ALGANT Master Program

Dipartimento di Matematica · Fakultät für Mathematik

K-theory of Higher Categories

Giacomo Bertizzolo

1242535

Advisors:

Prof. Denis-Charles Cisinski · Prof. Remke Kloosterman

Università degli Studi di Padova

Universität Regensburg

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ABSTRACT

The aim of this thesis is to present a contemporary treatment of K-theory for infinity categories, with an emphasis on stable ∞-categories. To achieve this, we construct both K-theory animae and spectra, and, more generally, we investigate the properties of additive, Verdier-localising, and Karoubi-localising functors. In particular, we prove additivity, universality, localising, and cofinality results for K-theory. In this process, we also construct non-connective K-theory spectra and certain Karoubi-localising functors.

CONTENTS

1 INTRODUCTION

1.1 MOTIVATIONS.

Algebraic K-theory is a vital tool for studying algebraic structures such as rings, schemes, categories, ∞-categories, and many others. It provides a way of associating sequences of abelian groups, called K-groups, to the algebraic structures. These groups encode essential information about the structure of the algebraic object in question. If one considers the generalised Eilenberg-Steenrod cohomology theories useful in algebraic topology, then it is reasonable to think that they might also be interesting in algebraic geometry; algebraic K-theory is in some sense the simplest and most widely studied among such theories.

Algebraic K-theory is highly complex, which poses difficulties in performing computations, but it enables us to establish a variety of useful theorems. A case in point is in the study of topology, where the analysis of chain complexes is fundamental, but the computation of their homology groups can be challenging. With K-theory, it is possible to extract additional information on the structure of the chain complex, such as Poincare duality, which is not visible through homology alone.

Algebraic K-theory originated from Grothendieck's attempts to extend the Riemann-Roch theorem to include algebraic varieties and projective morphisms between them. His goal was to provide a natural framework for the intersection theory on algebraic varieties. Motivated by the Grothendieck-Riemann-Roch theorem, Atiyah and Hirzebruch soon developed the topological K-theory, following their intuition that topological K-theory forms a multiplicative generalised cohomology theory. Compared to the topological case, defining algebraic K-theory is much more challenging. The concept of "lower" algebraic K-theory can be traced back to classical materials that connect class groups, unit groups, determinants, Brauer groups, and other related items for rings of integers, fields, and so on, and it encompasses a lot of local-to-global principles. Initially, defining higher algebraic K-theory appeared to be extremely difficult. With time passing, many correct but unsatisfying definitions of higher K-theory were given. Here, Quillen's intuitions come into play. He first realised that the K-groups of a ring R could be written as homotopy groups of a certain simplicial space $k(R)$ enter into places. This first construction of $k(R)$, through a process called $+$-construction, was working but was not too much satisfying; for example, $k_0(R)$ results separated by the other higher homotopy groups $k_n(R)$. Later work by Quillen, Segal, Waldhausen and others led to more refined constructions, including the S and Q-constructions we are using now.

Various definitions of K-theory exist, each with its own unique flavor and significance. These different definitions stem from the diverse needs that arise when tackling various problems in the field. Due to the significant advantages they offer, we have chosen to focus our study on stable ∞-categories rather than triangulated categories or Waldhausen 1-categories. Firstly, ∞-categories provide the natural setting to develop homotopy theory. Secondly, stable ∞-categories are particularly appealing to algebraic geometers, as they provide a richer and better-behaved way to consider the derived category of a ring or a scheme in contrast to the triangulated setting. For instance, the functor that associates to an affine scheme its derived ∞-categories satisfies Zariski descent, whereas this is not the case when considering the classical derived category in the triangulated setting. Moreover, algebraic K-theory offers a Zariski (or Nisnevich) sheaf of spectra on the category of schemes, which enables remarkable computations that would be challenging to perform without the use of stable ∞-categories. Another advantage of stable ∞-categories is that being stable is a property rather than a structure, which means we do not need to ask for additional data as we would in the case of triangulated, exact, or Waldhausen 1-categories. This is especially helpful when endowing them with a structure. For example, we do not need to verify compatibility conditions when endowing a stable ∞-category with a symmetric monoidal ∞-structure, whereas there are many conditions to be met in the case of triangulated categories. They are also well-behaved in families; for example, as they form a semi-additive presentable ∞-category $\mathbf{Cat}_\infty^{\mathbf{st}}$. In addition, the sub-$\infty$-categories of a stable ∞-category are all generalisations of the standard notion of additive 1-categories, being all additive ∞-categories.

We should also explain our choice of developing a K-theory of stable ∞-categories instead of Waldhausen ∞-categories or exact ∞-categories (à la Barwick [Bar15], [Bar16]). First of all, this theory is itself very satisfying. While many result holds in all of these, the one we worked on are very clean do to the symmetry that this case present. Additionally, in some cases where nice factorization hypotheses are present, these other approaches do not yield added generality; an example of this is shown in [BM10, Thm 1.3]. Finally, this definition allows for rich constructions and numerous applications of K-theory in various ways.

1.2 OVERVIEW.

We start with a small parenthesis on the construction of K_0 for a stable ∞-category in *chapter* 2. Next we deeply dive into the study of Verdier and Karoubi sequences and the functors that have a nice behavior on them in *chapter* 3. These will be the tools we are going to use throughout all the thesis. In particular, grouplike additive, Verdier-localising, and Karoubi-localising functors will be our central focus, being K-theory the central example for them. After this, we proceed in *chapter* 4 with the definition of higher K-theory through S- and Q-construction, and we show that they are indeed equivalent. The reason to why we give both construction is that, while S-construction appeared first and surely gives a good intuition on what is happening, the Q-construction is often more aesthetically (and computation-wise) pleasing. *Chapter* 5 is the core of the thesis, where we prove all the important standard results for K-theory such as additivity, universality, localising, and cofinality results for K-theory. Here, we also talk about the relative Q-construction and of a particular version of the Waldhausen fibration theorem. Throughout this chapter, we provide a more straightforward proof of the various statements to then move to a more complex proof for some of the additive or localising functors. The reason for this is, besides adding generality, the latter proofs usually give a better insight of what is happening. In the *appendix* A we resolve the problem of $K \circ \mathrm{Idem}$ not being Karoubi localising by introducing the non-connective K-theory spectrum. In the *appendix* B we give a brief report on t-structures and weight structures.

1.3 CONVENTIONS AND NOTATIONS.

We assume the reader has some familiarity with the concept of infinity-category (foundational references for this are [Lur09] and [Cis19]) and a few things of "brave new algebra", that is at least the concept of $\mathbf{E}_\infty$-space, $\mathbf{E}_\infty$-grouplike-space, and Spectra (references for these are [Lur17] and [GGN13]). We will try not to fix a model for infinity categories but rather work in the general case. The reader who is averse to this can fix their favourite model; we suggest quasi-categories. We call ANIMA what is usually referred also as *infinity groupoid* or *space*. The usual convention is to pluralise this word with "anima". However, we are going with the word ANIMAE; see this discussion on English-Stackexchange.

We list the names we give to objects we are going to work with: $\mathbf{Cat}_\infty$ is the ∞-category of ∞-categories; $\mathbf{An}$ is the ∞-category of animae; $\mathrm{s}\,\mathbf{An}$ is the ∞-category of simplicial animae; $\mathrm{S}\,\mathbf{An}$ is the ∞-category of Segal animae; $\mathrm{CS}\,\mathbf{An}$ is the ∞-category of complete Segal animae; $\mathrm{Cat}_1^{(2)}$ is the 2-category of 1-categories; $\mathbf{An}_{\leqslant 1}$ is the 2-category of 1-truncated animae; $\mathbf{Cat}_\infty^{\mathbf{st}}$ is the ∞-category of stable ∞-categories; $\mathbf{Cat}_\infty^{\mathbf{perf}}$ is the ∞-category of idempotent-complete stable ∞-categories; $\mathbf{Pr}^{\mathrm{L}}$ is the ∞-category of presentable ∞-categories; and $\mathbf{Pr}^{\mathrm{L}}_{\mathrm{st}}$ is the ∞-category of stable presentable ∞-categories. Moreover, we are going to denote functor categories with $\mathbf{Fun}$, mapping animae with $\mathbf{Map}$, and internal hom-objects with $\underline{\mathrm{Hom}}$ (or with $\mathbf{Fun}$, depending on the context). Also, k will denote the algebraic K-theory space, K the algebraic K-theory spectrum, K_n the K-groups, and $\mathbf{K}$ the non-connective algebraic K-theory spectrum.

1.4 SOME RECALLS.

Recall there is an equivalence of ∞-categories between *the ∞-category of ∞-categories* and the *∞-category of complete Segal animae*. Recall also we have a diagram

$$\begin{tikzcd}
\mathrm{Cat}_1^{(2)} \arrow[r, hook] \arrow[r, phantom, bend left=15, "\perp"] \arrow[d, bend right] \arrow[d, phantom, shift right=2, "\dashv"] \arrow[d, phantom, shift left=2, "\dashv"] \arrow[d, bend left] & \mathbf{Cat}_\infty \arrow[l, bend right=40, "\mathrm{ho}"'] \arrow[r, bend right, "\mathrm{N}^r"'] \arrow[r, phantom, "\simeq"] \arrow[d, bend right, "|\bullet|"'] \arrow[d, bend left, "\mathrm{core}"] \arrow[d, phantom, shift right=2, "\dashv"] \arrow[d, phantom, shift left=2, "\dashv"] & \mathrm{CS}\,\mathbf{An} \arrow[l, bend right, "\mathrm{asscat}"'] \arrow[d, bend right] \arrow[d, bend left] \arrow[d, phantom, shift right=2, "\dashv"] \arrow[d, phantom, shift left=2, "\dashv"] \\
\mathbf{An}_{\leqslant 1} \arrow[r, hook] \arrow[r, phantom, bend right=15, "\perp"] \arrow[u, hook] & \mathbf{An} \arrow[l, bend left=40] \arrow[r, bend right] \arrow[r, phantom, "\simeq"] \arrow[u, hook] & \mathrm{const}\,\mathrm{s}\,\mathbf{An} \arrow[l, bend right] \arrow[u, hook]
\end{tikzcd}$$

and a diagram

$$\begin{tikzcd}
\mathrm{N}(\mathrm{Set}) \arrow[r, hook] \arrow[r, phantom, shift left=1.5, "\perp"] & \mathbf{An} \arrow[l, dashed, bend right, "\pi_0"'] \arrow[r, hook] \arrow[r, phantom, shift left=1.5, "\perp"] & \mathbf{Cat}_\infty \arrow[l, dashed, bend right, "|\bullet|"'] \arrow[l, dashed, bend left, "\mathrm{core}"]
\end{tikzcd}.$$

Recall that an ∞-category is POINTED if it has a zero object, i.e. an object which is both initial and terminal. A stable ∞-category is a pointed ∞-category that admits pullbacks and pushouts in which cartesian squares and cocartesian squares coincide. $\mathbf{Cat}_\infty^{\mathbf{st}}$, the ∞-category of small stable ∞-category, is the sub-∞-category of $\mathbf{Cat}_\infty$ spanned by stable ∞-categories and exact functors between them; a functor is called exact if it preserves finite limits and finite colimits (asking one of these is enough, it implies both). $\mathbf{Cat}_\infty^{\mathbf{st}}$ and its full sub-∞-category $\mathbf{Cat}_\infty^{\mathbf{perf}}$ admit all limits and colimits; furthermore, they are compactly generated, so in particular presentable.

Stable ∞-categories are naturally enriched over spectra; if $\underline{\mathbf{Map}}_{\mathcal{C}}(x,y)$ is the mapping spectrum from x to y, objects of a stable ∞-category $\mathcal{C}$, then we have

$$\mathbf{Map}_{\mathcal{C}}(x,y) \simeq \Omega^\infty \underline{\mathbf{Map}}_{\mathcal{C}}(x,y).$$

Given a stable ∞-category, a sub-∞-category of it is is a *stable* sub-∞-subcategory if it is stable and the inclusion functor is exact, i.e. if the sub-∞-category is closed under taking finite limits and finite colimits. For two stable ∞-categories $\mathcal{C}$ and $\mathcal{D}$, $\mathbf{Fun}^{\mathrm{ex}}(\mathcal{C},\mathcal{D})$ denotes the full sub-∞-category of $\mathbf{Fun}(\mathcal{C},\mathcal{D})$ spanned by exact functors; moreover,

$\mathbf{Map}_{\mathbf{Cat}_\infty^{\mathbf{st}}}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{core}\,\mathbf{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$. When $\mathcal{C}$ and $\mathcal{D}$ are stable (actually, just the latter is enough), we have that $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ is stable and $\mathbf{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$ is a stable sub-∞-category of it.

An ∞-category is idempotent-complete if its essential image through the Yoneda lemma is closed under retracts. The existence of an idempotent-completion is discussed in [Lur17, sec. 5.1.4]. Taking the idempotent-completion of a (stable) ∞-category $\mathcal{C}$ is the same thing as adjoining all the retracts of the essential image of $\mathcal{C}$ through the Yoneda embedding. The idempotent completion functor is always fully faithful. Moreover

$$\mathbf{PSh}(\mathcal{C}) \simeq \mathbf{PSh}(\mathrm{Idem}(\mathcal{C})),$$

and if $\mathcal{C}$ is already idempotent, then the idempotent completion gives an equivalence

$$\mathcal{C} \simeq \mathrm{Idem}(\mathcal{C}).$$

$\mathbf{Cat}_\infty^{\mathbf{perf}}$ is a full sub-∞-category of $\mathbf{Cat}_\infty^{\mathbf{st}}$, and the idempotent-completion functor gives a left adjoint to the inclusion. Also, idempotent-completion functor $\mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{Cat}_\infty^{\mathbf{perf}}$ preserves limits and colimits. $\mathbf{Cat}_\infty^{\mathbf{st}}$ is closed under finite limits, but not finite colimits, in $\mathbf{Cat}_\infty$.

An ∞-category is SEMI-ADDITIVE or ADDITIVE if its homotopy category is semi-additive or additive, respectively. For reference [GGN13]. As examples, any sub-∞-category of a stable ∞-category is additive, while $\mathbf{Cat}_\infty^{\mathbf{st}}$ and $\mathbf{An}$ are semi-additive.

Consider a sequence of functors $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ in $\mathbf{Cat}_\infty^{\mathbf{st}}$. We say that the sequence has VANISHING COMPOSITION if $p \circ f : \mathcal{C} \to \mathcal{E}$ is a zero object in the stable ∞-category $\mathbf{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{E})$. Since the full sub-$\infty$-category of $\mathbf{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{E})$ spanned by zero objects is contractible, this means $p \circ f$ is equivalent in a (essentially) unique way to the functor $\mathcal{C} \to 0 \to \{0\} \subset \mathcal{E}$. This is therefore the same as asking that the following diagram commute

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "f"] \arrow[d, "!"'] & \mathcal{D} \arrow[d, "p"] \\
0 \arrow[r, "!"] & \mathcal{E}
\end{tikzcd}$$

The property of the sequence to be FIBRE or COFIBRE sequence refers to this diagram being cartesian or cocartesian, respectively.

2 GROTHENDIECK GROUP

In this brief chapter, we will present an introduction to K_9 for a stable ∞-category and demonstrate the proof of the Thomason theorem. This theorem establishes a correspondence similar to Galois theory between subgroups of K_0 and specific stable sub-∞-categories.

2.1 Grothendieck group.

2.1 Definition (Grothendieck Group)
Consider a stable ∞-category $\mathcal{C}$ and the monoid

$$\big(\pi_0(\mathrm{core}\,(\mathcal{C})), \oplus\big)$$

of connected components of the core of $\mathcal{C}$, with operation given by

$$[x] \oplus [y] := [x \oplus y],$$

where $[x]$ denotes the connected component of x . We define the GROTHENDIECK GROUP $K_0(\mathcal{C})$ of $\mathcal{C}$ as

$$K_0(\mathcal{C}) := \big(\pi_0(\mathrm{core}\,(\mathcal{C})), \oplus\big)/\sim$$

where $\sim$ is the equivalence relation given by

$$[x] = [x'] + [x''] \quad \text{whenever } x' \to x \to x''$$

is a cofibre sequence in $\mathcal{C}$.

2.2 Remark
$\pi_0(\mathrm{core}\,(\mathcal{C}))$ is the set of isomorphism classes of objects of $\mathcal{C}$.

2.3 Remark
The definition of K_0 implies immediately that

- $[0] = 0$.
- $[\Sigma x] = [\Omega x] = -[x]$; so $K_0(\mathcal{C})$ actually a group, not only a monoid.

- If $\mathcal{C}$ admits infinite coproducts, then $K_0(\mathcal{C})$ is trivial, since any $x \in \mathrm{Ob}(\mathcal{C})$ fits into a cofibre sequence

  $$x \rightarrow\rightarrow \oplus_{n\geqslant 0} x \rightarrow \oplus_{n\geqslant 1} x$$

  and the last two terms are equivalent.

Also, $K_0(\mathcal{C})$ is an abelian group, since, for any pair of objects $x, y \in \mathcal{C}$, there are cofibre sequences

$$x \to x \oplus y \to y, \quad y \to x \oplus y \to y,$$

from which we see

$$[x] + [y] = [x \oplus y] = [y] + [x].$$

2.4 Remark

There are some equivalent definitions for K_0. The other most common is: $K_0(\mathcal{C})$ is the *free abelian group* with generators $[x]$, where x is an objects of $\mathcal{C}$ (or an isomorphism class of objects of $\mathcal{C}$), modulo relations given by

$$[x] = [x'] + [x''] \quad \text{whenever } x' \to x \to x''$$

is a cofibre sequence in $\mathcal{C}$.

2.5 Remark

We obtain a functor

$$K_0 : \mathrm{ho}(\mathbf{Cat}_\infty^{\mathbf{st}}) \to \mathrm{Ab}.$$

2.2 THOMASON THEOREM.

The functor $K_0 : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathrm{Ab}$ is already interesting; indeed, fixed a small stable ∞-category $\mathcal{C}$, it gives a “Galois-like correspondence” between the subgroups of $K_0(\mathcal{C})$ and certain sub-∞-categories of $\mathcal{C}$. This theorem first appeared in [Tho97] for triangulated categories. To state the theorem properly we need some definitions.

2.6 Definition (Karoubi Equivalences)

An exact functor of stable ∞-categories $f : \mathcal{C} \to \mathcal{D}$ is a KAROUBI EQUIVALENCE if it is fully faithful and has dense image.

With “dense image” we mean that for any x object of $\mathcal{D}$ there exists an object y in the essential image of $\mathcal{C}$ such that y is a retract of x.

The main example of Karoubi equivalence is $\mathcal{C} \hookrightarrow \mathrm{Idem}(\mathcal{C})$, the inclusion of a stable ∞-category into its idempotent completion.
We will come back to discuss Karoubi equivalences in 3.2.5.

2.7 Definition
A REPLETE sub-∞-category of $\mathcal{C}$ is a sub-∞-category $\mathcal{A}$ of $\mathcal{C}$ such that if $x \simeq y$ in $\mathcal{C}$, then $x \in \mathcal{A}$ if and only if $y \in \mathcal{A}$.

2.8 Theorem (Thomason for stable ∞-categories)
Consider a small stable ∞-category $\mathcal{C}$. There is a one-to-one correspondence between replete full dense stable sub-∞-categories $\mathcal{A}$ in $\mathcal{C}$ and the subgroups H of the Grothendieck group $K_0(\mathcal{C})$.
To $\mathcal{A}$ corresponds the subgroup

$$H_{\mathcal{A}} := \mathrm{Im}(K_0(\mathcal{A}) \to K_0(\mathcal{C})) \subset K_0(\mathcal{C}).$$

To H corresponds the full sub-∞-category $\mathcal{C}^H$ of $\mathcal{C}$ spanned by those x in $\mathcal{C}$ such that $[x] \in H \subset K_0(\mathcal{C})$.
It follows that any Karoubi equivalences induce injections on $K_0 : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathrm{Ab}$.

Proof. As a first step, we have to prove that the maps are well-defined.

- $H_{\mathcal{A}}$ is clearly an image subgroup of $K_0(\mathcal{C})$. (We are fine with the category-side of the correspondence because we chose to work with replete sub-∞-categories; another solution would have been to consider sub-∞-categories up to equivalence.)

- We want to prove $\mathcal{C}^H$ is a replete full dense stable sub-∞-category of $\mathcal{C}$.
  From how we defined $\mathcal{C}^H$, it is clearly full.
  Next, if $x \in \mathcal{C}^H$ and $y \simeq x$ in $\mathcal{C}$, then $[y] = [x] \in H$, so $y \in \mathcal{C}^H$; therefore, $\mathcal{C}^H$ is replete.
  To show that it is stable, we want to prove it is closed under finite limits and colimits. Consider $x, y \in \mathcal{C}^H$, then

  $$[x \oplus y] = [x] + [y] \in H$$

  since it is the sum of elements of H. Similarly, if we have a fibre sequence

  $$x \to y \to z$$

  with y and z in $\mathcal{C}$, then $[x] = [y] - [z] \in H$. The closure under finite colimit follows from the additivity of $\mathcal{C}$.
  It remains us to show that $\mathcal{C}^H$ is dense in $\mathcal{C}$; for $x \in \mathcal{C}$, we need

to find a $y \in \mathcal{C}^H$ such that x is a retract of y. Choose $y := x \oplus \Sigma x$, then

$$[y] = [x \oplus \Sigma x] = [x] + [\Sigma x] = [x] - [x] = 0 \in H;$$

so $y \in \mathcal{C}^H$. Moreover, we know that $x \to x \oplus \Sigma x \simeq x \times \Sigma x \to x$ is the identity; hence x is a retracts of y. We conclude that $\mathcal{C}^H$ is dense in $\mathcal{C}$.

Next, we want to prove that these two maps of sets are inverses to each other.

- We want to prove $H_{\mathcal{C}^H} = H$. By definition of $\mathcal{C}^H$

  $$\mathrm{Im}(K_0(\mathcal{C}^H) \to K_0(\mathcal{C})) \subset H.$$

  Now, take any $h \in H \subset K_0(\mathcal{C})$; h is for sure the class of a x in $\mathcal{C}$. So $h = [x]$, hence $x \in \mathcal{C}^H$. Therefore

  $$\mathrm{Im}(K_0(\mathcal{C}^H) \to K_0(\mathcal{C})) = H.$$

- We want to prove $\mathcal{C}^{H_\mathcal{A}} = \mathcal{A}$. This means proving that $x \in \mathcal{A}$ if and only if $[x] \in H_\mathcal{A}$. The "only if" direction is clear. For the other one, consider the isomorphism classes of objects of $\mathcal{C}$ and define a relation as follows: $x \sim x'$ if there exists $y, y' \in \mathcal{A}$ such that $x \oplus y \simeq x' \oplus y'$. This is evidently an equivalence relation x, so we can consider the quotient

  $$G := \pi_0 \,\mathrm{core}\,(\mathcal{C})/\sim;$$

  we denote with $\langle x \rangle$ the class of any $x \in \mathrm{Ob}(\mathcal{C})$. G is an abelian group: the sum is given by the direct sum; $\langle 0 \rangle$ is the zero; if $\langle x \rangle \in G$, x is a direct summand on an object of $\mathcal{A}$, since $\mathcal{A}$ is dense in $\mathcal{C}$, say $x \oplus x' = y \in \mathcal{A}$, therefore

  $$\langle x \rangle + \langle x' \rangle = \langle y \rangle = 0.$$

  (i) Now we claim that $x \in \mathcal{A}$ if and only if $\langle x \rangle = 0$ in G. If $x \in \mathcal{A}$, then $x \oplus 0 = 0 \oplus x$, so $x \sim 0$. If $x \sim 0$, then $x \oplus y \simeq y' \in \mathcal{A}$. So $x \to x \oplus y \simeq y' \to y$ is a bifibre sequence in $\mathcal{C}$. But since both $y, y' \in \mathcal{A}$ and $\mathcal{A}$ is a stable sub-∞-category, $x \in \mathcal{A}$.

  (ii) We have a canonical monoid morphism $\alpha : \pi_0 \,\mathrm{core}\,(\mathcal{C}) \to G$. We want to show this descends to a morphism of groups $K_0(\mathcal{C})/H_\mathcal{A} \to G$. First, we have to show that α respects the relation that gives $K_0(\mathcal{C}$; to do this, we have to prove that for any bifibre sequence $x \to y \to z$, then $\langle y \rangle = \langle x \rangle + \langle z \rangle$. Since $\mathcal{A}$ is dense in $\mathcal{C}$, there exist x' and z' in $\mathcal{C}$ such that $x \oplus x', z \oplus z' \in \mathcal{A}$. Also,

  $$x \oplus x' \to y \oplus x' \oplus z' \to z \oplus z'$$

is a bifibre sequence. Now the two terms of this sequence are in $\mathcal{A}$, so also the middle term must be in $\mathcal{A}$ since $\mathcal{A}$ is a stable sub-∞-category. So we have that both $(x \oplus x' \oplus z \oplus z')$ and $(y \oplus x' \oplus z')$ belongs to $\mathcal{A}$. Therefore, we get

$$y \oplus (x \oplus x' \oplus z \oplus z') \simeq x \oplus z \oplus (y \oplus x' \oplus z'),$$

which implies $\langle y \rangle = \langle x \rangle + \langle z \rangle$.
Until now we have prove α descends to a morphism $K_0(\mathcal{C}) \to G$. To show it descends to a morphism $K_0(\mathcal{C})/H_{\mathcal{A}} \to G$, we have to show $H_{\mathcal{A}}$ is mapped to $0 \in G$. Let $[x] \in H_{\mathcal{A}} \subset K_0(\mathcal{C})$ and take $y \in \mathcal{C}$ such that $x \oplus y \in \mathcal{A}$ (or better for any choice of x in that equivalence class there exist such a y); then $\langle x \rangle = \alpha[x]$ and $\langle x \rangle + \langle y \rangle = \langle x \oplus y \rangle = 0 \in H_{\mathcal{A}}$.

(iii) Now, this morphism $\alpha : K_0(\mathcal{C})/H_{\mathcal{A}} \to G$ is clearly surjective, and we know $H_{\mathcal{A}} \subset \ker(\alpha)$. By (i) we have that $H_{\mathcal{A}} = \ker(\alpha)$.

We have therefore proved the correspondence.

It remains to prove that Karoubi equivalences induce injective morphisms. To cite Thomason

"The reader may find the indirectness of the proof of this useful corollary psychologically uncomfortable. "

Let $\mathcal{A}$ be a dense stable sub-∞-category of $\mathcal{C}$. Let $\tilde{\mathcal{A}}$ be the replete full stable sub-∞-category of $\mathcal{C}$ spanned by object in the essential image of $\mathcal{A}$. The inclusion $\mathcal{A} \hookrightarrow \tilde{\mathcal{A}}$ is (by definition) an equivalence of stable ∞-category, hence $K_0(\mathcal{A}) \simeq K_0(\tilde{\mathcal{A}})$. We can therefore assume $\mathcal{A}$ to be replete.
Take N to be the kernel of the morphism $K_0(\mathcal{A}) \to K_0(\mathcal{C})$. Since $\mathcal{A}$ is stable, we can also use the "Galois-like correspondence" for $\mathcal{A}$. We have the inclusion of dense stable sub-∞-categories

$$\mathcal{A}^0 \subset \mathcal{A}^N \subset \mathcal{A},$$

where $\mathcal{A}^0$ is the dense stable sub-∞-category of $\mathcal{A}$ corresponding to the trivial sub-group of $K_0(\mathcal{A})$. Now, both $\mathcal{A}^0$ and $\mathcal{A}^N$ are dense stable sub-∞-category of $\mathcal{C}$, and both $K_0(\mathcal{A}^0) \to K_0(\mathcal{C})$ and $K_0(\mathcal{A}^N) \to K_0(\mathcal{C})$ are trivial. Hence $\mathcal{A}^0 = \mathcal{A}^N$ as full sub-∞-category of $\mathcal{C}$. Therefore $K_0(\mathcal{A}^0) \to K_0(\mathcal{A})$ and $K_0(\mathcal{A}^N) \to K_0(\mathcal{A})$ have the same image. We conclude $N = 0$, so $K_0(\mathcal{A}) \to K_0(\mathcal{C})$ is injective.

$\square$

2.9 Remark

Using the fact that Karoubi equivalences induce monomorphisms, the “Galois-like correspondence” of the theorem can be explained better as a correspondence of sets

$$\begin{aligned} \{\text{replete stable subcategories of } \mathcal{C}\} &\xrightarrow{\sim} \{\text{subgroups of } K_0(\mathcal{C})\} \\ \mathcal{A} \subset \mathcal{C} &\mapsto K_0(\mathcal{A}) \\ \mathcal{C}^H := \{x \in \mathcal{C} \colon [x] \in H\} &\leftarrow\!\shortmid H \subset K_0(\mathcal{C}). \end{aligned}$$

3 ARROWS, SEQUENCES, AND INVARIANTS

This chapter aims to present an overview of the tools that will be utilised throughout the thesis. It will start with a brief introduction to the arrow and twisted arrows categories, followed by an exploration of localisation and Verdier quotients. In the interest of time, we have chosen to omit the various proofs; nevertheless, we assure that everything we state has a reference. Next, the concepts of Verdier and Karoubi sequences will be introduced, and we will examine functors that display favourable properties towards these sequences. These results are widely recognised and are based on established findings within triangulated categories. The proofs presented in this chapter are, in a way, based on [Cal+21b], although we have made efforts to expand and elaborate on certain aspects.

3.1 VARIOUS.

3.1.1 *Arrows and Twisted Arrows.*

Our definition of K-theory anima for stable ∞-categories will be based on the $S_\bullet$-construction and $Q_\bullet$-construction. For these it is necessary to consider the higher analogue of the classical arrow category and twisted arrow category.

Fix an ∞-category $\mathcal{C}$. The ARROW CATEGORY of $\mathcal{C}$ is

$$\mathrm{Arr}(\mathcal{C}) := \mathbf{Fun}([1], \mathcal{C})$$

and the TWISTED ARROW CATEGORY of $\mathcal{C}$ can be defined as the pullback

$$\begin{tikzcd} \mathrm{TwArr}(\mathcal{C}) \arrow[r] \arrow[d, "{(s,t)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & */\mathbf{An} \arrow[d] \\ \mathcal{C}^{op} \times \mathcal{C} \arrow[r, "\mathbf{Map}_{\mathcal{C}}"] & \mathbf{An} \end{tikzcd},$$

i.e.

$$(s, t) : \mathrm{TwArr}(\mathcal{C}) \to \mathcal{C}^{op} \times \mathcal{C}$$

is the left fibration classifying the **Map**-anima functor

$$\mathbf{Map}_{\mathcal{C}} : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{An}\,.$$

3.1 Remark (Definition of Twisted arrow category via simplicial set)

Alternatively (and more classically), $\mathrm{TwArr}(\mathcal{C})$ ca be defined level-wise as the simplicial set

$$\mathrm{TwArr}(\mathcal{C})_n := \mathrm{Hom}_{\mathrm{sSet}}(([n])^{op} \star [n], \mathcal{C}) \in \mathcal{S}et.$$

For every $n \geqslant 0$, there is a unique isomorphism of simplicial sets $([n])^{op} \star [n] \cong [2n+1]$; so we can identify n-simplices of $\mathrm{TwArr}(\mathcal{C})$ with $(2n+1)$-simplices of $\mathcal{C}$. Denote with σ an n-simplex of $\mathrm{TwArr}(\mathcal{C})$, and with $\overline{\sigma}$ the corresponding $(2n+1)$-simplex of $\mathcal{C}$. Then, in terms of face and degeneracy maps, we have

$$\overline{d_i\sigma} = d_{n-i}d_{n+i+1}\overline{\sigma}, \quad \text{and } \overline{s_i\sigma} = s_{n-i}s_{n+i+1}\overline{\sigma}.$$

There are canonical maps (induced level-wise by the two possible inclusion of $[n]$ in $([n])^{op} \star [n]$)

$$s : \mathrm{TwArr}(\mathcal{C}) \to \mathcal{C}^{op}, \quad \text{and } t : \mathrm{TwArr}(\mathcal{C}) \to \mathcal{C}.$$

We should prove that $\mathrm{TwArr}(\mathcal{C})$ is actually an ∞-category. This fact is a corollary of the following proposition.

3.2 Reference ([Lur17, sec. 5.2.1] or Kerodon)
The map

$$(s,t) : \mathrm{TwArr}(\mathcal{C}) \to \mathcal{C}^{op} \times \mathcal{C}$$

is a left fibration and it classifies the Map-anima functor

$$\mathbf{Map}_{\mathcal{C}} : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{An}\,.$$

This means that (s,t) is a left fibration and $\mathrm{TwArr}(\mathcal{C})$ fits into a cartesian square in $\mathbf{Cat}_\infty$

$$\begin{tikzcd}
\mathrm{TwArr}(\mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & */\mathbf{An} \arrow[d] \\
\mathcal{C}^{op} \times \mathcal{C} \arrow[r] & \mathbf{An}
\end{tikzcd}$$

Let us see explicitly what the objects and 1-morphisms of these two ∞-categories are.

- The objects of both are morphisms in $\mathcal{C}$.
- A 1-morphism $f \to g$ in $\mathrm{Arr}(\mathcal{C})$ is a (full) square

$$\begin{tikzcd} {} \arrow[r, "p"] \arrow[d, "f"'] & {} \arrow[d, "g"] \\ {} \arrow[r, "q"'] & {} \end{tikzcd}.$$

A morphism $f \to g$ in $\mathrm{TwArr}(\mathcal{C})$ is a (full) square

$$\begin{tikzcd} {} \arrow[d, "f"'] & {} \arrow[l, "p"'] \arrow[d, "g"] \\ {} \arrow[r, "q"'] & {} \end{tikzcd},$$

which we can think as a factorization of g through f.

3.3 Remark
There are relation between the arrow (∞)-)category, the twisted arrow (∞)-)category, and their 1-categorical counterparts. Consider a 1-category $\mathcal{C}$

- There exists an isomorphism of simplicial set

$$A : \mathrm{N}(\mathrm{Arr}(\mathcal{C})) \cong \mathrm{N}(\mathrm{Fun}([1], \mathcal{C})) \cong \mathbf{Fun}([1], \mathrm{N}(\mathcal{C})) \cong \mathrm{Arr}(\mathrm{N}(\mathcal{C})).$$

- There exists an isomorphism of simplicial set

$$T : \mathrm{N}(\mathrm{TwArr}(\mathcal{C})) \to \mathrm{TwArr}(\mathrm{N}(\mathcal{C})),$$

  uniquely determined by the properties $T(f) = f$ and (s, t) are consistent with $\mathrm{N}(\mathcal{C}^{op} \times \mathcal{C}) \cong \mathrm{N}(\mathcal{C})^{op} \times \mathrm{N}(\mathcal{C})$.

$\mathrm{Arr}([n])$ can be pictured as follows

$$\begin{tikzcd}
(0,0) \arrow[r] & (0,1) \arrow[r] \arrow[d] & (0,2) \arrow[r] \arrow[d] & (0,3) \arrow[r, dotted, no head] \arrow[d] & (0,n) \arrow[d] \\
& (1,1) \arrow[r] & (1,2) \arrow[r] \arrow[d] & (1,3) \arrow[r, dotted, no head] \arrow[d] & (1,n) \arrow[d] \\
& & (2,2) \arrow[r] \arrow[dr, dotted, no head] & (2,3) \arrow[r, dotted, no head] \arrow[d, dotted, no head] & (2,n) \arrow[d, dotted, no head] \\
& & & (n-1,n-1) \arrow[r] & (n-1,n) \arrow[d] \\
& & & & (n,n)
\end{tikzcd}$$

The twisted arrow category of $\mathcal{C}$ has to do with spans in $\mathcal{C}$. This becomes transparent when looking at $\mathrm{TwArr}([n])$. Its objects are arrows in $[n]$, that we can organise as follows

$$
\begin{tikzcd}[column sep=small]
 & & & & (0,n)\bullet & & & & \\
 & & & (0,n-1)\bullet \arrow[ur] & & (1,n)\bullet \arrow[ul] & & & \\
 & & (0,2)\bullet \arrow[ur, dotted, no head] & & \bullet \arrow[ul, dotted, no head] \arrow[rr, dotted, no head] & & \bullet \arrow[ul, dotted, no head] & & \\
 & (0,1)\bullet \arrow[ur] & & (1,2)\bullet \arrow[ul] \arrow[ur] & & \bullet \arrow[ul] \arrow[rr, dotted, no head] & & \bullet \arrow[ul] & \\
(0,0)\bullet \arrow[ur] & & (1,1)\bullet \arrow[ul] \arrow[ur] & & \bullet \arrow[ul] \arrow[ur] & & \bullet \arrow[ul] \arrow[rr, dotted, no head] & & \bullet(n,n) \arrow[ul]
\end{tikzcd}
$$

The object of $\mathrm{Arr}([n])$ and $\mathrm{TwArr}([n])$ will be denoted as $(i \leqslant j)$ and the image of $(i \leqslant j)$ through a functor $A : \mathrm{Arr}([n]) \to \mathcal{C})$ or $A : \mathrm{TwArr}([n]) \to \mathcal{C}$ will usually be written as $A_{i,j}$.

3.1.2 *Localisations.*

We will use two different notion of localisation of an ∞-category:

- With the term localisation, we mean Dwyer-Kan localisation; the DWYER-KAN LOCALISATION of an ∞-category $\mathcal{C}$ at set of morphism S is the essentially unique functor $\gamma : \mathcal{C} \to \mathcal{C}[S^{-1}]$, such that for any ∞-category $\mathcal{D}$, the functor (of functor ∞-categories)

  $$\gamma^* : \mathbf{Fun}(\mathcal{C}[S^{-1}]), \mathcal{D}) \to \mathbf{Fun}(\mathcal{C}, \mathcal{D}),$$

  is a fully faithful functor with essential image $\mathbf{Fun}_S(\mathcal{C}, \mathcal{D})$, which is the full sub-$\infty$-category of $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functor mapping the morphisms from S to equivalences.

- With the term LEFT (resp. RIGHT) BOUSFIELD LOCALISATION, we mean a left adjoint (resp. right adjoint) to a fully faithful functor. A left Bousfield localisation g, with fully faithful right adjoint f, is called accessible if g or $L := g \circ f$ are accessible functor [Lur09, sec 5.1.2], or, equivalently, if the essential image of $L\mathcal{C}$ is an accessible sub-∞-category of $\mathcal{C}$. A Bousfield localisation is a functor that has both a fully faithful right adjoint and a fully faithful left adjoint.

3.4 Lemma (When does Dwyer-Kan imply Bousfield? [Cal+21b, Lemma A.2.1])

Consider a small ∞-category $\mathcal{C}$, a collection of morphisms S in $\mathcal{C}$, and the canonical functor $p : \mathcal{C} \to \mathcal{C}[S^{-1}]$. p has a left adjoint (resp.

right adjoint), if and only if for any $x \in \mathcal{C}$ there exists $y \in \mathcal{C}$ and an equivalence $px \to py$ in $\mathcal{C}[S^{-1}]$ such that

$$\mathbf{Map}_{\mathcal{C}}(y, \bullet) \quad (\text{resp. } \mathbf{Map}_{\mathcal{C}}(\bullet, y))$$

sends morphisms of S to equivalences in **An**.
The Yoneda lemma combines the selection of objects y for all x in the ∞-category $\mathcal{C}$ in both cases. This combination results in the appropriate adjoint to the localisation functor. This adjoint is fully faithful, which automatically makes p into either a right or left Bousfield localisation, depending on the case.

3.5 Lemma (Bousfield always implies Dwyer-Kan, [Lur09, Prop. 5.2.7.12])
Consider a right Bousfield localisation (resp. a left Bousfield localisation) $p : \mathcal{C} \to \mathcal{D}$, i.e. p has a fully faithful left (resp. right) adjoint q. p is a Dwyer-Kan localisation at the class of morphisms $f : x \to y$ in $\mathcal{C}$ such that

$$f^* : \mathbf{Map}_{\mathcal{C}}(y, q\bullet) \to \mathbf{Map}(x, q\bullet) \quad (\text{resp. } \mathbf{Map}_{\mathcal{C}}(q\bullet, x) \to \mathbf{Map}(q\bullet, y))$$

is an equivalence of functors $\mathcal{D} \to$ **An**.

3.1.3 *Verdier Quotients.*

3.6 Definition
Consider an exact functors of stable ∞-categories $f : \mathcal{C} \to \mathcal{D}$. A morphism in $\mathcal{D}$ is called an EQUIVALENCE MODULO $\mathcal{C}$ if its fibre lies in the smallest *stable* sub-∞category spanned by the essential image of f. Let S denote the set of equivalences modulo $\mathcal{C}$. The VERDIER QUOTIENT $\mathcal{D}/\mathcal{C}$ of $\mathcal{D}$ by $\mathcal{C}$ is the localisation of $\mathcal{D}$ with respect to S.

3.7 Remark
To define equivalence modulo $\mathcal{C}$ we could ask that the cofibre of a morphism lies in the smallest stable sub-∞-category spanned by the essential image of f. Indeed we have a diagram

$$\begin{tikzcd}
\mathrm{fib} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & x \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dl, phantom, "\ulcorner", very near end] & 0 \arrow[d] \arrow[dl, phantom, "\ulcorner", very near end] \\
0 \arrow[r] & y \arrow[r] & \mathrm{cof}
\end{tikzcd}.$$

So by the pasting law of pushouts

$$\mathtt{cofib} \simeq \Sigma_{\mathcal{D}}(\mathtt{fib}),$$

and vice versa

$$\mathtt{fib} \simeq \Omega_{\mathcal{D}}(\mathtt{cofib}).$$

Therefore, if one belongs to the smallest stable subcategory spanned by the essential image of f, so does the other.

There is a proposition that can be found in [NS18, Thm. I.3.3.(i)] that gives another description of the Verdier quotient. The same proposition is also proved in [BGT13, Prop. 5.6]

3.8 Proposition

Let $f : \mathcal{C} \to \mathcal{D}$ be an exact functor of stable ∞-categories. Then:

(i) $\mathcal{D}/\mathcal{C}$ is stable and the localisation functor $\mathcal{D} \to \mathcal{D}/\mathcal{C}$ is exact.

(ii) For any $\mathcal{E}$ stable ∞-category, the restriction functor

$$\mathbf{Fun}^{\mathrm{ex}}(\mathcal{D}/\mathcal{C}, \mathcal{E}) \to \mathbf{Fun}(\mathcal{D}, \mathcal{E})$$

is fully faithful, with essential image those functors which vanish after pre-composing with f. In particular,

$$\mathcal{C} \to \mathcal{D} \to \mathcal{D}/\mathcal{C}$$

is cofibre in $\mathbf{Cat}_{\infty}^{\mathbf{st}}$.

3.2 VERDIER AND KAROUBI SEQUENCES.

This second section concerns Verdier, Split-Verdier, and Karoubi sequences and their properties.

3.2.1 *Verdier and Karoubi Sequence.*

3.9 Definition (Verdier Sequence)

Consider a sequence $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ in $\mathbf{Cat}_{\infty}^{\mathbf{st}}$ with vanishing composite. This is a VERDIER SEQUENCE if it is both fibre and cofibre in $\mathbf{Cat}_{\infty}^{\mathbf{st}}$. In this case, the map f is referred to as the Verdier inclusion, and p is referred to as the Verdier projection.

3.10 Remark
In $\mathbf{Cat}_\infty^{\mathbf{st}}$, it is really a property, and not a structure, to be a Verdier sequence.

3.11 Definition
A Verdier sequence $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ is called LEFT-SPLIT (resp. RIGHT-SPLIT, RESP. SPLIT) if p admits a left adjoint (resp. right adjoint, resp. both adjoints).

As we will see in lemma 3.21, it is equivalent to requiring f to have a left adjoint (resp. a right adjoint, resp. both adjoints) instead of p.

3.12 Definition
A sequence $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ in $\mathbf{Cat}_\infty^{\mathbf{st}}$ with vanishing composite is a KAROUBI SEQUENCE if the sequence

$$\mathrm{Idem}(\mathcal{C}) \to \mathrm{Idem}(\mathcal{D}) \to \mathrm{Idem}(\mathcal{E})$$

is both fibre and cofibre in $\mathbf{Cat}_\infty^{\mathbf{perf}}$. In this case, f is referred to as Karoubi inclusion, and p is referred ti as Karoubi projection.

3.2.2 *Properties of Verdier Sequences.*

3.13 Proposition
Consider an exact functor of stable ∞-categories $p : \mathcal{D} \to \mathcal{E}$. The following are equivalent:

(i) p is a Verdier projection.

(ii) p is the canonical map into a Verdier quotient of $\mathcal{D}$.

(iii) p is a localisation functor (at the morphisms it takes to equivalences).

Proof. (i) ⇒ (ii) If p is a Verdier projection, then it fits into a fibre-cofibre sequence

$$\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$$

in $\mathbf{Cat}_\infty^{\mathbf{st}}$, where $\to \mathcal{D}$ is the kernel of the functor p. Then by 3.8 (ii), $\mathcal{E}$ is equivalent to $\mathcal{D}/\mathcal{C}$ in an essentially unique way, and the functor p is therefore a canonical map into a Verdier quotient of $\mathcal{D}$.

(ii) ⇒ (iii) The Verdier quotient of a ∞-category is defined as a localisation.

(iii) $\Rightarrow$ *(i)* Consider $f : x \to y$ in $\mathcal{D}$. Then pf is an equivalence in $\mathcal{E}$ if and only if

$$p(\mathtt{cofib}(f)) \cong \mathtt{cofib}(pf) \cong 0$$

since p is exact. Hence, p is an equivalence if and only if $\mathtt{cofib}(f)$ belongs to $\ker(p)$. Therefore p is the localisation at the class of equivalences modulo $\ker(p)$, and again by 3.8 (ii) it is the cofibre of the inclusion $\ker(p) \hookrightarrow \mathcal{D}$. This means we have a cofibre sequence

$$\ker(p) \hookrightarrow \mathcal{D} \to \mathcal{E}.$$

But this sequence is fibre by definition of $\ker(p)$, and so it is bifibre in $\mathbf{Cat}_\infty^{\mathbf{st}}$. □

3.14 Corollary

Verdier projections are essentially surjective.

Proof. By our definition localisations are essentially surjective. □

3.15 Lemma

Consider an exact functor of stable ∞-categories $\mathcal{C} \to \mathcal{D}$. The kernel of the canonical map $p : \mathcal{D} \to \mathcal{D}/\mathcal{C}$ consists of all the objects of $\mathcal{D}$ which are retracts of objects in $\mathcal{C}$.

Proof. Let us identify $\mathcal{C}$ with its essential image in $\mathcal{D}$. First, we want to prove that $\mathcal{C} \subset \ker(p)$. Suppose $x \in \mathcal{C}$, then $\mathtt{fib}(x \to 0) \simeq x \in \mathcal{C}$; therefore, $p(x) \simeq 0$ in $\mathcal{D}/\mathcal{C}$, so $x \in \ker(p)$. Secondly, we want to show that any retract of an object of $\mathcal{C}$ lies in $\ker(p)$. Let $x \in \mathcal{D}$ be a retract of an object $y \in \mathcal{C}$, i.e. we have a commutative diagram in $\mathcal{D}$

$$\begin{tikzcd} x \arrow[r, "i"] \arrow[rr, bend right, "\mathrm{id}_x"] & y \arrow[r, "r"] & x \end{tikzcd}.$$

Now by applying p, we get a retract diagram

$$\begin{tikzcd} p(x) \arrow[r, "p(i)=0"] \arrow[rr, bend right, "\mathrm{id}_{p(x)}"] & 0 \arrow[r, "p(r)=0"] & p(x) \end{tikzcd}.$$

This immediately implies that $p(x) \simeq 0$, because $p(x) \to 0$ and $0 \to p(x)$ are inverses to each other.

Finally, let us prove that any object of the kernel of p is a retract of an object of $\mathcal{C}$. To start, we want to prove that if a functor $f : \mathcal{D} \to \mathbf{Spectra}$ which vanishes on $\mathcal{C}$, then it vanishes on $\ker(p)$. By proposition 3.8(ii), the following diagram commute

$$\begin{tikzcd}
\mathcal{C} \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{D} \arrow[r, "f"] \arrow[dd, "p"] & \textbf{Spectra} \\
& \ker(p) \arrow[ur] \arrow[dl] & & \\
0 \arrow[rr] \arrow[rrruu, bend right] & & \mathcal{D}/\mathcal{C} \arrow[uur, dashed] &
\end{tikzcd},$$

which immediately implies that f must vanished also on $\ker(p)$.
Next, consider for any $x \in \ker(p)$ the functor

$$\phi_x : \mathcal{D} \to \textbf{Spectra}, \quad \phi_x(y) := \operatorname*{colim}_{(\alpha: z \to y) \in \mathcal{C}_{/y}} \underline{\textbf{Map}}(x, \mathtt{cofib}(\alpha)),$$

where $\mathcal{C}_{/y}$ is the comma ∞-category $\mathcal{C} \times \mathcal{D}_{/y}$. We want to prove that ϕ_x vanishes on $\mathcal{C}$.
If $y \in \mathcal{C}$, then $\mathcal{C}_{/y}$ has a final object, namely the identity on y, so we can compute the colimit. Since $\mathtt{cofib}(\mathtt{id}_y) = 0$ we get

$$\begin{aligned}
\phi_x(y) &:= \operatorname*{colim}_{(\alpha: z \to y) \in \mathcal{C}_{/y}} \underline{\textbf{Map}}(x, \mathtt{cofib}(\alpha)) \\
&\cong \underline{\textbf{Map}}(x, \mathtt{cofib}(\mathtt{id}_y)) \\
&= \underline{\textbf{Map}}(x, 0) \\
&\cong 0.
\end{aligned}$$

Now, by our first step, we have that ϕ_x vanishes on $\ker(p)$, so in particular on x itself
At the moment, we know

$$\phi_x(x) := \operatorname*{colim}_{(\alpha: z \to x) \in \mathcal{C}_{/x}} \underline{\textbf{Map}}(x, \mathtt{cofib}(\alpha)) \simeq 0,$$

and $\{(\alpha : z \to y) \in \mathcal{C}_{/c}\}$ is non-empty, since for sure any zero morphism is inside; we claim

$$\underline{\textbf{Map}}(x, \mathtt{cofib}(\alpha)) \simeq 0$$

for any such α. To prove this, fix an $\overline{\alpha}$, take the identity on $\overline{\alpha}$ and the zero morphism $\alpha \to \overline{\alpha}$, for any other α in the collection; then, by universal property, we should have a diagram

$$\begin{tikzcd}
\underline{\mathbf{Map}}(x,\mathrm{cofib}(\overline{\alpha})) \arrow[dr, dotted, "0"] \arrow[drr, bend left, "\mathrm{id}"] & & \\
& 0 \arrow[r, dotted, "0"] & \underline{\mathbf{Map}}(x,\mathrm{cofib}(\overline{\alpha})) \\
\underline{\mathbf{Map}}(x,\mathrm{cofib}(\alpha)) \arrow[uu, "0"] \arrow[ur, dotted, "0"'] \arrow[urr, bend right, dotted, "0"'] & &
\end{tikzcd} ;$$

this tells us that the identity morphism on $\underline{\mathbf{Map}}(x,\mathrm{cofib}(\overline{\alpha}))$ is zero, hence $\underline{\mathbf{Map}}(x,\mathrm{cofib}(\overline{\alpha})) \simeq 0$ for any object of $\overline{\mathcal{C}_{/x}}$.
Now, consider the morphism

$$\mathrm{cofib}(0 \to x) \simeq x \to \mathrm{cofib}(\alpha : z \to x)$$

and the induced

$$\underline{\mathbf{Map}}(x,x) \to \underline{\mathbf{Map}}(x,\mathrm{cofib}(\alpha)) \simeq 0.$$

In particular, the identity map on x vanishes once we compose it with the map $x \to \mathrm{cofib}(\alpha)$; since $\mathcal{D}$ is stable, z is the kernel of $x \to \mathrm{cofib}(\alpha)$, so id_x must factor through z by universal property of the kernel; so the situation is as we see in the following diagram

$$\begin{tikzcd}
x \arrow[dr, dotted] \arrow[drr, bend left, "\mathrm{id}_x"] \arrow[ddr, bend right] & & \\
& z \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & x \arrow[d] \\
& 0 \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \mathrm{cofib}(\alpha)
\end{tikzcd} .$$

We can conclude id_x factors though z, hence x is a retract of $z \in \mathcal{C}$. □

3.16 Proposition
Consider an exact functor of stable ∞-categories $f : \mathcal{C} \to \mathcal{D}$. The following are equivalent:

(i) f is a Verdier inclusion.

(ii) f is fully faithful and its essential image is closed under retract in $\mathcal{D}$.

Proof. Let us prove the two sides.
(*(i)* ⇒ *(ii)*) Any Verdier inclusion is the inclusion of a kernel. Also, from lemma 3.15 we know that this is closed under retracts. Therefore, *(ii)* holds.

((*ii*) ⇒ (*i*)) Consider the cofibre of f, so that we obtain a cofibre sequence

$$\mathcal{C} \xrightarrow{f} \mathcal{D} \to \mathcal{D}/\mathcal{C}.$$

But by the lemma 3.15 f must be equivalent to the kernel of the projection. Therefore this is a fibre-cofibre sequence, hence f is a Verdier inclusion. □

It should now be clear that the following corollary, which summarises the discussion above, is true.

3.17 Corollary

Consider a sequence $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ in $\mathbf{Cat}_\infty^{\mathbf{st}}$ for which the composite vanishes The following are equivalent:

(i) The sequence is a Verdier sequence.

(ii) f is a fully faithful functor with essential image closed under retracts in $\mathcal{D}$, and p exhibits $\mathcal{E}$ as the Verdier quotient of $\mathcal{D}$ by $\mathcal{C}$.

(iii) p is a localisation functor, and f exhibits $\mathcal{C}$ as the kernel of p.

3.18 Lemma

The pullback of a Verdier projection (in $\mathbf{Cat}_\infty^{\mathbf{st}}$) is a Verdier projection.

Proof. Consider a pullback diagram in $\mathbf{Cat}_\infty^{\mathbf{st}}$

$$\begin{tikzcd} \mathcal{D} \arrow[r, "u"] \arrow[d, "p"'] & \mathcal{D}' \arrow[d, "p'"] \\ \mathcal{E} \arrow[r, "l"'] & \mathcal{E}' \end{tikzcd}$$

such that the right leg p' is a Verdier projection.
First of all, notice that p and p' share the same fibre, which we denote as $\mathcal{C}$. This follows immediately by the pasting law for pullbacks.

Our goal is to prove that $\mathcal{E} \simeq \mathcal{D}/\mathcal{C}$. p' is essentially surjective because it is a Verdier projection, then its pullback must be essentially surjective (this holds since we can take the pullback in $\mathbf{Cat}_\infty$) and since $\mathcal{D} \to \mathcal{D}/\mathcal{C}$ is essentially surjective, $\mathcal{D}/\mathcal{C} \to \mathcal{E}$ must be too. It remains to prove that $\mathcal{D} \to \mathcal{E}$ is fully faithful.

Consider a small stable ∞-category X, a small stable sub-∞-category of X called Y, and their Verdier quotient X/Y; consider also z, w objects of X, and denote as $\bar{z}, \bar{w}$ their projection on X/Y. [NS18, Theorem I.3.3

(ii)] gives us a way to compute the mapping anima in the Verdier quotient

$$\mathbf{Map}_{X/Y}(\bar{z},\bar{w}) \simeq \operatorname*{colim}_{\alpha:x\to w\in Y_{/x}} \mathbf{Map}(\bar{z},\mathrm{cofib}(\alpha)),$$

where the colimit is filtered.

We obtain, for d and d' objects of $\mathcal{D}$,

$$
\begin{aligned}
&\mathbf{Map}_{\mathcal{D}/\mathcal{C}}(\overline{d},\overline{d'}) \simeq \\
(1) \simeq &\operatorname*{colim}_{c\to d'\in\mathcal{C}_{/d'}} \mathbf{Map}_{\mathcal{D}}(d,\mathrm{cofib}(\underbrace{c}_{\in\mathcal{D}} \to d')) \\
(2) \simeq &\operatorname*{colim}_{c\to d'\in\mathcal{C}_{/d'}} \Big(\mathbf{Map}_{\mathcal{D}'}(u(d),u(\mathrm{cofib}(\underbrace{c}_{\in\mathcal{D}} \to d'))) \\
&\times_{\mathbf{Map}_{\mathcal{E}'}(lp(d),lp(\mathrm{cofib}(\underbrace{c}_{\in\mathcal{D}}\to d')))} \mathbf{Map}_{\mathcal{E}}(p(d),p(\mathrm{cofib}(c\to d')))\Big) \\
(3) \simeq &\operatorname*{colim}_{c\to d'\in\mathcal{C}_{/d'}} \Big(\mathbf{Map}_{\mathcal{D}'}(u(d),\mathrm{cofib}(\underbrace{c}_{\in\mathcal{D}'} \to u(d'))) \\
&\times_{\mathbf{Map}_{\mathcal{E}'}(lp(d),lp(d'))} \mathbf{Map}_{\mathcal{E}}(p(d),p(d'))\Big) \\
(4) \simeq &\operatorname*{colim}_{c\to u(d')\in\mathcal{C}_{/u(d')}} \Big(\mathbf{Map}_{\mathcal{D}'}(u(d),\mathrm{cofib}(\underbrace{c}_{\in\mathcal{D}'} \to u(d'))) \\
&\times_{\mathbf{Map}_{\mathcal{D}'/\mathcal{C}}(u(d),u(d'))} \mathbf{Map}_{\mathcal{E}}(p(d),p(d'))\Big) \\
(5) \simeq &\operatorname*{colim}_{c\to u(d')\in\mathcal{C}_{/u(d')}} \Big(\mathbf{Map}_{\mathcal{D}'}(u(d),\mathrm{cofib}(\underbrace{c}_{\in\mathcal{D}'} \to u(d')))\Big) \\
&\times_{\mathbf{Map}_{\mathcal{D}'/\mathcal{C}}(u(d),u(d'))} \mathbf{Map}_{\mathcal{E}}(p(d),p(d')) \\
(6) \simeq &\mathbf{Map}_{\mathcal{D}'/\mathcal{C}}(u(d),u(d')) \times_{\mathbf{Map}_{\mathcal{D}'/\mathcal{C}}(u(d),u(d'))} \mathbf{Map}_{\mathcal{E}}(p(d),p(d')) \\
(7) \simeq &\mathbf{Map}_{\mathcal{E}}(p(d),p(d'))
\end{aligned}
$$

where

(1) we apply the formula;

(2) we use the fact that $\mathcal{D}$ is a pullback;

(3) all maps are exact, so they commute with cofibrations; moreover, u maps c to c (the one in $\mathcal{D}$ to the one in $\mathcal{D}'$); $p(c) \cong 0$ necessarily, so $\mathrm{cofib}(lp(c) \to lp(d')) \cong lp(d')$ and $\mathrm{cofib}(p(c) \to p(d')) \cong p(d')$;

(4) we use the fact that $\mathcal{C}_{/d'} \simeq \mathcal{C}_{/u(d')}$; this is true because

$$\mathcal{C}_{/d'} \simeq \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/d'}$$

is equivalently the pullback of

$$\mathcal{C} \times \{d\} \to \mathcal{D} \times \mathcal{D} \leftarrow \mathrm{Arr}(\mathcal{D})$$

and by using the pasting law of pullbacks

$$\begin{tikzcd}
\mathcal{C}_{/d} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Arr}(\mathcal{D}) \arrow[r] \arrow[d, "ev_0 \times ev_1"] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Arr}(\mathcal{D}') \arrow[d, "ev_0 \times ev_1"] \\
\mathcal{C} \times \{d'\} \arrow[r, hook] \arrow[d, "\cong"'] & \mathcal{D} \times \mathcal{D} \arrow[r] & \mathcal{D}' \times \mathcal{D}' \\
\mathcal{C} \times \{u(d')\} \arrow[urr, bend right] & &
\end{tikzcd}$$

(5) the latter terms do not depend on c;

(6) we apply the formula again;

(7) we compute the pullback.

□

3.2.3 *Properties of Split-Verdier Sequences.*

We now want to show that split Verdier sequences satisfy similar properties to the one we have just proved. In order to do this, we need some criteria to recognise when a Verdier projection has a one-sided adjoint.
Consider an exact category $\mathcal{D}$ and a full sub-∞-category $\mathcal{C}$ of it. An object $y \in \mathcal{D}$ is called RIGHT (resp. LEFT) ORTHOGONAL to $\mathcal{C}$ if $\mathbf{Map}_{\mathcal{D}}(x, y) \cong 0$ (resp. $\mathbf{Map}_{\mathcal{D}}(y, x) \cong 0$) for any x object of $\mathcal{C}$. We denote by $\mathcal{C}^r$ (resp. $\mathcal{C}^l$) the full sub-∞-category of $\mathcal{D}$ spanned by right (resp. left) orthogonal objects to $\mathcal{C}$.

3.19 Lemma (Some adjunction rules)
Let $p : \mathcal{D} \to \mathcal{E}$ be an exact functor of stable ∞-categories. The following are equivalent:

(i) p is a Verdier-localisation and it admits a right (resp. left) adjoint.

(ii) p is a localisation and $\ker(p)^r$ (resp. $\ker(p)^l$) projects essentially surjectively to $\mathcal{E}$ via p.

(iii) p is a localisation and $p|_{\ker(p)^r}$ (resp. $p|_{\ker(p)^l}$) is an equivalence.

(iv) p admits a fully faithful right (resp. left) adjoint.

Proof. We are going to prove the right-adjoint variant; the other one is similar.
(i) $\Rightarrow$ *(iv)* Clear from 3.4.

(iv) ⇒ (i) Clear from 3.5.

