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Review of Lambert's Theorem, a two-body orbital boundary value problem

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Abstract

The problem of finding the orbit that connects two position vectors in a certain time came to light towards the end of the 18th century in connection with the determination of the orbits of celestial bodies from observation made from Earth. Because of Lambert's pioneering contributions (i.e. *Lambert's theorem*) the problem (his original motivation was the determination of cometary orbits) is typically referred to as *Lambert's problem*. Even Gauss (to cite explicitly [14, 10.1]), the Prince of Mathematicians, said that this particular problem is "to be considered among the most important in the theory of the motions of the heavenly bodies". He published the first formal solution to the problem in his treaty on the motion of celestial bodies in 1809 but some years before he was able to correctly determine the orbit of the newly-discovered object Ceres (It was firstly discovered by the astronomer Piazzi who observed it 24 times between 1 January and 11 February 1801 when observations were interrupted and the object was lost. On 31 December 1801 Ceres was recovered thanks to what was an orbit determination problem solved by Gauss).
In modern Astrodynamics and Celestial Mechanics, thanks to the rise of interplanetary exploration that has brought a renewed interest in *Lambert's problem*, it has application mainly on mission design: for example, to name a few, *Lambert's theorem* has been used to calculate the orbits of the Voyager interplanetary missions, which required a very accurate determination of orbital parameters in order to execute fly-bys of the outer planets and it was also "present" in the *Lambert Guidance Program* which guided the Apollo spacecrafts to their landing sites on the Moon [15, 7.2].

The purpose of this thesis work is to provide an introduction to *Lambert's theorem* and to do so the work has been divided into four parts. In Chapter 1 I analyze the two body problem and derive its properties in order to get to the orbit equation. In Chapter 2 the aim is to derive Kepler's equations, which relate position to time for the different kinds of orbit. Chapter 3 is the heart of the thesis in which *Lambert's theorem* is introduced and demonstrated. Finally, Chapter 4 represents an attempt (just one of many in literature) to get a unified form of *Lambert's theorem* which is valid for elliptic, hyperbolic and parabolic orbits.

Sommario

Il problema di determinare un'orbita dati due vettori posizione ed il relativo tempo di percorrenza è nato verso la fine del XVIII secolo dall'obbiettivo di voler determinare le orbite dei corpi celesti da osservazioni fatte dalla Terra. E' divenuto noto come *Problema di Lambert* in onore dei suoi pioneristici contributi (i.e. *Teorema di Lambert*) inizialmente rivolti verso le orbite cometarie. Anche Gauss (citando esplicitamente [14, 10.1]), ha definito il problema come "uno tra i più importanti nella teoria del moto dei corpi massivi". Nel suo trattato *Theoria motus corporum coelestium in sectionibus conicis solem ambientium* del 1809 ha pubblicato la prima soluzione formale al problema dopo che alcuni anni prima aveva correttamente determinato l'orbita dell'allora appena scoperto asteroide(pianeta nano) Cerere (Lo individuò per la prima volta l'astronomo Piazzi che fu in grado di compiere solo 24 osservazioni tra 1 gennaio e 11 febbraio 1801, prima di perderne le tracce. Il 31 dicembre 1801 l'oggetto venne ritrovato grazie alle previsioni di Gauss che ne aveva determinato l'orbita). Nell'ambito dell'Astrodinamica e Meccanica Celeste moderne, grazie alla nascita dell'esplorazione interplanetaria che ha rinnovato l'interesse verso questo tipo di problema, ha trovato la sua principale applicazione nel campo della progettazione di missione (rendezvous, targeting, determinazione preliminare dell'orbita, ecc...): per esempio, il teorema è stato utilizzato per calcolare le orbite delle missioni Voyager, le quali richiedevano la determinazione molto accurata dei parametri orbitali al fine di eseguire i fly-bys dei pianeti esterni ed era anche alla base del *Lambert Guidance Program* che ha guidato le sonde del programma Apollo sulla Luna [15, 7.2].

Lo scopo di questo lavoro di tesi è quello di presentare un'introduzione del *Teorema di Lambert*; il lavoro è stato diviso in quattro parti. Nel Capitolo 1 ho analizzato il problema dei due corpi e derivato le sue proprietà con lo scopo di arrivare all'equazione dell'orbita. Nel Capitolo 2 l'obbiettivo è quello di derivare l'equazione di Kepler, la quale mette in relazione la posizione con il tempo per i diversi tipi di orbita. Il Capitolo 3 rappresenta il cuore della tesi in cui viene introdotto e dimostrato il *Teorema di Lambert*. Infine, nel Capitolo 4 viene fatto il tentativo (seguendo uno dei tanti possibili metodi presenti in letteratura) di ottenere una forma unificata del *Teorema di Lambert* valida per i tre tipi di orbita (ellittica, iperbolica e parabolica).

Contents

1 Two-Body Problem

Summary This chapter introduces the problem of determining the motion of two bodies due solely to their own mutual gravitational attraction. We show that it can be treated as the motion of a point particle of mass equal to the *reduced mass* that is attracted by the origin and, with the demonstration of some fundamental laws of conservation, we highlight some important properties associated to this kind of motion. It ends with the derivation of the solution equation that represents conic sections whose shape is determined by the *eccentricity*.

1.1 N-Body Problem

According to Newton's law of gravitation, two particles attract each other with a force, acting along the line joining them, which is proportional to the product of their masses and inversely proportional to the square of the distance between them. Considering a system of $P_1, ..., P_n$ particles whose masses are $m_1, ..., m_n$, the position and velocity vectors of the i^{th} particle expressed with respect to an inertial coordinate system can be written as follow (Battin [1, pp.95])

$$\boldsymbol{r}_i = x_i \boldsymbol{i}_x + y_i \boldsymbol{i}_y + z_i \boldsymbol{i}_z \quad \text{and} \quad \boldsymbol{v}_i = \frac{d\boldsymbol{r}_i}{dt}$$

Furthermore, let

$$r_{ij} = |r_j - r_i| = \sqrt{(r_j - r_i)(r_j - r_i)}$$

denote the distance between P_i and P_j so that the magnitude of the force of attraction between the i^{th} an the j^{th} particles is $G\frac{m_i m_j}{r_{ij}^2}$ where G is the gravitational constant, amounting to $G = 6.67 \cdot 10^{-11} m^3 kg^{-1} s^{-2}$. The force acting on P_i due to P_j has the direction of $\frac{(r_j - r_i)}{r_{ij}}$ while the force on P_j due to P_i is oppositely directed (Newton's third law). Hence, the total force F_i affecting P_i, due to the presence of the other $n-1$ masses is

$$F_i = G\sum_{j=1}^{n} \frac{m_i m_j}{r_{ij}^3}(\boldsymbol{r}_j - \boldsymbol{r}_i) \quad \text{for i}\neq\text{j} \tag{1.1}$$

In accordance with Newton's second law of motion,

$$F_i = m_i \frac{d^2\boldsymbol{r}_i}{dt^2} = m_i \frac{d\boldsymbol{v}_i}{dt} \tag{1.2}$$

so that the n vector differential equations

$$\frac{d^2\boldsymbol{r}_i}{dt^2} = G\sum_{j=1}^{n} \frac{m_j}{r_{ij}^3}(\boldsymbol{r}_j - \boldsymbol{r}_i) \tag{1.3}$$

together with appropriate initial conditions, constitute a complete mathematical description of the motion of the system of n mass particles.
Let's then consider, for our purpose, a system of $n = 2$ point particles of mass m_1 and m_2 respectively and let (O, X, Y, Z) be an inertial frame with r_1 and r_2 their position vectors. Thus (Battin [1, p.108]) the motion of two bodies is fully described by the following pair of nonlinear second-order vector differential equations

$$\begin{aligned} m_1\ddot{\boldsymbol{r}}_1 &= G\frac{m_1 m_2}{r^3}\boldsymbol{r} \\ m_2\ddot{\boldsymbol{r}}_2 &= -G\frac{m_1 m_2}{r^3}\boldsymbol{r} \end{aligned} \tag{1.4}$$

where $\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1 = \boldsymbol{r}_{12}$ (which defines the position of P_2 with respect to P_1), together with the position vectors $\boldsymbol{r}_{1,2}(t)$ and the velocities vectors $\boldsymbol{v}_{1,2}(t)$ specified at some particular instant of time. Finding the positions and velocities at future times is the well known *two-body problem*.

