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Properties of extremal and supersymmetric horizons

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Abstract

String theory and holography have been able to provide the microstates accounting for the entropy of supersymmetric and extremal black objects, such as black holes and black strings. Gauged supergravity in different spacetime dimensions allows for rotating, charged black objects displaying inner horizons, in addition to the outer event horizon. The product of the areas of these horizons has been shown to depend on angular momenta and electric charges, but not on the mass. The thesis studies these features both from first principles and in concrete examples. The main aim is to implement the extremal and supersymmetric limit of the black hole thermodynamics. It is found that in this limit the area product formulae reproduce certain relations between the conserved charges that have been emphasized recently. A key role is played by the recently discovered extremization principle for the black hole thermodynamics in the supersymmetric and extremal limit. We show that this in fact captures the areas of all horizons as well as the area product formula. We give an explicit proof of these results in the context of black holes in five-dimensional $\mathcal{N} = 2$, $U(1)^3$ gauged supergravity and four-dimensional $\mathcal{N} = 2$, $U(1)^4$ gauged supergravity and argue that they hold more generally.

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Chapter 1

Introduction

It is well known that black holes are thermal objects, their thermodynamics is derived from quantities related to the outer event horizon. One of the aim of a fundamental theory of quantum gravity is to account for the thermodynamics of black objects, starting from a microscopic statistical description. This has been first achieved, in the framework of string theory, for a class of asymptotically flat black holes [1] and from there a lot of work has been done.

On the other hand, after the discovery of the AdS/CFT correspondence [2], the interest in the study of asymptotic Anti de Sitter (AdS) black holes has grown. The correspondence was originally formulated in type IIB string theory on $AdS_5 \times S^5$, whose low energy limit is described by a supergravity theory on the same background, and states that the theory is dual to a conformal field theory in four flat spacetime dimensions. One of the main features of the AdS/CFT correspondence is that it is a weak/strong duality, meaning that if one of the two holographic duals is a weakly coupled theory, the other is strongly coupled and vice versa. Many generalizations of the correspondence have been found later on (e.g. AdS_4/CFT_3 [3]).

In this context, studying black hole solutions on AdS_D backgrounds has proven to be a powerful tool to test the validity of the correspondence, these black holes are solutions of gauged D dimensional supergravities, which are obtained from dimensional reduction of the original eleven/ten dimensional supergravity [4]. Black holes are related to thermal states of the dual CFT [5] meaning that, assuming the validity of the correspondence, one can give a microscopic interpretation of the black hole thermodynamics in terms of an ensemble of states in the dual CFT.

Despite many (AdS) black hole solutions have been constructed in the last twenty years, only recently there have been some advances in proving the validity of the picture of black holes as made by microscopic constituents of the dual field theory, the reason is the difficulty in dealing with strongly coupled field theories. In [6, 7] the Bekenstein-Hawking entropy of a class of supersymmetric AdS_4 black holes has been reproduced from an exact localization computation in the dual field theory, considering the leading order in the large N expansion[1]. A similar result has been obtained later for supersymmetric AdS_5 black holes in [8, 9, 10].

The requirement of supersymmetry is usually needed to have a good control of the two sides of the holographic picture. On the gravity side, a peculiarity of these supersymmetric black hole solutions is that they are generally not well-behaved unless one further imposes extremality [11]. This happens because supersymmetric but non extremal black holes are characterized by the presence of a naked singularity, or by causal pathologies such as closed timelike curves (CTCs) outside the event horizon. We will call "BPS" the solutions that are both supersymmetric and extremal. BPS black holes in different dimensions and with different sets of charges seem

[1]Dual field theories are generally gauge theories with gauge group given by e.g. $SU(N)$. Requiring that N is large is needed to account for the large value of the black hole entropy (even in the zero temperature case), which implies a large number of microscopic degrees of freedom.

to share some general properties.

In particular it has been found in all cases that the Bekenstein-Hawking entropy can be obtained from an *extremization principle* [12, 13, 14]. One can define a rather simple, homogeneous function $I(\omega_i, \phi^I)$ of some variables (ω_i, ϕ^I) related to the chemical potentials of the black hole thermodynamics. These satisfy a linear constraint of the form:

$$\sum_i \omega_i - \sum_I \phi^I = \pm 2\pi i, \tag{1.1}$$

the entropy of the BPS black hole is then found by extremizing the following entropy function:

$$I(\omega_i, \phi^I) - \omega_i J_i - \phi^I Q_I, \tag{1.2}$$

with respect to the (ω_i, ϕ^I) variables subjected to the constraint **(1.1)**, (J_i, Q^I) are the conserved charges of the black hole solution, corresponding to angular momenta and electric-like charges.

The works cited above did not derive the entropy function, nor give a meaning to the (ω_i, ϕ^I) variables, which moreover, are found to be complex when solving the extremization equations. These explanations recently came from the work of [8] who *derived* the extremization principle for a class of five dimensional AdS black holes. They considered a complexified family of supersymmetric but not extremal solutions in Euclidean signature, this allowed to identify $I(\omega_i, \phi^I)$ as the Euclidean on-shell action in the supersymmetric solution, that is a saddle point of the gravitational path integral, and the (ω_i, ϕ^I) variables as a modified set of chemical potentials which, in the extremal and supersymmetric (BPS) limit allows to define a non-trivial thermodynamics. Moreover, condition **(1.1)** has been interpreted as a regularity condition for the killing spinor near the event horizon.

In this framework the extremization of the entropy function **(1.2)** corresponds to the (constrained) Legendre transform of the grand-canonical potential I in the supersymmetric solution. A key point in the discussion of [8], was to implement the constraint **(1.1)** in the Legendre transform via a Lagrange multiplier Λ, which is found to be determined (via the extremization equations) in terms of the charges from the cubic equation:

$$\Lambda^3 + p_2(J_i, Q_I)\Lambda^2 + p_1(J_i, Q^I)\Lambda + p_0(J_i, Q^I) = 0. \tag{1.3}$$

Moreover, the homogeneity of degree 1 of the Euclidean supersymmetric action I allows to obtain the entropy simply as $S = \pm 2\pi i \Lambda$ where Λ has to be chosen as one of the roots of **(1.3)**, requiring reality of S obtained in this way allows one to *derive* the BPS entropy.

In the last few years, it has been shown that the extremization principle is a general feature of many classes of static [6] and rotating AdS black holes in $D = 4, 5, 6, 7$ dimensions [15], AdS_4 black holes with acceleration [16], and it has been explored considering higher derivative corrections to the original five-dimensional theories [17] and also for asymptotically flat black holes [18].

In our work we are going to reconsider the extremization principle as discussed by [8], we will show that there is actually more to learn about it which regards the properties of the *inner* horizons of the black hole.

The idea of black hole solutions with multiple horizons is not a new one, e.g. four-dimensional Reissner-Nordstrom and Kerr-Newman black holes have an internal Cauchy horizon, together with the outer event horizon. Properties of these internal horizons have been studied since the eighties. A remarkable feature is that, in some classes of four and five dimensional multi charged, static and/or stationary asymptotically flat black holes, the wave equation for a minimally coupled scalar field in the black hole background is sensitive to the geometry near both horizons [19, 20]. Moreover the Bekenstein-Hawking entropy[2] for the outer and inner horizons can

[2] It should be noted that one can *formally* define quantities such as the entropy, or chemical potentials for the Cauchy horizon. The way to do it is to repeat the same calculations that one usually does for the outer horizon also for the Cauchy horizon. In this way one

be rewritten as [19, 20, 21] (regardless of supersymmetry and extremality):

$$S_\pm = \frac{A_\pm}{4} = \sqrt{N_R} \pm \sqrt{N_L}, \tag{1.4}$$

where $N_{R/L} = N_{R/L}(M, J_i, Q^I)$ are generally functions of all the charges. Remarkably the product of the two entropies $S_- S_+ = N_R - N_L$ turns out to be *independent of the mass M of the black hole*, which implies that it should be quantized at the quantum level, due to the quantization of the electric charges and the angular momenta.

The splitting of the entropy **(1.4)** into two distinct contributions and the properties of the scalar wave equation, led the authors of [19, 20, 21] to conjecture the possibility of explaining the microscopic origin of the thermodynamic properties of the black hole by considering an *effective* string model (or equivalently two-dimensional CFT). The two contributions appearing in the entropy S_+ are interpreted as deriving from the right/left moving modes on the string (or right/left moving sectors of a CFT_2), which has to be taken weakly interacting to allow for the splitting in **(1.4)**. In this picture the mass-independence of the entropy product formula, and the consequent quantization at the quantum level, played a special role as it was related to the level matching condition of the dual[3] CFT_2.

This conjecture has been proved valid in specific cases where the *near horizon geometry* contains an AdS_3 factor, which is holographycally dual to a CFT_2. This happens for specific types of black hole solutions [23] (here the inner horizon were not considered), and in the extremal case [24, 25, 26]. Very recently these ideas have been applied also to the AdS_5 black holes that we have discussed in the beginning [27, 28], where an agreement with the results obtained by studying the dual field theory on the boundary at infinity has been found. Outside of extremality, where the duality with the CFT_2 is not as well established, one usually exploit the thermodynamic properties of the outer and inner horizon, to infer the thermodynamic properties of the R/L sectors of the effective dual CFT_2 [28, 29, 30].

Inspired by the possibility to obtain a microscopic interpretation of the outer and inner horizons properties in terms of a dual CFT_2, some authors have studied the properties of the area product formula in more general settings, see e.g. [31]. In almost all cases the area product formula is found to be mass-independent. In particular in [32] some of the AdS black holes discussed at the beginning have been studied, again the independence of the area product formula on the mass of the solution holds, but only if one considers also more general horizons than the event and Cauchy horizons, specifically *complex horizons*. These "virtual" horizons emerge as complex zeroes of the function which determines the position of the "physical" event and Cauchy horizon as its *positive real* zeroes $r_\pm$. The authors of [32], treated these complex horizons as formal loci which however turn out to be necessary to obtain the area product formula. It is important to stress that in a general setting only the mass independence of the area product formula is recovered. Instead the splitting in Eq. **(1.4)** generally appears for black holes with *at most* two horizons.

Quoting [32], the independence of the area product formula: "would suggest the possibility of an explana-

can compute an area A_-, surface gravity κ_- and (in the stationary case) finding a null killing vector of the form $V = \partial_r + \Omega_- \partial_\phi$ for the Cauchy horizon, it is then natural to interpret these as thermodynamic quantities (e.g. $S_- = \frac{1}{4}A_-$ or $T_- = \frac{\kappa_-}{2\pi}$). Clearly, it should be kept in mind that these are only *formal* definitions, and one should then work to understand what is the meaning of these quantities as thermodynamics quantities.

[3] Some concerns about this conjecture have been raised by Visser [22], in particular the $N_{R/L}$ variables should be integers in order to be associated with a number of excitations. Moreover interpreting the area product mass-independence in terms of the level matching condition, implies that the product of the areas should be quantized as $S_- S_+ \propto L_P^4 N$ (L_P is the Planck length), again with N integer. These requirements do not follow immediately from the observations of [19, 20, 21] unless one considers the supersymmetric and extremal solution. For example the quantization of the area product in terms of *integers* does not follow generally from the mass independence, which simply states that $S_+ S_-$ *only depends on the quantized charges*. This additional requirement can be obtained provided that one tunes the elementary electric charges, upon which the charges Q^I are quantized, so that once $S_+ S_-$ is written in terms of the quantized charges and angular momenta, one correctly obtains $S_- S_+ \propto L_p^4 N$ with $N \in \mathbb{N}$. Moreover $N_{R/L} \in \mathbb{N}$ implies the quantization of the mass of the black hole solution.

tion for the microscopic behaviour of such black holes in terms of a field theory in more than two dimensions", where all horizons may be relevant to study the microscopic properties of the given black hole solutions. Unfortunately however, at least to our knowledge, very little to no work has been done in understanding these complex horizons, contrary to the Cauchy and event horizons as we have briefly discussed above.

The present thesis aims at making some steps forward in the understanding of the properties of these complex horizons. We will concentrate on the black hole solutions of four and five dimensional gauged supergravities discussed at the beginning, dual to thermal states of three and four dimensional field theories on the boundary of the spacetime. The end goal would be to obtain a microscopic interpretation of the properties of all the horizons. If possible, it is natural to assume that it would arise by considering the dual field theory on the boundary. This would be different from what we have briefly described above, the reason being that the dual CFT_2 description formally arises by studying certain near horizon geometry, while here we will consider the asymptotic AdS_D geometry. One may speculate that both pictures can be used and could be related, some indications may have been given by [28], which however did not consider the virtual horizons. Understanding if such an interpretation actually exists would be an ambitious result, which would certainly require much more work than what can be done in this thesis. Indeed, we will not directly address these problems, but rather give an analysis of the properties of the different horizons[4], concentrating on the gravitational side of what we hope is a broader holographic picture.

We will first consider the known area product formula and show that, in the BPS solution, it is equivalent to the BPS constraint on the charges, which allows us to get extremality from the supersymmetric solution. We will briefly argue why this is actually a non trivial result, and use this as our starting observation. In trying to frame this result in a more fundamental picture we will define a set of thermodynamic variables (entropies and chemical potentials) for each horizon, showing that they trivially satisfy a first law of thermodynamics, in doing so we will generalize some results previously obtained for other kind of black holes [29, 31, 30].

A key role in our discussion will be played by a sort of "exchange symmetry" which relates quantities defined for each horizon[5]. Using this symmetry we will deduce the possibility of defining a set of *universal* quantities which allows us to capture the properties of all horizons at once. This idea applied to the quantities in Eqs. **(1.1, 1.2)**, will allow us to show that the extremization principle is *universal* in the sense that it reproduces all the horizons entropies in the BPS limit. A key role is played by the Lagrange multiplier **(1.3)**. Indeed, we find that while the purely imaginary root of **(1.3)** gives the outer horizon area, the other roots provide the "entropy" associated with the other horizons. Promoting the extremization principle to an "*universal*" extremization principle, will explain our starting result regarding the area product formula in the BPS limit, and allows to unify the properties of all horizons in the BPS limit (and formally also in the supersymmetric but not extremal case) in a more fundamental picture. We will explicitly show that our results are valid for many classes of AdS black holes, in different dimensions and with different set of conserved charges, we will also prove it for a new generalization for a class of AdS_4 black hole solutions with acceleration. We will argue why this result can be seen as a starting point to obtain a holographic interpretation of the properties of all horizons, and also argue how these results may be extended to non-supersymmetric solutions. We leave for future work

[4]Studying these complex horizons, formally requires to consider complex solutions for the metric and gauge/scalar fields, with analytically continued radial variable or, equivalently, complexified parameters (see e.g. [8] Sec. **3.1**). These would not be allowed as Lorentzian solutions. However, we are interested in studying these horizons interpreted as *formal loci*, and in particular, their generalized thermodynamics (which is characterized by *formal* complex thermodynamic quantities), without addressing what physical meaning these quantities might have (this is not so clear neither for the "physical" Cauchy horizons). The motivation to do so is found in the properties of the area product formula. Moreover, one can regard these complexified solutions as complex saddle points of the gravitational path integral (hence as formal solutions of the equations of motion), we will make some quick comments about this when studying the five-dimensional black holes.

[5]This symmetry has already been observed for black holes with only an event and Cauchy horizon. In this case it has been argued that the exchange symmetry of the horizons is related to the $\mathcal{T}$-duality of the dual CFT_2 (See [19], in particular Sec. **4**).

a better explanation of these aspects.

The plan of the rest of this work is as follows. In chapter **2** we give a review of elementary aspects of supergravity in $D = 4, 5$ dimensions and on supersymmetric and BPS, asymptotically AdS black hole solutions. In chapter **3** we concentrate on two classes of solutions of gauged $D = 5$ supergravity. Starting from the single charged, double spinning black hole solution, we will first give a detailed review of its known properties (Sec. **3.1**); then we also present the extremization procedure discussed in [8] and the derivation of the area product formula of [32]. Next in Sec. **3.2** we prove our original results about, where we first analyse what kind of horizons does the solution admit, and then show in detail how the results that we have just anticipated are derived. In particular showing how the extremization principle can be promoted to a *universal* extremization principle for all horizons.

In Sec. **3.3** we move on discussing the consequences of the *universal* extremization principle for the general double spinning black hole with three charges, of the $U(1)^3$ gauged theory (STU model). We explicitly check the validity of these results for the simpler solution where the two angular momenta are set equal in Sec. **3.4**. For this black hole solution we follow a similar logic as for the single charged case, for this reason we don't go at the same level of detail.

In chapter **4** we generalize our results to the AdS black holes in four dimensions which arise from the $U(1)^4$ gauged theory. We first consider the spinning single charged solution, showing how the *universal* extremization principle is carried out in this case (Sec. **4.1, 4.2**), next we quickly discuss the pairwise equal charged case (Sec. **4.3**). Also in this case we discuss the consequences of the *universal* extremization principle for the most general solution with four independent charges, explicitly showing the validity of this universality for the specific solutions discussed before. Finally, in Sec. **4.5** we conclude discussing the generalization of the single charged solution with acceleration, and we explain the new features that arise when acceleration is present. For this solution the universal area product has never been discussed before, so we will derive it and show its equivalence with the BPS constraint between the electric charge and the angular momentum. Again we will show that the *universality* of the extremization principle works out also in this case.

Chapter 2

Review of supergravity and supersymmetric black hole solutions in AdS backgrounds

The main subjects of our investigations are asymptotically AdS black hole solutions in supergravity theories. It is therefore useful to give a brief introduction to supergravity before starting with the main subjects. A complete review on how to construct supergravity theories would go beyond the scope of the present thesis. Here we only provide a brief introduction to their structure, focusing on the features that will be relevant in the following chapters.

We are interested in $\mathcal{N} = 2$ gauged supergravity in $D = 4, 5$ dimensions, possibly coupled to a certain number of Abelian vector fields which generally allow to gauge a larger Abelian group $U(1)^n$, translating in the possibility of having black hole solutions with multiple electric-like charges. The reason why we are interested in these theories, for a certain number of vector multiplets, is that they can be obtained as consistent truncations of supergravity theories in ten or eleven dimensions where the AdS/CFT correspondence is naturally established. We will say a little more about this when introducing each specific theory, in the next chapters.

We are going to follow closely the presentation of [33], and discuss the on-shell, component formulation of the above supergravity theories, not considering auxiliary fields.

2.1 Elementary aspects of supersymmetry in $D = 4, 5$

The properties of supersymmetric theories depend on the number of spacetime dimensions where they are formulated in. On one hand, massless physical states in D (flat) dimensions, which are massless irreducible representations of the Poincaré group $ISO(1, D-1)$, are classified by representations of the $SO(D-2)$ group. In four,(five) dimensions massless particles are characterized by helicity,(spin) and are therefore described by $2, (2s+1)$ states.

This is important because massless particles are the "building blocks" of the irreducible representations of the super-Poincaré group, the supermultiplets, which must contain the same number of fermionic and bosonic degrees of freedom. This requirement implies that in higher dimensions, one has to consider longer supermultiplets, for example, the supergravity multiplet in $D = 4$ contains just the spin-2 graviton and the spin-$\frac{3}{2}$ gravitino $(g_{\mu\nu}, \psi_\mu)$, while in $D = 5$ the minimal supergravity multiplet contains also a gauge field $(g_{\mu\nu}, \psi_\mu, A_\mu)$.

Moreover, the properties of the spinorial representations of the Lorentz group are different in different dimensions [34], this directly influences the properties of the supercharges. In particular, remember that any irrep. of the Clifford algebra in D Lorentzian dimensions,

$$\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}, \qquad \eta_{ab} = \text{diag}(-1, +1, \cdots, +1), \tag{2.1}$$

is also a representation of the Lorentz algebra via the map:

$$\rho(\hat{M}_{ab}) = \frac{1}{4}[\Gamma_a, \Gamma_b], \tag{2.2}$$

here $(a, b = 0, 1, \cdots, D-1)$ and M_{ab} are the generators of the D-dimensional Lorentz algebra.

Generally these spinorial representations of the Lorentz group are reducible. Take, for example, the irrep. of the four-dimensional Clifford algebra given by Dirac matrices, the corresponding representation of the Lorentz group is the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$. An irreducible spinor representation can be obtained by imposing constraints on the spinors on which the wanted representation acts. These constraints can be *chirality conditions* or *reality conditions*, in four dimensions these give rise to Weyl or Majorana spinors.

What kind of constraints one can impose depends on the dimension D, due to the different properties of the Lorentz group in different dimensions, and halve the independent components of a spinor, hence influencing the minimal number of *real* independent supercharges one can have. This is reflected in the length of the supersymmetry representations, denoted as supermultiplets.

In $D = 4$ either a chirality or Majorana condition can be imposed on Dirac spinors to get an irrep. of the Lorentz group, but not both. The minimal amount of supersymmetry in $4D$ is achieved by considering, for example, one Majorana supercharge, which contains four independent real components.

In $D = 5$ one can show that two inequivalent irreps. of the Clifford algebra can be obtained by starting from the four-dimensional Γ_a matrices and adding one of $\pm\Gamma_5$. The corresponding representation of the Lorentz group acts again on four-components complex Dirac spinors. In $5D$ one cannot impose neither a chirality nor a reality condition, meaning that the minimal amount of supersymmetry is achieved considering one Dirac supercharge, containing eight independent real components. For this reason, the $D = 5$ supergravity multiplet contains more states with respect to the four-dimensional one.

However, in $D = 5$ there is a special type of condition that one can impose, called symplectic Majorana condition. First one has to double the number of Dirac spinors (e.g. the supercharges $\mathcal{Q}_i$) so that to have an even number of them $2N$, then one imposes the following condition:

$$(\mathcal{Q}_i)^\star = \Omega_{ij} B \mathcal{Q}_j, \tag{2.3}$$

where Ω_{ij} is a real antisymmetric matrix such that $\Omega^2 = -\mathbb{I}_{2N}$ and B is the matrix that allows us to express $(\Gamma_a)^\star$ as $(\Gamma_a)^\star = -B\Gamma_a B^{-1}$, the number of independent components does not change, as we had to double the number of supercharges first. This allows to make explicit the R-symmetry group, which in $D = 5$ is found to be $USp(2N) \equiv U(2N) \cap Sp(2N, \mathbb{R})$.

It is equivalent to work with N Dirac supercharges or $2N$ symplectic Majorana ones, but usually in the literature one refers to N extended $D = 5$ supersymmetry to the one with an even number of symplectic Majorana supercharges. Hence, the $N = 2$ supergravity we are going to consider is truly the minimal one despite the $N = 2$ designation. Similarly, when considering the $D = 5$ supergravity multiplet, the eight fermionic degrees of freedom can be thought of as distributed between two symplectic Majorana Dirac spinors ψ^i_μ so that the multiplet is given by $(g_{\mu\nu}, \psi^i_\mu, A_\mu)$, where the i indices are in the fundamental of $USp(2) \cong SU(2)_R$.

2.2 Pure supergravity

Supergravity can be thought of as a *gauge theory of supersymmetry* where the supersymmetry transformations with fermionic parameter ϵ are promoted to local transformations. To see how this works (following the discussion of [33], Ch. **2**), let us start with a supersymmetric theory with Lagrangian $\mathscr{L}$. Let us consider $D = 4$ and $\mathcal{N} = 1$ supersymmetry so that we have only one global fermionic symmetry with parameter ϵ, such that $\delta_\epsilon \mathscr{L} = \partial_\mu V^\mu$.

If we promote ϵ to a local parameter, we lose the invariance under supersymmetry as the Lagrangian now transforms as $\delta_\epsilon \mathscr{L} = \partial_\mu V^\mu + \partial_\mu \bar{\epsilon} J^\mu$, where J^μ is the conserved current associated with global supersymmetry transformations. Notice that the components of J^μ must be spinorial quantities.

Following the Noether procedure, supersymmetry is restored by adding to the Lagrangian a term like:

$$\mathscr{L}' = -\frac{1}{M_p}\bar{\psi}_\mu J^\mu, \qquad \delta_\epsilon \psi_\mu = M_p \partial_\mu \epsilon, \tag{2.4}$$

where M_p is a (for now arbitrary) mass parameter needed to get a mass-dimension 4 Lagrangian, and ψ_μ is the gauge field associated with the local supersymmetry transformation, which couples to the supercurrent J^μ of the original theory. Its components are Majorana spinors, so it carries both a spinorial and a Lorentz index $\psi_{\alpha\mu}$.

We are not finished yet, as generally one finds that the supercurrent is not gauge invariant, which means that we need to add more terms to the Lagrangian. In particular, one finds that (after some calculations):

$$\delta_\epsilon \mathscr{L}' = -\frac{1}{M_p}\delta_\epsilon \bar{\psi}_\mu\, J^\mu + \frac{1}{M_p}\psi_\mu\, \delta_\epsilon J^\mu \sim -\partial_\mu \bar{\epsilon}\, J^\mu + \frac{1}{M_p}\bar{\epsilon}\,\gamma_\mu \psi_\nu T^{\mu\nu} + \cdots\,, \tag{2.5}$$

the first term exactly cancels the variation of $\mathscr{L}$ by construction. Instead, the variation of the current gives rise to many terms, one of which is proportional to the stress-energy tensor of the original theory $T^{\mu\nu}$. The cancellation of this particular term can be achieved by adding to the Lagrangian a new coupling between $T^{\mu\nu}$ and a symmetric tensor field $g_{\mu\nu}$ such that:

$$\mathscr{L}'' = -g_{\mu\nu}T^{\mu\nu}, \qquad \delta_\epsilon g_{\mu\nu} \sim \frac{1}{M_p}\bar{\epsilon}\,\gamma_{(\mu}\psi_{\nu)}, \tag{2.6}$$

$g_{\mu\nu}$ can only be the spacetime metric, and therefore ψ_μ must be its superpartner, the spin-$\frac{3}{2}$ gravitino.

Promoting supersymmetry to a local symmetry requires coupling the original theory to the supergravity multiplet $(g_{\mu\nu}, \psi_\mu)$, which can be made dynamical by adding the corresponding kinetic terms. To fully restore supersymmetry in the original Lagrangian $\mathscr{L}$ (which contains also matter fields) one should add more terms. In this way, one would get a matter-coupled supergravity theory. However, what we have discussed above already allows us to obtain the supersymmetric Lagrangian for the *pure supergravity multiplet*, not coupled to matter.

It can be shown that this is nothing but the sum of the kinetic terms for the graviton and the gravitino, after having performed the required spacetime covariantizations[1]

$$S = \int d^4x\sqrt{-g}\left(\frac{M_p^2}{2}R - \frac{1}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho\right), \qquad \begin{cases}\delta_\epsilon\psi_\mu = M_p D_\mu\epsilon \\ \delta_\epsilon g_{\mu\nu} = \frac{1}{2M_p}\bar{\epsilon}\,\gamma_{(\mu}\psi_{\nu)}\end{cases}, \tag{2.7}$$

in these conventions, we have $M_p = (8\pi G_N)^{-1/2}$, which is the reduced Planck mass [33], moreover $\gamma^{\mu\nu\rho} = \gamma^{[\mu}\gamma^\nu\gamma^{\rho]}$ and $D_\mu = \partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}$ is the covariant derivative[2] acting on spinors via the spin connection ω_μ^{ab}, latin letters indicate flat indices and are related to the curved (greek) ones via the vielbein (e.g. $\gamma^a = e^a_\mu\gamma^\mu$).

The specific form of the kinetic term for the gravitino in **(2.7)**, allows to have the propagation of only the

[1] Having not included auxiliary fields, the supersymmetry algebra closes only on-shell, hence $[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\psi_\mu = \delta_{[\epsilon_1,\epsilon_2]}\psi_\mu$ holds only on-shell. $\delta_{[\epsilon_1,\epsilon_2]}\psi_\mu$ is the variation of ψ_μ under the element of the super-Poincaré group given by: $[\epsilon_1\mathcal{Q}, \bar{\epsilon}_2\bar{\mathcal{Q}}]$.

[2] The covariant derivative acting on the gravitino should also take into account the Lorentz index in ψ_μ so that one should have $\nabla_\mu\psi_\nu = D_\mu\psi_\nu - \Gamma^\rho_{\mu\nu}\psi_\rho$, but in the action **(2.7)**, it always appears the *antisymmetrized* covariant derivative of ψ_μ which satisfies $\nabla_{[\mu}\psi_{\nu]} = D_{[\mu}\psi_{\nu]}$, due to the torsionless condition on the Levi-Civita connection $\Gamma^\rho_{[\mu\nu]} = 0$.

physical spin-$\frac{3}{2}$ degrees of freedom contained in the ψ_μ field, and is called the Rarita-Schwinger action [35].

An important comment has to be made about the way the spin connection ω^{ab} is interpreted (let us also treat the vielbein and spin connection as 1-forms). Recall that in Einstein gravity (not coupled to matter), the spin connection is determined in terms of the vielbein 1-forms $\omega^{ab} = \omega^{ab}(e)$ via the torsionless condition $T^a = De^a = 0$, hence it is a *composite field depending on* e^a. This is known as the *second-order* formulation of general relativity [33, 36]. However, one can adopt a different point of view and treat ω^{ab} *as an independent field*. In this way the wanted gravitational degrees of freedom are described on-shell, as one finds that the equation of motion for the spin connection is precisely the torsionless condition $\frac{\delta S_{EH}}{\delta \omega_\mu^{ab}} = 0 \Leftrightarrow De^a = 0$. This is called first order (or Palatini) formalism [33, 36].

This distinction is important in supergravity theories, where fermions are always present. In a general gravity theory coupled to fermions, one finds that in the first order formalism, the equation of motion for the spin connection acquires an extra term that is bilinear in the fermions. Considering the example given in Eq. **(2.7)**, one would find [33]: $De^a - \frac{1}{4M_p^2}\bar{\psi} \wedge \gamma^a \psi = 0$. In this case, the symmetry under local supersymmetry for the theory in Eq. **(2.7)**, can be proven by fixing the susy transformation for ω^{ab} in such a way that $\delta_\epsilon S = 0$. Alternatively, adopting the second order formalism, one interprets ω^{ab} as a composite field which is now determined by the condition $T^a = De^a = \frac{1}{4M_p^2}\bar{\psi} \wedge \gamma^a \psi$ [33]. In this case, $\omega^{ab}(e, \psi) = \omega^{ab}(e) + K^{ab}(\psi)$ is not a torsionless connection anymore, due to $K^{ab}(\psi)$ which is quadratic in the fermions, but one can show that the equations of motion in the two formalisms are equivalent [36]. Notice also that one has to add interaction terms for the fermions in this case (due to the $K^{ab}(\psi)$ piece). Moreover, $\omega^{ab}(e, \psi)$ and the Levi-Civita connection $\Gamma^\rho_{\mu\nu}$ are now not equivalent, the latter still being torsionless as $\Gamma^\rho_{[\mu\nu]} = 0$, and only determining $\omega^{ab}(e)$ (this means that footnote **2** is still true). In this case, the susy transformation of $\omega^{ab}(e, \psi)$ is *fixed* by those of the vielbein and the gravitino. However, in order to prove the invariance under local supersymmetry in the action **(2.7)** one simply needs the condition $T^a = De^a = \frac{1}{4M_p^2}\bar{\psi} \wedge \gamma^a \psi$.

2.2.1 $\mathcal{N} = 2$ extended pure supergravity

Eq. **(2.7)** is the pure $\mathcal{N} = 1$ supergravity action in $D = 4$, but we are going to work in $\mathcal{N} = 2$ extended supergravities, which allows to have gaugings within the pure supergravity theory itself. Moreover, in $D = 5$ there is not a generalization of **(2.7)** due to the different field content of the minimal supergravity multiplet, which instead is similar to the four-dimensional $\mathcal{N} = 2$ one.

Let us stick to $D = 4$, in $\mathcal{N} = 2$ extended supersymmetry the supergravity multiplet is now given by: $(g_{\mu\nu}, \psi^i_\mu, A_\mu)$, where the two gravitini now form a doublet in the fundamental representation of the $SU(2)_R$ R-symmetry subgroup of the full $U(2)_R$ symmetry.

It is tempting to view the supergravity multiplet as a sort of combination of $\mathcal{N} = 1$ multiplets as $(g_{\mu\nu}, \psi^i_\mu, A_\mu) \equiv \big(g_{\mu\nu}, \psi^{(1)}_\mu\big) \oplus \big(\psi^{(2)}_\mu, A_\mu\big)$. The second one is the $\mathcal{N} = 1$ gravitino multiplet whose *globally* supersymmetric action is given by:

$$S = \int d^4x \Big(-\frac{1}{2}\bar{\psi}_\mu \gamma^{\mu\nu\rho}\partial_\nu \psi_\rho - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \Big) \qquad \begin{cases} \delta_\epsilon \psi_\mu = \frac{1}{4}\gamma^{\nu\rho}\gamma_\mu \epsilon F_{\nu\rho} \\ \delta_\epsilon A_\mu = \bar{\epsilon}\,\psi_\mu \end{cases} , \tag{2.8}$$

Indeed, by coupling pure supergravity $(g_{\mu\nu}, \psi^{(1)}_\mu)$ **(2.7)** with the above theory $(\psi^{(2)}_\mu, A_\mu)$ **(2.8)** interpreted as a sort of "matter theory", following the Noether procedure described in the last section, allows one to get a local supersymmetric theory, with parameter $\epsilon^{(1)}$, for the desired physical fields $(g_{\mu\nu}, \psi^i_\mu, A_\mu)$. One has to add interaction terms of the form $\bar{\psi}\psi F$ and ψ^4, and also covariantize **(2.8)** under space-time coordinates redefinitions. One then realizes that this new theory enjoys an $SO(2)$ symmetry which rotates the two gravitini

ψ^i_μ, this implies that the theory is also invariant under another local supersymmetry transformation $\epsilon^{(2)}$ which can be obtained by essentially swapping the role of the two gravitini.

In this way, one gets the pure $\mathcal{N} = 2$ supergravity theory [37, 38], only the $SO(2) \cong U(1) \in U(2)_R$ symmetry is evident if one uses Majorana spinors ψ^i_μ, ϵ^i, which can be interpreted as $SO(2)$ doublets, and it is the only symmetry that can be gauged by using the A_μ gauge field present in the supermultiplet, without adding matter.

We will need only the bosonic part of this theory when discussing black holes, which is simply given by summing the bosonic parts of **(2.7, 2.8)**, and the supersymmetry transformation rule of the gravitini:

$$\delta\psi_\mu = M_p D_\mu \epsilon - \frac{1}{4}\gamma^{\nu\rho}\gamma_\mu F_{\nu\rho}\,\epsilon + O(\psi^2) \tag{2.9}$$

where the two Majorana gravitini and supersymmetry parameters ϵ^i have been combined into one Dirac spinor, $\psi_\mu = \psi^{(1)}_\mu + i\psi^{(2)}_\mu$ [39] in this way, the $SO(2)$ symmetry that rotates the two real spinors, becomes an $U(1)$ symmetry for the complex Dirac spinor. The $O(\psi^2)$ terms are not going to be relevant for us, since the gravitino is set to zero in the background.

What about $D = 5$ pure supergravity? Unfortunately, one cannot repeat the procedure we have described above, and one has to resort to other methods [40]. Skipping the details, the pure $D = 5$, $\mathcal{N} = 2$ supergravity theory is actually similar to the four-dimensional one. For example, the bosonic part of the Lagrangian, which we are interested in, has the same structure but in five-dimensions one is allowed to add a Chern-Simons gauge invariant term proportional to the 5-form $F \wedge F \wedge A = -\frac{\sqrt{-g}}{4}\epsilon^{\mu\nu\rho\sigma\lambda}F_{\mu\nu}F_{\rho\sigma}A_\lambda$, it is easy to see that under a gauge transformation $A_\mu \to A_\mu + \partial_\mu\Lambda$ this term transforms as a total derivative.

Regarding the gravitino supersymmetry transformation, one finds again a similar result as in $D = 4$ **(2.9)**:

$$\delta\psi^i_\mu = M_p D_\mu \epsilon^i - i\frac{1}{4\sqrt{3}}\left(\Gamma_\mu^{\ \nu\rho} - 4\delta^\nu_\mu\Gamma^\rho\right)F_{\nu\rho}\epsilon^i + O(\psi^2), \tag{2.10}$$

M_p here is related to the five-dimensional Newton's constant, so it is formally different from the one used before. Remember that in this case the spinors are symplectic Majorana and form a doublet of the $USp(2)_R \cong SU(2)_R$ R-symmetry group of the five-dimensional Poincaré superalgebra. Also in five-dimensions the pure supergravity theory contains only one gauge field A_μ which allows to gauge only a global $U(1)$ symmetry of the theory (contained in the R-symmetry group).

2.2.2 Gauging and AdS background

We will now describe the procedure of gauging the above pure supergravity theories, specifically we want to give an idea of what kind of modifications are needed in the Lagrangian and in the gravitino supersymmetric transformation rule. The most important point for us is that upon gauging, a *negative* cosmological constant appears, meaning that the gauged version of the above supergravities have AdS_D and not $Mink_D$ as a background solution. Similarly, black hole solutions in these gauged supergravities are asymptotically AdS_D.

Let us consider again the four-dimensional case. As we have already said, we can only gauge the global $SO(2)$ symmetry rotating the two Majorana gravitini (and correspondingly the two supersymmetry parameters $\epsilon^{(i)}$, or equivalently the $U(1)$ symmetry associated with the Dirac gravitino $\psi_\mu \to e^{ig\alpha(x)}\psi_\mu$ where g is the gauge coupling constant. Local $U(1)$ symmetry is then restored by minimally coupling the gravitino with the gauge field A_μ (now seen as the gauge field associated with the local $U(1)$ symmetry), hence one promotes the Lorentz-covariant derivative of the gravitino to a Lorentz and gauge-covariant derivative:

$$D_\mu \longrightarrow \hat{D}_\mu = D_\mu - igA_\mu, \tag{2.11}$$

this implies that the gravitino susy rule **(2.9)** acquires a new $O(g)$ term of the form $\delta\psi_\mu^{\text{new}} = \delta\psi_\mu^{\text{old}} - iM_p\, gA_\mu\epsilon$, while the other supersymmetry transformations are unchanged.

The action that one gets at this stage is not locally supersymmetric anymore. To restore supersymmetry, one has to add other $O(g)$ and $O(g^2)$ terms in the Lagrangian and further modify the susy rule of the gravitino. For example, the supersymmetry variation of the kinetic term of the gravitino produces a new $O(g)$ term of the form: $igM_p\bar{\epsilon}F_{\mu\nu}\gamma^{\mu\nu\rho}\psi_\rho$, which appears as a consequence of the new $O(g)$ variation $\delta\bar{\psi}_\mu \sim igM_pA_\mu\bar{\epsilon}$, (an integration by parts is also required). This term can be (partially) cancelled by the $\bar{\psi}\psi F$ term cited above, if we further modify the gravitino susy rule as $\delta\psi_\mu^{\text{new}} = \delta\psi_\mu^{\text{old}} - igM_pA_\mu\epsilon + \frac{g}{2}M_p^2\gamma_\mu\epsilon$, [41].

This new piece in the gravitino susy rule produces yet new terms when considering the variation of the Lagrangian, for example, from the kinetic term of the gravitino one gets a new $O(g)$ term of the form $gM_p^2\bar{\psi}_\mu\gamma^{\mu\nu\rho}\gamma_\rho D_\nu\epsilon$, this is trivially removed if one adds to the Lagrangian a mass term for the gravitino of the form[3] $\mathscr{L}_m \propto gM_p\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu$, the variation of this piece cancels the above term upon considering $\delta\psi_\mu \sim M_pD_\mu\epsilon$.

