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The Tolman-Oppenheimer-Volkoff equations in unified models of dark matter and dark energy

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Abstract

In this thesis we will work on scalar field models for dark matter and dark energy. In particular, we will study unified haloes by means of a static and spherically symmetric metric which leads to the anisotropic Tolman-Oppenheimer-Volkoff (TOV) equation. In this framework, we will develop a procedure which allows us to compute an unified density profile starting from a generic cold dark matter one, or, equivalently, from its rotation velocity. In particular, we will focus our attention on finding a suitable Lagrangian of a given type, which provides us with a microscopical description of the problem.

Contents

Introduction

The greatest part of our universe is still unknown. While ordinary (baryonic) matter, visible since emitting electromagnetic radiation, is believed to be about the 5% of all the matter detectable from gravitational effects, the rest of the universe is thought to be composed by dark matter (25%) and dark energy (70%).
Since, as far as we know, dark matter interacts very weakly with ordinary matter, indications about its existence are given only by indirect effects, such as the effective speed of sound and the rotation velocity of spiral galaxies. First, the value of the speed of sound has to be sufficiently small to explain the formation of large scale structures (clustering) in the universe, and the observed pattern of Cosmic Microwave Background (CMB) temperature anisotropies. Second, even with General Relativity taken into account, visible matter cannot alone explain the rotation velocity far from the galactic centre, which is flat for about ten times the galaxy's radius, instead of going as the inverse squared distance from the origin. Therefore, we suppose galaxies to be plunged in vast cosmic areas of dark matter, known as haloes.
Although, this is not the end of the story, because another dark component needs to be taken into account: dark energy, which is responsible for cosmic acceleration at present. At first, driven by philosophical reasons, Einstein introduced the cosmological constant Λ in order to compensate the gravitational collapse due to ordinary matter and have a static universe, but Friedmann showed that Einstein's solution is unstable and the idea of a static universe was then abandoned. On the other hand, in the 30's Hubble's observations indicated that universe was actually expanding, while in the 90's the studies on type Ia supernovae (SNIa) redshift showed that the expansion was accelerating, legitimizing the reintroduction of the cosmological constant.

In the second half of the 20^{th} century the dominant model of large scale universe, known as ΛCDM model, emerged. Characterized by a FLRW (Friedmann-Lemaître-Robertson-Walker) metric and based on the Cosmological Principle, for which the universe is homogeneous and isotropic at large scale, this model reproduces correctly decades of astronomical observations and turns Einstein field equations into Friedmann equations, combining a cosmological constant Λ, which drives acceleration at the present epoch, with a Cold component of Dark Matter (CDM).
However, this paradigm cannot explain important anomalies. First of all, the so-called problem of "cosmic coincidence" has been pointed out. As a matter of fact, the energy densities of the main components of our universe (dark matter, dark energy and baryonic matter) are nowadays of the same order of magnitude, while they decay with different profiles. Moreover, ΛCDM predicts a way more dwarf galaxies than actually observed and a cosmological constant many orders of magnitude larger than what has been measured.

While ΛCDM model provides us with a description of the universe which preserves a separation between the Dark Matter (DM) and Dark Energy (DE) components, in the last years many Unified Dark Matter (UDM) models have emerged. These models are capable of describing the dark components with a single fluid which can reproduce the accelerated expansion at late times and the structures we observe today.
A class of unified models consists of the so-called k-fields. First introduced in inflationary cosmology, in the last decades they have been applied to unified models due to their flexibility in describing various cosmic fluids with negative pressure, known as k-essence. K-fields are very useful because they

allow us to give a microscopical explanation of the most relevant cosmic effects by means of a suitable Lagrangian which depends on a scalar field minimally coupled with gravity and its kinetic term. Many Lagrangian functions, however, are equally possible and lead to the same physical results, such as the flatness of the velocity profile of galactic haloes or sound propagation.
The first of this k-fields that has been widely studied is Chaplygin gas, a perfect fluid obeying the peculiar equation of state $p = -\Lambda^2/\rho$. This example is particularly interesting since it can be derived from the Nambu-Goto action in string theory [8], and it is the only known fluid, up to now, admitting a supersymmetric generalization [10, 11].

In this thesis, we focus our attention to unified haloes, which can be described by a static and spherically symmetric universe, filled with a fluid with anisotropic pressure terms (i.e. with a transverse pressure component different from the parallel one). From this, we will derive the anisotropic Tolman-Oppenheimer-Volkoff (TOV) equation, which is a differential equation that fixes a relation between energy density and pressure. The reader is referred to [9] for qualitative solutions of the isotropic (i.e. with equal pressure components) TOV equation in the presence of the generalized Chaplygin gas[1].
In chapter one, after a brief introduction to General Relativity and Einstein field equations from a Lagrangian point of view, we will derive the anisotropic TOV equation in the context of the most general static and spherically symmetric metric. Then, we will write down the most relevant features of scalar field models.
In chapter two, we will explore calculations already done in literature [4], namely solutions to the complete TOV equation for three categories of Lagrangian functions. In particular, we will focus our attention to constant energy solutions for a Lagrangian which does not depend only on the kinetic term.
In chapter three, we will perform calculations using the model. At first, we will derive the Lagrangian functions of two types from well known CDM density profiles neglecting the TOV equation. In particular, we will focus our attention to a purely kinetic Lagrangian and more general ones characterized by a simple function of the kinetic term.
Hence, we will remap these two density profiles into UDM ones by equating the respective rotation velocity profiles and by imposing an equation of state.
Last, we will neglect the equation of state and we will develop a method which allows us to obtain directly the UDM density profile from a general CDM one by means of a system of differential equations which include the anisotropic TOV equation in the case of a weak gravitational field.

In order to simplify calculations throughout the thesis, we set $8\pi G = c = 1$ and $\frac{d}{dr} = '$. The metric signature is $(- + + +)$ and if not explicitly expressed, we will work with the Weak Energy Condition (WEC)[2].

[1] which is characterized by the more general equation of state $p = -\Lambda^{\gamma+1}/\rho^{\gamma}$.
[2] i.e. $\rho \geq 0$ and $\rho + p \geq 0$.

Chapter 1

Scalar field models

In this chapter, after briefly discussing Einstein field equations, we follow [7] in order to get the TOV equation with the most general static and spherically symmetric metric. This equation describes universes filled with a fluid with anisotropic pressure components, and it is also known as stellar structure equation because it is widely used for studying compact objects like ordinary fusion stars and neutron stars. The model is the first step in studying haloes of dark matter and dark energy. In particular, it might allow us to obtain the energy density in function of the radial coordinate.
In section 1.3, we couple the gravitational field, obtained from Einstein-Hilbert action, with a scalar field that will provide us with an unified model for dark matter and dark energy.