(i) ⇒ (ii) Suppose we have a Verdier sequence

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] & \mathcal{D} \arrow[r, "p"] \arrow[r, phantom, shift right=2, "\perp"] & \mathcal{E} \arrow[l, bend left, "q"] \end{tikzcd}$$

so that $\mathcal{C} \simeq \ker(p)$, $\mathcal{E} \simeq \mathcal{D}/\mathcal{C}$, and p is a Verdier-localisation $\mathcal{D} \to \mathcal{E}$. By lemma 3.13 p is a localisation. The right orthogonal to $\ker(p)$ is the full sub-∞-category of $\mathcal{D}$ spanned by those $y \in \mathcal{D}$ such that $\mathbf{Map}_{\mathcal{D}}(x, y) \simeq 0$ for any $x \in \ker(p)$. Take $z \in \mathcal{E}$, since p is a localisation, hence essentially surjective, there exists $y \in \mathcal{D}$ such that $py \simeq z$. Let $x \in \ker(p)$. Then

$$\begin{aligned} &\mathbf{Map}_{\mathcal{D}}(x, qp(y)) \simeq \\ &\simeq \mathbf{Map}_{\mathcal{E}}(px, py) \\ &\simeq \mathbf{Map}_{\mathcal{E}}(px, z) \\ &\simeq \mathbf{Map}_{\mathcal{E}}(0, z) \\ &\simeq 0 \end{aligned}$$

Then $qpy \in \ker(p)^r$ and by adjunction $p(qpy) \simeq py \simeq z$. Therefore $\ker(p)^r$ projects essentially surjectively on $\mathcal{E}$.

(ii) ⇒ (i) We use 3.4, for $x \in \mathcal{D}$ take y the preimage of $p(x)$ through $p|_{\ker(p)^r}$, which is well-defined up to equivalence. Take $\alpha : a \to b \in S$, then

$$p(\mathrm{cofib}(a \to b)) \simeq \mathrm{cofib}(p(a \to b)) \simeq \mathrm{cofib}(p(a) \xrightarrow{\simeq} p(b)) \simeq 0;$$

so $\mathrm{cofib}(a \to b) \in \ker(p)$. This implies

$$\mathrm{fib}(\mathbf{Map}_{\mathcal{D}}(b, y) \to \mathbf{Map}_{\mathcal{D}}(a, y)) \simeq \mathbf{Map}(\mathrm{cofib}(a \to b), y) \simeq 0,$$

because $y \in \ker(p)^r$. Therefore $\mathbf{Map}_{\mathcal{D}}(b, y) \to \mathbf{Map}_{\mathcal{D}}(a, y)$ is an equivalence.

(iii) ⇒ (iv) Suppose $p : \mathcal{D} \to \mathcal{E}$ is a localisation at a class of morphisms S, and the restriction of p to $\ker(p)$ gives an equivalence with $\mathcal{E}$. Consider any $x \in \mathcal{D}$ and $y \in \ker(p)^r$ equivalent to $p(x)$ via the equivalence $\mathcal{E} \simeq \ker(p)^r$ (so that we have an equivalence $p(x) \to p(y)$ in $\mathcal{E}$). Consider $a \to b \in S$; then $\mathbf{Map}(b, y) \to \mathbf{Map}(a, y)$ has cofibre

$$\mathrm{cofib}(\mathbf{Map}(b, y) \to \mathbf{Map}(a, y)) \simeq \mathbf{Map}(\mathrm{fib}(a \to b), y) \simeq 0,$$

indeed the fibre belongs to $\ker(p)$, since $a \to b$ is mapped to an equivalence, and $y \in \ker(p)^r$. Therefore $\mathbf{Map}_{\mathcal{D}}(b, y) \to \mathbf{Map}_{\mathcal{D}}(a, y)$ and we can apply 3.4 and we are done.

(iv) ⇒ (iii) p is by definition a left Bousfield localisation, hence a Dwyer-Kan localisation by lemma 3.5. We denote with $q : \mathcal{E} \to \mathcal{D}$ the right adjoint to p; we claim this is the inverse of $p|_{\ker(p)^r}$.
We already know by adjunction rules that q and $p|_{\mathrm{EssIm}(q)}$ are inverses to each other; so we just have to prove $\mathrm{EssIm}(q) \simeq \ker(p)^r$.

Clearly $\mathcal{E} \simeq \mathrm{EssIm}(q)$; take $x \in \ker(p)$, $y \in \mathcal{E}$,

$$\mathbf{Map}(x, qy) \simeq \mathbf{Map}(px, y) \simeq \mathbf{Map}(0, y) \simeq 0.$$

So $\ker(p)^r \supset \mathrm{EssIm}(q) \simeq \mathcal{E}$.

For $x \in \ker(p)^r$, applying 3.4, we could choose both x and $qp(x)$ as a "y" for our x. Then Yoneda lemma gives us that this two must be equivalent.

□

3.20 Remark

In particular, from the last lemma we get that for a right-split Verdier sequence

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] & \mathcal{D} \arrow[r, "p"] \arrow[r, phantom, shift right=2, "\perp"] & \mathcal{E} \arrow[l, bend left, "q"] \end{tikzcd}$$

we get

$$\ker(p)^r \simeq \mathrm{EssIm}(q) \simeq \mathcal{E}.$$

If we have a left-split Verdier sequence

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] & \mathcal{D} \arrow[r, "p"'] \arrow[r, phantom, shift left=2, "\perp"] & \mathcal{E} \arrow[l, bend right, "q"'] \end{tikzcd}$$

we get

$$\ker(p)^l \simeq \mathrm{EssIm}(q) \simeq \mathcal{E}.$$

3.21 Lemma

Consider a sequence in $\mathbf{Cat}_\infty^{\mathbf{st}}$ with vanishing composite

$$(\star)\ \mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}.$$

The following are equivalent:

(i) It is a fibre sequence and p admits a fully faithful left (resp. right) adjoint q.

(ii) It is a cofibre sequence, f is fully faithful, and f admits a left (resp. right) adjoint g.

Furthermore, if these conditions hold, then both

$$(\star)\ \mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E} \quad \text{and} \quad (\star\star)\ \mathcal{E} \xrightarrow{q} \mathcal{D} \xrightarrow{g} \mathcal{C}$$

are Verdier sequences.

Proof. We are going to prove the existence of the left adjoints; the result for the right adjoints is then immediate just by replacing each ∞-category with its opposite.

(i) $\Rightarrow$ *(ii).* Suppose the sequence $(\star)$ is fibre and p admits a fully faithful left adjoint q.

First, f is the inclusion of a kernel, being $(\star)$ fibre, so it is fully faithful. We construct a functor $\tilde{g} : \mathcal{D} \to \mathcal{D}$ as the cofibre of the counit

$$qp \Rightarrow \mathrm{id}_{\mathcal{D}} : \mathcal{D} \to \mathcal{D} \in \mathrm{Fun}(\mathcal{D}, \mathcal{D}).$$

Consider the cocartesian square in $\mathbf{Fun}(\mathcal{D}, \mathcal{D})$ defining $\tilde{g}$

$$\begin{tikzcd} qp \arrow[r, "\mathrm{counit}"] \arrow[d] & \mathrm{id}_{\mathcal{D}} \arrow[d] \\ 0 \arrow[r] & \tilde{g} \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd}.$$

By applying p, which is exact, we obtain a cocartesian square in $\mathbf{Fun}(\mathcal{D}, \mathcal{E})$

$$\begin{tikzcd} p \arrow[r, "\mathrm{unit} \circ p"] \arrow[rr, bend right, dashed, "\mathrm{id}_p"'] & pqp \arrow[r, "p(\mathrm{counit})"] \arrow[d] & p \arrow[d] \\ & 0 \arrow[r] & p\tilde{g} \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd}$$

where, by adjointness, the red-coloured triangle is commutative. (That colour is actually called " engineering orange", but let us keep it simple and call it red.) Moreover, by assumption q is fully faithful, therefore the unit $\mathrm{id}_{\mathcal{E}} \to pq$ is an equivalence. By 2-out-of-3 the map $p(\mathrm{counit})$ is an equivalence, therefore $p\tilde{g}$ must vanish. So $\tilde{g}$ must factor (uniquely) through $\ker(p) \cong \mathcal{C} \xrightarrow{f} \mathcal{D}$. Define g as the unique functor $\mathcal{D} \to \mathcal{C}$ such that $\tilde{g} \simeq fg$.

Consider the canonical transformation

$$\chi : \mathrm{id}_{\mathcal{D}} \to \tilde{g} = \mathrm{cofib}(qp \to \mathrm{id}_{\mathcal{D}}) \simeq fg;$$

we claim χ turns out to be the unit of an adjunction $g \dashv f$.

Take $x \in \mathcal{D}$ and $y \in \mathcal{C}$. We want to prove that the composition of maps

$$\underline{\mathbf{Map}}_{\mathcal{C}}(g(x), y) \xrightarrow{f(\bullet)} \underline{\mathbf{Map}}_{\mathcal{D}}(fg(x), fy) \xrightarrow{\chi^*} \underline{\mathbf{Map}}_{\mathcal{D}}(x, f(y))$$

is an equivalence of spectra. Indeed:

- The first map is an equivalence because f is fully faithful.

- The second map is an equivalence because its fibre is

$$
\begin{aligned}
&\mathtt{fib}(\underline{\mathbf{Map}}_{\mathcal{D}}(fg(x), f(y)) \xrightarrow{\chi^*} \mathbf{Map}(x, f(y))) \simeq \\
&\simeq \underline{\mathbf{Map}}_{\mathcal{D}}(\mathtt{cofib}(x \xrightarrow{\chi} fg(x)), f(y)) \\
&\simeq \underline{\mathbf{Map}}_{\mathcal{D}}(\Sigma_{\mathcal{D}}\mathtt{fib}(x \xrightarrow{\chi} fg(x)), f(y)) \\
&\simeq \underline{\mathbf{Map}}_{\mathcal{D}}(\Sigma_{\mathcal{D}} qp(x), f(y)) \\
&\simeq \Omega\, \underline{\mathbf{Map}}_{\mathcal{D}}(qp(x), f(y)) \\
&\simeq \Omega\, \underline{\mathbf{Map}}_{\mathcal{E}}(p(x), pf(y)) \\
&\simeq \Omega\, \underline{\mathbf{Map}}_{\mathcal{E}}(p(x), 0) \\
&\simeq \Omega 0 \\
&\simeq 0.
\end{aligned}
$$

It remains us to show $(\star)$ is cofibre, or in other words that it is a Verdier sequence.
p has a fully faithful left adjoint, so, being a right-Bousfield localisation, it is a Verdier projection by 3.13. Therefore

$$\ker(p) \to \mathcal{D} \to \mathcal{E}$$

is a Verdier sequence. But this is exactly $(\star)$ since it is already fibre.

Now that we have showed (i) $\Rightarrow$ (ii) and that (i) $\Rightarrow$ "$(\star)$ is Verdier", let us prove that (i) $\Rightarrow$ "$(\star\star)$ is Verdier".
First of all, the composition vanishes

$$
\begin{aligned}
&\mathbf{Map}_{\mathcal{C}}(gqx, y) \simeq \\
&\simeq \mathbf{Map}(qx, fy) \\
&\simeq \mathbf{Map}(x, pfy) \\
&\simeq \mathbf{Map}(x, 0) \\
&\simeq 0
\end{aligned}
$$

for any $y \in \mathcal{C}$ and any $x \in \mathcal{E}$, hence $gq \simeq 0$. We want to show that $(\star\star)$ satisfies (i) for right adjoints. Clearly g admits f as a fully faithful right adjoint; therefore, we have to prove the sequence is fibre, i.e. that $\mathcal{E} \simeq \ker(g)$. If $x \in \ker(g) \subset \mathcal{D}$, then for any $y \in \mathcal{C}$,

$$\mathbf{Map}(x, fy) \simeq \mathbf{Map}(gx, y) \simeq \mathbf{Map}(0, y) \simeq 0.$$

So $\ker(g) \subset \mathcal{C}^{l}$. On the other side, if $x \in \mathcal{C}^{l}$, for any $y \in \mathcal{C}$, then

$$0 \simeq \mathbf{Map}(x, fy) \simeq \mathbf{Map}(gx, y);$$

so by definition gx is a zero object, hence $x \in \ker(g)$. But now $\mathcal{C} \simeq \ker(p)$, so $\ker(g) \simeq \ker(p)^{l} \simeq \mathrm{EssIm}(q) \simeq \mathcal{E}$.

(ii) $\Rightarrow$ (i). Suppose the sequence $(\star)$ is cofibre, f is fully faithful, and it admits a left adjoint g. First of all, g is a localisation by lemma 3.5.

Therefore, the essential image of f coincides with the right orthogonal to $\ker(g)$, by 3.19. Using proposition 3.13, since the sequence is cofibre, we know p exhibits $\mathcal{E}$ is the localisation of $\mathcal{D}$ at $\ker(p)$. Applying 3.15 we see that $\mathrm{EssIm}(f) \cong \ker(p)$, and so it is a Verdier sequence.
To prove that p admits a fully faithful left adjoint we need to use 3.4. We claim that, for $x \in \mathcal{D}$, we can choose as object the fibre of $x \to fg(x)$, which is mapped to px through p, since p is exact and $pfg(x) \simeq 0$. So we have to prove that for any $l : a \to b$ equivalence modulo $\mathcal{C}$, then

$$\mathbf{Map}_{\mathcal{D}}(\mathrm{fib}(x \to fg(x)), l) :$$
$$\mathbf{Map}_{\mathcal{D}}(\mathrm{fib}(x \to fg(x)), a) \to \mathbf{Map}_{\mathcal{D}}(\mathrm{fib}(x \to fg(x)), b)$$

is an equivalence of anime. But, since $\mathbf{Map}_{\mathcal{D}}(\mathrm{fib}(x \to fg(x)), \bullet)$ is left exact

$$\mathrm{fib}\Big(\mathrm{Map}_{\mathcal{D}}(\mathrm{fib}(x \to fg(x)), a) \to \mathbf{Map}_{\mathcal{D}}(\mathrm{fib}(x \to fg(x)), b)\Big) \simeq$$
$$\simeq \mathrm{Map}_{\mathcal{D}}(\mathrm{fib}(x \to fg(x)), \mathrm{fib}(a \to b);$$

hence it is enough to consider $c \in \mathcal{C}$ and show that

$$\mathrm{Map}_{\mathcal{D}}(\mathrm{fib}(x \to fg(x)), f(c)) \simeq 0.$$

For $c \in \mathcal{C}$ we have

$$\mathbf{Map}_{\mathcal{D}}(\mathrm{fib}(x \to fg(x)), f(c)) \simeq \mathrm{cof}\Bigg(\mathbf{Map}_{\mathcal{D}}(x, f(c)) \to \mathbf{Map}_{\mathcal{D}}(fg(x), f(c))\Bigg)$$

and since f is fully faithful

$$\mathbf{Map}_{\mathcal{D}}(fg(x), f(c)) \simeq \mathbf{Map}_{\mathcal{C}}(g(x), c)$$

which identifies with $\mathrm{Map}_{\mathcal{D}}(x, f(c))$ via the adjunction $f \dashv g$. Therefore

$$\mathrm{Map}_{\mathcal{D}}(\mathrm{fib}(x \to fg(x)), f(c)) \simeq \mathrm{cof}(\mathrm{id}) \sim 0,$$

and so the morphism l is sent to an equivalence.

□

3.22 Remark

To sum up what we have just seen in the proof:

- the left adjoint g to $f$identifies with the cofibre of the unit $\mathrm{id}_{\mathcal{D}} \Rightarrow pq$ “as a morphism” $\mathcal{D} \to \mathcal{C}$;
- the left adjoint q to p identifies with the fibre of the counit $fg \to \mathrm{id}_{\mathcal{D}}$ “as a morphism” $\mathcal{E} \to \mathcal{D}$.

The following remark should be clear from the construction of $\tilde{g}$ in the proof of lemma 3.21.

3.23 Remark
If

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"'] & \mathcal{D} \arrow[l, dashed, bend right, "g"'] \arrow[r, "p"'] & \mathcal{E} \arrow[l, dashed, bend right, "q"'] \end{tikzcd}$$

is a left split Verdier sequence, then for all $d \in \mathcal{D}$, there is a bifibre sequence

$$qp(d) \to d \to fg(d).$$

Furthermore

$$\begin{tikzcd} \mathcal{E} \arrow[r, "q"] & \mathcal{D} \arrow[l, dashed, bend left, "p"] \arrow[r, "g"] & \mathcal{C} \arrow[l, dashed, bend left, "f"] \end{tikzcd}$$

is a right split Verdier sequence.

If

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] & \mathcal{D} \arrow[l, dashed, bend left, "g'"] \arrow[r, "p"] & \mathcal{E} \arrow[l, dashed, bend left, "q'"] \end{tikzcd}$$

is a right split Verdier sequence, then for all $d \in \mathcal{D}$, there is a bifibre sequence

$$fg'(d) \to d \to q'p(d).$$

Furthermore

$$\begin{tikzcd} \mathcal{E} \arrow[r, "q'"'] & \mathcal{D} \arrow[l, dashed, bend right, "p"'] \arrow[r, "g'"'] & \mathcal{C} \arrow[l, dashed, bend right, "f"'] \end{tikzcd}$$

is a left split Verdier sequence.

The following two corollaries summarise the discussion.

3.24 Corollary
An exact functor of stable ∞-categories $p : \mathcal{D} \to \mathcal{E}$ is a left (resp. right) split Verdier projection if and only if it admits a fully faithful left (resp. right) adjoint.

Proof. Lemma 3.19. □

3.25 Corollary

An exact functor of stable ∞-categories $f : \mathcal{C} \to \mathcal{D}$ is a left (resp. right) split Verdier inclusion if and only if it is fully faithful and it admits a left (resp. right) adjoint.

Proof. Lemma 3.21 and 3.19. $\square$

Similarly, the same also holds for split Verdier projections and split Verdier inclusion by asking the existence of both adjoints.

As a consequence of this corollaries we have:

3.26 Corollary

The pullback of a left-split (resp. right-split) Verdier projection is a left-split (resp. right-split) Verdier projection.
The pullback of split verdier projection is a split Verdier projection.

Proof. Let us prove the statement for left-split Verdier projection. Consider a pullback diagram in $\mathbf{Cat}_\infty^{\mathbf{st}}$

$$\begin{tikzcd} \mathcal{D} \arrow[r, "u"] \arrow[d, "p"'] & \mathcal{D}' \arrow[d, "p'"] \\ \mathcal{E} \arrow[r, "l"'] & \mathcal{E}' \end{tikzcd}$$

such that the right leg p' is a Verdier projection.
Using the pasting law for pullbacks, p and p' share the same fibre that we denote as $\mathcal{C}$. So, considering p' has a left adjoint q', we have a diagram in $\mathbf{Cat}_\infty^{\mathbf{st}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{D} \arrow[r, "u"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{D}' \arrow[d, "p'"] \\ 0 \arrow[r] & \mathcal{E} \arrow[r, "l"'] & \mathcal{E}' \arrow[u, bend left, "q'"] \end{tikzcd}.$$

Consider now the pair of morphism $id_{\mathcal{E}}$ and $q'l : \mathcal{E} \to \mathcal{D}'$. Notice that

$$l \circ id_{\mathcal{E}} \simeq l \simeq id_{\mathcal{E}'} \circ l \simeq p'(q'l),$$

where $p'q' \simeq id_{\mathcal{E}'}$. By universal property of the pullback $\mathcal{D}$, there exists a (essentially unique) functor $q : \mathcal{E} \to \mathcal{D}$ such that

$$pq \simeq id_{\mathcal{E}}, \quad \text{and } uq \simeq q'l.$$

Let us verify that q is left adjoint to p. Take $x \in \mathcal{E}$ and $y \in \mathcal{D}$:

$$\begin{aligned}
&\mathbf{Map}_{\mathcal{D}}(qx, y) \simeq \\
(1) &\simeq \mathbf{Map}_{\mathcal{E}}(pqx, py) \times_{\mathbf{Map}_{\mathcal{E}'}(p'uqx, p'uy)} \mathbf{Map}_{\mathcal{D}'}(uqx, uy) \\
(2) &\simeq \mathbf{Map}_{\mathcal{E}}(x, py) \times_{\mathbf{Map}_{\mathcal{E}'}(p'q'lx, p'uy)} \mathbf{Map}_{\mathcal{D}'}(q'lx, uy) \\
(3) &\simeq \mathbf{Map}_{\mathcal{E}}(x, py) \times_{\mathbf{Map}_{\mathcal{D}'}(q'p'q'lx, uy)} \mathbf{Map}_{\mathcal{D}'}(q'lx, uy) \\
(4) &\simeq \mathbf{Map}_{\mathcal{E}}(x, py) \times_{\mathbf{Map}_{\mathcal{D}'}(q'lx, uy)} \mathbf{Map}_{\mathcal{D}'}(q'lx, uy) \\
(5) &\simeq \mathbf{Map}_{\mathcal{E}}(x, py)
\end{aligned}$$

where

(1) we used the fact the mapping anima of a pullback is pullback of mapping animae;

(2) we used the properties defining q;

(3) we used adjointness $q' \dashv p'$;

(4) we used $q'p'q' \simeq q'$ by adjointness;

(5) we used the property of pullback.

Moreover, q is fully faithful because

$$\mathbf{Map}_{\mathcal{D}}(qx, qy) \simeq \mathbf{Map}_{\mathcal{E}}(x, pqy) \simeq \mathbf{Map}(_{\mathcal{E}}x, y)$$

because $pq \simeq \mathrm{id}_{\mathcal{E}}$; or immediately because the unit of the adjunction $q \dashv p$ is an equivalence. $\square$

We can now state a criterions to recognise when a left-split/right split Verdier sequences.

3.27 Corollary

Consider a stable ∞-category $\mathcal{D}$ and two full stable subcategories $\mathcal{C}, \mathcal{E} \subset \mathcal{D}$ such that $\mathbf{Map}_{\mathcal{D}}(x, y) \simeq 0$ for any $x \in \mathcal{C}$ and $y \in \mathcal{E}$. Then the following are equivalent

(i) $\mathcal{C} \hookrightarrow \mathcal{D}$ admits a right adjoint $p : \mathcal{D} \to \mathcal{C}$ and the inclusion $\mathcal{E} \hookrightarrow \mathcal{C}^r$ is an equivalence.

(ii) $\mathcal{C} \hookrightarrow \mathcal{D}$ is a Verdier inclusion and the projection $\mathcal{E} \to \mathcal{D}/\mathcal{C}$ is an equivalence.

(iii) $\mathcal{E} \hookrightarrow \mathcal{D}$ admits a left adjoint $q : \mathcal{D} \to \mathcal{E}$ and the inclusion $\mathcal{C} \hookrightarrow \mathcal{E}^l$ is an equivalence.

(iv) $\mathcal{E} \hookrightarrow \mathcal{D}$ is a Verdier inclusion and the projection $\mathcal{C} \to \mathcal{D}/\mathcal{E}$ is an equivalence.

If these conditions are true, then

$$\mathcal{C} \hookrightarrow \mathcal{D} \xrightarrow{q} \mathcal{E}, \quad \text{and } \mathcal{E} \hookrightarrow \mathcal{D} \xrightarrow{p} \mathcal{C}$$

formed by the inclusions and their adjoints are right-split and left-split Verdier sequences, respectively.

Proof. The implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) are dual; also, the implications (iv) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) are dual. So if we prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), we are done proving the equivalences.

Assume (i): we have an adjunction

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook, bend left, "i"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "p"] \end{tikzcd}$$

and $\mathcal{E} \xhookrightarrow{\simeq} \mathcal{C}^r$. By assumption, i is a fully faithful functor with a right adjoint. By corollary 3.25, i is a right-split Verdier inclusion. By taking the cofibre, we obtain a right-split Verdier sequence

$$\mathcal{C} \to \mathcal{D} \xrightarrow{\pi} \mathcal{D}/\mathcal{C}.$$

But then

$$\mathcal{E} \simeq \mathcal{C}^r \simeq \ker(\pi)^r \simeq \mathcal{D}/\mathcal{C},$$

where the last equivalence comes from 3.19. In particular, the map $\mathcal{D}/\mathcal{E}$ induced by p is an equivalence.

Assume (ii): $\mathcal{C} \xhookrightarrow{j} \mathcal{D}$ is a Verdier inclusion and the projection $\mathcal{E} \to \mathcal{D}/\mathcal{C}$ is an equivalence. Then we have a Verdier sequence

$$\mathcal{C} \xhookrightarrow{j} \mathcal{D} \xrightarrow{\pi} \mathcal{D}/\mathcal{C}$$

where $\mathcal{C} \simeq \ker(\pi)^E$. Now π is a localisation, and $\mathcal{C}^r \simeq \ker(\pi)^r \supset \mathcal{E}$ projects surjectively on $\mathcal{D}/\mathcal{C}$ (by assumption the composition $\mathcal{E} \hookrightarrow \mathcal{D} \xrightarrow{\pi} \mathcal{D}/\mathcal{C}$ is an equivalence). Condition (ii) if lemma 3.19 is satisfied. Therefore, we find a left adjoint to the projection $\mathcal{D} \to \mathcal{D}/\mathcal{C}$. Moreover the inclusion of $\mathcal{E}$ into $\mathcal{C}^r \simeq \ker(\pi)^r$ is an equivalence. Since both project to $\mathcal{D}/\mathcal{C}$ by an equivalence. Applying lemma 3.21, we get a right adjoint q to i.

The fact that

$$\mathcal{C} \to \mathcal{D} \to \mathcal{E}, \quad \text{and} \quad \mathcal{E} \to \mathcal{D} \to \mathcal{C}$$

are Verdier sequences follows immediately from (ii) and (iv) combined with lemma 3.21.

$\square$

3.28 Lemma

Consider a split Verdier sequence

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "f"] & \mathcal{D} \arrow[l, dashed, bend right, "g"'] \arrow[l, dashed, bend left, "g'"] \arrow[r, "p"] & \mathcal{E} \arrow[l, dashed, bend right, "q"'] \arrow[l, dashed, bend left, "q'"]
\end{tikzcd}.$$

Then

$$gq' \simeq \mathrm{cofib}(q \Rightarrow q') \simeq \mathrm{cofib}(g' \Rightarrow g) \simeq \Sigma g' q.$$

Moreover, for any $d \in \mathcal{D}$ there are bicartesian squares

$$\begin{tikzcd}
d \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & fg(d) \arrow[d] \\
q'p(d) \arrow[r] & fgq'p(d) \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}$$

and

$$\begin{tikzcd}
fg'qp(d) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & fg'(d) \arrow[d] \\
qp(d) \arrow[r] & d \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}.$$

Proof. Since q and q' are fully faithful we have that the q-unit and the q'-counit functors

$$\mathrm{id}_{\mathcal{E}} \Rightarrow pq, \quad pq \Rightarrow \mathrm{id}_{\mathcal{E}}.$$

are natural equivalences. We want a canonical morphism $q \Rightarrow q'$. To obtain this, we notice that such kinds of functors correspond exactly to morphisms

$$qp \Rightarrow \mathrm{id}_{\mathcal{D}},$$

because by pre-composing with q' we obtain

$$q \simeq q(\mathrm{id}) \simeq qpq' \Rightarrow q'$$

and q' is fully faithful (so we can go back). We have a canonical morphism $qp \Rightarrow \mathrm{id}_{\mathcal{D}}$, which is the q-counit. Moreover,

$$\begin{aligned}
&p(\mathrm{cofib}(q \Rightarrow q')) \simeq \mathrm{cofib}(pq \to pq') \simeq \\
&\simeq \mathrm{cofib}\Big(\mathrm{id} : \mathrm{id}_{\mathcal{E}} \underset{q-\mathrm{unit}}{\overset{\simeq}{\Rightarrow}} pq \underset{pq(q'-\mathrm{counit}^{-1})}{\overset{\simeq}{\Rightarrow}} pqpq' \underset{p(q-\mathrm{counit})q'}{\Rightarrow} \\
&\quad \Rightarrow pq' \underset{q'-\mathrm{counit}}{\overset{\simeq}{\Rightarrow}} \mathrm{id}_{\mathcal{E}}\Big) \\
&\simeq \mathrm{cofib}(\mathrm{id} : \mathrm{id}_{\mathcal{E}} \Rightarrow \mathrm{id}_{\mathcal{E}}) \simeq 0
\end{aligned}$$

This implies that $\mathtt{cofib}(q \Rightarrow q')$ factors through f (and so $\mathcal{C}$).

Now consider the bifibre sequence in $\mathcal{D}$

$$qp(d) \to d \to fg(d),$$

which we have from the previous lemma. By plugging in $q'(e)$ in the place of d, for $e \in \mathcal{E}$, we obtain

$$q(e) \simeq qpq'(e) \to q'(e) \to fgq'(e),$$

in which the composition of the first two maps comes exactly from our morphism applied to $e \in \mathcal{E}$. Thus

$$\mathtt{cofib}(q(e) \to q'(e)) \simeq fgq'(e),$$

and since everything is natural in e we obtain

$$\mathtt{cofib}(q \Rightarrow q') \simeq fgq' \in \mathbf{Fun}^{\mathrm{ex}}(\mathcal{E} \to \mathcal{D}).$$

Nevertheless, as cofibre of natural transformation, we want it as a functor $\mathcal{E} \to \mathcal{C}$, so we can drop the f.

Now consider the bifibre sequence in $\mathcal{D}$

$$fg'(d) \to d \to q'p(d)$$

and plug in $d = q(e)$; we obtain

$$fg'q(e) \to q(e) \to q'pq(e) \simeq q'(e),$$

in which the composition of the last two maps comes exactly from our morphism applied to $e \in \mathcal{E}$. So, since everything is natural in e, we obtain that

$$\Omega\mathtt{cofib}(q \Rightarrow q') \simeq \mathtt{fib}(q \Rightarrow q') \simeq fg'q.$$

As above we drop f and get

$$\mathtt{cofib}(q \Rightarrow q') \simeq \Sigma g'q.$$

We want a similar construction for a canonical $g' \Rightarrow g$. Notice that any such a morphism corresponds to $fg' \Rightarrow \mathrm{id}$ by composing with g'. A canonical morphism like that is the g'-counit. Now

$$\mathtt{cofib}(g'q \Rightarrow gq \simeq 0) \simeq \Sigma g'q$$

and

$$\mathtt{cofib}(0 \simeq g'q' \Rightarrow gq') \simeq gq',$$

and we already know $\Sigma g'q \simeq gq'$. Notice that this factors through p: by precomposing with f, which is exact and fully faithful

$$\mathtt{cofib}(g'f \Rightarrow gf) \simeq \mathtt{cofib}(\mathrm{id} \to \mathrm{id}) \simeq 0.$$

Therefore

$$gq' \simeq \mathtt{cofib}(q \Rightarrow q') \simeq \mathtt{cofib}(g' \Rightarrow q)\sigma\Sigma g'q.$$

It just remain to show that the two diagrams above are bifibre squares in $\mathcal{D}$. Consider

$$
\begin{tikzcd}
 & fg'(d) \arrow[r] \arrow[d] & 0 \arrow[d] \\
qp(d) \arrow[r] \arrow[d] & d \arrow[r] \arrow[d] & q'p(d) \\
0 \arrow[r] & fg(d) &
\end{tikzcd},
$$

which we complete to a diagram in which each square is bicartesian

$$
\begin{tikzcd}
\Omega\star \arrow[r] \arrow[d] & fg'(d) \arrow[r] \arrow[d] & 0 \arrow[d] \\
qp(d) \arrow[r] \arrow[d] & d \arrow[r] \arrow[d] & q'p(d) \arrow[d] \\
0 \arrow[r] & fg(d) \arrow[r] & \star
\end{tikzcd}
$$

Now, consider the bifibre sequence

$$qp(d) \to d \to fg(d)$$

and plug in $d \simeq q'p(d')$ so that we obtain a bifibre sequence

$$qpq'p(d') \to q'p(d') \to fgq'p(d')$$

where the first morphism turns out to be the same as the composition of the central horizontal row of the diagram. Since everything is natural in d',

$$\star \simeq fgq'p$$

and

$$\star[-1] \simeq fgq'p[-1].$$

□

3.2.4 *Stable Recollements.*

The notion of recollement is developed in [Lur17, Appendix A.8] In the stable case, it coincides with split Verdier sequences.

3.29 Definition

Consider a stable ∞-category $\mathcal{D}$ and two stable sub-∞-categories $\mathcal{C}$ and $\mathcal{E}$. $\mathcal{D}$ is a STABLE RECOLLEMENT of $\mathcal{C}$ and $\mathcal{E}$ if

(a) Both inclusions have a left adjoint, which will denote $L_{\mathcal{C}}$ and $L_{\mathcal{E}}$, respectively.

(b) The composite

$$\mathcal{C} \to \mathcal{D} \xrightarrow{L_{\mathcal{E}}} \mathcal{E}$$

vanishes.

(c) If α is a morphism in $\mathcal{D}$ such that $L_{\mathcal{C}}(\alpha)$ and $L_{\mathcal{E}}(\alpha)$ are equivalences, then α is an equivalence. If the pair of functors $(L_{\mathcal{C}}, L_{\mathcal{E}})$ satisfies this property, we say it is *jointly conservative*.

We can portray a stable recollement as a diagram of the form

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[r, phantom, shift left=2, "\perp"] & \mathcal{D} \arrow[l, bend right, "L_{\mathcal{C}}"'] \arrow[r, "L_{\mathcal{E}}"] \arrow[r, phantom, shift right=2, "\perp"'] & \mathcal{E} \arrow[l, bend left] \end{tikzcd}$$

For the sake of completeness, let us give the definition of recollement in the general (non-stable) case.

3.30 Definition ([Lur17, Def A.8.1.])

Consider a stable ∞-category $\mathcal{D}$, which admits finite limits, and two full sub-∞-categories $\mathcal{C}$ and $\mathcal{E}$. $\mathcal{D}$ is a recollement of $\mathcal{C}$ and $\mathcal{E}$ if the following conditions are satisfied:

(a′) The full subcategories $\mathcal{C}$ and $\mathcal{E}$ are stable under equivalence;

(b′) Both inclusions have a left adjoint, which will denote $L_{\mathcal{C}}$ and $L_{\mathcal{E}}$, respectively. As a consequence, both $\mathcal{C}$ and $\mathcal{E}$ are closed under limits in $\mathcal{D}$, and therefore admit finite limits.

(c′) $L_{\mathcal{C}}$ and $L_{\mathcal{E}}$ are left exact.

(d′) $L_{\mathcal{E}}$ carries every object of $\mathcal{C}$ to the final object of $\mathcal{D}$.

(e′) If α is a morphism in $\mathcal{D}$ such that $L_{\mathcal{C}}(\alpha)$ and $L_{\mathcal{E}}(\alpha)$ are equivalences, then α is an equivalence.

3.31 Proposition

Consider a stable recollement $\mathcal{D}$ of $\mathcal{C}$ and $\mathcal{E}$. Then

$$\mathcal{C} \hookrightarrow \mathcal{D} \xrightarrow{L_{\mathcal{E}}} \mathcal{E}$$

is a split Verdier sequence.

Proof. By lemma 3.21 if we are able to prove that

$$\mathcal{C} \hookrightarrow \mathcal{D} \xrightarrow{L_{\mathcal{E}}} \mathcal{E}$$

is a fibre sequence, then, since $L_{\mathcal{E}}$ has a fully faithful right, the sequence is right-split Verdier. But then again by lemma 3.21, we immediately get that it is also left-split, since the inclusion $\mathcal{C} \hookrightarrow \mathcal{D}$ is fully faithful and has a left-adjoint. Let us therefore proceed in this way.

We want to prove that $\ker(L_{\mathcal{E}})$ is equivalent to $\mathcal{C}$. We already know that any object of $\mathcal{C}$ belongs to $\ker(L_{\mathcal{E}})$, since the vanishing of the composition is exactly condition (b). We have to prove that if x is an object of $\ker(L_{\mathcal{E}})$, then x belongs to the essential image of $\mathcal{C}$ in $\mathcal{D}$. Call $k : \ker(L_{\mathcal{E}}) \hookrightarrow \mathcal{D}$, and $c : \mathcal{C} \hookrightarrow \mathcal{D}$. Consider for $x \in \ker(L_{\mathcal{E}})$, the unit $k(x) \to cL_{\mathcal{C}}(k(x))$; then

$$L_{\mathcal{C}}(kx \to cL_{\mathcal{C}}(kx)) \simeq \Big(L_{\mathcal{C}}(kx) \to L_{\mathcal{C}}(cL_{\mathcal{C}}(kx))\Big) \simeq \Big(L_{\mathcal{C}}(kx) \xrightarrow{id} L_{\mathcal{C}}(kx)\Big)$$

$$L_{\mathcal{E}}(kx \to cL_{\mathcal{C}}(kx)) \simeq \Big(L_{\mathcal{E}}(kx) \to L_{\mathcal{E}}(cL_{\mathcal{C}}(k))\Big) \simeq (0 \xrightarrow{\simeq} 0).$$

By using (c), since $kx \to cL_{\mathcal{C}}(kx)$ is mapped to an equivalence by both $L_{\mathcal{C}}$ and $L_{\mathcal{E}}$, $kx \to cL_{\mathcal{C}}(kx)$ must be an equivalence. So we are done, being kx equivalent to the image of $L_{\mathcal{C}}(kx)$ through c. □

So, from one side, all stable recollement produce split Verdier sequence. The converse is also true.

3.32 Proposition
Consider a split Verdier sequence

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[r, bend left, dashed, leftarrow, "g"] \arrow[r, bend right, dashed, leftarrow, "g'"'] & \mathcal{D} \arrow[r, "p"] \arrow[r, bend left, dashed, leftarrow, "q"] \arrow[r, bend right, dashed, leftarrow, "q'"'] & \mathcal{E} \end{tikzcd}.$$

Then $\mathcal{D}$ is a stable recollement of the essential images of f and q.

Proof. (a) and (b) are immediately satisfied. We have to show g and p are jointly conservatives. Given that we are in a stable setting, proving that the functor p and g can jointly detect zero objects is sufficient. Indeed, supposed p and g have this property; we want to show that p and g detect isomorphisms. Let α be a morphism $x \to y$. Then

$$\mathrm{fib}(p(\alpha)) \simeq p(\mathrm{fib}(\alpha)).$$

Now α is an isomorphism if and only if $\mathrm{fib}(\alpha) \simeq 0$. Then clearly $p(\mathrm{fib}(\alpha)) \simeq 0$ and $g(\mathrm{fib}(\alpha)) \simeq 0$.
Suppose $p(\mathrm{fib}(\alpha)) \simeq 0$ and $g(\mathrm{fib}(\alpha)) \simeq 0$. By $p(\mathrm{fib}(\alpha)) \simeq 0$, we get

$\mathrm{fib}(\alpha)$ is in $\mathrm{EssIm}(f)$; so there exists a $c \in \mathcal{C}$ such that $\mathrm{fib}(\alpha) \simeq f(c)$. Then $0 \simeq g(\mathrm{fib}(\alpha)) \simeq gf(c) \simeq c$. Therefore $\mathrm{fib}(\alpha) \simeq f(c) \simeq 0$, and α is an equivalence.

Now the same reasoning with $x \in \mathcal{D}$ instead of $\mathrm{fib}(\alpha)$ gives us that they indeed jointly detect zero objects. □

3.33 Remark

The sequence

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "x \mapsto (x \to 0)"] & \mathrm{Arr}(\mathcal{C}) \arrow[l, bend right, "s"'] \arrow[l, bend left, "\mathrm{fib}"] \arrow[r, "t"'] & \mathcal{C} \arrow[l, bend right, "x \mapsto (0 \to x)"'] \arrow[l, bend left, "x \mapsto \mathrm{id}_x"]
\end{tikzcd}$$

clearly is a stable recollement, hence a split Verdier sequence.

The functor $c := gq'$ (or any other of the three equivalent definitions) is called CLASSIFYING FUNCTOR of the recollement. This name is justified by the following proposition.

3.34 Proposition

Consider a split Verdier sequence

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "f"] & \mathcal{D} \arrow[l, bend right, dashed, "g"'] \arrow[l, bend left, dashed, "g'"] \arrow[r, "p"] & \mathcal{E} \arrow[l, bend right, dashed, "q"'] \arrow[l, bend left, dashed, "q'"]
\end{tikzcd},$$

then the following diagram is cartesian

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "g \Rightarrow cp", "d \mapsto (g(d) \to cp(d))"'] \arrow[d, "p"'] & \mathrm{Arr}(\mathcal{C}) \arrow[d, "t\ (\text{target})"] \\
\mathcal{E} \arrow[r, "c"'] & \mathcal{C}
\end{tikzcd}.$$

Proof. We have a canonical map form $\mathcal{D}$ to the pullback P. The pullback P has as objects pairs

$$(e, a \to c(e)).$$

The canonical map sends d to $(p(d), g(d) \to cp(d))$. The map we now claim is the inverse sends this to

$$q'(e) \times_{f(c(e))} f(a),$$

where the map $q'(e) \to f(c(e))$ is the unit of the $(g \dashv f)$-adjunction (valued in $q'(e)$, and the map $f(a) \to f(c(e))$ is clear.
We have to show that the two compositions are equivalent to identities. First,

$$\begin{aligned} d \mapsto \Big(p(d), g(d) \to cp(d)\Big) &\mapsto q'p(d) \times_{fgq'p(d)} fg(d) \\ &\simeq fg(d) \\ &\simeq d \end{aligned}$$

and

$$\begin{aligned} &(e, a \to c(e)) \mapsto \\ &\mapsto q'(e) \times_{f(c(e))} f(a) \\ &\mapsto \Big(p(q'(e) \times_{f(c(e))} f(a)), \\ &\quad g(q'(e) \times_{f(c(e))} f(a)) \to cp(q'(e) \times_{f(c(e))} f(a))\Big) \\ &\simeq \Big(pq'(e) \times_{pfc(e)} pf(a), \\ &\quad gq'(e) \times_{gfgq'(e)} gf(a) \to gq'pq'(e) \times_{gq'pfc(e)} gq'pf(a) \\ &\simeq (e \times_0 0, c(e) \times_{c(e)} a \to c(e) \times_0 0) \end{aligned}$$

where we used $pq \simeq \mathrm{id}_{\mathcal{E}}$, $gf \simeq \mathrm{id}_{\mathcal{C}}$, $pf \simeq 0$, and we compute some pullbacks. □

3.2.5 *More on Karoubi Equivalences.*

Recall from definition 2.6 that a Karoubi Equivalence is a fully faithful exact functor of stable ∞-categories with dense image.

Morita equivalences are commonly discussed as well. A functor $f : \mathcal{A} \to \mathcal{B}$ is called a MORITA EQUIVALENCE if it becomes equivalent after undergoing idempotent completion. It is worth noting that these two concepts are actually the same.

3.35 Remark

There is also another notion of Morita equivalence, this last concerning rings. Two rings R and S are Morita equivalent if their categories of left modules are equivalent. These two notions of Morita equivalence do not coincide. For example, the (connective) K-theory of a ring is invariant under Morita equivalence (of rings) being defined via (derived) module categories; however, (connective) K-theory is not invariant under Karoubi-equivalences.

For the next results we need some notation; for a stable ∞-category $\mathcal{C}$ we denote

- $\mathrm{Idem}(\mathcal{C})$ the idempotent completion of $\mathcal{C}$;
- $\mathcal{C}^{min}$ the full sub-∞-category of $\mathcal{C}$ spanned by those object whose class in $K_0(\mathcal{C})$ is trivial.

By 2.8) $\mathcal{C}^{min}$ is the smallest dense sub-∞-category of $\mathcal{C}$.
Also, we denote with $\mathbf{Cat}_\infty^{\mathbf{st},\mathbf{K_0=0}}$ the full sub-∞-category of $\mathbf{Cat}_\infty^{\mathbf{st}}$ spanned by those stable ∞-categories $\mathcal{C}$ that have no dense stable sub-∞-categories, or equivalently, by 2.8, with $K_0(\mathcal{C}) = 0$.
In particular, we have

$$\begin{tikzcd} \mathbf{Cat}_\infty^{\mathbf{perf}} \arrow[r, hook] & \mathbf{Cat}_\infty^{\mathbf{st}} \arrow[l, bend right, "\mathrm{Idem}"'] & \mathbf{Cat}_\infty^{\mathbf{st},\mathbf{K_0=0}} \arrow[l, hook'] \end{tikzcd}$$

3.36 Proposition

The localisation functor $\mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{Cat}_\infty^{\mathbf{st}}[\mathrm{Kar.Eq.}^{-1}]$ is both a left and a right Bousfield localisation. The left adjoint is given the functor

$$L : \mathcal{C} \mapsto \mathcal{C}^{min};$$

the right adjoint is given by the functor

$$R : \mathcal{C} \mapsto \mathrm{Idem}(\mathcal{C}).$$

Proof. Consider the functor

$$(\bullet)^{min} : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{Cat}_\infty^{\mathbf{st}}, \mathcal{C} \mapsto \mathcal{C}^{min}.$$

We want to show this functor descends to a right adjoint of the inclusion $\mathbf{Cat}_\infty^{\mathbf{st},\mathbf{K_0=0}} \hookrightarrow \mathbf{Cat}_\infty^{\mathbf{st}}$. If we can prove this, then by definition $\mathbf{Cat}_\infty^{\mathbf{st},\mathbf{K_0=0}}$ is a right Bousfield localisation of $\mathbf{Cat}_\infty^{\mathbf{st}}$. Before doing this, we should prove that $(\bullet)^{min} : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{Cat}_\infty^{\mathbf{st}}, \mathcal{C} \mapsto \mathcal{C}^{min}$ is an actual functor; the it should also be clear that this functor comes with a pointwise fully faithful natural transformation $(\bullet)^{min} \Rightarrow \mathrm{id}$. To see this, we should consider an appropriate simplicial subset of the cocartesian un-straightening of the non-full inclusion $\mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{Cat}_\infty$. We skip this since it is easy but tedious.
There is a very important criterion, we can find in [Lur09, Prop. 5.2.7.4], to verify if a functor $L : \mathcal{D} \to \mathcal{D}$ descends to a left Bousfield localisation:

3.37 Reference

Let $L : \mathcal{D} \to \mathcal{D}$ be a functor of ∞-categories with a natural transformation $\eta : \mathrm{id}_\mathcal{C} \Rightarrow L$ such that

$$\eta_{Lx} : Lx \to LLx$$

and

$$L\eta_x : Lx \to LLx$$

are equivalences for all $x \in \mathcal{D}$. Then $L : \mathcal{C} \to \mathrm{Im}(\mathcal{C})$ is left-adjoint to the inclusion $\mathrm{Im}(L) \subset \mathcal{C}$ with unit η. In particular, L is a Bousfield localisation and $L\eta \simeq \eta L$ as natural transformations (by the triangle identities).

Thomason's theorem 2.8 implies that $(\bullet)^{min}$ satisfies the dual condition to the one just described, therefore it factor through the inclusion, giving a functor

$$(\bullet)^{min} : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{Cat}_\infty^{\mathbf{st}, \mathbf{K_0}=\mathbf{0}}$$

which is a right Bousfield localisation of $\mathbf{Cat}_\infty^{\mathbf{st}}$.
$(\bullet)^{min}$ clearly inverts Karoubi equivalences, since dense inclusions are mapped to essentially surjective inclusions, and so it must descend to a functor

$$(\bullet)^{min} : \mathbf{Cat}_\infty^{\mathbf{st}}[\mathrm{Kar.Eq.}^{-1}] \to \mathbf{Cat}_\infty^{\mathbf{st}, \mathbf{K_0}=\mathbf{0}}.$$

This functor is an equivalence, indeed the functor

$$\mathbf{Cat}_\infty^{\mathbf{st}, \mathbf{K_0}=\mathbf{0}} \hookrightarrow \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{Cat}_\infty^{\mathbf{st}}[\mathrm{K.E.}^{-1}]$$

is an inverse.
So we are in this situation

$$\begin{tikzcd}
\mathbf{Cat}_\infty^{\mathbf{st}} \arrow[rr, "(\bullet)^{min}"] \arrow[d, "\pi"'] \arrow[drr, dashed, "(\bullet)^{min}"] & & \mathbf{Cat}_\infty^{\mathbf{st}} \\
\mathbf{Cat}_\infty^{\mathbf{st}}[\mathrm{Kar.eq.}^{-1}] \arrow[rr, dashed, "(\bullet)^{min}"] & & \mathbf{Cat}_\infty^{\mathbf{st}, \mathbf{K_0}=\mathbf{0}} \arrow[u, hook] \arrow[ll, dashed, bend left, "\text{inverse}"]
\end{tikzcd}$$

Notice that $\mathcal{C}^{min}$ is Karoubi equivalent to $\mathcal{C}$.

This and the adjunction above prove that

$$(\bullet)^{min} : \mathbf{Cat}_\infty^{\mathbf{st}}[\mathrm{Kar.Eq.}^{-1}] \to \mathbf{Cat}_\infty^{\mathbf{st}, \mathbf{K_0}=\mathbf{0}} \subset \mathbf{Cat}_\infty^{\mathbf{st}}$$

is a left adjoint of

$$\mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{Cat}_\infty^{\mathbf{st}}[\mathrm{Kar.Eq.}^{-1}].$$

More explicitly, consider the following diagram (where we used a and b just to distinguish the two functors). Then we have

$$\begin{aligned}
&\mathbf{Map}(i\mathcal{C}_a^{min}, \mathcal{D}) \simeq \\
&\simeq \mathbf{Map}(\mathcal{C}_a^{min}, \mathcal{D}_b^{min}) \\
&\simeq \mathbf{Map}(\mathcal{C}, \pi i \mathcal{D}_b^{min}) \\
&\simeq \mathbf{Map}(\mathcal{C}, \pi \mathcal{D})
\end{aligned}$$

Notice that we had a left-right switch.

We already know, for example from [Lur17, Prop. 5.4.2.16], that

$$\mathrm{Idem} : \mathbf{Cat}^{\mathbf{st}}_{\infty} \to \mathbf{Cat}^{\mathbf{perf}}_{\infty}$$

form a left adjoint to the inclusion $\mathbf{Cat}^{\mathbf{perf}}_{\infty} \hookrightarrow \mathbf{Cat}^{\mathbf{st}}_{\infty}$. The functor $\mathrm{Idem} : \mathbf{Cat}^{\mathbf{st}}_{\infty} \to \mathbf{Cat}^{\mathbf{perf}}_{\infty}$ takes dense inclusions to equivalences. Now Idem descend to an equivalence (as above)

$$\mathrm{Idem} : \mathbf{Cat}^{\mathbf{st}}_{\infty}[\mathrm{Kar.Eq.}^{-1}] \to \mathbf{Cat}^{\mathbf{perf}}_{\infty} .$$

Therefore we have that

$$\mathrm{Idem} : \mathbf{Cat}^{\mathbf{st}}_{\infty}[\mathrm{Kar.Eq.}^{-1}] \to \mathbf{Cat}^{\mathbf{perf}}_{\infty} \subset \mathbf{Cat}^{\mathbf{st}}_{\infty}$$

is a right adjoint to $\mathbf{Cat}^{\mathbf{st}}_{\infty} \to \mathbf{Cat}^{\mathbf{st}}_{\infty}[\mathrm{Kar.qE.}^{-1}]$, where we are having the same switch we had above. □

3.38 Corollary

An exact functor $F : \mathcal{C} \to \mathcal{D}$ of stable ∞-categories is a Karoubi equivalence if and only if its is and equivalence after mininimalization or idempotent completion.

Proof. Notice that Karoubi equivalence are closed under 2-out-of-3 and that any equivalence is a Karoubi equivalence.

("If") We have a diagram, in which we denote Karoubi equivalences with $\sim$,

$$\begin{tikzcd}
\mathcal{C}^{min} \arrow[r, "\sim"'] \arrow[d] & \mathcal{C} \arrow[r, "\sim"'] \arrow[d] & \mathrm{Idem}(\mathcal{C}) \arrow[d] \\
\mathcal{D}^{min} \arrow[r, "\sim"'] & \mathcal{D} \arrow[r, "\sim"'] & \mathrm{Idem}(\mathcal{D})
\end{tikzcd} .$$

If one between $\mathcal{C}^{min} \to \mathcal{D}^{min}$ and $\mathrm{Idem}(\mathcal{C}) \to \mathrm{Idem}(\mathcal{D})$ is an equivalence, then by 2-out-of-3 $\mathcal{C} \to \mathcal{D}$ is a Karoubi equivalence.

("Only if") Suppose $\mathcal{C} \to \mathcal{D}$ is a Karoubi equivalence. Then $\mathcal{C}^{min} \to \mathcal{D}^{min}$ and $\mathrm{Idem}(\mathcal{C}) \to \mathrm{Idem}(\mathcal{D})$ are Karoubi equivalences. But $\mathcal{D}^{min}$ does not have any dense sub-∞-category, besides itself (and equivalent sub-∞-categories). Therefore $\mathcal{C}^{min} \to \mathcal{D}^{min}$ is an equivalence. Similarly, $\mathrm{Idem}(\mathcal{C})$ is already closed under taking retracts (every idempotent already splits), so $\mathrm{Idem}(\mathcal{C}) \to \mathrm{Idem}(\mathcal{D})$ must be an equivalence. □

3.39 Definition

A small stable ∞-category $\mathcal{C}$ for which $K_0(\mathcal{C}) \cong 0$ is called a MINIMAL. The functor $(\bullet)^{min} : \mathbf{Cat}^{\mathbf{st}}_{\infty} \to \mathbf{Cat}^{\mathbf{st}}_{\infty}$ given by the assignment $\mathcal{C} \mapsto \mathcal{C}^{min}$ is called MINIMALISATION.

To summarise, we have a Bousfield localisation

$$\begin{tikzcd} \mathbf{Cat}_\infty^{\mathbf{st}} \arrow[r, bend left, "(\bullet)^{min}"] \arrow[r, bend right, "\mathrm{Idem}(\bullet)"'] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathbf{Cat}_\infty^{\mathbf{st}}[\mathrm{Kar.Eq.}^{-1}] \arrow[l, hook'] \end{tikzcd},$$

a Bousfield localisation

$$\begin{tikzcd} \mathbf{Cat}_\infty^{\mathbf{st}} \arrow[r, bend left, "\mathrm{Idem}(\bullet)"] \arrow[r, bend right, "(\bullet)^{min}"'] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathbf{Cat}_\infty^{\mathbf{perf}} \arrow[l, hook'] \end{tikzcd},$$

and we obtain equivalences

$$\mathbf{Cat}_\infty^{\mathbf{st}}[\mathrm{Kar.Eq.}^{-1}] \simeq \mathbf{Cat}_\infty^{\mathbf{perf}} \simeq \mathbf{Cat}_\infty^{\mathbf{st},\mathbf{K_0=0}}.$$

Moreover we have an adjunction

$$\begin{tikzcd} \mathbf{Cat}_\infty^{\mathbf{st}} \arrow[r, bend left, "(\bullet)^{min}"] \arrow[r, phantom, "\perp"] & \mathbf{Cat}_\infty^{\mathbf{st}} \arrow[l, bend left, "\mathrm{Idem}(\bullet)"] \end{tikzcd}$$

We have also obtained a reason for why the idempotent-completion functor $\mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{Cat}_\infty^{\mathbf{perf}}$ preserves limits and colimits.

3.2.6 *Properties of Karoubi Sequences.*

Our definition of Karoubi sequence is a sequence which becomes fibre-cofibre after idempotent-completlion. By corollary 3.38, it is equivalent to ask the sequence to be fibre-cofibre after minimalisation, or that the sequence is fibre-cofibre after inverting Karoubi equivalences.

3.40 Proposition

Consider a sequence $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ in $\mathbf{Cat}_\infty^{\mathbf{st}}$ with vanishing composite. Then

(a) $\mathrm{Idem}(\mathcal{C}) \to \mathrm{Idem}(\mathcal{D}) \to \mathrm{Idem}(\mathcal{E})$ is fibre in $\mathbf{Cat}_\infty^{\mathbf{perf}}$ if and only if f becomes a Karoubi equivalence when regarded as a functor $\mathcal{C} \to \ker(p)$.

(b) $\mathrm{Idem}(\mathcal{C}) \to \mathrm{Idem}(\mathcal{D}) \to \mathrm{Idem}(\mathcal{E})$ is cofibre in $\mathbf{Cat}_\infty^{\mathbf{perf}}$ if and only if the Verdier quotient of $\mathcal{D}$ by the stable sub-∞-category generated by $\mathrm{Im}(f)$ is Karoubi equivalent to $\mathcal{E}$.

(c) $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ is a Karoubi sequence if and only if f is fully faithful and $\mathcal{D}/\mathcal{C} \to \mathcal{E}$ is a Karoubi equivalence.

In particular, Verdier sequences are Karoubi sequences.

Proof. (a) We know Idem : $\mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{Cat}_\infty^{\mathbf{perf}}$ preserves limits and that $\mathbf{Cat}_\infty^{\mathbf{perf}}$ is complete. Consider the functor $\tilde{f} : \mathcal{C} \to \ker(p)$ which arise as natural factorization of f through $\ker(p)$.
("only if") Then

$$\widetilde{\mathrm{Idem}(f)} \simeq \mathrm{Idem}(\tilde{f}) : \mathrm{Idem}(\mathcal{C}) \to \ \ker(\mathrm{Idem}(p)) \simeq \mathrm{Idem}(\ker(p))$$

is an equivalence. Therefore $f(x)$ is a Karoubi equivalence.
("if") On the other hand, if $\tilde{f}$ is a Karoubi equivalence, then $\widetilde{\mathrm{Idem}(f)}$ gives an equivalence with $\ker(\mathrm{Idem}(p))$. Since $\mathrm{Idem}(\mathcal{C})$ is closed under retracts in $\mathcal{D}$ (all retracts already split), then the sequence must be fibre.

(b) We already know Idem : $\mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{Cat}_\infty^{\mathbf{perf}}$ preserves colimits and $\mathbf{Cat}_\infty^{\mathbf{perf}}$ is cocomplete. Therefore $\mathrm{Idem}(\mathcal{D}/\overline{\mathcal{C}}) \simeq \mathrm{Idem}(\mathcal{D})/\overline{\mathrm{Idem}(\mathcal{C})}$, where $\overline{\bullet}$ is the stable closure of the essential image. So if $\mathcal{D}/\overline{\mathcal{C}} \to \mathcal{E}$, then

$$\mathrm{Idem}(\mathcal{E}) \simeq \mathrm{Idem}(\mathcal{D}/\overline{\mathcal{C}}) \simeq \mathrm{Idem}(\mathcal{D})/\overline{\mathrm{Idem}(\mathcal{C})}.$$

On the other hand, if it is cofibre, then

$$\mathrm{Idem}(\mathcal{E}) \simeq \mathrm{Idem}(\mathcal{D})/\overline{\mathrm{Idem}(\mathcal{C})} \simeq (\mathcal{D}/\overline{\mathcal{C}}).$$

So $\mathcal{E}$ is Karoubi equivalent to $\mathcal{D}/\overline{\mathcal{C}}$.

(c) ("if") Let f be a fully faithful functor, and $\mathcal{D}/\mathcal{C} \to \mathcal{E}$ and equivalence. Then $\ker(p) \simeq \ker(\pi)$ the kernel of the projection $\pi : \mathcal{D} \to \mathcal{D}/\mathcal{C}$. Then using lemma 3.15, the map $f : \mathcal{C} \to \ker(\pi)$ has dense essential image and therefore is a Karoubi equivalence.
("only if") If it is a Karoubi equivalence, then $\mathrm{Idem}(f)$ is fully faithful and essentially surjective. So f is essentially surjective by combining (i) and (ii). □

3.41 Corollary

An exact functor of stable ∞-categories is a Karoubi inclusion if and only if it is fully faithful.

3.42 Corollary

An exact functor of stable ∞-categories $p : \mathcal{D} \to \mathcal{E}$ is a Karoubi projection if and only if it has dense essential image and the induced functor $p : \mathcal{D} \to \mathrm{EssIm}(p)$ is a Verdier projection.

Combining this statement with Thomason's result above, we find

3.43 Corollary
Let $p : \mathcal{D} \to \mathcal{E}$ be a Karoubi projection. The following are equivalent:

(i) p is a Verdier projection.

(ii) p is essentially surjective.

(iii) $K_0(p) : K_0(\mathcal{D}) \to K_0(\mathcal{E})$ is surjective.

3.44 Lemma
The pullback of a Karoubi projection is a Karoubi projection.

Proof. Using lemma 3.18 and corollary 3.42, we just have to prove the pullback of a Karoubi equivalence is a Karoubi equivalence. Consider then a square

$$\begin{tikzcd} \mathcal{D} \arrow[r, "u"] \arrow[d, "p"'] & \mathcal{D}' \arrow[d, "p'"] \\ \mathcal{E} \arrow[r, "l"'] & \mathcal{E}' \end{tikzcd}$$

in which the right leg is a Karoubi equivalence; we want to prove $\mathcal{D} \to \mathcal{E}$ is a Karoubi equivalence. This pullback is given by the full subcategory spanned by those $x \in \mathcal{E}$ such that $l([x] \in K_0(\mathcal{D}') \subset K_0(\mathcal{E}')$. Now Thomason's theorem gives the claim. □

In the context of triangulated categories, the Thomason-Neeman localisation theorem, which appeared in [Nee92], provides a valuable criterion for detecting Karoubi-sequences. We now present the theorem after providing a brief review of the inductive completion of a small ∞-category that admits finite colimits. To fully understand the Ind-object and Ind-completion, a comprehensive overview is provided in [Lur09, sec. 5.3] and to some extent, in [BGT13].
Consider a small ∞-category $\mathcal{C}$ which admits finite colimits. The idempotent completion is a formal procedure to enlarge $\mathcal{C}$ to admit all colimits: we can replace $\mathcal{C}$ by the ∞-category $\mathrm{Ind}(\mathcal{C})$ of Ind-objects of $\mathcal{C}$. This enlargement can be characterised as follows:

(i) There is a fully faithful embedding $j : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$;

(ii) The ∞-category $\mathrm{Ind}(\mathcal{C})$ admits filtered colimits (in fact all colimits, and j preserves finite colimits);

(iii) every object of $\mathrm{Ind}(\mathcal{C})$ can be written as a filtered colimit $\varinjlim X_\alpha$, where each X_α belongs to $\mathcal{C}$;

(iv) every object of $\mathcal{C}$ is compact as an object of $\mathrm{Ind}(\mathcal{C})$.

3.45 Construction (Informal Construction of the Category of Ind-Objects)

At informal level, we can define $\mathrm{Ind}(\mathcal{C})$ as follows. For each filtered diagram $\{A_\alpha\}$ in the ∞-category $\mathcal{C}$, let $[\varinjlim A_\alpha]$ denote the colimit of the diagram in $\mathrm{Ind}(\mathcal{C})$. Morphisms in $\mathrm{Ind}(\mathcal{C})$ are given by

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}([\varinjlim A_\alpha], [\varinjlim B_\beta]) &\cong \varprojlim_{\alpha} \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(A_\alpha, [\varinjlim B_\beta]) \\ &\cong \varprojlim_{\alpha} \varinjlim_{\beta} \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(A_\alpha, B_\beta) \\ &\cong \varprojlim_{\alpha} \varinjlim_{\beta} \mathrm{Hom}_{\mathcal{C}}(A_\alpha, B_\beta),\end{aligned}$$

where the last two equivalences are obtained by (iv) and (i) respectively.