Finally, we arrive at the last relevant piece that one has to add to the Lagrangian, this is the $O(g^2)$ cosmological constant that essentially allows to cancel the $O(g^2)$ variation of the gravitino mass term $\mathscr{L}_m$ obtained by considering $\delta\psi_\mu \sim \frac{g}{2}M_p^2\gamma_\mu\epsilon$, so one has to add the following term[4] $\mathscr{L}' = \sqrt{-g}\; g^2M_p^4$. Where the use of the same letter for the gauge coupling constant and the determinant of the metric hopefully does not cause any confusion, given that the latter always appears under a square root.

Remarkably, these modifications are sufficient to restore the invariance under local supersymmetry. Notice that we had to add a cosmological constant term $\Lambda \propto -g^2M_p^2$ whose sign is *negative and fixed*, hence the background solution of this gauged supergravity theory is AdS_4.

The fact that the gauging procedure introduces a *negative* cosmological constant is a general fact for gauged suepergravities. For more complicated cases where there are also matter couplings and scalar fields, the mass term $\mathscr{L}_m$ and the cosmological constant will become scalar field dependent, producing a Yukawa-like coupling for ψ_μ and a scalar potential, which can be seen as an effective cosmological constant. It turns out that only vacua with negative or vanishing cosmological constant can be compatible with supersymmetry, while generally dS vacua break supersymmetry. The reason for this can be deduced from algebraic considerations (see e.g. [33]). Indeed, once a cosmological constant is introduced the background solution is either AdS or dS spacetime, the isometry group is not Poincaré anymore but $SO(1,4)$ or $SO(2,3)$ for dS or AdS in four dimensions. A consistent supersymmetric gravity theory with cosmological constant should then be obtained by promoting the symmetry groups of dS or AdS to supergroups. It turns out that this can be consistently done only for the AdS isometry group $SO(2,3)$ (in this way, the corresponding superalgebra closes).

These algebraic considerations also help to understand another peculiarity of the gauged theory that we have found above, which is the fact that the gravitino acquires a mass $m_\psi \sim gM_p$ while all other states in the supergravity multiplet stay massless. This would not be possible for the super-Poincaré group irreps. (each supermultiplet is degenerate in mass) but in the gauged case we should consider the irreps. of the super-$SO(2,3)$ group. In this case, one finds that the generators of spacetime translation P_a are such that[5] $[P^2, \mathcal{Q}] \neq 0$ and moreover, P^2 is not anymore a Casimir of the algebra. This means that the definition of the mass of the

[3]One can immediately see this by noticing that it holds $\gamma^{\mu\nu\rho}\gamma_\rho = -\gamma^{\mu\nu}$.

[4]This term does vary under local supersymmetric transformation as it depends on the metric via its determinant, one can then show that its variation cancels the $O(g^2)$ variation of the gravitino mass term [33].

[5]It is also true that $[P_a, \mathcal{Q}] \propto g \neq 0$, this is different from what happens in the super Poincaré group and is needed in order to satisfy the super Jacobi identities [33]. From this, one can show that the super Jacobi identities can be satisfied only for AdS space, using the properties of its isometry group. Moreover, the modification of the P_a, $\mathcal{Q}$ commutator is at the origin of the constant $\sim g\gamma_\mu\epsilon$ term appearing in the gravitino susy rule, which also makes the mass term $\mathscr{L}_m$ and the cosmological constant appear in the action. This is true also for *Pure* $\mathcal{N}=1$, $D=4$ AdS supergravity, where the g parameter is not a gauge coupling constant. However, if one couples this pure AdS supergravity with suitable supersymmetric matter, as we did above, one would find that one has to make the gravitini become charged under the gauge field, with g becoming the gauge coupling constant [38].

elementary particles in AdS is trickier, and does not need to be degenerate in the supermultiplets.

Finally ,the *bosonic part* of the action for $\mathcal{N} = 2$, $D = 4$ pure gauged supergravity relevant for our discussions is given by:

$$S = \int d^4x\sqrt{-g}\left(\frac{M_p^2}{2}R + 3g^2M_p^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right), \tag{2.12}$$

while the susy transformation of the gravitino is given by:

$$\delta\psi_\mu = M_p\Big(D_\mu - igA_\mu - \frac{g}{2}M_p\gamma_\mu\Big)\epsilon - \frac{1}{4}\gamma^{\nu\rho}\gamma_\mu F_{\nu\rho}\,\epsilon + O(\psi^2). \tag{2.13}$$

In five dimensions, the discussion follows the same. Remember now that there are two symplectic Majorana gravitini carrying an $SU(2)$ index i. We can consider the $U(1)$ transformation generated by the σ_3 generator of $SU(2)$ which acts on the gravitini as $\psi_\mu^{(1)} \to e^{ig\alpha(x)}\psi_\mu^{(1)}$ and $\psi_\mu^{(2)} \to e^{-ig\alpha(x)}\psi_\mu^{(2)}$. The gauging procedure produces similar modifications to the Lagrangian and gravitino susy transformation rule, also in this case a negative cosmological constant appears (see e.g. [42]).

The explicit gravitino susy transformation rule is given by:

$$\delta\psi_\mu^i = D_\mu\epsilon^i - i\frac{1}{4\sqrt{3}}\Big(\Gamma_\mu^{\ \nu\rho} - 4\delta_\mu^\nu\Gamma^\rho\Big)F_{\nu\rho}\epsilon^i - g\left(\frac{1}{2\sqrt{3}}\Gamma_\mu + iA_\mu\right)\epsilon^{ij}\epsilon^j + O(\psi^2), \tag{2.14}$$

remember that ψ_μ^i and ϵ^i are symplectic Majorana spinors, moreover $\epsilon^{ij} = -\epsilon^{ji}$ and $\epsilon^{12} = 1$.

2.3 Coupling pure supergravity to vector multiplets

As we have anticipated in the introduction, we are also interested in supergravity theories where we gauge larger Abelian $U(1)^n$ groups. This cannot be done in pure supergravity as there are not enough vector fields which can play the role of gauge fields. This problem can be solved by coupling the pure supergravity theory with vector multiplets, we are then going to give a rather brief and incomplete overview on this topic, hoping to convey some basic ideas on the structure of this matter coupled supergravities, (more details can be found, for instance, in [33] and [36]).

Let us again stick to $D = 4$. We want to couple the pure supergravity theory to $\mathcal{N} = 2$ vector multiplets. In $D = 4$ these contain a gauge field A_μ, two Majorana spin-$\frac{1}{2}$ gaugini λ^i whose i index transforms under the R symmetry group, and a complex scalar ϕ. $\mathcal{N} = 2$ matter coupled supergravity necessarily contains scalar fields, it is well known that when scalar fields are present in a generic supersymmetric theory, the properties of such a theory strongly depend on the geometry of the scalar field manifold[6] $\mathcal{M}$. The idea is that in a supersymmetric theory the n_s scalars $\{\phi^n\}_{n=1,\cdots,n_s}$ can be seen as (complex) coordinates for $\mathcal{M}$. However, the presence of other fields and the requirement of preserving supersymmetry impose some constraints on $\mathcal{M}$.

For example, considering the globally supersymmetric $\mathcal{N} = 1$ most general (non-renormalizable) Wess-Zumino model, there are also n_s (Majorana) fermions χ^n in the same multiplet as the scalars, whose susy transformation contains a term like: $\delta\chi_L^n \sim \partial\!\!\!/\phi^n\epsilon_R$ and $\delta\chi_R^{\bar{n}} \sim \partial\!\!\!/\phi^{\bar{n}}\,\epsilon_L$, using the notation of [33] we denote $\bar{\phi}^n = \phi^{\bar{n}}$. Supersymmetry relates the left handed component of χ to ϕ and the right handed one to $\phi^{\bar{n}}$, this structure must be preserved by the internal geometry of $\mathcal{M}$ which implies that it should be thought of as a

[6]Thinking at the most general non-renormalizable theory, it is allowed to have non-minimal kinetic terms, e.g. $g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j$, and possibly other kinds of non-minimal couplings between the scalars and other fields. Even without considering supersymmetry, in general non-renormalizable theories for scalar fields (non-linear sigma models) it is natural to view the scalars as coordinates parametrizing a (scalar)manifold $\mathcal{M}$. This allows to give a geometrical interpretation of the various couplings that one has in the theory. For example, in the non-minimal scalar kinetic term one identifies $g_{ij}(\phi)$ as the metric of the scalar field manifold.

complex manifold, covered by biholomorphic coordinate systems[7]. Other constraints can be deduced from this. For example, by studying the holonomy group of $\mathcal{M}$ and requiring that it does not mix ϕ^n and $\phi^{\bar{n}}$, one can derive that $\mathcal{M}$ must be a *Kähler manifold*.

Oversimplifying it, we may say that in this case the geometry of $\mathcal{M}$ is determined by a *real* function $K(\phi^n, \phi^{\bar{n}})$ called *Kähler potential* (for example the metric on $\mathcal{M}$ is $g_{m\bar{n}} = \partial_m \partial_{\bar{n}} K$), which is invariant under the Kähler transformations: $K \to K + h(\phi^n) + \bar{h}(\phi^{\bar{n}})$. These relate the different expressions of K on different coordinate patches on $\mathcal{M}$. Notice also that the susy transformation rules of the fermions indicate that they should be thought as tangent vectors on $\mathcal{M}$, hence one has to covariantize all derivatives of the χ fields with respect to scalar field redefinitions, which adds non minimal couplings between scalars and fermions.

What happens when supersymmetry is made local? Generally new structures and constraints have to be imposed on $\mathcal{M}$. For example, one can couple the above general non-renormzalizable Wess-Zumino model to the supergravity multiplet by generalizing the Noether procedure discussed previously. Restoring local supersymmetry requires adding many terms to the Lagrangian, together with the metric and the gravitino. One realizes that in this way *all* fermions must transform under Kähler transformations as (e.g. taking the gravitino): $\psi_\mu \to exp\big[\frac{-i}{2M_p} Im(h(\phi))\gamma_5\big]\psi_\mu$, this implies that all derivatives of the fermions must be made Kähler covariant. The new terms that appear in the Lagrangian in this way are precisely (part of) those needed to restore local supersymmetry.

Fermions should now be thought of as sections in a principal $U(1)$ bundle over $\mathcal{M}$, Kähler transformations on them can be seen as the maps that relate the different local definitions of the fermions on different coordinate patches on $\mathcal{M}$, similarly as happened for the Kahler potential. This additional structure on $\mathcal{M}$ constrains further the geometry of $\mathcal{M}$ which now becomes a so called *Kähler-Hodge* manifold. More details can be found in [33, 36]

Finally, let us consider extended (specifically $\mathcal{N} = 2$) supergravity coupled to matter. Assume that only $\mathcal{N} = 2$ Abelian vector multiplets are involved, in this case, scalars appear in the same multiplet as the vectors. Moreover, the number of vectors does not match the number of scalars as also the supergravity $\mathcal{N} = 2$ multiplet contains a vector, the graviphoton. The field content of $\mathcal{N} = 2$ supergravity coupled to n_V vector multiplets is then: $(g_{\mu\nu}, \psi^i_\mu, A^I_\mu, \lambda^{n,i}, \phi^n)$, where the graviphoton is generally not distinguished from the other n_V vectors. indeed, the I index runs from 0 to n_V, while the n indices run from 1 to n_V as there are n_V scalars. The i index is instead related to the usual $SU(2)_R$ doublets.

In this case, $\mathcal{M}$ turns out to be still a Hodge-Kähler manifold, but of special type, the reason is that once Abelian vector fields are involved (and we have not yet gauged any symmetry), the theory exhibits the phenomenon of electric-magnetic duality. These are a set of symmetries of the *equations of motion plus Bianchi identities* for the vector fields which, if the Lagrangian of the theory is $\mathscr{L}$, can be written as:

$$\begin{cases} \nabla^\mu \frac{\partial \mathscr{L}}{\partial F^{I\mu\nu}} = 0 \\ dF^I = 0 \end{cases} \Longrightarrow \begin{cases} dG_I = 0 \\ dF^I = 0 \end{cases}, \quad \text{where:} \quad \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} G_I^{\rho\sigma} = 2\frac{\partial \mathscr{L}}{\partial F^{I\mu\nu}}, \tag{2.15}$$

here we allow for the most general gauge kinetic terms in $\mathscr{L}$ of the form $\mathscr{L}_{vec} \sim \mathcal{P}_{IJ} F^I_{\mu\nu} F^{J\mu\nu} + \mathcal{R}_{IJ} F^I_{\mu\nu} \tilde{F}^{J\mu\nu} + 2\mathcal{O}^{\mu\nu}_I F^I_{\mu\nu}$, where the matrices $\mathcal{P}_{IJ}$ and $\mathcal{R}_{IJ}$ possibly depend on the scalar fields. Electric-magnetic duality acts as rotations of the following two-component vector:

$$\mathbb{F} \equiv \begin{bmatrix} F^I \\ G_J \end{bmatrix}, \qquad \mathbb{F}' = S\mathbb{F}, \tag{2.16}$$

[7] As chirality is a physical spacetime property, it must be preserved. From the susy transformation rules of the fermions this implies that ϕ^n and $\phi^{\bar{n}}$ should not "mix". Hence, we cannot view $\mathcal{M}$ as a real manifold of dimension $2n_s$, as parametrizing it with real coordinates necessarily requires to mix ϕ^n and $\phi^{\bar{n}}$.

the equations of motion plus the Bianchi identities and the definition of G_J in terms of the F^I field strengths **(2.15)** are left invariant by any $S \in Sp(2n_V, \mathbb{R})$.

Duality transformations generally *change the Lagrangian* but leave invariant the equations of motion, meaning that they can be seen as a way to describe the same underlying physics with different fields or explicit degrees of freedom[8]. Coming back to supersymmetry, if one uses a different set of gauge fields after a duality transformation, one may expect to also have a different set of scalar fields as they sit in the same multiplet as the vectors. Again, these new scalars would not be different degrees of freedom, but only a different way to describe the original ones given by ϕ^n.

Moreover, there is one more vector than the number of scalars, and also this aspect should be taken into account. The way to deal with this is to mimic what we have done in the $\mathbb{F}$ symplectic vector, and define the following scalar-field dependent symplectic vector $\mathcal{V}$:

$$\mathcal{V}(\phi) = \begin{bmatrix} X^I(\phi) \\ F_J(\phi) \end{bmatrix}, \qquad \mathcal{V}' = S\mathcal{V}, \quad \text{for a symplectic transformation: } S \in Sp(2n_V + 2, \mathbb{R}), \tag{2.17}$$

$X^I(\phi)$ and $F_J(\phi)$ are a set of $n_V + 1$ *auxiliary* scalar fields and their duals, which depend on the *physical* n_V scalars ϕ^n. The auxiliary fields make it easier to take into account the duality.

For consistency, one finds that $\mathcal{V}(\phi)$ should transform under Kähler transformations $\mathcal{V} \to exp[-h(\phi)]\mathcal{V}$, meaning that $\mathcal{V}(\phi)$ should be thought as a section of a flat holomorphic vector bundle with a symplectic structure group over $\mathcal{M}$. Moreover, by taking Kähler covariant derivatives of $\mathcal{V}$ one obtains quantities that carry both an I index and an m index, which again transform covariantly under Kähler transformations but also under scalar field reparametrizations. The auxiliary fields and their kähler covariant derivatives can be used to obtain quantities that are simultaneously, duality, Kahler and scalar field redefinition covariant. These are needed to consistently write (for example) the susy transformation rules of the fermions, which are considered invariant under duality transformations, but transform under Kähler and scalars redefinitions.

The structure of $\mathcal{M}$ is now determined from the auxiliary scalar fields[9]. This imposes some other constraints on $\mathcal{M}$ which now becomes a local (or "projective") special Kähler manifold[10].

The main message that we want to convey is the importance of the scalar field manifold in matter coupled supergravity theories, and the need to use the auxiliary scalar fields X once extended, matter-coupled supergravities are taken into account. In practice, however, we will always work in a *fixed symplectic frame* where all the physical degrees of freedom are described by the F^I field strengths and $X^I(\phi)$ auxiliary scalars[11]. Finally, having more vector fields at our disposal allows us to gauge a larger global symmetry group.

The considerations above can be repeated also in the five-dimensional theories, but in this case the structure of the scalar manifold is drastically different. The reason is that the $\mathcal{N} = 2$ vector multiplet in this case contains a *real* scalar field ϕ instead of a complex field. Moreover, the two spin$-\frac{1}{2}$ fields are now symplectic Majorana spinors. Again, scalars and vectors are in the same multiplets, this implies that the properties of $\mathcal{M}$ can be described by $n_V + 1$ *real* auxiliary scalars $X^I(\phi)$, in particular $\mathcal{M}$ is given as the hypersurface in the auxiliary $\mathbb{R}^{n_V+1}$ space parametrized by the X^I, defined via the constraint:

$$\mathcal{M} = \{X^I \in \mathbb{R}^{n_V+1} | C_{IJK} X^I X^J X^K = 1\}, \tag{2.18}$$

[8] For example, one may use only the electric field strengths F^I as is usually done, only the magnetic G_J ones or a combination of both.

[9] for example K is determined in a symplectic invariant way in terms of $\mathcal{V}$

[10] "projective" refers to the fact that we have to use the X auxiliary scalars, which provide a redundant description being the physical scalars one less in number.

[11] In this case X^I can be viewed as simple functions of the scalars, so we may as well write the actions in terms of only the physical scalars ϕ, knowing that under a duality transformation the explicit expression of the action may change.

where the C_{IJK} constants define the Chern-Simons term $C_{IJK}F^I \wedge F^J \wedge A^K$. One finds that the C_{IJK} constants completely determine the structure of the supergravity theory in this case.

It is a general fact that in different dimensions and in different $\mathcal{N}$-extended supergravities, the scalar field manifold may be completely different. This happens even when considering scalar fields in different multiplets, for example, the scalars in the four-dimensional $\mathcal{N} = 2$ hypermultiplets (generalizing the $\mathcal{N} = 1$ chiral multiplets), would produce a completely different scalar field manifold $\mathcal{M}_{hyper}$ with respect to the one associated with the scalars in the vector multiplets.

For the explicit actions of $\mathcal{N} = 2$, $D = 5$ ungauged and gauged supergravity coupled to n_V Abelian vector fields See [42, 43].

2.4 Supersymmetric, extremal and BPS black hole solutions in gauged supergravities

When discussing black hole solutions in supergravity theories, one usually set to zero all fermionic fields in the action. This allows to simplify a lot the equations of motion, which can be solved more easily. Doing this, the four and five dimensional pure $\mathcal{N} = 2$ gauged supergravity reduces to the Maxwell-Einstein theory with a cosmological constant (and Chern-Simons term in five dimensions).

Clearly, the general black hole solutions of these bosonic theories do not preserve supersymmetry, as the supersymmetric transformations would make non-zero fermionic fields appear. The standard way to restore at least part of the original supersymmetry in the above black hole solutions, is to require that there exists (at least one) *killing spinor* ϵ [39, 44, 45, 46]. A killing spinor is a spinorial field that satisfies the equation $\delta_{susy}\psi_\mu|_{\psi=0} = \hat{\nabla}_\mu \epsilon = 0$, which for the pure gauged supergravities described above is simply given by setting to zero Eqs. **(2.13, 2.14)**. If matter is present, one should require that also the susy transformation rule for all the fermions vanishes (see e.g. [47]). To fully restore the original supersymmetry, one would need more than one independent killing spinor, but generally one is able to restore only a fraction of it[12], for example, in $D = 5$ *gauged* $\mathcal{N} = 2$ supergravity the only solution that is maximally supersymmetric is empty AdS_5 spacetime [44].

Requiring the existence of at least one killing spinor imposes some constraints on the parameters of the solution, which usually arise from an *integrability condition* derived from the killing spinor equation [44, 48], this is a necessary but not sufficient condition. As an example, in $D = 4$ it is given by $[\hat{\nabla}_\mu, \hat{\nabla}_\nu]\epsilon = \mathcal{R}_{\mu\nu}\epsilon = 0$, where $\mathcal{R}_{\mu\nu}$ is called supercurvature, and one has to require that its determinant vanishes ($\mathcal{R}_{\mu\nu}$ is a 4×4 matrix in spinor space).

A different but equivalent way to derive the conditions that allow to restore supersymmetry, is to study the Bogomol'nyi matrix obtained from the anticommutator of the supercharges[13] $\mathcal{Q}$ evaluated on the given solution, and in particular requiring that it admits at least one vanishing eigenvalue, from similar considerations as in footnote **12** each vanishing eigenvalue allows to restore a fraction ($1/4$ in the $\mathcal{N} = 2$ case) of the original supersymmetry [11, 49]. The supersymmetric condition that is obtained in this way is a constraint on the conserved charges, but is equivalent to the one that is obtained from the integrability condition of the killing

[12] Remember that in the $\mathcal{N} = 2$ theories above, there are eight *real* supercharges that can be organized in four couples of operators that rise/lower the spin of a state in the supermultiplet. Requiring that one killing spinor exists can naively be seen as requiring that one of the four couples of real supercharges acts trivially on the fermions *in the specific solution considered*, so that once we set the fermions to 0 they do not appear under the action of this restricted set of supercharges. In this way we restore one quarter of the original supersymmetry per independent killing spinor .

[13] This is related to the integrability condition $[\hat{\nabla}_\mu, \hat{\nabla}_\nu]\epsilon = 0$. To see this, notice that the latter is essentially $[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\psi_\mu = 0$, then one can interpret the gravitino ψ_μ as the *gauge field* associated with the gauging of the supersymmetry parameters. This allows to derive the gravitino susy transformation rule from purely algebraic considerations interpreting the supergravity theory as a sort of Yang-Mills theory [33]. It is a general feature of Yang-Mills theories that $[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\psi_\mu = \delta_{[\epsilon_1,\epsilon_2]}\psi_\mu$, the latter is the transformation of the gravitino under the element $[\epsilon_1\mathcal{Q}, \epsilon_2\mathcal{Q}]$ of the given superalgebra, which is related to the anticommutator of the supercharges.

spinor, which is instead a constraint on the parameters of the solution.

An important point has to be made now. If one simply imposes the supersymmetry conditions described above, either requiring the existence of a killing spinor or vanishing of the eigenvalues of the Bogomol'nyi matrix, one gets a (partially) supersymmetric solution, which however, is generally a not well-behaved black hole solution. For example, if one considers the Lorentzian solution, one usually finds that the supersymmetric solution has a naked singularity or causal pathologies such as closed timelike curves outside of the event horizon [11], a different kind of problem arises in the Euclidean supersymmetric solution [8].

To regulate these pathologies, one has to further constrain the solution, for example, requiring the absence of closed timelike curves or of naked singularities will produce an additional constraint. Remarkably, the well behaved solution obtained in this way is also *extremal*. In the cases we are going to study we will always find that supersymmetric but not yet extremal solutions have naked singularities, requiring that an event horizon exists will always imply that the solution is also extremal, and in particular that the function determining the position of the horizons has a double root.

Chapter 3

Asymptotically AdS_5 black holes in 5D, $\mathcal{N} = 2$ gauged supergravity

In this chapter, we will discuss in detail the results that we anticipated in the introduction. We are going to consider asymptotically AdS_5 rotating and electrically charged black hole solutions of $5D$, $\mathcal{N} = 2$ minimal gauged and $U(1)^3$ gauged supergravity also known as STU model. The former can be thought of as a special case of the second one where we set the three gauge fields equal and the scalar fields to suitable constant values [50].

As anticipated in the introduction, this supergravity theory can be obtained as a consistent truncation on S^5 of type IIB supergravity on $AdS_5 \times S^5$ background to $\mathcal{N} = 2$ supersymmetry [4], in this way the original $SO(6)$ isometry group of S^5 is reduced to its Cartan subgroup $U(1)^3$, which is the gauge group of the supergravity theory we consider. By means of the AdS_5/CFT_4 correspondence, this supergravity is dual to a $\mathcal{N} = 4$ SYM field theory [2], if black holes are involved, the duality correlates the thermodynamics of the black hole and the dual field theory [51, 52, 53].

The generic black hole solution admits six independent conserved charges: two angular momenta associated with the two independent rotations along two orthogonal spatial planes[1], three electric charges associated with the $U(1)^3$ gauge group and the energy (or mass). We will directly consider the solution with one electric charge and two angular momenta [54] of minimal gauged supergravity, and the solution with one angular momentum and three electric charges [50] of the $U(1)^3$ gauged supergravity. Other non-extremal black hole solutions of this theory with different combinations of charges can be found in [55, 56, 57].

3.1 Review of the double spinning charged AdS_5 black hole solution

3.1.1 Review of the general non extremal and non supersymmetric solution

Following the conventions of [8] with non canonically normalized gauge kinetic term, the bosonic sector of minimal 5D gauged supergravity contains the metric and the graviphoton A, gauging the R symmetry group, the Lagrangian is given by [42, 43]:

$$\mathcal{L} = (R + 12g^2) \star 1 - \frac{2}{3g^2} F \wedge \star F + \frac{8}{27g^3} F \wedge F \wedge A, \tag{3.1}$$

[1] we are considering asymptotically AdS_5 spacetimes meaning that the asymptotic isometries are given by the $SO(2,4)$ group, rotations are generated by the subgroup $SO(4)$ which admits two commuting $U(1)$ rotations.

where $g>0$ is related to the cosmological constant and it is the inverse of the AdS radius. The general (non extremal) asymptotically AdS black hole solution with two unequal angular momenta was first found in [54], in the conventions of [8] the solution is:

$$\begin{aligned} ds^2 = &-\frac{\Delta_\theta[(1+g^2r^2)\rho^2 dt + 2q\nu]dt}{\Xi_a\Xi_b\rho^2} + \frac{2q\,\nu\omega}{\rho^2} + \frac{f}{\rho^4}\Big(\frac{\Delta_\theta}{\Xi_a\Xi_b}dt - \omega\Big)^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 \\ &+ \frac{r^2+a^2}{\Xi_a}\sin^2\theta\, d\phi^2 + \frac{r^2+b^2}{\Xi_b}\cos^2\theta\, d\psi^2, \end{aligned} \tag{3.2}$$

$$A = \frac{3gq}{2\rho^2}\Big(\frac{\Delta_\theta}{\Xi_a\Xi_b}dt - \omega\Big) + \alpha dt, \tag{3.3}$$

where:

$$\begin{aligned} \Delta_r &= \frac{(r^2+a^2)(r^2+b^2)(1+g^2r^2)+q^2+2abq}{r^2} - 2m, \\ \Delta_\theta &= 1 - a^2g^2\cos^2\theta - b^2g^2\sin^2\theta, \qquad \rho^2 = r^2 + a^2\cos^2\theta + b^2\sin^2\theta, \\ \Xi_a &= 1-a^2g^2, \quad \Xi_b = 1-g^2g^2, \qquad f = 2m\rho^2 - q^2 + 2abqg^2\rho^2, \\ \nu &= b\sin^2\theta\, d\phi + a\cos^2\theta\, d\psi, \qquad \omega = \frac{a\sin^2\theta}{\Xi_a}d\phi + \frac{b\cos^2\theta}{\Xi_b}d\psi. \end{aligned} \tag{3.4}$$

The explicit components of the metric can be found in [54], we allow for different gauge choices for the A field parametrized by the α parameter [8]. The solution is given in terms of Boyer-Lindquist type coordinates $(t, r, \theta \in [0, \frac{\pi}{2}], \phi \sim \phi+2\pi, \psi \sim \psi+2\pi)$ which are not rotating at infinity[2], and depends on the four parameters (a, b, q, m), which must satisfy $a^2g^2 < 1, b^2g^2 < 1$ to avoid faster than light speed rotations of the 4D Einstein universe on the $r\to\infty$ boundary where the dual CFT lives [52, 53].

Thermodynamics

The solution above admits four conserved charges associated with the three killing vectors ($\partial_t \to E, \partial_\phi \to J_1, \partial_\psi \to J_2$) and one electric charge Q. The angular momenta and the electric charge can be unambiguously evaluated by using the appropriate Komar integrals [54]. Following the conventions of [8], these quantities are given by:

$$J_1 = \frac{\pi[2am + qb(1+a^2g^2)]}{4\Xi_a^2\Xi_b}, \qquad J_2 = \frac{\pi[2bm + qa(1+b^2g^2)]}{4\Xi_a\Xi_b^2}, \qquad Q = \frac{\pi q}{2g\Xi_a\Xi_b}, \tag{3.5}$$

the calculation of the conserved energy/mass is more subtle due to the appearance of IR divergences caused by the infinite volume of the AdS_5 background, these can be removed via different procedures (background subtraction method [52], holographic renormalization [53, 58]). In our case, the conserved energy has been calculated by integrating the first law of BH thermodynamics [59]: $dE = TdS + \Omega_1 dJ_1 + \Omega_2 dJ_2 + \Phi dQ$, where the temperature, entropy, and angular/electric potentials will be defined momentarily[3], the expression for the energy that one obtains in this way is:

$$E = \frac{m\pi(2\Xi_a + 2\Xi_b - \Xi_a\Xi_b) + 2\pi qabg^2(\Xi_a+\Xi_b)}{4\Xi_a^2\Xi_b^2}. \tag{3.6}$$

[2] Taking the large r limit of the metric (**3.3**), one finds [8]: $ds^2 \approx \frac{dr^2}{r^2} + r^2(-\frac{\Delta_\theta}{\Xi_a\Xi_b}dt^2 + \frac{d\theta^2}{\Delta_\theta} + \frac{\sin^2\theta}{\Xi_a}d\phi^2 + \frac{\cos^2\theta}{\Xi_b}d\psi^2)$, which can be shown to be the metric of AdS_5 as seen by a non rotating observer, in non-standard coordinates [52].

[3] The conserved mass has also been obtained [60] from the integration of a conserved quantity $Q[K]$, associated with the asymptotic killing vector $K = \partial_t$, where $Q[K]$ is obtained following the AMD construction from the asymptotic Weyl tensor on the conformal boundary of the spacetime, and also more recently [8] via holographic renormalization which makes use of the boundary stress-energy tensor. This provides a connection between the "thermodynamic" definition of the mass and its definition as the conserved charge associated with an isometry of the spacetime.

The other quantities are expressed in terms of the outer horizon radius r_+, which is the largest real root of $\Delta_r(r)$. The Killing vector that vanishes on the horizon is $V = \partial_t + \Omega_1\partial_\phi + \Omega_2\partial_\psi$, where the two angular velocities relative to a non rotating frame at infinity are given by:

$$\Omega_1 = \frac{a(r_+^2+b^2)(1+g^2r_+^2)+bq}{(r_+^2+a^2)(r_+^2+b^2)+abq}, \qquad \Omega_2 = \frac{b(r_+^2+b^2)(1+g^2r_+^2)+aq}{(r_+^2+a^2)(r_+^2+b^2)+abq}, \tag{3.7}$$

the Hawking temperature, proportional to the surface gravity is:

$$T = \frac{r_+^4[1+g^2(2r_+^2+a^2+b^2)]-(ab+q)^2}{2\pi r_+[(r_+^2+a^2)(r_+^2+b^2)+abq]}, \tag{3.8}$$

the electrostatic potential on the horizon is:

$$\Phi = V^\mu A_\mu\big|_\infty^{r_+} = \frac{3gqr_+^2}{2[(r_+^2+a^2)(r_+^2+b^2)+abq]}, \tag{3.9}$$

finally the Bekenstein-Hawking entropy of the outer horizon is:

$$S = \frac{\pi^2[(r_+^2+a^2)(r_+^2+b^2)+abq]}{2\Xi_a\Xi_b r_+}. \tag{3.10}$$

These quantities obey the first law of thermodynamics by construction

$$dE = T\,dS + \Omega_1\,dJ_1 + \Omega_2\,dJ_2 + \Phi\,dQ, \tag{3.11}$$

the fact that we can integrate the first law to get the energy depends crucially on the definition of S as one quarter of the area of the outer horizon[4].

Finally, one can define a grand-canonical potential TI for the thermal ensemble at fixed chemical potentials, from the principles of Euclidean quantum gravity this is calculated from the Euclidean on-shell action I [61].

The Euclidean on-shell action I should be evaluated on the *regular* Euclidean section from r_+ to infinity, where the metric should be kept *real* and *positive definite*. Following [8, 60], this can be achieved by first performing the Wick rotation $t \to -i\tau$, then the additional analytic continuation[5] $a, b \to ia, ib$ allows to recover a real Euclidean metric, the m, q parameters remain real.

Regularity can be achieved by studying the near horizon Euclidean metric which, from [8, 54], is:

$$\begin{aligned} ds_5^2 \approx \frac{4\rho^2}{\Delta_r'}\left[dR^2 + R^2\left(\frac{2\pi}{\beta}\right)^2 d\tau^2\right] + \frac{\rho^2}{\Delta_\theta}\,d\theta^2 + g_{\phi\phi}(d\phi + i\Omega_1 d\tau)^2 + g_{\psi\psi}(d\psi + i\Omega_2 d\tau)^2 \\ + 2g_{\phi\psi}(d\phi + i\Omega_1 d\tau)(d\psi + i\Omega_2 d\tau), \end{aligned} \tag{3.12}$$

where $R^2 = r - r_+$, and notice that after the analytic continuation of the (a, b) parameters, $i\Omega_i$ are real.

(R, τ) parametrize an $\mathbb{R}^2$ plane in polar coordinates, and the spacetime comes to an end on the horizon at the origin of this plane at $R = 0$. A conical singularity at $R = 0$ is avoided if one performs the standard thermal identifications $\tau \sim \tau + \beta$. Moreover, the presence of rotation, and of the mixed $dtd\phi$ and $dtd\psi$ terms requires

[4]Other definitions of S as a monotonically increasing function of A would not produce a closed differential on the RHS of Eq. **(3.11)**, which would not be exact either.

[5]After the wick rotation, the $dtd\phi$ and $dtd\psi$ terms in the metric become imaginary, by making also a, b imaginary one recovers a real Euclidean metric. Notice that as m, q remain real, the gauge field necessarily remains imaginary in Euclidean signature.

us to impose the following identifications [61], when one goes around the Euclidean time circle:

$$(\tau,\,\phi,\,\psi) \sim (\tau+\beta,\,\phi - i\Omega_1\beta,\,\psi - i\beta\Omega_2). \tag{3.13}$$

A regularity condition also has to be imposed on the gauge field. As the Euclidean time circle shrinks to a point at $R=0$ one has to require that the component of the gauge field along the direction that shrinks vanishes. However, it is not sufficient to require that $\iota_{\partial_t} A\big|_{r\to r_+}=0$ because when one goes around the time circle, one also moves along the (ϕ,ψ) directions due to rotation.

To see what the correct regularity condition is, one can introduce the coordinates [8]:

$$\tau=\hat{\tau}, \qquad \phi=\hat{\phi}-i\Omega_1\hat{\tau}, \qquad \psi=\hat{\psi}-i\Omega_2\hat{\tau}, \tag{3.14}$$

in this way, the mixed $\phi,\psi/t$ terms in the near horizon metric **(3.12)** disappear, and the regularity conditions that one has to impose are the standard "untwisted" identifications: $(\hat{\tau},\hat{\phi},\hat{\psi}) \sim (\hat{\tau}+\beta,\hat{\phi},\hat{\psi})$.

A rotation along the Euclidean time circle is now not accompanied by rotations along the $\hat{\phi}$,$\hat{\psi}$ directions in the near horizon limit, hence the regularity condition for the gauge field is: $\iota_{\partial_{\hat{\tau}}} A\big|_{r\to r_+} = \iota_V A\big|_{r\to r_+}=0$, where one can easily see that the killing vector V that vanishes on the horizon is simply $V=i\partial_{\hat{\tau}}$ in these coordinates.

It is easy to see that this condition on the gauge field can be implemented by performing a gauge transformation and fixing the α parameter in Eq. **(3.3)** as [8, 61]:

$$\iota_V A\big|_{r\to r_+}=0 \quad\Longleftrightarrow\quad \alpha=-\Phi. \tag{3.15}$$

From the real, positive definite metric on the regular Euclidean section outside the horizon, one computes the on-shell action, which provides the semiclassical approximation of the full quantum gravity theory, and in particular of the *grand-canonical* partition function[6] $\mathcal{Z}=e^{-I}$. It then follows that TI can be considered as the grand-canonical potential.

The calculation of the Euclidean on-shell action was first performed by [60], who removed the long range divergences appearing in the calculation of the integral using the background subtraction method.

The result they found was:

$$I=\frac{\pi\beta}{4\Xi_a\Xi_b}\left[m-g^2(r_+^2+a^2)(r_+^2+b^2)-\frac{q^2r_+^2}{(r_+^2+a^2)(r_+^2+b^2)+abq}\right], \tag{3.16}$$

and one can show using Eqs. **(3.5)** to **(3.10)** that the quantum statistical relation (QSR) is satisfied in the form:

$$T\,I=E-S-\Omega_1\,J_1-\Omega_2\,J_2-\Phi Q, \tag{3.17}$$

justifying that TI is indeed the grand-canonical potential.

As a final remark, remember that the angular velocities and the electrostatic potential *relevant for the thermodynamics*, must be considered relatively to the observer at infinity [53, 59]. The metric **(3.3)** is non rotating at infinity, so the angular velocities of the horizon in Eqs. **(3.7)** are already the ones entering the thermodynamics.

[6] The thermal identifications **(3.13)** imply that a Euclidean QFT on this geometry describes thermal states at finite T and $\Omega_{1,2}$. To see why we also have a fixed electrostatic potential Φ one should look at the transformation properties of charged correlators when $\tau\to\tau+\beta$. However, as the graviphoton and the graviton are both neutral, these transformation properties do not appear on the metric or the gauge field. Alternatively, one notices that the regularity condition implies that $\alpha=-\Phi$ meaning that any matter outside the horizon is immersed in the same electric potential as the black hole. Moreover, the boundary gauge filed reads $A_{bdry}=i\Phi dt$, meaning that the thermodynamics of the black hole should be matched to one of the dual CFT at fixed electrostatic potential.

3.1.2 Supersymmetry and BPS solutions

Following the discussion in Sec. **2.4**, to restore at least part of the supersymmetry, one has to impose additional constraints on the parameters of the theory. In particular, on supersymmetric solutions one has to impose that a solution of the killing spinor equation $\delta_{susy}\psi_\mu = 0$ exists [44]. In this case, it turns out that the solution admits one killing spinor, hence restoring one quarter of the total supersymmetry, if the parameters satisfy the constraint:

$$q = \frac{m}{1 + ag + bg}, \tag{3.18}$$

the same condition was obtained in [54] following the discussion of [11], by requiring that the Bogomol'nyi matrix evaluated on the solution admits at least one vanishing eigenvalue, for this solution only one eigenvalue can be set to zero at a time (other cases are related by changing the signs of the parameters), this produces the following constraint on the conserved charges (which can be shown to be the same as Eq. **(3.18)**):

$$E - gJ_1 - gJ_2 - \frac{3}{2}gQ = 0, \tag{3.19}$$

Purely supersymmetric solutions are in general non physical due to the appearance of closed timelike curves (CTCs) [11, 54] and naked singularities in the Lorentzian solution or, if one passes to Euclidean signature, because one needs to consider a complexified family of solutions characterized by a complex value of the charges and potentials (See [8] in particular Sec.**3.1** and appendix A). Either requiring the absence of CTCs and naked singularities in the Lorentzian solution, or to have a real value for the charges and chemical potentials in the Euclidean case, one must impose the additional constraints on the parameters

$$m = \frac{1}{g}(a + b)(1 + ag)(1 + bg)(1 + ag + bg), \tag{3.20}$$

$$a + b + abg > 0, \tag{3.21}$$

the solution that one gets in this way can be shown to be also *extremal*. We will refer to "BPS" solution as the one for which both *supersymmetry* and *extremality* are realized.

