ALGANT Master Thesis in Mathematics

On Weyl's construction and Schur-Weyl duality for the symplectic group

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Introduction

The first person to introduce and study representation theory of finite groups was F. G. Frobenius in 1896. In his work [4], he introduced the notion of character for a finite non-abelian group, generalizing the notion of group character of finite abelian group developed by R. Dedekind in 1879. Afterwards, I. Schur, who was a student of Frobenius, continued his work in several directions, including presenting in his doctoral thesis [10] the duality between the general linear group $\mathrm{GL}_n\mathbb{C}$ and the symmetric group S_n. This duality was then developed by H. Weyl in his work [11]. For further historical information see [7].

The object of this thesis is to study the Schur-Weyl duality in the case of symplectic groups over the field $\mathbb{C}$. This arises from the will to give a detailed explanation of the main results in section 2 of paper [9]. We will refer mostly to the approach that R. Brauer illustrated in 1936 in his work [1]. Here Brauer introduced the algebra $\mathcal{B}_d(\delta)$ to describe the invariants of symplectic and orthogonal groups acting on $V^{\otimes d}$ for a finite-dimensional complex vector space V. In fact, in order to accomplish our goal, we will need to state the First Fundamental Theorem for the symplectic group (of which we will not give a proof).

The thesis is divided in five chapters. As the titles suggest, the second and the third chapters are devoted to the study of the general linear group $\mathrm{GL}_n\mathbb{C}$, namely the Schur-Weyl duality and the polynomial representations for $\mathrm{GL}_n\mathbb{C}$, while the last two chapters concern the symplectic group $\mathrm{Sp}_{2n}\mathbb{C}$.

In the first chapter, we will present, as prerequisites, all the results from classical representation theory of finite groups that we will use in all the following parts. Namely, we will introduce the notion of group algebra and Lie algebra that will be fundamental to work with the representations of the classical groups since, as we will show, studying representations of these algebraic structures will be equivalent to studying representations of the corresponding groups.

In the second chapter, we will present the Schur-Weyl duality for $\mathrm{GL}_n\mathbb{C}$. We will first provide a complete portrait of the irreducible representations of S_n and give a formula for their characters, the so-called Frobenius character formula. The proof of the duality will be based on an important result on semisimple algebras over an algebraically closed field known as Double Centralizer Theorem. We will finish this chapter computing the Weyl character formula for S_n.

The main goal of the third chapter is to finish the description of all the algebraic representations of $\mathrm{GL}_n\mathbb{C}$. For, we will first provide a general structure to analyse a semisimple Lie algebra, applying it to the special linear algebra $\mathfrak{sl}_n\mathbb{C}$ to obtain a description of its representations. As a consequence, we will have all the ingredients to list the irreducible representations of $\mathrm{GL}_n\mathbb{C}$.

In the fourth chapter, we will start to work with the symplectic group $\mathrm{Sp}_{2n}\mathbb{C}$. After recalling the structures of the symplectic group $\mathrm{Sp}_{2n}\mathbb{C}$ and the corresponding Lie algebra $\mathfrak{sp}_{2n}\mathbb{C}$, we will apply the previous algorithm to delineate the representations of $\mathfrak{sp}_{2n}\mathbb{C}$. Afterwards, we will give a complete description of Weyl's construction for symplectic groups using what we have developed in chapter 2.

In the fifth chapter, we will present without proof some important facts on invariant theory for symplectic groups. Next, we will apply the Double Centralizer Theorem to the symplectic case and we will characterize the centralizer of the natural factorwise action of $\mathrm{Sp}(V)$ on the tensor product $V^{\otimes d}$ for a finite-dimensional complex vector space V endowed with a suitable form $f\colon V \times V \to \mathbb{C}$. It will turn out that this centralizer is closely connected to the Brauer algebra $\mathcal{B}_d(-2n)$ and we will be able to prove the symplectic Schur-Weyl duality.

Each chapter contains a little summary and some references at the beginning.

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Chapter 1

Preliminaries

In this chapter, we will develop some of the general theory of representations of finite groups. In particular we will introduce the notions of character and group algebra that will be important in the following chapters.

All results appearing in the first four sections are proved from chapter 2 to the first sections of chapter 5 of [3]. Also, the same topics are treated in lectures 1 to 3 of [5] with some interesting examples. The results in the last section on Lie algebras can be found in [8].

1.1 Representations of finite groups

A representation of a finite group G (also called a G-module) on a finite-dimensional complex vector space V is a homomorphism $\rho\colon G \to \mathrm{GL}(V)$. We will often abuse notation by referring to V as the representation of G and write gv for $\rho(g)(v)$ whenever the map ρ is understood from context.

A G-map φ between two representations V and W of G is a vector space map $\varphi\colon V \to W$ such that $\varphi(gv) = g\varphi(v)$ for every $g \in G$ and $v \in V$. We say a G-map $\varphi\colon V \to W$ is an isomorphism of representations if it is an isomorphism of vector spaces. In this case, we say that V and W are isomorphic and denote this by $V \simeq W$. Let $\mathrm{Hom}_G(V, W)$ denote the set of all G-maps from V to W and define $\mathrm{End}_G(V) = \mathrm{Hom}_G(V, V)$. Notice that $\mathrm{Hom}_G(V, W)$ inherits the structure of vector space from $\mathrm{Hom}(V, W)$.

For any G-maps $\varphi\colon V \to W$ and $\psi\colon W \to Z$, the composition $\psi \circ \varphi\colon V \to Z$ is again a G-map. Indeed we have

$$(\psi \circ \varphi)(gv) = \psi(\varphi(gv)) = \psi(g\varphi(v)) = g\psi(\varphi(v)) = g(\psi \circ \varphi)(v).$$

This defines the composition of G-maps, which is clearly associative and has identity element id_V. So we may consider the category of representations of G, denoted $\mathrm{Rep}(G)$.

A subrepresentation of a representation V is a vector subspace W of V which is invariant under G, that is, $gw \in W$ for all $g \in G$ and $w \in W$. We say V is irreducible if it contains exactly two subrepresentations, namely, 0 and V itself.

Example 1.1.1. Let G be a finite group. We have:

i) The trivial representation $\mathbb{C}$ where $gv = v$ for all $g \in G$ and $v \in \mathbb{C}$. It is clearly always irreducible.

ii) The regular representation of G is the $\mathbb{C}$-vector space with basis G where G acts on itself by left multiplication.

If V and W are representations, the direct sum $V \oplus W$ and the tensor product $V \otimes W$ are also representations, respectively via

$$g(v \oplus w) = gv \oplus gw \quad \text{and} \quad g(v \otimes w) = gv \otimes gw,$$

for $v \in V$ and $w \in W$. For a representation V, the nth tensor power $V^{\otimes n}$ is thus a representation of G, the nth exterior power $\wedge^n(V)$ and the nth symmetric power $\mathrm{Sym}^n(V)$ are subrepresentations of it. Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation. The dual $V^* = \mathrm{Hom}(V, \mathbb{C})$ of V is also a representation, defining $\rho^*(g)\colon V^* \to V^*$ as $\rho^*(g)(f) = ((\rho^*(g)(f))(v) \mapsto f(\rho(g^{-1})(v)))$ for all $f \in V^*$ and $v \in V$. This forces ρ to respect the natural pairing $\langle \cdot, \cdot \rangle$ of V^* and V in the following sense: $\langle \rho^*(g)(\lambda), \rho(g)(v) \rangle = \langle \lambda, v \rangle$ for all $g \in G$, $\lambda \in V^*$, and $v \in V$. Having this, the action on $\mathrm{Hom}(V, W)$ is given by the identification $\mathrm{Hom}(V, W) = V^* \otimes W$ given by $f \otimes w \mapsto (v \mapsto f(v) \cdot w)$.

A representation is indecomposable if it cannot be expressed as a direct sum of proper subrepresentations. Clearly, if a representation is irreducible, then it is indecomposable. Remarkably, also the converse holds.

Theorem 1.1.2 (Maschke). *If W is a subrepresentation of a representation V of a finite group G, then there is a complementary invariant subspace W' of V, so that $V = W \oplus W'$. Consequently, every representation is a direct sum of irreducible representations.*

This property is called complete reducibility.

Lemma 1.1.3 (Schur). *Let $\varphi\colon V \to W$ be a nonzero G-map.*

i) If V is irreducible, then φ is injective.

ii) If W is irreducible, then φ is surjective.

iii) If $V = W$ is irreducible, then $\varphi = \lambda \cdot I$ for some nonzero $\lambda \in \mathbb{C}$, I the identity.

It follows proposition 1.8 in [5]:

Proposition 1.1.4. *For any representation V of a finite group G, there is a decomposition*

$$V = V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k},$$

where the V_i are distinct irreducible representations. The decomposition of V into a direct sum of the k factors is unique, as are the V_i that occur and their multiplicities a_i.

1.2 Character Theory

If V is a representation of G, its character $\chi_V\colon G \to \mathbb{C}$ is defined by

$$\chi_V(g) = \mathrm{Tr}(g|_V).$$

In particular, by the properties of trace, we have $\chi_V(hgh^{-1}) = \chi_V(g)$, so that χ_V is constant on the conjugacy classes of G; such a function is called a class function. Note that $\chi_V(1) = \dim V$. Since the trace of a linear transformation is the sum of the eigenvalues, the identities

$$\chi_{V \oplus W} = \chi_V + \chi_W, \qquad \chi_{V \otimes W} = \chi_V \cdot \chi_W, \qquad \chi_{V^*} = \bar{\chi}_V$$

easily follow.

For any representation V of a group G, we set $V^G = \{v \in V \mid gv = v \text{ for all } g \in G\}$. The map

$$\varphi = \frac{1}{|G|} \sum_{g \in G} g \in \operatorname{End}(V)$$

is a projection of V onto V^G. Let $\mathbb{C}_{\text{class}}(G) = \{\text{class functions on } G\}$ and define a Hermitian inner product on $\mathbb{C}_{\text{class}}(G)$ by

$$(\alpha, \beta) = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g).$$

We have the following fundamental theorem of character theory.

Theorem 1.2.1. *In terms of this inner product, the characters of the irreducible representations of G are orthonormal.*

Remark 1.2.2. Here are some important consequences of this theorem which follow almost immediately.

i) Any representation is determined by its character, i.e., $V \simeq W$ if and only if $\chi_V = \chi_W$.

ii) A representation V is irreducible if and only if $(\chi_V, \chi_V) = 1$.

iii) The multiplicity a_i of V_i in V is the inner product of χ_V with χ_{V_i}, i.e., $a_i = (\chi_V, \chi_{V_i})$.

iv) Any irreducible representation V of G appears in the regular representation $\dim V$ times. In particular, $|G| = \sum (\dim V_i)^2$.

Clearly, the image of g in $\operatorname{End}(V)$ is a G-map if and only if g is in the centre $Z(G)$. Let $\alpha \colon G \to \mathbb{C}$ be a function on the group G, for any representation V of G the map

$$\varphi_{\alpha,V} = \sum \alpha(g) \cdot g \in \operatorname{End}(V)$$

is a G-map for all V if and only if α is a class function. As an immediate consequence, we have

Proposition 1.2.3. *The number of irreducible representations of G is equal to the number of conjugacy classes of G. Equivalently, their characters $\{\chi_V\}$ form an orthonormal basis for $\mathbb{C}_{class}(G)$.*

The orthogonality in theorem 1.2.1 is called row orthogonality. Similarly, there is a corresponding column orthogonality.

Corollary 1.2.4. *For $g, h \in G$, we have*

$$\frac{1}{|G|} \sum_i \overline{\chi_{V_i}(g)} \chi_{V_i}(h) = \begin{cases} 1/c(g); & \textit{if } g \textit{ and } h \textit{ are conjugate,} \\ 0; & \textit{otherwise} \end{cases}$$

where the sum is over all distinct irreducible representations V_i of G and $c(g)$ is the size of the conjugacy class of g.

This proposition completes the description of the characters of a finite group in general. Now we introduce a notion that will be useful in the next chapter. A virtual representation of a finite group G is an integer linear combination of irreducible representations of G, $V = \sum n_i V_i$, $n_i \in \mathbb{Z}$. The character of V is $\chi_V = \sum n_i \chi_{V_i}$.

Lemma 1.2.5. *Let V be a virtual representation with character χ_V. Suppose $(\chi_V, \chi_V) = 1$ and $\chi_V(1) > 0$, then χ_V is a character of an irreducible representation of G.*

1.3 Induced representations

If $H \subset G$ is a subgroup, any representation V of G restricts to a representation of H, denoted $\mathrm{Res}_H^G V$. Conversely, suppose W is a representation of H. The induced representation $\mathrm{Ind}_H^G W$ is the representation of G with

$$\mathrm{Ind}_H^G W = \{f\colon G \to W \mid f(hx) = \rho_W(h)f(x) \text{ for all } x \in G \text{ and } h \in H\}$$

and the action $g(f)(x) = f(xg)$ for any $g \in G$. To compute the character $\chi_{\mathrm{Ind}_H^G W}$ of $\mathrm{Ind}_H^G W$ one has the following

Theorem 1.3.1 (Mackey formula)**.** *Let T be a right transversal for H in G. We have*

$$\chi_{Ind_H^G W}(g) = \sum_{t \in T : tgt^{-1} \in H} \chi_W(tgt^{-1}) = \frac{1}{|H|} \sum_{x \in G : xgx^{-1} \in H} \chi_W(xgx^{-1}).$$

Another important result about induced representations is the Frobenius Reciprocity Theorem.

Theorem 1.3.2 (Frobenius Reciprocity)**.** *Let W be a representation of H, and V be a representation of G. Then, as vector spaces, $Hom_G(V, Ind_H^G W)$ is naturally isomorphic to $Hom_H(Res_H^G V, W)$.*

Corollary 1.3.3. *If W is a representation of H, and V a representation of G, then we have $(\chi_{Ind_H^G W}, \chi_V)_G = (\chi_W, \chi_{Res_H^G V})_H$.*

1.4 Group Algebra

The group algebra $\mathbb{C}[G]$ of a finite group G is the associative $\mathbb{C}$-algebra with basis G and where multiplication is inherited from group multiplication, i.e.,

$$\left(\sum_{p \in G} a_p p\right)\left(\sum_{q \in G} b_q q\right) = \sum_{p,q \in G} a_p b_q pq = \sum_{g \in G} c_g g$$

where c_g is the sum of all $a_p b_q$ where $g = pq$.

We can generalize the notion of a representation to associative algebras with unit. A representation of an associative $\mathbb{C}$-algebra A on a finite-dimensional complex vector space V (also called a left A-module) is an algebra homomorphism $\rho\colon A \to \mathrm{End}(V)$. An A-map φ between two representations V and W of A is a vector space map $\phi\colon V \to W$ such that $\varphi(av) = a\varphi(v)$ for every $a \in A$ and $v \in V$.

We want to show now that we have an equivalence of categories between $\mathrm{Rep}(G)$ and the category of left $\mathbb{C}[G]$-modules.

Notice that given a representation of a finite group G, we can extend it linearly to get a representation of the group algebra $\mathbb{C}[G]$, namely $\rho_{\mathbb{C}[G]}(\sum_{g \in G} a_g g) = \sum_{g \in G} a_g \rho_G(g)$. And conversely, given a representation of $\mathbb{C}[G]$, we can restrict it to G to get back a representation of G.

Thus we define a covariant functor $F\colon \mathrm{Rep}(G) \to \mathbb{C}[G]$-Mod as follows: F assigns to every representation (V, ρ_G) the representation $(V, \rho_{\mathbb{C}[G]})$ defined as above and to every G-map $\varphi\colon V \to$

W the $\mathbb{C}[G]$-map $\varphi^*\colon V \to W$ such that $v \mapsto \varphi(v)$. This is indeed a $\mathbb{C}[G]$-map since

$$\begin{aligned}\varphi^*\left(\left(\sum_{g\in G} a_g g\right) v\right) &= \varphi\left(\sum_{g\in G} a_g (gv)\right) = \sum_{g\in G} a_g \varphi(gv) \\ &= \sum_{g\in G} a_g (g\varphi(v)) = \left(\sum_{g\in G} a_g g\right) \varphi^*(v).\end{aligned}$$

Easily, F respects the identity and compositions, so $F\colon \mathrm{Rep}(G) \to \mathbb{C}[G]$-Mod is in fact a covariant functor. Roughly speaking, we can say that a G-map $\varphi\colon V \to W$ is mapped to itself as linear map with the G-representations V, W mapped to $\mathbb{C}[G]$-modules as objects. Hence the functor F is clearly fully faithful. From what we have seen above, taking restriction and linear extension, F is also essentially surjective. Hence the covariant functor $F\colon \mathrm{Rep}(G) \to \mathbb{C}[G]$-Mod is an equivalence, that is representations of G and representations of $\mathbb{C}[G]$ are categorically equivalent notions.

In fact, we can notice that this is an isomorphism of categories. Indeed, we can consider the covariant functor $\tilde{F}\colon \mathbb{C}[G]\text{-Mod} \to \mathrm{Rep}(G)$ defined as follows: $\tilde{F}$ assigns to every representation $(V, \rho_{\mathbb{C}[G]})$ the representation (V, ρ_G) by restriction as above and to every $\mathbb{C}[G]$-map $\varphi\colon V \to W$ the G-map $\tilde{\varphi}\colon V \to W$ such that $v \mapsto \varphi(v)$ that is clearly a G-map. It is immediate to prove that $\tilde{F} \circ F = \mathrm{id}_{\mathrm{Rep}(G)}$ and $F \circ \tilde{F} = \mathrm{id}_{\mathbb{C}[G]\text{-Mod}}$.

Finally, we state the following proposition on the structure of $\mathbb{C}[G]$.

Proposition 1.4.1. *As algebras,*

$$\mathbb{C}[G] = \bigoplus_i End(V_i)$$

where the sum is over all distinct irreducible representations V_i of G.

1.5 Lie algebras

A complex vector space $\mathfrak{g}$ endowed with a bilinear map $[\cdot,\cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called Lie bracket, is called a Lie algebra over $\mathbb{C}$ if it satisfies the following axioms: $[x, x] = 0$ for all $x \in \mathfrak{g}$ and

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

for all $x, y, z \in \mathfrak{g}$. The last axiom is called Jacobi identity. It is easy to see that every associative $\mathbb{C}$-algebra A becomes a Lie algebra over $\mathbb{C}$ with the operation $[x, y] = xy - yx$, called commutator of x and y, for any $x, y \in A$. We will write $\mathcal{L}(A)$ for A viewed as a Lie algebra. In particular, for a finite-dimensional complex vector space W, $\mathrm{End}(W)$ becomes a Lie algebra over $\mathbb{C}$, denoted $\mathfrak{gl}(W)$, and we will call it the general linear algebra.

A linear map $\rho\colon \mathfrak{g} \to \mathfrak{h}$ that respects the Lie bracket, i.e., $\rho([x, y]_{\mathfrak{g}}) = [\rho(x), \rho(y)]_{\mathfrak{h}}$ for all $x, y \in \mathfrak{g}$, is called a Lie algebra homomorphism. A representation of a Lie algebra $\mathfrak{g}$ on a finite-dimensional complex vector space W is a Lie algebra homomorphism $\rho\colon \mathfrak{g} \to \mathfrak{gl}(W)$.

Example 1.5.1. If $x \in \mathfrak{g}$, then $y \mapsto [x, y]$ is an endomorphism of $\mathfrak{g}$, which we denote $\operatorname{ad} x$. In fact, we can rewrite the Jacobi identity in the form: $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$. The map $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ sending x to $\operatorname{ad} x$ is called the adjoint representation of $\mathfrak{g}$.

Let $\mathfrak{g}$ be a Lie algebra. We define the tensor algebra $\mathcal{T}(\mathfrak{g})$ as the complex vector space $\bigoplus_{n\geq 0}\mathfrak{g}^{\otimes n}$ where $\mathfrak{g}^{\otimes 0} := \mathbb{C}$ with multiplication given by the canonical isomorphism $\mathfrak{g}^{\otimes n}\otimes\mathfrak{g}^{\otimes m}\to\mathfrak{g}^{\otimes(n+m)}$. Now define the universal enveloping algebra $\mathcal{U}(\mathfrak{g}) = \mathcal{T}(\mathfrak{g})/I$ where I is the ideal generated by elements $x\otimes y - y\otimes x - [x,y]$ for all $x, y\in\mathfrak{g}\subseteq\mathcal{T}(\mathfrak{g})$. We claim now that

$$\mathrm{Hom}_{\mathrm{Alg}}(\mathcal{U}(\mathfrak{g}), \mathrm{End}(V)) \simeq \mathrm{Hom}_{\mathrm{LieAlg}}(\mathfrak{g}, \mathfrak{gl}(V)).$$

Let $\rho\colon\mathfrak{g}\to\mathfrak{gl}(V)$ be a Lie algebra representation. We can extend it to an algebra representation $\rho'\colon\mathcal{U}(\mathfrak{g})\to\mathrm{End}(V)$ by setting $\rho'(x_1\otimes\cdots\otimes x_n) = \rho(x_1)\dots\rho(x_n)$ on the basis tensors and then extending by linearity. Similarly, given an algebra representation $\rho'\colon\mathcal{U}(\mathfrak{g})\to\mathrm{End}(V)$, we can restrict it to $\mathfrak{g}\subseteq\mathcal{U}(\mathfrak{g})$ to get a Lie algebra representation $\rho\colon\mathfrak{g}\to\mathfrak{gl}(V)$ since

$$\begin{aligned}\rho([x,y]_{\mathfrak{g}}) &= \rho'([x,y]_{\mathfrak{g}}) = \rho'(x\otimes y - y\otimes x)\\ &= \rho'(x)\rho'(y) - \rho'(y)\rho'(x) = \rho(x)\rho(y) - \rho(y)\rho(x) = [\rho(x),\rho(y)]_{\mathfrak{gl}(V)}\end{aligned}$$

for all $x, y\in\mathfrak{g}$. We may conclude then that studying representations of $\mathfrak{g}$ is equivalent to studying representations of $\mathcal{U}(\mathfrak{g})$.

Chapter 2

Schur-Weyl duality for GL(V)

In this chapter, we will study the classical Schur-Weyl duality relating irreducible finite-dimensional representations of the general linear and symmetric groups. In section 2.1, we will study a construction of the irreducible representations of the symmetric group and a formula for their characters. Next, in section 2.2, we will prove the classical case of Schur-Weyl duality using the Double Centralizer Theorem. We will finally obtain the Frobenius character formula for the irreducible representations of S_n.

The presentation in this chapter is based mainly on chapter 5 of [3] and on some parts of lecture 6 of [5].

2.1 Representations of the symmetric group S_n

2.1.1 Irreducible representations for S_n

Recall that a partition λ of n is an integer sequence $\lambda = (\lambda_1, \dots, \lambda_k)$ so that $\lambda_1 \geq \cdots \geq \lambda_k \geq 1$ and $n = \lambda_1 + \cdots + \lambda_k$. We already know that the conjugacy classes of S_n are in bijection with the partitions of n. Let $p(n)$ denote the number of partitions of n. Since the number of conjugacy classes of S_n is $p(n)$, by Proposition 1.2.3 there are also $p(n)$ distinct irreducible representations of S_n.