While this construction is useful to understand what we are looking for, it is not very formal. Formally, we can define the ∞-category of Ind-object of a small ∞-category $\mathcal{C}$ as the smallest sub-∞-category of $\mathbf{Fun}(\mathcal{C}^{op}, \mathbf{An}) \simeq \mathbf{PSh}(\mathcal{C})$ containing all representable functors and closed under filtered colimits. Another characterization of this is the sub ∞-category of $\mathbf{PSh}(\mathcal{C})$ spanned by left exact functors. More generally we have the following definition.

3.46 Definition

Consider a small ∞-category $\mathcal{C}$ and a regular cardinal κ. Denote with $\mathrm{Ind}_\kappa(\mathcal{C})$ the full sub-∞-category of $\mathbf{PSh}(\mathcal{C})$ spanned by those functors $f : \mathcal{C}^{op} \to \mathbf{An}$ which are classified by right fibrations $\overline{\mathcal{C}} \to \mathcal{C}$, where $\overline{\mathcal{C}}$ is κ-filtered. We call $\mathrm{Ind}(\mathcal{C}) := \mathrm{Ind}_\omega(\mathcal{C})$ the ∞-category of Ind-objects of $\mathcal{C}$.

There are many things we can say on Ind-completions.

- The Ind completion preserves the stability of ∞-categories and the exactness of functors ([Lur09, Prop. 5.3.5.10]). The colimit preserving extension of suspension is suspension, and the extension of loops is its inverse.
- The extension of a fully faithful functor is again fully faithful [Lur09, Prop. 5.3.5.11].
- The Ind-completion commutes with Verdier quotients ([NS18, Prop. I.3.5].

More generally, there is an equivalence between

- the ∞-category of κ-compact stable presentable ∞-categories with colimit preserving functors;

- the ∞-category of κ-compact stable ∞-categories with exact κ-continuous functors.

We can now state the theorem.

3.47 Theorem (Thomason-Neeman localisation theorem for stable categories)

Consider a sequence $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ in $\mathbf{Cat}_\infty^{\mathbf{st}}$ with vanishing composite. It is a Karoubi sequence if and only if $\mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathcal{E})$ is a Verdier sequence (of non-necessarily small ∞-categories).

Proof. For the discussion above, an exact functor is a Karoubi equivalence if and only if it induces an equivalence on inductive completions. The "only if" direction is immediate. Conversely, given a Karoubi equivalence $\mathcal{C} \to \mathcal{D}$, $\mathrm{Ind}(\mathcal{C})$ is the kernel of by lemma 3.15; in fact cocomplete categories are idempotent complete by [Lur09, Corollary 4.4.5.16]. The claim now follows from our characterisation of Verdier (3.17) and Karoubi sequences (3.40). $\square$

3.48 Remark

This theorem is very important. In a certain sense, it facilitate the transition from the theory we are developing in this thesis to the one outlined in [BGT13], provided that it is applied with appropriate caution and in a particular manner.

3.3 ADDITIVE AND LOCALISING FUNCTORS.

Now that we have introduced the kind of sequences we are going to work with, let us talk about functors that behaves well on them.

For a stable ∞-category $\mathcal{C}$, we will denote with $\mathrm{Seq}(\mathcal{C})$ the ∞-category of bifibre sequences in $\mathcal{C}$, i.e. the full sub-∞-category of $\mathbf{Fun}([1] \times [1], \mathcal{C})$ spanned by those cartesian square with low-left corner 0. It is worth noting $\mathrm{Seq}(\mathcal{C})$ is equivalent to the category of arrows in $\mathcal{C}$, for example by taking pushouts; in particular,$\mathrm{Seq}(\mathcal{C})$ is a stable ∞-category.

3.3.1 *Squares.*

3.49 Definition

Let us introduce some terminology.

- A VERDIER SQUARE is a square of stable ∞-categories and exact functors that is cartesian and has Verdier projections as vertical maps (or, equivalently, localisations as vertical maps).

- A SPLIT VERDIER SQUARE is a square of stable ∞-categories and exact functors that is cartesian and has split Verdier projections as vertical maps.
- A KAROUBI SQUARE is a square of stable ∞-categories and exact functors that becomes a Verdier square after after idempotent-completion.

3.50 Remark

From the discussion in the subsection 3.2.5 , it should be clear that in the definition of Karoubi square, we could replace "after idempotent-completion" with "after inverting Karoubi equivalences" or 'after inverting minimimalazing".
To state more explicitly the condition we want to be satisfied: a Karoubi square is a square that becomes cartesian after localising idempotent completion and both its vertical maps are Karoubi projections.

3.51 Remark

For what we proved in 3.18, 3.26, 3.44, we can just ask the right vertical leg to be respectively a Verdier projection, a split Verdier projection, or a Karoubi projection.

3.52 Remark

(a) A Verdier square (resp. split-Verdier square, resp. Karoubi square) with lower left corner $0 \in \mathbf{Cat}^{\mathrm{st}}_{\infty}$ is a Verdier sequence (resp. split-Verdier sequence, resp. Karoubi sequence).
The converse also holds, any Verdier sequence (resp. split-Verdier sequence, resp. Karoubi sequence) is a Verdier square (resp. split-Verdier square, resp. Karoubi square) with lower left corner $0 \in \mathbf{Cat}^{\mathrm{st}}_{\infty}$. For example a split-Verdier sequence

$$\begin{tikzcd} \mathcal{C} \arrow[r] & \mathcal{D} \arrow[l, dashed, bend right] \arrow[l, dashed, bend left] \arrow[r] & \mathcal{E} \arrow[l, dashed, bend right] \arrow[l, dashed, bend left] \end{tikzcd}$$

is clearly a bicartesian square

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{D} \arrow[d] \\
0 \arrow[r] & \mathcal{E} \arrow[u, bend left, dashed] \arrow[u, bend right, dashed] \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}.$$

(b) $\mathcal{B} \to \mathcal{B} \to 0$ is a split Verdier sequence. Indeed, the square

$$\begin{tikzcd}
\mathcal{B} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{B} \arrow[d] \\
0 \arrow[r] & 0 \arrow[u, bend left, dashed] \arrow[u, bend right, dashed] \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

is clearly bicartesian and the adjoints to $\mathcal{B} \to 0$ are given by the unique map $0 \to \mathcal{B}$.

(c) For any $\mathcal{A}, \mathcal{B}$ stable ∞-categories, their products exists in $\mathbf{Cat}_\infty^{\mathbf{st}}$. Also, $\mathcal{B} \to 0$ is a split Verdier projection, so

$$\begin{tikzcd}
\mathcal{A} \times \mathcal{B} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{B} \arrow[d] \\
\mathcal{A} \arrow[r] & 0
\end{tikzcd}$$

is a split Verdier square.

(d) $\mathbf{Cat}_\infty^{\mathbf{st}}$ is semi-additive, so for any $\mathcal{A}, \mathcal{B}$ stable ∞-categories $\mathcal{A} \times \mathcal{B} \simeq \mathcal{A} \oplus \mathcal{B}$. We know, from the previous points, that $\mathcal{A} \times \mathcal{B} \to \mathcal{A}$ is a split-Verdier projection, so

$$\begin{tikzcd}
\mathcal{B} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{A} \times \mathcal{B} \arrow[d] \\
0 \arrow[r] & \mathcal{A}
\end{tikzcd}$$

is a split Verdier square. It follows that $\mathcal{B} \to \mathcal{A} \times \mathcal{B} \to \mathcal{A}$ is a Verdier sequence.

3.53 Remark

Any Verdier square is also cocartesian in $\mathbf{Cat}_\infty^{\mathbf{st}}$.
Consider a Verdier square

$$\begin{tikzcd} \mathcal{D} \arrow[r, "u"] \arrow[d, "p"'] & \mathcal{D}' \arrow[d, "p'"] \\ \mathcal{E} \arrow[r, "l"'] & \mathcal{E}' \end{tikzcd} .$$

As we have already seen above, by the pasting law of pullbacks, the two vertical maps have common fibre 0. So, we obtain a diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{D} \arrow[r, "u"] \arrow[d, "p"'] & \mathcal{D}' \arrow[d, "p'"] \\ 0 \arrow[r] & \mathcal{E} \arrow[r, "l"'] & \mathcal{E}' \end{tikzcd}$$

where all the squares are cartesian. So, both $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ and $\mathcal{C} \to \mathcal{D}' \to \mathcal{E}'$ are Verdier sequences. In particular, both p and p' are Verdier projection, so both the external and the left square are also cocartesian. Then by the pasting law for pushouts also the right square must be cocartesian.
Similarly, also split-Verdier squares are cocartesian, and Karoubi square are cocartesian after idempotent completion.

3.54 Remark

Every Verdier square is a Karoubi square, since the idempotent-completion functor preserves limits.
A Karoubi square involving idempotent-complete ∞-categories is a Verdier square if and only if its vertical maps are essentially surjective.

3.3.2 *Functors.*

3.55 Definition

A functor $F : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathcal{E}$ such that $F(0)$ is a terminal object of $\mathcal{E}$ is called *reduced*.

3.56 Definition

Consider an ∞-category with finite limits $\mathcal{E}$ and a reduced functor $F : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathcal{E}$.

- F is ADDITIVE (or split-Verdier-localising) if it sends split Verdier squares to Cartesian squares.
- F is VERDIER-LOCALISING if it sends Verdier squares to Cartesian squares.
- F is KAROUBI-LOCALISING if it sends Karoubi squares to Cartesian squares.

The full sub-∞-category of $\mathbf{Fun}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E})$ spanned by respectively by additive, Verdier-localising, and Karoubi-localising functor are denoted respectively by

$$\mathbf{Fun}^{add}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E}), \quad \mathbf{Fun}^{Vloc}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E}), \quad \mathbf{Fun}^{Kloc}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E}).$$

We clearly have inclusions

$$\mathbf{Fun}^{Kloc}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E}) \subset \mathbf{Fun}^{Vloc}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E}) \subset \mathbf{Fun}^{add}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E}) \subset \mathbf{Fun}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E}).$$

3.57 Remark

These sub-∞-categories of $\mathbf{Fun}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E})$ are usually not locally small. To work with locally small ∞-category we should restrict our attention only to accessible additive/Verdier-localising/Karoubi-localising functors. There is more: if we fixes a regular cardinal κ and restrict our attention to κ-accessible additive/Verdier-localising/Karoubi-localising functors then we obtain functor categories

$$\mathbf{Fun}^{Kloc}_{\kappa}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E}) \subset \mathbf{Fun}^{Vloc}_{\kappa}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E}) \subset \mathbf{Fun}^{add}_{\kappa}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E}) \subset \mathbf{Fun}_{\kappa}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E}).$$

All these are presentable. Any way, the most interesting examples of such functors for us is K-theory, which preserves ω-filtered colimits. Recall that a functor is (κ-)accessible if it preserves (κ-)filtered colimits.

3.58 Remark

An additive functor maps split Verdier sequences to fibre sequences. Indeed, any split Verdier sequences

$$\begin{tikzcd} \mathcal{C} \arrow[r] & \mathcal{D} \arrow[r] \arrow[l, dashed, bend right] \arrow[l, dashed, bend left] & \mathcal{E} \arrow[l, dashed, bend right] \arrow[l, dashed, bend left] \end{tikzcd}$$

forms uniquely a bicartesian square

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{D} \arrow[d] \\ 0 \arrow[r] \arrow[ur, phantom, very near end, "\ulcorner"] & \mathcal{E} \arrow[u, dashed, bend left] \arrow[u, dashed, bend right] \end{tikzcd}$$

where the right leg is a split Verdier projection. Therefore the square

$$\begin{tikzcd} F(\mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F(\mathcal{D}) \arrow[d] \\ * \arrow[r] & F(\mathcal{E}) \end{tikzcd}$$

is cartesian and hence is

$$F(\mathcal{C}) \to F(\mathcal{D}) \to F(\mathcal{E})$$

is a fibre sequence.
Similarly, a Verdier-localising functor maps Verdier sequence to fibre sequences, and a Karoubi-localising functor maps Karoubi sequences to fibre sequences.

If $\mathcal{E}$ is stable, the converse of this remark holds, as showed in the following proposition.

3.59 Proposition

Consider a reduced functor $F : \mathbf{Cat}^{\mathbf{st}}_{\infty} \to \mathcal{E}$, where $\mathcal{E}$ is a stable ∞-category. Then F is

- additive if and only if it takes split Verdier sequences to fibre sequences in $\mathcal{E}$.
- Verdier-localising if and only if it takes Verdier sequences to fibre sequences in $\mathcal{E}$.
- Karoubi-localising if and only if it takes Karoubi sequences to fibre sequences in $\mathcal{E}$.

Proof. The "only if" part is contained the previous remark.
Let us prove the "if" part only in the additive case, the other two cases are similar.
Suppose we have a functor which takes split Verdier sequences to exact sequence. Consider a split Verdier square

$$\begin{tikzcd} \mathcal{D} \arrow[r, "u"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{D}' \arrow[d, "p'"] \\ \mathcal{E} \arrow[r, "l"'] \arrow[u, dashed, bend left] \arrow[u, dashed, bend right] \arrow[ur, phantom, "\ulcorner", very near end] & \mathcal{E}' \arrow[u, dashed, bend left] \arrow[u, dashed, bend right] \end{tikzcd} .$$

Take the common fibre $\mathcal{C}$ of p and p'

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{D} \arrow[r, "u"] \arrow[d, "p"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{D}' \arrow[d, "p'"] \\
0 \arrow[r] & \mathcal{E} \arrow[r, "l"'] \arrow[u, dashed, bend left] \arrow[u, dashed, bend right] & \mathcal{E}' \arrow[u, dashed, bend left] \arrow[u, dashed, bend right]
\end{tikzcd}$$

where all three squares are cartesian. Apply F, so we get a diagram

$$\begin{tikzcd}
F(\mathcal{C}) \arrow[r] \arrow[d] & F(\mathcal{D}) \arrow[r, "F(u)"] \arrow[d, "F(p)"] & F(\mathcal{D}') \arrow[d, "F(p')"] \\
F(0) \arrow[r] & F(\mathcal{E}) \arrow[r, "F(l)"'] \arrow[u, dashed, bend left] \arrow[u, dashed, bend right] & F(\mathcal{E}') \arrow[u, dashed, bend left] \arrow[u, dashed, bend right]
\end{tikzcd}$$

The left and the external square are cartesian, since they form split-Verdier sequences. By the pasting lemma also the right square is cartesian. □

For non-stable E it is expected that the condition of being additive or Verdier-localising is strictly stronger than sending split Verdier or Verdier sequences to fibre sequences, and similarly for the condition of being Karoubi-localising.

3.60 Proposition

Additive functors preserve finite products. Therefore, Verdier-localising and Karoubi-localising functors preserve finite products.

Proof. It is enough to recall that for any $\mathcal{A}$, $\mathcal{B}$ stable ∞-categories, their product exists in $\mathbf{Cat}_\infty^{\mathbf{st}}$ and

$$\begin{tikzcd}
\mathcal{A} \times \mathcal{B} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{B} \arrow[d] \\
\mathcal{A} \arrow[r] & 0
\end{tikzcd}$$

is a split Verdier square, and that $\mathcal{B} \to 0$ is a Split-Verdier projection. Then

$$\begin{tikzcd}
F(\mathcal{A} \times \mathcal{B}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F(\mathcal{B}) \arrow[d] \\
F(\mathcal{A}) \arrow[r] & *
\end{tikzcd}$$

is a cartesian square, which implies

$$F(\mathcal{A} \times \mathcal{B}) \simeq F(\mathcal{A}) \times F(\mathcal{B}).$$

□

3.61 Remark
Recall $\mathbf{Cat}_\infty^{\mathbf{st}}$ is a semi-additive category.

3.62 Lemma
The forgetful functors

$$\begin{aligned}
&\mathbf{Fun}^{\mathrm{add}}(\mathbf{Cat}_\infty^{\mathbf{st}}, \mathrm{CMon}(\mathcal{E})) \to \mathbf{Fun}^{\mathrm{add}}(\mathbf{Cat}_\infty^{\mathbf{st}}, \mathcal{E}),\\
&\mathbf{Fun}^{\mathrm{Vloc}}(\mathbf{Cat}_\infty^{\mathbf{st}}, \mathrm{CMon}(\mathcal{E})) \to \mathbf{Fun}^{\mathrm{Vloc}}(\mathbf{Cat}_\infty^{\mathbf{st}}, \mathcal{E}),\\
&\mathbf{Fun}^{\mathrm{Kloc}}(\mathbf{Cat}_\infty^{\mathbf{st}}, \mathrm{CMon}(\mathcal{E})) \to \mathbf{Fun}^{\mathrm{Kloc}}(\mathbf{Cat}_\infty^{\mathbf{st}}, \mathcal{E})
\end{aligned}$$

are equivalences.

Proof. In [GGN13, Corollary 2.5.(iii)] is proved the following statement

3.63 Reference
Let $\mathcal{C}$ be an ∞-category with finite products and let $\mathcal{D}$ be a semi-additive ∞-category. Then there is an equivalence

$$\mathbf{Fun}^{\Pi}(\mathcal{D}, \mathrm{CMon}(\mathcal{C})) \to \mathbf{Fun}^{\Pi}(\mathcal{D}, \mathcal{C}).$$

The condition for this are clearly satisfied by considering $\mathcal{C} = \mathcal{E}$ and $\mathcal{D} = \mathbf{Cat}_\infty^{\mathbf{st}}$. By proposition 3.60, our ∞-categories are sub-∞-categories of $\mathbf{Fun}^{\Pi}(\mathbf{Cat}_\infty^{\mathbf{st}})$. This equivalence clearly restrict on both sides to the equivalences we want to prove. □

In particular any product preserving functor from $\mathbf{Cat}_\infty^{\mathbf{st}}$ to a category with finite limits $\mathcal{E}$ uniquely lifts to $\mathrm{CMon}(\mathcal{E})$ in such a way that

$$\begin{tikzcd}
\mathbf{Cat}_\infty^{\mathbf{st}} \arrow[r, "\tilde{F}"] \arrow[rr, bend right, "F"'] & \mathrm{CMon}(\mathcal{E}) \arrow[r, "\mathrm{Forg}"] & \mathcal{E}
\end{tikzcd}$$

commutes. We also have canonical equivalences

$$\mathbf{Fun}^{\mathrm{add}}(\mathbf{Cat}_\infty^{\mathbf{st}}, \mathcal{E}) \simeq \mathbf{Fun}^{\mathrm{add}}(\mathbf{Cat}_\infty^{\mathbf{st}}, \mathrm{CMon}(\mathcal{E})) \simeq \mathrm{CMon}(\mathbf{Fun}^{\mathrm{add}}(\mathbf{Cat}_\infty^{\mathbf{st}}, \mathcal{E})).$$

3.64 Definition
An additive functor $F : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathcal{E}$ is GROUPLIKE if it lifts to $\mathrm{CMon}(\mathcal{E})$ actually takes values in $\mathrm{CGrp}(\mathcal{E})$.

Recall that $\mathrm{CGrp}(\mathcal{E})$ is a full sub-∞-category of $\mathrm{CMon}(\mathcal{E})$, spanned by cartesian commutative groups. The full sub-∞-category of $\mathbf{Fun}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E})$ spanned by grouplike additive functor is denote as $\mathbf{Fun}^{grp}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E})$, and we have

$$\mathbf{Fun}^{grp}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E}) \subset \mathbf{Fun}^{add}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E}) \subset \mathbf{Fun}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E}).$$

This definition makes sense because we have

$$\begin{aligned}&\mathbf{Fun}^{grp}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E}) \simeq \mathbf{Fun}^{add}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathrm{CGrp}(\mathcal{E})) \simeq \mathrm{CGrp}(\mathbf{Fun}^{add}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E})) \subset \\ &\subset \mathrm{CMon}(\mathbf{Fun}^{add}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E})) \simeq \mathbf{Fun}^{add}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E}).\end{aligned}$$

The second equivalence can be proved as in 3.62, using a similar results for cartesian commutative groups that can be found in [GGN13]

3.65 Remark
If $\mathcal{E}$ is additive, then both forgetful functors

$$\mathrm{CGrp}(\mathcal{E}) \to \mathrm{CMon}(\mathcal{E}) \to \mathcal{E}$$

are equivalences. Therefore, any additive functor to $\mathcal{E}$ is grouplike.

3.66 Example
Let us give some examples.

- The core functor

  $$\mathrm{core} : \mathbf{Cat}^{\mathbf{st}}_{\infty} \to \mathbf{An}$$

  is Verdier-localising, since it takes all cartesian squares to cartesian squares. More generally, it preserves all limits since it is right adjoint.

- We will introduce below functors

  $$k : \mathbf{Cat}^{\mathbf{st}}_{\infty} \to \mathbf{An}$$

  and

  $$K : \mathbf{Cat}^{\mathbf{st}}_{\infty} \to \mathbf{Spectra}$$

  that associate to a stable ∞-category its algebraic K theory anima or spectrum. This functors are Verdier-localising and grouplike (this will follows from Waldhausen's additivity and fibration theorems). However, they are not Karoubi-localising, for this reason we will introduce the functor

  $$\mathbf{K} : \mathbf{Cat}^{\mathbf{st}}_{\infty} \to \mathbf{Spectra},$$

which to a stable ∞-category associate its non-connective K-theory spectrum.

- The functor

$$K \circ \mathrm{Idem} : \mathbf{Cat}^{\mathbf{st}}_{\infty} \to \mathbf{Spectra}$$

is an additive, but not a Verdier-localising functor.

- (The cofinality theorem will imply)

$$k \circ \mathrm{Idem} : \mathbf{Cat}^{\mathbf{st}}_{\infty} \to \mathbf{An}$$

is Karoubi-localising.

3.67 Lemma
The ∞-categories

$$\mathbf{Fun}^{add}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E}), \mathbf{Fun}^{Vloc}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E}), \mathbf{Fun}^{Kloc}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E})$$

are semi-additive.

Proof. In [GGN13, Cor. 2.4] it is proved the following statement.

3.68 Reference
Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories with finite products and suppose that either $\mathcal{C}$ or $\mathcal{D}$ is semi-additive. Then the ∞-category of product preserving functors $\mathbf{Fun}^{\Pi}(\mathcal{C}, \mathcal{D})$ is semi-additive.

From this, we get that $\mathbf{Fun}^{\Pi}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E})$ is semi-additive. $\mathbf{Fun}^{add}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E})$ is closed under finite products in $\mathbf{Fun}^{\Pi}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E})$, since the products of pullbacks is the pullbacks of products (because limits commute). Them we immediately get that $\mathbf{Fun}^{add}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E})$ is semi-additive. Similarly for Verdier-localising and Karoubi-localising functors. $\square$

3.69 Lemma
The ∞-category

$$\mathbf{Fun}^{grp}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E})$$

is semi-additive.

Proof. $\mathbf{Fun}^{add}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E})$ is semi-additive and $\mathbf{Fun}^{grp}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E}) \subset \mathbf{Fun}^{add}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E})$. The product of two grouplike additive functor is grouplike again since $\mathrm{CGrp}(\mathcal{E})$ is closed under limits. Therefore $\mathbf{Fun}^{grp}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E})$ is semi-additive. $\square$

In particular, notice that we have proved that all these ∞-categories are closed under finite limits in $\mathbf{Fun}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathcal{E})$.

3.70 Proposition

Consider a grouplike additive functor $F : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$. Then for every split Verdier sequence

$$\mathcal{A} \to \mathcal{B} \to \mathcal{C}$$

we have

$$F(\mathcal{B}) \simeq F(\mathcal{A}) \times F(\mathcal{C}).$$

Proof. $F(\mathcal{A}) \to F(\mathcal{B}) \to F(\mathcal{C})$ is a fibre sequence in **An**, whose elements are in $\mathrm{CGrp}(\mathbf{An})$, hence it is fibre also in $\mathrm{CGrp}(\mathbf{An})$. By taking one of the adjoint of $\mathcal{B} \to \mathcal{C}$ (it does not matter which one), we see that it induces a split of $F(\mathcal{B}) \to F(\mathcal{C})$, hence it implies

$$\pi_0(F(\mathcal{B})) \to \pi_0(F(\mathcal{C}))$$

is surjective. Hence, if we take the fibre of $F(\mathcal{B}) \to F(\mathcal{C})$ in **Spectra**, it must be connective too. Therefore it coincides with the fibre taken in $\mathrm{CGrp}(\mathbf{An})$, which is $F(\mathcal{A})$. Now every fibre sequence in **Spectra** with a split must be split, by the splitting lemma. □

3.3.3 *Universal K-equivalences and Extension-splitting Functors.*

3.71 Definition

Let KK^{ex} denote the 1-category whose objects are stable ∞-categories and whose mapping-sets are $\mathrm{Hom}_{\mathrm{KK}^{ex}}(\mathcal{C}, \mathcal{D}) := K_0(\mathbf{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D}))$ and composition induced by the composition of functors.

Consider now the functor of 1-categories

$$\mathrm{ho}(\mathbf{Cat}_\infty^{\mathbf{st}}) \to \mathrm{KK}^{ex}$$

defined as

$$\mathcal{C} \mapsto \mathcal{C}$$
$$[f : \mathcal{C} \to \mathcal{D}] \mapsto [f : \mathcal{C} \to \mathcal{D}] \in K_0(\mathbf{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})).$$

3.72 Definition

A UNIVERSAL K-EQUIVALENCE is an exact functor $\mathcal{C} \to \mathcal{D}$ that becomes an equivalence in KK^{ex}.

3.73 Example

If

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"'] \arrow[r, phantom, bend left, "\dashv"] & \mathcal{D} \arrow[l, dashed, bend right, "g"'] \arrow[r, "p"'] \arrow[r, phantom, bend left, "\dashv"] & \mathcal{E} \arrow[l, dashed, bend right, "q"'] \end{tikzcd}$$

is a left split Verdier sequence where $g \dashv f$ and $q \dashv p$ denote the left adjoints, then the functor

$$(g,p) : \mathcal{D} \to \mathcal{C} \oplus \mathcal{E}$$

is a universal K-equivalence with inverse functor

$$f + q : \mathcal{C} \oplus \mathcal{E} \to \mathcal{E}.$$

Indeed

- $\mathcal{C} \oplus \mathcal{E} \xrightarrow{f+q} \mathcal{D} \xrightarrow{(g,p)} \mathcal{C} \oplus \mathcal{E}$ is already the identity functor, as on components

  $$g(f+q) \simeq gf + gq \simeq \mathrm{id} + 0 \simeq \mathrm{id}$$

  and

  $$p(f,q) \simeq pf + pq \simeq 0 + \mathrm{id} \simeq \mathrm{id}.$$

- We recall that as lemma 3.21 in there is a fibre sequence

  $$pq \Rightarrow \mathrm{id}_{\mathcal{D}} \Rightarrow fg$$

  in **Fun**$(\mathcal{D}, \mathcal{D})$. This implies

  $$[(f+q) \circ (g,p)] = [fg + pq] = [\mathrm{id}_{\mathcal{D}}].$$

For the next definition we need some more notation. Consider a stable ∞-category $\mathcal{C}$; we define functors

$$\mathrm{fib}, \mathrm{m}, \mathrm{cof} : \mathrm{Seq}(\mathcal{C}) \to \mathcal{C},$$

by

$$\begin{aligned} \mathrm{fib} &: x' \to x \to x'' \mapsto x' \\ \mathrm{m} &: x' \to x \to x'' \mapsto x \\ \mathrm{cof} &: x' \to x \to x'' \mapsto x''. \end{aligned}$$

3.74 Definition

Consider an ∞-category with finite limits $\mathcal{E}$ and a reduced functor $F : \mathbf{Cat}^{\mathbf{st}}_{\infty} \to \mathcal{E}$. F is called EXTENSION SPLITTING if the map

$$(\mathrm{fib}, \mathrm{cof}) : \mathrm{Seq}(\mathcal{C}) \to \mathcal{C}^2$$

induces an equivalence $F(\mathrm{Seq}(\mathcal{C})) \cong F(\mathcal{C})^2$ for any stable ∞-category $\mathcal{C}$.

Notice that we have a commutative diagram in $\mathcal{E}$

$$\begin{tikzcd}
 & & F(\mathcal{C}) \arrow[dr, "F(\mathrm{incl}-1)"] & \\
F(\mathrm{Seq}(\mathcal{C})) \arrow[r] & F(\mathcal{C}^2) \arrow[ur, "F(\mathrm{proj}-1)"] \arrow[dr, "F(\mathrm{proj}-2)"'] \arrow[rr, dashed] & & F(\mathcal{C})^2 \arrow[ul, bend right, "\mathrm{proj}-1"'] \arrow[dl, bend left, "\mathrm{proj}-2"] \\
 & & F(\mathcal{C}) \arrow[ur, "F(\mathrm{incl}-2)"'] &
\end{tikzcd}.$$

3.75 Lemma
Extension splitting functors preserve product.

Proof. The composition

$$F(\mathrm{Seq}(\mathcal{C})) \to F(\mathcal{C}^2) \to F(\mathcal{C})^2$$

is an equivalence. This gives us a retraction of

$$F(\mathrm{Seq}(\mathcal{C})) \to F(\mathcal{C}^2)$$

given by

$$F(\mathcal{C}^2) \to F(\mathcal{C})^2 \overset{\simeq}{\to} F(\mathrm{Seq}(\mathcal{C})).$$

Moreover, $F(\mathrm{Seq}(\mathcal{C})) \to F(\mathcal{C}^2)$ admits a section induced by the functor

$$\mathcal{C}^2 \to \mathrm{Seq}(\mathcal{C}), \quad (x, y) \mapsto x \to x \oplus y \to y.$$

We have therefore proved that $F(\mathrm{Seq}(\mathcal{C})) \to F(\mathcal{C}^2)$ is an equivalence. It follows that also $F(\mathcal{C}^2) \to F(\mathcal{C})^2$ is an equivalence.
Now,

$$F(\mathcal{C} \oplus \mathcal{D}) \to F(\mathcal{C}) \times F(\mathcal{D})$$

is a retract of

$$F((\mathcal{C} \oplus \mathcal{D}) \oplus (\mathcal{C} \oplus \mathcal{D})) \to F(\mathcal{C} \oplus \mathcal{D}) \times F(\mathcal{C} \oplus \mathcal{D}),$$

which we proved is an equivalence. Indeed we have a diagram

$$\begin{tikzcd}
\begin{matrix} F(\mathcal{C}\oplus\mathcal{D}) \\ \simeq F(\mathcal{C}\times\mathcal{D}) \end{matrix} \arrow[r] \arrow[d, "F\left(\substack{\text{canon.}\\ \text{incl.1}}\right)"'] \arrow[dd, dotted, bend right, "\mathrm{id}"'] & F(\mathcal{C}) \times F(\mathcal{D}) \arrow[d, "F\left(\substack{\text{canon.}\\ \text{incl. 1}}\right) \times F\left(\substack{\text{canon.}\\ \text{incl. 2}}\right)"] \arrow[dd, dotted, bend left, "\mathrm{id}"] \\
\begin{matrix} F((\mathcal{C}\oplus\mathcal{D})\oplus(\mathcal{C}\oplus\mathcal{D})) \\ \simeq F((\mathcal{C}\oplus\mathcal{D})\times(\mathcal{C}\oplus\mathcal{D})) \end{matrix} \arrow[r] \arrow[d, "F\left(\substack{\text{canon.}\\ \text{proj.1}}\right)"'] & \begin{matrix} F(\mathcal{C}\oplus\mathcal{D})\times F(\mathcal{C}\oplus\mathcal{D}) \\ \simeq F(\mathcal{C}\times\mathcal{D})\times F(\mathcal{C}\times\mathcal{D}) \end{matrix} \arrow[d, "F\left(\substack{\text{canon.}\\ \text{proj. 1}}\right) \times F\left(\substack{\text{canon.}\\ \text{proj. 2}}\right)"] \\
F(\mathcal{C} \oplus \mathcal{D}) \arrow[r] & F(\mathcal{C}) \times F(\mathcal{D})
\end{tikzcd}$$

The retract of an equivalence is an equivalence, so we are done. $\square$

3.76 Proposition

Consider an ∞-category with finite limits $\mathcal{E}$ and a reduced functor $F : \mathbf{Cat}^{\mathbf{st}}_{\infty} \to \mathcal{E}$. Then the following are equivalent:

(i) F inverts universal K-equivalences and preserves pairwise products.

(ii) F is extension splitting.

(iii) F is additive and grouplike.

The proof of the proposition is based on following observations made in the original context by Waldhausen, and it is adapted to this context for example in [HLS22].

3.77 Lemma (Waldhausen additivity)

Consider an ∞-category with finite limits $\mathcal{E}$ and an extension splitting functor $F : \mathbf{Cat}^{\mathbf{st}}_{\infty} \to \mathcal{E}$. Then there is a canonical equivalence between the functors

$$m_* \text{ and } \mathrm{fib}_* + \mathrm{cof}_* : F(\mathrm{Seq}(\mathcal{C})) \to F(\mathcal{C}).$$

In particular, any extension splitting functor is grouplike with the inversion map of $F(\mathcal{C})$ induced by the shift functor

$$\Sigma_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}.$$

Proof. Consider the two functors $\mathrm{Seq}(\mathcal{C}) \to \mathrm{Seq}(\mathcal{C})$

$$\mathrm{id}_{\mathrm{Seq}(\mathcal{C})} \quad \text{and } \alpha : (x' \to x \to x'') \mapsto (x' \to x' \oplus x'' \to x'').$$

Notice that they have the same evaluation on the first and third term of the sequence, i.e. that

$$(\mathrm{fib}, \mathrm{cof}) \circ \mathrm{id}_{\mathrm{Seq}(\mathcal{C})} = (\mathrm{fib}, \mathrm{cof}) \circ \alpha : \mathrm{Seq}(\mathcal{C}) \to \mathcal{C}^2.$$

Applying F to these functors we obtain

$$F(\mathrm{id}_{\mathrm{Seq}(\mathcal{C})}) \simeq \mathrm{id}_{F(\mathrm{Seq}(\mathcal{C}))}$$

and $F(\alpha)$. Now

$$F(\alpha) : F(\mathrm{Seq}(\mathcal{C})) \xrightarrow[F(\mathrm{fib},\mathrm{cof})]{\simeq} \underbrace{F(\mathcal{C}^2) \xrightarrow{\simeq} F(\mathcal{C})^2 \xrightarrow{\simeq} F(\mathrm{Seq}(\mathcal{C}))}_{\text{Induced by } (x,z) \mapsto (x \to x \oplus z \to z)};$$

Here all maps are equivalences, as we have seen in 3.75 . By definition, this is equivalent to $\mathrm{id}_{F(\mathrm{Seq}(\mathcal{C}))}$; indeed, by using the equivalence

$$F(\mathrm{Seq}(\mathcal{C})) \simeq F(\mathcal{C})^2 \simeq F(\mathcal{C}^2)$$

these two are induced by the same map. Therefore

$$m^* = F(m) \simeq F(m) \circ \mathrm{id}_F \simeq F(m) \circ F(\alpha) \simeq$$
$$\simeq F(m \circ \alpha) \simeq F(\mathrm{fib} + \mathrm{cof}) \simeq \mathrm{fib}_* + \mathrm{cof}_*,$$

where in the last equivalence we use the fact that F preserves products.
Notice that the monoidal operation on $F(\mathcal{C})$ is given by

$$F(\mathcal{C})^2 \simeq F(\mathrm{Seq}(\mathcal{C})) \overset{m^*}{\to} F(\mathcal{C})$$

so the last claims follows from the existence of a natural bifibre sequence

$$x \to 0 \to \Sigma_{\mathcal{C}} x,$$

which implies $x + \Sigma_{\mathcal{C}} x \simeq 0$. □

3.78 Remark

Notice there are many equivalences of stable ∞-categories between $\mathrm{Seq}(\mathcal{C})$ and $\mathrm{Arr}(\mathcal{C})$, for $\mathcal{C}$ stable ∞-category. For example

$$\mathrm{Arr}(\mathcal{C}) \to \mathrm{Seq}(\mathcal{C}), \quad (x \to y) \mapsto (x \to y \to \mathrm{cofib})$$

and

$$\mathrm{Arr}(\mathcal{C}) \to \mathrm{Seq}(\mathcal{C}), \quad (x \to y) \mapsto (\mathrm{fib} \to x \to y).$$

In the next lemma we will need the equivalence given by

$$\mathrm{Seq}(\mathcal{C}) \to \mathrm{Arr}(\mathcal{C}), \quad (x \to y \to z) \mapsto (z \to \Sigma_{\mathcal{C}} x).$$

An inverse of this can be constructed by taking pullbacks.

3.79 Lemma

Consider an ∞-category with finite limits $\mathcal{E}$ and a reduced product preserving functor $F : \mathbf{Cat}^{\mathbf{st}}_\infty \to \mathcal{E}$. F is extension splitting if and only if the map $(s,t) : \mathrm{Arr}(\mathcal{C}) \to \mathcal{C}^2$ is mapped into an equivalence by F for any $\mathcal{C} \in \mathbf{Cat}^{\mathbf{st}}_\infty$.

Proof. For the "only if" part, we have

$$F(\mathrm{Arr}(\mathcal{C})) \simeq F(\mathrm{Seq}(\mathcal{C})) \simeq F(\mathcal{C})^2 \xrightarrow[\mathrm{switch}]{\simeq} F(\mathcal{C})^2 \simeq F(\mathcal{C})^2$$

which is induced by

$$\big(a \to b\big) \mapsto \big(b \to \mathrm{cof}(a \to b) \to \Sigma_{\mathcal{C}} a\big) \mapsto \big(b, \Sigma_C a\big) \mapsto \big(\Sigma_{\mathcal{C}} a, b\big) \simeq \big(a, b\big).$$

For the "if"' part, we have an equivalence

$$(\mathtt{cof}, \Sigma_{\mathcal{C}}\mathtt{fib})_* : F(\mathtt{Seq}(\mathcal{C})) \xrightarrow{\quad\simeq\quad} F(\mathrm{Arr}(\mathcal{C})) \xrightarrow[(s,t)]{\simeq} F(\mathcal{C})^2$$

induced by

$$a \to b \to c \mapsto c \to \Sigma_{\mathcal{C}} a \longmapsto (c, \Sigma_{\mathcal{C}} a$$

which gives that F is extension splitting. □

Proof of proposition 3.76. First of all, notice there is a left-split Verdier sequence

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, bend right, "\bullet \overset{id}{\to} \bullet \to 0"'] \arrow[rr, phantom, "\perp"] & & \mathtt{Seq}(\mathcal{C}) \arrow[ll, bend right, "\mathrm{fib}"'] \arrow[rr, bend right, "\mathrm{cof}"'] \arrow[rr, phantom, "\perp"] & & \mathcal{C} \arrow[ll, bend right, "0 \to \bullet \overset{id}{\to} \bullet"']
\end{tikzcd}$$

(i) $\Rightarrow$ *(ii)* Suppose F satisfies (i). In particular, $F(\mathcal{C}^2) \simeq F(\mathcal{C})^2$. Also by the example 3.73

$$(\mathtt{fib}, \mathtt{cof}) : \mathtt{Seq}(\mathcal{C}) \to \mathcal{C}^2$$

is a universal K-equivalence. Therefore, applying f to $(\mathtt{fib}, \mathtt{cof})$ we get an equivalence $F(\mathtt{Seq}(\mathcal{C})) \simeq F(\mathcal{C})^2$.

(ii) $\Rightarrow$ *(i)* Suppose F is extension splitting. We proved in lemma 3.75 that F preserve pairwise products. It is then enough to prove that

$$hF : h\,\mathbf{Cat}_\infty^{\mathbf{st}} \to h\mathcal{E}$$

factors through KK^{ex}; then clearly it would invert universal K-equivalences. Let $f \to g \to h$ form a bifibre sequence of exact functors $\mathcal{C} \to \mathcal{D}$, so in $\mathbf{Fun}^{ex}(\mathcal{C}, \mathcal{D})$. We want to show that $[F(f)] + [F(h)] = [F(g)]$ in $\pi_0(\mathbf{Map}_{\mathcal{E}}(F(\mathcal{C}), F(\mathcal{D})))$. But now this follows from 3.77 applied to $\mathbf{Fun}^{ex}(\mathcal{C}, \mathcal{D})$.

(iii) $\Rightarrow$ *(ii)* Suppose F is additive and consider the left-split fibre sequence

$$F(\mathcal{C}) \to F(\mathtt{Seq}(\mathcal{C})) \to F(\mathcal{C}).$$

By the splitting lemma we obtain exactly that $(\mathtt{fib}, \mathtt{cof})$ induces an equivalence $F(\mathtt{Seq}(\mathcal{C})) \simeq F(\mathcal{C})^2$.

(ii) $\Rightarrow$ *(iii)* Suppose F is extension splitting. Lemma 3.77 implies that F is grouplike. Suppose there is a split Verdier sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$. Then by the example 3.73, there is a universal K-equivalence $\mathcal{D} \to \mathcal{C} \oplus \mathcal{E}$. But then we have a sequence

$$F(\mathcal{C}) \to F(\mathcal{D}) \simeq F(\mathcal{C} \oplus \mathcal{E}) \simeq F(\mathcal{C}) \times F(\mathcal{E}) \ \to F(\mathcal{E})$$

which is fibre. □

3.80 Proposition ([Cal+21b, Prop. 1.5.11])
Consider a Verdier sequence

$$\mathcal{C} \xrightarrow{f} \mathcal{C}' \xrightarrow{p} \mathcal{C}''$$

and a grouplike Verdier-localising functor $F : \mathbf{Cat}^{\mathbf{st}}_{\infty} \to \mathcal{E}$. Assume that the Verdier projection p admits a section $s : \mathcal{C}'' \to \mathcal{C}'$ in $\mathbf{Cat}^{\mathbf{st}}_{\infty}$. Then f and s together induce an equivalence

$$(\star) \quad (f, s) : F(\mathcal{C}) \oplus F(\mathcal{C}'') \to F(\mathcal{C}').$$

If also the Verdier inclusion f admits a retraction, $r : \mathcal{C}' \to \mathcal{C}$, in $\mathbf{Cat}^{\mathbf{st}}_{\infty}$, then p and r together induce an equivalence

$$(\star\star) \quad p \oplus r : F(\mathcal{C}') \to F(\mathcal{C}) \oplus F(\mathcal{C}'').$$

$(\star\star)$ is inverse to the $(\star)$ when $r \circ s$ vanishes.

3.81 Remark
It is worth noting that $\mathbf{Cat}^{\mathbf{st}}_{\infty}$ is only semi-additive, but not additive. Therefore, in general, the middle term in a Verdier sequence, admitting a split as in the proposition, does not split as a direct sum before applying F. In example 3.73 we showed that in the case of left (or right) split-Verdier sequences, there is a relation between the middle term and the extremes, but this is, in general, only a K-equivalence.

To prove the proposition we need a version of the splitting lemma for additive ∞-categories. Before giving the ∞-version, let us recall the classical version.

3.82 Lemma (Classical Splitting Lemma)
Consider an *additive* category $\mathcal{A}$ and a short exact sequence in $\mathcal{A}$

$$0 \to A \xrightarrow{f} B \xrightarrow{p} C \to 0.$$

The following three statements are equivalent:

(a) There exists a section of p;

(b) There exists a retract of i;

(c) There exists an isomorphism of sequences between our original sequence and the exact sequence given by

$$0 \to A \xrightarrow{\text{canon}} A \oplus C \xrightarrow{\text{canon}} C \to 0.$$

Although the splitting lemma is usually demonstrated for abelian categories, but is also applicable to additive categories. The proof presented in [Bor94, Prop. 1.8.7] is applicable to both types of categories. We choose to state it in this manner because the abelian case is insufficient for our needs; indeed, we are going to work in the homotopy categories of an additive ∞-category.

3.83 Lemma (Splitting Lemma, [Cal+21b, Lemma 1.5.12])
Consider an additive ∞-category which admits fibres and cofibres $\mathcal{A}$ and a retract diagram.

$$x \xrightarrow{i} y \xrightarrow{r} x.$$

Then the followings statements are true:

(i) There is an equivalence $x \oplus \mathrm{fib}(r) \xrightarrow{(i,\mathrm{canon.})} y$.

(ii) There is an equivalence $y \xrightarrow{(r,\mathrm{canon.})} x \oplus \mathrm{cof}(i)$.

(iii) The fibre sequence $\mathrm{fib}(r) \to y \to x$ is also a cofibre sequence.

(iv) The cofibre sequence $x \to y \to \mathrm{cof}(i)$ is also a fibre sequence.

(v) The composite map $\mathrm{fib}(r) \to y \to \mathrm{cof}(i)$ is an equivalence.

Proof of the lemma 3.83. First of all, notice that (i) and (ii) are equivalent up to passing to the opposite ∞-category $\mathcal{A}^{\mathrm{op}}$, which is still additive. Same for (iii) and (iv).

(i) We claim that we can work in $\mathrm{Ho}(\mathcal{A})$. If this is the case, then the classical splitting lemma for additive categories ((a) $\Rightarrow$ (c)) applied to

$$\mathrm{fib}(r) \to y \xrightarrow{r} x$$

gives us the result. Let us prove the claim. For any $z \in \mathcal{A}$, $\mathbf{Map}_{\mathcal{A}}(z, \bullet)$ preserves limits. Therefore, we obtain a fibre sequence in **An**

$$\mathbf{Map}_{\mathcal{A}}(z, \mathrm{fib}(r)) \to \mathbf{Map}(z, y) \xrightarrow{r_*} \mathbf{Map}(z, x).$$

The map

$$\pi_1(\mathbf{Map}(z, y)) \to \pi_1(\mathbf{Map}(z, x)),$$

induced by r, is surjective because r has a section. For this, the long exact sequence of homotopy groups ends with a fibre sequence of sets

$$\pi_0(\mathbf{Map}_{\mathcal{A}}(z, \mathrm{fib}(r))) \to \pi_0(\mathbf{Map}(z, y)) \to \pi_0(\mathbf{Map}(z, x)).$$

This implies that, in $\mathrm{ho}(\mathcal{A})$, $\mathrm{fib}(r)$ is still the fibre of r. Now, since $\mathrm{ho}(\mathcal{A})$ is an additive category, and since direct sums of objects are preserved when passing to $\mathrm{ho}(\mathcal{A})$, we can state the condition equivalently in $\mathrm{ho}(\mathcal{A})$.

From (i) we get a retraction of $y \to \mathrm{fib}(r)$ which vanishes when composed with $i : x \to y$. So we have a commutative diagram in $\mathcal{A}$

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathtt{fib}(r) \arrow[r] \arrow[d] & 0 \arrow[d] \\
x \arrow[r, "i"] \arrow[d] & y \arrow[r, "r"] \arrow[d] & x \arrow[d] \\
0 \arrow[r] & \mathtt{fib}(r) \arrow[r] & 0
\end{tikzcd}$$

in which both the middle column and the middle row are retract diagrams.

(iii) Applying the pasting law for pushouts to the two upper squares, we get that

$$\mathtt{fib}(r) \to y \to x$$

is a cofibre.

(v) Applying the pasting law for pushouts to the two left squares, we get

$$x \to y \to \mathtt{fib}(r)$$

is cofibre; hence, there is an equivalence $\mathtt{fib}(r) \to \mathtt{cof}(i)$. But this map is the same as the one $\mathtt{fib}(r) \to \mathtt{cof}(i)$ obtained from the composition

$$\mathtt{fib}(r) \to y \to \mathtt{cof}(i)$$

because the $\mathtt{fib}(r) \to y \to \mathtt{fib}(r)$ is a retract diagram. □

Proof of proposition 3.80. Assume that the Verdier projection p admits a section $s : \mathcal{C}'' \to \mathcal{C}'$ in $\mathbf{Cat}_\infty^{\mathbf{st}}$. Then we have a retract diagram in $\mathbf{Cat}_\infty^{\mathbf{st}}$

$$\mathcal{C}'' \xrightarrow{s} \mathcal{C}' \xrightarrow{p} \mathcal{C}'',$$

which induce a retract diagram

$$F(\mathcal{C}'') \xrightarrow{s_*} F(\mathcal{C}') \xrightarrow{p_*} F(\mathcal{C}'')$$

in the additive ∞-category $\mathrm{CGrp}(\mathcal{E})$. Then by (i) we obtain

$$F(\mathcal{C}'') \oplus \mathtt{fib}(p_*) \xrightarrow{\simeq} F(\mathcal{C}');$$

since F is Verdier-localising, we can identify the fibre of p_* with $F(\mathcal{C})$, so we obtain an equivalence

$$F(\mathcal{C}'') \oplus F(\mathcal{C}) \xrightarrow{\simeq} F(\mathcal{C}').$$

By (iii) we also obtain that the fibre sequence

$$F(\mathcal{C}) \xrightarrow{f_*} F(\mathcal{C}') \xrightarrow{p_*} F(\mathcal{C}'')$$

is also cofibre.

Assume now the Verdier inclusion f admits a retraction, $r : \mathcal{C}' \to \mathcal{C}$. Then identifying $F(\mathcal{C}'')$ with $\mathrm{cof}(f_*)$ and applying (ii) to the retract diagram

$$F(\mathcal{C}) \xrightarrow{f_*} F(\mathcal{C}') \to \xrightarrow{r_*} F(\mathcal{C}''),$$

we obtain an equivalence

$$F(\mathcal{C}') \xrightarrow{\simeq} F(\mathcal{C}'') \oplus F(\mathcal{C}).$$

Let us prove now the final statement. Since we already know the two functors are equivalences, for them to be inverses it is equivalent to be one-sided inverses. Composing in one of the verses we get

$$F(\mathcal{C}'') \oplus F(\mathcal{C}) \xrightarrow{\simeq} F(\mathcal{C}') \xrightarrow{\simeq} F(\mathcal{C}'') \oplus F(\mathcal{C})$$

which component-wise is given by the matrix

$$\begin{pmatrix} \mathrm{id} & 0 \\ r_* s_* & \mathrm{id} \end{pmatrix}.$$

So the composition is equivalent to the identity if and only if $r_* s_* \simeq (rs_*) \simeq 0$, and so if and only if $r \circ s \simeq 0$. ☐

3.3.4 *Additive vs Verdier-localising Functors.*

Let us use the following notation. Consider a stable ∞-category $\mathcal{D}$, a full stable sub-∞-category $\mathcal{C}$ of $\mathcal{D}$, and a small ∞-category I. We denote with $\mathbf{Fun}^{\mathcal{C}}(I, \mathcal{D})$ the full sub-∞-category of $\mathbf{Fun}(I, \mathcal{D})$ spanned by those functor that take maps in I to equivalences modulo $\mathcal{C}$.

The connection between additive and Verdier-localising functors was essentially discussed in Waldhausen's research on the fibration theorem.

3.84 Theorem (Waldhausen's fibration theorem)

Consider a Verdier sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{D}/\mathcal{C}$ and a grouplike additive functor $F : \mathbf{Cat}^{\mathbf{st}}_{\infty} \to \mathbf{An}$. For any $[n] \in \mathbf{\Delta}$, consider also the constant map

$$\mathrm{const} : \mathcal{D} \to \mathbf{Fun}^{\mathcal{C}}([n], \mathcal{D})$$

defined on objects by

$$b \mapsto \tilde{b}_n : [n] \to \mathcal{D}$$

collapsing everything to b. (Here the cofibre of any arrow in $[n]$ through $\tilde{b}_n$ is given by the zero object of $\mathcal{D}$, which is in $\mathcal{C}$; so $\tilde{b}_n$

actually belongs to $\mathbf{Fun}^{\mathcal{C}}([n], \mathcal{D})$. The constant map induces a bifibre sequence of $\mathbf{E}_\infty$-groups

$$F(\mathcal{C}) \to F(\mathcal{D}) \to |F(\mathbf{Fun}^{\mathcal{C}}(\bullet, \mathcal{D}))|, \tag{3.84.1}$$

where $F(\mathbf{Fun}^{\mathcal{C}}(\bullet, \mathcal{D})) : N(\Delta)^{op} \to \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$ is regarded as a simplicial $\mathbf{E}_\infty$-group and

$$|F(\mathbf{Fun}^{\mathcal{C}}(\bullet, \mathcal{D}))| := \operatorname*{colim}_{\Delta} F(\mathbf{Fun}^{\mathcal{C}}([n], \mathcal{D})) \in \mathbf{An}$$

is its realization.

Recall the following definitions. An ∞-category $\mathcal{D}$ is *sifted* if equivalently:

- All colimits of diagrams $\mathcal{D} \to \mathbf{An}$ commutes with finite products, i.e. for any $F : \mathcal{D} \times S \to \mathbf{An}$ with S discrete and finite ∞-category, we have

$$\operatorname*{colim}_{d \in \mathcal{D}} \prod_{s \in S} F(d, s) \xrightarrow{\simeq} \prod_{s \in S} \operatorname*{colim}_{d \in \mathcal{D}} F(d, s).$$

- (As in [Lur09, Def.5.5.8.1]) There exists a quasi-category $K \in \mathrm{sSet}$ that models $\mathcal{D}$, such that
  (a) The simplicial set K is non-empty.
  (b) The diagonal map $K \to K \times K$ is cofinal.

A *sifted colimit* is the colimit of a diagram $\mathcal{D} \to \mathcal{C}$ where $\mathcal{D}$ is sifted.

3.85 Remark

Sifted colimits in $\mathrm{CGrp}(\mathbf{An})$ are preserved by the forgetful functor $\mathrm{CGrp}(\mathbf{An}) \to \mathbf{An}$; indeed the forgetful functor is a right adjoint. In particular, $|F(\mathbf{Fun}^{\mathcal{C}}(\bullet, \mathcal{D}))|$ is just the geometric realization of the simplicial anima $F(\mathbf{Fun}^{\mathcal{C}}(\bullet, \mathcal{D}))$.

3.86 Remark

A sequence $(\star)$ $L \to M \to N$ in $\mathrm{CGrp}(\mathbf{An})$ is bifibre if and only if the underlying sequence of animae is a fibre sequence (over the unit of N) and $\pi_0(M) \to \pi_0(N)$ is surjective.
The idea to prove this, that we will also use during the proof of 3.84, is the following. Let us call U the forgetful functor from $\mathrm{CGrp}(\mathbf{An}) \to \mathbf{An}$. This functor is fully faithful, and it has as right adjoint the group-completion functor. So we have an adjunction

$$\begin{tikzcd} \mathbf{An} \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathrm{CGrp}(\mathbf{An}) \arrow[l, hook', "U"] \end{tikzcd}$$

or even better a diagram

$$\begin{tikzcd}
\mathbf{An} \arrow[r, bend left, "(\bullet)^{\infty-\mathrm{cmon}}"] & \mathrm{CMon}(\mathbf{An}) \arrow[l, "\simeq", "u_m"'] \arrow[r, bend left, "(\bullet)^{\infty-\mathrm{cgrp}}"] & \mathrm{CGrp}(\mathbf{An}) \arrow[l, "\perp", "u_g"'] \arrow[ll, bend left, "u"]
\end{tikzcd}$$

(If) Suppose the sequence $(\star)$ is bifibre, then U being a right adjoint preserve limits so

$$U(L) \to U(M) \to U(N)$$

is fibre. Moreover being cofibre, surely $\pi_0 M \to \pi_0 N$ is surjective.

(Only if) Suppose the sequence of underlying animae of $(\star)$ if fibre and $\pi_0(M) \to \pi_0(N)$ is surjective. Since L, M, N are in $\mathrm{CGrp}(\mathbf{An})$,and $\mathrm{CGrp}(\mathbf{An})$ is a sub-∞-category of $\mathbf{An}$ closed under taking limits, the sequence $(\star)$ is for sure fibre in $\mathrm{CGrp}(\mathbf{An})$. Using the equivalence $\mathrm{CGrp}(\mathbf{An}) \simeq \mathbf{Spectra}_{\geqslant 0}$ we have this is a fibre sequence of spectra connective spectra. Since $\pi_0(M) \to \pi_0(N)$ we have this is also a fibre sequence in $\mathbf{Spectra} \supset \mathbf{Spectra}_{\geqslant 0}$. But $\mathbf{Spectra}$ is stable, so it is also a cofibre sequence in $\mathbf{Spectra}$ and hence in $\mathbf{Spectra}_{\geqslant 0}$. By equivalence we conclude the sequence is bifibre in $\mathrm{CGrp}(\mathbf{An})$.

3.87 Remark

Before proving the theorem, let us introduce the DÉCALAGE functor, which will be central in the proof. Fix an ∞-category $\mathcal{C}$. We define the following functor of simplicial objects in $\mathcal{C}$,

$$\mathrm{dec} : \mathbf{Fun}(N(\boldsymbol{\Delta}^{\mathrm{op}}), \mathcal{C}) \to \mathbf{Fun}(N(\boldsymbol{\Delta}^{\mathrm{op}}), \mathcal{C})$$

as the functor induced by $[0] \star \bullet : \boldsymbol{\Delta} \to \boldsymbol{\Delta}$. The inclusions

$$[n] \hookrightarrow [0] \star [n] \cong [1+n] \quad \text{and } [0] \hookrightarrow [0] \star [n] \cong [1+n]$$

induce natural transformations

$$\mathrm{dec} \Rightarrow \mathrm{id}, \quad \text{and } \mathrm{dec} \Rightarrow ev_0.$$

In particular, if $X : N(\boldsymbol{\Delta})^{\mathrm{op}} \to \mathcal{C}$, then

$$\mathrm{dec}(X)_\bullet = X_{[0]\star\bullet},$$

and we have maps

$$\mathrm{dec}(X)_\bullet \to X_\bullet \quad \text{and} \quad \mathrm{dex}(X)_\bullet \to \mathrm{const}(X_0).$$

Also, $\mathrm{dec}(X)$ is always a split simplicial object over X_0; with this we mean we have an adjunction

$$\mathrm{dec}(X) \rightleftarrows \mathrm{const}(X_0)$$

where one map is given by the evaluation of the natural transformation $\mathrm{dec} \to ev_0$ and the other is given by the composition of the 0-th degeneracies map s_0 of X (so for example, in level n, this is $s_0^n \circ s_0^{n-1} \circ \cdots \circ s_0^2 \circ s_0^1$). In particular, X_0 is a colimit of $\mathrm{dec}(X)$ by [Lur17, Lemma 6.1.3.16], i.e.

$$X_0 \simeq \operatorname*{colim}_{\Delta} \mathrm{dec}(X)_n.$$

3.88 Remark

In the proof of theorem 3.84, we also need the following criterion: a square of $\mathbf{E}_\infty$-groups with right vertical map π_0-surjective is cartesian if and only if the induced map on vertical fibres over 0 is an equivalence.
Since the right vertical map π_0-surjective, its fibres in $\mathbf{Spectra}_{\geqslant 0}$ and **Spectra** coincide; but then the sequence is also cofibre. Since the fibres of the vertical right map and vertical left map are equivalent, they are equivalent also in **Spectra**. Then by pasting law of pushouts and stability of **Spectra** we can conclude.

3.89 Definition

A natural transformation of functors $\tau : Y \Rightarrow W : I \to \mathcal{C}$ is EQUIFIBRED if, for any $f : i \to j \in I$,

$$\begin{tikzcd} Y(i) \arrow[r, "\tau_i"] \arrow[d, "Y(f)"'] & W(i) \arrow[d, "W(f)"] \\ Y(j) \arrow[r, "\tau_j"] & W(j) \end{tikzcd}$$

3.90 Lemma (Equifibrancy lemma of Segal and Rezk, from [Cal+21b, Lemma 3.3.14])

Consider a cartesian square of functors from some small category I to **An**

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d, "\tau"] \\ Z \arrow[r] & W \end{tikzcd}$$

such that the natural transformation $\tau : Y \Rightarrow W$ is equifibred. Then the square

$$\begin{tikzcd} \operatorname{colim} X \arrow[r] \arrow[d] & \operatorname{colim} Y \arrow[d] \\ \operatorname{colim} Z \arrow[r] & \operatorname{colim} W \end{tikzcd}$$

is cartesian.