BPS solutions have a fixed value of all the chemical potentials[7]:

$$\beta \to \infty, \qquad \Omega_1 \to \Omega_1^\star = g, \qquad \Omega_2 \to \Omega_2^\star = g, \qquad \Phi \to \Phi^\star = \frac{3}{2}g. \tag{3.22}$$

The condition $\Delta_r(r) = 0$ can now be easily solved and gives as largest root:

$$r_\star = \sqrt{\frac{1}{g}(a + b + abg)}, \tag{3.23}$$

which is real due to Eq. **(3.21)**, the conserved charges and entropy then reads:

$$J_1^\star = \frac{\pi(a + b)(2a + b + abg)}{4g(1 - ag)^2(1 - bg)}, \qquad J_2^\star = \frac{\pi(a + b)(a + 2b + abg)}{4g(1 - ag)(1 - bg)^2}, \qquad Q^\star = \frac{\pi(a + b)}{2g^2(1 - ag)(1 - bg)}, \tag{3.24}$$

$$S^\star = \frac{\pi^2(a + b)r_\star}{2g(1 - ag)(1 - bg)} = \pi\sqrt{3(Q^\star)^2 - \frac{\pi}{g^3}(J_1^\star + J_2^\star)}, \tag{3.25}$$

the energy can be obtained from the supersymmetry condition on the charges Eq. **(3.19)**, while the other three charges are not all independent, remember that the BPS solution has only two free parameters left, hence only

[7] In the BPS solution we only impose two conditions on the parameters **(3.18, 3.20)**, so one may expect that only two chemical potentials are fixed. However, a consistent BPS partition function can be obtained only via a specific limit of all chemical potentials, which enforces the constraint **(3.19)** and allows to have states with non-zero statistical weight despite $\beta \to \infty$ [62].

two charges are independent. Indeed, a non-linear relation for the conserved charges can be found and is given by:

$$(Q^\star)^3+\frac{2\pi}{g^3}J_1^\star J_2^\star=\left(3Q^\star+\frac{\pi}{2g^3}\right)\left(3(Q^\star)^2-\frac{\pi}{g^3}(J_1^\star+J_2^\star)\right), \tag{3.26}$$

as anticipated in the introduction, this condition will turn out to be equivalent to the area product formula, when considered in the BPS limit. In anticipation of the explicit check of this fact, notice that in the RHS of Eq. **(3.26)** appears the BPS entropy squared $(S^\star)^2$. We will return to this important point later.

3.1.3 BPS limit of black hole thermodynamics

In this section, we will give a review on how to get a consistent BPS thermodynamics, using the complexified family of supersymmetric and Euclidean solutions introduced by [8]. As anticipated in the introduction, this will also provide the basis to *derive* the extremization principle.

As a starting point, notice that the temperature vanishes and $\beta\to\infty$ in BPS solutions due to extremality. This makes quantities like I **(3.16)** diverge making the thermodynamics in the BPS solution apparently ill-defined. Notice however, that the grand-potential $T\,I$ should remain finite, suggesting that it is possible to define a consistent thermodynamics in the extremal limit, where a quantum statistical relation should still hold.

However, remember that BPS solutions also require to impose supersymmetry. This makes all the chemical potentials take a fixed value in the BPS solution **(3.22)**. This means that if we want to study the grand-canonical thermal ensemble for the BPS black hole, we cannot use the chemical potentials **(3.22)** as thermodynamic variables.

A possible solution for these problems, and especially for the triviality of the chemical potentials **(3.22)**, has been explored by some authors [8, 62]. The key idea is to consider the BPS solution via a *limiting* procedure along a specific trajectory in the parameter space, this is usually chosen to be non extremal so that to avoid the divergence of β but preserving one of the two conditions which define the BPS solution. For example [8] considered a trajectory preserving supersymmetry, while [62] considered a trajectory preserving the condition $q=q_\star$ **(3.18)**, parametrized via a parameter μ such that $m=m_\star+\mu$, so that the BPS solution is obtained in the $\mu\to 0$ limit ($q_\star$ and $m_\star$ are the BPS parameters obtained by **(3.18, 3.20)**). As was pointed out in [8] there are actually infinite possible trajectories that one can consider, because the BPS solution is obtained by imposing two independent conditions.

By doing things carefully, one is able to define a non trivial thermodynamics for the BPS solution in this limit, and particularly to define some modified chemical potentials that remain non-trivial in the BPS limit, this also allows to remove any possible divergences caused by β.

As anticipated, we will follow [8]. We first need to consider supersymmetry alone, which means that we must impose the constraint, setting $g=1$:

$$q=\frac{m}{1+a+b}\,, \tag{3.27}$$

it is convenient to parametrize the supersymmetric trajectory using as a parameter the outer horizon radius r_+, this can be done by trading m with r_+ using Eq. **(3.60)**, obtaining a relation $m=m(r_+)$, inserting this relation in the supersymmetric constraint **(3.27)** one gets a quadratic equation in q with solutions:

$$q=-ab+(1+a+b)r_+^2\pm ir_+(r_+^2-r^{\star 2})\,, \tag{3.28}$$

remember that $r_\star^2=a+b+ab$, is the BPS value of the outer horizon radius.

In [8] it was chosen to keep r_+ positive and real, then one must necessarily promote q to a complex parameter, unless one is exactly considering the BPS solution[8] $r_+ = r^\star$. This happens because in the pure supersymmetric case, positive real solutions of the $\Delta_r = 0$ condition do not exist, and hence there is not an outer horizon. This will be explicitly checked later.

A complex q (but also m) parameter implies that also the metric and the gauge field are complex, so effectively one is considering a complexified family of solutions parametrized by the real parameter r_+. This would not be allowed in Lorentzian signature, however, one finds that these solutions formally satisfy the requirement of preserving supersymmetry (a killing spinor exists also in this complexified background) provided that one rotates to Euclidean signature [8]. These solutions are clearly unphysical, as one realizes by considering the Lorentzian solution, but are nevertheless useful as they reduce to the physical BPS solution in the $r_+ \to r^\star$ limit, in such a way that one is able to obtain a consistent thermodynamics in the limit. Moreover, some relevant quantities will take a particularly simple form if supersymmetry is first imposed.

Moving on, the next step is to rewrite the chemical potentials and the conserved charges of the supersymmetric solution in terms of the (a, b, r_+) parameters:

$$\Omega_1 = \frac{(1 \pm ir_+)(r^{\star 2} \mp iar_+)}{(a \pm ir_+)(r^{\star 2} \mp ir_+)}, \qquad \Omega_2 = \frac{(1 \pm ir_+)(r^{\star 2} \mp ibr_+)}{(b \pm ir_+)(r^{\star 2} \mp ir_+)}, \qquad \Phi = \frac{3}{2}\frac{r_+(r_+ \mp i)}{r^{\star 2} \mp ir_+},$$

$$T = \frac{(r^{\star 2} - r_+^2)[2(1+a+b)r_+ \mp i(r^{\star 2} - 3r_+^2)]}{-2\pi(a \pm ir_-)(b \pm ir_-)(r^{\star 2} \mp ir_-)}, \tag{3.29}$$

the supersymmetric chemical potentials of the outer horizon are now complex, the sign ambiguity is related to the one in Eq. **(3.28)**. When $r_+ \to r^\star$, the chemical potentials become real and take the fixed values **(3.22)**.

Similarly, the conserved charges and entropy take the form:

$$\begin{aligned}
J_1 &= \frac{\pi(2a+b+ab)}{4(1-a)(1-a^2)(1-b^2)}\left[-ab + (1+a+b)r_+^2 \mp ir_+(r^{\star 2} - r_+^2)\right],\\
J_2 &= \frac{\pi(a+2b+ab)}{4(1-b)(1-a^2)(1-b^2)}\left[-ab + (1+a+b)r_+^2 \mp ir_+(r^{\star 2} - r_+^2)\right],\\
Q &= \frac{\pi}{2(1-a^2)(1-b^2)}\left[-ab + (1+a+b)r_+^2 \mp ir_+(r^{\star 2} - r_+^2)\right],\\
E &= J_1 + J_2 + \frac{3}{2}Q, \qquad S = \frac{\pi^2(a \pm ir_+)(b \pm ir_+)(\mp ir^{\star 2} - r_+)}{2(1-a^2)(1-b^2)},
\end{aligned} \tag{3.30}$$

again, the charges are complex unless $r_+ = r^\star$ in this case they take the BPS value **(3.24)**.

Finally, one can show that the chemical potentials in Eqs. **(3.29)** satisfy the following constraint:

$$\beta(1 + \Omega_1 + \Omega_2 - 2\Phi) = \mp 2\pi i, \tag{3.31}$$

this is one of the results anticipated in the introduction **(1.1)**.

Eq. **(3.31)** is a consequence of the fact that we are reducing the number of independent parameters by imposing the supersymmetry constraint **(3.27)**, this is also reflected in the chemical potentials, which are not all independent. The same is true for the charges Eq. **(3.19)**. If one chooses different trajectories that satisfy different conditions one would still find some conditions that would be different from **(3.31, 3.19)** (see [8]).

Condition **(3.31)** appears frequently in the discussion of supersymmetric black holes, and in [8] it was shown to be related to a regularity condition of the killing spinor near the outer horizon in Euclidean signature.

[8] As we are using r_+ as parameter for the supersymmetric trajectory, then in the limit $r_+ \to r^\star$ one obtains the BPS solution where all physical quantities *must be real*.

New chemical potentials

We have not solved the problem of finding a set of suitable chemical potentials, that allows to non-trivially describe the thermodynamics in the BPS limit, yet.

Following [8, 62], these quantities are given by:

$$\omega_1 = \beta(\Omega_1 - \Omega_1^\star), \qquad \omega_2 = \beta(\Omega_2 - \Omega_2^\star), \qquad \phi = \beta(\Phi - \Phi^\star),$$

$$\omega_1 + \omega_2 - 2\phi = \mp 2\pi i, \tag{3.32}$$

one can explicitly check that, in the BPS limit $r_+ \to r^\star$ equivalent to $T \to 0$, the quantities in Eq. **(3.32)** take a non-trivial value in terms of the parameters (a, b) [9]:

$$\omega_i^\star(a,b) = \lim_{r_+ \to r^\star} \omega_i(r_+, a, b), \qquad \phi^\star(a,b) = \lim_{r_+ \to r^\star} \phi(r_+, a, b), \qquad \omega_1^\star + \omega_2^\star - 2\phi^\star = \mp 2\pi i, \tag{3.33}$$

the explicit expressions are not important and can be found in [8].

Remarkably, using the new variables **(3.32)** the supersymmetric Euclidean on-shell action I can be rewritten as:

$$I = \frac{2\pi}{27}\frac{\phi^3}{\omega_1\omega_2}, \tag{3.34}$$

this is the supersymmetric Euclidean on-shell action appearing in the entropy function **(1.2)**.

The rather simple and *homogeneous* expression **(3.34)** is a peculiarity of having chosen to preserve supersymmetry. If instead one considers a different trajectory to define the BPS limit, one would generally get a more complicated expression for the Euclidean action I which cannot be easily rewritten in terms of the variables (ω_i, ϕ), this will turn out to be very important later.

In terms of the new variables **(3.33)**, the QSR can be rewritten as follows:

$$I = -S - \omega_1 J_1 - \omega_2 J_2 - \phi Q. \tag{3.35}$$

Notice that the dependence on β has disappeared, and one can safely take the extremal limit $T \to 0$ in the above quantities. Moreover, in this limit, the quantities **(3.32)** take a non-trivial value in terms of the (a, b) parameters, which means that we can obtain a well defined and non-trivial BPS on-shell action I and QSR. Essentially Eqs. **(3.34, 3.35)** still holds when considering the "star" BPS quantities.

It has been argued in [8] that the quantities in Eq. **(3.33)** can be interpreted as the chemical potentials associated with the J_i, Q charges, once β is interpreted as the conjugate variable to the charge associated with the supersymmetric Hamiltonian $\{\mathcal{Q}, \bar{\mathcal{Q}}\} = E - J_1 - J_2 - \frac{3}{2}Q$.

This interpretation can be demonstrated by performing the extremization principle, which consists in evaluating the Legendre transform of the action I (which is interpreted as the grand-potential) with respect to the $(\omega_i \phi)$ variables, in order to obtain the entropy as a function of the charges $S(J_i, Q)$. This can formally be done without taking the BPS limit, which instead can be *derived* by imposing reality of the relevant physical quantities, such as the entropy and the charges[10].

In practice, one needs to remember that the chemical potentials, both in the supersymmetric and not extremal and in the BPS solution, are not independent but satisfy the constraint **(3.33)** meaning that they cannot be varied independently. To solve the problem, one can either parametrize (ω_i, ϕ) in terms of three auxiliary

[9]From their definition, it is easy to realize that $(\omega_i^\star, \phi^\star)$ are the sub-leading order terms in the expansion of the chemical potentials (Ω_i, Φ) around the BPS value $\Omega = \Omega_i^\star + \frac{1}{\beta}\omega_i^\star$ and similarly for the electrostatic potential.

[10]Notice that contrary to the charges and entropy, the supersymmetric chemical potentials (ω_i, ϕ) remain complex even in the BPS solution.

variables in such a way that the constraint **(3.33)** is automatically satisfied, and then consider variations of these auxiliary parameters [12], or add a Lagrange multiplier Λ which enforces the constraint [8].

What we have just described is precisely the extremization principle that we anticipated in the introduction. As promised, we are now going to review how it is carried out, following [8].

3.1.4 The extremization principle

From what we have said above, the entropy (as a function of the charges) can be obtained from the following constrained Legendre transform:

$$S(Q, J_1, J_2) = ext_{\{\omega_i,\phi,\Lambda\}}\Big[-I(\omega_i, \phi) - \omega_1\, J_1 - \omega_2\, J_2 - \phi\, Q - \Lambda\big(\omega_1 + \omega_2 - 2\phi \pm 2\pi i\big)\Big], \tag{3.36}$$

notice the appearance of the entropy function as anticipated in the introduction **(1.2)**, plus the Lagrange multiplier Λ enforcing the constraint **(3.33)**. Notice also that we have not taken the BPS limit yet, meaning that we are formally dealing with a complex value of the charges and the entropy.

The extremization is carried out as usual by evaluating the RHS after having expressed the chemical potentials and the Lagrange multiplier in terms of the charges, this is done by solving the extremization equations:

$$\frac{\partial I}{\partial \omega_i} = -J_i - \Lambda, \qquad \frac{\partial I}{\partial \phi} = -Q + 2\Lambda, \qquad \omega_1 + \omega_2 - 2\phi = \mp 2\pi i, \tag{3.37}$$

notice that as the action I **(3.34)** depends only on the chemical potentials, one can easily solve the extremization equations by means of some algebraic manipulations. One would have to do much more calculations if I would have been expressed in terms of the (a, b, r_+) parameters, as it generally happens when a different trajectory is chosen. The expressions of the chemical potentials in terms of the charges and λ can be found in [8] and will not be needed here.

Obviously, one of the equations in **(3.37)** has to be used to find Λ as a function of the charges, it is however much simpler to notice that the following identity holds:

$$\left(\frac{\partial I}{\partial \phi}\right)^3 - 2\pi\left(\frac{\partial I}{\partial \omega_1}\right)\left(\frac{\partial I}{\partial \omega_2}\right) = 0 \quad \longrightarrow \quad (-Q + 2\Lambda)^3 - 2\pi(J_1 + \Lambda)(J_2 + \Lambda) = 0, \tag{3.38}$$

this is the cubic equation for Λ anticipated in the introduction **(1.3)**:

$$\Lambda^3 + p_2\Lambda^2 + p_1\Lambda + p_0 = 0, \tag{3.39}$$

with coefficients:

$$p_0 = -\frac{1}{8}\big(Q^3 + 2\pi J_1 J_2\big), \qquad p_1 = \frac{1}{4}\big(3Q^2 - \pi(J_1 + J_2)\big), \qquad p_2 = -\frac{1}{2}\Big(3Q + \frac{\pi}{2}\Big), \tag{3.40}$$

remember that these coefficients are complex, as the charges in the non extremal solution are also complex.

For the purpose of finding the entropy $S = S(J_1, J_2, Q)$, it is sufficient to only consider the solution of **(3.39)** $\Lambda = \Lambda(J_1, J_2, Q)$, the reason is that the entropy is *proportional* to Λ. To see this, notice that I is an homogeneous function of degree 1 of the chemical potentials, then by using the Euler theorem one finds that:

$$I = \omega_i\,\frac{\partial I}{\partial \omega_i} + \phi\,\frac{\partial I}{\partial \phi} = \omega_i(-J_i - \Lambda) + \phi(-Q + 2\Lambda), \tag{3.41}$$

inserting this into **(3.36)** one finds:

$$S = ext_\Lambda\big[\pm 2\pi i\Lambda\big], \tag{3.42}$$

where Λ has to be taken from the solutions of **(3.39)**.

This discussion also allows us to see what conditions we should impose to obtain a physical solution, particularly from Eq. **(3.42)** we see that a physical *real* value for the entropy can be obtained if Λ is purely imaginary, this is always true if the following condition on the coefficients of the cubic polynomial **(3.39)** is satisfied:

$$p_0 = p_1 p_2, \tag{3.43}$$

moreover, one should require that the charges are all real. In this way, the cubic polynomial factorizes as follows:

$$(\Lambda^2 + p_1)(\Lambda + p_2) = 0, \tag{3.44}$$

meaning that there are two conjugated imaginary roots given by $\Lambda = \pm i\sqrt{p_1}$, choosing the one that produces a positive entropy from Eq. **(3.42)** one finds:

$$S = 2\pi\sqrt{p_1} = \pi\sqrt{3(Q)^2 - \pi(J_1 + J_2)}, \tag{3.45}$$

which is real provided that the charges are real and exactly correspond to the expression for the BPS entropy that we have found before **(3.25)**, with $g = 1$.

Moreover, notice that the condition **(3.43)** which allows to get a real entropy, is exactly the BPS non-linear constraint **(3.26)**, once we explicitly write the p_i coefficients in terms of the charges. This means that in this way we are truly reproducing the entropy of the BPS solution as we have already imposed supersymmetry, and the BPS non-linear constraint is an equivalent way to express the extremality condition **(3.20)**.

Actually, the full BPS thermodynamics is reproduced, as one may check that the expressions for $(\omega_i^\star, \phi^\star)$ **(3.33)** are the saddles that solve the extremization equations **(3.37)**, once we substitute $\Lambda = \Lambda(J_1^\star, J_2^\star, Q^\star)$ and we rewrite the BPS chemical potentials and the BPS charges in terms of the (a, b) parameters.

As pointed out in [8] this is a non-trivial result as one has to consider a specific BPS limit along a trajectory which preserves supersymmetry. Precisely supersymmetry allows to obtain the constraint **(3.32)** and the supersymmetric Euclidean on-shell action **(3.34)** in the simple form that we have shown. This is crucial to obtain the results that we have discussed above.

As anticipated in the introduction the extremization principle has been considered in many other classes of black holes (See [6, 15, 16, 18]).

To conclude the actual review of the known results about this black hole solution (that are relevant for us) we should address the universal area product formula of [32]. However, we find more useful to first introduce the notion of physical and virtual (or "complex") horizons that we will use throughout all our work. This will make it much easier to correctly interpret the universal area product formula, and also to understand our first main results regarding the area product formula and the BPS non-linear constraint **(3.26)**.

3.2 Properties of the general horizons for the single charged, double spinning black hole solution

Let us now consider the properties of the general horizons of the above black hole solutions. We start by first clarifying what these general horizons are, and how we interpret them. Next, we derive a formula for the inner virtual horizon entropy as a function of the charges $S_-^\star(Q^\star, J_1^\star, J_2^\star)$ valid in the BPS limit[11]. This allows us to explicitly show the equivalence of the universal area product formula and the BPS non-linear constraint

[11] It is understood that by "extremality" we always mean vanishing of the outer horizon temperature. To avoid confusion, we clarify right away that we will never consider "extremality" in terms of vanishing of the inner horizons temperature.

(3.26). We then derive a set of thermodynamic quantities for each horizon, discussing the symmetry that relates the horizons. Using this symmetry, we prove the validity of the first law of thermodynamics for each horizon, and give a proposal for a definition of a grand-canonical potential for each horizon thermodynamics. This is equivalently obtained by imposing the validity of the QSR. We will not give an independent derivation of these generalized grand-potentials, accordingly, we will not rely much on them.

Based on these considerations, we conclude that the extremization principle of Sec. **3.1.4**, can be promoted to a *universal* extremization principle. In doing so, we derive the BPS entropies of all horizons *from one calculation*, and provide an independent derivation of the area product formula. In this framework, the equivalence between the area product formula and the BPS non-linear constraint is explained and is related to the factorization condition **(3.43)**.

The results that we are going to prove represent our original contribution in the study of the above black hole solution.

3.2.1 Physical and virtual horizons

Following [32], we will consider as "horizon" any constant r_i hypersurface, where r_i is *any* root of the Δ_r radial function, being it real or complex. Hence the terms "horizon" or "root of Δ_r" can then be used interchangeably.

Formally, any horizon defined in this way is a null hypersurface ($g^{rr} = \frac{\Delta_r}{\rho^2}$ vanishes if evaluated at r_i), and it exists a killing vector $V = \frac{\partial}{\partial x^0}$, in an appropriate coordinate system, whose norm vanishes on all the horizons. To check this one has to consider the $g_{t't'}$ component of the metric (in the appropriate coordinate system where $V = \frac{\partial}{\partial x^0}$), and evaluate it at $r = r_i$. One realizes that $g_{t't'}(r_i) = 0$ whenever it holds $\Delta_r(r_i) = 0$, which is the definition of a general horizon. This is an instance of the symmetry that relates the horizons, the idea is that in many cases it is only needed that $\Delta_r(r_i) = 0$ is satisfied to perform a given calculation, this does not distinguish the horizons, which are then treated symmetrically.

An important comment has to be made now. Despite some horizons are associated with an imaginary, or even complex, root r_i this will generally not represent an obstruction for our calculations, which however should be treated as *formal*. An example has been given above when we stated that virtual horizons can be seen as null hypersurfaces at constant r_i. Despite being formally true (g^{rr} vanishes regardless of the fact that r_i can be complex), this implies to consider the metric for complex values of the radial coordinate, and hence consider an analytically continued solution (equivalently one can keep a real radial coordinate and complexify the parameters of the solution [8]). As already said in footnote **4**, we are interested in considering these complex horizons as formal loci satisfying the $\Delta_r = 0$ condition, and in studying the properties of the generalized (complex) thermodynamics that can be defined for them. We are not going to directly address the problem of giving a physical interpretation for the complex horizons, and simply make some comments about it.

A similar discussion holds also for the various thermodynamic quantities that we are going to derive for them, which should be thought of as *formal* quantities. These quantities can be derived by simply using the properties of the Δ_r function and the symmetry that relates the horizons, starting from those of the outer horizon. However, these quantities are the same that one would derive by studying the complexified metric (e.g. by studying the near horizon metric).

In the following, we will refer to the complex horizons as *virtual* or *inner* horizons, while by *physical* horizons we mean the event or Cauchy horizons associated with real and positive roots of Δ_r.

Let us now quickly discuss what kind of horizons we may expect for the present black hole, we need to study the solutions of $\Delta_r(r) = 0$ (Eq. **(3.4)**) whichis a cubic polynomial in the r^2 variable[12] meaning that there

[12]The metric **(3.3)** depends only on r^2, hence it enjoys a symmetry under exchange of $r \to -r$ with curvature singularity located at

are three horizons. In Fig. **3.1**, the curves $\Delta_r = 0$ are depicted in the (r^2, m) plane for a fixed value of the other parameters, whose specific value does not change the qualitative behaviour shown.

When $m > m_{ext}$ there are two physical horizons associated with the $r_+^2 > r_0^2 \in \mathbb{R}^+$ roots, while a virtual horizon always exists and is associated with the $r_-^2 < 0$ root. For this reason this black hole solution is special, because the virtual horizon appears for a purely imaginary value of r_-, or better, a negative value of r_-^2 (and not for a genuine complex r_-). Notice that when $m = q = 0$, the inner horizon is associated with the root $r_-^2 = -\frac{1}{g^2}$, meaning that an inner horizon is present also in pure AdS_5 space. This can be trivially seen by considering that the metric **(3.3)** reduces to the one of AdS_5 space if one sets $m = q = 0$ and performs a suitable change of coordinates, which essentially removes the dependence on the (a, b) parameters [52]

$$ds^2_{AdS_5} = -(1 + g^2 r^2)dt^2 + \frac{dr^2}{(1 + g^2 r^2)} + r^2 d\Omega_3, \tag{3.46}$$

if we allow for $r^2 < 0$, a virtual horizon appears at $r^2 = -\frac{1}{g^2}$.

If one wants to take seriously the spacetime at $r^2 < 0$, one may first notice that the metric **(3.46)** has signature $(-,+,-,-,-)$ for small values of $|r^2|$ and $(+,-,-,-,-)$ for large values of $|r^2|$. When r^2 becomes negative the signature of the metric changes, this happens because one has to cross the $r^2 = 0$ point where the metric degenerates. One could interpret these two sections as separate spacetimes which both solve the Einstein equations and are connected to different spatial infinities for $r^2 \to +\infty$ or $r^2 \to -\infty$, the latter happens to also contain an event horizon. This observation could be relevant to give a physical meaning to the virtual horizon.

We can immediately realize that the inner horizon is characterized by an imaginary value of the area (or entropy with abuse of terminology). This immediately follows if one generalizes the formula of the entropy for the outer horizon **(3.10)**, for $r_-^2 < 0$. Instead, interpreting the virtual horizon as a genuine hypersurface, one may notice that the determinant of the metric induced on the (θ, ϕ, ψ) coordinates:

$$det(g|_{\theta,\phi,\psi}) = \frac{\left(ab(ab+q) + (a^2+b^2)r^2 + r^4\right)^2 \cos^2\theta \sin^2\theta}{(1-a^2g^2)^2(1-b^2g^2)^2 r^2}, \tag{3.47}$$

is negative as soon as $r^2 < 0$. The area of the inner horizon evaluated at r_- is therefore necessarily imaginary[13]. Notice that at $r_-^2 = -g^{-2}$ the above determinant takes the value $det(g|_{\theta,\phi,\psi}) = -\frac{1}{g^6}\cos^2\theta \sin\theta^2$, which formally is the surface element of a three-sphere of radius $r_- = \frac{i}{g}$.

As a final comment, notice that if one wants to consider the extension of the black hole metric **(3.3)** for $r^2 < 0$ seriously, one has to face the problem that a new curvature singularity appears. Indeed, the function $\rho^2 = r^2 + a^2\cos\theta^2 + b^2\sin\theta^2$ can now vanish, producing some divergences in the metric. This is actually a genuine curvature singularity as one can see that the Ricci scalar diverges when $\rho^2 = 0$. Indeed, taking the trace of the Einstein equations, one gets the following expression for the Ricci scalar:

$$R = \frac{1}{12}F^2 - 20g^2 \quad \longrightarrow \quad R = -\left(20g + 2q^2\frac{\Delta_\theta^2}{\rho^4}\right), \tag{3.48}$$

however, one can show that the inner horizon never touches this new curvature singularity, as it holds:

$$\begin{cases} \rho^2 = 0 \\ r_+^2 + r_0^2 + r_-^2 = -(a^2 + b^2 + g^-2) \end{cases} \longrightarrow \begin{cases} r^2 = -a^2\cos^2\theta - b^2\sin^2\theta \\ r_-^2 < -(a^2+b^2) \end{cases} \longrightarrow \begin{cases} |r^2| < a^2 + b^2 \\ |r_-^2| > a^2 + b^2 \end{cases}. \tag{3.49}$$

$r = 0$ [55], it is then reasonable to consider r^2 rather than r as a more natural radial coordinate.

[13] One may argue that one should consider the root of the modulus of the determinant of the metric so that a real value for the area is re-established. This however breaks the symmetry under exchange of horizon radius r_i.

As we have already said, we will not elaborate more on these considerations.

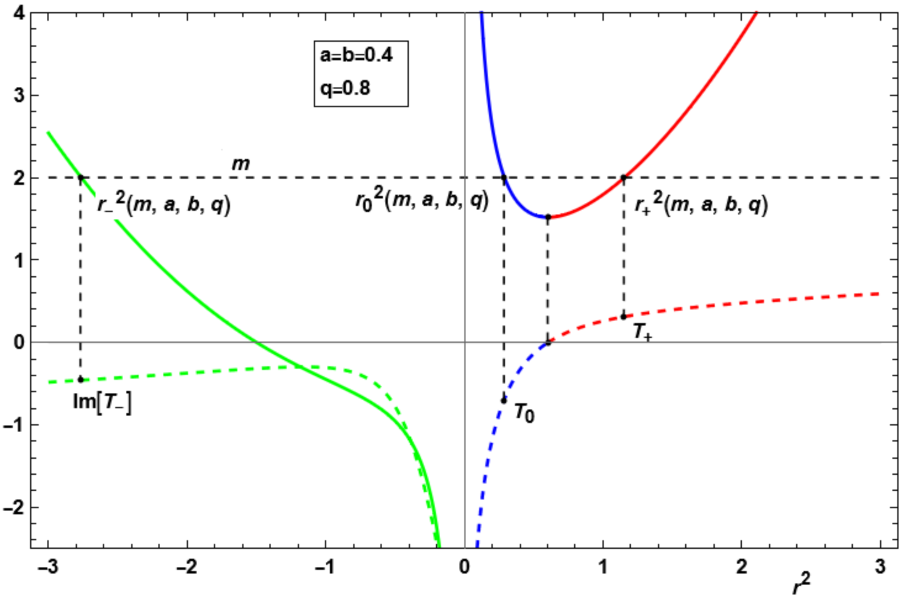

Figure 3.1: The solid lines represent the curves $\Delta_r(r^2; m) = 0$ in the (r^2, m) plane (for them the vertical axes represents m), for a fixed value of the other parameters and $g = 1$. The qualitative behaviour is not influenced much by the specific value of the parameters. The different colors differentiate the three horizons, one can recognize the extremal solution, where $r_0^2 = r_+^2$. Notice that an inner horizon $r_-^2 < 0$ always exists , even when $m = 0$ as we have previously observed. The dashed curves instead represent the behaviour of the temperature $T(r^2, a, b, q)$ Eq. **(3.8)**, where r^2 should be thought of as one of the horizons radii, substituting m via Eq. **(3.60)** (for this curve the vertical axes represents T). When $r^2 < 0$ the imaginary part of T is shown, as the temperature is imaginary in this case. The specific value of the temperature, for a fixed value of the m parameter, of the three horizons is shown, again the colours differentiate the three horizons. Notice that at the extremal configuration, the temperature vanishes as expected, and is negative for the intermediate horizon.

3.2.2 Inner horizon radius and area in the BPS limit

Let us now consider again the supersymmetric and extremal (BPS) solution, we are interested in finding an expression for the area of the inner horizon in terms of only the conserved BPS charges. From now on, we will use "area" and "entropy" interchangeably assuming that $S = \frac{A}{4}$. Remember that the expression for A_- in the general solution is given by Eq. **(3.10)** with r_- substituting r_+. The idea now is to determine r_- in terms of the parameters (a, b, m, q), in this way one gets $A_-(a, b, m, q)$ and then use the definition of the conserved charges to write $A_-(E, J_1, J_2, Q)$. In the BPS solution this can be easily done.

Notice that by rewriting the radial function Δ_r as:

$$r^2\Delta_r(r) = (r^2 + a^2)(r^2 + b^2)(1 + g^2r^2) + q^2 + 2abq - 2mr^2 = g^2 \prod_{i=0}^{2}(r^2 - r_i^2), \tag{3.50}$$

one can obtain a system of equations that determine the horizon radii r_i:

$$\begin{cases} r_+^2 + r_0^2 + r_-^2 = -(a^2 + b^2 + g^{-2}) \\ r_+^2r_0^2 + r_0^2r_-^2 + r_-^2r_+^2 = g^{-2}(a^2 + b^2 + a^2b^2g^2 - 2m) \ , \\ r_+^2r_0^2r_-^2 = -g^{-2}(ab + q)^2 \end{cases} \tag{3.51}$$

in the BPS case, this system can be solved analytically in terms of the two independent parameters (a, b), remember that q and m are fixed by Eqs. **(3.18, 3.20)**, and are real being the BPS solution physical.

The BPS values of the inner and outer horizon radii are given by:

$$r^\star = \sqrt{\frac{1}{g}(a+b+abg)}\ , \qquad r_-^\star = \pm\frac{i}{g}(1+ag+bg)\ , \tag{3.52}$$

inserting this result in the formula for A_-, after some algebraic , one can prove that in the BPS limit it holds:

$$A_-^\star = -\frac{2\pi^2 i}{(1-ag)(1-bg)g^3}\Big[(1+ag+bg)+g(a+b+abg)\Big], \tag{3.53}$$

where we have chosen the plus sign in the expression for $r_-^\star$ **(3.52)**, the other sign choice can be obtained by sending $i \to -i$. As expected $A_-^\star$ is imaginary as a, b, g are real.

Finally, using the definition of $Q^\star$ **(3.24)** one is able to find the following expression for $A_-^\star$:

$$A_-^\star = -4\pi i\left(3Q^\star + \frac{\pi}{2g^3}\right), \tag{3.54}$$

notice that $A_-^\star$ depends only on $Q^\star$ and not on the angular momenta as we might have expected, and receives a constant contribution from the cosmological constant parameter g. This contribution survives in the $Q^\star = 0$ case where $a = b = m = q = 0$ and the metric trivially reduces to the one of static AdS_5 spacetime **(3.46)**, which has a virtual horizon at $r^2 = -g^{-2}$ as we have seen before. A direct computation of the area of this virtual horizon in AdS_5 spacetime gives exactly $A_- = -\frac{2\pi^2 i}{g^3}$, which is the area of an S^3 sphere of radius $r = ig^{-1}$. It is not clear what the meaning of this contribution is, and whether it can be thought of as a contribution from empty AdS_5 spacetime. However, this observation might be an indication that we have to consider the spacetime with $r^2 < 0$ more seriously.

Notice also that the BPS value of the virtual horizon area (or entropy)**(3.54)** is quantized in terms of the electric charge, at the quantum level. This may indicate the possibility of finding a microscopic interpretation for this quantity, which could also explain why it turns out to be independent on the angular momenta $J_1^\star, J_2^\star$.

One could have somehow expected a similar result for $A_-^\star$, indeed, the BPS solution is described by only two independent parameters so it is natural to expect that BPS quantities depend only on (Q, J_1, J_2) up to the BPS non-linear condition. The only non trivial result in this sense is that we are actually able to invert the relations that define the charges, in terms of the other parameters, in a closed manner. This is easily done by noticing that one can obtain the following relations:

$$r^\star = \frac{\pi\sqrt{3(Q^\star)^2 - \pi(J_1^\star + J_2^\star)}}{\pi Q^\star}, \qquad r_-^\star = i\frac{Q^\star(12Q^\star + \pi) - 2\pi(J_1^\star + J_2^\star)}{Q^\star(\pi + 4Q^\star)}, \tag{3.55}$$

from which one can obtain all other quantities in terms of the conserved charges, remembering the dependence of $r^\star$ and $r_-^\star$ on a and b, and from this all the results that we have found above follow.

3.2.3 Universal area product

We will now consider the area product formula obtained by [32], let us first give a review of how it can be derived in the general solution, and then discuss its implications when considering the BPS solution.

Following [32] one notices that the radial function Δ_r can be rewritten as in **(3.50)**, this allows us to calculate products of the form $\prod_i (c^2 - r_i^2) = g^{-2} r^{-2} \Delta(c)$ which appear in the product of the three areas.

To exploit this observation, one rewrites the horizon areas as [32]:

$$A_i = \frac{2\pi^2[(r_i^2 + a^2)(r_i^2 + b^2) + abq]}{\Xi_a \Xi_b r_i} = -\frac{2\pi^2(2m + abqg^2)}{\Xi_a\Xi_b(1 + g^2 r_i^2) r_i}\left[\frac{q(q+ab)}{2m + abqg^2} - r_i^2\right], \tag{3.56}$$

the product of the areas can then be computed in terms of the parameters (a, b, q, m) using Eq. **(3.50)** to trade products of the roots r_i. Then, using the expressions for the conserved charges **(3.5)** one can show that the product of the areas can be rewritten in terms of only the (J_1, J_2, Q) charges, and takes the expression[14]:

$$A_+ A_0 A_- = -i(4\pi)^3 \left(Q^3 + \frac{2\pi}{g^3} J_1 J_2 \right), \tag{3.57}$$

this result holds for the general non supersymmetric and non extremal solution.

Even if we will not use much this fact, it is important to stress that the product of the areas does not depend on the mass of the solution, as we have discussed in the introduction.

Equivalence between the universal area product and the BPS non-linear condition in the BPS limit

We are finally ready to prove one of the main results of this thesis. Consider the universal area product found above **(3.57)**, in the BPS solution the areas of the physical horizons are equal $A_0^\star = A_+^\star = A^\star$ and can be rewritten in terms of only the BPS charges using **(3.25)**. We also found that the inner horizon BPS area $A_-^\star$ is a function of the BPS charges **(3.54)**, if one inserts these informations in the area product formula one finds:

$$-i(4\pi)^3 \left((Q^\star)^3 + \frac{2\pi}{g^3} J_1^\star J_2^\star \right) = A_-^\star (A^\star)^2 = -i(4\pi)^3 \left(3Q^\star + \frac{\pi}{2g^3} \right) \left(3(Q^\star)^2 - \frac{\pi}{g^3} (J_1^\star + J_2^\star) \right), \tag{3.58}$$

the RHS is precisely the BPS non-linear constraint **(3.26)**.

We have shown that considering the universal area product formula in the BPS limit allows to *derive* the BPS non-linear constraint in a different way. One could have somehow expected this result as in the BPS solution the universal area product formula represents a constraint on J_1, J_2, Q, once we rewrite the areas in terms of them. However, we might have found that this constraint was trivially satisfied, for example $A_-^\star (A^\star)^2$ might have been identically equal to the RHS of **(3.57)** after having expressed the BPS horizons areas in terms of the BPS charges and finding the less exciting result $1 = 1$.

For this reason, this is a non trivial result, this is also true because *we had to consider the virtual inner horizon on the same footing as the physical event and Cauchy horizons*, indeed the explicit expression of $A_-^\star$ in terms of the charges is crucial to obtain the equivalence between the area product and the BPS non-linear constraint.

The main task we will pursue now is to understand if this relation between the area product formula and the BPS non-linear constraint can be obtained from an independent, and more fundamental calculation. As we have anticipated, this will lead us to discover that the extremization principle can be promoted to an *universal* extremization principle, which is able to reproduce the BPS thermodynamics of all horizons in a unified way.

It is worth mentioning that the results we have found above are valid in the BPS limit, moreover, part of what we will find in the following depends crucially on the presence of supersymmetry, or better, on the constraint that supersymmetry imposes. However, remember that when supersymmetry is present, there is also a relation of the type (**3.19**), meaning that the mass-independence of certain quantities (for example the BPS entropies) is a less striking result. In any case, the mass-independence implies the quantization of such quantities at the quantum level. It would be interesting to understand if in the BPS solution, one could account for the (quantized) entropies of all horizons (and not only the outermost one) starting from a microscopic description, similarly as it was done for black holes with two horizons or, in this case, by considering the description in terms of the dual field theory on the boundary.

However, we need to remember that the area product formula remains mass-independent also in the non-

[14]The result found in [32] is different with respect to the one we present here due to the different normalization choice regarding the electric charge used in [8].

supersymmetric solution, in this case, this is a non trivial result. It would then be interesting to explore the implications of this fact without having to resort on supersymmetry, which forces all quantities to be mass independent by means of **(3.19)**. The price to pay is that in this case a holographic description (in terms of the boundary field theory) is not well established. A possible way to study this problem could be to consider different BPS limits along different trajectories that do not preserve supersymmetry, as we have discussed in Sec. **3.1.3**. In this way, at least in the BPS limit, one recovers the holographic description and may be able to understand something about the non supersymmetric solutions.

As for the microscopic accounting of all BPS entropies, we may leave a possible answer to these questions for future works.

