{s\bar{n}\bar{n}} D_4^{s\bar{n}}(0,t) \\
&+ \big(\tilde{p}_\zeta^{\bar{s}}(0,t)\tilde{\rho}_0^s\tilde{\rho}_0^{\bar{s}} - \tilde{\rho}_0^s\tilde{\rho}_0^{\bar{s}}\tilde{p}_\zeta^{\bar{s}}(0,t)\big) C_{\bar{s}nn} D_4^{\bar{s}n}(0,t) \\
&+ \big(\tilde{p}_\zeta^{\bar{s}}(0,t)\tilde{\rho}_0^s\tilde{\rho}_0^{\bar{s}} - \tilde{\rho}_0^s\tilde{\rho}_0^{\bar{s}}\tilde{p}_\zeta^{\bar{s}}(0,t)\big) C_{\bar{s}\bar{n}\bar{n}} D_4^{\bar{s}\bar{n}}(0,t) \\
&- \big(\tilde{p}_\zeta^s(0,t)\tilde{\rho}_0^s\tilde{\rho}_0^{\bar{s}} - \tilde{\rho}_0^s\tilde{\rho}_0^{\bar{s}}\tilde{p}_\zeta^s(0,t)\big) C_{nns} D_3^{sn}(0,t) \\
&- \big(\tilde{p}_\zeta^s(0,t)\tilde{\rho}_0^s\tilde{\rho}_0^{\bar{s}} - \tilde{\rho}_0^s\tilde{\rho}_0^{\bar{s}}\tilde{p}_\zeta^s(0,t)\big) C_{\bar{n}\bar{n}s} D_3^{s\bar{n}}(0,t) \\
&- \big(\tilde{p}_\zeta^{\bar{s}}(0,t)\tilde{\rho}_0^s\tilde{\rho}_0^{\bar{s}} - \tilde{\rho}_0^s\tilde{\rho}_0^{\bar{s}}\tilde{p}_\zeta^{\bar{s}}(0,t)\big) C_{nn\bar{s}} D_3^{\bar{s}n}(0,t) \\
&- \big(\tilde{p}_\zeta^{\bar{s}}(0,t)\tilde{\rho}_0^s\tilde{\rho}_0^{\bar{s}} - \tilde{\rho}_0^s\tilde{\rho}_0^{\bar{s}}\tilde{p}_\zeta^{\bar{s}}(0,t)\big) C_{\bar{n}\bar{n}\bar{s}} D_3^{\bar{s}\bar{n}}(0,t) \Big\} \\
&= -\frac{\rho\mu a(t)}{\sqrt{2}(2\pi)^{3/2}} \Big\{ \big(\tilde{p}_\zeta^s(0,t)\tilde{\rho}_0^s - \tilde{\rho}_0^s\tilde{p}_\zeta^s(0,t)\big) C_{snn} D_4^{sn}(0,t) \\
&+ \big(\tilde{p}_\zeta^s(0,t)\tilde{\rho}_0^s - \tilde{\rho}_0^s\tilde{p}_\zeta^s(0,t)\big) C_{s\bar{n}\bar{n}} D_4^{s\bar{n}}(0,t) \\
&+ \underbrace{\left(\tilde{\rho}_0^s \, \mathrm{Tr}_{\left\{ {\bar{s} \atop 0} \right\}} \left(\tilde{p}_\zeta^{\bar{s}}(0,t)\tilde{\rho}_0^{\bar{s}}\right) - \tilde{\rho}_0^s \, \mathrm{Tr}_{\left\{ {\bar{s} \atop 0} \right\}} \left(\tilde{\rho}_0^{\bar{s}}\tilde{p}_\zeta^{\bar{s}}(0,t)\right) \right)}_{0} C_{\bar{s}nn} D_4^{\bar{s}n}(0,t) + \underbrace{\ldots}_{0} \\
&- \big(\tilde{p}_\zeta^s(0,t)\tilde{\rho}_0^s - \tilde{\rho}_0^s\tilde{p}_\zeta^s(0,t)\big) C_{nns} D_3^{sn}(0,t) \\
&- \big(\tilde{p}_\zeta^s(0,t)\tilde{\rho}_0^s - \tilde{\rho}_0^s\tilde{p}_\zeta^s(0,t)\big) C_{\bar{n}\bar{n}s} D_3^{s\bar{n}}(0,t) - \underbrace{\ldots}_{0} \Big\} \\
&= -\frac{\rho\mu a(t)}{\sqrt{2}(2\pi)^{3/2}} \Big\{ \big[\tilde{p}_\zeta^s(0,t), \tilde{\rho}_0^s\big] \Big(C_{snn} D_4^{sn}(0,t) - C_{s\bar{n}\bar{n}} D_4^{s\bar{n}}(0,t) \\
&- C_{nns} D_3^{sn}(0,t) - C_{\bar{n}\bar{n}s} D_3^{s\bar{n}}(0,t) \Big) \Big\}.
\end{aligned} \tag{5.54}$$

Replacing this expression in the equation for the purity we obtain:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Tr}_{\left\{ {s \atop 0} \right\}} \left(\tilde{\rho}_0^{s2} \right) = -\frac{\rho\mu a(t)}{\sqrt{2}(2\pi)^{3/2}} \Big\{ & \underbrace{\mathrm{Tr}_{\left\{ {s \atop 0} \right\}} \tilde{\rho}_0^s \big[\tilde{p}_\zeta^s(0,t), \tilde{\rho}_0^s\big]}_{0} \Big(C_{snn} D_4^{sn}(0,t) - C_{s\bar{n}\bar{n}} D_4^{s\bar{n}}(0,t) \\
& - C_{nns} D_3^{sn}(0,t) - C_{\bar{n}\bar{n}s} D_3^{s\bar{n}}(0,t) \Big) \Big\} = 0.
\end{aligned} \tag{5.55}$$

The same result is obtained if we consider all the other combinations of Dirac delta contained in (5.24) and (5.25) because the reason why every term goes to zero is the presence of the commutator, together with the ciclicity property of the trace.