1.1 Gravitation

Before deriving the TOV equation, we would like to present Einstein field equations, starting from a Lagrangian point of view. However, due to the limited space, we will not discuss General Relativity in detail. We consider the reader familiar with it, so we will only exhibit the main results. An excellent textbook on tensor calculus is [1], while two of the most appreciated textbooks on General Relativity are [6, 13].
General Relativity is built on two fundamental ideas. General Covariance Principle and Equivalence Principle. The former asserts that, in a field theory, we must substitute all partial derivatives with covariant ones, to take into account the curve geometry of space, which tells the matter how to move. The latter asserts that, it sounds trivial, in weak gravitational fields Einstein equations reduce to Newton equation for gravity.
In the Lagrangian approach, we start from a suistable choice of an action, also known as Einstein-Hilbert action. It can be shown [6] that the only non trivial scalar built from the metric, making use of only its first and second derivatives, is the scalar curvature $R = g^{\mu\nu}R_{\mu\nu}$, where $R_{\mu\nu}$ is the Ricci tensor. Therefore, it is natural that a suitable action has to include it.
The action takes the form of $S = S_H + S_m$, where

$$S_H = \frac{1}{2}\int \sqrt{-g}R \, d^4x \tag{1.1}$$

is called Einstein-Hilbert action, while S_m is the action associated with matter.
If we vary Einstein-Hilbert action with to respect to the metric $g^{\mu\nu}$, then we get the Einstein field equations in vacuum:

$$\frac{1}{\sqrt{-g}}\frac{\delta S_H}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0 \; . \tag{1.2}$$

The Einstein tensor is defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \tag{1.3}$$

and it satisfies the contracted Bianchi identities

$$\nabla^{\mu} G_{\mu\nu} = 0 \; . \tag{1.4}$$

If we define the stress-energy tensor, relative to the matter part, as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}} \; , \tag{1.5}$$

we finally get the Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu} \; . \tag{1.6}$$

They are, in fact, an equivalence between the Einstein tensor (1.3) and the stress-energy tensor (1.5). The physical interpretation is straightforward. Once we have the metric for the universe, we get the energy and matter distribution (they are the same). On the contrary, we can also define the matter (or the energy) and then obtain the metric (and then the curvature etc.) of our universe.
The first constraint that has to be imposed is one of the energy conditions, which are discussed in detail in [6]. In this thesis we will use the WEC.
The most general static and spherically symmetric metric [6] is

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega^2 \; , \tag{1.7}$$

where $d\Omega^2 = d\theta^2 + \sin^2(\theta) d\varphi^2$ is the metric on S^2 (the unitary sphere on $\mathbb{R}^3$), while $e^{\alpha(r)}$ and $e^{\beta(r)}$ are two non-negative functions which depend only on the radial coordinate.
The Christoffel symbols can be computed [6] as

$$\begin{aligned}
&\Gamma^{t}{}_{tr} = \alpha' \; , && \Gamma^{r}{}_{tt} = e^{2(\alpha-\beta)} \alpha' \; , && \Gamma^{r}{}_{rr} = \beta' \; , \\
&\Gamma^{\theta}{}_{\theta r} = \frac{1}{r} \; , && \Gamma^{r}{}_{\theta\theta} = -r e^{-2\beta} \; , && \Gamma^{\varphi}{}_{r\varphi} = \frac{1}{r} \; , \\
&\Gamma^{r}{}_{\varphi\varphi} = -r e^{-2\beta} \sin^2\theta \; , && \Gamma^{\theta}{}_{\varphi\varphi} = -\sin\theta \, \cos\theta \; , && \Gamma^{\varphi}{}_{\theta\varphi} = \frac{\cos\theta}{\sin\theta} \; .
\end{aligned} \tag{1.8}$$

We compute the non-vanishing terms of the Einstein tensor (1.3) for the general static spherically symmetric metric (1.7). So, we get

$$\begin{aligned}
G^t_t &= e^{-2\beta} \left[-\frac{2}{r} \beta' + \frac{1}{r^2} \left(1 - e^{2\beta}\right) \right] , \\
G^r_r &= e^{-2\beta} \left[\frac{2}{r} \alpha' + \frac{1}{r^2} \left(1 - e^{2\beta}\right) \right] , \\
G^\theta_\theta = G^\varphi_\varphi &= e^{-2\beta} \left[\frac{1}{r} \left(\alpha' - \beta'\right) + \alpha'' + \alpha'^2 - \alpha'\beta' \right] .
\end{aligned} \tag{1.9}$$

1.2 Anisotropic Tolman-Oppenhaimer-Volkoff equation

First of all, we can simplify the Einstein tensor (1.3) by setting

$$e^{-2\beta(r)} = 1 - \frac{m(r)}{4\pi r} \; . \tag{1.10}$$

We claim, as the authors of [7] demonstrated , that $2\alpha(r)$ represents the gravitational acceleration towards the origin of the coordinates in the weak-field limit, while $m(r)$ is the total mass (actual mass, gravitational energy etc.) contained in the sphere of radius r, centred in the origin. So, the metric becomes

$$ds^2 = -e^{2\alpha(r)} dt^2 + \frac{dr^2}{1 - m(r)/4\pi r} + r^2 d\Omega^2 \; . \tag{1.11}$$

By inserting the new expression for $e^{2\beta}$ (1.10) into the Einstein tensor (1.9), the only non-vanishing terms are given by

$$\begin{aligned} G^t{}_t &= -\frac{m'}{4\pi r^2} \,, \\ G^r{}_r &= -\frac{2}{r^2}\left[\frac{m}{8\pi r} - r\alpha'\left(1-\frac{m}{4\pi r}\right)\right] \,, \\ G^\theta{}_\theta = G^\varphi{}_\varphi &= -\frac{(m'r-m)(1+r\alpha')}{8\pi r^3} + \left(1-\frac{m}{4\pi r}\right)\left[\frac{\alpha'}{r}+\alpha'^2+\alpha''\right] . \end{aligned} \tag{1.12}$$

Since the Einstein tensor (1.12) is diagonal for the metric (1.11), in the comovent frame of reference the stress-energy tensor has to be diagonal too. Thanks to the rotational symmetry, the (θ,θ) component of the stress-energy tensor is equal to the (φ,φ) component. The most general stress-energy tensor for a non-perfect fluid that sources a spherically symmetric gravitational field [7] is given by

$$T^\alpha{}_\beta = \begin{bmatrix} -\rho & 0 & 0 & 0 \\ 0 & p_\parallel & 0 & 0 \\ 0 & 0 & p_\perp & 0 \\ 0 & 0 & 0 & p_\perp \end{bmatrix} , \tag{1.13}$$

where ρ is the energy density measured by an observer at rest, $p_\parallel$ is the pressure in the radial direction, and $p_\perp$ is the pressure in the transverse direction.
By equating the (t,t) component of the Einstein tensor (1.12) to the same component of the stress-energy tensor (1.13), it is easy to obtain the mass equation

$$m' = 4\pi\rho r^2 \,. \tag{1.14}$$

Since ρ is the energy density, we can interpret $m(r)$ as the total "energy" contained in a sphere of radius r. We will restrict our analysis to the null baryonic mass case, therefore we can set $m(0) = 0$.
Now, we want to obtain an expression for the derivative with respect to r of the radial pressure starting from the Einstein tensor and the above matrix representation of the stress-energy tensor. The former has a bad inconvenient, because it contains the second derivative of the gravitational potential. Indeed, it would be difficult to handle that expression. Nonetheless, we can pursue another direction. We remind that in a general field theory, we have to impose certain conditions to the stress-energy tensor. Such conditions are covariant conservation and symmetry. The former is contained in the contracted Bianchi identities (1.4).
Einstein field equations (1.6) provide us with 10 equations, out of which six are off-diagonal. These are $'0 = 0'$ identities, provided that we are working in a coordinate system in which the metric is diagonal. Moreover, the (θ,θ) component of the Einstein tensor (1.12) is equal to its (φ,φ) component, due to spherical symmetry. The latter also satisfies the contracted Bianchi identities, due to the Einstein equations. So, we can substitute the radial component of Einstein equations into the radial part of the covariant conservation of the stress-energy tensor.
The covariant derivative of a tensor T of the type $(1,1)$ reads

$$\nabla_\mu T^\alpha_\beta = \partial_\mu T^\alpha_\beta + \Gamma^\alpha_{\mu\rho} T^\rho_\beta - \Gamma^\rho{}_{\mu\beta} T^\alpha_\rho \,. \tag{1.15}$$

From the matrix expression (1.13) and the Christoffel symbols (1.8), we have:

$$\begin{aligned} 0 = \nabla_\mu T^\mu_r &= \partial_\mu T^\mu_r + \Gamma^\mu_{\mu\rho} T^\rho_r - \Gamma^\rho{}_{\mu r} T^\mu_\rho = \\ &= p'_\parallel + \left(\alpha' + \frac{2}{r} + \beta'\right) p_\parallel + \alpha'\rho - \beta' p_\parallel - \frac{2}{r} p_\perp \,. \end{aligned} \tag{1.16}$$