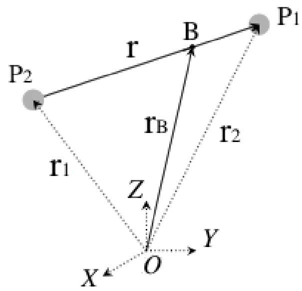

Figure 1.1: The position vectors of two point particles and of their center of mass are shown relative to an inertial frame of reference XYZ.

1.2 Integrals Of Motion

Equations (1.4) represents a system of 6 scalar differential equations of order 2 (we have a differential equation of order 2 for each of the 3 components of the two vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$; the equations are coupled to each other). Now (Celletti [2]), summing these two equations one gets that

$$m_1\ddot{\boldsymbol{r}}_1 + m_2\ddot{\boldsymbol{r}}_2 = 0 \tag{1.5}$$

Since the masses are considered to be constant in our consideration, this equation can be integrated twice:

$$m_1\dot{\boldsymbol{r}}_1 + m_2\dot{\boldsymbol{r}}_2 = C_1 \quad , \quad m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2 = C_1 t + C_2 \tag{1.6}$$

with C_1, C_2 constant vectors.
Let M be the total mass, namely $M = m_1 + m_2$. The location of the barycenter is given by $M\boldsymbol{r}_B = m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2$ where $\boldsymbol{r}_B$ is the position vector of the barycenter B of the system; it behaves like a point particle of mass equal to the total mass M of the system under the influence of the total force given by (1.5). Hence we obtain the equations

$$M\dot{\boldsymbol{r}}_B = C_1 \quad , \quad M\boldsymbol{r}_B = C_1 t + C_2 \tag{1.7}$$

which express that the barycenter moves with a constant rectilinear velocity (proportional to C_1). The non accelerating center of mass B of a two-body system is then a good choice for the origin of an inertial frame.
Quantities which remain constant during the motion (called integrals of motion) can be used to reduce the degree of freedom of the system of differential equations. For example, dividing by m_1 and m_2 in the first and second equation of (1.4) respectively, subtracting the resulting equations and accounting for the conservation of the integral of the center of mass of the two-body system, one gets

$$\ddot{\boldsymbol{r}} + \mu\frac{\boldsymbol{r}}{r^3} = 0 \tag{1.8}$$

and this is a system of 3 scalar differential equations of order 2 (components of $\boldsymbol{r}$ are defined by a system of 3 coupled equations of order 2) where $\mu = GM$ and $\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$ now referred to the new reference frame with origin in B.
These are the equations of relative motion, naturally termed as the equations of motion of the *reduced two-body problem*. They can be seen as the equation of motion of a point particle of mass equal to the *reduced mass* $\frac{m_1 m_2}{(m_1+m_2)}$ attracted by the origin O.

1.2.1 Integrals of angular momentum and energy

Before attempting to solve the equation (1.8) we shall derive some useful information about the nature of that motion. For example, by taking the vector product of (1.8) with the position vector $\boldsymbol{r}$ we have (Celletti [2])

$$\boldsymbol{r} \times \ddot{\boldsymbol{r}} = 0 \tag{1.9}$$

and, remembering that $\frac{d(\boldsymbol{r}\times\dot{\boldsymbol{r}})}{dt} = \boldsymbol{r} \times \ddot{\boldsymbol{r}}$ (since $\dot{\boldsymbol{r}} \times \dot{\boldsymbol{r}} = 0$), by integrating obtain

$$\boldsymbol{r} \times \dot{\boldsymbol{r}} = \boldsymbol{h} \tag{1.10}$$

where $\boldsymbol{h}$ is the integration constant that represents the angular momentum.
Since the cross product of the position vector and its velocity stays constant (property true for any central force[1]), they must lie in the same plane, orthogonal to $\boldsymbol{h}$ (this means that the two bodies move at any instant on what is called the *orbit plane*). Notice also, as it will be useful in a while to find another integral of motion, that not only the direction but also the magnitude of vector $\boldsymbol{h}$ is conserved ($|\boldsymbol{h}| = const$).
Now, using the previous results, (without loss of generality) we can restrict ourselves to consider the motion of a point particle of mass equal to the *reduced mass* in the orbit plane and consider (r, θ) as the polar coordinate system.
Let's denote $\boldsymbol{e}_r = (\cos\theta \boldsymbol{e}_x + \sin\theta \boldsymbol{e}_y)$ and $\boldsymbol{e}_\theta = (-\sin\theta \boldsymbol{e}_x + \cos\theta \boldsymbol{e}_y)$ the unit vectors of the radial and orthogonal components respectively, $\boldsymbol{e}_z$ the unit vector normal to the orbital plane and remember that $\frac{d}{dt}\boldsymbol{e}_r = \dot{\theta}\boldsymbol{e}_\theta$ while $\frac{d}{dt}\boldsymbol{e}_\theta = -\dot{\theta}\boldsymbol{e}_r$. Then the position, velocity and acceleration vectors can be written in polar coordinates as

$$\boldsymbol{r} = r\boldsymbol{e}_r, \quad \dot{\boldsymbol{r}} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta, \quad \ddot{\boldsymbol{r}} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{e}_r + [\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})]\boldsymbol{e}_\theta \tag{1.11}$$

and calculating

$$det\begin{vmatrix} \boldsymbol{e}_r & \boldsymbol{e}_\theta & \boldsymbol{e}_z \\ r & 0 & 0 \\ \dot{r} & r\dot{\theta} & 0 \end{vmatrix}$$

one obtains that $h = |\boldsymbol{h}| = |r^2\dot{\theta}\boldsymbol{e_z}| = r^2\dot{\theta} = const.$
Now let's consider the radius vector $\boldsymbol{r}$ in an interval dt, it sweeps out the angle $d\theta = \dot{\theta}dt$ and the area

$$dA = \frac{1}{2}r(t)r(t+dt)\sin(d\theta) \quad .$$

The variation of A with respect to the time is then given by

$$\frac{dA}{dt} = \frac{1}{2}r(t)r(t+dt)\frac{\sin(d\theta)}{d\theta}\frac{d\theta}{dt}$$

and in the limit $dt \to 0$ one gets

$$\dot{A} = \frac{1}{2}r^2\dot{\theta} = \frac{1}{2}h \quad . \tag{1.12}$$

[1]A type of motion characterized by (1.9) $\forall t$

Since h is a constant, this implies that equal areas are swept out in equal times: hence this is the mathematical form of Kepler's second law of planetary motion.
Now, considering the scalar product of (1.8) with $\dot{\boldsymbol{r}}$ we have

$$\dot{\boldsymbol{r}}\ddot{\boldsymbol{r}} = -\mu\frac{\dot{\boldsymbol{r}}\boldsymbol{r}}{r^3} \tag{1.13}$$

For the right-hand side, we know that

$$\boldsymbol{r}\cdot\boldsymbol{r} = r^2$$

so that

$$\frac{d}{dt}(\boldsymbol{r}\cdot\boldsymbol{r}) = 2r\frac{dr}{dt}$$

But we also can observe that

$$\frac{d}{dt}(\boldsymbol{r}\cdot\boldsymbol{r}) = \boldsymbol{r}\frac{d\boldsymbol{r}}{dt} + \frac{d\boldsymbol{r}}{dt}\boldsymbol{r} = 2\boldsymbol{r}\frac{d\boldsymbol{r}}{dt}$$