Proof of Theorem 3.84. Consider now, for $[n] \in \mathbf{\Delta}$, the functor

$$d_0^{n+1} : \mathbf{Fun}^{\mathcal{C}}([1+n], \mathcal{D}) \to \mathbf{Fun}^{\mathcal{C}}([n], \mathcal{D})$$

which maps "sequences"

$$x_0 \to x_1 \to \cdots \to x_n$$

such that $\mathrm{cof}(x_i \to x_{i+1}) \in \mathcal{C}$, to

$$x_1 \to \cdots \to x_n.$$

The kernel of this is $\mathbf{Fun}^{\mathcal{C}}([0], \mathcal{D}) \simeq \mathcal{C}$. So we obtain a fibre sequence

$$\mathcal{C} \hookrightarrow \mathbf{Fun}^{\mathcal{C}}([1+n], \mathcal{D}) \xrightarrow{d_0^{n+1}} \mathbf{Fun}^{\mathcal{C}}([n], \mathcal{D})$$

where the first arrow is the inclusion of $y \in \mathcal{C}$ as $y \to 0 \to \cdots \to 0$. d_0^{n+1} has a fully faithful right adjoint, so by lemma 3.21 this is a right-split Verdier sequence. The right adjoint is given by the map

$$\Big(x_0 \to x_1 \to \cdots \to x_{n-1}\Big) \mapsto \Big(x_0 \to x_0 \to x_1 \to \cdots \to x_{n-1}\Big).$$

We now want to prove that the natural transformation

$$\mathrm{dec} F\, \mathbf{Fun}^{\mathcal{C}}([\bullet], \mathcal{D}) \to F\, \mathbf{Fun}^{\mathcal{C}}([\bullet], \mathcal{D}) : \mathrm{N}(\mathbf{\Delta})^{\mathrm{op}} \to \mathbf{An}$$

is EQUIFIBRED, i.e. that for every $f : [m] \to [n] \in \mathbf{\Delta}$

$$\begin{tikzcd} F\,\mathbf{Fun}^{\mathcal{C}}([1+n], \mathcal{D}) \arrow[r, "\mathrm{dec}(F\,\mathbf{Fun}^{\mathcal{C}}(f, \mathcal{D}))"] \arrow[d, "d_0^*"'] & F\,\mathbf{Fun}^{\mathcal{C}}([1+m], \mathcal{D}) \arrow[d, "d_0^*"] \\ F\,\mathbf{Fun}^{\mathcal{C}}([n], \mathcal{D}) \arrow[r, "(F\,\mathbf{Fun}^{\mathcal{C}}(f, \mathcal{D}))"'] & F\,\mathbf{Fun}^{\mathcal{C}}([m], \mathcal{D}) \end{tikzcd}$$

is cartesian. To prove this we use the remark 3.88. In our case the map is even π_0-split-surjective, since we have a fully faithful left adjoint, and the vertical fibres both identify with $F(\mathcal{C})$.
(It is worth noting that also that before applying F, the square above is not necessarily cartesian, for example if $f = d_0$, and so not a Verdier square; therefore we actually need the remark).
Now apply the equifibrancy lemma of Segal and Rezk: consider the cartesian square of functors $N(\mathbf{\Delta})^{op} \to \mathbf{An}$

$$\begin{tikzcd} \mathrm{const}_{F(A)} \arrow[r] \arrow[d] & \mathrm{dec} F\, \mathbf{Fun}^{\mathcal{C}}([\bullet], \mathcal{D}) \arrow[d] \\ \mathrm{const}_{F(0)} \arrow[r] & F\, \mathbf{Fun}^{\mathcal{C}}([\bullet], \mathcal{D}) \end{tikzcd}$$

where we know the right vertical leg to be equifibred; then

$$\begin{tikzcd} |\mathrm{const}_{F(A)}| \arrow[r] \arrow[d] & |\mathrm{dec} F\, \mathbf{Fun}^{\mathcal{C}}([\bullet], \mathcal{D})| \arrow[d] \\ 0 \simeq |\mathrm{const}_{F(0)}| \arrow[r] & |F\, \mathbf{Fun}^{\mathcal{C}}([\bullet], \mathcal{D})| \end{tikzcd}$$

is cartesian, which means that

$$F(\mathcal{C}) \to F(\mathcal{D}) \to |F\, \mathbf{Fun}^{\mathcal{C}}([\bullet], \mathcal{D})|$$

is a fibre sequence.

To see that this is a bifibre sequence, notice that the right hand map is the inclusion of the 0-simplices into the realization. This induces a surjection on π_0 for every simplicial anima, so we can use the same reasoning as above involving $\mathbf{Spectra}_{\geqslant 0}$ and $\mathbf{Spectra}$, and therefore conclude. □

Given a Verdier sequence $\mathcal{C} \to \mathcal{D} \xrightarrow{p} \mathcal{D}/\mathcal{C}$, the restriction of the projection

$$\mathbf{Fun}^{\mathcal{C}}([n], \mathcal{D}) \hookrightarrow \mathbf{Fun}([n], \mathcal{D}) \xrightarrow{p_*} \mathbf{Fun}([n], \mathcal{D}/\mathcal{C})$$

maps a diagram

$$x = x_0 \to \cdots \to x_n$$

with cofibres in $\mathcal{C}$ to a diagram

$$x = x_0 \to \cdots \to x_n$$

in which every $x_i \to x_j$ is an equivalence in $\mathcal{D}/\mathcal{C}$; therefore, it takes values in the sub-∞-category of functors that send maps in $[n]$ to equivalences in $\mathcal{D}/\mathcal{C}$. Since $[n]$ is contractible (clearly $|[n]| \cong *$), the image of the projection is the essential image if the constant functor

$$\mathrm{const} : \mathcal{D}/\mathcal{C} \to \mathbf{Fun}([n], \mathcal{D}/\mathcal{C}), \quad x \mapsto x \xrightarrow{\mathrm{id}} x \to \cdots \to x,$$

which is clearly fully faithful. For any $F : \mathbf{Cat}^{\mathbf{st}}_\infty \to \mathbf{An}$ we therefore obtain a map

$$|F(\mathbf{Fun}^{\mathcal{C}}([\bullet], \mathcal{D}))| \to F(\mathcal{D}/\mathcal{C}).$$

3.91 Corollary

A grouplike additive functor $F : \mathbf{Cat}^{\mathbf{st}}_\infty \to \mathbf{An}$ is Verdier-localising if and only if the following two conditions are satisfied:

(i) For every Verdier sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{D}/\mathcal{C}$, the map we constructed above, $|F(\mathbf{Fun}^{\mathcal{C}}([\bullet], \mathcal{D}))| \to F(\mathcal{D}/\mathcal{C})$, is an inclusion of path components.

(ii) For every Verdier square

$$\begin{tikzcd}
\mathcal{D} \arrow[r] \arrow[d] & \mathcal{D}' \arrow[d] \\
\mathcal{D}/\mathcal{C} \arrow[r, "f"] & \mathcal{D}'/\mathcal{C}
\end{tikzcd}$$

then

$$\mathrm{im}\big(\pi_0 F(\mathcal{D}) \to \pi_0 F(\mathcal{D}/\mathcal{C})\big) = f^{-1}\Big(\mathrm{im}\big(\pi_0 F(\mathcal{D}') \to \pi_0 F(\mathcal{D}'/\mathcal{C})\big)\Big).$$

As a consequence, $|F(\mathbf{Fun}^{\mathcal{C}}([\bullet], \mathcal{D}))| \to F(\mathcal{D}/\mathcal{C})$ is an equivalence for any Verdier sequence if and only if F is Verdier-localising and

$$\pi_0 F(\mathcal{D}) \to \pi_0 F(\mathcal{D}/\mathcal{C})$$

is surjective for all Verdier sequences, since we know $\pi_0 F(\mathcal{D}) \to \pi_0 |F\,\mathbf{Fun}^{\mathcal{C}}([\bullet], \mathcal{D})$ is always surjective.

3.92 Corollary

Consider an additive functor $F : \mathbf{Cat}^{\mathbf{st}}_\infty \to \mathbf{Spectra}$. F is Verdier-localising if and only if the canonical map

$$|F(\mathbf{Fun}^{\mathcal{C}}([\bullet], \mathcal{D}))| \to F(\mathcal{D}/\mathcal{C})$$

is an equivalence for every Verdier sequence.

Proof. Let us start by applying theorem 3.84 to $\Omega^\infty F$, $\Omega^{\infty-1} F$, etc. Recall that the spectrification functor preserves colimits, since it is left adjoint to the functor Ω^∞. We get that the cofibre of $F(\mathcal{C}) \to F(\mathcal{D})$ is given by the spectrification of

$$\big(|\Omega^\infty F\,\mathbf{Fun}^{\mathcal{C}}([\bullet], \mathcal{D})|, |\Omega^{\infty-1} F\,\mathbf{Fun}^{\mathcal{C}}([\bullet], \mathcal{D})|, \dots\big).$$

But this is exactly $|F\,\mathbf{Fun}^{\mathcal{C}}([\bullet], \mathcal{D})|$, so we get a bifibre sequence

$$F(\mathcal{C}) \to F(\mathcal{D}) \to |F\,\mathbf{Fun}^{\mathcal{C}}([\bullet], \mathcal{D}).$$

Then by applying proposition 3.59 we finish. □

3.3.5 *Verdier-localising vs. Karoubi-localising Functors.*

3.93 Proposition

Consider a reduced functor $F : \mathbf{Cat}^{\mathbf{st}}_\infty \to \mathcal{E}$ where $\mathcal{E}$ is a stable ∞-category. F is Karoubi-localising if and only if it is Verdier-localising and inverts Karoubi equivalences.

Proof. For the "only if" part. We already from remark 3.54 that F is Verdier-localising, since every Verdier square is a Karoubi square. F also inverts Karoubi equivalences since

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d, hook] & 0 \arrow[d] \\ \mathrm{Idem}(\mathcal{C}) \arrow[r] & 0 \end{tikzcd}$$

is a Karoubi square.

For the "if" part. Inverting Karoubi-equivalences makes Karoubi square becomes a Verdier square, hence it is sent to a cartesian square. □

Consider any functor $F : \mathbf{Cat}^{\mathbf{st}}_\infty \to \mathcal{E}$. We have a universal approximation of F from the right by a functor inverting Karoubi equivalences, defined as

$$F \circ \mathrm{Idem} : \mathbf{Cat}^{\mathbf{st}}_\infty \to \mathcal{E}.$$

If F is additive, then $F \circ \mathrm{Idem}$ is additive again; this follows immediately, because the square remains cartesian and we still have the adjunctions, making the square split Verdier. We would also like $F \circ \mathrm{Idem}$ to be Karoubi-localising whenever F is Verdier-localising. However this is not true, it may fail to be even Verdier-localising; the connective K-spectrum functor is a counter-example for this.

3.94 Lemma

Consider a Verdier-localising functor $F : \mathbf{Cat}^{\mathbf{st}}_\infty \to \mathcal{E}$ where $\mathcal{E}$ is a stable ∞-category. Then $F \circ \mathrm{Idem} : \mathbf{Cat}^{\mathbf{st}}_\infty \to \mathcal{E}$ is Karoubi-localising if it takes pullbacks square in $\mathbf{Cat}^{\mathbf{st}}_\infty$, whose vertical legs are dense inclusions, to pullbacks in $\mathcal{E}$.

Proof. We just have to prove that $F \circ \mathrm{Idem}$ is again Verdier-localising, then it is clearly Karoubi-localising. Consider a Verdier square

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] & \mathcal{A}' \arrow[d] \\ \mathcal{B} \arrow[r] & \mathcal{B}' \end{tikzcd},$$

with common vertical fibre $\mathcal{C}$, and also the diagram

$$\begin{tikzcd} \mathrm{Idem}\,\mathcal{A} \arrow[r] \arrow[d] & \mathrm{Idem}\,\mathcal{A}' \arrow[d] \\ (\mathrm{Idem}\,\mathcal{A})/\mathcal{C} \arrow[r] \arrow[d] & (\mathrm{Idem}\,\mathcal{A}')/\mathcal{C} \arrow[d] \\ \mathrm{Idem}\,\mathcal{B} \arrow[r] & \mathrm{Idem}\,\mathcal{B}' \end{tikzcd}.$$

The outer square is cartesian, since $\mathrm{Idem}(\bullet)$ preserves limits. The two lower vertical maps are dense inclusions; indeed

- $\mathcal{A}/\mathcal{C} \to \mathrm{Idem}(\mathcal{A})/\mathcal{C}$ and $\mathcal{A}/\mathcal{C} \to \mathcal{B}$ are dense inclusions;
- $\mathcal{A}'/\mathcal{C} \to \mathrm{Idem}(\mathcal{A}')/\mathcal{C}$ and $\mathcal{A}'/\mathcal{C} \to \mathcal{B}'$ are dense inclusions.

Now, the lower square is a pullback, from which follows that also the upper square is a pullback for pasting laws. Moreover, the upper vertical maps are Verdier projections, so F maps both the upper and the lower square to pullbacks in $\mathcal{E}$; by pasting laws, the outer square is mapped to a pullback. $\square$

4 K-THEORY OF STABLE INFINITY CATEGORIES

In this chapter, we will construct K-theory using two different methods, namely the S-construction and the Q-construction. We will also demonstrate that these two approaches are equivalent. Additionally, this chapter will feature a brief discussion on Segal animae and span. We require the former throughout the chapter, while the latter is necessary for the next one, but it fits very well here.

4.1 THE WALDHAUSEN'S (OR SEGAL'S) S-CONSTRUCTION.

Consider a stable ∞-category $\mathcal{C}$ and $[n] \in \mathbf{\Delta}$. The category $S_n(\mathcal{C})$ of $[n]$-*gapped object* of $\mathcal{C}$ is the full sub-∞-category of $\mathbf{Fun}(\mathrm{Arr}([n]), \mathcal{C})$ spanned by those functors $A : \mathrm{Arr}([n]) \to \mathcal{C}$ with the following properties:

(i) For every $i \in [n]$, $A(i \leqslant i)$ is a zero object of $\mathcal{C}$.

(ii) $A(i \leqslant j) \to A(i \leqslant k) \to A(j \leqslant k)$ is a cofibre sequence in $\mathcal{C}$, or, equivalently, the square

$$\begin{tikzcd} A(i \leqslant j) \arrow[r] \arrow[d] & A(i \leqslant k) \arrow[d] \\ A(j \leqslant j) = 0 \arrow[r] & A(j \leqslant k) \end{tikzcd}$$

is a cocartesian in $\mathcal{C}$. (Note that, since $\mathcal{C}$ is stable this sequences is immediately bifibre).

An $[n]$-gapped object A of $\mathcal{C}$ can be pictured as a diagram

$$\begin{tikzcd}
0 \arrow[r] & A_{0,1} \arrow[r] \arrow[d] & A_{0,2} \arrow[r] \arrow[d] & A_{0,3} \arrow[r] \arrow[d] & A_{0,4} \arrow[r, dotted, no head] \arrow[d] & A_{0,n-1} \arrow[r] \arrow[d] & A_{0,n} \arrow[d] \\
 & 0 \arrow[r] & A_{1,2} \arrow[r] \arrow[d] & A_{1,3} \arrow[r] \arrow[d] & A_{1,4} \arrow[r, dotted, no head] \arrow[d] & A_{1,n-1} \arrow[r] \arrow[d] & A_{1,n} \arrow[d] \\
 & & 0 \arrow[r] & A_{2,3} \arrow[r] \arrow[d] & A_{2,4} \arrow[r, dotted, no head] \arrow[d] & A_{2,n-1} \arrow[r] \arrow[d] & A_{2,n} \arrow[d] \\
 & & & 0 \arrow[r] \arrow[dr, dotted, no head] & A_{3,4} \arrow[r, dotted, no head] \arrow[d, dotted, no head] & A_{3,n-1} \arrow[r] \arrow[d, dotted, no head] & A_{3,n} \arrow[d, dotted, no head] \\
 & & & & 0 \arrow[r] & A_{n-2,n-1} \arrow[r] \arrow[d] & A_{n-2,n} \arrow[d] \\
 & & & & & 0 \arrow[r] & A_{n-1,n} \arrow[d] \\
 & & & & & & 0
\end{tikzcd}$$

in which each square is cocartesian.

4.1 Example

Let us study the first cases of $S_\bullet\mathcal{C}$.

- $S_0(\mathcal{C})$ is the full sub-∞-category of $\mathcal{C}$ spanned by the zero objects of $\mathcal{C}$. Since $\mathcal{C}$ is pointed, this is a contractible anima.

- $S_1(\mathcal{C})$ is the ∞-category whose objects are diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[d] \\
 & 0
\end{tikzcd}$$

  where A is an object of $\mathcal{C}$ and 0 is a zero object. This is an ∞-category equivalent to $\mathcal{C}$.

- $S_2(\mathcal{C})$ is the ∞-category of arrows of $\mathcal{C}$; however, it is better to picture it as the ∞-category whose objects are diagrams

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[d] \\ & 0 \arrow[r] & A'' \arrow[d] \\ & & 0 \end{tikzcd}$$

We can think this as a cofibre sequence

$$A' \to A \to A''.$$

In this picture, the face maps $d_0, d_1, d_2 : S_2(\mathcal{C}) \to S_1(\mathcal{C})$ are respectively the quotient, target, and source maps. Hence they give A'', A, and A', respectively.

4.2 Remark

An $[n]$-gapped object A is determined by the sequence of maps in the top row of the diagram (or better, the red-coloured part of the upper row)

$$A_{0,1} \to A_{0,2} \to A_{0,3} \to A_{0,4} \to \cdots \to A_{0,n-1} \to A_{0,n}.$$

This because the condition (ii) implies the rest of the diagram can be recovered by forming cofibres: $A_{i,j}$ is the cofibre of the map

$$A_{0,i} \to A_{0,j} := A_{0,i} \to A_{0,i+1} \to \cdots \to A_{0,j}.$$

It follows that, for any $n \in \mathbf{N}$, the restriction of A to the top row gives an equivalence of ∞-categories

$$S_n(\mathcal{C}) \simeq \mathbf{Fun}([n-1], \mathcal{C}).$$

Since $\mathcal{C}$ is stable, also $S_n\mathcal{C}$ is stable.

4.3 Remark

$\mathrm{Arr}([n])$ is functorial in $[n]$. Moreover, $\mathbf{Fun}(\mathrm{Arr}([n]), \mathcal{C})$ is functorial in both $[n]$ and $\mathcal{C}$.

4.4 Remark

The $S_n(\mathcal{C})$ assemble into simplicial stable ∞-category

$$S_\bullet(\mathcal{C}) : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathbf{Cat}_\infty^{\mathbf{st}}, \quad [n] \mapsto S_n(\mathcal{C}).$$

To prove this, we have to describe the face and degeneracy maps, and check that they actually preserve the sub-∞-categories $S_n(\mathcal{C})$ (e.g. that restricting a degeneracy map s_k to $S_n(\mathcal{C})$ we actually land in $S_{n+1}(\mathcal{C})$). Also we have to check that they satisfies the simplicial identities. In the "row description" (through the equivalence) these are given as follows.

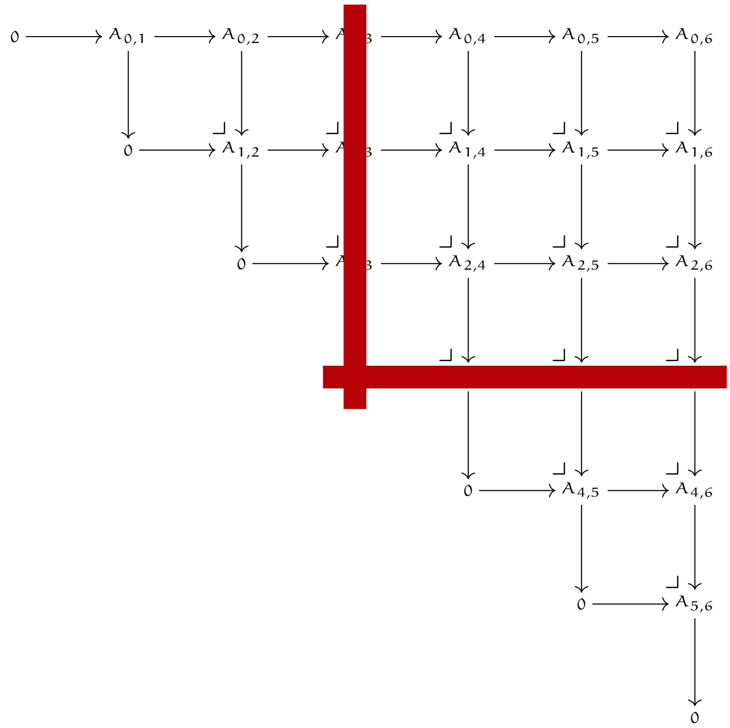
- The k-th face map, for $i \neq 0$, is given by forgetting the object $A_{0,k}$ in the diagram.
- The 0-th face map is given by taking quotient $A_{1,j}$ in the diagram.
- The k-th degeneracy map, for $k \neq 0$, is given by adding the morphism $\mathrm{id}_{A_{0,k}}$ in the diagram.
- The 0-th degeneracy map, is given by adding the morphism 0 in front the diagram.

From the description of these maps it should be clear that everything we want is satisfied.

To be complete, we also give the full description of the face and degeneracy maps.

- The k-th face map, for $k \neq 0$, is given by forgetting k-th column and the k-th row; so we forget all objects $A_{i,k}$ for $0 \leqslant i \leqslant k$ and $A_{k,j}$ for $k \leqslant j \leqslant n$.
- The 0-th face map is given by removing the 0-th row.
- The k-th degeneracy map, for $k \neq 0$, is given by adding in between the k-th and $(k+1)$-th column a copy of the column extended by 0 below, and then extending on the right by copying the element above.
- The 0-th degeneracy map by adding of the diagram a copy of the 0-th raw extended by 0 on the left.

For example, in the case $n = 6$, the 3-rd face map would be



4.5 Remark

It also should be clear that $S_n(\mathcal{C})$ is functorial in $\mathcal{C}$. If $F : \mathcal{C} \to \mathcal{D}$ is an exact functor, then we have a functor $S_n(\mathcal{C}) \to S_n(\mathcal{D})$, which applies F object-wise (and this preserves cocartesian squares being exact); this is also compatible with the degeneracy and face maps, so for any morphism $f : [m] \to [n] \in \boldsymbol{\Delta}$ we get a commutative diagram

$$\begin{tikzcd}
S_m(\mathcal{C}) \arrow[r, "S_m(F)"] \arrow[d, "f_*"'] & S_m(\mathcal{D}) \arrow[d, "f_*"] \\
S_n(\mathcal{C}) \arrow[r, "S_n(F)"'] & S_n(\mathcal{D})
\end{tikzcd}.$$

We get a functor

$$S_\bullet : \mathbf{Cat}_\infty^{\mathbf{st}} \to s\,\mathbf{Cat}_\infty^{\mathbf{st}}, \quad \mathcal{C} \mapsto S_\bullet(\mathcal{C}).$$

This is called $S_\bullet$-CONSTRUCTION.

4.6 Remark (A note on the name of the S-construction)

The S-construction is frequently known as Waldhausen's S-construction (probably as it first appeared in [Wal85]). However, Waldhausen named it Segal's S-construction. Indeed, the S here stands for Segal, indicating his pivotal role in its development.

Composing with the core functor we obtain

$$\mathrm{core}\ S_\bullet : \mathbf{Cat}_\infty^{\mathbf{st}} \to s\,\mathbf{An}\,.$$

4.7 Remark

The anima $|\mathrm{core}\ S(\mathcal{C})|$ has a canonical base point (up to contractible ambiguity) given by the map

$$\mathrm{core}\ S_0(\mathcal{C}) \to |\mathrm{core}\ S(\mathcal{C})|;$$

this because the left hand side is just the anima of zero objects in $\mathcal{C}$. More generally, for any simplicial anima X,

$$\pi_0 X_0 \to \pi_0|X|$$

is surjective. Therefore, we also get that

$$0 \simeq \pi_0\ \mathrm{core}\ S_0(\mathcal{C}) \to \pi_0|\mathrm{core}\ S_\bullet(\mathcal{C})|$$

is surjective, hence $|\mathrm{core}\ S_\bullet(\mathcal{C})|$ is connected.

4.8 Definition (Algebraic K-theory Anima)

Consider a stable ∞-category $\mathcal{C}$. We define the ALGEBRAIC K-THEORY ANIMA (or algebraic K-theory space, or projective anima class) of $\mathcal{C}$ to be

$$k(\mathcal{C}) := \Omega|\mathrm{core}\ S_\bullet(\mathcal{C})|,$$

where the loop is formed on the canonical base point.

We have defined in 2.1 the Grothendieck group, or 0-th K-group, of a stable ∞-category. Now we can define the higher version on this.

4.9 Definition (Higher K-groups)

Consider a stable ∞-category $\mathcal{C}$. For $n \geqslant 1$ we define the n-th K-group of $\mathcal{C}$ as the abelian group $K_n(\mathcal{C}) := \pi_n\, k(\mathcal{C})$.

4.10 Remark

If the reader wishes to choose a specific model, here they can work either with topological spaces or simplicial sets. Let us just recall that, when dealing with the simplicial sets context, the geometric realization of a simplicial anima $X_\bullet : [n] \mapsto X_n$ is given by the diagonal colimit $[n] \to (X_n)_n$.

4.11 Remark

Consider two stable ∞-categories $\mathcal{C}$ and $\mathcal{D}$, and an exact functor f between them. f induces a map of simplicial animae $S_\bullet(\mathcal{C}) \to S_\bullet(\mathcal{D})$, so also a map of K-theory animae $k(\mathcal{C}) \to k(\mathcal{D})$. We obtain the K-*theory functor*

$$k : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}\,.$$

In the following remark we are showing that $K_0(\mathcal{C})$, as defined in 2.1, is equivalent to $\pi_0\, k(\mathcal{C})$.

4.12 Remark

The anima $|\text{core } S_\bullet\mathcal{C}|$ can be written as a colimit of partial geometric realizations

$$\text{sk}_0|\text{core } S_\bullet\mathcal{C}| \to \text{sk}_1|\text{core } S_\bullet\mathcal{C}| \to \text{sk}_2|\text{core } S_\bullet\mathcal{C}| \to \cdots$$

The partial geometric realization is given by

$$\text{sk}_n|\text{core } S_\bullet(\mathcal{C})| = |S_{\bullet\leqslant n}(\mathcal{C})|.$$

The 0-skeleton $\text{sk}_0|\text{core } S_\bullet(\mathcal{C})|$ is equivalent to $|\text{core } S_0(\mathcal{C})|$, and so it is contractible. Using the classical cocartesian square relating the n-skeleton with the $(n+1)$-skeleton, the 1-skeleton is obtained by attaching core $S_1\mathcal{C} \times [1]$ to the 0-skeleton. So it is equivalent to the suspension of $|\text{core } S_1\mathcal{C}| \simeq |\text{core } \mathcal{C}|$. In particular we obtain an inclusion

$$\text{sk}_1|\text{core } S_\bullet(\mathcal{C})| \simeq \mathbf{S}^1 \wedge |\text{core } \mathcal{C}| \to |\text{core } S_\bullet(\mathcal{C})|.$$

It is worth noting two things

- This inclusion is 1-connected; in fact, all the inclusion $\text{sk}_n|\text{core } S_\bullet| \hookrightarrow |\text{core } S_\bullet|$ are n-connected. Recall that a map $f : X \to Y$ of animae is n-connected if $\pi(f)$ is an isomorphism for any $0 \leqslant i < n$ and $\pi_n(f)$ is surjective.
- By adjointness we have an inclusion

  $$|\text{core } \mathcal{C}| \to \Omega|\text{core } S_\bullet\mathcal{C}| = k(\mathcal{C}).$$

  (As Waldhausen points out in [Wal85] this bears resemblance to Segal's technique for group completion).

Since the inclusion is 1-connected we have a surjection of fundamental groups

$$\pi_1(\mathbf{S}^1 \wedge |\mathrm{core}\ \mathcal{C}|) \to \pi_1|\mathrm{core}\ S_\bullet(\mathcal{C})|.$$

The left hand side is the free group with generators the connected components of core $(\mathcal{C})$, modulo the single relation $[0] = 1$ unit of the group.
The inclusion $sk_2|\mathrm{core}\ S_\bullet(\mathcal{C})| \to |\mathrm{core}\ S_\bullet(\mathcal{C})|$ is 2-connected, hence all the relations in $\pi_1|\mathrm{core}\ S_\bullet(\mathcal{C})|$ come from connected components of core $S_2\mathcal{C}$. But we can identify core $S_2\mathcal{C}$ with cofibre sequences in $\mathcal{C}$. So the relations are exactly

$$[x] = [x'] + [x'']$$

whenever $x' \to x \to x''$ is a cofibre sequence in $\mathcal{C}$. Consequently, we obtain

$$K_0(\mathcal{C}) \simeq \pi_1(|\mathrm{core}\ S_\bullet\mathcal{C}|) \simeq \pi_0\, k(\mathcal{C}).$$

4.2 A small parenthesis on Segal animae and complete Segal animae.

Prior to discussing the Q-construction, it would be useful to review some information about Segal animae and complete Segal animae. It is important to note that this chapter will not provide any proofs and instead refer the interested reader to [Rez00] for the original source on them and to [Ber18] for a text-book account on them.

Remember that a SIMPLICIAL ANIMA is a functor

$$X : N(\mathbf{\Delta})^{op} \to \mathbf{An}\,.$$

Also, recall this classical theorem (for example [Lur09, Thm 5.1.5.6]).

4.13 Theorem

Consider a small ∞-category $\mathcal{C}$ and a cocomplete ∞-category $\mathcal{D}$. The Yoneda embedding $よ_{\mathcal{C}} : \mathcal{C} \to \mathbf{PSh}(\mathcal{C})$ gives an equivalence

$$よ_{\mathcal{C}}^* : \mathbf{Fun}^L(\mathbf{PSh}(\mathcal{C}), \mathcal{D}) \to \mathbf{Fun}(\mathcal{C}, \mathcal{D}),$$

where $\mathbf{Fun}^L(\mathbf{PSh}(\mathcal{C}), \mathcal{D})$ is the full sub-∞-category of $\mathbf{Fun}(\mathbf{PSh}(\mathcal{C}), \mathcal{D})$ spanned by colimit preserving functors.
Furthermore, any functor in $\mathbf{Fun}^L(\mathbf{PSh}(\mathcal{C}), \mathcal{D})$ has a right adjoint.

This theorem is very important, and it allows us, for example, to construct useful adjunction starting from a functor $\mathcal{C} \to \mathcal{D}$.

4.14 Lemma

The inclusion of $N(\mathbf{\Delta}) \hookrightarrow \mathbf{Cat}_\infty$, i.e. considering $N(\mathbf{\Delta})$ as the full sub-∞-category spanned by the ∞-categories $[n]$, gives an adjunction

$$\mathrm{asscat} : s\,\mathbf{An} \to \mathbf{Cat}_\infty : N^r,$$

where the left adjoint is called ASSOCIATED CATEGORY functor and the right adjoint is the REZK NERVE. If $\mathcal{C}$ is an ∞-category and $n \in \mathbf{N}$, then

$$N^r_n(\mathcal{C}) \simeq \mathbf{Map}_{\mathbf{Cat}_\infty}([n], \mathcal{C}) \simeq \mathrm{core}\,\mathbf{Fun}([n], \mathcal{C}).$$

It is worth noting that if $\mathcal{C}$ is a 1-category, then $N^r(\mathcal{C})$ is different from the ordinary nerve

$$N_\bullet(\mathcal{C}) \simeq \mathrm{Hom}_{\mathrm{Cat}_1}([\bullet], \mathcal{C})$$

since $\mathrm{Cat}_1 \hookrightarrow \mathrm{Cat}_1^{(2)}$ is not a fully faithful functor. The difference is that $N(\mathcal{C})$ is always a discrete simplicial anima, while $N^r(\mathcal{C})$ is usually not.

4.15 Remark

For us $[n]$ will denote either the category (or poset)

$$[n] := \{0 < 1 < \cdots < n\} \in \mathbf{\Delta}$$

or the ∞-category $N([n])$. Instead, Δ^n) will denote the simplicial set, which we considered as a simplicial anima. These two are related by

$$N^r([n]) \simeq \Delta^n.$$

Indeed

$$N^r_m([n]) \simeq \mathbf{Map}_{\mathbf{Cat}_\infty}([m], [n]) \simeq \mathrm{core}\,\mathbf{Fun}([m], [n]);$$

since there are no non-identity equivalences in $[n]$ any isomorphism of functors in $\mathbf{Fun}([n], [m])$ must be an identity. Therefore, the right hand side must be the discrete set $(\Delta^n)_m$.

4.16 Definition

A SEGAL ANIMA is a simplicial anima $X : N(\mathbf{\Delta})^{op} \to \mathbf{An}$ for which the inclusion of the n-th spine $sp^n \hookrightarrow \Delta^n$ induces an equivalence

$$X_n \simeq \mathbf{Map}_{s\,\mathbf{An}}(\Delta^n, X) \xrightarrow{\simeq} \mathbf{Map}_{s\,\mathbf{An}}(sp^n, X) \simeq X_1 \times_{X_0} \cdots \times_{X_0} X_1,$$

for all n.
A COMPLETE SEGAL ANIMA is a Segal anima $X : N(\mathbf{\Delta})^{op} \to \mathbf{An}$ for which the diagram

$$\begin{tikzcd} X_0 \arrow[r, "\Delta"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_0 \times X_0 \arrow[d, "(s,s)"] \\ X_3 \arrow[r, "(d_{\{0,2\}}, d_{\{1,3\}})"'] & X_1 \times X_1 \end{tikzcd}$$

is Cartesian.

Segal anima can be thought of as simplicial anima with a unique spine lifting, meaning that there is a unique (up to a contractible choice) lifting against $sp_n \hookrightarrow \Delta^n$.

There are many equivalent conditions for completeness; the following are the most common.

4.17 Proposition

For a Segal anima $X : N(\mathbf{\Delta})^{op} \to \mathbf{An}$ the following are equivalent:

(a) The diagram

$$\begin{tikzcd} X_0 \arrow[r, "\Delta"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_0 \times X_0 \arrow[d, "(s,s)"] \\ X_3 \arrow[r, "(d_{\{0,2\}}, d_{\{1,3\}})"'] & X_1 \times X_1 \end{tikzcd}$$

is Cartesian.

(b) The morphism $N^r(0 \xrightarrow{\simeq} 1) \to \Delta^0$ induces an equivalence

$$X_0 \simeq \mathbf{Map}_{s\,\mathbf{An}}(\Delta^0, X) \xrightarrow{\simeq} \mathbf{Map}(N^r(0 \xrightarrow{\simeq} 1), X).$$

(c) Before giving the condition we want to define a collection of path components $X_1^{\times} \subset X_1$. For any $w, z \in X_0$, we define $P_{w,z}$ as the pullback

$$\begin{tikzcd} P_{w,z} \arrow[r, "\Delta"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_1 \arrow[d, "(d_1, d_0)"] \\ * \arrow[r, "(w,z)"'] & X_0 \times X_0 \end{tikzcd} .$$

Consider $g \in X_1$, the "composition map"

$$X_1 \times_{d_1, X_0, d_0} X_1 \xleftarrow[(d_0, d_2)]{\simeq} X_2 \xrightarrow[d_1]{} X_1$$

induces maps

$$g_* : P_{w, d_1(g)} \to P_{w, d_0(g)}$$

and

$$g^* : P_{d_0(g), z} \to P_{d_1(g), z}$$

(that should be thought as post-composition and pre-composition) simply by restricting to

$$P_{w, d_1(g)} \times_{d_1(g)} g \simeq P_{w, d_1(g)} \quad \text{and} \quad g \times_{d_0(g)} P_{d_0(g), z} \simeq P_{d_0(g), z}.$$

We say that $g \in X_1^\times$ if g_* and g^* are equivalences for any $w, z \in X_0$.
Finally the condition is the following: the degeneracy map $s : X_0 \to X_1^\times$ is an equivalence.

(d) The simplicial sub-anima $X^\times$, which defined level-wise as the collection of path components whose all edges lies in $X_1^\times$, is constant. (Notice that $X_0^\times = X_0$, so it is constant on X_0.)

The completeness condition can be understood as saying that higher simplices must have equivalences that correspond to degenerate edges.

4.18 Proposition

The Rezk nerve $N^r : \mathbf{Cat}_\infty \to s\,\mathbf{An}$ is fully faithful and has as essential image the full sub-∞-category $\mathbf{CS\,An}$ spanned by complete animae.

4.19 Remark

It immediately follows by the $(\text{asscat} \dashv N^r)$-adjunction that for any ∞-category $\mathcal{C}$

$$\text{asscat}\, N^r(\mathcal{C}) \simeq \mathcal{C}.$$

Morevoer,

$$\text{core asscat}\,(X) \simeq |X^\times|,$$

and so

$$\text{core asscat}\, N^r(\mathcal{C}) \simeq |N^r(\mathcal{C})|.$$

Since

$$\operatorname{core} \mathcal{C} \simeq N_0^r(\mathcal{C}),$$

we get that $N^r(\mathcal{C})^\times$ is constant on $N_0^r(\mathcal{C})$; therefore the colimit over the category $\boldsymbol{\Delta}^{op}$ is just $|N^r(\mathcal{C}^\times)| \simeq N_0^r(\mathcal{C})$.

4.20 Lemma

The completion functor

$$\mathrm{comp} : s\,\mathbf{An} \overset{\text{asscat}}{\longrightarrow} \mathbf{Cat}_\infty \overset{N^r}{\longrightarrow} \mathrm{CS}\,\mathbf{An}$$

is a left adjoint to the inclusion **CS An** $\hookrightarrow$ s **An**.

4.21 Remark

Define $s\,\mathbf{An}_{\mathrm{const}}$ as the full sub-∞-category of $s\,\mathbf{An}$ spanned by constant simplicial anima (i.e. $X_n \simeq X_0$ for any n). By restricting the $(\text{asscat} \dashv N^r)$-adjunction we obtain an equivalence

$$\begin{tikzcd} s\,\mathbf{An}_{\mathrm{const}} \arrow[rr, bend left, "ev_0"] \arrow[rr, phantom, "\simeq"] & & \mathbf{An} \arrow[ll, bend left, "\mathrm{const}"] \end{tikzcd}.$$

Indeed, for $A \in \mathbf{An}$ and $[n] \in \boldsymbol{\Delta}$,

$$N_n^r(A) \simeq \operatorname{core} \mathbf{Fun}([n], A) \simeq \mathbf{Fun}(|[n]|, A) \simeq \mathbf{Fun}(*, A) \simeq A.$$

Finally, let us state a lemma we are going to use later, which relates the Rezk nerve of a category of twisted arrows in a ∞-category $\mathcal{C}$ and the Rezk nerve of $\mathcal{C}$.

4.22 Lemma

Consider an ∞-category $\mathcal{C}$; then

$$\begin{aligned} &N_n^r(\mathrm{TwArr}(\mathcal{C})) \simeq \\ &\simeq \mathbf{Map}_{\mathbf{Cat}_\infty}([n] \star [n]^{op}, \mathcal{C}) \simeq \\ &\simeq \mathbf{Map}_{s\,\mathbf{An}}((\Delta^n)^{op} \star \Delta^n, N^r(\mathcal{C})), \end{aligned}$$

where $(\Delta^n)^{op} \star \Delta^n$ is formed as a join in $s\mathcal{S}et$ and moved to $s\,\mathbf{An}$.

4.3 QUILLEN'S Q-CONSTRUCTION.

Consider a stable ∞-category $\mathcal{C}$ and $[n] \in \boldsymbol{\Delta}$. We define the ∞-category $Q_n(\mathcal{C})$ as the full sub-∞-category of $\mathbf{Fun}(\mathrm{TwArr}([n])^{op}, \mathcal{C})$ spanned by functors that take each square in $\mathrm{TwArr}([n])^{op}$ to cartesian squares in $\mathcal{C}$. A functor $A : \mathrm{TwArr}([n])^{op} \to \mathcal{C}$ can be pictured as a diagram

$$\begin{tikzcd}[column sep=small]
 & & & & A_{0,n} \arrow[dl] \arrow[dr] & & & & \\
 & & & A_{0,n-1} \arrow[dll, dash, dotted] \arrow[dr, dash, dotted] & & A_{1,n} \arrow[dr, dash, dotted] & & & \\
 & & A_{0,2} \arrow[dl] \arrow[dr] & & A_{1,3} \arrow[dl] \arrow[dr] \arrow[rr, dash, dotted] & & A_{n-2,n} \arrow[dr] & & \\
 & A_{0,1} \arrow[dl] \arrow[dr] & & A_{1,2} \arrow[dl] \arrow[dr] & & A_{2,3} \arrow[dl] \arrow[dr] \arrow[rr, dash, dotted] & & A_{n-1,n} \arrow[dr] & \\
A_{0,0} & & A_{1,1} & & A_{2,2} & & A_{3,3} \arrow[rr, dash, dotted] & & A_{n,n}
\end{tikzcd}$$

where all the squares

$$\begin{tikzcd}[column sep=small]
 & A_{i,l} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" very near start] & \\
A_{i,k} \arrow[dr] & & A_{j,l} \arrow[dl] \\
 & A_{j,k} &
\end{tikzcd}$$

are cartesian.

4.23 Remark

$\mathrm{TwArr}(\bullet)$ is functorial in $\bullet$; this is showed for example in [Ker, Rmk. 8.1.1.4] $\mathbf{Fun}(\mathrm{TwArr}([n])^{op}, \mathcal{C})$ is functorial in both $[n]$ and $\mathcal{C}$.

4.24 Remark

The $Q_n(\mathcal{C})$ assemble into a simplicial stable ∞-category

$$Q_\bullet(\mathcal{C}) : N(\mathbf{\Delta})^{op} \to \mathbf{Cat}_\infty^{\mathbf{st}}.$$

To prove this, we have to describe the face and degeneracy maps, and check that they actually preserve the sub-∞-categories $Q_n(\mathcal{C})$ (e.g. that restricting a face map d_k from $Q_n(\mathcal{C})$ we actually land inside $Q_{n-1}(\mathcal{C})$). Also, we have to check that they satisfy the simplicial identities. From the description of these maps, we should immediately see that everything we want is satisfied. The maps are given as follows.

- The face map, for $k \neq 0$, $d_k : Q_n(\mathcal{C}) \to Q_{n-1}(\mathcal{C})$, forgets the k-th diagonal

  $$A_{0,k} \to A_{1,k} \to \cdots \to A_{k,k}$$

  and the k-th anti-diagonal

  $$A_{k,k} \leftarrow A_{k,k+1} \leftarrow \cdots \leftarrow A_{k,n}.$$

- The face map, for $k = 0$, $d_0 : Q_n(\mathcal{C}) \to Q_{n-1}(\mathcal{C})$, forgets the 0-th anti-diagonal

$$A_{0,n} \to A_{0,n-1} \to \cdots \to A_{0,0}.$$

- The degeneracy map, for $k \neq 0$, $s_k : Q_n(\mathcal{C}) \to Q_{n+1}(\mathcal{C})$, adds, after the k-th diagonal,

$$A_{0,k} \to A_{1,k} \to \cdots \to A_{k,k} \to A_{k,k},$$

and after the k-th anti-diagonal

$$A_{k,k} \leftarrow A_{k,k+1} \leftarrow \cdots \leftarrow A_{k,n}.$$

- The degeneracy map, for $k = 0$, $s_0 : Q_n(\mathcal{C}) \to Q_{n+1}(\mathcal{C})$, adds, before the 0-th anti-diagonal, a copy of the 0-th anti-diagonal extended by $A_{0,0}$ below.

4.25 Remark

$Q_\bullet(\mathcal{C})$ is functorial also in $\mathcal{C} \in \mathbf{Cat}_\infty^{\mathbf{st}}$; moreover, this is compatible with that face and degeneracy map, i.e. for any $F : \mathcal{C} \to \mathcal{D}$ exact functor of stable ∞-categories and for any morphism $f : [m] \to [n] \in \mathbf{\Delta}$ we get a commutative diagram

$$\begin{tikzcd} Q_m(\mathcal{C}) \arrow[r, "Q_m(F)"] \arrow[d, "f_*"'] & Q_m(\mathcal{D}) \arrow[d, "f_*"] \\ Q_n(\mathcal{C}) \arrow[r, "Q_n(F)"'] & Q_n(\mathcal{D}) \end{tikzcd} .$$

Therefore we obtain a functor

$$Q : \mathbf{Cat}_\infty^{\mathbf{st}} \to s\,\mathbf{Cat}_\infty := \mathbf{Fun}(N(\mathbf{\Delta})^{op}, \mathbf{Cat}_\infty).$$

The process that assigns to a stable ∞-category $\mathcal{C}$ the simplicial ∞-category $Q(\mathcal{C})$ is called the QUILLEN'S Q-CONSTRUCTION. Applying the core functor, we obtain

$$\mathrm{core} \circ Q : \mathbf{Cat}_\infty^{\mathbf{st}} \to s\,\mathbf{An}\,.$$

4.26 Proposition

Consider an ∞-category with pullbacks $\mathcal{C}$. Then the simplicial ∞-category

$$Q_\bullet(\mathcal{C}) : N(\boldsymbol{\Delta}) \to \mathbf{Cat}_\infty$$

is a complete Segal object in $\mathbf{Cat}_\infty$; this mean that $Q_\bullet(\mathcal{C})$ satisfies the Segal and the completness conditions. In particular, the simplicial anima

$$\operatorname{core} Q_\bullet(\mathcal{C}) : N(\boldsymbol{\Delta})^{op} \to \mathbf{An}$$

is a complete Segal anima.

Before proceeding with the proof, let us give some remarks.

4.27 Remark
What we want to verify is the following:

- *Segal Condition.* The Segal maps

$$e_i : [1] \to [n], \quad 0 \mapsto i, 1 \mapsto i+1$$

  induces an equivalence

$$Q_n(\mathcal{C}) \xrightarrow{\simeq} Q_1(\mathcal{C}) \times_{Q_0(\mathcal{C})} Q_1(\mathcal{C}) \times_{Q_0(\mathcal{C})} \cdots \times_{Q_0(\mathcal{C})} Q_1(\mathcal{C})$$

- *Completeness Condition.* The following square is cartesian

$$\begin{tikzcd}
Q_0(\mathcal{C}) \arrow[r, "\Delta"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & Q_0(\mathcal{C}) \times Q_0(\mathcal{C}) \arrow[d, "{(s,s)}"] \\
Q_3(\mathcal{C}) \arrow[r, "{(d_{\{0,2\}}, d_{\{1,3\}})}"'] & Q_1(\mathcal{C}) \times Q_1(\mathcal{C})
\end{tikzcd}.$$

Define J_n as the full sub-∞-category of $\mathrm{TwArr}([n])^{op}$ spanned by objects $(i \leqslant j)$ such that $j \leqslant i+1$. In the case $n = 4$ we get J_4 as the red-coloured part here below.

$$\begin{tikzcd}[column sep=tiny]
 & & & & 0 \leqslant 4 \arrow[dl] \arrow[dr] & & & & \\
 & & & 0 \leqslant 3 \arrow[dl] \arrow[dr] & & 1 \leqslant 4 \arrow[dl] \arrow[dr] & & & \\
 & & 0 \leqslant 2 \arrow[dl] \arrow[dr] & & 1 \leqslant 3 \arrow[dl] \arrow[dr] & & 2 \leqslant 4 \arrow[dl] \arrow[dr] & & \\
 & 0 \leqslant 1 \arrow[dl, red] \arrow[dr, red] & & 1 \leqslant 2 \arrow[dl, red] \arrow[dr, red] & & 2 \leqslant 3 \arrow[dl, red] \arrow[dr, red] & & 3 \leqslant 4 \arrow[dl, red] \arrow[dr, red] & \\
0 \leqslant 0 & & 1 \leqslant 1 & & 2 \leqslant 2 & & 3 \leqslant 3 & & 4 \leqslant 4
\end{tikzcd}.$$

We will reserve the name J_n for this particular ∞-category all throughout the thesis.

4.28 Lemma

Consider an ∞-category $\mathcal{C}$ with pullbacks. A functor

$$F : \mathrm{TwArr}([n])^{op} \to \mathcal{C}$$

belongs to $Q_n(\mathcal{C})$ if and only if it coincide with the right Kan extension of its restriction $F|_{J_n}$ along the inclusion of J_n in $\mathrm{TwArr}([n])^{op}$.

Proof. First, note that we can factor the inclusion $J_n \hookrightarrow \mathrm{TwArr}([n])^{op}$ in $\frac{n(n-1)}{2}$ inclusions, adding in each step a single vertex. In our example, we add first $0 \leqslant 2$, then $1 \leqslant 3$, then $2 \leqslant 4$, then $0 \leqslant 3$, etc. Then the Kan extension can be compute in steps, as the composition of the Kan extension given by adding a single vertex; this is possible because these right adjoints compose to give the searched right adjoint.

We claim that the classical *pointwise limit formula for right Kan extensions* shows that each of these Kan extension along a single vertex is given by a pullback.

To prove this claim, recall the proposition giving the limit formula for right Kan extensions.

4.29 Reference ([Heba, Ch.1])

Consider a small ∞-category $\mathcal{A}$, ∞-categories $\mathcal{B}$ and $\mathcal{Z}$, and a functor $f : \mathcal{A} \to \mathcal{Z}$.

- If $\mathcal{B}$ is complete (resp. cocomplete) then the functor

$$f^* : \mathbf{Fun}(\mathcal{Z}, \mathcal{B}) \to \mathbf{Fun}(\mathcal{A}, \mathcal{B})$$

  has a right adjoint Ran_f (resp. left adjoint Lan_f) and this must satisfy

$$\mathrm{Ran}_f F(z) \simeq \lim_{(a \in \mathcal{A}, z \to f(a) \in \mathcal{Z}) \in z/f} F(a)$$

$$\left(\text{resp. } \mathrm{Lan}_f F(z) \simeq \operatorname*{colim}_{(a \in \mathcal{A}, f(a) \to z \in \mathcal{Z}) \in f/z} F(a)\right)$$

  for any $F \in \mathbf{Fun}(\mathcal{A}, \mathcal{B})$.

- If $\mathcal{B}$ is not complete (resp. not cocomplete) but the limits (resp. colimits) exist for all $z \in \mathcal{Z}$, for a fixed functor $F : \mathcal{A} \to \mathcal{B}$, then they build up a functor $\mathrm{Ran}_f F : \mathcal{Z} \to \mathcal{B}$ (resp. $\mathrm{Lan}_f F : \mathcal{Z} \to \mathcal{B}$).

Working on the example $n = 4$, let us prove that the right Kan extension of a functor $F : J_n \to \mathcal{C}$ along $J_n \hookrightarrow J_n \coprod (0 \leqslant 2)$ is given by a pullback. In this case $\mathcal{A} = J_n$, $\mathcal{B} = \mathcal{C}$, and $\mathcal{Z} = J_n \coprod (0 \leqslant 2)$. Of course if we take $z \in \mathcal{A}$ we get $\mathrm{Ran} F(z) \simeq F(z)$. If we take $z = (0 \leqslant 2)$, then z/f is given by the part of the diagram under $(0 \leqslant 2)$; so by the red coloured part in this diagram

$$\begin{tikzcd}[column sep=small]
 & & 0 \leqslant 2 \arrow[dl] \arrow[dr] & & & & & & \\
 & 0 \leqslant 1 \arrow[dl] \arrow[dr] & & 1 \leqslant 2 \arrow[dl] \arrow[dr] & & 2 \leqslant 3 \arrow[dl] \arrow[dr] & & 3 \leqslant 4 \arrow[dl] \arrow[dr] & \\
0 \leqslant 0 & & 1 \leqslant 1 & & 2 \leqslant 2 & & 3 \leqslant 3 & & 4 \leqslant 4 .
\end{tikzcd}$$

So we have to compute the limit in $\mathcal{C}$ of the diagram

$$\begin{tikzcd}[column sep=small]
 & F(0 \leqslant 1) \arrow[dl] \arrow[dr] & & F(1 \leqslant 2) \arrow[dl] \arrow[dr] & \\
F(0 \leqslant 0) & & F(1 \leqslant 1) & & F(2 \leqslant 2) .
\end{tikzcd}$$

Now, the pullback of the span $F(0 \leqslant 1) \to F(1 \leqslant 1) \leftarrow F(1 \leqslant 2)$ is clearly the limit of this diagram.
We can apply the same reasoning to $(1 \leqslant 3)$ and so on for that row. This reasoning can also be applies to $(0 \leqslant 3)$ and so on for any other vertex of our diagram. More generally we can do this for any of the vertex added to the diagram J_n, for any n, to get $\mathrm{TwArr}([n])^{op}$.
This indeed happen because the larger sub-∞-category in which we compute the limit has the span in which we compute the pullback as a cofinal (i.e. ´´limit preserving") sub-∞-category.

Now the "if and only if" should be clear because the condition that a functor $F : \mathrm{TwArr}([n])^{op} \to \mathcal{C}$ must satisfy to be in $Q_n(\mathcal{C})$ is precisely to map each square in $\mathrm{TwArr}([n])^{op}$ to a cartesian square. $\square$

Let us give a useful lemma.

4.30 Lemma

Consider functor of ∞-categories $\iota : \mathcal{A} \to \mathcal{Z}$. Then for any $F : \mathcal{A} \to \mathcal{B}$ with $\mathcal{B}$ complete and cocomplete (or at least such that the limits and colimits we want to compute exist)

$$F \simeq \iota^* \mathrm{Lan}_\iota(F) \simeq \mathrm{Lan}_\iota F \circ \iota$$

and

$$F \simeq \iota^* \mathrm{Ran}_\iota(F) \simeq \mathrm{Ran}_\iota F \circ \iota.$$

In particular, Lan_ι and Ran_ι are fully faithful.

Proof. The first natural transformation is induced by the unit

$$\mathrm{id}_{\mathbf{Fun}(\mathcal{A},\mathcal{B})} \Rightarrow \iota^* \mathrm{Lan}_\iota$$

of the adjunction $\mathrm{Lan}_\iota \dashv \iota^*$. The second natural transformation is induced by the counit

$$\iota^* \mathrm{Ran}_\iota \Rightarrow \mathrm{id}_{\mathbf{Fun}(\mathcal{A},\mathcal{B})}$$

of the adjunction $\iota^* \dashv \mathrm{Ran}_\iota$. Therefore we can just check object-wise if they are equivalences. Using the colimit formula for left Kan extensions we know

$$\mathrm{Lan}_\iota F(\iota(a)) \simeq \operatorname*{colim}_{(x\in\mathcal{A},\iota(x)\to\iota(a)\in\mathcal{Z})\in\iota/\iota(a)} F(x)$$

but the ∞-category $\iota/\iota(a)$ in which we are computing the colimit has $(a, \mathrm{id}_{\iota(a)})$ as a terminal object, hence

$$\mathrm{Lan}_\iota F(\iota(a)) \simeq \operatorname*{colim}_{(x\in\mathcal{A},\iota(x)\to\iota(a)\in\mathcal{Z})\in\iota/\iota(a)} F(x) \simeq F(a).$$

Similarly for the right Kan extension.
The fact that they are fully faithful follows from the fact that the unit and counit, respectively, are equivalences. □

What we obtain from this discussion is also the following.

4.31 Corollary
The restriction of $\mathbf{Fun}(\mathrm{TwArr}([n])^{op}, \mathcal{C})$ along $J_n \hookrightarrow \mathrm{TwArr}([n])^{op}$ gives an equivalence

$$Q_n(\mathcal{C}) \xrightarrow{\simeq} \mathbf{Fun}(J_n, \mathcal{C}).$$

Proof. We already showed a side of the equivalence. The other one comes from 4.30. □

Proof of the Proposition. Once we have proved that the first part holds, the "in particular" part follows because the core functor preserve limits, since it is a right adjoint.

For $[n] \in \mathbf{\Delta}$ fixed, consider the Segal maps

$$e_i : [1] \to [n], \quad 0 \mapsto i, \quad 1 \mapsto i+1;$$

these maps induce an equivalence

$$J_n \simeq J_1 \sqcup_{J_0} J_1 \sqcup_{J_0} \cdots \sqcup J_0 J_1$$

in $\mathbf{Cat}_\infty$, hence

$$\begin{aligned} Q_n(\mathcal{C}) &\simeq \mathbf{Fun}(J_n, \mathcal{C}) \\ &\simeq \mathbf{Fun}(J_1 \sqcup_{J_0} J_1 \sqcup_{J_0} \cdots \sqcup_{J_0} J_1, \mathcal{C}) \\ &\simeq \mathbf{Fun}(J_1, \mathcal{C}) \sqcup_{\mathbf{Fun}(J_0,\mathcal{C})} \cdots \sqcup_{\mathbf{Fun}(J_0,\mathcal{C})} \mathbf{Fun}(J_1, \mathcal{C}) \\ &\simeq Q_1(\mathcal{C}) \times_{Q_0(\mathcal{C})} \cdots \times_{Q_0(\mathcal{C})} \times Q_1(\mathcal{C}). \end{aligned}$$

This proves $Q_\bullet(C)$ is a Segal object in $\mathbf{Cat}_\infty$.

To prove the completeness, consider the commutative square

$$\begin{tikzcd}
Q_0(\mathcal{C}) \arrow[r, "\Delta"] \arrow[d] & Q_0(\mathcal{C}) \times Q_0(\mathcal{C}) \arrow[d, "(s,s)"] \\
Q_3(\mathcal{C}) \arrow[r, "(d_{\{0,2\}}, d_{\{1,3\}})"'] & Q_1(\mathcal{C}) \times Q_1(\mathcal{C})
\end{tikzcd}$$

and take the pullback P. We have a map $Q_0(\mathcal{C}) \to P$, which we want to show is an equivalence. Let us start by the fully faithfulness. The degeneracy map $Q_0(\mathcal{C}) \to Q_3(\mathcal{C})$ is fully faithful; the canonical map $P \to Q_3(\mathcal{C})$ is fully faithful since it is the pullback of $(s,s) : Q_0(\mathcal{C}) \times Q_0(\mathcal{C}) \to Q_1(\mathcal{C}) \times Q_1(\mathcal{C})$, which is fully faithful as $s : Q_0(\mathcal{C}) \to Q_1(\mathcal{C})$ is. We notice that by definition the triangle

$$\begin{tikzcd}
Q_0(\mathcal{C}) \arrow[r, "\text{canon.}"] \arrow[dr, "s"'] & P \arrow[d, "\text{canon.}"] \\
& Q_3(\mathcal{C})
\end{tikzcd}$$

commute, and conclude by 2-out-of-3 that also $Q_0(\mathcal{C}) \to P$ is fully faithful.

For the essential surjectivity, let us start by describing P. The objects of P are functors $F : \mathrm{TwArr}([3])^{op} \to \mathcal{C}$, which maps squares in $\mathrm{TwArr}([3])^{op}$ to cartesian squares, and it satisfies

- $d_{\{0,2\}}F$ is equivalent to the image of $s : Q_0(\mathcal{C}) \to Q_1(\mathcal{C})$; s maps an object x of $Q_0(\mathcal{C}) \simeq \mathcal{C}$ to the diagram

$$\begin{tikzcd}
& x \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & \\
x & & x
\end{tikzcd}.$$

  Therefore we must have a diagram

$$\begin{tikzcd}
& F(1 \leqslant 3) \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] & \\
F(1 \leqslant 1) & & F(3 \leqslant 3)
\end{tikzcd}$$

  in which the two arrows are equivalences.

- $d_{\{1,3\}}F$ is equivalent to the image of $s : Q_0(\mathcal{C}) \to Q_1(\mathcal{C})$. Therefore we must have a diagram

$$\begin{tikzcd}
& F(0 \leqslant 2) \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] & \\
F(0 \leqslant 0) & & F(2 \leqslant 2)
\end{tikzcd}$$

in which the two arrows are equivalences.

Therefore we have a diagram

$$\begin{tikzcd}[column sep=tiny]
 & & & F(0 \leqslant 3) \arrow[dl] \arrow[dr] & & & \\
 & & F(0 \leqslant 2) \arrow[dl] \arrow[dr] \arrow[ddll, red, thick] \arrow[ddrr, red, thick] & & F(1 \leqslant 3) \arrow[dl] \arrow[dr] \arrow[ddll, red, thick] \arrow[ddrr, red, thick] & & \\
 & F(0 \leqslant 1) \arrow[dl] \arrow[dr] & & F(1 \leqslant 2) \arrow[dl] \arrow[dr] & & F(2 \leqslant 3) \arrow[dl] \arrow[dr] & \\
F(0 \leqslant 0) & & F(1 \leqslant 1) & & F(2 \leqslant 2) & & F(3 \leqslant 3)
\end{tikzcd},$$

in which all the red coloured arrows are equivalence.
We can say more; the red coloured square square

$$\begin{tikzcd}[column sep=tiny]
 & & & F(0 \leqslant 3) \arrow[ddll, red, thick] \arrow[dr, red, thick] & & & \\
 & & & & F(1 \leqslant 3) \arrow[ddll, red, thick] \arrow[dr] & & \\
 & F(0 \leqslant 1) \arrow[dl] \arrow[dr, red, thick] & & & & F(2 \leqslant 3) \arrow[dl] \arrow[dr] & \\
F(0 \leqslant 0) & & F(1 \leqslant 1) & & F(2 \leqslant 2) & & F(3 \leqslant 3)
\end{tikzcd}$$

is cartesian. Since the arrow $F(0 \leqslant 3) \to F(2 \leqslant 3)$ is the pullback of an equivalence, it is an equivalence. Similarly $F(0 \leqslant 3) \to F(2 \leqslant 3)$ is an equivalence.
Furthermore, we know equivalences satisfy 2-out-of-6, so all the red coloured arrows in the following diagram are equivalences

$$\begin{tikzcd}[column sep=tiny]
 & & & F(0 \leqslant 3) \arrow[dl, red] \arrow[dr, red] & & & \\
 & & F(0 \leqslant 2) \arrow[dl, red] \arrow[dr] \arrow[ddrr, red, thick] & & F(1 \leqslant 3) \arrow[dl] \arrow[dr, red] \arrow[ddll, red, thick] & & \\
 & F(0 \leqslant 1) \arrow[dl, red] \arrow[dr] & & F(1 \leqslant 2) & & F(2 \leqslant 3) \arrow[dl] \arrow[dr, red] & \\
F(0 \leqslant 0) & & F(1 \leqslant 1) & & F(2 \leqslant 2) & & F(3 \leqslant 3)
\end{tikzcd}.$$

Then considering again the red coloured square

$$\begin{tikzcd}[column sep=tiny]
 & & & F(0 \leqslant 3) \arrow[ddll, red, thick] \arrow[dr, red, thick] & & & \\
 & & & & F(1 \leqslant 3) \arrow[ddll, red, thick] \arrow[dr] & & \\
 & F(0 \leqslant 1) \arrow[dl] \arrow[dr, red, thick] & & & & F(2 \leqslant 3) \arrow[dl] \arrow[dr] & \\
F(0 \leqslant 0) & & F(1 \leqslant 1) & & F(2 \leqslant 2) & & F(3 \leqslant 3)
\end{tikzcd}$$

we obtain by 2-out-of-3 that $F(0 \leqslant 1) \to F(1 \leqslant 1)$ is an equivalence.
Similarly $F(2 \leqslant 3) \to F(2 \leqslant 2)$ is an equivalence.
Then by considering the cartesian squares

$$\begin{tikzcd}
 & F(0 \leqslant 2) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & & F(1 \leqslant 3) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\
F(0 \leqslant 1) \arrow[dr] & & F(1 \leqslant 2) \arrow[dl] & F(1 \leqslant 2) \arrow[dr] & & F(2 \leqslant 3) \arrow[dl] \\
 & F(1 \leqslant 1) & & & F(2 \leqslant 2) &
\end{tikzcd}$$

we obtain that $F(0 \leqslant 2) \to F(1 \leqslant 2)$ and $F(1 \leqslant 3) \to F(1 \leqslant 2)$ are equivalences.
Again by 2-out-of-3, we obtain that $F(1 \leqslant 2) \to F(1 \leqslant 1)$ and $F(1 \leqslant 2) \to F(2 \leqslant 2)$ are equivalences.
The image of $Q_0(\mathcal{C}) \to P$ consists of those $F : \mathrm{TwArr}([3])^{op} \to \mathcal{C} \in Q_3(\mathcal{C})$ such that F is constant. Therefore, the essential image contains of all functors F which maps the edges of $\mathrm{TwArr}([3])^{op}$ to equivalences. It is now clear that the functor is essentially surjective. □

4.32 Remark (Important)
We actually proved something more specific. We know

$$Q_n(\mathcal{C}) \simeq \mathbf{Fun}(J_n, \mathcal{C})$$

and the latter is a stable ∞-category for any $n \in \mathbf{N}$ and for any stable ∞-category $\mathcal{C}$. Therefore the Q-construction gives a functor

$$Q_\bullet : \mathbf{Cat}_\infty^{\mathbf{st}} \to s\,\mathbf{Cat}_\infty^{\mathbf{st}},$$

and since $\mathbf{Cat}_\infty^{\mathbf{st}}$ is closed under finite limits in $\mathbf{Cat}_\infty$, $Q_\bullet(\mathcal{C})$ is a complete Segal stable ∞-category, for any $\mathcal{C}$ stable ∞-category.