Hence

$$p'_\parallel = -(\rho + p_\parallel)\alpha' + \frac{2(p_\perp - p_\parallel)}{r} \,. \tag{1.17}$$

By the (r, r) component of Einstein equations (1.6), we are now able to compute α', given our expression for the Einstein tensor in eq. (1.12):

$$\alpha' = \frac{\frac{m}{8\pi} + \frac{p_\| r^3}{2}}{r^2(1 - m/4\pi r)} \; . \tag{1.18}$$

So, finally we obtain the anisotropic TOV equation [7]

$$p'_\| = -\frac{(p_\| + \rho)(\frac{m}{8\pi} + \frac{1}{2}p_\| r^3)}{r^2(1 - m/4\pi r)} + \frac{2(p_\perp - p_\|)}{r} \; . \tag{1.19}$$

We easily notice that in the case of isotropic pressure components, i.e. $p_\perp = p_\|$, this equation reduces to the simpler one (in which the second term disappears)

$$p' = -\frac{(p + \rho)(\frac{m}{8\pi} + \frac{1}{2}pr^3)}{r^2(1 - m/4\pi r)} \; , \tag{1.20}$$

whose solutions have been widely investigated in [9] for the generalised Chaplygin gas .
A clarification about nomenclature has to be made. In this thesis we will always call eq. (1.19) the anisotropic TOV equation, while in literature equations (1.14) and (1.19) are usually referred together as the anisotropic TOV equations.

1.3 Scalar field with non-canonical kinetic term

We start by describing the general framework for a scalar field ϕ, minimally coupled with gravity. Let us consider the action

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{2} + \mathcal{L}(\phi, X) \right] \; , \tag{1.21}$$

where $\mathcal{L}(\phi, X)$ is the general Lagrangian depending on the scalar field and

$$X = -\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi \tag{1.22}$$

is the kinetic term.
By varying the action with respect to the metric, we obtain the stress-energy tensor relative to the scalar field as

$$T^\phi_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \frac{\partial \mathcal{L}(\phi, X)}{\partial X} \nabla_\mu \phi \nabla_\nu \phi + \mathcal{L}(\phi, X) g_{\mu\nu} \; . \tag{1.23}$$

We see that for the metric (1.7) the Einstein tensor is diagonal, so the stress-energy tensor has to be diagonal too. Using (1.23), we compute $T_{tr} = (\partial \mathcal{L}/\partial X) \partial_t \phi \partial_r \phi$, which has to be zero. Therefore, we can either have $\partial_r \phi = 0$ or $\partial_t \phi = 0$, unless $\partial \mathcal{L}/\partial X = 0$. We will exclude the latter and get, in the static case, $\phi = \phi(r)$. Therefore, from (1.22) and (1.10) the kinetic term becomes

$$X = -\frac{1}{2} e^{-2\beta(r)} \phi' = -\frac{1}{2} \left[1 - \frac{m(r)}{4\pi r} \right] \phi'^2 \; . \tag{1.24}$$

The equation of motion reads

$$\nabla_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right] = \frac{\partial \mathcal{L}}{\partial \phi} \; . \tag{1.25}$$

If we consider a spatially inhomogeneous static scalar field, i.e. $X < 0$, the stress-energy tensor does not describe a perfect fluid. Indeed, let us consider the time-like four-vector $n_\mu = \nabla_\mu \phi / \sqrt{-X}$, which is proportional to the gradient along the radial direction of the scalar field. The stress-energy tensor reads

$$T^\phi_{\mu\nu} = (p_\| + \rho) n_\mu n_\nu - \rho g_{\mu\nu} \; , \tag{1.26}$$

where

$$p_{\parallel} = \mathcal{L} - 2X\frac{\partial \mathcal{L}}{\partial X} \tag{1.27}$$

and

$$\rho = -p_{\perp} = -\mathcal{L} \ . \tag{1.28}$$

So, provided that the fluid is no longer perfect, we have that in the pressure in radial direction is different from the transverse pressure.
Equations (1.27) and (1.28) are useful for getting the physically significant quantities after obtaining the Lagrangian of the scalar field; but what is the correct one? One could as well find a large set of equivalent Lagrangian functions, which give the same physical results.
One dynamic variable we may consider to select Lagrangian functions is the rotation velocity. We remind that in our model we are studying spherical haloes with anisotropic pressure components, composed by dark matter and dark energy, by means of an additional scalar field minimally coupled with gravity. Haloes like this could be detected only by effects on large scale, for example by their rotation velocity v_c around the centre of gravity. It can be shown [3] that for the isotropic model the velocity curve v_c takes a simple form, in a weak-field limit[1].

$$v_c^2 = \frac{m(r)}{8\pi r} + \frac{pr^2}{2} \ . \tag{1.29}$$

Please notice that we are not dealing with the Newtonian limit. Sure enough, in deriving it, the authors simplified an expression in the slow velocity limit $v_c \ll c$[2], but by still making use of Einstein equations.
Expression (1.29) is valid in the case we have only a radial pressure component. In our model, we also have a transverse pressure component, but the expression remains the same [4], if we substitute p with $p_{\parallel}$:

$$v_c^2 = \frac{m(r)}{8\pi r} + \frac{p_{\parallel} r^2}{2} \ . \tag{1.30}$$

Another important physical quantity to be considered is the speed of sound, which in the inhomogeneous case can be defined [3,5] as

$$c_s^2 = \frac{\delta p_{\parallel}}{\delta \rho} = \frac{2X\partial^2\mathcal{L}/\partial X^2 + \partial \mathcal{L}/\partial X}{\partial \mathcal{L}/\partial X} \qquad (X < 0) \ . \tag{1.31}$$

Three regimes are possible:

1. $c_s^2 < 0$
   This is the case which leads to quantum mechanically unstable solutions. Therefore, we will neglect it in our analysis.

2. $0 < c_s^2 < 1$
   This is the underluminal case. In order to explain the clustering of the structures we observed today, c_s^2 must be sufficiently small compared to the speed of light.

3. $c_s^2 > 1$
   This is the superluminal case. It could be also possible because it does not lead to a violation of causality, since the k-field propagates in the "light cone" of a different metric [3].

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[1] which is defined by the conditions: $\frac{m(r)}{8\pi r} \ll 1$ and $\frac{p_{\parallel} r^2}{2} \ll 1$.
[2] typical rotation velocities are in the order of magnitude of 200 $km/s \ll c$.

Chapter 2

Solutions to the complete TOV

In this chapter, we repeat calculations already done by the authors of [4] for a purely kinetic Lagrangian and constant energy solutions.
Equation (1.28) provides us with an useful relation between ρ and $p_\perp$, so we can write (1.17) as

$$p'_\parallel = -\left(\frac{2}{r} + \alpha'\right)(\rho + p_\parallel) \ . \tag{2.1}$$

If we define $R = \ln(r^2 e^\alpha)$, it is immediate to see that

$$\frac{dR}{dr} = \alpha' + \frac{2}{r} \ . \tag{2.2}$$

Therefore

$$\frac{dp_\parallel}{dR} = -(p_\parallel + \rho) \ , \tag{2.3}$$

which is valid for $\alpha' + \frac{2}{r} > 0$.
Is it immediate to see that this equation is invariant under the transformation $\{\rho \to \rho + K, p_\parallel \to p_\parallel - K\}$, where K is a constant. Hence, by adding the cosmological constant $K = \Lambda$ to the Lagrangian we can describe DM + Λ as an unified fluid. From (1.27), we get

$$\frac{d}{dR}\left(\mathcal{L} - 2X\frac{\partial \mathcal{L}}{\partial X}\right) = 2X\frac{\partial \mathcal{L}}{\partial X} \ . \tag{2.4}$$

In the following of this section, we take $\mathcal{L} < 0$ (when $X < 0$) in order to have a positive energy density. Therefore, we set

$$\chi = -X \ , \qquad g_S(\chi) = -\mathcal{L}(X) > 0 \ . \tag{2.5}$$

From (2.4) we want to derive a simple expression that connects X with r [4] for a purely kinetic Lagrangian.
By setting $y = \frac{\partial \mathcal{L}}{\partial X} = \frac{\partial g_S}{\partial \chi}$, we get

$$\frac{dg_S}{dR} - \frac{d}{dR}\left(2\chi y\right) = 2\chi y \ . \tag{2.6}$$

Since $\mathcal{L}$ depends only on X and X is a function of only r, we deduce that X can be seen as a function of only R, after remapping it. We get the separable equation

$$\frac{2\ dy/dR}{y} = -\frac{2\chi + d\chi/dR}{\chi} \ . \tag{2.7}$$

After solving it, we get

$$\chi\left(\frac{dg_S}{d\chi}\right)^2 = \frac{k}{(r^2 e^\alpha)^2} \ , \tag{2.8}$$

where k is a positive constant of integration.