3.2.4 Thermodynamics quantities and relations for the other horizons

Chemical potentials and first law for the other horizons

We are now going to give a definition for a set of thermodynamic quantities, specifically chemical potentials, associated with each horizon. The conserved charges are instead considered *universal*, hence, the same for all horizons.

Inspired by [30, 31] the idea is that once we get the thermodynamic quantities associated with the outer horizon, the ones associated with the other horizons can be obtained simply by swapping r_+ with r_i. This only regards the chemical potentials $(T,\Omega_1,\Omega_2,\Phi)$ and the area, which are intrinsic properties of the horizons, while the charges (E,J_1,J_2,Q) are the same for all horizons, indeed, they are calculated with suitable integrals at infinity so it is natural to assume their independence with respect to the horizon we consider. Accordingly, one notices from Eqs.**(3.5)** that the conserved charges can be expressed without the need to use the horizons radii r_i, unlike the chemical potentials. This identification of the horizons' properties seems standard in the literature up to a sign difference in the definition of the temperature, which we will discuss briefly.

In our case, the chemical potentials for the outer horizon are given by Eqs. **(3.7)** to **(3.9)**, the dependence on r_+ is already explicit and allows to get the chemical potentials for the other horizons without further work.

We can be more precise about this statement. Take the temperature as an example. It can be calculated from the surface gravity, which is obtained from the following scalar function, where V is the null killing vector on the horizon

$$\kappa(r,\theta;m,a,b,q)=\sqrt{-\frac{1}{2}\nabla_\mu V_\nu\nabla^\mu V^\nu}\,, \tag{3.59}$$

which also depends on the parameters of the black hole solution. The surface gravity of (say) the outer horizon is obtained by taking the limit $r\to r_+(m,a,b,q)$ and one gets a constant depending only on the parameters of the solution.

In our case, we do not know the explicit form of $r_+(m,a,b,q)$, we solve this problem by trading one of the parameters of the solution with r_+ using the defining condition of the horizon position $\Delta_r(r_+)=0$, in this way we trivially get:

$$m=\frac{(r_+^2+a^2)(r_+^2+b^2)(1+g^2r_+^2)+q^2+2abq}{2r_+^2}, \tag{3.60}$$

the surface gravity of the outer horizon is then evaluated by the limit[15]:

$$\kappa_+(r_+,a,b,q)=\lim_{r\to r_+}\kappa\big(r,\theta;\,m(r_+,a,b,q),a,b,q\big), \tag{3.61}$$

[15]Generally, it is sufficient to naively impose the condition **(3.60)** and substitute r with r_+ to get the correct result, but in some cases, like the calculation of the surface gravity, one needs to take the limit with care, for example by setting $\Delta_r=\epsilon$ and $m=(\cdots)-\frac{\epsilon}{2}$ and taking the $\epsilon\to 0$ limit at the end.

providing the result in Eq. **(3.8)**.

Notice that there are two "sources" for the r_+ dependence in the final expression for κ_+ coming from the explicit r dependence in the scalar function Eq. **(3.59)** and from the exchange $m \leftrightarrow r_+$. Turning to the evaluation of the surface gravity for the other horizons, we can still use Eq. **(3.59)** and take the suitable $r \to r_{0;-}$ limit in Eq. **(3.61)**, the dependence on r_+ that still remains in $m(r_+)$ can be traded with a dependence on $r_{0;-}$ by noticing that Eq. **(3.60)** holds regardless of which solution of $\Delta_r(r_i) = 0$ is used.

In this sense, the surface gravity (and consequently the temperature) of each horizon is obtained simply by replacing r_+ with r_0 or r_-.

$$\kappa_i(r_i, a, b, q) = \lim_{r \to r_i} \kappa\big(r, \theta;\, m(r_i, a, b, q), a, b, q\big), \tag{3.62}$$

The same argument can also be made for the other chemical potentials.

Notice that we only have to use the symmetry under the exchange of the horizons and the properties of the Δ_r polynomial, to define the chemical potentials for each horizon. However, if we consider the Cauchy horizon, these quantities are exactly the ones that can be derived from the metric, and the same would still be true for the inner horizon if we regard it as an hypersurface in the space with $r_-^2 < 0$.

One may notice a sign difference in the definition of the temperature with respect to the usual identifications found in literature ([28, 31]) which we will discuss later. The chemical potentials on each horizon are then defined starting from those of the outer horizon as:

$$T_i = T_+(r_+ \to r_i), \qquad \Omega_{1/2;i} = \Omega_{1/2;+}(r_+ \to r_i), \qquad \Phi_i = \Phi_+(r_+ \to r_i), \tag{3.63}$$

Verification of the validity of the first law for each horizon

Having clarified how to evaluate the thermodynamic quantities on each horizon, we are now ready to verify the first law of thermodynamics for each horizon. This can actually be done almost effortlessly as follows.

Let us consider a generic horizon i, first we trade the parameter m with r_i from the definition of the charges **(3.5)** then it is only a matter of using the chain rule to rewrite the first law that we want to check as:

$$\begin{aligned} dE &= T_i\, dS_i + \Omega_{1;i}\, dJ_1 + \Omega_{2;i}\, dJ_2 + \Phi_i\, dQ \\ &= \frac{dE}{dr_i}\, dr_i + \frac{dE}{da}\, da + \frac{dE}{db}\, db + \frac{dE}{dq}\, dq, \end{aligned} \tag{3.64}$$

one immediately realizes that checking the first law for a given horizon is only a matter of relabelling the r_i parameter[16] on the various chemical potentials and on the entropy appearing in the above equation, then the same relabelling can be made also in the charges as m is equivalently expressed in terms of each r_i. This means that checking the first law for each horizon can be made equivalent to a global relabelling of the r_i parameter in the above equations which cannot alter its validity.

From this discussion, one can conclude that the first law is in some sense a *universal* relation valid for each horizon. The key relation that one has to satisfy is $\Delta_r(r) = 0$ which is independent on the particular solution r_i we choose, this independence ultimately reflects on the first law, provided that we identify the chemical potentials of the horizons as done above.

Notice also that these arguments are valid only if we assume as first law for the intermediate and inner horizons the one in Eq. **(3.64)** and not the modified first law that is most commonly found in the literature,

[16]Remember that even if we could have a complex value for r_i, all quantities that are involved depend only on r_i and not $\bar{r}_i$, hence are holomorphic functions of r_i which can be differentiated without problems.

which, for the intermediate horizon would read:

$$dE = -TdS + \cdots \ , \tag{3.65}$$

the reason for this difference is that the temperature for the intermediate horizon (usually referred to as "inner horizon" [28, 30, 31] for black holes with only two physical horizons) should be negative. An intuitive reason is that the intermediate and outer horizons have *mirroring* behaviours. Think for example about what happens to the areas of the two horizons when the black hole approaches extremality [31].

This implies that along the same thermodynamic transformation, dS_+ and dS_0 have an opposite sign, which also implies that the first law of the intermediate horizon is satisfied with a negative value of the temperature, see also [63] for a discussion. In the literature, it is usually chosen to keep the temperature positive, which forces to use the modified first law Eq. **(3.65)**.

In our case, from the definition of the intermediate horizon temperature in Eq. **(3.63)**, one actually finds that T_0 is negative. This can be seen, for example, graphically in Fig. **(3.1)**.

A similar analysis can be done for the inner horizon, associated with $r_-^2 < 0$. In this case, the first law is verified provided that the inner horizon temperature is taken to be purely *imaginary* as can be easily seen from the dependence of T_- on $\frac{1}{r_-}$. Again, insisting on having a positive temperature also for the inner horizon would force us to modify the first law in a similar way as before, introducing an i factor.

Recalling that also S_- is purely imaginary, we can then recover a first law involving purely real quantities. Remember that all other thermodynamic quantities depend on r_-^2 only

$$dE = \pm i|T_-|\, dS_- + \cdots = \pm|T_-|\, d|S_-| + \cdots \ . \tag{3.66}$$

Area product formula and first laws

The first law for the three horizons, allows to get another interesting relation by starting from the universal area product formula **(3.57)**, in particular the fact that the product of the areas is independent of the energy of the BH can be equivalently expressed as a relation involving $S_i T_i$ terms.

To get this relation, one considers the variation of the product of the areas with respect to variations of the conserved charges:

$$d(S_+S_0S_-) = \left(\frac{dS_+}{dE}S_0S_- + S_+\frac{dS_0}{dE}S_- + S_+S_0\frac{dS_-}{dE}\right)dE + \cdots \ , \tag{3.67}$$

using the first laws **(3.64)**, we have the standard relation $\frac{dS_i}{dE} = \frac{1}{T_i}$. The independence of the area product from the energy can then be rewritten as the following condition:

$$0 = \frac{S_0S_-}{T_+} + \frac{S_+S_-}{T_0} + \frac{S_+S_0}{T_-} = \frac{(S_0T_0)(S_-T_-) + (S_+T_+)(S_-T_-) + (S_+T_+)(S_-T_-)}{T_+T_0T_-}, \tag{3.68}$$

where the temperatures are those defined through Eq. **(3.63)**, again using the positive temperatures would require to modify the above relation with suitable signs or i factors.

We want to check if the above relation holds in the present case, to do that we notice that the product T_iS_i is explicitly given by:

$$T_iS_i = \frac{\pi}{4\Xi_a\Xi_b}\frac{r_i^4[1+g^2(2r_+^2+a^2+b^2)] - (ab+q)^2}{r_i^2}, \tag{3.69}$$

then, using Eqs. **(3.51)** which define the horizon positions we can substitute in the above expression $a^2+b^2 = -(r_+^2+r_0^2+r_-^2+g^{-2})$ and $(ab+q)^2 = -g^2r_+^2r_0^2r_-^2$.

After some easy simplifications, one gets the following result

$$T_+S_+ = \frac{g^2\pi}{4\Xi_a\Xi_b}\left(r_+^4 - r_+^2 r_0^2 - r_+^2 r_-^2 + r_0^2 r_-^2\right) = \frac{g^2\pi}{4\Xi_a\Xi_b}\left(r_+^2 - r_0^2\right)\left(r_+^2 - r_-^2\right), \tag{3.70}$$

the relations for the other horizons are easily obtained by appropriately permuting the indices. Using this result, one can easily verify the validity of the condition Eq. **(3.68)** as the numerator of the RHS vanishes identically.

In [30, 63], similar relations appeared for black holes with two horizons for which the area product is independent of the mass, in that case, it reads $T_+S_+ + T_-S_- = 0$.

Notice also that Eqs. **(3.68, 3.70)** agree with the ones shown in [63] (Eqs. **5.34**, **5.35**) for the Wu black hole with three independent electric charges and two angular momenta once we reduce to the solution that we are studying here, after some basic algebraic manipulations.

Grand-canonical potential for each horizon and quantum statistical relations

The *universality* of the thermodynamic properties of each horizon described above, allows us to give a proposal for a definition of the thermodynamic potential for each horizon, this is essentially obtained by imposing that a suitable quantum statistical relation holds.

To do so, one can start from the grand-canonical potential of the outer horizon **(3.16)**, and trivially substitute r_+ with r_i, then our proposal for the *universal* grand-canonical potentials $T_i\, I_i$ is:

$$T_i\, I_i = \frac{\pi}{4\Xi_a\Xi_b}\left[m - g^2(r_i^2 + a^2)(r_i^2 + b^2) - \frac{q^2 r_i^2}{(r_i^2 + a^2)(r_i^2 + b^2) + abq}\right], \tag{3.71}$$

this definition could be equivalently obtained by imposing that the following QSR holds:

$$I_i = -S_i - \Omega_{1,i}\, J_1 - \Omega_{2,i} J_2 - \Phi_i Q, \tag{3.72}$$

once we trade the m parameter with the horizon radius r_i in the definition of the charges.

Notice that with this definition, one should be careful about the sign of the potential for the intermediate horizon, and one would definitely get an imaginary value for the potential of the inner horizon, due to our definition of β_i we have $\beta_0 < 0$ and β_- is imaginary.

It would be useful if we were actually able to get Eq. **(3.71)** from an independent calculation, in the same way as we can get I_+ as the Euclidean on-shell action on the appropriate regular Euclidean section, hoping also to be able to develop a physical interpretation of these new quantities in the process. We have explored the possibility of performing such a calculation, but we have not found a simple way to do it. We leave for a future work the task of finding a better explanation of these generalized grand-potentials I_i. However, we can make some considerations.

To calculate I_0 it may be possible to follow a similar strategy as for I_+. In this case, one should also consider the spacetime between these two horizons and evaluate the Euclidean on-shell action from r_0 to infinity. The reason why one should integrate up to infinity and not up to r_+, for example, can be guessed from the fact that the chemical potentials for r_0 are all relative to their value at infinity, following the definitions in Eq. **(3.63)**, similarly the conserved charges are evaluated from integrals at infinity.

One should also remember that in Euclidean signature, the spacetime "comes to an end" on the horizons, in the sense that the Euclidean time circle shrinks to a point. Integrating from r_0 to infinity then means to consider the Euclidean section contained between r_0 and r_+ plus the one from r_+ to infinity, the on shell action evaluated on the latter gives the known result I_+ in [60], so we should only evaluate the on shell action in the intermediate Euclidean section $I_{0\to+}$ and so:

$$I_0 = I_{0\to+} + I_+, \tag{3.73}$$

this calculation must be done carefully as, despite not having long range divergences as the spacetime is finite between r_0 and r_+, there are two horizons and hence two sets of regularity conditions to impose on the Euclidean time circle to avoid conical singularities. This can be done only if the temperatures and the other chemical potentials of the two horizons are the same[17], which however should not be the case as one can easily see by looking at the behaviour of the temperatures for the two horizons in Fig. **3.1**. Another fact to take in consideration is that, when performing the Wick rotation $t \to -i\tau$ the spacetime between the two horizons has signature $(-,-,+,+,+)$. Clearly, we expect $I_{0\to r_+}$ to vanish in the extremal limit.

The situation is much trickier for I_-, as the inner horizon is located at $r_-^2 < 0$, this means that integrating from r_-^2 to infinity one encounters the black hole (ring) singularity, but also the other singularity appearing at $\rho = 0$. A possible way out could be to analytically continue the solution and smoothly change the integration path in such a way to avoid the singularity, but it is not clear how to do it. A physical interpretation of the nature of the inner horizon would be helpful in this case.

3.2.5 Universality of the extremization principle

We are finally ready to discuss the generalization of the extremization principle as a *universal* extremization principle, able to reproduce the thermodynamics of all horizons in the BPS solution (but also formally in the supersymmetric but not extremal one).

The idea now is that we can repeat the procedure discussed in Sec. **3.1.3** separately for each horizon[18]. By virtue of the symmetry under the exchange of horizons radii r_i as parameters, the results that we have already discussed generalize to all horizons. However, there are some caveats that one has to take care.

The first step is to parametrize the supersymmetric trajectory whose limit is the BPS solution in terms of r_i, this can be done by simply trading r_+ for r_i in the supersymmetric constraint **(3.28)**:

$$q = \frac{m}{1+a+b} = -ab + (1+a+b)r_i^2 \pm i r_i (r_i^2 - r^{\star 2}), \tag{3.74}$$

where $r^\star$ is the BPS value of the outer horizon radius.

We have already seen that if we want to keep r_+ (but also r_0) real, then one is forced to introduce a complex q parameter [8]. However, if we use the inner horizon radius r_- in **(3.74)** one would get a *real* q parameter because r_- is imaginary. One can understand why this happens by looking at Fig. **3.1**. In practice, the condition $\Delta_r(r^2) = 0$ can always be solved for any *real* value of the parameters if one considers $r^2 < 0$, hence, one can always find $r_-^2 < 0$ which produces any given real value of m (also in the supersymmetric solution where m is fixed by q, a, b) this, however, is not always true for the outer or intermediate horizons, for example if one considers the supersymmetric but non-extremal solution.

This produces an apparent contradiction with what we have said so far, as it seems that using r_+ (or r_0) as a parameter and using r_- produces different results (complex or real q parameter,) breaking the symmetry under the exchange of r_i. This is solved if one remembers that the symmetry exists only among the roots of Δ_r and we clearly cannot compare the roots of different Δ_r polynomials. What was done in [8] was to consider a

[17] This is true provided that the regularity conditions on the intermediate horizon are related to its chemical potentials in the same way as it is for the outer horizon. This may be checked by a near horizon limit analysis, but one can expect that this is actually the case due to the symmetry under exchange of r_+ and r_0.

[18] we are going to change perspective in a moment and rather think in terms of *universal* quantities instead of treating each horizon separately. For now, it is useful to proceed in this way.

family of supersymmetric solutions with complex q and m parameters, in such a way that the complexified Δ'_r radial function had two real positive roots r'_+ and r'_0, but in this case the third root r'_- is not purely imaginary but complex, so that also q from Eq. **(3.74)** is complex.

The other possibility is instead to *keep all parameters real*, this forces us to consider complex conjugated r_+ and r_0 variables which produce a real value of q from Eq. **(3.74)**, which is the same that we would obtain using the imaginary root r_-.

For a simple matter of convenience, we are going to to consider the second choice and keep real a, b, q, m parameters, which also allows to maintain a real value of the conserved charges in the supersymmetric and non-extremal solution, the entropies $S_{+,0}$ are still complex. It is important to stress that this is only a matter of choosing a parametrization to describe the supersymmetric trajectory, and accordingly, the complex variables r_+ and r_0 should be thought of as the *parameters describing such supersymmetric trajectories*, or equivalently as "convenient parameters" that can be used instead of m, rather than an analytically continued radial coordinate. We will briefly discuss what constraints one has to impose on these parameters to truly parametrize our supersymmetric solutions. Clearly, the BPS solution is obtained when $r_+ = r_0 \to r^\star$ or equivalently $r_- \to r^\star_-$.

Coming back to our discussion, notice that if we impose q to be real in **(3.74)**, then the requirement $q = \bar{q}$ implies that once we choose the sign convention for r_+, we also fix the sign convention for r_0 to be the opposite. The same happens for r_- because $\sqrt{(r_-^2 - r^{\star 2})^2} = -(r_-^2 - r^{\star 2})$.We choose the negative sign for r_+ as in [8], then Eq. **(3.74)** for the other parameters reads:

$$q = \begin{cases} -ab + (1+a+b)r_+^2 - ir_+(r_+^2 - r^{\star 2}) \\ -ab + (1+a+b)r_{0,-}^2 + ir_{0,-}(r_{0,-}^2 - r^{\star 2}) \end{cases}, \tag{3.75}$$

we can now fully generalize the results of Sec. **3.1.3**, and for each horizon we can parametrize the various supersymmetric thermodynamic quantities in terms of (a, b, r_i), the expressions that one would find are exactly equal to Eqs. **(3.29, 3.30)**, we can then define the "lower case" chemical potentials:

$$\omega_{1,i} = \beta_i(\Omega_{1,i} - 1), \qquad \omega_{2,i} = \beta_i(\Omega_{2,i} - 1), \qquad \phi_i = \beta_i\Big(\Phi_i - \frac{3}{2}\Big), \tag{3.76}$$

$$\omega_{1,i} + \omega_{2,i} - 2\phi_i = \mp 2\pi i, \tag{3.77}$$

the upper sign is related to the outer horizon quantities, while the lower one is related to the intermediate and inner ones.

For the inner horizon, this redefinition of the chemical potentials would not be needed, indeed, one can check that $\Omega^\star_{i,-}, \Phi^\star_-$ do not take fixed values in the BPS solution, as happens for the outer and intermediate horizons, so the interpretation of these new chemical potentials that we have developed in Sec. **3.1.3**, does not hold if we consider the inner horizon.

This redefinition is useful as it allows to rewrite all the grand-potentials I_i as we have already seen **(3.34)**

$$I_i = \frac{2\pi}{27}\frac{\phi_i^3}{\omega_{1,i}\,\omega_{2,i}}, \qquad I_i = -S_i - \omega_{1,i}\,J_1 - \omega_{2,i}\,J_2 - \phi_i\,Q_i, \tag{3.78}$$

one can explicitly check these results, but they are automatically true starting from those of the outer horizon.

Combining Eqs. **(3.77, 3.78)** one may repeat the extremization procedure singularly for each horizon, the calculations that one has to do are exactly the same as those that we have already discussed in Sec. **3.1.4**, and one should recover the BPS thermodynamics for each horizon.

We are now going to show this, but adopting a different point of view. Notice that in the discussion above we have distinguished between each horizon (essentially, we have kept the i indices). However, one does not need to do so, as we have seen that we eventually arrive at expressions that are *universal* **(3.77, 3.78)**. Then, instead of performing three separate extremizations for each horizon, we will define an *universal* supersymmetric grand-potential I defined in terms of *universal* chemical potentials $(\omega_1, \omega_2, \phi)$, these have to be thought of as not associated with a specific horizon and satisfy the relations found above.

We will then perform one *universal* extremization procedure considering the universal quantities given above (this has essentially already been done in Sec. **3.1.4**), and we will be able to reproduce the BPS thermodynamics of all horizons at once. This has the advantage that one does not have to necessarily use the I_i grand-potentials, instead one simply needs to consider the universal quantities.

Let us see how this works.

Proof of the universality of the extremization principle

The starting point is given by the universal relations:

$$I = \frac{2\pi}{27}\frac{\phi^3}{\omega_1\omega_2}, \qquad I = -S - \omega_1\, J - \omega_2\, J_2 - \phi\, Q, \qquad \omega_1 + \omega_2 - 2\phi = \pm 2\pi i\,, \tag{3.79}$$

from the extremization procedure discussed in Sec. **3.1.4**, the entropy is given by:

$$S = ext_\Lambda\big[\pm 2\pi i\Lambda\big], \tag{3.80}$$

where Λ has to be taken from the solutions of the cubic polynomial **(3.39)**:

$$\Lambda^3 + p_2\Lambda^2 + p_1\Lambda + p_0 = 0, \tag{3.81}$$

where the coefficients can be found in **(3.40)**.

Our claim now is that the three roots Λ_i reproduce the thermodynamics of the three horizons, for example, the entropies are given by:

$$S_+ = 2\pi i\Lambda_1, \qquad S_0 = -2\pi i\bar{\Lambda}_1, \qquad S_- = -2\pi i\Lambda_3, \tag{3.82}$$

the sign difference is related to the one in **(3.74)**.

Notice that with our choice of parametrization of the supersymmetric solution, the charges and hence the coefficients of the cubic polynomial are real, meaning that there will always be a real root Λ_3 and two complex conjugated roots Λ_1. It is natural to associate Λ_3 to the inner horizon as it always produces an imaginary entropy as we expect.

Following our claim, the three roots Λ_i of **(3.39)** also determine the supersymmetric chemical potentials of each horizon $(\omega_{1,i}, \omega_{2,i}, \phi_i)$ as, possibly complex, functions of the charges. This can be seen by starting with the universal supersymmetric chemical potentials $(\omega_1, \omega_2, \phi)$, which, upon solving the extremization equations, can be written as functions of the charges and Λ (see [8]). Inserting one of the three roots Λ_i in these functions reproduces the chemical potentials of the horizon associated with the Λ_i root:

$$\omega_{1,i} = \omega_1(Q, J_1, J_2, \Lambda_i), \qquad \omega_{2,i} = \omega_2(Q, J_1, J_2, \Lambda_i), \qquad \phi_i = \phi_i(Q, J_1, J_2, \Lambda_i) \tag{3.83}$$

To verify the validity of our claim (considering for example the entropies), we should compare the expressions that we get from the extremization principle (Eq. **(3.82)**), with the ones that we already know in the supersymmetric **(3.10)** or BPS case **(3.25, 3.54)**. The same check should be performed on the supersymmetric chemical potentials.

It is easier to do these checks in the BPS case, where the entropies and the chemical potentials for each horizon acquire distinct expressions in terms of the parameters (a, b). In this case, one can easily distinguish between the various horizons. Instead, in the supersymmetric but not extremal case, the above quantities are expressed in terms of the r_i parameters. Then, the symmetry under the exchange of r_i does not allow to distinguish the horizons, unless one is able to find an *explicit* expression for the r_i radii in terms of the other parameters in the supersymmetric solution. Luckily, one needs less to distinguish the horizons in the susy case, we will discuss this later.

Notice that by assuming the validity of Eqs. **(3.82)**, the universal area product formula immediately follows:

$$S_+ S_0 S_- = (2\pi i)^3 \Lambda_1 \bar{\Lambda}_1 \Lambda_3 = -(2\pi i)^3 p_0 = -i\pi^3 \Big(Q^3 + 2\pi J_1 J_2 \Big). \tag{3.84}$$

Validity of our claim in the BPS case

As we have already discussed, the BPS solution can be recovered from the extremization principle if one imposes the condition **(3.43)**, which makes the cubic polynomial in Λ factorize as:

$$p_0 = p_1 p_2 \longrightarrow \begin{cases} (\Lambda + p_2)(\Lambda^2 + p_1) = 0 \\ (Q^\star)^3 + 2\pi J_1^\star J_2^\star = \Big(3Q^\star + \frac{\pi}{2} \Big) \Big(3(Q^\star)^2 - \pi(J_1^\star + J_2^\star) \Big) \end{cases} , \tag{3.85}$$

the charges are real, meaning that the two roots $\Lambda = \pm i\sqrt{p_1}$ are purely imaginary. By means of our definition of the entropies S_+ and S_0 **(3.82)** one gets the same positive real value[19] for them as expected.

The other root of the cubic polynomial instead is real and is given by $\Lambda = -p_2$ and has to be associated with the inner horizon entropy. By substituting the explicit expression of the parameters p_i in terms of the charges (Eqs. **(3.40)**), one gets for the three entropies:

$$S_+^\star = S_0^\star = 2\pi\sqrt{p_1} = \pi\sqrt{3(Q^\star)^2 - \pi(J_1^\star + J_2^\star)}, \qquad S_-^\star = 2\pi i p_2 = -\pi i \Big(3Q^\star + \frac{\pi}{2} \Big), \tag{3.86}$$

remarkably, these are precisely the BPS entropies for the various horizons that we have already found before **(3.25, 3.54)**, this result was already known for the outer horizon but is new for the outer horizons.

Notice also that if we define the entropies as in **(3.82)**, we automatically derive the universal area product formula, but we also automatically find that in the BPS solution the area product formula and the BPS non-linear constraint are equivalent due to the factorization condition **(3.85)**.

As anticipated, the extremization principle is able to reproduce all the horizon entropies in the BPS limit. Furthermore, one can explicitly check that the extremization equations Eqs. **(3.37)** for a given horizon are satisfied in the BPS limit if one chooses the correct root Λ_i. This can be quite easily checked as the BPS value of every relevant quantity is easily expressed in terms of (a, b) using Eqs. **(3.24)**, **(3.55)** and **(3.34)**, this proves that the full thermodynamics of each horizon is reproduced.

Validity of our claim in the supersymmetric but non extremal case

One can show that the universality of the extremization principle discussed above also holds in the supersymmetric but not extremal case (which we will refer to as simply "supersymmetric" now on), meaning that the thermodynamic quantities for each horizon are all reproduced at once by the extremization principle, also in this case. These results must be interpreted as *formal* results, as the pure supersymmetric configurations are non physical if one does not impose extremality. However, there are cases where proving the validity of

[19]For this to happen, it is important that the S_+ and S_0 entropies have an opposite sign in their definition in terms of the Λ_1 roots **(3.82)**, this happens precisely due to the sign difference in **(3.74)**

the universality of the extremization principle *along the full trajectory and not only in the BPS limit*, may be physically relevant. This is the case if the BPS solution is reached via a trajectory that always contains physical solutions (that are causally well-behaved and do not allow for naked singularities). For an example, see [62].

In the pure susy case there is no constraint on the coefficients of the cubic polynomial **(3.39)**, and generally there are two complex conjugate roots and one real root Λ_i as we have already discussed.

To compare the definitions of the entropies obtained from the Λ roots with the expected supersymmetric expressions for the entropies **(3.30)**, we should formally find the explicit solutions for the three horizon radii r_i. Indeed, to make the comparison, both **(3.30)** and the Λ_i have to be expressed in terms of the same parameters (r_i, a, b), but the symmetry under exchange of r_i would not allow us to truly differentiate between the horizons, and we could not check which of the three roots Λ_i have to be associated with which horizon.

Luckily, it turns out that it is sufficient to find the range of validity for each horizon radius r_i interpreted as a parameter describing the solution. This information is sufficient to differentiate the various horizons. For example, the simple requirement that r_- should be imaginary and r_+,r_0 complex allows to differentiate these two sets of horizons. To find the range of validity for each parameter r_i we can use the requirement that the relation $m = m(r_i)$ is a bijective function.

Let us start with the inner horizon, one can immediately associate the real Λ_3 root with the inner horizon, as in this way one correctly gets an imaginary value of the inner entropy. Λ_3 is given in terms of the conserved supersymmetric charges (J_1, J_2, Q) via the complicated cubic root formula, and by using Eqs. **(3.30)** one gets Λ_3 in terms of the parameters (r_-, a, b), which can be directly compared with **(3.30)**:

$$S_- = \frac{-i\pi^2(a+ir_-)(b+ir_-)(r^{\star 2}-ir_-)}{2(1-a^2)(1-b^2)}, \tag{3.87}$$

the two expressions do not agree if one allows r_- to take any imaginary value, see Fig. **(3.2)** on the left. The problem is that for small values of m the condition $\Delta_r = 0$ admits three imaginary r_i (or equivalently negative r_i^2) roots[20]. This can be immediately seen in the $q = 0 \to m = 0$ case, where $\Delta_r = 0$ has the three trivial solutions: $r^2 = -1$, $r^2 = -a^2$,$r^2 = -b^2$, or by looking at Fig. **3.2** where the behaviour of the $\Delta_r(r, m) = 0$ supersymmetric curves has been shown for small values of m and for imaginary values of r.

If we associate *all* the pure imaginary roots of $\Delta_r = 0$ with the inner horizon r_-, the function $m(r_-)$ would not be invertible. We can make it invertible by restricting the range of validity for $Im[r_-] = -ir_-$ as shown by the green curve in Fig. **3.2** in the center, where remember that the BPS solution is located at $Im[r_-^\star] = 1 + a + b > 1$ hence on the right branch of the curve.

From a direct computation, we find that:

$$Im[r_-] \in (-\infty, r_C) \cup (r_A, \infty), \quad \text{where:} \quad \begin{cases} r_A = \frac{1}{3}\Big(1+a+b+\sqrt{a^2+b^2-ab-a-b+1}\Big) \\ r_C = \frac{1}{3}\Big(1+a+b-2\sqrt{a^2+b^2-ab-a-b+1}\Big) \end{cases}, \tag{3.88}$$

by taking this into account one finds a perfect agreement between the two definitions of the entropies, as shown in Fig. **(3.2)**.

The same check can be performed for the complex root Λ_1 which is associated with either the outer or intermediate horizon. Let us consider the intermediate horizon, as the $m(r_0)$ relation has the same sign choice as for the inner one **(3.74)**. Again, Λ_1 can be rewritten in terms of the parameters (r_0, a, b) and the expression we get can be compared to the expected supersymmetric entropy of the intermediate horizon, which has exactly the same form as Eq. **(3.87)** (after we substitute r_- for r_0).

[20] This means that for small values of the m parameter, there are three imaginary values that satisfy the relation $m = m(r_i)$.

In this case, one has to take into account only the complex values of r_0 that produce real m, q parameters. In other words, we have to take r_0 such that it solves $\Delta_r = 0$ with real coefficients. The requirement that m, q are real imposes a constraint between the real and imaginary parts of r_0 (that can be obtained by studying $q(r_0)$ **(3.74)**). By taking this into account, r_0 is given in terms of a variable $x \in \mathbb{R}$ as:

$$r_0 = \pm\sqrt{ab + a + b - 2x(1 + a + b) + 3x^2} + ix\,, \quad \longrightarrow \quad q = \frac{m}{1 + a + b} = (a+1-2x)(b+1-2x)(a+b-2x), \tag{3.89}$$

if $x = 0$, and choosing the positive sign in front of the root, one gets the BPS outer horizon radius $r^\star \in \mathbb{R}$, and the BPS value of the q parameter as expected. The sign of q does not depend on the sign of the real part of r_0.

Notice that r_0 becomes purely imaginary if $x \in [r_B, r_A]$, where:

$$r_B = \frac{1}{3}\left(1 + a + b - \sqrt{a^2 + b^2 - ab - a - b + 1}\right), \tag{3.90}$$

because the term under the square root is negative, meaning that $x = Im[r_0]$ only if $x \notin [r_B, r_A]$.

One can see that the function $m(r_0(x))$ **(3.89)** is not injective as a function of x precisely when $x \in [r_B, r_A]$, meaning that in this region, x is not a good parameter to use. Instead of x one can use $Im[r_0] = -ir_0 \in \mathbb{R}$ when $x \in [r_B, r_A] \to r_0 \in i\mathbb{R}$. This is shown in Fig. **3.2** on the right, representing the function:

$$m\Big(r_0\big(Im[r_0]\big)\Big) = \begin{cases} m\big(r_0(x)\big), & \text{when: } x \notin [r_B, r_A] \\ m\big(-ir_0\big), & \text{when: } x \in [r_B, r_A] \end{cases}\,, \tag{3.91}$$

hence, the mass parameter interpreted as a function of $Im[r_0]$. It is demonstrated graphically that this is indeed a bijective function, hence $Im[r_0]$ can be used as a good, real parameter to trade m.

Moreover, we immediately see that in the region where r_0 is purely imaginary (blue line) the plot of $m(r_0)$ exactly matches the corresponding piece that we have found when discussing the inner horizon as a parameter, this confirms the fact that not all purely imaginary roots r_i have to be associated with the inner horizon.

Keeping in mind the above considerations, one finds that the definition of the entropy from Λ_1 and from Eq. **(3.87)** agree, provided that one chooses the right sign for the real part of r_0, which turns out to be positive for the left branch of the curve at $Im[r_0] < r_B$ (this is also required to recover the BPS horizon radius when $Im[r_0] = 0$), and negative for the right branch for $Im[r_0] > r_A$

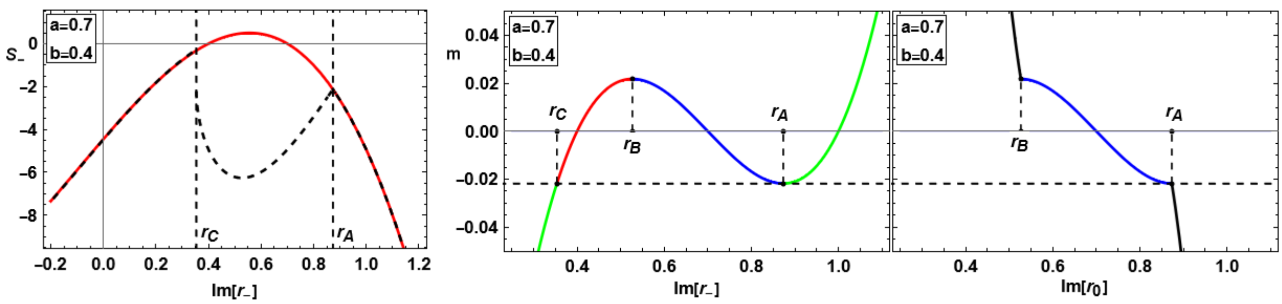

Figure 3.2: (*Left plot:*) Comparison between the definitions of the inner horizon supersymmetric entropy, from Λ_3 (dashed curve), and from Eq. **(3.87)** (solid red curve), the two definitions agree only for the expected values of $Im[r_-]$. (*Central plot:*) behaviour of $m(r)$ when $r \in i\mathbb{R}$. For small values of m there are three imaginary roots r_i, these must be associated with r_- (green curve), r_0 (blue curve) and r_+ (red curve), in order for the three horizons radii to be good parameters. Notice that when $m = 0$ there appear the three expected solutions: $r_- = i$, $r_0 = ai$, $r_+ = bi$.
(*Right plot:*) Plot of Eq. **(3.91)**, the black line being the $m(r_0(x))$ function and the blue line being $m(-ir_0)$. This definition is needed for $m(r_0)$ to be bijective. When $r_0 \in i\mathbb{R}$ (blue line) the $m(r_0)$ curve matches the corresponding piece in the central plot, meaning that those imaginary values of r_i must be associated with the intermediate horizon.

Similar considerations hold true for the outer horizon. When r_0 is imaginary also r_+ must be imaginary and its imaginary part must take values in $[r_C, r_B]$ (red curve in the central plot of Fig. **3.2**.

Discussion

As we have claimed, the thermodynamics of all the horizons can be reproduced from one *universal* extremization principle, which is also able to reproduce the area product formula, the BPS non-linear constraint, and explain the equivalence between these two formulas in the BPS limit. These are interpreted by means of the factorization condition **(3.43)** for the cubic polynomial, which allows to get real entropies. As new results, we have derived the inner and intermediate horizons areas, the universal area product, and its equivalence with the BPS non-linear constraint, *directly from the extremization principle*.

A key role is played by the constraint on the chemical potentials **(3.32)**, this makes the Legendre transform determining the entropy a *constrained Legendre transform*, which can be carried over by introducing the Lagrange multiplier Λ which, being determined by the roots of a cubic polynomial, reproduces all horizons thermodynamics.

The constraint **(3.32)** takes that specific form because we are considering a BPS limit along trajectories that preserve supersymmetry. The advantage of preserving supersymmetry is that the action I in terms of chemical potentials takes a rather simple form **(3.71)**. This makes it much easier to solve the constrained Legendre transform. Moreover, remember that the Euclidean on-shell action I, has been reproduced from the dual field theory only in a supersymmetric setting up to now [8].

Another comment regards the possibility of finding the cubic polynomial **(3.39)** in a different way. Indeed, one could generalize the procedure to derive the product of the areas of Sec. **(3.2.3)**, and derive formulas for the sum and mixed products of the entropies $S_- + S_0 + S_+$, and $S_+S_0 + S_+S_- + S_0S_-$ in terms of the charges. This would allow one to derive a Christodulou-Ruffini-like formula:

$$(S - S_-)(S - S_0)(S - S_+) = 0, \qquad \xrightarrow[\text{BPS}]{} \qquad (S - S_-^\star)(S - S^\star)^2 = 0, \tag{3.92}$$

this relation holds regardless of supersymmetry or extremality and will generally be mass-dependent. Moreover, in the BPS limit, this formula should be equivalent (but not the same due to **(3.82)**) to the cubic polynomial **(3.39)**. A Christodulou-Ruffini formula has been given in [63] for a class of $4D$ black holes, by manipulating the horizon equation. This may be possible to do also in the $5D$ case, but one has to deal with rather complicated expressions of the (m, a, b, q) parameters, which means that rewriting the sum and partial products of the entropies in terms of only the charges would be difficult.

Another comment regards the expression of the BPS inner horizon entropy as a function of the charges. We have already noted that it does not depend on the angular momenta $A_-^\star = -4\pi i\big(3Q^\star + \frac{\pi}{2}\big)$, this can be understood by looking at the structure of the cubic polynomial **(3.39)** and the fact that $A_-^\star$ is related to the p_2 parameter, which is the coefficient of the quadratic term. One can immediately see from the factorized form of the polynomial in Eq. **(3.44)** that the angular momenta can only appear in the p_1 and p_0 coefficients.

This is a consequence of the structure of the Euclidean action, which is related to the fact that this particular black hole solution only allows for two independent angular momenta. Moreover, the electrostatic potential must appear cubed in the Euclidean action (for a reason that we will explain briefly). Then the requirement of having an homogeneous function of degree 1 of the chemical potentials, as it is found in the supersymmetric case, fixes the form of the action, which in turn determines the cubic polynomial.

To see why the electrostatic potential must appear cubed, one has to view this black hole solution as a particular class of solutions of $\mathcal{N} = 2$, $U(1)^n$ gauged supergravity (See [47, 64]), whose general black hole solutions admit n independent charges and two angular momenta. Our case can be obtained by setting the n charges equal.