We found that the non-linear part does not affect the evolution of the purity. This result is not what we expected naively at the beginning of this work, since, as we explained at the beginning of this chapter, we expected that the non-linear interaction would have spoiled the perfect recoherence we observe in the linear model, favouring the decoherence. Instead we found that the new interaction has no effect on the quantum properties of the system. There are at least two ways to physically interpret this result: the first one is simply that the new interaction is a self interaction in the environment sector and it looks like it does not affect the system; this explanation seems quite improbable, since the new interaction should have the effect that every mode of the system couple to all the modes of the environment through the quadratic coupling and it seems unreasonable to think that this has no effect on the system. The second interpretation is that actually the non-linear interaction has a double effect, but these effects cancel each other out. Indeed, the non-linear coupling leads to a mode-mixing between system and environment, amplifying the number of environment modes with which every system mode can couple. We expect that this effect leads to a more effective transfer of the information from the system to the environment, leading to an increase of the decoherence. At the same time, the non-linear interaction will generate some form of non-gaussianities in the environment (differently from the linear coupling which yields a gaussian model), and this can be seen as an increase of the information that can flow from the environment to the system, favouring the process of recoherence. Since the cause of these two opposite effects is the same, i.e. the introduction of a non-linear coupling in the environment sector, we could think that the two effects compensate exactly each other, leading to a null net effect.

Chapter 6

Conclusions and future prospects

In this thesis project I tried to study the phenomenon of quantum decoherence in a cosmological setting, considering a model with both a linear interaction term and a non-linear term between the so called system and environment bipartition (within the formalism of Open Quantum System).

First of all I gave an introduction about the inflation mechanism, starting with the presentation of the shortcomings of the Hot Big Bang model, like the *horizon*, the *flatness* and the *unwanted relics* problem, showing how an accelerated expansion of the primordial Universe provides a solution to these issues. Then I focused my attention to the simplest mechanism that can drive this inflationary period, which is the slow-roll single-field inflation, where we assume the dominance of a scalar field, the inflaton, during the early times of the Universe. We studied the dynamics of this scalar field, considering both the background dynamics (in a completely homogeneous and isotropic Universe) and the fluctuations dynamics. We are mainly interested in the latter because the inflaton quantum fluctutations can be considered as the seeds of the large scale structure of the Universe we observe today, indeed they can be directly connected to the temperature CMB anisotropies. The question that arises at this point is: how did the transition from quantum to classical fluctuations happen? This is the so-called quantum-to-classical transition problem and one of the main obstruction to the direct observation of quantum properties in our Universe is due to the phenomenon of quantum decoherence.

Quantum decoherence is widely studied in the context of open quantum systems and in this thesis, inspired by other works such as [28, 8, 34], I wanted to apply these techniques in a cosmological setting. In chapter 3 we derived the quantum state of the primordial fluctuations in the case of a free scalar inflaton field, showing its peculiar behaviour that historically led to interpret it as a highly quantum state or a highly classical state. Then we introduced the system-environment bipartition, typical of the OQS context, defining in particular the reduced density matrix which describe the state of the system once the environment has been integrated out. At this point we introduced the so-called *master equations* that are evolution equations for the reduced density matrix, starting with the Nakajima-Zwanzig master equation which is exact and then implementing some approximations leading to the perturbative NZ_n and the TCL_2 equation. The latter is a non-Markovian master equation and it is the equation we used to study our model in chapter 4 and 5. Finally we derived the Lindblad equation, the most known Markovian master equation, and showed various techniques to monitor the quantumness of the system, in particular in this thesis we used the purity, defined as $\gamma = \mathrm{Tr}_\varepsilon(\rho_{red}^2)$.

In chapter 4 we studied a model for the inflaton fluctuations which was proposed in [31] and was studied in [8]. This model contains a linear interaction term, $\zeta' \mathcal{F}$, with ζ the curvature perturbation and $\mathcal{F}$ an entropic field, leading to a Gaussian dynamics. Given the linearity of the evolution this model could be studied both with an exact approach and with a OQS approach and given the gaussianity of the dynamics, all the relevant information on the state of the system are contained in the covariance matrix, $\mathbf{\Sigma}$, which can be simply related to the purity, γ, through $\gamma = 1/(4 \det \mathbf{\Sigma})$. Computing the purity we found that the system undergoes an initial period of decoherence, followed by a growth of the purity which saturates to 1. This phenomenon, called *recoherence*, is due to the smallness of the environment, that has dimensions comparable to those of the system in the linear case, and is able to give back the information to the system. In particular in a Minkowski Universe the purity is observed to oscillate, alternating periods of decoherence with period of recoherence, but in an expanding Universe, the coupling between system and environment is quenched off after a while, so we see a first period of decoherence, followed by a phase of recoherence and then the purity saturates to 1, due to the turn-off of the interaction. The recoherence is a non-Markovian phenomenon, so in an OQS context can only be faced using a non-Markovian master equation, like the TCL_2.

Finally, in chapter 5, we gave our original contribution trying to generalize the previous model, inserting a cubic term in the entropic sector, in order to make this model more realistic. In this case we could not use an exact approach, but only the master equation method and given the non-gaussianity of the dynamics the purity was not simply connected to the covariance matrix of the system, so we used another method to compute the purity evolution, which was already introduced in chapter 3 where it was used to track the decoherence using the Lindblad equation. We derived a new non-Markovian master equation, containing the linear term we alredy had and a non-linear term and surprisingly we found that the non-linear term does not contribute to the purity evolution. The interpretation to this result is that the new term provides two opposite effects that cancel each others out: on one hand the non-linear interaction increases the number of modes the system is coupled with and this favours the information flow from the system to the environment, increasing the decoherence phenomenon. On the other hand, the introduction of a non-linear term lead to the production of some non-Gaussianities in the environment, that can be transferred to the system, increasing the information that can flow from the environment to the system and so the recoherence.