4.33 Definition
The ∞-category of spans in $\mathcal{C}$ is

$$\mathcal{S}pan(\mathcal{C}) \simeq \text{asscat core } Q(\mathcal{C}).$$

4.34 Remark
The K-theory anima of an ∞-category $\mathcal{C}$ could be described in term of the Q-construction. Indeed, the Q-construction is a sort of cleaned-up version of Segal's edgewise subdivision of S-construction.

In order to get a relation between the Q- and the S-construction we need to work a bit. Consider the functor

$$\mathbf{\Delta}^{op} \to \mathbf{\Delta}^{op}, \quad [n] \to [n] \star [n]^{op}.$$

Although we know that $[n] \star [n]^{op} \simeq [2n+1]$, we should not interpret it as the poset

$$(0 < 1 < \cdots < 2n < 2n+1),$$

but rather as the poset

$$(0_l < 1_l < \cdots < n_l < n_r < \cdots < 1_r < 0_r),$$

where $\bullet_l$ refers to the left part associated with $[n]$ and $\bullet_r$ refers to the right part associated with $[n]^{op}$. This functor induces the EDGEWISE SUBDIVISION functor

$$\bullet^{esd} : s\,\mathbf{An} \to s\,\mathbf{An}\,.$$

4.35 Lemma
For any ∞-category $\mathcal{C}$, then

$$N^r(\mathcal{C})^{esd} \simeq N^r(\mathrm{TwArr}(\mathcal{C}^{op})).$$

Proof. For $n \in \mathbf{N}$, we have

$$\begin{aligned}
(N^r(\mathcal{C})^{esd})_n &\simeq \\
&\simeq (\mathbf{Map}_{\mathbf{Cat}_\infty}([\bullet], \mathcal{C}))^{esd}_n \\
&\simeq \mathbf{Map}_{\mathbf{Cat}_\infty}([n] \star [n]^{op}, \mathcal{C}) \\
&\simeq N^r(\mathrm{TwArr}(\mathcal{C}^{op}))_n,
\end{aligned}$$

where the last equivalence comes from lemma 4.22. Therefore, we obtain the wanted equivalence. □

4.36 Lemma
Consider a simplicial anima X; we have

$$|X| \simeq |X^{esd}|.$$

Proof. Both sides are colimit preserving functors $s\,\mathbf{An} \to \mathbf{An}$; therefore, it is enough to prove that they coincide on standard simplices (this comes from 4.13). For any $n \in \mathbf{N}$:

- the left hand side is $|\Delta^n| \simeq *$;
- the right hand side $|(\Delta^n)^{esd}|$ and

$$(\Delta^n)^{esd} \simeq (N^r([n]))^{esd} \simeq N^r(\mathrm{TwArr}([n])^{op}).$$

We know $\mathrm{TwArr}([n])^{op}$ is contractible, because it has an initial object, therefore

$$|(\Delta^n)^{esd}| \simeq *.$$

□

4.37 Proposition

For any stable ∞-category $\mathcal{C}$, the collection of functors

$$\begin{aligned}\chi_n : \mathrm{TwArr}([n])^{op} &\to \mathrm{Arr}([n] \star [n]^{op}),\\ (i \leqslant j) &\mapsto (i_l \leqslant j_r)\end{aligned}$$

induces an equivalence

$$S_\bullet(\mathcal{C})^{esd} \xrightarrow{\simeq} Q_\bullet(\mathcal{C}).$$

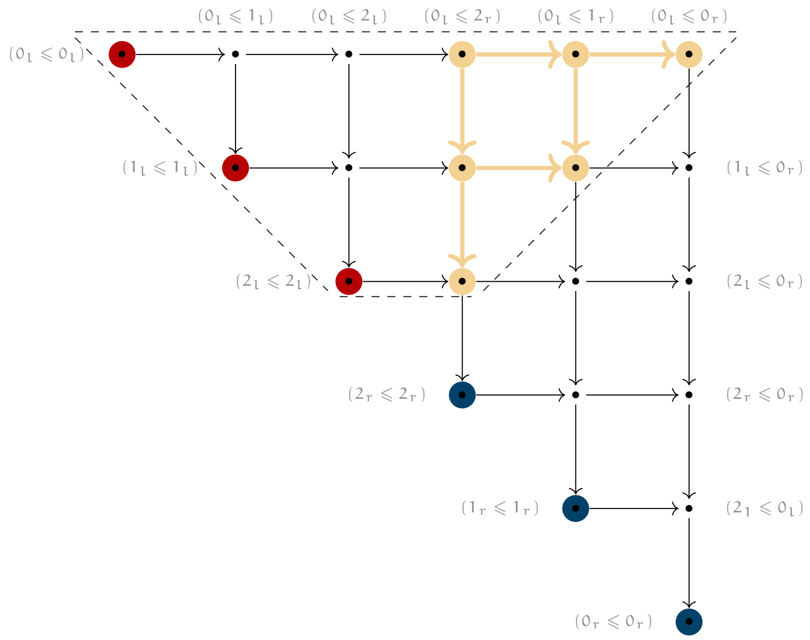
In particular

$$k(\mathcal{C}) \simeq \Omega|\mathrm{Span}(\mathcal{C})|.$$

Proof. Let us describe with a picture what the functor

$$\chi_n : \mathrm{TwArr}([n]^{op}) \to \mathrm{Arr}([n] \star [n]^{op})$$

does. We draw this in the case $n = 2$, the other cases are similar just "larger". For $n = 2$ we can portray $\mathrm{Arr}([2] \star [2]^{op})$ as a diagram:



Let us establish some notation. We call:

- I_n the image of χ_n; in our picture it is the yellow part.
- Δ_n^l the sub-poset spanned by $\{(0_l \leqslant 0_l), \dots (n_l, n_l)\}$; in our picture it is the red part.
- Δ_n^r the sub-poset spanned by $\{(n_r \leqslant n_r), \dots (0_r, 0_r)\}$; in our picture it is the indigo part.
- H_n be the sub-poset spanned by

$$\{(i_l \leqslant j_l) : i \leqslant j\} \cup \{(i_l \leqslant j_r) : i \leqslant j\};$$

in our picture it is the part inside the dotted line.

- K_n the whole diagram.

To start we have to prove that, for any $n \in \mathbf{N}$, χ_n is a map of ∞-categories. Both these categories

$$\mathrm{TwArr}([n])^{op} \quad \text{and} \quad \mathrm{Arr}([n] \star [n]^{op})$$

are the nerve of their 1-categorical correspondent notion. Since the nerve functor is fully faithful, χ_n is induced by a 1-categorical map, which for simplicity will be called χ_n.
The value of χ_n on the object $(i \leqslant j) \in \mathrm{TwArr}([n])^{op}$.

$$\chi_n(i \leqslant j) = i_l \leqslant j_r.$$

A morphism $(i \leqslant j) \to (a \leqslant b)$ in $\mathrm{TwArr}([n])^{op}$ is a square in $[n]^{op}$

$$\begin{tikzcd} i & a \arrow[l] \arrow[d] \\ j \arrow[u, leftarrow] \arrow[r] & b \end{tikzcd}$$

so in $[n]$ it is the square

$$\begin{tikzcd} i \arrow[r] & a \\ j \arrow[u] & b \arrow[l] \arrow[u] \end{tikzcd}.$$

This square is sent by χ_n to

$$\begin{tikzcd} i_l \arrow[r] \arrow[d] & a_l \arrow[d] \\ j_r \arrow[r] & b_r \end{tikzcd}$$

which is a morphism in $\mathrm{Arr}([n] \star [n]^{op})$.

Secondly, we want to show these χ_n build up a natural transformation of functors

$$\mathrm{TwArr}([\bullet])^{op} \to \mathrm{Arr}([\bullet] \star [\bullet]^{op}).$$

To prove this, it is enough to verify they are compatible with "face" and "degeneracy" maps (in quotations marks because these are not proper face or degeneracy maps), i.e. that, for $d^n_k : [n-1] \to [n]$ and $s^n_k : [n] \to [n-1]$, the squares

$$\begin{tikzcd}
\mathrm{TwArr}([n])^{op} \arrow[r, "\chi_n"] \arrow[d, "d_k^*"'] & \mathrm{Arr}([n] \times [n]^{op}) \arrow[d, "d_k^*"] \\
\mathrm{TwArr}([n-1])^{op} \arrow[r, "\chi_{n-1}"'] & \mathrm{Arr}([n-1] \times [n-1]^{op})
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{TwArr}([n-1])^{op} \arrow[r, "\chi_{n-1}"] \arrow[d, "s_k^*"'] & \mathrm{Arr}([n-1] \times [n]^{op}) \arrow[d, "s_k^*"] \\
\mathrm{TwArr}([n])^{op} \arrow[r, "\chi_n"'] & \mathrm{Arr}([n] \times [n]^{op})
\end{tikzcd}$$

commute.
We already know how these maps are on $\mathrm{TwArr}([n])^{op} \simeq \mathrm{TwArr}([n])^{op}$; they are basically the same as the one we have described to show that $Q_\bullet(\mathcal{C})$ form a simplicial anima.
For $\mathrm{Arr}([n] \star [n]^{op})$, fix a $k \neq 0$; counting from 0, the k-th face map forgets

- the k-th column and the k-th row,
- the $(2n-k)$-th column and the $(2n-k)$-th row,

simply by composing the morphisms. The k-th degenracy map adds

- After the k-th column, a replica of this extended with a 0; also it adds on the right of this 0 a row which is a copy of what it has above.
- After the $(2n-k)$-th column, a replica of this extended with a 0; also it adds on the right of this 0 a row which is a copy of what it has above.

The 0-th face map forgets the first row and the last column. The 0-th degeneracy map adds

- Add a 0-th row which is a copy of the first one extended by 0 on the left;
- Add a $2n$-th column which is a copy of the last one extended by 0 below;
- Add a final element on the top-right which is the same one of the the previous top-right one.

From this description is then immediate that the χ_n define a natural transformation

$$\chi : \mathrm{TwArr}([\bullet])^{op} \Rightarrow \mathrm{Arr}([\bullet] \star [\bullet]^{op}).$$

Applying $\mathbf{Fun}(\bullet, \mathcal{C})$ to χ we obtain a natural transformations

$$\chi^* : \mathbf{Fun}(\mathrm{Arr}([\bullet] \star [\bullet]^{op}), \mathcal{C}) \Rightarrow \mathbf{Fun}(\mathrm{TwArr}([\bullet])^{op}, \mathcal{C})$$

of functors $N(\mathbf{\Delta})^{op} \to \mathbf{Cat}_\infty$. Now notice that $S_\bullet(\mathcal{C})^{esd} \subset \mathbf{Fun}(\mathrm{Arr}([\bullet] \star [\bullet]^{op})$, i.e. that for any $n \in \mathbf{N}$

$$S_n(\mathcal{C})^{esd} \subset \mathbf{Fun}(\mathrm{Arr}([n] \star [n]^{op}.$$

We would like to prove that the image of the $S_\bullet(\mathcal{C})^{esd}$ through χ^* lands inside $Q_\bullet(\mathcal{C})$.
In other words, for any $n \in \mathbf{N}$, we have

$$\chi_n^* : \mathbf{Fun}(\mathrm{Arr}([n] \star [n]^{op}), \mathcal{C}) \to \mathbf{Fun}(\mathrm{TwArr}([n])^{op}, \mathcal{C}),$$

and we want to show that the restriction $\chi_n^*|_{S_n(\mathcal{C})^{esd}}$ has image is inside $Q_n(\mathcal{C})$.

This is true, indeed $(S_\bullet(\mathcal{C}))_n^{esd} \simeq S_{2n+1}(\mathcal{C})$, and any $A \in S_{2n+1}(\mathcal{C})$ is represented by the diagram similar to K_n; applying χ_n means restricting to I_n. This remaining part of the diagram diagram clearly represent an object of $Q_n(\mathcal{C})$. By compatibility with face and degeneracy maps, this extends to a map $S_\bullet(\mathcal{C})^{esd} \to Q_\bullet(\mathcal{C})$.

What we want to do now is to construct a "pseudo-inverse". First, recall that a functor of $\mathbf{Fun}(\mathrm{Arr}([n] \star [n]^{op}), \mathcal{C}))$ belongs to $S_{2n+1}(\mathcal{C})$ if it maps all the square to cocartesian (and hence cartesian) squares and the elements of the diagonal to 0. A functor of $\mathbf{Fun}(I_n, \mathcal{C})$ belongs to $Q_n(\mathcal{C})$ if it maps all the square of I_n to cartesian (and hence cocartesian) squares. Consider the Kan extensions

$$\begin{tikzcd}
\mathbf{Fun}(I_n, \mathcal{C}) \arrow[dr, "\text{Left Kan Ext.}"'] \arrow[rrrrdddd, dotted, bend left] & & & & \\
& \mathbf{Fun}(I_n \cup \Delta^n_l, \mathcal{C}) \arrow[dr, "\text{Right Kan Ext.}"'] & & & \\
& & \mathbf{Fun}(I_n \cup \Delta^n_l \cup \Delta^n_r, \mathcal{C}) \arrow[dr, "\text{Right Kan Ext.}"'] & & \\
& & & \mathbf{Fun}(H_n, \mathcal{C}) \arrow[dr, "\text{Left Kan Ext.}"'] & \\
& & & & \mathbf{Fun}(\mathrm{Arr}([n] \star [n]^{op}), \mathcal{C})
\end{tikzcd}$$

where we use $\cup$ to denote the full sub-∞ category of $\mathrm{Arr}([n] \star [n]^{op})$ spanned by the union of objects.

All of the Kan extensions are along fully faithful embeddings of subcategories. Therefore, by 4.30 all these Kan extensions are fully faithful, thus

$$\mathbf{Fun}(I_n, \mathcal{C}) \to \mathbf{Fun}(\mathrm{Arr}([n] \star [n]^{op}), \mathcal{C})$$

is fully faithful.

We know want to show that the essential image of $Q_n(\mathcal{C}) \subset \mathbf{Fun}(I_n, \mathcal{C})$ through the Kan extension is $S_{2n+1}(\mathcal{C})$. To do so we use the limit and colimit formulas for, respectively, right and left Kan extensions.

- For any $(i_l \leqslant i_l) \in \Delta^l_n$ the colimit to compute is on an empty diagram, therefore it is the zero object.
- For any $(i_r \leqslant i_r) \in \Delta^r_n$ the limit to compute is on an empty diagram, therefore it is the zero object.

We have proved it extends as 0 on the diagonal.

- For any $(i_l \leqslant j_l) \in J_n$, the pullback of the span

  $$0 \simeq (j_l, j_l) \to (j_l \leqslant n_r) \leftarrow (i_l \leqslant n_r)$$

  already satisfies the universal property of the limit we have to compute. Recalling that $\mathcal{C}$ is stable, all the squares we have completed are cocartesian.
- In a similar fashion, all the rest of the extensions are pushouts.

For the last part, by lemma 4.36

$$|\text{core } S(\mathcal{C})| \simeq |(\text{core } S(\mathcal{C}))^{esd}| \simeq |\text{core } (S(\mathcal{C})^{esd})| \simeq |\text{core } Q(\mathcal{C})|.$$

Moreover, for any X simplicial anima

$$|\text{asscat } X| \simeq |X|.$$

Indeed, both sides are colimit preserving functors so for theorem 4.13 we can just verify they are equivalent on standard simplices

$$|\text{asscat } \Delta^n| \simeq |\text{asscat } N^r([n])| \simeq |[n]| \simeq *$$

and

$$|\Delta^n| \simeq *$$

for any $n \in \mathbf{N}$. It follows

$$|\text{core } Q(\mathcal{C})| \simeq |\text{asscat core } Q(\mathcal{C}) \simeq |Span(\mathcal{C})|,$$

and so

$$k(\mathcal{C}) := \Omega|\text{core } S(\mathcal{C})| \simeq \Omega|Span(\mathcal{C})|.$$

□

4.4 A SMALL PARENTHESIS ON SPANS.

4.38 Definition
For any additive functor $F : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$, we define

$$\mathrm{Span}^F(\mathcal{C}) := \mathrm{asscat}(F(Q(\mathcal{C}))) \in \mathbf{Cat}_\infty .$$

As an example,

$$\mathrm{Span}^{\mathrm{core}}(\mathcal{C}) \simeq \mathrm{Span}(\mathcal{C}).$$

4.39 Lemma
Consider a stable ∞-category $\mathcal{C}$. For any $0 \leqslant i \leqslant n$, the pushout square in $\mathbf{\Delta}$

$$\begin{tikzcd}
{[0]} \arrow[r, "0"] \arrow[d, "i"'] \arrow[dr, phantom, "\lrcorner", very near start] & {[n-i]} \arrow[d, "\mathrm{incl.}+i"] \\
{[i]} \arrow[r, "\mathrm{incl.}"'] & {[n]}
\end{tikzcd} .$$

induces a split Verdier square

$$\begin{tikzcd}
Q_n(\mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Q_i(\mathcal{C}) \arrow[d] \\
Q_{n-i}(\mathcal{C}) \arrow[r] & Q_0(\mathcal{C}) \simeq \mathcal{C} \arrow[u, dotted, bend left] \arrow[u, dotted, bend right]
\end{tikzcd} .$$

Proof. The square is a cartesian square, because by the Segal condition we get

$$\begin{tikzcd}
\underbrace{Q_1(\mathcal{C}) \times_{Q_0(\mathcal{C})} \cdots \times_{Q_0(\mathcal{C})} Q_1(\mathcal{C})}_{n} \arrow[r, "\text{Proj. on first } i."] \arrow[d, "\text{Proj. on last } n-i"] & \underbrace{Q_1(\mathcal{C}) \times_{Q_0(\mathcal{C})} \cdots \times_{Q_0(\mathcal{C})} Q_1(\mathcal{C})}_{n-i} \arrow[d, "\text{Proj. on last}"] \\
\underbrace{Q_1(\mathcal{C}) \times_{Q_0(\mathcal{C})} \cdots \times_{Q_0(\mathcal{C})} Q_1(\mathcal{C})}_{i} \arrow[r, "\text{Proj. on first}"'] & \mathcal{C}
\end{tikzcd} ,$$

which is clearly cartesian. Let us prove it is split Verdier. Any inclusion of an interval $\iota : [m] \hookrightarrow [n]$, induces a fully faithful functor $J_m \to J_n$. Since all the J_i are finite, and $\mathcal{C}$ has finite limits and colimits, then

$$\iota^* : \mathbf{Fun}(J_n, \mathcal{C}) \to \mathbf{Fun}(J_m, \mathcal{C})$$

has both adjoints, given by the left and right Kan extension, which by lemma 4.30 are given by fully faithful functors. Thus $Q_n\mathcal{C} \to Q_m\mathcal{C}$ is a split Verdier projection. $\square$

4.40 Remark

We already proved that for any stable ∞-category $\mathcal{C}$, the simplicial ∞-category $Q(\mathcal{C})$ is a complete Segal ∞-category. Now, we also have that

$$\begin{tikzcd} Q_n(\mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Q_i(\mathcal{C}) \arrow[d] \\ Q_{n-i}(\mathcal{C}) \arrow[r] & Q_0(\mathcal{C}) \simeq \mathcal{C} \arrow[u, dotted, bend left] \arrow[u, dotted, bend right] \end{tikzcd} .$$

is a split Verdier square.

We have already seen core $Q(\mathcal{C})$ is a complete Segal anima. In fact, for any additive functor $F : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$,

$$F(Q(\mathcal{C}))$$

is a Segal anima, which is complete if F preserves pullbacks. To prove this, notice that the square

$$\begin{tikzcd} Q_n(\mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Q_1(\mathcal{C}) \arrow[d] \\ Q_{n-1}(\mathcal{C}) \arrow[r] & Q_0(\mathcal{C}) \simeq \mathcal{C} \arrow[u, dotted, bend left] \arrow[u, dotted, bend right] \end{tikzcd}$$

is split Verdier; from this it follows that

$$F(Q_n(\mathcal{C})) \simeq F(Q_1) \times_{F(Q_0(\mathcal{C}))} F(Q_{n-1}(\mathcal{C})).$$

Iterating we obtain

$$F(Q_n(\mathcal{C})) \simeq F(Q_1) \times_{F(Q_0(\mathcal{C}))} \cdots \times_{F(Q_0(\mathcal{C}))} F(Q_1).$$

For the completeness, recall that $Q(\mathcal{C})$ is a complete Segal ∞-category, so the following is by definition a cartesian square in $\mathbf{Cat}_\infty^{\mathbf{st}}$

$$\begin{tikzcd}
Q_0(\mathcal{C}) \arrow[r, "\Delta"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Q_0(\mathcal{C}) \times Q_0(\mathcal{C}) \arrow[d, "{(s,s)}"] \\
Q_3(\mathcal{C}) \arrow[r, "{(d_{\{0,2\}}, d_{\{1,3\}})}"'] & Q_1(\mathcal{C}) \times Q_1(\mathcal{C})
\end{tikzcd}.$$

If F preserves pullback, then is cartesian, and $F(Q(\mathcal{C}))$ is a complete Segal anima.
It is also worth noting that that

$$|\mathrm{Span}^F \mathcal{C}| \simeq |\text{asscat } FQ\mathcal{C}| \simeq |FQ\mathcal{C}|.$$

4.41 Proposition
$\mathrm{Span}^F(\mathcal{C})$ is connected for every stable ∞-category $\mathcal{C}$ and any additive functor $F : \mathbf{Cat}^{\mathbf{st}}_\infty \to \mathbf{An}$.

Proof. Notice that

$$\begin{aligned}
&|\mathrm{Span}^F \mathcal{C}| \simeq \\
&\simeq |\text{asscat}\, F(Q(\mathcal{C}))| \\
&\simeq |F(Q(\mathcal{C}))| \\
&\simeq |(F(S(\mathcal{C}))^{esd})| \\
&\simeq |(F(S(\mathcal{C})))^{esd}| \\
&\simeq |F(S(\mathcal{C}))|.
\end{aligned}$$

We have a surjective map

$$\pi_0 F(S_0(\mathcal{C})) \to \pi_0 |F(S(\mathcal{C}))|;$$

in fact, more generally, for any simplicial anima X the map

$$\pi_0 X_0 \to \pi_0 X$$

is surjective. Since $S_0(\mathcal{C}) \simeq *$, it follows immediately that

$$\pi_0 |F(S(\mathcal{C}))| = 0,$$

so $|\mathrm{Span}^F(\mathcal{C})|$ is connected. $\square$

5 RESULT ON THE K-THEORY OF STABLE INFINITY CATEGORIES

This central chapter of the thesis is dedicated to establishing various properties of K-theory. We begin by proving the additivity and universality theorems for K-theory, which immediately implies that the K-theory anima functor descends to a (connective) spectrum-valued functor. Once this is established, we prove similar but more general properties for grouplike additive functors. (Although we do not provide additional examples, it is worth noting that there are many other significant additive and Verdier localising functors, such as topological Hochschild homology, topological cyclic homology, and others. Moreover, similar investigations are conducted in Hermitian K-theory, which presents a vast array of opportunities for exploration.) Next, we present another construction of the K-theory spectrum, this time through iterated S- and Q-constructions. We also discuss the localisation and cofinality properties, both in the context of K-theory and more generally. Additionally, we introduce the relative Q-construction and provide a particular statement of the Waldhausen fibration theorem.

All the proofs for these properties are already available in [HLS22], [Heba], and similar proofs to those found in sections 5.4 and 5.5 can be found in [Cal+21b] concerning Hermitian K. As always, we have endeavored to expand and provide comprehensive coverage of various aspects, while making no claims of originality for the presented content.

5.1 ADDITIVITY.

5.1.1 *Additivity and Proof.*

This short proof of the additivity theorem comes from in [HLS22].

Theorem A (Additivity Theorem)
For every stable ∞-category $\mathcal{C}$, the source-target projection

$$(s,t) : |\mathrm{Span}(\mathrm{Arr}(\mathcal{C}))| \to |\mathrm{Span}(\mathcal{C})|^2$$

induces an equivalence of animae.

We break the proof of the theorem in two propositions.

5.1 Proposition
Consider a stable ∞-category $\mathcal{C}$. There are canonical equivalences

$$\mathrm{Span}(\mathcal{C}) \simeq \mathrm{Span}(\mathcal{C}^{op}), \quad \text{and} \quad \mathrm{Span}(\mathrm{Arr}(\mathcal{C})) \simeq \mathrm{Span}(\mathrm{TwArr}(\mathcal{C})^{op}).$$

These equivalences fit into a commutative diagram

$$\begin{tikzcd}
\mathrm{Span}(\mathrm{Arr}(\mathcal{C})) \arrow[r, "\simeq"] \arrow[d, "(s,t)"'] & \mathrm{Span}(\mathrm{TwArr}(\mathcal{C})^{op}) \arrow[d, "(s,t)"] \\
\mathrm{Span}(\mathcal{C}) \times \mathrm{Span}(\mathcal{C}) \arrow[r, "\simeq"] & \mathrm{Span}(\mathcal{C}) \times \mathrm{Span}(\mathcal{C}^{op})
\end{tikzcd}.$$

Proof. (First equivalence).
Let us call α the functor

$$\mathrm{Span}(\mathcal{C}) \longrightarrow \mathrm{Span}(\mathcal{C}^{op})$$

that we want to prove is an equivalence. α is given

- On objects by the identity.
- On a morphisms by

$$\begin{tikzcd}[column sep=small]
 & s \arrow[dl] \arrow[dr] & & & & s \arrow[dl] \arrow[dr] & & & & & \\
x & & y & \longmapsto & x \arrow[dr] & & y \arrow[dl] & \longmapsto & x \arrow[dr] & & y \arrow[dl] \\
 & & & & & x \oplus_s y & & & & x \oplus_s y &
\end{tikzcd}.$$

To define α on higher cells we do the following.
We define with $\hat{Q}_n(\mathcal{C})$ as the full sub-∞-category of $\mathbf{Fun}([n] \times [n]^{op}, \mathcal{C})$ spanned by those diagrams A such that each square

$$\begin{tikzcd}
A_{i,j+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A_{i+1,j+1} \arrow[d] \\
A_{i,j} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & A_{i+1,j}
\end{tikzcd}$$

is bicartesian, for any $0 \leqslant i, j \leqslant n$. A functor $A \in \hat{Q}_n(\mathcal{C})$ can be portrayed as a diagram

$$\begin{tikzcd}
A_{0,0} \arrow[d] & A_{0,1} \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner" very near start] & \cdots \arrow[l] \arrow[d] & A_{0,n} \arrow[l] \arrow[d] \\
A_{1,0} \arrow[d] \arrow[ur, phantom, "\urcorner" very near start] & A_{1,1} \arrow[l] \arrow[d] & \cdots \arrow[l] \arrow[d] & A_{1,n} \arrow[l] \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[l] \arrow[d] & \ddots \arrow[l] \arrow[d] & \vdots \arrow[l] \arrow[d] \\
A_{n,0} & A_{n,1} \arrow[l] & \cdots \arrow[l] & A_{n,n} \arrow[l]
\end{tikzcd}$$

in which all squares are bicartesian. Notice that the restriction of $[n] \times [n]^{op}$ to the (i, j) such that $i \leqslant j$ we obtain $\mathrm{TwArr}[n]^{op}$. The canonical functor

$$(s, t) : \mathrm{TwArr}[n]^{op} \to [n] \times [n]^{op}$$

induces a functor

$$\mathbf{Fun}([n] \star [n]^{op}, \mathcal{C}) \to \mathbf{Fun}(\mathrm{TwArr}([n])^{op}, \mathcal{C})$$

that restricts to a functor

$$\hat{Q}_n(\mathcal{C}) \to Q_n(\mathcal{C}).$$

We want to show this map is an equivalence of ∞-categories. We can do this in two ways.

- Let us call J_n the full sub-∞-category of $[n] \times [n]^{op}$ (equivalently of $\mathrm{Arr}([n])$) spanned by $(i \leqslant j)$ such that $j \leqslant i + 1$. We already showed that a functor $A : \mathrm{TwArr}([n])^{op} \to \mathcal{C}$ belongs to $Q_n(\mathcal{C})$ if and only if it is a right Kan extension of $A|_{J_n}$ along the fully faithful embedding $J_n \hookrightarrow \mathrm{TwArr}([n])^{op}$. We have also proved $\mathbf{Fun}(J_n, \mathcal{C}) \simeq Q_n(\mathcal{C})$.
  On the other hand $B : [n] \times [n]^{op} \to \mathcal{C}$ belongs to $\hat{Q}_n(\mathcal{C})$ if and only if it is left Kan extension of $B|_{J_n}$ along $J_n \hookrightarrow \mathrm{TwArr}([n])^{op}$ followed by a left Kan extension of $\mathrm{Lan}_{J_n \hookrightarrow \mathrm{TwArr}([n])^{op}} B|_{J_n}$ along $\mathrm{TwArr}([n])^{op} \hookrightarrow [n] \times [n]^{op}$. Therefore, also $\hat{Q}_n(\mathcal{C})$ is equivalent to $\mathbf{Fun}(J_n, \mathcal{C})$.

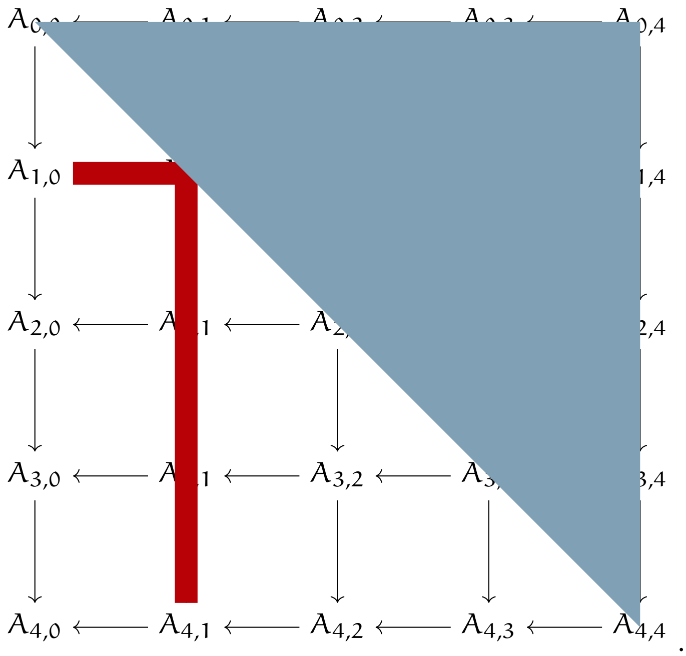
- Call I_n the full sub-∞-category of $[n] \times [n]^{op}$ spanned by $(0, i)$ and (i, n) for any $0 \leqslant i \leqslant n$. Then $A : \mathrm{TwArr}([n])^{op} \to \mathcal{C}$ belongs to $Q_n(\mathcal{A})$ if and only if it is a left Kan extension of $A|_{I_n}$ along $I_n \hookrightarrow \mathrm{TwArr}([n])^{op}$, and $B : [n] \times [n]^{op} \to \mathcal{C}$ belongs to $\hat{Q}_n\mathcal{C}$ if and only if it is left Kan extension of $B|_{I_n}$ along $I_n \hookrightarrow [n] \times [n]^{op}$. Similarly at above,

$$\hat{Q}_n(\mathcal{C}) \simeq \mathbf{Fun}(I_n, \mathcal{C}) \simeq Q_n(\mathcal{C}).$$

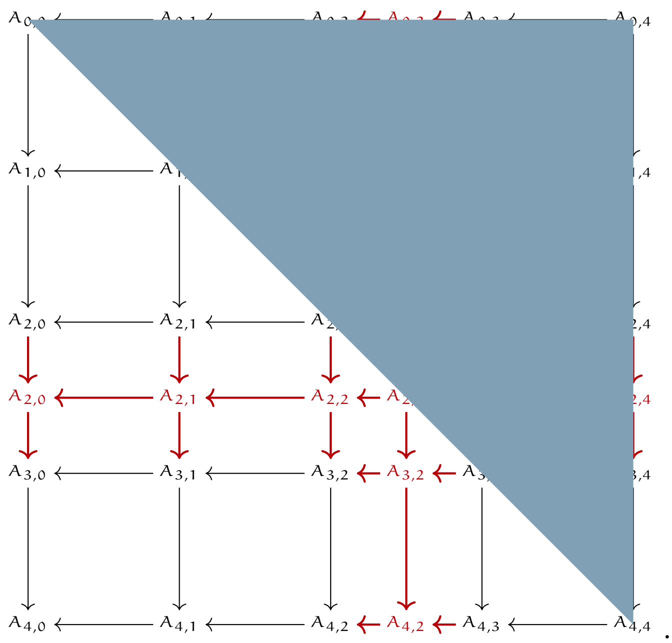
These two last equivalences precisely compose to the morphism

$$\hat{Q}_n(\mathcal{C}) \to Q_n(\mathcal{C}),$$

showing it is an equivalence. Moreover, this is natural in n. Let us provide examples for this, the general case is totally analogous.
Let us describe d_2^4. In the following diagram the top-right part is $\mathrm{TwArr}([n])^{op}$ and the red lines is what d_2^4 forgets:



An example of degeneracy map s_2^4 is



We also have equivalences

$$
\begin{tikzcd}[column sep=tiny]
\mathbf{Fun}([n]\times[n]^{op},\mathcal{C}) \arrow[dr, "\bullet^{op}"] & & \\
& \mathbf{Fun}(([n]\times[n]^{op})^{op},\mathcal{C}^{op})^{op} \arrow[dr, "\mathrm{switch}"] & \\
& & \mathbf{Fun}(([n]\times[n]^{op}),\mathcal{C}^{op})^{op}
\end{tikzcd}
$$

where the second equivalence is induced by the equivalence

$$\mathrm{switch}\colon [n]\times[n]^{op}\to[n]^{op}\times[n],(i,j)\mapsto(j,i).$$

This restrict to an equivalence (natural in n)

$$\hat{Q}_n(\mathcal{C})\simeq\hat{Q}_n(\mathcal{C}^{op})^{op};$$

hence we have

$$
\begin{aligned}
&Q_n(\mathcal{C})\simeq\hat{Q}_n(\mathcal{C})\simeq\hat{Q}_n(\mathcal{C}^{op})^{op}\simeq Q_n(\mathcal{C}^{op})^{op}\Rightarrow\\
&\Rightarrow \text{core }Q_n(\mathcal{C})\simeq\text{core}\left(Q_n(\mathcal{C}^{op})^{op}\right)\simeq\text{core }Q_n(\mathcal{C}^{op}).
\end{aligned}
$$

Both these equivalences are clearly natural in n. Therefore, we have the equivalence α on all the higher cells:

$$
\begin{aligned}
&\text{core }Q_\bullet(\mathcal{C})\simeq\text{core }Q_\bullet(\mathcal{C}^{op})\Rightarrow\\
&\Rightarrow\text{asscat core }Q_\bullet(\mathcal{C})\simeq\text{asscat core }Q_\bullet(\mathcal{C}^{op})\\
&\Rightarrow \mathrm{Span}(\mathcal{C})\simeq\mathrm{Span}(\mathcal{C}^{op}).
\end{aligned}
$$

(Second equivalence).
Let us call β the functor

$$\mathrm{Span}(\mathrm{Arr}(\mathcal{C}))\to\mathrm{Span}(\mathrm{TwArr}(\mathcal{C}))$$

that we want to prove is an equivalence. β is given by

- On the objects by the identity.
- On morphisms by

$$
\begin{tikzcd}
x \arrow[d] & s \arrow[l] \arrow[r] \arrow[d] & x' \arrow[d] & & x \arrow[d] & s \arrow[l] \arrow[r] \arrow[d] & x' \arrow[d] \\
y & t \arrow[l] \arrow[r] & y' \arrow[r, phantom, "\longmapsto"] & {} & y \arrow[r] & y\oplus_t y' & y' \arrow[l]
\end{tikzcd}
$$

To define β on higher cells we do the following. Consider the map χ_n defined as

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{Cat}_\infty}([n] \times [n]^{op} \times [1], \mathcal{C}) \arrow[dr, "\mathrm{TwArr}(\bullet)^{op}"] & & \\
& \mathrm{Hom}_{\mathbf{Cat}_\infty}(\mathrm{TwArr}([n] \times [n]^{op} \times [1]), \mathrm{TwArr}(\mathcal{C})^{op}) \arrow[dr, "(i^{op})^*"] & \\
& & \mathrm{Hom}_{\mathbf{Cat}_\infty}(\mathrm{TwArr}[n]^{op}, \mathrm{TwArr}(\mathcal{C})^{op})
\end{tikzcd}$$

where the map $(i^{op})^*$ is induced by the map

$$\begin{aligned} i : \mathrm{TwArr}([n]) &\hookrightarrow \mathrm{TwArr}([n] \times [n]^{op} \times [1]) \\ (i \leqslant j) &\mapsto (i \leqslant j, j \leqslant i, 0 \leqslant 1) \equiv (i, j, 0) \to (j, i, 1), \end{aligned}$$

which sends a morphism $(i \leqslant j) \to (a \leqslant b)$, i.e. a square

$$\begin{tikzcd}
i \arrow[d] & a \arrow[l] \arrow[d] \\
j \arrow[r] & b
\end{tikzcd},$$

into

$$\begin{tikzcd}
(i,j,0) \arrow[d] & (a,b,0) \arrow[l] \arrow[d] \\
(j,i,1) \arrow[r] & (b,a,1)
\end{tikzcd},$$

We have inclusions

$$\begin{aligned} \hat{Q}_n(\mathrm{Arr}(\mathcal{C})) &\subset \mathbf{Fun}([n] \times [n]^{op}, \mathrm{Arr}(\mathcal{C})) \\ &\simeq \mathbf{Fun}([n] \times [n]^{op}, \mathbf{Fun}([1], \mathcal{C})) \\ &\simeq \mathbf{Fun}([n] \times [n]^{op} \times [1], \mathcal{C}) \end{aligned}$$

and

$$Q_n(\mathrm{TwArr}(\mathcal{C})) \subset \mathbf{Fun}(\mathrm{TwArr}[n]^{op}, \mathrm{TwArr}(\mathcal{C})).$$

χ_n restricts to a map

$$\hat{Q}_n(\mathrm{Arr}(\mathcal{C})) \to Q_n(\mathrm{TwArr}(\mathcal{C})^{op}).$$

In a similar fashion as how we proved the naturality in the *first equivalence* α, these functors χ_n are natural in n; hence they build up a map of simplicial anima

$$\chi : \hat{Q}_\bullet(\mathrm{Arr}(\mathcal{C})) \to Q_\bullet(\mathrm{TwArr}(\mathcal{C})).$$

By applying asscat core we obtain our searched map β

$$\begin{aligned}
&\mathsf{Span}(\mathrm{Arr}(\mathcal{C})) \simeq \\
&\simeq \text{asscat core } Q_\bullet(\mathrm{Arr}(\mathcal{C})) \simeq \\
&\simeq \text{asscat core } \hat{Q}_\bullet(\mathrm{Arr}(\mathcal{C})) \to \\
&\to \text{asscat}\,(\text{core } Q_\bullet(\mathrm{TwArr}(\mathcal{C})^{op})) \simeq \\
&\simeq \mathsf{Span}(\mathrm{TwArr}(\mathcal{C})^{op})
\end{aligned}$$

β is an equivalence, in fact, χ is an equivalence. We can check this levelwise, but then, because of the Segal condition, we can just verify the cases $n = 0$ and $n = 1$. But we already saw that on the β is an equivalence.

Finally, the diagram commute because of how we have defined these equivalences α and β. $\square$

5.2 Proposition

Consider an ∞-category $\mathcal{C}$ with finite limits, finite colimits, and a zero object. Then

$$\mathsf{Span}(s,t) : \mathsf{Span}(\mathrm{TwArr}(\mathcal{C})^{op}) \to \mathsf{Span}(\mathcal{C} \times \mathcal{C}^{op})$$

is cofinal.

Before proving this, we need to discuss a useful lemma.

5.3 Lemma

Consider two ∞-categories with finite limits $\mathcal{C}$ and $\mathcal{D}$, a left exact right fibration $f : \mathcal{C} \to \mathcal{D}$ between them, and a object $x \in \mathcal{D}$. Then the functor given on objects by

$$(f/x)^{op} \to x/\mathsf{Span}(f), \quad (w, f(w) \xrightarrow{\phi} x) \mapsto (w, x \xleftarrow{\phi} f(w) \xrightarrow{id} f(w))$$

has a right adjoint given by

$$(w, x \xleftarrow{\chi} y \xrightarrow{\psi} f(w)) \mapsto (\hat{y}, f(\hat{y}) \simeq y \xrightarrow{\chi} x)$$

where $\hat{y} \to w$ is a lift of $y \xrightarrow{\psi} f(w)$.

Proof of the Lemma. First of all, notice that the lift $\hat{y} \to w$ of $y \xrightarrow{\psi} f(w)$ exists and it is well-defined because f is a right fibration.
Call **A** the morphism $(f/x)^{op} \to x/\mathsf{Span}(f)$, and **B** its supposed right adjoint. What we want to prove is that

$$\mathbf{Map}_{x/\mathsf{Span}(f)}(\mathbf{A}(w, f(w) \xrightarrow{\phi} x), (v, x \xleftarrow{\chi} y \xrightarrow{\psi} f(v))) \simeq$$
$$\left(\simeq \mathbf{Map}_{x/\mathsf{Span}(f)}((w, x \xleftarrow{\phi} f(w) \xrightarrow{id} f(w)), (v, x \xleftarrow{\chi} y \xrightarrow{\psi} f(v))) \right)$$

is equivalent to

$$\begin{aligned}
&\mathbf{Map}_{(f/x)^{\mathrm{op}}}((w, f(w) \xrightarrow{\phi} x), \mathbf{B}(v, x \xleftarrow{\chi} y \xrightarrow{\psi} f(v))) \simeq \\
&\Big(\simeq \mathbf{Map}_{(f/x)^{\mathrm{op}}}((w, f(w) \xrightarrow{\phi} x), (\hat{y}, f(\hat{y} \simeq y \xrightarrow{\chi} x))) \\
&\simeq \mathbf{Map}_{(f/x)}((\hat{y}, f(\hat{y} \simeq y \xrightarrow{\chi} x)), (w, f(w) \xrightarrow{\phi} x)) \Big)
\end{aligned}$$

naturally in $(w, f(w) \xrightarrow{\phi} x)$.

The first mapping anima $\mathbf{Map}_{x/\mathrm{Span}(f)}(\dots)$ fits into a cartesian square $(\star)$

$$\begin{tikzcd}
\mathbf{Map}_{x/\mathrm{Span}(f)}(\mathbf{A}(w, f(w) \xrightarrow{\phi} x), (v, x \xleftarrow{\chi} y \xrightarrow{\psi} f(v)))) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Map}_{\mathrm{Span}(\mathcal{C})}(w, v) \arrow[d, "(x \xleftarrow{\phi} f(w) \xrightarrow{\mathrm{id}} f(w))^* \circ \mathrm{Span}(f)"'] \arrow[dr, dotted, "\mathrm{Span}(f)"] & \\
* \arrow[r, "(x \xleftarrow{\chi} y \xrightarrow{\psi} f(v))"] & \mathbf{Map}_{\mathrm{Span}(\mathcal{D})}(x, fv) & \mathbf{Map}_{\mathrm{Span}(\mathcal{D})}(fw, fv) \arrow[l, dotted, "(x \xleftarrow{\phi} f(w) \xrightarrow{\mathrm{id}} f(w))^*"]
\end{tikzcd}$$

By definition

$$\begin{aligned}
&\mathbf{Map}_{\mathrm{Span}(\mathcal{C})}(w, v) \simeq \\
&\simeq \mathrm{core}\,(\mathbf{Fun}(\mathrm{TwArr}([1])^{\mathrm{op}}, \mathcal{C}) \times_{\mathcal{C}^{\mathrm{op}} \times \mathcal{C}} \{w, v\}) \\
&\simeq \mathrm{core}\,\mathbf{Fun}(\mathrm{TwArr}([1])^{\mathrm{op}}, \mathcal{C}) \times_{\mathrm{core}\,(\mathcal{C}^{\mathrm{op}} \times \mathcal{C})} \{w, v\} \\
&\simeq \mathrm{core}\,(\mathcal{C}/w) \times_{\mathrm{core}\,\mathcal{C}} \mathrm{core}\,(\mathcal{C}/v).
\end{aligned}$$

Similarly,

$$\mathbf{Map}_{\mathrm{Span}(\mathcal{D})}(x, f(v)) \simeq \mathrm{core}\,(\mathcal{D}/x) \times_{\mathrm{core}\,\mathcal{D}} \mathrm{core}\,(\mathcal{D}/f(v)).$$

So from $(\star)$ we obtain a pullback square

$$\begin{tikzcd}
\mathbf{Map}_{x/\mathrm{Span}(f)}(\mathbf{A}(w, f(w) \xrightarrow{\phi} x), (v, x \xleftarrow{\chi} y \xrightarrow{\psi} f(v)))) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{core}\,(\mathcal{C}/w) \times_{\mathrm{core}\,\mathcal{C}} \mathrm{core}\,(\mathcal{C}/v) \arrow[d] \\
* \arrow[r, "(\chi, \psi)"] & \mathrm{core}\,(\mathcal{D}/x) \times_{\mathrm{core}\,\mathcal{D}} \mathrm{core}\,(\mathcal{D}/f(v))
\end{tikzcd},$$

where the right vertical is now given by

$$\mathcal{C}/w \xrightarrow{f} \mathcal{D}/f(w) \xrightarrow{\phi_*} \mathcal{D}/x \quad \text{and} \qquad \mathcal{C}/v \xrightarrow{f} \mathcal{D}/f(v).$$

Now we can rearrange the order of the pullbacks, and since the fibre of the right hand map is contractible, because f is a right fibration, we re-write $(\star)$ as

$$\begin{tikzcd}
\mathbf{Map}_{x/\mathcal{S}pan(f)}(\mathbf{A}(w, f(w) \xrightarrow{\phi} x), (v, x \xleftarrow{\chi} y \xrightarrow{\psi} f(v))) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{core}(\mathcal{C}/w) \times_{\operatorname{core} \mathcal{D}/x} \{\psi\} \arrow[d] \\
* \arrow[r, "\hat{y}"] & \operatorname{core} \mathcal{C} \times_{\operatorname{core} \mathcal{D}} \{y\}
\end{tikzcd}$$

where $\hat{y} \to v$ is the lift of ψ in $\mathcal{C}$. Switching again the order of the pullbacks we obtain a pullback square

$$\begin{tikzcd}
\mathbf{Map}_{x/\mathcal{S}pan(f)}(\mathbf{A}(w, f(w) \xrightarrow{\phi} x), (v, x \xleftarrow{\chi} y \xrightarrow{\psi} f(v))) \arrow[r] \arrow[dd] \arrow[ddr, phantom, "\lrcorner", very near start] & \mathbf{Map}_{\mathcal{C}}(\hat{y}, w) \arrow[d, "f"] \\
 & \mathbf{Map}_{\mathcal{D}}(y, f(w)) \arrow[d, "\phi_*"] \\
* \arrow[r, "\chi"] & \mathbf{Map}_{\mathcal{D}}(y, x)
\end{tikzcd}.$$

But now, this pullback describe

$$\mathbf{Map}_{f/x}((\hat{y}, f(\hat{y}) \simeq y \xrightarrow{\chi} x), (w, f(w) \xrightarrow{\phi} x))$$

and the whole procedure is natural in $(w, \phi : f(w) \to x)$. □

Proof of the Proposition. One of the many equivalent condition for the functor $\mathcal{S}pan(s,t)$ to be cofinal is to ask that, for any $(c,d) \in \mathcal{S}pan(\mathcal{C} \times \mathcal{C}^{op})$,

$$|(c,d)/\mathcal{S}pan(s,t)|$$

is connected. We are going to prove more, namely that

$$|(c,d)/\mathcal{S}pan(s,t)| \simeq *, \quad \text{for any } (c,d) \in \mathcal{C} \times \mathcal{C}^{op}.$$

We want to use the lemma we have just proved with

$$x = (c,d) \quad \text{and} \quad f = (s,t);$$

the fact that $f = (s,t) : \mathrm{TwArr}(\mathcal{C})^{op} \to \mathcal{C} \times \mathcal{C}^{op}$ is a right fibration is proved in [Ker, Proposition 8.1.1.11]. Therefore we obtain an adjunction

$$\begin{tikzcd}
((s,t)/(c,d))^{op} \arrow[r, bend left] \arrow[r, phantom, "\perp"] & (c,d)/\mathcal{S}pan(s,t) \arrow[l, bend left]
\end{tikzcd}$$

and so we have an equivalence of animae

$$|(c,d)/\mathcal{S}pan(s,t)| \simeq |(s,t)/(c,d)|.$$

We want to prove $|(s,t)/(c,d)| \simeq *$. This is true because $(s,t)/(c,d)$ has an initial object; specifically, the identity arrow id_0 serves as the initial object, and since the pair (s,t) preserves it, the slices of $(s,t)/(c,d)$ inherit the initial object as well. □

5.1.2 *Consequences.*

5.4 Corollary (Additivity Theorem for K-theory, Version 1)
Consider a stable ∞-category $\mathcal{C}$. The functor

$$(s, t) : \mathrm{Arr}(\mathcal{C}) \to \mathcal{C}^2$$

induces an equivalence

$$k(\mathrm{Arr}(\mathcal{C})) \to k(\mathcal{C}^2) \simeq k(\mathcal{C})^2.$$

Proof. We already showed in the subsection 5.2.2 that k preserve finite products. In theorem A, we proved $|\mathrm{Span}\,\mathrm{Arr}\,\mathcal{C}| \simeq |\mathrm{Span}\mathcal{C}|^2$. By taking loops in both side, and using the fact that Ω preserves product, we obtain

$$k(s, t) : k(\mathrm{Arr}(\mathcal{C})) \xrightarrow{\simeq} k(\mathcal{C})^2 \simeq k(\mathcal{C}^2).$$

□

5.5 Corollary
The K-theory anima functor $k : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$ is additive and grouplike.

Proof. Our goal is to prove k is extension splitting; indeed we know from proposition 3.76 that is its equivalent for a reduced functor to be extension splitting or grouplike additive, and it should be clear that

$$k(0) \simeq \Omega|\mathrm{core}\ S_\bullet(0)| \simeq 0,$$

so k is reduced.

We already showed in the subsection 5.2.2 that k preserve finite products; by lemma 3.79, proving that, for any stable ∞-category $\mathcal{C}$,

$$(s, t) : \mathrm{Arr}(\mathcal{C}) \to \mathcal{C}^2$$

is mapped to an equivalence, is enough to prove that k is extension splitting. We conclude by 5.4. □

5.6 Corollary (Additivity Theorem for K-theory, Version 2)
Consider a stable ∞-category $\mathcal{C}$. The functor

$$\chi : \mathrm{Arr}(\mathcal{C}) \to \mathcal{C}^2, \quad (\alpha : x \to y) \mapsto (x, \mathrm{cofib}(\alpha))$$

induces an equivalence

$$k(\mathrm{Arr}(\mathcal{C})) \to k(\mathcal{C}^2) \simeq k(\mathcal{C})^2.$$

5.7 Corollary
Consider stable ∞-categories $\mathcal{C}$ and $\mathcal{D}$, and a cofibre sequence

$$F' \xrightarrow{\alpha} F \xrightarrow{\beta} F''$$

of exact functors $\mathcal{C} \to \mathcal{D}$. Then

$$k\,F = k\,F' + k\,F''.$$

Proof. Consider the functors

$$m, \mathrm{fib} + \mathrm{cofib} : \mathrm{Seq}(\mathbf{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})) \to \mathbf{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$$

and notice that $\mathbf{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$ is a stable ∞-category. k is extension splitting, so by lemma 3.77 we have

$$k(m) \simeq k(\mathrm{fib}) + k(\mathrm{cofib}).$$

We apply this to the sequence $F' \xrightarrow{\alpha} F \xrightarrow{\beta} F''$ and we are done. □

5.8 Corollary
Consider a stable ∞-category $\mathcal{C}$. Then the suspension functor

$$\Sigma_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$$

induces the map $k(\mathcal{C}) \to k(\mathcal{C})$ given by multiplication by -1.

Proof. Apply lemma 3.77 to the sequence the Verdier functors $\mathcal{C} \to \mathcal{C}$

$$\mathrm{id} \to 0 \to \Sigma.$$

Then we get

$$k(\mathrm{id}) \oplus k(\Sigma) \simeq k(0) \simeq 0.$$

□

5.2 K-THEORY SPECTRA.

In our investigation of the K-theory of a stable ∞-category, we now proceed to establish the definition of a K-theory spectrum. There are multiple equivalent ways to achieve this. We will address two of these techniques now and postpone two for later.
Throughout all the section $\mathcal{C}$ and $\mathcal{D}$ will denote stable ∞-categories.
Before starting, let us recall that a spectrum is connective if its homotopy groups in all negative degrees are trivial and that there is an equivalence between grouplike $\mathbf{E}_\infty$-anima and connective spectra.

5.2.1 *Through Additive Theorem.*

Our first construction of the K-theory spectrum is immediate. The corollary 5.5 of the *additivity theorem* A tell us that k is in fact grouplike. Therefore, it descends a functor

$$\mathrm{k} : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathrm{CGrp}(\mathbf{An})$$

and since $\mathrm{CGrp}(\mathbf{An}) \simeq \mathbf{Spectra}_{\geqslant 0}$ we obtain the algebraic K-theory spectrum functor

$$\mathrm{K} : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{Spectra}_{\geqslant 0} \subset \mathbf{Spectra}\,.$$

5.9 Remark
Considering the $(\Sigma^\infty \dashv \Omega^\infty) : \mathbf{Spectra}_{\geqslant 0} \leftrightarrows \mathrm{CGrp}(\mathbf{An})$-adjunction and recalling that Σ^∞ is fully faithful, we can recover the K-theory anima as

$$\mathrm{k}(\mathcal{C}) \simeq \Omega^\infty \mathrm{K}(\mathcal{C}).$$

5.2.2 *Through Monoidal Structure.*

Our second description concern with explicitly constructing the "cartesian commutative group structure" on $\mathrm{k}(\mathcal{C})$. While this construction may seem less formal, it is for sure more explicit.
We want to prove $\mathrm{k}(\bullet)$ preserves finite products. The morphisms

$$\mathcal{C} \leftarrow \mathcal{C} \times \mathcal{D} \to \mathcal{D}$$

induce a natural morphism

$$\mathrm{k}(\mathcal{C} \times \mathcal{D}) \to \mathrm{k}(\mathcal{C}) \times \mathrm{k}(\mathcal{D}),$$

which turns out to be an equivalence. This is true because

$$\mathrm{k}(\mathcal{C} \times \mathcal{D}) \simeq \Omega|\mathrm{core}\ \mathrm{S}_\bullet(\mathcal{C} \times \mathcal{D})|,$$

and we know Ω, $|\bullet|$, and core preserve finite products, indeed Ω is given by a finite limit, $|\bullet|$ is a sifted colimit, and core is a right adjoint functor. Therefore if we prove $S_\bullet$ preserve products we are done. This immediately follows from the equivalence

$$\mathbf{Fun}(\mathrm{Arr}([\bullet]), \mathcal{C} \times \mathcal{D}) \simeq \mathbf{Fun}(\mathrm{Arr}([\bullet]), \mathcal{C}) \times \mathbf{Fun}(\mathrm{Arr}([\bullet]), \mathcal{D})$$

that clearly restricts to an equivalence

$$S_\bullet(\mathcal{C} \times \mathcal{D}) \simeq S_\bullet(\mathcal{C}) \times S_\bullet(\mathcal{D}).$$

Therefore k is a product preserving functor $\mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$, and so it naturally descends to a functor

$$k : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathrm{CMon}(\mathbf{An}),$$

since $\mathbf{Cat}_\infty^{\mathbf{st}}$ is additive. Hence, for any $\mathcal{C}$ stable ∞-category, $k(\mathcal{C})$ has a symmetric cartesian monoidal structure.

If we want to express this symmetric cartesian monoidal structure more esplicitely, we can do ass follow. Since $\mathcal{C}$ is stable, it is additive (i.e. finite products and coproducts are equivalent and the shear map is an equivalence). The coproduct $\oplus_\mathcal{C} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves finite colimits, therefore it induces a multiplication

$$m : k(\mathcal{C}) \times k(\mathcal{C}) \simeq k(\mathcal{C} \times \mathcal{C}) \xrightarrow{k(\oplus)} k(\mathcal{C}).$$

The coproduct on $\mathcal{C}$ is coherently commutative and associative, therefore, the multiplication m is also coherently commutative and associative. This is exactly the symmetric cartesian monoid structure on $k(\mathcal{C})$ we wanted to express.

The structure on $k(\mathcal{C})$ endow the set $\pi_0\, k(\mathcal{C}) =: K_0(\mathcal{C})$ with a commutative monoid structure, which coincides with the abelian group structure that we gave it in 2.1. Consequently, $k(\mathcal{C})$ is a grouplike $\mathbf{E}_\infty$-anima, and so a connective spectrum.
Thanks to this equivalence, since exact functors preserve coproducts, we obtain the algebraic K-theory spectrum functor

$$K : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{Spectra}_{\geqslant 0} \subset \mathbf{Spectra}\,.$$

These first two methods produce equivalent K-theory spectrum functors, since they both concern with the canonical commutative cartesian group structure on $k(\mathcal{C})$.

5.3 UNIVERSALITY.

To start this section, we want to show there exists a canonical natural transformation

$$\tau : \mathrm{core} \Rightarrow k\,.$$

For any $\mathcal{C}$ stable ∞-category, this map comes from

$$\mathrm{core}\,(\mathcal{C}) \simeq \mathbf{Map}_{Span(\mathcal{C})}(0,0) \to \mathrm{Map}_{|Span(\mathcal{C})|}(0,0) \simeq \Omega|Span(\mathcal{C})|.$$

The right equivalence should be clear. For the left equivalence, $Span(\mathcal{C})$ is an ∞-category, so by definition

$$\begin{tikzcd}
\mathbf{Map}_{Span(\mathcal{C})}(0,0) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Arr}(Span(\mathcal{C})) \arrow[d, "(ev_0, ev_1)"] \\
* \arrow[r, "(0,0)"'] & Span(\mathcal{C}) \times Span(\mathcal{C})
\end{tikzcd}.$$

It is also true that for every (not necessarily complete) Segal anima X, for any x, y objects of X_0 there is a pullback diagram (in **An**)

$$\begin{tikzcd}
\mathbf{Map}_{\mathrm{asscat}\ X}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_1 \arrow[d, "(d_1^1, d_0^1)"] \\
* \arrow[r, "(x,y)"'] & X_0 \times X_0
\end{tikzcd}.$$

(For clearness, recall that the object of asscat X are precisely the objects of X_0, hence the object of $Span(\mathcal{C})$ are the object of $\mathcal{C}$). Applying this to $X \simeq \mathrm{core}\ Q(\mathcal{C})$ we obtain

$$\begin{tikzcd}
\mathbf{Map}_{Span(\mathcal{C})}(0,0) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{core}\ Q_1(\mathcal{C}) \arrow[d, "(d_1^1, d_0^1)"] \\
* \arrow[r, "(0,0)"'] & \mathcal{C} \times \mathcal{C}
\end{tikzcd},$$

and this square clearly shows $\mathrm{core}\,(\mathcal{C})$ as the pullback.
The morphism is clearly natural in $\mathcal{C}$; therefore, we obtain a natural transformation

$$\mathrm{core} \Rightarrow k \in \mathbf{Fun}(\mathbf{Cat}_\infty^{\mathbf{st}}, \mathbf{An}),$$

which descends to a natural transformation

$$\mathrm{core} \Rightarrow k \in \mathbf{Fun}(\mathbf{Cat}_\infty^{\mathbf{st}}, \mathrm{CMon}(\mathbf{An})) \simeq \mathrm{CMon}(\mathbf{Fun}(\mathbf{Cat}_\infty^{\mathbf{st}}, \mathbf{An})).$$

Indeed,

- core preserves all limits;
- k preserve finite products.

5.10 Remark
We know (for example from [Lur17] and [GGN13]) that there exists a left adjoint to the forgetful functor

$$\mathrm{CGrp}(\mathbf{An}) \hookrightarrow \mathrm{CMon}(\mathbf{An}),$$

which we call

$$(\bullet)^{\infty-\mathrm{grp}} : \mathrm{CMon}(\mathbf{An}) \to \mathrm{CGrp}(\mathbf{An}).$$

By composing with this left adjoint, we can lift any additive functor $F : \mathbf{Cat}^{\mathbf{st}}_{\infty} \to \mathbf{An}$ (or in general any functor which preserves finite products) to a functor

$$F^{\infty-\mathrm{grp}} : \mathbf{Cat}^{\mathbf{st}}_{\infty} \to \mathrm{CGrp}(\mathbf{An}).$$

The question now is whether or not this is still an additive functor. The answer is yes, and we are going to show this in the section 5.4. Let us direct our attention, for a moment, to the core functor case, which is far less complicated than the general one.

The goal for this section is to prove the following theorem.

Theorem B (Universality Theorem - Blumberg, Gepner, Tabuada)
The functor $k : \mathbf{Cat}^{\mathbf{st}}_{\infty} \to \mathbf{An}$ is the initial grouplike additive functor under $\mathrm{core} : \mathbf{Cat}^{\mathbf{st}}_{\infty} \to \mathbf{An}$.