To a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of n, we attach the Young diagram of λ, which is a collection of n left-justified cells with λ_i cells on the ith row. The conjugate partition $\lambda' = (\lambda'_1, \dots, \lambda'_r)$ to the partition λ is defined by interchanging rows and columns in the Young diagram. Given the Young diagram of λ, we define a tableau on it to be a numbering of the boxes by the integers $1, \dots, n$. Numbering the boxes consecutively as shown in the following tableau for (4,2,1)

1	2	3	4
5	6		
7			

,

we get the so-called canonical labeling. S_n acts naturally on tableaux by permuting the labels. Given the canonical tableau for a partition λ of n, we define the following subgroups of the symmetric group

$$P_\lambda = \{g \in S_n \mid g \text{ preserves each row of } \lambda\},$$

and

$$Q_\lambda = \{g \in S_n \mid g \text{ preserves each column of } \lambda\}.$$

Moreover, we define the following elements in $\mathbb{C}[S_n]$:

$$a_\lambda = \sum_{g \in P_\lambda} g \quad \text{and} \quad b_\lambda = \sum_{g \in Q_\lambda} \operatorname{sgn}(g) g.$$

Finally, we call their product $c_\lambda = a_\lambda b_\lambda \in \mathbb{C}[S_n]$ the Young symmetrizer of λ.

Remark 2.1.1. i) For the canonical tableau, $S_\lambda := P_\lambda$ is the subgroup of elements in S_n stabilizing the sets $\{1, \dots, \lambda_1\}, \{\lambda_1+1, \dots, \lambda_1+\lambda_2\}, \dots, \{\lambda_1+\dots+\lambda_{k-1}+1, \dots, \lambda_1+\dots+\lambda_k\}$.

ii) $P_\lambda \cap Q_\lambda = \{1\}$.

iii) Let T be a Young tableau corresponding to λ and $g \in S_n$. If $T' = gT$, then $P'_\lambda = gP_\lambda g^{-1}$ and $Q'_\lambda = gQ_\lambda g^{-1}$.

iv) Because $P_\lambda \cap Q_\lambda = \{1\}$, we have no cancellations in the sum

$$\sum_{h \in P_\lambda, k \in Q_\lambda} \operatorname{sgn}(k) hk,$$

so $c_\lambda \neq 0$.

The irreducible representations of S_n are described by the following theorem.

Theorem 2.1.2. *The subspace $V_\lambda = \mathbb{C}[S_n]c_\lambda$ of $\mathbb{C}[S_n]$ is an irreducible representation of S_n under left multiplication. Moreover, every irreducible representation of S_n is isomorphic to V_λ for some unique partition λ of n.*

The module V_λ is called the Specht module corresponding to λ.

Remark 2.1.3. Different Young symmetrizers, given by different labelings of the same partition, give rise to isomorphic Specht modules. For let T and T' be tableaux of the same partition. Then $T' = gT$ for some $g \in S_n$. By iii) of the previous remark, we have

$$\mathbb{C}[S_n]c_{T'} = \mathbb{C}[S_n]c_{gTg^{-1}} = \mathbb{C}[S_n]c_{Tg^{-1}}.$$

Thus the map $\mathbb{C}[S_n]c_T \to \mathbb{C}[S_n]c_{T'}$ sending $x \mapsto xg^{-1}$ is an isomorphism of S_n-modules.

In this section, we will always work with tableaux with the canonical labeling. Note that, as a corollary, each irreducible representation of S_n can be defined over the rational numbers since c_λ is in the rational group algebra $\mathbb{Q}[S_n]$.

Before proving the theorem, let us compute all the distinct irreducible representations of S_3.

Example 2.1.4. We have the canonical tableaux

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$$

Clearly, we have $P_{(3)} = S_3$ and $Q_{(3)} = \{1\}$ for $\lambda = (3)$, $P_{(2,1)} = \{1, (12)\}$ and $Q_{(2,1)} = \{1, (13)\}$ for $\lambda = (2,1)$, and $P_{(1,1,1)} = \{1\}$ and $Q_{(1,1,1)} = S_3$ for $\lambda = (1,1,1)$. Hence we get

$$c_{(3)} = a_{(3)}b_{(3)} = \sum_{g \in S_3} g,$$

$$c_{(2,1)} = a_{(2,1)}b_{(2,1)} = (1 + (12))(1 - (13)) = 1 + (12) - (13) - (132),$$

$$c_{(1,1,1)} = a_{(1,1,1)}b_{(1,1,1)} = \sum_{g \in S_3} \operatorname{sgn}(g) g.$$

It is immediate to see that, for any $h \in S_3$, we have $hc_{(3)} = c_{(3)}$ and $hc_{(1,1,1)} = \text{sgn}(h)c_{(1,1,1)}$. Hence we conclude that $V_{(3)} = \mathbb{C}[S_3]c_{(3)}$ is the trivial representation and $V_{(1,1,1)} = \mathbb{C}[S_3]c_{(1,1,1)}$ is the alternating representation. Finally, by easy computations, $V_{(2,1)} = \mathbb{C}[S_3]c_{(2,1)}$ is the span of $c_{(2,1)}$ and $(13)c_{(2,1)}$.

Remark 2.1.5. Consider the natural permutation representation, in which S_n acts on $\mathbb{C}^n$ by permuting the coordinates. This representation is not irreducible: the line spanned by the sum $(1, \dots, 1)$ of the basis vectors is invariant, with complementary subspace $V = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_1 + \cdots + z_n = 0\}$. This $(n-1)$-dimensional representation V is easily seen to be irreducible, we call it the standard representation of S_n.

In general, $V_{(n)}$ is the trivial representation, $V_{(n-1,1)}$ is the standard representation, and $V_{(1,\dots,1)}$ is the sign representation.

We are now going to prove theorem 2.1.2. In the following, let λ be a partition of n.

Lemma 2.1.6. *There exists a unique $t_\lambda \in \mathbb{C}[S_n]^* = Hom(\mathbb{C}[S_n], \mathbb{C})$ such that $a_\lambda g b_\lambda = t_\lambda(g) c_\lambda$ for all $g \in \mathbb{C}[S_n]$. In particular for all $g \in S_n$ we have*

$$t_\lambda(g) = \begin{cases} 0 & \text{if } g \notin P_\lambda Q_\lambda \\ sgn(q) & \text{if } g = pq, \text{ with } p \in P_\lambda \text{ and } q \in Q_\lambda \end{cases}.$$

Proof. Since S_n forms a basis of $\mathbb{C}[S_n]$, it suffices to show the statement for $g \in S_n$. We start with the following easy observation: if $p \in P_\lambda$ and $q \in Q_\lambda$, then

$$a_\lambda p = \sum_{g \in P_\lambda} (gp) = a_\lambda$$

and

$$qb_\lambda = \sum_{g \in Q_\lambda} \text{sgn}(g)(qg) = \text{sgn}(q) \sum_{g \in Q_\lambda} \text{sgn}(qg)(qg) = \text{sgn}(q) b_\lambda.$$

Notice if $g \in P_\lambda Q_\lambda$, then it has a unique representation as $g = pq$ with $p \in P_\lambda$ and $q \in Q_\lambda$. Thus,

$$a_\lambda g b_\lambda = (a_\lambda p)(q b_\lambda) = a_\lambda(\text{sgn}(q) b_\lambda) = \text{sgn}(q) c_\lambda.$$

Now if $g \notin P_\lambda Q_\lambda$, notice that the coefficient for the identity 1 in $a_\lambda g b_\lambda$ is 0 because if $pgq = 1$ for some $p \in P_\lambda$ and $q \in Q_\lambda$, then $g = p^{-1}q^{-1} \in P_\lambda Q_\lambda$. Since the coefficient for the identity in c_λ is 1, we must have that $t_\lambda(g) = 0$. Thus, we must show that $a_\lambda g b_\lambda = 0$. So it suffices to show that there exists a transposition $q \in Q_\lambda$ so that $p = gqg^{-1} \in P_\lambda$ because then

$$a_\lambda g b_\lambda = (a_\lambda p) g (\text{sgn}(q) q b_\lambda) = -a_\lambda (gqg^{-1}) gq b_\lambda = -a_\lambda g b_\lambda$$

and, hence, $a_\lambda g b_\lambda = 0$. Consider the tableau $T' = gT$ where T is the given tableau. Notice that p is a row-preserving transposition in T because $p \in P_\lambda$, and $p = gqg^{-1}$ is a column-preserving transposition in T' because $q \in Q_\lambda$. So it suffices to show that there exists two distinct integers which lie in the same row in T and in the same column in T'.

Suppose there were not two such integers. We can find a row-preserving permutation $p_1 \in P_\lambda$ of T and a column-preserving permutation $q_1' \in gQ_\lambda g^{-1}$ of T' so that $p_1 T$ and $q_1' T'$ have the same first row. Continuing like this, we could find $p \in P_\lambda$ and $q' = gqg^{-1} \in gQ_\lambda g^{-1}$ so that $pT = q'T' = q'gT = gqT$. But then $p = gq$ and, hence, $g = pq^{-1} \in P_\lambda Q_\lambda$, a contradiction. □

We order partitions lexicographically: $\lambda > \mu$ if the first non-vanishing $\lambda_i - \mu_i$ is positive.

Lemma 2.1.7. *If $\lambda > \mu$, then $a_\lambda g b_\mu = 0$ for all $g \in \mathbb{C}[S_n]$. In particular, if $\lambda > \mu$, then $c_\lambda c_\mu = 0$.*

Proof. Similarly to the previous lemma, it suffices to find a transposition $q \in Q_\mu$ so that $p = gqg^{-1} \in P_\lambda$ for all $g \in S_n$. The computation is completely similar to the previous proof considering T and gT' where T is the tableau used to construct a_λ and T' is the tableau used to construct b_μ. □

Lemma 2.1.8. *We have that* $c_\lambda c_\lambda = n_\lambda c_\lambda$ *where* $n_\lambda = n!/\dim V_\lambda$.

Proof. Lemma 2.1.6 implies that $c_\lambda c_\lambda = n_\lambda c_\lambda$ for some $n_\lambda \in \mathbb{C}$. Consider the map $F\colon \mathbb{C}[S_n] \to V_\lambda$ where $x \mapsto xc_\lambda$. F multiplies by n_λ on V_λ, while F multiplies by 0 on $\operatorname{Ker} F$. Hence $\operatorname{tr} F = n_\lambda \dim V_\lambda$.

On the other hand, the coefficient of g in $F(g) = gc_\lambda$ is 1 because, as we have already noticed, the coefficient for the identity in c_λ is 1. Hence we also have that $\operatorname{tr} F = \dim \mathbb{C}[S_n] = n!$. □

Lemma 2.1.9. *For each partition* λ *of* n*,* V_λ *is an irreducible representation of* S_n.

Proof. Using lemma 2.1.6 we get that $c_\lambda V_\lambda \subset \mathbb{C}c_\lambda$. Let $W \subseteq V_\lambda$ be a subrepresentation. There are two cases: either $c_\lambda W = 0$ or $c_\lambda W = \mathbb{C}c_\lambda$. If $c_\lambda W = 0$, then $W \cdot W \subset V_\lambda \cdot W = 0$. We claim that this implies $W = 0$. Indeed, by Maschke's theorem there exists a complementary subrepresentation W' of $\mathbb{C}[S_n]$ such that $\mathbb{C}[S_n] = W \oplus W'$. Define a projection p from $\mathbb{C}[S_n]$ onto W by $p(w + w') = w$. Since W, W' are invariant subspaces of V we have $p(gx) = gp(x)$ for all $g \in S_n$ and $x \in \mathbb{C}[S_n]$. But this means that p is given by right multiplication by an element $\varphi \in W$. Indeed, for any $g \in S_n$ we have $p(g) = p(g \cdot 1) = gp(1)$ with $p(1) \in W$ and so, by $\mathbb{C}$-linearity, $p(x) = xp(1)$ for all $x \in \mathbb{C}[S_n]$. Thus , take $\varphi = p(1) \in W$. Now since $p^2 = p$ we get $\varphi = \varphi^2 \in W \cdot W = 0$, so that $\varphi = 0$. Since p is surjective, we must have $W = 0$.

If $c_\lambda W = \mathbb{C}c_\lambda$, then

$$V_\lambda = \mathbb{C}[S_n]c_\lambda = \mathbb{C}[S_n](c_\lambda W) = (\mathbb{C}[S_n]c_\lambda)W \subseteq W$$

because W is a representation of S_n. Therefore, V_λ is irreducible. □

Lemma 2.1.10. *If* $\lambda \neq \mu$*, then* $V_\lambda \not\simeq V_\mu$.

Proof. Without loss of generality, assume $\lambda > \mu$. Then by the previous proof along with lemma 2.1.7, we have that $c_\lambda V_\lambda = \mathbb{C}c_\lambda$ but $c_\lambda V_\mu = c_\lambda \mathbb{C}[S_n]c_\mu = 0$. □

Now using lemmas 2.1.9 and 2.1.10 and the fact that partitions of n list all conjugacy classes of S_n, which by Proposition 1.2.3, list all irreducible representations of S_n, we get theorem 2.1.2.

2.1.2 Induced representations for S_n

Let U_λ be the representation of S_n induced by the trivial representation of P_λ, so that

$$U_\lambda = \operatorname{Ind}_{P_\lambda}^{S_n}\mathbb{C} = \mathbb{C}[S_n] \otimes_{\mathbb{C}[P_\lambda]} \mathbb{C} = \mathbb{C}[S_n/P_\lambda],$$

where S_n acts on the last $\mathbb{C}$-vector space through its action by left multiplication on S_n/P_λ.

Proposition 2.1.11. *We have* $U_\lambda \simeq \mathbb{C}[S_n]a_\lambda$ *as* $\mathbb{C}[S_n]$*-modules.*

Proof. If $g, g' \in S_n$, we have $ga_\lambda = g'a_\lambda$ if and only if $gP_\lambda = g'P_\lambda$. Let $(g_i)_{i\in I}$ be a system of representatives of S_n/P_λ. Recall that the support of an element $x = \sum_{g\in S_n} a_g g \in \mathbb{C}[S_n]$ is the set of $g \in S_n$ such that $a_g \neq 0$. Then the $g_i a_\lambda$ have support in pairwise disjoint subsets of S_n, so they are linearly independent over $\mathbb{C}$. So the $(g_i a_\lambda)_{i\in I}$ form a $\mathbb{C}$-basis of $\mathbb{C}[S_n]a_\lambda$. In particular, we can define a $\mathbb{C}[S_n]$-linear map $u\colon U_\lambda = \mathbb{C}[S_n/P_\lambda] \to \mathbb{C}[S_n]a_\lambda$ by sending gP_λ to ga_λ, for every $g \in S_n$. This sends the basis $(g_i P_\lambda)_{i\in I}$ of $\mathbb{C}[S_n/P_\lambda]$ to the basis $(g_i a_\lambda)_{i\in I}$ of $\mathbb{C}[S_n]a_\lambda$ that we have just defined, and so it is an isomorphism. □

Lemma 2.1.12. *Let A be a k-algebra over any field k and e be an idempotent in A. Then for any left A-module M, we have $Hom_A(Ae, M) \cong eM$ (namely, $x \in eM$ corresponds to $f_x \colon Ae \to M$ given by $f_x(a) = ax$, $a \in Ae$).*

Proof. Let us prove that this map is injective. Let $x, y \in eM$ such that $f_x = f_y$. Then $0 = f_x(e) - f_y(e) = e(x - y)$. As $x - y \in eM$ and $e^2 = e$, we have $e(x - y) = x - y$, and so $x - y = 0$. Now we prove that the map $x \mapsto f_x$ is surjective. Let $f \in \mathrm{Hom}_A(Ae, M)$, let $x = f(e)$. Then $x = f(e^2) = ef(e) \in eM$. Also, for every $a \in Ae$, $f(a) = f(ae) = af(e) = ax$. Hence $f = f_x$. □

Proposition 2.1.13. *We have $Hom(U_\lambda, V_\mu) = 0$ for $\mu < \lambda$, and moreover* $\dim Hom(U_\lambda, V_\lambda) = 1$. *Thus,*

$$U_\lambda = \oplus_{\mu \geq \lambda} K_{\mu\lambda} V_\mu$$

where $K_{\mu\lambda}$ are non-negative integers and $K_{\lambda\lambda} = 1$.

The integers $K_{\mu\lambda}$ are called the Kostka numbers.

Proof. By proposition 2.1.11 and lemmas 2.1.8 and 2.1.12 (notice that $a_\lambda \cdot a_\lambda = |P_\lambda| \cdot a_\lambda$, so a_λ is idempotent up to a factor $|P_\lambda|$),

$$\mathrm{Hom}(U_\lambda, V_\mu) = \mathrm{Hom}(\mathbb{C}[S_n]a_\lambda, \mathbb{C}[S_n]a_\mu b_\mu) = a_\lambda \mathbb{C}[S_n]a_\mu b_\mu,$$

and the result follows from lemmas 2.1.6 and 2.1.7. □

Now let us compute the character $\psi_\lambda = \chi_{U_\lambda}$ of U_λ. For $\mathbf{i} = (i_1, \dots, i_d)$ a d-tuple of non-negative integers with $\sum \alpha i_\alpha = n$, denote by $\mathrm{C}_\mathbf{i} \subset S_n$ the conjugacy class consisting of elements made up of i_1 1-cycles, i_2 2-cycles,..., i_d d-cycles. For $N \geq k$ (where k is the number of parts of λ), set $\lambda_p = 0$ for $k + 1 \leq p \leq N$ and let $x_1, \dots, x_N$ be independent variables. Consider the power sum polynomials

$$P_j(x) = x_1^j + \dots + x_N^j$$

and the power sum symmetric polynomial

$$P^{(\mathbf{i})} = \prod_{j=1}^{k} P_j(x)^{i_j} = (x_1 + \dots + x_k)^{i_1} \cdot (x_1^2 + \dots + x_k^2)^{i_2} \cdot \dots \cdot (x_1^d + \dots + x_k^d)^{i_d}.$$

If $f(x) = f(x_1, \dots, x_N)$ is a formal power series, and $(l_1, \dots, l_N)$ is an N-tuple of non-negative integers, let

$$[f(x)]_{(l_1, \dots, l_N)} = \text{coefficient of } x_1^{l_1} \dots x_N^{l_N} \text{ in } f.$$

If $g \in S_n$, we write $Z_g = \{h \in S_n \mid hg = gh\}$ for the centralizer of g in S_n.

Proposition 2.1.14. *Let $g \in C_\mathbf{i} \subset S_n$. Then*

$$|Z_g| = \prod_{j \geq 1} i_j! j^{i_j}$$

and

$$|C_\mathbf{i}| = \frac{n!}{\prod_{j \geq 1} i_j! j^{i_j}}.$$

Proof. Let $h \in Z_g$. Then h sends the support of each cycle of g to the support of any other cycle of the same length, and it must also respect the cyclical order given by g on the support of these cycles. This gives an isomorphism

$$Z_g \simeq \prod_{j\geq 1} S_{i_j} \ltimes (\mathbb{Z}/jZ)^{i_j}$$

where S_{i_j} acts on $(\mathbb{Z}/jZ)^{i_j}$ by permuting the entries of the i_j-tuples. Hence

$$|Z_g| = \prod_{j\geq 1} i_j! j^{i_j}.$$

Now note that $\mathrm{C}_{\mathbf{i}} = S_n/Z_g$. So we get

$$|\mathrm{C}_{\mathbf{i}}| = \frac{n!}{\prod_{j\geq 1} i_j! j^{i_j}}.$$

□

Denote $X^\lambda = x_1^{\lambda_1} \cdot \ldots \cdot x_k^{\lambda_k}$.

Theorem 2.1.15. *We have*

$$\psi_\lambda(C_{\mathbf{i}}) = [P^{(\mathbf{i})}]_\lambda = \textit{coefficient of } X^\lambda \textit{ in } P^{(\mathbf{i})}.$$

Proof. By theorem 1.3.1, we get

$$\psi_\lambda(\mathrm{C}_{\mathbf{i}}) = \frac{1}{|P_\lambda|} \sum_{h\in S_n | hgh^{-1}\in P_\lambda} 1 = \frac{1}{|P_\lambda|} |Z_g||P_\lambda \cap \mathrm{C}_{\mathbf{i}}|,$$

for a representative $g \in \mathrm{C}_{\mathbf{i}}$.

First, we have $P_\lambda = S_\lambda \simeq S_{\lambda_1} \times \cdots \times S_{\lambda_k}$, so $|P_\lambda| = \prod_{p=1}^k \lambda_p! = \prod_{p=1}^N \lambda_p!$. Second, by Proposition 2.1.14,

$$|Z_g| = \prod_{j\geq 1} i_j! j^{i_j}.$$

Finally, we have to calculate $|P_\lambda \cap \mathrm{C}_{\mathbf{i}}|$. The conjugacy class $\mathrm{C}_{\mathbf{i}}$ is the set of permutations in S_n that have i_j cycles of length j for every $j \geq 1$. So its intersection with P_λ is a finite disjoint union of the following conjugacy classes in $P_\lambda \simeq S_{\lambda_1} \times \cdots \times S_{\lambda_k}$: the product for $p = 1, \ldots, k$ of the conjugacy classes in S_{λ_p} of permutations with r_{pj} cycles of length j for every $j \geq 1$, for every family $(r_{pj})_{1\leq p\leq k, j\geq 1}$ such that, for every $j \geq 1$, $i_j = \sum_{p=1}^k r_{pj}$ and for every $p \in \{1, \ldots, k\}$, $\lambda_p = \sum_{j\geq 1} j r_{pj}$. The cardinality of this product of conjugacy classes is

$$\prod_{p=1}^k \frac{\lambda_p!}{\prod_{j\geq 1} r_{pj}! j^{r_{pj}}},$$

by proposition 2.1.14. We can actually take p in $\{1, \ldots, N\}$ without changing the result, because $\lambda_p = 0$ for $p > k$.

Put all this together, we get

$$\psi_\lambda(\mathrm{C}_{\mathbf{i}}) = \frac{1}{\prod_{p=1}^N \lambda_p!} \prod_{j\geq 1} i_j! j^{i_j} \sum_{(r_{pj})} \prod_{p=1}^N \frac{\lambda_p!}{\prod_{j\geq 1} r_{pj}! j^{r_{pj}}},$$

where the sum is over families (r_{pj}) as above. This is equal to

$$\sum_{(r_{pj})} \prod_{p=1}^{N} \frac{i_j!}{\prod_{j\geq 1} r_{pj}}.$$

On the other hand, for every $j \geq 1$, we have

$$P_j(x)^{i_j} = \left(\sum_{p=1}^{N} x_p^j\right)^{i_j} = \sum_{i_j = r_{1j}+\cdots+r_{Nj}} \frac{i_j!}{\prod_{p=1}^{N} r_{pj}!} \prod_{p=1}^{N} x_p^{jr_{pj}}!.$$

So the coefficient of $\prod_{p=1}^{N} x_p^{\lambda_p}$ in $P^{\mathbf{i}} = \prod_{j\geq 1} P_j(x)^{i_j}$ is indeed equal to $\sum_{(r_{pj})} \prod_{p=1}^{N} \frac{i_j!}{\prod_{j\geq 1} r_{pj}!}$, where the sum is over families (r_{pj}) as above. □

For example, if $n = 4$, $\lambda = (3, 1)$, and $\mathrm{C}_{\mathbf{i}}$ is the conjugacy class of (123)(4), i.e., $i_1 = 1$, $i_2 = 0$, $i_3 = 1$, $i_4 = 0$, then

$$\psi_{(3,1)}(\mathrm{C}_{\mathbf{i}}) = [(x_1 + x_2)(x_1^3 + x_2^3)]_{(3,1)} = 1.$$

2.1.3 The Frobenius character formula

In this section we turn to Frobenius' formula for the character χ_λ of V_λ.

