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Final Dissertation

On the Wormhole - Black Hole correspondence

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Abstract

Euclidean wormholes play a key role in Quantum Gravity (QG), and have been studied in relation with various aspects, as for instance the possible loss of quantum coherence, the black hole information paradox, the absence of global symmetries in quantum gravity and various 'swampland' criteria characterising the effective field theories (EFTs) compatible with QG.

The thesis will investigate a large class of axionic wormholes in four-dimensional EFTs preserving $\mathcal{N} = 2$ supersymmetry, their relation to black holes in five dimensions, and their explicit realisation in string theory. Understanding this relation may provide a new perspective on the quantum gravity effects generated by wormholes, and their connection with other non-perturbative effects appearing in string theory.

Contents

Introduction

General relativity has been a tremendously inspiring branch of physics since its dawn. The idea that the very fabric of spacetime can be curved and deformed is so appealing that it has shaped over the decades a completely new perspective on our reality. Along with quantum mechanics it is also the most successful and solid theory of modern physics and it has allowed an outstanding amount of progress in our comprehension of astrophysics and cosmology. General relativity is, at the present day, the most complete theory of classical mechanics we have, and its beauty may be ultimately found in its mathematical power: spacetime is a manifold, the most general object we can measure distances on, and there is no reason why it should have a priori this or that structure. As it is always said, "gravity is geometry".

This very simple idea has quite deep and counter-intuitive consequences. One of the most celebrated solutions in general relativity is Schwarzschild spacetime, which is popularly known as the "black hole solution" although it simply describes the geometry outside a spherical distribution of matter. However if the body is sufficiently dense it really is a black hole: an object around which the spacetime is so curved that nothing - not even light - can escape. Recent pictures of M87 and Sagittarius A have allowed us to concretely observe these weird beasts for the first time.

Pop culture has fuelled the veil of mystery around black holes and the science fiction enthusiast has probably read or seen lots of misconceptions about superluminal motion and travel between multiverses. According to a popular idea, black holes are associated to wormholes, tunnels in spacetime that would enable a space traveler to reach distant regions of the universe, or maybe a completely different universe.

Are things in theoretical physics as exciting as the common picture makes us believe? Well, yes, and maybe even more! Wormholes do in fact exist as sensible solutions in general relativity, and some of them could also be traversable by a human observer. They are fascinating objects whose research is only at the very beginning, since their study is related to the deepest open problems in quantum gravity. As it is widely known, some troubles are met when one tries to quantize Einstein's gravity (for a good summary see [1]), and the final answer about a theory of quantum gravity is still missing. However, many tools have been developed over the years to get a glimpse on quantum mechanical properties of gravitational systems. String theory is of course the most prominent one, but lots of efforts have been made to extract information directly from the gravitational path integral in a purely low-energy perspective. It turns out that wormholes could contribute in a non-trivial way to the partition function, as much as instantons do in ordinary quantum mechanics and quantum field theory. There is, however, a crucial difference: instantons can be thought of as localized objects, whereas wormholes connect two points in spacetime. Thus, they seem to introduce an intrinsic non-local contribution in the low energy effective action, spoiling the basic requirements we would like to fulfill in any safe and sound quantum field theory.

Pioneers of quantum gravity, mainly Hawking and Coleman, argued that this is not necessarily the case. We can wash out non-locality if we pay a high price, that is, if we accept that fundamental constants of our world are random variables. Some years before the perspective of string landscape and swampland, physicists had to deal for the first time with a multiverse scenario, a pool of worlds with very different constants and very different physics accordingly. This picture is troublesome: correlation functions in our quantum field theory should then be interpreted as probabilistic correlations of a "multiverse ensemble" rather than well defined quantum quantities. So, what is really happening? Are wormholes really so catastrophic that we should rethink about everything we know of holography and black hole quantum physics? Is there any hope to recover a local and unitary quantum field theory even in the presence of wormholes that allow warping between disconnected spacetime regions? Do we really need to know a UV theory of quantum gravity to address such problems? These questions still remain unanswered.

This thesis will revise some key features of a typical theory exhibiting wormhole solutions. Our goal is to find out what is the relation between four dimensional wormholes and five dimensional supergravity solutions. This should shed some light on the true nature of wormholes and how they emerge from string theory and M-theory, and it could also lay the basis for future studies about factorization in supergravity.

In section 1 we will revise the main features of wormholes, from their origins to the state of art of current research. We will begin by recalling some history, starting from their usual definition in Lorentzian spacetime. We will then pass to their Euclidean counterpart, define the Euclidean path integral and review how instantons emerge in quantum mechanics and quantum field theory. The similarities and differences with wormholes will be highlighted in order to write a low energy partition function incorporating topology changes. The resulting action is non-local, but locality can be restored introducing the so-called α parameters, which however appear to irremediably compromise the unitarity of our theory. Two main solutions could save the day: Coleman's α vacua, or the swampland baby universe conjecture. Finally, the factorization puzzle will be briefly mentioned.

In section 2 we will summarize the prominent features of the two low energy theories we will need in the work, namely $\mathcal{N} = 2$ supergravity in four and five dimensions. We will remind the particle content and the bosonic structure of the two theories. Then we will discuss in general the idea of Kaluza-Klein reduction and how it can be used to derive 5D supergravity from M-theory, and then 4D supergravity from 5D supergravity. This two steps reduction will make clear how typical axio-dilaton lagrangians admitting wormhole solutions naturally emerge from M-theory.

In section 3 we will study the theory thus obtained. We will see how solutions can be organized in three different classes and what are their properties. The core and original part of this work is the analysis of the uplift of such solutions, namely what they correspond to in the five dimensional perspective. We will make contact with other results in the literature and discuss how this theory provides useful insights about links between wormholes, black holes and anti-de Sitter vacua.

1 Wormholes, Black Holes and Swampland

1.1 Lorentzian wormholes

This entire work will be devoted to the study of *wormholes*. So the first question that should come to our mind is: what is a wormhole?

The word "wormhole" was used for the first time in 1957 by Wheeler and Misner in [2] when discussing Maxwell's equations in curved spacetime and paradoxical effects that could arise when electromagnetic fields are immersed in a non-trivial geometry. The authors consider a situation where the lines of force of an electric field pass through a handle and argue that, even in absence of sources, an observer near the mouth of this handle would incorrectly infer the existence of some charge. The particular concentration of the flux lines might make us believe that they end somewhere; instead, they are trapped inside this handle and they go on forever. Wheeler's original drawing is reported in figure 1.

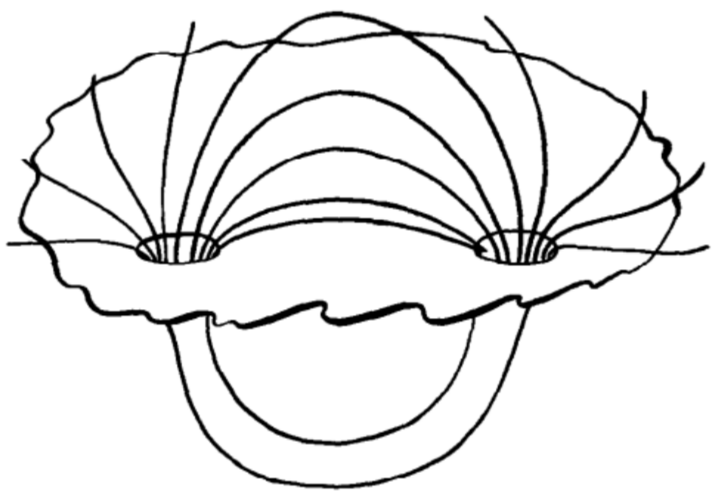

Figure 1: Wheeler's first drawing of a wormhole, from [2].

In some sense the electric flux and the topology of this strange handle, which Wheeler calls "wormhole", are strictly tied to each other. The entrapment of the electric field inside the wormhole guarantees that its lines cannot change in time unless the topology itself is changed, providing geometrical stability to this configuration. We could say that the wormhole is "supported" by the existence of this electric flux.

Despite coining the term, Wheeler was not the first one to discover these bizarre topological objects in general relativity. Forty years before, Schwarzschild had found his famous solution to Einstein's equations describing the space-time outside a spherical object of mass M (for a complete discussion on black holes, see [3] and [4]). Using spherical coordinates the metric reads:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2, \tag{1.1}$$

where $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the two-sphere volume element. This metric is not well defined for $r = 2GM =: r_s$, which is called "Schwarzschild radius", and $r = 0$. However, it can be proven that r_s is not a real singularity and the degeneracy of the metric at this point is just due to a poor choice of the coordinates. It is convenient to introduce the Kruskal-Szekeres coordinates:

$$\begin{aligned} UV &= -\left(\frac{r-2GM}{2GM}\right)e^{\frac{r}{2GM}} \\ \frac{U}{V} &= e^{-\frac{t}{2GM}}. \end{aligned} \tag{1.2}$$

Under this coordinate redefinition the metric becomes:

$$ds^2 = -\frac{32G^3M^3}{r}e^{-\frac{r}{2GM}}dUdV + r^2 d\Omega_2^2, \tag{1.3}$$

where here r must be thought as a function of U and V. In these coordinates the metric is perfectly regular at $r = 2GM$ and the only genuine singularity is manifestly $r = 0$. Most importantly, these coordinates can be defined far beyond the Schwarzschild radius and they represent the maximal extension of this spacetime. The diagram of the Kruskal-Szekeres spacetime is reported in figure 2.

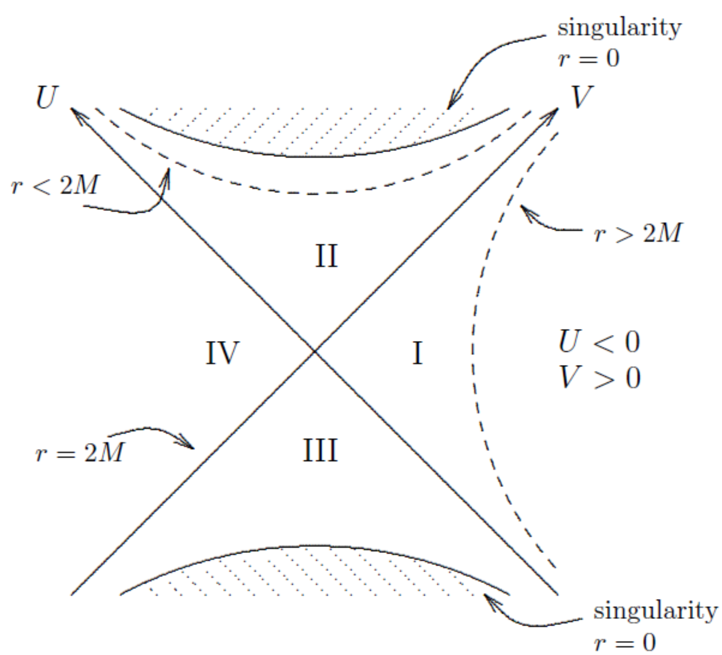

Figure 2: Kruskal-Szekeres spacetime in units $G = 1$, from [4].

The lines $U = 0, V = 0$ divide the Kruskal spacetime into four regions. Region I is the one covered by our orignal coordinates (t, r); region II is the interior of the black hole, where every object is doomed to fall towards the singularity; region III is the time-reversed version of region II (a *white* hole); region IV is an exact copy of region I. However, such portion of spacetime cannot be reached by an observer in region I, since there is no timelike path connecting the two regions. Still, there is something interesting happening between I and IV: the spacelike sections really look like Wheeler's handles! They are the first example of wormholes that was ever known and they are called "Einstein-Rosen bridges". They are indeed tube-shaped surfaces linking the two asymptotically flat regions I and IV. To better understand the properties of these hypersurfaces we introduce the isotropic coordinate ρ. This coordinate is such that the spatial part of the metric satisfies:

$$ds^2|_{dt=0} = f^2(\rho)(d\rho^2 + \rho^2 d\Omega_2^2) \Rightarrow r = \left(1 + \frac{GM}{2\rho}\right)^2 \rho, \tag{1.4}$$

so the metric becomes:

$$ds^2 = -\left(\frac{1 - \frac{GM}{2\rho}}{1 + \frac{GM}{2\rho}}\right)^2 dt^2 + \left(1 + \frac{GM}{2\rho}\right)^4 (d\rho^2 + \rho^2 d\Omega_2^2). \tag{1.5}$$

There are two values of ρ that correspond to the same value of r. These two values are exchanged under the transformation $\rho \to G^2M^2/4\rho$, which has a fixed point at $\rho_* = GM/2$. This has a precise physical meaning: the ρ coordinate parametrizes region I and IV of our space-time, and the transformation relating the two values can be seen as a $\mathbb{Z}_2$ isometry of our wormhole. As ρ approaches 0 or ∞ we reach the two asymptotically flat spacetimes of region I and IV, and at ρ_* we have a two-sphere of minimal radius that corresponds to the wormhole neck. To better visualize the wormhole we introduce the concept of embedding diagram, i.e. the geometry of each slice $t = \text{const.}, \theta = \pi/2$ as if it was immersed in a Euclidean space endowed with the metric:

$$ds^2 = dz^2 + dr^2 + r^2 d\phi^2, \tag{1.6}$$

so that the wormhole is a surface $z = z(r)$ with cylindrical symmetry. This is clarified in figure 3.

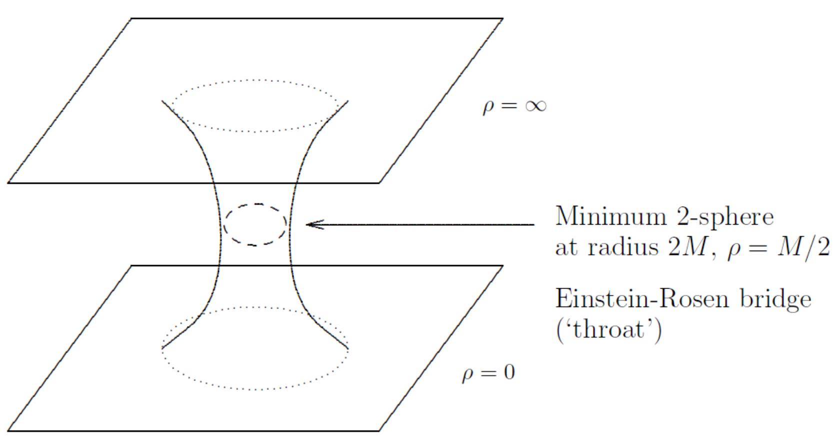

Figure 3: The Einstein-Rosen bridge in isotropic coordinates. Figure taken from [4].

As already mentioned, these wormholes cannot be traversed: this was shown for the first time by Wheeler and Fuller in [5]. By taking space-like hypersurfaces of constant $T := (U+V)/2$ in the Kruskal spacetime, at very early times the wormhole is still closed, then opens up for a bit but closes soon enough to prevent any object from going through it. This process is clarified in figure 4.

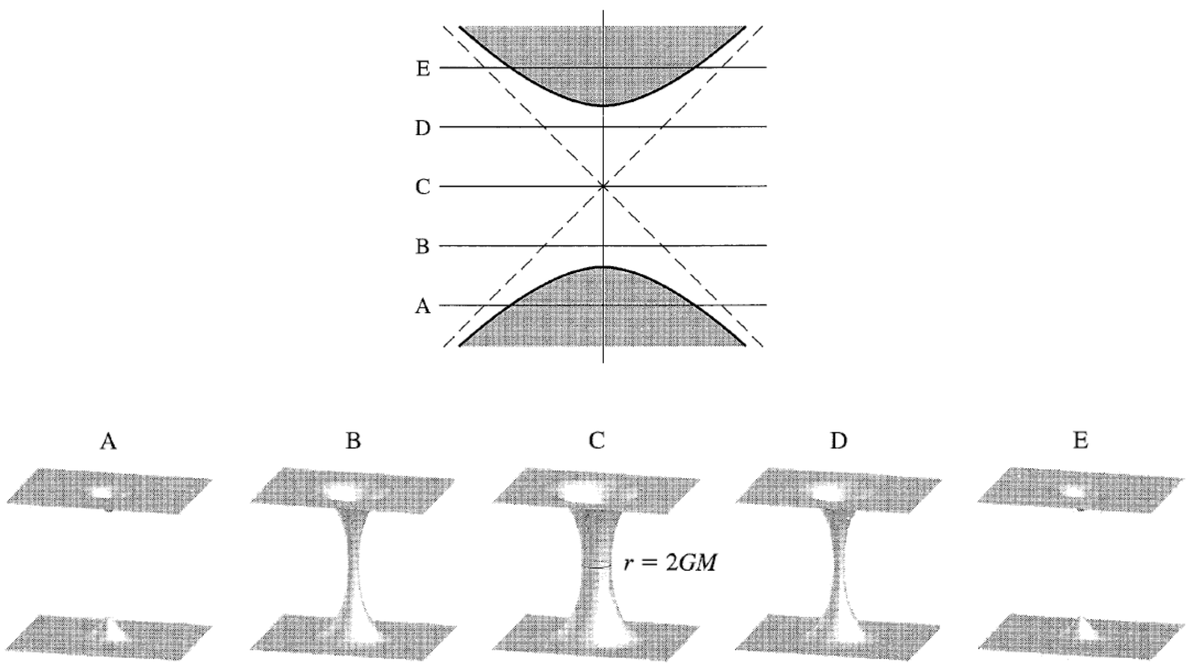

Figure 4: Constant T slicing in Kruskal-Szekeres spacetime. The wormhole connecting two asymptotic regions is closed at early times, then opens but closes again before any object can traverse it. Figure taken from [3].

Einstein-Rosen bridges emerge in many classes of solutions in general relativity. An example which will be very relevant for us is given by the Reissner-Nordström black hole, which is the solution of the Einstein-Maxwell theory. Now we will be more general and work in D space-time dimensions, since this will be useful later:

$$S = \frac{1}{16\pi G_D} \int d^D x \sqrt{-g}\, (R - F_{\mu\nu}F^{\mu\nu})\,, \tag{1.7}$$

where the D-dimensional Newton constant G_D has the dimension of length^{D-2}. The spherically symmetric solution of this system can be characterized in terms of two quantities, the black hole mass M and its charge Q, defined as suitable surface integrals. The metric as written in [6] reads:

$$ds^2 = -\left(1 - \frac{r_+^{D-3}}{r^{D-3}}\right)\left(1 - \frac{r_-^{D-3}}{r^{D-3}}\right) dt^2 + \left[\left(1 - \frac{r_+^{D-3}}{r^{D-3}}\right)\left(1 - \frac{r_-^{D-3}}{r^{D-3}}\right)\right]^{-1} dr^2 + r^2 d\Omega_{D-2}^2, \tag{1.8}$$

where we have defined:

$$r_{\pm}^{D-3} := \hat{M} \pm \sqrt{\hat{M}^2 - \hat{Q}^2}, \quad \hat{M} := G_D M, \;\; \hat{Q} := G_D^{\frac{2D-6}{2D-4}} Q. \tag{1.9}$$

In order to avoid naked singularities, the argument of the square root must be non negative. This implies that there are two coordinate singularities, which reduce to one in the extremal limit $\hat{M} = \hat{Q}$, while $r = 0$ is still a curvature singularity. To remove the coordinate singularities one could proceed as in Schwarzschild and define Kruskal-Szekeres coordinates. However, now we will have two sets of such coordinates: the first one allows us to describe the portion of spacetime covered by the horizon r_+ but not to go beyond r_-, while the second patch enables us to venture as close as we can get to the singularity $r = 0$. Differently from Schwarzschild space-time, the singularity at $r = 0$ is timelike and can be avoided; furthermore, it turns out that this spacetime can be extended infinitely if we keep gluing copies of itself, even though this will not happen in a realistic context of gravitational collapse. The properties of this space-time are summed up in figure 5.

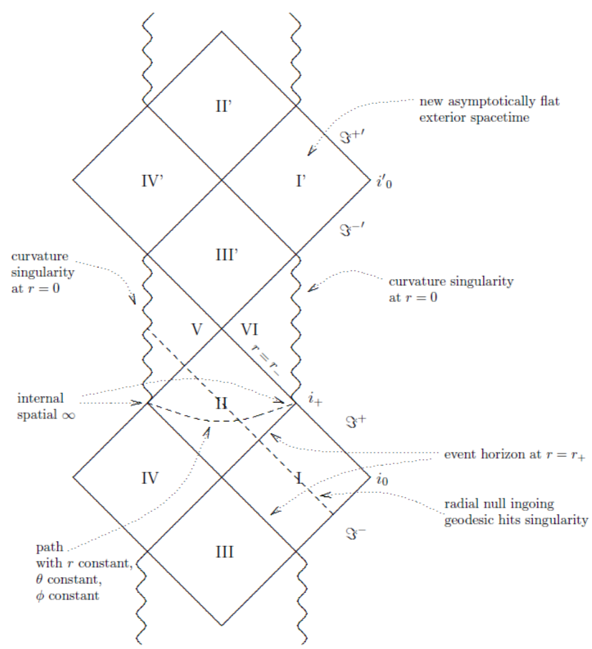

Figure 5: The maximally extended Reissner-Nordström spacetime. This extension goes on forever in both directions. Figure taken from [4].

Like before we can introduce again an isotropic coordinate ρ:

$$r = \left(\rho^{D-3} + \hat{M} + \frac{\hat{M}^2 - \hat{Q}^2}{4\rho^{D-3}} \right)^{\frac{1}{D-3}}, \tag{1.10}$$

and the metric now reads:

$$ds^2 = -\frac{1}{r(\rho)^{2(D-3)}} \left(\rho^{D-3} - \frac{\hat{M}^2 - \hat{Q}^2}{4\rho^{D-3}} \right)^2 dt^2 + \frac{r(\rho)^2}{\rho^2} (d\rho^2 + \rho^2 d\Omega_{D-2}^2). \tag{1.11}$$

Just like before, the spatial part of the metric describes an Einstein-Rosen bridge connecting regions I and IV, with an isometry $\rho^{D-3} \to \frac{\hat{M}^2-\hat{Q}^2}{4\rho^{D-3}}$ (see also [6]). In the extremal limit $\hat{M} = \hat{Q}$ the isometry loses its meaning, as the fixed point is virtually sent at $\rho = 0$. This implies that the wormhole neck becomes infinitely long and that only one of the two regions I and IV survives in this limit, as our spacetime is geodesically complete without the need to add a mirror image to the asymptotically flat region. This marks an important difference with respect to the Schwarzschild spacetime and it is a result of the existence of a conserved charge.

Of course, also in this case Einstein-Rosen bridges cannot be traversed. This is a common feature to other more general solutions of Einstein's equations, as the Kerr-Newman black hole. Nevertheless, in 1987 Morris and Thorne in [7] conjectured a solution of general relativity which could display a traversable wormhole. They considered a spherically symmetric ansatz of the form:

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1-\frac{b(r)}{r}} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{1.12}$$

The function $b(r)$ is called the "shape function" since it determines the wormhole form, while $\Phi(r)$ controls the redshift. The Morris-Thorne wormhole is then a surface with spherical symmetry $z = z(r)$ whose embedding diagram satisfies:

$$\frac{dz}{dr} = \pm\left(\frac{r}{b(r)} - 1\right)^{-\frac{1}{2}}. \tag{1.13}$$

Birkhoff's theorem guarantees that the only spherically symmetric solution to Einstein's equations in the vacuum is given by Schwarzschild spacetime. This implies that the Morris-Thorne geometry must be supported by some kind of energy. To ensure that we are really describing a wormhole we have to fulfill some requirements. There must be a minimum radius r_*, at which $r_* = b(r_*)$; furthermore we require that throughout spacetime $1 - b/r \geq 0$ and that $b/r \to 0$ as $r \to \infty$. We also demand that there are no horizons, so Φ must be everywhere finite: this is a necessary condition to have a traversable wormhole.

As it is described in detail in [7], nothing a priori forbids us to write down a space-time with these characteristics. However, Einstein's equations pose serious constraints on the kind of energy-momentum tensor that should generate such configuration. If ρ and p represent respectively the energy density and the radial pressure, then the requirement that the wormhole has no horizon at the throat and that it can be connected to an asymptotically flat spacetime implies that:

$$\rho(r_*) + p(r_*) < 0. \tag{1.14}$$

This is quite troublesome. As discussed in [8], a reasonable and general assumption in relativity is the weak energy condition, namely that $\rho + p \geq 0$. Systems that violate this condition are called "exotic" and they were thought to be impossible, at least classically. Evidence was found that some quantum systems can violate this condition (e.g. due to Casimir energy) as well as classical systems where fields are not minimally coupled to gravity.

Morris and Thorne's work marked an important milestone in the study of wormhole physics and led to renewed interest in this area of research. From that day on much progress has been made and now we know examples of traversable wormholes that do not require exotic matter, see for example [9]. Recent results also concern wormholes which could be traversed by a human being, as described in [10]. We are not going to enter into further details about this topic, since we will not need wormholes more complicated than those we have listed. Moreover, we are interested in a different and more modern conception of wormholes.

1.2 Euclidean wormholes and quantum gravity

So far the wormholes we have described had two main features: they were *Lorentzian* wormholes, namely particular geometries embedded in a Lorentzian spacetime, where there exists a timelike direction, and they arose in a purely classical context. These objects conform to our common conception of wormholes as passages between two universes and historically they were the first kind of wormholes to be ever discussed.

In more recent times, however, the developments in quantum gravity have brought the research to consider a different kind of wormholes, namely *Euclidean* wormholes. To better understand their role in theories of quantum gravity we will first revise some basic ingredients of the path integral approach and semiclassical configurations. This general discussion is based on [11], [12] and [13].

1.2.1 The gravitational path integral

Given a physical theory living in the classical realm there are two main ways to quantize it. The first one is canonical quantization, which consists in promoting dynamical quantities to operators on some Hilbert space satisfying appropriate (anti)commutation relations. The second one is the celebrated Dirac-Feynman path integral, which instead consists in summing on all possible configurations of our theory. If $S[\phi]$ is the action functional that describes the classical theory we can define the partition function:

$$Z = \int \mathcal{D}\phi \, e^{i\frac{S[\phi]}{\hbar}}. \tag{1.15}$$

The partition function in principle allows us to calculate quantum mechanical transition amplitudes, and not only that: the path integral has proven an incredibly useful tool to investigate the relation between classical and quantum physics. If we focus our attention on values of S that are large compared to $\hbar$ (or formally, if we let $\hbar \to 0$) we expect that only tiny fluctuations around classical solutions are relevant in our path integral. This is the key idea to understand the semiclassical behaviour of our theory and it will be relevant for the following.

However, as nice as it is, the partition function is not very manageable due to the oscillatory nature of the exponent. It is thus convenient to analytically extend the time to imaginary values via a Wick rotation $t = -i\tau$ and define a Euclidean partition function:

$$Z_E = \int \mathcal{D}\phi \, e^{-\frac{S_E[\phi]}{\hbar}}. \tag{1.16}$$

This in general guarantees a better convergence of the path integral. However, the situation is far more complicated when we try to quantize gravity. This can be seen immediately if we write down the euclidean gravity action:

$$S_E = -\frac{1}{16\pi G_D}\int_{\mathcal{M}} d^D x\sqrt{g}\,(R-2\Lambda) - \frac{1}{8\pi G_D}\int_{\partial\mathcal{M}} d^{D-1}x\sqrt{h}\left(K-K^0\right). \tag{1.17}$$

The first term is the standard Einstein-Hilbert action with the addition of a cosmological constant, while the second one is the Gibbons-Hawking-York boundary term that has to be added to make the variational problem well defined on the manifold boundary. h is the determinant of the metric induced on the boundary, K is the trace of the second fundamental form at the boundary and K^0 is the same trace but evaluated when the boundary is embedded in flat space. The problem with this action is that it is linear in the curvature, in contrast to what happens with Yang-Mills actions, and since the curvature is not bounded from below the action will not be either, as one can check by Weyl-rescaling the metric with a sufficiently large conformal factor. This means that in general the gravitational Euclidean path integral is not well defined. A possible way out of this is presented in [14]: the idea is to divide the possible metrics in equivalence classes under conformal transformations and then to integrate over a metric with finite Ricci scalar in each class.

However, we will not need to deal with these issues in the following, as we will not integrate over all possible metric configurations. We are interested in a semiclassical approach: the strategy is to find the saddles of the Euclidean action and expand around them. If one stays sufficiently close to these configurations the path integral is fine and no ad-hoc prescription is needed. This is the way we will follow from now on.

1.2.2 Path integral and instantons

We previously mentioned that the path integral is an incredible tool for investigating the links between classical and quantum mechanics. As a warm up we will now go through a couple of examples of these properties, and even though they seem to have nothing to do with wormholes they will turn out to be relevant for our goals.

The first example is provided by the the double well potential in single particle mechanics. We will follow [11] and [15]. Consider the action:

$$S = \int dt\left(\frac{1}{2}\left(\frac{dq}{dt}\right)^2 + \frac{\mu^2}{2}q^2 - \frac{\lambda}{4}q^4\right). \tag{1.18}$$

We wish to quantize this system by means of the path integral. If we Wick-rotate $t=-i\tau$ the action becomes:

$$S_E = \int d\tau\left(\frac{1}{2}\left(\frac{dq}{d\tau}\right)^2 - \frac{\mu^2}{2}q^2 + \frac{\lambda}{4}q^4\right). \tag{1.19}$$

Importantly the sign of the potential is inverted in the Euclidean action, so now the potential is flipped over. We want to calculate within the semiclassical approximation the correlation function:

$$\langle q(\tau_1)q(\tau_2)...q(\tau_n)\rangle = \frac{1}{Z}\int \mathcal{D}q(\tau)q(\tau_1)q(\tau_2)...q(\tau_n)e^{-S_E[q]}. \tag{1.20}$$

The first thing to do is then to find the saddles of our action. The equation of motion

$$\frac{d^2q}{d\tau^2} = -\mu^2 q + \lambda q^3 \tag{1.21}$$

admits constant solutions $q^{(1)} = 0, q^{(2)}_{\pm} = \pm\sqrt{\mu^2/\lambda}$. $q^{(1)}$ is a local maximum of the original potential, while $q^{(2)}_{\pm}$ are its two global minima. There is however another important solution we have to take into account. If we define the conserved "energy":

$$\mathcal{E} = \frac{1}{2}\left(\frac{dq}{d\tau}\right)^2 + \frac{\mu^2}{2}q^2 - \frac{\lambda}{4}q^4 \tag{1.22}$$

it is easy to check that:

$$q^{(3)} = \pm\sqrt{\frac{\mu^2}{\lambda}}\tanh\left(\frac{\mu(\tau-\tau_0)}{\sqrt{2}}\right) \tag{1.23}$$

is a solution of the equations of motion with $\mathcal{E} = 0$ for any constant τ_0. The solution with the plus sign (respectively minus) is called an instanton (anti-instanton) and it has some interesting properties: it interpolates between $q^{(2)}_-$ and $q^{(2)}_+$ as τ goes from $-\infty$ and $+\infty$ (viceversa for the anti-instanton). The variation in the profile of $q^{(3)}$ is limited in a very narrow region centered in τ_0, hence the name of "instanton". By explicit computation, it can be also shown that the on-shell value of the euclidean action S_E evaluated on $q^{(3)}$ is non-vanishing and independent on τ_0, which is a defining feature of instantons. The parameter τ_0 is completely free and is called the "collective coordinate" of the instanton: it represents the arbitrariness in our choice of the instanton location.