It has then been conjectured in [13] that the BPS outer horizon entropy, can be obtained by extremizing the following function of the chemical potentials ($I, J, K = \{1, \cdots, n\}$)

$$I = \mu \frac{C_{IJK}\phi^I\phi^J\phi^K}{\omega_1\omega_2}, \tag{3.93}$$

where ϕ^I is the electrostatic potential associated with the $n-th$ gauge field A^I.

We also have to impose the usual constraint[21] $\omega_1 + \omega_2 + \sum_I \phi^I = 1$, here μ is an arbitrary parameter that depends on the normalization of the chemical potentials, while C_{IJK} is the completely symmetric tensor appearing in the Chern-Simons term, which fixes the structure of the whole Lagrangian. The extremization procedure in this case has been carried out in [15], showing that the outer horizon entropy is the same as the one originally found in [65].

The structure of the supersymmetric Euclidean action I in Eq. **(3.93)** is fixed by the supergravity theory properties, in particular the C_{IJK} tensor, which imposes that I depends on cubic monomials of the potentials ϕ^I. This ultimately reflects in the way the entropies depend on the various conserved charges.

As a final comment, notice that *the extremization principle gives a method to derive the thermodynamic properties of all horizons (in the supersymmetric case), starting from the dual field theory via the AdS/CFT correspondence*. The reason is that the universal Euclidean action I, which appears in the extremization principle, can be computed in a supersymmetric setting from the dual CFT on the boundary. This is arguably one of our most important observations, as it may directly connect the properties of all horizons with the dual field theory description.

This concludes the analysis of the single charged black hole solution. In the next sections, we are going to extend the results we found to other black hole solutions, following a similar logic. The next case we are going to consider is the single spinning black hole solution with three electric charges of $U(1)^3$ gauged $D = 5$ supergravity. Before analizing it, we consider the consequences of the universality of the extremization principle for the general solution with three charges and two angular momenta. Then we specialize to the single spinning case, showing that the results obtained via the universal extremization principle are correct.

3.3 Universality of the extremization principle and $U(1)^3$ gauged supergravity general black hole solution

The extremization principle has been derived for a large class of black hole solutions, see [8, 15] for the $U(1)^3$ and $U(1)^n$ gauged cases. It is always able to correctly reproduce the outer horizon BPS entropy, but what about the other horizons? Luckily, generalizing the results found above is quite straightforward. We discuss the case of the general $U(1)^3$ gauged black hole solution, concentrating on the BPS case[22].

The supersymmetric Euclidean action and the supersymmetric constraint now read ([8, 12]):

$$\begin{cases} I = 2\pi \frac{\phi^1\phi^2\phi^3}{\omega_1\omega_2} \\ \omega_1 + \omega_2 - 2\phi^1 - 2\phi^2 - 2\phi^3 = \pm 2\pi i \end{cases}, \tag{3.94}$$

where ϕ^I are the electrostatic potentials associated with the A^I_μ independent gauge fields.

The coefficient μ and the normalization of the chemical potentials has been fixed in such a way that by

[21] Remember that one can always rescale the chemical potentials in order to make the constraint have the same form as the one discussed here.

[22] To study the pure supersymmetric case one would have to find the range of validity of each horizon radius r_i intended as a parameter. This would require studying the properties of the metric, which we are not going to do here.

setting $\phi^1 = \phi^2 = \phi^3 = \frac{1}{3}\phi$, one recovers the previous results for the one charge case (Eqs. **(3.34, 3.31)**). Instead, by setting $\omega_1 = \omega_2 = \frac{1}{2}\omega$ and $\phi_i \to \frac{\phi_i}{2}$ one recovers the case with three charges and one angular momentum discussed in [15] (Eqs. **(2.51, 2.53)**).

The extremization procedure goes through as before [15], and we simply review the main steps. First, one defines the entropy function whose extrema give the entropy as a function of the charges, the supersymmetric constraint is taken into account by adding the usual Lagrange multiplier:

$$S = ext_{\{\omega_i,\phi_i,\Lambda\}}\left[-I - \omega_i\, J_i - \phi^I\, Q_I - \Lambda(\omega_1 + \omega_2 - 2\phi^1 - 2\phi^2 - 2\phi^3 \pm 2\pi i)\right], \tag{3.95}$$

at this stage, one should admit the possibility of having complex charges. Then, by imposing that the physical quantities are real, we can recover the BPS limit.

One should now write the chemical potentials and Λ as functions of the conserved charges using the extremization equations, however Λ can be obtained by exploiting the identity:

$$0 = (2\Lambda - Q_1)(2\Lambda - Q_2)(2\Lambda - Q_3) - 2\pi(\Lambda + J_1)(\Lambda + J_2), \tag{3.96}$$

which is a cubic equation for Λ:

$$\Lambda^3 + p_2\Lambda^2 + p_1\Lambda + p_0 = 0, \tag{3.97}$$

with coefficients:

$$\begin{cases} p_2 = -\frac{1}{2}\Big(Q_1 + Q_2 + Q_3 + \frac{\pi}{2}\Big) \\ p_1 = \frac{1}{4}\Big(Q_1Q_2 + Q_2Q_3 + Q_3Q_1 - \pi(J_1 + J_2)\Big) \\ p_0 = -\frac{1}{8}\Big(Q_1Q_2Q_3 + 2\pi J_1J_2\Big) \end{cases}, \tag{3.98}$$

the homogeneity of degree 1 of the Euclidean action I simplifies the calculation of the entropy, which is given by $S = ext_\Lambda[\pm 2\pi i\Lambda]$, where Λ has to be chosen among the solutions of the cubic polynomial. The claim is that the three roots represent the entropies of the three horizons following Eqs. **(3.82)**.

Let us now specialize to the BPS case, the entropies obtained from the roots of the cubic polynomial are generally complex, in order to get a real value for the outer and intermediate horizons entropies, one has to impose the factorization condition:

$$p_0 = p_1p_2 \to \begin{cases} (\Lambda + p_2)(\Lambda^2 + p_1) = 0 \\ Q_1^\star Q_2^\star Q_3^\star + 2\pi J_1^\star J_2^\star = \Big(Q_1^\star + Q_2^\star + Q_3^\star + \frac{\pi}{2}\Big)\Big(Q_1^\star Q_2^\star + Q_2^\star Q_3^\star + Q_3^\star Q_1^\star - \pi(J_1^\star + J_2^\star)\Big) \end{cases}, \tag{3.99}$$

this condition is the generalization of the BPS non-linear condition to the three charges, two angular momenta case, this indicates that requiring reality for the entropies is equivalent to considering the BPS case.

The three BPS entropies are then given by:

$$\begin{cases} S_+^\star = S_0^\star = 2\pi\sqrt{p_1} = \pi\sqrt{Q_1^\star Q_2^\star + Q_2^\star Q_3^\star + Q_3^\star Q_1^\star - \pi(J_1^\star + J_2^\star)} \\ S_-^\star = 2\pi i p_2 = -\pi i\Big(Q_1^\star + Q_2^\star + Q_3^\star + \frac{\pi}{2}\Big) \end{cases}. \tag{3.100}$$

In the general supersymmetric case, the universal area product is also derived, and from the factorization condition **(3.99)** follows its equivalence with the BPS constraint in the BPS limit.

$$S_+S_0S_- = -(2\pi i)^3 p_0 = -i\pi^3\Big(Q_1^\star Q_2^\star Q_3^\star + 2\pi J_1^\star J_2^\star\Big), \tag{3.101}$$

this result for the area product formula, agrees with the one found in [66].

In the next section **(3.4)**, we verify the validity of these results for the black hole solution with three charges and one angular momenta. In particular, we re-derive the BPS horizons entropies, the BPS non-linear constraint, and the universal area product formula. Then, from the discussion made in this section, the proof of the universality of the extremization principle directly follows.

Remember that the original results that we find, are related to the fact that we consider all horizons rather than just the outer horizon, as was done in the literature.

3.4 Single spinning $U(1)^3$ charged AdS_5 black hole solution

Motivated by the observations of the previous sections, we want to verify if those results hold for the black hole solution with one angular momentum and three unequal charges, arising from the $\mathcal{N} = 2$, $5D$, $U(1)^3$ gauged supergravity theory, also known as STU model.

We follow the same logic as for the single charged solution, and for this reason we are not going to go at the same level of detail. First, we give a review of the general black hole solution of [50], following the discussion of [15]. We then discuss the generalization of the thermodynamics of all the horizons and derive the formulae involving the TS products. Then we obtain the area product formula, which has not been directly derived yet but follows from the analysis of [66], showing that it agrees with the result that we have obtained from the extremization principle of Sec. **3.3**.

Next, we move on studying the BPS solution, deriving the relation $A^\star_-(J^\star, Q^\star_I)$. In the process, we show that the properties of all three BPS horizons, the area product formula, and the BPS non-linear constraint, are the same as those derived from the universal extremization principle, hence proving the validity of our claim also in this case.

3.4.1 Review of the solution

The bosonic part of the Lagrangian is [42, 43](following the conventions of [15, 50]):

$$\mathcal{S} = \frac{1}{16\pi}\int\left[\left(R+4g^2\sum_{I=1}^{3}(X^I)^{-1}-\frac{1}{2}\partial\vec{\phi}^2\right)\star 1-\frac{1}{2}\sum_{I=1}^{3}(X^I)^{-2}F^I\wedge\star F^I-\frac{1}{6}|\epsilon_{IJK}|A^I\wedge F^J\wedge F^k\right], \quad (3.102)$$

The A^I fields are three Abelian gauge fields and the $\vec{\phi} = (\phi^1, \phi^2)$ are the real physical scalars, while the $\{X^I\}$ are the auxiliary scalars given by:

$$X^1 = e^{-\frac{1}{\sqrt{6}}\phi^1-\frac{1}{\sqrt{2}}\phi^2}, \qquad X^2 = e^{-\frac{1}{\sqrt{6}}\phi^1+\frac{1}{\sqrt{2}}\phi^2}, \qquad X^3 = e^{\frac{2}{3}\phi^1}, \quad (3.103)$$

the scalar field manifold is given by the submanifold in the $\{X^I\}$ field space given by $X^1X^2X^3 = 1$.

The black hole solution with three independent electric charges and one angular momentum can be obtained by setting the two independent angular momenta equal. The solution can be expressed in terms of coordinates $(t, r, \theta, \phi, \psi)$, these coordinates are different from the ones used in the previous section ("old") and are related by:

$$\phi = \psi^{\text{old}} - \phi^{\text{old}}, \qquad \psi = \psi^{\text{old}} + \phi^{\text{old}}, \qquad \theta = \frac{1}{2}\theta^{\text{old}}, \quad (3.104)$$

with this set of coordinates the rotation only occurs along the ψ coordinate.

It is convenient to use the following left-invariant 1-forms of S^3:

$$\begin{aligned}\sigma_1 + i\sigma_2 &= e^{-i\psi}(d\theta + i\sin\theta d\phi),\\ \sigma_3 &= d\psi + \cos\theta d\phi\,.\end{aligned} \tag{3.105}$$

The black hole solution is given by:

$$ds_5^2 = \big(H_1H_2H_3\big)^{1/3}\left[-\frac{r^2Y}{f_1}dt^2 + \frac{r^4}{Y}dr^2 + \frac{r^2}{4}(\sigma_1^2+\sigma_2^2) + \frac{f_1}{4r^4H_1H_2H_3}\left(\sigma_3 - \frac{2f_2}{f_1}dt\right)^2\right], \tag{3.106}$$

$$X^I = \frac{(H_1H_2H_3)^{1/3}}{H_I}, \qquad A^I = A^I_t dt + A^I_\psi \sigma_3, \tag{3.107}$$

and it depends on the parameters $(m, a, \delta_1, \delta_2, \delta_3)$. If one sets $\delta_1 = \delta_2 = \delta_3$ this black hole solution reduces to the one with one angular momentum and one electric charge of minimal $\mathcal{N}=2$ supergravity [50], as the three gauge fields A^I become equal, and the three auxiliary scalars X^I become constants.

The functions (H_I, f_1, f_2, f_3, Y) are defined as:

$$\begin{aligned}H_I &= 1 + \frac{2ms_I^2}{r^2},\\ f_1 &= r^6H_1H_2H_3 + 2ma^2r^2 + 4m^2a^2\Big[2(c_1c_2c_3 - s_1s_2s_3)s_1s_2s_3 - s_1^2s_2^2 - s_2^2s_3^2 - s_3^2s_1^2\Big],\\ f_2 &= 2ma\big(c_1c_2c_3 - s_1s_2s_3\big)r^2 + 4m^2as_1s_2s_3,\\ f_3 &= 2ma^2(1+g^2r^2) + 4g^2m^2a^2\Big[2(c_1c_2c_3 - s_1s_2s_3)s_1s_2s_3 - s_1^2s_2^2 - s_2^2s_3^2 - s_3^2s_1^2\Big],\\ Y &= f_3 + g^2r^6H_1H_2H_3 + r^4 - 2mr^2,\end{aligned} \tag{3.108}$$

with $s_I = \sinh\delta_I$, $c_I = \cosh\delta_I$. The components of the A^I fields will not be needed and can be found in [15].

The position of the outer horizon is given by the largest root r_+ of the Y function, this is a Killing horizon associated with the killing vector:

$$V = \partial_t + 2\frac{f_2(r_+)}{f_1(r_+)}\partial_\psi, \tag{3.109}$$

it is trivial to check that this vector has a vanishing norm on the horizon. As anticipated, rotation occurs only along psi in these coordinates.

Thermodynamics

The solution has five conserved charges (E, J, Q_I), in [50] the electric charges and angular momentum were obtained from the usual Komar integrals, while the energy from the integration of the first law. The same results were obtained in the framework of holographic renormalization by [15] up to a constant shift of the conserved energy, this shift is related to the renormalization scheme that is used to regulate the divergent integral defining the energy.

The conserved charges are found to be:

$$E = E_0 + \frac{m\pi}{4}\Big(3 + a^2g^2 + 2s_1^2 + 2s_2^2 + 2s_3^2\Big), \qquad J = \frac{ma\pi}{2}(c_1c_2c_3 - s_1s_2s_3), \qquad Q_I = \frac{m\pi}{2}s_Ic_I, \tag{3.110}$$

E_0 is the constant shift cited above, it can be interpreted as a contribution to the energy from empty AdS_5 space, as it survives when $m = 0$.

Again, the chemical potentials for the black hole thermodynamics are expressed in terms of the outer horizon radius r_+ as:

$$T = \frac{dY(r_+)}{dr}\left(4\pi r_+\sqrt{f_1(r_+)}\right)^{-1}, \quad \Omega = 2\frac{f_2(r_+)}{f_1(r_+)}, \quad \Phi^I = \frac{2m}{r_+^2 H_I(r_+)}\Big(s_I c_I + \frac{1}{2}a\,\Omega(c_I s_J s_K - s_I c_J c_K)\Big), \tag{3.111}$$

where T is obtained as usual from the surface gravity , Ω is the horizon angular velocity relative to a non rotating observer at infinity, and Φ^I are the electrostatic potentials, defined as $\Phi^I = \iota_V A^I|_{r_+}^{\infty}$.

Finally, the Bekenstein-Hawking entropy of the outer horizon is:

$$S = \frac{\pi^2}{2}\sqrt{f_1(r_+)}, \tag{3.112}$$

notice that one has to require $f_1(r_+) > 0$ in order to have a real value for the temperature and entropy.

These quantities satisfy the first law of thermodynamics:

$$dE = T\,dS + \Omega\,dJ + \Phi^I\,Q_I, \tag{3.113}$$

and the quantum statistical relation, in the following form [15]:

$$I = \beta E - S - \beta\,\Omega\,J - \beta\,\Phi^I\,Q_I, \tag{3.114}$$

where the Euclidean on-shell action was calculated in [15] using holographic renormalization. They showed that also in the definition of the action it appears a constant term $I_0 = \beta E_0$, that is interpreted as the on-shell action of the empty AdS_5 space.

From the same considerations that we did in the single charge case, we can interpret the on-shell action I as minus the logarithm of the grand-canonical partition function.

Supersymmetry and setup of the extremization principle

In [11], it was found that the solution preserves one quarter of the original supersymmetry if the parameters satisfy:

$$ag = e^{-(\delta_1+\delta_2+\delta_3)}, \tag{3.115}$$

this condition is equivalent to the following one on the charges:

$$E - E_0 = 2g\,J + g\big(Q_1 + Q_2 + Q_3\big), \tag{3.116}$$

which is obtained by requiring that the Bogomol'nyi matrix has at least one vanishing eigenvalue [11].

For simplicity, we set $g = 1$ now on. Supersymmetric solutions are naturally parametrized by the four parameters (m, δ_I). Equivalently, one can trade the δ_I for the new parameters μ_I defined as [15]:

$$e^{4\delta_1} = \frac{\mu_1(\mu_2+2)(\mu_3+2)}{(\mu_1+2)\mu_2\mu_3}, \quad e^{4\delta_2} = \frac{\mu_2(\mu_3+2)(\mu_1+2)}{(\mu_2+2)\mu_3\mu_1}, \quad e^{4\delta_3} = \frac{\mu_3(\mu_1+2)(\mu_2+2)}{(\mu_3+2)\mu_1\mu_2}, \tag{3.117}$$

in terms of these parameters the susy condition **(3.115)** becomes:

$$a = \left(\frac{\mu_1\mu_2\mu_3}{(\mu_1+2)(\mu_2+2)(\mu_3+2)}\right)^{1/4}. \tag{3.118}$$

In anticipation of performing the extremization principle, we want to parametrize the supersymmetric solutions with the r_+ parameter by trading m using the $Y=0$ condition[23]. In doing this, one faces the problem that Y is of third order in m, making the relation $m(r_+,\delta_I)$ quite complicated. The situation improves if one defines the new radial coordinate [15]:

$$r^2 = R^2 + \frac{m}{m^\star}\Big(r^{\star 2} - \mu_1\Big), \tag{3.119}$$

where $m^\star$ is the BPS value of the m parameter which will be given below. Because of this coordinate redefinition, $Y(R)$ is only quadratic in m, so that one can more easily invert the $Y=0$ relation:

$$Y(R,\delta_I;m) = 0 \quad\longrightarrow\quad m = m(R,\delta_I), \tag{3.120}$$

notice that the symmetry under exchange of horizon radius r_i is not altered by the change of coordinates given above, and in any case, the condition $Y(R)=0$ still does not differentiate between the roots R_i. Therefore, all the results discussed by [15] generalize for all horizons, as we have seen before.

Again, one finds that either m or R_+ are generally complex. In particular, this happens close to the BPS solution $R_+^{\star 2}=\mu_1$, and one has to choose which variable to complexify. For large values of R_+ instead, both m and R_+ can be chosen real, However, when moving towards the BPS solution, one always needs to complexify one of the two parameters. We will not study the pure susy case for this black hole solution, so we will not elaborate more on this.

It was then shown in [15], that the modified supersymmetric chemical potentials (ω,ϕ^I) satisfy the key relation:

$$\omega - \phi^1 - \phi^2 - \phi^3 = \mp 2\pi i, \quad \text{where:} \quad \omega = \beta(\Omega - 2), \quad \phi^I = \beta(\Phi^I - 1), \tag{3.121}$$

where the sign ambiguity is a reflection of the sign ambiguity in the definition of $m(R_+,\delta_I)$ as the root of a quadratic polynomial, it is important to remember that once the sign choice for the outer horizon is fixed, the sign choice for the other two horizons is fixed as well, and one has to take the opposite sign.

The supersymmetric action takes the form:

$$I - I_0 = \pi\frac{\phi^1\phi^2\phi^3}{\omega^2}, \qquad I - I_0 = -S - \omega J - \phi^I Q_I, \tag{3.122}$$

these quantities allow to define a non-trivial thermodynamics in the BPS limit.

Remember that even if we have derived these quantities for the outer horizon, we prefer to think about these as *universal* quantities, not related to a specific horizon yet. The distinction emerges only after having performed the extremization principle. Our results are also true if one performs the three extremizations distinguishing each horizon from the beginning. This implies, however, to rely on the non-fundamental definitions of I_0 and I_-.

This serves as the setup for the extremization principle. Then, the entropy of the outer horizon is given by the following constrained Legendre transform, as shown in [8, 15]:

$$S = ext_{\{\omega,\phi^I,\Lambda\}}\Big[-(I-I_0) - \omega J - \phi^I Q_I - \Lambda\big(\omega - \phi^1 - \phi^2 - \phi^3 \pm 2\pi i\big)\Big]. \tag{3.123}$$

Following our claim, this constrained Legendre transform should also reproduce the entropies of the other horizons. We have already explored the consequences of this assumption in Sec. **3.3**, where we notice that the general relation Eq. **(3.95)** reduces to the one relevant for this black hole solution **(3.123)** once we set $\omega_1 = \omega_2 = \frac{\omega}{2}$ and rescale $\phi^I \to \frac{\phi^I}{2}$ and also $Q_I \to 2Q_I$. This will allow us to compare the results obtained in

[23] Notice that it would be easier to trade a instead of m for r_+, due to the structure of the Y function. However, a is fixed by the susy condition **(3.115)**.

Sec. **3.3** with those that we are going to obtain by studying the solution directly, in the next sections.

The BPS solution

General supersymmetric solutions have causal pathologies, these can be avoided by imposing other conditions on the parameters, which also make the solution extremal.

In particular, one finds that CTCs can be avoided by imposing the constraints [47]:

$$m = m^\star \equiv \frac{1}{2}\sqrt{\mu_1\mu_2\mu_3(\mu 1+2)(\mu_2+2)(\mu_3+2)}\,, \qquad \mu_I > 0,$$

$$4\mu_1\mu_2\mu_3(\mu_1+\mu_2+\mu_3+1) > (\mu_1\mu_2+\mu_2\mu_3+\mu_3\mu_1)^2\,, \tag{3.124}$$

one can see that by imposing also this condition one gets an extremal solution and that the outer horizon position is real and positive $r_\star > 0$.

The BPS value of the charges is given by [47]:

$$\begin{aligned} J^\star &= \frac{\pi}{8}(2\mu_1\mu_2\mu_3+\mu_1\mu_2+\mu_2\mu_3+\mu_3\mu_1), \\ Q_1^\star &= \frac{\pi}{8}(2\mu_1+\mu_1\mu_2+\mu_1\mu_3-\mu_2\mu_3), \end{aligned} \tag{3.125}$$

the other electric charges $Q_I^\star$, can be obtained from the expression of $Q_1^\star$ by cyclic permutations of the indices, while the conserved energy can be obtained by using the susy constraint on the charges **(3.116)**.

The BPS outer horizon entropy reads:

$$\begin{aligned} S^\star &= \frac{\pi^2}{4}\sqrt{4\mu_1\mu_2\mu_3(\mu_1+\mu_2+\mu_3+1)-(\mu_1\mu_2+\mu_2\mu_3+\mu_3\mu_1)^2}\,, \\ &= 2\pi\sqrt{Q_1^\star Q_2^\star+Q_2^\star Q_3^\star+Q_3^\star Q_1^\star-\frac{\pi}{2}J^\star}\,, \end{aligned} \tag{3.126}$$

the chemical potentials take the fixed values $\Omega^\star = 2$ and $\Phi^{I\star} = 1$, justifying the definition of the modified supersymmetric chemical potentials (ω, ϕ^I) given in **(3.121)**.

A non-linear relation between the three BPS electric charges and the angular momentum can also be found in this case. Indeed, after imposing susy and extremality, only three independent parameters are left, meaning that only three conserved charges are independent. The non-linear relation can be found to be:

$$Q_1^\star Q_2^\star Q_3^\star + \frac{\pi}{4}J^{\star 2} = \underbrace{\left(Q_1^\star Q_2^\star+Q_2^\star Q_3^\star+Q_3^\star Q_1^\star-\frac{\pi}{2}J^\star\right)}_{S^{\star 2}/(2\pi)^2}\left(Q_1^\star+Q_2^\star+Q_3^\star+\frac{\pi}{4}\right). \tag{3.127}$$

Notice that the outer horizon BPS entropy **(3.126)** is the same as the one found via the extremization principle **(3.100)** upon suitably manipulating the charges, and the same holds true for the BPS non-linear condition **(3.127)** as compared to **(3.99)**.

Notice that in the RHS of Eq. **(3.127)** it appears the BPS entropy squared $(S^\star)^2$. This has been explained in Sec. **3.3** as being a consequence of the equivalence between the universal area product formula (LHS of **(3.127)**), and the BPS non-linear constraint.

We now proceed to study the properties of the general horizons of this solution. In particular, we want to show that the universal area product formula and the BPS inner horizons entropies, as obtained by studying the solution directly, agree with the definitions given in Sec. **3.3**.

3.4.2 Properties of the general horizons

Let us start by giving a brief analysis of what kind of horizons we may expect for the above black hole solution. The situation is similar to the single charged case (Sec. **3.2.1**) but we will find some qualitative differences.

As Y is a cubic polynomial in r^2, there will be three horizons once more. The usual set of relations that determine the three radii r_i can be found by rewriting:

$$Y(r) = g^2 \prod_{i=1}^{3} (r^2 - r_i^2), \tag{3.128}$$

by matching the coefficients with the expression in Eq. **(3.108)** one gets:

$$\begin{cases} r_+^2 + r_0^2 + r_-^2 = -2m(s_1^2 + s_2^2 + s_3^2) - g^{-2} \\ r_+^2 r_0^2 + r_-^2 r_+^2 + r_-^2 r_+^2 = 4m^2(s_1^2 s_2^2 + s_2^2 s_3^2 + s_3^2 s_1^2) + 2m(a^2 - g^{-2}) \\ r_+^2 r_0^2 r_-^2 = -8m^3 s_1^2 s_2^2 s_3^2 - g^{-2} f_3(r=0) \end{cases} . \tag{3.129}$$

To understand what kind of horizons we may expect, consider the curves $Y(r^2, m) = 0$ in the (r^2, m) plane. In this case, these curves are qualitatively different from those of the single charged solution because Y is cubic in m and not linear, see Fig. **3.3**. In particular, for appropriate fixed values of the (δ_I, a) parameters, there are two extremal configurations $m_{ext,\pm}$ associated with a minimum and maximum value of m. When these two extremal values coincide, the BPS solution is obtained. To see this, we can fix the δ_I parameters, which parametrize the general BPS solution, and follow a trajectory where a approaches its BPS value **(3.115)**. This is shown in Fig. **3.3**, where the black dot represents the BPS solution for the given δ_I parameters.

If one considers any possible value of (δ_I, a), one would find qualitatively different behaviours for the $Y = 0$ curves. However, we shall only consider the ones described above, as they generally admit a BPS limit. In this case, there are two physical horizons located at real and positive radii r_+ and r_0 (the event and Cauchy horizons), and a virtual horizon located at a negative value of r_-^2. Similarly to the single charged black hole solution, an inner horizon with $r_-^2 < 0$ always exists for every value of the parameters[24]. This is not true for the outer and intermediate horizons, for example, in the supersymmetric but not extremal solutions.

Giving a complete analysis of the regularity conditions for this black hole solution is quite involved [50], and goes beyond the scopes of our work. For this reason, we limit ourselves to the points discussed above, which are sufficient for what we are going to show next.

Thermodynamics of the other horizons and TS product formulae

The same reasoning used in the single charged case can also be applied here. The chemical potentials and the entropies of each horizon can be obtained from those of the outer horizon **(3.111, 3.112)** by the replacement: $r_+ \to r_i$. The same holds for the on-shell action, so we could give a proposal for an "on-shell" action, or better, a thermodynamic potential, for each horizon but we will not use these generalized definitions as already discussed.

The conserved charges are the same for each horizon. This is confirmed by looking at the definitions **(3.110)** where one does not need to use r_i as parameter.

Then, proving the validity of the first law and the QSR for each horizon is only a matter of relabelling the r_+ parameter in the expressions associated with the outer horizon. This is true after we have traded one of the

[24]Because $Y(r)$ is a cubic polynomial in r^2, there must be at least one real root, provided that the parameters of the solution are kept real.

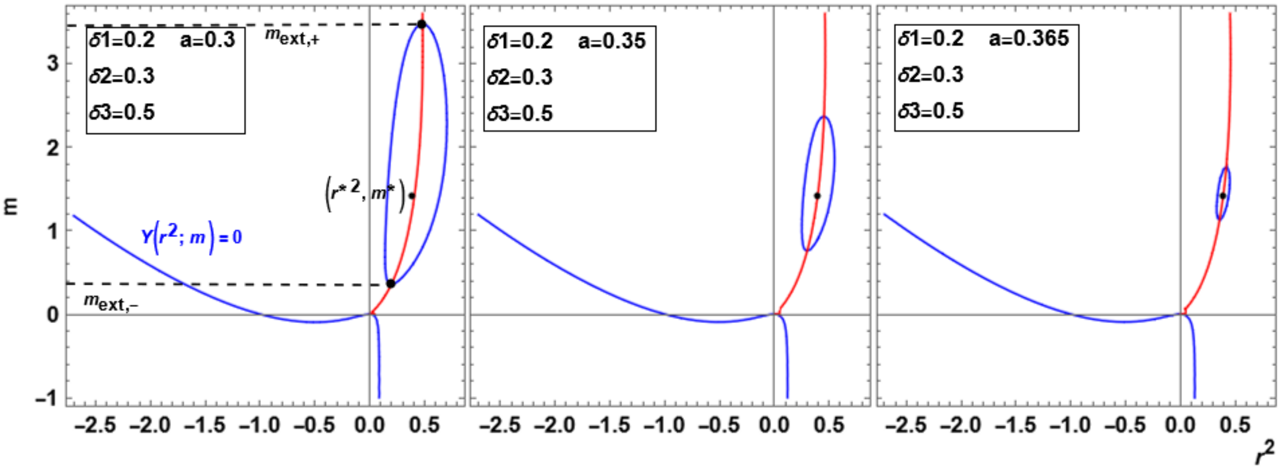

Figure 3.3: The solid (blue) lines represent the curves $Y(r^2;m) = 0$ in the (r^2, m) plane for a fixed value of the δ_I parameters, while the a parameter tends to the BPS value, which in this case is approximately given by $a \approx 0.368$. The BPS solution is represented by the black dot. The solid (red) line represents the locus where $T(r^2;m) = 0$. The qualitative behaviour of the $Y = 0$ curves is different from that of the single charged black hole in Fig. **3.1**. In this case, there are two physical horizons for suitable values of the (δ_I, a) parameters, if m is contained between two extremal values $m_{ext,\pm}$. For both of these extremal configurations, the temperature vanishes. This behaviour does not appear for every possible value of the (δ_I, a) parameters. However, in these cases, we find that the solutions do not admit a BPS limit, or they are characterized by the presence of a naked singularity.

parameters for a given horizon radius r_i using the condition $Y(r) = 0$. $Y(r)$ is linear in[25] a^2, so we can easily trade a in favour of r_i:

$$Y(r; a, m, \delta_I) = 0 \longrightarrow a^2 = a^2(r_i, m, \delta_I). \tag{3.130}$$

With these definitions, we should find a negative temperature for the intermediate horizon, and an imaginary temperature for the inner horizon.

The validity of the first law for each horizon allows us to derive a relation of the form of Eq. **(3.68)**, which expresses the mass-independence of the product of the areas in an equivalent way. Indeed, the products $(T_i)(S_i)$ can be rewritten as in Eq. **(3.69)**:

$$T_i S_i = \frac{\pi}{8}\frac{Y'(r_i)}{r_i} = \frac{\pi g^2}{4}\left(r_i^2 - r_j^2\right)\left(r_i^2 - r_k^2\right), \quad \text{with:} \quad j,k \neq i, \tag{3.131}$$

this is the analogue of Eq. **(5.34)** of [63], and were also found in [15]. Using this result and the first laws, it is trivial to check that it holds:

$$0 = \underbrace{\frac{S_0 S_-}{T_+} + \frac{S_+ S_-}{T_0} + \frac{S_+ S_0}{T_-}}_{d(S_+S_-S_0)/dE} = \frac{(S_0T_0)(S_-T_-) + (S_+T_+)(S_-T_-) + (S_+T_+)(S_-T_-)}{T_+T_0T_-}. \tag{3.132}$$

3.4.3 Area product formula

We now turn our attention to the area product formula. This has not been directly derived yet but can be easily obtained as a special case of [66] and Eq. **(3.101)**. In particular, by setting $J_1 = J_2$ and rescaling $Q_I \to 2Q_I$ in Eq. **(3.101)**, one expects to find the following entropy product formula for this case:

$$S_- S_0 S_+ = -i\left(\frac{2\pi}{g}\right)^3\left(Q_1 Q_2 Q_3 + \frac{\pi}{4}J^2\right), \tag{3.133}$$

[25] for each value of the parameters one finds two solutions $\pm a$ related to R/L handed rotating Black Holes

as expected, the product of the areas does not depend on the mass of the solution, which is exactly what is expressed by Eq. **(3.132)**.

We now verify the validity of Eq. **(3.133)** following the procedure of [32]. From the definitions of Y, f_1, f_3 in Eqs. **(3.108)**, we can rewrite the Y function as follows:

$$Y(r) = f_3 + g^2 r^6 H_1 H_2 H_3 + r^4 - 2mr^2 = g^2 f_1(r) + r^4 + 2m(a^2 - r^2), \tag{3.134}$$

if we evaluate the above equality on a given root r_i of Y we immediately find the following result:

$$g^2 f_1(r_i) = -r_i^4 + 2m(r_i^2 - a^2) = -(c^2 - r_i^2)(d^2 - r_i^2), \tag{3.135}$$

where, in the second equality we have rewritten $g^2 f_1(r_i)$, which is a quadratic polynomial in r_i^2, in terms of its roots (c^2, d^2), which then satisfy:

$$\begin{cases} c^2\, d^2 = 2ma^2 \\ c^2 + d^2 = 2m \end{cases} , \qquad \begin{cases} Y(c) = g^2 f_1(c) + c^4 - 2m(c^2 - a^2) = g^2 f_1(c) \\ Y(d) = g^2 f_1(d) + d^4 - 2m(d^2 - a^2) = g^2 f_1(d) \end{cases} , \tag{3.136}$$

the second set of equations is obtained because (c^2, d^2) are the roots of the polynomial that appears in the definition of $Y(r)$, if we use **(3.134)**. Eq. **(3.135)** allows us to rewrite the entropy of a general horizon as:

$$S_i = \frac{\pi^2}{2}\sqrt{f_1(r_i)} = \frac{\pi^2}{2g}\sqrt{-(c^2 - r_i^2)(d^2 - r_i^2)}\ , \tag{3.137}$$

therefore:

$$S_- S_0 S_+ = -\frac{\pi^6 i}{8g^3}\prod_i (c^2 - r_i^2)^{1/2}(d^2 - r_i^2)^{1/2} = -\frac{\pi^6 i}{8g^5}\sqrt{Y(c)Y(d)} = -\frac{\pi^6 i}{8g^3}\sqrt{f_1(c) f_1(d)}\,, \tag{3.138}$$

where we have used Eqs. **(3.128, 3.136)**.

The expression obtained in this way is rather complicated, but it can be explicitly checked that it agrees with the expected formula in terms of charges **(3.133)**. To see this, it is convenient to trade the (m, a) parameters for the (c^2, d^2) in the definition of the charges. This can be done easily using Eqs. **(3.136)**.

3.4.4 Independent derivation of $S_-^\star$

The final step to prove the universality of the extremization principle, is the derivation of $S_-^\star$ by directly studying the black hole solution. We should recover the result in Eq. **(3.100)**, once we put $J_1^\star = J_2^\star$.

We need to find the BPS value of the inner horizon $r_-^\star$. First, it is convenient to find the expression of the s_I parameters in terms of the μ_I. After a bit of algebra, we find:

$$s_1^2 = \frac{1}{4}\left(\sqrt{\frac{\mu_1(\mu_2+2)(\mu_3+2)}{(\mu_1+2)\mu_2\mu_3}} + \sqrt{\frac{(\mu_1+2)\mu_2\mu_3}{\mu_1(\mu_2+2)(\mu_3+2)}} - 2\right), \tag{3.139}$$

the RHS is always positive because of the condition **(3.124)**. The other s_I parameters are found via obvious permutations of the indices.

Then, using the known value of the outer horizon BPS radius:

$$r^{\star 2} = \frac{1}{2}\left(\sqrt{\mu_1\mu_2\mu_3(\mu_1+2)(\mu_2+2)(\mu_3+2)} - \mu_1\mu_2\mu_3 - \mu_1\mu_2 - \mu_2\mu_3 - \mu_3\mu_1\right), \tag{3.140}$$

and exploiting the first of Eqs. **(3.129)**, one is able to derive the following expression for the inner horizon radius:

$$r_-^{\star 2} = \frac{1}{2}\left(\sqrt{\mu_1\mu_2\mu_3(\mu_1+2)(\mu_2+2)(\mu_3+2)} - \mu_1\mu_2\mu_3 - \mu_1\mu_2 - \mu_2\mu_3 - \mu_3\mu_1 - 2\mu_1 - 2\mu_2 - 2\mu_3 - 2\right), \tag{3.141}$$

Setting $\mu_1 = \mu_2 = \mu_3 = 0$ and thus reducing to empty AdS_5 spacetime, we find $r_-^{\star 2} = -1$ in $g = 1$ units, which is the radius of the virtual horizon appearing in the AdS_5 space discussed in the single charged case.

To evaluate the inner horizon entropy, we can use the same trick that we used in the discussion of the universal area product, this allows to simplify the expression for the Bekenstein-Hawking entropy initially given in Eq. **(3.112)**, to the one in Eq. **(3.137)**:

$$S_i = \frac{\pi^2}{2}\sqrt{-r_i^4 + 2m(r_i^2 - a^2)}, \tag{3.142}$$

Specializing to the inner horizon, using the definition **(3.142)** for the entropy and remembering the BPS value of the a and m parameters in Eqs. **(3.118, 3.124)**, one can show that:

$$\begin{aligned} S_-^\star &= -2\pi^2 i\Big[1 + \mu_1 + \mu_2 + \mu_3 + \frac{1}{2}\big(\mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1\big)\Big], \\ &= -2\pi i\left(Q_1^\star + Q_2^\star + Q_3^\star + \frac{\pi}{4}\right), \end{aligned} \tag{3.143}$$

proving the validity of the result that we have already found from the extremization principle.

As a final test of the validity of the extremization principle's universality, we have checked that the extremization equations are actually solved in the BPS case. From Eq. **(3.123)** the equations that one has to verify are:

$$\frac{\partial(I - I_0)}{\partial\omega} = -2\pi\frac{\phi^1\phi^2\phi^3}{\omega^3} = -J - \Lambda, \qquad \frac{\partial(I - I_0)}{\partial\phi^I} = \pi\frac{\phi^J\phi^K}{\omega^2} = -Q_I + \Lambda, \tag{3.144}$$

if one considers the explicit expression for a given root (in the BPS solution) $\Lambda_i(J^\star, Q_I^\star)$, and the expressions of the chemical potentials (ω_i, ϕ_i^I) that can be obtained from Eqs. **(3.111, 3.121)**, then, the extremization equations are found to hold in the BPS solution, after we express all quantities in terms of only the μ_I parameters.

This concludes the explicit proof of the universality of the extremization principle for the single spinning, triple charged black hole solution. We are now going to turn our attention to a different class of (AdS) black holes in four dimensions.

Chapter 4

Asymptotically AdS_4 black holes in 4D, $\mathcal{N} = 2$ gauged supergravity

In this chapter, we are going to turn our attention to some special cases of the general asymptotically AdS_4 black hole solutions of $U(1)^4$ gauged $\mathcal{N} = 2$, $D = 4$ supergravity.