In this thesis we tried to generalize a model which was already studied through the introduction of a non-linear interaction term in the environment. Another case of study would be to consider another kind of interaction of the type $\mathcal{L}_{int} \sim (\partial\zeta)^2\mathcal{F}$, which introduce a non-linear interaction between the system and the environment and not only on the environment. Actually I have started to study this kind of interaction but the computations got much more involved than our case, so I leave this idea for future work. More generally it would be interesting to understand if there are other type of interactions leading to the recoherence phenomenon. Finally, since non-linear interaction terms are related to non-Gaussianities, it would be interesting to see what type of bispectrum and trispectrum are generated by the term we have considered. Some references that work in this direction are [33, 34, 35, 36]. In the end, the computation of equation (5.46) should be performed to verify the validity of the previous method to compute the purity evolution. [1]

[1]Some recent papers [37, 38] appeared to investigate similar issues of this Master Thesis work.

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Appendix A

Explicit computations

A.1 Transport equation

Here we make explicit the computations whose result was shown in section (3.4.6). The first term in the transport equation for the covariance matrix is:

$$
\begin{aligned}
&-\frac{i}{2}\operatorname{Tr}_S\left[\left\{\hat{z}_{\zeta,a},\hat{z}_{\zeta,b}\right\}\left[\hat{H}_S(t)+\hat{H}_{LS}(t),\hat{\rho}_{red}(t)\right]\right] = -\frac{i}{4}\operatorname{Tr}_S\left[\left\{\hat{z}_{\zeta,a},\hat{z}_{\zeta,b}\right\}\left[\hat{z}_{\zeta,i}\big(\hat{H}_0+\Delta\big)_{ij}\hat{z}_{\zeta,j},\hat{\rho}_{red}(t)\right]\right]\\
&= -\frac{i}{4}\operatorname{Tr}_S\left[\hat{z}_{\zeta,a}\hat{z}_{\zeta,b}\left[\hat{z}_{\zeta,i}\big(\hat{H}_0+\Delta\big)_{ij}\hat{z}_{\zeta,j},\hat{\rho}_{red}(t)\right]+\hat{z}_{\zeta,b}\hat{z}_{\zeta,a}\left[\hat{z}_{\zeta,i}\big(\hat{H}_0+\Delta\big)_{ij}\hat{z}_{\zeta,j},\hat{\rho}_{red}(t)\right]\right]\\
&= -\frac{i}{4}\big(\hat{H}_0+\Delta\big)_{ij}\operatorname{Tr}_S\Bigg[\hat{z}_{\zeta,a}\hat{z}_{\zeta,b}\hat{z}_{\zeta,i}\left[\hat{z}_{\zeta,j},\hat{\rho}_{red}(t)\right]+\hat{z}_{\zeta,a}\hat{z}_{\zeta,b}\left[\hat{z}_{\zeta,i},\hat{\rho}_{red}(t)\right]\hat{z}_{\zeta,j}\\
&+\hat{z}_{\zeta,b}\hat{z}_{\zeta,a}\hat{z}_{\zeta,i}\left[\hat{z}_{\zeta,j},\hat{\rho}_{red}(t)\right]+\hat{z}_{\zeta,b}\hat{z}_{\zeta,a}\left[\hat{z}_{\zeta,i},\hat{\rho}_{red}(t)\right]\hat{z}_{\zeta,j}\Bigg]\\
&= -\frac{i}{4}\big(\hat{H}_0+\Delta\big)_{ij}\operatorname{Tr}_S\Bigg[\hat{z}_{\zeta,a}\hat{z}_{\zeta,b}\hat{z}_{\zeta,i}\hat{z}_{\zeta,j}\hat{\rho}_{red}(t)-\hat{z}_{\zeta,a}\hat{z}_{\zeta,b}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t)\hat{z}_{\zeta,j}+\hat{z}_{\zeta,a}\hat{z}_{\zeta,b}\hat{z}_{\zeta,i}\\
&-\hat{z}_{\zeta,a}\hat{z}_{\zeta,b}\hat{\rho}_{red}(t)\hat{z}_{\zeta,i}\hat{z}_{\zeta,j}+\hat{z}_{\zeta,b}\hat{z}_{\zeta,a}\hat{z}_{\zeta,i}\hat{z}_{\zeta,j}\hat{\rho}_{red}(t)-\hat{z}_{\zeta,b}\hat{z}_{\zeta,a}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t)\hat{z}_{\zeta,j}+\hat{z}_{\zeta,b}\hat{z}_{\zeta,a}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t)\hat{z}_{\zeta,j}\\
&-\hat{z}_{\zeta,b}\hat{z}_{\zeta,a}\hat{\rho}_{red}(t)\hat{z}_{\zeta,i}\hat{z}_{\zeta,j}\Bigg]\\
&= -\frac{i}{4}\big(\hat{H}_0+\Delta\big)_{ij}\operatorname{Tr}_S\Bigg[\hat{z}_{\zeta,a}\hat{z}_{\zeta,b}\hat{z}_{\zeta,i}\hat{z}_{\zeta,j}\hat{\rho}_{red}(t)-\hat{z}_{\zeta,i}\hat{z}_{\zeta,j}\hat{z}_{\zeta,a}\hat{z}_{\zeta,b}\hat{\rho}_{red}(t)\\