2.1 Constant energy density solutions

In general, if the Lagrangian has two degrees of freedom, the kinetic term and the scalar field, we have to impose extra conditions. It is common to impose that $\mathcal{L} = -\Lambda$ along classical trajectories . Moreover, we want a positive energy density, which explains the minus sign before Λ.
In this section we set $g_S(\chi) = -g(X) > 0$ and the potential $V(\phi)$ is a positive function of the scalar field.

2.1.1 Lagrangian of the type $\mathcal{L} = f(\phi)g(X)$

Starting from equation (2.4), since $\mathcal{L} = -f(\phi)g_S(\chi)$, we first have

$$\frac{d}{dR}\left[-g_S(\chi)f(\phi) + 2\chi\frac{dg_S(\chi)}{d\chi}f(\phi)\right] = -2\chi\frac{dg_S(\chi)}{d\chi}f(\phi)\ . \tag{2.9}$$

Remembering the expression $\mathcal{L}$ along the solutions, by setting $h(X) = 2\chi\frac{dg_S(\chi)}{d\chi}\frac{1}{g_S(\chi)}$, we obtain a simple differential equation:

$$\frac{dh(\chi)}{dR} = -h(\chi) \qquad\qquad (\mathcal{L} = -\Lambda)\ . \tag{2.10}$$

The general solution of (2.10) is $h(\chi) = ke^{-R}$, where k is a positive constant of integration. By restoring the expression of $h(X)$, we have

$$2\chi\frac{d\ln(g_S(\chi))}{d\chi} = \frac{k}{r^2e^\alpha}\ . \tag{2.11}$$

From(1.27), we have

$$p_\parallel = g_S(\chi)f(\phi)\left[2\chi\frac{d\ln(g_S(\chi))}{d\chi} - 1\right]\ . \tag{2.12}$$

So, plugging (2.11) in (2.12), we finally obtain

$$p_\parallel = \frac{\Lambda k}{r^2e^\alpha} - \Lambda\ . \tag{2.13}$$

Then, the rotation velocity becomes

$$v_c^2 = \frac{\Lambda k/2}{e^\alpha} - \frac{\Lambda r^2}{3}\ . \tag{2.14}$$

Provided that the RHS of (2.14) is positive, there exists a maximum radius r_{max}, such that

$$r_{max}^2 = \frac{3k}{2e^\alpha}\ , \tag{2.15}$$

and the rotation curve is approximately flat for all $r < r_{max}$, if e^α is constant and if $r_{max}^2 \ll 3k/2e^\alpha$.
Now we are able to solve for α and then obtain an expression for the radial part of of pressure that depends entirely on r.
Previously, we imposed that $\rho = \Lambda$ along solutions, so by integration from equation (1.14), we obtain

$$m(r) = \frac{4\pi}{3}\Lambda r^3 \tag{2.16}$$

and from (1.10)

$$e^{-2\beta} = 1 - \frac{\Lambda r^2}{3}\ . \tag{2.17}$$

It is straightforward to see there is a maximum radius for the validity of our solution. In fact, the solution found so far is valid for $r < r_{ds} = \sqrt{3/\Lambda}$, where r_{ds} is de Sitter radius. Since $\Omega_\Lambda = \Lambda/3H_0^2 \approx 0.704$, where $H_0 \approx 68\ km/s/Mpc$ is the Hubble constant at present [2], we get

$$r_{ds} \approx 5\ Gpc\ , \tag{2.18}$$

which is even larger than visible universe.
By inserting (2.16) in equation (1.18), we obtain a linear differential equation in e^α:

$$\frac{de^\alpha}{dr} + \frac{\Lambda r}{3 - \Lambda r^2} e^\alpha = \frac{3\Lambda k}{2r(3 - \Lambda r^2)} \ . \tag{2.19}$$

This can be easily solved. Then, we have

$$e^\alpha = \frac{\Lambda k}{2} \left[\left(1 - \frac{\Lambda r^2}{3}\right)^{1/2} \left[\frac{2\kappa}{\Lambda k} - \ln \frac{(\Lambda/3)^{1/2} r}{1 - (1 - \Lambda r^2/3)^{1/2}} \right] + 1 \right] , \tag{2.20}$$

where κ is a positive constant of integration, which should satisfy the following inequality for the RHS of (2.20) to be positive:

$$\left(\frac{\Lambda}{3}\right)^{1/2} r > \left[\cosh\left(\frac{2\kappa}{\Lambda k} + 1 \right) \right]^{-1} . \tag{2.21}$$

Therefore, this inequality fixes a minimum radius for the halo.
Finally, we obtain the rotation velocity [4] as

$$v_c^2 = \left\{ \left(1 - \frac{\Lambda r^2}{3}\right)^{1/2} \left[\frac{2\kappa}{\Lambda k} - \ln \frac{(\Lambda/3)^{1/2} r}{1 - (1 - \Lambda r^2/3)^{1/2}} \right] + 1 \right\}^{-1} . \tag{2.22}$$

But what is the physical meaning of this expression? We see immediately that the rotation velocity is approximately flat for $2\kappa/\Lambda k \approx 10^6 \ll 3/\Lambda r^2_{max}$ in the region of interest, as required, while it diverges for radii in the order of magnitude of de Sitter radius.

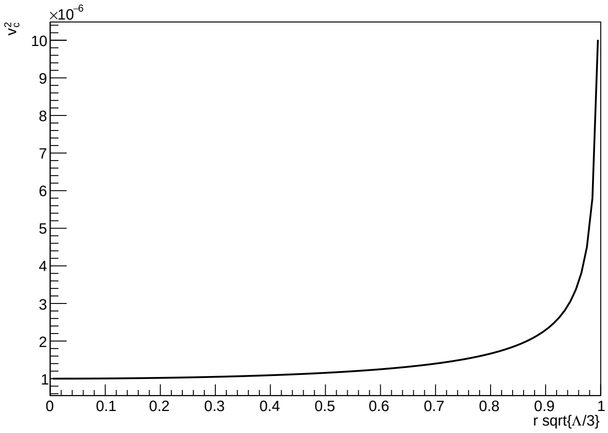

Figure 2.1: The rotation velocity for $2\kappa/\Lambda k = 10^6$.

2.1.2 Lagrangian of the type $\mathcal{L} = g(X) - f(\phi)$

We start by rewriting the Lagrangian as $\mathcal{L} = -g_S(\chi) - V(\phi)$, where $g_S(\chi) > 0$ in order to have ρ positive.
From equation (2.4), we get

$$\frac{d}{dR}\left[V(\phi) + g_S(\chi) - 2\chi \frac{dg_S(\chi)}{d\chi} \right] = 2\chi \frac{dg_S(\chi)}{d\chi} \qquad (\mathcal{L} = -\Lambda) \ . \tag{2.23}$$

Along classical solutions, we have $V(\phi) = -g_S(\chi) + \Lambda$. Then, by defining $h(\chi) = 2\chi \frac{dg_S(\chi)}{d\chi}$, equation (2.10) holds. From (2.23), we get [4]

$$\chi \frac{dg_S(\chi)}{d\chi} = \frac{k/2}{r^2 e^\alpha};, \tag{2.24}$$

where k is a positive constant of integration.
By plugging eq. (2.24) into (1.28), we get the same expressions for $p_\parallel$ and v_c found in section 2.1.1.