To proceed with the semiclassical calculation, we have to sum over fluctuations around the saddles we have found, so we expand $q(\tau) = q(\tau)_{saddle} + \delta q$. Let us first discuss what would happen if we only considered the two saddles $q^{(2)}_{\pm}$. The semiclassical partition function would be:

$$Z[\delta q] = Z_+[\delta q^{(2)}_+, J_+] + Z_-[\delta q^{(2)}_-, J_-], \tag{1.24}$$

where $J_{\pm}$ are source terms for the two fluctuations $\delta q^{(2)}_{\pm}$. This partition function represents two disconnected systems localized around the two global minima of the potential. This implies that any mixed correlator of the kind $\langle \delta q^{(2)}_+ ...\delta q^{(2)}_- ...\rangle$ vanishes: there is no mixing between the two subsystems. Of course, the situation changes if we also include the contribution of the instanton saddle:

$$Z[\delta q] = Z_+[\delta q^{(2)}_+, J_+] + Z_-[\delta q^{(2)}_-, J_-] + Z_{inst}[\delta q^{(3)}, J_3]. \tag{1.25}$$

Since $\delta q^{(3)}$ is expected to get closer to $\delta q^{(2)}_{\pm}$ for large τ, the instanton saddle introduces a correlation between the degrees of freedom around the two global minima and we will have in general $\langle \delta q_+^{(2)} ... \delta q_-^{(2)} \rangle \neq 0$. The instantonic configuration is then describing a *tunnelling* process between $q_-^{(2)}$ and $q_+^{(2)}$: this is the most important result of our semiclassical analysis and shows how instantons furnish some insight on the links between classical and quantum world.

As a concluding remark, we stress that this analysis is very qualitative and fails to calculate correlators in a precise way. A better result is obtained by summing over multi-instanton processes, where the particle bounces back and forth N times. This calculation shares a lot of similarities with a thermodynamic system of N weakly interacting particles, since instantonic bounces cannot be too many in a finite time interval, so this system is often called a "dilute instanton gas". To obtain physical quantities we have to sum over N and to integrate over all the possible locations of instantonic bounces. The position correlator reads:

$$\langle q(0)q(\tau) \rangle = \frac{\mu^2}{\lambda} e^{-\Delta E \tau}, \quad \Delta E \sim e^{-\frac{\mu^3}{\lambda}}, \tag{1.26}$$

where ΔE represents the splitting between the energies of the ground and first excited state. This result shows an essential singularity in λ, which signals the non-perturbative nature of instantons.

Despite its simplicity, this toy model contains all the details one has to deal with when encountering instanton calculations. Wormholes make no exception.

1.2.3 Instantons in quantum field theory

Before passing to wormholes we briefly discuss how instanton calculus is generalized to a quantum field theory, following [13]. We will consider the pure Yang-Mills theory:

$$S_E = \int d^4x \frac{1}{2g^2} \mathrm{Tr} F_{\mu\nu} F^{\mu\nu}. \tag{1.27}$$

For simplicity we will take $SU(2)$ as gauge group. We will not go through the full solution of the equations of motion, but it can be proven that there are classical solutions with finite action and a collective coordinate x_0 that rapidly go to zero as $|x - x_0|$ becomes large. These configurations are indeed instantons and they are identified by their topological charge:

$$\frac{1}{8\pi^2} \int \mathrm{Tr} F \wedge F = N \in \mathbb{Z}, \tag{1.28}$$

which is analogous to the number of instantonic bounces of the previous section. Configurations with positive N are instantons, those with negative N are anti-instantons. It must be stressed that in quantum field theory instantons are controlled by an additional parameter that determines their size, i.e. how quickly they die off from x_0. If this size is too large the dilute gas approximation fails; a possible way out is to Higgs the theory at $M \gg \Lambda$, where Λ is the confinement scale, so that the largest instantons have size $\sim 1/M$.

Another relevant example comes from stringy instantons (for a detailed discussion on instantons in string theory see [16]). If we have a gauge theory defined on $\mathbb{R}^4 \times S^1$ we can imagine tunnelling processes where a particle-antiparticle pair is created out of the vacuum and annihilates after passing around S^1 in opposite directions. From a string theory point of view such process corresponds to a D(-1) Euclidean brane wrapping the S^1 at some point $x_0 \in \mathbb{R}^4$, see figure 6.

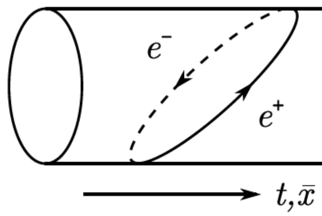

Figure 6: Euclidean brane instanton as particle-antiparticle fluctuation wrapping S^1. Figure taken from [13].

In both the above examples we expect a new scalar θ to couple to instantons. This scalar is called *axion*, it is endowed with a classical shift symmetry $\theta \to \theta + c$ and it is periodic in field space. In the case of Yang-Mills theories the axion is QCD-like and couples to instantons with a term like $\sim \theta\, \mathrm{Tr} F \wedge F$; in the case of string theory the axions are called fundamental and they arise from dimensional reductions of p-forms (more on this in sections 2 and 3). Instantons of this kind are as well called *fundamental* and they are zero-dimensional objects; in this context axions are viewed as zero-form gauge potentials which couple to instantons by evaluating them at the point where the instanton is inserted.

The partition function for fundamental instantons is obtained by summing over all topological charges, both positive and negative, and integrating over all collective coordinates:

$$\begin{aligned} Z = & \int \mathcal{D}\phi \mathcal{D}\theta e^{-S[\phi,\theta]} \sum_{N=1}^{\infty} \sum_{\overline{N}=1}^{\infty} \frac{1}{N!} \frac{1}{\overline{N}!} \prod_{i=1}^{N} \left(\int d^4 x_i \, M^4 e^{-S_I + i\theta(x_i)} \right) \prod_{\bar{i}=1}^{\overline{N}} \left(\int d^4 x_{\bar{i}} \, M^4 e^{-S_I - i\theta(x_{\bar{i}})} \right) = \\ = & \int \mathcal{D}\phi \mathcal{D}\theta \exp \left(-S[\phi,\theta] + \int d^4 x \, 2M^4 e^{-S_I} \cos\theta(x) \right). \end{aligned} \tag{1.29}$$

In this expression ϕ denotes all the other fields and S_I is the instanton action; the product $M^4 e^{-S_I}$ may be thought as the instanton density. The dilute gas approximation lets us resum the instanton contributions to obtain a correction to the effective action. This contribution, of pure non-perturbative origin, explicitly breaks the axion shift symmetry. We are familiar with this result from the QCD case, where the axion potential (and thus its mass) stems from the chiral anomaly.

1.2.4 Wormholes as instantons: the Giddings-Strominger solution

What does all this discussion about instantons have to do with wormholes?

It occurs that in the gravitational path integral we should include also the configurations that lead to topological changes in our manifold. This is forbidden classically in a Lorentzian setting, as shown in [17], since the resulting spacetimes are highly singular and/or exhibit causality problems. Nonetheless we expect that in Euclidean signature such topology changes may be possible via quantum tunnelling: we can imagine a process in which a baby universe with an S^3 topology splits off a parent universe. At this point this baby universe can either end on another parent universe or asymptotically rejoin the same parent universe, depending on how we glue the two halves of our configuration (figure 7).

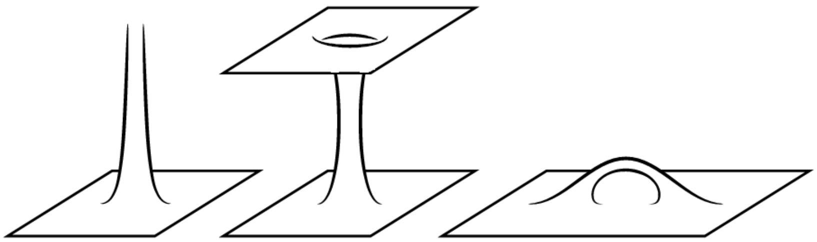

Figure 7: On the left, the birth of a baby universe; at the center, a wormhole connecting two different asymptotic universes; on the right, a wormhole connecting two parts of the same universe. Figure taken from [13].

As we previously mentioned, in a semiclassical analysis tunnelling processes between asymptotic regions, such as $q_{\pm}^{(2)}$ in the double well potential, are mediated by instantons, i.e. saddles of the Euclidean path integral with a non trivial topology. This is also what happens in quantum gravity, where such Euclidean saddles are wormholes. They really are wormholes in the sense that they connect two asymptotic regions as it happens for Lorentzian wormholes, but now they arise in a purely Euclidean theory and are not embedded in a Lorentzian spacetime. Indeed, these wormholes are rightfully called "gravitational instantons", even though there are some important distinctions between ordinary instantons and Euclidean wormholes, which exhibit a rich phenomenology to be later explored.

The first example of wormholes in Euclidean quantum gravity was given by Giddings and Strominger in [18], though we will follow the discussion of [13]. In order to have instanton configurations we cannot simply rely on pure gravity since the Einstein-Hilbert action shows no saddles (for a proof see [19]), so the idea of Giddings and Strominger was to add an axion:

$$S_E = -\frac{M_p^2}{2}\int *R + \frac{f^2}{2}\int d\theta \wedge * d\theta, \tag{1.30}$$

where f is the axion decay constant. However, it is convenient to pass to a dual formulation, where the axion is traded for a two-form $\mathcal{B}_2$ which satisfies $\mathcal{H}_3 = d\mathcal{B}_2 = -f^2 * d\theta$. This dualization must be done in the path integral (see e.g. [20]) and the result reads:

$$S_E = -\frac{M_p^2}{2}\int *R + \frac{1}{2f^2}\int \mathcal{H}_3 \wedge * \mathcal{H}_3. \tag{1.31}$$

This formulation has some advantages. The three-form $\mathcal{H}_3$ is real in the Euclidean setting, while the axion θ is imaginary, and this introduces some subtleties about complex saddles (again, more details in [20]), so while the formulation in terms of $\mathcal{B}_2$ is well posed the one in terms of θ may not be. We will ignore these issues and just focus on the solution.

Assuming radial symmetry, if an instanton couples to an axion the three form carries a non-zero flux across any three-sphere surrounding the instanton:

$$\frac{1}{2\pi}\int_{S^3}\mathcal{H}_3 = q \in \mathbb{Z}. \tag{1.32}$$

Here q is the analogous of the instanton charge N of the previous example, but since it is now an axionic charge supporting the wormhole we will stick to this new notation. Assuming that the instanton is localized at the origin, by radial symmetry the three-form must be:

$$\mathcal{H}_3 = \frac{q}{\pi}\mathrm{vol}_{S^3}, \tag{1.33}$$

where vol_{S^3} is the volume form of the three-sphere. The metric reads:

$$ds^2 = \left(1 - \frac{C}{r^4}\right)^{-1} dr^2 + r^2 d\Omega_3^2, \qquad C = \frac{q^2}{6\pi^2 f^2 M_p^2}. \tag{1.34}$$

This metric profile is common to many axionic theories; a full derivation will be given in section 3. It is asymptotically flat and it displays a coordinate singularity at $r^4 = C$, which coincides with the wormhole neck where the three-sphere radius is minimum. Because of this singularity this metric actually describes a semiwormhole (see figure 7), but we can glue two of such profiles to obtain the full wormhole.

Thus, the Giddings-Strominger wormhole represents an explicit realization of a gravitational instanton, a saddle of the Euclidean partition function. Now the question is: what is the deep meaning of these objects?

1.3 The α multiverse

1.3.1 The bilocal path-integral

Even though the analogy between instantons and wormholes is well-motivated, there is a crucial feature that makes the two objects very different: while an instanton is something localized at one point, a wormhole relates two spacetime points, because a baby universe is emitted at a point x and reabsorbed at a different point y. The consequence is that in the low-energy theory instantons will produce local corrections to the effective action, while wormhole will not.

Hence, we expect the instanton partition function (1.29) to be generalized to wormholes in the following way, taking the contribution of one wormhole for simplicity:

$$Z_{1w} = \int \mathcal{D}g\mathcal{D}\phi e^{-S[g,\phi]} \int d^4x\, d^4y\, \sqrt{g(x)}\,\sqrt{g(y)}\, e^{-S_w[x,y,\phi,g]}. \tag{1.35}$$

Now ϕ represents all the matter content of our theory, including axions. $S_w[x, y, \phi, g]$ is the action of a single wormhole connecting x and y and evaluated on the background field ϕ. We are integrating over metrics g that are soft on the topologically trivial background universe into which the wormhole is inserted, where "soft" means that the metric varies on scales larger than the wormhole length L.

$S_w[x,y,\phi,g]$ is in general a very complicated expression and would require us to know the wormhole action in a given background field configuration, which is practically impossible. However, if we assume that all the fields are soft with respect to the wormhole size we can expand S_w as a double series in a basis of gauge-invariant local operators $\mathcal{O}_i$:

$$S_w[x,y,g,\phi] = S_w^{(0)} + \sum_{i,j} C_{ij}\mathcal{O}_i(x)\mathcal{O}_j(y), \tag{1.36}$$

where $S_w^{(0)}$ is the wormhole action in absence of background field, and the coefficients C_{ij} are assumed to be independent of x and y, which is true in a dilute wormhole approximation where the two points are very far apart. We can also write:

$$e^{-S_w[x,y,g,\phi]} = \frac{1}{2}\sum_{i,j}\Delta_{ij}\mathcal{O}_i(x)\mathcal{O}_j(y) \tag{1.37}$$

because when we expand $\exp(C_{ij}\mathcal{O}_i(x)\mathcal{O}_j(y))$ we get all the powers $\mathcal{O}^n$; but if $\mathcal{O}_i$ is a complete basis of operators, the term $\mathcal{O}^n$ is already included in such basis. The coefficients Δ_{ij} are then linear combinations of C_{ij} with suitable rescalings of $\exp(-S_w^{(0)})$. We have thus obtained a "bilocal" expansion for the wormhole action, reducing the complexity of $S_w[x,y,g,\phi]$. If we insert this expression in (1.35) and we sum over all wormholes an exponential appears as in the case of instantons and we get:

$$Z_w = \int \mathcal{D}g\mathcal{D}\phi\, e^{-S[g,\phi]+I}, \qquad I = \frac{1}{2}\int d^4x\, d^4y\sqrt{g(x)}\,\sqrt{g(y)}\sum_{i,j}\Delta_{ij}\mathcal{O}_i(x)\mathcal{O}_j(y). \tag{1.38}$$

1.3.2 Coleman's α parameters

The above result confirms both the deep analogies and differences with the instantonic partition function (1.29): we were able to resum all the contributions of a diluted gas of wormholes to a correction to the low-energy effective action, but this correction depends on two points rather than one. We are not very much pleased with non-local interactions and we would like to know whether we can write everything in terms of local operators. The answer is yes, using the Hubbard-Stratonovitch transformation (we will omit sum symbols from now on):

$$\exp\left(\frac{1}{2}\Delta_{ij}V_iV_j\right) = \int\prod_k d\alpha_k \exp\left(-\frac{1}{2}\Delta^{-1}_{ij}\alpha_i\alpha_j\right)\exp\left(\alpha_iV_i\right). \tag{1.39}$$

We have introduced the auxiliary parameters α_i, which importantly do not depend on spacetime coordinates, while Δ^{-1}_{ij} is of course the inverse matrix of Δ_{ij}. By inserting this in (1.38) we obtain:

$$Z_w = \int \mathcal{D}g\mathcal{D}\phi\prod_k d\alpha_k \exp\left(-\frac{1}{2}\Delta^{-1}_{ij}\alpha_i\alpha_j\right)\exp\left(-S[g,\phi]+\alpha_i\int d^4x\sqrt{g}\mathcal{O}_i(x)\right). \tag{1.40}$$

To clarify the nature of these α parameters we can expand the action $S[g,\phi]$ in the same operatorial basis we used for the wormhole action:

$$S[g,\phi,\lambda] = \lambda_i \int d^4x \sqrt{g}\, \mathcal{O}_i(x), \tag{1.41}$$

where we have made explicit the dependence on the coupling constants λ_i. So the partition function can be cast in the form:

$$\begin{aligned} Z_w &= \int \mathcal{D}g\mathcal{D}\phi \prod_k d\alpha_k \, \exp\left(-\frac{1}{2}\Delta^{-1}_{ij}\alpha_i\alpha_j\right) \exp\left(-(\lambda_i - \alpha_i)\int d^4x \sqrt{g}\mathcal{O}_i(x)\right) \\ &= \int \mathcal{D}g\mathcal{D}\phi \prod_k d\alpha_k \exp\left(-\frac{1}{2}\Delta^{-1}_{ij}\alpha_i\alpha_j\right) \exp\left(-S[g,\phi,\lambda-\alpha]\right). \end{aligned} \tag{1.42}$$

The simple mathematical trick we have used turns out to have meaningful physical consequences. We have traded the non-locality of our action for a new set of parameters α_i, which represent an effective shift to the coupling constants λ_i of our original theory without wormholes. This shift is weighted by a gaussian factor that may be interpreted as a probability distribution for the α parameters. The coupling constants of our theory have then become random variables! This randomness cannot be measured, since any measurement of a coupling constant λ_i in our universe will always give the same value. The probability distribution should instead be interpreted as an ensemble of universes with different initial conditions, and in each universe a given value of λ will be more or less likely depending on such distribution. One could then object that this probability distribution is actually useless, since the coupling constants in our universe are set once and for all and should just be measured. But the point is that we can extract statistical information on the couplings and hope they are peaked at some value, for example the cosmological constant - again, see [13] for all the details.

Until now, all this story about α parameters seems to be a mere mathematical artifact arising from a convenient Hubbard-Stratonovitch transformation. But they must be related to wormholes somehow: after all, α parameters have been introduced as a useful way to reabsorb non-locality, which was an intrinsic feature of wormholes. To elucidate of α parameters we can define a Fock space of baby universes endowed with the usual bosonic creation/annihilation operators:

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \quad [a_i, a_j^\dagger] = \delta_{ij}. \tag{1.43}$$

A state with n baby universes of type i is then obtained applying n times the corresponding operator to the vacuum state, where no baby universe is present:

$$|n_i\rangle = \frac{(a_i^\dagger)^{n_i}}{\sqrt{n_i!}}|0\rangle. \tag{1.44}$$

For simplicity we will now assume that there is only a single type of baby universe created by the operator $a^\dagger$, then the results will be immediately generalized to the case of an arbitrary number of operators. We can define the "position" operator and its eigenvectors:

$$\hat{\alpha} = a + a^\dagger, \qquad \hat{\alpha}|\alpha\rangle = \alpha|\alpha\rangle. \tag{1.45}$$

The wormhole action in this case is written as:

$$e^I = \int d\alpha \exp\left(-\frac{1}{2}\Delta^{-1}\alpha^2 + \alpha\int d^4x\,\sqrt{g}\,\mathcal{O}(x)\right) \to \int \frac{d\alpha}{2\pi}\exp\left(-\frac{1}{2}\alpha^2 + \sqrt{\Delta}\alpha\int d^4x\,\sqrt{g}\,\mathcal{O}(x)\right) \tag{1.46}$$

where we have rescaled $\alpha \to \sqrt{\Delta}\alpha$ and divided the measure by 2π for future convenience. This action represents the sum of processes where an arbitrary number of baby universes is created and annihilated. In fact, we are going to show that e^I can be interpreted as a vacuum-to-vacuum transition of the baby universe state. By usual arguments of the harmonic oscillator the vacuum state can be written as:

$$|0\rangle = \int \frac{d\alpha}{\sqrt{2\pi}}\exp\left(-\frac{\alpha^2}{4}\right)|\alpha\rangle. \tag{1.47}$$

By the definition of α-eigenstates it follows that:

$$\langle 0|\,\exp\left(\hat{\alpha}\tilde{\mathcal{O}}\right)|0\rangle = \int \frac{d\alpha}{2\pi}\exp\left(-\frac{\alpha^2}{2} + \alpha\tilde{\mathcal{O}}\right) = e^I, \tag{1.48}$$

where we have defined $\tilde{\mathcal{O}} := \sqrt{\Delta}\int d^4x\,\sqrt{g}\,\mathcal{O}$. So the wormhole action can be thought of as the matrix element of a process in which at the initial time there are no baby universes, an arbitrary number of them is created at intermediate times, and finally all of them get annihilated, returning to the vacuum state.

This reasoning can be generalized to any initial and final state and to any number of baby universe types. The transition amplitude between an initial state $|\psi_i\rangle$ and a final state $|\psi_f\rangle$ in this scenario will be given by:

$$\langle\psi_f|\,\exp\left(\sqrt{\Delta_{ii}}\int d^4x\,\sqrt{g}\,\mathcal{O}_i(x)(a_i + a^\dagger_{i^*})\right)|\psi_i\rangle. \tag{1.49}$$

We have chosen a diagonal basis of operators. Here i^* represents the CPT conjugate of i: we cannot distinguish between the annihilation of a baby universe of type i and the creation of a baby universe of type i^*, so only the combination $\hat{\alpha}_i := a_i + a^\dagger_{i^*}$ can appear in every matrix element. From this expression we can infer that $\mathcal{O}_i(x)(a_i + a^\dagger_{i^*})$ represents the hamiltonian governing the evolution of the state from initial to final time. It was first derived by Coleman in [21] by summing over all wormhole and semiwormhole configurations.

1.3.3 α vacua

The $\hat{\alpha}_i$ operators mutually commute, so they can be simultaneously diagonalized on convenient eigenstates, the so-called α vacua:

$$\hat{\alpha}_i|\alpha\rangle = \alpha_i|\alpha\rangle. \tag{1.50}$$

Importantly, Coleman's hamiltonian commutes with the $\hat{\alpha}_i$, so the states $|\alpha\rangle$ are unchanged under time evolution. This solves an important problem linked to loss of quantum coherence; we will follow Coleman's example in [21]. Suppose we have only one operator in our effective hamiltonian of the form $\phi^4(a+a^\dagger)$, where ϕ is a scalar field describing some meson. Suppose we make a scattering experiment with two mesons. For some reason we know that the initial state contains only one baby universe. After the scattering, the final state will be a superposition of states with zero and two baby universes. The situation can be summed up in the following way:

$$|\text{in}\rangle = |\phi_{\text{in}}, 1\rangle \;\Rightarrow |\text{fin}\rangle = |\phi_0, 0\rangle + |\phi_2, 2\rangle \tag{1.51}$$

where ϕ_0 and ϕ_2 are meson states and are in general different. An observer will not be able to measure the baby universe state, so effectively we have to trace it out. But this implies that the aforementioned observer will see a pure state before the scattering and a mixed state soon after! This apparent loss of unitarity is solved if we assume that we live in an α state, which is untouched by time evolution. Each set of α parameters then represents a different superselection sector and a unitary quantum field theory, each distinguished by the others in the values of the coupling constants.

But what if the initial state is instead a combination of α states? In this case the relative phases between α states would change after the scattering. But these relative phases could not be known from the start, so the quantum coherence that has been lost was never possessed to begin with.

1.4 The Swampland Baby Universe Conjecture

Coleman's solution to loss of unitarity is appealing. However, it could leave us unsatisfied for some good reasons: α parameters are a priori arbitrary, and this clearly contradicts our experience with quantum gravity, where no free parameters are allowed. In string theory this is motivated by one of the core concepts of the Swampland program (see [22] for a detailed review): the no global symmetry conjecture states that in string theory every symmetry must be either broken or gauged. Indeed, this immediately implies that there are no free parameters in string theory and every coupling constant is the expectation value of some field. Free parameters may be seen as "-1 - form symmetries" whose conserved current is a 0-form. For instance, if we have a Lagrangian of the form $\lambda_i \mathcal{L}_i(x)$ we should be able to gauge this expression to obtain $\lambda_i(x)\mathcal{L}_i(x)$, where λ has been promoted to a dynamical field. This is precisely the kind of situation we ran into when discussing Coleman's hamiltonian. So, can we view α parameters as expectation values of some dynamical field $\alpha(x)$ and conclude that Coleman's theory is safe from a stringy perspective? Unfortunately, the answer is no. If this was the case we should be able to deform $\alpha(x)$ so that in some finite region of space it differs from its expectation value (this is guaranteed by Swampland cobordism triviality, [23]). But the α parameters are the eigenvalues of the operator $\hat{\alpha}$ that is spacetime independent, otherwise a baby universe would carry energy and momentum. So the α parameters are a global -1 - form symmetry, which is forbidden by the no global symmetry conjecture. In conclusion, a theory in the landscape cannot have randomly distributed α parameters.

There is perhaps only one way to make baby universe physics consistent with the Swampland program, which was proposed mainly in [23] and [25]. This conjecture can be summarized in the following statement:

In a unitary theory of quantum gravity in $d > 3$, the baby universe Hilbert space $\mathcal{H}_{BU}$ is one-dimensional.

If this was indeed the case, no arbitrary α parameter could show up and the theory would remain in the landscape. But how could it be that $\mathcal{H}_{BU}$ has dimension one if wormholes can start and end wherever they want? We have to keep in mind that until now we have studied low energy theories. In a more complete theory if one defines an inner product via the path integral using an approach à la Wightman it may happen that there are states with zero norm which we should quotient out in the physical Hilbert space. These null states could so many that the resulting Hilbert space ends up being one-dimensional.

It is also true that viceversa $\dim\mathcal{H}_{BU} = 1$ implies the no global symmetry conjecture. By contradiction, suppose we have a theory of gravity with a global symmetry. Baby universes can carry a global charge, as argued by Coleman in [21]. So the baby universe Hilbert space is decomposed into mutually orthogonal subspaces with different charge. This immediately implies that $\dim\mathcal{H}_{BU} > 1$.

1.4.1 The factorization puzzle

There are other good reasons to believe to the baby universe conjecture. Many come from holography: baby universes would destroy the cluster decomposition principle, which in the language of AdS/CFT corresponds to factorization of CFT partition functions. If X_1 and X_2 are two disjoint sets, we expect that $Z(X_1 \sqcup X_2) = Z(X_1)Z(X_2)$. But the disconnected partition function receives contributions also from connected bulk geometries, so there must be miraculous cancellations to ensure the expected factorization. Showing that this indeed happens is highly non-trivial and is still a matter of current research. However, the baby universe conjecture is a sufficient condition to ensure the factorization, as proven in [23] and [25]. The idea is to start from the no-boundary Hartle-Hawking state $|\mathrm{HH}\rangle$ (for a rigorous definition see [24]) as a vacuum state. One then defines a family of operators $\hat{Z}(X)$ acting on the vacuum by adding an additional boundary X, so that the matrix elements of $\hat{Z}(X)$ are obtained evaluating the path integral with the boundary X. It can be shown that the eigenstates of $\hat{Z}(X)$ are analogous to Coleman's α vacua, i.e. $\hat{Z}(X)|\alpha\rangle = Z_\alpha(X)|\alpha\rangle$. Therefore the partition function is:

$$Z(X) = \langle \mathrm{HH}| \hat{Z}(X) |\mathrm{HH}\rangle = \sum_\alpha |\langle \alpha|\mathrm{HH}\rangle|^2 Z_\alpha(X). \tag{1.52}$$

If we insert two boundaries instead of one we get:

$$Z(X_1 \sqcup X_2) = \langle \mathrm{HH}| \hat{Z}(X_1)\hat{Z}(X_2) |\mathrm{HH}\rangle = \sum_\alpha |\langle \alpha|\mathrm{HH}\rangle|^2 Z_\alpha(X_1)Z_\alpha(X_2), \tag{1.53}$$

so the partition function does not factorize because the Hartle-Hawking state is not an eigenstate of the boundary operator. However, if we considered an α vacuum instead of $|\text{HH}\rangle$ state the factorization would indeed take place! This is a restatement of Coleman's idea that each α vacuum defines a local and unitary quantum field theory. But if the baby universe Hilbert space was one-dimensional the Hartle-Hawking state would be its only state and factorization would always happen.

Of course, this is a formal argument. To ensure that factorization actually occurs one needs several cancellations between contributions to the path integral with different topologies, which looks a fine tuning problem from the low energy perspective. The point is that wormholes come inevitably with α parameters, which codify a disorder over possible theories. So if wormholes give a net non-zero contribution to the path integral we are effectively describing a classical average of quantum systems. This is not compatible with the standard AdS/CFT picture, where quantum systems are unique and do not exhibit ensemble properties.

Some progress has been made on the factorization of some quantities, mainly in [26]. In that paper the authors show two important results: first, that the gravitational path integral receives perturbative contributions from a limited class of smooth saddles obeying suitable boundary conditions; second, that the factorization issue is solved for the Witten index:

$$\text{Index}(\beta) := \text{Tr}_Q \left[(-1)^F e^{-\beta H}\right], \tag{1.54}$$

where Tr_Q computes the trace in the sector of charge Q, $(-1)^F$ is the fermion number operator and H is the hamiltonian in Euclidean time. More specifically, the Witten index of a four-dimensional black hole does not receive contribution from topological changes in the AdS_2 near horizon region, so this index factorizes between disconnected boundaries even in presence of wormholes. Both these remarkable results are guaranteed by the presence of a gravitino zero mode, which makes the one-loop determinant vanish on each connected boundary. This work has relevant links with JT supergravity, a two-dimensional model that includes the coupling of a scalar Φ to gravity of the form ΦR. Dimensional reduction has proven an essential tool for understanding the relations between the partition functions of different theories.