&+\hat{z}_{\zeta,b}\hat{z}_{\zeta,a}\hat{z}_{\zeta,i}\hat{z}_{\zeta,j}\hat{\rho}_{red}(t)-\hat{z}_{\zeta,i}\hat{z}_{\zeta,j}\hat{z}_{\zeta,b}\hat{z}_{\zeta,a}\hat{\rho}_{red}(t)\Bigg]\\
&= -\frac{i}{4}\big(\hat{H}_0+\Delta\big)_{ij}\operatorname{Tr}_S\left[\left\{\hat{z}_{\zeta,a},\hat{z}_{\zeta,b}\right\}\hat{z}_{\zeta,i}\hat{z}_{\zeta,j}\hat{\rho}_{red}(t)-\hat{z}_{\zeta,i}\hat{z}_{\zeta,j}\left\{\hat{z}_{\zeta,a},\hat{z}_{\zeta,b}\right\}\hat{\rho}_{red}(t)\right]\\
&= -\frac{i}{4}\big(\hat{H}_0+\Delta\big)_{ij}\operatorname{Tr}_S\left\{\left[\left\{\hat{z}_{\zeta,a},\hat{z}_{\zeta,b}\right\},\hat{z}_{\zeta,i}\hat{z}_{\zeta,j}\right]\hat{\rho}_{red}(t)\right\}\\
&= -\frac{i}{4}\big(\hat{H}_0+\Delta\big)_{ij}\operatorname{Tr}_S\left\{\left[\hat{z}_{\zeta,a}\hat{z}_{\zeta,b},\hat{z}_{\zeta,i}\hat{z}_{\zeta,j}\right]\hat{\rho}_{red}(t)+(a\leftrightarrow b)\right\}\\
&= -\frac{i}{4}\big(\hat{H}_0+\Delta\big)_{ij}\operatorname{Tr}_S\Big\{\hat{z}_{\zeta,a}\hat{z}_{\zeta,i}\left[\hat{z}_{\zeta,b},\hat{z}_{\zeta,j}\right]\hat{\rho}_{red}(t)+\left[\hat{z}_{\zeta,a},\hat{z}_{\zeta,i}\right]\hat{z}_{\zeta,j}\hat{z}_{\zeta,b}\hat{\rho}_{red}(t)\\
&+\hat{z}_{\zeta,a}\left[\hat{z}_{\zeta,b}\hat{z}_{\zeta,i}\right]\hat{z}_{\zeta,j}\hat{\rho}_{red}(t)+\hat{z}_{\zeta,i}\left[\hat{z}_{\zeta,a},\hat{z}_{\zeta,j}\right]\hat{z}_{\zeta,b}\hat{\rho}_{red}(t)+(a\leftrightarrow b)\Big\}\\
&= -\frac{i}{4}\big(\hat{H}_0+\Delta\big)_{ij}\operatorname{Tr}_S\Big\{i\omega_{bj}\hat{z}_{\zeta,a}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t)+i\omega_{ai}\hat{z}_{\zeta,j}\hat{z}_{\zeta,b}\hat{\rho}_{red}(t)+i\omega_{bi}\hat{z}_{\zeta,a}\hat{z}_{\zeta,j}\hat{\rho}_{red}(t)\\
&+i\omega_{aj}\hat{z}_{\zeta,i}\hat{z}_{\zeta,b}\hat{\rho}_{red}(t)+i\omega_{aj}\hat{z}_{\zeta,b}\hat{z}_{\zeta,i}\hat{\rho}_{red}(t)+i\omega_{bi}\hat{z}_{\zeta,j}\hat{z}_{\zeta,a}\hat{\rho}_{red}(t)+i\omega_{ai}\hat{z}_{\zeta,b}\hat{z}_{\zeta,j}\hat{\rho}_{red}(t)\\
&+i\omega_{bj}\hat{z}_{\zeta,i}\hat{z}_{\zeta,a}\hat{\rho}_{red}(t)\Big\}
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
&= -\frac{i}{4}\big(\hat{H}_0+\Delta\big)_{ij}\,\mathrm{Tr}_S\,\Big\{i\omega_{bj}\big\{\hat{z}_{\zeta,a},\hat{z}_{\zeta,i}\big\}\hat{\rho}_{red}(t)+i\omega_{ai}\big\{\hat{z}_{\zeta,b},\hat{z}_{\zeta,j}\big\}\hat{\rho}_{red}(t)\\
&+i\omega_{bi}\big\{\hat{z}_{\zeta,a},\hat{z}_{\zeta,j}\big\}\hat{\rho}_{red}(t)+i\omega_{aj}\big\{\hat{z}_{\zeta,b},\hat{z}_{\zeta,i}\big\}\hat{\rho}_{red}(t)\Big\}\\
&= \frac{1}{4}\,\mathrm{Tr}_S\,\Big[\big\{\hat{z}_{\zeta,a},\hat{z}_{\zeta,i}\big\}\hat{\rho}_{red}(t)\big(\hat{H}_0+\Delta\big)_{ij}\underbrace{\omega_{bj}}_{-\omega_{jb}}+\omega_{ai}\big(\hat{H}_0+\Delta\big)_{ij}\big\{\hat{z}_{\zeta,b},\hat{z}_{\zeta,j}\big\}\hat{\rho}_{red}(t)\\
&+\omega_{bi}\big(\hat{H}_0+\Delta\big)_{ij}\big\{\hat{z}_{\zeta,a},\hat{z}_{\zeta,j}\big\}\hat{\rho}_{red}(t)+\big\{\hat{z}_{\zeta,b},\hat{z}_{\zeta,i}\big\}\hat{\rho}_{red}(t)\big(\hat{H}_0+\Delta\big)_{ij}\underbrace{\omega_{aj}}_{-\omega_{ja}}\Big]\\
&= \frac{1}{2}\Big\{-\boldsymbol{\Sigma}\big(\hat{\boldsymbol{H}}_0+\boldsymbol{\Delta}\big)\boldsymbol{\omega}+\boldsymbol{\omega}\big(\hat{\boldsymbol{H}}_0+\boldsymbol{\Delta}\big)\boldsymbol{\Sigma}+\boldsymbol{\omega}\big(\hat{\boldsymbol{H}}_0+\boldsymbol{\Delta}\big)\boldsymbol{\Sigma}-\boldsymbol{\Sigma}\big(\hat{\boldsymbol{H}}_0+\boldsymbol{\Delta}\big)\boldsymbol{\omega}\Big\}\\
&= \boldsymbol{\omega}\big(\hat{\boldsymbol{H}}_0+\boldsymbol{\Delta}\big)\boldsymbol{\Sigma}-\boldsymbol{\Sigma}\big(\hat{\boldsymbol{H}}_0+\boldsymbol{\Delta}\big)\boldsymbol{\omega}.
\end{aligned}
\tag{A.2}
$$