This supergravity theory arises as a consistent truncation of $SO(8)$ gauged $\mathcal{N} = 8$, $D = 4$ supergravity [4], the procedure reduces the original $SO(8)$ gauge group to its Cartan $U(1)^4$ subgroup. Furthermore, the $\mathcal{N} = 8$ theory can be obtained as a consistent truncation on S^7 of $D = 11$ supergravity on $AdS_4 \times S^7$. By means of the AdS/CFT correspondence [3], the thermodynamics of the black holes that we are going to consider should be reproduced by a dual three-dimensional conformal field theory [6, 67].

The most general black hole solution is described by six conserved charges [68]: the energy E, one angular momentum J, and four electric charges associated with the four $U(1)$ gauge symmetries. We are going to directly consider the solution with two independent charges of [68], and the one with one charge [69] which is also a solution of pure $\mathcal{N} = 2$ supergravity. We are also going to consider the generalization with acceleration and magnetic charge [16, 70] of the latter.

We discuss the three black hole solutions separately, first giving a review of the known properties of the solutions and then generalizing the results that we have found in the previous sections to the current ones. For the black hole solution with acceleration, we also derive the area product formula, which has never been discussed before. We also discuss the consequences of the universality of the extremization principle in the general solution with four charges, starting from the supersymmetric entropy function proposed by [14]. In this case, we will not provide an explicit proof of the universality of the extremization principle.

4.1 Review of the single charged, spinning AdS_4 black hole solution

Let us start discussing the single charged solution of pure $\mathcal{N} = 2$ gauged supergravity. This was originally found in [71], but we will follow the presentation of [69].

The bosonic part of the action for $D = 4$ minimal $\mathcal{N} = 2$ gauged supergravity is given by:

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left(R + 6 - \frac{1}{4} F^2 \right), \tag{4.1}$$

where we have fixed the cosmological constant so that the AdS_4 solution has radius 1 [69].

This theory admits an electrically charged, spinning, asymptotically AdS_4 black hole solution parametrized by three constants (m, a, δ). In Boyer-Lindquist type coordinates (t, r, θ, ϕ), the metric and the graviphoton gauge field describing the solution are given by:

$$ds_4^2 = -\frac{\Delta_r}{W}\left(dt - \frac{a\sin^2\theta}{\Xi}d\phi\right)^2 + W\left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\right) + \frac{\Delta_\theta \sin^2\theta}{W}\left(a dt - \frac{(\tilde{r}^2+a^2)}{\Xi}d\phi\right)^2,$$
$$A = \frac{2m\tilde{r}\sinh 2\delta}{W}\left(dt - \frac{a\sin^2\theta}{\Xi}d\phi\right), \tag{4.2}$$

where the gauge field can always be modified by adding a pure gauge term of the form αdt, which is required when discussing the regularity of the solution in the Euclidean signature.

The functions that appear in the metric and the gauge field are defined as:

$$\tilde{r} = r + 2m\sinh^2\delta, \qquad \Delta_r = r^2 + a^2 - 2mr + \tilde{r}^2(\tilde{r}^2 + a^2), \qquad \Xi = 1 - a^2,$$
$$\Delta_\theta = 1 - a^2\cos^2\theta, \qquad W = \tilde{r}^2 + a^2\cos^2\theta, \tag{4.3}$$

the position of the horizons is given by the roots of the Δ_r quartic polynomial.

We need to assume $a^2 < 1$ so that the Einstein universe on the conformal boundary does not rotate faster than the speed of light [53], and we can assume without loss of generality that all parameters are non negative [69], $a, m, \delta \geq 0$.

The black hole solution is characterized by three conserved charges, its energy E, charge Q and angular momentum J which are given by:

$$E = \frac{m\cosh 2\delta}{\Xi^2}, \qquad Q = \frac{m\sinh 2\delta}{\Xi}, \qquad J = \frac{ma\cosh 2\delta}{\Xi^2}, \tag{4.4}$$

these are all positive, due to our assumptions on the parameters.

The Bekenstein-Hawking entropy of the outer horizon is:

$$S = \frac{\pi(\tilde{r}_+^2 + a^2)}{\Xi}, \tag{4.5}$$

where clearly r_+ is the outer horizon radius, hence the largest real root of the Δ_r polynomial.

The chemical potentials of the outer horizon are given by:

$$T = \frac{\Delta_r'(r_+)}{4\pi(\tilde{r}_+^2 + a^2)}, \qquad \Omega = \frac{a(1+\tilde{r}_+^2)}{\tilde{r}_+^2 + a^2}, \qquad \Phi = \frac{m\tilde{r}_+\sinh 2\delta}{\tilde{r}_+^2 + a^2}, \tag{4.6}$$

where one needs to remember that the angular velocity relevant for the thermodynamics is not directly the one of the horizon Ω_H, obtained by requiring the vanishing of the norm of the killing vector $V = \partial_t + \Omega_H \partial_\phi$ on the horizon, but rather the difference between Ω_H and the angular velocity of the asymptotic observer $\Omega = \Omega_H - \Omega_\infty$ [53]. Similarly, the electrostatic potential is defined as $\Phi = V^\mu A_\mu|_\infty^{r_+}$.

With these definitions, one can easily show that the first law of thermodynamics is satisfied, together with the quantum statistical relation:

$$dE = T\,dS + \Omega\,dJ + \Phi\,dQ, \qquad I = \beta\,E - S - \beta\,\Omega\,J - \beta\,\Phi\,Q, \tag{4.7}$$

where the Euclidean on-shell action is given by:

$$I = \frac{\beta}{2(a^2-1)}\left[r_+^3 + 6ms^2r_+^2 + r_+\left(a^2 + 12m^2s^4\right) + 2ms^2\left(a^2 + 4m^2s^4 - 1\right) - m + \frac{4m^2c^2s^2\tilde{r}_+}{R_+^2 + a^2}\right] \tag{4.8}$$

with s, c used as a shorthand notation for $\sinh\delta$ and $\cosh\delta$. This can be obtained from the more general black hole solution discussed in [15] with two pairwise equal charges, by setting $\delta_1 = \delta_2 = \delta$ in such a way to reduce to the present one charged solution.

4.1.1 Supersymmetric and BPS solution

It can be shown that half of the total $\mathcal{N} = 2$ supersymmetry can be restored by imposing the following constraint on the conserved charges [72] :

$$E = J + Q \;\longleftrightarrow\; \begin{cases} a = \frac{2}{e^{4\delta}-1} \\ e^{\delta} > \sqrt[4]{3} \end{cases} , \tag{4.9}$$

this follows as usual from the requirement that the Bogomol'nyi matrix has a vanishing eigenvalue.

It can be shown [69] that by imposing the supersymmetry constraint **(4.9)** in the definition of the chemical potentials Eqs. **(4.6)** one finds that they satisfy the usual constraint:

$$\beta(1 + \Omega - 2\Phi) = \pm 2\pi i, \tag{4.10}$$

in the spirit of [8], these should be thought of as functions of the (r_+, δ) parameters, where m has been traded for r_+ using the $\Delta_r(r) = 0$ condition. We discuss this in more detail later. This will also allow us to explain the origin of the sign ambiguity in the above relation.

As usual, one defines the modified chemical potentials (ω, ϕ) as:

$$\omega = \beta(\Omega - 1) = \pm\frac{2\pi i(a-1)}{1 + a \pm 2i\tilde{r}_+}, \qquad \phi = \beta(\Phi - 1) = \pm\frac{2\pi i(a \pm i\tilde{r}_+)}{1 + a \pm 2i\tilde{r}_+},$$
$$\omega - 2\phi = \pm 2\pi i, \tag{4.11}$$

these provide the non-trivial intensive parameters describing the thermodynamics of the BPS solution.

The quantum statistical relation and the supersymmetric action for each horizon now take the form:

$$I = S - \omega J - \phi Q, \qquad I = \mp\frac{i}{2}\frac{\phi^2}{\omega} \tag{4.12}$$

from this, one can develop the extremization principle, and obtain the black hole entropy from the constrained Legendre transform:

$$S(Q, J) = ext_{\{\omega,\phi,\Lambda\}}\Big[-I - \omega J - \phi Q + \Lambda(\omega - 2\phi \mp 2\pi i)\Big]. \tag{4.13}$$

It was shown in [69] that this extremization principle correctly reproduces the outer horizon thermodynamics in the supersymmetric solution. The same result can also be obtained from the discussion of [14] after having reduced to the present solution.

Extremality and BPS solution

Pure supersymmetric solutions are characterized by causal pathologies in Lorentzian signature, as usual. Moreover, the function Δ_r does not have positive real zeros. However, one can obtain a regular and supersymmetric solution free from causal pathologies, by imposing also the condition [69]:

$$m = a(1 + a)\sqrt{2 + a} = \frac{2\sqrt{2}e^2\left(e^4 + 1\right)}{\left(e^4 - 1\right)^{5/2}}, \tag{4.14}$$

in this limit, one can show that the Δ_r polynomial has a double real root corresponding to the physical outer horizon, and the black hole is extremal.

In this case, one can check that the outer horizon chemical potentials take a fixed value[1] $\Omega^\star = \Phi^\star = 1$ and $T^\star = 0$.

In the BPS limit, the charges are given by:

$$Q^\star = \frac{\sqrt{2}\sqrt{e^4 - 1}}{e^4 - 3}, \qquad J^\star = \frac{2\sqrt{2}\sqrt{e^4 - 1}}{(e^4 - 3)^2}, \tag{4.15}$$

and they satisfy the following constraint:

$$J^\star = \frac{Q^\star}{2}\left(\sqrt{1 + 4Q^{\star 2}} - 1\right), \tag{4.16}$$

in addition to **(4.9)**.

The Bekenstein Hawking entropy for the outer horizon can be shown to be equal to:

$$S^\star = \frac{\pi}{2}\left(\sqrt{1 + 4Q^{\star 2}} - 1\right). \tag{4.17}$$

One may notice that now the BPS constraint **(4.16)** has a different structure with respect to the ones that we have found in the five-dimensional case. However, the equivalence between the area product formula and the BPS constraint continues to hold, the only difference being that in this case it is not directly evident.

4.2 Properties of the general horizons of the single charged solution

Let us now turn to our original analysis of the properties of the general horizons of this black hole solution. The main task of this section is to show the validity of our claim regarding the *universality* of the extremization principle also for this solution. This universality holds true also in this case.

We also discuss the relation between the universal area product, which we have directly derived, and the BPS condition, showing again that in the BPS limit they are equivalent, provided that one knows the expressions of the BPS entropies of all the horizons.

4.2.1 Physical and virtual horizons

Let us consider the Δ_r polynomial **(4.3)** and rewrite it as:

$$\Delta_r = r^2 + a^2 - 2mr + \tilde{r}^2(\tilde{r}^2 + a^2) = \prod_i (r - r_i), \tag{4.18}$$

where remember that $\tilde{r} = r + 2ms^2$.

As we have already noticed, there are four roots r_i of Δ_r, which satisfy (using r or $\tilde{r}$):

$$\begin{cases} r_1 + r_2 + r_3 + r_4 = -8ms^2 \\ r_1 r_2 + r_1 r_3 + \cdots = 1 + a^2 + 24m^2 s^4 \\ r_1 r_2 r_3 + \cdots = 2m(1 - 2a^2 s^2 - 16m^2 s^6) \\ r_1 r_2 r_3 r_4 = a^2 + 4m^2 s^4 (1 + 4m^2 s^4) \end{cases}, \qquad \begin{cases} \tilde{r}_1 + \tilde{r}_2 + \tilde{r}_3 + \tilde{r}_4 = 0 \\ \tilde{r}_1 \tilde{r}_2 + \tilde{r}_1 \tilde{r}_3 + \cdots = a^2 + 1 \\ \tilde{r}_1 \tilde{r}_2 \tilde{r}_3 + \cdots = 2m(1 + 2s^2) \\ \tilde{r}_1 \tilde{r}_2 \tilde{r}_3 \tilde{r}_4 = a^2 + 4m^2 s^2 (1 + s^2) \end{cases}, \tag{4.19}$$

[1] This justify the definition in Eq. **(4.11)**.

from the first equation, involving the sum of the roots, we conclude that at least one root must be negative. However, it is easy to see that the polynomial $\Delta_r(r)$ is strictly positive if $r < 0$ and we assume all parameters to be non negative, as we did at the beginning. This means that there are no real negative roots. The only possibility is that there are at least two complex conjugated roots whose real part is negative.

This means that there will always be at most two physical horizons associated with the real positive roots (r_0, r_+), and at least two virtual horizons associated with a couple of complex conjugated roots $(r_-, \bar{r}_-)$. Giving a physical meaning to these complex roots is even more difficult than it would be in the five-dimensional case, where the virtual horizon is associated with the real root $r_-^2 < 0$. However, we will see that some of the observations that we have made for the inner horizon in relation to the spacetime with $r^2 < 0$ are also valid here.

We could argue, for example, that the virtual horizons are somehow associated with the virtual horizon that appears in the empty AdS_4 spacetime when we allow r^2 to be negative. Indeed, notice that when we set $m = \delta = 0$, so that to recover empty AdS_4 spacetime[2], the polynomial Δ_r reduces to a quadratic in the r^2 coordinate, with $r_-^2 = -1$ and $r^2 = -a$ as zeroes. The latter can be removed via the change of coordinates [52] that removes the a parameter and puts the metric **(4.2)** into the standard form of static AdS_4 spacetime. Instead, $r_-^2 = -1$ is the genuine virtual horizon (remember that we have set the AdS_4 radius to 1), notice that it is associated with the complex conjugate pair of roots $r_- = \pm i$.

As soon as we let $m > 0$, the polynomial **(4.18)** becomes of quartic order in r, and one can check that already at first order in m the root r_-^2 splits into two complex conjugate solutions of $\Delta_r = 0$, given by:

$$r_- = \pm i - \frac{m\big(2s^2(1-a^2)+1\big)}{1-a^2} + O(m^2), \tag{4.20}$$

notice that the real part of r_- is negative as expected.

Similarly to the five-dimensional case, the virtual horizons reduce to the one of pure AdS_4 space in the $m \to 0$ limit. The difference being that in the five-dimensional case, the polynomial determining the horizon radii is always cubic in r^2, allowing us to use it as a coordinate. In this way, even if there are two complex conjugated horizons also in the five-dimensional case, at $r_- = \pm i(\cdots)$, we can regard them as the same horizon, related to the same value of r_-^2.

From a more detailed analysis of the roots of the Δ_r polynomial, one realizes that there are two physical horizons if $m \in [m_{ext,-}, m_{ext+}]$ meaning that there are two extremal values for the m parameter. When m takes one of the extremal values, the solution is extremal. This is similar to what we saw for the five-dimensional black hole solution with three independent charges in Sec. **3.4.2**, and happens because the $\Delta_r(r)$ polynomial is of fourth order in m.

This can be seen more easily by passing to the $\tilde{r}$ radial coordinate, which makes $\Delta_r(\tilde{r})$ only quadratic in m. The condition $\Delta_r(\tilde{r}; m) = 0$ can be equivalently expressed by considering the relation $m = m(\tilde{r})$ given by:

$$m(\tilde{r}) = \frac{\tilde{r}^2\left(2s^2+1\right) \pm \sqrt{\tilde{r}^2\left(2s^2+1\right)^2 - 4s^2\left(s^2+1\right)\left(\tilde{r}^2+1\right)\left(a^2+\tilde{r}^2\right)}}{4s^2\left(s^2+1\right)}, \tag{4.21}$$

plotting these curves, one finds the behaviour shown in Fig. **4.1** which is similar to what we have seen in Fig. **3.3**. where one can also see that for $m = m_{extr,\pm}$ the solution is extremal. For other values of m there are always four complex roots if we keep the parameters of the solution real.

As a final note, one can explicitly check that the BPS solution can be equivalently obtained by simultane-

[2]As previously discussed in the five-dimensional case, by setting $m = \delta = 0$ one gets the metric of AdS_4 spacetime but in non-standard coordinates due to the presence of the a parameter (and hence rotation). The standard metric of the AdS_4 spacetime can be obtained via a suitable change of coordinates [52].

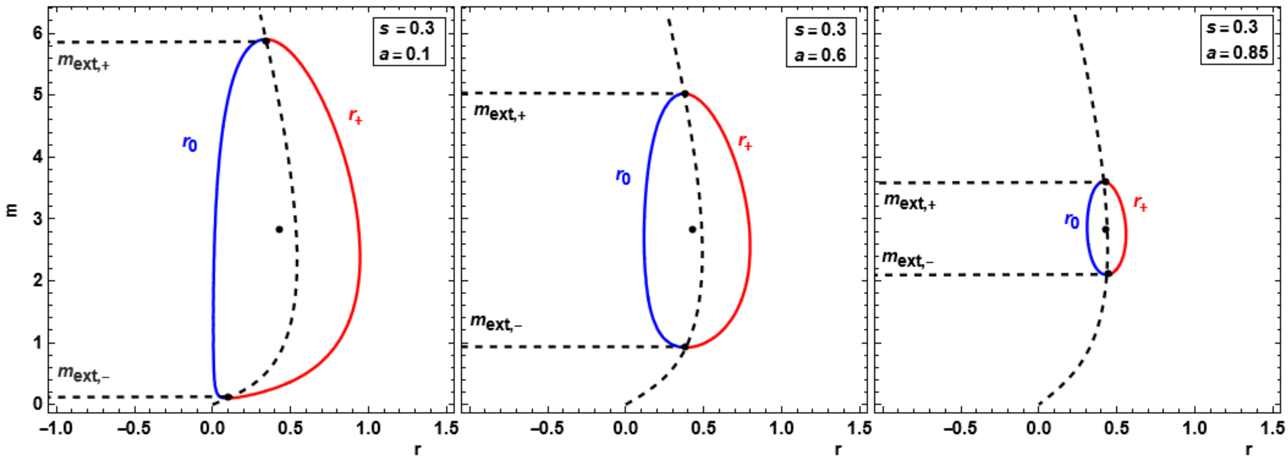

Figure 4.1: Behaviour of the $\Delta_r(r;m)=0$ curves (solid line) in the (r,m) plane, for a fixed value of the s parameter, while a tends to the BPS value, which in this case is approximately $a\approx 0.88$. The dashed line represents the locus where $T_+(r_+;m)=0$, while the black dot represents the BPS solution associated with the given value of the s parameter. The situation is the same as for the five-dimensional black hole solution with three charges **3.3**. There are two extremal configurations associated with $m_{ext,\pm}$, which coincide and produce the BPS solution once $a=a_\star$. The qualitative behaviour does not change if we consider different ranges of the parameters.

ously requiring that $m_{ext,+}=m_{ext,-}=m_\star$ by using the general relation **(4.21)** and imposing extremality.

Supersymmetric and BPS horizons

From now on we will use R instead of $\tilde{r}$ to make the notation lighter.

It is important to consider what happens in the supersymmetric case, following [8] we want to use an horizon radius $\tilde{r}_i$ as a parameter describing the supersymmetric solutions, we need to study Eq. **(4.21)** after having imposed the constraint **(4.9)**, in this way the relation[3] $m=m(R_i)$ becomes:

$$m=\frac{2e^2\Big[R_i(1+e^4)\pm i\big(R_i^2(e^4-1)-2\big)\Big]}{R_i^4-1}, \tag{4.22}$$

remember that $e=e^\delta$.

If one forces the horizon radius to be real, then one must introduce a complexified family of solutions characterized by a complex value of the m parameter, one also has to pass to Euclidean signature for consistency. This is analogous to what we have seen in the five-dimensional case [8].

However, we will follow the same conventions that we used when discussing the five-dimensional case and keep the parameters of the solution *real*. This implies that in the general supersymmetric solution we will have four virtual horizons, associated with two pairs of complex conjugated roots $(R,\bar{R})$ and $(R_-,\bar{R}_-)$. This is only a choice of parametrization that is more convenient to us, as it allows us to keep a real value of the charges. Moreover, in this way we keep the symmetry under exchange of horizon radius R_i when used as a parameter describing the solution.

Notice that by keeping m real and fixing one of the two signs in **(4.22)** when using one of the complex roots R, we are forced to choose the opposite sign when considering the same relations in terms of $\bar{R}_i$. This happens in order to satisfy $m=\bar{m}$. The sign differences in Eq. **(4.22)** propagates in many places when discussing the universal extremization principle, and determines which sign appears in the expression determining a given horizon entropy in terms of the corresponding Λ root.

[3]Remember that this relation does not distinguish between the horizons, meaning that all roots solving $\Delta_r(R;m)=0$ can be used here.

Finally, notice that in the BPS solution, obtained after having imposed both **(4.9)** and **(4.14)**, one is able to explicitly find all the roots of the Δ_r polynomial, which are given by:

$$R^\star = \sqrt{a}, \qquad R^\star_- = -\sqrt{a} \pm i(a+1), \tag{4.23}$$

clearly $R^\star$ is a double root. This solution corresponds to the point in Fig.**4.1**. Having found the explicit value of $R^\star_-$, we can directly derive the expressions of the inner horizons entropies, which are obtained as usual from the generalization of Eq. **(4.5)**. Remembering the expression of the BPS electric charge **(4.15)** we find:

$$S^\star = \frac{\pi}{2}\Big(\sqrt{1+4Q^{\star 2}} - 1\Big), \qquad S^\star_- = -\frac{\pi}{2}\Big(1 + 4iQ^\star + \sqrt{1+4Q^{\star 2}}\Big), \tag{4.24}$$

where we have also given the BPS value of the physical BPS entropy for completeness.

Notice that when we set $Q^\star = 0$, hence reducing to empty AdS_4 spacetime, the BPS entropies reduce to $S^\star = 0$ and $S^\star_- = -\pi$. The latter is precisely one quarter of the area of a 2-sphere with radius $r^2 = -1$ that appears in empty AdS_4 space when we allow $r^2 < 0$. This is the same observation that we made in the five-dimensional case in Sec. **3.2.2**. This observation reinforces the idea that we might have to treat the spacetime with $r^2 < 0$ more seriously, as it seems to be a general feature for these AdS black holes which does not depend on the number of dimensions of the spacetime, nor on the number of independent charges that characterize the solution.

These results can be used to show the equivalence between the BPS constraint **(4.14)** with the universal area product, which we are now going to derive.

4.2.2 Universal area product formula

The universal area product has not been derived yet for this specific solution, however, it should trivially follow from the results of [32] for the pairwise equal charged case.

We can proceed as in Sec. **3.4.3**, first notice that the condition $\Delta_r(r) = 0$ is equivalent to:

$$\tilde{r}^2 + a^2 = \frac{-r^2 + 2mr - a^2}{(r+2ms^2)^2} = -\frac{(r-b)(r-d)}{(r+2ms^2)^2}, \tag{4.25}$$

where:

$$b + d = 2m, \qquad bd = a^2 \tag{4.26}$$

moreover, Δ_r evaluated on either b or d reads:

$$\Delta_r(b) = (b+2ms^2)^2\big((b+2ms^2)^2 + a^2\big), \tag{4.27}$$

exploiting this result, the product of the four entropies can be rewritten as (using $S_i = \frac{\pi}{\Xi}(\tilde{r}_i^2 + a^2)$, Eq. **(4.5)**):

$$\prod_{i=1}^{4} S_i = \Big(\frac{\pi}{1-a^2}\Big)^4 \prod_{i=1}^{4} \frac{(\tilde{r}_i - b)(\tilde{r}_i - d)}{(\tilde{r}_i + 2ms^2)^2} = \Big(\frac{\pi}{1-a^2}\Big)^4 \prod_{i=1}^{4} \frac{\Delta_r(b)\Delta_r(d)}{\Delta_r(-2ms^2)^2}, \tag{4.28}$$

rearranging the above expression in such a way that only terms depending on bd and $b+d$ appear one finds that the product of the entropies reads (also using Eqs. **(4.4)**):

$$\prod_{i=1}^{4} S_i = \frac{4m^2\pi^4}{(1-a^2)^4}\Big(4m^2 s^4 c^4 + a^2(s^2+c^2)\Big) = \pi^4\Big(Q^4 + 4J^2\Big), \tag{4.29}$$

this result agrees with the expected one found in [32] for the pairwise equal case, once we reduce to the one charged case and suitably rescale (Q, J).

We can now specialize to the BPS solution, where $S_+^\star = S_0^\star = S^\star$ and the BPS entropies are given by Eqs. **(4.24)**. By inserting these informations in the area product formula one finds:

$$\pi^4 Q^{\star 2}\Big(2+5Q^{\star 2}-2\sqrt{1+4Q^{\star 2}}\Big) = S_-^\star\, \bar{S}_-^\star\, S^{\star 2} = \pi^4\Big(Q^{\star 4}+4J^{\star 2}\Big), \tag{4.30}$$

this is precisely the BPS constraint **(4.14)**. To see this, one can view the above equation as an equation for $J^\star$, whose solution is precisely **(4.14)**.

In anticipation of what we have shown, also in this case the equivalence of the area product formula and the BPS constraint can be interpreted by means of a factorization condition, needed to obtain real entropies from the extremization principle.

4.2.3 Thermodynamics of all the horizons

Following the logic of the five dimensional case, we will now proceed to define the generalized thermodynamics for each horizon. Again, we can do this in such a way that the first law and the quantum statistical relations are automatically satisfied. This, will also prepare the ground to show the universality of the extremization principle.

Following Sec. **(3.2.4)** it is quite easy to define the relevant thermodynamic quantities for the other horizons. In particular, the conserved charges remain fixed while the chemical potentials, the entropy, and the on-shell action are simply obtained by replacing the explicit dependence on r_+ with the appropriate r_i root in the various definitions associated with the outer horizon **(4.5, 4.6, 4.8)**. Clearly, the two virtual horizons are characterized by a complex value of these quantities, which we treat as formal quantities.

The first law of thermodynamics and the QSR for each horizon, are immediately derived from those of the outer horizon once we trade one parameter of the solution with a given R_i (e.g. we can trade m using Eq. **(4.21)**). It does not matter which horizon radius is used until $\Delta_r = 0$ is satisfied, and again checking the first law or the QSR for each horizon is only a matter of relabelling the r_i parameter. For the intermediate horizon at r_0 one would get a negative temperature.

We can now easily generalize equations **(3.68, 3.69)**. From the mass-independence of the entropy product formula, combined with the generalized first law for each horizon, we get:

$$\begin{cases} \frac{d(S_1S_2S_3S_4)}{dE} = 0 \\ \frac{dS_i}{dE} = \frac{1}{T_i} \end{cases}, \quad \text{At fixed } Q \text{ and } J, \tag{4.31}$$

implying the following identity which generalizes Eq. **(3.68)** in the case where there are 4 horizons:

$$(S_2T_2)(S_3T_3)(S_4T_4)+(S_1T_1)(S_3T_3)(S_4T_4)+(S_1T_1)(S_2T_2)(S_4T_4)+(S_1T_1)(S_2T_2)(S_3T_3)=0, \tag{4.32}$$

this relation can be easily checked by noticing that the products $T_i\, S_i$ for a given horizon can be rewritten as (using Eqs. **(4.19)**):

$$T_1S_1 = \frac{m(1+2s^2)-R_1(2R_1^2+1+a^2)}{2\Xi} = \frac{1}{4\Xi}\Big((R_1-R_2)(R_1-R_3)(R_1-R_4)+3R_1^2\underbrace{(R_1+R_2+R_3+R_4)}_{0}\Big), \tag{4.33}$$

this generalizes Eq. **(3.69)** and allows to easily check the validity of Eq. **(4.32)**.

Supersymmetric case and setup of the universal extremization principle

As a final step, before moving on proving the universality of the extremization principle, let us consider the supersymmetric chemical potentials for each horizon. These are obtained by imposing the supersymmetric constraint and trading m for the appropriate R_i parameter in Eqs. **(4.6)**, one gets:

$$T_i = \frac{e^8 R_i^2 \left(2R_i \mp i\right) + e^4 \left(-4R_i^3 - 4R_i \pm 2i\right) + 2R_i^3 + 4R_i \pm i(R_i^2+2)}{2\pi\left(\left(e^4-1\right)^2 R_i^2 + 4\right)}, \tag{4.34}$$

$$\Omega_i = \frac{2\left(e^4-1\right)\left(R_i^2+1\right)}{\left(e^4-1\right)^2 R_i^2+4}, \qquad \Phi_i = \frac{\left(e^4-1\right)\left(1 \pm iR_i\right)R_i}{\left(e^4-1\right)R_i \pm 2i}, \qquad \frac{1+\Omega_i-2\Phi_i}{T_i} = \mp 2\pi i, \tag{4.35}$$

the sign ambiguity is correlated to the one in the $m = m(R_i)$ relation **(4.22)**. As we have already discussed, once we choose a sign convention for a given horizon, we have to choose the opposite sign for the conjugated horizon. These relations are the same *in form* if we choose to work with complex horizons radii R_i and real m or if we force (R_0, R_+) to be real and complexify m (as done in [8]). However, the two choices give different chemical potentials, because the two possibilities above correspond to two different ways that can be used to parametrize the supersymmetric solution.

Next we introduce the usual modified chemical potentials (ω_i, ϕ_i):

$$\omega_i = \beta_i(\Omega_i - 1), \qquad \phi_i = \beta_i(\Phi_i - 1), \qquad \omega_i, -2\phi_i = \mp 2\pi i, \tag{4.36}$$

by virtue of the symmetry under exchange of horizon radius R_i, these have the same expression as the ones given in Eq. **(4.11)**. Remember that only for the outer and intermediate horizons these represent, in the $T_+ \to 0$ or $T_0 \to 0$ limit, the subleading order contributions in the expansion of the potentials around the BPS solution, following a supersymmetric trajectory.

The quantum statistical relation and the supersymmetric action for each horizon now take the form:

$$I_i = S_i - \omega_i\, J - \phi_i\, Q, \qquad I_i = \mp\frac{i}{2}\frac{\phi_i^2}{\omega_i}. \tag{4.37}$$

We are now ready to define the universal action I, depending on the universal chemical potentials (ω, ϕ) not associated with any particular horizon. As we have already discussed in the five-dimensional case, this allows us to not depend on the definitions of the I_i actions.

Notice that the sign ambiguity appearing in the relation $m = m(R_i)$, which we have seen to be fixed between two conjugated horizons, has propagated to the supersymmetric constraint **(4.36)**, but also to the action I_i **(4.37)**, this means that one has to formally define two universal actions, one associated with the horizons located at (R, R_-) and one, with opposite sign, associated with the conjugated horizons at $(\bar{R}, \bar{R}_-)$.

We are now ready to prove the universality of the extremization principle.

4.2.4 Universal extremization principle

In [69] the extremization principle was briefly discussed (concentrating only on the outer horizon). We are going to consider their discussion, extending their calculations and taking all the horizons in considerations.

The starting point is given by the following universal relations:

$$\omega - 2\phi = \mp 2\pi i, \qquad I = \mp\frac{i}{2}\frac{\phi^2}{\omega}, \tag{4.38}$$

the fact that the two couples of conjugated horizons (R, R_-) and $(\bar{R}, \bar{R}_-)$ have a sign difference in the definition of the action, implies that we should formally treat the two couples separately. This is different from what we have done in the five-dimensional case and, indeed, the procedure is slightly different in this case. However, we are going to show that one can derive the properties of all four horizons from one calculation, which directly generalizes what we have done in the five-dimensional case.

The entropy $S = S(Q, J)$ can be obtained from the following constrained Legendre transform:

$$S = ext_{\{\omega,\phi,\Lambda\}}\Big[-I - \omega\, J - \phi\, Q + \Lambda(\omega - 2\phi \pm 2\pi i)\Big], \tag{4.39}$$

following our choice of parametrization, the charges are real.

The extremization equations are:

$$\frac{\partial I}{\partial \omega} = -J + \Lambda, \qquad \frac{\partial I}{\partial \phi} = -Q - 2\Lambda, \tag{4.40}$$

together with the supersymmetric constraint, they should be used to determine the chemical potentials and Λ.

Combining the second extremization equation and the supersymmetric constraint gives:

$$\phi = \frac{2\pi i}{2(Q + 2\Lambda) - i}, \qquad \omega = 2\pi i\bigg(\frac{2 - 2(Q + 2\Lambda) - i}{2(Q + 2\Lambda) - i}\bigg), \tag{4.41}$$

these depend on J through Λ, once we have solved the extremization equations to determine $\Lambda = \Lambda(Q, J)$.

The Lagrange multiplier can be obtained by noticing that the following combination of derivatives of the supersymmetric action vanishes:

$$\bigg(\frac{\partial I}{\partial \phi}\bigg)^2 \mp 2i\frac{\partial I}{\partial \omega} = 0 \quad\longrightarrow\quad \big(Q + 2\Lambda\big)^2 \mp 2i\big(-J + \Lambda\big) = 0, \tag{4.42}$$

the Euclidean action is an homogeneous function of degree one of the chemical potentials, meaning that it still holds:

$$S_i = \pm 2\pi i \Lambda_i, \tag{4.43}$$

where Λ_i is a root of the polinomial in Eq. **(4.42)**.

To associate the right root Λ_i to its corresponding horizon, one should choose the root that satisfies the extremization equations Eqs. **(4.40)**, once we consider the explicit expressions for the chemical potentials associated with that given horizon, and we express everything in terms of the (a, e) parameters. This can be easily done in the BPS limit and, with some care, also in the general supersymmetric case. The discussion is similar to that of the five-dimensional solution.

Notice that in this case, we have two quadratic polynomials determining the Lagrange multiplier Λ. The two pairs of roots (Λ, Λ_-) and $(\bar{\Lambda}, \bar{\Lambda}_-)$ are separately associated with the horizons at (R, R_-) and $(\bar{R}, \bar{R}_-)$. This happens because of the sign difference in the definition of the supersymmetric action.

However, we can recover a completely unified description of all four horizons by considering the following quartic polynomial determining the Λ roots:

$$(Q+2\Lambda)^4 + 4(-J+\Lambda)^2 = 0 \quad\longleftrightarrow\quad \Big[\big(Q+2\Lambda\big)^2 - 2i\big(-J+\Lambda_i\big)\Big]\Big[\big(Q+2\Lambda\big)^2 + 2i\big(-J+\Lambda_i\big)\Big] = 0. \tag{4.44}$$

Even if one could formally work by using only the quadratic polynomial in Λ it is worth considering also the procedure with the quartic polynomial, which directly generalizes the one discussed in the five-dimensional case. Moreover, we will need it in the following, when discussing the extremization principle for the general solution with four charges.

Extremization principle using the quadratic polynomial

Let us choose the $+$ sign in Eq. **(4.42)**, one gets the following:

$$\Lambda^2 + \Lambda\frac{1}{2}(2Q - i) + \frac{1}{4}(2iJ + Q^2) = 0, \tag{4.45}$$

the roots are given by:

$$\Lambda = \frac{1}{4}\Big(-2Q + i \pm i\sqrt{1 + 4i(Q + 2J)}\Big), \tag{4.46}$$

these are complex, as the charges are real, but this is coherent with the fact that in the general supersymmetric solution all horizons are complex. The BPS case is obtained by imposing that at least one of the two Λ roots is purely imaginary, we cannot require to have two conjugated imaginary roots as the quadratic polynomial has complex coefficients, this confirms our interpretation that each quadratic polynomial reproduces either (R, R_-) or its conjugated pair of horizons, as one of the two must always be complex.

One can see that only the root with the $-$ sign in **(4.46)** can be made purely imaginary by requiring[4]

$$Im\Big[\sqrt{1 + 4i(Q + 2J)}\Big] = 2Q \;\longrightarrow\; \Big(1 + 16(Q + 2J)^2\Big)^{\frac{1}{4}} \sin\Big[\frac{1}{2}\arctan\big[4(Q + 2J)\big]\Big] = 2Q. \tag{4.47}$$

A solution can be found by exploit the trigonometric identities $\sin(\frac{x}{2}) = \sqrt{\frac{1-\cos x}{2}}$ and $\cos[\arctan x] = (x^2 + 1)^{-\frac{1}{2}}$. This allows us to rewrite the above condition as the following quadratic equation in J:

$$J^2 + JQ - Q^4 = 0, \quad \longrightarrow \quad J = \frac{Q}{2}\Big(\sqrt{1 + 4Q^2} - 1\Big), \tag{4.48}$$

whose solution is exactly the BPS condition **(4.15)**. Imposing the above constraint, the two roots of the quadratic polynomial becomes[5]:

$$\Lambda^\star = -\frac{i}{4}\Big(\sqrt{1 + 4Q^{\star 2}} - 1\Big), \qquad \Lambda^\star_- = \frac{i}{4}\Big(1 + 4iQ^\star + \sqrt{1 + 4Q^{\star 2}}\Big), \tag{4.49}$$

by using the definition of the entropy in terms of the Lagrange multiplier $S = 2\pi i\Lambda$, we immediately see that we correctly reproduce the results already found in the BPS solution for the entropies Eqs. **(4.17)**. The results for the conjugated horizons can be easily obtained by replacing $i \to -i$ in all expressions.

Finally, notice that exploiting Eqs. **(4.49)**, the expressions for the BPS charges **(4.15)** and the expressions for the BPS chemical potentials for the two horizons, one can check that the extremization equations are solved, this confirms that the two roots Λ are correctly associated with the expected horizon.

The product formula for the entropies can also be derived from the extremization principle, the constant term in the quadratic polynomial gives the product of two horizons meaning that:

$$\prod_i S_i = (4\pi)^4 \Lambda\Lambda_-\bar{\Lambda}\bar{\Lambda}_- = \pi^4(Q^2 + 2iJ)(Q^2 - 2iJ) = \pi^4\big[Q^4 + 4J^2\big]. \tag{4.50}$$

Supersymmetric but non extremal case

Eq. **(4.46)** allows us to find a formal expression for the supersymmetric entropies S, S_- as a function of only the conserved charges. The entropy product formula is always satisfied by means of Eq. **(4.50)**.

We can check if the entropies obtained from this definition agree with the ones that one would get from the

[4]Remember that we have initially fixed the parameters to be positive, meaning that also the charges are positive. One finds that the other root can become, real but only if the charges are negative.

[5]To find the expression for the complex root one needs to exploit the identity $1 + 4iQ\sqrt{1 + 4Q^2} = (2iQ + \sqrt{1 + 4Q^2})^2$.

formula of the area Eq. **(4.5)**. As we have seen in the five-dimensional case, to correctly associate each Λ root to the corresponding horizon entropy:

$$\begin{aligned}\Lambda &= \frac{1}{4}\Big(-2Q+i-i\sqrt{1+4i(Q+2J)}\Big)\Big|_{Q(e,R),J(e,R)} &&\longleftrightarrow \quad \pi\frac{(R^2+a^2)}{1-a^2}\\ \Lambda_- &= \frac{1}{4}\Big(-2Q+i+i\sqrt{1+4i(Q+2J)}\Big)\Big|_{Q(e,R_-),J(e,R_-)} &&\longleftrightarrow \quad \pi\frac{(R_-^2+a^2)}{1-a^2},\end{aligned} \tag{4.51}$$

we need to find the range of validity for R and R_- interpreted as "good" parameters that substitute m, where with "good" we simply mean that the inverting relation $m=m(R_i)$ is bijective, remember also that $m>0$.

With our parametrization, we have $m\in\mathbb{R}$. This requirement imposes a constraint on the real and imaginary part of the complex horizon radii, in particular, using the inversion relation Eq. **(4.22)** one easily finds that it must hold:

$$R=\sqrt{\frac{(e^4-1)\,x^2-(e^4+1)\,x+2}{e^4-1}}+ix, \qquad R_-=-\sqrt{\frac{(e^4-1)\,x^2-(e^4+1)\,x+2}{e^4-1}}+ix \tag{4.52}$$

The $+$ sign is associated with the R root while the $-$ sign is associated with the R_- root whose real part must be negative. Setting $x=0$ we recover the real BPS solution for the physical outer horizon radius $R^\star=\sqrt{a}$, and similarly setting $x=1+a$ we recover the BPS solution for the virtual horizon radius $R_-^\star=-\sqrt{a}+i(1+a)$.