Chapter 3

General solutions

In Chapter 2, after founding an interesting relation for the purely kinetic Lagrangian (2.8) , in subsections 2.1.1 and 2.1.1 we imposed that ρ must be constant along the classical trajectories. In this Chapter, we are going to weaken this condition, letting ρ vary with r.
Starting from a given density profile for a CDM halo, which does not have a radial pressure component, we will derive the purely kinetic Lagrangian and the more general one of the type:

$$L(\phi, X) = X - V(\phi) , \tag{3.1}$$

which resembles the classical Lagrangian (kinetic energy - potential energy).
Then, we will reconstruct the scalar potential $V(\phi)$.

The authors of [12] found a peculiar potential for a Lagrangian with a standard kinetic term by studying a Friedmann-Robertson-Walker (FRW) cosmology of a universe filled with a Chaplygin gas.
At first, we will make a similar analysis for the two well-known CDM density profiles: the polynomial density profile and the Navarro-Frenk-White (NFW) [14] density profile.
Successively, from a CDM density profile in which there is no a proper term of pressure (i.e. $p_{\parallel} = 0$), we will remap the density function into a UDM one by equating the rotation velocity profiles, and by imposing an equation of state . We will reconstruct the Lagrangian if possible.

However, for a complete analysis one should also make use of the Einstein field equations as it has been done by the authors of [3] in the weak-field limit, which in particular yields $\alpha' \approx 0$. With this approximation, they were able to solve analytically the TOV equation by choosing only an equation of state.
Instead, we will use the weak-field limit in order to remap a general CDM density profile into a UDM one. This procedure has the advantage to give us directly an expression for the UDM density and the radial pressure, by taking into account only the physical effects, like the rotation velocity. We could, as well, choose a suitable rotation profile by investigating cosmological observations and then get a density profile with a null radial pressure component.

3.1 Cold dark matter haloes

In this section, we consider simple haloes composed by only dark matter, whose energy density profile is given. The radial pressure is set to zero.

From (1.27) and (1.28) we notice that if the Lagrangian is of the purely kinetic type, we need only an equation of state in order to reconstruct it. Since $p_{\parallel} = 0$, we get

$$\mathcal{L} = -A\sqrt{-X} \qquad (X < 0) , \tag{3.2}$$

where A is a positive constant of integration.

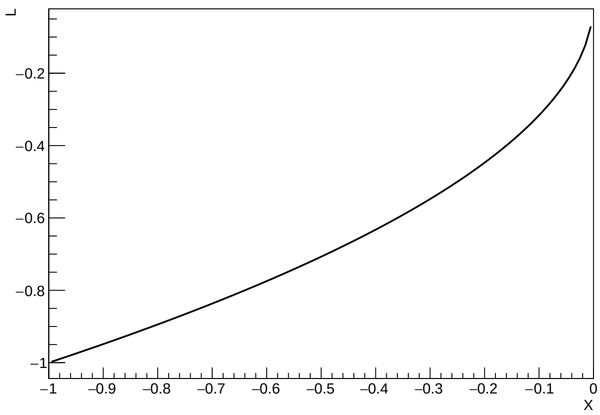

Figure 3.1: The CDM purely kinetic Lagrangian with $A = 1$.

If, however, the Lagrangian is of the type (3.1), from equations (1.27) and (1.28) we get

$$\rho = -2X = 2V(\phi) \tag{3.3}$$

Now, we want to find an explicit expression for $V(\phi)$.
From (3.3), by means of the expression for the kinetic term (1.24), we get the following integral:

$$\phi = \int e^{\beta(r)} \rho(r)^{1/2} \, dr \, . \tag{3.4}$$

If explicitly solvable, one should be able to invert the expression to find $r = r(\phi)$ and then insert into the second equation of (3.3) to get the desired expression.

3.1.1 Polynomial density profile

We could start from a simple density profile of the kind:

$$\rho = \rho_0 \left(\frac{r}{r_0} \right)^{\gamma} , \tag{3.5}$$

where $\rho_0 > 0$ is the density at the scale radius $r_0 > 0$ and $\gamma \in \mathbb{R}$.

Let us focus our attention to the case $\gamma > -3$ at moment.

From (1.14) we get

$$m(r) = \frac{4\pi\rho_0}{r_0^{\gamma}(\gamma+3)} r^{\gamma+3} = \frac{8\pi c}{2} \, r^{\gamma+3} \, , \tag{3.6}$$

where $c = \frac{\rho_0}{r_0^{\gamma}(\gamma+3)} > 0$. From (1.24), the kinetic term becomes

$$X = -\frac{1}{2}\left(1 - c r^{\gamma+2}\right)\phi'^2 \tag{3.7}$$

and the Lagrangian becomes

$$\mathcal{L} = -\frac{1}{2}\left(1 - c r^{\gamma+2}\right)\phi'^2 - V(\phi) \, . \tag{3.8}$$

Provided that $p_{\|} = 0$, the rotation velocity (1.30) becomes

$$v_c^2 = \frac{\rho_0}{r_0^\gamma} \frac{1}{2(\gamma+3)} r^{\gamma+2} \,, \tag{3.9}$$

which is limited as r tends to infinity only if $\gamma \leq -2$. In particular, $\gamma = -2$ is the only value for which the velocity profile does not depend on r.

The integral (3.4) becomes

$$\phi(r) = K \int \frac{r^{\gamma/2}}{\sqrt{1 - c r^{\gamma+2}}}\, dr = K \frac{2 \arcsin(\sqrt{c}\, r^{\gamma/2+1})}{(\gamma+2)\sqrt{c}} + \phi_0 \qquad (\gamma \neq -2) \,, \tag{3.10}$$

where we have defined $K = \sqrt{\rho_0 / r_0^\gamma}$, and ϕ_0 is a constant of integration.
We notice that for $\gamma = -2$ the potential becomes

$$\phi(r) = \frac{K}{\sqrt{1-c}} \ln(r) + \phi_0 \qquad (\gamma = -2) \,. \tag{3.11}$$

From (3.3), we get

$$V(\phi(r)) = \frac{\rho}{2} = \frac{1}{2} \frac{\rho_0}{r_o^\gamma} r^\gamma \,. \tag{3.12}$$

By inverting equation (3.10), we find:

$$r^\gamma = \left\{ \frac{\sin\left[(\phi - \phi_0) \frac{(\gamma+2)\sqrt{c}}{2K}\right]}{\sqrt{c}} \right\}^{\frac{2\gamma}{\gamma+2}} \qquad (\gamma \neq -2) \,. \tag{3.13}$$

Hence, the potential becomes

$$V(\phi) = \frac{c(\gamma+3)}{2} \left\{ \frac{\sin\left[(\phi - \phi_0) \frac{(\gamma+2)\sqrt{c}}{2K}\right]}{\sqrt{c}} \right\}^{\frac{2\gamma}{\gamma+2}} \qquad (\gamma \neq -2) \,. \tag{3.14}$$

Instead, by inverting eq. (3.11), we have

$$V(\phi) = \frac{\rho_0 r_0^2}{2} \exp\left[-\frac{2}{K} \sqrt{1-c} (\phi - \phi_0) \right] \qquad (\gamma = -2) \,. \tag{3.15}$$