Thus, it is clear that $\dot{\boldsymbol{r}}\cdot\boldsymbol{r} = \dot{r}r$.
For the left-hand side, it holds

$$\dot{\boldsymbol{r}}\cdot\ddot{\boldsymbol{r}} = \frac{1}{2}\frac{d}{dt}(\dot{\boldsymbol{r}}\dot{\boldsymbol{r}}) = \frac{1}{2}\frac{d}{dt}\dot{r}^2 = \frac{d}{dt}(\frac{\dot{r}^2}{2})$$

Substituting the previous results in (1.13) and integrating, it yields

$$\frac{1}{2}\dot{\boldsymbol{r}}^2 - \frac{\mu}{r} = \varepsilon \tag{1.14}$$

where ε is a constant while the first and the second terms are the kinetic and the potential energy per unit mass respectively. This equation is a statement of the conservation of energy, namely, that the specific mechanical energy is the same at all points of the trajectory. It is also known as the *vis viva* equation.
Recalling a system of n particles, as we started, no further integrals are obtainable in general for the n-body problem.

1.3 Equation Of Orbit

In order to solve the equation for the relative motion, we substitute the expression for $\ddot{\boldsymbol{r}}$ in (1.11) into (1.8). The radial component gives

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2} \quad , \tag{1.15}$$

while the orthogonal component is equal to

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \tag{1.16}$$

since we are dealing with a central force. Such an equation, written in the form $\frac{d}{dt}(r^2\dot{\theta}) = 0$, provides again the constancy of the angular momentum $\boldsymbol{h}$.
To solve (1.15) instead, since we are interested in r as a function of θ (an equation for the orbit), there is a trick to change it into a much simpler differential equation. Let's replace r with a reciprocal radial variable $u(\theta) = 1/r(t(\theta))$ (the inverse function is well-defined since $\dot{\theta}(t) > 0$ always; planets don't backtrack along their orbits). Differentiating r with respect to time using the chain rule and eliminating the time by making use of the constant $h = r^2\dot{\theta}$ we obtain

$$\dot{r} = -\frac{1}{u^2}\frac{du}{d\theta}\dot{\theta} = -h\frac{du}{d\theta} \quad \text{and} \quad \ddot{r} = -h\frac{d^2u}{d\theta^2}\dot{\theta} = -h^2u^2\frac{d^2u}{d\theta^2} \tag{1.17}$$

and hence (1.15) can be written

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} \tag{1.18}$$

This is the *Binet equation*, a second-order linear differential equation (the equation of an harmonic oscillator subject to a constant perturbation $\frac{\mu}{h^2}$); solution of this equation is a linear superposition of a general solution of the homogeneous equation and a particular solution of the inhomogeneous one:

$$u = \frac{\mu}{h^2} + B\cos(\theta - \omega) \tag{1.19}$$

where B and ω are constants of integration (the amplitude and the phase of the oscillator respectively) depending on the initial conditions.
This solution[2] looks similar to a well-known analytic geometry expression for the reciprocal distance from a point on a conic to one of its foci,

$$\frac{p}{r} = 1 + e\cos f \tag{1.20}$$

where f is the true anomaly, that is, the angular separation of the point from the periapsis, the point of the orbit where the distance $r = r_{min} = \frac{p}{(1+e)}$ takes its minimal value. The quantity $p = a(1-e^2)$ is a constant parameter instead, called *semilatus rectum*, expressed

[2] A variant of this proof could have been achieved using the Laplace–Runge–Lenz vector $\dot{\boldsymbol{r}}\times\boldsymbol{h} - \frac{\mu}{r}\boldsymbol{r} = \boldsymbol{C}$ which is another constant of motion for the *two body problem* only. It lies in the orbital plane, so it's perpendicular to $\boldsymbol{h}$, and points toward the periapsis. Using $\boldsymbol{C}$, the true anomaly f is defined as the angle between the eccentricity vector $\boldsymbol{e}$ and the position vector $\boldsymbol{r}$, where $\boldsymbol{e} = \boldsymbol{C}/\mu$.

in terms of the semi-major axis a and eccentricity e.

To convert the resemblance to equivalence, one must choose the constants B and h in

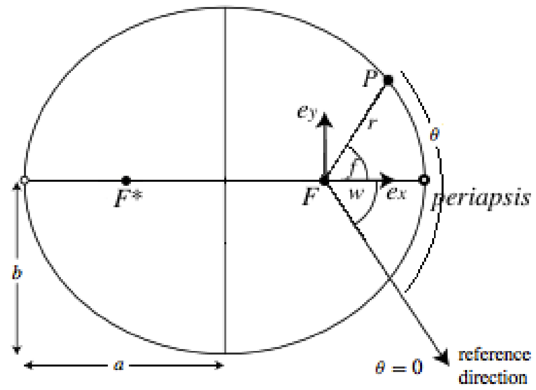

Figure 1.2: Position of the periapsis and the angular variables f, θ and ω.

the following form

$$B = \frac{e}{a(1-e^2)} = \frac{e}{p} \quad \text{and} \quad h = \sqrt{\mu p} \tag{1.21}$$

and equate the angular variables $f = \theta - \omega$. After these identifications, the orbital elements e and p turn out to be interconnected with the integrals of motion via formulae

$$p = \frac{h^2}{\mu} \quad \text{and} \quad e = \sqrt{1 + \frac{2\varepsilon h^2}{\mu^2}} \tag{1.22}$$

so the constant B becomes

$$B = \frac{\mu}{h^2}\sqrt{1 + \frac{2\varepsilon h^2}{\mu^2}} \tag{1.23}$$

and the integral of the reduced total energy

$$\varepsilon = \frac{\mu}{2p}(e^2 - 1) \quad . \tag{1.24}$$

For different conics, the parameters of the orbit are defined as

conic	*eccentricity*	*parameter*
circle	$e = 0$	$p = r = const$
ellipse	$0 < e < 1$	$p = a(1-e^2)$
parabola	$e = 1$	$p = 2q$
hyperbola	$e > 1$	$p = a(e^2 - 1)$

Since the orbit equation describes conic sections, including ellipses, it is a mathematical statement of Kepler's first law, namely that the planets follow elliptical paths around the Sun, having it as one of the foci (two-body orbits are often referred to as Keplerian orbits). The case of a parabola is exceptional in that its eccentricity $e = 1$, and the *semilatus rectum* is defined as $p = 2q$, where q is the minimal distance of the orbit to the gravitating center at body's closest approach.

Equation for p in (1.22) demonstrates that the angular momentum h of the orbit only depends on the focal parameter p and is never null for any type of orbit

$$h^2 = \mu p > 0 \tag{1.25}$$

the only case for which $h = 0$ is *rectilinear* motion, when position and velocity vectors are parallel.
On the other hand, (1.24) reveals that the reduced total energy ε of the two-body system depends only on the semi-major axis a and has positive, negative or zero value for different types of conics

conic	*energy*
circle	$\varepsilon = -\frac{\mu}{2r} < 0$
ellipse	$\varepsilon = -\frac{\mu}{2a} < 0$
parabola	$\varepsilon = 0$
hyperbola	$\varepsilon = \frac{\mu}{2a} > 0$

2 Kepler's equation

SUMMARY From the previous chapter we know that the solution of the problem is represented by conic sections. Now, in this chapter, we express the position as a function of time going through the various forms of Kepler's equation. To do so we introduce the concepts of *eccentric anomaly* and *mean anomaly* relating them to the *true anomaly* already encountered previously.