A.2 Master equation coefficients

Here we provide the explicit computation of the master equation coefficients whose final form is shown in (4.2.2):

$$
\begin{aligned}
\Delta_{12}(z) &= -\rho^2 \frac{k}{zH} \operatorname{Re}\Big[p_\zeta(\eta) v_{\mathcal{F}}(\eta) \int_{z_0}^{z} \frac{dz'}{k} \frac{k}{z'H} p^*_\zeta(\eta') v^*_{\mathcal{F}}(\eta') \\
&- p_\zeta(\eta) v^*_{\mathcal{F}}(\eta) \int_{z_0}^{z} \frac{dz'}{k} \frac{k}{z'H} p^*_\zeta(\eta') v_{\mathcal{F}}(\eta')\Big] \\
&= -\frac{\rho^2}{H^2}\frac{k}{z} \operatorname{Re}\Big[p_\zeta(\eta) v_{\mathcal{F}}(\eta) \int_{z_0}^{z} \frac{dz'}{z'} p^*_\zeta(\eta') v^*_{\mathcal{F}}(\eta') - p_\zeta(\eta) v^*_{\mathcal{F}}(\eta) \int_{z_0}^{z} \frac{dz'}{z'} p^*_\zeta(\eta') v_{\mathcal{F}}(\eta')\Big] \\
\Delta_{22}(\eta) &= \frac{4\rho^2}{4i^2} a(\eta) \int_{\eta_0}^{\eta} d\eta' a(\eta') \big[\big(v_\zeta(\eta) p^*_\zeta(\eta') - v^*_\zeta(\eta) p_\zeta(\eta')\big)\big(v_{\mathcal{F}}(\eta) v^*_{\mathcal{F}}(\eta') - v^*_{\mathcal{F}}(\eta) v_{\mathcal{F}}(\eta')\big)\big] \\
&= -\rho^2 a(\eta) \int_{\eta_0}^{\eta} d\eta' a(\eta') \Big[v_\zeta(\eta) v_{\mathcal{F}}(\eta) p^*_\zeta(\eta') v^*_{\mathcal{F}}(\eta') - v^*_\zeta(\eta) v_{\mathcal{F}}(\eta) p_\zeta(\eta') v^*_{\mathcal{F}}(\eta') \\
&- v_\zeta(\eta) v^*_{\mathcal{F}}(\eta) p^*_\zeta(\eta') v_{\mathcal{F}}(\eta') + v^*_\zeta(\eta) v^*_{\mathcal{F}}(\eta) p_\zeta(\eta') v_{\mathcal{F}}(\eta')\Big] \\
&= -\rho^2 a(\eta) \Bigg[v_\zeta(\eta) v_{\mathcal{F}}(\eta) \int_{\eta_0}^{\eta} d\eta' p^*_\zeta(\eta') v^*_{\mathcal{F}}(\eta') \\
&- v_\zeta(\eta) v^*_{\mathcal{F}}(\eta) \int_{\eta_0}^{\eta} d\eta' p^*_\zeta(\eta') v_{\mathcal{F}}(\eta') + \text{h.c.}\Bigg] \\
&= -2\rho^2 a(\eta) \operatorname{Re}\Big[v_\zeta(\eta) v_{\mathcal{F}}(\eta) \int_{\eta_0}^{\eta} d\eta' p^*_\zeta(\eta') v^*_{\mathcal{F}}(\eta') - v_\zeta(\eta) v^*_{\mathcal{F}}(\eta) \int_{\eta_0}^{\eta} d\eta' p^*_\zeta(\eta') v_{\mathcal{F}}(\eta')\Big] \\
&= 2\frac{\rho^2}{H^2}\frac{k}{z} \operatorname{Re}\Big[v_\zeta(z) v_{\mathcal{F}}(z) \int_{z_0}^{z} \frac{dz'}{z'} p^*_\zeta(z') v^*_{\mathcal{F}}(z') - v_\zeta(z) v^*_{\mathcal{F}}(z) \int_{z_0}^{z} \frac{dz'}{z'} p^*_\zeta(z') v_{\mathcal{F}}(z')\Big] \\
D_{12}(\eta) &= -\frac{2\rho^2}{4i} a(\eta) \int_{\eta_0}^{\eta} d\eta' a(\eta') \big[\big(p_\zeta(\eta) p^*_\zeta(\eta') - p^*_\zeta(\eta) p_\zeta(\eta')\big)\big(v_{\mathcal{F}}(\eta) v^*_{\mathcal{F}}(\eta') + v^*_{\mathcal{F}}(\eta) v_{\mathcal{F}}(\eta')\big)\big] \\
&= -\frac{\rho^2}{2i} a(\eta) \int_{\eta_0}^{\eta} d\eta' a(\eta') \Bigg[p_\zeta(\eta) v_{\mathcal{F}}(\eta) p^*_\zeta(\eta') v^*_{\mathcal{F}}(\eta') + p_\zeta(\eta) v^*_{\mathcal{F}}(\eta) p^*_\zeta(\eta') v_{\mathcal{F}}(\eta') \\
&- p^*_\zeta(\eta) v_{\mathcal{F}}(\eta) p_\zeta(\eta') v^*_{\mathcal{F}}(\eta') - p^*_\zeta(\eta) v^*_{\mathcal{F}}(\eta) p_\zeta(\eta') v_{\mathcal{F}}(\eta')\Bigg] \\
&= -\rho^2 a(\eta) \operatorname{Im}\Big[p_\zeta(\eta) v_{\mathcal{F}}(\eta) \int_{\eta_0}^{\eta} d\eta' a(\eta') p^*_\zeta(\eta') v^*_{\mathcal{F}}(\eta') + p_\zeta(\eta) v^*_{\mathcal{F}}(\eta) \int_{\eta_0}^{\eta} d\eta' a(\eta') p^*_\zeta(\eta') v_{\mathcal{F}}(\eta')\Big] \\
&= \frac{\rho^2}{H}\frac{k}{z} \operatorname{Im}\Big[p_\zeta(z) v_{\mathcal{F}}(z) \int_{z_0}^{z} \frac{dz'}{z'} p^*_\zeta(z') v^*_{\mathcal{F}}(z') + p_\zeta(z) v^*_{\mathcal{F}}(z) \int_{z_0}^{z} \frac{dz'}{z'} p^*_\zeta(z') v_{\mathcal{F}}(z')\Big] \\
D_{22}(\eta) &= \frac{4\rho^2}{4i} a(\eta) \int_{\eta_0}^{\eta} d\eta' a(\eta') \big[\big(v_\zeta(\eta) p^*_\zeta(\eta') - v^*_\zeta(\eta) p_\zeta(\eta')\big)\big(v_{\mathcal{F}}(\eta) v^*_{\mathcal{F}}(\eta') + v^*_{\mathcal{F}}(\eta) v_{\mathcal{F}}(\eta')\big)\big] \\
&= \frac{\rho^2}{i} a(\eta) \int_{\eta_0}^{\eta} d\eta' a(\eta') \Big[v_\zeta(\eta) v_{\mathcal{F}}(\eta) p^*_\zeta(\eta') v^*_{\mathcal{F}}(\eta') + v_\zeta(\eta) v^*_{\mathcal{F}}(\eta) p^*_\zeta(\eta') v_{\mathcal{F}}(\eta') \\
&- v^*_\zeta(\eta) v_{\mathcal{F}}(\eta) p_\zeta(\eta') v^*_{\mathcal{F}}(\eta') - v^*_\zeta(\eta) v^*_{\mathcal{F}}(\eta) p_\zeta(\eta') v_{\mathcal{F}}(\eta')\Big] \\
&= 2\rho^2 a(\eta) \operatorname{Im}\Big[v_\zeta(\eta) v_{\mathcal{F}}(\eta) \int_{\eta_0}^{\eta} d\eta' a(\eta') p^*_\zeta(\eta') v^*_{\mathcal{F}}(\eta') + v_\zeta(\eta) v^*_{\mathcal{F}}(\eta) \int_{\eta_0}^{\eta} d\eta' a(\eta') p^*_\zeta(\eta') v_{\mathcal{F}}(\eta')\Big] \\
&= -2\frac{\rho^2}{H}\frac{k}{z} \operatorname{Im}\Big[v_\zeta(z) v_{\mathcal{F}}(z) \int_{z_0}^{z} \frac{dz'}{z'} p^*_\zeta(z') v^*_{\mathcal{F}}(z') + v_\zeta(z) v^*_{\mathcal{F}}(z) \int_{z_0}^{z} \frac{dz'}{z'} p^*_\zeta(z') v_{\mathcal{F}}(z')\Big].
\end{aligned}
\tag{A.3}
$$