The case $\gamma < -3$ is analogous. The kinetic term (1.24) becomes

$$X = -\frac{1}{2} \left(1 + c r^{\gamma+2}\right) \phi'^2 \,, \tag{3.16}$$

where now we have $c = -\frac{\rho_0}{r_0^\gamma (\gamma+3)} > 0$.
By repeating all the calculations, we get

$$\phi(r) = K \int \frac{r^{\gamma/2}}{\sqrt{1 + c r^{\gamma+2}}}\, dr = K \frac{2\, \mathrm{arcsinh}(\sqrt{c}\, r^{\gamma/2+1})}{(\gamma+2)\sqrt{c}} + \phi_0 \qquad (\gamma < -3) \tag{3.17}$$

and

$$V(\phi) = \frac{c(\gamma+3)}{2} \left\{ \frac{\sinh\left[(\phi - \phi_0) \frac{(\gamma+2)\sqrt{c}}{2K}\right]}{\sqrt{c}} \right\}^{\frac{2\gamma}{\gamma+2}} \qquad (\gamma < -3) \,. \tag{3.18}$$

To conclude, let us briefly examine the case $\gamma = -3$. From (1.14), the mass profile is now slightly different:

$$m(r) = 4\pi \rho_0 r_0^3 \, \ln\left(\frac{r}{r_0}\right) \,. \tag{3.19}$$

We easily get the integral

$$\phi(r) = K \int \frac{1}{r^3 - \rho_0 r_0^3 \, r^2 \, \ln(r)} dr + \phi_0 \ , \tag{3.20}$$

where $K = \sqrt{\rho_0 r_o^3}$ and ϕ_0 is a constant of integration, as before.

Unfortunately, this integral is not analytically solvable. In principle one should solve it, inverting the relation to find $r(\phi)$ and then substitute to find $V(\phi)$.

3.1.2 NFW density profile

A more general density profile has been investigated by the authors of [14]. This profile reads

$$\frac{\rho}{\rho_{crit}} = \frac{\delta_c}{\frac{r}{r_s}\left(1 + \frac{r}{r_s}\right)^2} \ , \tag{3.21}$$

where r_s is a scale radius, δ_c is a characteristic (dimensionless) density, and $\rho_{crit} = 3H_0^2/8\pi G$ the critical density for closure.

From (1.14), we compute the mass profile as a function of the radius as:

$$m(r) = \int 4\pi\rho r^2 \, dr = 4\pi\delta_c\rho_{crit} r_s^3 \left[\ln\left(1 + \frac{r}{r_s}\right) + \frac{1}{1 + \frac{r}{r_s}} \right] - 4\pi\delta_c\rho_{crit} r_s^3 \ , \tag{3.22}$$

then (1.10) becomes

$$e^{-2\beta} = 1 - \frac{m(r)}{4\pi r} = 1 - \frac{a}{t}\left[\ln(1+t) - \frac{t}{1+t}\right] \ , \tag{3.23}$$

where we have set $a = \sqrt{\delta_c \rho_{crit}} \, r_s$ and $t = r/r_s$; thus

$$v_c^2 = \frac{a^2}{t}\left[\ln(1+t) - \frac{t}{1+t}\right] \ . \tag{3.24}$$

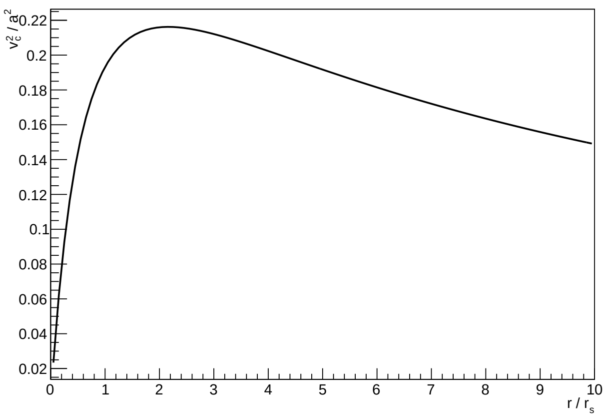

Figure 3.2: The rotation velocity profile for NFW with a null radial pressure component.

We see that the rotation velocity profile is approximately flat in the region of interest.

From (3.4), we get the following integral

$$\phi(r) = a \int \frac{\sqrt{1+t}}{\sqrt{t(t+1) - a\left[(t+1)\ln(t+1) - t\right]}} \, dt + \phi_0 \ , \tag{3.25}$$

which is not analytically solvable.

3.2 From cold dark matter to unified models

In this section, we are going to remap the two CDM density profiles considered before in order to get UDM density profiles which exhibit the same rotation velocity.
Since haloes composed by only dark matter do not have a radial pressure component, from eq. (1.30), by equating the rotation velocity profiles, we get

$$v_c^2 = \frac{m(r)}{8\pi r} + \frac{p_{\|} r^2}{2} = \frac{m_C(r)}{8\pi r} \ , \tag{3.26}$$

where in the LHS of second equation a UDM profile is considered.
Since $m(r)$ satisfies the differential equation (1.14), by multiplying both members of (3.26) for $r > 0$ and by differentiation, we get the following differential equation:

$$\rho + p_{\|}' r + 3p_{\|} = \rho_C \ , \tag{3.27}$$

where $p_{\|}$ is the pressure in the UDM model.

3.2.1 Barotropic equation of state

In this section, we consider a simple relation that connects $p_{\|}$ with ρ, which leads to barotropic Lagrangian functions:

$$p_{\|} = \frac{\rho}{w} \tag{3.28}$$

From (1.31) we have that the squared speed of sound is

$$c_s^2 = 1/w \ . \tag{3.29}$$

Solutions with $w < 0$ are, however, unstable [3]. If it is not explicitly mentioned, we will neglect the superluminal case (i.e. $|w| < 1$) [1].
From (1.27) and (1.28) we have already noticed that if the Lagrangian is of the purely kinetic type, we need only an equation of state in order to reconstruct it. Hence

$$\mathcal{L} = -A(-X)^{(w+1)/2w} \qquad (X < 0) \ , \tag{3.30}$$

where A is a positive constant of integration.

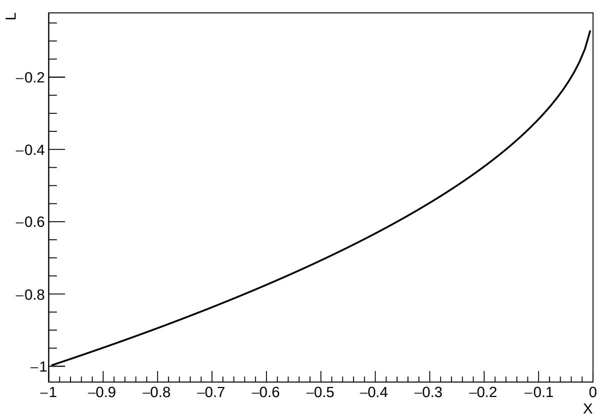

Figure 3.3: The barotropic Lagrangian for $w = 3$.

[1] however, we have already pointed out that this case does not lead to a violation of causality.