Although very important, this achievement is just a first step in the difficult problem of factorization. The analysis was carried out in the case of pure $\mathcal{N} = 2$ supergravity, without any additional vector multiplet or hypermultiplet, so it would be interesting to understand whether it holds with a more general matter content. Furthermore, the calculation is definitely harder in higher dimensions or in geometries far from extremality.

From this discussion it should be clear that the importance of the gravitational path integral can be hardly overestimated. The most relevant features needed to investigate the properties of a gravitational system are the saddles - and fluctuations around them - and non-perturbative objects like wormholes, which must be carefully inspected to see whether factorization is realized or not.

This is why in the next sections we are going to study a concrete model of low energy effective field theory admitting wormhole solutions. The questions we would like to examine are the following: what are the characteristics of these saddles? How are wormholes related to higher dimensional theories? What are the links between Euclidean and Lorentzian wormholes?

What are the relations of these solutions to the Euclidean path integral and the machinery coming from holographic arguments?

In section 2 we will revise some essential concepts of supergravity in four and five dimensions, which are the effective theories where our model takes place. We will see how a typical axio-dilaton wormhole action studied in [34] can be obtained by dimensional reduction from five dimensions. In section 3 we will discuss the relevant solutions of this model and see what are their corresponding configurations in the five-dimensional theory. This system shows a lot of connections to other well-known results, like BPS black-holes, and proves an incredible laboratory for future developments in holography, JT gravity, black holes and axion physics.

2 N=2 supergravity in four and five dimensions

Since the main concern of this work is to study some gravitational solutions in four dimensions and see how they can uplift to five dimensions, it is useful to summarize the prominent features of $\mathcal{N}=2$ supergravity in four and five dimensions and the relation between the two. We will also discuss how 5D supergravity may be obtained from dimensional reduction of the more fundamental M-theory. This discussion is largely taken from [27].

2.1 N=2 5D supergravity

Let us start our journey with 5D supergravity [1]. We have to remember that in five dimensions there is no notion of chirality and that one cannot impose Majorana or Weyl conditions on fermions. We can however introduce symplectic Majorana spinors: if we have an even number of spinors $\chi^i, i = 1...2m$ and a non-singular antisymmetric matrix ε^{ij} we can impose the condition:

$$\chi^i = \varepsilon^{ij}(\chi^j)^C \tag{2.1}$$

where χ^C denotes the charge-conjugate of χ, i.e. $\chi^C = (iC\gamma^0)^{-1}\chi^*$ with C the charge conjugation matrix obeying $C^t = -C = C^{-1}$ (our conventions are taken from [28]). In five-dimensional supergravity one always has to deal with an even number of fermions, so we can safely impose this symplectic Majorana condition.

Keeping this in mind, the particle content is made up of three main ingredients:

- the gravity multiplet $(e^a_\mu, \psi^1_\mu, \psi^2_\mu, A_\mu)$, which contains the graviton, two gravitini and a vector field called graviphoton. The gravitini sit in the fundamental representation of the $SU(2)$-R symmetry group;

- the vector multiplet $(A'_\mu, \lambda^1, \lambda^2, \phi)$, which contains a gauge field, two fermions named gaugini and a real scalar. Similarly to the gravitini, also the gaugini are a doublet under $SU(2)_R$;

- the hypermultiplet $(\zeta^{1,2}, q^{1,2,3,4})$, which contains two fermions named hyperini and four real scalar fields. The hyperini are unaffected by $SU(2)_R$, while the four hyperscalars transform as two doublets.

In the effective field theory we are considering we are looking at the Coulomb branch of the moduli space, the space of vacua of our theory. In this context, the gauge group reduces to an abelian one, and since we are interested in an effective description all massive excitations will be discarded. Thus we will consider a theory with n_V $U(1)$ vector multiplets coupled to gravity, while typically hypermultiplets will not be taken into account. This is possible because the scalar manifold is the direct product $\mathcal{M}_V \times \mathcal{M}_H$, where $\mathcal{M}_V$ is the manifold of the vector scalars and $\mathcal{M}_H$ is the manifold of the hyperscalars. We thus learn that there is no mixing in the kinetic matrices of the two sectors and we can simply set the hypermultiplets to zero.

[1] When we say $\mathcal{N}=2$ supergravity in 5D we mean minimal supergravity.

Let us now remind some geometric properties of $\mathcal{M}_V$. This manifold lives in a n_V+1 dimensional ambient space parametrized by coordinates $Y^a, a = 0...n_V$. We define the prepotential

$$\mathcal{V} = \frac{1}{6}C_{abc}Y^aY^bY^c \tag{2.2}$$

where C_{abc} is a symmetric symbol encoding integer Chern-Simons coefficients, which in M-theory count intersections of the Calabi-Yau threefold, see section 2.4 and the appendix 4. The scalar manifold is the level set $\mathcal{V} = 1$, admitting a system of n_V local coordinates which can be identified with the scalar fields ϕ^i; basically, we are solving the constraint $\mathcal{V} = 1$ by writing the $n_V + 1$ coordinates Y^a in the ambient space in terms of the n_V coordinates ϕ^i living in the physical space. Indeed, the index a has one more component with respect to i, namely $a = 0$, stemming from the graviphoton. Following the conventions of [29], the bosonic part of the action is then the most general one we can write with the features we have listed above:

$$\begin{aligned} S^{(5d)}_{bosonic} = \frac{2\pi}{l_5^3}\int\left[*R - \frac{1}{2}\tilde{G}_{ij}(\phi)d\phi^i\wedge *d\phi^j\right] - \frac{1}{4\pi l_5}\int G_{ab}(\phi)F^a\wedge *F^b \\ - \frac{1}{6(2\pi)^2}\int C_{abc}\,A^a\wedge F^b\wedge F^c\,. \end{aligned} \tag{2.3}$$

All the kinematic quantities are derived from the prepotential via the following relations. In particular, the metric for the vector fields is given by:

$$G_{ab} = -\frac{\partial}{\partial Y^a}\frac{\partial}{\partial Y^b}\log\mathcal{V}|_{\mathcal{V}=1} = \left[\frac{\partial\mathcal{V}}{\partial Y^a}\frac{\partial\mathcal{V}}{\partial Y^b} - \frac{\partial^2\mathcal{V}}{\partial Y^a\partial Y^b}\right]|_{\mathcal{V}=1} = \frac{1}{4}C_{ade}Y^dY^eC_{bfg}Y^fY^g - C_{abd}Y^d, \tag{2.4}$$

while the metric for the scalar fields is simply the pullback of G_{ab} on the space of physical coordinates ϕ^i:

$$\tilde{G}_{ij} = G_{ab}\frac{\partial Y^a}{\partial\phi^i}\frac{\partial Y^b}{\partial\phi^j}|_{\mathcal{V}=1}. \tag{2.5}$$

In order to have a physical sensible theory we need to restrict ourselves to the region of the moduli space in which the kinetic matrices G_{ab} and $\tilde{G}_{ij}$ are positive definite.

It is meaningful to cite some identities that will be useful later on, which can be also found in [37]. First of all we introduce the dual coordinate:

$$\tilde{Y}_a := \frac{1}{2}C_{abc}Y^bY^c. \tag{2.6}$$

By the definition of the metric and the constraint of the prepotential it follows immediately that:

$$d(C_{abc}Y^aY^bY^c) = 0 \Rightarrow C_{abc}dY^aY^bY^c = 0 \Rightarrow Y^a\frac{\partial\tilde{Y}_a}{\partial\phi^i} = \tilde{Y}_a\frac{\partial Y^a}{\partial\phi^i} = 0. \tag{2.7}$$

The metric can be also cast in the form:

$$G_{ab} = -C_{abc}Y^c + \tilde{Y}_a\tilde{Y}_b \tag{2.8}$$

from which it immediately follows that:

$$\tilde{Y}_a = G_{ab}Y^b \;\rightarrow\; G_{ab}Y^aY^b = 3 \tag{2.9}$$

and finally we can relate the differentials of Y^a and $\tilde{Y}_a$:

$$d\tilde{Y}_a = -G_{ab}dY^b. \tag{2.10}$$

Let us now make a few remarks on the normalization choices. The field strengths F^a are normalized so that they obey the quantization condition:

$$\frac{1}{2\pi}\oint_{S^2} F^a \in \mathbb{Z}. \tag{2.11}$$

The electric charges q_a of our theory are quantized, so we can imagine they live in a discrete vector space. It is sometimes useful to define an index-free notation in the following way: we introduce the abstract vector space $V_{\mathbb{R}}$ of dimension $n_V + 1$ with its dual space $V^*_{\mathbb{R}}$. We denote with $V_{\mathbb{Z}}$ the subset of $V_{\mathbb{R}}$ made up of vectors with integer components. We can always choose the basis vectors of $V_{\mathbb{R}}$ $\{v_a\}_{a=0}^{n_V}$ so that they are also a basis for $V_{\mathbb{Z}}$. This choice on $V_{\mathbb{R}}$ induces a discrete basis on the dual lattice $V^*_{\mathbb{Z}}$ denoted as $\{w^a\}_{a=0}^{n_V}$ satisfying the canonical pairing condition $\langle w^i, v_j\rangle = \delta^i_j$. The charge vector is then $\mathbf{q} = q_a w^a$.

As a further remark, we have normalized all the terms in the action (2.3) in terms of the five-dimensional Planck length l_5. The relation between l_5, the five dimensional Newton constant G_5 and the constant k_5 of [29] is:

$$\frac{1}{16\pi G_5} = \frac{1}{2k_5^2} = \frac{2\pi}{l_5^3} \tag{2.12}$$

This is convenient because of the particularly simple form of the BPS bounds. Consider a massive charged particle coupled to our theory whose action is:

$$S_{part} = -\int m(\phi)d\tau + q_a\int A^a. \tag{2.13}$$

The mass of this particle will depend on the asymptotic values of the field in moduli space and is subject to a BPS constraint.

Indeed, if we define the central charge:

$$Z := q_aY^a(\phi) = \langle \mathbf{q}, \mathbf{Y}\rangle \tag{2.14}$$

we can write the BPS bound providing the least possible mass of the particle in our theory:

$$m(\phi) \geq \frac{2\pi}{l_5}|Z(\phi)| = \frac{2\pi}{l_5}|q_aY^a(\phi)| = \frac{2\pi}{l_5}|\langle \mathbf{q}, \mathbf{Y}\rangle|. \tag{2.15}$$

There is another BPS bound. In this setup we should consider also magnetically charged strings, which from M-theory perspective come from the wrapping of M5 branes on the internal

Calabi-Yau space. If we introduce the string charge:

$$\tilde{q}^a = \frac{1}{2\pi} \oint_{S^2} F^a \tag{2.16}$$

there is an associated central charge $\tilde{Z} = \tilde{q}^a \tilde{Y}_a$ and the BPS bound on string tension is:

$$\mathcal{T}(\phi) \geq \frac{2\pi}{l_5^2} |\tilde{Z}(\phi)| = \frac{2\pi}{l_5^2} |\tilde{q}^a \tilde{Y}_a| = \frac{2\pi}{l_5^2} |\langle \tilde{\mathbf{Y}}, \tilde{\mathbf{q}} \rangle|. \tag{2.17}$$

The geometry we just outlined is called "very special real geometry". We will see that when this theory is reduced to four dimensions it will give rise to a special Kähler manifold with cubic prepotential.

We will not consider the contribution of hypermultiplets, since we have seen that they are completely decoupled from vector multiplets. Suffice it to say that they live in a quaternionic Kähler manifold whose tangent vectors may be identified with the hyperini.

2.2 N=2 4D supergravity

The other supergravity structure we will need is $\mathcal{N} = 2$ theory in four space-time dimensions. This theory is particularly rich and a complete discussion is beyond the scope of this thesis. We will recap the main aspects we will need in order to understand the structure of the theory we will consider.

Let us sum up the particle content of $\mathcal{N} = 2$ supergravity in four dimensions:

- the gravity multiplet $(e^a_\mu, \psi^1_\mu, \psi^2_\mu, A_\mu)$, which has the same structure as the one in 5D, namely it contains the graviton, two gravitini and the graviphoton;
- the vector multiplet $(A'_\mu, \lambda^1, \lambda^2, z)$; similarly to the vector multiplet in 5D it contains a vector, two gaugini and a scalar, but this time the scalar is complex;
- the hypermultiplet $(\zeta^{1,2}, q^{1,2,3,4})$, equal to the one in 5D, with two hyperini and four real scalars transforming as a doublet under the $SU(2)_R$ symmetry.

As it happened in five dimensions, the scalar manifold decomposes into the direct product $\mathcal{M}_V \times \mathcal{M}_H$ of the vector manifold and the hypermultiplet manifold. However here the situation is a bit different, since the scalars of the vector multiplets are complex and the manifold is modified accordingly: we are describing a projective special Kähler geometry.

If n_V is the number of vector multiplets, $\mathcal{M}_V$ is a Kähler manifold equipped with complex coordinates z^m and with metric $g_{m\overline{n}}$. This manifold is also endowed with a symplectic structure: over $\mathcal{M}_V$ is defined a vector bundle whose section in each patch of $\mathcal{M}_V$ may be written as:

$$V(z) = \begin{pmatrix} Z^I(z) \\ \mathcal{F}_J(z) \end{pmatrix}. \tag{2.18}$$

Here $V(z)$ is a symplectic vector with $2(n_V + 1)$ components: indeed just as it occurred before the addition of the graviphoton provides an extra coordinate on our manifold, so the indices

I, J run from 0 to n_V. $V(z)$ depends holomorphically on the complex coordinates z^m and transforms in the fundamental representation of the $2n_V + 2$ symplectic group.

The Z^I and the $\mathcal{F}_J$ are a priori independent functions which must be thought as components of an abstract section of a fiber bundle. However it can be shown that if $n_V > 1$ in certain frames the $\mathcal{F}_I$ are the derivatives with respect to Z^I of a function $\mathcal{F}$, called prepotential, which must be a homogeneous function of degree two (i.e. $\mathcal{F}(\alpha Z) = \alpha^2 \mathcal{F}(Z)$).

It is worth saying that in such frames we can interpret the Z^I as homogeneous coordinates of a projective space and set $z^m = Z^m/Z^0$. In practice in these homogeneous coordinates we can evaluate all the expressions containing Z^I by eventually setting $Z^0 = 1$; of course this identification is possible only locally on the manifold. We will always consider this choice as understood in the following.

The vector field $V(z)$ naturally possesses an "inner product":

$$\langle \overline{V}(\overline{z}), V(z) \rangle := \overline{Z^I}(\overline{z}) \mathcal{F}_I(Z(z)) - \overline{\mathcal{F}_I}(\overline{Z}(\overline{z})) Z^I(z) = \overline{V}(\overline{z})^T \Omega V(z), \tag{2.19}$$

where Ω is the symplectic unity in $2n_V + 2$ dimensions. Of course this inner product is invariant under $\mathrm{Sp}(2n_V + 2, \mathbb{R})$.

The Kähler potential is then defined starting from this inner product as:

$$K = -\log(i \langle \overline{V}(\overline{z}), V(z) \rangle). \tag{2.20}$$

We can finally write down the bosonic part of the lagrangian for our theory, neglecting hyper-multiplets:

$$\begin{aligned} S^{(4d)}_{bosonic} = & \frac{M_p^2}{2} \int *R - \frac{M_p^2}{2} \int g_{m\overline{n}}(z)\, dz^m \wedge *d\overline{z}^{\overline{n}} + \\ & + 2 \int \mathrm{Im}(\mathcal{N}_{IJ})(z)\, F^I \wedge *F^J + 2 \int \mathrm{Re}(\mathcal{N}_{IJ})(z)\, F^I \wedge F^J. \end{aligned} \tag{2.21}$$

We have defined the Kähler metric $g_{m\overline{n}} := \partial_m \partial_{\overline{n}} K$, while $\mathcal{N}$ is the gauge kinetic matrix defined as:

$$\mathcal{N}_{IJ} = \overline{\mathcal{F}_{IJ}} + 2i \frac{\mathrm{Im}(\mathcal{F}_{IK}) Z^K \mathrm{Im}(\mathcal{F}_{JL}) Z^L}{\mathrm{Im}(\mathcal{F}_{KL}) Z^K Z^L}, \tag{2.22}$$

where $\mathcal{F}_{KL} = \partial_K \partial_L \mathcal{F}$, and the derivatives are taken with respect to Z^I and setting $Z^0 = 1$ at the end.

Notice that the kinetic term of the vector fields is determined by the imaginary part of the gauge kinetic matrix, while its real part determines a term not coupled to gravity, analogous to the Chern-Simons term in five dimensions. It is not hard to guess that this term arises precisely from reduction of the CS lagrangian.

As we previously hinted, when we reduce from five to four dimensions the prepotential and thus the Kähler potential cannot be arbitrary but are determined by the five-dimensional prepotential, which is a cubic polynomial with integer coefficients. We will show this explicitly in the next sections but we anticipate that the four-dimensional prepotential is:

$$\mathcal{F}(Z) = \text{const}\, C_{mnl} \frac{Z^m Z^n Z^l}{Z^0}. \tag{2.23}$$

This prepotential is homogeneous of degree two as it is required by the structure of 4D supergravity. It is no surprise that the prepotential has precisely this form: a large class of four dimensional supergravities can be obtained from compactification of IIA supergravity on a Calabi-Yau threefold, which is known to yield this prepotential; on the other hand, five dimensional supergravity is obtained through the same kind of compactification from M-theory. But M-theory and IIA supergravity are related by circle compactification, so the dimensional reduction order does not matter and we end up with the same $\mathcal{F}(Z)$.

2.3 Kaluza-Klein reduction

One of the key ideas in string theory is the one of dimensional reduction. If we start from a theory living in D space-time dimensions we may assume that some of these dimensions are not observable: they live on a manifold that is curled up ("compactified") in such a way that it is practically invisible from a low-energy perspective. If l is the typical length scale of the invisible manifold for energies much lower than l^{-1} we have no way to probe this additional structure. This idea, called Kaluza-Klein reduction, is particularly appealing for many reasons. Superstring theories by consistency have to live in ten dimensions, but if we had any wish to make predictions in our four dimensional world then there must be six invisible dimensions.

When we perform a reduction the degrees of freedom are rearranged depending on the structure on the internal manifold, so we may end up with a theory that looks quite different with respect to the original one, even though the physics is the same. If we are interested in an effective field theory approach we can discard all massive excitations and keep only the massless fields; this is tremendously useful because, with a clever ansatz, we can study a simpler theory in a lower number of dimensions requiring some symmetry properties on this reduced theory. After we have found some interesting solutions in the lower dimensional theory we can make contact with the theory in the original number of dimensions. This is the very core concept of this thesis.

To make things a little more precise we assume to have a theory in D dimensions living on a manifold $\mathcal{M}_D$ that may be decomposed as:

$$\mathcal{M}_{\mathcal{D}} = \mathcal{M}_d \times \mathcal{M}_{D-d}, \tag{2.24}$$

where $\mathcal{M}_d$ is the visible manifold of the reduced theory and $\mathcal{M}_{D-d}$ is the invisible internal manifold. We will denote with $x^\mu, \mu = 0...d-1$ the coordinates on $\mathcal{M}_d$ and with $y^m, m = 1...D-d$ the coordinates on $\mathcal{M}_{D-d}$. The metric on the total manifold will then be a sum of two pieces belonging to each space. The covariant laplacian will decompose accordingly:

$$\Delta^{(D)} = \nabla_\mu \nabla^\mu + \nabla_m \nabla^m. \tag{2.25}$$

This tells us how to perform the reduction of a scalar field ϕ. Assuming that this was massless at the beginning its equation of motion will be:

$$\nabla_M \nabla^M \phi = \nabla_\mu \nabla^\mu \phi + \nabla_m \nabla^m \phi = 0. \tag{2.26}$$

We would like to disentangle the contribution of the two pieces. The most natural choice is to decompose the scalar field in the following way:

$$\phi(x, y) = \sum_\alpha \phi_\alpha(x) Y_\alpha(y), \tag{2.27}$$

where Y_α is a complete set of functions on $\mathcal{M}_{D-d}$ obeying the eigenvalue equation:

$$\nabla_m \nabla^m Y_\alpha(y) = -\alpha^2 Y_\alpha(y) \tag{2.28}$$

and the eigenvalues belong to some set $\alpha \in A$; it is known that for a compact manifold the Laplace operator has $\alpha^2 \geq 0$ and each eigenspace is finite-dimensional. Inserting this ansatz in the equation of motion:

$$\nabla_\mu \nabla^\mu \phi_\alpha(x) - \alpha^2 \phi_\alpha(x) = 0. \tag{2.29}$$

This is Klein-Gordon equation for a massive particle with mass $m^2 = \alpha^2$. By reducing a scalar field we are producing an infinite number of particles with mass proportional to the mode number: this is the so-called Kaluza-Klein tower. There are also massless excitations in the spectrum, corresponding to the modes that are annihilated by the internal Laplace operator, which are addressed as "zero modes".

For dimensional reasons the eigenvalues α^2 will be proportional to the inverse of the square of a length, which will be the characteristic size of the internal manifold. So, as we mentioned before, we will not be able to create the massive excitations if the length of $\mathcal{M}_{D-d}$ is small enough. If we are interested in a low-energy theory $E \ll l^{-1}$ we can discard the massive particles and keep only the zero modes in the lower dimensional theory. This marks the important distinction between compactification, where both massless and massive modes are retained, and dimensional reduction, where only zero modes survive. In this thesis we will always deal with the latter, but the terms "compactification" and "reduction" will be sloppily used as synonyms.

Of course we have to be sure that truncating the massive modes is consistent, namely that the solutions of the equations of motion for $\phi_{\alpha=0}$ correspond to solutions of the full D-dimensional theory. This will always be the case for the reductions we will take into account.

To conclude this detour on Kaluza-Klein reduction we make the simplest example: a theory in $D+1$ dimensions reduced on a circle. The manifold will be $\mathcal{M}_D \times S^1$, with metric:

$$ds^2_{D+1} = \eta_{\mu\nu} dx^\mu dx^\nu + l^2 dy^2, \tag{2.30}$$

where we have made explicit the compactification radius l and $y \sim y + 2\pi$. A real scalar field admits the Fourier decomposition:

$$\phi(x, y) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \phi_n(x) e^{iny} \tag{2.31}$$

with $\phi_n(x) = \phi^*_{-n}(x)$. Plugging this in the action for a free scalar field we get:

$$
\begin{aligned}
S &= -\frac{1}{2}\int d^D x \int_0^{2\pi} dy (\partial_\mu \phi(x,y)\partial^\mu \phi(x,y) + \frac{1}{l^2}\partial_y \phi(x,y)\partial_y \phi(x,y)) = \\
&= -\frac{1}{2}\int d^D x \frac{1}{2\pi}\int_0^{2\pi} dy \sum_{n,m=-\infty}^{\infty} (\partial_\mu \phi_n \partial^\mu \phi_m e^{i(m+n)y} - \frac{1}{l^2} nm \phi_n \phi_m e^{i(m+n)y}) = \\
&= -\frac{1}{2}\int d^D x \sum_{n=-\infty}^{\infty} (\partial_\mu \phi_{-n}(x)\partial^\mu \phi_n(x) + \frac{n^2}{l^2}\phi_{-n}(x)\phi_n(x))
\end{aligned}
\tag{2.32}
$$

which gives the expected result: a massless field and an infinite tower of massive fields with $m^2 = \frac{n^2}{l^2}$.

2.4 5D supergravity from M-theory

A lot of supergravity theories may be obtained starting from a somewhat more fundamental theory, that is M-theory. M-theory lives in 11 space-time dimensions and its UV completion is at the present day poorly understood; it can be related through dualities to ten dimensional string theories, but we lack further knowledge.

However, just relying on the counting of supersymmetric degrees of freedom and gauge symmetries we can build out the effective lagrangian of M-theory, which is just 11D supergravity. This is also the maximum number of space-time dimensions where usual supergravity makes sense at all: if we started from a theory in $D \geq 12$ we would end up with too many supercharges and this would result in particles with spin $s \geq 5/2$. Let us then briefly analyze the structure of this 11D theory.

Following [38], the field content of 11D SUGRA includes the graviton G_{MN}, which in 11D sits in the traceless symmetric tensor representation of $SO(9)$, and as such amounts to 44 degrees of freedom. We then have the gravitino Ψ_M, sitting in the $\mathbf{9} \times \mathbf{16} = \mathbf{128} \bigoplus \mathbf{16}$ representation of *Spin*(9); however, gauge invariance ensures that only 128 degrees of freedom are physical. The remaining bosonic degrees of freedom are encoded in a three-form A_{MNP} with 84 degrees of freedom that together with the graviton precisely match the fermionic content.

The action is purely geometrical (it contains no coupling constants) and it is strongly constrained by gauge symmetries. The only pieces in the bosonic action we can write are:

$$
S^{(11d)}_{bosonic} = \frac{2\pi}{l_{11}^9}\int_{\mathcal{M}_{11}} (*R - \frac{1}{2}F_4 \wedge *F_4 - \frac{1}{6}A_3 \wedge F_4 \wedge F_4), \tag{2.33}
$$

where $F_4 = dA_3$. The action is simply an Einstein-Maxwell theory with the addition of a topological Chern-Simons term that does not depend on the metric and will be directly responsible for the topological term in the 5D theory. We may be tempted to say that F_4 satisfies the quantization condition:

$$
\frac{1}{l_{11}^3}\oint F_4 \in \mathbb{Z}, \tag{2.34}
$$

but this is not always the case, as pointed out by Witten in [31], because of a global anomaly: the fermions on the world-volume of M2 branes have no chirality and this creates an ambiguity in the path integral measure. This ambiguity can be removed by suitably modifying the holonomy of the A_3 field by requiring the quantization condition on the manifold $\mathcal{M}_{11}$:

$$[F_4] - \frac{p(\mathcal{M}_{11})}{4} \in H^4(\mathcal{M}_{11}, \mathbb{Z}), \tag{2.35}$$

where $p(\mathcal{M}_{11})$ is the first Pontryagin class of the manifold. In other words, this condition is telling us that a particular shift of the flux of F_4 must be integer, but not necessarily F_4 itself. However, as explained in [32], we should not worry about this in the case of our interest, namely compactification on Calabi-Yau threefolds, because there the flux is indeed integer.

If 11D supergravity undergoes dimensional reduction to ten dimensions one will find the massless degrees of freedom of IIA supergravity, so the graviton G, the dilaton ϕ, the NS-NS two-form B and the R-R one and three-forms A_1, A_3. But what we are really interested in now is the compactification of 11D supergravity to five dimensions.

We start assuming that the eleven dimensional space-time can be decomposed as the direct product of two manifolds:

$$\mathcal{M}_{11} = \mathcal{M}_5 \times Y. \tag{2.36}$$

Here $\mathcal{M}_5$ is the five-dimensional visible space-time of the reduced theory, while Y is an internal six-dimensional space which is assumed to have the structure of a Calabi-Yau threefold (see appendix 4 for a recap on these manifolds). This manifold is characterized by the Hodge numbers $h^{(1,1)}$ and $h^{(2,1)}$, counting the number of independent harmonic (1,1) and (2,1) forms. Labelling the 11D coordinates $(x \in \mathcal{M}_5, y \in Y)$, the metric ansatz has the standard form:

$$ds_{11}^2 = e^{2A} ds_5^2 + l_{11}^2 ds_Y^2, \tag{2.37}$$

where the factor of l_{11} has been included to make the internal metric dimensionless. This means that lengths in the Calabi-Yau space are implicitly measured in units of l_{11}. The factor in front of ds_5^2 is called "warp factor".

As it is known, when a theory is compactified on a Calabi-Yau threefold the resulting lower dimensional effective field theory is obtained by keeping the zero-modes of the starting theory, since the other Kaluza-Klein particles will acquire a mass depending on the mode number. The lower dimensional fields emerge as deformations of the very same structure of the Calabi-Yau manifold. For example, when one compactifies IIA theory one usually defines a complexified Kähler form $\mathcal{J} = B + iJ$, where B is the NS-NS two-form and J is the Kahler form the manifold is naturally endowed with. The deformations of this form in moduli space are encoded in $h^{(1,1)}$ complex fields, identified as scalar fields of vector multiplets.

Compactification of M-theory works a little differently. To begin, let us consider the Kähler form J. Being a real harmonic (1,1) form, it can be decomposed in a basis of real harmonic (1,1) forms spanning the cohomology class $H^{(1,1)}$ which we name V_a and are conveniently normalized:

$$J = \sum_{a=1}^{h^{(1,1)}} M^a V_a \tag{2.38}$$

where M^a are the Kähler moduli. The volume of the manifold is given by:

$$\mathcal{V}(M) = \frac{1}{6}\int_Y J \wedge J \wedge J = \frac{1}{6}\sum_{a,b,c} M^a M^b M^c \int_Y V_a \wedge V_b \wedge V_b \equiv \frac{1}{6} C_{abc} M^a M^b M^c. \tag{2.39}$$

These integrals define the so-called intersection numbers C_{abc}, which can be proven to be integers. They are precisely the Chern-Simons coefficients that determine the entire structure of $\mathcal{N} = 2$ five-dimensional supergravity. The volume of the CY manifold is also used to relate the five and the eleven dimensional Planck lengths. The following relation must hold by consistency:

$$\frac{2\pi}{l_{11}^9}\int_{\mathcal{M}_5 \times Y} \sqrt{-g^{(11)}} R^{(11)} = \frac{2\pi}{l_5^3}\int_{\mathcal{M}_5} \sqrt{-g^{(5)}} R^{(5)} + ... \tag{2.40}$$

where the dots denote other terms coming from the reduction of the Einstein-Hilbert term. On the other hand it also holds:

$$\sqrt{-g^{(11)}} = e^{5A} l_{11}^6 \sqrt{-g^{(5)}}\sqrt{-g^{(Y)}} \tag{2.41}$$

so by comparing the two expressions:

$$e^{3A} = \frac{l_{11}^3}{l_5^3 \mathcal{V}}. \tag{2.42}$$

When reducing the theory on a Calabi-Yau space the most important piece of information is the geometry of the moduli space, namely the space of the deformations of complex or Kähler structure encoding the vevs of the scalar fields. In M-theory there is no Kalb-Ramond field $B_{\mu\nu}$; as a consequence, the Kähler form is not complexified, so its deformation are real. This is in agreement with the fact that in the reduced 5D theory the vector multiplets contain real scalars.