The roots become purely imaginary if the radical is negative, this happens if $x\in\big(\frac{2}{e^4-1},1\big)$, this is similar to what happened to the r_0 root in the five-dimensional case. However, we cannot allow x to take these values as it would produce an imaginary $m=m(R)$. To see this, let us insert Eqs. **(4.52)** into the inverting relation **(4.22)**, obtaining:

$$m\big(R_i(x)\big)=\pm\frac{2e^2\sqrt{(x-1)\big((e^4-1)x-2\big)}\,(e^4(1-2x)+2x+1)}{(e^4-1)^{5/2}}, \tag{4.53}$$

the sign depends on which horizon radius R or R_-) is considered. Notice that precisely when $x\in(\frac{2}{e^4-1},1)$ the mass parameter is imaginary, hence we should exclude those values from the range of validity of x. There are two separate branches of allowed values for the x parameter, one finds that each branch must be associated with either R or R_- in such a way to satisfy the condition $m>0$ see Fig. **(4.2)**.

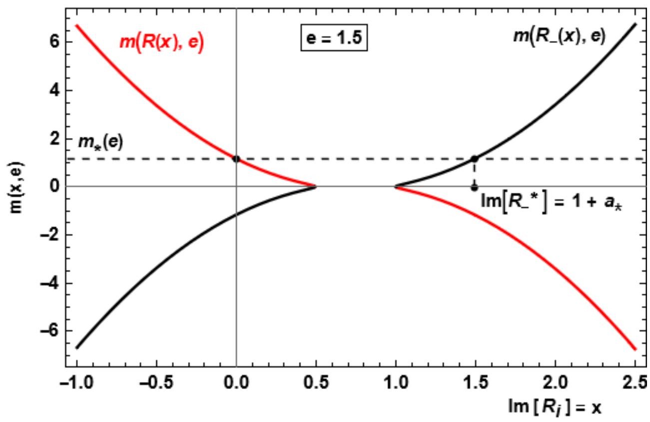

Figure 4.2: Behaviour of $m=m\big(R_i(x)\big)$ in Eq. **(4.53)**, associated with R in red, and R_- in black. We see that, imposing $m>0$ implies that the two allowed regions for the $x=Im[R_i]$ parameter are associated with either one of the two horizons. Notice that the BPS value for m **(4.14)** (with $e=1.5$ we have $m_\star(e)\approx1.16$) is obtained precisely when R is real ($x=0$) and when $Im[R_-]=x=1+a_\star(e)$.

In particular, one finds that the complex variable $R(x)$ (associated with the outer horizon) is related to a positive value for m if $x < \frac{2}{e^4-1}$. Instead, $R_-(x)$ (associated with the inner horizon), produces a positive value for m if $x > 1$.

After having understood which numerical (complex) values R and R_- can take, one can easily check numerically that the relations in Eqs. **(4.51)** are satisfied, provided that one evaluates them in the correct range of the x parameter.

Extremization principle using the quartic polynomial

Consider now the quartic polynomial obtained by multiplying the two quadratic polynomials generating the two couples of roots, from this the thermodynamics of all four horizons can be reproduced at once.

$$\left(Q+2\Lambda\right)^4+4\left(-J+\Lambda\right)^2=\left[\left(Q+2\Lambda\right)^2-2i\left(-J+Q\right)\right]\left[\left(Q+2\Lambda\right)^2+2i\left(-J+Q\right)\right], \tag{4.54}$$

explicitly, it is given by:

$$\Lambda^4+p_3\Lambda^3+p_2\Lambda^2+p_1\Lambda+p_0, \quad \text{where:} \quad \begin{cases} p_3=2Q \\ p_2=\frac{1}{4}\left(1+6Q^2\right) \end{cases}, \quad \begin{cases} p_1=\frac{1}{2}\left(Q^3-J\right) \\ p_0=\frac{1}{16}\left(Q^4+4J^2\right) \end{cases}. \tag{4.55}$$

The entropy product formula can be immediately read off from the p_0 parameter. Proceeding as usual, the BPS solution can be imposed requiring that the quartic polynomial has two conjugated imaginary roots, this can be done as the polynomial has real coefficients. This is achieved by imposing the following condition on the p_i coefficients:

$$\frac{p_1}{p_3}+\frac{p_0p_3}{p_1}=p_2 \quad \to \quad \frac{\left(4Q^2+1\right)\left(J^2+JQ-Q^4\right)}{4\left(Q^4-JQ\right)}=0 \tag{4.56}$$

and again, one easily sees that this condition is equivalent to imposing the BPS constraint **(4.15)**. The above condition is the factorization condition which puts the quartic polynomial in the form:

$$\left(\Lambda^2+\frac{p_1}{p_3}\right)\left(\Lambda^2+p_3\Lambda+p_2-\frac{p_1}{p_3}\right), \tag{4.57}$$

then in the BPS limit one finds for the outer physical horizon that[6]:

$$S^\star=2\pi\sqrt{\frac{p_1}{p_3}}=\frac{\pi}{2}\left(\sqrt{1+4Q^{\star 2}}-1\right) \tag{4.58}$$

while the BPS value of the two virtual horizons entropies, are obtained by considering the roots Λ_- of the quadratic polynomial[7] $\Lambda^2+p_3\Lambda+p_2-\frac{p_1}{p_3}$:

$$S_-^\star=\pm\pi i\left(-p_3\pm\sqrt{4\frac{p_1}{p_3}-4p_2+p_3}\right)=-\frac{\pi}{2}\left(1\pm 4iQ+\sqrt{1+4Q^2}\right). \tag{4.59}$$

These are exactly the results found in Eqs. **(4.24)**, moreover, the extremization equations can be found to be satisfied for each horizon by choosing the appropriate root for Λ, this has essentially already been checked when discussing the case generated by the quadratic polynomial.

As a final comment, notice that the equivalence between the universal area product formula and the BPS constraint, can be derived using the definitions of the BPS entropies in terms of the p_i parameters **(4.58, 4.59)**,

[6]One needs to use $(\sqrt{1+4Q^2}-1)^2=-1-2Q^2+\sqrt{1+4Q^2}$.

[7]One needs to use: $(\sqrt{1+4Q^2}+1)^2=2+4Q^2+2\sqrt{1+4Q^2}$.

and rewriting the factorization condition **(4.56)** as:

$$p_0 = \frac{p_1}{p_3}\Big(p_2 - \frac{p_1}{p_3}\Big) \quad \longrightarrow \quad p_0 = S^{\star 2}\, S_-^{\star}\, \bar{S}_-^{\star}. \tag{4.60}$$

This concludes our discussion for the present black hole solution.

4.3 Pairwise equal charged, spinning AdS_4 black hole solution

In this section, we are going to generalize some of the previous results also to the pairwise equal charged solution. We will mostly concentrate on the properties of the BPS horizons in order to prove the universality of the extremization principle. We will not elaborate on the details of the solution as it does not bring any novelty with respect to the single charged case, which can be trivially obtained from the present solution.

Let us start with an overview of the properties of the solution.

Review of the black hole solution

Consider the black hole solution of [68], we simply give a review the relevant features of the solution skipping all the details that are not directly needed.

This is a particular solution of $U(1)^4$ gauged $\mathcal{N}=2$ supergravity, the bosonic fields of this theory are: the metric $g_{\mu\nu}$, four Abelian gauge fields A^I which are then set pairwise equal $A^1 = A^2$ and $A^3 = A^4$ and two real scalar fields, the axion and the dilaton[8]. One can find in [15, 68] the specific expressions for the metric and the other fields characterizing the solution.

The solution is given using coordinates (t, r, θ, ϕ), with $\theta \in [0, \pi]$ and $\phi \sim \phi + 2\pi$ parametrizing a 2-sphere, and depends on four parameters $(m, a, \delta_1, \delta_2)$ which can be roughly associated with the mass, angular momenta, and the two independent electric charges. It is interesting to notice that the scalar fields depend on the combination $r_1 - r_2$ where $r_i = r + 2m\sinh^2\delta_i$, meaning that by setting $\delta_1 = \delta_2 = \delta$ the two scalar fields can be set to zero, while the two gauge fields become equal. One can easily check that in this way one recovers the single charged solution discussed in Sec. **4.1**.

The function determining the position of the horizons is now given by:

$$\Delta_r = r^2 + a^2 - 2mr + r_1 r_2\big(r_1 r_2 + a^2\big), \tag{4.61}$$

where we have set the AdS_4 radius to 1 as usual, notice that it trivially reduces to **(4.3)** if $\delta_1 = \delta_2$.

The solution has four independent conserved charges:

$$E = \frac{m}{\Xi^2}(1 + s_1^2 + s_2^2), \qquad J = \frac{ma}{\Xi^2}(1 + s_1^2 + s_2^2),$$
$$Q_1 = Q_2 = \frac{m\, s_1\, c_1}{2\Xi}, \qquad Q_3 = Q_4 = \frac{m\, s_2\, c_2}{2\Xi}, \tag{4.62}$$

where $s_i = \sinh\delta_i$, $c_i = \cosh\delta_i$ and $\Xi = 1 - a^2$. These have been obtained via the standard Maxwell and Komar integrals, and by integrating the first law to get the energy in [11], or via holographic renormalization in [15].

The chemical potentials of the outer horizon, which is located at the largest positive root r_+ of Δ_r, are

[8] The original $U(1)^4$ gauged $\mathcal{N}=2$ supergravity theory is obtained by coupling the $\mathcal{N}=2$ supergravity multiplet with three vector multiplets, hence the theory would contain a total of three complex scalars, or equivalently, six real scalars. After having set the four gauge fields pairwise equal, one also sets four of the total scalar fields to constants so that only two non-trivial real scalar fields [68] remain. Equivalently, one realizes that the field content described above is consistent with the one that one gets when considering pure supergravity coupled to one vector multiplet

given by the following expressions:

$$T = \frac{\Delta'_r(r_+)}{4\pi(r_1\,r_2 + a^2)}, \quad \Omega = \frac{a(1 + r_1\,r_2)}{r_1\,r_2 + a^2}, \quad \Phi^1 = \Phi^2 = \frac{2m\,s_1\,c_1\,r_2}{r_1\,r_2 + a^2}, \quad \Phi^3 = \Phi^4 = \frac{2m\,s_2\,c_2\,r_1}{r_1\,r_2 + a^2}, \quad (4.63)$$

these have been calculated in a non-rotating frame at infinity with coordinates $(t' = t, \phi' = \phi + at)$ (see [15]) hence the angular velocity is both the horizon angular velocity (the killing vector generating the horizon is $V = \partial_{t'} + \Omega\partial_{\phi'}$) and the thermodynamic one. Moreover, having set the gauge fields pairwise equal, also makes the electrostatic potentials pairwise equal as they are defined via the usual relation $\Phi^I = \iota_V A^I\big|_{r_+}^{\infty}$.

Finally the Bekenstein-Hawking entropy is:

$$S = \frac{\pi(r_1\,r_2 + a^2)}{\Xi}. \quad (4.64)$$

Notice that all the thermodynamic quantities reduce to the ones of Sec. **4.1** upon setting $\delta_1 = \delta_2$, with the exception of the electric charges that reduce to $Q_1 = Q_3 = \frac{Q}{4}$ where Q has been defined in **(4.4)**.

The above thermodynamic quantities satisfy the first law of thermodynamics in the form:

$$dE = T\,dS + \Omega\,dJ + 2\Phi^1\,dQ_1 + 2\Phi^3\,dQ_3, \quad (4.65)$$

and the quantum statistical relation:

$$I = \beta E - S - \beta\,\Omega\,J - 2\beta\,\Phi^1\,Q_1 - 2\beta\,\Phi^3\,Q_3, \quad (4.66)$$

where I is the Euclidean on-shell action that was calculated in [15] by using holographic renormalization.

Supersymmetric solutions

Supersymmetry is (partially) restored if the above solution satisfies the condition[9] [15]:

$$a = \frac{2}{e^{2(\delta_1+\delta_2)} - 1}, \quad (4.67)$$

again, one finds that the supersymmetric chemical potentials and charges satisfy the standard supersymmetric constraints:

$$\beta(1 + \Omega - \Phi^1 - \Phi^3) = \mp 2\pi i, \qquad E - J - 2Q_1 - 2Q_3 = 0, \quad (4.68)$$

where as usual, one should think of these quantities as parametrized by the $(r_+, \delta_1, \delta_2)$ parameters, with r_+ substituting m by using the $\Delta_r(r_+) = 0$ condition. Also in this case Δ_r **(4.61)** is quartic in m so to effectively trade m for the outer horizon radius, one needs to introduce the new coordinate $R = r + 2m\,s_1^2$ which makes $\Delta_r(R)$ only quadratic in m. This is similar to what we did before (Sec. **(4.1.1)**) and was discussed in [15].

One can introduce the modified chemical potentials:

$$\omega = \beta(\Omega - 1), \qquad \phi^I = \beta(\Phi^I - 1), \qquad \omega - \phi^1 - \phi^3 = \mp 2\pi i, \quad (4.69)$$

in terms of which, the supersymmetric Euclidean action and the QSR take the usual form:

$$I = \mp\frac{i}{2}\frac{\phi^1\phi^3}{\omega}, \qquad I = -S - \omega\,J - 2\,\phi^1\,Q_1 - 2\,\phi^3\,Q_3. \quad (4.70)$$

[9] Notice that if we impose to have real parameters, this condition satisfies the bound $a^2 < 1$ only provided that $a > 1$ because $e^{2(\delta_1+\delta_2)} > 0$, moreover one has to impose $\delta_1 + \delta_2 > \log(\sqrt{3})$.

The quantities **(4.69, 4.70)** remain well defined in the BPS limit and allow to describe a non trivial BPS thermodynamics. Moreover, starting from these relations it can be shown [14] that the constrained Legendre transform of the supersymmetric action I, with respect to all chemical potential is able to reproduce the BPS entropy, after having imposed the usual reality condition.

Extremality and BPS solution

As usual, supersymmetric solutions have causal pathologies, and the condition $\Delta_r(r) = 0$ determining the position of the horizons only has complex roots. One can obtain a well-behaved solutions by imposing another constraint on the parameters[10] [11]:

$$m^2 = m_\star^2 \equiv \frac{\cosh^2(\delta_1+\delta_2)}{4e^{\delta_1+\delta_2}\sinh^3(\delta_1+\delta_2)\, c_1\, s_1\, c_2\, s_2}, \tag{4.71}$$

it can be shown that in this way the solution becomes extremal and Δ_r has a real double root at:

$$r_\star = \frac{2m_\star\, s_1\, s_2}{\cosh(\delta_1+\delta_2)}, \tag{4.72}$$

the chemical potentials of the outer horizon take the fixed values: $\Omega^\star = 1$, $\Phi^{1\star} = \Phi^{3\star} = 1$ and $T = 0$, while the charges now depend only on (δ_1, δ_2) and satisfy the constraint [14]:

$$J^\star = (Q_1^\star + Q_3^\star)\left(\sqrt{1+64Q_1^\star Q_3^\star} - 1\right), \tag{4.73}$$

together with the one in **(4.68)**.

The BPS outer horizon entropy is given in terms of the charges as [14]:

$$S^\star = \frac{\pi}{2}\left(\sqrt{1+64Q_1^\star Q_3^\star} - 1\right), \tag{4.74}$$

again the results for the single charged case in Sec. **4.1.1** can be recovered by setting $Q_1^\star = Q_3^\star = \frac{Q^\star}{4}$.

4.3.1 Properties of the BPS horizons

In this section, we are going to derive the BPS entropies and radii for the virtual inner horizons. Notice that also in this case the Δ_r function **(4.61)** is a quartic polynomial in r, hence there are four horizons (possibly complex conjugated).

It is useful to work with the parameters $e_i = e^{\delta_i}$. In terms of these, one can easily find that the supersymmetry and BPS conditions on the (a, m) parameters **(4.67, 4.71)** become:

$$a = \frac{2}{e_1^2\, e_2^2 - 1}, \qquad m = \frac{2\sqrt{2}\, e_1\, e_2(1+e_1^2\, e_2^2)}{(e_1^2\, e_2^2 - 1)^{\frac{3}{2}}\sqrt{(e_1^4-1)(e_2^4-1)}}. \tag{4.75}$$

We need to find the BPS value for the inner horizon radius $R_-^\star = r_-^\star - 2m\, s_1^2$. This can be done by rewriting: $\Delta_r(R) = \prod_i (R - R_i)$, where R_i are the general horizons radii. In this way, one gets the usual

[10] A real mass parameter is obtained provided that $\delta_1\delta_2 > 0$ so that $s_1 s_2 > 0$, and remember that $\delta_1 + \delta_2$ has already been set positive when considering the supersymmetry condition **(4.67)**. The combination of these 2 bounds forces δ_1 and δ_2 to be positive. Then, if we also require $r_\star > 0$ this forces also m to be positive, meaning that the BPS solution is characterized by a positive value of all parameters and hence charges from Eqs. **(4.62)**.

relations between the roots:

$$\begin{cases} R_1 + R_2 + R_3 + R_4 = 4m(s_1^2 - s_2^2) \equiv A_0 \\ R_1 R_2 + \cdots = 1 + a^2 + 4m^2(s_1^2 - s_2^2)^2 \equiv A_1 \\ R_1 R_2 R_3 + \cdots = 2m\big(1 + 2s_1^2 + a^2(s_1^2 - s_2^2)\big) \equiv A_2 \\ R_1 R_2 R_3 R_4 = a^2 + 4m^2 s_1^2(1 + s_1^2) \equiv A_3 \end{cases} \xrightarrow[R_3=R_4=R^\star]{\text{BPS}} \begin{cases} 2Re[R_-^\star] + 2R^\star = A_0 \\ |R_-^\star|^2 + 2Re[R_-^\star]R^\star + R_-^{2\star} = A_1 \\ 2|R_-^\star|^2 R^\star + 2Re[R_-^\star]R^{2\star} = A_2 \\ |R_-^\star|^2 R^{\star 2} = A_3 \end{cases} \tag{4.76}$$

where clearly one should impose the constraints **(4.75)** on the A_i coefficients, and we have written the inner virtual horizons radii as $R_1 = \bar{R}_2 = R_-$.

Combining the first and the last equations above, one gets:

$$|R_-^\star|^2 = \frac{A_3}{R^{\star 2}}, \qquad Re\big[R_-^\star\big] = \frac{1}{2}A_0 - R^\star, \tag{4.77}$$

which, after some calculations, and using the known result for $R^\star$ that follows from Eq. **(4.72)**, one finds that:

$$R^\star = \frac{e_2\sqrt{2(e_1^4 - 1)}}{e_1\sqrt{(e_2^4 - 1)(e_1^2 e_2^2 - 1)}}, \qquad R_-^\star = -\frac{e_1\sqrt{2(e_2^4 - 1)}}{e_2\sqrt{(e_1^4 - 1)(e_1^2 e_2^2 - 1)}} + i\frac{e_1^2 e_2^2 + 1}{e_1^2 e_2^2 - 1}, \tag{4.78}$$

the other equations hold with these definitions. Everything reduces to the previous single charged BPS solution upon setting $\delta_1 = \delta_2$.

Inserting the result for $R_-^\star$ in the definition of the entropy **(4.64)**, one finds an expression for the virtual horizons BPS entropies in terms of the (e_1, e_2) parameters. We will not explicitly show it as it is quite long, but it can be used to explicitly check that the virtual horizon BPS entropy is equivalent to the following expression:

$$S_-^\star = -\frac{\pi}{2}\Big[1 + 8i(Q_1^\star + Q_3^\star) + \sqrt{1 + 64 Q_1^\star Q_3^\star}\Big]. \tag{4.79}$$

Also in this case, one can notice that by setting $Q_1^\star = Q_3^\star = 0$ and reducing to AdS_4 space, one obtains $R_-^\star = \pm i$ and $S_-^\star = -\pi$. As we have seen in the single charged case.

As a final comment, notice that by using the universal area product formula, which was derived in [32] for this solution, and the expressions for the BPS horizons entropies one can recover the BPS constraint **(4.73)**:

$$S_-^\star \bar{S}_-^\star S^{2\star} = (2\pi)^4\left[\frac{J^{\star 2}}{4} + 16 Q_1^{\star 2} Q_3^{\star 2}\right] \quad \longrightarrow \quad J^\star = J^\star\big(Q_i^\star, S_i^\star(Q_i^\star)\big). \tag{4.80}$$

Instead of discussing the universality of the extremization principle for this solution, which is a rather trivial generalization of the procedure discussed in **4.2.4**, we are going to discuss it for the more general four-charged solution in the next section, and then reducing those results to the pairwise equal case.

4.4 Universal extremization principle for the four-charge black holes

The extremization principle for the general black hole solution was discussed in [14] who concentrated, as usual, on the physical outer horizon. We are going to revisit their discussion following a different procedure, analogous to the one in Sec. **4.2.4**, and extending their results by explicitly solving the extremization equations and deriving the universal area product and BPS constraint on the charges for the general four-charged solution. We will also consider the inner horizons. Finally, we will reduce to the pairwise equal case showing the validity of our claim for this solution as well.

The form of the supersymmetric on-shell action proposed in [14] is the following:

$$I = \pm 2i \frac{\sqrt{\phi^1\phi^2\phi^3\phi^4}}{\omega}, \tag{4.81}$$

here we work in units where the Newton's constant and the AdS_4 radius are both one, we also consider a different proportionality constant with respect to [14].

The proposed action depends on the (ϕ^I, ω) variables, which we interpret as the supersymmetric chemical potentials generalizing the ones defined in Eq. **(3.32)**, and satisfying the usual constraint given by:

$$\omega - \phi^1 - \phi^2 - \phi^3 - \phi^4 = \pm 2\pi i, \tag{4.82}$$

the ambiguity in the signs has been clarified when discussing the single charged solution, remember that of the four supersymmetric horizons, two are associated with the complex roots (R, R_-) for which one can choose one of the two sign conventions. Then, the conjugated horizons at $(\bar{R}, \bar{R}_-)$ must be associated with the opposite sign convention[11].

The entropies of all horizons should then be reproduced by the following constrained Legendre transform:

$$S = ext_{\{\omega,\phi^I,\Lambda\}} \Big[-I - \omega J - \phi^I Q_I + \Lambda(\omega - \phi^1 - \phi^2 - \phi^3 - \phi^4 \mp 2\pi i) \Big], \tag{4.83}$$

where clearly J and Q_I are the supersymmetric charges of the black hole, which at this stage should be allowed to be complex, similarly as for the chemical potentials.

Performing the extremization principle with respect to the chemical potentials requires solving the following extremization equations:

$$\frac{\partial I}{\partial \omega} = \Lambda - J, \qquad \frac{\partial J}{\partial \phi^I} = -\Lambda - Q_I, \tag{4.84}$$

plus the constraint **(4.82)**, in order to find the chemical potentials and Λ in terms of the charges. We find a solution to the above equations to be:

$$\omega = \frac{2\pi\sqrt{\tilde{Q}_1\tilde{Q}_2\tilde{Q}_3\tilde{Q}_4}}{\Theta}, \qquad \phi^I = -\frac{\tilde{Q}_J\tilde{Q}_K\tilde{Q}_L}{\Theta}, \quad \text{with:} \qquad I \neq J \neq K \neq L,$$

$$\Theta = \sum_{I<J<K} \tilde{Q}_I\tilde{Q}_J\tilde{Q}_K \mp i\sqrt{\tilde{Q}_1\tilde{Q}_2\tilde{Q}_3\tilde{Q}_4} \tag{4.85}$$

where $\tilde{Q}_I = Q_I + \Lambda$, the dependence on J is hidden in the definition of $\Lambda = \Lambda(Q_I, J)$.

As before, one can find an equation that determines Λ by noticing that a suitable combination of derivatives of the action I vanishes, this observation leads to the following equation determining Λ:

$$0 = (\Lambda + Q_1)(\Lambda + Q_2)(\Lambda + Q_3)(\Lambda + Q_4) + \frac{1}{4}(\Lambda - J)^2 = \Lambda^4 + p_3\Lambda^3 + p_2\Lambda^2 + p_1\Lambda + p_0, \tag{4.86}$$

notice that in the general case it is not possible to find a quadratic equation determining Λ as for the single charged case.

[11] This should be true also in the four-charged solution, for which we don't have the explicit solution. However, the condition $\Delta_r = 0$ should still be of quartic order in r, and moreover, by making the usual choice of keeping the parameters of the solution real in the pure supersymmetric solution, the structure described above for the four roots should hold.

The coefficients of the quartic polynomial are given by:

$$\begin{cases} p_3 = P_1 \\ p_2 = \frac{1}{4} + P_2 \\ p_1 = -\frac{J}{2} + P_3 \\ p_0 = \frac{J^2}{4} + P_4 \end{cases}, \quad \text{where:} \quad \begin{cases} P_1 = Q_1 + Q_2 + Q_3 + Q_4 \\ P_2 = \sum_{I<J} Q_I Q_J \\ P_3 = \sum_{I<J<K} Q_I Q_J Q_K \\ P_4 = Q_1 Q_2 Q_3 Q_4 \end{cases}. \tag{4.87}$$

Being the Euclidean action I an homogeneous function of degree 1 of the chemical potentials, it still holds:

$$S_i = \pm 2\pi i \Lambda_i, \tag{4.88}$$

where Λ_i should be taken among the roots of Eq. **(4.86)**. Notice that by assuming the validity of this definition for the supersymmetric entropies of each horizon, one immediately derives the universal area product formula, but formally only in the supersymmetric case:

$$\prod_i S_i = (2\pi)^4 \left(Q_1 Q_2 Q_3 Q_4 + \frac{J^2}{4} \right). \tag{4.89}$$

Generally, the roots of the quartic polynomial **(4.86)** are complex, arranged in conjugated pairs if we work in a parametrization that leaves the charges real. Following the same procedure of Sec. **(4.2.4)** we should recover the BPS solution by requiring that the quartic polynomial admits two conjugated purely imaginary roots (when the charges are all real). This is always the case provided that we can factorize the polynomial as in **(4.56)**:

$$\left(\Lambda^2 + \frac{p_1}{p_3} \right) \left(\Lambda^2 + p_3 \Lambda + p_2 - \frac{p_1}{p_3} \right), \tag{4.90}$$

this is possible provided that on requires the factorization condition:

$$\frac{p_1}{p_3} + \frac{p_0 p_3}{p_1} = p_2 \quad \longrightarrow \quad \frac{1}{4} P_1^2 \left(J^2 + 4P_4 \right) - \frac{1}{8} P_1 (4P_2 + 1)(2P_3 - J) + \left(P_3 - \frac{J}{2} \right)^2 = 0, \tag{4.91}$$

which, as we know, allows us to derive the BPS constraint on the charges. Indeed, this is a quadratic equation in J which can be solved to give:

$$J = \frac{8P_3 - 4P_1 P_2 - P_1 \pm |P_1| \sqrt{X}}{4(1 + P_1)^2}, \tag{4.92}$$

where:

$$X = \Big[\big(1 + 4(Q_2 + Q_3)(Q_1 + Q_4) \big) \big(1 + 4(Q_1 + Q_3)(Q_2 + Q_4) \big) \big(1 + 4(Q_1 + Q_2)(Q_3 + Q_4) \big) \Big], \tag{4.93}$$

remember that imposing Eq. **(4.92)**, one should recover the BPS solution where the charges are real, and hence one must have $X \geq 0$. Moreover, we will see that the correct BPS constraint is reproduced in the pairwise equal and single charged case provided that one choose the plus sign in Eq. **(4.92)**.

The BPS entropy of the outer horizon is then obtained from the imaginary root $\Lambda_i = i\sqrt{\frac{p_1}{p_3}}$

$$\Lambda^2 = \frac{p_1}{p_3} = \frac{1 + 4P_2 + 8P_1 P_3 - \sqrt{X}}{8(1 + P_1^2)}, \tag{4.94}$$

where the BPS condition **(4.92)** has already been imposed.

Finally, one can find the entropies of the remaining two horizons by solving the quadratic equation $\Lambda^2 + p_3\Lambda + p_2 - \frac{p_1}{p_3} = 0$. The virtual horizons entropies $(S_-, \bar{S}_-)$ are then proportional to the roots:

$$\Lambda_- = \frac{1}{2}\left[-p_3 \pm \sqrt{4\frac{p_1}{p_3} - 4p_2 + p_3}\right] = \frac{1}{2}\left[-P_1 \mp i\sqrt{\frac{1 - 2P_1^4 + 8P_2P_1^2 - 8P_3P_1 + 4P_2 + \sqrt{X}}{2P_1^2 + 2}}\right], \quad (4.95)$$

again we have already imposed the BPS condition **(4.92)**.

Finally, following the same logic as in **(4.60)** one can show that the universal area product and the BPS constraint are equivalent in the BPS solution.

Reduction to the pairwise equal and single charged cases

Let us see what results we find when we reduce to the pairwise equal or single charged solutions. We can get these cases by imposing the following redefinitions for the pairwise equal case:

$$\begin{cases} \phi^1 = \phi^2 \to \frac{1}{2}\phi^1, \quad \phi^3 = \phi^4 \to \frac{1}{2}\phi^3 \\ Q_1 = Q_2 \to 2Q_1, \quad Q_3 = Q_4 \to 2Q_3 \end{cases} \longrightarrow \begin{cases} I = \pm\frac{i}{2}\frac{\phi^1\phi^3}{\omega}, \quad \omega - \phi^1 - \phi^3 = \pm 2\pi i \\ I = -S - \omega J - 2\phi^1 Q_1 - 2\phi^3 Q_3 \end{cases}, \quad (4.96)$$

while the single charged case is obtained by imposing the redefinitions:

$$\begin{cases} \phi^I \to \frac{1}{2}\phi \\ Q_I \to \frac{1}{2}Q \end{cases} \longrightarrow \begin{cases} I = \pm\frac{i}{2}\frac{(\phi)^2}{\omega}, \quad \omega - 2\phi = \pm 2\pi i \\ I = -S - \omega J - \phi Q \end{cases}, \quad (4.97)$$

in these two cases, the main results that we have found above simplify a lot. Starting from the BPS condition **(4.92)** one gets:

$$J = (Q_1 + Q_3)\left(\sqrt{1 + 64Q_1Q_3} - 1\right), \qquad J = \frac{Q}{2}\left(\sqrt{1 + 4Q^2} - 1\right) \quad (4.98)$$

which are exactly the results found in **(4.15, 4.73)** , then the outer horizon BPS entropy is found to be proportional to the purely imaginary root $\Lambda = \pm i\sqrt{\frac{p_3}{p_1}}$ which is given by:

$$\Lambda^2 = \frac{1}{8}\left(1 + 32Q_1Q_3 - \sqrt{1 + 64Q_1Q_3}\right) = \frac{1}{16}\left(\sqrt{1 + 64Q_1Q_3} - 1\right)^2, \quad (4.99)$$

$$\Lambda^2 = \frac{1}{8}\left(1 + 2Q - \sqrt{1 + 4Q^2}\right) = \frac{1}{16}\left(\sqrt{1 + 4Q^2} - 1\right)^2. \quad (4.100)$$

Similarly, one finds that also the roots associated with the virtual horizons Λ_- simplify, they are given by:

$$\Lambda_- = -2(Q_1 + Q_3) \mp \frac{i}{4}\left(\sqrt{1 + 64Q_1Q_3} + 1\right), \qquad \Lambda_- = -Q \mp \frac{i}{4}\left(\sqrt{1 + 4Q^2} + 1\right), \quad (4.101)$$

for the pairwise equal case one can easily check that Λ_- correctly reproduces the expected value for the BPS inner horizon entropy Eq. **(4.80)**.

This proves the universality of the extremization principle also for the pairwise equal charged solution.

4.5 Single charged, spinning AdS_4 black hole with acceleration

As our last example, we are going to consider the generalization of the black hole solution of minimal $D = 4, \mathcal{N} = 2$ gauged supergravity discussed in Sec. **4.1**, with acceleration and magnetic charge [70]. This solution depends on two more parameters p, α related to the magnetic charge and acceleration.

The presence of the acceleration makes the event horizon, and more generally all constant (t, r) slices have conical singularities, these stretch up to the AdS boundary and are usually referred to as *cosmic strings* [16]. These singularities can be completely removed if one embeds the $D = 4$ solution in $D = 11$ supergravity [73] and imposes some constraints on the parameters of the $D = 4$ solution. In particular, one needs to *quantize* the conical deficits on the north and south poles of the horizon, this makes the horizon a *spindle* whose topology is determined by two different integers $n_\pm$, related to the N/S pole singularities. For the purposes of our discussion, the only new property that we are going to use about the horizon geometry is its Euler characteristics χ. Other details about the geometry of the horizons will not be needed.

From the constraints that one has to impose to get a regular solution, after the uplifting, one finds that the magnetic charge is fixed and depends on the properties of the spindle, in particular $G_{(4)}Q_m = (n_- - n_+)/(4n_-n_+)$ where $G_{(4)}$ is the four-dimensional Newton's constant. This implies that one necessarily has to consider a magnetically charged solution (to have regularity after the uplifting). We will also see that the presence of the magnetic charge is necessary in order to impose supersymmetry.

Unless explicitly stated, we will work with fixed spindle topology, hence fixed $n_\pm$ parameters, and in the assumption that the solution is regular under the uplifting. In this way, it has been shown [70] that the new parameters p, α are fixed in terms of the spindle topology. This means that the solution we are going to consider is described by the same three parameters (or charges) of Sec. **4.1**, plus two *fixed* constants $n_\pm$ or equivalently $(G_{(4)}Q_m, \chi)$ which are treated on the same footing as the AdS radius[12] g. However, some of the results that we are going to find are true also in the general case, where we do not make any assumption on the new parameters or the horizon topology.

In this framework, we want to see if the results of Sec. **4.1** generalizes to the present case, we will give first a review of the solution and explain the role of the new parameters, next we will discuss the properties of all the horizons of this solution and prove the universality of the extremization principle. We will also derive the universal area product formula which has never appeared before in the literature for this solution.

4.5.1 Review of the solution

We will quickly review the black hole solution following [16], the metric is given by:

$$ds^2 = \frac{1}{H^2}\Bigg[-\frac{Q}{\Sigma}\left(\frac{1}{\kappa}dt - a\sin^2\theta d\phi\right)^2 + \frac{\Sigma}{Q}dr^2 + \frac{\Sigma}{P}d\theta^2 + \frac{P}{\Sigma}\sin^2\theta\left(\frac{a}{\kappa}dt - (r^2+a^2)d\phi\right)^2\Bigg], \tag{4.102}$$

where we have defined:

$$\begin{aligned} P(\theta) &= 1 - 2\alpha m\cos\theta + \Big(\alpha^2(a^2+e^2+p^2) - a^2\Big)\cos^2\theta, \\ Q(r) &= (r^2 - 2mr + a^2 + e^2 + p^2)(1 - \alpha^2 r^2) + r^2(a^2 + r^2), \\ H(r,\theta) &= 1 - \alpha r\cos\theta, \qquad \Sigma(r,\theta) = r^2 + a^2\cos^2\theta. \end{aligned} \tag{4.103}$$

[12] One can relax these assumptions and still obtain a consistent thermodynamics by considering the AdS radius, the magnetic charge, and the cosmic strings, as thermodynamic variables [16], the reason why we want to fix all these parameters is that in this way we get a fixed geometry on the conformal boundary where the dual CFT lives, remember that the cosmic strings "touch" the conformal boundary because all constant (t, r) slices have conical singularities. Moreover, the presence of a magnetic charge influences the properties of the gauge field at infinity.

In order to avoid confusion with the notation of the previous sections, we denote the parameter associated with the magnetic charge as p instead of g as was done in [16].

The expression of the gauge field can be found in the references. The solution depends on the already mentioned parameters (m, e, p, a, α) which we consider to be all positive following [16]. κ is a normalization constant for the time coordinate, it turns out that it is related to the acceleration parameter α in order to have a first law of the correct form [16]. Notice that the parameters that we are using here are not exactly the same as the ones in Sec. **4.1**, as one can easily see by setting $p = \alpha = 0$ in the above metric. However, in this case it can be shown that the solution reduces to the one discussed in Sec. **4.1** [70].

The positions of the horizons are given by the roots of the $Q(r)$ quartic polynomial, so there are four horizons (either physical or virtual). In this case, we need to require for the outer horizon radius, the following condition:

$$0 < r_+ < \frac{1}{\alpha}, \tag{4.104}$$

which ensures that the outer horizon is located between the singularity at $r = 0$ and the conformal boundary at $H(r, \theta) = 0$, and that the horizon does not touch the conformal boundary at $\theta = 0$.

Horizon topology and regularity

To understand how the $n_\pm$ parameters appear, we need to study the constant (t, r) slices Σ parametrized by (θ, ϕ). These have the topology of a 2-sphere but with conical singularities at the poles. As usual, θ can be taken with range $0 \leq \theta \leq \pi$, next let us consider the restriction of the metric on the (θ, ϕ) coordinates and evaluate it near the north/south poles at $\theta_{+,-} = 0, \pi$.

$$\begin{aligned} ds^2|_{\theta,\phi} &= \frac{1}{H^2}\left[\frac{\Sigma}{P}d\theta^2 + \frac{P}{\Sigma}\left((r^2+a^2)^2 - \frac{Q}{P}a^2\sin^2\theta\right)\sin^2\theta d\phi^2\right] \\ &\xrightarrow[\theta\approx 0,\pi]{} \frac{(r^2+a^2)}{H^2 P_\pm}\left[d\theta^2 + \sin^2\theta P_\pm^2 d\phi^2\right], \end{aligned} \tag{4.105}$$

where

$$P_\pm = P(\theta)|_{0,\pi} = \Xi \pm 2\alpha m, \qquad \Xi = 1 + \alpha^2(a^2+e^2+p^2) - a^2. \tag{4.106}$$

Eq. **(4.105)** is the metric of a 2-sphere (near the poles), which is regular only provided that the ϕ coordinate is chosen with periodicity $\Delta_\phi = 2\pi/P_\pm$.

Clearly, unless $\alpha m = 0$ we have $P_+ \neq P_-$, meaning that one cannot avoid at least one conical singularity on Σ. This singularity stretches up to the boundary generating the "cosmic string", as it is present for all constant (t, r) slices.

Following [16, 73] we choose the periodicity Δ_ϕ to be:

$$\frac{\Delta_\phi}{2\pi} = \frac{1}{n_+ P_+} = \frac{1}{n_- P_-}, \tag{4.107}$$

this implies that the conical defects at the poles have deficit angles given by $2\pi\left(1 - \frac{1}{n_\pm}\right)$, and one has $\Sigma = S^2$ if $n_+ = n_- = 1$, which is possible only provided that $\alpha m = 0$.

Notice that Eq. **(4.107)** fixes Δ_ϕ but also represents a constraint on the parameters *once we fix both* $n_\pm$. The idea is that both conical singularities are fixed by Δ_ϕ, as it is expressed by the above equation. This also implies that once we fix the conical singularity on one pole, say the n_+ parameter, the conical singularity on the other pole is fixed by $n_- = \frac{P_+}{P_-}n_+$ and cannot be chosen freely. We have to tune the parameters of the solution in such a way that after we have independently chosen n_+ and n_-, it holds the relation **(4.107)**, and specifically:

$$\frac{n_+}{n_-} = \frac{P_-}{P_+} = \frac{\Xi - 2\alpha m}{\Xi + 2\alpha m}, \tag{4.108}$$

remember that this happens only if we fix both n_+ and n_- as it was done in [16, 73], this condition will be used later to fix the α parameter in terms of $n_\pm$.

For later convenience, it is useful to define the tensions of the "cosmic strings" as the parameters:

$$\mu_\pm = \frac{1}{4G_{(4)}}\Big[1 - \frac{1}{n\pm}\Big] = \frac{1}{4G_{(4)}}\left[1 - (\Xi \pm 2\alpha m)\frac{\Delta_\phi}{2\pi})\right]. \tag{4.109}$$

and the Euler characteristic of Σ:

$$\chi = \frac{1}{n_+} + \frac{1}{n_-} = 2 - 4G_{(4)}(\mu_- + \mu_+), \tag{4.110}$$

which computes to $\chi = 2$ if $n_- = n_+ = 1$ as expected for a sphere.