Firstly, we start from the polynomial density profile (3.5), for which the mass profile has already been computed in (3.6), with the barotropic equation of state (3.28). Since $m(r)$ satisfies the differential equation (1.14), from (3.27) we get the following differential equation of the first order in ρ:

$$r\rho'(r) + \left[w+3\right]\rho(r) = \frac{\rho_0 w}{r_0^\gamma} r^\gamma \, . \tag{3.31}$$

We solve this equation and get the UDM profile

$$\rho(r) = \frac{w\rho_0}{3+\gamma+w}\left(\frac{r}{r_0}\right)^\gamma + c\rho_0\left(\frac{r}{r_0}\right)^{-(3+w)} \qquad (3+\gamma+w \neq 0) \, , \tag{3.32}$$

where c is a positive constant of integration. For $\gamma \neq -3$, from (1.14) we get

$$m(r) = \frac{4\pi w\rho_0}{3+\gamma+w}\frac{r^{\gamma+3}}{(\gamma+3)r_0^\gamma} - \frac{4\pi c\rho_0 r_o^{w+3}}{wr^w} \qquad (\gamma \neq -3) \, , \tag{3.33}$$

whereas for $\gamma = -3$

$$m(r) = \frac{4\pi w\rho_0 r_0^3}{w}\ln\left(\frac{r}{r_0}\right) + c\rho_0\left(\frac{r}{r_0}\right)^{-(3+w)} \qquad (\gamma = -3) \tag{3.34}$$

and the rotation velocity is (3.9).
If we repeat the procedure for the NFW density profile (3.21), we get the following differential equation

$$r^2\rho(r)' + \rho(r)r\left[w+3\right] = \delta_c\rho_{crit}r_s w\frac{1}{(1+r/r_s)^2} \tag{3.35}$$

which is not analytically solvable.
However, if we restrict our attention to large radii ($r/r_s \gg 1$), eq. (3.35) becomes solvable and yields

$$\rho(r) = \frac{\delta_c\rho_{crit}}{(r/rs)^3} + d\delta_c\rho_{crit}\left(\frac{r}{r_s}\right)^{-(3+w)} \, , \tag{3.36}$$

where d is a constant of integration and the rotation velocity is (3.24).

3.2.2 Polynomial density profile with $\gamma = -2$ and constant pressure -Λ

In section 3.1.1, we found that the for a polynomial density profile with a barotropic equation of state the rotation velocity (3.9) is constant only if $\gamma = -2$. Something similar can be found considering a constant parallel pressure equal to $-\Lambda$ for the unified halo. The physical interpretation for such a choice is that we want our solution to asymptotically approach a constant positive energy density and a negative pressure.
From (1.27) it is immediate to get the purely kinetic Lagrangian (see Figure 3.4)

$$\mathcal{L} = -\Lambda - k\sqrt{-X} \qquad (X < 0) \, , \tag{3.37}$$

where k is a positive constant of integration.

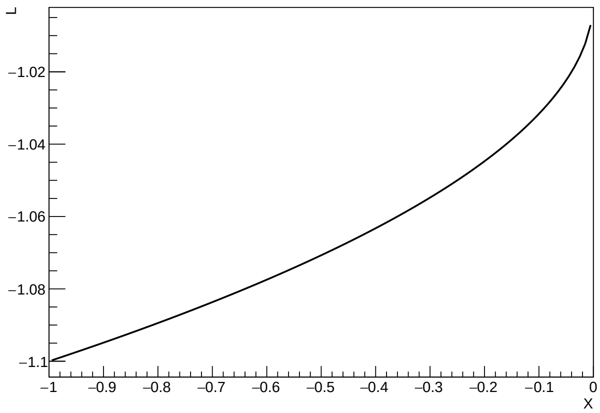

Figure 3.4: The purely kinetic Lagrangian for a UDM polynomial density profile and constant pressure. The plot is done in units of $\Lambda = 1$ with $k = 0.1$.

From (3.27), a CDM polynomial density profile with $\gamma = -2$ (3.5) is remapped into the UDM profile

$$\rho = \rho_0 \left(\frac{r}{r_0}\right)^{-2} + 3\Lambda \ . \tag{3.38}$$

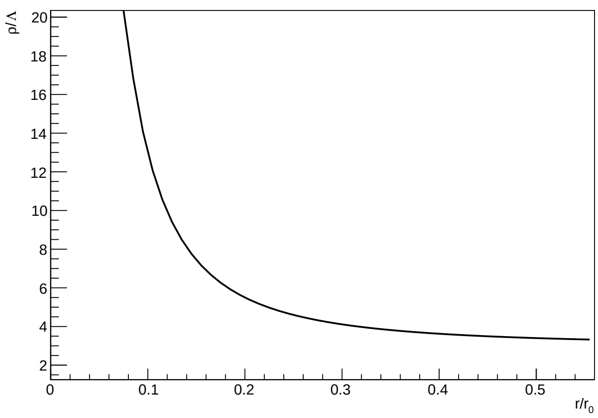

Figure 3.5: The UDM density profile with constant pressure obtained from a CDM polynomial profile with $\rho/\Lambda = 0.1$.

Hence, the mass profile reads

$$m(r) = 4\pi\rho_0 r_0^2 r + 4\pi\Lambda r^3 \tag{3.39}$$

and the rotation velocity profile (3.26) yields

$$v_c^2 = \frac{\rho_0 r_0^2}{2} \ , \tag{3.40}$$

which is flat by construction.
For a Lagrangian of the type (3.1), by using (1.27) and (1.28), we get

$$X = -\Lambda - \frac{\rho_0 r_0^2}{2r^2} \ , \qquad V(\phi) = 2\Lambda + \frac{\rho_0 r_0^2}{2r^2} \ , \tag{3.41}$$

and from the expression of the kinetic term (1.24) we have

$$\phi(r) = \int \frac{\sqrt{\rho_0 r_0^2/r^2 + 2\Lambda}}{\sqrt{1 - \rho_0 r_0^2 - \Lambda r^2}} \, dr + \phi_0 \tag{3.42}$$

where ϕ_0 is a constant of integration[2].

3.2.3 Polytropic equation of state

In this section, we briefly consider the so-called polytropic equation of state [3]

$$p_{\|} = p_* \left(\frac{\rho}{\rho_*}\right)^{\delta} , \tag{3.43}$$

where $\rho_*, p_* > 0$ and $\delta \in \mathbb{R}$. This general expression incorporates also the generalised Chaplygin gas for $p_* < 0$ and $\delta < 0$.
The speed of sound (1.31) is

$$c_s^2 = \delta \frac{p_{\|}}{\rho} . \tag{3.44}$$

Since $\rho > 0$ from the WEC, this expression allows us to fix the sign of $\gamma p_{\|} > 0$ in order to have quantum mechanically stable solutions.
As usual, from (1.27) and (1.28) we easily obtain the separable equation:

$$2X \frac{d\rho}{dX} = \rho + p_* \left(\frac{\rho}{p_*}\right)^{\delta} . \tag{3.45}$$

Therefore

$$\mathcal{L} = -\left[(-M^4 X)^{(1-\delta)/2} - \frac{p_*}{\rho_*^{\delta}}\right]^{1/(1-\delta)} , \tag{3.46}$$

where M is a constant of integration and the minus sign in front guarantees the positiveness of the energy density, while the one inside the parentheses guarantees the definiteness for all space-like $X < 0$.

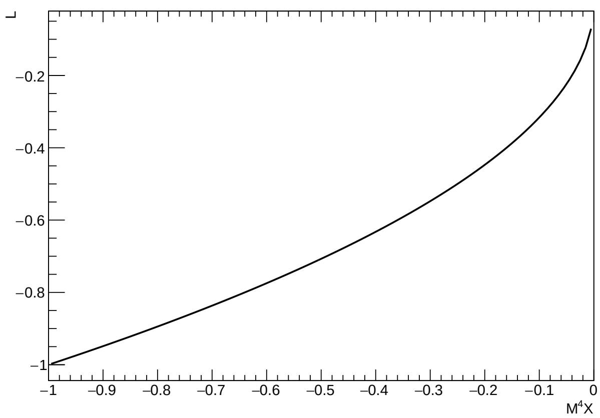

Figure 3.6: The polytropic Lagrangian with parameters: $p_*/\rho_*^{\delta} = 1$, $\delta = 0.1$.

If we insert (3.43) into (3.27) we get the non-linear differential equation in ρ

$$\rho + \frac{p_*}{\rho_*^{\delta}} \delta \rho^{\delta-1} \rho' r + 3 \frac{p_*}{\rho_*^{\delta}} \rho^{\delta} = 3\rho_C , \tag{3.47}$$

which is not analytically solvable, even by considering simple cases like $\delta = 2$ or the Chaplygin gas ($p_* < 0$ and $\delta = -1$) with a CDM density profile which decays with the inverse squared radius.