Let us explain what this means. Consider the ∞-category

$$\mathrm{core}/\,\mathbf{Fun}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathbf{An}),$$

which is the ∞-category of functors $\mathbf{Cat}^{\mathbf{st}}_{\infty} \to \mathbf{An}$ with natural transformation from the core functor.
We define

$$(\mathrm{core}\,/\,\mathbf{Fun}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathbf{An}))_{\mathrm{grp,add}}$$

as the full sub-∞-category of $\mathrm{core}/\,\mathbf{Fun}(\mathbf{Cat}^{\mathbf{st}}_{\infty}, \mathbf{An})$ spanned by grouplike additive functor.
The theorem shows that the natural map $\tau : \mathrm{core} \Rightarrow k : \mathbf{Cat}^{\mathbf{st}}_{\infty} \to \mathbf{An}$ is an initial object for this ∞-category, which means

$$|\mathbf{Map}_{\mathrm{core}}(k, F)| \simeq *$$

for any grouplike additive functor $F : \mathbf{Cat}^{\mathbf{st}}_{\infty} \to \mathbf{An}$, or equivalently that

$$\tau^* : \mathrm{Nat}(k, F) \xrightarrow{\simeq} \mathrm{Nat}(\mathrm{core}, F)$$

is an equivalence.

There are many versions of this theorem, first established in [BGT13]; a version for higher Waldhausen category is presented by [Bar16], and a proof for Hermitian K-theory is present in [Cal+21b, Sec. 2.7]. In all these references, we generally find proofs of more general statements concerning additive functor to **An** (preserving filtered colimits in [BGT13]), which result in theorem 2 when specified to core. Our proof comes from [HLS22], where is itself adapted from a proof in [Ste17]. This version of the proof is for sure easier and shorter but loose much generality. With our restriction, it is not possible to prove that given an additive functor $F : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$, the functor $\Omega|FQ-|$ is again additive and that it is a "group-completion" of F, i.e. that $\Omega|FQ-|$ is the initial grouplike additive functors under F. The reason why this proof does not extend is that it makes use of the fact that the core functor is defined on $\mathrm{TwArr}(\mathcal{C})$ (which is non-stable). We will see this the general version in section 5.5.

Proof. Our goal is to prove that, for any grouplike additive functor $F : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$,

$$\tau^* : \mathrm{Nat}(k, F) \to \mathrm{Nat}(\mathrm{core}, F)$$

is an equivalence. The inclusion

$$\mathcal{C} \to Q_1(\mathcal{C}), \quad x \mapsto (0 \leftarrow x \to 0)$$

induces a map

$$F(\mathcal{C}) \to F(Q_1(\mathcal{C})) \to \Omega|FQ\mathcal{C}|$$

natural in $\mathcal{C}$. This induces a natural transformation

$$F\bullet \Rightarrow \Omega|FQ\bullet|,$$

and since this is natural in F, we get a natural transformation of grouplike additive functors $\mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$

$$\eta : \mathrm{id} \Rightarrow G(-) := \Omega|(-) \circ Q\bullet|.$$

(If the existence of the mapping function η is not apparent to the reader, they can refer to the next remark.) There is now a diagram

$$\begin{tikzcd}
\mathrm{Nat}(G\,\mathrm{core}, F) \arrow[r, "(\eta_{\mathrm{core}})^*"] \arrow[d, "G"] \arrow[dd, bend right=60, "(\eta_F)_*"'] & \mathrm{Nat}(\mathrm{core}, F) \arrow[d, "G"'] \arrow[dd, bend left=60, "(\eta_F)_*"] \\
\mathrm{Nat}(GG\,\mathrm{core}, GF) \arrow[r, "(G\eta_{\mathrm{core}})^*"] \arrow[d, "(\eta_{G\,\mathrm{core}})^*"] & \mathrm{Nat}(G\,\mathrm{core}, GF) \arrow[d, "(\eta_{\mathrm{core}})^*"'] \\
\mathrm{Nat}(G\,\mathrm{core}, GF) \arrow[r, "(\eta_{\mathrm{core}})^*"] & \mathrm{Nat}(\mathrm{core}, GF)
\end{tikzcd}$$

where

- the upper square commute because G is a functor;
- the other square, and the two triangles (left and right) commute as a consequence of the naturality of η.

Suppose the red-coloured arrows are equivalences, then

(i) The upper-left vertical arrow "G" is an equivalence by 2-out-of-3;

(ii) The composition of the upper horizontal arrow with the upper-right vertical arrow "$G \circ (\eta_{\mathrm{core}})^*$" is an equivalence by 2-out-of-3;

(iii) Both the upper horizontal arrow "$(\eta_{\mathrm{core}})^*$", the upper-right vertical arrow "G", and the lower-right vertical arrow "$(\eta_{\mathrm{core}})^*$" are equivalences by 2-out-of-6;

(iv) The lower horizontal arrow "$(\eta_{\mathrm{core}})^*$" is an equivalence by 2-out-of-3.

In particular, the upper arrow

$$\mathrm{Nat}(k, F) \to \mathrm{Nat}(\mathrm{core}, F)$$

is an equivalence. The fact that the red-coloured arrow are equivalences follows from the following two propositions.

5.11 Proposition
Consider an additive grouplike functor $F : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$. Then

$$\eta_F : F \Rightarrow \Omega|FQ_\bullet|$$

is an equivalence.

This implies $(\eta_F)_*$ and $(G\eta_{\mathrm{core}})^*$ are equivalences.

5.12 Proposition
The two natural transformations

$$\eta_{GF}, G\eta_F : GF \to GGF$$

differ by an automorphism of the target.

This implies $(\eta_{G\,\mathrm{core}})^*$ is an equivalence. $\square$

Before proving these propositions, let us gives a definition.

5.13 Definition
Consider a stable ∞-category $\mathcal{C}$. We define the simplicial ∞-category

$$\mathrm{Null}(\mathcal{C}) : N(\boldsymbol{\Delta})^{op} \to \mathbf{Cat}_\infty$$

as

$$\mathrm{fib}\Big(\mathrm{dec}(Q(\mathcal{C})) \overset{(\mathrm{dec}\Rightarrow ev_0)_{Q(\mathcal{C})}}{\longrightarrow} \mathrm{const}Q_0(\mathcal{C})\Big),$$

where the fibre is formed over $0 \in Q_0(\mathcal{C}) \simeq \mathcal{C}$.

The natural transformation $\mathrm{dec} \Rightarrow \mathrm{id}$ induces a map $\mathrm{Null}(\mathcal{C}) \to Q(\mathcal{C})$, so we have a diagram

$$\begin{tikzcd}
\mathrm{Null}\mathcal{C} \arrow[r] \arrow[d] \arrow[rr, bend left, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{dec}Q\mathcal{C} \arrow[r, "(\mathrm{dec}\Rightarrow \mathrm{id})_{Q(\mathcal{C})}"'] \arrow[d, "(\mathrm{dec}\Rightarrow ev_0)_{Q(\mathcal{C})}"] & Q\mathcal{C} \\
* \arrow[r, "0"] & \mathrm{const}Q_0\mathcal{C} &
\end{tikzcd}$$

The right vertical map has a right adjoint, given by the composition of 0-th degeneracy maps, as we so previously. It follows that also $\mathrm{Null}(\mathcal{C})$ is split over 0.

5.14 Remark
Since $\mathrm{Null}(\mathcal{C})$ is split over 0, for any reduced (non-necessarily) additive functor $F : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$,

$$|F(\mathrm{Null}\mathcal{C})| \simeq *.$$

Consider the cartesian square

$$\begin{tikzcd}
\mathrm{const}\mathcal{C} \arrow[r] \arrow[d] & \mathrm{Null}\mathcal{C} \arrow[d] \\
0 \arrow[r] & Q\mathcal{C}
\end{tikzcd}$$

where the upper-horizontal map is given on objects by

$$x \mapsto (0 \leftarrow x \rightarrow 0).$$

By applying F and the geometric realization that we obtain a commutative diagram

$$\begin{tikzcd}
F\mathcal{C} \arrow[rr] \arrow[dr, dashed] \arrow[dd] & & {|FNull\mathcal{C}| \simeq 0} \arrow[dd] \\
& \Omega|FQ\mathcal{C}| \arrow[ur] \arrow[dl] \arrow[dr, phantom, "\lrcorner", very near start] & \\
0 \arrow[rr] & & {|FQ\mathcal{C}|}
\end{tikzcd}$$

The map $F\mathcal{C} \to \Omega|FQ\mathcal{C}|$ is exactly $\eta_F(\mathcal{C})$ by construction.

5.15 Lemma

$\mathrm{Null}(\mathcal{C})$ is a complete Segal ∞-category.

Proof of the lemma. Let us start by proving it is a Segal ∞-category. By passing to the levelwise cartesian square we obtain

$$\begin{tikzcd}
\mathrm{Null}_n\mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {\begin{array}{c} \mathrm{dec}(Q\mathcal{C})_n \simeq Q_{1+n}\mathcal{C} \simeq \\ \simeq Q_1\mathcal{C} \times_{Q_0\mathcal{C}} \cdots \times_{Q_0\mathcal{C}} Q_1\mathcal{C} \end{array}} \arrow[d, "(\mathrm{dec} \Rightarrow ev_0)_{Q(\mathcal{C})}"] \\
0 \arrow[r] & (\mathrm{const}Q_0\mathcal{C})_n \simeq \mathcal{C}
\end{tikzcd}$$

Recall that an element of $Q_{1+n}\mathcal{C}$ can be represented by a diagram A

$$\begin{tikzcd}[column sep=small]
& A_{0,1} \arrow[dl] \arrow[dr] & & A_{1,2} \arrow[dl] \arrow[dr] & & A_{n,n+1} \arrow[dl] \arrow[dr] & \\
A_{0,0} & & A_{1,1} & & \cdots & & A_{n+1,n+1}
\end{tikzcd}$$

where the rest can be recover by taking pullbacks.
The map $(\mathrm{dec} \Rightarrow ev_0)_{Q(\mathcal{C})}$ sends A to $A_{0,0}$. Since $\mathrm{Null}_n\mathcal{C}$ is the fibre over 0, we get that $\mathrm{Null}_n\mathcal{C}$ consists of the full sub-∞-category of $Q_{1+n}\mathcal{C}$ spanned by diagrams with $A_{0,0} \simeq 0$. In particular,

- $A \in \mathrm{Null}_0(\mathcal{C})$ corresponds to

  $$0 \leftarrow A_{0,1} \rightarrow A_{1,1}$$

  which is equivalently an object of $\mathrm{Arr}(\mathcal{C})$. In fact, we actually have $\mathrm{Null}_0\mathcal{C} \simeq \mathrm{Arr}(\mathcal{C})$.

- $A \in \mathrm{Null}_1(\mathcal{C})$ corresponds to

  $$\begin{tikzcd} & A_{0,1} \arrow[dl] \arrow[dr] & & A_{1,2} \arrow[dl] \arrow[dr] & \\ 0 & & A_{1,1} & & A_{2,2} \end{tikzcd};$$

  furthermore, d_0 maps this to $A_{1,2} \to A_{2,2}$ and d_1 maps it to $A_{0,1} \to A_{1,1}$.

Now it should be clear that

$$\mathrm{Null}_n(\mathcal{C}) \simeq \mathrm{Null}_1(\mathcal{C}) \times_{\mathrm{Null}_0(\mathcal{C})} \mathrm{Null}_1(\mathcal{C}) \times_{\mathrm{Null}_0(\mathcal{C})} \cdots \times_{\mathrm{Null}_0(\mathcal{C})} \mathrm{Null}_1(\mathcal{C}).$$

To prove it is a complete Segal object in $\mathbf{Cat}_\infty^{\mathbf{st}}$, we should prove

$$\begin{tikzcd} \mathrm{Null}_0(\mathcal{C}) \arrow[r, "\Delta"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Null}_0(\mathcal{C}) \times \mathrm{Null}_0(\mathcal{C}) \arrow[d, "(s,s)"] \\ \mathrm{Null}_3(\mathcal{C}) \arrow[r, "(d_{\{0,2\}}, d_{\{1,3\}})"'] & \mathrm{Null}_1(\mathcal{C}) \times \mathrm{Null}_1(\mathcal{C}) \end{tikzcd}$$

is Cartesian. The description of $d_{\{0,2\}}$ and $d_{\{1,3\}}$ makes this clear. An object of $\mathrm{Null}_3(\mathcal{C})$ correspond to a diagram of the form

$$\begin{tikzcd} & A_{0,1} \arrow[dl] \arrow[dr] & & A_{1,2} \arrow[dl] \arrow[dr] & & A_{2,3} \arrow[dl] \arrow[dr] & & A_{3,4} \arrow[dl] \arrow[dr] & \\ 0 & & A_{1,1} & & A_{2,2} & & A_{3,3} & & A_{4,4} \end{tikzcd}.$$

$d_{\{0,2\}}$ maps it to

$$\begin{tikzcd} & A_{1,2} \arrow[dl] \arrow[dr] & & A_{2,4} \arrow[dl] \arrow[dr] & \\ 0 & & A_{2,2} & & A_{4,4} \end{tikzcd}.$$

$d_{\{1,3\}}$ maps it to

$$\begin{tikzcd} & A_{0,1} \arrow[dl] \arrow[dr] & & A_{1,3} \arrow[dl] \arrow[dr] & \\ 0 & & A_{1,1} & & A_{3,3} \end{tikzcd}.$$

In case this does not convince the reader, the condition of completeness involving the free living isomorphism category $N^r(0 \xrightarrow{\simeq} 1)$ should be self-evident. $\square$

The following remarks are not strictly necessary for the proof of universality, but they fits very well with our discussion and it helps us make the statements more clear. Proofs for the various statements can be found in [Cal+21b, Lemma 2.4.7].

5.16 Remark

It is possible to define a décalage functor for ∞-categories, by setting

$$\mathrm{dec}(\mathcal{C}) \simeq \mathrm{core}\,(\mathcal{C}) \times_{\mathcal{C}} \mathrm{Arr}(\mathcal{C})$$

for any ∞-category $\mathcal{C}$.
There is a canonical equivalences

$$N^r(\mathrm{dec}(\mathcal{C})) \simeq \mathrm{dec}\, N^r(\mathcal{C}),$$

natural in the $\mathcal{C}$. Moreover, this equivalence makes the Rezk nerve of the functors

$$\mathrm{dec}(\mathcal{C}) \to \mathrm{core}\,(\mathcal{C}) \quad \text{and} \quad \mathrm{dec}(\mathcal{C}) \to \mathcal{C}$$

correspond to the canonical functors

$$N^r_{1+n}(\mathcal{C}) \to N^r_0(\mathcal{C}) \quad \text{and} \quad N^r_{1+n}(\mathcal{C}) \to N^r_n(\mathcal{C}).$$

Moreover, for any ∞-category $\mathcal{C}$ and $x \in \mathcal{C}$, there is an equivalence

$$N^r(x/\mathcal{C}) \simeq \mathrm{fib}_x(\mathrm{dec}(N^r(\mathcal{C})) \to N^r_0(\mathcal{C})),$$

natural in $\mathcal{C}$ and x. On the other hand, for any Segal anima X and $x \in X_0$, then

$$x/\,\mathrm{asscat}(X) \simeq \mathrm{asscat}\,(\mathrm{fib}_x(\mathrm{dec}(X) \to \mathrm{const}X_0)),$$

in such a way that they fit into a commutative diagram

$$\begin{tikzcd}
x/\,\mathrm{asscat}(X) \arrow[rr, "\simeq"] \arrow[dr, "t"'] & & \mathrm{asscat}\,(\mathrm{fib}_x(\mathrm{dec}(X) \to \mathrm{const}X_0)) \arrow[dl, "d_0"] \\
& \mathrm{asscat}(X) &
\end{tikzcd}.$$

5.17 Remark

Before starting with the proof, we would like to show that

$$\mathrm{asscat}\; F(\mathrm{Null}(\mathcal{C})) \simeq 0/\mathrm{Span}^F(\mathcal{C}).$$

We have

$$F(\mathrm{Null}(\mathcal{C})) \simeq \mathrm{fib}_0(\mathrm{dec}(FQ(\mathcal{C}))) \to \mathrm{const}(F(\mathcal{C}))).$$

Indeed, we know

$$\begin{tikzcd} \mathrm{Null}(\mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{decQ}(\mathcal{C}) \arrow[d] \\ 0 \arrow[r] & \mathrm{const}\mathcal{C} \end{tikzcd}$$

is a split Verdier square, so F being additive maps it to the cartesian square

$$\begin{tikzcd} F(\mathrm{Null}(\mathcal{C})) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F(\mathrm{decQ}(\mathcal{C})) \simeq \mathrm{decF}(Q(\mathcal{C})) \arrow[d] \\ F(0) \simeq * \arrow[r] & F(\mathrm{const}\mathcal{C}) \simeq \mathrm{constF}(\mathcal{C}) \end{tikzcd}$$

The two equivalences on the right hand side come from the fact that the equivalences are clear level-wise.

Then, applying 5.16,

$$\begin{aligned} &\text{asscat } F(\mathrm{Null}(\mathcal{C})) \simeq \\ &\simeq \text{asscat } F(\mathrm{fib}_0(\mathrm{decQ}(\mathcal{C}) \to \mathrm{constQ}_0(\mathcal{C}))) \\ &\simeq \text{asscat}\,(\mathrm{fib}_0(\mathrm{decFQ}(\mathcal{C}) \to \mathrm{constFQ}_0(\mathcal{C}))) \\ &\simeq 0/\mathrm{asscatFQ}\mathcal{C} \\ &\simeq 0/\mathrm{Span}^F(\mathcal{C}). \end{aligned}$$

Now that we have some more notation, let us give a better version of the statement of 5.11 in the following theorem.

5.18 Theorem

Consider a stable ∞-category $\mathcal{C}$ and a grouplike additive functor $F : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$. Consider also the cartesian square in $\mathbf{Cat}_\infty$

$$\begin{tikzcd} \mathbf{Map}_{\mathrm{Span}^F(\mathcal{C})}(0,0) \simeq \Omega\mathrm{Span}^F(\mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0/\mathrm{Span}^F(\mathcal{C}) \arrow[d, "t"] \\ * \arrow[r, "0"] & \mathrm{Span}^F(\mathcal{C}) \end{tikzcd}.$$

By applying the realization functor $|\bullet|\,\mathbf{Cat}_\infty \to \mathbf{An}$, we get a cartesian square

$$\begin{tikzcd}
F(\mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\
* \arrow[r] & |\mathrm{Span}^F(\mathcal{C})|
\end{tikzcd}.$$

In particular

$$F(\mathcal{C}) \simeq \Omega|\mathrm{Span}^F(\mathcal{C})|.$$

Proof of the theorem 5.18. We would like to prove that the following is a cartesian diagram

$$\begin{tikzcd}
\mathrm{const}F(\mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F(\mathrm{Null}(\mathcal{C})) \arrow[d, "F(d_0)"] \\
\mathrm{const}* \arrow[r, "0"] & F(Q(\mathcal{C}))
\end{tikzcd}$$

with right vertical map equifibred. If we prove this, we can then apply the Segal-Rezk equifibrancy lemma 3.90 and conclude.

Consider the fibre over 0 of $\mathrm{Null}_n(\mathcal{C}) \xrightarrow{d_0} Q_n(\mathcal{C})$. It consists of diagrams of the form (or better, the completion after taking pullbacks)

$$\begin{tikzcd}[column sep=small]
 & x \arrow[dl] \arrow[dr] & & 0 \arrow[dl] \arrow[dr] & & 0 \arrow[dl] \arrow[dr] & \\
0 & & 0 & & \cdots & & 0
\end{tikzcd},$$

and hence it is equivalent to $\mathcal{C}$. Moreover, $\mathrm{Null}(\mathcal{C}) \xrightarrow{d_0} Q(\mathcal{C})$ is a split Verdier projection, with fully faithful left and right given by left and right Kan extensions. Therefore F maps this split Verdier sequence into a fibre sequence. This builds up into our cartesian square of Segal animae

$$\begin{tikzcd}
F(\mathrm{const}\mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F(\mathrm{Null}(\mathcal{C})) \arrow[d] \\
* \arrow[r, "0"] & F(Q(\mathcal{C}))
\end{tikzcd}$$

Now, we want to show $F(d_0)$ is equifibred, i.e that, for any arrow $[n] \to [m] \in \mathbf{\Delta}$, the following square is cartesian

$$\begin{tikzcd}
F(\mathrm{Null}_m(\mathcal{C})) \arrow[r] \arrow[d, "F(d_0^m)"'] \arrow[dr, phantom, "\lrcorner", very near start] & F(\mathrm{Null}_n(\mathcal{C})) \arrow[d, "F(d_0^n)"] \\
F(Q_m(\mathcal{C})) \arrow[r] & F(Q_n(\mathcal{C}))
\end{tikzcd}.$$

The Segal condition and the fact that there is a split Verdier square such that $Q_n\mathcal{C} \simeq Q_{n-1}\mathcal{C} \times_{\mathcal{C}} Q_1\mathcal{C}$ reduce this to show that

$$\begin{tikzcd}
F(\mathrm{Null}_2(\mathcal{C})) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F(\mathrm{Null}_1(\mathcal{C})) \arrow[d, "d_i"] \\
F(Q_2(\mathcal{C})) \arrow[r] & F(Q_1(\mathcal{C}))
\end{tikzcd}$$

for $i = 0, 1, 2$. For $i = 1$ and $i = 2$, the square without F is split Verdier, therefore the square with F is cartesian. For $i = 0$, the square the square without F is given as follows, in $\mathcal{C}$

$$\begin{array}{ccc}
\begin{tikzcd}[column sep=small, row sep=small]
 & & & a \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee", very near start] & & & \\
 & & b \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee", very near start] & & c \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee", very near start] & & \\
 & d \arrow[dl] \arrow[dr] & & e \arrow[dl] \arrow[dr] & & f \arrow[dl] \arrow[dr] & \\
0 & & g & & h & & i
\end{tikzcd}
& \longmapsto &
\begin{tikzcd}[column sep=small, row sep=small]
 & & c \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee", very near start] & & \\
 & e \arrow[dl] \arrow[dr] & & f \arrow[dl] \arrow[dr] & \\
g & & h & & i
\end{tikzcd}
\\
\Big\downarrow & & \Big\downarrow \\
\begin{tikzcd}[column sep=small, row sep=small]
 & & a \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee", very near start] & & \\
 & b \arrow[dl] \arrow[dr] & & f \arrow[dl] \arrow[dr] & \\
0 & & h & & i
\end{tikzcd}
& \longmapsto &
\begin{tikzcd}[column sep=small, row sep=small]
 & f \arrow[dl] \arrow[dr] & \\
h & & i
\end{tikzcd}
\end{array}.$$

The fibre over 0 of the left map is given by

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & & 0 \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee", very near start] & & & \\
 & & 0 \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee", very near start] & & y \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee", very near start] & & \\
 & x \arrow[dl] \arrow[dr] & & y \arrow[dl] \arrow[dr] & & 0 \arrow[dl] \arrow[dr] & \\
0 & & x \oplus y & & 0 & & 0
\end{tikzcd},$$

which is equivalent to $\mathcal{C}^2$.
The fibre over 0 of the right map is given by

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & x \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee", very near start] & & \\
 & x \arrow[dl] \arrow[dr] & & z \arrow[dl] \arrow[dr] & \\
0 & & 0 & & 0
\end{tikzcd},$$

which is equivalent to $\mathrm{Arr}(\mathcal{C})$. The induced map between the fibres is a right inverse to the equivalence in animae

$$(s, \mathrm{cof}) : F(\mathrm{Arr}(\mathcal{C})) \to F(\mathcal{C})^2.$$

Now we recall remark 3.88 and finish. □

Proof of proposition 5.12. We want to show that η_{GF} and $G\eta_F$ differ by an automorphism on GGF. Let us decompose the definition of $\eta_{GF}, G\eta_F : GF \to GGF$. (The different colors should help us)

$$\eta_{GF} : GF(-) = \Omega|FQ-| \to \Omega|(\Omega|FQ-|)(Q-)| =$$
$$= \Omega|\Omega|FQQ-|| = \Omega|\Omega|FQ^{(2)}_{\bullet,\bullet}||.$$

Also,

$$G\eta_F : \Omega|FQ-| \to \Omega|\Omega|FQQ-||=$$
$$= \Omega|\Omega|FQ^{(2)}_{\bullet,\bullet}-||$$

Therefore, this two maps are induces by maps into different Ω and Q. Using the limits-colimits interchange map, to take out the internal Ω, we obtain

$$\Omega|FQ-| \overset{\eta_{GF}}{\Rightarrow} \Omega|\Omega|FQ^{(2)}_{\bullet,\bullet}|| \overset{\star}{\Rightarrow} \Omega\Omega|FQ^{(2)}_{\bullet,\bullet}-|$$

and

$$\Omega|FQ-| \overset{G\eta_F}{\Rightarrow} \Omega|\Omega|FQ^{(2)}_{\bullet,\bullet}-|| \overset{\star}{\Rightarrow} \Omega\Omega|FQ^{(2)}_{\bullet,\bullet}-|.$$

The target of these two morphism are equivalent through the natural transformation given by flipping both Ω and Q. The morphism $\star$ is not always an equivalence, indeed the geometric realization is not always compatible with pullbacks. However, we claim it is, and this will finish the proof.

Consider, for $[k] \in \mathbf{\Delta}$, we have an obvious fibre sequence

$$\Omega|FQQ_k\mathcal{C}| \to 0 \to |FQQ_k\mathcal{C}|.$$

This sequence is also a cofibre sequence of $\mathbf{E}_\infty$-groups, or equivalently $\pi_0|FQQ_k\mathcal{C}| = 0$. This is true, indeed, more generally for any stable ∞-category $\mathcal{D}$,

$$\pi_0|FQ\mathcal{D}| \simeq 0.$$

To prove this aim notice that there is a surjective map

$$0 \simeq \pi_0|FS_0\mathcal{D}| \to \pi_0|FS\mathcal{D}| \simeq \pi_0|FS^{esd}\mathcal{D}| \simeq \pi_0|FQ\mathcal{D}|.$$

Another equivalent method to prove this (the one used in [HLS22]) is to notice that

$$Q_0\mathcal{D} \simeq \mathcal{D} \to Q_1\mathcal{D}, \quad x \mapsto (0 \leftarrow 0 \to x)$$

composed with d_0 gives the identity, and composed with d_1 gives the null-morphism. This immediately implies $\pi_0|FQ\mathcal{D}| \simeq 0$ as a consequence of F being reduced.

Then it follows that we have a bifibre sequence of $\mathbf{E}_\infty$-groups (for the usual reasoning on spectra)

$$|\Omega|FQ^{(2)}\mathcal{C}|| \to 0 \to |FQ^{(2)}\mathcal{C}|,$$

so in particular a fibre sequence of the underlying anima. We obtain

$$|\Omega|FQ^{(2)}\mathcal{C}|| \simeq \Omega|FQ^{(2)}\mathcal{C}|$$

and by looping again, the equivalence

$$\Omega|\Omega|FQ^{(2)}\mathcal{C}|| \simeq \Omega\Omega|FQ^{(2)}\mathcal{C}|$$

that we were searching. □

5.19 Remark

Our methodology presents a more direct demonstration of the additivity theorem and the universal property of algebraic Ktheory. This is achieved without having to refer to non-commutative motives, which is instead what is done in [BGT13].

5.4 ADDITIVITY, ONCE MORE.

In section 5.1, we have showed that

$$k \simeq \Omega|\mathrm{Span}(\bullet)| \simeq \Omega|\mathrm{core}\ Q\bullet|$$

is an additive functor $\mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$. This works fine, but, as we have already claimed, more is true. For any additive functor $F : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$, it is true that

$$\Omega|FQ\bullet|$$

is additive.
Before proving the theorem, we need to discuss some things.

5.4.1 *Waldhausen Additivity Theorem.*

We can finally discuss the theorem.

Theorem C (Waldhausen Additivity for additive functors)
Consider an additive functor $F : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$. Then

$$|\mathrm{Span}^F(\bullet)| \simeq |FQ\bullet| : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$$

is additive too.

The proof is divided in seven steps.

STEP 1. *If $p : \mathcal{C} \to \mathcal{D}$ is a split Verdier projection, then p is a bicartesian fibration.*
Recall that bicartesian fibration means both cartesian and cocartesian fibration. To prove this we need the following lemma.

5.20 Lemma
Consider an adjunction of ∞-categories

$$\begin{tikzcd} \mathcal{B} \arrow[rr, bend left, "g"] \arrow[rr, phantom, "\perp"] & & \mathcal{A} \arrow[ll, bend left, "f"] \end{tikzcd}$$

with counit $\epsilon : gf \Rightarrow \mathrm{id}_{\mathcal{A}}$. Then

(i) An arrow $\phi : x \to y \in \mathcal{A}$ is f-cocartesian if and only if the square

$$
\begin{tikzcd}
gf(x) \arrow[r, "gf(\phi)"] \arrow[d, "\epsilon"'] & gf(y) \arrow[d, "\epsilon"] \\
x \arrow[r, "\phi"'] & y \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
$$

is cocartesian in $\mathcal{A}$.

(ii) Suppose $\mathcal{A}$ admits pushouts, f preserves pushouts, and g is fully faithful. Then f is a cocartesian fibration.

Proof. (i) "If" part.
We want to prove that for any $\psi : x \to z$ and any $u : p(y) \to p(z)$ such that $f(\phi)u \simeq f(\psi)$, then there exists a essentially unique $\hat{u} : y \to z$ such that $f(\hat{u}) \simeq u$. Therefore, we have the following

$$
\begin{tikzcd}
\underline{z} \arrow[d, dotted] & \underline{x} \arrow[l, "\forall\psi"'] \arrow[r, "\phi"] \arrow[d, dotted] & \underline{y} \arrow[d, dotted] \\
f(z) & f(x) \arrow[l, "f(\psi)"'] \arrow[r, "f(\phi)"] & f(y) \arrow[ll, dashed, bend left, "u"]
\end{tikzcd}
.
$$

By applying g to the diagram below we get a diagram

$$
\begin{tikzcd}
gf(z) \arrow[d, "\psi"'] & gf(x) \arrow[l, "gf(\psi)"'] \arrow[r, "gf(\phi)"] \arrow[d, "\epsilon"'] & gf(y) \arrow[d, "\epsilon"] \arrow[ll, dashed, bend right, "g(u)"'] \\
z & x \arrow[l, "\psi"'] \arrow[r, "\phi"] & y \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
,
$$

where the right square is cocartesian by assumption. By universal property there exist an essentially unique morphism $y \to z$.

"Only if" part.
If ϕ is a cocartesian morphism, then also $gf(\phi)$ is a cocartesian morphism. If

$$\begin{tikzcd}
gf(x) \arrow[r, "gf(\phi)"] \arrow[d, "\epsilon"'] & gf(y) \arrow[d] \\
x \arrow[r] & z
\end{tikzcd}$$

is another cocone under $x \leftarrow gf(x) \rightarrow gf(y)$, then we have the triangle

$$\begin{tikzcd}
f(x) \simeq fgf(x) \arrow[r, "f(\phi) \simeq fgf(\phi)"] \arrow[dr] & f(y) \simeq fgf(y) \arrow[d] \\
& f(z)
\end{tikzcd} .$$

Now, we have another

$$\begin{tikzcd}
gf(x) \arrow[r, "gf(\phi)"] \arrow[dr] & gf(y) \arrow[d] \\
& z
\end{tikzcd}$$

and a horn

$$\begin{tikzcd}
x \arrow[r, "\phi"] \arrow[dr] & y \arrow[d] \\
& z
\end{tikzcd} .$$

which gives a unique filler $y \rightarrow z$ by the fact that ϕ is cocartesian. But now we have two fillers of

$$\begin{tikzcd}
gf(x) \arrow[r, "gf(\phi)"] \arrow[d, "\epsilon"'] & gf(y) \\
x \arrow[r] & z
\end{tikzcd} ,$$

the original $gf(y) \rightarrow z$ and the new $gf(y) \rightarrow y \rightarrow z$. Since $gf(phi)$ is cocartesian, the two fillers must be equal. Since the morphism such that all these commute in unique, we get that the original square was a pushout by universal property.

(ii) Fix a morphism $\phi' : x' \to y' \in \mathcal{B}$ and a $x \in \mathcal{A}$ such that $f(x) \simeq x'$. One of such x for surely exists, for example $g(x')$; indeed, $fg(x') \simeq x'$ because g is a fully faithful left adjoint to f.
Consider now the morphism

$$f(x) \simeq x' \xrightarrow{\phi'} y';$$

by applying g we obtain a morphism

$$gf(x) \simeq g(x') \xrightarrow{g(\phi)} g(y').$$

Forming the pushout, which exists by assumption, we obtain

$$\begin{tikzcd} gf(x) \simeq g(x') \arrow[r, "g(\phi')"] \arrow[d, "\epsilon"'] & g(y') \arrow[d] \\ x \arrow[r] & y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd},$$

and we call ϕ the morphism $x \to y$. By applying f, which preserves pushout by assumption, we obtain

$$\begin{tikzcd} fgf(x) \simeq x' \arrow[r, "fg(\phi') \simeq \phi'"] \arrow[d, "f(\epsilon)"'] & fg(y') \simeq y' \arrow[d] \\ f(x) \simeq x' \arrow[r, "f(\phi)"'] & f(y) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

where the left vertical maps is equivalent to the identity on x'. Therefore, $y' \to f(y)$ must be an equivalence and $f(\phi) \simeq \phi'$. Therefore $\phi : x \to y$ is a lift of ϕ'. By applying (i), we get that ϕ is f-cocartesian, so f is a cocartesian fibration (it has all lift to f-cocartesian morphisms). □

To prove *STEP 1*, we just have to apply the lemma to p and its fully faithful left adjoint to obtain that p is cocartesian, and the dual lemma (by considering the opposite of everything) to p and its fully faithful right adjoint to obtain that p is cartesian. Notice that all functors are exact between stable ∞-categories, so all condition are verified.

STEP 2. *[Barwick] Consider an exact bicartesian fibration $p : \mathcal{C} \to \mathcal{D}$ between stable ∞-categories. Then a morphism $x \leftarrow y \to z$ is* *(i)* $\mathrm{Span}(p)$*-cocartesian if and only if* $y \to x$ *is* p*-cartesian and* $y \to z$ *is* p*-cocartesian.*

(ii) $\mathrm{Span}(p)$*-cartesian if and only if* $y \to x$ *is* p*-cocartesian and* $y \to z$ *is* p*-cartesian.*

Reference for this are [Spectral Mackey functors and equivariant algebraic K-theory (I), Theorem 12.2] and [Two-variable fibrations, factorization systems and infinity-categories of spans, Theorem 3.1]

But let us prove the *if* part of (i) anyway, since that is actually what we need.
Fix $\phi : x \to z$ in $\mathrm{Span}(\mathcal{C})$, i.e. a span $x \leftarrow y \to z$ in $\mathcal{C}$, such that $x \leftarrow y$ is p-cartesian and $y \to z$ is p-cocartesian. We want to prove for any $\psi : x \to a \in \mathrm{Span}(\mathcal{C})$ and any $\mu : z \to a \in \mathrm{Span}(\mathcal{C})$ such that $p(\phi) \circ \mu \simeq p(\psi)$, there exists an (essentially unique) $\rho : z \to a$ lifting μ.
The two morphism ϕ and ψ form a diagram in $\mathcal{C}$

$$\begin{tikzcd}
 & b \arrow[dl] \arrow[dr] & & y \arrow[dl] \arrow[dr] & \\
a & & x & & z
\end{tikzcd}.$$

With μ, this gives a diagram in $\mathcal{D}$

$$\begin{tikzcd}
 & & u \arrow[dddll, bend right] \arrow[dddrr, bend left] & & \\
 & & & & \\
 & p(b) \arrow[dl] \arrow[dr] & & p(y) \arrow[dl] \arrow[dr] & \\
p(a) & & p(x) & & p(z)
\end{tikzcd},$$

and the commutativity implies that actually the diagram is of this form (notice the rearrangement we had)

$$\begin{tikzcd}
 & & p(b) \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\vee"] & & \\
 & p(y) \arrow[dl] \arrow[dr] & & u \arrow[dl] \arrow[dr] & \\
p(x) & & p(z) & & p(a)
\end{tikzcd}.$$

Then the original diagram is better represented as

$$\begin{tikzcd}
 & & b \arrow[dll, bend right] \arrow[dl, dotted] \arrow[dr, dotted] \arrow[drr, bend left] \arrow[d, phantom, "(a)\quad(b)"] & & \\
 & y \arrow[dl] \arrow[dr] & & \star \arrow[dl, dotted] \arrow[dr, dotted] \arrow[d, phantom, "(c)\quad(d)"] & \\
x & & z & & a
\end{tikzcd},$$

where we aim to fill in the missing objects denoted by the dotted arrows. Since by assumption $x \leftarrow y$ is p-cartesian, we can fill the dotted arrow (a). p is a cocartesian fibration, so we can lift $p(b) \to u$ to a p-cocartesian edge $b \to \star$, filling (b). Now considering the map $b \to a$ we can fill (d), and by considering $b \to y \to z$ we can fill (c). Now we claim the middle square is a pullback if and only if b to$\star$ is cocartesian. Consider the central square and suppose $b \to \star$ is a cocartesian morphism. z satisfies the universal property for the pushout. Indeed, take $y \to z'$ and $\star \to z'$ such that the square commutes. Then since $y \to z$ is cocartesian, there exists a unique $z \to z'$ such that the triangle commutes. But since $b \to \star$ is cocartesian, there is a unique filling of $b \to y \to z'$ such that these commute, which must be the original morphism, so also the right triangle commute. Since $\mathcal{C}$ is stable, this is also a pullback.
Conversely, if the square is cartesian, and we have a morphism $b \to z'$, then by universal property of the pullback, we have (essentially unique) morphisms $y \to z'$ and $\star \to z'$. Applying p to the cartesian square, which remains cartesian since p is exact, we get that the morphism $p(\star) \simeq u \to p(z')$ such that all the triangle formed by $p(b)$, u, and $p(z')$ commute, is essentially unique; moreover this is equivalent to the image of the essentially unique morphism $\star \to z'$ we already found. Therefore, $b \to \star$ is p-cocartesian.

STEP 3. *Consider a split Verdier projection* $p : \mathcal{C} \to \mathcal{D}$. *Then*

$$\mathrm{Span}(p) : \mathrm{Span}(\mathcal{C}) \to \mathrm{Span}(\mathcal{D})$$

is a bicartesian fibration.
By *STEP 1*, p is a bicartesian fibration. Consider a morphism $x' \to z'$ in $\mathrm{Span}(\mathcal{D})$, i.e. a span $x' \leftarrow y' \to z'$ in $\mathcal{D}$. Fix $x \in \mathcal{D}$ such that $p(x) \simeq x'$, which surely exists because p is essentially surjective.
We can lift the arrow $x' \leftarrow y'$ to a p-cartesian arrow $x \leftarrow y$ (so now we have a fixed y), and the arrow $y' \to z'$ to a p-cocartesian arrow $y \to z$. Then *STEP 2* implies this is a $\mathrm{Span}(p)$-cocartesian morphism, and so $\mathrm{Span}(p)$ is a cocartesian fibration. Applying the dual of this, we obtain $\mathrm{Span}(p)$ is a cartesian fibration.

***STEP 4**. Consider a split Verdier projection $p : \mathcal{C} \to \mathcal{D}$. Then*

$$\mathrm{Span}^F(p) : \mathrm{Span}^F(\mathcal{C}) \simeq \operatorname{asscat} FQ(\mathcal{C}) \to \mathrm{Span}^F(\mathcal{D}) \simeq \operatorname{asscat} FQ(\mathcal{D})$$

is a bicartesian fibration for any $F : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$ additive.
Let $\mathcal{E} \subset Q_1(\mathcal{C}) \simeq \mathrm{Fun}(J_1, \mathcal{C})$ denote the full sub-∞-category of $Q_1(\mathcal{C})$ spanned by $\mathrm{Span}(p)$-cocartesian morphisms of $\mathrm{Span}(\mathcal{C})$, i.e. those with left arrow p-cartesian and right arrow p-cocartesian. We claim the following square $(\star)$ is split Verdier

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "d_1"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C} \arrow[d, "p"] \\
Q_1(\mathcal{D}) \arrow[r, "d_1", "(x' \leftarrow y' \to z') \mapsto x'"'] & D
\end{tikzcd}.$$

Fix P the pullback of the square. We call η canonical map : $\mathcal{E} \to P$.

- η is essentially surjective for the following reason. Any span $p(x) \leftarrow y' \to z' \in Q_1(\mathcal{D})$ admits an essentially uniquely lift to $x \leftarrow y \to z \in \mathcal{E}$. The "uniqueness" comes from the fact that the left arrow is required to be a p-cartesian lift and the right one a p-cocartesian lifts.

- The fully faithfulness follows from the computation of the mapping animae

  $$\mathbf{Map}_{\mathcal{E}}(\alpha, \beta) \quad \text{and} \quad \mathbf{Map}_P(\eta(\alpha), \eta(\beta)),$$

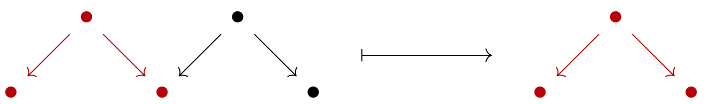
  using the fact that $\mathbf{Map}_P$ can be computed as a pullback of mapping animae, and that since $\mathcal{E}$ is a full sub-∞-category of $Q_1(\mathcal{C}) \simeq \mathbf{Fun}(J_1, \mathcal{C})$, we have

  $$\mathrm{Map}_{\mathcal{E}}(\alpha, \beta) \simeq \lim_{(x \to y) \in \mathrm{Ob}(\mathrm{TwArr}(J_1) \simeq J_2^{op})} \mathbf{Map}_{\mathcal{C}}(\alpha x, \beta y)$$

  and similarly for the mapping anima in $Q_1(\mathcal{D})$.

$(\star)$ is split Verdier because the right vertical leg $p : \mathcal{C} \to \mathcal{D}$ is split Verdier.

Next, consider the pullback square $\mathcal{E} \times_{Q_1(\mathcal{C})} Q_2(\mathcal{C})$, where the map $Q_2(\mathcal{C}) \to Q_1(\mathcal{C})$ is d_2, which can be described as



We claim the following square $(\star\star)$ is split Verdier

$$
\begin{tikzcd}
\mathcal{E} \times_{Q_1(\mathcal{C})} Q_2(\mathcal{C}) \arrow[rr, "d_1"] \arrow[d, hook] \arrow[drr, phantom, "\lrcorner", very near start] & & \mathcal{E} \times_{\mathcal{C}} Q_1(\mathcal{C}) \arrow[d, hook] \\
Q_2(\mathcal{C}) \arrow[d, "p"'] & & Q_1(\mathcal{C}) \times_{\mathcal{C}} Q_1(\mathcal{D}) \arrow[d, "p"] \\
Q_2(\mathcal{D}) \arrow[rr] & & Q_1(\mathcal{D}) \times_{\mathcal{D}} Q_1(\mathcal{D})
\end{tikzcd}
.
$$

Fix R the pullback. The fact that the canonical map $\rho : \mathcal{E} \times_{Q_1(\mathcal{C})} Q_2(\mathcal{C}) \to R$ is essentially surjective is basically contained in the proof of *STEP 2*. The fully faithfulness follows from the computation of the mapping anima in $\mathcal{E} \times_{Q_1(\mathcal{C})} Q_2(\mathcal{C})$ and R using similar limit formula as above.
Next, we want to prove that the map

$$\mathcal{E} \times_{\mathcal{C}} Q_1(\mathcal{C}) \to Q_1(\mathcal{D}) \times_{\mathcal{D}} Q_1(\mathcal{D})$$

is split Verdier. We have

$$
\begin{tikzcd}
\mathcal{E} \times_{\mathcal{C}} Q_1(\mathcal{C}) \arrow[r, "\simeq", "\text{1st square}"'] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & (Q_1(\mathcal{D}) \times_{\mathcal{D}} \mathcal{C}) \times_{\mathcal{C}} \times Q_1(\mathcal{C}) \arrow[r, "\simeq"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Q_1(\mathcal{D}) \times_{\mathcal{D}} Q_1(\mathcal{C}) \arrow[r, "\text{canon.}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Q_1(\mathcal{C}) \arrow[d, "p"'] \\
Q_1(\mathcal{D}) \times_{\mathcal{D}} Q_1(\mathcal{D}) \arrow[r, "id"] & Q_1(\mathcal{D}) \times_{\mathcal{D}} Q_1(\mathcal{D}) \arrow[r, "id"] & Q_1(\mathcal{D}) \times_{\mathcal{D}} Q_1(\mathcal{D}) \arrow[r, "\text{canon. on 2nd}"] & Q_1(\mathcal{D})
\end{tikzcd}
.
$$

Therefore the map $\mathcal{E} \times_{\mathcal{C}} Q_1(\mathcal{C}) \to Q_1(\mathcal{D}) \times_{\mathcal{D}} Q_1(\mathcal{D})$ is the pullback of $Q_1(\mathcal{C}) \xrightarrow{p} Q_1(\mathcal{D})$. This is a split Verdier projection because the fully faithful left and right adjoints of $p : \mathcal{C} \to \mathcal{D}$ extend to $Q_1(\mathcal{D}) \to Q_1(\mathcal{C})$.

Being split Verdier squares, both $(\star)$ and $(\star\star)$ are mapped by pullback square by any additive F. So in **An** we have pullbacks squares

$$
\begin{tikzcd}
F(\mathcal{E}) \arrow[r, "F(d_1)"] \arrow[d, "F(p)"'] \arrow[dr, phantom, "\lrcorner", very near start] & F(\mathcal{C}) \arrow[d, "F(p)"] \\
F(Q_1(\mathcal{D})) \arrow[r, "F(d_1)"] \arrow[u, bend left, dotted] \arrow[u, bend right, dotted] & F(D) \arrow[u, bend left, dotted] \arrow[u, bend right, dotted]
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
F(\mathcal{E} \times_{Q_1(\mathcal{C})} Q_2(\mathcal{C})) \arrow[r] \arrow[d, "F(p)"'] \arrow[dr, phantom, "\lrcorner", very near start] & F(\mathcal{E} \times_{\mathcal{C}} Q_1(\mathcal{C})) \arrow[d, "F(p)"] \\
F(Q_2(\mathcal{C})) \arrow[r] \arrow[u, bend left, dotted] \arrow[u, bend right, dotted] & F(Q_1(\mathcal{D}) \times_{\mathcal{D}} Q_1(\mathcal{D})) \arrow[u, bend left, dotted] \arrow[u, bend right, dotted]
\end{tikzcd}
.
$$

The combination of these two diagrams shows that $\mathrm{Span}^F(p)$ is a cocartesian fibration.

- The second diagram exhibits the image of $F(\mathcal{E}) \to \mathbf{Map}_{\mathbf{Cat}_\infty}([1], \mathrm{Span}^F(\mathcal{C}))$ as spanned by $\mathrm{Span}^F(p)$-cocartesian edges.
- The first diagram demonstrates the existence of the different liftings.

STEP 5. *The realization functor* $|\bullet| : \mathbf{Cat}_\infty \to \mathbf{An}$ *sends pullbacks squares that have as right vertical leg a bicartesian fibration to pullback squares in* $\mathbf{An}$.

Consider a cartesian square in $\mathbf{Cat}_\infty$

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "u"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C} \arrow[d, "f"] \\ \mathcal{D}' \arrow[r, "l"'] & \mathcal{D} \end{tikzcd}$$

such that $f : \mathcal{C} \to \mathcal{D}$, and consequently $f' : \mathcal{C}' \to \mathcal{D}'$ are bicartesian fibration. We denote with St^{cocart} the cocartesian straightening, and with Un^{cocart} the cocartesian unstraightening. Then we call

$$F := \mathrm{St}^{cocart}(\mathcal{C} \to \mathcal{D}) : \mathcal{D} \to \mathbf{Cat}_\infty$$

the cocartesian straightening of $\mathcal{C} \to \mathcal{D}$. We obtain

$$\mathrm{Un}^{cocart}(F) \simeq \mathcal{C} \quad \text{and} \quad \mathrm{Un}^{cocart}(F \circ l) \simeq \mathcal{C}',$$

because unstraightening turns compositions into pullbacks.

We have a diagram

$$\begin{tikzcd} \mathcal{D} \arrow[r, "F"] \arrow[d] & \mathbf{Cat}_\infty \arrow[d, "|\bullet|"] \\ |\mathcal{D}| & \mathbf{An} \end{tikzcd}.$$

It is worth noting that any morphism $\phi : x \to y$ in $\mathcal{D}$ is mapped to a left-adjoint functor $F(\phi) : F(x) \to F(y)$ by F, because F is the straightening of a bicartesian fibration. This gives rise to an equivalence $|F(\phi)| : |F(x)| \to |F(y)|$. As a result, we have an arrow $|\mathcal{D}| \to \mathbf{An}$, which complete the diagram to a commutative square, due to the universal property of localisations of ∞-categories.

We denote with Un^{left} the left unstraightening. Now unstraightening the functor into $\mathbf{An}$, we obtain a cartesian square

$$\begin{tikzcd} \mathrm{Un}^{left}(|F| \circ |l|) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Un}^{left}(|F| \circ) \arrow[d] \\ |\mathcal{D}'| \arrow[r, "|l|"'] & |\mathcal{D}| \end{tikzcd}.$$

We aim to prove that there is an equivalence

$$\mathrm{Un}^{\mathrm{left}}(|F| \circ |\iota|) \simeq |\mathcal{C}'|,$$

or better an equivalence

$$\mathrm{Un}^{\mathrm{left}}(|F| \circ |\iota|) \simeq |\mathrm{Un}^{\mathrm{cocart}}(F \circ p)|,$$

and similarly another equivalence

$$\mathrm{Un}^{\mathrm{left}}(|F|) \simeq |\mathrm{Un}^{\mathrm{cocart}}(F)| \simeq |\mathcal{C}|.$$

To achieve this, we need the following proposition from [Hebb, Thm XI.23].

5.21 Reference

Consider a diagram $F : I \to \mathbf{Cat}_\infty$. Then

$$\operatorname*{colim}_I F \simeq \mathrm{Un}^{\mathrm{cocart}}(F)[\{\text{cocart.edges}\}^{-1}]$$

and

$$\lim_I F \simeq \Gamma_{\mathrm{cocart}}(\mathrm{Un}^{\mathrm{cocart}}(F)).$$

Here, for $p : E \to S$ a cocartesian fibration, $\Gamma(p)$ is the ∞-category of sections of p defined by the pullback

$$\begin{tikzcd} \Gamma(p) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Fun}(S, E) \arrow[d, "p_*"] \\ \{\mathrm{id}_S\} \arrow[r] & \mathbf{Fun}(S, S) \end{tikzcd},$$

and $\Gamma_{\mathrm{cocart}}(p)$ is the full sub-∞-category of $\Gamma(p)$ spanned by sections that take all edges in S to p-cocartesian edges.

Now, $\operatorname*{colim}_{\mathcal{D}}(F)$ is a localisation of $\mathrm{Un}^{\mathrm{cocart}}(F)$, therefore

$$|\operatorname{colim}(\mathcal{D} \xrightarrow{F} \mathbf{Cat}_\infty)| \simeq |\mathrm{Un}^{\mathrm{cocart}}(F)|.$$

On the other hand, since $|\bullet|$ is a left adjoint, it preserves colimits, therefore

$$\begin{aligned} &|\operatorname{colim}\left(\mathcal{D} \xrightarrow{F} \mathbf{Cat}_\infty\right)| \simeq \\ &\simeq \operatorname{colim}|(\mathcal{D} \xrightarrow{F} \mathbf{Cat}_\infty)| \\ &\simeq \operatorname{colim}\left(\mathcal{D} \xrightarrow{F} \mathbf{Cat}_\infty \xrightarrow{|\bullet|} \mathbf{An}\right) \\ &\simeq \operatorname{colim}\left(|\mathcal{D}| \xrightarrow{F} \mathbf{An}\right), \end{aligned}$$

where the last equivalence is true because the localisation map $\mathcal{D} \to |\mathcal{D}|$ is cofinal. Then it follows

$$|\mathrm{Un}^{\mathrm{cocart}}(F)| \simeq \operatorname{colim}\left(|\mathcal{D}| \xrightarrow{F} \mathbf{An}\right) \simeq \mathrm{Un}^{\mathrm{left}}(|F|).$$

Similarly for $\mathrm{Un}^{\mathrm{left}}(|F| \circ |l|)$.

***STEP 6**. $\mathrm{Span}^F(\bullet) : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{Cat}_\infty$ is an additive functor.*
Consider a split Verdier square

$$\begin{tikzcd}
\mathcal{C}' \arrow[r, "u"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C} \arrow[d, "p"] \\
\mathcal{D}' \arrow[r, "l"'] \arrow[u, dotted, bend left] \arrow[u, dotted, bend right] & \mathcal{D} \arrow[u, dotted, bend left] \arrow[u, dotted, bend right]
\end{tikzcd}.$$

For any $n \in \mathbf{N}$, $Q_n(\bullet) \simeq \mathbf{Fun}(J_n, \bullet)$ preserves pullbacks, and in fact all limits. Moreover the fully faithful adjoints to p and p' gives fully faithful adjoints to $Q_n(p)$ and $Q_n(p')$; we have so a split verdier square

$$\begin{tikzcd}
Q_n(\mathcal{C}') \arrow[r, "Q_n(u)"] \arrow[d, "Q_n(p')"'] \arrow[dr, phantom, "\lrcorner", very near start] & Q_n(\mathcal{C}) \arrow[d, "Q_n(p)"] \\
Q_n(\mathcal{D}') \arrow[r, "Q_n(l)"'] \arrow[u, dotted, bend left] \arrow[u, dotted, bend right] & Q_n(\mathcal{D}) \arrow[u, dotted, bend left] \arrow[u, dotted, bend right]
\end{tikzcd}.$$

Applying F we obtain a pullback square

$$\begin{tikzcd}
F(Q_n(\mathcal{C}')) \arrow[r, "F(Q_n(u))"] \arrow[d, "F(Q_n(p'))"'] \arrow[dr, phantom, "\lrcorner", very near start] & F(Q_n(\mathcal{C})) \arrow[d, "F(Q_n(p))"] \\
F(Q_n(\mathcal{D}')) \arrow[r, "F(Q_n(l))"'] & F(Q_n(\mathcal{D}))
\end{tikzcd}.$$

Since we can check equivalences levelwise, we get a pullback square in $s\,\mathbf{An}$

$$\begin{tikzcd}
F(Q(\mathcal{C}')) \arrow[r, "F(Q(u))"] \arrow[d, "F(Q(p'))"'] \arrow[dr, phantom, "\lrcorner", very near start] & F(Q(\mathcal{C})) \arrow[d, "F(Q(p))"] \\
F(Q(\mathcal{D}')) \arrow[r, "F(Q(l))"'] & F(Q(\mathcal{D}))
\end{tikzcd}.$$

Recalling the remark 4.40, this is a cartesian square of Segal animae. Unfortunately, asscat : $s\,\mathbf{An} \to \mathbf{Cat}_\infty$ do not necessarily preserve pullbacks of non-complete Segal animae, and F do not necessarily preserve pullbacks so that proposition is useless. We have a more tricky way to show that this is anyway a cartesian square.
Consider three Segal animae $X, Y, Z \in s\,\mathbf{An}$. Then the (essentially unique) canonical map

$$\mathrm{asscat}(X \times_Y Z) \to \mathrm{asscat}(X) \times_{\mathrm{asscat}(Y)} \mathrm{asscat}(Z)$$

is always fully faithful. To see this, consider the diagram

$$\begin{tikzcd}
\mathrm{asscat}(X \times_Y Z) \arrow[dr, "\Phi"] \arrow[drr, bend left, "\mathrm{asscat}(\pi_Z)"] \arrow[ddr, bend right, "\mathrm{asscat}(\pi_X)"'] & & \\
& \mathrm{asscat}(X) \times_{\mathrm{asscat}(Y)} \mathrm{asscat}(Z) \arrow[r, "\pi_{\mathrm{asscat}(Z)}"] \arrow[d, "\pi_{\mathrm{asscat}(X)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{asscat}(Z) \arrow[d, "g"] \\
& \mathrm{asscat}(X) \arrow[r, "f"'] & \mathrm{asscat}(Y)
\end{tikzcd}.$$

Recall that for any Segal animae S the mapping anima in S can be be computed as

$$\mathbf{Map}_{\mathrm{asscat}(X)}(x, y) \simeq \{x, y\} \times_{X_0 \times X_0} X_1.$$

Therefore

$$\begin{aligned}
&\mathbf{Map}_{\mathrm{asscat}(X) \times_{\mathrm{asscat}(Y)} \mathrm{asscat}(Z)}(\Phi a, \Phi b) \simeq \\
&\simeq \mathbf{Map}_{\mathrm{asscat}(X)}(\pi_X a, \pi_X b) \times_{\mathbf{Map}_{\mathrm{asscat}(Y)}(f\pi_X a, f\pi_X b)} \mathbf{Map}_{\mathrm{asscat}(Z)}(\pi_Z a, \pi_Z b) \\
&\simeq (\{\pi_X a, \pi_X b\} \times_{X_0 \times X_0} X_1) \times_{(\{f\pi_X a, f\pi_X b\} \times_{Y_0 \times Y_0} Y_1)} (\{\pi_Z a, \pi_Z b\} \times_{Z_0 \times Z_0} Z_1) \\
&\simeq \{a, b\} \times_{(X_0 \times X_0) \times_{Y_0 \times Y_0} (Z_0 \times Z_0)} (X_1 \times_{Y_1} Z_1)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{Map}_{\mathrm{asscat}(X \times_Y Z)}(a, b) &\simeq \\
&\simeq \{a, b\} \times_{(X_0 \times_{Y_0} Z_0) \times X_0 \times_{Y_0} Z_0)} (X_1 \times_{Y_1} Z_1) \\
&\simeq \{a, b\} \times_{(X_0 \times X_0) \times_{Y_0 \times Y_0} (Z_0 \times Z_0)} (X_1 \times_{Y_1} Z_1).
\end{aligned}$$

It remains us to prove that that Φ is essentially surjective. Recall

$$\mathrm{core}\ \mathrm{asscat}(X) \simeq |X^\times|.$$

Φ is essentially surjective if and only if

$$\begin{aligned}
&\mathrm{core}\,(\mathrm{asscat}(X \times_Y Z)) \simeq \\
&\simeq |(X \times_Y Z)^\times| \\
&\simeq |X^\times \times_{Y^\times} Z^\times| \to \\
&\to \mathrm{core}\,(\mathrm{asscat}(X) \times_{\mathrm{asscat}(Y)} \mathrm{asscat}(Z)) \simeq \\
&\simeq \mathrm{core}\,(\mathrm{asscat}(X)) \times_{\mathrm{core}\,(\mathrm{asscat}(Y))} \mathrm{core}\,(\mathrm{asscat}(Z)) \\
&\simeq |X^\times| \times_{|Y^\times|} |Z^\times|
\end{aligned}$$

is essentially surjective. This is equivalent to the map

$$\pi_0(|X^\times \times_{Y^\times} Z^\times|) \to \pi_0(|X^\times| \times_{|Y^\times|} |Z^\times)$$

being surjective.
We claim this is surjective if $\mathrm{asscat}(X) \to \mathrm{asscat}(Y)$ is a bicartesian fibration, which is is enough for our case. The connected component on $|X^\times| \times_{|Y^\times|} |Z^\times|$ is represented by

$$\{(x, z, \gamma) \colon x \in X_0, z \in Z_0, \gamma \text{ path connecting } x \text{ and } z \text{ in } |Y^\times|\}$$

where we clearly have to quotient by $(x, z, \gamma) \sim (x', z', \gamma')$ if x and x' are in the same path component, z and z' are in the same path component, and γ is equivalent to γ'. A path in $|Y^\times|$ connecting the images in Y of $x \in X_0$ and $z \in Z_0$, can be represented by a sequence of edges $y_i \in Y_1^\times$ and a path w in Y_0 of the form

$$\begin{tikzcd}[column sep=small]
 & y_1 \arrow[dl, "d_1"'] \arrow[dr, "d_0"] & & y_2 \arrow[dl, "d_1"'] \arrow[dr] & & y_n \arrow[dl] \arrow[dr, "d_0"] & & \\
x & & s_1 & & \cdots & & s_n \arrow[r, squiggly, "w"] & z
\end{tikzcd},$$

since

$$\pi_1 sk_1 Y^\times \to \pi_1 |Y^\times|$$

is surjective. Using now the assumption that $\mathrm{asscat}(X) \to \mathrm{asscat}(Y)$ is a bicartesian fibration, we can lift this to a sequence in X which gives a connected component in

$$\pi_0 |X^\times \times_{Y^\times} Z^\times|.$$

Therefore the square in $\mathbf{Cat}_\infty$

$$\begin{tikzcd}
\mathrm{Span}^F(\mathcal{C}') \arrow[r, "u_*"] \arrow[d, "p'_*"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Span}^F(\mathcal{C}) \arrow[d, "p_*"] \\
\mathrm{Span}^F(\mathcal{D}') \arrow[r, "l_*"'] & \mathrm{Span}^F(\mathcal{D})
\end{tikzcd}$$

is cartesian, and so $\mathrm{Span}^F(\bullet)$ is an additive functor.
STEP 7. *$|\mathrm{Span}^F(\bullet)|$ is additive.*
Consider a split Verdier square

$$\begin{tikzcd}
\mathcal{C}' \arrow[r, "u"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C} \arrow[d, "p"] \\
\mathcal{D}' \arrow[r, "l"'] & \mathcal{D}
\end{tikzcd} .$$

By *Step 4*, $\mathrm{Span}^F(p)$ and $\mathrm{Span}^F(p')$ are bicartesian fibration. By *Step 6*, the induced square is cartesian. Therefore, by *Step 5*, also

$$\begin{tikzcd}
|\mathrm{Span}^F(\mathcal{C}')| \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & |\mathrm{Span}^F(\mathcal{C})| \arrow[d] \\
|\mathrm{Span}^F(\mathcal{D}')| \arrow[r] & |\mathrm{Span}^F(\mathcal{D})|
\end{tikzcd}$$

is cartesian. So we are finally done done.

5.4.2 *K-theory is Additive.*

5.22 Corollary (K-theory is additive, once more)
The K-theory anima functor

$$k : \mathbf{Cat}^{\mathbf{st}}_{\infty} \to \mathbf{An}$$

is additive.