A.3 Non-linear purity evolution

Here we show explicitly the computations for the purity evolution in case of non-linear interaction, whose result is shown in chapter 5.

$$
\begin{aligned}
\frac{\mathrm{d}\tilde{\rho}_{red}(t)}{\mathrm{d}t} &= -\int_{t_0}^{t} dt' \,\mathrm{Tr}_\varepsilon \left[\tilde{H}_{int}(t), \left[\tilde{H}_{int}(t'), \tilde{\rho}_{red}(t)\otimes\hat{\rho}_\varepsilon^{(0)}\right]\right] \\
&= -\int_{t_0}^{t} \mathrm{Tr}_\varepsilon \Bigg[\tilde{H}_{int}(t)\tilde{H}_{int}(t')\tilde{\rho}_{red}(t)\otimes\hat{\rho}_\varepsilon^{(0)} - \tilde{H}_{int}(t)\tilde{\rho}_{red}(t)\otimes\hat{\rho}_\varepsilon^{(0)}\tilde{H}_{int}(t') \\
&- \tilde{H}_{int}(t')\tilde{\rho}_{red}(t)\otimes\hat{\rho}_\varepsilon^{(0)}\tilde{H}_{int}(t) + \tilde{\rho}_{red}(t)\otimes\hat{\rho}_\varepsilon^{(0)}\tilde{H}_{int}(t')\tilde{H}_{int}(t)\Bigg] \\
&= -\int_{t_0}^{t} dt' \,\mathrm{Tr}_\varepsilon \Bigg[\int d^3\mathbf{k}d^3\mathbf{p}\rho^2 a(t)a(t')\sum_{s,l}\Big(\tilde{v}^s_{\mathcal{F}}(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{v}^l_{\mathcal{F}}(\mathbf{p},t')\tilde{p}^l_\zeta(\mathbf{p},t')\tilde{\rho}_{red}(t)\otimes\hat{\rho}_\varepsilon^{(0)} \\
&- \tilde{v}^s_{\mathcal{F}}(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\otimes\hat{\rho}_\varepsilon^{(0)}\tilde{v}^l_{\mathcal{F}}(\mathbf{p},t')\tilde{p}^l_\zeta(\mathbf{p},t') - \tilde{v}^l_{\mathcal{F}}(\mathbf{p},t')\tilde{p}^l_\zeta(\mathbf{p},t')\tilde{\rho}_{red}(t)\otimes\hat{\rho}_\varepsilon^{(0)}\tilde{v}^s_{\mathcal{F}}(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t) \\
&+ \tilde{\rho}_{red}(t)\otimes\hat{\rho}_\varepsilon^{(0)}\tilde{v}^l_{\mathcal{F}}(\mathbf{p},t')\tilde{p}^l_\zeta(\mathbf{p},t')\tilde{v}^s_{\mathcal{F}}(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t)\Big) \\
&+ \int d^3\mathbf{k}d^3\mathbf{p}\Bigg[\frac{\rho\mu a(t)}{\sqrt{2}a^3(t')}\frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{p}'\sum_{s,\bar{l},n,\bar{n}} C_{\bar{l}n\bar{n}}\Big(\tilde{v}^s_{\mathcal{F}}(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{v}^{\bar{l}}_{\mathcal{F}}(\mathbf{p},t')\tilde{v}^n_{\mathcal{F}}(\mathbf{p}',t')\tilde{v}^{\bar{n}}_{\mathcal{F}}(\mathbf{p}+\mathbf{p}',t') \\
&\times\tilde{\rho}_{red}(t)\otimes\hat{\rho}_\varepsilon^{(0)} - \tilde{v}^s_{\mathcal{F}}(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\otimes\hat{\rho}_\varepsilon^{(0)}\tilde{v}^{\bar{l}}_{\mathcal{F}}(\mathbf{p},t')\tilde{v}^n_{\mathcal{F}}(\mathbf{p}',t')\tilde{v}^{\bar{n}}_{\mathcal{F}}(\mathbf{p}+\mathbf{p}',t') \\
&- \tilde{v}^{\bar{l}}_{\mathcal{F}}(\mathbf{p},t')\tilde{v}^n_{\mathcal{F}}(\mathbf{p}',t')\tilde{v}^{\bar{n}}_{\mathcal{F}}(\mathbf{p}+\mathbf{p}',t')\tilde{\rho}_{red}(t)\otimes\hat{\rho}_\varepsilon^{(0)}\tilde{v}^s_{\mathcal{F}}(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t) \\
&+ \tilde{\rho}_{red}(t)\otimes\hat{\rho}_\varepsilon^{(0)}\tilde{v}^{\bar{l}}_{\mathcal{F}}(\mathbf{p},t')\tilde{v}^n_{\mathcal{F}}(\mathbf{p}',t')\tilde{v}^{\bar{n}}_{\mathcal{F}}(\mathbf{p}+\mathbf{p}',t')\tilde{v}^s_{\mathcal{F}}(\mathbf{k},t)\tilde{p}^s_\zeta(\mathbf{k},t)\Big) \\
&+ \frac{\rho\mu a(t')}{\sqrt{2}a^3(t)}\frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{k}'\sum_{l,\bar{s},q,\bar{q}} C_{\bar{s}q\bar{q}}\Big(\tilde{v}^{\bar{s}}_{\mathcal{F}}(\mathbf{k},t)\tilde{v}^q_{\mathcal{F}}(\mathbf{k}',t)\tilde{v}^{\bar{q}}_{\mathcal{F}}(\mathbf{k}+\mathbf{k}',t)\tilde{v}^l_{\mathcal{F}}(\mathbf{p},t')\tilde{p}^l_\zeta(\mathbf{p},t')\tilde{\rho}_{red}(t)\otimes\hat{\rho}_\varepsilon^{(0)} \\
&- \tilde{v}^{\bar{s}}_{\mathcal{F}}(\mathbf{k},t)\tilde{v}^q_{\mathcal{F}}(\mathbf{k}',t)\tilde{v}^{\bar{q}}_{\mathcal{F}}(\mathbf{k}+\mathbf{k}',t)\tilde{\rho}_{red}(t)\otimes\hat{\rho}_\varepsilon^{(0)}\tilde{v}^l_{\mathcal{F}}(\mathbf{p},t')\tilde{p}^l_\zeta(\mathbf{p},t') \\