Let us then count the degrees of freedom of the reduced theory as zero modes on the Calabi-Yau (see e.g. [30]); the indices on $\mathcal{M}_5$ are greek and those on Y are latin, with an overline for the complex conjugate components. The metric produces the 5D graviton $G_{\mu\nu}$; from $G_{i\bar{j}}$ we get $h^{(1,1)}$ real scalars, while the pure components G_{ij} correspond to $h^{(2,1)}$ moduli of the complex structure. The three-form A_3 produces a real scalar ($A_{\mu\nu\rho}$), $h^{(1,1)}$ vectors ($A_{\mu i\bar{j}}$), $h^{(2,1)}$ complex scalars ($A_{ij\bar{k}}$) and one complex scalar ($A_{ijk} = \epsilon_{ijk} C$, with ϵ_{ijk} the 3D Levi-Civita symbol). These are the bosonic degrees of freedom of $h^{(2,1)} + 1$ hypermultiplets, $h^{(1,1)} - 1$ vector multiplets and of the gravity multiplet in 5D supergravity. This statement is not completely obvious because of a particular rearrangement of the degrees of freedom. To understand what is going on we trade the Kähler moduli M^a for other coordinates:

$$M^a \rightarrow (Y^a := \frac{M^a}{\mathcal{V}^{\frac{1}{3}}}, \mathcal{V}). \tag{2.43}$$

The new coordinates Y^a satisfy the constraint $\mathcal{V}(Y) = 1$, thus they are not independent and encode $h^{(1,1)} - 1$ degrees of freedom, while we keep the volume $\mathcal{V}$ as a distinct modulus; of course the coordinates Y^a are the same we used in section 2.1. The volume modulus combines with the real scalar $A_{\mu\nu\rho}$ and the complex scalar C to give the bottom component of a universal hypermultiplet, which appears in any compactification regardless of the particular internal space.

The decomposition in the basis of harmonic (1,1) forms can be used also to show how the Chern-Simons term arises in the 5D action. We can expand the three form A_3 in the following way:

$$A_3 = A^a \wedge V_a \Rightarrow F_4 = F^a \wedge V_a \tag{2.44}$$

where the A^a are the 5D vector fields, and the implication holds because the V_a are closed. Then:

$$\int_{\mathcal{M}_5 \times Y} A_3 \wedge F_4 \wedge F_4 = \int_{\mathcal{M}_5} C_{abc} A^a \wedge F^b \wedge F^c \tag{2.45}$$

which is precisely the topological term appearing in (2.3).

As a concluding remark, it must be stressed that the supergravity approximation works well provided that the Calabi-Yau volume is large enough with respect to l_{11}^6, otherwise the effective theory at two derivatives would not be reliable.

2.5 Dimensional reduction from 5D to 4D

The next thing we would like to investigate is the link between five and four-dimensional supergravities: indeed we will show that 4D $\mathcal{N} = 2$ supergravity can be obtained by dimensional reduction of 5D supergravity. Of course, as 5D supergravity is characterized by a specific prepotential, the reduced theory must have a particular prepotential as well. This is consistent with string theory: we know that our 5D supergravity is given by Calabi-Yau compactification of M-theory; on the other hand, 4D supergravity may be obtained from Calabi-Yau compactification of IIA theory. It is also true that IIA theory is obtained from reduction of M-theory on a circle. Reducing first from 11D to 10D and then from 10D to 4D must be the same as reducing from 11D to 5D and then from 5D to 4D; since in the first case the prepotential is of the form $\mathcal{F} \approx C_{abc} Z^a Z^b Z^c / Z^0$ the same must occur also in the second situation. We will verify explicitly that this is indeed what happens.

We compactify our theory on S^1, with a dimensionless variable y of unit period $y \sim y + 1$. The metric ansatz we consider is (see also [33]):

$$ds_5^2 = \frac{l_5^2 M_p^2}{4\pi} e^{-\varphi} ds_4^2 + l_5^2 e^{2\varphi} \left(dy - \frac{A^0}{2\pi} \right)^2 . \tag{2.46}$$

We will denote $x^M = (x^\mu, y)$, where x^μ are the coordinates on the four-dimensional space.

In (2.46) $A^0 = A^0_\mu dx^\mu$ is the Kaluza-Klein vector and φ is a real scalar called dilaton. The exponential factors are put in order to obtain the action in Einstein frame in four dimensions, as (2.21). Furthermore, all the fields are assumed to depend only on the x coordinates. In matrix form the metric reads:

$$g_{5\,MN} = \begin{pmatrix} \frac{l_5^2 M_p^2}{4\pi} e^{-\varphi} g_{\mu\nu} + l_5^2 e^{2\varphi} \frac{A^0_\mu A^0_\nu}{4\pi^2} & -l_5^2 e^{2\varphi} \frac{A^0_\mu}{2\pi} \\ -l_5^2 e^{2\varphi} \frac{A^0_\mu}{2\pi} & l_5^2 e^{2\varphi} \end{pmatrix} \tag{2.47}$$

while the inverse metric is:

$$g_5^{MN} = \begin{pmatrix} \frac{4\pi}{l_5^2 M_p^2} e^{\varphi} g^{\mu\nu} & \frac{2}{l_5^2 M_p^2} e^{\varphi} A^{0\mu} \\ \frac{2}{l_5^2 M_p^2} e^{\varphi} A^{0\mu} & \frac{e^{-2\varphi}}{l_5^2} + \frac{e^{\varphi}}{l_5^2 M_p^2 \pi} A^0_\mu A^{0\mu} \end{pmatrix} \tag{2.48}$$

where the indices of A^0 are raised with the four dimensional metric.

Before passing to the actual calculation it is worth performing a quick check of the degrees of freedom we expect upon reduction. If we start from a theory in 5D with a gravity multiplet and n_V abelian vector multiplets the initial bosonic degrees of freedom are one graviton, $n_V + 1$ vectors and n_V real scalars. Then:

- the reduction of the metric $g^{(5d)}_{MN}$ will give the 4D metric $g^{(4d)}_{\mu\nu}$, an abelian vector A^0 and a scalar φ;
- the reduction of each vector A^a_M will give a vector A^a_μ and a real scalar a^a, called axion;
- the reduction of each scalar ϕ^i will give just a scalar.

So we end up with one graviton, $n_V + 2$ vectors and $2(n_V + 1)$ scalars. These are precisely the degrees of freedom of a $\mathcal{N} = 2$ 4D theory with one gravity multiplet and $n_V + 1$ vector multiplets, correctly with $n_V + 1$ complex scalars. However, the identification of the four-dimensional degrees of freedom in terms of the five-dimensional ones is not immediate because a rearrangement occurs. Let us see more in detail how this works.

We start with the reduction of the Einstein-Hilbert term. With the previous ansatz the metric determinant is:

$$\sqrt{|g_5|} = \left(\frac{l_5^2 M_p^2}{4\pi}\right)^2 l_5 e^{-\varphi} \sqrt{|g_4|}. \tag{2.49}$$

So the reduction of the Einstein-Hilbert term yields:

$$\begin{aligned} S^{(5d)}_{EH} =& \frac{2\pi}{l_5^3} \int_{\mathcal{M}_4 \times S^1} d^4x\, dy \sqrt{|g_5|}\, R_5 = \\ &= \frac{2\pi}{l_5^3} \int_{\mathcal{M}_4} d^4x \sqrt{|g_4|} \left(\frac{l_5^2 M_p^2}{4\pi}\right)^2 l_5 \left[\frac{4\pi}{l_5^2 M_p^2} R_4 - \frac{6\pi}{l_5^2 M_p^2} \partial_\mu\varphi \partial^\mu \varphi - \frac{1}{l_5^2 M_P^4} e^{3\varphi} F^0_{\mu\nu} F^{0\mu\nu}\right] = \\ &= \frac{M_p^2}{2} \int_{\mathcal{M}_4} \left(*R_4 - \frac{3}{2} d\varphi \wedge *d\varphi\right) - \frac{1}{4\pi} \int_{\mathcal{M}_4} e^{3\varphi} F^0 \wedge *F^0, \end{aligned} \tag{2.50}$$

where $F^0 = dA^0$. We have thrown away boundary terms proportional to $\nabla_\mu \nabla^\mu \varphi$.

The reduction of the scalars yields:

$$\begin{aligned} S^{(5d)}_{scalars} &= -\frac{2\pi}{l_5^3}\int_{\mathcal{M}_4\times S^1} \frac{1}{2}\, d^4x\, dy \sqrt{|g_5|}\, \tilde{G}_{ij}\, g_5^{MN}\, \partial_M \phi^i \partial_N \phi^j \\ &= -\frac{\pi}{l_5^3}\int_{\mathcal{M}_4\times S^1} d^4x\, dy \sqrt{|g_5|}\, G_{ab}\, g_5^{MN}\, \partial_M Y^a(\phi) \partial_N Y^b(\phi) = \\ &= -\frac{M_p^2}{4}\int_{\mathcal{M}_4} G_{ab} dY^a(\phi)\wedge * dY^b(\phi), \end{aligned} \tag{2.51}$$

where we have used that $\tilde{G}_{ij}$ is the pullback of G_{ab}.

We can rearrange the scalar part of (2.50) together with (2.51) in a convenient way. We define the saxions:

$$s^a := e^{\varphi} Y^a; \tag{2.52}$$

consider then the following expression:

$$\begin{aligned} & e^{-2\varphi} G_{ab} \partial_\mu s^a \partial^\mu s^b = e^{-2\varphi} G_{ab} (\partial_\mu \varphi\, e^{\varphi}\, Y^a + e^{\varphi}\, \partial_\mu Y^a)(\partial^\mu \varphi\, e^{\varphi}\, Y^b + e^{\varphi}\, \partial^\mu Y^b) = \\ & = 3\partial_\mu \varphi \partial^\mu \varphi + G_{ab} \partial_\mu Y^a \partial^\mu Y^b \end{aligned} \tag{2.53}$$

where we have used $G_{ab} Y^a Y^b = 3$. Note that the mixed term vanishes because of (2.7).

This saxionic kinetic term correctly reproduces the kinetic terms of both the dilaton and the scalars Y^a. The saxionic metric is simply a rescaling of G:

$$\mathcal{G}_{ab}(\phi) := \frac{1}{2} e^{-2\varphi} G_{ab}(\phi). \tag{2.54}$$

As a consequence of this redefinition, the constraint of the scalar fields Y induces an analogous constraint on the saxions:

$$\mathcal{V}(Y) = \frac{1}{6} C_{abc} Y^a Y^b Y^c = 1 \quad \Rightarrow \quad \mathcal{V}(s) = \frac{1}{6} C_{abc} s^a s^b s^c = e^{3\varphi}. \tag{2.55}$$

This also tells us that the new metric $\mathcal{G}_{ab}$ can be derived as it is done for G_{ab} with the new constraint:

$$\mathcal{G}_{ab} = -\frac{1}{2}\frac{\partial}{\partial s^a}\frac{\partial}{\partial s^b} \log \mathcal{V}(s)|_{\mathcal{V}=e^{3\varphi}}. \tag{2.56}$$

Finally we can write that:

$$S_{saxions} = -\frac{M_p^2}{2}\int_{\mathcal{M}_4} \mathcal{G}_{ab}(s)\, ds^a \wedge * ds^b. \tag{2.57}$$

For the vector fields the reduction is a bit more subtle. We expect that the reduction of a 5D vector produces a 4D vector plus a scalar a^a, which corresponds to the compactified direction of the vector and will be called axion. This is correct, but one has to be careful. The naive guess for the reduction ansatz would be to take $A^a_{(5d)} = A^a_\mu dx^\mu + a^a dy$. However, with such an ansatz we would end up with an action which is not gauge invariant, since it would mix the Kaluza-Klein vector with the axions. To better understand what is the correct guess we have to keep in mind that our starting theory was invariant under a general transformation of the full set of the five coordinates. When we reduce our theory this invariance seems to be lost due to the smaller set of coordinates. Nevertheless, if we consider a transformation $x'^\mu = x^\mu, y' = y + \Lambda(x)$ it is easy to verify that:

$$\hat{g}'_{MN}(x') = \frac{\partial x^R}{\partial x'^M}\frac{\partial x^S}{\partial x'^N} g_{RS}(x) \Rightarrow A'^0_\mu = A^0_\mu - 2\pi\partial_\mu\Lambda \tag{2.58}$$

meaning that the Kaluza-Klein vector inherits its gauge invariance from the diffeomorphism invariance of the original theory. To preserve gauge invariance in the reduced action the correct ansatz is:

$$A^a_{(5d)} = A^a + 2\pi a^a \left(dy - \frac{A^0}{2\pi}\right). \tag{2.59}$$

The normalization is chosen so that:

$$a^a = \frac{1}{2\pi}\oint_{S^1} A^a_{(5d)}, \tag{2.60}$$

and the periodicity of a^a is $a^a \sim a^a + 1$. So these are truly axions in the particle physics sense, as they are (pseudo)scalar fields endowed with a perturbative shift symmetry. As we are going to verify the classical action preserves this shift symmetry, although this may be broken by non-perturbative corrections.

From $A^a_{(5d)}$ we can get $F^a_{(5d)}$:

$$F^a_{(5d)} = F^a - a^a F^0 + 2\pi da^a \wedge \left(dy - \frac{A^0}{2\pi}\right). \tag{2.61}$$

Since the axions are invariant under discrete transformations $a^a \to a^a + n^a$ the same must happen for $F^a_{(5d)}$. This is fulfilled if and only if we require that $A^a \to A^a + n^a A^0$. Then the object:

$$\tilde{F}^a := F^a - a^a F^0 \tag{2.62}$$

is invariant under axion shifts. The components of the field strength are:

$$\begin{aligned} F^a_{(5d)\,\mu\nu} &= \tilde{F}^a_{\mu\nu} - \partial_\mu a^a A^0_\nu + \partial_\nu a^a A^0_\mu \\ F^a_{(5d)\,\mu y} &= 2\pi\partial_\mu a^a. \end{aligned} \tag{2.63}$$

Then we get:

$$
\begin{aligned}
&G_{ab}\, g_5^{MN} g_5^{RS} F^a_{(5d)\,MN} F^b_{(5d)\,RS} = \\
&= G_{ab}(g_5^{\mu\nu} g_5^{\rho\sigma} F^a_{(5d)\,\mu\rho} F^b_{(5d)\,\nu\sigma} + 2 g_5^{\mu\nu} g_5^{yy} F^a_{(5d)\,\mu y} F^b_{(5d)\,\nu y} + \\
&+ 2 g_5^{\mu y} g_5^{y\nu} F^a_{(5d)\,\mu y} F^b_{(5d)\,y\nu} + 4 g_5^{\mu\nu} g_5^{\rho y} F^a_{(5d)\,\mu\rho} F^b_{(5d)\,\nu y}) = \\
&= G_{ab}\left(\frac{16\pi^2}{l_5^4 M_P^4} e^{2\varphi} g^{\mu\nu} g^{\rho\sigma} \tilde{F}^a_{\mu\rho} \tilde{F}^b_{\nu\sigma} + \frac{32\pi^3}{l_5^4 M_p^2} e^{-\varphi} g^{\mu\nu} \partial_\mu a^a \partial_\nu a^b \right).
\end{aligned} \tag{2.64}
$$

So the reduced action for the vectors is:

$$
\begin{aligned}
S^{(5d)}_{vectors} &= -\frac{1}{4\pi l_5} \int_{\mathcal{M}_4 \times S^1} G_{ab}(s) F^a_{(5d)} \wedge * F^b_{(5d)} = \\
&- \frac{M_p^2}{4} \int_{\mathcal{M}_4} e^{-2\varphi}\, G_{ab}(s)\, da^a \wedge * da^b - \frac{1}{4\pi} \int_{\mathcal{M}_4} e^{\varphi}\, G_{ab}(s)\, \tilde{F}^a \wedge * \tilde{F}^b.
\end{aligned} \tag{2.65}
$$

We can combine the kinetic term of the axions with the one of saxions of (2.57) to get:

$$
S^{(4d)}_{scalars} = -\frac{M_p^2}{2} \int_{\mathcal{M}_4} \mathcal{G}_{ab}(da^a \wedge * da^b + ds^a \wedge * ds^b) = -\frac{M_p^2}{2} \int_{\mathcal{M}_4} \mathcal{G}_{ab} dt^a \wedge * d\bar{t}^b, \tag{2.66}
$$

where we have defined $t^a := a^a + \mathrm{i} s^a$. Thus the axions and the saxions combine into a complex scalar that is the bottom component of a $\mathcal{N} = 2$ vector multiplet.

The last thing to do is the reduction of the Chern-Simons term:

$$
\begin{aligned}
&C_{abc} A^a_{(5d)} \wedge F^b_{(5d)} \wedge F^c_{(5d)} = C_{abc}(A^a - a^a A^0 + 2\pi a^a dy) \wedge \\
&(F^b - a^b F^0 + 2\pi da^b \wedge dy - da^b \wedge A^0) \wedge (F^c - a^c F^0 + 2\pi da^c \wedge dy - da^c \wedge A^0) = \\
&= C_{abc}(6\pi a^a F^b \wedge F^c \wedge dy - 6\pi a^a a^b F^c \wedge F^0 \wedge dy + 2\pi a^a a^b a^c F^0 \wedge F^0 \wedge dy).
\end{aligned} \tag{2.67}
$$

The calculation is quite involved and requires some steps:

- Terms containing $A^0 \wedge A^0$ vanish since the square of an odd form is zero;
- Terms containing $C_{abc} da^a \wedge da^b$ vanish since they are the contraction of a symmetric expression with an antisymmetric one;
- All the terms which do not contain exactly one dy vanish. Indeed the result must be a 5-form; as all fields depend only on dx^μ there cannot be a term with five dx^μ because their wedge product is zero;
- We have exploited integration by parts using graded Leibniz's rule and then thrown away total differentials. This step is quite delicate and cannot be performed in general. In particular, if there are background topological fluxes related to the vector fields neglecting boundary terms may break gauge invariance. However, since we will switch off all the vectors, integration by parts is legitimate.

Then the Chern-Simons action becomes:

$$
\begin{aligned}
S^{(5d)}_{CS} &= -\frac{1}{6(2\pi)^2}\int_{\mathcal{M}_4\times S^1} C_{abc}A^a_{(5d)}\wedge F^b_{(5d)}\wedge F^c_{(5d)} = \\
&= -\frac{1}{12\pi}\int_{\mathcal{M}_4} C_{abc}(3a^aF^b\wedge F^c - 3a^aa^bF^c\wedge F^0 + a^aa^ba^cF^0\wedge F^0).
\end{aligned} \tag{2.68}
$$

We started from a topological term in five dimensions and we ended up with an axion-vector coupling in four dimensions. This is a theta-term, much alike the one we meet when we study axions in QCD. To sum up, the reduced action reads:

$$
\begin{aligned}
S^{(4d)} &= S_{EH} + S_{scalars} + S_{vectors} + S_{CS} = \\
&\frac{M_p^2}{2}\int_{\mathcal{M}_4}(*R - \mathcal{G}_{ab}(s)dt^a\wedge *d\bar{t}^b) + \\
&-\frac{1}{4\pi}\int_{\mathcal{M}_4}\mathcal{V}(s)\left(F^0\wedge *F^0 + 2\mathcal{G}_{ab}(s)\tilde{F}^a\wedge *\tilde{F}^b\right) + \\
&-\frac{1}{12\pi}\int_{\mathcal{M}_4} C_{abc}(3a^aF^b\wedge F^c - 3a^aa^bF^c\wedge F^0 + a^aa^ba^cF^0\wedge F^0).
\end{aligned} \tag{2.69}
$$

Now we would like to match this expression with the typical 4D supergravity structure. First, as we already underlined the metric $\mathcal{G}$ is Kähler, with the Kähler potential determined by the five-dimensional prepotential:

$$
K = -\log\mathcal{V}(s) + \log\pi \;\Rightarrow\; \mathcal{G}_{ab} = \frac{1}{2}\frac{\partial^2 K}{\partial s^a\partial s^b}|_{\mathcal{V}(s)=e^{3\varphi}} = 2\frac{\partial^2 K}{\partial t^a\partial\bar{t}^b}. \tag{2.70}
$$

The inconsequential term with π is added in order to obtain that the mass of a particle in moduli space is:

$$
M_{\mathbf{q}} = M_p e^{\frac{1}{2}K}\langle\mathbf{q},\mathbf{s}\rangle. \tag{2.71}
$$

It can be immediately verified that the Kähler potential can be derived from a specific prepotential:

$$
\mathcal{F} = -\frac{1}{8\pi}C_{abc}\frac{Z^aZ^bZ^c}{6Z^0}, \tag{2.72}
$$

where Z^a are homogeneous coordinates on the scalar moduli space, and $a = 1...n^V+1$. The extra coordinate Z^0 on the projective space can be traced back to the appearance of the Kaluza-Klein vector A^0. At the end of each calculation we are allowed to take $Z^0 = 1$ and $Z^a = t^a$.

We have to check that the Kähler potential obtained via the general formula of $\mathcal{N} = 2$ 4D supergravity agrees with the one we have obtained by the matching to 5D supergravity. This is easily done using that $s^a = (t^a - \bar{t}^a)/2i$:

$$
\begin{aligned}
K &= -\log\left[\frac{C_{abc}}{6}\left(\frac{t^a-\bar{t}^a}{2i}\right)\left(\frac{t^b-\bar{t}^b}{2i}\right)\left(\frac{t^c-\bar{t}^c}{2i}\right)\right]+\log\pi \\
&= -\log\left[\frac{i}{48}C_{abc}\left(t^at^bt^c-3t^at^b\bar{t}^c+3t^a\bar{t}^b\bar{t}^c-\bar{t}^a\bar{t}^b\bar{t}^c\right)\right]+\log\,\pi.
\end{aligned}
\tag{2.73}
$$

This must be compared to the general formula (2.20):

$$
\begin{aligned}
K &= -\,\log\left[i\left(\bar{Z}^0\frac{\partial\mathcal{F}}{\partial Z^0}+\bar{Z}^a\frac{\partial\mathcal{F}}{\partial Z^a}-Z^0\frac{\partial\bar{\mathcal{F}}}{\partial\bar{Z}^0}-Z^a\frac{\partial\bar{\mathcal{F}}}{\partial\bar{Z}^a}\right)\right]|_{Z^0=1,\,Z^a=t^a} = \\
&= -\,\log\left[\frac{i}{48}C_{abc}\left(t^at^bt^c-3\bar{t}^at^bt^c-\bar{t}^a\bar{t}^b\bar{t}^c+3t^a\bar{t}^b\bar{t}^c\right)\right]+\log\,\pi
\end{aligned}
\tag{2.74}
$$

which by symmetry of C_{abc} agrees with (2.73).

The final thing to do is to check that the kinetic and Chern-Simons terms for the vectors are obtained from the prepotential (2.72) using the general formula (2.22). The prepotential gives the following entries of the gauge kinetic matrix:

$$
\begin{aligned}
&\mathrm{Im}\mathcal{N}_{00} = -\frac{1}{8\pi}e^{3\varphi}-\frac{1}{8\pi}e^{\varphi}G_{ab}a^aa^b \qquad \mathrm{Re}\mathcal{N}_{00} = -\frac{1}{24\pi}C_{abc}a^aa^ba^c \\
&\mathrm{Im}\mathcal{N}_{0a} = \frac{1}{8\pi}e^{\varphi}G_{ab}a^b \qquad \mathrm{Re}\mathcal{N}_{0a} = \frac{1}{16\pi}C_{abc}a^ba^c \\
&\mathrm{Im}\mathcal{N}_{ab} = -\frac{1}{8\pi}e^{\varphi}G_{ab} \qquad \mathrm{Re}\mathcal{N}_{ab} = -\frac{1}{8\pi}C_{abc}a^c.
\end{aligned}
\tag{2.75}
$$

Then the 4D vector action in the $\mathcal{N}=2$ formalism is:

$$
\begin{aligned}
&2\int\mathrm{Im}\mathcal{N}_{IJ}F^I\wedge *F^J+2\int\mathrm{Re}\mathcal{N}_{IJ}F^I\wedge F^J = \\
&-\frac{1}{4\pi}\int(e^{3\varphi}+e^{\varphi}G_{ab}a^aa^b)F^0\wedge *F^0-\frac{1}{12\pi}\int C_{abc}a^aa^ba^cF^0\wedge F^0+ \\
&+\frac{1}{2\pi}\int e^{\varphi}G_{ab}a^aF^b\wedge *F^0+\frac{1}{4\pi}\int C_{abc}a^aa^bF^c\wedge F^0+ \\
&-\frac{1}{4\pi}\int e^{\varphi}G_{ab}F^a\wedge *F^b-\frac{1}{4\pi}\int C_{abc}a^aF^b\wedge F^c.
\end{aligned}
\tag{2.76}
$$

This expression agrees with (2.69) upon expanding $\tilde{F}^a = F^a - a^aF^0$. Thus we have learnt that the dimensional reduction of a 5D supergravity action produces a structure compatible with 4D supergravity with a specific choice of the prepotential, directly induced from the higher-dimensional theory. Moreover, since all the 5D theory is determined by the coefficients C_{abc}, we can conclude that the entire 4D theory is encoded in the Calabi-Yau geometry that determines the intersection numbers.

3 Axionic wormholes

In this section, which represents the core of this work, we are going to analyze a large class of axionic wormholes. They emerge as classical solutions of four-dimensional effective field theories providing non-perturbative corrections to the euclidean gravitational path integral. Besides their relevance for wormhole physics, they are important because they can break the axionic shift symmetry we required at the classical level to ensure a vanishing axion mass. After building the setup, we will present classical solutions to the equations of motion, following mainly [34]. Then we will try to elucidate some of the deep links between these wormhole configurations and their higher dimensional counterparts, mainly five-dimensional black holes. The results will be found to have an intriguing unity upon uplifting thanks to the structure of 5D supergravity.

3.1 Fundamental axions

The theory we are going to consider is an effective field theory (EFT) with an arbitrary and possibly large number of axions, namely (pseudo)scalar fields which at the classical level enjoy a shift symmetry. The axions we will focus on are *fundamental* axions, in the sense that they cannot be described as angular components of more fundamental fields (see [44]). These are precisely the kind of axions we have already met and they are understood as zero modes of a higher dimensional gauge field when we take its Wilson lines around extra-dimensional cycles during compactification, as we did in the previous section. The other kind of axions is perhaps the one particle physicists are more familiar with and is simply the (pseudo) Nambu-Goldstone boson of a global symmetry spontaneously broken at high energies, e.g. Peccei-Quinn symmetry. The two families of axions are both realized in string theory: fundamental axions are realized in closed string compactifications, while Peccei-Quinn-like axions are realized as modes of open strings in D-brane dynamics.

The difference between the two does not end here. As conjectured by Ooguri and Vafa in [45], there is no non-trivial 1-cycle with minimum length within a given homotopy class in moduli space. This immediately implies that each axion must have a partner, since axions have compact moduli space of S^1 and define an homotopy class. Then the partner encodes the direction along which the axion decay constant (i.e. period) changes and eventually goes to zero, meaning that the axionic S^1 shrinks to a point. In this sense it is a radial partner, which is the saxion in supersymmetric theories. Here lies the main difference between a Nambu-Goldstone axion and a fundamental axion: while for the former the point where the decay constant vanishes lies at finite distance in the moduli space, for the latter such point lies at infinite distance. An example is provided by circle compactification, where the radial partner is the radius r and defines a metric in moduli space is $ds^2 = (dr/r)^2$. The distance from a given point r_0 scales like $\log(r/r_0)$; it is then clear that a vanishing decay constant corresponds to an infinite volume limit of the compact manifold. This degeneration is signalled by the singularity of the kinetic term, which instead is well behaved in the case of Peccei-Quinn axions.

Then it does not come as a surprise that when we discussed dimensional reduction from five to four dimensions we found that the radial partners of the axions are $s^a = e^{\varphi} Y^a$. By construction the radius of the compact dimension is $r = l_5 e^{\varphi}$ and the metric is proportional to r^{-2}, so the degeneration is in agreement with the previous discussion.

3.2 Axionic effective field theory

The setup we will consider is a four-dimensional EFT truncated at two derivatives level whose matter content consists of gravity and axion-saxion partners, preserving the axionic shift symmetry. The most general action we can write with this information is (in *Lorentzian* signature):

$$S_{EFT} = \frac{M_p^2}{2} \int_{\mathcal{M}_4} *R - \frac{M_p^2}{2} \int_{\mathcal{M}_4} \mathcal{G}_{ab}(s)(da^a \wedge *da^b + ds^a \wedge *ds^b). \tag{3.1}$$

This setup is very general and emerges in many situations in string theory. Of course in a supersymmetric context axions and saxions are real and imaginary part of a complex scalar and $\mathcal{G}$ must be a Kähler metric, with a generic Kähler potential K. It is immediate to see that an explicit realization of this EFT comes from dimensional reduction from five dimensions as described in section 2.5, so the details of the Kähler structure derive from the chosen Calabi-Yau upon which reduction from M-theory is performed. In particular, the EFT action is a consistent truncation of (2.69) when all vector fields are set to zero. However, many of the results we will recapitulate in the following can be easily generalized to a Kähler potential of the form:

$$K = -\log P(s), \tag{3.2}$$

where $P(s)$ is a homogeneous function of degree n. In string theory and M-theory models n is an integer number that goes from 1 to 7; in our case $P(s) = \mathcal{V}(s)$. Actually, as we will discuss later on, n must be at least equal to 3 in order to have everywhere meaningful solutions. The case $n = 3$, occurring in 5D compactifications, is somehow special and will be further discussed.