The orbit formula (1.20), written in the form $r(f) = p/(1 + e\cos f)$, represents the relationship between position and true anomaly for the two-body problem. However, for many practical reasons, it is more useful to be able to determine the position as a function of time.
The problem we want to solve can be stated as follow: given measurements of the position of a body moving in the gravitational field of another one at various times, how can the orbit be determined? This is referred as *Lambert's problem* and stated in another way, the problem is the *boundary value problem* for the (1.8). It can be faced in two different ways: through *Kepler's equation* or using *Lambert's theorem*. Let's start deriving the first one as it offers a way to demonstrate the theorem topic of this work.
As we are working in the orbital plane, we are just interested in the eccentricity e, the semimajor axis a and the true anomaly f since the first two parameters define the shape and size of the trajectory and the latter one the position of the orbiting body along the trajectory. The other three parameters define the orientation of the orbital plane (i.e. of the trajectory in the reference frame) instead and are not required for the proof; they are the inclination i, the longitude of the ascending node Ω and the argument of periapsis ω. Thus, the one equation we have that relates true anomaly directly to time is $h = r^2\dot{f}$ which, written as

$$\frac{df}{dt} = \frac{h}{r^2}$$

substituting $r = p/(1 + e\cos f)$ and separating variables, gives

$$\frac{\mu^2}{h^3}dt = \frac{df}{(1 + e\cos f)^2} \tag{2.1}$$

where we used $\dot{f}$ instead of $\dot{\theta}$ since $f = \theta - \omega$ and ω is a constant. After integration it yields the *Kepler's equation* in true anomaly

$$\frac{\mu^2}{h^3}(t - t_p) = \int_0^f \frac{df}{(1 + e\cos f)^2} \tag{2.2}$$

where the constant t_p (usually set equal to 0 for convenience) is the time at periapsis passage, where $f = 0$ by definition. This equation is of fundamental importance in celestial mechanics but cannot be directly inverted in terms of simple functions (that is no algebraic solution) in order to determine the position as a function of time. However, an extra substitution can be added to simplify the problem but we have to deal with three different cases as pointed out at the end of the previous chapter: $0 \leq e < 1$, $e = 1$ and $e > 1$ or equivalently $\varepsilon < 0$, $\varepsilon = 0$ and $\varepsilon > 0$.

2.1 Elliptic orbits

Let's introduce the so-called *eccentric anomaly* E (a parametrization of the polar angle). Considering figure 2.1, we can define it as the angle between the major axis of the ellipse and the radius from the centre O to the intersection point Q on the circumscribed circle. It follows that $E = 0$ corresponds to $f = 0$ and $E = \pi$ corresponds to $f = \pi$.

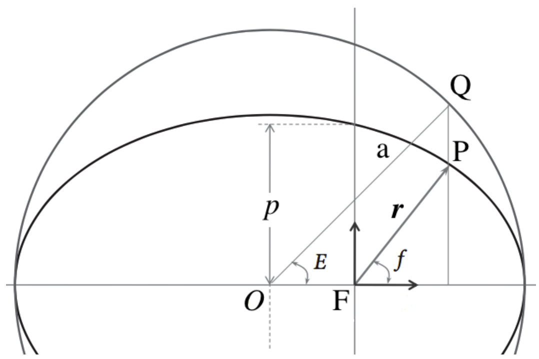

Figure 2.1: Relationship between the true anomaly f and the eccentric anomaly E. The semimajor axis of the ellipse a is also the radius of the circumscribed and concentric circle.

Trough some geometric considerations (Murray [12]), we can easily get

$$\cos f = \frac{\cos E - e}{(1 - e\cos E)} \tag{2.3}$$

Using the fact that $\sin^2 f + \cos^2 f = 1$, it is then easy to calculate also

$$\sin f = \frac{\sqrt{1-e^2}\sin E}{(1-e\cos E)} \quad . \tag{2.4}$$

A simpler relationship between E and f can be derived by writing

$$1-\cos f = \frac{(1+e)(1-\cos E)}{(1-e\cos E)} \quad , \quad 1+\cos f = \frac{(1-e)(1+\cos E)}{(1-e\cos E)} \tag{2.5}$$

With the aid of the half-angle formulae, these relations can be rewritten as

$$\sin^2\frac{f}{2} = \frac{1+e}{(1-e\cos E)}\sin^2\frac{E}{2} \quad , \quad \cos^2\frac{f}{2} = \frac{1-e}{(1-e\cos E)}\cos^2\frac{E}{2} \tag{2.6}$$

hence another elegant relationship between the true and eccentric anomaly

$$\tan\frac{f}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2} \quad . \tag{2.7}$$

Now, rewriting (2.3) as

$$\cos E = \frac{\cos f + e}{(1+e\cos f)} \tag{2.8}$$

so that

$$1-e\cos E = \frac{1-e^2}{(1+e\cos f)} \tag{2.9}$$

we get also

$$\sin E = \frac{\sqrt{1-e^2}\sin f}{(1+e\cos f)} \quad . \tag{2.10}$$

Taking the derivative of (2.8) with respect to E it gives

$$\sin E dE = \frac{(1-e^2)\sin f}{(1+e\cos f)^2}df \quad . \tag{2.11}$$

that's the Kepler equation (2.1) in terms of the eccentric anomaly E.
The next step is to substitute $\sin f$ with (2.4) and rewrite it as

$$\frac{df}{(1+e\cos f)^2} = (1-e^2)^{-\frac{3}{2}}(1-e\cos E)dE \tag{2.12}$$

then, using the Kepler equation (2.1) and remembering the expression for the *semilatus rectum* $p = a(1-e^2)$, we are led to

$$\frac{\mu^2}{h^3}dt = (1-e^2)^{-3/2}(1-e\cos E)dE \quad . \tag{2.13}$$

Finally, with p in (1.22) in mind, it can be written as

$$(1-e\cos E)dE = \sqrt{\frac{\mu}{a^3}}dt \quad . \tag{2.14}$$

Introducing then a new quantity $n = \sqrt{\mu/a^3}$, called *mean motion*, and integrating we get the well known *Kepler's equation* in the case of an elliptic orbit

$$(E - e\sin E) = M = n(t-\tau) \tag{2.15}$$

where $M = n(t-\tau)$ represents the *mean anomaly* (a parametrization of time). Solving E as a function of M is the way to calculate the position of the object in two-body motion but, since equation (2.15) is transcendent, it should be solved numerically. Analyzing these methods is not the purpose of this dissertation but, before going through the other cases, we can still derive some useful informations.

Let's first analyze the *mean anomaly*. Since τ represents the time at which $M = 0$, it is easy to see from (2.15) that when $t = \tau$ the body is situated in its periapsis, i.e. $M = E = 0$. In the special case of a bound orbit, that is when the conic is a circle or an ellipse[1], this result yields to an important property. In fact, since the interval between two successive periapsis passages, that is from $M = 0$ to $M = 2\pi$, gives $nP = 2\pi$, using $n = \sqrt{\mu/a^3}$, we get

$$\frac{2\pi}{P} = \sqrt{\frac{\mu}{a^3}}$$

where P is the orbital period while the previous n is also known as *mean angular frequency* as it represents the pulsation of that period ($P = 2\pi/n$). In a more common form it gives

$$P^2 = \frac{4\pi a^3}{GM} \tag{2.16}$$

that is Kepler's third law in the two-body problem: the square of the orbital period of a body is directly proportional to the cube of the semi-major axis of its orbit.