&- \tilde{v}^l_{\mathcal{F}}(\mathbf{p},t')\tilde{p}^l_\zeta(\mathbf{p},t')\tilde{\rho}_{red}(t)\otimes\hat{\rho}_\varepsilon^{(0)}\tilde{v}^{\bar{s}}_{\mathcal{F}}(\mathbf{k},t)\tilde{v}^q_{\mathcal{F}}(\mathbf{k}',t)\tilde{v}^{\bar{q}}_{\mathcal{F}}(\mathbf{k}+\mathbf{k}',t) \\
&+ \tilde{\rho}_{red}(t)\otimes\hat{\rho}_\varepsilon^{(0)}\tilde{v}^l_{\mathcal{F}}(\mathbf{p},t')\tilde{p}^l_\zeta(\mathbf{p},t')\tilde{v}^{\bar{s}}_{\mathcal{F}}(\mathbf{k},t)\tilde{v}^q_{\mathcal{F}}(\mathbf{k}',t)\tilde{v}^{\bar{q}}_{\mathcal{F}}(\mathbf{k}+\mathbf{k}',t)\Big)\Bigg]\Bigg] \\
&= -\int_{t_0}^{t} dt' \Bigg\{\int d^3\mathbf{k}d^3\mathbf{p}\rho^2 a(t)a(t')\sum_{s,l}\Bigg(\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{p}^l_\zeta(\mathbf{p},t')\tilde{\rho}_{red}(t)\,\mathrm{Tr}_\varepsilon\left[\tilde{v}^s_{\mathcal{F}}(\mathbf{k},t)\tilde{v}^l_{\mathcal{F}}(\mathbf{p},t')\hat{\rho}_\varepsilon^{(0)}\right] \\
&- \tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\tilde{p}^l_\zeta(\mathbf{p},t')\,\mathrm{Tr}_\varepsilon\left[\tilde{v}^s_{\mathcal{F}}(\mathbf{k},t)\hat{\rho}_\varepsilon^{(0)}\tilde{v}^l_{\mathcal{F}}(\mathbf{p},t')\right] - \tilde{p}^l_\zeta(\mathbf{p},t')\tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t)\,\mathrm{Tr}_\varepsilon\left[\tilde{v}^l_{\mathcal{F}}(\mathbf{p},t')\hat{\rho}_\varepsilon^{(0)}\tilde{v}^s_{\mathcal{F}}(\mathbf{k},t)\right] \\
&+ \tilde{\rho}_{red}(t)\tilde{p}^l_\zeta(\mathbf{p},t')\tilde{p}^s_\zeta(\mathbf{k},t)\,\mathrm{Tr}_\varepsilon\left[\hat{\rho}_\varepsilon^{(0)}\tilde{v}^l_{\mathcal{F}}(\mathbf{p},t')\tilde{v}^s_{\mathcal{F}}(\mathbf{k},t)\right]\Bigg) \\
&+ \int d^3\mathbf{k}d^3\mathbf{p}\Bigg[\frac{\rho\mu a(t)}{\sqrt{2}a^3(t')}\frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{p}'\sum_{s,\bar{l},n,\bar{n}} C_{\bar{l}n\bar{n}}\Bigg(\tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\,\mathrm{Tr}_\varepsilon\left[\tilde{v}^s_{\mathcal{F}}(\mathbf{k},t)\tilde{v}^{\bar{l}}_{\mathcal{F}}(\mathbf{p},t')\tilde{v}^n_{\mathcal{F}}(\mathbf{p}',t')\right] \\
&\times\tilde{v}^{\bar{n}}_{\mathcal{F}}(\mathbf{p}+\mathbf{p}',t')\hat{\rho}_\varepsilon^{(0)} - \tilde{p}^s_\zeta(\mathbf{k},t)\tilde{\rho}_{red}(t)\,\mathrm{Tr}_\varepsilon\left[\tilde{v}^s_{\mathcal{F}}(\mathbf{k},t)\hat{\rho}_\varepsilon^{(0)}\tilde{v}^{\bar{l}}_{\mathcal{F}}(\mathbf{p},t')\tilde{v}^n_{\mathcal{F}}(\mathbf{p}',t')\tilde{v}^{\bar{n}}_{\mathcal{F}}(\mathbf{p}+\mathbf{p}',t')\right] \\
&- \tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t)\,\mathrm{Tr}_\varepsilon\left[\tilde{v}^{\bar{l}}_{\mathcal{F}}(\mathbf{p},t')\tilde{v}^n_{\mathcal{F}}(\mathbf{p}',t')\tilde{v}^{\bar{n}}_{\mathcal{F}}(\mathbf{p}+\mathbf{p}',t')\hat{\rho}_\varepsilon^{(0)}\tilde{v}^s_{\mathcal{F}}(\mathbf{k},t)\right] \\
&+ \tilde{\rho}_{red}(t)\tilde{p}^s_\zeta(\mathbf{k},t)\,\mathrm{Tr}_\varepsilon\left[\hat{\rho}_\varepsilon^{(0)}\tilde{v}^{\bar{l}}_{\mathcal{F}}(\mathbf{p},t')\tilde{v}^n_{\mathcal{F}}(\mathbf{p}',t')\tilde{v}^{\bar{n}}_{\mathcal{F}}(\mathbf{p}+\mathbf{p}',t')\tilde{v}^s_{\mathcal{F}}(\mathbf{k},t)\right]\Bigg) \\
&+ \frac{\rho\mu a(t')}{\sqrt{2}a^3(t)}\frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{k}'\sum_{l,\bar{s},q,\bar{q}} C_{\bar{s}q\bar{q}}\Bigg(\tilde{p}^l_\zeta(\mathbf{p},t')\tilde{\rho}_{red}(t)\,\mathrm{Tr}_\varepsilon\left[\tilde{v}^{\bar{s}}_{\mathcal{F}}(\mathbf{k},t)\tilde{v}^q_{\mathcal{F}}(\mathbf{k}',t)\tilde{v}^{\bar{q}}_{\mathcal{F}}(\mathbf{k}+\mathbf{k}',t)\tilde{v}^l_{\mathcal{F}}(\mathbf{p},t')\hat{\rho}_\varepsilon^{(0)}\right] \\
&- \tilde{\rho}_{red}(t)\tilde{p}^l_\zeta(\mathbf{p},t')\,\mathrm{Tr}_\varepsilon\left[\tilde{v}^{\bar{s}}_{\mathcal{F}}(\mathbf{k},t)\tilde{v}^q_{\mathcal{F}}(\mathbf{k}',t)\tilde{v}^{\bar{q}}_{\mathcal{F}}(\mathbf{k}+\mathbf{k}',t)\hat{\rho}_\varepsilon^{(0)}\tilde{v}^l_{\mathcal{F}}(\mathbf{p},t')\right]
\end{aligned}
\tag{A.4}
$$