Since we are dealing with an EFT it is worth asking what is its validity, even more so if we consider that quantum gravity demands stringent constraints on the low-energy theories. Indeed, while EFT structure tells us that there is a UV cutoff Λ suppressing higher-dimensional operators, string theory structure tells us that there is a maximum distance in field space where this theory makes sense. We will focus on this second point.

We have stressed several times that the axions are endowed with a perturbative shift symmetry. This is a global symmetry, and we know that by swampland conjectures global symmetries are not allowed in a consistent theory of quantum gravity. It is clear that the axion shift symmetry is valid only perturbatively and must be broken by non-perturbative effects, which are actually exponentially suppressed like $e^{-A/g}$. We must also take into account that in string theory there are no coupling constants, so g is a function of the fields in moduli space $g = g(\phi)$. It follows that if we want to build an EFT where shift symmetry is a good approximation we have to specify a region in field space where non-perturbative corrections are negligible.

A first naive estimate of the EFT domain of validity is given by a simple argument in [34]. The saxions and the axions have zero mass dimension and restoring $\hbar$ by dimension analysis it follows that $[a^a] = [s^a] = [\hbar]$. Then we can define:

$$\alpha_* := \frac{1}{s_*} \tag{3.3}$$

as a sort of "fine structure constant" that controls the perturbative expansion. Here s_* is some linear combination of saxions to be better specified, and of course α_* cannot depend on the axions because of shift symmetry, which is what we want to be (approximately) preserved. Then the size of the non-perturbative corrections is expected to be of the order of $e^{-A/\alpha_*} = e^{-As_*}$, meaning that they exponentially decrease as the saxions increase. Then our EFT of fundamental axions makes sense in an asymptotic region in field space, where non-perturbative effects are very small and shift symmetry is quite good. However, string theory prevents us from physically reaching such regions because of the swampland distance conjecture: starting from a point p_0 and moving in moduli space we will encounter theories with an infinite tower of light particles with mass exponentially suppressed by the distance. In the limit of infinite distance, then, the number of particles with mass below a given scale will be infinite. This implies that the EFT has a maximal cutoff associated with the appearance of light states, which can be determined precisely only if we know the UV details of the theory.

Of course, this is not the end of the story. There are several important mass scales in the UV, like the species scale, often defined as the mass scale at which any EFT breaks down. Once again, we invite to read [34] for a complete and physically meaningful discussion.

3.3 Dual formulation

As it happened for the Giddings-Strominger theory, we can give an alternative and completely equivalent formulation of the action (3.1) in terms of dual fields. We can trade the saxions for *dual-saxions* defined as:

$$\ell_a := -\frac{1}{2}\frac{\partial K}{\partial s^a}. \tag{3.4}$$

The associated dual potential F is obtained by taking the Legendre transform of the Kähler potential:

$$F = K + 2\ell_a s^a. \tag{3.5}$$

Of course the Legendre transform can be inverted to obtain the saxions:

$$s^a = \frac{1}{2}\frac{\partial F}{\partial \ell_a}, \tag{3.6}$$

and we can also write the inverse metric in terms of dual saxions:

$$\mathcal{G}^{ab} = -\frac{1}{2}\frac{\partial^2 F}{\partial \ell_a \partial \ell_b}. \tag{3.7}$$

That this is truly the inverse metric can be immediately verified:

$$\mathcal{G}^{ab}\mathcal{G}_{bc} = -\frac{1}{4}\frac{\partial^2 F}{\partial \ell_a \partial \ell_b}\frac{\partial^2 K}{\partial s^b \partial s^c} = -\frac{1}{4}\frac{\partial}{\partial \ell_a}\left(\frac{\partial F}{\partial \ell_b}\right)\frac{\partial}{\partial s^b}\left(\frac{\partial K}{\partial s^c}\right) = \frac{\partial s^b}{\partial \ell_a}\frac{\partial \ell_c}{\partial s^b} = \delta^a_c. \tag{3.8}$$

Furthermore, if $P(s)$ is homogeneous of degree n, from Euler's theorem for homogeneous functions we have $\ell_a s^a = n$ so the potential and its dual differ by a constant that can be safely ignored. Thus we can write:

$$F(\ell) = \log \tilde{P}(\ell) := \log \left[P(s(\ell))\right]^{-1}. \tag{3.9}$$

Note that if $P(s)$ is homogeneous of degree n the same is true for $\tilde{P}(\ell)$ due to the definition of the dual-saxions.

To complete the dualization, as in section 1.2.4 we trade the axions for the dual three-form defined as:

$$\mathcal{H}_{3,a} = d\mathcal{B}_{2,a} = -M_p^2 \mathcal{G}_{ab} * da^b. \tag{3.10}$$

It can be shown that this dual map preserves the supersymmetric structure. There is a subtlety, since the axion dualization produces a boundary term of the form $a^a \mathcal{H}_{3,a}$, but we will not need to write this unless it is strictly necessary.

We can finally write the action in terms of dual fields. For our purposes it is more convenient to Wick rotate to Euclidean signature, since we will study Euclidean solutions to the equations of motion:

$$S_{EFT} = -\frac{M_p^2}{2}\int_{\mathcal{M}_4} *R + \frac{M_p^2}{2}\int_{\mathcal{M}_4} \mathcal{G}^{ab}(\ell) d\ell_a \wedge *d\ell_b + \frac{1}{2M_p^2}\int_{\mathcal{M}_4} \mathcal{G}^{ab}(\ell)\mathcal{H}_{3,a} \wedge *\mathcal{H}_{3,b}. \tag{3.11}$$

As we previously remarked, when we perform the dualization at the Euclidean path integral level we find real two-forms $\mathcal{B}_2$ but imaginary axions, so it is convenient to explicitly show their complex nature:

$$da^a = \frac{i}{M_p^2}\mathcal{G}^{ab} * \mathcal{H}_{3,b}. \tag{3.12}$$

There are also terms that do not affect the equations of motion and should be included in the action, as they can contribute to the on-shell action and give non trivial contributions to the Euclidean path integral. The most prominent example is the Gibbons-Hawking-York boundary term, which we already discussed in section 1.2.1. There are also topological terms and semi-topological terms one should keep track of. The former expression denotes operators written as integrals of total derivatives, whereas the latter includes the possibility that the coefficient of a topological operator depends on the fields. An interesting example of a topological operator is the Gauss-Bonnet term, which is discussed at length in [34]:

$$E_{GB} := \frac{1}{32\pi^2}\left(R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2\right). \tag{3.13}$$

A semi-topological operator consistent with axionic shift symmetry is:

$$\int d^4x\sqrt{-g}\,\frac{\pi}{6}\tilde{C}_a s^a E_{GB} \tag{3.14}$$

where $\tilde{C}_a \in \mathbb{Z}$. The Gauss-Bonnet operator is the most important higher-curvature term and receives both perturbative and non-perturbative corrections. We will not dwell on a complete discussion, since we will assume that such corrections are small in the regime we are studying.

3.4 Equations of motion

In this section we will find the metric solution for the Euclidean lagrangian.

In particular we are looking for solutions with rotational $SO(4)$ symmetry, so all the fields and the components of the metric are allowed to depend only on the radial distance. The metric ansatz is:

$$ds_4^2 = \frac{1}{M_p^2}\left[e^{2A(\eta)}d\eta^2 + e^{2B(\eta)}d\Omega_3^2\right], \tag{3.15}$$

where η is a dimensionless variable defined on an interval $I \subset \mathbb{R}$ and $d\Omega_3^2$ is the line element on the unit three-sphere $d\Omega_3^2 = d\Psi^2 + \sin^2\Psi(d\theta^2 + \sin^2\theta d\phi^2)$. By reparametrization invariance only one of the two function A and B is independent, therefore we gauge-fix η conveniently:

$$e^{B(\eta)} = \eta := M_p r, \tag{3.16}$$

and r is now a coordinate with the dimension of a length. This parametrization is well defined on the domain $\eta > 0$.

Radial symmetry enforces the field strength to take the form:

$$\mathcal{H}_{3,a} = \frac{q_a}{\pi}\mathrm{vol}_{S^3}, \tag{3.17}$$

where vol_{S^3} is the volume three-form on S^3. To see why this is the case note that the conserved current under axion shift symmetry is $J_{a\,\mu} = -M_p^2\mathcal{G}_{ab}\partial_\mu a^b$. If the axions depend only on the radial coordinate, the current 1-form is $J_a = -M_p^2\mathcal{G}_{ab}\partial_\eta a^b d\eta$, so the dual form depends only on the spherical differentials. The conserved Nöther charge is:

$$\frac{1}{2\pi}\int_{S^3} *J_a = \frac{1}{2\pi}\int_{S^3}\mathcal{H}_{3,a} = q_a \in \mathbb{Z} \tag{3.18}$$

which is obeyed by (3.17). Therefore we can interpret q_a as wormhole charges, which must be quantized because of the periodicity of the axions. Of course, the sign of q_a is arbitrary, so we expect that for every solution with charge q_a there is a corresponding solution with charge $-q_a$.

Then, we evaluate the action on the ansatz:

$$\begin{aligned} S|_{ansatz} = &-6\pi^2\int_I d\eta\left[e^{A+B} + \left(\frac{dB}{d\eta}\right)^2 e^{3B-A}\right] + 6\pi^2\left[e^{2B}\right]_{\partial I} + \\ &+\int_I d\eta\left[e^{3B-A}\pi^2\mathcal{G}^{ab}\frac{d\ell_a}{d\eta}\frac{d\ell_b}{d\eta} + e^{A-3B}\mathcal{G}^{ab}q_a q_b\right], \end{aligned} \tag{3.19}$$

and extremizing with respect to A we get the constraint:

$$e^{6B-2A}\pi^2\mathcal{G}^{ab}\frac{d\ell_a}{d\eta}\frac{d\ell_b}{d\eta}-\mathcal{G}^{ab}q_aq_b=6\pi^2\left(\frac{dB}{d\eta}\right)^2e^{6B-2A}-6\pi^2e^{4B}. \tag{3.20}$$

It is better to replace η with an "affine" coordinate τ that satisfies:

$$d\tau=\pm\frac{1}{\pi}e^{A-3B}d\eta. \tag{3.21}$$

The plus/minus sign includes the two possible orientations: as we will see τ covers a larger patch than η. We will denote $\dot{\ell}_a := d\ell_a/d\tau$. Using this reparametrization the dual-saxions action becomes:

$$S_\ell=2\pi\int d\tau\left[\frac{1}{2}\mathcal{G}^{ab}(\ell)\dot{\ell}_a\dot{\ell}_b+\frac{1}{2}\mathcal{G}^{ab}(\ell)q_aq_b\right]. \tag{3.22}$$

This is the lagrangian of a classical point particle with kinetic matrix $\mathcal{G}^{ab}$ and potential

$$V_{\mathbf{q}}(\ell):=-\frac{1}{2}\mathcal{G}^{ab}(\ell)q_aq_b:=-\frac{1}{2}\left\|\mathbf{q}\right\|^2. \tag{3.23}$$

The dual-saxions equations of motion are immediately deduced:

$$\frac{\partial^2F}{\partial\ell_a\partial\ell_b}\ddot{\ell}_b=\frac{1}{2}\frac{\partial^3F}{\partial\ell_a\partial\ell_b\partial\ell_c}\left(q_bq_c-\dot{\ell}_b\dot{\ell}_c\right). \tag{3.24}$$

However, we will not solve directly these equations of motion in general: a few simplifications can be done. Indeed, given the lagrangian nature of the problem, there is a conserved "energy":

$$E:=\frac{1}{2}\mathcal{G}^{ab}\dot{\ell}_a\dot{\ell}_b+V_{\mathbf{q}}(\ell). \tag{3.25}$$

Since the potential is negative definite, the conserved energy can take any real value. An important distinction must be made in the three cases $E \gtreqless 0$, as the wormhole geometry is very different. We can interpret the dual-saxions dynamics as a scattering process: if τ is defined on a symmetric interval $[-\tau_\infty;+\tau_\infty]$, the dual-saxions start with a value ℓ_∞ at $-\tau_\infty$, they climb the potential $V_{\mathbf{q}}$ until they reach a value ℓ_* at $\tau=0$, and eventually they roll back to ℓ_∞ at τ_∞. Of course, this explanation holds for $E<0$; if $E>0$ the dual-saxions manage to escape the potential and they develop a singular behaviour at finite distance. We will return on this point later on.

If we insert the conserved energy (3.25) in (3.20) and we use the gauge (3.16) we get:

$$2E=6\pi^2M_p^4r^4(e^{-2A}-1)\quad\Rightarrow\quad e^{2A}=\frac{1}{1+\frac{E}{3\pi^2M_p^4r^4}} \tag{3.26}$$

so the metric in this gauge is:

$$ds^2=\frac{1}{1+\frac{E}{3\pi^2M_p^4r^4}}dr^2+r^2d\Omega_3^2. \tag{3.27}$$

What we have found really looks like a gravitational instanton of the Giddings-Strominger kind: it is a solution of the classical equations of motion that has a narrow localization in space and interpolates among two vacua. From this formula it is clear that the sign of E has deep consequences on the wormhole geometry. The peculiarities of the three possible situations will be discussed in the next sections. The differences in the geometry depend only on the sign of E whose absolute value simply sets the characteristic size beyond which the space becomes flat. E plays the role of the instanton size M we mentioned in section 1.2.3 and is a collective coordinate of our solution.

The three cases $E > 0, E < 0, E = 0$ are different "superselection" sectors which cannot be deformed one into another, although as we will see they are connected by suitable limits. As it will be analyzed later, the connections between these sectors are clearer in a higher dimensional perspective. This is similar to the discussion of [6], where the role of E in determining the underlying geometry is played by q^2. As we will further explain, the lagrangian studied in [6] is a particular case of the one reported here obtained from the reduction of pure supergravity (possibly with an additional dilaton), so it is natural to expect a correspondence between the two frameworks (the matching will be given in section 3.9).

In the next sections we will consider one by one the three possibilities $E \gtreqless 0$. First we will describe the peculiarities of each solution in four dimensions, discussing their definiteness and singularities. Then we will see how they can be uplifted to five dimensions using the dictionary developed in section 2. The result is a surprising unity in the description of a large class of 5D supergravity solutions, which points at a deep correspondence between wormholes and black holes.

3.5 Extremal solution

3.5.1 The E=0 solution and BPS limits

We will start the analysis of the wormhole solutions with the case $E = 0$, which will be addressed as "extremal". This solution is tackled first as it is the easiest one in the four dimensional perspective and allows us to give an easy argument about the size of non-perturbative corrections. Moreover it is natural to expect that this is a limiting case of both the $E > 0$ and $E < 0$ solutions by taking appropriate limits, thus it will serve us as a guide to understand what happens when we deform it to finite energy. As we will discuss this case is a "bridge" to non-extremal solutions.

The energy is identically vanishing if we take $\dot{\ell}_a = -q_a$, or equivalently $\dot{\ell}_a = q_a$, since the charge sign is arbitrary. The metric in this case is manifestly flat:

$$ds^2_{extr} = dr^2 + r^2 d\Omega_3^2, \tag{3.28}$$

while the equation of motion for the dual-saxions can be immediately integrated to:

$$\ell_a(\tau) = \ell_{a\,\infty} + q_a(\tau_\infty - \tau), \tag{3.29}$$

where $\ell_{a\,\infty}$ are the values of the dual-saxions at infinity. In this case it would seem improper to call such solution a wormhole, since it does not connect two asymptotically flat spaces. As hinted in the introduction, there is another possible meaning we can give to this solution:

even though this is not a wormhole in the sense of gravitational instantons it could still be a wormhole in the higher dimensional Lorentzian sense. We postpone the discussion of the uplift to five dimension to the next section.

It is convenient to write the dual-saxions in terms of the radial coordinate. From (3.21), using $e^A = 1$, we get:

$$d\tau = \pm \frac{dr}{\pi M_p^2 r^3} \Rightarrow \int_\tau^{\tau_\infty} d\tau' = \pm \frac{1}{M_p^2 \pi} \int_r^\infty \frac{dr'}{r'^3}. \tag{3.30}$$

In the case of extremal wormholes the space we are describing is one half of the space of non-extremal wormholes since the coordinate change degenerates (see section 3.8). The $\pm$ sign should then be interpreted not as a result of the patch described by ρ, but as the choice of which of the two sides of the wormhole we are retaining in the limit $E \to 0$. Choosing the plus sign so that τ and r have the same orientation we find:

$$\ell_a(r) = \ell_{a\,\infty} + \frac{q_a}{2\pi M_p^2 r^2}. \tag{3.31}$$

The solution is singular at $r = 0$. This is the typical form of extremal (BPS) instantons known in the literature, equal to what is found in [6], see section 3.9 for further details about the comparison with our results.

The BPS structure of the solution can be made more manifest if we look at the scalar part of our action (3.11), as discussed in [36]:

$$S_{scal} = \frac{1}{2} \int \mathcal{G}^{ab} \left(M_p^2 d\ell_a \wedge * d\ell_b + \frac{1}{M_p^2} \mathcal{H}_{3,a} \wedge * \mathcal{H}_{3,b} \right). \tag{3.32}$$

We can rewrite this in the manifestly BPS form:

$$S_{scal} = \frac{M_p^2}{2} \int \left[\mathcal{G}^{ab} \left(d\ell_a \pm \frac{1}{M_p^2} * \mathcal{H}_{3,a} \right) \wedge * \left(d\ell_b \pm \frac{1}{M_p^2} * \mathcal{H}_{3,b} \right) \mp \frac{2}{M_p^2} d \left(s^a \mathcal{H}_{3,a} \right) \right]. \tag{3.33}$$

We have used that the double Hodge dual of a p-form in a n-dimensional space is $* * \omega_p = (-1)^{p(n-p)} \omega_p$, and also the identity $\omega_p \wedge * \eta_p = \eta_p \wedge * \omega_p$. The only non trivial step to prove is that the differential cancels the double product stemming from the first term. For this purpose we need to relate the differential of s^a with that of ℓ_a. From the definitions of section 3.3 we have:

$$ds^a = \frac{1}{2} d \left(\frac{\partial F}{\partial \ell_a} \right) = \frac{1}{2} \frac{\partial^2 F}{\partial \ell_a \partial \ell_b} d\ell_b = -\mathcal{G}^{ab} d\ell_b, \tag{3.34}$$

thus as promised:

$$d(s^a \mathcal{H}_{3,a}) = ds^a \mathcal{H}_{3,a} + s^a d\mathcal{H}_{3,a} = -\mathcal{G}^{ab} d\ell_a \mathcal{H}_{3,b}. \tag{3.35}$$

The first part of the action (3.33) is manifestly non-negative, so the action is greater or equal than the boundary term. The BPS bound is saturated by the condition:

$$d\ell_a = \pm \frac{1}{M_P^2} * \mathcal{H}_{3,a} \tag{3.36}$$

or equivalently:

$$d\ell_a = \pm i \mathcal{G}_{ab} da^b, \tag{3.37}$$

where the $\pm$ reflects the ambiguity of the charge sign (equivalently, of what we call instanton or anti-instanton). Starting from (3.17) this BPS condition translates into:

$$d\ell_a = \pm \frac{q_a}{M_p^2 \pi} * \mathrm{vol}_{S^3} = \pm \frac{q_a}{\pi M_p^2 r^3} dr \tag{3.38}$$

which is precisely what we found above. So as it happens usually in field theory the BPS bound provides an alternative and explicit method to construct particular configurations that stationarize the action.

Now we want to show that the size of non-perturbative corrections is indeed compatible with the naive estimate we gave at the beginning of this discussion. Starting from (3.11) the on-shell action can be computed using the traced Einstein equation:

$$S_{on-shell} = \frac{1}{M_p^2} \int \mathcal{G}^{ab} \mathcal{H}_{3,a} \wedge * \mathcal{H}_{3,b} = 2\pi \int d\tau \, \|\mathbf{q}\|^2 . \tag{3.39}$$

Actually the relevant quantity we will consider is half the full on-shell action, i.e. the contribution of a semi-wormhole. Of course in the extremal case we get this automatically since the "wormhole" is one-sided. If we now evaluate the real part of the on-shell action for the extremal solution with $\dot{\ell}_a = -q_a$ we get:

$$\begin{aligned} S_{extr} &= 2\pi \int_0^{\tau_\infty} d\tau \mathcal{G}^{ab} q_a q_b = -2\pi \int_0^{\tau_\infty} d\tau \mathcal{G}^{ab} q_a \frac{d\ell_b}{d\tau} \\ &= 2\pi \int_0^{\tau_\infty} d\tau q_a \frac{ds^a}{d\tau} = 2\pi \langle \mathbf{q}, \mathbf{s}(\infty) \rangle - 2\pi \langle \mathbf{q}, \mathbf{s}(0) \rangle . \end{aligned} \tag{3.40}$$

At this point we could argue that the second term vanishes: since the dual-saxions diverge at $r = 0$ the saxions will go to zero. However, our procedure so far has been a little too naive. The divergence at $r = 0$ is a serious problem and could point at a breaking of our EFT. Furthermore, we assumed that $d\mathcal{H}_{3,a} = 0$ which enabled us to write $\mathcal{H}_{3,a} = d\mathcal{B}_{2,a}$. This is *almost* true, likewise the electric field of a point particle is *almost* divergence-free, except for the location of the particle, where Gauss's law exhibits a delta singularity. The same situation is occurring here, as the source located at $r = 0$ will modify Bianchi identity for $\mathcal{H}_{3,a}$ to $d\mathcal{H}_{3,a} = 2\pi q_a \delta_0^{(4)}$. To keep track of this point-like instanton source we need to add a piece to our action that includes contribution of localized BPS instantons. It is then clear that our discussion holds provided we stay away from $r = 0$; for example, if Λ is a UV cutoff, the minimum distance we can probe is $r_\Lambda = \Lambda^{-1}$. Thus our on-shell action is regulated as:

$$S_{extr}^{\Lambda} = 2\pi \langle \mathbf{q}, \mathbf{s}(\infty) - \mathbf{s}(\Lambda^{-1}) \rangle . \tag{3.41}$$

Now we also need a counterterm that describes the contribution of a BPS instanton localized at distances of order Λ^{-1}. It can be proven that this action is of the form (see section 5.3 of [34] and appendix E of [36]):

$$S^{\Lambda}_{loc} = 2\pi\langle \mathbf{q}, \mathbf{s}(\Lambda^{-1})\rangle. \tag{3.42}$$

The sum of the two contributions yields precisely $2\pi\langle \mathbf{q}, \mathbf{s}(\infty)\rangle$, which is the form we expect for the action of a BPS instanton. In truth, this is only the real part of the action, since by supersymmetry there is also an imaginary piece for the axions that ensures that the full on-shell action is holomorphic. However, the size of the non-perturbative corrections is set by the saxions, as we could expect. We can conclude that extremal wormholes are manifestation in the low energy EFT of fundamental BPS instantons.

As a final remark, it is worth stressing that the BPS bound may be violated in some situations, because for non-extremal wormholes with $E < 0$ one does not have to insert the local counterterm encoding the presence of fundamental BPS instantons (see section 5.5 of [34]).

3.5.2 Uplift of extremal solution

Now that we have understood the basic features of extremal configurations, we ask ourselves what is the corresponding solution in the five dimensional theory. To this purpose we have to make contact between the dual description in the four dimensional lagrangian and the scalar parametrization of the five dimensional geometry (2.3). In the context of dimensional reduction the dual-saxions have an immediate interpretation in terms of five-dimensional coordinates; using that the Kähler potential is (2.70) we have:

$$\ell_a = -\frac{1}{2}\frac{\partial K}{\partial s^a} = \frac{1}{2}\frac{\partial}{\partial s^a}\log\left(C_{def}s^d s^e s^f\right) = \frac{1}{2}e^{-\varphi}\tilde{Y}_a. \tag{3.43}$$

This means that the dual-saxions are nothing but rescalings of the dual coordinates $\tilde{Y}_a$ defined in (2.6). In general, by homogeneity of $P(s)$ it follows that $\ell_a s^a = n/2$. In the case of dimensional reduction this constraint is inherited by the structure of the 5D manifold since it holds $\ell_a s^a = \tilde{Y}_a Y^a/2 = 3/2$.

The dual potential is again $F = \log(\tilde{P}(\ell)) := \log(P(s(\ell)))^{-1}$. Since $P(s) = e^{3\varphi}$ we then have:

$$\tilde{P}(\ell) = e^{-3\varphi}, \tag{3.44}$$

and so $\tilde{P}(\tilde{Y}) = 8$.

The scalar fields are determined by the following system of equations:

$$\begin{cases} \ell_a = \frac{1}{2}e^{-\varphi}\tilde{Y}_a = \ell_{a\,\infty} + \frac{q_a}{2\pi M_p^2 r^2} \\ \frac{1}{6}C_{abc}Y^aY^bY^c = 1 \end{cases} \tag{3.45}$$

where the $\tilde{Y}_a$ are written as functions of the Y^a. Consistently, this is a system of n_V+2 equations in $n_V + 2$ variables, and the Y^a are eventually expressed in terms of some parametrization of the physical scalar fields ϕ.

We can then write the uplifted five-dimensional metric:

$$\begin{aligned} ds_5^2 &= \frac{l_5^2 M_p^2}{4\pi} e^{-\varphi} ds_4^2 + l_5^2 e^{2\varphi} dy^2 = \\ &= \frac{l_5^2 M_p^2}{4\pi} \tilde{P}(\ell)^{\frac{1}{3}} (dr^2 + r^2 d\Omega_3^2) + l_5^2 \tilde{P}(\ell)^{-\frac{2}{3}} dy^2. \end{aligned} \tag{3.46}$$

We would like to write the metric in a form that is asymptotically flat. In general the dual-saxions can have any value at infinity, provided they stay in the perturbative domain. Thus the elements of the metric tend to a constant which is different from one; to fix this issue we define a new radial coordinate:

$$\hat{r} := \frac{l_5 M_p}{\sqrt{4\pi}} \tilde{P}(\ell_\infty)^{\frac{1}{6}} r, \tag{3.47}$$

along with a new Euclidean time:

$$\sigma := l_5 \tilde{P}(\ell_\infty)^{-\frac{1}{3}} y, \tag{3.48}$$

so the metric becomes:

$$ds_5^2 = \left(\frac{\tilde{P}(\ell)}{\tilde{P}(\ell_\infty)} \right)^{\frac{1}{3}} (d\hat{r}^2 + \hat{r}^2 d\Omega_3^2) + \left(\frac{\tilde{P}(\ell)}{\tilde{P}(\ell_\infty)} \right)^{-\frac{2}{3}} d\sigma^2. \tag{3.49}$$

Note that the periodicity of y induces the one of σ:

$$y \sim y + 1 \ \Rightarrow \sigma \sim \sigma + l_5 \tilde{P}(\ell_\infty)^{-\frac{1}{3}}. \tag{3.50}$$

Now if we define:

$$e^{2U} := \left(\frac{\tilde{P}(\ell)}{\tilde{P}(\ell_\infty)} \right)^{\frac{1}{3}} \tag{3.51}$$

and we Wick rotate:

$$\sigma = it \tag{3.52}$$

the metric takes the form:

$$ds_5^2 = -e^{-4U} dt^2 + e^{2U} (d\hat{r}^2 + \hat{r}^2 d\Omega_3^2). \tag{3.53}$$

It is easy to see that this is precisely the metric of a BPS black-hole in five dimensions, as described in [37]. Recall that this kind of solution is characterized by harmonic functions H_a that satisfy:

$$e^{2U} \tilde{Y}_a = H_a(\hat{r}) = h_a + \frac{Q_a}{M_p^2 \hat{r}^2}. \tag{3.54}$$

(Pay attention that the dual coordinates X_I defined in [37] differ from our $\tilde{Y}_a$ by a factor of 1/3.) Here h_a are numbers related to the values of the scalar at infinity and Q_a are rescalings of our charges q_a. From (3.51) it follows that:

$$H_a = \frac{2\ell_a}{\tilde{P}(\ell_\infty)^{\frac{1}{3}}} = \frac{2\ell_{a\,\infty}}{\tilde{P}(\ell_\infty)^{\frac{1}{3}}} + \frac{l_5^2 q_a}{(2\pi\hat{r})^2}, \tag{3.55}$$

and this gives the identification between our parameters and those used in [37]. Using homogeneity and $\tilde{P}(\tilde{Y}) = 8$, the relation between the warp factor and the harmonic functions is:

$$e^{2U} = \frac{1}{2}\tilde{P}(H)^{\frac{1}{3}}. \tag{3.56}$$

Notice also that:

$$e^{2U}\tilde{Y}_a = H_a \Rightarrow e^{2U} = \frac{1}{3}H_a Y^a \tag{3.57}$$

which will be useful later. We can also find out what is the uplifted field strength, recalling that the only non-vanishing component is the one determined by the axions:

$$\begin{aligned} F^a_{(5d)} &= 2\pi da^a \wedge dy = -2\pi i \mathcal{G}^{ab} d\ell_b \wedge dy = \\ &= -4\pi i e^{2\varphi} G^{ab} d\ell_b \wedge dy = -2\pi i e^{2\varphi} G^{ab} \tilde{P}(\ell_\infty)^{\frac{1}{3}} dH_b \wedge dy = \\ &= -2\pi i e^{-4U} G^{ab} \tilde{P}(\ell_\infty)^{-\frac{1}{3}} dH_b \wedge dy = -\frac{2\pi i}{l_5} e^{-4U} G^{ab} dH_b \wedge d\sigma, \end{aligned} \tag{3.58}$$

and now if we Wick rotate back to Lorentzian signature $d\sigma = idt$ we get:

$$F^a_{(5d)} = \frac{2\pi}{l_5} e^{-4U} G^{ab} dH_b \wedge dt \tag{3.59}$$

which apart from normalization constants is consistent with [37].

We have thus learnt that there is a deep correspondence between extremal wormholes in four dimensions and BPS black holes in five dimensions, since the two solutions are related via uplift/dimensional reduction. To show this result we heavily relied on the structure of 5D supergravity and the induced homogeneity of degree 3 of the prepotential. Therefore it would be interesting to understand whether this result holds in other dimensions, beyond the simplest case of pure supergravity where all kinematic quantities are trivial. Indeed, as shown in [6], which considers precisely this case, the uplift to physical black holes (i.e. free from naked singularities) holds for any dimension $D > 2$.