Let $\Delta(x) = \prod_{1\leq i<j\leq N}(x_i - x_j)$. This is equal to the Vandermonde determinant:

$$\begin{vmatrix} 1 & x_N & \dots & x_N^{N-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_1 & \dots & x_1^{N-1} \end{vmatrix} = \sum_{\sigma\in S_N} (\mathrm{sgn}(\sigma)) x_N^{\sigma(1)-1} \cdots x_1^{\sigma(N)-1}.$$

Given a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of n, set

$$l_1 = \lambda_1 + k - 1, \quad l_2 = \lambda_2 + k - 2, \dots, l_k = \lambda_k,$$

a strictly decreasing sequence of k non-negative integers, and denote $l = (l_1, \dots, l_k)$. The character of V_λ computed in $g \in \mathrm{C}_{\mathbf{i}}$ is given by

Theorem 2.1.16 (Frobenius Formula). *We have*

$$\chi_\lambda(C_{\mathbf{i}}) = \left[\Delta(x) \cdot \prod_j P_j(x)^{i_j}\right]_{(l_1,\dots,l_k)}.$$

We need some lemmas.

Lemma 2.1.17. *Let $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{Z}^N$ be such that $\lambda_1 \geq \cdots \geq \lambda_N$. Let $\sigma \in S_N$, and let μ be the N-tuple of integers $(\lambda_1 + \sigma(1) - 1, \dots, \lambda_N + \sigma(N) - N)$, rearranged to be in non-increasing order. Then $\mu \geq \lambda$, and we have $\mu = \lambda$ if and only if $\sigma = 1$.*

Proof. Let $i_0 \in \{0, \dots, N\}$ be an integer such that $\lambda_i = \mu_i$ for every $1 \leq i \leq i_0$. We claim that $\sigma(i) = i$ for every $1 \leq i \leq i_0$ and that, if $i_0 \leq N - 1$, then $\mu_{i_0+1} \geq \lambda_{i_0+1}$. Applying this claim to the biggest i_0 with the above property, the lemma follows.

We proceed by induction on i_0. If $i_0 = 0$, the claim is trivial. Now let $i_0 \geq 1$ and suppose that the result holds for $i_0 - 1$. We have $\mu_{i_0} \geq \lambda_i + \sigma(i) - i$ for $i_0 \leq i \leq N$, so $\mu_{i_0} \geq \lambda_{i_0} + \sigma(i_0) - i_0$.

As $\mu_{i_0} = \lambda_{i_0}$, this gives $\sigma(i_0) \leq i_0$. But $\sigma(i_0) \in \{i_0, \ldots, N\}$, so $\sigma(i_0) = i_0$. Now, if $i_0 \leq N-1$, we have

$$\mu_{i_0+1} = \sup_{i_0+1 \leq i \leq N} (\lambda_i + \sigma(i) - i) \geq \lambda_{i_0+1} + \sigma(i_0+1) - (i_0+1) \geq \lambda_{i_0+1},$$

because $\sigma(i_0+1) \in \{i_0+1, \ldots, N\}$ as $\sigma_{|\{1,\ldots,i_0\}} = \mathrm{id}$. □

We state next a combinatorial lemma.

Lemma 2.1.18 (Cauchy)**.** *Let* $A_N = \left(\frac{1}{x_i - y_j}\right)_{1 \leq i,j \leq N}$. *We have*

$$\det(A_N) = \frac{\prod_{1 \leq i < j \leq N}(x_i - x_j)(y_j - y_i)}{\prod_{1 \leq i,j \leq N}(x_i - y_j)}.$$

Proof. See corollary 5.15.4 of [3]. □

Proof of Theorem 2.1.16. Let θ_λ be the class function defined on the right-hand side of the equality in the theorem. We claim that this function has the property $\theta_\lambda = \sum_{\mu \geq \lambda} L_{\mu\lambda}\chi_\mu$, where $L_{\mu\lambda}$ are integers and $L_{\lambda\lambda} = 1$. By definition, $\theta_\lambda(\mathrm{C}_{\mathbf{i}})$ is the coefficient of $\prod_{p=1}^N x_p^{\lambda_p+N-p}$ in $\Delta(x)\prod_{j\geq 1} P_j(x)^{i_j}$. As $\Delta(x)$ is equal to the Vandermonde determinant, we have

$$\Delta(x) = \sum_{\sigma \in S_N} \mathrm{sgn}(\sigma) \prod_{p=1}^N x_p^{N-\sigma(p)}.$$

So

$$\begin{aligned}\theta_\lambda(\mathrm{C}_{\mathbf{i}}) &= \sum_{\sigma \in S_N} \mathrm{sgn}(\sigma) \left[\prod_{p=1}^N x_p^{N-\sigma(p)} \prod_{j \geq 1} P_j(x)^{i_j}\right]_{(\lambda_1+N-1, \lambda_2+N-2, \ldots, \lambda_N)} \\ &= \sum_{\sigma \in S_N} \mathrm{sgn}(\sigma) \left[\prod_{j\geq 1} P_j(x)^{i_j}\right]_{(\lambda_1-1+\sigma(1), \ldots, \lambda_N-N+\sigma(N))}.\end{aligned}$$

Let $\sigma \in S_N$, and denote $\mu_\sigma = (\mu_{\sigma,1}, \ldots, \mu_{\sigma,N})$ the N-tuple of integers $(\lambda_1 + \sigma(1) - 1, \ldots, \lambda_N + \sigma(N) - N)$, rearranged to be in non-increasing order. Observe that the polynomial $\prod_{j\geq 1} P_j(x)^{i_j}$ is symmetric in the variables x_i, because all the $P_j(x)$ are. So the coefficient of $\prod_{p=1}^N x_p^{\lambda_p - p + \sigma(p)}$ in $\prod_{j\geq 1} P_j(x)^{i_j}$ is equal to the coefficient of $\prod_{p=1}^N x_p^{\mu_{\sigma,p}}$. Also, if one of the $\mu_{\sigma,p}$ is negative, then this coefficient is zero, because there are no negative powers of the x_i in $\prod_{j\geq 1} P_j(x)^{i_j}$. Note that saying that none of the $\mu_{\sigma,p}$ is negative is equivalent to say that μ_σ is a partition of n (because of course $\sum_{p=1}^N \mu_{\sigma,p} = \sum_{p=1}^N (\lambda_p + \sigma(p) - p) = \sum_{p=1}^N \lambda_p$).

So we get that

$$\theta_\lambda(\mathrm{C}_{\mathbf{i}}) = \sum_{\sigma \in S_N} \mathrm{sgn}(\sigma) \left[\prod_{j \geq 1} P_j(x)^{i_j}\right]_{(\mu_{\sigma,1}, \ldots, \mu_{\sigma,N})} = \sum_{\sigma \in S_N} \mathrm{sgn}(\sigma)\psi_\mu(\mathrm{C}_{\mathbf{i}}).$$

Note also that, by lemma 2.1.17, for every $\sigma \in S_N$, we have $\mu_\sigma \geq \lambda$, and that $\mu_\sigma = \lambda$ if and only if $\sigma = 1$. Hence $\theta_\lambda = \psi_\lambda + \sum_{\mu > \lambda} J_{\mu\lambda}\psi_\mu$, for some integers $J_{\mu\lambda} \in \mathbb{Z}$. Using the decomposition $U_\mu = V_\mu \oplus_{\nu > \mu} K_{\nu\mu} V_\nu$ of Proposition 2.1.13, we get that

$$\theta_\lambda = \chi_\lambda + \sum_{\mu > \lambda} L_{\mu\lambda}\chi_\mu,$$

for some integers $L_{\mu\lambda} \in \mathbb{Z}$.

Therefore, to show that $\theta_\lambda = \chi_\lambda$, by lemma 1.2.5, it suffices to show that $(\theta_\lambda, \theta_\lambda) = 1$. We have

$$(\theta_\lambda, \theta_\lambda) = \frac{1}{n!} \sum_{\mathbf{i}} |\mathrm{C}_{\mathbf{i}}| \theta_\lambda(\mathrm{C}_{\mathbf{i}})^2.$$

Recall that, by Proposition 2.1.14,

$$|\mathrm{C}_{\mathbf{i}}| = \frac{n!}{\prod_{j \geq 1} i_j! j^{i_j}}.$$

Note that, if we take an arbitrary family $(i_j)_{j \geq 1}$ of non-negative integers that are almost all zero, then $\theta_\lambda(\mathrm{C}_{\mathbf{i}}) = 0$ unless $\sum_{j \geq 1} j i_j = \sum_{p=1}^{N} \lambda_p$ for degree reasons. So we can take the sum over all families $(i_j)_{j \geq 1}$ of non-negative integers that are almost all zero, and conclude that $(\theta_\lambda, \theta_\lambda)$ is the coefficient of $x^l y^l$ in $R(x, y) = \Delta(x)\Delta(y)S(x, y)$, where

$$\begin{aligned} S(x,y) &= \sum_{\mathbf{i}} \prod_j \frac{P_j(x)^{i_j} P_j(y)^{i_j}}{i_j! j^{i_j}} = \sum_{\mathbf{i}} \prod_j \frac{(\sum_m x_m^j)^{i_j} (\sum_n y_n^j)^{i_j}}{i_j! j^{i_j}} \\ &= \sum_{\mathbf{i}} \prod_j \frac{1}{i_j!} \left(\sum_{m,n} \frac{x_m^j y_n^j}{j} \right)^{i_j} = \prod_j \exp\left(\sum_{m,n} \frac{x_m^j y_n^j}{j} \right) \\ &= \exp\left(-\sum_{m,n} \log(1 - x_m y_n) \right) = \prod_{m,n} (1 - x_m y_n)^{-1}. \end{aligned}$$

Thus,

$$R(x,y) = \frac{\prod_{m<n} (x_m - x_n)(y_m - y_n)}{\prod_{m,n} (1 - x_m y_n)}.$$

So by lemma 2.1.18, $R(x, y)$ is the determinant of the $N \times N$ matrix $\left(\frac{1}{1 - x_m y_n} \right)$, and we have

$$R(x,y) = \sum_{\sigma \in S_N} \mathrm{sgn}(\sigma) \prod_m \frac{1}{1 - x_m y_{\sigma(m)}}.$$

Recall that $(\theta_\lambda, \theta_\lambda)$ is the coefficient of $x^l y^l$ in this formal power series. If $\sigma \neq 1$, then there exists $j \in \{1, \ldots, N\}$ such that $q = \sigma(j) > j$. In the formal power series expansion of $\prod_m \frac{1}{1 - x_m y_{\sigma(m)}}$, x_j and y_q must have the same exponent in each term. In particular, $x^l y^l$ does not appear in this expansion, because the exponent $\lambda_j + N - j$ of x_j in this product is greater than the exponent $\lambda_q + N - q$ of y_q. So $(\theta_\lambda, \theta_\lambda)$ is the coefficient of $x^l y^l$ in $\prod_m \frac{1}{1 - x_m y_m}$, i.e. 1, and we are done. □

For example, if $n = 4$, $\lambda = (3, 1)$, and $\mathrm{C}_{\mathbf{i}}$ is the conjugacy class of (123)(4), i.e., $i_1 = 1$, $i_2 = 0$, $i_3 = 1$, $i_4 = 0$, then

$$\chi_{(3,1)}(\mathrm{C}_{\mathbf{i}}) = [(x_1 - x_2)(x_1 + x_2)(x_1^3 + x_2^3)]_{(3,1)} = 0.$$

Let us use the Frobenius formula to compute the dimension of V_λ. The conjugacy class of the identity corresponds to $\mathbf{i} = (n)$, so

$$\dim V_\lambda = \chi_\lambda(C_{(n)}) = [\Delta(x) \cdot (x_1 + \cdots + x_k)^n]_{(l_1, \ldots, l_k)}.$$

Recall that $\Delta(x) = \sum_{\sigma\in S_k} \operatorname{sgn}(\sigma) x_k^{\sigma(1)-1} \dots x_1^{\sigma(k)-1}$ and

$$(x_1 + \cdots + x_k)^n = \sum_{r_1+\cdots+r_k=n} \frac{n!}{r_1! \dots r_k!} x_1^{r_1} \dots x_k^{r_k},$$

The coefficient of $x_1^{l_1} \dots x_k^{l_k}$ in the product is then

$$\sum \operatorname{sgn}(\sigma) \frac{n!}{(l_1 - \sigma(k) + 1)! \dots (l_k - \sigma(1) + 1)!},$$

the sum is taken over $\sigma \in S_k$ such that $l_{k-i+1} - \sigma(i) + 1 \geq 0$ for all $1 \leq i \leq k$. Equivalently,

$$\begin{aligned} &\frac{n!}{l_1! \dots l_k!} \sum_{\sigma\in S_k} \operatorname{sgn}(\sigma) \prod_{j=1}^{k} l_j(l_j - 1) \dots (l_j - \sigma(k - j + 1) + 2) \\ &= \frac{n!}{l_1! \dots l_k!} \begin{vmatrix} 1 & l_k & l_k(l_k - 1) & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 1 & l_1 & l_1(l_1 - 1) & \cdots \end{vmatrix}. \end{aligned}$$

By column reduction this determinant reduces to the Vandermonde determinant, so

$$\dim V_\lambda = \frac{n!}{l_1! \dots l_k!} \prod_{i<j} (l_i - l_j), \tag{2.1}$$

with $l_i = \lambda_i + k - i$.

We can also express the dimensions of the V_λ in terms of hook lengths. The hook length $h(i, j)$ of a box (i, j) (i.e., the cell in the ith row and jth column) in a Young diagram is the number of cells that are directly to the right or directly below the box, including the box once. For instance, labeling each box by its hook length we have:

7	6	4	2	1
4	3	1		
2	1			

Theorem 2.1.19 (Hook length formula)**.** *We have*

$$\dim V_\lambda = \frac{n!}{\prod_{j\leq\lambda_i} h(i, j)}.$$

Proof. By (2.1) we have to show that

$$\prod_{p=i+1}^{k} (l_i - l_p) \prod_{j=1}^{\lambda_i} h(i, j) = l_i! \quad \text{for each } i$$

Notice that the product on the left-hand side is a product of $\lambda_i + k - i = l_i$ terms. Thus it is enough to show that these factors are precisely $1, 2, \dots, l_i$. We have

$$l_i - l_k > l_i - l_{k-1} > \cdots > l_i - l_{j+1}$$

and

$$h(i, 1) > h(i, 2) > \cdots > h(i, \lambda_i).$$

Since λ has k parts and λ'_1 is the length of the first column, $\lambda'_1 = k$, and $h(i,1) = \lambda_i$. So each factor is $\leq l_i$. Notice that $h(i,j) = \lambda_i + r - i - j + 1$ for any i, j. Thus it suffices to show that $h(i,j) \neq l_i - l_p$ for any j, p. If $r = \lambda'_j$ then $\lambda_r \geq j$ and $\lambda_{r+1} < j$, so we get

$$h(i,j) - l_i + l_r = (\lambda_j + r - i - j + 1) - (\lambda_i + k - i) + (\lambda_r + k - r) = \lambda_r + 1 - j > 0,$$

and

$$h(i,j) - l_i + l_{r+1} = (\lambda_j + r - i - j + 1) - (\lambda_i + k - i) + (\lambda_{r+1} + k - r - 1) = \lambda_{r+1} - j < 0.$$

So $l_i - l_r < h(i,j) < l_i - l_{r+1}$. □

For the above partition $5+3+2$ of 10, the dimension of the corresponding representation of S_{10} is therefore $\dim V_{(5,3,2)} = \frac{10!}{7\cdot 4\cdot 2\cdot 6\cdot 3\cdot 4\cdot 2} = 450$.

2.2 Schur-Weyl duality

Given a finite-dimensional complex vector space V, consider the nth tensor power space $V^{\otimes n}$. We have two natural actions on this space. There is a natural right action of S_n by permuting the factors, that is, for all $\sigma \in S_n$, we have

$$(v_1 \otimes v_2 \otimes \cdots \otimes v_n)\sigma = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}.$$

Moreover, we have the natural left factorwise action of $\mathrm{GL}(V)$ given by

$$g(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = g(v_1) \otimes g(v_2) \otimes \cdots \otimes g(v_n)$$

for all $g \in \mathrm{GL}(V)$. Notice that these two actions commute with each other.

Recall that a module is said to be simple if it contains exactly two submodules, namely 0 and itself, and it is said to be semisimple if it can be decomposed as a direct sum of simple submodules. We say that an algebra is semisimple if all of its finite-dimensional modules are semisimple. The following is an important theorem on the structure of semisimple algebras.

Theorem 2.2.1. *Let A be a finite-dimensional k-algebra over an algebraically closed field k. Then A has finitely many simple modules U_i up to isomorphism. These simple modules are finite-dimensional. Moreover, A is semisimple if and only if as an algebra*

$$A \simeq \bigoplus_i End(U_i),$$

where U_i runs over all simple A-modules.

Proof. See theorem 3.5.4 of [3]. □

Example 2.2.2. It is perhaps worth noting that theorem 2.2.1 does not hold if the field is not algebraically closed. First, note that, as in the proof of proposition 3.5.8 in [3], if theorem 2.2.1 holds, A semisimple implies $A \cong \bigoplus_i \mathrm{Mat}_{d_i}(k)$ for some d_i. Thus, as an example, we take

$$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$$

where $i^2 = j^2 = -1$, $ij = -ji = k$, the quaternions. For all $x \in \mathbb{H} \setminus \{0\}$ there exists $y \in \mathbb{H}$ such that $xy = yx = 1$, i.e., $\mathbb{H}$ is a division $\mathbb{R}$-algebra. It follows then that $\mathbb{H}$ is a simple $\mathbb{H}$-module and hence $\mathbb{H}$ is a semisimple algebra (recall that A is a semisimple algebra if and only if A is a semisimple A-module). However, $\mathbb{H}$ cannot be isomorphic to $\mathrm{Mat}_{d_1}(\mathbb{R}) \times \cdots \times \mathrm{Mat}_{d_s}(\mathbb{R})$ for any $d_1, \ldots, d_s$ since there must be a $d_i \geq 2$ (as $\mathbb{H}$ is not commutative) and then

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

which contradicts $\mathbb{H}$ being a division algebra.

The following is theorem 5.18.1 in [3].

Theorem 2.2.3 (Double Centralizer Theorem)**.** *Let V be a finite-dimensional k-vector space over an algebraically closed field k, A be a semisimple subalgebra of $End(V)$, and $B = End_A(V)$ (i.e., the algebra of all A-module endomorphisms of V). Then:*

i) B is semisimple.

ii) $A = End_B(V)$ (i.e., the centralizer of the centralizer of A is A).

iii) As a module of $A \otimes B$, we have the decomposition

$$V \simeq \bigoplus_i U_i \otimes W_i$$

where U_i are all the simple modules of A and $W_i := Hom_A(U_i, V)$ are all the simple modules of B.

Proof. Since A is semisimple, by theorem 2.2.1 we have $A \simeq \bigoplus_i \mathrm{End}(U_i)$. By Schur's lemma we get the A-module decomposition

$$V \simeq \bigoplus_i U_i \otimes W_i \tag{2.2}$$

given (from right to left) by $u \otimes f = f(u)$, where $W_i := \mathrm{Hom}_A(U_i, V)$ and $U_i \otimes W_i$ is an A-module by the action $a(u \otimes f) = au \otimes f$ for any $a \in A$. Moreover, we get a module isomorphism

$$\begin{aligned} B = \mathrm{End}_A(V) &\simeq \mathrm{Hom}_A(\bigoplus_i U_i \otimes W_i, V) \simeq \bigoplus_i \mathrm{Hom}_A(U_i \otimes W_i, V) \\ &\overset{\varphi}{\simeq} \bigoplus_i \mathrm{Hom}(W_i, \mathrm{Hom}_A(U_i, V)) = \bigoplus_i \mathrm{End}(W_i). \end{aligned}$$

where φ (from right to left) is given by $f \mapsto (u \otimes w \mapsto f(w)u)$. We claim that W_i are simple B-modules. Let U be a simple A-module and fix a nonzero $u \in U$. Since U is simple and Au is a nonzero submodule of U, any map $f \in \mathrm{Hom}_A(U, V)$ is uniquely determined by its evaluation in u. Let $f, f' \in \mathrm{Hom}_A(U, V)$ with $f(u) = v$ and $f'(u) = v'$ with $v, v' \in V$. Since Av is an invariant subspace of V, we can write $V = (Av) \oplus W$ for a subspace W of V. Define $\theta \colon V \to V$ by $\theta(av) = av'$ for $av \in Av$ and $\theta(w) = w$ for $w \in W$. Notice that $\theta(v) = \theta(1 \cdot v) = 1 \cdot v' = v'$. Hence $\theta \in B$ and it is such that $\theta \circ f = f'$. It follows that B acts transitively on the nonzero maps in $\mathrm{Hom}_A(U, V)$, so that our claim follows.

By theorem 2.2.1, we get that B is semisimple, and so we have i). Now we can repeat the same argument with (2.2) as a decomposition of V into simple B-modules W_i and $U_i = \mathrm{Hom}_B(W_i, V)$. Hence ii) and iii) follow. □

Remark 2.2.4. To complete the previous proof of the Double Centralizer Theorem, one should prove that the module isomorphism $B \simeq \bigoplus_i \mathrm{End}(W_i)$ is in fact an isomorphism of rings and that W_i are all the simple modules of B.

We will now apply the Double Centralizer Theorem to the following situation: V is the nth tensor product of the complex vector space considered above and A is the image of $\mathbb{C}[S_n]$ in $\mathrm{End}(V^{\otimes n})$. Let us now characterize the algebra B. Recall that we may identify $\mathfrak{gl}(V)$ with $\mathrm{End}(V)$ endowed with the Lie bracket given by $[a,b] = ab - ba$ for all $a, b \in \mathrm{End}(V)$. The action of $b \in \mathfrak{gl}(V)$ on $v_1 \otimes v_2 \otimes \cdots \otimes v_n \in V^{\otimes n}$ is defined by

$$b(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = \sum_{i=1}^{n} v_1 \otimes \cdots \otimes bv_i \otimes \cdots \otimes v_n$$

for any $v_1, v_2, \ldots, v_n \in V$.

Lemma 2.2.5. *The image of* $\mathcal{U}(\mathfrak{gl}(V))$ *in* $End(V^{\otimes n})$ *is* $B = End_{S_n}(V^{\otimes n})$.

Proof. We have that the image of b in $\mathrm{End}(V^{\otimes n})$ is

$$\Delta_n(b) := b \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id} + \mathrm{id} \otimes b \otimes \cdots \otimes \mathrm{id} + \cdots + \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes b.$$

Clearly, the image of $\mathfrak{gl}(V)$ and, thus the image of $\mathcal{U}(\mathfrak{gl}(V))$, is contained in B.