$$
\begin{aligned}
&- \tilde{p}^{l}_{\zeta}(\mathbf{p}, t')\tilde{\rho}_{red}(t) \, \mathrm{Tr}_{\varepsilon} \left[\tilde{v}^{l}_{\mathcal{F}}(\mathbf{p}, t')\hat{\rho}^{(0)}_{\varepsilon} \tilde{v}^{\bar{s}}_{\mathcal{F}}(\mathbf{k}, t)\tilde{v}^{q}_{\mathcal{F}}(\mathbf{k}', t)\tilde{v}^{\bar{q}}_{\mathcal{F}}(\mathbf{k} + \mathbf{k}', t) \right] \\
&+ \tilde{\rho}_{red}(t)\tilde{p}^{l}_{\zeta}(\mathbf{p}, t') \, \mathrm{Tr}_{\varepsilon} \left[\hat{\rho}^{(0)}_{\varepsilon} \tilde{v}^{l}_{\mathcal{F}}(\mathbf{p}, t')\tilde{v}^{\bar{s}}_{\mathcal{F}}(\mathbf{k}, t)\tilde{v}^{q}_{\mathcal{F}}(\mathbf{k}', t)\tilde{v}^{\bar{q}}_{\mathcal{F}}(\mathbf{k} + \mathbf{k}', t) \right] \Bigg) \Bigg] \Bigg\}.
\end{aligned}
$$

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