This correspondence shows how singularities in lower dimensions can be reinterpreted upon uplifting to higher dimensions. In four dimensions $r = 0$ is a point at finite spatial distance that codifies the present of localized instantons coupled to the three-forms $\mathcal{H}_3$. At this point the dual-saxions diverge, so as we already pointed out we cannot trust this classical solution throughout all space: when r tends to zero, the dual-saxions go to infinite distance in field space. From string theory experience we know that this infinite boundary cannot be reached for many reasons, so realistically the solution captures the correct physics for $r > \Lambda^{-1}$. For smaller distances the theory will enter in a strongly-coupled regime, the EFT will break down and corrections, both perturbative and non-perturbative, will alter the metric.

The situation is different in the five-dimensional theory. $\hat{r} = 0$ now corresponds to the horizon of the BPS black-hole, which lies at infinite spatial distance. We seem to have introduced a pathological behaviour in the metric, which was absent at the four-dimensional level. However, in the five-dimensional theory the scalars are perfectly regular: it is well known that in BPS black-holes they tend to a value at the horizon that is completely independent on asymptotic information and only depends on the black-hole charges. This is the famous attractor mechanism, which is better understood as a flow towards a fixed point related to the central charge of the supergravity algebra. Let us then study more in detail the near-horizon limit.

The values towards which the scalars flow are obtained taking the horizon limit:

$$\tilde{Y}_a(\phi_*) := \lim_{\hat{r}\to 0} \frac{2\ell_a}{\tilde{P}(\ell)^{\frac{1}{3}}} = \frac{2q_a}{\tilde{P}(q)^{\frac{1}{3}}}. \tag{3.60}$$

These values are important because they determine the near-horizon geometry of our spacetime. The quantity we need is the black-hole central charge:

$$Z = q_a Y^a. \tag{3.61}$$

This central charge is useful as the near-horizon geometry is completely captured by it:

$$e^{2U} = \frac{1}{3}H_a Y^a = \frac{1}{3}\frac{l_5^2 q_a}{(2\pi\hat{r})^2}Y^a(\phi_*) + ... = \frac{1}{3}\frac{l_5^2 Z_{hor}}{(2\pi\hat{r})^2} + ... \tag{3.62}$$

Where the dots denote subleading terms in the limit $\hat{r} \to 0$. Using (3.56) we find that equivalently:

$$e^{2U} = \frac{1}{2}\tilde{P}(H)^{\frac{1}{3}} = \frac{1}{2}\frac{l_5^2}{(2\pi\hat{r})^2}\tilde{P}(q)^{\frac{1}{3}} + ... \tag{3.63}$$

Comparing the two equations one gets an explicit expression for the central charge at the horizon:

$$Z_{hor} = \frac{3}{2}\tilde{P}(q)^{\frac{1}{3}} \tag{3.64}$$

which as expected depends only on the black-hole charges. Note that $Z_{hor}^{(Sabra)} = l_5^2 Z_{hor}^{(here)}/4\pi^2 M_p^2$.

We now define the quantity:

$$R^2 := \frac{l_5^2}{8\pi^2}\tilde{P}(q)^{\frac{1}{3}}, \tag{3.65}$$

then the near-horizon metric in Euclidean signature becomes:

$$\begin{aligned} ds_5^2 &= e^{-4U}d\sigma^2 + e^{2U}(d\hat{r}^2 + \hat{r}^2 d\Omega_3^2) \\ &\simeq \frac{\hat{r}^4}{R^4}d\sigma^2 + \frac{R^2}{\hat{r}^2}d\hat{r}^2 + R^2 d\Omega_3^2. \end{aligned} \tag{3.66}$$

We define a new variable:

$$u := \frac{\hat{r}^2}{2R} \tag{3.67}$$

so that:

$$ds_5^2 \simeq \frac{4u^2}{R^2}d\sigma^2 + \frac{R^2}{4u^2}du^2 + R^2 d\Omega_3^2. \tag{3.68}$$

So in the near-horizon we have manifestly an $AdS_2 \times S^3$ metric, which is a known result for supersymmetric black-holes (it is analogous to the Bertotti-Robinson metric in four dimensions). We stress that the anti-de Sitter and the three-sphere radii correctly differ by a factor of 2, a feature that is characteristic of five dimensional black-holes, while e.g. in four dimensions the two spaces have the same radius of curvature.

The space we have found is a bit peculiar, because in Euclidean signature the AdS_2 part has a periodic direction due to compactification conditions. The length of the compactification circle depends on u and is simply the product between the circle radius and the σ periodicity, so in the near-horizon we have:

$$L_{NH}(S^1) = \frac{2u}{R} l_5 \tilde{P}(\ell_\infty)^{-\frac{1}{3}} = \frac{4\sqrt{2}\pi u}{\tilde{P}(q)^{\frac{1}{6}}\tilde{P}(\ell_\infty)^{\frac{1}{3}}}, \tag{3.69}$$

and it vanishes in the limit $u/R \to 0$, i.e. when $\hat{r}/l_5$ goes to zero.

This type of BPS solution can be understood in M-theory perspective. In M-theory the fundamental entities are M2 branes, which are extended objects with a three-dimensional worldvolume. These branes can wrap cycles of the internal Calabi-Yau space, providing non-perturbative corrections to the reduced theory; for a discussion on this topic, see e.g. [39]. One kind of membrane instanton is realized if the M2 wraps a three-cycle of the internal space. The resulting correction to the reduced low energy action depends on the volume of the Calabi-Yau space. But from the discussion of section 2.4 we know that this volume is part of a universal hypermultiplet, so this implies that non-perturbative corrections arising from wrapping of three-cycles can affect only the hypermultiplet geometry. Since we have set hypermultiplets to zero from the beginning, this is not the correction we are looking for. But there is another kind of membrane instanton we should take into account when we perform the additional S^1 reduction to four dimensions: in this case the membrane has also the possibility to wrap around the S^1 and a Calabi-Yau two-cycle. These instantons now correct the moduli space of the vector multiplets, as the S^1 factor explicitly appears in the saxions. From a four dimensional perspective they are point-like objects and reduce to worldsheet instantons breaking axionic shift symmetry. In particular, since we are describing a BPS configuration, these M2 branes wrap supersymmetric cycles that in this case are holomorphic cycles, meaning that the embedding $X(\sigma)$ of the M2 brane satisfies:

$$\overline{\partial} X^a = 0, \qquad \partial X^{\overline{a}} = 0 \tag{3.70}$$

(see the appendix 4 for a recap on Calabi-Yau spaces). The metric (3.46) then represents the backreaction of such branes.

That is why we need that the compactification length is sufficiently large: instantonic contributions are schematically given by $e^{-L(S^1)}$, so non-perturbative corrections are strong for short worldlines. In the limit $u \to 0$ such corrections become of order 1, and this is the regime we previously identified as problematic because of the scalar divergencies.

3.5.3 Marginally degenerate E=0 solution

A special treatment is deserved by the solution with all the asymptotic values of the 4D dual-saxions set to zero $\ell_{a\,\infty} = 0$, corresponding to a field space point lying at infinite distance. This situation will be addressed as "marginally degenerate" case. The dual-saxions are simply:

$$\ell_a = \frac{q_a}{2\pi M_p^2 r^2}. \tag{3.71}$$

The three-forms are the same as before, as in (3.17), while the 5D scalars are constant throughout all space and they are fixed to their horizon $\phi^i = \phi^i_*$. Indeed recall that the horizon values of the 5D scalars are obtained sending $r \to 0$, so practically switching off all the $\ell_{a\,\infty}$ and keeping only the terms proportional to the charges. But that is precisely what we are doing here in an exact way, so we get constant scalars. This implies that the dilaton is obtained by taking the scalar product with Y^a:

$$\ell_a = \frac{1}{2} e^{-\varphi} \tilde{Y}_a = \frac{q_a}{2\pi M_p^2 r^2} \;\Rightarrow e^{-\varphi} = \frac{2}{3} \frac{Z_{hor}}{2\pi M_p^2 r^2} = \frac{\tilde{P}(q)^{\frac{1}{3}}}{2\pi M_P^2 r^2}. \tag{3.72}$$

The uplift works a bit differently with respect to the non-degenerate case, since now $\tilde{P}(\ell_\infty) = 0$ and we cannot rescale the coordinate as we previously did. The five-dimensional metric is explicitly:

$$\begin{aligned} ds_5^2 &= \frac{l_5^2 M_p^2}{4\pi} e^{-\varphi} ds_4^2 + l_5^2 e^{2\varphi} dy^2 = \\ &= \frac{R^2}{r^2} dr^2 + R^2 d\Omega_3^2 + l_5^2 \frac{(2\pi M_p^2 r^2)^2}{\tilde{P}(q)^{\frac{2}{3}}} dy^2. \end{aligned} \tag{3.73}$$

Now if we introduce the rescaled radial coordinate:

$$\tilde{r} := \frac{l_5 M_p}{\sqrt{4\pi}} r \tag{3.74}$$

and the Euclidean time:

$$\tilde{\sigma} := l_5 y \;\sim\; \tilde{\sigma} + l_5 \tag{3.75}$$

the metric becomes:

$$ds_5^2 = \frac{\tilde{r}^4}{R^4} d\tilde{\sigma}^2 + \frac{R^2}{\tilde{r}^2} d\tilde{r}^2 + R^2 d\Omega_3^2. \tag{3.76}$$

Finally as before we define:

$$u := \frac{\tilde{r}^2}{2R} \tag{3.77}$$

and we obtain:

$$ds_5^2 = \frac{4u^2}{R^2}d\tilde{\sigma}^2 + \frac{R^2}{4u^2}du^2 + R^2 d\Omega_3^2. \tag{3.78}$$

This is an exact $AdS_2 \times S^3$ spacetime, not just a near-horizon limit. This result is quite peculiar, in the sense that it holds for the special situation in which all the dual-saxions vanish at infinity. Nevertheless this solution is very useful since it will enable us to make contact with solutions with $E < 0$.

In this case the compactification circle has length:

$$L(S^1) = \frac{2u}{R} l_5 \tag{3.79}$$

which again vanishes in the limit $u/R \to 0$, while the decompactification limit is reached in the limit $u/R \to \infty$. However, unlike the non-degenerate case this time the periodicity of $\tilde{\sigma}$ is physically irrelevant. This happens because the metric is invariant under the simultaneous redefinitions $\tilde{\sigma} \to a\tilde{\sigma}$, $u \to u/a$, where a is a positive constant. Of course such rescaling will modify the periodicity of $\tilde{\sigma}$ into al_5, so this tells us that the period is completely arbitrary and has no physical effect. This freedom of rescaling the Euclidean time can be interpreted as the dilation symmetry of the dual CFT_1, which is simply quantum mechanics (0 + 1 dimensions), so it is nothing but time reparametrization invariance.

Now that we have described how $AdS_2 \times S^3$ solutions emerge naturally from uplift of extremal wormholes it is worth investigating more in detail their geometric properties. In particular, we want to show what are the peculiarities due to the periodicity of the compactified dimension, which acts as a kind of "thermal" direction. It is then useful to recall the basic properties of Euclidean AdS_2 space; we will mainly follow [46]. In general, a Euclidean AdS_{p+2} space can be obtained by embedding the hyperboloid:

$$X_0^2 - X_E^2 - \sum_{i=1}^{p+1} X_i^2 = \alpha^2 \tag{3.80}$$

in a flat $p+3$-dimensional space with metric:

$$ds_E^2 = -dX_0^2 + dX_E^2 + \sum_{i=1}^{p+1} dX_i^2 \tag{3.81}$$

where α is the radius of curvature (in our case $\alpha = R/2$). When dealing with anti-de Sitter spacetimes one usually employs two system of coordinates. The first one is given by global coordinates, in which the metric takes the form:

$$ds_E^2 = \alpha^2(\cosh^2\rho\, d\tau_E^2 + d\rho^2 + \sinh^2\rho\, d\Omega_p^2) \tag{3.82}$$

where $d\Omega_p^2$ is the p-sphere volume element and τ_E is the Euclidean time. If we perform the change of coordinates $\tan\theta = \sinh\rho$ the metric becomes:

$$ds_E^2 = \frac{\alpha^2}{\cos^2\theta}(d\tau_E^2 + d\theta^2 + \sin^2\theta d\Omega_p^2). \tag{3.83}$$

The coordinate θ in general takes values in the range $0 \leq \theta < \pi/2$, so we are describing a space that can be mapped into $\mathbb{R}$ times a hemisphere of S^{p+1}. When $\theta = \pi/2$ we reach the boundary of the space, which is the equator of the hemisphere and has the topology of S^p. The case of AdS_2 is special, since $p = 0$ and we have S^0, which consists just of two points. This means that θ now ranges from $-\pi/2$ to $\pi/2$, and these two points represent two separate boundaries of the space.

Another useful parametrization is yielded by the Poincaré coordinates:

$$ds_E^2 = \alpha^2 \left(\frac{du^2}{u^2} + u^2(dt_E^2 + dx^i dx^i)\right). \tag{3.84}$$

Note that although these coordinates cover only a part of the full AdS space in Lorentzian signature they actually cover the full Euclidean space.

Now we focus our attention on AdS_2. This space is the upper half-plane with the Poincaré metric. In particular by setting $u = 1/z$ we have the two equivalent parametrizations:

$$ds_E^2 = \alpha^2 \left(u^2 dt_E^2 + \frac{du^2}{u^2}\right) = \alpha^2 \frac{dt_E^2 + dz^2}{z^2}. \tag{3.85}$$

This space can be mapped into a disk. Indeed, the boundary $z = 0$ (i.e. $u = \infty$) can be mapped into a circle with one point removed, using for example stereographic projection. We recover the full boundary if we add a point at infinity that corresponds to $z = \infty$ (i.e. $u = 0$). We clarify this mapping in figure 8.

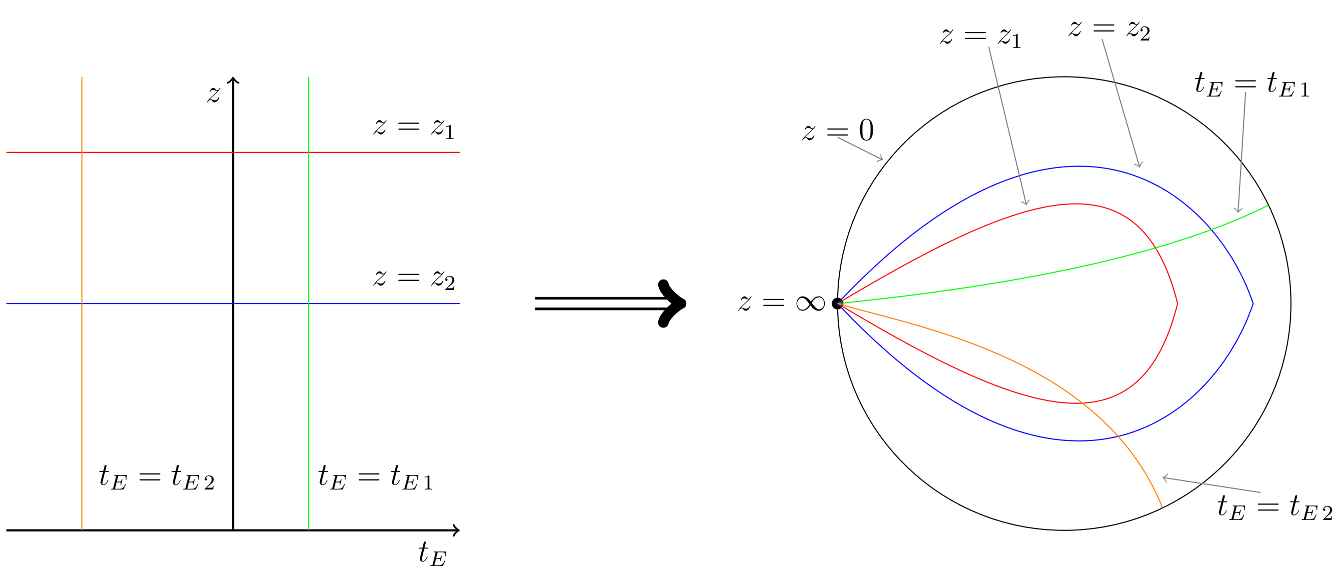

Figure 8: The Euclidean AdS_2 space as Poincaré plane which is then compactified on a disk.

The lines of constant z start at infinity, reach the line $t_E = 0$ and eventually go back to infinity, whereas the lines of constant t_E start at the boundary $z = 0$ and reach infinity. In this description the horizontal coordinate t_E could take any real value. This ensures that horizontal lines are closed in the compactified geometry and that the space-time is everywhere regular.

However, the geometry arising from the uplift of degenerate extremal wormholes is different. The horizontal coordinate is now periodic, so we have to restrict our Poincaré plane to a strip and identify the points at the boundary of this strip. We will then have a conical tip at infinity, which is typical of periodic configurations. We illustrate the situation in figure 9.

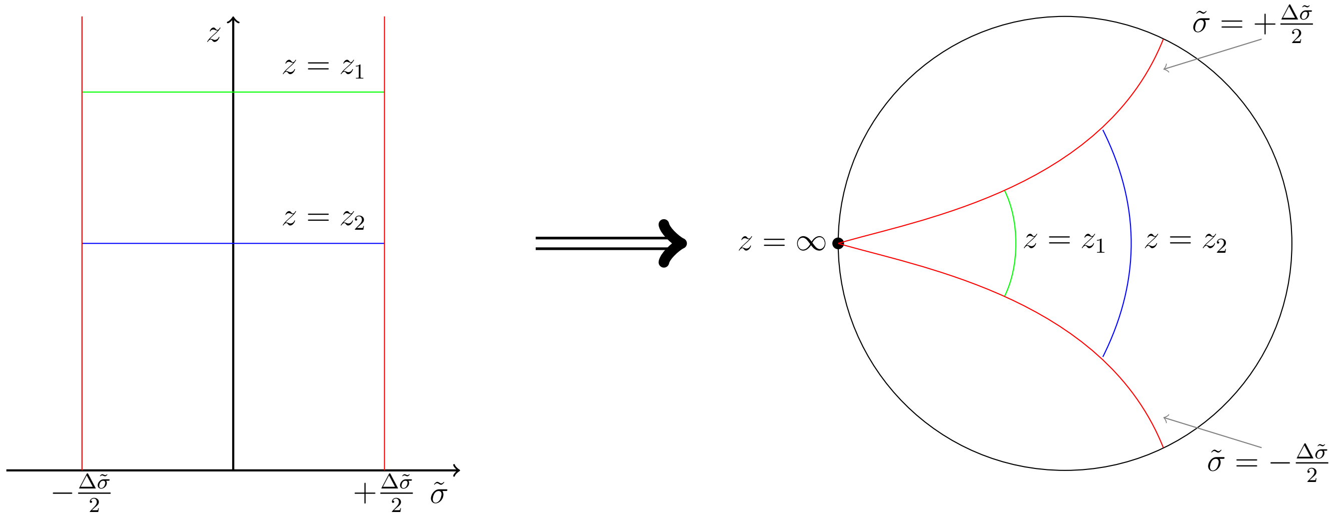

Figure 9: The "thermal" AdS_2 space as a strip in Poincaré plane and its disk compactification.

The width of the strip is equal to the period of the $\tilde{\sigma}$ variable, which we label $\Delta\tilde{\sigma}$. As we previously mentioned, this periodicity is physically irrelevant, even though it is fixed in any given coordinate system. For the sake of clearness we chose a symmetric strip centered at the origin, but of course we can move it wherever we want. The two vertical lines represent the boundary of the strip and they are identified, so every horizontal line comes back on itself after $\Delta\tilde{\sigma}$. Pictorially the conical tip at $z = \infty$ represents the wormhole throat at infinite spatial distance. This "thermal" AdS_2 is known to have important links to the Witten index $(-1)^F e^{-\beta H}$, where H is the generator of translations in the $\tilde{\sigma}$ direction; this index counts the degeneracy of ground states of the associated supersymmetric black hole. It would then be interesting to clarify the nature of 4D wormholes concerning the thermodynamic properties of such black holes. Interesting discussions regarding also string theory may be found in [47], where the relation between thermal and euclidean AdS_2 is emphasized, as in figure 10.

3.6 Homogeneous solution

So far we have described solutions in which the metric and the dual-saxions took a particularly simple form. This is due to the "mechanical" nature of our model, which allows us to find a quantity E that is conserved along the solutions. The case $E = 0$ is quite special, since all

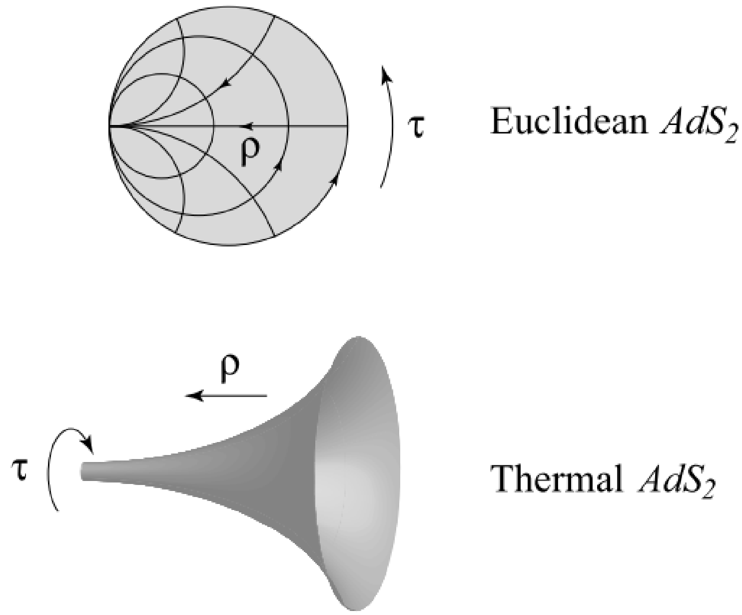

Figure 10: The difference between Euclidean and thermal AdS_2 as reported in [47].

the dual-saxions proceed with constant affine velocity, equal to the axionic charges (or their opposite). However, in general the situation is much more complicated, especially if $P(s)$ is not factorizable and thus the metric $\mathcal{G}$ is not diagonal, so one cannot find solutions in closed form.

Despite these difficulties, among all non-extremal solutions there is one which is quite peculiar. Such a solution is understood if we get a little more insight into the properties of the potential $V_{\mathbf{q}}$. If the potential was zero, all the radial directions $\ell_a(\tau) = v_a\tilde{\ell}(\tau)$ for any constant vector $\mathbf{v}$ would be geodesics of the metric $\mathcal{G}^{ab}$. The potential is proportional to $\|\mathbf{q}\|^2$ and therefore breaks isotropy identifying a preferred direction dictated by $\mathbf{q}$. It is then reasonable that a profile of the form:

$$\ell_a(\tau) = q_a\tilde{\ell}(\tau) \tag{3.86}$$

will be a solution, which we will call homogeneous. If any initial value lies on the ray generated by $\mathbf{q}$ as in (3.86) the "particle" will feel a force proportional to the charge, namely

$$F^a_{\mathbf{q}}|_{\ell_a=q_a\tilde{\ell}} = -\frac{\partial V_{\mathbf{q}}}{\partial \ell_a}|_{\ell_a=q_a\tilde{\ell}} = -\mathcal{G}^{ab}\frac{q_b}{\tilde{\ell}}, \tag{3.87}$$

so the solution will remain on this line. Indeed in [34] it is shown that the potential has a hill shape whose crest is situated along $\mathbf{q}$. The hill becomes steeper and steeper as one goes towards $\ell = 0$, as drawn in figure 11.

The peculiarities of this solution are related to the particular rearrangement of the dimensional reduction. The (dual) saxions are homogeneous variables of the constrained five-dimensional field Y^a, with the homogeneity factor represented by the dilaton. It is then intuitive that a solution of this kind is privileged in the sense that it respects the structure of the saxions and one could rightfully guess that $e^{-\varphi} \sim \tilde{\ell}$.

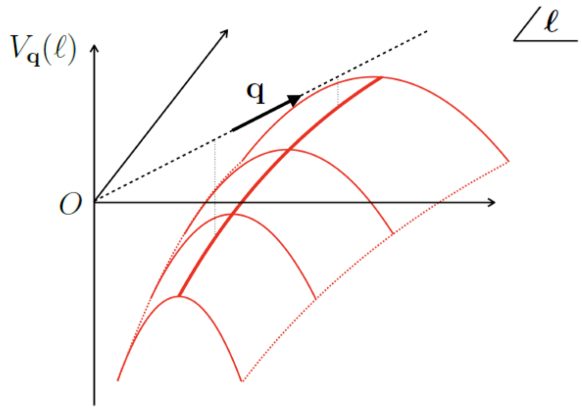

Figure 11: A representation of the potential $V_{\mathbf{q}}$. The crest $\mathbf{q}$ identifies a locus of local maxima in the directions orthogonal to $\mathbf{q}$, so the line $\mathbf{q}$ is preserved by the dynamics. Figure taken from [34].

The particular form of the homogeneous solution enables us to reduce effectively to a one dimensional dynamics encoded in $\tilde{\ell}$. Plugging (3.86) inside (3.22) we obtain:

$$S_{hom} = 2\pi \int d\tau \left[\frac{n}{4\tilde{\ell}^2} \left(\frac{d\tilde{\ell}}{d\tau} \right)^2 + \frac{n}{4\tilde{\ell}^2} \right] \tag{3.88}$$

where we have used $\mathcal{G}^{ab}(\mathbf{q}) q_a q_b = n/2$. This system can be immediately integrated using energy conservation:

$$\frac{n}{4\tilde{\ell}^2} \left(\frac{d\tilde{\ell}}{d\tau} \right)^2 - \frac{n}{4\tilde{\ell}^2} = E \tag{3.89}$$

which yields:

$$\frac{d\tilde{\ell}}{d\tau} = \pm \sqrt{\frac{4}{n} E\tilde{\ell}^2 + 1} \Rightarrow \frac{d\tilde{\ell}}{\sqrt{\frac{4}{n} E\ell^2 + 1}} = \pm d\tau. \tag{3.90}$$

At this point we need to know whether $E > 0$ or $E < 0$ to proceed with the integration. The sign of the energy will have relevant consequences on the metric and the profile of the dual-saxions. Can we find a correspondence between these two quite different solutions and five-dimensional black-holes as in the $E = 0$ case?

3.7 Wormholes, instantons and black holes

3.7.1 The E>0 solution

Let us start from the case $E > 0$. The four-dimensional metric becomes:

$$ds_4^2 = \frac{1}{1+\frac{L^4}{r^4}} dr^2 + r^2 d\Omega_3^2, \tag{3.91}$$

where we have defined the wormhole parameter:

$$L^4 := \frac{E}{3\pi^2 M_p^2}. \tag{3.92}$$

To understand the properties of this metric it is convenient to pass to isotropic coordinates. Such coordinates satisfy the relation:

$$ds_4^2 = f^2(\rho)(d\rho^2 + \rho^2 d\Omega_3^2) \tag{3.93}$$

for some function $f(\rho)$ to be determined. Comparing the two expressions for the metric we learn that:

$$\rho^2 f^2(\rho) = r^2 \Rightarrow \frac{d\rho}{\rho} = \pm \frac{dr}{r\sqrt{1+\frac{L^4}{r^4}}}. \tag{3.94}$$

The relative sign is arbitrary, since what really matters are the squares of the coordinates. We will pick the + sign so that all the quantities described below are positive. The solution to (3.94) is:

$$r^2 = L^2 \frac{k^4\rho^4 - 1}{2k^2\rho^2} \tag{3.95}$$

where k is an integration constant with the dimension of an inverse length. In isotropic coordinates the metric becomes:

$$ds_4^2 = \left(\frac{L^2k^2}{2} - \frac{L^2}{2k^2\rho^4}\right)(d\rho^2 + \rho^2 d\Omega_3^2). \tag{3.96}$$

To fix the integration constant we require this metric to be asymptotically flat, so we set $k^2 = 2/L^2$. We arrive at the result:

$$ds_4^2 = \left(1 - \frac{L^4}{4\rho^4}\right)(d\rho^2 + \rho^2 d\Omega_3^2). \tag{3.97}$$

If one rescales the metric (3.97) by an overall factor $ds_4^2 \to a^2 ds_4^2$ and then requires asymptotic flatness by rescaling $\rho^2 \to \rho^2/a^2$ the net effect is to replace $L^4 \to a^4 L^4$, so we are describing another solution with a different L which however stays in the same sector. This solution is equal to what is found in [6]; see section 3.9 for the details about the matching with our results.

Note that this metric presents a curvature singularity at $\rho^2 = L^2/2$, which corresponds to $r = 0$. The nature of this singularity will be better explored when the uplift is discussed.

Now we will find the explicit solution for the dual saxion profile $\tilde{\ell}$. From (3.90) we get:

$$\sqrt{\frac{3}{4E}} \operatorname{arcsinh}\left(\sqrt{\frac{4}{3}} E\tilde{\ell}\right) = \pm\tau + c, \tag{3.98}$$

where c is an arbitrary integration constant. Now we need to express τ in terms of the isotropic coordinate ρ. From the definition of τ we have:

$$d\tau = \pm \frac{1}{M_p^2 \pi} \frac{\rho}{\rho^4 - \frac{L^4}{4}} d\rho \tag{3.99}$$

which integrates to:

$$\tau + c' = \pm \frac{1}{2\pi M_p^2 L^2} \log \left(\frac{2\rho^2 - L^2}{2\rho^2 + L^2} \right) \tag{3.100}$$

and plugging this in (3.98) we arrive at:

$$\tilde{\ell} = \pm \frac{1}{2\pi M_p^2 L^2} \sinh \left(\log \frac{2\rho^2 - L^2}{2\rho^2 + L^2} + C \right) \tag{3.101}$$

where we have combined the two integration constants c and c' into C. The dual-saxions will then simply be given by $\ell_a = q_a \tilde{\ell}$. Concerning the sign ambiguity in (3.101), we will assume that τ and ρ have the same orientation; then we will choose the minus sign in (3.101), using the parity of sinh to invert the fraction and renaming $C \to -C$. It makes sense to choose the negative solution in (3.90), since it means to require that the dual-saxions are everywhere positive and decreasing with the radial distance.