By the fundamental theorem on symmetric functions, there exists a polynomial p with rational coefficients in the power sum symmetric polynomials $P_j(T) = t_1^j + t_2^j + \cdots + t_n^j$ such that $p(P_1(t), P_2(t), \ldots, P_n(t)) = t_1 t_2 \ldots t_n$. Then

$$b \otimes b \otimes \cdots \otimes b = p(\Delta_n(b), \Delta_n(b^2), \ldots, \Delta_n(b^n)).$$

Thus, elements of the form $b^{\otimes n}$ for $b \in \mathrm{End}(V)$ are generated by the images of elements in $\mathcal{U}(\mathfrak{gl}(V))$. And since these elements span

$$\mathrm{Sym}^n \mathrm{End}(V) \simeq (\mathrm{End}(V)^{\otimes n})^{S_n} \simeq (\mathrm{End}(V^{\otimes n}))^{S_n} = \mathrm{End}_{S_n}(V^{\otimes n}),$$

where the first and the last isomorphism follow by definition, while the second one is given by the map $\mathrm{End}(V)^{\otimes n} \to \mathrm{End}(V^{\otimes n})$ defined as $f_1 \otimes \cdots \otimes f_n \mapsto (v_1 \otimes \cdots \otimes v_n \mapsto f_1(v_1) \otimes \cdots \otimes f_n(v_n))$. Then the image of $\mathcal{U}(\mathfrak{gl}(V))$ in $\mathrm{End}(V^{\otimes n})$ is B. □

Proposition 2.2.6. *The images of* $\mathbb{C}[S_n]$ *and* $\mathcal{U}(\mathfrak{gl}(V))$ *in* $End(V^{\otimes n})$ *are centralizers of each other.*

Proof. Since $\mathbb{C}[S_n]$ is semisimple and the fact that homomorphic images of semisimple rings are semisimple, A is semisimple. By the Double Centralizer Theorem 2.2.3, we are done. □

Lemma 2.2.7. *The span of the image of* $GL(V)$ *in* $End(V^{\otimes n})$ *is* B.

Proof. Since $\mathrm{GL}(V)$ commutes with S_n, the image of $\mathrm{GL}(V)$, and thus its span, is contained in $B = \mathrm{End}_A(V^{\otimes n})$.

Conversely, let $b \in \mathrm{End}(V)$ and B' be the span of the image of the elements $g^{\otimes n}$ for $g \in \mathrm{GL}(V)$. For all but finitely many $t \in \mathbb{C}$, $tI + b$ is invertible. Thus the polynomial $(tI + b)^{\otimes n}$ is in B' for all but finitely many t. But this implies that this holds for all t. In particular, for $t = 0$, we have that $b^{\otimes n} \in B'$. As in the previous lemma, these elements span $B = \mathrm{End}_A(V^{\otimes n})$. □

Therefore, by the Double Centralizer Theorem 2.2.3, we have the following theorem.

Theorem 2.2.8 (Schur-Weyl duality for $\mathrm{GL}(V)$)**.** *We have a decomposition*

$$V^{\otimes n} \simeq \bigoplus_{|\lambda|=n} V_\lambda \otimes \mathbb{S}_\lambda V,$$

as a representation of $S_n \otimes GL(V)$ *where* V_λ *runs through all the irreducible representations of* S_n *and each* $\mathbb{S}_\lambda V := Hom_{S_n}(V_\lambda, V^{\otimes n})$ *is an irreducible representation of* $GL(V)$ *or is zero.*

Let us calculate the characters of the representations $\mathbb{S}_\lambda V$. For a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of n, and $N \geq k$, set

$$D_\lambda(x) = \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) \prod_{i=1}^{N} x_{\sigma(i)}^{\lambda_i + N - i} = |x_j^{\lambda_i + N - i}|.$$

for independent variables $x_1, \dots, x_N$ and $x = (x_1, \dots, x_N)$. These are alternating polynomials by properties of the determinant and therefore they must be divisible by the Vandermonde determinant $\Delta(x) = |x_j^{N-i}| = \prod_{i<j}(x_i - x_j)$. The Schur polynomials are defined as the ratio

$$S_\lambda(x) := \frac{D_\lambda(x)}{\Delta(x)} = \frac{|x_j^{\lambda_i+N-i}|}{|x_j^{N-i}|}.$$

These are symmetric functions because the numerator and denominator are both alternating, and a polynomial since all alternating polynomials are divisible by the Vandermonde determinant.

Proposition 2.2.9. *We have*

$$\prod_j (x_1^j + \cdots + x_N^j)^{i_j} = \sum_{\lambda : k \leq N} \chi_\lambda(C_{\mathbf{i}}) S_\lambda(x).$$

Proof. Write $\Delta(x) \prod_j (x_1^j + \cdots + x_N^j)^{i_j} = \sum_{k_1, \dots, k_N \geq 0} C_{k_1, \dots, k_N} x_1^{k_1} \dots x_N^{k_N}$. Since $\Delta(x) \prod_j (x_1^j + \cdots + x_N^j)^{i_j}$ is alternating, the coefficients $C_{k_1, \dots, k_N}$ are alternating in $\{k_1, \dots, k_N\}$. In particular, $C_{k_1, \dots, k_N} = 0$ when $k_m = k_n$ for some $m \neq n$. If that is not the case, we can write $(k_j)_j = \sigma(\lambda_j + N - j)_j$ for a unique partition λ of n and a unique $\sigma \in S_N$. This gives

$$\Delta(x) \prod_j (x_1^j + \cdots + x_N^j)^{i_j} = \sum_\lambda C_{\lambda_1 + N - 1, \dots, \lambda_N} \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) \prod_{j=1}^{N} x_{\sigma(j)}^{\lambda_j + N - j}.$$

Now the identity follows from the Frobenius character formula. □

Let us compute certain special values of Schur polynomials. Namely, using the Vandermonde determinant, it follows that

$$S_\lambda(1, t, \dots, t^{N-1}) = \prod_{1 \leq i < j \leq N} \frac{t^{\lambda_i + M - i} - t^{\lambda_j + N - j}}{t^{N-i} - t^{N-j}}.$$

As $t \to 1$, this gives

$$S_\lambda(1, 1, \dots, 1) = \prod_{1 \leq i < j \leq N} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

Let $\dim V = N$, $g \in \mathrm{GL}(V)$, and $x_1, \dots, x_N$ be the eigenvalues of g on V. To find the character $\chi_{\mathbb{S}_\lambda V}(g)$, we must compute $\operatorname{tr}_{V^{\otimes n}}(g\sigma)$, where $\sigma \in S_n$. Suppose $\sigma \in \mathrm{C}_{\mathbf{i}}$. Choosing a basis $(e_k)_{k=1}^N$ in V, we write $ge_k = \sum_j g_{jk} e_j$. Then,

$$(g\sigma)(e_{k_1} \otimes \cdots \otimes e_{k_n}) = \sum_{j_1, \dots, j_n = 1}^{N} g_{j_1 k_{\sigma^{-1}(1)}} \cdots g_{j_n k_{\sigma^{-1}(n)}} e_{j_1} \otimes \cdots \otimes e_{j_n}.$$

It follows that

$$\operatorname{tr}_{V^{\otimes n}}(g\sigma) = \sum_{j_1, \dots, j_n = 1}^{N} g_{j_{\sigma(1)} j_1} \cdots g_{j_{\sigma(n)} j_n}.$$

Since

$$\operatorname{tr}(g^k) = \sum_{j_1,\ldots,j_k} g_{j_1 j_2} g_{j_2 j_3} \cdots g_{j_{k-1} j_k} g_{j_k j_1},$$

this can be written as

$$\operatorname{tr}_{V^{\otimes n}}(g\sigma) = \prod_j \operatorname{tr}_V(g^j)^{i_j} = \prod_j P_j(x)^{i_j}.$$

On the other hand, by the Schur-Weyl duality

$$\operatorname{tr}_{V^{\otimes n}}(g\sigma) = \sum_\lambda \chi_\lambda(\mathrm{C}_{\mathbf{i}}) \operatorname{tr}_{\mathbb{S}_\lambda V}(g).$$

Comparing this to proposition 2.2.9 and using linear independence of columns of the character table of S_n, we obtain

Theorem 2.2.10 (Weyl character formula for S_n)**.** *Let* $\dim V = N$*, let* $\lambda = (\lambda_1, \ldots, \lambda_k)$ *be a partition of* n *and* $g \in GL(V)$ *have eigenvalues* $x_1, \ldots, x_N$*. Then the representation* $\mathbb{S}_\lambda V = 0$ *if and only if* $N < k$*. Moreover, if* $N \geq k$*, the character of* $\mathbb{S}_\lambda V$ *is the Schur polynomial* $S_\lambda(x)$*, i.e.,*

$$tr_{\mathbb{S}_\lambda V}(g) = S_\lambda(x_1, \ldots, x_N),$$

where on the right-hand side we complete λ *to an* N*-dimensional vector by defining* $(\lambda_1, \ldots, \lambda_N) = (\lambda_1, \ldots, \lambda_k, 0, \ldots, 0)$*. In particular, the dimension of* $\mathbb{S}_\lambda V$ *is given by the formula*

$$\dim \mathbb{S}_\lambda V = \prod_{1 \leq i < j \leq N} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

This shows that irreducible representations of $\mathrm{GL}(V)$ which occur in $V^{\otimes n}$ for some n are labeled by Young diagrams with any number of squares but at most $N = \dim V$ rows.

We can now define a covariant functor $\mathbb{S}_\lambda \colon \mathrm{FinVect} \to \mathrm{FinVect}$ in the following way: if $f \colon V \to W$ is a linear map, we define $\mathbb{S}_\lambda f \colon \mathbb{S}_\lambda V \to \mathbb{S}_\lambda W$ by $(\mathbb{S}_\lambda f)(\psi) = f^{\otimes n} \circ \psi$. It can be verified that

$$\mathbb{S}_\lambda(f \circ g) = (\mathbb{S}_\lambda f) \circ (\mathbb{S}_\lambda g) \quad \text{and} \quad \mathbb{S}_\lambda \mathrm{id}_V = \mathrm{id}_{\mathbb{S}_\lambda}.$$

We call the covariant functor $\mathbb{S}_\lambda$ the Schur functor of λ. Notice that all representations W of S_n are self-dual since

$$\chi_{W^*}(g) = \overline{\chi_W(g)} = \chi_W(g^{-1}) = \chi_W(g)$$

for all $g \in S_n$ because g and g^{-1} have the same cycle type and, thus, are in the same conjugacy class, so that $W \simeq W^*$. Thus we get the following more constructive description of the Schur functor.

$$\begin{aligned}\mathbb{S}_\lambda V &= \mathrm{Hom}_{S_n}(V_\lambda, V^{\otimes n}) \simeq (V_\lambda)^* \otimes_{\mathbb{C}[S_n]} V^{\otimes n} \simeq V_\lambda \otimes_{\mathbb{C}[S_n]} V^{\otimes n} \\ &= \mathbb{C}[S_n] c_\lambda \otimes_{\mathbb{C}[S_n]} V^{\otimes n} = \mathbb{C}[S_n] \otimes_{\mathbb{C}[S_n]} V^{\otimes n} c_\lambda \simeq V^{\otimes n} c_\lambda.\end{aligned}$$

So Schur-Weyl duality tells us that

$$V^{\otimes n} \simeq \bigoplus_{|\lambda| = n} (\mathbb{S}_\lambda V)^{\oplus f_\lambda}$$

as a representation of $\mathrm{GL}(V)$ where $f_\lambda = \dim V_\lambda$.

Chapter 3

Algebraic Representations of $\mathrm{GL}_n\mathbb{C}$

This chapter describes all irreducible algebraic representations of $\mathrm{GL}_n\mathbb{C}$. The aim of section 3.1 is to find a procedure to study the structure of a general finite-dimensional representation of a semisimple Lie algebra and to obtain more information about the structure of the Lie algebra itself. In section 3.2 we will apply this general paradigm to the Lie algebras $\mathfrak{sl}_n\mathbb{C}$. Finally, in section 3.3 we will describe all the polynomial representations of $\mathrm{GL}_n\mathbb{C}$.

In the first section, we will follow a "third" way mixing the root systems construction developed in [8] and the algorithm illustrated in [5]. The last two sections reproduce the main concepts in lecture 15 of [5].

3.1 Analyzing Semisimple Lie Algebras in General

A subspace $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is called an ideal if $[x, y] \in \mathfrak{h}$ for all $x \in \mathfrak{h}, y \in \mathfrak{g}$. Clearly 0 and $\mathfrak{g}$ itself are ideals of $\mathfrak{g}$. Another example is the center $Z(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, y] = 0 \text{ for all } y \in \mathfrak{g}\}$. We say that $\mathfrak{g}$ is abelian if and only if $Z(\mathfrak{g}) = \mathfrak{g}$. Notice that $\mathfrak{g}$ is abelian if all brackets are zero. A Lie algebra $\mathfrak{g}$ is said to be simple if $\dim \mathfrak{g} > 1$ and it contains no ideals except itself and 0. Further, we say that $\mathfrak{g}$ is solvable if there exists a sequence of Lie subalgebras $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_k = 0$, such that $\mathfrak{g}_{i+1}$ is an ideal in $\mathfrak{g}_i$ and $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is abelian.

Next we assemble a few simple observations about solvability.

Remark 3.1.1. Let $\mathfrak{g}$ be a Lie algebra.

i) If $\mathfrak{g}$ is solvable, then so are all subalgebras and homomorphic images of $\mathfrak{g}$.

ii) If $\mathfrak{h}$ is a solvable ideal of $\mathfrak{g}$ such that $\mathfrak{g}/\mathfrak{h}$ is solvable, then $\mathfrak{g}$ itself is solvable.

iii) If $\mathfrak{h}, \mathfrak{l}$ are solvable ideals of $\mathfrak{g}$, then so is $\mathfrak{h} + \mathfrak{l}$.

It follows that the sum of all solvable ideals in $\mathfrak{g}$ is a maximal solvable ideal, called the radical of $\mathfrak{g}$ and denoted $\mathrm{Rad}(\mathfrak{g})$. $\mathfrak{g}$ is called semisimple if $\mathrm{Rad}(\mathfrak{g}) = 0$. Notice that the quotient $\mathfrak{g}/\mathrm{Rad}(\mathfrak{g})$ is semisimple. Now we state without proof Lie's theorem.

Theorem 3.1.2 (Lie's Theorem)**.** *Let V be a nonzero finite-dimensional complex vector space and $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a complex solvable Lie algebra. Then there exists a nonzero vector $v \in V$ that is an eigenvector of x for all $x \in \mathfrak{g}$.*

Proof. See theorem 9.11 in [5]. □

We define the Killing form as the bilinear map $\kappa\colon \mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ such that $\kappa(x,y)=\operatorname{tr}(\operatorname{ad}x\circ\operatorname{ad}y)$ for $x,y\in\mathfrak{g}$. Notice that κ is also associative, in the sense that $\kappa([x,y],z)=\kappa(x,[y,z])$. One can prove that a Lie algebra $\mathfrak{g}$ is semisimple if and only if its Killing form is non-degenerate.

Throughout the remaining part of this section $\mathfrak{g}$ denotes a nonzero complex semisimple Lie algebra. Recall that $x\in\mathfrak{g}$ is said to be semisimple if $\operatorname{ad}x$ is diagonalizable (see example 1.5.1). Fix a maximal toral subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, i.e., a maximal subalgebra consisting of semisimple elements. In general, there is no reason for such a subalgebra to exist. However, in the case of $\mathfrak{g}$ being a complex semisimple Lie algebra it always exists. One can prove that $\mathfrak{h}$ is abelian, so that $\operatorname{ad}_{\mathfrak{g}}\mathfrak{h}$ is a commuting family of semisimple endomorphisms of $\mathfrak{g}$. It follows that $\mathfrak{g}$ is the direct sum of the subspaces $\mathfrak{g}_\alpha=\{x\in\mathfrak{g}\mid [h,x]=\alpha(h)x \text{ for all } h\in\mathfrak{h}\}$, with $\alpha\in\mathfrak{h}^*$. One can see that $\mathfrak{g}_0$ is simply $\mathfrak{h}$. The set of all nonzero $\alpha\in\mathfrak{h}^*$ for which $\mathfrak{g}_\alpha\neq 0$ is denoted by R; the elements of R are called the roots of $\mathfrak{g}$ relative to $\mathfrak{h}$ and are in finite number. Thus we have a root space decomposition (or Cartan decomposition):

$$\mathfrak{g}=\mathfrak{h}\oplus\Big(\bigoplus_{\alpha\in R}\mathfrak{g}_\alpha\Big).$$

Since the restriction of κ to $\mathfrak{h}$ is non-degenerate, we can identify $\mathfrak{h}$ with $\mathfrak{h}^*$ in the following way: to $\phi\in\mathfrak{h}^*$ corresponds the unique element $t_\phi\in\mathfrak{h}$ satisfying $\phi(h)=\kappa(t_\phi,h)$ for all $h\in\mathfrak{h}$. In particular, R corresponds to the subset $\{t_\alpha\mid\alpha\in R\}$ of $\mathfrak{h}$. This allows us to extend the Killing form to $\mathfrak{h}^*$: if $\alpha,\beta\in\mathfrak{h}^*$, set $\kappa(\alpha,\beta)=\kappa(t_\alpha,t_\beta)$.

We can now state propositions 8.3 and 8.4 in [8].

Proposition 3.1.3 (Orthogonality properties)**.** *We have:*

i) *R generates a lattice $\Lambda_R\subset\mathfrak{h}^*$ of rank equal to the dimension of $\mathfrak{h}$.*

ii) *If $\alpha\in R$, the only scalar multiples of α which are roots are α and $-\alpha$.*

iii) *Let $\alpha\in R$, $x\in\mathfrak{g}_\alpha$, $y\in\mathfrak{g}_{-\alpha}$. Then $[x,y]=\kappa(x,y)t_\alpha$.*

iv) *If $\alpha\in R$, then $[\mathfrak{g}_\alpha,\mathfrak{g}_{-\alpha}]$ is one dimensional, with basis t_α.*

v) *$\alpha(t_\alpha)=\kappa(t_\alpha,t_\alpha)\neq 0$, for $\alpha\in R$.*

vi) *If $\alpha\in R$ and x_α is any nonzero element of $\mathfrak{g}_\alpha$, then there exists $y_\alpha\in\mathfrak{g}_{-\alpha}$ such that $x_\alpha,y_\alpha,h_\alpha=[x_\alpha,y_\alpha]$ span a three dimensional simple subalgebra $\mathfrak{s}_\alpha$ of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ via $x_\alpha\mapsto\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $y_\alpha\mapsto\begin{pmatrix}0&0\\1&0\end{pmatrix}$, $h_\alpha\mapsto\begin{pmatrix}1&0\\0&-1\end{pmatrix}$.*

vii) *$h_\alpha=\frac{2t_\alpha}{\kappa(t_\alpha,t_\alpha)}$ and $h_\alpha=-h_{-\alpha}$.*

Proposition 3.1.4 (Integrality properties)**.** *We have:*

i) *$\alpha\in R$ implies $\dim\mathfrak{g}_\alpha=1$. In particular, $\mathfrak{s}_\alpha=\mathfrak{g}_\alpha\oplus\mathfrak{g}_{-\alpha}\oplus\mathfrak{h}_\alpha$ (where $\mathfrak{h}_\alpha=[\mathfrak{g}_\alpha,\mathfrak{g}_{-\alpha}]$), and for any given nonzero $x_\alpha\in\mathfrak{g}_\alpha$, there exists a unique $y_\alpha\in\mathfrak{g}_{-\alpha}$ satisfying $[x_\alpha,y_\alpha]=h_\alpha$.*

ii) *If $\alpha,\beta\in R$, then $\beta(h_\alpha)\in\mathbb{Z}$, and $\beta-\beta(h_\alpha)\alpha\in R$. (The numbers $\beta(h_\alpha)$ are called Cartan integers.)*

iii) *If $\alpha,\beta,\alpha+\beta\in R$, then $[\mathfrak{g}_\alpha,\mathfrak{g}_\beta]=\mathfrak{g}_{\alpha+\beta}$.*

iv) *Let $\alpha,\beta\in R$, $\beta\neq\pm\alpha$. Let r,q be (respectively) the largest integers for which $\beta-r\alpha$, $\beta+q\alpha$ are roots. Then all $\beta+i\alpha\in R$ $(-r\le i\le q)$, and $\beta(h_\alpha)=r-q$.*

v) *$\mathfrak{g}$ is generated (as Lie algebra) by the root spaces $\mathfrak{g}_\alpha$.*

In particular, we can pick a basis $x_\alpha \in \mathfrak{g}_\alpha, y_\alpha \in \mathfrak{g}_{-\alpha}$ and $h_\alpha \in \mathfrak{h}_\alpha$ satisfying the standard commutation relations for $\mathfrak{sl}_2\mathbb{C}$; x_α and y_α are not determined by this, but h_α is, being uniquely characterized by the requirements that $h_\alpha \in \mathfrak{h}_\alpha$ and $\alpha(h_\alpha) = 2$.

Recall that the subalgebra $\mathfrak{h}$ acts diagonally on any representation of $\mathfrak{g}$. So, if we consider the eigenspaces of the action of $\mathfrak{h}$ on V, we get the following decomposition:

$$V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha$$

where $h(v) = \alpha(h) \cdot v$ for all $h \in \mathfrak{h}$ and $v \in V_\alpha$. Each index $\alpha \in \mathfrak{h}^*$ is called weight and the dimension of V_α is said the multiplicity of the relative weight. Let us analyse the action of every $\mathfrak{g}_\beta$: for any root β, $h \in \mathfrak{h}$, $x \in \mathfrak{g}_\beta$ and $v \in V_\alpha$

$$h(x(v)) = x(h(v)) + [h, x](v) = x(\alpha(h)(v)) + \beta(h)x(v) = (\alpha(h) + \beta(h))x(v),$$

so that $\mathfrak{g}_\beta \colon V_\alpha \to V_{\alpha+\beta}$.

Finally, define the weight lattice $\Lambda_W = \{\beta \in \mathfrak{h}^* \mid \beta(\mathfrak{h}_\alpha) \in \mathbb{Z} \quad \text{for all } \alpha \in R\}$. By ii) of proposition 3.1.4, we have $R \subset \Lambda_W$, and thus $\Lambda_R \subset \Lambda_W$.

Fix a root $\alpha \in \mathfrak{h}^*$. Consider the hyperplane

$$\Omega_\alpha = \{\beta \in \mathfrak{h}^* \mid \beta(h_\alpha) = 0\}$$

and the subspace $\mathbb{C} \cdot \alpha$ generated by α in $\mathfrak{h}^*$. Define $W_\alpha \colon \mathfrak{h}^* \to \mathfrak{h}^*$ as

$$W_\alpha(\beta) = \beta - \frac{2\beta(h_\alpha)}{\alpha(h_\alpha)}\alpha = \beta - \beta(h_\alpha)\alpha.$$

Let $\mathfrak{W}$ be the group generated by these operators, called the Weyl group of the Lie algebra $\mathfrak{g}$.

Now we want to define a notion of direction in $\mathfrak{h}^*$. We say that a subset B of R is called a base if B is a basis of $\mathfrak{h}^*$ and each root β can be written as $\beta = \sum_{\alpha \in B} k_\alpha \alpha$ with integral coefficients k_α all non-negative or all non-positive. The roots in B are called simple. We define the height of a root (relative to B) as $\operatorname{ht}\beta = \sum_{\alpha \in B} k_\alpha$. If all $k_\alpha \geq 0$ (resp. all $k_\alpha \leq 0$), we call β positive (resp. negative). Denote R^+ (resp. R^-) the set of positive (resp. negative) roots relative to B. Clearly, we have $R = R^+ \cup R^-$ and $R^- = -R^+$.