[2] we do not report the explicit expression because it is neither nice nor useful for getting an expression for the scalar potential, since it is not invertible.

3.3 Weak gravitational field

Since dark matter is by definition pressure-less, in section 3.1 we started from a CDM energy density and a null radial pressure component in order to get the Lagrangian. Successively, in section 3.2 we obtained UDM density profiles from CDM ones by imposing the equality between the rotation velocity profiles with a barotropic equation of state. We did not take into account, however, the TOV equation (1.19).
In this section, we consider a weak gravitational field which is enough to explain the observed rotation velocity profiles (since $v_c^2 \ll c$). It is characterized by the conditions [3]

$$\frac{m(r)}{8\pi r} \ll 1 \,, \qquad \frac{p_\parallel r^2}{2} \ll 1 \,, \tag{3.48}$$

from which the TOV equation reduces to

$$p'_\parallel \approx -\frac{2}{r}(p_\parallel + \rho) \,, \tag{3.49}$$

where we consider the term

$$\alpha' = \frac{m/8\pi + p_\parallel r^3/2}{r^2(1 - m/4\pi r)} \approx \frac{m/8\pi + p_\parallel r^2/2}{r^3} = \frac{v_c^2}{r} \tag{3.50}$$

negligible with respect to $2/r$.
Then, from eq. (3.27) we get the system

$$\begin{aligned} p'_\parallel r + 2p_\parallel &= -2\rho \,, \\ p'_\parallel r + 3p_\parallel + \rho &= \rho_C \end{aligned} \tag{3.51}$$

which yields

$$p_\parallel = \frac{2}{r^4}\int \rho_C(r) r^3 \, dr + \frac{c\rho_0 r_0^4}{r^4} \tag{3.52}$$

and

$$\rho = p_\parallel - \rho_C \,, \tag{3.53}$$

where c is a constant of integration.
These equations, from a CDM energy density profile or, equivalently, from a given rotation velocity, allow us to obtain the energy density and the radial pressure for the UDM model, which incorporates also the dark energy.
Now we consider, for example, a polynomial density profile with $\gamma = -2$ for the CDM halo. From the equations (3.52) and (3.53) we get

$$p_\parallel = \frac{\rho_0 r_0^2}{r^2} + \frac{c\rho_0 r_0^4}{r^4} \tag{3.54}$$

and

$$\rho = \frac{c\rho_0 r_0^4}{r^4} \,, \tag{3.55}$$

which satisfy the WEC if we choose a non-negative c. Moreover, by construction the rotation velocity profile is flat and reads $v_c^2 = \frac{\rho_0 r_0^2}{2}$.
This solution is clearly singular at the origin, but of course the weak-field conditions are there violated and so we do not expect our solution to be valid in this regime.
Moreover, we have already pointed out (see chapter 2) that the TOV equation is invariant under the transformation $\{\rho \to \rho + K,\, p_\parallel \to p_\parallel - K\}$, hence one could also add a cosmological constant to the density and subtract it from the radial pressure in order to match de Sitter universe at large radii.
By the elimination of r from the equations (3.54) and (3.55), we get the following equation of state

$$p_\parallel = \rho + A\sqrt{\rho} \,, \tag{3.56}$$

where we have defined $A = \sqrt{\rho_0/c}$.
The speed of sound is (1.31)

$$c_s^2 = 1 + \frac{A}{2\sqrt{\rho}} , \tag{3.57}$$

which leads to quantum mechanically stable solutions, but it is superluminal and diverges as $r \to \infty$. For a purely kinetic Lagrangian, since we want positive energy density solutions (i.e. $\mathcal{L} < 0$ when $X < 0$), from (1.27) we finally get

$$\mathcal{L} = -\left[\frac{\sqrt{-kX} - A}{2}\right]^2 \qquad (X < 0) , \tag{3.58}$$

where k is a positive constant of integration.

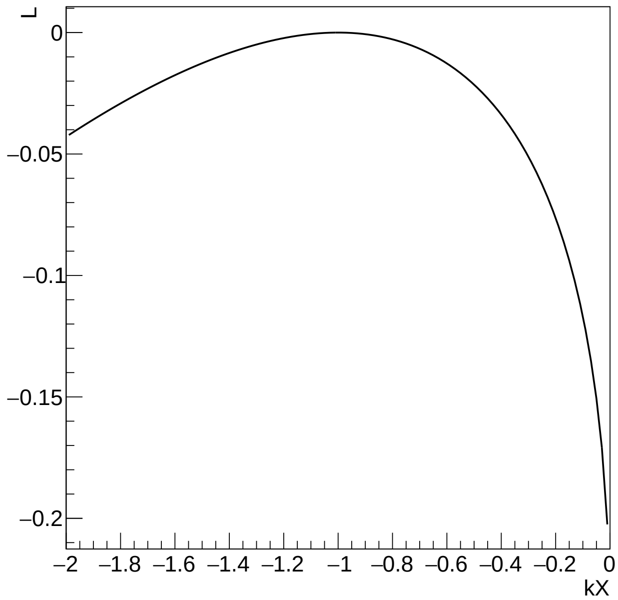

Figure 3.7: The purely kinetic Lagrangian for a UDM model obtained starting from a polynomial CDM density profile with parameter $A = 1$.

Conclusions

In this thesis, we investigated unified haloes of dark matter and dark energy. After a brief introduction to General Relativity, in the first chapter we wrote down the anisotropic Tolman-Oppenheimer-Volkoff equation, which allowed us to study static and spherically symmetric haloes with anisotropic pressure components. We chose a general Lagrangian depending on a non-canonical scalar field and its kinetic term. This scalar field is the candidate for the unified fluid.
In chapter two, we first analysed the general solution for a purely kinetic Lagrangian and we found an interesting expression that can allow us to give the kinetic term in function of r if the explicit form of the Lagrangian is known. Afterwards, we restricted our attention to constant energy solutions and we found out that the rotation velocity profile flattens in the region of interest by a suitable choice of a constant of integration.
In chapter three, we reconstructed the scalar potential in the case of a simple Lagrangian for two CDM density profiles. Then, we remapped these profiles by imposing the equivalence between the rotation velocities and choosing an equation of state. In the above analysis we used nothing but the mass equation. Instead, further constraints can be gained by inserting the UDM profiles obtained so far into the complete TOV equation or its expression in the weak-field limit. For example, an interesting bond on w could be achieved by solving eq. (3.31) in the case of $3 + \gamma + w = 0$.
The main result of this thesis is a general procedure for obtaining an energy density profile for an unified fluid starting from a CDM one. That has been made by equating the rotation velocities and using the anisotropic TOV equation in the case of a weak gravitational field, without imposing a priori an equation of state, as it was done in [3]. In particular, from a CDM profile that decays with the inverse squared radius and reproduces a flat rotation velocity function we reconstructed the energy density and the radial pressure for the unified fluid. This method provided us with an equation of state from which we reconstructed the purely kinetic Lagrangian. However, this choice for the energy density is not able to reproduce a suitable speed of sound.
To generalise the thesis, other Lagrangian functions can be considered. In particular, the simple Lagrangian (3.1) seems to be promising. Otherwise, one could either start from a different CDM profile, e.g. the NWF [14], or a given velocity function which triggers the astronomical observations.
A detailed analysis, however, should necessarily take into account the complete anisotropic TOV equation. In particular, qualitative solutions to the TOV by imposing a priori an equation of state could be illuminating, as it has been proven by the authors of [9] for the isotropic case in the presence of the generalized Chaplygin gas. Furthermore, a computational approach is inevitable because the complete TOV equation seems to be too much complicated to be solved analytically. For example, starting from a given energy density profile, one should be able to compute numerically the radial pressure and from that obtain an equation of state which would give us the purely kinetic Lagrangian and the speed of sound.

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