Again, the five-dimensional scalars ϕ_i are constant and fixed to their horizon values. This happens every time we deal with a homogeneous solution irrespective of the particular form of $\tilde{\ell}$: as explained in [37] the particular form of the function H_a is irrelevant from an algebraic point of view (thus it is irrelevant also for the supersymmetry properties of the solution). Therefore if in 4D we have $\ell_a = q_a \tilde{\ell}$ the 5D scalars will be constant throughout all space $\tilde{Y}_a \equiv \tilde{Y}_a(\phi_*)$; this was true in $E = 0$ case, where $\tilde{\ell} \sim r^{-2}$ and it is true now as well. The important point is the proportionality between $\tilde{Y}_a$ and the charges q_a.

We can thus conclude that the dilaton is given by:

$$e^{-\varphi} = \frac{2}{3} Z_{hor} \tilde{\ell} = \tilde{P}(q) \tilde{\ell}, \tag{3.102}$$

where again $Z_{hor} = q_a Y^a_{hor}$ is the central charge of the corresponding BPS black hole at the horizon. From now on we will omit the subscript and Z will be understood as this particular value of the central charge.

Now we want to find the explicit form of the axions. The vielbein for our metric is:

$$\begin{gathered} e^{\underline{1}} = \sqrt{1 - \frac{L^4}{4\rho^4}} d\rho, \quad e^{\underline{2}} = \rho \sqrt{1 - \frac{L^4}{4\rho^4}} d\Psi, \\ e^{\underline{3}} = \rho \sqrt{1 - \frac{L^4}{4\rho^4}} \sin(\Psi) d\theta, \quad e^{\underline{4}} = \rho \sqrt{1 - \frac{L^4}{4\rho^4}} \sin(\Psi) \sin(\theta) d\phi. \end{gathered} \tag{3.103}$$

This leads to:

$$\mathcal{H}_{3,a} = \frac{q_a}{\pi} \frac{1}{\rho^3(1-\frac{L^4}{4\rho^4})^{\frac{3}{2}}}\, e^{\underline{2}} \wedge e^{\underline{3}} \wedge e^{\underline{4}}. \tag{3.104}$$

So exploiting the relation between axions and three-forms we have:

$$\begin{aligned} da^a &= \frac{i}{M_p^2}\mathcal{G}^{ab} * \mathcal{H}_{3,b} = \frac{2i}{M_p^2} e^{2\varphi} G^{ab} * \mathcal{H}_{3,b} = \\ &= -\frac{2i}{M_p^2} e^{2\varphi} G^{ab} \frac{q_b}{\pi} \frac{1}{\rho^3(1-\frac{L^4}{4\rho^4})^{\frac{3}{2}}}\, e^{\underline{1}} = \\ &= -\frac{2i}{M_p^2} e^{2\varphi} G^{ab} \frac{q_b}{\pi} \frac{1}{\rho^3(1-\frac{L^4}{4\rho^4})}\, d\rho \end{aligned} \tag{3.105}$$

which will be useful in the next section to find the uplifted field strength.

3.7.2 Uplift of E>0 homogeneous solution

Now we want to see what the homogeneous solution with $E > 0$ uplifts to; this result is already known from [6] in the case of a single axio-dilaton, but it is interesting to discuss it in the case of multiple scalars, since such a generalization has not been carried out so far to our knowledge.

It is convenient to rewrite the dilaton by means of hyperbolic identities. In addition so sum identities we will use that $\text{arctanh}x = \frac{1}{2}\log(\frac{1+x}{1-x})$. We also need the expression $\cosh^2(\text{arctanh}\frac{L^2}{2\rho^2})$, which is easily found:

$$\tanh^2\theta = \frac{L^4}{4\rho^4} \Rightarrow \frac{\cosh^2\theta - 1}{\cosh^2\theta} = \frac{L^4}{4\rho^4} \Rightarrow \cosh^2\theta = \frac{1}{1-\frac{L^4}{4\rho^4}}, \tag{3.106}$$

then the dilaton is:

$$e^{-\varphi} = \frac{Z}{3\pi M_p^2 L^2} \frac{4\rho^4 \sinh C + 4L^2\rho^2 \cosh C + L^4 \sinh C}{4\rho^4 - L^4}. \tag{3.107}$$

We can then evaluate the uplifted metric:

$$\begin{aligned} ds_5^2 &= \frac{l_5^2 M_p^2}{4\pi} e^{-\varphi} ds_4^2 + l_5^2 e^{2\varphi} dy^2 = \\ &= \frac{l_5^2 M_p^2}{4\pi}\left(\frac{Z}{3\pi M_p^2 L^2} \frac{4\rho^4 \sinh C + 4L^2\rho^2 \cosh C + L^4 \sinh C}{4\rho^4}\right)(d\rho^2 + \rho^2 d\Omega_3^2) + \\ &+ l_5^2 \left(\frac{Z}{3\pi M_p^2 L^2} \frac{4\rho^4 \sinh C + 4L^2\rho^2 \cosh C + L^4 \sinh C}{4\rho^4 - L^4}\right)^{-2} dy^2. \end{aligned} \tag{3.108}$$

Now we rescale the radial variable:

$$\tilde{\rho} := \frac{l_5 M_p}{\sqrt{4\pi}} \rho := \gamma\rho \tag{3.109}$$

and also the compactified coordinate, which is then Wick-rotated:

$$\tilde{\sigma} := l_5 y, \quad \tilde{\sigma} = it \tag{3.110}$$

so that the metric is cast in the form:

$$\begin{aligned} ds_5^2 = &\left(\frac{Z}{3\pi M_p^2 L^2} \frac{4\tilde{\rho}^4 \sinh C + 4\gamma^2 L^2 \tilde{\rho}^2 \cosh C + \gamma^4 L^4 \sinh C}{4\tilde{\rho}^4} \right) (d\tilde{\rho}^2 + \tilde{\rho}^2 d\Omega_3^2) \\ &- \left(\frac{Z}{3\pi M_p^2 L^2} \frac{4\tilde{\rho}^4 \sinh C + 4\gamma^2 L^2 \tilde{\rho}^2 \cosh C + \gamma^4 L^4 \sinh C}{4\tilde{\rho}^4 - \gamma^4 L^4} \right)^{-2} dt^2. \end{aligned} \tag{3.111}$$

It is not hard to see that this metric describes a Reissner-Nordström black hole, which has the form:

$$\begin{aligned} ds_{RN}^2 = &\left(\frac{4\tilde{\rho}^4 + 4\hat{M}\tilde{\rho}^2 + \hat{M}^2 - \hat{Q}^2}{4\tilde{\rho}^4} \right) (d\tilde{\rho}^2 + \tilde{\rho}^2 d\Omega_3^2) \\ &- \left(\frac{4\tilde{\rho}^4 + 4\hat{M}\tilde{\rho}^2 + \hat{M}^2 - \hat{Q}^2}{4\tilde{\rho}^4 - (\hat{M}^2 - \hat{Q}^2)} \right)^{-2} dt^2. \end{aligned} \tag{3.112}$$

The comparison between the two metrics immediately yields the identifications:

$$\begin{aligned} &\frac{Z}{3\pi M_p^2 L^2} \sinh C = 1 \\ &\frac{Z}{3\pi M_p^2} \gamma^2 \cosh C = \hat{M} \\ &\gamma^4 L^4 = \hat{M}^2 - \hat{Q}^2. \end{aligned} \tag{3.113}$$

Not surprisingly the quantity $\hat{M}^2 - \hat{Q}^2$ is proportional to the conserved energy E. Physical Reissner-Nordström black holes should have $\hat{M}^2 \geq \hat{Q}^2$ to avoid naked singularities. The case $\hat{M}^2 > \hat{Q}^2$ corresponds to a non-extremal black hole and is then the uplift of $E > 0$ wormholes, so there is consistency between the allowed range of E and the good behaviour of our spacetime. As we remarked before, the four-dimensional wormholes are badly singular at $\rho^4 = L^4/4$; this singularity is solved upon uplifting, as this point corresponds to the outer event horizon of the black hole, where there is a coordinate singularity but no curvature singularity. Note also that from the previously equality we can extract the black hole charge:

$$\hat{Q}^2 = -(\hat{M}^2 - \hat{Q}^2) + \hat{M}^2 = \frac{Z^2}{9\pi^2 M_p^2} \gamma^4, \tag{3.114}$$

so the black hole charge is completely determined by the central charge of the corresponding extremal solution.

It is also easy to verify that the uplifted field strength yields the correct result: in spherical coordinates we have to obtain the Coulomb field in five dimensions. The relation between the isotropic coordinate $\tilde{\rho}$ and the spherical coordinate r was given in (1.10) (pay attention that r is different in four and five dimensions), so using (3.105) we have:

$$\begin{aligned} F^a_{(5d)} &= 2\pi da^a \wedge dy = -\frac{4i}{M_p^2} G^{ab} q_b \gamma^2 \frac{4\tilde{\rho}(4\tilde{\rho}^4 - \hat{M}^2 + \hat{Q}^2)}{(4\tilde{\rho}^4 + 4\hat{M}\tilde{\rho}^2 + \hat{M}^2 - \hat{Q}^2)} d\tilde{\rho} \wedge dy = \\ &= -iG^{ab} \frac{q_b}{\pi} \frac{l_5^2}{r^3} dr \wedge dy = G^{ab} \frac{q_b}{\pi} \frac{l_5}{r^3} dr \wedge dt, \end{aligned} \tag{3.115}$$

which indeed is the right result, since by Bianchi identities the electric field is expected to decay like $1/r^{D-2}$ in D spacetime dimensions.

The result we have obtained is quite interesting. Solutions with positive energy look meaningless in the four dimensional sense, as the dual-saxions are "scattered away" to infinite distance and the metric develops a nasty singularity. Nevertheless the reshuffling of fields in five dimensions eliminates this behaviour and produces a regular metric. Four dimensional $E > 0$ solutions cannot be interpreted as Euclidean wormholes/gravitational instantons but they make sense in higher dimensional perspective as Lorentzian wormholes, i.e. spacelike sections of a black-hole metric.

From this discussion one could expect that nothing should prevent us from considering also the case $L = 0$, which should (and indeed does) correspond to an extremal black hole. However, the limit $L \to 0$ is a bit delicate, since we cannot simply set L to zero in our solution (3.107) keeping the other parameters fixed, otherwise we would get a divergent result. As already pointed out in [6] and suggested in (3.113), the key is to send both L and C to zero while keeping the quantity $\frac{Z}{3\pi M_p^2 L^2} \sinh C$ fixed to 1; this will also guarantee automatically that the uplifted metric is asymptotically flat. Starting from (3.107) and Taylor expanding for small L we get:

$$\begin{aligned} e^{-\varphi} &= \frac{Z}{3\pi M_p^2 L^2} \frac{1}{4\rho^4} \left(1 + \frac{L^4}{4\rho^4} + ...\right) (4\rho^4 \sinh C + 4L^2\rho^2 \cosh C + L^4 \sinh C) \\ &= 1 + \frac{Z}{3\pi M_p^2 \rho^2} + O\left(\frac{L^4}{\rho^4}\right) \end{aligned} \tag{3.116}$$

while of course taking $L = 0$ in the 4d metric results in a flat space. So the uplift reads:

$$ds_5^2 = \frac{l_5^2 M_p^2}{4\pi} \left(1 + \frac{Z}{3\pi M_p^2 \rho^2}\right) \left(d\rho^2 + \rho^2 d\Omega_3^2\right) + l_5^2 \left(1 + \frac{Z}{3\pi M_p^2 \rho^2}\right)^{-2} dy^2. \tag{3.117}$$

Apart from obvious rescalings, this is the metric of a double-extreme black hole, as shown is [37]. This metric is obtained from the general BPS solution by fixing all the scalars to their horizon values. Of course, this solution is not the most general BPS configuration one can obtain and is the result of homogeneity: it is a particular subclass of the configurations previously discussed. As explained in [48], the solution we have just found for $E > 0$ is the non-extremal deformation of a double-extreme black hole, which rightfully gives a Reissner-Nordström spacetime.

The upshot of our discussion so far is that $E \geq 0$ wormholes in four dimensions admit a natural interpretation in terms of five dimensional black holes with a smooth transition from non-extremal to extremal. It appears that uplifted $E < 0$ solutions should display naked singularities and have nothing to do with black holes. We will see that this is not entirely correct.

3.8 Axionic wormholes and AdS vacua

3.8.1 The E<0 solution

Now we will discuss the solution with negative conserved energy. The metric now reads:

$$ds_4^2 = \frac{1}{1 - \frac{L^4}{r^4}} dr^2 + r^2 d\Omega_3^2, \tag{3.118}$$

where we have defined:

$$L^4 := \frac{|E|}{3\pi^2 M_p^4}. \tag{3.119}$$

These solutions are wormholes in the Euclidean gravity sense. They are everywhere regular and connect two asymptotically flat spaces in the four-dimensional framework. The value $r = L$ is a coordinate singularity and identifies the wormhole throat, where the radius of S^3 is minimal.

Like before we pass to isotropic coordinates:

$$\frac{d\rho}{\rho} = \frac{dr}{r\sqrt{1 - \frac{L^4}{r^4}}} \Rightarrow r^2 = L^2 \frac{k^4 \rho^4 + 1}{2k^2 \rho^2}, \tag{3.120}$$

so again to have asymptotic flatness we impose $k^2 = 2/L^2$. In isotropic coordinates the metric reads:

$$ds_4^2 = \left(1 + \frac{L^4}{4\rho^4}\right)(d\rho^2 + \rho^2 d\Omega_3^2) \tag{3.121}$$

which shows that the metric is everywhere well-behaved, except for $\rho = 0$, in contrast to the $E > 0$ case. From (3.121) it can be easily shown that the minimum of the three-sphere radius is indeed L.

The τ parametrization will turn out to be the most useful to understand the properties of the wormhole and its uplift. We can integrate (3.21) with the condition that $\tau(r = L) = 0$. The result is reported in [34] and reads:

$$\tau = \pm \frac{1}{2\pi M_p^2 L^2} \left[\frac{\pi}{2} - \arcsin\left(\frac{L^2}{r^2}\right)\right] \iff \frac{L^2}{r^2} = \cos\left(2\pi M_p^2 L^2 \tau\right). \tag{3.122}$$

So if $r \in (L, +\infty)$ then $\tau \in (-\tau_\infty, \tau_\infty)$, with $\tau_\infty = \frac{1}{4M_p^2 L^2}$. The τ coordinate hence brings infinite radial distance to finite affine distance and enables us to describe the complete wormhole, not just one half. Integrating energy conservation and choosing the - sign as before we have:

$$\tilde{\ell} = -\frac{1}{2\pi M_p^2 L^2} \sin\left(2\pi M_p^2 L^2 \tau + C\right). \tag{3.123}$$

Since we want the profile to be symmetric and non-negative in the allowed τ range we set $C = -\pi/2$. This results in:

$$\tilde{\ell} = \frac{1}{2\pi M_p^2 L^2} \cos\left(2\pi M_p^2 L^2 \tau\right) = \frac{1}{2\pi M_p^2 r^2} = \frac{1}{2\pi M_p^2} \frac{4\rho^2}{4\rho^4 + L^4}. \tag{3.124}$$

Importantly, in the r coordinate the dual-saxion has the same form as in the case $E = 0$. This solution is marginally degenerate, since all the dual-saxions vanish at infinity. This is a peculiarity of the homogeneous solution stemming from a $P(s)$ of degree 3. The homogeneous solution with $P(s)$ of degree $n = 1, 2$ degenerates at finite distance, hence it is not physically acceptable, while in the cases $n > 3$ dual-saxions reach a finite value at infinity.

3.8.2 Uplift of E<0 homogeneous solution

As also discussed in [6] the solution we have just found cannot be uplifted to a black hole. This can be also understood by looking at the previous uplifts, which described physical black holes for $E \geq 0$ solutions. Furthermore the fact the dual-saxions are marginally degenerate tells us that the five-dimensional geometry cannot be asymptotically flat, since asymptotic flatness can be achieved only via a suitable fixing of integration constants as we did before. Is there any chance to push the wormhole - black hole correspondence also to such situation?

To find out let us write the uplifted metric:

$$\begin{aligned} ds_5^2 &= \frac{l_5^2 M_p^2}{4\pi} e^{-\varphi} ds_4^2 + l_5^2 e^{2\varphi} dy^2 = \\ &= \frac{l_5^2 \tilde{P}(q)^{\frac{1}{3}}}{8\pi^2 r^2} \frac{1}{1 - \frac{L^4}{r^4}} dr^2 + \frac{l_5^2 \tilde{P}(q)^{\frac{1}{3}}}{8\pi^2} d\Omega_3^2 + l_5^2 \frac{4\pi^2 M_p^4 r^4}{\tilde{P}(q)^{\frac{2}{3}}} dy^2 = \\ &= \frac{R^2}{\tilde{r}^2} \frac{d\tilde{r}^2}{1 - \frac{l_5^4 M_p^4 L^4}{16\pi^2 \tilde{r}^4}} + R^2 d\Omega_3^2 + \frac{\tilde{r}^4}{R^4} d\tilde{\sigma}^2 = \\ &= \frac{R^2}{4u^2} \frac{du^2}{1 - \frac{l_5^4 M_p^4 L^4}{64\pi^2 R^2 u^2}} + R^2 d\Omega_3^2 + \frac{4u^2}{R^2} d\tilde{\sigma}^2, \end{aligned} \tag{3.125}$$

where we have used the same redefinitions we exploited in section 3.5.3, passing first to $\tilde{r}$ and then to u. The metric we have found is of course very similar to the one of the marginally degenerate $E = 0$ case, except for the additional factor controlled by the wormhole parameter L. In the limit $u \to \infty$, away from the wormhole neck, this factor approaches one and we recover the $AdS_2 \times S^3$ spacetime we found in section 3.5.3. However this metric displays a coordinate singularity at the minimum radius $r = L$ as it also happened in four dimensions, so this coordinate system is not the most suited to describe the whole wormhole. The idea is to exploit the affine coordinate τ we have used to integrate the equations of motion, so we define:

$$\xi := 2\pi M_p^2 L^2 \tau \;\Rightarrow\; \frac{L^2}{r^2} = \cos\xi. \tag{3.126}$$

The coordinate ξ takes values in $(-\frac{\pi}{2}, \frac{\pi}{2})$. The metric becomes:

$$ds_5^2 = \frac{R^2}{4}\frac{d\xi^2}{\cos^2\xi} + R^2 d\Omega_3^2 + \frac{M_p^4 L^4 l_5^4}{16\pi^2 R^4}\frac{d\tilde{\sigma}^2}{\cos^2\xi}. \tag{3.127}$$

If we define the coordinate:

$$\zeta := \frac{2\tilde{\sigma}}{R}\frac{M_p^2 L^2 l_5^2}{4\pi R^2} \sim \zeta + \frac{2l_5}{R}\frac{M_p^2 L^2 l_5^2}{4\pi R^2} := \zeta + \Delta\zeta \tag{3.128}$$

the metric is:

$$ds_5^2 = \frac{R^2}{4\cos^2\xi}(d\xi^2 + d\zeta^2) + R^2 d\Omega_3^2, \tag{3.129}$$

while the uplifted field strength reads:

$$\begin{aligned} F^a &= 2\pi da^a \wedge dy = \frac{2\pi i}{M_p^2}\mathcal{G}^{ab} * \mathcal{H}_{3,b} \wedge dy = -\frac{2\pi i}{M_p^2}\frac{\mathcal{G}^{abq_b}}{2\pi L^2} d\xi \wedge dy = \\ &= -\frac{2i}{M_p^2 L^2} e^{2\varphi} G^{ab} q_b d\xi \wedge dy = -i\frac{l_5}{4\pi R\cos^2\xi} G^{ab} q_b d\xi \wedge d\zeta. \end{aligned} \tag{3.130}$$

This is an $AdS_2 \times S^3$ Euclidean space in global coordinates, but importantly it is an exact result and not just a large distance limit. Furthermore, the ξ coordinate enables us to describe both sides of the wormhole, not just one as it occurs using the r coordinate.

Let us make a few remarks about this result. In section 3.5.3 we argued that the period of the coordinate $\tilde{\sigma}$ was physically irrelevant because a rescaling $\tilde{\sigma} \to a\tilde{\sigma}$ could be always compensated by redefining $u \to u/a$. This is not the case now: if we start from the last line of (3.125) and perform the mentioned rescalings we find the same metric with L^4 replaced by $a^2 L^4$, which still is a sensible solution of our theory. This is connected to the observation we made previously and suggested by [6]: in the sectors $E > 0$ and $E < 0$ a coordinate rescaling results in a solution in the same sector with a different value of E. We could say that this is a sort of spontaneous breaking of the accidental scale symmetry. This reflects ultimately on different periods of the Euclidean AdS_2 time ζ.

Note that in the limit $L \to 0$, the coordinate change (3.126) becomes ill-defined: there is no way to have a variable which describes the entire wormhole. In this limit the wormhole throat closes and the two sides become two entirely disconnected spaces, each described by (3.78). This situation is analogous to the extremal limit of a Reissner-Nordström black hole, where the wormhole throat becomes infinitely long and the spacetime is geodesically complete without the need of a mirror-like region on the other side. We could then state a bit naively that the solution with $E < 0$ is obtained from the marginally degenerate solution with $E = 0$ upon opening the wormhole neck and bringing another copy of our universe to finite distance. We illustrate pictorially this limit in figure 12.

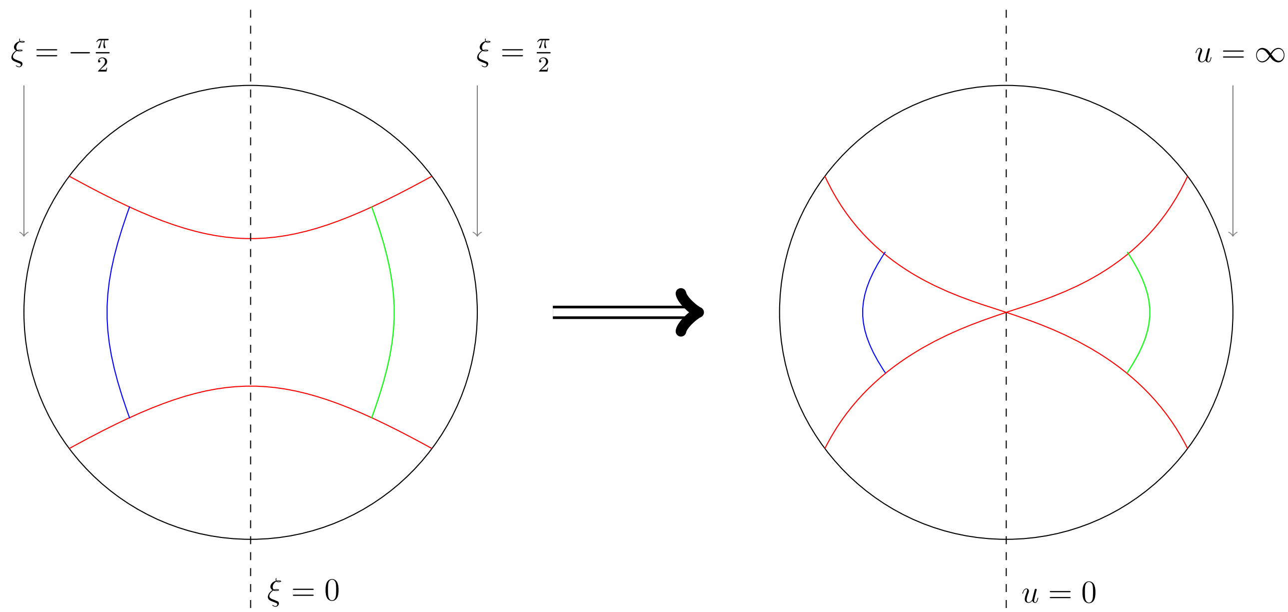

Figure 12: The uplift with $E < 0$ and its limit as $L \to 0$. In this limit the wormhole throat closes and only one of the halves survives as a geodesically complete space (for example, the right-hand side).

The configuration we have obtained from this uplift is an $AdS_2 \times S^3$ vacuum of the very same kind of a near-horizon of an extremal black hole. To show this it is convenient to Wick rotate $\zeta \to i\zeta$ and to pass to Poincaré coordinates with the following transformation:

$$T = \frac{R}{2}\frac{\sin\zeta}{\sin\xi + \cos\zeta}, \quad U = \frac{R}{2}\frac{\cos\zeta + \sin\xi}{\cos\xi} \tag{3.131}$$

and the metric takes the familiar near-horizon form:

$$ds^2 = -\frac{4U^2}{R^2}dT^2 + \frac{R^2}{4U^2}dU^2 + R^2 d\Omega_3^2. \tag{3.132}$$

Note that these Poincaré coordinates (U, T) do not coincide with those we previously defined in (3.125). In that case u represented the distance from the wormhole neck, and in fact the metric showed a coordinate singularity at $r = L$. The new coordinate U is a function both of the AdS_2 time and global spatial coordinate, thus it does not admit an immediate interpretation from the four dimensional perspective. In these new coordinates the metric seems to represent a near-horizon of a black hole, which however is not the same black hole we were describing in the $E = 0$ situation. In that case the $AdS_2 \times S^3$ spacetime was a genuine near-horizon of a physical black hole, the same we got from the uplift of $E > 0$ and then taking the extremal limit. We also know from [6] and previous discussions that the solution with $E < 0$ cannot be uplifted to physical black holes. But this configuration appears to satisfy all the feature we expect from an $AdS_2 \times S^3$ solution in supergravity. We know that this spacetime is supported by constant flux, so we look at the transformation of the field strength:

$$\begin{aligned} F^a &= \frac{l_5}{4\pi R \cos^2 \xi} G^{ab} q_b d\xi \wedge d\zeta = \frac{l_5}{4\pi R \cos^2 \xi} \left(\frac{\partial \xi}{\partial U} \frac{\partial \zeta}{\partial T} - \frac{\partial \zeta}{\partial U} \frac{\partial \xi}{\partial T} \right) G^{ab} q_b dU \wedge dT = \\ &= \frac{l_5}{\pi R^3} G^{ab} q_b dU \wedge dT, \end{aligned} \tag{3.133}$$

which is indeed constant as we expected. Thus we have all the ingredients for a sensible AdS vacuum, with constant flux and scalars fixed to their horizon values.

The result we have obtained tells us that, despite the observations made in the previous sections could suggest the opposite, a sort of wormhole - black hole correspondence holds even in the case of $E < 0$, even though in a more subtle way. This correspondence between four and five dimensions is clear in the case $E \geq 0$, where the four dimensional solution is reinterpreted as a Lorentzian wormhole, a spacelike section of a black hole that eventually becomes infinitely long when $E \to 0$. The uplift of $E < 0$ solutions cannot be related immediately to the previous cases: the coordinate u measuring the distance from the wormhole neck in the $E = 0$ sector now degenerates and must be replaced with the global coordinate ξ that describes the two sides of the wormhole. This situation resembles the $\mathbb{Z}_2$ isometry we mentioned in section 1, but its nature is different since we are no more talking about a Reissner-Nordström black hole. The Poincaré coordinate u of the $E = 0$ solution is now a redefinition of the *global* coordinate ξ. The resulting global AdS_2 space has its own Poincaré coordinates (U, T), in which the geometry is identical to the near-horizon of an extremal black hole that has nothing to do with the one of the $E = 0$ solution.

As explicitly shown in [40], the $AdS_2 \times S^3$ solution preserves all the supercharges, since the supersymmetric Killing spinors coincide with the geometric Killing spinors, which are purely mathematical objects encoding information about the manifold and can be viewed as "square roots" of Killing vectors. Indeed, in Poincaré coordinates (3.132) the Killing spinor equations read:

$$\begin{aligned} AdS_2: &\quad \delta\Psi_\mu = \left(\nabla_\mu - \frac{1}{R}\gamma_\mu \right) \varepsilon_{AdS} = 0 \\ S^3: &\quad \delta\Psi_a = \left(\nabla_a + \frac{1}{2R}\gamma_a \right) \varepsilon_{S^3} = 0 \end{aligned} \tag{3.134}$$

for some convenient representation of the γ matrices. It is worth saying that this simple factorization is possible only in Poincaré coordinates, where the field strength F is constant. In other parametrizations the equations for S^3 contain the AdS_2 coordinates and one does not manage to disentangle the two parts.

Of course, this discussion holds in Lorentzian signature. In Euclidean signature with periodic boundary conditions for ζ the situation is less intelligible. To make clearer the distinction between Lorentzian and Euclidean we set $\zeta_L = -i\zeta_E$. The change of variables is now:

$$T_L = -\frac{R}{2}\frac{i\sinh\zeta_E}{\sin\xi + \cosh\zeta_E} =: -iT_E, \quad U = \frac{R}{2}\frac{\cosh\zeta_E + \sin\xi}{\cos\xi}. \tag{3.135}$$

The Euclidean structure is preserved by the coordinate change and we can identify T_E as the new Euclidean time, so that the Euclidean metric has the same form as (3.132). But there is a problem: ζ_E is a periodic variable and the functions $\cosh, \sinh$ are not well defined on periodic coordinates. This poses some troubles: it means that the results depend on the particular interval we choose for ζ_E, which does not make much sense for a periodic coordinate; in addition, T_E itself would not be periodic in this scenario. This change of coordinates then does not appear completely legitimate in Euclidean signature. This is a manifestation of a deeper fact: when we reduce over y our metric ansatz breaks the symmetry group of our theory from Diffeomorphisms($\mathcal{M}_5$) to Diffeomorphisms($\mathcal{M}_4$) $\times$ Diffeomorphisms(S^1). Coordinate changes that mix y with the reduced coordinates do not respect such structure.

The study of the quantum properties of our uplift is made harder by these difficulties. In [26] a prescription is given to understand whether a given saddle contributes to the path integral or not. One needs to identify the correct boundary conditions for the metric and the gravitinos around the reduction circle, then one has to take into account fluctuations and study zero modes. There are two kinds of zero modes: gauged zero modes, which leave the boundary conditions unchanged and provide a non-vanishing contribution to the partition function, and physical zero modes, which instead change the boundary values and do not contribute to the path integral. This analysis is done in Poincaré coordinates with T_E as a periodic, thermal coordinate. It would be non trivial to repeat the same calculation in global coordinates with ζ_E as a periodic variable, since it would change the boundary conditions and thus the class of saddles with a finite one loop determinant. We will not attempt such calculation here, but the problem remains.