In fact, one can prove that a base of R always exists (for details see theorem 10.1 of [8]). Now we are in the position to state theorem 10.3 of [8].

Proposition 3.1.5. *Let B be a base of R.*

i) If B' is another base of R, then $W(B') = B$ for some $W \in \mathfrak{W}$ (so $\mathfrak{W}$ acts transitively on bases).

ii) If α is a root, there exists $W \in \mathfrak{W}$ such that $W(\alpha) \in B$.

iii) $\mathfrak{W}$ is generated by the W_α ($\alpha \in B$).

iv) If $W(B) = B$, $W \in \mathfrak{W}$, then $W = 1$ (so $\mathfrak{W}$ acts simply transitively on bases).

Now let V be any finite-dimensional representation of $\mathfrak{g}$. A nonzero vector $v \in V_\alpha$ killed by all $\mathfrak{g}_\beta$ for all $\beta \in R^+$ is called highest weight vector of weight α in V. We have

Proposition 3.1.6. *For any semisimple complex Lie algebra $\mathfrak{g}$,*

i) every finite-dimensional representation V of $\mathfrak{g}$ possesses a highest weight vector;

ii) the subspace $W \subset V$ generated by the images of a highest weight vector v under successive applications of root spaces $\mathfrak{g}_\beta$ for $\beta \in R^-$ is an irreducible subrepresentation;

iii) a finite-dimensional irreducible representation V possesses a unique highest weight vector up to scalars.

Proof. See proposition 14.13 in [5].

□

The weight α of the highest weight vector of an irreducible representation is called the highest weight of that representation. Define the Weyl chamber $\mathcal{W}$ associated to B as the set of roots such that $\alpha(h_\gamma) \geq 0$ for every $\gamma \in R^+$. Now we may state the following fundamental existence and uniqueness theorem.

Theorem 3.1.7. *For any α in the intersection of the Weyl chamber $\mathcal{W}$ associated to B with the weight lattice Λ_W, there exists a unique irreducible, finite-dimensional representation Γ_α of $\mathfrak{g}$ with highest weight α; this gives a bijection between $\mathcal{W} \cap \Lambda_W$ and the set of irreducible representations of $\mathfrak{g}$. The weights of Γ_α will consist of those elements of the weight lattice congruent to α modulo the root lattice Λ_R and lying in the convex hull of the set of points in $\mathfrak{h}^*$ conjugate to α under the Weyl group.*

Uniqueness-proof. See theorem 14.18 in [5] for the proof of uniqueness. We will see the explicit construction of the existence part for the cases we will be interested in. □

Finally, we define the fundamental weights as the elements $\omega_1, \ldots, \omega_n \in \mathfrak{h}^*$ such that $\omega_i(h_{\alpha_j}) = \delta_{i,j}$, where $\alpha_1, \ldots, \alpha_n$ are the simple roots (in some base). Every highest weight vector can be expressed uniquely as a non-negative integral linear combination of fundamental weights. We will write $\Gamma_{a_1,\ldots,a_n}$ for the irreducible representation with highest weight $a_1\omega_1 + \cdots + a_n\omega_n$.

3.2 Representations of $\mathfrak{sl}_n\mathbb{C}$

In this section, we will deduce the existence part of theorem 3.1.7 for $\mathfrak{sl}_n\mathbb{C}$. Define the special linear algebra $\mathfrak{sl}_n\mathbb{C}$ as the set of linear transformations of $\mathrm{End}(V)$ with trace zero. Since $\mathrm{tr}\,(xy) = \mathrm{tr}\,(yx)$ and $\mathrm{tr}\,(x+y) = \mathrm{tr}\,(x) + \mathrm{tr}\,(y)$, $\mathfrak{sl}_n\mathbb{C}$ is a Lie subalgebra of $\mathfrak{gl}(V)$.

First, we have to find a maximal toral subalgebra $\mathfrak{h}$ of $\mathfrak{sl}_n\mathbb{C}$. Writing $h_i = E_{ii}$ where E_{ij} is the endomorphism of $\mathbb{C}^n$ sending e_j to e_i and sending all the other e_k to zero, we consider the subalgebra of diagonal matrices

$$\mathfrak{h} = \{a_1h_1 + a_2h_2 + \cdots + a_nh_n \mid a_1 + a_2 + \cdots + a_n = 0\}.$$

Clearly, $\mathfrak{h}$ is toral. Let us prove that $\mathfrak{h}$ is also maximal. Let $\mathfrak{h}' \supset \mathfrak{h}$ be another toral subalgebra. Then for all $h' \in \mathfrak{h}'$ we get $\mathrm{ad}(\mathfrak{h})(h') = 0$. We have

$$0 = \mathrm{ad}(a_1h_1 + a_2h_2 + \cdots + a_nh_n)(h') = -\sum_{i,j}(a_i - a_j)h'_{i,j}(E_{ij})$$

if and only if $h' \in \mathfrak{h}$. Hence $\mathfrak{h}$ is a maximal toral subalgebra.

The dual space can be written as

$$\mathfrak{h}^* = \mathbb{C}\{l_1, l_2, \ldots, l_n\}/(l_1 + l_2 + \cdots + l_n = 0),$$

where $l_i(h_j) = \delta_{i,j}$. Let us see how the diagonal matrices act on $\mathfrak{sl}_n\mathbb{C}$: we have

$$\begin{aligned}\operatorname{ad}(a_1h_1 + a_2h_2 + \cdots + a_nh_n)(E_{ij}) &= a_i[h_i, E_{ij}] + a_j[h_j, E_{ij}] \\ &= a_i(h_i(E_{ij})) + a_j(E_{ij}(-h_j)) \\ &= (a_i - a_j)(E_{ij}).\end{aligned}$$

Thus E_{ij} is an eigenvector with eigenvalue $l_i - l_j$, and the roots of $\mathfrak{sl}_n\mathbb{C}$ are precisely these pairwise differences of the l_i. Thus the root lattice of $\mathfrak{sl}_n\mathbb{C}$ can be described as

$$\Lambda_R = \left\{\sum a_i l_i \mid a_i \in \mathbb{Z}, \sum a_i = 0\right\} / \left(\sum l_i = 0\right).$$

Our next goal is to find the subalgebras $\mathfrak{s}_\alpha$. The root space $\mathfrak{g}_{l_i - l_j}$ is generated by E_{ij}, so $\mathfrak{s}_{l_i - l_j}$ is generated by E_{ij}, E_{ji} and $[E_{ij}, E_{ji}] = h_i - h_j$. The action of $h_i - h_j$ on E_{ij} has eigenvalue $(l_i - l_j)(h_i - h_j) = 2$, by the same computation above, so that $h_{l_i - l_j} = h_i - h_j$.

A linear functional $\beta = \sum a_i l_i \in \mathfrak{h}^*$ has integral values on all $\mathfrak{h}_\alpha$ if and only if all the a_i are congruent to one another modulo $\mathbb{Z}$. Since $\sum l_i = 0$ in $\mathfrak{h}^*$, we have

$$\Lambda_W = \mathbb{Z}\{l_1, \dots, l_n\} / \left(\sum l_i = 0\right).$$

The Weyl group is generated by the reflections in the hyperplane perpendicular to the root $l_i - l_j$. In particular, $W_{l_i - l_j}$ will exchange l_i and l_j in $\mathfrak{h}^*$ and leave the other l_k alone. The Weyl group $\mathfrak{W}$ is then the group S_n, acting on the generators l_i of $\mathfrak{h}^*$.

Finally, we choose a base of R and describe the corresponding Weyl chambers. We have $R = \{l_i - l_j \mid i \neq j\}$. The vectors $l_i - l_{i+1}$ $(1 \leq i \leq n-1)$ are independent, and $l_i - l_j = (l_i - l_{i+1}) + (l_{i+1} - l_{i+2}) + \cdots + (l_{j-1} - l_j)$ if $i < j$, which shows that $B = \{l_i - l_{i+1} \mid 1 \leq i \leq n-1\}$ form a base of R. The corresponding ordering of the roots will then be $R^+ = \{l_i - l_j \mid i < j\}$ and $R^- = \{l_i - l_j \mid j < i\}$. Thus the simple negative roots for this ordering are the roots $l_{i+1} - l_i$. The Weyl chamber associated to this ordering is then $\mathcal{W} = \{\sum a_i l_i \mid a_1 \geq a_2 \geq \cdots \geq a_n\}$.

Clearly, the fundamental weights (relative to B) are $\omega_i = l_1 + l_2 + \cdots + l_i$ for $i = 1, \dots, n-1$. Notice that the intersection of the Weyl chamber $\mathcal{W}$ with the weight lattice Λ_W is in fact a free semigroup $\mathbb{N}^{n-1}$ generated by the fundamental weights $\omega_i = l_1 + \cdots + l_i$. Thus, for an arbitrary $(n-1)$-tuple of natural numbers $(a_1, \dots, a_{n-1}) \in \mathbb{N}^{n-1}$ we will denote by $\Gamma_{a_1,\dots,a_{n-1}}$ the irreducible representation of $\mathfrak{sl}_n\mathbb{C}$ with highest weight $a_1l_1 + a_2(l_1 + l_2) + \cdots + a_{n-1}(l_1 + \cdots + l_{n-1}) = (a_1 + \cdots + a_{n-1})l_1 + (a_2 + \cdots + a_{n-1})l_2 + \cdots + a_{n-1}l_{n-1}$.

Let $1 \leq k \leq n$ and let V be the standard representation of $\mathfrak{sl}_n\mathbb{C}$. We want to prove that the exterior power $\bigwedge^k V$ contains a highest weight vector of weight $l_1 + \cdots + l_k$. Recall that the standard basis vectors e_i of $\mathbb{C}^n$ are eigenvectors for the action of $\mathfrak{h}$, with eigenvalues l_i. Consider the vector $e_1 \wedge e_2 \wedge \cdots \wedge e_k$, and let α be a positive root. Hence $\alpha = l_i - l_j$ for $i < j$, so it takes e_i to e_j for $j \leq i$. Thus, the action of $\mathfrak{g}_\alpha$ must either take some e_i in $e_1 \wedge e_2 \wedge \cdots \wedge e_k$ to 0, or to some e_j already in the term, and so must be zero. Thus, $e_1 \wedge e_2 \wedge \cdots \wedge e_k$ is a highest weight vector of weight $\omega_k = l_1 + \cdots + l_k$ in $\bigwedge^k V$.

Now, since every vector in $\mathcal{W} \cap \Lambda_W$ is a non-negative integral linear combination of the vectors $\omega_k = l_1 + \cdots + l_k$ and $\bigwedge^k V$ has highest weight vectors with weights $\omega_k = l_1 + \cdots + l_k$, it follows that the tensor product

$$\operatorname{Sym}^{a_1} V \otimes \operatorname{Sym}^{a_2}(\bigwedge^2 V) \otimes \cdots \otimes \operatorname{Sym}^{a_{n-1}}(\bigwedge^{n-1} V)$$

contains a highest weight vector with weight $a_1l_1 + a_2(l_1 + l_2) + \cdots + a_{n-1}(l_1 + \cdots + l_{n-1})$, and hence a copy of the irreducible representation $\Gamma_{a_1,\dots,a_{n-1}}$ with this highest weight.

3.3 Representations of $\mathrm{GL}_n\mathbb{C}$

Let us introduce some notations. A finite-dimensional representation W of $\mathrm{GL}(V)$ is algebraic if, given the corresponding map $\rho\colon \mathrm{GL}(V) \to \mathrm{GL}(W)$, the coordinates of $\rho(A)$ are rational functions of the coordinates of $A \in \mathrm{GL}(V)$. Any such rational function is, in fact, in $\mathbb{C}[a_{ij}][1/\det A]$. It will turn out that the Schur functors $\mathbb{S}_\lambda$, seen in chapter 2, only give us representations which are polynomial, that is, the coordinates of $\rho(A)$ are polynomials of the coordinates of $A \in \mathrm{GL}(V)$. Thus, we could never hope to get the algebraic representation $\det^{-1}\colon \mathrm{GL}(V) \to \mathrm{GL}(\mathbb{C})$ where $\det^{-1}(A) = 1/\det A$. But this is essentially it.

Let $V = \mathbb{C}^n$ be the standard representation for $\mathrm{GL}_n\mathbb{C}$ and let $\det^k$ denote the one-dimensional representation of $\mathrm{GL}_n\mathbb{C}$ given by the k-th power of the determinant for $k \in \mathbb{Z}$. Notice that

$$\det{}^1 \cong \bigwedge^n V, \quad \det{}^k \cong (\det{}^1)^{\otimes k} \text{ if } k \geq 0, \quad \det{}^k \cong (\det{}^{-k})^* \text{ if } k \leq 0.$$

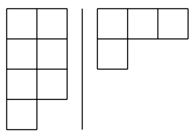
Thus, if $k \geq 0$, $\det^k = (\bigwedge^n V)^{\otimes k}$ and $\det^{-k}$ is the dual $(\det^k)^*$. Let V be the standard representation of $\mathrm{GL}_n\mathbb{C}$. For any partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of n, consider the representation $\psi_\lambda := \mathbb{S}_\lambda V$ of $\mathrm{GL}_n\mathbb{C}$. We saw in chapter 2 that $\mathbb{S}_\lambda V$ is an irreducible representation of $\mathrm{GL}_n\mathbb{C}$. We claim the following

Proposition 3.3.1.

$$\psi_{\lambda_1+k,\ldots,\lambda_n+k} = \psi_{\lambda_1,\ldots,\lambda_n} \otimes \det{}^k.$$

Note that in lecture 15 of [5] the proof of the previous proposition is omitted. Let us develop it in details.

First we need a slightly technical result. Let V be an m-dimensional complex vector space and let n be a positive integer. Take a partition λ of n such that its Young diagram is partitioned into two non-empty parts, say of sizes i and $j = n - i$, by a vertical wall. For example, if $\lambda = (5, 3, 2, 1)$ is a partition of 11 we could have



with $i = 7$ and $j = 4$. Let t_λ be a tableau whose entries in the left-hand part are $\{1, 2, \ldots, i\}$. Let μ be the corresponding partition of i, so that $t_\mu = t_{\lambda|\mu}$. Let ν be the partition of j corresponding to the right-hand part, with tableau t_ν. This is a map

$$t_\nu\colon \{\text{Young diagram of } \nu\} \to \{1', \ldots, j'\} \quad \text{where } k' = k + i \text{ for } 1 \leq k \leq j.$$

Lemma 3.3.2. *There exists a surjective* $\mathbb{C}[GL(V)]$*-map*

$$(V^{\otimes i} c_\mu) \otimes (V^{\otimes j} c_\nu) \to V^{\otimes n} c_\lambda. \tag{3.1}$$

Proof. Let $S_i = \mathrm{Aut}\{1, \ldots, i\}$ and $S_j = \mathrm{Aut}\{1', \ldots, j'\}$. Since S_i and S_j are embedded in S_n and disjoint, we regard $\mathbb{C}[S_i]$ and $\mathbb{C}[S_j]$ as subsets of $\mathbb{C}[S_n]$ which commute. Also

$$Q_\lambda = Q_\mu \times Q_\nu \quad \text{and} \quad H = P_\mu \times P_\nu \leqslant P_\lambda$$

and H permutes each side of the wall. Take a transversal $P_\lambda = \bigcup_i g_i H$. Then

$$c_\lambda = a_\lambda b_\lambda = \sum_i g_i a_\mu a_\nu b_\mu b_\nu = \sum_i g_i c_\mu c_\nu.$$

So,

$$c_\lambda c_\mu c_\lambda = \sum_i g_i c_\mu^2 c_\nu^2 = c_\lambda k$$

where $k = \frac{i!j!}{\dim V_\mu \cdot \dim V_\nu}$ by lemma 2.1.8. We thus have a $\mathbb{C}[\mathrm{GL}(V)]$-map $V^{\otimes i} \otimes V^{\otimes j} \to V^{\otimes n} c_\lambda$ given by right multiplication by c_λ. The restriction of this map to the left-hand side of (3.1) is surjective, since

$$(x \otimes y)c_\lambda = \frac{1}{k}(xc_\mu \otimes yc_\nu) \cdot c_\lambda.$$

□

Lemma 3.3.3. *If $\lambda_{m+1} = 0$ and $\lambda_m > 0$ then*

$$V^{\otimes n} c_\lambda \cong V^{\otimes n-m} c_{\lambda_1 - 1, \ldots, \lambda_m - 1} \otimes \bigwedge^m V.$$

Proof. Put a wall in in the Young diagram of λ between the first column and the rest. Let t_λ be a tableau whose first column consists of $\{1, \ldots, m\}$. By lemma 3.3.2 there exists a surjection $V^{\otimes n-m} c_\nu \otimes \bigwedge^m V \to V^{\otimes n} c_\lambda$, where $\nu = (\lambda_1 - 1, \ldots, \lambda_m - 1)$. Both $V^{\otimes n-m} c_\nu$ and $V^{\otimes n} c_\lambda$ are nonzero, hence they are irreducible $\mathbb{C}[\mathrm{GL}(V)]$-modules. Since $\bigwedge^m V$ is one-dimensional, both sides are irreducible and, by Schur's lemma, the map is an isomorphism. □

Now iterating lemma 3.3.3 we obtain, for the standard representation V of $\mathrm{GL}_n\mathbb{C}$,

$$V^{\otimes n} c_{\lambda_1 + k, \ldots, \lambda_n + k} = V^{\otimes n} c_{\lambda_1, \ldots, \lambda_n} \otimes \left(\bigwedge^n V\right)^{\otimes k}.$$

Thus we have proved proposition 3.3.1. This allows us to define ψ_λ for any index λ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, even if some of the λ_i are negative: we simply take

$$\psi_{\lambda_1, \ldots, \lambda_n} = \psi_{\lambda_1 + k, \ldots, \lambda_n + k} \otimes \det{}^{-k}$$

for any sufficiently large k. By the non-triviality of $\det^k$ for $k \neq 0$, ψ_λ is isomorphic to $\psi_{\lambda'}$ if and only if $\lambda = \lambda'$. Thus, to complete our description of the irreducible finite-dimensional representations of $\mathrm{GL}_n\mathbb{C}$, we just have to check that these are all the irreducible representations of $\mathrm{GL}_n\mathbb{C}$, that is we want to prove the following theorem.

Theorem 3.3.4. *Every irreducible algebraic complex representation of $GL_n\mathbb{C}$ is isomorphic to ψ_λ for a unique $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.*

Notice that $\mathfrak{gl}_n\mathbb{C} = \mathfrak{sl}_n\mathbb{C} \times \mathbb{C}$, where $\mathbb{C}$ is identified with the one-dimensional ideal of $\mathfrak{gl}_n\mathbb{C}$ formed by the scalar matrices aI with $a \in \mathbb{C}$. In particular, $\mathbb{C}$ is the radical of $\mathfrak{gl}_n\mathbb{C}$ and $\mathfrak{sl}_n\mathbb{C}$ is the semisimple part.

Remark 3.3.5. In the following we will use that $\mathrm{SL}_n\mathbb{C}$ is simply connected. For a proof of this fact see section 23.1 of [5]. From results on Lie group theory, it follows that there is a one-to-one correspondence between irreducible representations of $\mathrm{SL}_n\mathbb{C}$ and irreducible representations of $\mathfrak{sl}_n\mathbb{C}$ (for details see lecture 8 in [5]).

We need the following lemma.

Lemma 3.3.6. *Every irreducible representation of $\mathfrak{gl}_n\mathbb{C}$ is a tensor product of an irreducible representation of $\mathfrak{sl}_n\mathbb{C}$ and a one-dimensional representation.*

Proof. Let V be an irreducible representation of $\mathfrak{gl}_n\mathbb{C}$. Since $\mathrm{Rad}(\mathfrak{gl}_n\mathbb{C}) = \mathbb{C}$ is solvable and $\mathrm{Rad}(\mathfrak{gl}_n\mathbb{C}) \subset \mathfrak{gl}(V)$, Lie's theorem 3.1.2 implies that there exists a nonzero vector $v \in V$ that is an eigenvector for all $x \in \mathrm{Rad}(\mathfrak{gl}_n\mathbb{C})$. Hence there exists $\lambda \in (\mathrm{Rad}(\mathfrak{gl}_n\mathbb{C}))^*$ such that $W = \{v \in V : x(v) = \lambda(x) \cdot v \quad \text{for all } x \in \mathrm{Rad}(\mathfrak{gl}_n\mathbb{C})\}$ is non-empty. So, since every element of $\mathfrak{gl}_n\mathbb{C}$ can be written in the form $x + y$ with $x \in \mathrm{Rad}(\mathfrak{gl}_n\mathbb{C})$ and $y \in \mathfrak{sl}_n\mathbb{C}$, we get $x \cdot y \cdot w = y \cdot x \cdot w + [x, y] \cdot w)\lambda(x)(y \cdot w)$ for all $w \in W$. It follows that $y \cdot w \in W$, so that W is a nonzero subrepresentation of V. Hence $W = V$. Extend λ to a linear functional on $\mathfrak{gl}_n\mathbb{C}$ and let L be the one-dimensional representation of $\mathfrak{gl}_n\mathbb{C}$ determined by λ, that is, $x \cdot z = \lambda(x) \cdot z$ for all $x \in \mathfrak{gl}_n\mathbb{C}$ and $z \in L$. Then $V \otimes L^*$ is a representation that is trivial on $\mathrm{Rad}(\mathfrak{gl}_n\mathbb{C})$, and so it comes from an irreducible representation of $\mathfrak{sl}_n\mathbb{C}$. □

For any partition λ of n, denote $W_\lambda = \mathbb{S}_\lambda(\mathbb{C}^n)$ the representation of $\mathfrak{sl}_n\mathbb{C}$ determined by the partition λ. We can extend it to $\mathfrak{sl}_n\mathbb{C} \times \mathbb{C}$ by acting trivially on the second factor. For any $w \in \mathbb{C}^*$, denote $L(w)$ the one-dimensional representation of $\mathfrak{sl}_n\mathbb{C} \times \mathbb{C}$ which is zero on the first factor and acts by multiplication by w on the second. By the previous lemma, every irreducible representation of $\mathfrak{gl}_n\mathbb{C} = \mathfrak{sl}_n\mathbb{C} \times \mathbb{C}$ is isomorphic to a tensor product $W_\lambda \otimes L(w)$. By remark 3.3.5, going back to the corresponding groups this remains true for the simply connected group $\mathrm{SL}_n\mathbb{C} \times \mathbb{C}$. Consider now the exact sequence

$$1 \to \mathrm{Ker}\,(\rho) \hookrightarrow \mathrm{SL}_n\mathbb{C} \times \mathbb{C} \xrightarrow{\rho} \mathrm{GL}_n\mathbb{C} \to 1,$$

given by $\rho(g \times z) = e^z \cdot g$. The kernel of ρ is generated by $e^s \cdot I \times (-s)$, where $s = 2\pi i/n$ since $e^s \cdot I \in \mathrm{SL}_n\mathbb{C}$. Notice that every irreducible representation of $\mathrm{GL}_n\mathbb{C}$ can be lifted to an irreducible representation on $\mathrm{SL}_n\mathbb{C} \times \mathbb{C}$ by acting trivially on $\mathrm{Ker}\,(\rho)$ since $\mathrm{SL}_n\mathbb{C} \times \mathbb{C} = \mathrm{GL}_n\mathbb{C} \times \mathrm{Ker}\,(\rho)$.