Proof. We have

$$k(\bullet) \simeq \Omega|\mathrm{Span}(\bullet)| \simeq \Omega|\mathrm{Span}^{\mathrm{core}}(\bullet)|.$$

core is additive, because it is right adjoint, so $|\mathrm{Span}^{\mathrm{core}}(\mathcal{C})|$ is additive. Ω preserves all limits because it is a right adjoint. □

More generally, for any additive functor F,

$$\Omega|\mathrm{Span}^F(\bullet)| \simeq \Omega|FQ\bullet|$$

is again additive.

5.5 UNIVERSALITY, ONCE MORE.

5.23 Lemma

Consider a stable ∞-category $\mathcal{C}$ and a grouplike additive functor $F : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$. Then

$$F(\mathrm{Arr}(\mathcal{C})) \simeq F(\mathcal{C}) \times F(\mathcal{C}).$$

Proof. Consider the split exact sequence

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "x \mapsto (x \to 0)"] & \mathrm{Arr}(\mathcal{C}) \arrow[l, dashed, bend right, "s"'] \arrow[l, dashed, bend left, "s"] \arrow[r, "t"] & \mathcal{C} \arrow[l, dashed, bend right, "x \mapsto (0 \to x)"'] \arrow[l, dashed, bend left, "x \mapsto (0 \to x)"]
\end{tikzcd},$$

then for any $F : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$ grouplike and additive, using proposition 3.80 we can immediately conclude. □

5.24 Lemma

Consider a stable ∞-category $\mathcal{C}$ and a grouplike additive functor $F : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$. Then

$$F(Q_n(\mathcal{C})) \simeq F(\mathcal{C})^{2n+1}.$$

Proof. We have showed in 4.26 that $Q_\bullet\mathcal{C}$ is a complete Segal object, and in 4.39 that there is a split Verdier square such that

$$Q_n(\mathcal{C}) \simeq Q_{n-1}(\mathcal{C}) \times_{Q_0(\mathcal{C})} Q_1(\mathcal{C}).$$

Therefore

$$F(Q_n\mathcal{C}) \simeq F(Q_{n-1}(\mathcal{C})) \times_{F(Q_0(\mathcal{C}))} F(Q_1(\mathcal{C}))$$

because F is additive. Iterating this we get

$$F(Q_n\mathcal{C}) \simeq F(Q_1(\mathcal{C})) \times_{F(\mathcal{C})} F(Q_1(\mathcal{C})) \times_{F(\mathcal{C})} \cdots \times_{F(\mathcal{C})} F(Q_1(\mathcal{C})).$$

There is a split Verdier sequence

$$\begin{tikzcd}
\mathcal{C} \arrow[r] & Q_1(\mathcal{C}) \arrow[l, dashed, bend right] \arrow[l, dashed, bend left] \arrow[r, "(x \leftarrow y \to z) \mapsto (y \to z)"] & \mathrm{Arr}(\mathcal{C}) \arrow[l, dashed, bend right, "(a \to b) \mapsto (a \leftarrow a \to b)"'] \arrow[l, dashed, bend left, "(a \to b) \mapsto (0 \leftarrow a \to b)"]
\end{tikzcd}$$

But then we have, by proposition 3.80,

$$F(Q_1(\mathcal{C})) \simeq F(\mathcal{C}) \times F(\mathrm{Arr}(\mathcal{C})) \simeq F(\mathcal{C})^3.$$

By plugging this above we get

$$\begin{aligned} &F(Q_n\mathcal{C}) \simeq \\ &\simeq F(Q_1(\mathcal{C})) \times_{F(\mathcal{C})} F(Q_1(\mathcal{C})) \times_{F(\mathcal{C})} \cdots \times_{F(\mathcal{C})} F(Q_1(\mathcal{C})) \\ &\simeq F(\mathcal{C})^3 \times_{F(\mathcal{C})} F(\mathcal{C})^3 \times_{F(\mathcal{C})} \cdots \times_{F(\mathcal{C})} F(\mathcal{C})^3 \\ &\simeq F(\mathcal{C})^{2n+1}. \end{aligned}$$

□

In section 5.3, we showed that k is the initial grouplike additive functor under core. As we already claimed in 5.10, more than this is true: for any additive functor $F : \mathbf{Cat}^{\mathbf{st}}_\infty \to \mathbf{An}$, $\Omega|\mathrm{Span}^F(\bullet)|$ is the grouplike additive functor under F.

Theorem D (Universality - Blumberg, Gepner, Tabuada)
The inclusion functor

$$\mathbf{Fun}^{grp}(\mathbf{Cat}^{\mathbf{st}}_\infty, \mathbf{An}) \hookrightarrow \mathbf{Fun}^{add}(\mathbf{Cat}^{\mathbf{st}}_\infty, \mathbf{An})$$

admits a left adjoint $(\bullet)^{grp}$, given by

$$F^{grp} \simeq \Omega|\mathrm{Span}^F(\bullet)|$$

In particular,

$$\mathrm{core}^{\,grp} \hookrightarrow k,$$

i.e. the projective anima class functor $k : \mathbf{Cat}^{\mathbf{st}}_\infty \to \mathbf{An}$ is the initial grouplike additive functor under the anima-core functor core : $\mathbf{Cat}^{\mathbf{st}}_\infty \to \mathbf{An}$.

Therefore, for any $F : \mathbf{Cat}^{\mathbf{st}}_\infty \to \mathbf{An}$ additive functor, we have a functor F^{grp} which is the initial grouplike functor under F, i.e

$$\mathrm{unit} : F \Rightarrow F^{grp}$$

has the property that, for any other grouplike additive functor G $\mathbf{Cat}^{\mathbf{st}}_\infty \to \mathbf{An}$,

$$\mathrm{Nat}(F, G) \simeq \mathrm{Nat}(F^{grp}, G).$$

5.25 Remark
We proved in theorem C that, for any additive functor $F : \mathbf{Cat}^{\mathbf{st}}_\infty \to \mathbf{An}$, $|\mathrm{Span}^F(\bullet)|$ is additive, and consequently also $\Omega|\mathrm{Span}^F(\bullet)|$ is additive. In order to prove the universality theorem, we know need a more detailed study of the functors Ω and $|\bullet|$.

5.26 Reference ([Heba, Cor. II.21])
The functor $\Omega : \mathrm{CMon}(\mathbf{An}) \to \mathrm{CMon}(\mathbf{An})$ actually takes values in $\mathrm{CGrp}(\mathbf{An})$.

By this it should be clear that the functor $\Omega|\mathrm{Span}^{F}(\bullet)|$ is grouplike and additive.

It is maybe the right moment to add some more information about the functor. There exists a functor

$$B = |\bullet| : \mathrm{CMon}(\mathbf{An}) \to \mathrm{CMon}(\mathbf{An})$$

which is a left adjoint to Ω. Also B takes values in $\mathrm{CGrp}(\mathbf{An})$, and when restricted to a functor $\mathrm{CGrp}(\mathbf{An}) \to \mathrm{CMon}(\mathbf{An})$ the functor B is fully faithful. Furthermore, the functor

$$\Omega B : \mathrm{CMon}(\mathbf{An}) \to \mathrm{CGrp}(\mathbf{An})$$

forms a left adjoint to the inclusion

$$\mathrm{CGrp}(\mathbf{An}) \hookrightarrow \mathrm{CMon}(\mathbf{An})$$

Proof of Universality. What we want to prove is that the functor

$$\Omega|\mathrm{Span}^{-}(\bullet)| : \mathbf{Fun}^{add}(\mathbf{Cat}_{\infty}^{\mathbf{st}}, \mathbf{An}) \to \mathbf{Fun}^{add,grp}(\mathbf{Cat}_{\infty}^{\mathbf{st}}, \mathbf{An})$$

is a left adjoint to the inclusion

$$\mathbf{Fun}^{add,grp}(\mathbf{Cat}_{\infty}^{\mathbf{st}}, \mathbf{An}) \hookrightarrow \mathbf{Fun}^{add}(\mathbf{Cat}_{\infty}^{\mathbf{st}}, \mathbf{An}).$$

To do this, we want to apply [Lur09, Prop. 5.2.7.4] to the functor

$$L \simeq \Omega|\mathrm{Span}^{(-)}(\bullet)| : \mathbf{Fun}^{add}(\mathbf{Cat}_{\infty}^{\mathbf{st}}, \mathbf{An}) \to \mathbf{Fun}^{add,grp}(\mathbf{Cat}_{\infty}^{\mathbf{st}}, \mathbf{An}).$$

Therefore we just have to prove there is a natural transformation

$$\eta : \mathrm{id} \Rightarrow L$$

such that

$$\eta_L F : LF \to LLF$$

and

$$L\eta_F : L \to LLF$$

are equivalences for all $F \in \mathbf{Fun}^{add}(\mathbf{Cat}_{\infty}^{\mathbf{st}}, \mathbf{An})$.
Consider map

$$\mathbf{Map}_{\mathrm{Span}^{F}(\mathcal{C})}(0,0) \to \mathbf{Map}_{|\mathrm{Span}^{F}(\mathcal{C})|}(0,0),$$

which is natural in F.

- The right hand side is equivalent to $\Omega|\mathrm{Span}^F(\mathcal{C})|$.
- We claim the left hand side is equivalent to $F(\mathcal{C})$. Consider the square

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & Q_1(\mathcal{C}) \arrow[d, "{(d_1, d_0)}"] \\ 0 \arrow[r] & Q_0(\mathcal{C}) \times Q_0(\mathcal{C}) \end{tikzcd}.$$

This is a split Verdier square. The fact that it is cartesian should be clear. The adjoints to (d_1, d_0) coincide and are given by

$$(x, y) \in Q_0(\mathcal{C}) \times Q_0(\mathcal{C}) \to x \leftarrow x \oplus y | y \in Q_1(\mathcal{C}).$$

It should be clear this are fully faithful, for example from

$$(d_1, d_0)(x \leftarrow x \oplus y \to y) \simeq (x, y).$$

Applying F additive functor we obtain a cartesian square

$$\begin{tikzcd} F(\mathcal{C}) \arrow[r] \arrow[d] & F(Q_1(\mathcal{C})) \arrow[d] \\ 0 \arrow[r] & F(Q_0(\mathcal{C})) \times F(Q_0(\mathcal{C})) \end{tikzcd};$$

by the description of mapping anima of associated category to a Segal anima, we get

$$F(\mathcal{C}) \simeq \mathbf{Map}_{\mathrm{Span}^F(\mathcal{C})}(0, 0).$$

Now the map becomes

$$F(\mathcal{C}) \simeq \mathbf{Map}_{\mathrm{Span}^F(\mathcal{C})}(0, 0) \to \mathbf{Map}_{|\mathrm{Span}^F(\mathcal{C})|}(0, 0) \simeq \Omega|\mathrm{Span}^F(\mathcal{C})|;$$

since it is natural in both F and $\mathcal{C}$, we define

$$\eta_F : F(\bullet) \simeq \mathbf{Map}_{\mathrm{Span}^{(F)}(\bullet)}(0, 0) \to \mathbf{Map}_{|\mathrm{Span}^F(\bullet)|}(0, 0) \simeq \Omega|\mathrm{Span}^F(\bullet)|.$$

Theorem 5.18 implies

$$\eta L, L\eta : L \Rightarrow L \circ L$$

are both equivalences.

Therefore L is a left adjoint to

$$\mathrm{Im}(L) \hookrightarrow \mathrm{Fun}^{add}(\mathbf{Cat}^{\mathbf{st}}_\infty, \mathbf{An}).$$

$\mathrm{Im}(L) \simeq \mathrm{Fun}^{add,grp}(\mathbf{Cat}^{\mathbf{st}}_\infty, \mathbf{An})$ because

- For sure the essential image is contained in the grouplike additive functors as checked;
- Every grouplike additive functor F is equivalent to $\Omega|\mathrm{Span}^F(\bullet)|$ by 5.18.

$\square$

5.5.1 *Co-representability.*

5.27 Lemma
The evaluation at the *sphere spectrum* $\mathbf{S} \in \mathbf{Spectra}^{\omega}$ exhibits $\mathbf{Spectra}^{\omega}$ as co-representing the anima core functor

$$\mathrm{core} : \mathbf{Cat}_{\infty}^{\mathbf{st}} \to \mathbf{An},$$

this means

$$\mathbf{Map}(\mathbf{Spectra}^{\omega}, \mathcal{D}) \simeq \mathrm{core}\,(\mathcal{D})$$

for any stable ∞-category $\mathcal{D}$.

Recall a functor $F : \mathcal{C} \to \mathcal{D}$ between two ∞-categories $\mathcal{C}$ to $\mathcal{D}$, where $\mathcal{C}$ has all pushouts, is called EXCISIVE if it takes pushouts to pullbacks.

Proof. Recall that the ∞-category of finite animae is given by $\mathbf{An}^{fin} \simeq L_{whe}(CW^{fin})$, i.e. it is the localisation of finite CW-complex (i.e. CW-complexes with finite number of cells) at equivalences. Then the ∞-category of finite spectra can be defined as

$$\mathbf{Spectra}^{fin} := SW(\mathbf{An}^{fin}),$$

i.e. as the Spanier-Whitehead stabilization of finite animae. For spectra we have

$$\mathbf{Spectra}^{fin} \simeq \mathbf{Spectra}^{\omega},$$

(but not for animae where

$$\mathbf{An}^{fin} \subsetneq \mathbf{An}^{\omega}\,.)$$

For any $\mathcal{D}$ stable ∞-category,

$$\begin{aligned}
&\mathbf{Fun}^{\mathrm{ex}}(\mathbf{Spectra}^{fin}, \mathcal{D}) \simeq \\
&\simeq \mathbf{Fun}^{rex}(\mathbf{An}_*^{fin}, \mathcal{D}) \\
&\simeq \mathrm{Excisive}(\mathbf{An}_*^{fin}, \mathcal{D}) \\
&\simeq \mathrm{Sp}(\mathcal{D}) := \lim(\cdots \to \mathcal{D} \to \mathcal{D}) \simeq \mathcal{D};
\end{aligned}$$

where

- The first equivalence comes from [Spectral Algebraic Geometry, Lurie, C.1.1.7];
- The second equivalence comes from [HA, rmk. 1.4.2.2];
- The last two equivalences comes from [HA, section. 1.4.2].

Then we have

$$\mathbf{Map}_{\mathbf{Cat}_\infty^{\mathbf{st}}}(\mathbf{Spectra}^{fin}, \mathcal{D}) \simeq \text{core } \mathbf{Fun}^{\text{ex}}(\mathbf{Spectra}^{fin}, \mathcal{D}) \simeq \text{core } \mathcal{D}$$

naturally in $\mathcal{D}$. Then by the Yoneda lemma

$$\mathbf{Map}_{\mathbf{Cat}_\infty^{\mathbf{st}}}(\mathbf{Spectra}^{fin}, \bullet) \simeq \text{core }(\bullet).$$

□

We have proved in D that there exists a groupification functor

$$(-)^{grp} \simeq \Omega - \bullet Q| : \mathbf{Fun}^{add}(\mathbf{Cat}_\infty^{\mathbf{st}}, \mathbf{An}) \to \mathbf{Fun}^{grp,add}(\mathbf{Cat}_\infty^{\mathbf{st}}, \mathbf{An})$$

which is left-adjoint to the inclusion

$$\mathbf{Fun}^{grp,add}(\mathbf{Cat}_\infty^{\mathbf{st}}, \mathbf{An}) \hookleftarrow \mathbf{Fun}^{add}(\mathbf{Cat}_\infty^{\mathbf{st}}, \mathbf{An}).$$

By the co-Yoneda lemma, we have a functor

$$\text{co}\,よ : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{Fun}(\mathbf{Cat}_\infty^{\mathbf{st}}, \mathbf{An}), \quad \mathcal{C} \mapsto \mathbf{Map}_{\mathbf{Cat}_\infty^{\mathbf{st}}}(\mathcal{C}, \bullet) \simeq \text{core } \mathbf{Fun}^{\text{ex}}(\mathcal{C}, \bullet).$$

In the last lemma we have seen that core is co-represented by finite spectra (through this functor)
Also, the functor lifts to $\text{co}\,よ : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{Fun}^{add}(\mathbf{Cat}_\infty^{\mathbf{st}}, \mathbf{An})$ since, for any $\mathcal{C} \in \mathbf{Cat}_\infty^{\mathbf{st}}$, $\mathbf{Map}(\mathcal{C}, \bullet)$ preserves all limits. But then we have a "Yoneda-like" functor defined as

$$g\,よ^{grp} : \mathbf{Cat}_\infty^{\mathbf{st}} \xrightarrow{\text{co}\,よ} \mathbf{Fun}^{add}(\mathbf{Cat}_\infty^{\mathbf{st}}, \mathbf{An}) \xrightarrow{\bullet^{grp}} \mathbf{Fun}^{add,grp}(\mathbf{Cat}_\infty^{\mathbf{st}}, \mathbf{An}),$$

$$\begin{aligned}\mathcal{C} \mapsto (\mathbf{Map}_{\mathbf{Cat}_\infty^{\mathbf{st}}}(\mathcal{C}, \bullet))^{grp} &\simeq \Omega|\mathbf{Map}_{\mathbf{Cat}_\infty^{\mathbf{st}}}(\mathcal{C}, Q.\bullet)| \\ &\simeq \Omega \operatorname*{colim}_{n \in \mathbf{N}} \mathbf{Map}_{\mathbf{Cat}_\infty^{\mathbf{st}}}(\mathcal{C}, Q_n \bullet)\end{aligned}$$

The functor K-theory anima functor k is co-represented by finite spectra under this functor; indeed

$$\mathbf{Spectra}^{fin} \mapsto \Omega \operatorname*{colim}_{n \in \mathbf{N}} \left(\text{core } \mathbf{Fun}^{\text{ex}}(\mathbf{Spectra}^{fin}, Q_n \bullet) \right) \simeq \Omega|\text{core } Q\bullet| \simeq k(\bullet).$$

Similarly the K-theory spectrum functor K is co-represented by finite spectra under the composition of this functor and Σ_+^∞.

In particular we obtain that for any grouplike additive functor $F : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$, then

$$\mathrm{Nat}(k, F) \simeq \mathrm{Nat}(\mathbf{Map}(\mathbf{Spectra}^{fin}, \bullet), F) \simeq F(\mathbf{Spectra}^{fin}),$$

by the Yoneda's lemma.

5.6 K-THEORY SPECTRA, ONCE MORE.

We will now proceed to construct an explicit delooping of the anima $k(\mathcal{C})$ by iterating the S– and Q– constructions.

5.6.1 *Through Iterated S-construction.*

Let us start with the one that first appeared and was first seen in [Wal85].

By the (product $\dashv$ internal Hom)-adjunction we can do it in two equivalent ways.

For any stable ∞-category $\mathcal{C}$, $S_\bullet(\mathcal{C})$ is a simplicial stable ∞-category, so we can apply the $S_\bullet$-construction to it. Indeed, we define

$$S_\bullet(S_\cdot(\mathcal{C})) \in \mathbf{Fun}(N(\boldsymbol{\Delta})^{op}, s\,\mathbf{An}),$$

$$S_\bullet(S_\cdot(\mathcal{C})) \subseteq \mathbf{Fun}(\mathrm{Arr}[\bullet], S_\cdot(\mathcal{C})) \subset \mathbf{Fun}(\mathrm{Arr}[\bullet], \mathbf{Fun}(\mathrm{Arr}[\cdot], \mathcal{C}))$$

just by asking the vanishing condition of the diagonal and the cartesian condition of the squares to be respected level-wise .
Iterating this construction we obtain a multi-simplicial ∞-category $S_\bullet^{(\bullet)}(\mathcal{C})$, where, for any $q \in \mathbf{N}$,

$$S_\bullet^{(q)} : (N(\boldsymbol{\Delta})^{op})^q \to \mathbf{Cat}_\infty^{\mathbf{st}}, \quad ([n_1], \dots, [n_q]) \mapsto \S_{n_q}(\cdots(S_{n_1}(\mathcal{C}))).$$

For the other construction, let us denote with

$$A_{i_1,j_1;\dots;i_q,j_q}$$

the values of a functor

$$A : \mathrm{Arr}([n_1]) \times \cdots \times \mathrm{Arr}([n_q]) \to \mathcal{C}$$

at $((i_1 \leqslant j_1), \dots, (i_q \leqslant j_q))$. Define $S_{n_1,\dots,n_q}(\mathcal{C})$ as the full sub-∞-category of

$$\mathbf{Fun}(\mathrm{Arr}([n_1]) \times \cdots \times \mathrm{Arr}([n_q]), \mathcal{C})$$

spanned by those functors $\mathcal{A}$ such that

- $A_{i_1,j_1;\dots;i_q,j_q} = 0$ whenever there exists $k \in \{1, \dots, q\}$ such that $i_k = j_k$.
- For any $(i_1 \leqslant j_1; \dots; i_q \leqslant j_q)$ in $\mathrm{Arr}([n_1]) \times \cdots \times \mathrm{Arr}([n_q])$, for any $1 \leqslant r \leqslant q$, and for any k such that $j_r \leqslant k \leqslant n_r$

$$\begin{tikzcd} A_{i_1,j_1;\dots;i_r,j_r;\dots;i_1,j_q} \arrow[r] \arrow[d] & A_{i_1,j_1;\dots;i_r,k;\dots;i_q,j_q} \arrow[d] \\ 0 \arrow[r] & A_{i_1,j_1;\dots;j_r,k;\dots;i_1,j_q} \end{tikzcd}$$

is cocartesian.

This builds up to be a multi-simplicial ∞-category $S_\bullet^{(\bullet)}\mathcal{C}$ where, for any $q \in \mathbf{N}$,

$$S_\bullet^{(q)}\mathcal{C} : (N(\mathbf{\Delta})^{op})^q \to \mathbf{Cat}_\infty, ([n_1], \dots, [n_q]) \mapsto S^{(q)}_{n_1,\dots,n_q}\mathcal{C}.$$

5.28 Remark

We can identify $S^{(0)}\mathcal{C}$ with $\mathcal{C}$ and $S_n^{(1)}\mathcal{C}$ with $S_n\mathcal{C}$.

5.29 Lemma

Consider an additive functor $F : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$. Then

$$\Omega|F(S(\bullet)) \simeq \Omega^i|F(S^i(\bullet))|,$$

where Ω^i means i-times Ω. Moreover, if F is group-like, then both corresponds to F.

Proof. We know for any additive functor $F : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$,

$$|FQ(\bullet)|$$

is again additive, by Waldahusen additivity theorem C. But then also

$$\begin{aligned} &|FS(\bullet)| \simeq \\ &\simeq |FS(\bullet)^{esd}| \\ &\simeq |FS^{esd}(\bullet)| \\ &\simeq |FQ(\bullet)| \end{aligned}$$

is also additive. Then we can apply this again to get that

$$||FS(-)|\Big(S(\bullet\Big)| \simeq ||FSS(\bullet)||,$$

is still additive. Iterating we get that

$$|FS^i(\bullet)|$$

is additive. This is also grouplike since

$$0 \simeq \pi_0(|FS_0S^{i-1}(\mathcal{C})| \to \pi_0|FSS^{i-1}(\mathcal{C})|$$

is surjective. Notice also that

$$|F(S^{esd}S^{esd}\cdots S^{esd}(\mathcal{C})| \simeq |F(S^i(\mathcal{C}))| \simeq |FQ^i(\mathcal{C})|.$$

We apply therefore theorem 5.18

$$|FS^i(\bullet)| \simeq$$
$$|FQ^i(\bullet)|$$
$$\simeq \Omega|Span^{|FQ^i(-)|}(\bullet)|$$
$$\simeq \Omega|FQ^{i+1}(\bullet)|$$
$$\simeq \Omega|FS^{i+1}(\bullet)|.$$

Then inductively

$$\Omega^{i+1}|FS^{i+1}(\bullet)| \simeq \Omega^i|FS^i(\bullet)| \simeq \cdots \simeq \Omega|FS(\bullet)|.$$

If F is already grouplike, we have proved

$$F(\bullet) \simeq \Omega|FQ(\bullet)| \simeq \Omega|FS^{esd}(\bullet)| \simeq \Omega|FS(\bullet)|.$$

□

Now for any $F : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$ additive functor descents to a pre-spectrum valued functor

$$\mathbf{Cat}_\infty^{\mathbf{st}} \to \mathrm{Pre}\,\mathbf{Spectra}, \quad \mathcal{C} \to \Big(F(\mathcal{C}), |F(S(\mathcal{C}))|, \Omega|F(S^2(\mathcal{C}))|, \dots\Big).$$

Since the construction is functorial in F, we get a functor

$$\mathbf{Fun}^{add}(\mathbf{Cat}_\infty^{\mathbf{st}}, \mathbf{An}) \to \mathbf{Fun}(\mathbf{Cat}_\infty^{\mathbf{st}}, \mathrm{Pre}\,\mathbf{Spectra}).$$

If F is group-like then

$$\Big(F(\mathcal{C}), |F(S(\mathcal{C}))|, \Omega|F(S^2(\mathcal{C}))|, \dots\Big)$$

is a spectrum, because

$$F(\bullet) \simeq \Omega|FS(\bullet)| \simeq \Omega^2|FS^2(\bullet)| \simeq \dots$$

by the lemma. We obtain than a functor

$$\mathbf{Fun}^{add,grp}(\mathbf{Cat}_\infty^{\mathbf{st}}, \mathbf{An}) \to \mathbf{Fun}(\mathbf{Cat}_\infty^{\mathbf{st}}, \mathbf{Spectra}).$$

5.30 Definition

Consider a stable ∞-category $\mathcal{C}$. We define the K-THEORY SPECTRUM of $\mathcal{C}$ as the connective spectrum

$$K(\mathcal{C}) \simeq \Big(k(\mathcal{C}), |k(S_\bullet\mathcal{C})|, \Omega|k(S_\bullet^2(\mathcal{C}))|, \Omega^2|k(S_\bullet^3(\mathcal{C}))|, \dots\Big).$$

Notice that we have a sequence of equivalence

$$k(\mathcal{C}) \simeq \Omega|k(S_\bullet(\mathcal{C}))| \simeq \Omega^2|k(S_\bullet^2(\mathcal{C}))| \simeq \dots,$$

or more explicitly

$$k(\mathcal{C}) \simeq \Omega|\Omega|\mathrm{core}(S_\bullet(\mathcal{C}))|| \simeq \Omega^2|\Omega|\mathrm{core}\,(S_\bullet^2(\mathcal{C}))|| \simeq \dots.$$

5.31 Remark

Another equivalent description of this K-spectrum, surely more aesthetically pleasing, is

$$K(\mathcal{C}) \simeq \Big(k(\mathcal{C}), k(S(\mathcal{C})), \Omega\, k(S^2(\mathcal{C})), \Omega^2\, k(S^3(\mathcal{C})), \dots \Big).$$

This are equivalent because level-wise, for $n > 0$, the animae are

$$\Omega^{n-1} |\Omega| \mathrm{core}\, (S^n_\bullet(\mathcal{C}))|| \quad \text{and} \quad \Omega^{n-1}\, k(S^n(\mathcal{C})) \simeq \Omega^n |\mathrm{core}\; S^n(\mathcal{C})|,$$

and exactly as in the proof of proposition 5.12 the limit-colimit interchange map, which take Ω out of the realization, is an equivalence. (Notice that this is not free, we actually need that argument.) These maps build up an equivalence between these two spectra.

5.32 Remark

For any stable ∞-category $\mathcal{C}$, the K-theory spectrum for $\mathcal{C}$ constructed in 5.2, call it $\widetilde{K}(\mathcal{C})$, and the K-theory spectrum $K(\mathcal{C})$ that we just defined in 5.30 are equivalent. Indeed the former one is

$$\widetilde{K}(\mathcal{C}) = \Big(k(\mathcal{C}), \Sigma\, k(\mathcal{C}), \Sigma^2\, k(\mathcal{C}), \dots \Big)$$

and the latter one is

$$K(\mathcal{C}) = \Big(k(\mathcal{C}), |k(S_\bullet \mathcal{C})|, \Omega |k(S^2_\bullet(\mathcal{C}))|, \Omega^2 |k(S^3_\bullet(\mathcal{C}))|, \dots \Big).$$

Now recalling that the infinite loop-space functor Ω^∞ is conservative when restricted to connective spectra we can immediately conclude that these are indeed equivalent.

5.6.2 *Through Iterated Q-construction.*

Let us talk now about the iterated Q-construction. The proof of the main lemma of subsection is (almost) identical to the one before, but we left it for completeness.

We denote with Q^i the i-FOLD Q-CONSTRUCTION. Q^i is a functor that at each stable ∞-category associates an i-simplicial ∞-category; more precisely

$$\begin{aligned} Q^i : \mathbf{Cat}^{\mathbf{st}}_\infty &\to \mathbf{Fun}(N(\mathbf{\Delta}^{op})^i, \mathbf{Cat}^{\mathbf{st}}_\infty) =: s^{(i)}\, \mathbf{Cat}^{\mathbf{st}}_\infty \\ \mathcal{C} &\mapsto \underbrace{Q_\bullet \cdots Q_\bullet}_{i-\text{times}} \mathcal{C}. \end{aligned}$$

There is, of course, a realization functor

$$|\bullet| : s^{(i)}\, \mathbf{Cat}_\infty \to \mathbf{An}, \quad X \mapsto \operatorname*{colim}_{(\mathbf{\Delta}^{op})^i}.$$

5.33 Lemma
Consider an additive functor $F : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$. Then

$$\Omega|F(Q(\bullet))| \simeq \Omega^i|F(Q^i(\bullet))|,$$

where Ω^i means i-times Ω. Moreover, if F is grouplike, then both sides correspond to F.

Proof. For any additive functor $F : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$,

$$|\mathcal{S}pan^F(\bullet)| \simeq |FQ(\bullet)|$$

is again additive, by Waldhausen additive theorem. Therefore, we can apply Waldhausen additivity theorem C again and get an additive functor

$$||FQ(-)|\Big(Q(\bullet)\Big)| \simeq ||FQQ(\bullet)||.$$

By iterating this process, we obtain that

$$|FQ^i(\bullet)|$$

is additive. This is also grouplike since, for any $\mathcal{C}$ stable ∞-category,

$$\begin{aligned}
&\pi_0|FQ^i(\mathcal{C})| \simeq \\
&\simeq \pi_0|FQQ^{i-1}(\mathcal{C})| \\
&\simeq \pi_0|FS^{esd}Q^{i-1}(\mathcal{C})| \\
&\simeq \pi_0|(FSQ^{i-1}(\mathcal{C}))^{esd}| \\
&\simeq \pi_0|FSQ^{i-1}(\mathcal{C})|
\end{aligned}$$

and

$$0 \simeq \pi_0(|FS_0Q^{i-1}(\mathcal{C})|) \to \pi_0|FSQ^{i-1}(\mathcal{C})|$$

is surjective. Therefore, applying theorem 5.18

$$\begin{aligned}
&|FQ^i(\bullet)| \simeq \\
&\simeq \Omega|\mathcal{S}pan^{|FQ^i(-)|}(\bullet)| \\
&\simeq \Omega|FQ^{i+1}(\bullet)|
\end{aligned}$$

But then

$$\begin{aligned}
&\Omega^{i+1}||FQ^{i+1}(\bullet)| \simeq \\
&\simeq \Omega^i|FQ^i(\bullet)| \\
&\simeq \Omega^{i-1}|FQ^{i-1}(\bullet)| \\
&\dots \\
&\simeq \Omega|FQ(\bullet)|
\end{aligned}$$

If F is also grouplike, then we already know

$$\Omega|FQ(\bullet)| \simeq F(\bullet).$$

□

For what we have just proved, any $\mathsf{F} : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$ additive functor descends to a pre-spectrum valued functor

$$\mathbf{Span}^{\mathsf{F}}(\bullet) : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathrm{Pre}\,\mathbf{Spectra}, \mathcal{C} \mapsto \Big(\mathsf{F}(\mathcal{C}), |\mathsf{F}(\mathsf{Q}(\mathcal{C}))|, \Omega|\mathsf{F}(\mathsf{Q}^2(\mathcal{C}))|, \dots\Big).$$

The construction is clearly functorial in F, so we obtain a functor

$$\mathbf{Span}^{(-)} : \mathbf{Fun}^{add}(\mathbf{Cat}_\infty^{\mathbf{st}}, \mathbf{An}) \to \mathbf{Fun}(\mathbf{Cat}_\infty^{\mathbf{st}}, \mathrm{Pre}\,\mathbf{Spectra}).$$

If F is grouplike, then

$$\mathbf{Span}^{\mathsf{F}}(\mathcal{C}) = \Big(\mathsf{F}(\mathcal{C}), |\mathsf{F}(\mathsf{Q}(\mathcal{C}))|, \Omega|\mathsf{F}(\mathsf{Q}^2(\mathcal{C}))|, \dots\Big)$$

is a spectrum, because

$$\mathsf{F}(\bullet) \simeq \Omega|\mathsf{F}\mathsf{Q}(\bullet)| \simeq \Omega^2|\mathsf{F}\mathsf{Q}^2(\bullet)| \simeq \dots$$

by the previous lemma. So we obtain

$$\mathbf{Span}^{(-)} : \mathbf{Fun}^{add,grp}(\mathbf{Cat}_\infty^{\mathbf{st}}, \mathbf{An}) \to \mathbf{Fun}(\mathbf{Cat}_\infty^{\mathbf{st}}, \mathbf{Spectra}).$$

5.34 Definition

Consider a stable ∞-category $\mathcal{C}$, we define (again) the K-theory spectrum, or projective spectrum class, of $\mathcal{C}$ as

$$\mathsf{K}(\mathcal{C}) := \mathbf{Span}^{\mathsf{k}}(\mathcal{C}).$$

The equivalence with the definition coming from iterated $\mathsf{S}_\bullet$-construction should be clear.

5.7 LOCALISATION.

This section goal is to prove the following theorem.

Theorem E (Localisation Theorem)
The algebraic K-functors

$$k : \mathbf{Cat}^{\mathbf{st}}_{\infty} \to \mathbf{An} \quad \text{and} \quad K : \mathbf{Cat}^{\mathbf{st}}_{\infty} \to \mathbf{Spectra}$$

are Verdier-localising.

By the corollary 3.91 of the Waldhausen fibration theorem, a grouplike additive functor $F : \mathbf{Cat}^{\mathbf{st}}_{\infty} \to \mathbf{An}$ is Verdier-localising if the following condition, which we will call $(\star)$, holds: for any Verdier sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$, the canonical map $|F(\mathbf{Fun}^{\mathcal{C}}([\bullet], \mathcal{D}))| \to F(\mathcal{E})$ is an equivalence.
Our goal is to prove that k satisfies $(\star)$. The following proposition tells us that to prove this it is enough to see that the core functor satisfies $(\star)$.

5.35 Proposition
For any additive functor $F : \mathbf{Cat}^{\mathbf{st}}_{\infty} \to \mathbf{An}$ satisfying $(\star)$, $|FQ(\bullet)|$ satisfies $(\star)$.

We get more than the result on k. Indeed, for any additive functor $F : \mathbf{Cat}^{\mathbf{st}}_{\infty} \to \mathbf{An}$ satisfying $(\star)$, both $|FQ(\bullet)|$ and $\Omega|FQ(\bullet)|$ are grouplike additive functor satisfying $(\star)$, hence grouplike Verdier-localising functor.

To prove the proposition, we need the following lemma.

5.36 Lemma
Verdier sequences are preserved by the functor

$$\mathbf{Fun}(N(P), \cdot) : \mathbf{Cat}^{\mathbf{st}}_{\infty} \to \mathbf{Cat}^{\mathbf{st}}_{\infty}$$

for any poset P.

Proof. More generally, one can call an ∞-category $\mathcal{I}$ STRONGLY FINITE if

- $\mathcal{I}$ is finite, i.e. categorically equivalent to a finite simplicial set, so a a simplicial set with finitely many non-degenerate simplices.
- For any i, j objects of $\mathcal{I}$, then $\mathbf{Map}_{\mathcal{I}}(i, j)$ is a finite anima.

In particular, any poset is strongly finite. Denote

$$\mathcal{C}^{\mathcal{I}} := \mathbf{Fun}(\mathcal{I}, \mathcal{C}), \quad \mathcal{C}_{\mathcal{I}} := \mathbf{Fun}(\mathcal{I}^{\mathrm{op}}, \mathcal{C}),$$

so that we have functors $(\bullet)^{\mathcal{I}} : \mathbf{Cat}^{\mathbf{st}}_{\infty} \to \mathbf{Cat}^{\mathbf{st}}_{\infty}$, and $(\bullet)_{\mathcal{I}} : \mathbf{Cat}^{\mathbf{st}}_{\infty} \to \mathbf{Cat}^{\mathbf{st}}_{\infty}$. Then

$$(\bullet)_{\mathcal{I}} \dashv (\bullet)^{\mathcal{I}} \dashv (\bullet)_{\mathcal{I}}.$$

Indeed, for the first adjunction we want to prove $\mathbf{Map}(\mathcal{C}_{\mathcal{I}}, \mathcal{D}) \simeq \mathcal{M}ap(\mathcal{C}, \mathcal{D}^{\mathcal{I}})$. We have

$$\begin{aligned} &\mathbf{Map}(\mathcal{C}_{\mathcal{I}}, \mathcal{D}) \simeq \text{core } \mathbf{Fun}(\mathcal{C}_{\mathcal{I}}, \mathcal{D}) \\ &\simeq \text{core } \mathbf{Fun}(\mathbf{Fun}(\mathcal{I}^{op}, \mathcal{C}), \mathcal{D}) \\ &\simeq \text{core } \mathbf{Fun}(\mathcal{I}, \mathbf{Fun}(\mathcal{C}, \mathcal{D})) \end{aligned}$$

and

$$\begin{aligned} &\mathbf{Map}(\mathcal{C}, \mathcal{D}^{\mathcal{I}}) \simeq \text{core } \mathbf{Fun}(\mathcal{C}, \mathcal{D}^{\mathcal{I}}) \\ &\simeq \text{core } \mathbf{Fun}(\mathcal{C}, \mathbf{Fun}(\mathcal{I}, \mathcal{D})) \\ &\simeq \text{core } \mathbf{Fun}(\mathcal{I}, \mathbf{Fun}(\mathcal{C}, \mathcal{D})) \end{aligned}$$

For the other adjunction see that $(\bullet)^{\mathcal{I}^{op}} \simeq (\bullet)_{\mathcal{I}}$, so we obtain $(\bullet)^{\mathcal{I}^{op}} \dashv (\bullet)_{\mathcal{I}^{op}}$ and by choosing $(\mathcal{I}^{op})$ as $\mathcal{I}$ we are done. □

Proof of Proposition 5.35. Suppose F is a grouplike additive functor satisfying $(\star)$. We want to prove

$$||FQ|(\mathbf{Fun}^{\mathcal{C}}([\bullet], \mathcal{D}))| \to |FQ|(\mathcal{E}), \quad \text{i.e.} \quad ||F(\mathbf{Fun}^{Q\mathcal{C}}([\bullet], Q\mathcal{D}))|| \to |FQ(\mathcal{E})|$$

is an equivalence. This is the same as proving that in each simplicial degree (of the Q-construction), the map of animae

$$|F(\mathbf{Fun}^{Q_k\mathcal{C}}([\bullet], Q_k\mathcal{D}))| \to FQ_k(\mathcal{E}).$$

is an equivalence. By the lemma 5.36,

$$Q_k\mathcal{C} \to Q_k\mathcal{D} \to Q_k\mathcal{E}$$

is also a Verdier sequence, indeed

$$Q_k\bullet \simeq \mathbf{Fun}(J_k, \bullet).$$

Since F satisfies $(\star)$ and $Q_k\mathcal{C} \to Q_k\mathcal{D} \to Q_k\mathcal{E}$ is a split Verdier sequence, we get that

$$|F(\mathbf{Fun}^{Q_k\mathcal{C}}([\bullet], Q_k\mathcal{D}))| \to F(Q_k\mathcal{E})$$

is an equivalence. □

Now, if the core functor satisfies $(\star)$, then also the K-theory functor satisfies $(\star)$. Consider a stable categories $\mathcal{D}$ such that $\mathcal{C}$ and a stable sub-∞-category of $\mathcal{D}$. Then we get

$$\text{core } \mathbf{Fun}^{\mathcal{C}}([\bullet], \mathcal{D}) \simeq \text{core } \mathbf{Fun}([\bullet], \mathcal{D}_{\mathcal{C}}) \simeq \mathbf{Map}_{\mathbf{Cat}_{\infty}}([\bullet], \mathcal{D}_{\mathcal{C}}) \simeq N^r(\mathcal{D}_{\mathcal{C}})$$

where $\mathcal{D}_{\mathcal{C}}$ is the (wide) sub-∞-category of $\mathcal{D}$ spanned by equivalences modulo $\mathcal{C}$ in $\mathcal{D}$.
It is well known fact (since it already holds for any simplicial anima) that the canonical map $|N^r(\mathcal{D}_{\mathcal{C}})| \to |\mathcal{D}_{\mathcal{C}}|$ is an equivalence. Therefore, there is an equivalence

$$|\text{core}\,\mathbf{Fun}^{\mathcal{C}}([\bullet], \mathcal{D})| \simeq |\mathcal{D}_{\mathcal{C}}|.$$

The last piece of the puzzle consists in proving that $|\mathcal{D}_{\mathcal{C}}| \simeq \text{core}\,(\mathcal{D}/\mathcal{C})$.

5.37 Proposition
Consider a stable ∞-category $\mathcal{D}$, a stable sub-∞-category $\mathcal{C} \subset \mathcal{D}$, and $\mathcal{D}_{\mathcal{C}}$ the ∞-category of equivalence modulo $\mathcal{C}$. Then

$$|\mathcal{D}_{\mathcal{C}}| \to \text{core}\,(\mathcal{D}/\mathcal{C})$$

is faithful. If, furthermore, $\mathcal{C} \hookrightarrow \mathcal{D}$ is a Verdier inclusion, then it is an equivalence.

This proposition is a special case of the following proposition. Once we proved this following proposition, by taking $S = \mathcal{D}_{\mathcal{C}}$ we are done.

5.38 Proposition
Consider $\mathcal{D}$ an ∞-category and S a sub-∞-category which is closed under 2-out-of-3 and pushouts in $\mathcal{D}$. Then

$$|S| = S[S^{-1}] \to \mathcal{D}[S^{-1}]$$

is faithful. Also, the following are equivalent

(i) $|S| \hookrightarrow \text{core}\,\mathcal{D}[S^{-1}]$ is fully faithful.

(ii) The morphisms of S satisfy 2-out-of-6 in $\mathcal{D}$.

(iii) A morphism of $\mathcal{D}$ lies in S if and only if its source and target are in S and it is invertible in $\mathcal{B}[S^{-1}]$.

Proof of the Proposition 5.38. The proof of the theorem is based on the following formula.

5.39 Reference ([Cis19, Sec. 7.2] or [Nui16])
Consider an ∞-category $\mathcal{D}$ and a sub-∞-category S of $\mathcal{D}$ such that S contains all the equivalences in $\mathcal{D}$ and it is closed under pushouts in $\mathcal{D}$ (i.e. any pushout in $\mathcal{D}$ of a morphism in S exists and the corresponding map lies in S).
Then, the canonical map

$$\text{colim}_{(y \to y') \in y/S}\, \mathbf{Map}_{\mathcal{D}}(x, y') \to \mathbf{Map}_{\mathcal{D}[S^{-1}]}(x, y)$$

is an equivalence, where y/S is the full subcategory of y/D spanned by maps in S.
Furthermore, this gives us an equivalence

$$\mathbf{Map}_{\mathcal{D}[S^{-1}]}(x, y) \simeq |x/t_y|,$$

where $t_y : y/S \to D$ is the functor mapping arrows to their target. This equivalence, which arise inside the proof, describes

$$\mathbf{Map}_{\mathcal{D}[S^{-1}]}(x, y)$$

as an anima of zig-zags

$$x \to z \leftarrow y$$

with $z \leftarrow y \in S$. This gives the homotopy categories of the localisation a calculus of fractions.

Apply the formula to $\mathcal{D} \supset S$ and $S \supset S$. We obtain that the canonical map

$$\mathbf{Map}_{S[S^{-1}]}(x, y) \to \mathbf{Map}_{\mathcal{D}[S^{-1}]}(x, y)$$

is induced by a map between colimits

$$\operatorname{colim}_{(y \to y') \in y/S} \mathbf{Map}_S(x, y') \to \operatorname{colim}_{(y \to y') \in y/S} \mathbf{Map}_{\mathcal{D}}(x, y');$$

this map is induced by inclusion maps

$$\mathbf{Map}_S(x, y') \to \mathbf{Map}_{\mathcal{D}}(x, y')$$

for any $y \to y' \in y/S$. Therefore it is a direct colimit of faithful inclusions, so it is a faithful map.

To prove the equivalence between the three condition, let us start by noticing that all three are conditions on the respective homotopy categories.

(i) $\Rightarrow$ (iii) Take f with source and target in S which becomes invertible in $\mathcal{D}[S^{-1}]$. Then for (i), it is represented by a zig-zag in $\mathrm{Ho}(S)$. By calculus of fractions and 2-out-of-3, we conclude that f does belong to S.

(iii) $\Rightarrow$ (ii) Equivalences satisfy 2-out-of-6.

(ii) $\Rightarrow$ (i) Assume the morphisms of S satisfy 2-out-of-6. We want to show the $|S| \hookrightarrow D[S^{-1}]$ is full; so take f invertible morphism in $B[S^{-1}]$ between objects of S. We want to show that f is represented by a zig-zag in S. Assume (we clearly can) that f is a morphisms in B. By calculous of fractions, f is a split monomorphism in $B[S^{-1}]$ if and only if there exists g in B such that gf lies in S. Since f is an equivalence in $B[S^{-1}]$, so it is g. Applying this argument to g we find h in B such that hg lies in S. Then by 2-out-of-6, f belongs to S.

□

Proof of the Proposition 5.37. First, we want to prove

$$|\mathcal{D}_{\mathcal{C}}| \to \mathrm{core}\,(\mathcal{D}/\mathcal{C})$$

is faithful by using proposition 5.38. Take $S = \mathcal{D}_{\mathcal{C}}$. $\mathcal{D}_{\mathcal{C}}$ is closed under pushouts; take $f : a \to b \in \mathcal{D}_{\mathcal{C}}$, then we have a diagram

$$\begin{tikzcd} a \arrow[r] \arrow[d] & a' \arrow[r] \arrow[d] & 0 \arrow[d] \\ b \arrow[r] & b' \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r] & c \in \mathcal{C} \end{tikzcd}$$

where the left and external square are cocartesian. By pasting law of pushouts the right square is cocartesian, hence $a' \to b' \in \mathcal{D}_{\mathcal{C}}$. Again by pasting laws, $\mathcal{D}_{\mathcal{C}}$ satisfies 2-out-of-3.

By the proposition

$$|\mathcal{D}_{\mathcal{C}}| \to \mathcal{D}/\mathcal{C}$$

is faithful, and this functor must factor through the faithful functor

$$\mathrm{core}\,(\mathcal{D}/\mathcal{C}) \hookrightarrow \mathcal{D}/\mathcal{C}.$$

So we end up with a faithful functor

$$|\mathcal{D}_{\mathcal{C}}| \to \mathrm{core}\,(\mathcal{D}/\mathcal{C}).$$

Now assume $\mathcal{C} \hookrightarrow \mathcal{D}$ is a Verdier inclusion; in particular $\mathcal{C}$ is closed under retracts. Assume $f \in \mathcal{D}$ is an equivalence in $\mathcal{D}/\mathcal{C}$ (the condition on source and target does not impose something, since $\mathcal{D}_{\mathcal{C}}$ is a wide subcategory). Then is a retract of a morphism in $\mathcal{D}_{\mathcal{C}}$, but this must be already in it, since $\mathcal{C}$ is closed under retracts. □

5.40 Remark

As a consequence of proposition 5.37, we obtain that, for any dense stable inclusion of stable ∞-categories $\mathcal{C} \subset \mathcal{D}$, the anima $|\mathcal{D}_{\mathcal{C}}|$ is discrete. Moreover

$$\pi_0(|\mathcal{D}_{\mathcal{C}}|) \simeq \pi_0(\mathrm{core}\,(\mathcal{D}))/\pi_0(\mathrm{core}\,(\mathcal{C})) \simeq K_0(\mathrm{core}\,(\mathcal{D}))/\,K_0(\mathrm{core}\,(\mathcal{C})).$$

Proof of Localisation. We proved

$$\begin{aligned} &|\mathrm{core}\,\mathbf{Fun}^{\mathcal{C}}([\bullet], \mathcal{D})| \simeq \\ &\simeq |N^r(\mathcal{D}_{\mathcal{C}})| \\ &\simeq |D_{\mathcal{C}}| \\ &\simeq \mathrm{core}\,(\mathcal{D}/\mathcal{C}). \end{aligned}$$

So the core functor satisfies ($\star$). From the proposition 5.35, we conclude that $|\text{core } Q\bullet|$ also satisfies ($\star$). Therefore $|\text{core } Q|$ is a Verdier-localising functor, and so is k.
For K, recall that a fibre sequence of $\mathbf{E}_\infty$-groups gives rise to a fibre sequence of spectra if and only if it is surjective on π_0; any verdier projection induces isomorphisms in K_0, since $\pi_0|\text{core } Q(\mathcal{C})| = 0$ for any $\mathcal{C} \in \mathbf{Cat}_\infty^{\mathbf{st}}$ and using the fact that $|\text{core } Q|$ is localising. □

5.7.1 *Waldhausen Fibration Theorem.*

In a similar way to what we just proved, we could also prove the following.

5.41 Theorem (A version of Waldhausen's Fibration Theorem)
Consider a stable ∞-category $\mathcal{D}$ and stable sub-∞-category $\mathcal{C}$ of $\mathcal{D}$. Consider also a finite ∞-category K. Then the map

$$\Gamma : |\mathbf{Fun}^{\mathcal{C}}(K, \mathcal{D})| \to \mathbf{Fun}(K, \mathcal{D}/\mathcal{C}).$$

is faithful, i.e. it induces inclusions of path components on mapping-animae.
Moreover

(a) If $\mathcal{C} \hookrightarrow \mathcal{D}$ is a Verdier inclusion, then Γ is an equivalence onto

$$\text{core } \mathbf{Fun}(K, \mathcal{D}/\mathcal{C}).$$

(b) If $\mathcal{C}$ is dense in $\mathcal{D}$, then $|\mathbf{Fun}^{\mathcal{C}}(K, \mathcal{D})|$ is equivalent to the discrete group

$$K_0(\mathbf{Fun}(K, \mathcal{D}))/\, K_0(\mathbf{Fun}(K, \mathcal{C})).$$

5.8 RELATIVE Q-CONSTRUCTION.

5.42 Definition

Consider an exact functor of stable ∞-categories $f : \mathcal{C} \to \mathcal{D}$. We defined the RELATIVE Q-CONSTRUCTION $Q(f) \in s\,\mathbf{Cat}^{\mathrm{st}}_\infty$ as the pullback in $s\,\mathbf{Cat}^{\mathrm{st}}_\infty$ of

$$\begin{tikzcd} Q(f) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Null}(\mathcal{D}) \arrow[d, "d_0"] \\ Q(\mathcal{C}) \arrow[r, "f"] & Q(\mathcal{D}) \end{tikzcd}$$

We portray $Q_n(f)$ as the completion by taking pullbacks of the diagram

$$\begin{tikzcd}[column sep=tiny] & d \arrow[dl] \arrow[dr] & & f(c_{0,1}) \arrow[dl] \arrow[dr] & & f(c_{1,2}) \arrow[dl] \arrow[dr] & & f(c_{n-1,n}) \arrow[dl] \arrow[dr] & \\ 0 & & f(c_{0,0}) & & f(c_{1,1}) & & \cdots & & f(c_{n,n}) \end{tikzcd}$$

in $\mathcal{D}$, where the red-coloured part comes from an element of $Q_n(\mathcal{C})$ of the form

$$\begin{tikzcd}[column sep=tiny] & c_{0,1} \arrow[dl] \arrow[dr] & & c_{1,2} \arrow[dl] \arrow[dr] & & c_{n-1,n} \arrow[dl] \arrow[dr] & \\ c_{0,0} & & c_{1,1} & & \cdots & & c_{n,n} \end{tikzcd}.$$

5.43 Remark

The relative S-construction already appeared in Waldhausen's work [Wal85]. There he defined There, for and exact functor $f\,\mathcal{C} \to \mathcal{D}$ between Waldhausen 1-categories he defined $S(f)$

$$\begin{tikzcd} S(f) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{dec}(S(\mathcal{D}) \arrow[d, "d_0"] \\ S(\mathcal{C}) \arrow[r, "f"] & S(\mathcal{D}) \end{tikzcd},$$

and $S(f)$ turns out to be a Waldhausen 1-categories as well.

5.44 Corollary

Consider an exact functor of stable ∞-categories $f : \mathcal{C} \to \mathcal{D}$, and a grouplike additive functor $F : \mathbf{Cat}^{\mathrm{st}}_\infty \to \mathbf{An}$. Then there is a fibre sequence in $\mathrm{CGrp}(\mathbf{An})$

$$(\star) \quad |F(Q(f))| \to |F(Q(\mathcal{C}))| \to |F(Q(\mathcal{D}))|.$$

In particular there is a bifibre sequence

$$(\star\star) \quad F(\mathcal{C}) \to F(\mathcal{D}) \to |F(Q(f))|.$$

Proof. The square defining $Q(f)$ is split Verdier, being d_0 a split-Verdier projection. Moreover, we have already proved in 5.18 that the functor

$$F(d_0) : F(\mathrm{Null}(\mathcal{D})) \to F(Q(\mathcal{D}))$$

is equifibred. Therefore, by 3.90 we obtain a cartesian square

$$\begin{tikzcd}
{|F(Q(f))|} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {|F(\mathrm{Null}(\mathcal{D}))|} \arrow[d, "d_0"] \\
{|F(Q(\mathcal{C}))|} \arrow[r, "f"] & {|F(Q(\mathcal{D}))|}
\end{tikzcd}.$$

We also know that

$$|F(\mathrm{Null}(\mathcal{D}))| \simeq |0/\mathrm{Span}^F(\mathcal{D}) \simeq *,$$

therefore there is a fibre sequence

$$|F(Q(f))| \to ||F(Q(\mathcal{C}))| \to ||F(Q(\mathcal{D}))|.$$

Computing the pullback twice and using

$$\Omega|F(Q(\mathcal{C})| \simeq F(\mathcal{C}) \quad \text{and} \Omega|F(Q(\mathcal{D})| \simeq F(\mathcal{D}),$$

we obtain that $(\star\star)$ is a fibre sequence. To prove this is also a cofibre sequence, it is enough to show that

$$\pi_0(F(Q(\mathcal{C})) \simeq 0,$$

and we have already proved many times this fact. □

5.45 Proposition ([HLS22, Prop. 8.3])
Consider a fully faithful and exact functor $f : \mathcal{C} \to \mathcal{D}$ of stable ∞-categories. Then there is an equivalence of simplicial categories

$$\mathbf{Fun}^{\mathcal{C}}([\bullet], \mathcal{D}) \simeq Q(f)$$

which fits into a commutative diagram

$$\begin{tikzcd}
\mathrm{const}\mathcal{D} \arrow[r] \arrow[d] & \mathbf{Fun}^{\mathcal{C}}([\bullet], \mathcal{D}) \arrow[d] \\
Q(i) \arrow[r] \arrow[ur] & \mathrm{const}\mathcal{D}/\mathcal{C}
\end{tikzcd}.$$

5.46 Corollary
Consider a grouplike additive functor $F : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$. F is Verdier-localising if and only if, for all Verdier inclusions $i : \mathcal{A} \hookrightarrow \mathcal{B}$, the canonical map

$$|F(Q(i))| \to F(\mathcal{B}/\mathcal{A})$$

is an equivalence.

5.9 COFINALITY.

The goal of this section is to prove the following.

Theorem F (Cofinality theorem)
Let $\mathcal{C} \hookrightarrow \mathcal{D}$ be a dense inclusion of stable ∞-category. The map induced on K-theory fits into a fibre sequence

$$k(\mathcal{C}) \to k(\mathcal{D}) \to G,$$

where G is the abelian group $K_0(\mathcal{D})/K_0(\mathcal{C})$ regarded as discrete simplicial set.
In particular,

$$K_i(\mathcal{C}) \to K_i(\mathcal{D})$$

is an isomorphism for any $i > 0$, and there is a short exact sequence (as we already know from Thomason theorem 2.8)

$$0 \to K_0(\mathcal{C}) \to K_0(\mathcal{D}) \to K_0(\mathcal{D})/K_0(\mathcal{C}) \simeq \pi_0 \operatorname{core}(\mathcal{D})/\pi_0 \operatorname{core}(\mathcal{C}) \to 0.$$

There are two ways to prove the cofinality theorem. The first method, which is shorter, builds upon our work to prove the localisation theorem. The second method (which we will investigate in the next section) is more general and only relies on the existence of a grouplike additive functor under an additive functor. Both of our proofs strictly follows [HLS22].

Proof. In previous sections we have demonstrated that

$$|\operatorname{core} Q\bullet|$$

is a grouplike additive functor $\mathbf{Cat}^{\mathbf{st}}_\infty \to \mathbf{An}$. By applying 3.84 to the Verdier sequence

$$\mathcal{C} \to \mathcal{D} \to \mathcal{D}/\mathcal{C}$$

we obtain a fibre sequence as follows

$$|\operatorname{core} Q\mathcal{C}| \to |\operatorname{core} Q\mathcal{D}| \to ||\operatorname{core} Q\, \mathbf{Fun}^{\mathcal{C}}(\bullet, \mathcal{D})||.$$

Furthermore, as bisimplicial animae, we can identify

$$\operatorname{core} Q\, \mathbf{Fun}^{\mathcal{C}}(\bullet, \mathcal{D})$$

with

$$\operatorname{core} \mathbf{Fun}^{Q\mathcal{C}}(\bullet, Q\mathcal{D})$$

as we did in the last section. Using the identification above, we can also identify

$$|\operatorname{core} \mathbf{Fun}^{Q_k\mathcal{C}}(\bullet, Q_k\mathcal{D})| \simeq |N^r\, Q_k\mathcal{D}_{Q_k\mathcal{C}}| \simeq |Q_k\mathcal{D}_{Q_k\mathcal{C}}|.$$

We know $\mathcal{C} \to \mathcal{D}$ is dense, so $Q_n\mathcal{C} \to Q_n\mathcal{D}$ is also dense. This implies $Q_n\mathcal{D}/Q_n\mathcal{C} \simeq 0$. We also have an equivalence

$$|\mathcal{D}_{\mathcal{C}}| \xrightarrow{\simeq} \mathrm{core}\,(\mathcal{D}/\mathcal{C}).$$

According to proposition 5.40, $|Q_n(\mathcal{D})_{Q_n(\mathcal{C})}|$ is discrete with components $K_0(Q_n(\mathcal{D}))/K_0(Q_n(\mathcal{C}))$.
$K_0(Q(\mathcal{D}/\mathcal{C}))$ is the edgewise subdivision of $Bar(K_0(\mathcal{D}/\mathcal{C}))$. Therefore

$$||\mathrm{core}\ Q\,\mathbf{Fun}^{\mathcal{C}}([\bullet], \mathcal{D})|| \simeq |Bar(K_0(\mathcal{D})/K_0(\mathcal{C}))|$$

is an Eilenber-Mac Lane anima in degree 1. Now, looping the first sequence (and hence shifting the homotopy groups of Eilenber-Mac Lane anima the down by one), we obtain a fibre sequence

$$\Omega|\mathrm{core}\ Q\mathcal{C}| \to \Omega|\mathrm{core}\ Q\mathcal{D}| \to \Omega||\mathrm{core}\ Q\,\mathbf{Fun}^{\mathcal{C}}(\bullet, \mathcal{D})||,$$

which it is exactly the fibre sequence

$$k(\mathcal{C}) \to k(\mathcal{D}) \to \mathcal{K}(K_0(\mathcal{D})/\,K_0(\mathcal{C}), 0)$$

(where $\mathcal{K}$ denotes the Eilenberg-Mac Lane anima). $\square$

5.10 COFINALITY, ONCE MORE.

The second proof of the cofinality theorem is applicable to any additive functor $F : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$ and it provides a clearer reason for why the theorem is referred to as the cofinality theorem.

5.47 Definition
A map $F : X \to Y \in \mathrm{CMon}(\mathbf{An})$ is called COFINAL if

(a) It is an inclusion of a collection of path components, i.e.

$$\pi_0(F) : \pi_0(X) \to \pi_0(Y)$$

is an inclusion.

(b) For each $x \in \pi_0(X)$, there exists a $x' \in \pi_0(X)$ such that

$$x + x' \in \pi_0(Y).$$

A cofinal map is called DENSE if in addition

(c) The sequence of monoids

$$0 \to \pi_0(X) \to \pi_0(Y) \to \pi_0(Y)/\pi_0(X) \to 0$$

is exact.

or, equivalently, if

(c′) $x \in \pi_0(Y)$ belongs to $\pi_0(X)$ if there exists $y \in \pi_0(X)$ such that $x + y \in \pi_0(X)$.

To prove the cofinality theorem we need the following technical lemma.

5.48 Lemma ([HLS22, Lemma 7.5])
If $F : X \to Y \in \mathrm{CMon}(\mathbf{An})$ is a cofinal map, then its cofibre Y/X in $\mathrm{CMon}(\mathbf{An})$ is a discrete group.

5.49 Definition
An additive functor $F : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$ is called KAROUBIAN if

(i) it turns dense inclusion of stable ∞-categories dense map $F(\mathcal{A}) \to F(\mathcal{B}) \in \mathrm{CMon}(\mathbf{An})$ of commutative cartesian monoid in $\mathbf{An}$;

(ii) it preserves pullback squares in $\mathbf{Cat}_\infty^{\mathbf{st}}$ whose vertical maps are dense.

5.50 Remark
The core functor is Karoubian.

- Condition (ii) is satisfied because core commutes with all limits.
- Condition (i). Given $\mathcal{A} \to \mathcal{B}$ dense inclusion of stable ∞-categories core $(A) \to$ core (B) is clearly cofinal. Moreover, say there is b object of $\mathcal{B}$ (or better, an isomorphism class of objects of $\mathcal{B}$) and a in $\mathcal{A}$ such that $a \oplus b \in \mathcal{A}$; then $b \simeq \mathrm{fib}(a \oplus b \to a)$ belongs to $\mathcal{A}$ too, so core is cofinal.