[1] Proposition For $e \in [0,1)$ the motion is periodic. Demonstration We want to prove that for $k > 0$ such that $\boldsymbol{r}(t+k) = \boldsymbol{r}(t)$, it is true that $\dot{\boldsymbol{r}}(t+k) = \dot{\boldsymbol{r}}(t)$. Using (1.10): $\boldsymbol{r}(t+k) \times \dot{\boldsymbol{r}}(t+k) = \boldsymbol{h}(t+k) = \boldsymbol{h}(t) = \boldsymbol{r}(t) \times \dot{\boldsymbol{r}}(t) = \boldsymbol{r}(t+k) \times \dot{\boldsymbol{r}}(t)$, hence $\dot{\boldsymbol{r}}(t+k) = \dot{\boldsymbol{r}}(t)$.

The second point to highlight is that $\forall\ M\ \exists!\ E$ such that (2.15) is true. It's of fundamental importance knowing about existence and uniqueness of a solution when one is faced with solving a transcendental equation. We would like to know if our numerical method might converge to a solution and if that is the one we are seeking. The proof can be done as follow (Klioner [9]).
Rewrite (2.15) in the form $E - e \sin E - M = 0$ and keep in mind that $0 \leq e < 1$. The solution for E is then a zero of the nonlinear continuous function $F(E) = E - e \sin E - M$, $F : \mathbb{R} \to \mathbb{R}$.

- $F'(E) = 1 - e \cos E > 0$, i.e. $F(E)$ is a monotonically increasing function in the domain, hence it is invertible;

- since for every fixed M

$$\lim_{E \to -\infty} F(E) = -\infty \qquad \lim_{E \to +\infty} F(E) = +\infty \tag{2.17}$$

  then $0 \in Range(F)$ for every fixed M and hence for every fixed M there is a unique solution of $F(E) = E - e \sin E - M = 0$

□

Hence, at a given time t, we can calculate $M(t)$, then using a numerical method we can solve (2.15) to find the value of $E(M)$ and finally, through (2.3) and (1.20) calculate the position. This is not the problem we want to solve (*Kepler's equation*, as it is, describes an *Initial Value Problem*) but we can use this result for our goal. If we know the angle E between E_1 and E_2 and the time interval $t_2 - t_1$ associated then (2.15) can be rewritten as

$$M_2 - M_1 = (E_2 - E_1 - e(\sin E_2 - \sin E_1) \tag{2.18}$$

or, expliciting the time interval and using Kepler's third law,

$$t_2 - t_1 = \frac{P}{2\pi}[(E_2 - E_1 - e(\sin E_2 - \sin E_1)] \tag{2.19}$$

It is then clear that the parameters that describe an elliptic orbit, the eccentricity e and the semi-major axis a, can be simply determined by making two measurements. The first one to get the orbital period P, which gives the semi-major axis a. The second one is about the time interval $t_2 - t_1$ that it takes the object in the orbit to move through the angle $E_2 - E_1$ and makes possible to determine the eccentricity e of the ellipse.
Once the orbital parameters are known, it is possible to follow $r(t)$ at any given instant.

2.2 Hyperbolic orbits

The interest for the hyperbolic version of the Kepler's equation is based on many different reasons. First of all, from the point of view of applied mathematics the solution of any highly transcendental equation (as we are going to see) is interesting, in itself, but it has an important role in many investigations in celestial mechanics as well as in other fields. For example, from the correct determination of position and velocity of cometary orbits to gravitational simulations (*Rein & Tamayo (2015) MNRAS, 452, 376*), quantum mechanics (*Rauh & Parisi (2011) Phys. Rev. A, 83, 042101*) and many others.

To derive Kepler's equation in the case of elliptic orbits we have used the fact that angular momentum is conserved. The same approach can be followed in the case of hyperbolic orbits but now (that $e > 1$) in the relations (2.8) and (2.10) $\sin E$ is imaginary while $\cos E$ is real but can exceed unity. This means that for hyperbolic motion E is imaginary. However, working directly with complex numbers it is not convenient and a quantity analogous to the eccentric anomaly E must be defined.
Since $e^{ix} = \cos x + i \sin x$, remembering the exponential form of the trigonometric functions and substituting $x = iy$ it's easy to find

$$\cos iy = \cosh y \qquad \text{and} \qquad \sin iy = i \sinh y$$

We can now introduce the new hyperbolic eccentric anomaly H defined as $H = -iE$, so that

$$\cos E = \cosh H \qquad \text{and} \qquad \sin E = -i \sinh H \quad . \tag{2.20}$$

In analogy to the elliptical case, it is now possible to write the hyperbolic Kepler's equation as

$$(e \sinh H - H) = M_{hyp} = \sqrt{\frac{\mu}{|a|^3}}(t - \tau) \tag{2.21}$$

where $M_{hyp} = \sqrt{\frac{\mu}{|a|^3}}(t - \tau)$ is the hyperbolic mean anomaly and the signs are chosen so that the body moves in the positive direction of the Y-axis for $t = \tau$.
For the same considerations in the previous section, $\forall\ M_{hyp}\ \exists!\ F$ such that the relationship is verified. We can then write an analogous equation to (2.19) for the hyperbolic orbit

$$t_2 - t_1 = \frac{P}{2\pi}[(e(\sinh F_2 - \sinh F_1) - (F_2 - F_1)] \tag{2.22}$$

where the parameter P is still valid even though a body moving in a hyperbolic orbit has no "period" in the usual sense.

2.3 Parabolic orbits

A particularly interesting case is the parabolic orbit, it's a type of orbit rarely found in nature although the orbits of some comets have been observed to be very close to parabolic. It's mathematically interesting because it represents the boundary between the bounded and unbounded orbit forms, its eccentricity is equal to 1 and the total energy of the system is exactly zero. For these reasons it represents an example of nonpredictable orbits: slight changes in the initial conditions result in greatly different orbits.
The polar equation for the form of the orbit can be rewritten as

$$r = \frac{p}{1+\cos f} = q(1+\tan^2 \frac{f}{2}) \tag{2.23}$$

where $q = p/2$ is the periapsis of the parabolic orbit ($r(0) = p/2$).
Once again, the conservation of angular momentum permits the derivation of an equation analogous to Kepler's equation. Consider

$$r^2 \frac{f}{dt} = h = \sqrt{\mu p} \tag{2.24}$$

it follows that

$$4\sqrt{\frac{\mu}{p^3}} dt = \sec^4 \frac{f}{2} df. \tag{2.25}$$

To perform the integration remember that $\sec x = 1/\cos x$, so that

$$\begin{aligned} \int \frac{dx}{\cos^4 x} &= \int \frac{\sin^2 x}{\cos^4 x} dx + \int \frac{\cos^2 x}{\cos^4 x} dx \\ &= \frac{1}{3} \tan^3 x + \tan x. \end{aligned} \tag{2.26}$$

Finally, Kepler's equation in the parabolic case, better known as *Barker's equation*, is

$$\frac{1}{3} \tan^3 \frac{f}{2} + \tan \frac{f}{2} = M_{par} = 2\sqrt{\frac{\mu}{p^3}}(t-\tau) \tag{2.27}$$

where $M_{par} = 2\sqrt{\frac{\mu}{p^3}}(t-\tau)$ is the parabolic mean anomaly. The parabolic motion can be represented by an explicit analytical formula using, for example, the Cardan formulas. The solution for f requires the root of a cubic equation in $\tan \frac{f}{2}$ and it is easy to show that one and only one real root exists. To prove it (Battin [1]), let's put

$$\tan \frac{f}{2} = z - z^{-1} \tag{2.28}$$

we get a quadratic equation in z^3

$$z^6 - 3M_{par}z^3 - 1 = 0 \tag{2.29}$$

for which

$$z = \left(\frac{3}{2}M_{par} \pm \sqrt{(\frac{3}{2}M_{par})^2 + 1}\right)^{1/3} \tag{2.30}$$

and either sign produces the same solution for $\tan\frac{f}{2}$.