Lemma 3.3.7. *The representation $W_\lambda \otimes L(w)$ of $SL_n\mathbb{C} \times \mathbb{C}$ acts trivially on $Ker(\rho)$ if and only if $w = \sum \lambda_i + kn$ for $k \in \mathbb{Z}$.*

Proof. Notice that $e^s \cdot I$ acts on $(\mathbb{C}^n)^{\otimes d}$ by multiplication by e^{sd}, where $d = \sum \lambda_i$. We can restrict this action to $\mathbb{S}_\lambda(\mathbb{C}^n)$. Moreover, $-s$ acts on $L(w)$ by multiplication by e^{-sw}, so the action of $e^s \cdot I \times \{-s\}$ on $W_\lambda \otimes L(w)$ by e^{sd-sw}, which is trivial if and only if $sd - sw \in 2\pi i\mathbb{Z}$. Since $s = 2\pi i/n$, this happens precisely when $w = \sum \lambda_i + kn$ for $k \in \mathbb{Z}$. □

To prove theorem 3.3.4, it remains only to show that every representation $W_\lambda \otimes L\left(\sum \lambda_i + kn\right)$ comes from the representation $\psi_{\lambda_1+k,\ldots,\lambda_n+k}$ of $\mathrm{GL}_n\mathbb{C}$. But this holds since both the representations restrict to the same representation on $\mathrm{SL}_n\mathbb{C}$ and to multiplication by $e^{wz} = e^{(\sum \lambda_i + nk)z}$ on $\mathbb{C}$.

Chapter 4

Symplectic Lie algebras

This chapter presents the symplectic Lie algebras on which we will work also in the next chapter. In section 4.1, we will first describe in general the structure of a symplectic Lie algebra and we will then compute the representations of $\mathfrak{sp}_{2n}\mathbb{C}$. In the final section we will describe Weyl's construction of the irreducible representations of the symplectic groups.

This chapter explains most of the general concepts illustrated in lectures 16 and 17 of [5] but with more detailed explanations in some parts.

4.1 The structure of $\mathrm{Sp}_{2n}\mathbb{C}$ and $\mathfrak{sp}_{2n}\mathbb{C}$

Let V be a $2n$-dimensional complex vector space, and let $f\colon V \times V \to \mathbb{C}$ be a non-degenerate, skew-symmetric bilinear form on V. It can be shown that even dimensionality is a necessary condition for existence of a non-degenerate bilinear form satisfying $f(v,w) = -f(w,v)$. We define the symplectic group as $\mathrm{Sp}_{2n}\mathbb{C} = \{A \in \mathrm{Aut}(V) \mid f(Av, Aw) = f(v,w) \quad \text{for all } v, w \in V\}$, and the symplectic Lie algebra as $\mathfrak{sp}_{2n}\mathbb{C} = \{A \in \mathrm{End}(V) \mid f(Av, w) + f(v, Aw) = 0 \quad \text{for all } v, w \in V\}$.

Let $e_1, \ldots, e_{2n}$ be a basis for V. In the following, we will always consider f to be the bilinear form given by

$$f(e_i, e_{i+n}) = 1, \quad f(e_{i+n}, e_i) = -1, \quad \text{and} \quad f(e_i, e_j) = 0 \quad \text{if } j \neq i \pm n.$$

In matrix form, f can be expressed as $f(x,y) = {}^t x \cdot s \cdot y$, where s is the $2n \times 2n$ matrix given by

$$s = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Thus we have $\mathrm{Sp}_{2n}\mathbb{C} = \{a \in \mathrm{Mat}_{2n}(\mathbb{C}) \mid s = {}^t a \cdot s \cdot a\}$ and $\mathfrak{sp}_{2n}\mathbb{C} = \{x \in \mathfrak{gl}_{2n}\mathbb{C} \mid {}^t x \cdot s + s \cdot x = 0\}$. Writing x in block form as

$$x = \begin{pmatrix} m & l \\ p & q \end{pmatrix}$$

with $m, l, p, q \in \mathfrak{gl}_n\mathbb{C}$, we have

$${}^t x \cdot s = \begin{pmatrix} -{}^t p & {}^t m \\ -{}^t q & {}^t l \end{pmatrix} \quad \text{and} \quad s \cdot x = \begin{pmatrix} p & q \\ -m & -l \end{pmatrix}$$

Hence the condition for x to be symplectic is that the off-diagonal blocks l and p are symmetric, and the diagonal blocks m and q of x are negative transposes of each other. Notice that the last condition forces $\operatorname{tr}(x) = 0$. We can now compute a basis for $\mathfrak{sp}_{2n}\mathbb{C}$. Let E_{ij} be the matrix having 1 in the (i,j)-position and 0 elsewhere for $1 \le i, j \le 2n$. Notice that, since $E_{ij}E_{kl} = \delta_{jk}E_{il}$, it follows that

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj}. \tag{4.1}$$

Take the diagonal matrices $E_{ii} - E_{n+i,n+i}$ $(1 \le i \le n)$. Add to these all $E_{ij} - E_{n+j,n+i}$ $(1 \le i \ne j \le n)$. For l we use the matrices $E_{i,n+i}$ $(1 \le i \le n)$ and $E_{i,n+j} - E_{j,n+i}$ $(1 \le i < j \le n)$, and similarly for the positions in p. Summing the number of these elements, we get $\dim \mathfrak{sp}_{2n}\mathbb{C} = n + n^2 - n + 2\left(n + \frac{1}{2}n(n-1)\right) = 2n^2 + n$.

The obvious candidate for maximal toral subalgebra $\mathfrak{h}$ in $\mathfrak{sp}_{2n}\mathbb{C}$ is the subalgebra of matrices diagonal in this representation; in fact, this works exactly as in the case of $\mathfrak{sl}_n\mathbb{C}$. As a subalgebra of $\mathfrak{sp}_{2n}\mathbb{C}$, $\mathfrak{h}$ is spanned by the n matrices $h_i = E_{i,i} - E_{n+i,n+i}$ whose action on V is to fix e_i, send e_{n+i} to its negative, and kill all the remaining basis vectors. Moreover, recall the dual basis l_j with $l_i(h_j) = \delta_{i,j}$. We already know how $\mathfrak{h}$ acts on $\mathfrak{sp}_{2n}\mathbb{C}$ since the latter is a subalgebra of $\mathfrak{sl}_{2n}\mathbb{C}$. Since the following arguments are very similar to those for the group $\mathfrak{sl}_n\mathbb{C}$ explained in 3.2, we will sketch most of the computations. The full discussion can be found in section 16.1 of [5].

Let us compute the action of $\mathfrak{h}$ on the basis vectors of $\mathfrak{sp}_{2n}\mathbb{C}$. For instance, for $1 \le i, j \le n$, keeping in mind formula (4.1), we have

$$\begin{aligned}
&\operatorname{ad}(a_1h_1 + a_2h_2 + \cdots + a_nh_n)(E_{ij} - E_{n+j,n+i}) \\
&\quad= \operatorname{ad}\left(\sum a_kh_k\right)(E_{ij}) - \operatorname{ad}\left(\sum a_kh_k\right)(E_{n+j,n+i}) \\
&\quad= (a_i - a_j)(E_{ij}) - (a_i - a_j)(E_{n+j,n+i}) \\
&\quad= (a_i - a_j)(E_{ij} - E_{n+j,n+i}).
\end{aligned}$$

Hence $E_{ij} - E_{n+j,n+i}$ is an eigenvector for the action of $\mathfrak{h}$, with eigenvalue $l_i - l_j$. Similarly, for $i \ne j$, $E_{i,n+j} - E_{j,n+i}$ and $E_{n+i,j} - E_{n+j,i}$ are eigenvectors with eigenvalues $l_i + l_j$ and $-l_i - l_j$, respectively. Finally, $E_{i,n+i}$ and $E_{n+i,i}$ are eigenvectors with eigenvalues $2l_i$ and $-2l_i$, respectively. Therefore, the roots of the Lie algebra $\mathfrak{sp}_{2n}\mathbb{C}$ are the vectors $\pm l_i \pm l_j \in \mathfrak{h}^*$.

Now we can find the distinguished subalgebras $\mathfrak{s}_\alpha$ isomorphic to $\mathfrak{sl}_2\mathbb{C}$, and the corresponding elements $h_\alpha \in \mathfrak{h}$. Considering the eigenvalues $l_i - l_j$ and $l_j - l_i$ corresponding to the elements $E_{ij} - E_{n+j,n+i}$ and $E_{ji} - E_{n+i,n+j}$, by (4.1) we have

$$\begin{aligned}
&[E_{ij} - E_{n+j,n+i}, E_{ji} - E_{n+i,n+j}] \\
&\quad= [E_{ij}, E_{ji}] - [E_{ij}, E_{n+i,n+j}] - [E_{n+j,n+i}, E_{ji}] + [E_{n+j,n+i}, E_{n+i,n+j}] \\
&\quad= [E_{ij}, E_{ji}] + [E_{n+j,n+i}, E_{n+i,n+j}] \\
&\quad= E_{ii} - E_{jj} + E_{n+j,n+j} - E_{n+i,n+i} \\
&\quad= h_i - h_j.
\end{aligned}$$

Thus, the distinguished element $h_{l_i-l_j}$ is a multiple of $h_i - h_j$. Since

$$\begin{aligned}
\operatorname{ad}(h_i - h_j)(E_{ij} - E_{n+j,n+i}) &= ((l_i - l_j)(h_i - h_j)) \cdot (E_{ij} - E_{n+j,n+i}) \\
&= 2(E_{ij} - E_{n+j,n+i}),
\end{aligned}$$

we conclude that

$$h_{l_i-l_j} = h_i - h_j.$$

Analogously, we may compute that $h_{l_i+l_j} = h_i + h_j$, $h_{-l_i-l_j} = -h_i - h_j$, $h_{2l_i} = h_i$, and $h_{-2l_i} = -h_i$. Thus, the distinguished elements $\{h_\alpha\} \subset \mathfrak{h}$ are $\{\pm h_i \pm h_j, \pm h_i\}$. In particular, the weight lattice is $\Lambda_W = \mathbb{Z}\{l_1, \ldots, l_n\}$.

Finally we choose a base of the set of roots R and describe the corresponding Weyl chambers. We have $R = \{\pm l_i \pm l_j \mid 1 \leq i, j \leq n\}$. The vectors $l_i - l_{i+1}$ $(1 \leq i \leq n-1)$ and $2l_n$ are independent. Moreover, $l_i + l_j = (l_i - l_{i+1}) + (l_{i+1} - l_{i+2}) + \cdots + (l_{j-1} - l_j) + 2(l_j - l_{j+1}) + \cdots + 2(l_{n-1} - l_n) + 2l_n$ if $i \leq j$ and $l_i - l_j = (l_i - l_{i+1}) + (l_{i+1} - l_{i+2}) + \cdots + (l_{j-1} - l_j)$ if $i < j$, which shows that $B = \{l_i - l_{i+1}\}_{i=1,\ldots,n-1} \cup \{2l_n\}$ form a base of R. The corresponding ordering of the roots will then be $R^+ = \{l_i + l_j\}_{i \leq j} \cup \{l_i - l_j\}_{i<j}$ and $R^- = \{-l_i - l_j\}_{i \leq j} \cup \{l_i - l_j\}_{j<i}$. Thus the simple negative roots for this ordering are the roots $l_{i+1} - l_i$ and $-2l_n$. The corresponding Weyl chamber is

$$\mathcal{W} = \{a_1 l_1 + a_2 l_2 + \cdots + a_n l_n \mid a_1 \geq a_2 \geq \cdots \geq a_n \geq 0\}.$$

Again, the fundamental weights are the weights $\omega_i = l_1 + \cdots + l_i$ for $i = 1, \ldots, n$, and, similarly to the case of the special linear Lie algebras in chapter 3, these n fundamental weights generate as a semigroup the intersection of the Weyl chamber $\mathfrak{W}$ with the weight lattice Δ_W. It follows that, by the existence and uniqueness theorem, for any n-tuple of natural numbers $(a_1, \ldots, a_n) \in \mathbb{N}^n$ there will be a unique irreducible representation with highest weight

$$a_1\omega_1 + a_2\omega_2 + \cdots + a_n\omega_n = (a_1 + \cdots + a_n)l_1 + (a_2 + \cdots + a_n)l_2 + a_n l_n,$$

denoted by $\Gamma_{a_1,\ldots,a_n}$.

Now let $V = \mathbb{C}^{2n}$ be the standard representation of $\mathfrak{sp}_{2n}\mathbb{C}$. Let us find the irreducible representation $V^{(k)} = \Gamma_{0,\ldots,0,1,0,\ldots,0}$, with 1 at the kth place, with highest weight $l_1 + \cdots + l_k$. It will be contained in the kth exterior power $\bigwedge^k V$. Moreover, consider the natural contraction map

$$\varphi_k \colon \bigwedge^k V \to \bigwedge^{k-2} V$$

defined by

$$\varphi_k(v_1 \wedge \cdots \wedge v_k) = \sum_{i<j} f(v_i, v_j)(-1)^{i+j-1} v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_k.$$

Since the representation $\bigwedge^{k-2} V$ does not have the weight $l_1 + \cdots + l_k$, the irreducible representation with this highest weight must be contained in the kernel of this map. To conclude the list of all irreducible representations of $\mathfrak{sp}_{2n}\mathbb{C}$, we state without proof theorem 17.5 of [5] which claims that also the converse holds.

Theorem 4.1.1. *For $1 \leq k \leq n$, the kernel of the map φ_k is exactly the irreducible representation $V^{(k)} = \Gamma_{0,\ldots,0,1,0,\ldots,0}$ with highest weight $l_1 + \cdots + l_k$.*

Proof. See theorem 17.5 of [5]. □

4.2 Weyl's construction for Symplectic Groups

In this section we will give a detailed picture of Weyl's construction following the description in section 17.3 of [5] providing more details, solving some exercises and giving some other results.

Let $V = \mathbb{C}^{2n}$ be the standard representation for $\mathrm{GL}_{2n}\mathbb{C}$. Let $d \in \mathbb{N}$. For each pair $J = (p, q)$ of integers such that $1 \leq p < q \leq d$, the symplectic form f determines a linear map

$$\Phi_J \colon V^{\otimes d} \to V^{\otimes(d-2)},$$

$$v_1 \otimes \cdots \otimes v_d \mapsto f(v_p, v_q) v_1 \otimes \cdots \otimes \hat{v}_p \otimes \cdots \otimes \hat{v}_q \otimes \cdots \otimes v_d.$$

The maps Φ_J are called contractions. Let $V^{<d>} \subset V^{\otimes d}$ denote the intersection of the kernels of all these contractions. Notice that the subspaces $V^{<d>}$ are S_d-invariant. Since $\mathrm{Sp}_{2n}\mathbb{C}$ preserves f, we get $\Phi_J(g \cdot v) = g \cdot \Phi_J(v)$ for all $v \in V^{\otimes d}$. Hence $V^{<d>}$ is $\mathrm{Sp}_{2n}\mathbb{C}$-invariant. Let λ be a partition of d. We define the $\mathfrak{sp}_{2n}\mathbb{C}$-module

$$\mathbb{S}_{<\lambda>}V = V^{<d>} \cap \mathbb{S}_\lambda V.$$

Let $\mathcal{B} = \{e_1, \dots, e_{2n}\}$ be the canonical basis of V and denote the dual basis of $\mathcal{B}$ relative to f by $\{u_1, \dots, u_{2n}\}$. So $f(e_i, u_j) = \delta_{ij}$ for all i, j. Let

$$\psi = \sum_{i=1}^{2n} e_i \otimes u_i,$$

For each pair $J = (p, q)$ of integers such that $1 \le p < q \le d$, we define

$$\Psi_J \colon V^{\otimes(d-2)} \to V^{\otimes d},$$

$$v_1 \otimes \cdots \otimes v_{d-2} \mapsto \sum_{i=1}^{2n} v_1 \otimes \cdots \otimes v_{p-1} \otimes \underbrace{e_i}_{p\text{th}} \otimes v_p \otimes \cdots \otimes v_{q-2} \otimes \underbrace{u_i}_{q\text{th}} \otimes v_{q-1} \otimes \cdots \otimes v_{d-2},$$

Notice that, since $\sum_{i=1}^{2n} f(e_i, u_i) = 2n$, we get $\Phi_J \circ \Psi_J = 2nI$. It follows that Ψ_J is injective and Φ_J is surjective. Moreover we define

$$\vartheta_J = \Psi_J \circ \Phi_J \colon V^{\otimes d} \to V^{\otimes d}.$$

Since Ψ_J is injective, $\mathrm{Ker}\,(\vartheta_J) = \mathrm{Ker}\,(\Phi_J)$.

Let $(\cdot,\cdot) \colon V \times V \to \mathbb{C}$ be the standard Hermitian product on $V = \mathbb{C}^{2n}$. This extends to an Hermitian product on $V^{\otimes d}$ by

$$(v_1 \otimes \cdots \otimes v_d, w_1 \otimes \cdots \otimes w_d) = (v_1, w_1) \dots (v_d, w_d)$$

for all $v_i, w_i \in V$. Clearly, for all $v, w \in V^{\otimes d}$ and $\sigma \in S_d$ we have $(v \cdot \sigma, w \cdot \sigma) = (v, w)$. Let us solve now Exercise 17.13 of [5] to get a direct sum decomposition of $V^{\otimes d}$.

Lemma 4.2.1. *Let $J = (p, q)$ as above. Then:*

i) $(v \otimes w, \psi) = f(v, w)$ for all $v, w \in V$,

ii) Ker $(\Phi_J) =$ Im$(\Psi_J)^\perp$.

Proof. i) Let $v = \sum_j a_j e_j$ and $w = \sum_j b_j u_j$ be arbitrary elements of V. Thus

$$(v \otimes w, \psi) = \sum_i (v, e_i)(w, u_i) = \sum_i a_i b_i = f(v, w).$$

ii) For the sake of simplicity we consider only the case $J = (1, 2)$. Let $\Phi = \Phi_{(1,2)}$ and $\Psi = \Psi_{(1,2)}$. Since

$$\dim(\mathrm{Im}\Psi)^\perp = \dim V^{\otimes d} - \dim(\mathrm{Im}\Psi) = \dim V^{\otimes d} - \dim V^{\otimes(d-2)} = \dim(\mathrm{Ker}\,\Phi),$$

it suffices to prove just one inclusion to get the equality $\mathrm{Ker}\,(\Phi_J) = \mathrm{Im}(\Psi_J)^\perp$. Let us prove $\subset$. Let

$$x = \sum_j v_j^1 \otimes \cdots \otimes v_j^d \in \mathrm{Ker}\,\Phi, \qquad y = w_3 \otimes \cdots \otimes w_d$$

with $w_3, \dots, w_d \in V$. We have

$$\begin{aligned}(x, \Psi(y)) &= \sum_{i,j} (v_j^1, e_i)(v_j^2, u_i)(v_j^3, w_3)\dots(v_j^d, w_d) \\ &= \sum_j (v_j^1 \otimes v_j^2, \psi)(v_j^3 \otimes \dots \otimes v_j^d, y) = \sum_j f(v_j^1, v_j^2)(v_j^3 \otimes \dots \otimes v_j^d, y) \\ &= (\Phi(x), y) = 0.\end{aligned}$$

Hence $x \in \mathrm{Im}(\Psi_J)^\perp$. □

Corollary 4.2.2. *We have*

$$V^{\otimes d} = V^{<d>} \oplus \sum_J Im(\Psi_J).$$

Proof. It is a general fact that, if $A_1, \dots, A_r$ are subspaces of a finite-dimensional Hilbert space, then

$$\left(\bigcap_i A_i\right)^\perp = \sum_i A_i^\perp.$$

Consequently,

$$\left(\bigcap_J \mathrm{Ker}\,(\Phi_J)\right)^\perp = \sum_J (\mathrm{Ker}\,(\Phi_J))^\perp = \sum_J \mathrm{Im}(\Psi_J).$$

□

Remark 4.2.3. Since $\sum_J \mathrm{Im}(\Psi_J)$ is the orthogonal complement of an S_d-invariant space with respect to an S_d-invariant Hermitian product, it is S_d-invariant.

Lemma 4.2.4. *Let λ be a partition of d. Then*

$$\mathbb{S}_{<\lambda>}V = V^{<d>} c_\lambda.$$

Proof. We have

$$\mathbb{S}_\lambda V = V^{\otimes d} c_\lambda = \left(V^{<d>} \oplus \sum_J \mathrm{Im}(\Psi_J)\right) c_\lambda = (V^{<d>} c_\lambda) \oplus \left(\sum_J \mathrm{Im}(\Psi_J) c_\lambda\right)$$

Taking the intersection with $V^{<d>}$, we are done. □

Remark 4.2.5. As in remark 2.1.3, it follows that $\mathbb{S}_{<\lambda_T>}V$ and $\mathbb{S}_{<\lambda_{T'}>}V$ are isomorphic as $\mathfrak{sp}_{2n}\mathbb{C}$-modules for tableaux T and T' of the same partition λ.

We need now the following fact on invariant theory. We will prove it in chapter 5 assuming the first fundamental theorem of invariant theory for the symplectic groups.

Theorem 4.2.6 (Invariant Theory Fact)**.** *Any endomorphism of $V^{\otimes d}$ that commutes with all permutations in S_d and all the operators ϑ_J is a finite $\mathbb{C}$-linear combination of operators of the form $g \otimes \dots \otimes g$, for $g \in Sp_{2n}\mathbb{C}$.*

Now let B be the algebra of all endomorphisms of the space $V^{<d>}$ that are $\mathbb{C}$-linear combinations of operators of the form $g \otimes \dots \otimes g$, for $g \in \mathrm{Sp}_{2n}\mathbb{C}$.

Proposition 4.2.7. *The algebra B is precisely the algebra of all endomorphisms of $V^{<d>}$ commuting with all permutations in S_d, i.e., $B = End_{S_d}(V^{<d>})$.*

Proof. Let $B' = \mathrm{End}_{S_d}(V^{<d>})$. Clearly $B \subset B'$. Let $\beta \in B'$. From the decomposition $V^{\otimes d} = V^{<d>} \oplus \sum_J \mathrm{Im}(\Psi_J)$, we can extend β to a linear map $\tilde{\beta}\colon V^{\otimes d} \to V^{\otimes d}$ such that $\tilde{\beta}(x) = 0$ for all $x \in \sum_J \mathrm{Im}(\Psi_J)$. Since both $V^{<d>}$ and $\sum_J \mathrm{Im}(\Psi_J)$ are S_d-invariant, $\tilde{\beta}$ is an endomorphism that commutes with all permutations in S_d. We want to prove that $\tilde{\beta}$ also commutes with all the operators ϑ_J to apply theorem 4.2.6.