3.9 Relation with D-instantons

The setup and results we have discussed are a generalization and extension of [6] (see also [35] for a review). This section will be devoted to a brief review of that paper and will make clear the links with our work.

The main motivation of [6] is the study of non-extremal deformations of the D-instanton, which in type IIB supergravity is the electromagnetic dual of the circular D7 brane. The results are actually very general and can be extended beyond their original scope to arbitrary dimensions and dilaton couplings. We will show that those solutions are fully compatible with ours, albeit our theory stems from the very different context of IIA supergravity.

The authors start from the following Einstein-Maxwell-dilaton action in a $D+1$ Minkowskian spacetime:

$$S_{D+1} = \frac{M_p^2}{2}\int\left[*R_{D+1} - \frac{1}{2}d\hat{\phi}\wedge *d\hat{\phi} - \frac{1}{2}e^{a\hat{\phi}}F_{D+1}\wedge *F_{D+1}\right], \tag{3.136}$$

where $F_{D+1} = dA_{D+1}$. This action is then reduced over the time coordinate with the ansatz:

$$ds^2_{D+1} = e^{2\alpha\varphi} ds^2_D - e^{2\beta\varphi} dt^2, \qquad A_{D+1} = \chi dt \tag{3.137}$$

which is analogous to the reduction we performed in section 2.5, although we preferred to reduce over a compact spacelike dimension and Wick-rotate *after* performing the uplift. Here χ is the Wick rotated axion and the coefficients α and β are chosen to obtain the Einstein frame action and the canonically normalized dilaton φ in D dimensions, i.e. $\alpha^2 = (2(D-1)(D-2))^{-1}, \beta = (2-D)\alpha$. The result of the reduction is in Euclidean signature:

$$S_D = -\frac{M_p^2}{2} \int \left[*R_D - \frac{1}{2} d\hat{\phi} \wedge *d\hat{\phi} - \frac{1}{2} d\varphi \wedge *d\varphi + \frac{1}{2} e^{a\hat{\phi} - 2\beta\varphi} d\chi \wedge *d\chi \right]. \tag{3.138}$$

Note that in this setting the axion has the "wrong" sign: this is connected to the remark we made when discussing the imaginary nature of axions in Euclidean signature. At this point one can perform a field redefinition by rotating $(\varphi, \hat{\phi})$ so that only one of the dilatons couples to the axion:

$$S_D = -\frac{M_p^2}{2} \int \left[*R_D - \frac{1}{2} d\hat{\phi} \wedge *d\hat{\phi} - \frac{1}{2} d\varphi \wedge *d\varphi + \frac{1}{2} e^{b\varphi} d\chi \wedge *d\chi \right], \tag{3.139}$$

where the constant b is given by:

$$b = \sqrt{a^2 + \frac{2(D-2)}{D-1}} := \sqrt{a^2 + \frac{4}{c^2}}. \tag{3.140}$$

To make contact with our model, where no dilaton was present in five dimensions, we set $a = 0$ and truncate $\hat{\phi}$. This amounts to fixing $bc = 2$, which is the least possible value b can take in the context of dimensional reduction. So we have the action:

$$S_D = -\frac{M_p^2}{2} \int \left[*R_D - \frac{1}{2} d\varphi \wedge *d\varphi + \frac{1}{2} e^{b\varphi} d\chi \wedge *d\chi \right]. \tag{3.141}$$

It is quite clear that this action is the specialization of (3.1) to the case of one axio-dilaton, arising from the reduction of pure supergravity. In our case the dilaton was not canonically normalized per se whereas the saxions were. Notice also that in our action (2.69) the axions were multiplied by a factor of $e^{-2\varphi}$ absorbed in the metric $\mathcal{G}_{ab}$. By comparing with (3.141) we can identify $e^{b\varphi/2} \to e^{-\varphi}$ in our situation. Furthermore since in the general case with n_V vector multiplets the dual-saxions are $\ell_a = \frac{1}{2} e^{-\varphi} \tilde{Y}_a$, in the case of pure supergravity it is natural to expect that for us ℓ coincides with $e^{b\varphi/2}$.

The action (3.141) has a manifest $SL(2, \mathbb{R})$ symmetry as it happens in the scalar part of IIB supergravity. Such symmetry produces a matrix of conserved charges transforming in the adjoint of $SL(2, \mathbb{R})$:

$$Q_E = \begin{pmatrix} q_3 & iq_+ \\ iq_- & -q_3 \end{pmatrix}. \tag{3.142}$$

However we trade the constant q_+ for a more useful one, which is defined as $\mathrm{q}^2 := -\det(Q_E) = q_3^3 - q_+ q_-$ and can take any real value. Notice that this charge is actually invariant under

$SL(2,\mathbb{R})$ transformations so it corresponds to another symmetry, which is an accidental invariance under constant rescalings of the equations of motion. The classical solutions to the equations of motion will then be split in three different conjugacy classes depending on the sign of q^2, precisely as it happened with the conserved energy E, so it is natural to expect a link between these two quantities. The crucial point discussed in [6] is that q^2 is the parameter of a spontaneously broken scale symmetry, so that rescalings of the metric result in an effective rescaling of q^2.

Let us first consider the solution with $\mathrm{q}^2 > 0$ specialized to the case $D = 4$. The metric in isotropic coordinates reads:

$$ds_4^2 = \left(1 - \frac{\mathrm{q}^2}{M_p^4 \rho^4}\right)\left(d\rho^2 + \rho^2 d\Omega_3^2\right). \tag{3.143}$$

Comparing this with what we found (3.97), we see that the solutions agree upon identifying $\mathrm{q}^2/M_p^4 = L^4/4$, i.e. $\mathrm{q}^2 = E/12\pi^2$; not surprisingly the conserved energy is identified with the instanton charge and the relevant information is contained only in the sign of such charge. Then E is both a non-extremality parameter quantifying the deviation from the BPS case and a generalization of the $SL(2,\mathbb{R})$ invariant to the case of non-vanishing vector multiplets in 5D.

Now we want to show that the homogeneous solution (3.101) is the natural generalization of the dilaton given in [6], which is:

$$e^{\frac{b\varphi}{2}} = \frac{q_-}{\mathrm{q}} \sinh\left(\frac{bc}{2} \log \frac{M_p^2\rho^2 + \mathrm{q}}{M_p^2\rho^2 - \mathrm{q}} + C_1\right). \tag{3.144}$$

If we compare with the solution $\ell_a = q_a \tilde{\ell}$ found in (3.101) and we recall that $bc = 2$ we find a perfect agreement if we identify $q_{-\,(there)} = q_{a\,(here)}/4\pi$, so the homogeneous solution correctly reduces to the single axio-dilaton solution presented in [6]. A similar calculation shows that our axion solution (3.105) agrees, apart from normalizations, with the one found in [6].

For the case $\mathrm{q}^2 = 0$ the 4D solution becomes:

$$ds^2 = dr^2 + r^2 d\Omega^2, \qquad e^{\frac{b\varphi}{2}} = g_s^{b/2} + \frac{2q_-}{M_p^2 r^2} \tag{3.145}$$

which agrees with (3.31) with the above identification of the charge q_-; the value of the dual-saxion at infinity corresponds to the string coupling $g_s^{b/2}$. In [6] the on-shell action for this BPS configuration is calculated in equation 92 and agrees with (3.40). One has to be a bit careful with the substitutions: there is an overall factor of 3 because our dilaton is not canonically normalized, and the combination $|bcq_-|/g_s^{b/2}$ in our conventions has its natural generalization in $\langle \mathbf{q}, \mathbf{s} \rangle / 2\pi$. Our solution represents an extension of this result in the sense that we showed explicitly the compatibility between the 4D solutions and the framework of BPS black holes in 5D, which involves the algebraic structure of the scalar manifold absent in the case of a single axio-dilaton.

The $\mathrm{q}^2 < 0$ solution is obtained from the $\mathrm{q}^2 > 0$ one by analytic continuation $\mathrm{q} \to i\tilde{\mathrm{q}}$:

$$ds_4^2 = \left(1 + \frac{\tilde{\mathrm{q}}^2}{M_p^4 \rho^4}\right)\left(d\rho^2 + \rho^2 d\Omega_3^2\right) \tag{3.146}$$

which agrees with (3.121) with $\tilde{\mathrm{q}}^2 = |E|/12\pi^2$. Finally if we consider the solution for the dilaton presented in [6] with $C_1 = 0$ we have:

$$
\begin{aligned}
e^{\frac{b\varphi}{2}} =& \frac{q_-}{\tilde{\mathrm{q}}} \sin\left(2\arctan\frac{\tilde{\mathrm{q}}}{M_p^2\rho^2}\right) = 2\frac{q_-}{\tilde{\mathrm{q}}} \sin\left(\arctan\frac{\tilde{\mathrm{q}}}{M_p^2\rho^2}\right)\cos\left(\arctan\frac{\tilde{q}}{M_p^2\rho^2}\right) = \\
& 2\frac{q_-}{\tilde{\mathrm{q}}}\frac{\tilde{\mathrm{q}}}{M_p^2\rho^2}\frac{M_p^4\rho^4}{M_p^4\rho^4+\tilde{\mathrm{q}}^2} = 2q_-\frac{M_p^2\rho^2}{M_p^4\rho^4+\tilde{\mathrm{q}}^2}
\end{aligned}
\tag{3.147}
$$

which agrees with what we found in (3.124) identifying again $q_{-\,(there)} = q_{a\,(here)}/4\pi$.

We have thus learnt that the homogeneous solution is the natural generalization of the non-extremal D-instanton to a theory with an arbitrary number of scalars and that the $SL(2,\mathbb{R})$ charges can be reinterpreted in terms of wormhole parameters.

3.10 Discussion and outlook

The results we have obtained have interesting physical consequences. The splitting of the solutions in three conjugacy classes, which can be regarded as distinct superselection sectors, seems to be a characteristic feature of axio-dilatonic theories. The Giddings-Strominger theory is purely axionic and does not allow for solutions with $E > 0$, namely it cannot be uplifted to black holes. This is consistent with the interpretation of the dilaton as the warp factor in the five dimensional metric.

Note that the three families of solutions may be obtained from each other by means of consecutive limits. We start from a Reissner-Nordström black hole with $E > 0$ and we take the extremal limit $\hat{M} \to \hat{Q}$, thus obtaining a double extreme black hole, which does not admit a wormhole connecting two asymptotic regions. But if we go in the near horizon we obtain an $AdS_2 \times S^3$ space in Poincaré coordinates, and finally we can "open" the wormhole throat to obtain the global AdS_2 space. We have started from a Lorentzian-like wormhole and we have ended up with a Euclidean wormhole, with a complete factorization of the angular part. This is remarkable: the particular reorganization of the degrees of freedom in dimensional reduction has allowed us to describe in a single system both Lorentzian and Euclidean wormholes. The transition between the two is related to a Wick rotation of the collective coordinate $L^2 \to iL^2$. This kind of complex-to-real transition shares a few similarities with other models: an interesting example is discussed in [49]. In the paper the authors study the Sachdev-Ye-Kitaev model, consisting in a system of N fermions in zero dimension. This theory admits a dual description in terms of JT gravity with a wormhole solution equal to (3.129) and exhibits a Lorentzian to Euclidean phase transition. It would then be relevant to understand if similar dualities can be extended to our case and if the transition we have studied can be found in other contexts.

Interestingly enough, the rigid structure of five dimensional supergravity forces a very specific prepotential, which ultimately leads to $n = 3$ marginally degenerate solutions and the uplift to an $AdS_2 \times S^3$ space for $E < 0$ wormholes. But five-dimensional supergravity arises from compactification of M-theory on Calabi-Yau threefolds, so the lowest order EFT we described is the only possible pure axio-dilaton system in this kind of compactification. It would be interesting to study other compactifications and see to which extent this discussion can be generalized; we can say that the cancellation of the $1/r^2$ factor coming from the dilaton and the r^2 of the metric is truly a peculiarity of the $n = 3$ case. Thus we do not expect that in other cases the uplifted geometry is factorized in such an easy way.

Another aspect which would be interesting to investigate is the extension of the above results to more general wormholes. We have restricted ourselves to the very simple case of the homogeneous solution, which has allowed us to trace the system back to a one-dimensional action, analogous to the case examined in [6]. So to say, we have not exploited the full dimensionality of our theory but we have just relied on the peculiar form of the potential, which breaks the radial symmetry maintaining a preferred direction selected by the charge vector. As remarked in [34], for $n = 3$ small perturbations to the homogeneous solution lead to:

$$\ell_i = \frac{q_i}{2\pi M_p^2 r^2} + \frac{f_{*\,i}}{4M_P^2 L^2}\left[1 + O(L^4/r^4)\right] \tag{3.148}$$

for some set of constants $f_{*\,i}$. This shows that small deformations at short distances will break, upon uplifting, the $AdS_2 \times S^3$ geometry.

As we hinted before, the natural continuation of this thesis is the investigation of the saddles from a path integral point of view. It would be interesting to figure out which ones of the solutions we have obtained really contribute to the partition function and if the results of [26] can be generalized to our case, i.e. $\mathcal{N} = 2$ 5D supergravity with an arbitrary number of vector multiplets. The main issue concerns the boundary conditions in Euclidean signature since they determine the smooth saddles we need to keep track of in evaluating the partition function. The structure of our reduction seems to force the identification $\zeta_E \sim \zeta_E + \Delta\zeta_E$. Given this boundary condition, do genuine wormholes with $E < 0$ contribute to the five-dimensional partition function? Do they cause non-factorization for the Witten index? A possible way is to investigate the compactification of our theory to two dimensions reducing on the trivial S^3 sphere with the ansatz:

$$ds_5^2 = \Phi^{-\frac{2}{3}} ds_2^2 + \Phi^{\frac{2}{3}}(dy^i + K^i_m \mathcal{A}^m)^2 \tag{3.149}$$

Here Φ is the dilaton, y^i are the spherical coordinates, K^i are Killing vectors on S^3 and $\mathcal{A}^m$ are $SO(4)$ gauge fields that describe rotating configurations and may be set to zero in our case. The full reduction of our 5D action yields a supersymmetrized version of JT gravity, a two dimensional theory where the dilaton is non-minimally coupled to gravity with a term $\Phi\, R$ (for a good review of this topic and its applications see also [41], [42], [43]). The goal of this reduction is to elucidate the nature of zero modes. The saddle with constant dilaton coincides with the $AdS_2 \times S^3$ solution we found, and expanding around it gives a near-horizon action that can be studied using the full machinery of JT gravity.

4 Conclusions

In this thesis we have revised some important questions related to wormhole physics in supergravity and string theory.

We have shown how a concrete axio-dilaton theory in four dimensions may be explicitly realized starting from eleven dimensional supergravity and performing Kaluza-Klein reduction on a Calabi-Yau threefold and then on a circle. We have observed that the bosonic degrees of freedom of 5D supergravity recombine in such a way that the structure of 4D supergravity naturally emerges: the complex scalar field t in four dimension has for real part the axion and for imaginary part the saxion $s^a = e^{\varphi} Y^a$. The constrained 5D scalars Y^a define a manifold which is linearly deformed in field space by the dilaton, so the saxions have a homogeneous structure that is crucial to make contact between the 4D and 5D language.

Then, motivated by wormhole conjectures and axion physics, mainly thanks to the works [6] and [34], we have turned our efforts to the axio-dilaton model obtained by consistent truncation of the resulting theory. We have stressed that radially symmetric metric solutions are classified by a parameter E and separated in three different classes depending on the sign of E. We have also paid specific attention to a particular solution, the homogeneous one, which identifies a linear direction in field space. The importance of this solution can be understood from a five-dimensional perspective: it is a privileged solution in the sense that it respects the homogeneous structure of the dual-saxions.

The uplift of these solutions is the core of the work. The solution with $E > 0$ presents a curvature singularity that looks very bad in the four dimensional theory, so that these solutions cannot be interpreted as wormholes in the Giddings-Strominger sense. However, upon uplifting the homogeneous solution, we have discovered that these solutions are indeed wormholes but in the Lorentzian sense, namely they are space-like section of a Reissner-Nordström metric. This shows that such solution represents a multiscalar generalization of the results found in [6].

The solution of $E = 0$ is understood as a "one-sided wormhole" and was shown to admit a general uplift to five-dimensional BPS black holes as described in [37]. This is an important result: even if we were completely unaware of the structure of 5D BPS black holes, we could pass to the reduced system and find the simple solutions with $E = 0$, which can be integrated exactly. Then if we uplift we automatically find a black hole preserving one half of supersymmetry. This approach shows how dimensional reduction can tell us a lot about the original theory.

The solution of $E < 0$ is analogous to the Giddings-Strominger gravitational instanton, so it is a wormhole in the Euclidean gravity sense and connects two asymptotic regions. Quite miraculously, the uplift of the homogeneous solution is an $AdS_2 \times S^3$ space, where in Euclidean setting the AdS_2 time is periodic. To our knowledge this result is completely new and could serve as a basis for future investigations about the properties of our theory, especially in its links with two dimensional Jackiw-Teitelboim gravity and further developments in the gravitational path integral.

Appendix: review on complex geometry

Calabi-Yau manifolds from string reduction

In this appendix we are going to review a few concepts about complex manifold and their importance in string theory. The initial part of this discussion is mainly based on [38].

Superstring theories live in ten spacetime dimensions because of gravitational anomaly cancellation. If we wish to describe a theory that makes any sense in our four-dimensional world we need to make an ansatz about the structure of the ten dimensional spacetime $\mathcal{M}_{10}$. As we explained in section 2 our assumption is:

$$\mathcal{M}_{10} = \mathcal{M}_4 \times \mathcal{M}_6. \tag{A.1}$$

Here $\mathcal{M}_4$ is a non-compact manifold, our four-dimensional spacetime, while $\mathcal{M}_6$ is a compact space. As before the coordinates will be denoted as $x^M = (x^\mu, y^m)$. Some simplifying assumptions are usually made, e.g. all fluxes are supposed to vanish and the dilaton is taken constant, but it is also possible to consider more general backgrounds.

Since superstring theories enjoy some supersymmetries it would be pleasant that at least a part of them is preserved in the reduced theory. This requirement is fulfilled if the gravitino variation vanishes, giving the Killing spinor equation:

$$\delta\Psi_M = \nabla_M \varepsilon = 0 \tag{A.2}$$

with ε the local supersymmetry parameter. Killing spinors must also satisfy the integrability condition:

$$[\nabla_M, \nabla_N]\varepsilon = \frac{1}{4} R_{MNRS}\Gamma^{RS}\varepsilon = 0 \tag{A.3}$$

where Γ are the ten-dimensional gamma matrices. Since our geometry is a product space the spinor may be decomposed as:

$$\varepsilon = \varepsilon^{(4)}(x) \otimes \varepsilon^{(6)}(y). \tag{A.4}$$

It can be proven that if we assume maximal symmetry for $\mathcal{M}_4$ then the integrability condition forces this manifold to be flat. $\varepsilon^{(4)}$ is not just covariantly constant but truly constant and is then the parameter of the unbroken supersymmetry on the visible spacetime. On the other hand the internal manifold $\mathcal{M}_6$ must be Ricci flat:

$$R_{mn} = 0. \tag{A.5}$$

$\mathcal{M}_6$ must be a six-dimensional spin manifold with a covariant spinor. In such a case the spinors have eight components that can be decomposed in the sum of $SU(4)$ irreducible representations $\mathbf{8} = \mathbf{4} \oplus \overline{\mathbf{4}}$ according to their chirality. The largest subgroup of $SU(4)$ under which a four dimensional spinor of definite chirality can be invariant is $SU(3)$; this implies that $\mathcal{M}_6$ must have $SU(3)$ holonomy.

So $\mathcal{M}_6$ must be a Ricci flat manifold with $SU(3)$ holonomy and a covariantly costant spinor. In other words, $\mathcal{M}_6$ is a Calabi-Yau threefold. In the following we will better explain how these manifolds can be constructed.

Complex manifolds

This discussion is mainly taken from [50]. Our starting point is a n-dimensional complex manifold $\mathcal{M}$, namely a topological space with an holomorphic atlas $\{U_j, z_j\}_{j\in A}$, where U_j is an open set on $\mathcal{M}$ and z_j are functions from U_j to an open set in $\mathbb{C}^n$; in every non-empty overlap $U_j \cap U_k$ the function $z_j z_k^{-1}$ is required to be holomorphic. In each patch U_j we may write $z_j^\mu = x_j^m + iy_j^m$ in terms of real coordinates. We will use greek indices for complex coordinates and latin indices for real coordinates. On each point is defined a tangent space that admits the basis:

$$\frac{\partial}{\partial z^\mu} := \frac{1}{2}\frac{\partial}{\partial x^m} - \frac{i}{2}\frac{\partial}{\partial y^m}, \qquad \frac{\partial}{\partial \overline{z}^\mu} := \frac{\partial}{\partial x^m} + \frac{i}{2}\frac{\partial}{\partial y^m}. \tag{A.6}$$

The dual space is the cotangent space with canonical basis $dz^\mu = dx^m + idy^m, d\overline{z}^\mu = dx^m - idy^m$ satisfying the pairing conditions $\langle dz^\mu, \frac{\partial}{\partial z^\nu}\rangle = \langle d\overline{z}^\mu, \frac{\partial}{\partial \overline{z}^\nu}\rangle = \delta^\mu_\nu$, $\langle dz^\mu, \frac{\partial}{\partial \overline{z}^\nu}\rangle = \langle d\overline{z}^\mu, \frac{\partial}{\partial z^\nu}\rangle = 0$.

We can endow $\mathcal{M}$ with an almost complex structure, a (1,1) tensor defined as:

$$J := i\, dz^m \otimes \frac{\partial}{\partial z^m} - i\, d\overline{z}^m \otimes \frac{\partial}{\partial \overline{z}^m}. \tag{A.7}$$

It is easily proven that J squares to minus the identity, i.e. ${J_m}^n {J_n}^p = -{\delta_m}^p$. This tensor is invariant under holomorphic coordinate changes and is thus globally defined on $\mathcal{M}$.

Is the viceversa also true? If a $2n$-dimensional manifold admits a tensor with these properties is it a complex manifold? The answer is yes provided that the *Nijenhuis tensor* vanishes:

$$N_{ij}{}^k = \partial_{[j} J_{i]}{}^k - J_{[i}{}^p J_{j]}{}^q \partial_q J_p{}^k = 0 \tag{A.8}$$

and in this case there exists an atlas in which J is the standard symplectic matrix.

The existence of an almost complex structure allows one to extend the de Rham cohomology to complex forms. A (p, q) form is an object like the following:

$$\omega^{(p,q)} = \omega_{\mu_1\ldots\mu_p\nu_1\ldots\nu_q} dz^{\mu_1} \wedge \ldots \wedge dz^{\mu_p} \wedge d\overline{z}^{\nu_1} \wedge \ldots \wedge d\overline{z}^{\nu_q}. \tag{A.9}$$

On a (p, q) form we can define the Dolbeault operators:

$$\begin{aligned} \partial\omega^{(p,q)} &:= \frac{\partial\omega_{\mu_1\ldots\mu_p\nu_1\ldots\nu_q}}{\partial z^\mu} dz^\mu \wedge dz^{\mu_1} \wedge \ldots \wedge dz^{\mu_p} \wedge d\overline{z}^{\nu_1} \wedge \ldots \wedge d\overline{z}^{\nu_q} \\ \overline{\partial}\omega^{(p,q)} &:= \frac{\partial\omega_{\mu_1\ldots\mu_p\nu_1\ldots\nu_q}}{\partial \overline{z}^\nu} d\overline{z}^\nu \wedge dz^{\mu_1} \wedge \ldots \wedge dz^{\mu_p} \wedge d\overline{z}^{\nu_1} \wedge \ldots \wedge d\overline{z}^{\nu_q} \end{aligned} \tag{A.10}$$

which define respectively a $(p+1, q)$ and $(p, q+1)$ form. If $H^i(\mathcal{M})$ is the i-th cohomology group of the manifold we can decompose it in cohomology groups of (p, q) forms:

$$H^i(\mathcal{M}) = \bigoplus_{p+q=i} H^{(p,q)}(\mathcal{M}). \tag{A.11}$$

The dimension of H^k is the *k-th Betti number* b_k, while the dimension of $H^{(p,q)}$ is the *Hodge number* $h^{(p,q)}$. We also recall the definition of the Euler characteristic:

$$\chi(\mathcal{M}) = \sum_{i=0}^{\dim\mathcal{M}} (-1)^i b_i. \tag{A.12}$$

$\mathcal{M}$ is called *hermitean* if it possesses a metric with only mixed indices:

$$ds^2 = g_{\mu\bar{\nu}}\, dz^\mu\, d\bar{z}^\nu. \tag{A.13}$$

Every hermitean manifold admits a unique *hermitean connection*, which has the property that J and g are covariantly constant and the connection coefficients Γ are pure in their indices. We can define the Ricci form:

$$\mathcal{R} = \frac{1}{4} R_{ijkl} J^{kl} dx^i \wedge dx^j = i\partial\,\bar{\partial}\log\sqrt{g}, \tag{A.14}$$

which is closed because of the properties of ∂. However, it is not exact in general as $\sqrt{g}$ is not a coordinate scalar. This is why we define the *first Chern class*, i.e. the equivalence class of the Ricci form:

$$c_1 = \left[\frac{1}{2\pi}\mathcal{R}\right]. \tag{A.15}$$

Of course, if c_1 vanishes then $\mathcal{R}$ is exact.

On a hermitean manifold we can define the antisymmetric form:

$$J = \frac{1}{2} J_{mn} dx^m \wedge dx^n, \quad J_{mn} = J_m{}^l g_{ln} = -J_{nm}. \tag{A.16}$$

If $dJ = 0$ then $\mathcal{M}$ is a *Kähler manifold*. In this case in each patch U_j the metric can be locally expressed as the second derivative of a function, the *Kähler potential*:

$$g_{\mu\bar{\nu}} = \frac{\partial^2 \varphi_j}{\partial z^\mu \partial \bar{z}^\nu} \Rightarrow J = i\partial\bar{\partial}\varphi_j. \tag{A.17}$$

Kähler potentials in the overlap of different patches $U_j \cap U_k$ are related by $\varphi_j = \varphi_k + f_{jk}$, for some holomorphic function f_{jk}. Importantly, the volume of a Kähler manifold of n complex dimensions is given by:

$$V = \frac{1}{n!}\int_{\mathcal{M}} \underbrace{J \wedge \ldots \wedge J}_{\text{n times}}. \tag{A.18}$$

One of the most important results about Kähler manifold is Yau's theorem: if $\mathcal{M}$ is Kähler with metric g and $c_1(\mathcal{M}) = 0$ there is a unique metric g' on $\mathcal{M}$ which is Ricci flat and whose Kähler form J' is in the same cohomology class as J. This theorem motivates the definition of a Calabi-Yau manifold, which is a compact Kähler manifold with vanishing first Chern class. We will focus on Calabi-Yau manifolds with three complex dimensions.

A first important result is that a Calabi-Yau space with non-vanishing Euler number has just two non-zero Hodge numbers, $h^{(1,1)}$ and $h^{(1,2)}$. These numbers are not sufficient to completely characterize the manifold, but they are extremely important in string theory because they determine the number of particles in the reduced theory. It is customary to summarize cohomological information through the Hodge diamond:

$$\begin{array}{ccccccc} & & & h^{(0,0)} & & & \\ & & h^{(1,0)} & & h^{(0,1)} & & \\ & h^{(2,0)} & & h^{(1,1)} & & h^{(0,2)} & \\ h^{(3,0)} & & h^{(2,1)} & & h^{(1,2)} & & h^{(0,3)} \\ & h^{(3,1)} & & h^{(2,2)} & & h^{(1,3)} & \\ & & h^{(3,2)} & & h^{(2,3)} & & \\ & & & h^{(3,3)} & & & \end{array} \quad = \quad \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{(1,1)} & & 0 & \\ 1 & & h^{(2,1)} & & h^{(2,1)} & & 1 \\ & 0 & & h^{(1,1)} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{A.19}$$

In fact complex conjugation implies $h^{(p,q)} = h^{(q,p)}$, whereas Poincaré duality forces $h^{(p,q)} = h^{(n-q,n-p)}$, so the Hodge diamond is symmetric alongside both the horizontal and vertical axis. Furthermore, if $\chi(\mathcal{M}) \neq 0$ Poincaré-Hopf theorem ensures $h^{(1,0)} = 0$. It can also be proven that any Calabi-Yau admits a holomorphic three-form Ω that is unique modulo rescalings, so $h^{(3,0)} = 1$. Thus we are left with just two Hodge numbers.

Now we come to the most important part of this detour, which is holonomy. A Calabi-Yau manifold admits a pair of globally defined spinors $\zeta, \overline{\zeta}$ of opposite chirality and a one-form $A = A_m dx^m$ such that $\mathcal{R} = dA$ and:

$$\begin{aligned} \left(\nabla_m - \frac{i}{2}A_m\right)\zeta &= 0 \\ \left(\nabla_m + \frac{i}{2}A_m\right)\overline{\zeta} &= 0, \end{aligned} \tag{A.20}$$

where $\nabla_m := \partial_m + \frac{1}{4}\omega_m{}^{ab}\gamma_{ab}$ is the spinor covariant derivative.
This theorem tells us that the holonomy group of a Calabi-Yau threefold is $SU(3) \times U(1)$. Four dimensional spinors of given chirality decompose with respect to $SU(3)$ as $\mathbf{4} = \mathbf{3} \oplus \mathbf{1}$. The spinors $\zeta, \overline{\zeta}$ are charged with respect to $U(1)$: in (A.20) the A_m cancels the $U(1)$ part of the holonomy and the operator acting on ζ is the $SU(3)$ covariant derivative.

If the manifold is Ricci flat and $b_1 = 0$, the form A is exact $A = da$ and its effect is just to multiply ζ by a phase. Reabsorbing the phase factor, we can conclude that a Ricci-flat Calabi-Yau space has $SU(3)$ holonomy and a covariantly constant spinor, which is precisely what we required in a supersymmetry preserving reduction.

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