3 Lambert's theorem

SUMMARY The aim in this chapter is to demonstrate *Lambert's theorem* and to achieve this result we use *Kepler's equation* derived in Chapter 2. It is firstly prove for the elliptical path and then generalized to the hyperbolic and parabolic cases.

In 1761, J. H. Lambert developed another method that can be used to establish orbits from measurements made from Earth, the original motivation was to determine the orbits of comets and it is now known as *Lambert's theorem*. In modern celestial mechanics it has important applications in the area of orbit determination not to mention in that of orbit design, space rendezvous and interception, space debris correlation. To be more precise, since it is based on the two-body equations of motion, it must be considered a *preliminary* orbit determination technique because the actual orbit is perturbed over time by other phenomena (e.g. gravitational forces, the asymmetric shape and non-uniform mass distribution of objects, atmospheric drag, stellar winds, etc...).

Orbits along conic sections present many surprising properties, maybe the most surprising is that some of the more important orbital quantities are independent on the eccentricity. The fact that the period of an elliptic motion depends only on the semimajor axis is an example ((2.16)), the total energy of the orbit (i.e. the velocity as a function of the radius given by the vis-viva equation ((1.14) together with 1.3)) is another one.
Lambert discovered another remarkable theorem in this connection with regard to the time to traverse an elliptic arc. Actually the theorem has been proved to be true for a general conic.
It can be derived from Kepler's equation (2.19) and it is generically the same kind of relationship, the only difference is essentially the choice of origin of the coordinate system in which the measurements to determine the orbits are made. In the case of Kepler's equation, a time interval between two positions and the value of the eccentric anomaly at these two times are measured. The use of the eccentric anomaly is not always convenient and the advantage of Lambert's theorem is that the measurements that must be made to determine the orbital parameters can be performed from the focus of the elliptic orbit (more useful in some applications).

Theorem 1. *The time required to traverse an elliptic arc between specified endpoints depends only on the semi-major axis a of the ellipse and on two geometric properties of the*

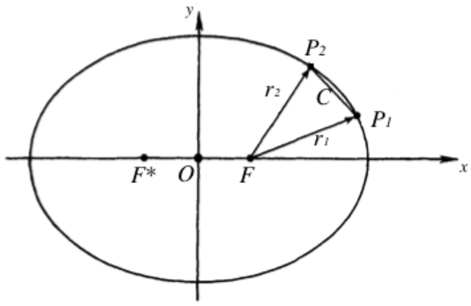

Figure 3.1: Geometry of the boundary-value problem.

space triangle, namely the chord length C and the sum of the radii $r_1 + r_2$ from the focus to the initial and final points (see figure 3.1).

If $t_2 - t_1 = \Delta t$ is the time to describe the arc from P_1 to P_2, then Lambert's theorem states that

$$\Delta t = \Delta t(a, C, r_1 + r_2) \tag{3.1}$$

To prove it (Szebehely's [15] and Battin's [1] approaches have been used to analyze the problem), let's start considering figure 3.1. From geometrical considerations, we can write the following relations for the radius r and the chord C:

$$r = a(1 - e\cos E) \tag{3.2}$$

$$C^2 = a^2(\cos E_2 - \cos E_1)^2 + a^2(1 - e^2)(\sin E_2 - \sin E_1)^2 \tag{3.3}$$

where (3.2) is analogous to (1.20) since it represents the equation of an ellipse but now in terms of the eccentric anomaly E (for a detailed geometrical derivation of (3.2) and (3.3) see [15, pp.102,116]). Then, using (3.2), we have that

$$\begin{aligned} r_1 + r_2 &= a(1 - e\cos E_1) + a(1 - e\cos E_2) \\ &= 2a(1 - \frac{1}{2}e(\cos E_2 + \cos E_1)) \end{aligned} \tag{3.4}$$

and remembering the prosthaphaeresis formulae, we can rewrite

$$\cos E_2 + \cos E_1 = 2(\cos(\frac{1}{2}(E_2 + E_1))(\cos(\frac{1}{2}(E_2 - E_1)) \tag{3.5}$$

so that, defining $E_+ = \frac{1}{2}(E_2 + E_1)$ and $E_- = \frac{1}{2}(E_2 - E_1)$, we get

$$r_1 + r_2 = 2a(1 - e\cos E_+ \cos E_-) \tag{3.6}$$

The previous results ((3.6) and (3.3)) come from a purely geometrical analysis of the elliptic orbit. It's time now to consider the dynamic (that is, time dependent) relationships for the remainder of the derivation.
Remember, from the previous chapter, that the time difference $t_2 - t_1$ is one of the measured quantities.
Kepler's equation associated to a general point P, representing the dynamic of the motion, is

$$t = \frac{P}{2\pi}(E - e \sin E) \tag{3.7}$$

If we calculate it in two different points P_1 and P_2, then we can write

$$t_2 - t_1 = \frac{P}{2\pi}((E_2 - E_1) - e(\sin E_2 - \sin E_1)) \tag{3.8}$$

and, again through the prosthaphaeresis formulae

$$\sin E_2 - \sin E_1 = 2(\cos(\frac{1}{2}(E_2 + E_1))(\sin(\frac{1}{2}(E_2 - E_1)) \tag{3.9}$$

we get

$$t_2 - t_1 = \frac{P}{2\pi}((E_2 - E_1) - 2e \cos E_+ \sin E_-) \tag{3.10}$$

This equation represents the measured quantity from the dynamics of the system, that is, the moving body (asteroid, comet, planet, satellite, etc...).
Rewriting (3.3) in a more convenient form (through prosthaphaeresis formulae, in particular $(\cos E_2 - \cos E_1) = -2(\sin(\frac{1}{2}(E_2 + E_1))(\sin(\frac{1}{2}(E_2 - E_1)))$

$$C^2 = 4a^2(\sin^2 E_-)(1 - e^2 \cos^2 E_+) \tag{3.11}$$

we can now solve it together with (3.6) and (3.10) to obtain Lambert's theorem.
Since our aim is to write the time difference as a function of a, C and $r_1 + r_2$, the next step is to get rid of the eccentric anomaly E. First of all, let's introduce

$$\xi = e \cos E_+ \tag{3.12}$$

so that (3.11), (3.6) and (3.10) can be rewritten as

$$t_2 - t_1 = \frac{P}{2\pi}(2E_- - 2\xi \sin E_-) \tag{3.13}$$

$$r_1 + r_2 = 2a(1 - \xi \cos E_-) \tag{3.14}$$

$$C^2 = 4a^2(\sin^2 E_-)(1 - \xi^2) \tag{3.15}$$

Now (3.14) and (3.15) can be combined to obtain

$$\begin{aligned} r_1 + r_2 + C &= 2a(1 - \xi \cos E_- + \sqrt{1-\xi^2}) \sin E_- \\ r_1 + r_2 - C &= 2a(1 - \xi \cos E_- - \sqrt{1-\xi^2}) \sin E_- \end{aligned} \tag{3.16}$$

To achieve a solution we are now obliged to perform another variable change

$$\xi = \cos \frac{1}{2}(\alpha + \beta) = \cos Q_+ \quad \text{and} \quad E_- = \frac{1}{2}(\alpha - \beta) = Q_- \tag{3.17}$$

which leads to

$$\begin{aligned} r_1 + r_2 + C &= 2a(1 - \cos Q_+ \cos Q_- + \sin Q_+ \sin Q_-) \\ &= 2a(1 - \cos \alpha) = 4a \sin^2 \frac{\alpha}{2} \\ r_1 + r_2 - C &= 2a(1 - \cos Q_+ \cos Q_- - \sin Q_+ \sin Q_-) \\ &= 2a(1 - \cos \beta) = 4a \sin^2 \frac{\beta}{2} \end{aligned} \tag{3.18}$$

where the last two equalities are obtained using prosthaphaeresis and half-angle formulae respectively. With variables defined in (3.17) and considering the prosthaphaeresis formulae again, (3.13) can be rewritten as

$$t_2 - t_1 = \frac{P}{2\pi}((\alpha - \beta) - (\sin \alpha - \sin \beta)) \tag{3.19}$$

where the parameters α and β are defined using (3.18)

$$\sin \frac{\alpha}{2} = \frac{1}{2}\left(\frac{r_1 + r_2 + C}{a}\right)^{1/2} \quad \text{and} \quad \sin \frac{\beta}{2} = \frac{1}{2}\left(\frac{r_1 + r_2 - C}{a}\right)^{1/2} \quad . \tag{3.20}$$

Equation (3.19), also known as *Lambert's equation*, proves the theorem: it is clear the relationship of the time interval $t_2 - t_1$ from a, $r_1 + r_2$ and C trough the parameters α and β. As Kepler's equation, also (3.19) is a transcendental equation and some numerical analysis must be applied for the resolution.