Let $J = (p, q)$ with $1 \leq p < q \leq d$. Let $v = v_1 + v_2$ with $v_1 \in V^{<d>}$ and $v_2 \in \sum_J \mathrm{Im}(\Psi_J)$. Recalling that $V^{<d>} \subset \mathrm{Ker}\,(\Phi_J) = \mathrm{Ker}\,(\vartheta_J)$ and $\mathrm{Im}(\vartheta_J) \subset \sum_J \mathrm{Im}(\Psi_J) \subset \mathrm{Ker}\,(\tilde{\beta})$, we have

$$(\tilde{\beta} \circ \vartheta_J)(v) = (\tilde{\beta} \circ \vartheta_J)(v_1 + v_2) = \tilde{\beta}(0 + \vartheta_J(v_2)) = 0 = \vartheta_J(\beta(v_1)) = (\vartheta_J \circ \tilde{\beta})(v).$$

By theorem 4.2.6, we deduce that $\tilde{\beta}$ is a finite $\mathbb{C}$-linear combination of operators of the form $g \otimes \cdots \otimes g$ with $g \in \mathrm{Sp}_{2n}\mathbb{C}$. It follows that $\beta \in B$, so that $B' \subset B$. □

Let us now state the following lemma about general group algebras. We will sketch the proof. The full proof can be found in section 6.2 of [5].

Lemma 4.2.8. *Let G be any finite group and $A = \mathbb{C}[G]$. Let U be a finite-dimensional right A-module and $B = Hom_G(U, U)$ be the centralizer algebra. Notice that B acts on U on the left, commuting with the right action of A. Moreover, if W is any left A-module, the tensor product $U \otimes_A W$ is a left B-module by acting on the first factor: $b \cdot (v \otimes w) = (b \cdot v) \otimes w$. We have:*

i) For any $c \in A$, the canonical map $U \otimes_A Ac \to Uc$ is an isomorphism of left B-modules.

ii) If $W = Ac$ is an irreducible left A-module, then $U \otimes_A W = Uc$ is an irreducible left B-module.

iii) If $W_i = Ac_i$ are the distinct irreducible left A-modules, with m_i the dimension of W_i, then

$$U \cong \bigoplus_i (U \otimes_A W_i)^{\oplus m_i} \cong \bigoplus_i (Uc_i)^{\oplus m_i}$$

is the decomposition of U into irreducible left B-modules.

Sketch of the proof. i) For $c \in A$, the map $U \otimes_A Ac \to Uc$ sending $u \otimes a \mapsto u \cdot a$ has inverse given by $uc \mapsto u \otimes c$. This is well-defined because $u \otimes c = uc \otimes 1 = u'c \otimes 1 = u' \otimes c$ for any $u, u' \in U$ are such that $uc = u'c$.

ii) Let $W = Ac$ be an irreducible left A-module. Consider first the case where U is an irreducible A-module, so $B = \mathbb{C}$. Since, by Maschke's theorem, A is semisimple, we can identify A with a direct sum $\bigoplus_{i=1}^r M_{m_i}(\mathbb{C})$ of r matrix algebras (see [3], proposition 3.5.8) and W with a minimal left ideal of A. By general results on matrix rings, one get $\dim(U \otimes_A W) \leq 1$ and we are done. Now let $U = \bigoplus_i U_i^{\oplus n_i}$ where all U_i are irreducible right A-modules. Thus $U \otimes_A W$ is either zero or $U \otimes_A W = \bigoplus_i (U_i \otimes_A W)^{\oplus n_i} = \mathbb{C}^{\oplus n_k}$ for some k, which is irreducible over $B = \bigoplus_j M_{n_j}(\mathbb{C})$. Indeed, only the factor $M_{n_k}(\mathbb{C})$ acts on $U \otimes_A W (= \mathbb{C}^{n_k})$, and the action is by left multiplication. Let S be a non-zero $M_{n_k}(\mathbb{C})$-submodule of $\mathbb{C}^{n_k}$. Let $x \in S$ with $x_i \neq 0$ for some $1 \leq i \leq n_k$. Then for $1 \leq j \leq n_k$ we have $e_{ji}x = x_i e_j \in S$, so that $e_j \in S$. Hence $\mathbb{C}^{n_k}$ is an irreducible $M_{n_k}(\mathbb{C})$-module.

iii) Using i) and the isomorphism $A \cong \bigoplus_i W_i^{\oplus m_i}$ we get

$$U \cong U \otimes_A A \cong U \otimes_A (\bigoplus_i W_i^{\oplus m_i}) \cong \bigoplus_i (U \otimes_A W_i)^{\oplus m_i}.$$

□

Hence we can deduce the following theorem on the irreducible representations of $\mathrm{Sp}_{2n}\mathbb{C}$.

Theorem 4.2.9. *Let λ be a partition of d. The representations $\mathbb{S}_{<\lambda>}V$ are either 0 or irreducible representations of $Sp_{2n}\mathbb{C}$. For each partition λ, let m_λ denote the dimension of the corresponding representation $V_\lambda = \mathbb{C}[S_d]c_\lambda$ of S_d. We have*

$$V^{<d>} \simeq \bigoplus_{|\lambda|=d} (\mathbb{S}_{<\lambda>}V)^{m_\lambda}$$

over $Sp_{2n}\mathbb{C}$.

Proof. Since, by proposition 4.2.7, B is the centralizer algebra to $A = \mathbb{C}[S_d]$ acting on the space $V^{<d>}$ and V_λ is an irreducible left $\mathbb{C}[S_d]$-module, lemma 4.2.8 implies that $S_{<\lambda>}V = V^{<d>}c_\lambda$ is an irreducible left B-module. Again by proposition 4.2.7, a subspace of $V^{<d>}$ is $\mathrm{Sp}_{2n}\mathbb{C}$-invariant if and only if it is B-invariant. Hence $S_{<\lambda>}V$ is irreducible over $\mathrm{Sp}_{2n}\mathbb{C}$. Finally, by part iii) of lemma 4.2.8, we get

$$V^{<d>} \simeq \bigoplus_{|\lambda|=d} (V^{<d>}c_\lambda)^{m_\lambda} = \bigoplus_{|\lambda|=d} (\mathbb{S}_{<\lambda>}V)^{m_\lambda}.$$

□

The following is theorem 17.11 in [5]

Theorem 4.2.10. *The space $\mathbb{S}_{<\lambda>}V$ is nonzero if and only if the Young diagram of λ has at most n rows, i.e., $\lambda_{n+1} = 0$. In this case, $\mathbb{S}_{<\lambda>}V$ is the irreducible representation of $\mathfrak{sp}_{2n}\mathbb{C}$ with highest weight $\lambda_1 l_1 + \cdots + \lambda_n l_n$.*

For $r = 1, \ldots, \lfloor d/2 \rfloor$, let

$$V^{<d>}_{d-2r} = \sum_{J_1,\ldots,J_r} (\psi_{J_1} \circ \cdots \circ \psi_{J_r})(V^{<d-2r>}).$$

Since the expansions operator $\psi_{J_1}, \ldots, \psi_{J_r}$ are all injective, $V^{<d>}_{d-2r}$ is isomorphic to the direct sum of several copies of $V^{<d>}$. Hence the irreducible $\mathrm{Sp}(V)$-module occurring in $V^{<d>}_{d-2r}$ as a summand is exactly the same as these occurring in $V^{<d>}$.

Proposition 4.2.11. *The tensor power $V^{\otimes d}$ decomposes into a direct sum*

$$V^{\otimes d} = V^{<d>} \oplus V^{<d>}_{d-2} \oplus \cdots \oplus V^{<d>}_{d-2k},$$

with $k = \lfloor d/2 \rfloor$.

Proof. It follows from corollary 4.2.2 and by induction on r that $V^{\otimes d}$ is equal to the sum $V^{<d>} + V^{<d>}_{d-2} + \cdots + V^{<d>}_{d-2k}$. It follows from theorem 4.2.9 and theorem 4.2.10 that $V^{<d>}_{d-2i}$ and $V^{<d>}_{d-2j}$ contain different irreducible $\mathrm{Sp}(V)$-modules for $i \neq j$. Hence the intersection $V^{<d>}_{d-2i} \cap V^{<d>}_{d-2j} = 0$ when $i \neq j$. Therefore it is a direct sum. □

Chapter 5

Schur-Weyl duality for Sp(V)

In this chapter we will illustrate the Schur-Weyl duality between the symplectic group and the Brauer algebra over the complex field $\mathbb{C}$. In section 5.1 we will present without proof (but giving appropriate references to the proof) an important fact on invariant theory for symplectic groups. In the last section, starting with the Double Centralizer Theorem and characterizing the commutator of the action of $\mathrm{Sp}(V)$ on the tensor product $V^{\otimes d}$, we will we able to prove and state the aforementioned duality.

The first section is based on lecture notes [2]. Also, chapter 5 of [6] treats the same topic with much more details. The last section follows closely chapter 10 of [6].

5.1 Preliminaries on Invariant Theory

In the following we will need a fact from invariant theory, namely the first fundamental theorem of invariant theory for $\mathrm{Sp}(V)$. We present it now.

Let V be a finite-dimensional representation of a finite group G and let $m \in \mathbb{Z}_{\geq 1}$. Taking the usual pointwise product of functions, we can define the $\mathbb{C}$-algebra $\mathbb{C}[V^m]$ of functions $F\colon V^m \to \mathbb{C}$ generated by the elements of $(V^m)^*$ viewed as functions on V^m. Its elements are called polynomial functions on V^m. Let $d = (d_1, \dots, d_m) \in \mathbb{N}^m$. We call a polynomial function $F \in \mathbb{C}[V^m]$ homogeneous of degree d if $F(a_1v_1, \dots, a_mv_m) = a_1^{d_1} \dots a_m^{d_m} F(v_1, \dots, v_m)$ for all $a_1, \dots, a_m \in \mathbb{C} \setminus \{0\}$ and $v_1, \dots, v_m \in V$. A polynomial function $F \in \mathbb{C}[V^m]$ is called G-invariant if $F(g \cdot v) = F(v)$ for all $g \in G$ and $v \in V^m$. The action of G on V^m induces an action on the polynomial functions on V^m by defining $(g \cdot F)(v) = F(g^{-1} \cdot v)$ for all $g \in G$, $F \in \mathbb{C}[V^m]$ and $v \in V^m$. Now assume that $G = \mathrm{Sp}(V)$ and let f be the defining non-degenerate, skew-symmetric bilinear form on V. For each pair (i, j) of integers such that $1 \leq i < j \leq m$, we define a function $f_{(i,j)}\colon V^m \to \mathbb{C}$ by

$$f_{(i,j)}(v_1, \dots, v_m) = f(v_i, v_j).$$

Now we can state the following fact from invariant theory (for details, see proposition F.13 in [5]).

Theorem 5.1.1 (First Fundamental Theorem for $\mathrm{Sp}(V)$)**.** *The polynomial invariants of $Sp(V)$ acting on V^m can be written as polynomials in functions $f_{(i,j)}$ for $1 \leq i < j \leq m$.*

Let $\mathbb{C}[V^{2d}]_{\mathbf{1}}$ be the space of homogeneous polynomials of degree $\mathbf{1} = (1, \dots, 1)$. As a consequence, applying the theorem to the case of homogeneous polynomials of degree $\mathbf{1} = (1, \dots, 1)$, one can see that the $\mathbf{1}$-homogeneous polynomial invariants of $\mathrm{Sp}(V)$ acting on V^m are all linear combinations of products

$$f_{(\sigma(1),\sigma(2))}f_{(\sigma(3),\sigma(4))}\cdots f_{(\sigma(m-1),\sigma(m))} \tag{5.1}$$

for permutations $\sigma \in S_m$ such that $\sigma(2i-1) < \sigma(2i)$ for $1 \le i \le m/2$.

5.2 Symplectic Schur-Weyl duality

Let V be a $2n$-dimensional complex vector space endowed with a non-degenerate, skew-symmetric bilinear form $f\colon V \times V \to \mathbb{C}$ and let $\mathrm{Sp}(V)$ be the corresponding symplectic group of V. By restricting the action in section 2.2, we have the natural factorwise action $\rho\colon \mathbb{C}[\mathrm{Sp}(V)] \to \mathrm{End}(V^{\otimes d})$ given by

$$g(v_1 \otimes v_2 \otimes \cdots \otimes v_d) = g(v_1) \otimes g(v_2) \otimes \cdots \otimes g(v_d),$$

for $g \in \mathrm{Sp}(V), v_i \in V$. To determine the symplectic Schur-Weyl duality, we can use a similar approach to the one followed in section 2.2. Consider the centralizer algebra

$$\begin{aligned} B &= \mathrm{End}_{\mathrm{Sp}(V)}(V^{\otimes d}) \\ &= \{x \in \mathrm{End}(V^{\otimes d}) \mid x\rho(g) = \rho(g)x \text{ for all } g \in \mathrm{Sp}(V)\}. \end{aligned}$$

We apply the Double Centralizer Theorem 2.2.3 to the following situation: $V = V^{\otimes d}$, and A is the image of $\mathbb{C}[\mathrm{Sp}(V)]$ in $\mathrm{End}(V^{\otimes d})$. Since B is by definition $\mathrm{End}_{\mathrm{Sp}(V)}(V^{\otimes d})$, the Double Centralizer Theorem gives a decomposition

$$V^{\otimes d} \simeq \bigoplus_i U_i \otimes W_i$$

as a module of $\mathbb{C}[\mathrm{Sp}(V)] \otimes B$, where U_i are all the irreducible representations of $\mathbb{C}[\mathrm{Sp}(V)]$ and $W_i = \mathrm{Hom}_{\mathrm{Sp}(V)}(U_i, V^{\otimes d})$ are all the irreducible representations of B. Thus now we want to determine the structure of the algebra B.

Consider the isomorphisms

$$\mathrm{End}(V^{\otimes d}) \simeq (V^{\otimes d})^* \otimes (V^{\otimes d}) \simeq (V^*)^{\otimes d} \otimes (V^{\otimes d})$$

as modules for $\mathrm{Sp}(V)$ where the first isomorphism follows from chapter 1 and the last one is given by the $\mathrm{Sp}(V)$-isomorphism $(V^*)^{\otimes d} \to (V^{\otimes d})^*$ mapping $\varphi_1 \otimes \cdots \otimes \varphi_d$ to the linear map $V^{\otimes d} \to \mathbb{C}$ such that $v_1 \otimes \cdots \otimes v_d \mapsto \varphi_1(v_1) \dots \varphi_d(v_d)$. Hence

$$B = \mathrm{End}_{\mathrm{Sp}(V)}(V^{\otimes d}) \simeq (\mathrm{End}(V^{\otimes d}))^{\mathrm{Sp}(V)} \simeq [(V^*)^{\otimes d} \otimes (V^{\otimes d})]^{\mathrm{Sp}(V)} \tag{5.2}$$

as vector spaces. Since B contains $\mathrm{End}_{\mathrm{GL}(V)}(V^{\otimes d})$, the image of $\mathbb{C}[S_d]$ in $\mathrm{End}(V^{\otimes d})$ is contained in B, by the results seen in section 2.2.

Since, by definition, $\mathrm{Sp}(V)$ leaves invariant the non-degenerate bilinear form f on V, we have an $\mathrm{Sp}(V)$-module isomorphism $V \simeq V^*$ given by the map $v \mapsto f(\cdot, v)$, and hence an isomorphism

$$(V^*)^{\otimes d} \otimes (V^{\otimes d}) \simeq V^{\otimes 2d}$$

of $\mathrm{Sp}(V)$-modules. The linear map $(V^*)^{\otimes 2d} \to \mathbb{C}[V^{2d}]_{\mathbf{1}}$ defined by

$$\varphi_1 \otimes \cdots \otimes \varphi_{2d} \mapsto [(v_1, \dots, v_{2d}) \mapsto \varphi_1(v_1) \dots \varphi_{2d}(v_{2d})]$$

is an isomorphism of $\mathrm{Sp}(V)$-modules.

Combining all these $\mathrm{Sp}(V)$-module isomorphisms, we obtain a chain of $\mathrm{Sp}(V)$-isomorphisms

$$\mathbb{C}[V^{2d}]_{\mathbf{1}} \simeq V^{\otimes 2d} \simeq \mathrm{End}(V^{\otimes d}).$$

In particular, denote $T\colon V^{\otimes 2d} \to \mathrm{End}(V^{\otimes d})$ the last $\mathrm{Sp}(V)$-module isomorphism above, which we take in the following explicit form: if $u = u_1 \otimes \cdots \otimes u_{2d}$ with $u_i \in V$, then $T(u)$ is the linear transformation

$$T(u)(v_1 \otimes \cdots \otimes v_d) = f(v_1, u_2) f(v_2, u_4) \ldots f(v_d, u_{2d}) u_1 \otimes u_3 \otimes \cdots \otimes u_{2d-1}$$

for $v_i \in V$. Extending f to a non-degenerate bilinear form on $V^{\otimes d}$ for every d by

$$f(x_1 \otimes \cdots \otimes x_d, y_1 \otimes \cdots \otimes y_d) = \prod_{i=1}^{d} f(x_i, y_i),$$

we can write

$$T(u_1 \otimes \cdots \otimes u_{2d})(v) = f(v, u_2 \otimes u_4 \otimes \cdots \otimes u_{2d}) u_1 \otimes u_3 \otimes \cdots \otimes u_{2d-1}$$

for $v \in V^{\otimes d}$. Thus we have a vector space isomorphism

$$T\colon (V^{\otimes 2d})^{\mathrm{Sp}(V)} \to \mathrm{End}_{\mathrm{Sp}(V)}(V^{\otimes d}).$$

Given the canonical basis $e_1, \ldots, e_{2n}$ of V and its dual basis $u_1, \ldots, u_{2n}$ relative to f, recall the element

$$\psi = \sum_{i=1}^{2n} e_i \otimes u_i.$$

Clearly, the identity operator $I_{V^{\otimes d}}$ is $\mathrm{Sp}(V)$-invariant. It follows that the preimage under T of $I_{V^{\otimes d}}$,

$$\psi_d = \underbrace{\psi \otimes \cdots \otimes \psi}_{d} = \sum_{i_1, \ldots, i_d = 1}^{2n} e_{i_1} \otimes u_{i_1} \otimes \cdots \otimes e_{i_d} \otimes u_{i_d},$$

is $\mathrm{Sp}(V)$-invariant. In particular, ψ is $\mathrm{Sp}(V)$-invariant. Since the action of $\mathrm{Sp}(V)$ on $V^{\otimes 2d}$ commutes with the action of S_{2d}, the tensors $\psi_d \cdot \sigma^{-1}$ are also $\mathrm{Sp}(V)$-invariants, for any $\sigma \in S_{2d}$.

Now consider the $\mathrm{Sp}(V)$-module isomorphism $S\colon V^{\otimes 2d} \to \mathbb{C}[V^{2d}]_{\mathbf{1}}$ sending each $v_1 \otimes \cdots \otimes v_{2d}$ to the polynomial function

$$(w_1, \ldots, w_{2d}) \mapsto f(w_1, v_1) f(w_2, v_2) \ldots f(w_{2d}, v_{2d}).$$

We claim that the preimage under S of the function (5.1) is $\psi_d \cdot \sigma^{-1}$. For, notice that for all $v, w \in V$ we have $f(v \otimes w, \psi) = f(v, w)$. Indeed, for $v = \sum_j a_j e_j$ and $w = \sum_j b_j u_j$ we have

$$\begin{aligned} f(v \otimes w, \psi) &= \sum_{i=1}^{2n} f(v, e_i) f(w, u_i) = \sum_{i=1}^{n} f(v, e_i) f(w, u_i) + \sum_{i=n+1}^{2n} f(v, e_i) f(w, u_i) \\ &= \sum_{i=1}^{n} (-a_{i+n})(-b_{i+n}) + \sum_{i=n+1}^{2n} a_{i-n} b_{i-n} = \sum_{i=1}^{2n} a_i b_i = f(v, w). \end{aligned}$$

Hence we get

$$\begin{aligned}S(\psi_d\cdot\sigma^{-1})(w_1,\dots,w_{2d}) &= S(\psi\cdot\sigma^{-1}\otimes\cdots\otimes\psi\cdot\sigma^{-1})(w_1,\dots,w_{2d})\\ &= f(w_1\otimes w_2,\psi\cdot\sigma^{-1})\dots f(w_{2d-1}\otimes w_{2d},\psi\cdot\sigma^{-1})\\ &= f(w_1\cdot\sigma\otimes w_2\cdot\sigma,\psi)\dots f(w_{2d-1}\cdot\sigma\otimes w_{2d}\cdot\sigma,\psi)\\ &= f(w_1\cdot\sigma,w_2\cdot\sigma)\dots f(w_{2d-1}\cdot\sigma,w_{2d}\cdot\sigma),\end{aligned}$$

and we are done. Since the functions (5.1) span $\mathbb{C}[V^{2d}]_{\mathbf{1}}$, we obtain the following characterization.

Theorem 5.2.1. $[V^{\otimes 2d}]^{Sp(V)}$ *is spanned by the elements* $\psi_d\cdot\sigma^{-1}$ *with* $\sigma\in S_{2d}$.

Now let $\tilde{S}_d\subset S_{2d}$ denote the subgroup that permutes the ordered pairs $\{(1,2),\dots,(2d-1,2d)\}$:

$$\tilde{\sigma}\colon (2i-1,2i)\mapsto(2\sigma(i)-1,2\sigma(i))$$

for $i=1,\dots,d$ with $\sigma\in S_d$. The map $S_d\to S_{2d}$ given by $\sigma\mapsto\tilde{\sigma}$ is clearly injective. Notice that $\psi_d\cdot\tilde{\sigma}=\psi_d$ for all $\tilde{\sigma}\in\tilde{S}_d$.

Let $N_d\subset S_{2d}$ be the subgroup generated by the transpositions $(2j-1\quad 2j)$ for $j=1,\dots,d$. Clearly $N_d\simeq\mathbb{Z}_2^d$. Since $\psi=-\sum_i u_i\otimes e_i$, we get that $\psi\cdot\sigma=\pm\psi_d$ for all $\sigma\in N_d$. Moreover, N_d is normalized by $\tilde{S}_d$, so that $B_d=\tilde{S}_dN_d$ is a subgroup of S_{2d}. In particular we have $\psi_d\cdot\sigma=\pm\psi_d$ for all $\sigma\in B_d$. Hence theorem 5.2.1 has the following equivalent version:

Theorem 5.2.2. $[V^{\otimes 2d}]^{Sp(V)}$ *is spanned by the elements* $\psi_d\cdot\sigma^{-1}$ *with* $\sigma\in\Upsilon_d$*, where* Υ_d *is any collection of representatives for the cosets* S_{2d}/B_d.

Moreover, we define the homomorphism $\tau\colon S_d\to S_{2d}$ by

$$\tau(\sigma)(2i-1)=2\sigma(i)-1,\qquad \tau(\sigma)(2i)=2i,$$

for $i=i,\dots,d$ and $\sigma\in S_d$. It is clearly injective. Hence $\tau(\sigma)$ permutes $\{1,3,\dots,2d-1\}$ and fixes $\{2,4,\dots,2d\}$ pointwise.

For any two subgroups H,K of a group G, denote $H\backslash G/K$ the quotient subset $G/\sim$ of G given by the equivalence relation $\sim$ in G defined by $g\sim g'$ if and only if $g=hgk$ for some $h\in H$ and $k\in K$. Its elements are called the (H,K)-double cosets in G. We say that a subset $X\subset G$ is a set of representatives for the (H,K)-double cosets in G if each of these double cosets contains exactly one element of X. Finally, denote $\pi\colon S_d\to\mathrm{GL}(V^{\otimes d})$ the natural action of S_d on $\mathrm{GL}(V^{\otimes d})$ with $\pi(\sigma)=\pi_\sigma$ for any $\sigma\in S_d$.