We can now clarify in which sense we *quantize* the conical defects on the horizon, this simply means that we have to choose $n_\pm$ as coprime positive integers, this makes Σ a weighted projective space, which is an orbifold also known as a spindle [16]. This quantization condition is one of the requirements that allow to obtain a regular solution after the uplifting[13].

The quantization condition is not sufficient to describe a regular solution, one also needs to impose the following condition on the parameters[14]:

$$p = \alpha\, m, \tag{4.111}$$

this will turn out to be also one of the supersymmetry constraints.

Before moving on, let us recap what we have found so far for these "regular" solutions. We have the parameters[15] $(m, e, p, a, \alpha; n_\pm)$, plus two constraints **(4.108, 4.111)**, the latter can be directly implemented, while the first one can be solved more easily by first introducing the parameter [16]:

$$\mu \equiv \frac{1 - 2G_{(4)}(\mu_- + \mu_+)}{2G_{(4)}(\mu_- - \mu_+)} = \frac{n_- + n_+}{n_- - n_+}, \tag{4.112}$$

this allows us to rewrite the condition **(4.108)** as follows:

$$\frac{n_-}{n_+} = \frac{\Xi + 2\alpha m}{\Xi - 2\alpha m} \quad \longrightarrow \quad \Xi = 2\alpha m\mu \quad \xrightarrow[\alpha m = p]{} \quad \Xi = 2p\mu, \tag{4.113}$$

remembering the definition of Ξ we can easily find for α the following:

$$\alpha^2 = \frac{2p\mu - 1 + a^2}{a^2 + e^2 + p^2}, \tag{4.114}$$

this fixes α in terms of the spindle data $n_\pm$ and the other parameters.

Notice that this condition and **(4.111)** are then easily implemented if one uses p as a parameter describing the solution, while α and m are fixed in terms of the others. In the following, we may want to use as parameter m instead and consider α, p fixed. This can formally be done but one has to deal with a cumbersome expression for α (and therefore p) as now the condition **(4.114)** (or equivalently **(4.112)**), is a fourth order equation in α.

[13] From now on, we will simply refer to these regular solutions, which can be obtained after the uplift, as "regular solutions"

[14] There would also be another technical condition to impose on the geometry of the $D = 11$ supergravity theory, which is obtained after the uplift, but it does not play any role in our discussion so we will ignore it.

[15] Clearly, we assume the AdS radius g as fixed, and we do not include the κ parameter which turns out to be fixed in terms of the others.

Either way, we will refer to the above conditions **(4.112, 4.114)** as "regularity" conditions, it is understood that when we impose these we are thinking $n_\pm$ as taking fixed values, hence fixing the spindle geometry.

We will consider this case in more detail later but for now it is sufficient to keep in mind that α and p can formally be expressed in terms of m and the other parameters.

Thermodynamics

From now on we will set also $G_{(4)}=1$.

In this section, we provide the relevant quantities entering the thermodynamics of this black hole solution, more details can be found in[16] [16, 73] but they are not essential for our discussion.

The solution admits four conserved charges (E, J, Q_e, Q_m):

$$Q_e=\frac{e\Delta_\phi}{2\pi},\qquad Q_m=\frac{p\Delta_\phi}{2\pi},\qquad J=am\left(\frac{\Delta_\phi}{2\pi}\right)^2,\qquad E=\frac{m\Delta_\phi}{2\pi\kappa}\frac{(\Xi+a^2)(1-a^2\Xi)}{\Xi(1+\alpha^2a^2)},\tag{4.115}$$

these definitions are true in the general solution, where we have not imposed the regularity conditions yet.

These quantities were calculated in [16] using holographic renormalization. In order to do so, one needs to remember that the conformal boundary of the metric **(4.102)**, where the charges are calculated as surface integrals by using the boundary stress-tensor, is located at $H(r,\theta)=0$. Furthermore, the energy E of the solution must be calculated considering the asymptotic killing vector $\partial_{\bar t}=\partial_t+\Omega_\infty\frac{\Delta_\phi}{2\pi}\partial_\phi$, where $\Omega_\infty=-\frac{2\pi}{\kappa\Delta_\phi}\frac{a(1-\alpha^2\Xi)}{\Xi(1+a^2\alpha^2)}$. These definitions allow us to obtain a set of charges that satisfy standard thermodynamics relations.

Notice that *once we impose the regularity conditions*, from the definition of the magnetic charge Q_m and $p=\alpha m$ one can show that:

$$Q_m=\mu_--\mu_+=\frac{n_--n_+}{4n_-n_+}=\frac{1}{2n_+(1+\mu)},\tag{4.116}$$

hence, the magnetic charge is completely determined by the spindle data $n_\pm$ in the regular solution[17]. This allows us to equivalently use the combinations (n_+,n_-),(μ,n_+) or (Q_m,χ) to express the spindle data.

Moving on with the thermodynamics, the chemical potentials associated with the above charges are given by:

$$\Omega=\underbrace{\frac{2\pi}{\kappa\Delta_\phi}\frac{a}{r_+^2+a^2}}_{\Omega_H}-\Omega_\infty,\qquad T=\frac{Q'(r_+)}{4\pi\kappa(a^2+r_+^2)},\qquad \Phi_e=\frac{er_+}{\kappa(r_+^2+a^2)},\tag{4.117}$$

where V is the null killing vector on the horizon: $V=\partial_t+\frac{\Delta_\phi}{2\pi}\Omega_H\partial_\phi$, one may also define a magnetostatic potential Φ_m, which one has to take into account when considering the thermodynamics of the general solution, but it does not play any role for us.

The Bekenstein-Hawking entropy reads:

$$S=\frac{\Delta_\phi}{2}\frac{r_+^2+a^2}{1-\alpha^2r_+^2},\tag{4.118}$$

[16] See also [74] for a discussion of the thermodynamics of this black hole solution with vanishing magnetic charge.

[17] The idea is that in the regular solution all topological data (the cosmic strings and the magnetic charge) are fixed in such a way that after the uplifting the singularities are removed [16].

while the Euclidean on-shell action associated with the outer horizon is given by [16]:

$$I = \frac{\beta\Delta_\phi}{16\pi\kappa}\left[-4r_+\left(\frac{a^2+r_+^2}{(\alpha^2 r_+^2-1)^2}+\frac{e^2-p^2}{a^2+r_+^2}\right)+4m\left(1-2\alpha^2-\frac{2\alpha^4(e^2+p^2)}{1+\alpha^2 a^2}\right)\right]. \tag{4.119}$$

One can show that these quantities satisfy the following quantum statistical relation, independently from the value of κ, and also without imposing the regularity conditions

$$I = -S + \beta\, E - \beta\,\Omega\, J - \beta\,\Phi_e\, Q_e, \tag{4.120}$$

while the usual first law is recovered only provided that one fixes $\kappa = \frac{\sqrt{(\Xi+a^2)(1-\alpha^2\Xi)}}{1+a^2\alpha^2}$ and imposes the regularity conditions, so that the thermodynamics of the solution depends on the 3 charges (E, J, Q_e), and their corresponding chemical potentials defined above

$$dE = T\, dS + \Omega\, dJ + \Phi_e\, dQ_e, \tag{4.121}$$

notice that if the acceleration is absent $\alpha = 0$ then $\kappa = 1$, hence this parameter is intrinsically related to the acceleration.

4.5.2 Supersymmetry and BPS solutions

Let us consider the regular solution in this section (fixed spindle topology). Following [16, 73] the solution admits a killing (Dirac) spinor, and hence preserves supersymmetry provided that the parameters satisfy the conditions:

$$\begin{aligned} p &= \alpha m, \\ 0 &= \alpha^2(e^2+p^2)(\Xi+a^2)-(p-a\alpha e)^2, \end{aligned} \tag{4.122}$$

the first condition is also one of the two regularity conditions, and can only be satisfied provided that $p \neq 0$.

The supersymmetric solution is also extremal if the parameters satisfy the additional constraint [73]:

$$ap^2(a\alpha e-p)(e+a\alpha p)+\alpha^3 e^2(e^2+p^2)^2 = 0, \tag{4.123}$$

unfortunately, implementing these conditions is rather cumbersome.

Luckily, the situation simplifies if one introduces the new set of parameters (b, c, s) given by [16]:

$$e = \frac{bs}{\alpha^2 c}, \quad p = \frac{s}{\alpha^2 c}, \quad m = \frac{s}{\alpha^3 c}, \quad a = \frac{s}{\alpha}, \quad \Longleftrightarrow \quad b = \frac{e}{p}, \quad c = \frac{a}{p\alpha}, \quad s = \alpha a, \tag{4.124}$$

we need to assume $a, \alpha \neq 0$. Remembering that we are considering all parameters to be non-negative then we have $b \geq 0$ and $c, s > 0$. The first supersymmetry condition is automatically implemented in these definitions. In the following, we will provide only the explicit expressions for the quantities that we will directly use, and briefly summarize how the BPS solution is obtained. More details can be found in [16].

In terms of the new parameters one finds that the charges take the simple expressions:

$$Q_e = b\, Q_m, \qquad J = c\, Q_m^2, \qquad M = \frac{Q_m\sqrt{(2cQ_m-\chi s)(2csQ_m+\chi)}}{\chi\sqrt{s}}, \tag{4.125}$$

positivity of the mass M implies that $c > 2s\mu$.

The supersymmetry condition in Eq. **(4.122)** reduces to a linear equation for c, which is solved by:

$$c = \frac{2(1+b^2)s\mu}{1-2bs-s^2}, \tag{4.126}$$

the extremality condition Eq. **(4.123)** now becames:

$$b^2(1+b^2)^2 s = c^2(1-bs)(b+s) \xrightarrow[\text{Eq. (4.126)}]{} b^2(b^2+1) = c(c+2b\mu), \tag{4.127}$$

unfortunately, this condition is still not practical to implement. We will need to exploit a different method, to obtain the BPS solution, which we are going to discuss later.

Remember that α is determined from the other parameters as:

$$\alpha^2 = \frac{2p\mu - 1 + a^2}{a^2e^2 + p^2} = \frac{s[2c\mu + s(c^2-1-b^2)]}{c^2(1+s^2)} \xrightarrow[\text{susy}]{} \frac{4\mu^2(1-bs)^2 - (1-2bs-s^2)^2}{4\mu^2(1+b^2)(1+s^2)}, \tag{4.128}$$

the supersymmetric solution is now parametrized by (b,s), plus the two constants (χ, Q_m). Accordingly, only two charges are independent and indeed, they satisfy the usual relation:

$$M = \frac{2}{\chi}J + Q_e, \tag{4.129}$$

which may be expected to be a consequence of the supersymmetry algebra evaluated on the solution [16].

The supersymmetric trajectory

As usual, we want to use one of the horizon radii as parameter for the supersymmetric solution by inverting the relation $Q(r) = 0$. This will also allow us to more easily obtain the BPS solution.

Following [16], it is convenient to define first:

$$r \equiv \frac{s}{\alpha}\rho, \tag{4.130}$$

where it is understood that we can formally work with any horizon radius r_i, solution of $Q(r_i) = 0$.

The new quartic polynomial $\mathcal{Q}(\rho)$, despite being more complicated than the starting one (Eq. **(4.103)**), is now only quadratic in b. The condition $\mathcal{Q}(\rho)$ is thus easily solved as:

$$b = \frac{2\mu\rho}{\rho^2-1} + \frac{(1-s^2 \pm 2i\mu s)}{2s(\rho^2-1)\big(\rho^2s^2 - 1 \mp i\mu s(\rho^2+1)\big)}B(\rho,s), \tag{4.131}$$

where:

$$B(\rho,s) = (1-\rho^2)(1-\rho^2s^2) + 2\mu(1+\rho^2)\rho s, \tag{4.132}$$

imposing that b remains real, as we have done before, forces us to consider complex ρ radii, which are organized in two complex conjugated pairs as $Q(\rho)$ has real coefficients. This is analogous to the discussion in Sec. **4.2.1**, remember that one has to choose the opposite sign choice in **(4.131)** when considering the conjugated horizons.

Alternatively, one can keep $\rho \in \mathbb{R}$ and introduce complex b and s parameters as in [16]. Eq. **(4.131)** is the usual equation that allows to trade one of the parameters of the solution (in this case b) for one of the horizons radii.

The chemical potentials can be shown to satisfy the usual relation:

$$\beta\Big(1 + \frac{\chi}{2}\Omega - 2\Phi_e\Big) = \mp 2\pi i, \tag{4.133}$$

the sign is correlated to the sign choice in Eq. **(4.131)**. The modified chemical potentials (ω, ϕ) are:

$$\omega = \beta\left(\Omega - \frac{2}{\chi}\right), \qquad \phi = \beta(\Phi_e - 1), \qquad \frac{\chi}{4}\omega - \phi = \mp\pi i, \tag{4.134}$$

notice that the same relations of the non-accelerating case (Eqs. **(4.11)**) can be obtained by setting $\chi = 2$.

In terms of the supersymmetric chemical potentials, the *universal* euclidean action can be rewritten as follows:

$$I = -S - \omega\, J - \phi\, Q_e = \mp\frac{i}{2}\left[\frac{\phi^2}{\omega} + Q_m^2\, \omega\right], \tag{4.135}$$

with our choice of parametrization, we have real charges, while the entropy and the chemical potentials are complex. Notice that the above result reduces to the non-accelerating one in Eq. **(4.37)** by setting $Q_m = 0$.

This discussion was originally made for the outer horizon, but we can generalize it with no effort to any horizon. Hence we can immediately regard the chemical potentials **(4.134)** and the supersymmetric Euclidean action **(4.135)**, as the universal ones.

The BPS solution

Let us now consider how the BPS solution can be derived. We need to impose extremality on the solution, which is achieved whenever the quartic polynomial $\mathcal{Q}$ has a vanishing first derivative w*hen evaluated on a real value* of ρ, which then becomes the radius of the BPS outer horizon. This is equivalent to requiring that $\mathcal{Q}$ has a double *real* root ρ_+.

$$\mathcal{Q}'(\rho) = \frac{4s^2\mu^2(1+s^2)^2(\rho \mp i)\big(\pm i\rho s^2 + \mu(\rho \mp i)s - 1\big)}{\big(1 - \rho^2 s^2 \pm i\mu(1+\rho^2)s\big)^2} B(\rho, s) = 0, \tag{4.136}$$

where we have used the definition of the b parameter in Eq. **(4.131)**.

By solving the extremality condition *for a real value of* ρ, we clearly break the invariance under exchange of horizon radius as parameter, as requiring reality forces us to consider the outer (physical) horizon radius.

The only allowed *real* solution of Eq. **(4.136)** is obtained if $B(\rho_+, s) = 0$ which can be seen as a quadratic equation in s, the other possibility would need to require $\rho_+ s = 1$ and $\mu = 1$ which is not possible due to the definition of μ.

In this way, the BPS solution is parametrized in terms of ρ_+ and μ as follows:

$$s^\star = \frac{-\mu(1+\rho_+^2) + \sqrt{\mu^2(1+\rho_+^2) + (\rho_+^2 - 1)^2}}{\rho_+(\rho_+^2 - 1)}, \quad c^\star = \frac{2\mu\rho}{\rho_+^2 - 1}\left[-\mu + \frac{\sqrt{\mu^2(\rho_+^2+1)^2 + (\rho_+^2-1)^2}}{\rho_+^2 - 1}\right],$$

$$b^\star = \frac{2\mu\rho_+}{\rho_+^2 - 1}, \tag{4.137}$$

from the constraints $0 < \alpha r_+ = s\rho_+ < 1$ and $s > 0$ one immediately finds that $0 < s^\star < 1$ and $\rho_+ > 1$.

Equivalently, after some calculations, one can re-express the BPS solution in terms of b and μ as follows:

$$\rho_+ = \frac{\mu + \sqrt{b^2 + \mu^2}}{\mu}, \;\; s^\star = \frac{\left(-\mu + \sqrt{b^2+\mu^2}\right)\left(\mu + \sqrt{1 + b^2 + \mu^2}\right)}{b} - b, \;\; c^\star = b\left(\sqrt{1 + b^2 + \mu^2} - \mu^2\right), \tag{4.138}$$

in both cases, one can check that the supersimmetry and extremality conditions Eqs. **(4.126, 4.127)** are both satisfied, and that $\rho_+(b;\mu)$ is a root of the $\mathcal{Q}$ polynomial.

One can show that in the BPS limit, the conserved charges satisfy the additional constraint:

$$J^\star = \frac{Q_e^\star}{4}\left(-\chi + \sqrt{\chi^2 + 16\big(Q_e^{\star 2} + Q_m^\star\big)^2}\right), \tag{4.139}$$

and the outer horizon chemical potentials take the fixed value:

$$T^\star = 0, \qquad \Omega^\star = \frac{2}{\chi}, \qquad \Phi_e^\star = 1, \qquad \Phi_m^\star = \frac{1}{b^\star}, \tag{4.140}$$

finally the outer horizon BPS entropy takes the following expression:

$$S^\star = \pi Q_m s^\star c^\star \frac{\rho_+^2 + 1}{1 - s^{\star 2}\rho_+^2} = \frac{\pi}{4}\left(\sqrt{\chi^2 + 16\big(Q_e^{\star 2} + Q_m^2\big)} - \chi\right), \tag{4.141}$$

notice that in all expressions involving only the charges, by setting $\chi = 2, Q_m = 0$ one again obtains exactly the solution of the non-accelerating case discussed before[18].

4.5.3 Properties of the general horizons of the black hole solution

We turn our attention to the other horizons of this black hole solution, following closely the presentation in Sec. **4.2**, as the results that we are going to find are the generalizations of those of the non-accelerating solution. Many calculations will be only briefly described as they represent a trivial generalization of those of Sec. **4.2**.

Qualitative behaviour of the 4 roots of $Q(r)$ in the regular solution

Let us concentrate on the regular solution in the following discussion.

In order to study what kind of horizons characterize this black hole solution, we need to study the $Q(r)$ polynomial. Remember that we have to impose the following condition on the position of the outer horizon:

$$0 < r_+ < \frac{1}{\alpha}. \tag{4.142}$$

The relations that determine the four roots are obtained as usual by rewriting the $Q(r)$ polynomial as:

$$Q(r) = (1-\alpha^2)\prod_i (r - r_i) \;\longrightarrow\; \begin{cases} r_1 + r_2 + r_3 + r_4 = -\frac{2m\alpha^2}{1-\alpha^2} \\ r_1 r_2 + \cdots = \frac{1 + a^2(1-\alpha^2) - \alpha^2(e^2+p^2)}{1-\alpha^2} \\ r_1 r_2 r_3 + \cdots = \frac{2m}{1-\alpha^2} \\ r_1 r_2 r_3 r_4 = \frac{a^2 + e^2 + p^2}{1-\alpha^2} \end{cases}. \tag{4.143}$$

Due to the acceleration, it is not true anymore that two of the four roots are necessarily complex. Indeed, by looking at the expression of $Q(r)$ we see that negative roots are now possible provided that $|r_i| > \frac{1}{\alpha}$, and the argument that we have made in Sec. **4.1** breaks down. However, also in this case, there are generally only two physical horizons (when they exists) located at $0 < r_i < \frac{1}{\alpha}$ while the other two horizons are complex conjugated, hence virtual.

First of all, notice that we can have at least one physical horizon located at $r_+ < \frac{1}{\alpha}$ only provided that the acceleration parameter satisfies the bound:

$$0 \le \alpha < 1, \tag{4.144}$$

[18] The non accelerating limit is more delicate if one wants to take it also for the b, c, s parameters as from their definition, if $\alpha = 0$ these parameters become infinite

this can be easily seen by imposing $p = \alpha m$ and noticing that $Q(r)$ is strictly positive if $a \geq 1$ and $r < \frac{1}{\alpha}$ as:

$$\begin{aligned} Q(r) &= (r^2 - 2mr + a^2 + e^2 + \alpha^2 m^2)(1 - \alpha^2 r^2) + r^2(a^2 + r^2) \\ &> \underbrace{(r^2 - 2mr + \alpha^2 m^2)}_{>0}\underbrace{(1 - \alpha^2 r^2)}_{>0}, \end{aligned} \tag{4.145}$$

the quadratic polynomial in the second line vanishes if $\alpha = b(1 \pm \sqrt{1-\alpha^2})$, meaning that if $\alpha > 1$ it is never zero and has a fixed positive sign. From this discussion we also see that as α approaches 1, the other parameters must be closer to 0 in order to have a physical horizon that does not touch the boundary.

We should now also impose the condition (**4.114**) in order to truly parametrize the "regular" solution, it would be more convenient to use the p parameter, but we use m instead, which allows us to more directly compare the following results with those of Sec. **4.1**.

Using $p = \alpha m$ into Eq.**(4.114)** produces a quartic equation for α:

$$\alpha^2 = \frac{2\alpha m\mu - 1 + a^2}{a^2 e^2 + \alpha^2 m^2 a^2}, \tag{4.146}$$

for suitable values of the $(m, e, a; \mu)$ parameters, there are two real and positive solutions $\alpha = \alpha_{1,2}(m, e, a; \mu)$ of **(4.146)**. The greater of the two (say α_2) can possibly be bigger or smaller than 1 and one can check graphically that it is never associated with a solution with physical horizons at $r_+ < \frac{1}{\alpha_2}$, for any value of the $(m, e, a; \mu)$ parameters. Instead, the smaller solution $\alpha_1(m)$ generally allows to find a black hole solution that admits two physical horizons $r_+ > r_0$, where r_+ satisfies the bound $r_+ < \frac{1}{\alpha_1}$. In this case, when the two physical horizons exist, the other two horizons are always found to be complex. Moreover, these configurations reduce to the BPS solutions discussed in Sec. **4.5.2** for suitable values of the parameters that satisfy the supersymmetric and extremality constraints.

The qualitative behaviour of the $Q(r, m; \alpha_{1,2}(m)) = 0$ curves is shown in Fig.**4.3**.

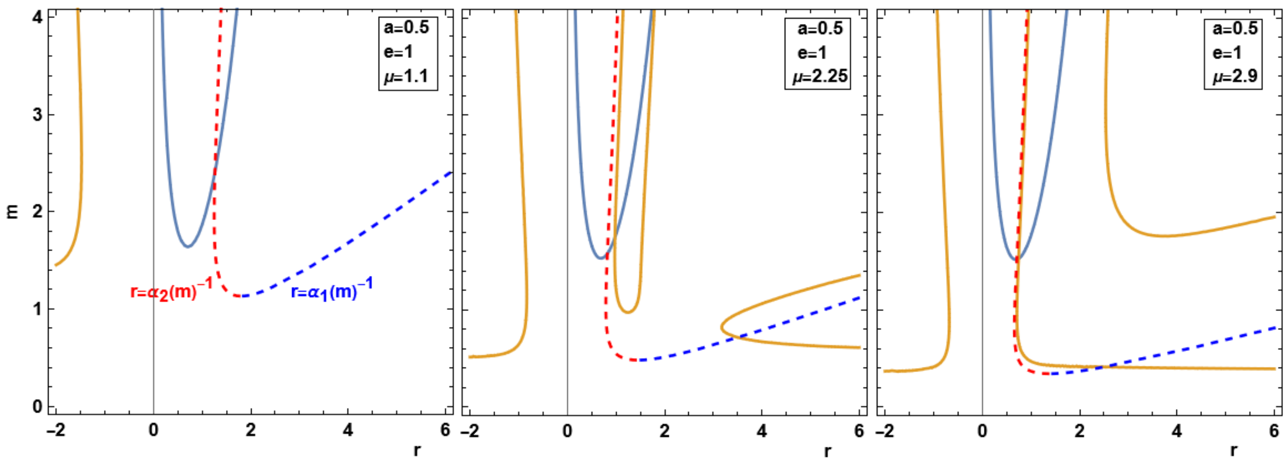

Figure 4.3: The solid lines represent the graphs of $Q(r, m; \alpha_i(m)) = 0$, associated with the regular solution, obtained by setting $p = \frac{\alpha}{m}$, and $\alpha = \alpha_i(m)$ as given by the two real solution of **(4.114)** (when they exists). The blue/orange solid lines are obtained considering the two real roots $\alpha_i(m)$, respectively the smaller(α_1)/largest(α_2) one. The dashed lines represent the $r = \alpha_i(m)^{-1}$ curves (position of the boundary at $\theta = 0$), again the blue/red one is associated with the smaller/largest root $\alpha_i(m)$. Choosing the smaller solution $\alpha_1(m)$ of **(4.114)** produces a solution with two physical horizons that never touch the boundary (for every fixed value of m, the points on the blue solid line always satisfy $r < \alpha_1(m)^-1$). Moreover, these solutions admit a BPS limit, meaning that we can continuoulsy change the parameters to obtain a BPS solution. Instead, choosing the largest root $\alpha_2(m)$ never produces a physical solution as the horizons always stretch beyond the boundary. Moreover, these solutions are not connected to BPS configurations. The qualitative behaviour does not change if we vary the parameters.

Finally, notice the different qualitative behaviour with respect to the case studied in Sec. **4.1**. In particular, in this case there is only one lower extremal solution (if one takes into account the cases where physical outer and intermediate horizons exist). For suitable values of the parameters (which satisfy the supersymmetry and BPS constraints), this extremal configuration is a BPS solution.

Virtual horizons BPS entropies

As we have discussed above, the BPS solutions are characterized by the presence of two physical horizons (which coincide, being the solution extremal) and two additional virtual horizons, associated with the two complex conjugated roots of the $Q(r)$ or $\mathcal{Q}(\rho)$ polynomial. Let us then call ρ_- the complex root associated with the virtual horizon with entropy $S_-^\star$, which is given by the same formula as the one for the outer horizon BPS entropy:

$$S_-^\star = \pi Q_m s^\star c^\star \frac{\rho_-^2+1}{1-s^{\star 2}\rho_-^2}, \tag{4.147}$$

we want to find an expression for $S_-^\star$ depending only on the charges. To do so we need to find ρ_-, which we parametrize as $\rho_- = A+iB$. Then, we can obtain the usual relations which determine the four roots ρ_i of $\mathcal{Q}(\rho)$ by rewriting: $\mathcal{Q}(\rho)=\prod_i(\rho-\rho_i)$, remembering that there is a double root ρ_+, as the solution is extremal, one finds:

$$\begin{cases} A+\rho_+ = -p_3 \\ A^2+B^2+4A\rho_+ +\rho_+^2 = p_2 \\ 2\rho_+(A^2+B^2+A\rho_+) = -p_1 \\ (A^2+B^2)\rho_+^2 = p_0 \end{cases} \qquad \longrightarrow \qquad \begin{cases} A = -\frac{p_3}{2}-\rho_+ \\ B = \pm\sqrt{\frac{p_0}{\rho_+^2}-A^2} \end{cases}, \tag{4.148}$$

The $\pm$ sign in front of the B coefficient is associated with the fact that we have two complex conjugated roots. Let us take the root with a positive imaginary part, by finding the parameters p_i from $\mathcal{Q}(\rho)$ and using either Eqs. **(4.137)** or Eqs. **(4.138)** to parametrize the BPS solution in terms of $(\rho_+;\mu)$ or $(b;\mu)$, one can show that the following relation holds for the virtual horizon:

$$S_-^\star = -\frac{\pi}{4}\left[\chi + 8iQ_e^\star + \sqrt{\chi^2+16\big(Q_e^{\star 2}+Q_m^2\big)}\right], \tag{4.149}$$

which again correctly reduces to the expression found for the non-accelerating case (Eq. **(4.17)**) in the usual limit.

We can use this result, together with the expression for the BPS outer horizon entropy **(4.141)**, to show the equivalence between the BPS constraint **(4.139)** and the area product formula, which we are going to derive now.

Universal area product formula

The area product formula for this black hole solution has not been derived yet, but it should reduce to the one found before, Eq. **(4.29)**, when we set $(Q_m=0,\chi=2)$. Accordingly, following the same steps as in Sec. **4.2.2**, and remembering the definition of the conserved charges (Eqs. **(4.115)**) one is able to find the following result for the product of the entropies:

$$\prod_i S_i = \pi^4\Big[(Q_e^2+Q_m^2)^2+4J^2\Big], \tag{4.150}$$

the calculations are exactly the same as those we have already done for the non accelerating solution, so we will not repeat them here.

We find that the area product formula is the natural generalization of the one previously found for the non-accelerating case **(4.29)**. This result actually holds in the general solution where we do not have imposed the regularity conditions yet. In this case Q_m can be chosen freely and does not depend on the topology of the horizon or acceleration. This means that in the general solution we are allowed to consider a non-vanishing acceleration and also $Q_m = 0$, in this case we see that the area product formula for the accelerating **(4.150)** and non-accelerating **(4.29)** black hole solutions are the same. This implies that the universal area product formula in the general solution is independent of both the energy, and the acceleration of the solution.

The situation is qualitatively different in the regular solution, in this chase Q_m is fixed in terms of the spindle data $n_\pm$ and hence depends on acceleration. However, the area product formula is still valid in the form **(4.150)**, and depends *only* on Q_m. This implies that, despite having formally added two new constants $n_\pm$, which characterize a specific family of accelerating black holes by fixing the topology of the Σ hypersurfaces, the area product formula formally depends on only one of them.

Remember in fact that the (n_+, n_-) constants can be traded for (Q_m, χ) in the regular solution, but the area product formula does not *explicitly* depend on χ. This means that the product formula is invariant under any transformation of the $n_\pm$ constants that leaves Q_m invariant[19]. Remembering that $Q_m = \mu_- - \mu_+$ we see that the transformation $\mu_\pm \to \mu_\pm + k$ leaves the magnetic charge invariant, which translates in a transformation of the spindle parameters $n_\pm$ of the form:

$$n_\pm \longrightarrow \frac{n_\pm}{1-4n_\pm k}, \quad \text{where:} \quad k \in \mathbb{Q} \text{ s.t} \begin{cases} \frac{n_+}{1-4n_+k} \in \mathbb{N} \\ \frac{n_-}{1-4n_-k} \in \mathbb{N} \end{cases}, \tag{4.151}$$

Finally, it is quite easy to show that using Eqs. **(4.147, 4.149)** for the horizons BPS entropies, and the area product formula Eq. **(4.150)** one can derive the BPS non-linear condition Eq. **(4.139)**.

$$J^{\star 2} = \frac{4}{\pi^4}\prod_i S_i^\star - 4\big(Q_e^{\star 2} + Q_m^2\big)^2 \quad \Longleftrightarrow \quad J^\star = \frac{Q_e^\star}{4}\left(-\chi + \sqrt{\chi^2 + 16\big(Q_e^{\star 2} + Q_m^2\big)^2}\right). \tag{4.152}$$

4.5.4 Universal extremization principle

The universal chemical potentials and thermodynamic potential I have been computed in Sec. **4.5.2** following the discussion of [16], they also showed that the outer horizon BPS entropy **(4.147)** is reproduced in the usual way from the extremization principle. We are now going to briefly discuss the extremization principle, partially discussed by **(4.147)**, and extending their discussion also for the other horizons.

The starting point is given by the usual universal quantities:

$$I = \mp\frac{i}{2}\Big[\frac{\phi^2}{\omega} + Q_m^2\,\omega\Big], \qquad \frac{\chi}{4}\omega - \phi = \mp\pi i, \tag{4.153}$$

we are considering the supersymmetric but non-extremal solution which is characterized by four complex horizons, organized in two coupled of conjugated horizons. In this framework, remember that the sign ambiguity in **(4.153)**, is associated with the sign ambiguity in the inverting relation $b = b(\rho_i)$ **(4.131)**, and one has to choose the opposite sign when considering an horizon and its conjugated.

The entropy is obtained from the constrained Legendre transform:

$$S = ext_{\{\omega,\phi,\Lambda\}}\left[-I - \omega J - \phi Q_e + \Lambda\left(\frac{\chi}{4}\omega - \phi \pm \pi i\right)\right], \tag{4.154}$$

[19] We also must require that Q_e and J are left invariant, but this can be easily done by suitably changing the parameters e and a, remember the definition of the charges in Eq. **(4.115)**.

by solving the extremization equations, plus the supersymmetric constraint,

$$\frac{\partial I}{\partial \omega} = -J + \frac{\chi}{4}\Lambda, \qquad \frac{\partial I}{\partial \phi} = -Q_e - \Lambda, \quad \longrightarrow \quad \begin{cases} \omega = \frac{\mp \pi i}{\frac{\chi}{4} \pm i(Q_e+\Lambda)} \\ \phi = \pm \pi i + \frac{\chi}{4}\omega \end{cases}, \tag{4.155}$$

where as usual, being the action I an homogeneous function of degree 1 of the chemical potentials, implies:

$$S_i = \pm \pi i \Lambda_i, \tag{4.156}$$

where Λ_i are the roots of either one of the two following quadratic polynomials or of the quartic polynomial:

$$\Lambda^2 + \Lambda\left(2Q_e \mp i\frac{\chi}{2}\right) + Q_e^2 + Q_m^2 \pm 2iJ = 0, \tag{4.157}$$

$$\Lambda^4 + \underbrace{4Q_e}_{p_3}\Lambda^3 + \underbrace{\left(6Q_e^2 + 2Q_m^2 + \frac{\chi^2}{4}\right)}_{p_2}\Lambda^2 + \underbrace{\left(4Q_e^3 + 4Q_eQ_m^2 - 2J\chi\right)}_{p_1}\Lambda + \underbrace{4J^2 + \left(Q_e^2 + Q_m^2\right)^2}_{p_0} = 0, \tag{4.158}$$

selecting the appropriate root allows to reproduce the thermodynamics of the corresponding horizon. Notice that by setting $Q_m = 0, \chi = 2$ and rescaling $\Lambda \to 2\Lambda$ one gets exactly the same results as in the non-accelerating case.

Notice also that by using the quartic polynomial, one can immediately read from the p_0 coefficient the entropy product formula Eq. **(4.150)** (up to a π^4 coefficient).

The BPS limit can be obtained by imposing that the above quadratic(quartic) polynomial has one(two) purely imaginary roots. If we consider the quadratic, polynomial this condition has to be directly imposed on one of the two roots Λ_i of the quadratic polynomial **(4.157)**:

$$\Lambda_i = \pm i\frac{\chi}{4} - Q_e \pm \eta\frac{i}{4}\sqrt{\chi^2 + 4Q_m^2 \pm 8i(\chi Q_e + 4J)}, \tag{4.159}$$

where η takes the values ± 1 and differentiates the two roots of each quadratic polynomial, essentially choosing η corresponds to choosing the horizon at ρ_+ or ρ_-, (and similarly for the conjugated horizons). The calculations are similar to those of the non-accelerating case.

Alternatively, the BPS solution can be obtained from the quartic polynomial by requiring that the coefficients of **(4.158)** satisfy the factorization condition:

$$\frac{p_1}{p_3} + \frac{p_0 p_3}{p_1} = p_2 \tag{4.160}$$

in both cases, one can show that imposing these conditions is equivalent to imposing the non-linear BPS condition Eq. **(4.139)**, and hence one correctly obtains the BPS solution.

Finally, in this limit, it is easy to prove that the four roots Λ_i correctly reproduce the BPS value for the entropies shown in Eqs. **(4.147, 4.149)**, hence showing that the *universality* of the extremization principle can actually be generalized also in the accelerating case as expected. We do not show the calculations, which are the direct generalization of those of the non-accelerating case.

Chapter 5

Conclusions

In this thesis, we have explicitly shown the equivalence between the universal area product formula [32] in the BPS limit, and the BPS constraint on the charges, for different classes of asymptotically AdS black hole solutions in gauged supergravity. This was one of our main results and we have briefly argued why it can be regarded as a non-trivial fact.

This result has motivated us to explore the properties of the general horizons for the above black hole solutions. These were interpreted as formal loci, satisfying a condition of the form $\Delta_r(r) = 0$ for either real or complex values of the radial coordinate (horizon radius r_i). The area product formula can be derived only if one considers also these complex horizons, and treats them on the same footing as the real Cauchy and event horizons.

Following this logic, we have given a definition for a set of thermodynamic variables for each horizon, valid in the general non-supersymmetric solution. We generalized some results that were previously obtained for other classes of black holes [30, 31]. In particular, we have found that the conserved charges are universal quantities, in the sense that they are defined asymptotically and do not depend on which horizon is considered. Instead, the entropy and the chemical potentials are intrinsic properties of each horizon. These can be defined exploiting the symmetry that exchanges the various horizons, which we have discovered being a consequence of the $\Delta_r = 0$ condition not distinguishing between its roots (hence between the various horizons). Exploiting this symmetry, we have shown that, once the thermodynamics associated with the outer horizon is defined, one can define a generalized (formal) thermodynamics for all horizons. The quantities defined in this way agree with the ones that one would formally get by studying the metric (e.g. in the near horizon limit for each horizon). In this way, the first law of thermodynamics is trivially satisfied for each horizon, together with a quantum statistical relation. The latter can be used to define a grand-canonical potential for each horizon. However, we have not been able to relate these generalized grand-potentials to independent quantities, calculated directly in the supergravity theory. Indeed, one may expect they are related to a suitably defined on-shell action, as it is for the grand-canonical potential associated with the outer horizon thermodynamics.

Using the first law of thermodynamics for each horizon, we have rephrased the mass independence of the area product formula by means of a condition involving $T_i\, S_i$ terms. These were obtained in [63] for the five-dimensional black hole solution with two angular momenta and three charges, but in a different context. Similar relations have also appeared in [30] for black holes with only two horizons.

The symmetry under exchange of horizon radius inspired us to define a set of universal thermodynamic quantities, in particular, the chemical potentials and the Euclidean action. This idea allowed us to consider the extremization principle as discussed in [8] in a different way. The extremization principle has been explored for many (supersymmetric) AdS black hole solutions [10, 13, 14, 15, 16], and it is always able to reproduce the outer horizon BPS entropy by means of a constrained Legendre transform. We have proven that the extremization principle is able to reproduce the BPS entropies of all horizons in the BPS limit (and formally also in the general supersymmetric solution), and not only of the outermost one. For this reason, we have said that the extremization principle can be promoted to a universal extremization principle. We have also seen that the area product formula can be derived from the extremization principle in the general supersymmetric solution. Moreover, in the BPS case, the requirement of having purely imaginary solutions for the Lagrange multiplier $\Lambda = \Lambda(J_i, Q_I)$ (and hence real BPS entropies), given by the factorization condition, is equivalent to the BPS constraint on the charges. The factorization condition also directly shows the equivalence between the area product formula in the BPS limit, and the BPS constraint.

We have explicitly shown these results for some classes of AdS_5 [50, 54] and AdS_4 black holes [68, 69], also considering AdS_4 black holes with acceleration [70]. We have discussed the consequences of the universality of the extremization principle for the most general black hole solutions (with no restrictions on the independent charges) for each theory. Our work seems to indicate that, when the extremization reproduces the outer horizon BPS entropy, it can also be used to describe all horizons thermodynamics, by viewing it as an universal extremization principle.

It would be interesting to explore these results for other AdS black holes in $D = 6, 7$, or for asymptotically flat black holes [18], or by taking into account higher derivatives corrections in the original theory [17]. It would also be interesting to understand if an universal extremization principle can be found without having to resort to supersymmetry.

On a more fundamental level, it would be interesting to explain the results that we have obtained by considering the dual field theory description. In this sense, the extremization principle may play a central role as the supersymmetric Euclidean action, which appears in it, has been obtained from calculations in the dual field theory via the AdS/CFT correspondence. This means that, by using the extremization principle, we have a method to derive the entropies for all horizons (in the supersymmetric case) starting from the dual field theory perspective. The end goal would be to account for the thermodynamics of all the horizons, by means of microstates of the dual field theory. This may also give some indications on how these complex horizons (but also the Cauchy horizon) can be physically interpreted.

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