Let's end this section with a simple case (Szebehely[15]): in 3.1, consider P_1 with a true anomaly $f_1 = 0$ and P_2 with a true anomaly $f_2 = \pi/2$.
We get

$$r_1 = a(1 - e) \quad \text{and} \quad r_2 = a(1 - e^2) = p \quad , \tag{3.21}$$

it follows that

$$C = a(1 - e)\sqrt{2 + 2e + e^2} \tag{3.22}$$

while for the time interval we have

$$t_2 - t_1 = \frac{P}{2\pi}(\arccos e - e\sqrt{1-e^2})) \tag{3.23}$$

where α and β are obtained trough *Lambert's equation*. Applying some numerical methods for the resolution, orbital parameters can then be obtained from these last two equations.

3.1 Hyperbolic and parabolic cases

A similar proof shows the theorem to be true for the hyperbola as well. As in the elliptical case, we need the sum of the radial distances, the length of the chord and the hyperbolic semi-major axis.
Now, keeping in mind the use of the hyperbolic functions, the radial distance is written as

$$r = a(e\cosh F - 1) \tag{3.24}$$

and the chord length becomes

$$C^2 = a^2(\cosh F_2 - \cosh F_1)^2 + a^2(e^2-1)(\sinh F_2 - \sinh F_1)^2 \quad . \tag{3.25}$$

Remembering Kepler's equation in the hyperbolic case (2.22), with the analogous path done to demonstrate the elliptic Lambert's equation and rewriting the terms $\sin(\alpha/2)$ and $\sin(\beta/2)$ in (3.20) with the corresponding hyperbolic functions

$$\sinh\frac{\gamma}{2} = \frac{1}{2}\left(\frac{r_1+r_2+C}{a}\right)^{1/2} \quad \text{and} \quad \sinh\frac{\delta}{2} = \frac{1}{2}\left(\frac{r_1+r_2-C}{a}\right)^{1/2} \tag{3.26}$$

we get the Lambert's equation for the hyperbolic orbits case

$$t_2 - t_1 = \frac{P}{2\pi}((\sinh\gamma - \sinh\delta) - (\gamma - \delta)) \tag{3.27}$$

The special case of Lambert's theorem for the parabola can be proven following the idea (Szebehely [15]) that the parabolic orbit can be seen as a stretched ellipse with an infinite semi-major axis a. This approach, which implies the use of the small-angle approximation for α and β in (3.19), gives us the possibility to eliminate the transcendental terms and to

achieve an analytic solution just as in the Kepler's equation case.
Equations in (3.20) can then be rewritten as

$$
\begin{aligned}
\sin\frac{\alpha}{2} &\approx \frac{\alpha}{2} \quad \text{with} \quad \alpha \approx \frac{1}{2}\left(\frac{r_1+r_2+C}{a}\right)^{1/2} \\
\sin\frac{\beta}{2} &\approx \frac{\beta}{2} \quad \text{with} \quad \beta \approx \frac{1}{2}\left(\frac{r_1+r_2-C}{a}\right)^{1/2}
\end{aligned}
\tag{3.28}
$$

Now, to obtain Lambert's equation we first need to evaluate the previous expansions till the third order (to avoid a null result)

$$
\sin\alpha = \alpha - \frac{\alpha}{3!} \quad \text{and} \quad \sin\beta = \beta - \frac{\beta}{3!} \tag{3.29}
$$

then, substituting in (3.19) we get

$$
t_2 - t_1 = \frac{P}{12\pi}((\frac{r_1+r_2+C}{a})^{3/2} - (\frac{r_1+r_2-C}{a})^{3/2}) \quad . \tag{3.30}
$$

At this point, we make use of Kepler Third Law written in this way (keeping in mind that a parabolic orbit is characterized by an infinite period P and an infinite semi-major axis a)

$$
\frac{P}{2\pi}\frac{1}{a^{3/2}} = \frac{1}{\sqrt{\mu}} \tag{3.31}
$$

so that (3.30) can be rewritten as

$$
t_2 - t_1 = \frac{1}{6\sqrt{\mu}}((r_1+r_2+C)^{3/2} \mp (r_1+r_2-C)^{3/2}) \tag{3.32}
$$

with the upper or lower sign taking effect if the transfer angle (angle between r_1 and r_2) is < or > than $180°$ respectively.

We conclude this analysis noting another practical information that can be provided by the theorem (Szebehely [15, p.123]): the radius of curvature, i.e. if the object is in an external or internal orbit with respect to the Earth.

4 Literature on Lambert's problem

Summary In this last chapter we want to write a unique equation that is valid for all the three orbital cases presented previously. The method we use is that of the *Universal variables* formulation but since it's just one of many different ways to solve the problem we also give a general overview of the large literature about it without dig deep to each technique.

In Chapter 2 we studied Kepler's equation in the three different cases in which an orbit can be found and in Chapter 3 we proved Lambert's theorem to be true for each one of those cases. The complication here is that those relations are valid for a particular type of orbit only and in our problem we cannot know a priori which type of conic will describe the orbit of the object. Then, it is necessary a general formulation valid for any type of orbit. This approach uses the *Universal variables* formulation and it's just one of several different methods. It utilizes those variables, parameters or formulas that can be used with any of the two-body conic section orbits. It's a useful method since it permits a nonsingular transition between orbits of different type and also because it reduces the programming necessary to treat an orbit that changes shape (for example during a mission).

We follow the idea presented by *E. R. Lancaster & R. C. Blanchard* [11] whose works represented an outstanding contribution to the subject and inspired some later authors (*Gooding* [7] before and *Izzo* [8] later, to name a few, who focused on the computational aspects).
Their method is particularly useful as it reduces the solution of Lambert's problem to require the computation of one only inverse trigonometric or hyperbolic function. This is made possible by the appropriate choice of an independent variable x and a parameter q such that the time interval between the initial and final points (see figure 3.1) $\Delta t = \Delta t_q(x)$ is a single-valued function of x for each value of q. The parameter q depends only upon known quantities while $\Delta t = \Delta t_q(x)$ is known so the problem is reduced to find x (which will define the type of orbit) for given values of q and Δt.
Since Δt is a monotonic function of x for each value of q (in the case of less than one revolution) there is a unique solution x of the problem (analogous to the demonstration of the existence and uniqueness of the solution done in Chapter 2).