Proposition 5.2.3. *Let* $\Gamma\subset S_{2d}$ *be a set of representatives for the* $(\tau(S_d),B_d)$*-double cosets in* S_{2d}. *Then*

$$End_{Sp(V)}(V^{\otimes d})=span\{\pi_{\sigma^{-1}}\circ T(\psi_d\cdot\gamma^{-1})\mid\sigma\in S_d,\gamma\in\Gamma\}.$$

Proof. Applying theorem 5.2.1 we have

$$\mathrm{End}_{\mathrm{Sp}(V)}(V^{\otimes d})=\mathrm{span}\{T(\psi_d\cdot\gamma^{-1})\mid\gamma\in\Upsilon_d\},\tag{5.3}$$

where Υ_d is any set of representatives for the cosets S_{2d}/B_d. It follows from the definition of T that

$$\pi_\sigma\circ T(v)=T(v\cdot\tau(\sigma))\tag{5.4}$$

for all $\sigma\in S_d$ and $v\in V^{\otimes 2d}$. Suppose now that $\gamma_1,\gamma_2\in\Upsilon_d$ are such that $\gamma_1=\tau(\sigma)\gamma_2\zeta$ for some $\sigma\in S_d$ and $\zeta\in B_d$. Then, since $\psi_d\cdot\zeta=\pm\psi_d$, by (5.4) we get

$$T(\psi_d\cdot\gamma^{-1})=T(\psi_d\cdot\zeta^{-1}\gamma_2^{-1}\tau(\sigma^{-1}))=\pm\pi_{\sigma^{-1}}\circ T(\psi_d\cdot\gamma_2^{-1}).$$

The result now follows by (5.3). □

Denote

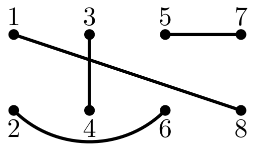
$$\tilde{\mathcal{B}}_d(-2n) = \mathrm{End}_{\mathrm{Sp}(V)}(V^{\otimes d}).$$

Now we want to find a particular set Γ of representatives for the $(\tau(S_d), B_d)$-double cosets in S_d to describe the multiplicative structure of $\tilde{\mathcal{B}}_d(-2n)$.

Let X_d denote the set of all 2-partitions of $\{1, 2, \ldots, 2d\}$, that is partitions of $\{1, 2, \ldots, 2d\}$ into d pointwise disjoint subsets containing two elements each. The Brauer diagram associated to a 2-partition $P \in X_d$ consists of two rows of d dots labeled $1, 3, \ldots, 2d-1$ on the top row and $2, 4, \ldots, 2d$ on the bottom row where the dots i, j are joined by an edge if and only if $\{i, j\} \in P$. For example, if $d = 4$ and P is

$$\{\{1, 8\}, \{2, 6\}, \{3, 4\}, \{5, 7\}\}, \tag{5.5}$$

the associated Brauer diagram is

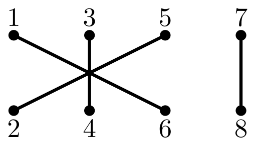


(5.6)

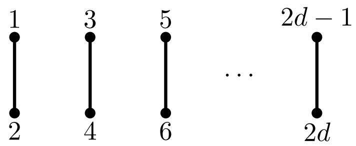
There is a natural action of S_{2d} on X_d given by

$$\{\{i_1, j_1\}, \ldots, \{i_d, j_d\}\} \mapsto \{\{\sigma(i_1), \sigma(j_1)\}, \ldots, \{\sigma(i_d), \sigma(j_d)\}\},$$

for $\sigma \in S_{2d}$. For example, if $\sigma = (1\,7\,2) \in S_8$ and x is (5.5), then $\sigma \cdot x$ is represented by



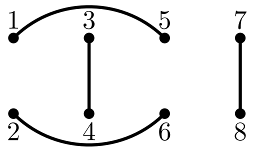
Let x_0 be the 2-partition corresponding to the Brauer diagram



By the transitivity of the action of S_{2d} on X_d, we have $X_d = S_{2d} \cdot x_0$ and the stabilizer of x_0 is precisely B_d. Thus we may identify X_d with the quotient space S_{2d}/B_d via the map $\sigma \cdot x_0 \mapsto \sigma B_d$. Notice that the subgroup $\tau(S_d)$ permutes the top row of dots and fixes each dot in the bottom row. This action can be seen, via the previous identification, as an action on S_{2d}/B_d by left multiplication, with orbits coinciding with the $(\tau(S_d), B_d)$-double cosets in S_{2d}.

We call $x \in X_d$ an r-bar diagram if there are r edges connecting two dots on the top row (or, equivalently, on the bottom row). Such edges are called top bars (or, resp., bottom bars). All diagrams in the $\tau(S_d)$-orbit of x are also r-bar diagrams.

Each $\tau(S_d)$-orbit on X_d has a unique representative z such that if $\{2i, 2j\} \in z$ then $\{2i-1, 2j-1\} \in z$, and if $\{2i-1, 2j\} \in z$ then $i = j$. We call such a Brauer diagram normalized and we denote the set of all such diagrams by Z_d. For instance, (5.6) is not normalized. Its normalized representative is



via the action of $\sigma = (1\,7) = \tau((1\,4)) \in \tau(S_4)$. If both $\{2i, 2j\}$ and $\{2i-1, 2j-1\}$ belong to $x \in X_d$, we say that x contains the (i,j)-bar.

We now determine the element of $\mathrm{End}(V^{\otimes d})$ corresponding to each normalized Brauer diagram. For each pair $J = (p,q)$ of integers such that $1 \le p < q \le d$, recall the linear map $\vartheta_J \colon V^{\otimes d} \to V^{\otimes d}$ given by

$$v_1 \otimes \cdots \otimes v_d \mapsto \sum_{i=1}^{2n} f(v_p, v_q) v_1 \otimes \cdots \otimes v_{p-1} \otimes e_i \otimes \cdots \otimes v_{q-1} \otimes u_i \otimes \cdots \otimes v_d.$$

Here $\{e_i\}$ and $\{u_i\}$ are bases for V with $f(e_i, u_j) = \delta_{ij}$. Let us begin with the 1-bar diagram z containing the (i,j)-bar. Notice that $z = (2i\ \ 2j-1) \cdot x_0$ and $(2i\ \ 2j-1)^{-1} = (2i\ \ 2j-1)$, so that z corresponds to the tensor $\psi_d \cdot (2i\ \ 2j-1)$. Let $C_p(w)$ denote the p-th coefficient of w with respect to the basis $e_1, \ldots, e_{2n}$ of V. For all $w_1, \ldots, w_d \in V$ we have

$$\begin{aligned}
&T(\psi \cdot (2i\ \ 2j-1))(w_1 \otimes \cdots \otimes w_d) \\
&= T\left(\sum_{p_1,\ldots,p_d=1}^{2n} e_{p_1} \otimes u_{p_1} \otimes \cdots \otimes \underbrace{e_{p_j}}_{2i\text{-th}} \otimes \cdots \otimes \underbrace{u_{p_i}}_{(2j-1)\text{-th}} \otimes \cdots \otimes e_{p_d} \otimes u_{p_d} \right) \\
&\qquad\qquad (w_1 \otimes \cdots \otimes w_d) \\
&= \sum_{p_1,\ldots,p_d=1}^{2n} f(w_1, u_{p_1}) \ldots f(w_i, e_{p_i}) \ldots f(w_d, u_{p_d}) e_{p_1} \otimes \cdots \otimes u_{p_j} \otimes \cdots \otimes e_{p_d} \\
&= \sum_{p_1,\ldots,p_d=1}^{2n} f(w_i, e_{p_i}) C_{p_1}(w_1) \ldots C_{p_{i-1}}(w_{i-1}) C_{p_{i+1}}(w_{i+1}) \ldots C_{p_d}(w_d) \cdot \\
&\qquad\qquad \cdot e_{p_1} \otimes \cdots \otimes u_{p_j} \otimes \cdots \otimes e_{p_d} \\
&= f(w_i, w_j) \sum_{p_j=1}^{2n} w_1 \otimes \cdots \otimes e_{p_j} \otimes \cdots \otimes u_{p_j} \otimes \cdots \otimes w_d \\
&= \vartheta_{(i,j)}(w_1 \otimes \cdots \otimes w_d).
\end{aligned}$$

Thus

$$T(\psi \cdot (2i\ \ 2j-1)) = \vartheta_{(i,j)}. \tag{5.7}$$

Suppose now that $z \in Z_d$ is an r-bar diagram containing the $(i_1, j_1), \ldots, (i_r, j_r)$-bars. Let $\gamma_z \in S_{2d}$ be the product of the transpositions $(2i_p\ \ 2j_p - 1)$ for $p = 1, \ldots, r$. Then $z = \gamma_z \cdot x_0$ corresponds to the tensor $\psi_d \cdot \gamma_z^{-1}$. Since the transpositions do not intersect, we have

$$T(\psi_d \cdot \gamma_z^{-1}) = T(\psi_d \cdot \gamma_z) = \vartheta_{(i_1,j_1)} \circ \cdots \circ \vartheta_{(i_r,j_r)}$$

by the same calculation that gives (5.7). For such a $z \in Z_d$, denote $\vartheta_z = \vartheta_{(i_1,j_1)} \circ \cdots \circ \vartheta_{(i_r,j_r)}$.

Proposition 5.2.4. *Let $2n = \dim V$. The algebra $\tilde{\mathcal{B}}_d(-2n) = End_{Sp(V)}(V^{\otimes d})$ is spanned by the set of operators $\pi_{\sigma^{-1}} \cdot \vartheta_z$ for $\sigma \in S_d$ and $z \in Z_d$.*

Proof. Given $z \in Z_d$ take $\gamma_z \in S_{2d}$ as above. Since Z_d is a set of representatives for the $\tau(S_d)$-orbits on X_d, $\Gamma = \{\gamma_z \mid z \in Z_d\}$ is a set of representatives for the $(\tau(S_d), B_d)$-double cosets in S_{2d}. Now it suffices to apply proposition 5.2.3. □

As a consequence, applying the Double Centralizer Theorem 2.2.3 with $A = \{g \otimes \cdots \otimes g \mid g \in \mathrm{Sp}(V)\} \subset \mathrm{End}(V^{\otimes d})$ and B the subalgebra of $\mathrm{End}(V^{\otimes d})$ spanned by the operators $\vartheta_{(p,q)}$ $(1 \le p < q \le d)$ and π_σ $(\sigma \in S_d)$, proposition 5.2.4 claims that $B = \mathrm{End}_A(V^{\otimes d})$ and thus $A = \mathrm{End}_B(V^{\otimes d})$, that is theorem 4.2.6.

We next study the relations in the algebra $\tilde{\mathcal{B}}_d(-2n)$.

Lemma 5.2.5. *The operators $\vartheta_{(i,j)}$ satisfy the following relations, where $2n = \dim V$:*

i) $\vartheta^2_{(i,j)} = 2n\vartheta_{(i,j)}$.

ii) $\vartheta_{(i,j)}\vartheta_{(j,l)} = \pi_{(i\,l)}\vartheta_{(j,l)}$ *for* $1 \le i < j < l \le d$.

iii) $\pi_{\sigma^{-1}}\vartheta_{(i,j)}\pi_\sigma = \vartheta_{(\sigma(i),\sigma(j))}$ *for all* $\sigma \in S_d$.

iv) $\pi_{(i\,j)}\vartheta_{(i,j)} = -\vartheta_{(i,j)}$.

Proof. Recall that $\Phi_{(i,j)} \circ \Psi_{(i,j)} = 2nI$. This implies property i). To verify ii), note that

$$\begin{aligned}&\vartheta_{(i,j)}\vartheta_{(j,l)}(v_1 \otimes \cdots \otimes v_d)\\ &\quad = f(v_j, v_l)\sum_{p,q} f(v_i, e_p) v_1 \otimes \cdots \otimes \underbrace{e_q}_{i\text{th}} \otimes \cdots \otimes \underbrace{u_q}_{j\text{th}} \otimes \cdots \otimes \underbrace{u_p}_{l\text{th}} \otimes \cdots \otimes v_d.\end{aligned}$$

But

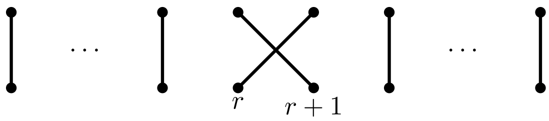
$$\begin{aligned}&\sum_p f(v_i, e_p) v_1 \otimes \cdots \otimes \underbrace{e_q}_{i\text{th}} \otimes \cdots \otimes \underbrace{u_q}_{j\text{th}} \otimes \cdots \otimes \underbrace{u_p}_{l\text{th}} \otimes \cdots \otimes v_d\\ &\quad = -v_1 \otimes \cdots \otimes \underbrace{e_q}_{i\text{th}} \otimes \cdots \otimes \underbrace{u_q}_{j\text{th}} \otimes \cdots \otimes \underbrace{v_i}_{l\text{th}} \otimes \cdots \otimes v_d,\end{aligned}$$

which gives ii). For iii), we have

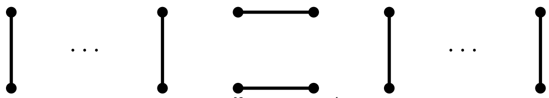
$$\begin{aligned}&(\pi_{\sigma^{-1}}\vartheta_{(i,j)}\pi_\sigma)(v_1 \otimes \cdots \otimes v_d)\\ &\quad = \pi_{\sigma^{-1}}\left(\sum_p f(v_{\sigma(i)}, v_{\sigma(j)}) v_{\sigma(1)} \otimes \cdots \otimes \underbrace{e_p}_{i\text{th}} \otimes \cdots \otimes \underbrace{u_p}_{j\text{th}} \otimes \cdots \otimes v_{\sigma(d)}\right)\\ &\quad = \sum_p f(v_{\sigma(i)}, v_{\sigma(j)}) v_1 \otimes \cdots \otimes \underbrace{e_p}_{\sigma(i)\text{-th}} \otimes \cdots \otimes \underbrace{u_p}_{\sigma(j)\text{-th}} \otimes \cdots \otimes v_d\\ &\quad = \vartheta_{(\sigma(i),\sigma(j))}(v_1 \otimes \cdots \otimes v_d).\end{aligned}$$

Finally, iv) follows from the fact that $\sum_i e_i \otimes u_i = -\sum_i u_i \otimes e_i$. □

We can describe the generators of $\tilde{\mathcal{B}}_d(-2n)$ in terms of Brauer diagrams. Let $\sigma_r \in S_d$ be the transposition $(r \quad r+1)$ corresponding to the Brauer diagram



Recall that S_d is generated by $\sigma_1, \dots, \sigma_{d-1}$. Let $z_r \in Z_d$ be the 1-bar Brauer diagram containing the $(r, r+1)$-bar



From what we have seen, it corresponds to the operator $\vartheta_{(r,r+1)}$. From proposition 5.2.4 and property iii) in lemma 5.2.5, it follows that the algebra $\tilde{\mathcal{B}}_d(-2n)$ is generated by the operators $\sigma_1, \dots, \sigma_{d-1}$ and $z_1, \dots, z_{d-1}$. Thus we can define the multiplication in $\tilde{\mathcal{B}}_d(-2n)$ in terms of concatenations of Brauer diagrams in the following way: for any $x, y \in \tilde{\mathcal{B}}_d(-2n)$ we place the diagram for x above the diagram for y, we join the lower row of dots in x to the upper row of dots in y and we perform the following operations:

- Multiply by a factor (2n) for each closed loop.
- Multiply by a factor -1 for every path beginning and ending on the same row after the concatenation.

We see now the definition of Brauer algebra.

Definition 5.2.6. Let $d \in \mathbb{Z}_{>0}$ and $\delta \in \mathbb{C}$. The Brauer algebra $\mathcal{B}_d(\delta)$ is the associative unital $\mathbb{C}$-algebra generated by the elements s_i, e_i for $1 \le i \le d-1$, subject to the relations

$$s_i^2 = 1, \quad s_i s_j = s_j s_i, \quad s_k s_{k+1} s_k = s_{k+1} s_k s_{k+1}, \tag{5.8}$$

$$e_i^2 = \delta e_i, \quad e_i e_j = e_j e_i, \quad e_k e_{k+1} e_k = e_k, \quad e_{k+1} e_k e_{k+1} = e_{k+1}, \tag{5.9}$$

$$s_i e_i = e_i = e_i s_i, \quad s_k e_{k+1} e_k = s_{k+1} e_k, \quad s_{k+1} e_k e_{k+1} = s_k e_{k+1}, \tag{5.10}$$

for $1 \le i, j \le d-1$, $|i-j| > 2$, and $1 \le k \le d-2$.

We can define a right action of $\mathcal{B}_d(-2n)$ on $V^{\otimes d}$ on generators in the following way. For any integer $1 \le i \le 2n$, we set $i' = 2n+1-i$. We fix an ordered basis $\{v_1, v_2, \dots, v_{2n}\}$ of V such that

$$(v_i, v_j) = 0 = (v_{i'}, v_{j'}),\ (v_i, v_{j'}) = \delta_{ij} = -(v_{j'}, v_i) \quad \text{for all } 1 \le i, j \le 2n.$$

For any $i, j \in \{1, 2, \dots, 2n\}$, let

$$\epsilon_{ij} = \begin{cases} 1 & \text{if } j = i' \text{ and } i < j, \\ -1 & \text{if } j = i' \text{ and } i > j, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$(v_{i_1} \otimes \cdots \otimes v_{i_d}) s_j = -(v_{i_1} \otimes \cdots \otimes v_{i_{j-1}} \otimes v_{i_{j+1}} \otimes v_{i_j} \otimes \cdots \otimes v_{i_d}),$$

$$(v_{i_1} \otimes \cdots \otimes v_{i_d}) e_j = -\sum_{k=1}^{2n} \epsilon_{i_j i_{j+1}} v_{i_1} \otimes \cdots \otimes v_{i_{j-1}} \otimes v_k \otimes v_{k'} \otimes v_{i_{j+2}} \otimes \cdots \otimes v_{i_d}.$$

Lemma 5.2.7. *The operators π_σ for $\sigma \in S_{2d}$ and $\vartheta_{(i,j)}$ satisfy the relations (5.8), (5.9) and (5.10) with $s_i = -\pi_{(i\,i+1)}$ and $e_i = -\vartheta_{(i,i+1)}$.*

Proof. Relations (5.8) follow from general results on transpositions in S_{2d}. The first relation in (5.9) is condition i) in lemma 5.2.5. We already know that $\vartheta_{(i,i+1)}$ commutes with $\vartheta_{(j,j+1)}$ if $|i-j|>1$. Let us prove

$$\vartheta_{(r,r+1)}\vartheta_{(r+1,r+2)}\vartheta_{(r,r+1)} = \vartheta_{(r,r+1)}.$$

For $v_1,\dots,v_d \in V$ we have

$$\begin{aligned}
&(\vartheta_{(r,r+1)}\vartheta_{(r+1,r+2)}\vartheta_{(r,r+1)})(v_1\otimes\cdots\otimes v_d)\\
&= (\vartheta_{(r,r+1)}\vartheta_{(r+1,r+2)})\left(\sum_p f(v_r,v_{r+1})v_1\otimes\cdots\otimes e_p\otimes u_p\otimes\cdots\otimes v_d\right)\\
&= f(v_r,v_{r+1})\vartheta_{(r,r+1)}\left(\sum_{p,q} f(u_p,v_{r+2})v_1\otimes\cdots\otimes e_p\otimes e_q\otimes u_q\otimes\cdots\otimes v_d\right)\\
&= -f(v_r,v_{r+1})\sum_{p,q,s} f(u_p,v_{r+2})f(e_p,u_q)v_1\otimes\cdots\otimes e_s\otimes u_s\otimes e_q\otimes\cdots\otimes v_d\\
&= -f(v_r,v_{r+1})\sum_{p,s} f(u_p,v_{r+2})v_1\otimes\cdots\otimes e_s\otimes u_s\otimes e_p\otimes\cdots\otimes v_d\\
&= f(v_r,v_{r+1})\sum_s v_1\otimes\cdots\otimes e_s\otimes u_s\otimes v_{r+2}\otimes\cdots\otimes v_d\\
&= \vartheta_{(r,r+1)}(v_1\otimes\cdots\otimes v_d).
\end{aligned}$$

The last relation in (5.9) is computed similarly. The first relation in (5.10) is condition iv) in lemma 5.2.5. The remaining relations in (5.10) follow by similar calculations. ☐

Since, as we have seen, $\tilde{\mathcal{B}}_d(-2n)$ is spanned by the operators $\pi_{(i\,i+1)}, \vartheta_{(i,i+1)}$ we have

Corollary 5.2.8. *$\tilde{\mathcal{B}}_d(-2n)$ is the image of the Brauer algebra $\mathcal{B}_d(-2n)$ via the assignment $s_i \mapsto -\pi_{(i\,i+1)}$ and $e_i \mapsto -\vartheta_{(i,i+1)}$.*

Notice that the action of $\mathcal{B}_d(-2n)$ on $V^{\otimes d}$ is precisely

$$\begin{aligned}
(v_1\otimes\cdots\otimes v_d)s_r &= -\pi_{(r,r+1)}(v_1\otimes\cdots\otimes v_d),\\
(v_1\otimes\cdots\otimes v_d)e_r &= -\vartheta_{(r,r+1)}(v_1\otimes\cdots\otimes v_d).
\end{aligned}$$

Summarizing, we have proved that the representations

$$\mathbb{C}[\mathrm{Sp}(V)] \xrightarrow{\rho} \mathrm{End}(V^{\otimes d}) \xleftarrow{\nu} \mathcal{B}_d(-2n),$$

where ν is the $\mathbb{C}$-algebra homomorphism induced by the above action, satisfy the following

Theorem 5.2.9 (Symplectic Schur-Weyl duality)**.** *Let V be a complex vector space of dimension $2n$ endowed with a non-degenerate, skew-symmetric bilinear form $f\colon V\times V\to\mathbb{C}$.*

i) The natural left action of $Sp(V)$ on $V^{\otimes d}$ commutes with the right action of the Brauer algebra $\mathcal{B}_d(-2n)$. Moreover,

$$\begin{aligned}
\nu(\mathcal{B}_d(-2n)) &= End_{Sp(V)}(V^{\otimes d}),\\
\rho(\mathbb{C}[Sp(V)]) &= End_{\mathcal{B}_d(-2n)}(V^{\otimes d}).
\end{aligned}$$

ii) We have a decomposition

$$V^{\otimes d} \simeq \bigoplus_{\substack{|\lambda|\leq d \\ |\lambda|\equiv 2 \pmod d}} \mathbb{S}_{<\lambda>}V \otimes \beta_{\lambda,d}$$

where $\mathbb{S}_{<\lambda>}V$ *are all the irreducible representations of* $\mathbb{C}[Sp(V)]$ *corresponding to* λ *and* $\beta_{\lambda,d} = Hom_{Sp(V)}(\mathbb{S}_{<\lambda>}V, V^{\otimes d})$ *are all the irreducible representations of* $\mathcal{B}_d(-2n)$.

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