Theorem G (General Cofinality Theorem)
Consider an additive and Karoubian functor $F : \mathbf{Cat}_{\infty}^{\mathbf{st}} \to \mathbf{An}$, and, as always, denote with F^{grp} the "group completion" $\Omega|FQ(\bullet)|$ of F. For any dense inclusions $\mathcal{A} \hookrightarrow \mathcal{B}$ of stable ∞-categories, the canonical map

$$F(\mathcal{B})/F(\mathcal{A}) \to F^{grp}(\mathcal{B})/F^{grp}(\mathcal{A})$$

is an equivalence. Therefore, F^{grp} induces isomorphisms

$$\pi_n F^{grp}(\mathcal{A}) \xrightarrow{\simeq} \pi_n F^{grp}(\mathcal{B})$$

for any $n > 0$, and a short exact sequence of abelian groups

$$0 \to \pi_0 F^{grp}(\mathcal{A}) \to F^{grp}(\mathcal{B}) \to \pi_0 F(\mathcal{B})/\pi_0 F(\mathcal{A}) \to 0,$$

where the last term is a discrete commutative monoid.

Proof Sketch. For the first part we want to prove

$$F(\mathcal{B})/F(\mathcal{A}) \to F^{grp}(\mathcal{B})/F^{grp}(\mathcal{A})$$

is an equivalence. Let us start by considering the sequence of dense inclusions

$$\mathcal{A} \hookrightarrow \mathcal{B} \hookrightarrow \mathrm{Idem}(\mathcal{A}) \simeq \mathrm{Idem}(\mathcal{B}).$$

By pasting law we obtain that all the square in the following diagram are cocartesians

$$\begin{tikzcd}
F(\mathcal{A}) \arrow[r] \arrow[d] & F(\mathcal{B}) \arrow[r] \arrow[d] & F(\mathrm{Idem}(\mathcal{A})) \arrow[d] \\
0 \arrow[r] & F(\mathcal{B})/F(\mathcal{A}) \arrow[r] \arrow[d] & F(\mathrm{Idem}(\mathcal{A}))/F(\mathcal{A}) \arrow[d] \\
 & 0 \arrow[r] & F(\mathrm{Idem}(\mathcal{A}))/F(\mathcal{B})
\end{tikzcd},$$

and therefore a cofibre sequence

$$F(\mathcal{B})/F(\mathcal{A}) \to F(\mathrm{Idem}(\mathcal{A}))/F(\mathcal{A}) \to F(\mathrm{Idem}(\mathcal{A}))/F(\mathcal{B}).$$

We have a similar result for F^{grp}, so we are reduced to consider the case consider the case $\mathcal{B} \simeq \mathrm{Idem}(\mathcal{A})$.

Next, we should prove that the functor $\widetilde{F} : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathrm{CMon}(\mathbf{An})$ given by

$$\widetilde{F}(\mathcal{A}) := F(\mathrm{Idem}(\mathcal{A}))/F(\mathcal{A})$$

- represents the quotient of the functors $F \circ \mathrm{Idem}$ and F in the ∞-category of additive functors;
- is grouplike.

(We already noticed that $F \circ \mathrm{Idem}$ is still additive, indeed $\mathrm{Idem} : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{Cat}_\infty^{\mathbf{st}}$ is right adjoint to the $(\bullet)^{min}$ functor and we still have the adjoints.) Furthermore, we should prove that $\widetilde{F^{grp}}$

- represents the quotient of functors $F^{grp} \circ \mathrm{Idem}$ and F^{grp} in the ∞-category of grouplike additive functors $\mathbf{Cat}_\infty^{\mathbf{st}} \to \mathrm{CGrp}(\mathbf{An})$.

To prove these two claims, it is enough to show that $\widetilde{F}$ and $\widetilde{F^{grp}}$ are both functors.

To prove the claim that $\widetilde{F}$ is additive we can combine proposition 3.76, lemma 3.77, and the splitting lemma to obtain that the split Verdier sequence

$$\mathcal{A} \to \mathrm{Arr}(\mathcal{A}) \to \mathcal{A}$$

is mapped by F' to a split fibre sequence. By lemma 5.48, it remains only to prove that the following sequence

$$(\star) \quad 0 \to \pi_0(\widetilde{F}(\mathcal{A})) \to \pi_0(\widetilde{F}(\mathrm{Arr}(\mathcal{A}))) \to \pi_0(\widetilde{F}(\mathcal{A})) \to 0$$

is an exact sequence of abelian groups. To prove this, notice first that short exact sequences of monoids are in particular cofibre sequences of monoids. By commuting quotients and using the fact that F and $F \circ \mathrm{Idem}$ are additive, we see that the sequence $(\star)$ is a cofibre sequence of abelian groups. In particular the sequence is right exact, and since the first map is split injective, the sequence is indeed split exact.

To prove the claim that $\widetilde{F^{grp}}$ is additive we want to prove that

$$(\star\star) \quad 0 \to \pi_n(\widetilde{F^{grp}}(\mathcal{A})) \to \pi_n(\widetilde{F^{grp}}(\mathrm{Arr}(\mathcal{A}))) \to \pi_n(\widetilde{F^{grp}}(\mathcal{A})) \to 0$$

is a short exact sequence for any $n \in \mathbf{N}$. For $n = 0$ it is proved as above; for $n > 0$ it follows from the fact that for any map $\mathcal{G} \to \mathcal{H}$ of commutative cartesian groups in $\mathbf{An}$

$$\mathrm{fib}(\mathcal{G} \to \mathcal{H}) \simeq \Omega(\mathcal{G}/\mathcal{H})$$

(by the usual argument in spectra and pasting laws.)

We proceed by noticing that the canonical map

$$F \circ \mathrm{Idem} \to F^{grp} \circ \mathrm{Idem}$$

is a group completion; indeed, for every grouplike additive functor G

$$\begin{aligned} \mathrm{Nat}(F \circ \mathrm{Idem}, G) &\simeq \mathrm{Nat}(F, G \circ (\bullet)^{min}) \\ &\simeq \mathrm{Nat}(F^{grp}, G \circ (\bullet)^{min}) \\ &\simeq \mathrm{Nat}(F^{grp} \circ \mathrm{Idem}, G). \end{aligned}$$

It follows that $\tilde{F} \to \tilde{F^{grp}}$ is a group completion, but $\tilde{F}$ is already grouplike, therefore the map must be an equivalence. □

We have not yet used the second condition for being a Karoubian functor, nor the density of the functor; just the first condition and the cofinality.

5.51 Corollary

For any an additive and Karoubian functor F, also F^{grp} is also Karoubian.

In particular if F^{grp} is Verdier-localising, then F^{grp} is Karoubi-localising.

Proof. The cofinality theorem immediately implies that F^{grp} sends dense functors of stable ∞-categories to dense maps of $\mathbf{E}_\infty$-monoid animae. Next we have to show that any cartesian square in $\mathbf{Cat}^{\mathbf{st}}_\infty$

$$\begin{tikzcd} \mathcal{C}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{D} \arrow[d] \\ \mathcal{C}' \arrow[r] & \mathcal{D} \end{tikzcd}$$

with dense inclusion as vertical maps is mapped to a cartesian square by F^{grp}. This is true for F (by assumption) and also for $\pi_0 F$. We can then deduce that canonical the map of quotient monoids

$$\pi_0 F(\mathcal{D}')/\pi_0 F(\mathcal{C}') \to \pi_0 F(\mathcal{D})/\pi_0 F(\mathcal{C})$$

is injective, by the density of the maps and the fact that the square (with $\pi_0 F$) is cartesian. By the cofinality theorem, this identifies with

$$\pi_0 F^{grp}(\mathcal{D}')/\pi_0 F^{grp}(\mathcal{C}') \to \pi_0 F^{grp}(\mathcal{D})/\pi_0 F^{grp}(\mathcal{C}).$$

We have therefore that the square with $\pi_0 F^{grp}$ is cartesian. Finally, by the cofinality theorem, F^{grp} maps the square to a pullback square in **An**. □

5.52 Corollary
The functor

$$k \circ \mathrm{Idem} : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$$

is Karoubi-localising.

5.53 Remark
On the other hand, the functor

$$k \circ (\bullet)^{min} : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$$

is not even Verdier-localising. A counter-example (presented in [HLS22]) is the following: the Verdier projection

$$D^p(\mathbf{Z}) \to D^p(\mathbf{Q})$$

with kernel the torsion complexes, does not yield an exact sequence on K-groups after minimalisation, since

$$k_1(D^p(\mathbf{Z})) \to k_1(D^p(\mathbf{Q}))$$

is not surjective.

5.54 Remark
Corollary 5.52 immediately implies the following. For any a stable ∞-category $\mathcal{D}$ and a dense stable ∞-subcategory $\mathcal{C} \subset \mathcal{D}$, Then, for any $i > 0$,

$$K_i(\mathcal{C}) \to K_i(\mathcal{D})$$

is an equivalence.

A

NON-CONNECTIVE K-THEORY OF STABLE INFINITY CATEGORIES

This appendix provides a concise overview of non-connective K-theory and its relevance to our study. To achieve this, we will present a construction similar to the axiomatic framework introduced by Schlichting [Sch06], which was further developed in [BGT13]. (However, our approach is considerably simpler and was developed based on a valuable discussion I had with Denis-Charlse Cisinski.) To maintain clarity and brevity, we will omit some proofs; nevertheless, these proofs should not be too challenging given the foundation we have already established and the discussion we are going to have throughout the chapter.

In this chapter, to keep the notation more clean, we are going to denote the idempotent completion of a stable ∞-category $\mathcal{C}$ as $\mathcal{C}^{\#}$ instead of $\mathrm{Idem}(\mathcal{C})$, for shortness. The idempotent completion functor will therefore be denote as

$$(\bullet)^{\#} = \mathrm{Idem} : \mathbf{Cat}_{\infty}^{\mathbf{st}} \to \mathbf{Cat}_{\infty}^{\mathbf{perf}}$$

and it will have all the properties discussed in 3.2.5.

A.1 Definition

Consider a stable ∞-category $\mathcal{C}$.We define the DERIVED CATEGORY of $\mathcal{C}$ as

$$\mathfrak{D}(\mathcal{C}) := \mathbf{Fun}^{\mathrm{lex}}(\mathcal{C}^{\mathrm{op}}, \mathbf{An}) \simeq \mathbf{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}}, \mathbf{Spectra}).$$

With this notation, we can now identify the idempotent completion of a stable ∞-category $\mathcal{C}$ with the ∞-category of compact objects in $\mathfrak{D}(\mathcal{C})$.

Let us outline some of the issues that give rise to non-connective K-theory.

- We have showed in 5.10 that $\mathrm{k} \circ (\bullet)^{\#}$ is a Karoubi-localising functor, i.e. that given a Karoubi square

$$\begin{tikzcd} \mathcal{D}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{D} \arrow[d] \\ \mathcal{C}' \arrow[r] & \mathcal{C} \end{tikzcd},$$

  $\mathrm{k} \circ (\bullet)^{\#}$ maps this into a cartesian square in $\mathrm{CGrp}(\mathbf{An})$

$$\begin{tikzcd}
k(\mathcal{D}'^{\#}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & k(\mathcal{D}^{\#}) \arrow[d] \\
k(\mathcal{C}'^{\#}) \arrow[r] & k(\mathcal{C}^{\#})
\end{tikzcd}.$$

It follows that the following square is cartesian in the ∞-category $\mathbf{Spectra}_{\geqslant 0}$ of connective spectra

$$\begin{tikzcd}
K(\mathcal{D}'^{\#}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & K(\mathcal{D}^{\#}) \arrow[d] \\
K(\mathcal{C}'^{\#}) \arrow[r] & K(\mathcal{C}^{\#})
\end{tikzcd}.$$

However, this is false when considering the diagram inside **Spectra**. (Indeed the inclusion $\mathbf{Spectra}_{\geqslant 0} \hookrightarrow \mathbf{Spectra}$ is right exact, not left exact.)

- Another reason comes from algebraic geometry. In this context the non-connective K functor have been known from a long time; for example, it is the Bass K-theory functor "K^B" considered by Thomason and Trobaugh in [TT90]. For example, considering a Zarinski cover of a scheme X given by open subschemes U and V, by Mayer-Vietoris one obtains a long exact sequence

$$\begin{aligned} K_i(X) \to K_i(U) \oplus K_i(V) \to K_i(U \cap V) \to \cdots \to \\ \to K_0(X) \to K_0(U) \oplus K_0(V) \to K_0(U \cap V) \end{aligned}$$

that however results incomplete.

Our aim is to construct a functor $\mathbf{K} : \mathbf{Cat}^{\mathbf{st}}_{\infty} \to \mathbf{Spectra}$ that is Karoubi-localising and such that,for any stable ∞-category $\mathcal{C}$,

$$\Omega^{\infty}\mathbf{K}(\mathcal{C}) \simeq k(\mathrm{Idem}(\mathcal{C})).$$

Once we have constructed the spectrum $\mathbf{K}(\mathcal{C})$ we will denote with $\mathbf{K}_n(\mathcal{C})$ its homotopy groups.

We can start by defining the 0-th non-connective K-group of a stable ∞-category $\mathcal{C}$ as

$$\mathbf{K}_0(\mathcal{C}) := K_0(\mathcal{C}^{\#}) \in \mathcal{A}b.$$

A.2 Remark

By the Thomason theorem 2.8, we know that $K_0(\mathcal{C})$ is a subgroup of $\mathbf{K}_0(\mathcal{C}^\#)$ since $\mathcal{C}$ is a replete full stable sub-∞-category of $\mathcal{C}^\#$.

A.3 Definition

A stable ∞-category $\mathcal{C}$ is called FLASQUE if there exist an exact functor $T : \mathcal{C} \to \mathcal{C}$ such that there is a cofibre sequence

$$Tx \to Tx \to x \in \mathcal{C},$$

naturally in $x \in \mathcal{C}$.

By the additivity theorem 1, this implies that

$$[Tx] \simeq [Tx] \oplus [x],$$

from which immediately follows

$$[x] = 0 \in K(\mathcal{C})$$

for any $x \in \mathcal{C}$, so that $k(\mathcal{C})$ vanishes.

A.4 Proposition (Eilenberg Swindle)

Consider a stable ∞-category $\mathcal{C}$ with countable coproducts. Then, for any grouplike additive functor $F : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$,

$$F(\mathcal{C}) \simeq 0.$$

In particular, such ∞-categories are all flasque and their K-theory always vanish.

Proof. Consider the functor

$$T : \mathcal{C} \to \mathcal{C}, \quad x \mapsto \bigoplus_{\mathbf{N}} x.$$

Then there is cofibre sequence of functors in $\mathbf{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{C})$

$$\mathrm{id}_{\mathcal{C}} \Rightarrow T \Rightarrow T.$$

So by additivity theorem 1

$$F(\mathrm{id}_{\mathcal{C}}) \oplus F(T) \simeq F(T),$$

which implies $F(\mathrm{id}_{\mathcal{C}}) \simeq 0$, and so $F(\mathcal{C}) \simeq 0$. □

A.5 Construction

Consider a stable ∞-category $\mathcal{C}$. We define a stable ∞-category

$$\mathcal{F}(\mathcal{C})$$

as the full stable sub-∞-category of $\mathfrak{D}(\mathcal{C})$

- containing the essential image of $\mathcal{C}$ through the Yoneda embedding

$$\text{よ} : \mathcal{C} \to \mathfrak{D}(\mathcal{C});$$

- stable under countable sums.

Notice that, if $\mathcal{C}$ is already stable under countable sums in $\mathfrak{D}(\mathcal{C})$ then $\mathcal{C}$ is idempotent complete. This because if $x \in \mathcal{C}$ and $p : x \to x$ is any functor such that $p^2 \simeq p$, then

$$\operatorname*{colim}_n (x \xrightarrow{p} x \xrightarrow{p} \dots) = \mathrm{Im}(p)$$

exists; indeed it fits into cofibre sequence

$$\oplus_n x \to \oplus_n x \to \mathrm{Im}(p).$$

Therefore there is a retraction

$$x \to \mathrm{Im}(p) \to x,$$

and so every idempotent morphism would already split.

We have inclusions

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[d, hook] & \mathcal{F}(\mathcal{C}) \arrow[d, "\simeq"] \\ \mathcal{C}^{\#} \arrow[r] & \mathcal{F}(\mathcal{C}^{\#}) \end{tikzcd}$$

where the right vertical arrow is an equivalence since

$$\mathfrak{D}(\mathcal{C}^{\#}) \simeq \mathfrak{D}(\mathcal{C}).$$

A.6 Remark

Consider a Karoubi sequence of stable ∞-categories

$$\mathcal{A} \to \mathcal{B} \to \mathcal{C}.$$

Then, by definition, there is an equivalence

$$(\mathcal{B}/\mathcal{A})^{\#} \simeq \mathcal{C}^{\#},$$

and we have a pullback squares of anima

$$\begin{tikzcd} k(\mathcal{A}^{\#}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & k(\mathcal{B}^{\#}) \arrow[d] \\ * \arrow[r] & k(\mathcal{C}^{\#}) \end{tikzcd}$$

Applying this to the Karoubi sequence

$$\mathcal{C} \to \mathcal{F}(\mathcal{C}) \to \mathcal{F}(C)/\mathcal{C}$$

we obtain a pullback square

$$\begin{tikzcd} k(\mathcal{C}^{\#}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & k(\mathcal{F}(\mathcal{C})) \simeq 0 \arrow[d] \\ 0 \arrow[r] & k((\mathcal{F}(\mathcal{C})/\mathcal{C})^{\#}) \end{tikzcd}.$$

Therefore,

$$k(\mathcal{C}^{\#}) \simeq \Omega\, k((\mathcal{F}(\mathcal{C})/\mathcal{C})^{\#}).$$

In particular we have

$$K_1(\mathcal{F}(\mathcal{C})/\mathcal{C}) \simeq \mathbf{K}_0(\mathcal{C}) := K_0(\mathcal{C}^{\#}),$$

and for any $n \geqslant 0$, we could define

$$\mathbf{K}_n(\mathcal{C}) \simeq K_{n+1}(\mathcal{F}(\mathcal{C})/\mathcal{C})$$

Recall that, by cofinality, $K_i(\mathcal{C}) \simeq K_i(\mathcal{C}^{\#})$ for any $i > 0$.

A problem that we meet after all this discussion is that $\mathcal{F}(\bullet)$ does not preserve filtered colimits, which would turn out to be very useful.

A.7 Remark

Recall, for example from [BGT13], that there exists a tensor product on the $\mathbf{Cat}_\infty^{\mathbf{st}}$ (and by restricton, one in $\mathbf{Cat}_\infty^{\mathbf{perf}}$), which unit $\mathbb{1}$ is

$$\mathbf{Spectra}^{\omega} \simeq \mathbf{Spectra}^{fin} := SW(\mathbf{An}_*^{fin}).$$

This can for example be deduced after constructing a tensor product on the ∞-categories $\mathbf{Pr}^{\mathrm{L}}_{\mathrm{st}}$ of stable presentable ∞-categories.

Consider stable ∞-categories $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$. We define $\mathrm{Bil}(\mathcal{A}, \mathcal{B}, \mathcal{C})$ as the full sub-∞-category of $\mathbf{Fun}(\mathcal{A} \times \mathcal{B}, \mathcal{C})$ spanned by those functor $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$, which are exact in both variables once we fix the other. It turns out that the functor

$$\text{core } \mathrm{Bil}(A, B, \bullet) : \mathbf{Cat}_{\infty}^{\mathbf{st}} \to \mathbf{An}$$

is precisely represented by the tensor product of $\mathcal{A}$ and $\mathcal{B}$ in the sense that there exists an exact functor

$$\gamma : \mathcal{A} \times \mathcal{B} \to \mathcal{A} \otimes \mathcal{B}, \quad (a, b) \mapsto a \otimes b,$$

such that there is an equivalence

$$\mathrm{core}\, \mathbf{Fun}^{\mathrm{ex}}(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) \to \text{core } \mathrm{Bil}(\mathcal{A}, \mathcal{B}, \mathcal{C}), \quad F \mapsto F \circ \gamma.$$

There is an adjunction $\left(\mathrm{Bil}(\bullet, \bullet, \mathcal{C}) \dashv \otimes\right)$. Furthermore, the tensor product is compatible with Karoubi equivalence and Karoubi sequences in both variables.

A.8 Remark

Recall there is an equivalence

$$\mathbf{Fun}^{L}(\mathbf{Spectra}, \mathcal{C}) \simeq \mathcal{C}, F \mapsto F(\mathbf{S}).$$

A.9 Proposition

For any exact fully faithful functor $f : \mathcal{A} \to \mathcal{B}$ of stable ∞-categories and for any stable ∞-category $\mathcal{C}$, the functor

$$f \otimes \mathrm{id}_{\mathcal{C}} : \mathcal{A} \otimes \mathcal{C} \to \mathcal{B} \otimes \mathcal{C}$$

is fully faithful.

This is not too difficult to prove just by studying the function induced between the mapping spectrum on both sides and the tensor product. In order to not loose to much time on the study of this tensor product, we omit the proof.

A.10 Theorem

If

$$\mathcal{A} \to \mathcal{B} \to \mathcal{C}$$

is an Karoubi sequence of stable ∞-categories, then so is

$$\mathcal{A} \otimes \mathcal{D} \to \mathcal{B} \otimes \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$$

for any stable ∞-category $\mathcal{D}$.

A.11 Remark

Consider an exact fully faithful embedding of stable ∞-categories $i : \mathcal{A} \hookrightarrow \mathcal{B}$ and another a stable ∞-category $\mathcal{C}$. We define

$$\mathbf{Fun}^{\mathrm{ex}}_{\mathcal{A}}(\mathcal{B}, \mathcal{C})$$

as the full sub-∞-category of $\mathbf{Fun}^{\mathrm{ex}}(\mathcal{B}, \mathcal{C})$ spanned by those functors F which vanishes on $\mathcal{A}$, i.e. such that $F(x) \simeq 0$ for all $x \in \mathcal{A}$. Then the canonical morphism

$$\gamma : \mathcal{B} \to \mathcal{B}/\mathcal{A}$$

induces an equivalence

$$\mathbf{Fun}^{\mathrm{ex}}(\mathcal{B}/\mathcal{A}, \mathcal{C}) \simeq \mathbf{Fun}^{\mathrm{ex}}_{\mathcal{A}}(\mathcal{B}, \mathcal{C}).$$

There is a cocartesian square in $\mathbf{Cat}^{\mathbf{st}}_{\infty}$

$$\begin{tikzcd} \mathcal{A} \otimes \mathcal{C} \arrow[r, "i \otimes \mathrm{id}_{\mathcal{C}}"] \arrow[d] & \mathcal{B} \otimes \mathcal{C} \arrow[d] \\ * \otimes \mathcal{C} \arrow[r] & (\mathcal{B}/\mathcal{A}) \otimes \mathcal{C} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}.$$

Since i exact fully faithful functor, also $i \otimes \mathrm{id}_{\mathcal{C}}$ is an exact fully faithful functor which implies there is an equivalence

$$(\mathcal{B}/\mathcal{A}) \otimes \mathcal{C} \simeq (\mathcal{B} \otimes \mathcal{C})/(\mathcal{A} \otimes \mathcal{C}).$$

A.12 Definition

We define $\mathbf{\Sigma}(\mathbb{1})$ as the pushout

$$\begin{tikzcd} \mathbb{1} = SW(\mathbf{An}^{fin}_{*}) \arrow[r, hook] \arrow[d] & \mathcal{F}(\mathbb{1}) \arrow[d] \\ * \arrow[r] & \Sigma(\mathbb{1}) := \mathcal{F}(\mathbb{1})/\mathbb{1} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}.$$

A.13 Remark

If a stable ∞-category $\mathcal{C}$ is flasque, so is $\mathcal{A} \otimes \mathcal{C}$ for any $\mathcal{A}$.

Indeedm, if $\mathsf{T} : \mathcal{C} \to \mathcal{C}$ is the functor for $\mathcal{C}$, then $\mathrm{id}_{\mathcal{A}} \otimes \mathsf{T} : \mathrm{id}_{\mathcal{A}} \otimes \mathcal{C} \to \mathrm{id}_{\mathcal{A}} \otimes \mathcal{C}$ gives a sequence

$$\mathrm{id}_{\mathcal{A}} \otimes \mathsf{T} \to \mathrm{id}_{\mathcal{A}} \otimes \mathsf{T} \to \mathrm{id}\mathcal{A} \otimes \mathrm{id}_{\mathcal{C}} \simeq \mathrm{id}_{\mathcal{A} \otimes \mathcal{C}}$$

in

$$\mathbf{Fun}^{\mathrm{ex}}(\mathcal{A} \otimes \mathcal{C}, \mathcal{A} \otimes \mathcal{C}) \simeq \mathbf{Fun}^{\mathrm{ex}}(\mathcal{A}, \mathbf{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{A} \otimes \mathcal{C})) \simeq \mathrm{Bil}(\mathcal{A} \times \mathcal{C}, \mathcal{A} \otimes \mathcal{C}).$$

A.14 Remark

Consider a stable ∞-category $\mathcal{A}$. There is a cocartesian square

$$\begin{tikzcd}
\mathcal{A} \simeq \mathcal{A} \otimes \mathbb{1} \arrow[r, hook, "\mathrm{id}_{\mathcal{A}} \otimes \mathrm{i}"] \arrow[d] & \mathcal{A} \otimes \mathcal{F}(\mathbb{1}) \arrow[d] \\
0 \simeq \mathcal{A} \otimes 0 \arrow[r] & \mathcal{A} \otimes \Sigma(\mathbb{1}) \arrow[ul, phantom, very near start, "\ulcorner"]
\end{tikzcd}$$

We obtain a diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r, hook, "\mathcal{A} \otimes \mathrm{i}"] \arrow[d, hook] & \mathcal{A} \otimes \mathcal{F}(\mathbb{1}) \arrow[r] \arrow[d] & \mathcal{A} \otimes \Sigma\mathbb{1} \arrow[dd] \\
\mathcal{F}(\mathcal{A}) \arrow[r] \arrow[d] & \mathcal{F}(\mathcal{A}) \otimes \mathcal{F}(\mathbb{1}) \arrow[dr] & \\
\mathcal{F}(\mathcal{A})/\mathcal{A} \arrow[rr] & & (\mathcal{F}(\mathcal{A}) \otimes \mathcal{F}(\mathbb{1}))/\mathcal{A}
\end{tikzcd}$$

which gives us equivalences

$$\mathrm{k}(\mathcal{F}(\mathcal{A})/\mathcal{A}) \simeq \mathrm{k}(\mathcal{F}(\mathcal{A}) \otimes \mathcal{F}(\mathbb{1})/\mathcal{A}) \simeq \mathrm{k}(\mathcal{A} \otimes \boldsymbol{\Sigma}(\mathbb{1})).$$

We can iterate this construction to get consecutive cocartesian squares of the form

$$
\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[d] & \mathcal{A} \otimes \mathcal{F}(\mathbb{1}) \arrow[r] \arrow[d] & 0 \arrow[d] & & & \\
0 \arrow[r] & \mathcal{A} \otimes \mathbf{\Sigma}(\mathbb{1}) \arrow[r] \arrow[d] \arrow[ul, phantom, "\ulcorner", very near start] & \mathcal{A} \otimes \mathbf{\Sigma}(\mathbb{1}) \otimes \mathcal{F}(\mathbb{1}) \arrow[r] \arrow[d] & 0 \arrow[d] & & \\
& 0 \arrow[r] & \mathcal{A} \otimes \mathbf{\Sigma}(\mathbb{1}) \otimes \mathbf{\Sigma}(\mathbb{1}) \arrow[r] \arrow[d] \arrow[ul, phantom, "\ulcorner", very near start] & \cdots \arrow[d] & & \\
& & 0 \arrow[r] & \cdots \arrow[dr, dotted, no head] & & \\
& & & \cdots \arrow[r] & \mathcal{A} \otimes \mathbf{\Sigma}(\mathbb{1})^{\otimes n} \arrow[r] \arrow[d] & \cdots \arrow[d] \\
& & & & \cdots \arrow[r] & \cdots
\end{tikzcd}
$$

Applying the $\mathrm{K}((\bullet)^{\#})$ to this, we get a diagram

$$
\begin{tikzcd}
\mathrm{K}((\mathcal{A})^{\#}) \arrow[r] \arrow[d] & \mathrm{K}((\mathcal{A} \otimes \mathcal{F}(\mathbb{1}))^{\#}) \simeq 0 \arrow[d] & & & \\
0 \arrow[r] & \mathrm{K}((\mathcal{A} \otimes \mathbf{\Sigma}(\mathbb{1}))^{\#}) \arrow[r] \arrow[d] & \mathrm{K}((\mathcal{A} \otimes \mathbf{\Sigma}(\mathbb{1}) \otimes \mathcal{F}(\mathbb{1}))^{\#}) \simeq 0 \arrow[d] & & \\
& 0 \arrow[r] & \mathrm{K}((\mathcal{A} \otimes \mathbf{\Sigma}(\mathbb{1}) \otimes \mathbf{\Sigma}(\mathbb{1}))^{\#}) \arrow[dr, dotted, no head] & & \\
& & & \mathrm{K}((\mathcal{A} \otimes \mathbf{\Sigma}(\mathbb{1})^{\otimes n})^{\#}) \arrow[dr, dotted, no head] & \\
& & & & {}
\end{tikzcd}
$$

This is the same as giving the diagonal and maps from each element of the diagonal to th loop of the next one, such as

$$\mathrm{K}(\mathcal{A}^{\#}) \to \Omega\,\mathrm{K}((\mathcal{A} \otimes \mathbf{\Sigma}(\mathbb{1}))^{\#}).$$

A.15 Definition

Consider a stable ∞-category $\mathcal{A}$. We define the non-connective

K-theory spectrum $\mathbf{K}(\mathcal{A})$ of $\mathcal{A}$ (sometimes also called Bass-Karoubi-spectrum) as the colimit indexed by $\mathbf{N}$ in **Spectra** of

$$\mathrm{K}(\mathcal{A}^\#) \to \Omega\,\mathrm{K}((\mathcal{A}\otimes\mathbf{\Sigma}(\mathbb{1}))^\#) \to \cdots \to \Omega^n\,\mathrm{K}((\mathcal{A}\otimes\mathbf{\Sigma}(\mathbb{1})^{\otimes n})^\#) \to \ldots$$

i.e.

$$\mathbf{K}(\mathcal{A}) := \operatorname*{colim}_n \Omega^n\,\mathrm{K}((\mathcal{A}\otimes\mathbf{\Sigma}(\mathbb{1})^{\otimes n})^\#).$$

A.16 Remark

This construction is functorial in $\mathcal{A}$ and gives the non-connective K-theory spectrum functor

$$\mathbf{K} : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{Spectra}\,.$$

Recall that there exists a right adjoint to the inclusion

$$\mathbf{Spectra}_{\geqslant 0} \hookrightarrow \mathbf{Spectra}$$

which is given by a truncation, usually called the CONNECTIVE COVER FUNCTOR

$$\tau_{\geqslant 0} : \mathbf{Spectra} \to \mathbf{Spectra}_{\geqslant 0}\,.$$

The counit of this adjunction, which for any $X \in$ **Spectra**, is given by

$$\tau_{\geqslant 0}(X) \to X \in \mathbf{Spectra},$$

induces an isomorphism of stable homotopy groups for any $n \geqslant 0$, i.e.

$$\pi_n(\tau_{\geqslant 0}X) \to \pi_n X$$

for any $n \geqslant 0$. However

$$\pi_n(\tau_{\geqslant 0}X) = 0$$

for any $n < 0$.

It follows that, fixed $\mathcal{A}$ stable ∞-category, the map

$$\mathrm{K}(\mathcal{A}^\#) \to \mathbf{K}(\mathcal{A})$$

factors through the counit $\tau_{\geqslant 0}\mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$. The next proposition then follows by the fact that π_n : **Spectra** $\to Ab$ commutes with filtered colimits.

A.17 Proposition

For any stable ∞-category $\mathcal{C}$

$$\tau_{\geqslant 0}\mathbf{K}(\mathcal{C}) \simeq \mathrm{K}(\mathcal{C}^\#).$$

A.18 Theorem
For any Karoubi sequence of stable ∞-categories

$$\mathcal{A} \to \mathcal{B} \to \mathcal{C}$$

there is a fibre sequence of spectra

$$\mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{B}) \to \mathbf{K}(\mathcal{C})$$

and thus a long exact sequence of non-connectiveK-theory groups

$$\mathbf{K}_n(\mathcal{A}) \to \mathbf{K}_n(\mathcal{B}) \to \mathbf{K}_n(\mathcal{C}) \to \mathbf{K}_{n-1}(\mathcal{A})$$

for any $n \in \mathbf{Z}$.

Proof. Let us start by noticing that, for any $n \geqslant 0$,

$$\Sigma^n \mathbf{K}(\mathcal{A}) \simeq \mathbf{K}(\mathcal{A} \otimes \mathbf{\Sigma}(\mathbb{1})^{\otimes n}).$$

We also know from theorem A.10, that

$$\mathcal{A} \otimes \mathbf{\Sigma}^{\otimes n} \to \mathcal{B} \otimes \mathbf{\Sigma}^{\otimes n} \to \mathcal{C} \otimes \mathbf{\Sigma}^{\otimes n}$$

is a Karoubi sequence for any $n \in \mathbf{N}$. Then, for any $i \geqslant n$, by applying $\mathbf{K}$ and the i-th homotopy groups we get

$$\begin{aligned} &K_i((\mathcal{A} \otimes \mathbf{\Sigma}(\mathbb{1})^{\otimes n})^\#) \simeq \pi_i \mathbf{K}(\mathcal{A} \otimes \mathbf{\Sigma}(\mathbb{1})^{\otimes n}) \simeq \pi_{i-n}(\mathbf{K}(\mathcal{A})) \to \\ &\to \pi_{i-n}(\mathbf{K}(\mathcal{B})) \to \pi_{i-n}(\mathbf{K}(\mathcal{C})) \end{aligned}$$

which know fits in an long exact sequence of abelian groups. By 5-lemma we can conclude that the canonical map

$$\mathbf{K}(\mathcal{A}) \xrightarrow{\simeq} \mathrm{fib}(\mathbf{K}(\mathcal{B}) \to \mathbf{K}(\mathcal{C}))$$

is an equivalence. $\square$

A.19 Theorem
$\mathcal{A}$ stable ∞-category, then

$$\mathbf{K}_{-1}(\mathcal{A}) = 0$$

if and only if for any exact fully faithful embedding $\mathcal{A} \hookrightarrow \mathcal{B}$ with $\mathcal{B}$ stable idempotent complete ∞-category, the Verdier quotient $\mathcal{B}/\mathcal{A}$ is idempotent complete, i.e. $\mathcal{B}/\mathcal{A} \simeq (\mathcal{B}/\mathcal{A})^\#$.

Proof. Assume that $\mathbf{K}_{-1}(\mathcal{A}) = 0$ and consider a Karoubi sequence

$$\mathcal{A} \to \mathcal{B} \to \mathcal{C}.$$

Then we have an exact sequence

$$\cdots \to \mathbf{K}_0(\mathcal{A}) \to \mathbf{K}_0(\mathcal{B}) \to \mathbf{K}_0(\mathcal{C}) \to \mathbf{K}_{-1}(\mathcal{A}) = 0 \to \ldots,$$

that is equivalently an exact sequence

$$\cdots \to \mathrm{K}_0(\mathcal{A}^\#) \to \mathrm{K}_0(\mathcal{B}^\#) \to \mathrm{K}_0(\mathcal{C}^\#) \simeq \mathrm{K}_0((\mathcal{B}/\mathcal{A})^\#) \to 0 \to \ldots$$

Recall that we have an inclusion

$$\mathcal{B}^\#/\mathcal{A}^\# \hookrightarrow (\mathcal{B}^\#/\mathcal{A}^\#)^\# \simeq (\mathcal{B}/\mathcal{A})^\#.$$

This induces by 2.8 an injective morphism in K_0

$$\mathrm{K}_0(\mathcal{B}^\#/\mathcal{A}^\#) \to \mathrm{K}_0((\mathcal{B}/\mathcal{A})^\#).$$

By 5-lemma this is an equivalence; so by 2.8 and the fact that $\mathcal{B}$ is idempotent complete, we get an equivalence

$$\mathcal{B}^\#/\mathcal{A}^\# \overset{\simeq}{\hookrightarrow} (\mathcal{B}/\mathcal{A})^\# \simeq \mathcal{B}/\mathcal{A}.$$

Conversely, fix $\mathcal{B} \simeq \mathcal{A} \otimes \mathcal{F}(\mathbb{1})$, so that $\mathcal{B}/\mathcal{A} \simeq \mathcal{A} \otimes \mathbf{\Sigma}(\mathbb{1})$ is idempotent complete. Then we have an exact sequence abelian group

$$\cdots \to \mathrm{K}_0(\mathcal{A}^\#) \to \mathrm{K}_0(\mathcal{A} \otimes \mathcal{F}(\mathbb{1})) \to \mathrm{K}_0(\mathcal{A} \otimes \mathbf{\Sigma}(\mathbb{1})) \to 0$$

and so an exact sequence of abelian groups

$$\cdots \to \mathbf{K}_(\mathcal{A}) \to \underbrace{\mathbf{K}_0(\mathcal{A} \otimes \mathcal{F}(\mathbb{1}))}_{=0} \to \underbrace{\mathbf{K}_0(\mathcal{A} \otimes \mathbf{\Sigma}(\mathbb{1}))}_{\simeq \mathbf{K}_{-1}(\mathcal{A})} \to 0,$$

which implies $\mathbf{K}_{-1}(\mathcal{A}) = 0$. □

Similarly to the connective case we have theorems like the followings.

A.20 Theorem
Consider a stable ∞-category $\mathcal{A}$. Then the map

$$(\mathtt{fib}, \mathtt{cofib}) : \mathtt{Seq}(\mathcal{A}) \to \mathcal{A}^2, \quad (x \to y \to z) \mapsto (x, z)$$

induces an isomorphism of spectra

$$\mathbf{K}(\mathtt{Seq}(\mathcal{A})) \to \mathbf{K}(\mathcal{A})^2.$$

A.21 Theorem
Consider stable ∞-categories $\mathcal{A}$ and $\mathcal{B}$. Consider also an Karoubi sequence of functors

$$F' \to F \to F''$$

in $\mathbf{Fun}^{\mathrm{ex}}(\mathcal{A}, \mathcal{B})$. Then

$$\mathbf{K}(F) \simeq \mathbf{K}(F') + \mathbf{K}(F'')$$

in $\pi_0(\mathbf{Map}_{\mathbf{Spectra}}(\mathbf{K}(\mathcal{A}), \mathbf{K}(\mathcal{B})))$.

We also have the following theorem.

A.22 Theorem

The functor $\Omega^\infty : \mathbf{Fun}(\mathbf{Cat}_\infty^{\mathbf{st}}, \mathbf{Spectra}) \to \mathbf{Fun}(\mathbf{Cat}_\infty^{\mathbf{st}}, \mathbf{An})$ induces an equivalence between the full sub-∞-category of Karoubi-localising functor on both sides.

In particular, under this equivalence, the non-connective K-thoery functor

$$\mathbf{K} : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{Spectra}$$

correspond to the approximation of the K-theory anima functor to a Karoubi localising functor

$$\mathrm{k} \circ \mathrm{Idem} : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}\,.$$

The way we can construct an inverse to this functor is similar to what we have done to construct the non-connective K-theory spectrum. Consider a Karoubi-localising functor $\mathsf{F} : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{An}$ and its grouplike completion $\Omega|\mathsf{F}\mathsf{Q}(\bullet)|$. The latter functor is still Karoubi-localising. We denote with $\widetilde{\mathsf{F}}$ the functor corresponding to $\Omega|\mathsf{F}\mathsf{Q}(\bullet)| : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathsf{CGrp}(\mathbf{An})$ valued into connective spectra. Then we define the Karoubi-localising functor $\mathbf{F} : \mathbf{Cat}_\infty^{\mathbf{st}} \to \mathbf{Spectra}$, as

$$\mathbf{F}(\mathcal{C}) := \operatorname*{colim}_{n \in \mathbf{N}} \Omega^n \widetilde{\mathsf{F}}(\mathcal{C} \otimes \mathbf{\Sigma}(\mathbb{1})^{\otimes n})$$

for any stable ∞-category $\mathcal{C}$.

B T-STRUCTURES AND WEIGHT STRUCTURED

In this appendix we are going to briefly discuss t-Structures and Weight Structured. This are generalizations of truncations of chain complexes to stable ∞-categories.

B.1 T-STRUCTURES.

A t-structure is a way of decomposing an triangulated category or stable ∞-category into two subcategories, which abstract the idea of complexes whose cohomology vanishes in positive and negative degrees. The concept of t-structures first appeared in [BBD82]. In the ∞-categorical setting they arise as torsion/torsion-free classes associated to suitable factorization systems on a stable infinity-category $\mathcal{C}$

Let us start by giving the classical definition of t-structure on a triangulated category.

B.1 Definition (t-structure on Triangulated categories)
Consider a triangulated category $\mathcal{T}$. A T-STRUCTURE on $\mathcal{T}$ is defined as a pair of full subcategories $\mathcal{T}_{\leqslant 0}, \mathcal{T}_{\geqslant 0} \subset \mathcal{T}$ such that

(tsT-1) $\mathcal{T}_{\leqslant 0}, \mathcal{T}_{\geqslant 0}$ are both closed under isomorphisms.

(tsT-2) If $x \in \mathcal{T}_{\geqslant 0}$ and $y \in \mathcal{T}_{\leqslant 0}$, then $\mathrm{Hom}_{\mathcal{T}}(x, y[-1]) \simeq 0$.

(tsT-3) $\mathcal{T}_{\geqslant 0}[1] \subset \mathcal{T}_{\geqslant 0}$.

(tsT-4) $\mathcal{T}_{\leqslant 0}[-1] \subset \mathcal{T}_{\leqslant 0}$.

(tsT-5) If $y \in \mathcal{T}$, then there exists a distinguished triangle

$$x \to y \to z \to x[1]$$

where $x \in \mathcal{T}_{\geqslant 0}$ and $z \in \mathcal{T}_{\leqslant 0}[-1]$.

If $\mathcal{T}$ is a triangulated category equipped with a t-structure we define, for any $m, n \in \mathbf{Z}$,

$$\mathcal{T}_{\geqslant n} := \mathcal{T}_{\geqslant 0}[n],$$

$$\mathcal{T}_{\leqslant n} := \mathcal{T}_{\leqslant 0}[n],$$

$$\mathcal{T}_{[m,n]} := \mathcal{T}_{\geqslant m} \cap \mathcal{T}_{\leqslant n},$$

(where $\cap$ means the largest full subcategory spanned by the objects in both).
We also define the heart of the t-structure as the subcategory

$$\mathcal{C}^{\heartsuit} := \mathcal{C}_{[0,0]}.$$

B.2 Remark

With this notation we can re-write the properties that have to be satisfied

(tsT-2) If $x \in \mathcal{T}_{\geqslant 0}$ and $y \in \mathcal{T}_{\leqslant -1}$, then $\mathrm{Hom}_{\mathcal{T}}(x, y) \simeq 0$.

(tsT-3) $\mathcal{T}_{\geqslant 1} \subset \mathcal{T}_{\geqslant 0}$.

(tsT-4) $\mathcal{T}_{\leqslant -1} \subset \mathcal{T}_{\leqslant 0}$.

(tsT-5) If $y \in \mathcal{T}$, then there exists a distinguished triangle

$$x \to y \to z \to x[1]$$

where $x \in \mathcal{T}_{\geqslant 0}$ and $z \in \mathcal{T}_{\leqslant -1}$.

B.3 Remark

Each of the full subcategories $\mathcal{T}_{\geqslant 0}, \mathcal{T}_{\leqslant 0} \subset \mathcal{T}$ determines the other. If $x \in \mathcal{T}$, then $x \in \mathcal{T}_{\leqslant -1}$ if and only if

$$\mathrm{Hom}_{\mathcal{T}}(x, y) \simeq 0$$

for all $y \in \mathcal{T} \geqslant 0$.

B.4 Definition (t-structure on Stable infinity-categories)

Consider a stable ∞-category $\mathcal{C}$. A t-STRUCTURE on $\mathcal{C}$ is defined as a pair of full subcategories $\mathcal{C}_{\leqslant 0}, \mathcal{C}_{\geqslant 0} \subset \mathcal{C}$ such that

(tsS-1) $\mathcal{C}_{\geqslant 0}$ is closed under pushout;

(tsS-2) $\mathcal{C}_{\leqslant 0}$ is closed under pullbacks;

(tsS-3) $\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0}$ are closed under retracts;

(tsS-4) If $x \in \mathcal{C}_{\geqslant 0}$ and $y \in \mathcal{C}_{\leqslant 0}$, then the spectrum $\underline{\mathbf{Map}}_{\mathcal{C}}(x, y)$ is co-connective, i.e. all of its homotopy groups of positive degrees vanishes.

(tsS-5) If $y \in \mathcal{C}$, then there exists

$$x \to y \to z$$

bifibre sequence with $x \in \mathcal{C}_{\geqslant 1}$ and $z \in \mathcal{C}_{\leqslant 0}$.

In the definition we have used the following notation. For any $m, n \in \mathbf{Z}$,

$$\begin{aligned} \mathcal{C}_{\geqslant n} &:= \Sigma^n(\mathcal{C}_{\geqslant 0}) = \mathcal{C}_{\geqslant 0}[n], \\ \mathcal{C}_{\leqslant n} &:= \Sigma^n(\mathcal{C}_{\leqslant 0}) = \mathcal{C}_{\leqslant 0}[n], \\ \mathcal{C}_{[m,n]} &:= \mathcal{C}_{\geqslant m} \cap \mathcal{C}_{\leqslant n}, \end{aligned}$$

(where $\cap$ means the largest full sub-∞-infinity category spanned by the objects in both).

B.5 Definition

Consider a stable ∞-category $\mathcal{C}$ with a t-structure. We define the HEART of the t-structure as

$$\mathcal{C}^{\heartsuit} := \mathcal{C}_{[0,0]}.$$

B.6 Definition

A t-structure on $\mathcal{C}$ is called BOUNDED if

$$\mathcal{C} \simeq \cup_{n \in \mathbf{N}} \mathcal{C}_{[-n,n]},$$

(where $\cup$ means the smallest full sub-∞-infinity category spanned by the objects in one of these.)

B.7 Remark

Consider a stable ∞-category $\mathcal{C}$. A t-structure on $\mathcal{C}$ is equivalent to a t-structure on its homotopy category $\operatorname{ho}\mathcal{C}$, where we define$\mathcal{C}_{\geqslant n}$ (resp. $\mathcal{C}_{\leqslant n}$) to the full sub-$\infty$-category of $\mathcal{C}$ spanned by objects in

$$\operatorname{ho}\mathcal{C}_{\geqslant n} \quad (\text{resp. } \operatorname{ho}\mathcal{C}_{\leqslant n}).$$

The most standard example of a stable ∞-category with a t-structure is the following.

B.8 Example

Consider a ring R. Then we can impose a t-structure on $\mathfrak{D}(R)$, the derived stable ∞-category of R, which consists of full sub-∞-categories $\mathfrak{D}_{\geqslant 0}(R)$, consisting of complexes whose homology is concentrated in non-negative degrees, and $\mathfrak{D}_{\leqslant 0}$, consisting of complexes whose homology is concentrated in non-positive degrees.

B.9 Remark

Notice that a t-structure is an additional *structure* of a stable ∞-category, and that the same stable ∞-category can have many different t-structures.

B.10 Proposition ([Lur17, Prop. 1.2.1.5])

Consider a stable ∞-category $\mathcal{C}$ with a t-structure. For any $n \in \mathbf{N}$

- $\mathcal{C}_{\leqslant n}$ is left Bousfield localisation of $\mathcal{C}$.
- $\mathcal{C}_{\geqslant n}$ is right Bousfield localisation of $\mathcal{C}$.

Let us denote

- with $\tau_{\leqslant n}$ the left adjoint to the inclusion $\mathcal{C}_{\leqslant n} \hookrightarrow \mathcal{C}$;
- with $\tau_{\geqslant n}$ the right adjoint to the inclusion $\mathcal{C}_{\geqslant n} \hookrightarrow \mathcal{C}$.

B.11 Corollary

For any $n \in \mathbf{N}$,

- $\mathcal{C}_{\leqslant n} \hookrightarrow \mathcal{C}$ is stable under limits that exists in $\mathcal{C}$.
- $\mathcal{C}_{\geqslant n} \hookrightarrow \mathcal{C}$ is stable under colimits that exists in $\mathcal{C}$.

We have that given $y \in \mathcal{C}$, the fibre sequence in (tsS-5) must be of the form

$$\tau_{\geqslant 1} y \to y \to \tau_{\leqslant 0} y$$

hence it must be (essentially) unique. Moreover, this process if functorial.

The proof of the following theorem can be found in [Lur17, Sec. 1.2.1], where all the concept of t-structure is well and carefully explained

B.12 Theorem

The heart $\mathcal{C}^{\heartsuit}$ is equivalent to (the nerve of) an abelian 1-category.

B.1.1 *The Theorem of the Heart.*

B.13 Theorem (Heart, [Bar15, Thm. 6.1])

Consider a small stable ∞-category $\mathcal{C}$ equipped with a bounded t-structure $(\mathcal{C}_{\leqslant 0}, \mathcal{C}_{\geqslant 0})$. The inclusions

$$\mathcal{C}^{\heartsuit} \hookrightarrow \mathcal{C}_{\geqslant 0} \hookrightarrow \mathcal{C}$$

induce equivalences

$$k(\mathcal{C}^{\heartsuit}) \xrightarrow{\simeq} k(\mathcal{C}_{\geqslant 0}) \xrightarrow{\simeq} k(\mathcal{C}).$$

Given two small stable ∞-categories each equipped with a t-structure, $\mathcal{C}$ with $(\mathcal{C}_{\leqslant 0}, \mathcal{C}_{\geqslant 0})$ and $\mathcal{D}$ with $(\mathcal{D}_{\leqslant 0}, \mathcal{D}_{\geqslant 0})$, a functor $F : \mathcal{C} \to \mathcal{D}$ is called t-exact if

$$F(\mathcal{C}_{\leqslant 0}) \subset \mathcal{D}_{\leqslant 0} \quad \text{and} \quad F(\mathcal{C}_{\geqslant 0}) \subset \mathcal{D}_{\geqslant 0}.$$

B.14 Theorem (Simil-Dévissage)
Consider two small stable ∞-categories each equipped with a t-structure, $\mathcal{C}$ with $(\mathcal{C}_{\leqslant 0}, \mathcal{C}_{\geqslant 0})$ and $\mathcal{D}$ with $(\mathcal{D}_{\leqslant 0}, \mathcal{D}_{\geqslant 0})$. Consider also an exact and t-exact functor between them $F : \mathcal{C} \to \mathcal{D}$. Suppose that

- the restriction

  $$F^{\heartsuit} : \mathcal{C}^{\heartsuit} \to \mathcal{D}^{\heartsuit}$$

  is fully faithful;
- the essential image closed under sub-objects and quotients;
- such that every object in $d \in \mathcal{D}^{\heartsuit}$ has a finite filtration

  $$0 = d_0 \subset d_1 \subset \cdots \subset d_n = d$$

  whose sub-quotients d_{i+1}/d_i are in the essential image of $F^{\heartsuit}$.

Then F induces an equivalence $F : k(\mathcal{C}) \to k(\mathcal{D})$.

A very short proof of this theorem, using the theorem of the Heart, is given in [Heba] and is itself based on a proof from Charles Weibel's K-book [Wei13].

B.2 WEIGHT STRUCTURES.

A similar concept to that of a t-structure is known as a weight structure.

B.15 Definition (Weight Structure)
Consider a stable ∞-category $\mathcal{C}$. A WEIGHT STRUCTURE on $\mathcal{C}$ is defined as a pair full sub-∞-categories $\mathcal{C}_{\leqslant 0}, \mathcal{C}_{\geqslant 0} \subset \mathcal{C}$ such that

(ws-1) $\mathcal{C}_{\geqslant 0}$ is closed under pushout;

(ws-2) $\mathcal{C}_{\leqslant 0}$ is closed under pullbacks;

(ws-3) $\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0}$ are closed under retracts;

(ws-4) If $x \in \mathcal{C}_{\leqslant 0}$ and $y \in \mathcal{C}_{\geqslant 0}$, then $\mathbf{Map}_{\mathcal{C}}(x, y)$ is a connective spectrum.

(ws-5) If $y \in \mathcal{C}$, then there exists

$$x \to y \to z$$

bifibre sequence with $x \in \mathcal{C}_{\leqslant 0}$ and $z \in \mathcal{C}_{\geqslant 1}$.

In the definition we used the following notation: for any $m, n \in \mathbf{Z}$,

$$\mathcal{C}_{\geqslant n} := \Sigma^n(\mathcal{C}_{\geqslant 0}) = \mathcal{C}_{\geqslant 0}[n], \mathcal{C}_{\leqslant n} := \quad \Sigma^n(\mathcal{C}_{\leqslant 0}) = \mathcal{C}_{\leqslant 0}[n], \mathcal{C}_{[m,n]} := \mathcal{C}_{\geqslant m} \cap \mathcal{C}_{\leqslant n},$$

(where $\cap$ means the largest full sub-∞-infinity category spanned by the objects in both.)

B.16 Remark
It is worth noting that, despite being similar, the two notion differ. This should be immediately clear by the fact that in (ws-5) and (tsS-5), $\leqslant$ and $\geqslant$ are inverted. However, we can think both as generalization of the notion of "truncation of chain complexes", where t-structure is the more sophisticated generalization of the two. Also, the choice of a fibre sequence for the weight-structure is non-canonical, while the choice of one for the t-structure (which will become a d.t. in the homotopy category) is canonically given by the truncation functors $\tau_{\leqslant n}$ and $\tau_{\geqslant n}$.

B.17 Definition
Consider a stable ∞-category $\mathcal{C}$ with a weight structure. We define the HEART of the weight structure as

$$\mathcal{C}^{\heartsuit} := \mathcal{C}_{[0,0]}.$$

B.18 Definition

A weight structure on a stable ∞-category $\mathcal{C}$ is EXHAUSTIVE if

$$\mathcal{C} \simeq \cup_{n \in \mathbf{N}} \mathcal{C}_{[-n,n]},$$

(where $\cup$ means the smallest full sub-∞-infinity category spanned by the objects in one of these).

B.19 Remark

Notice that both $\mathcal{C}_{\leqslant 0}$ and $\mathcal{C}_{\geqslant 0}$ are non empty because of (ws-5). Also, 0 belongs to both, because of (ws-3), being 0 a retract of any object.

In a similar fashion to the t-structure case, we could prove the following lemmas, which comprise many results on weight structure.

B.20 Lemma

Consider a stable ∞-category $\mathcal{C}$ with a weight structure.

- Consider an object $x \in \mathcal{C}$; $x \in \mathcal{C}_{\leqslant 0}$ if and only if $\mathbf{Map}_{\mathcal{C}}(x, y)$ is a connective spectrum for all $y \in \mathcal{C}_{\geqslant 0}$.
- Consider an object $x \in \mathcal{C}$; $x \in \mathcal{C}_{\geqslant 0}$ if and only if $\mathbf{Map}_{\mathcal{C}}(y, x)$ is a connective spectrum for all $y \in \mathcal{C}_{\leqslant 0}$.

B.21 Lemma

Consider a stable ∞-category $\mathcal{C}$ with a weight structure. Both $\mathcal{C}_{\leqslant 0}$ and $\mathcal{C}_{\geqslant 0}$ are closed under extension, i.e. if there is a (bi)fibre sequence

$$x \to y \to z$$

with $x, z \in \mathcal{C}_{\leqslant 0}$ (resp. $\mathcal{C}_{\geqslant 0}$), then also $y \in \mathcal{C}_{\leqslant 0}$ (resp. $\mathcal{C}_{\geqslant 0}$).

B.22 Lemma

Consider a stable ∞-category $\mathcal{C}$ with a weight structure. Then $\mathcal{C}^{\heartsuit}$ is an additive ∞-category. Furthermore, a fibre sequence $x \to y \to z$ in $\mathcal{C}$ splits if x, y, z lies in $\mathcal{C}^{\heartsuit}$.

A proof of the following theorem can be found (in the case of Poincaré categories) in [HS21]. This theorem should be thought as a less sophisticated version of the theorem of the Heart.

B.23 Theorem (Resolution Theorem, Gillet-Waldhausen)
Consider a stable ∞-category $\mathcal{C}$ with an exhaustive weight structure. Then

$$\operatorname{core}(\mathcal{C}^{\heartsuit})^{\infty-grp} \to \operatorname{core}(\mathcal{C})^{\infty-grp} \to k(\mathcal{C})$$

is an equivalence of animae.

[Bar13] Clack Barwick. *On the Q construction for exact quasicategories*. 2013. arXiv: 1301.4725 [math.KT].

[Bar15] Clark Barwick. *On exact infinity-categories and the Theorem of the Heart*. Vol. Compositio Mathematica, Vol. 151, n.11, pp. 2160–2186. Wiley, 2015.

[Bar16] Clark Barwick. *On the algebraic K-theory of higher categories*. Vol. Journal of Topology, Vol.9, n.1, pp. 245-347. Wiley, 2016.

[BBD82] A. A. Beilinson, J. Bernstein, and P. Deligne. *Faisceaux Pervers*. Vol. Asterisque 100, Volume 1. 1982.

[Ber18] Julia Bergner. *The homotopy theory of* $(\infty, 1)$*-categories*. Cambridge University Press, 2018.

[BGT13] Andrew J. Blumberg, David Gepner, and Gonçalo Tabuada. *A universal characterization of higher algebraic K–theory*. Vol. Geometry & Topology, Vol. 17, pp. 733 – 838. 2013.

[BM10] Andrew J. Blumberg and Michael A. Mandell. *Algebraic K-theory and abstract homotopy theory*. 2010. arXiv: 0708.0206 [math.KT].

[Bor94] F. Borceux. *Handbook of categorical algebra: Volume 2, categories and structures*. 1994.

[Cal+21a] Baptiste Calmès, Emanuele Dotto, Yonatan Harpaz, Fabian Hebestreit, Markus Land, Kristian Moi, Denis Nardin, Thomas Nikolaus, and Wolfgang Steimle. *Hermitian K-theory for stable* ∞*-categories I: Foundations*. 2021. arXiv: 2009.07223 [math.KT].

[Cal+21b] Baptiste Calmès, Emanuele Dotto, Yonatan Harpaz, Fabian Hebestreit, Markus Land, Kristian Moi, Denis Nardin, Thomas Nikolaus, and Wolfgang Steimle. *Hermitian K-theory for stable* ∞*-categories II: Cobordism categories and additivity*. 2021. arXiv: 2009.07224 [math.KT].

[Cis19] Denis-Charles Cisinski. *Higher Categories and Homotopical Algebra*. Cambridge University Press, 2019.

[GGN13] David Gepner, Moritz Grot, and Thomas Nikolaus. *Universality of Multiplicative Infinite Loop Space Machine*. 2013.

[Heba] Fabian Hebestreit. *Algebraic and Hermitian K-Theory. Lecture held at the University of Bonn*. URL: https://sites.google.com/view/fabian-hebestreit/home/lecture-notes?authuser=0.

[Hebb] Fabian Hebestreit. *Higher Categories II. Lecture held at the University of Bonn.* URL: https://sites.google.com/view/fabian-hebestreit/home/lecture-notes?authuser=0.

[HLS22] Fabian Hebestreit, Andrea Lachmann, and Wolfgang Steimle. *The Localisation Theorem for the K-theory of Stable infinity-Categories*. 2022. arXiv: arXiv:2205.06104v2 [math.KT].

[HS21] Fabian Hebestreit and Wolfgang Steimle. *Stable moduli spaces of hermitian forms*. 2021. arXiv: 2103.13911 [math.KT].

[Joh02] Peter T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium*. Vol. 1 & 2. Oxford University Press, 2002.

[Lur09] Jacob Lurie. *Higher Topos Theory*. Princeton University Press, 2009.

[Lur17] Jacob Lurie. *Higher Algebra*. Princeton University Press, 2017.

[Ker] Jacob Lurie and Others. *Kerodon*. URL: https://kerodon.net.

[Nee92] A. Neeman. *The connection between the K-theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel*. Vol. Annales Scientifiques de l'École Normale Supérieure (4) 25, no. 5, 547–566. 1992.

[NS18] Thomas Nikolaus and Peter Scholze. *On topological cyclic homology*. 2018. arXiv: 1707.01799 [math.AT].

[Nui16] Joost Nuiten. *Localizing infinity-categories with hypercovers*. 2016. arXiv: 1612.03800 [math.AT].

[Rez00] Charles Rezk. *A model for the homotopy theory of homotopy theory*. 2000. arXiv: math/9811037 [math.AT].

[Sch06] Marco Schlichting. *Negative K-theory of derived categories*. Vol. Math. Z. 253, 97–134. 2006.

[Ste17] Wolfgang Steimle. *On the universal property of Waldhausen's K-theory*. 2017. arXiv: 1703.01865 [math.KT].

[TT90] R. W. Thomason and Thomas Trobaugh. *Higher Algebraic K-Theory of Schemes and of Derived Categories*. Vol. Part of Modern Birkhäuser Classics book series, volume 88. 1990.

[Tho97] R.W. Thomason. *The classification of triangulated subcategories*. Vol. Composition Mathematica, Vol. 105, n. 1, pp. 1–27. Cambridge University Press, 1997.

[Wal85] Fiedhelm Waldhausen. *Algebraic K-theory of Spaces*. Vol. Lecture notes in mathematics ; 1126. Springer: 318-419. 1985.

[Wei13] Charles A. Weibel. *Higher Topos Theory*. Vol. Vol. 145. Graduate Studies in Mathematics. American Mathematical Society, 2013.

Useful discussions were found in

- This mathoverflow answer by Denis Nardin.
- This mathoverflow answer by Denis Nardin.
- This mathoverflow answer by Denis-Charles Cisinski.
- This whole mathoverflow conversation.
- This whole mathoverflow conversation.