In the case of a multi-revolution object instead, the time interval has a single minimum for each q. In this case (3.19) can be written as

$$t_2 - t_1 = \frac{P}{2\pi}(2m\pi + (\alpha - \beta) - (\sin\alpha - \sin\beta)) \tag{4.1}$$

where m is the number of complete revolutions made during the transfer from r_1 to r_2. As in the previous chapter, without loss of generality for our purpose, we will consider in the following the case $m = 0$ since $m \neq 0$ makes sense for the elliptic case only

$$t_2 - t_1 = \frac{P}{2\pi}((\alpha - \beta) - (\sin\alpha - \sin\beta)) \tag{4.2}$$

Let's first introduce the parameter q defined as

$$q = \frac{(r_1 r_2)^{1/2}}{s}\cos\frac{\theta}{2} \tag{4.3}$$

where $\theta \in [0, 2\pi]$ is the angle between the two position vectors in figure 3.1 and it is linked to the chord C by the expression $C^2 = {r_1}^2 + {r_2}^2 - 2r_1r_2\cos\theta$ (refer to [11]).
It can also be written as (Izzo [8])

$$q^2 = \frac{s - C}{s} \tag{4.4}$$

where $s = (r_1 + r_2 + C)/2$ while the parameter $q \in [-1, 1]$ is positive when $\theta \in [0, \pi]$ and negative when $\theta \in [\pi, 2\pi]$. Values of q^2 close to unity represent a curious case since corresponds to a chord of zero length, an interesting case in interplanetary trajectory design as it is linked to the design of resonant transfers (Izzo [8]).
It is also convenient to introduce a normalized time-of-flight defined as

$$T = \sqrt{\frac{2\mu}{s^3}}(t_2 - t_1) \tag{4.5}$$

Now we can proceed with the new formulation. With reference to Lambert's equations (previous chapter) it's convenient to consider a new independent variable

$$x = \begin{cases} \cos\frac{\alpha}{2} & -1 \leq x \leq 1 \\ \cosh\frac{\gamma}{2} & x > 1 \end{cases} \tag{4.6}$$

so we have for elliptic, hyperbolic and parabolic (in this case $x = 1$) transfer

$$x^2 - 1 = -\frac{s}{2a} = W \quad . \tag{4.7}$$

The following relations can be easily derived using (3.20) and (3.26). Analyzing first the elliptic case, let's define

$$y = \sin\frac{\alpha}{2} = (-W)^{\frac{1}{2}} \tag{4.8}$$

$$z = \cos\frac{\beta}{2} = (1 + q^2 W)^{\frac{1}{2}} \tag{4.9}$$

$$f = \sin\frac{1}{2}(\alpha - \beta) = y(z - qx) \tag{4.10}$$

$$g = \cos\frac{1}{2}(\alpha - \beta) = xz - qW \tag{4.11}$$

$$h = \frac{1}{2}(\sin\alpha - \sin\beta) = y(x - qz) \tag{4.12}$$

$$\lambda = \arctan\frac{f}{g} \quad . \tag{4.13}$$

Considering these new variables, (3.19) can be rewritten in the following new formulation

$$T = \frac{2(\lambda - h)}{y^3} \quad . \tag{4.14}$$

For the hyperbolic case instead, let's rewrite the previous variables considering the hyperbolic functions

$$y = \sinh\frac{\gamma}{2} = W^{\frac{1}{2}} \tag{4.15}$$

$$z = \cosh\frac{\gamma}{2} = (1 + q^2 W)^{\frac{1}{2}} \tag{4.16}$$

$$f = \sinh\frac{1}{2}(\gamma - \delta) = y(z - qx) \tag{4.17}$$

$$g = \cosh\frac{1}{2}(\gamma - \delta) = xz - qW \tag{4.18}$$

$$h = \frac{1}{2}(\sinh\gamma - \sinh\delta) = y(x - qz) \tag{4.19}$$

$$\chi = \tanh^{-1}\frac{f}{g} \quad . \tag{4.20}$$

Then for the hyperbolic case we get

$$T = \frac{2(h - \chi)}{y^3} \quad . \tag{4.21}$$

Finally it is clear that, given x and q values, we can get the normalized time T for all cases trough

$$T = \frac{2(x - qz - \frac{d}{y})}{W} \tag{4.22}$$

where

$$d = \begin{cases} \lambda & W < 0 \\ \chi & W > 0 \end{cases} \tag{4.23}$$

With (4.22) it is clear the advantage of using a universal variable (in this case x) that's independent on the type of the orbital path: $x > 1$ implies hyperbolic motion, $x = 1$ corresponds to parabolic motion while $x < 1$ to elliptic motion. The case of $x = 0$ corresponds to the minimum energy ellipse (particularly useful in applications) since (Izzo [8])

$$1 - x^2 = \sin^2 \frac{\alpha}{2} = \frac{s}{2a} = \frac{a_{min}}{a} \quad . \tag{4.24}$$

where a_{min} represents the semi-major axis of the lowest energy ellipse.
Finally it's interesting to note that x it's not just an universal variable but it is also a *Lambert invariant parameter* because different Lambert's problems having identical q values (i.e. same C/s) share also the same x value. To clarify, we introduce the concept of *Lambert invariance* which derives directly from *Lambert's theorem* and the triangles equivalence classification that this makes possible (cited Gooding [7]).
To be more precise, there are two different equivalence classifications based on two different possible relations: triangles that share the same values of s and C are defined as *L-congruent* while (as in geometry the congruence relation suggests the similarity relation, in fact) those that have in common the same value of C/s ratio are defined as *L-similar*. On this basis, we can also give a definition of a *Lambert invariant parameter* (geometric or dynamic) which is then a variable, for either relation, that keep the same value for all the problems that belong to the same equivalence class.
Two different problems are equivalent, on a congruent basis, if they share the same s, C and Δt values; on a similarity basis instead, for two problems to be equivalent it is sufficient that only the C/s ratio and the adimensional Δt are the same. Then it is clear that the last one is normally the more useful type of equivalence (it's the one that we encounter in *Lancaster and Blanchard* paper [11]).

Solutions to Lambert's problem abound in literature since Lambert's time (who provided the equations to determine the minimum-energy orbit), noteworthy are also those by Lagrange and Gauss (who formed his formulation trying to rediscover Ceres a year after its last observation (Vallado [17])). We conclude presenting all the methods (refer to the cited authors for a deeper description of each method) grouped (*de la Torre Sangrà & Fantino* [16]) into the major lines of research on the basis of the free *parameter* *adopted* (indicated in *italic*); the first author of every line represents that of the *progenitor idea* while the successive made improvements on it:

- *Universal variables*: (refer to the beginning of the Chapter for a description)
  - Lancaster & Blanchard [11]; Gooding [7]; Izzo [8].
  - Bate; Vallado [17]; Luo; Thomson; Arora.
  - Battin-Vaughan; Loechler; Shen; MacLellan.
- *Semi-major axis*: the "simple" application of Lambert's theorem for which there is a particular value of the semi-major axis associated with a single conic transfer arc that will uniquely satisfy the associated boundary conditions.
  - Lagrange; Thorne; Prussing; Chen; Wailliez.
- *Semi-latus rectum (p-iteration)*: this method consists of guessing a trial value of p from which we can compute the other two unknowns, a and E.
  - Bate.
- *Eccentricity vector*: (to cite (*Avanzini, "A Simple Lambert Algorithm"*)) this parametrization makes use of a property of the eccentricity vector $\boldsymbol{e}$, i.e. that it has a constant component in the direction of the chord $\boldsymbol{r}_1 - \boldsymbol{r}_2$. The idea is that the transfer time is a monotonic function of e_T (transverse eccentricity component) so that it is possible to consider e_T as the unknown for solving the problem.
  - Avanzini; He; Zhang; Wen.
- *Kustaanheimo-Stiefel (K-S) regularized coordinates*: the Kustaanheimo-Stiefel transformation turns a gravitational two-body problem into an harmonic oscillator, by going to four dimensions. In addition to the mathematical-physics interest, for example, it has proved to be very useful in N-body simulations, where it helps handle close encounters.
  - Kriz; Jezewsky.

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