UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Fisica e Astronomia “Galileo Galilei”

Corso di Laurea Triennale in Fisica

Tesi di Laurea

Forma lineare delle equazioni di Friedmann e probabilità quasi-classica

Linear form of Friedmann’s equations and quasi-classical probability

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Anno Accademico 2022/2023

Abstract

Following [1, 2], we formulate Friedmann's equations as a pair of second-order linear differential equations. This is done using techniques related to the Schwarzian derivative and its symmetry under $\mathrm{PSL}(2, \mathbb{C})$. A particular linear combination of the two Friedmann's equations is proportional to the Schwarzian derivative $\{t_\beta, t\}$, where $t_\beta := \int_0^t dt'\ a^{-2\beta}$ and $a(t)$ is the scale factor. Therefore General Relativity hides an underlying linearity at a cosmological level, where symmetry requests lead to Friedmann-Lemaître-Robertson-Walker metric and consequently to Friedmann's equations. For a vanishing spatial curvature there exists an infinite number of pairs of equivalent linear forms for Friedmann's equations (one pair for each value of $\beta \in \mathbb{R}$). For arbitrary curvature it turns out that Friedmann's equations are equivalent to

$$O_{1/2}\psi = \frac{\Lambda}{12}\psi \qquad\qquad O_1 a = \frac{\Lambda}{3}a$$

where $O_\beta(\rho, p)$ are Klein-Gordon space-independent operators depending only on energy density and pressure and Λ is the cosmological constant. The above pair of equations is the unique possible linear form for arbitrary curvature and such a uniqueness selects the conformal time $\eta \equiv t_{1/2}$ among all the t_βs.

A generic solution for $O_{1/2}\psi = \frac{\Lambda}{12}\psi$ is $\psi = \sqrt{a}\exp\left(\pm i\frac{\sqrt{\kappa}}{2}\eta(t)\right)$, with κ the spatial curvature. This is strongly reminiscent of WKB approximation in non-relativistic Quantum Mechanics. We will heuristically derive the equation which leads to this approximation, solve it for some simple expressions of $a(t)$, find a wave function $\psi(t)$ and discuss how $\psi(t)\psi(t)^*$ can be related to the evolution of the Universe. Although these simple expressions are not physically relevant, we will use them as toy models to find exact solutions and to show how it is possible to eliminate singularities in Universe's evolution as given by $\psi(t)\psi(t)^*$. The latter and other peculiar behaviours suggest considering the existence of physical models manifesting quantum effects that are not expressed by Friedmann's equations.

In questa tesi, seguendo [1, 2], riformuliamo le equazioni di Friedmann come una coppia di equazioni differenziali lineari al secon'ordine. Questo verrà fatto sfruttando tecniche associate alla derivata Schwarziana e alla sua simmetria sotto $\mathrm{PSL}(2, \mathbb{C})$. Una speciale combinazione lineare delle due equazioni di Friedmann è proporzionale alla derivata Schwarziana $\{t_\beta, t\}$, dove $t_\beta := \int_0^t dt'\ a^{-2\beta}$ e $a(t)$ rappresenta il fattore di scala. Pertanto la Relatività Generale nasconde, in un contesto cosmologico, una linearità sottostante, dove richieste di simmetria portano alla metrica di Friedmann-Lemaître-Robertson-Walker e conseguentemente alle equazioni di Friedmann. Per una curvatura spaziale nulla, esiste un infinito numero di coppie di forme lineari equivalenti delle equazioni di Friedmann (una coppia per ogni valore di $\beta \in \mathbb{R}$). Per una curvatura arbitraria le equazioni di Friedmann sono equivalenti a

$$O_{1/2}\psi = \frac{\Lambda}{12}\psi \qquad\qquad O_1 a = \frac{\Lambda}{3}a$$

dove gli $O_\beta(\rho, p)$ sono operatori di Klein-Gordon spazio-indipendenti che dipendono esclusivamente dalla densità di energia e dalla pressione e Λ è la costante cosmologica. La coppia di equazioni sopra riportata è l'unica forma lineare possibile nel caso di una curvatura arbitraria e tale unicità selziona il tempo conforme $\eta \equiv t_{1/2}$ tra tutti i t_β.

Una generica soluzione di $O_{1/2}\psi = \frac{\Lambda}{12}\psi$ è $\psi = \sqrt{a}\exp\left(\pm i\frac{\sqrt{\kappa}}{2}\eta(t)\right)$, con κ la curvatura spaziale. Questa soluzione ricorda fortemente l'approssimazione WKB in Meccanica Quantistica non relativistica. Deriveremo euristicamente l'equazione da cui emerge tale approssimazione, risolvendola per semplici espressioni di $a(t)$, trovando una funzione d'onda $\psi(t)$ e discutendo come $\psi(t)\psi(t)^*$ possa essere associata all'evoluzione dell'Universo. Nonostante queste semplici espressioni non siano fisicamente rilevanti, le useremo come toy models per trovare soluzioni esatte e per mostrare come sia possibile eliminare le singolarità nell'evoluzione dell'Universo data da $\psi(t)\psi(t)^*$. Quest'ultimo ed altri comportamenti peculiari suggeriscono di considereare l'esistenza di modelli fisici manifestanti effetti quantistici, i quali non sono espressi dalle equazioni di Friedmann.

Contents

1 Introduction

Unlike other fundamental laws of nature, Einstein's field equations [3]

$$\mathrm{Ric}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1.1}$$

are non linear. On the other hand, Quantum Mechanics (QM) is intrinsically linear. This innate difference makes General Relativity (GR) and QM hard to be described jointly by an unique theory. It is not certain that gravity, as given by GR, should admit a quantum description at all. Nevertheless, we will focus on GR when it is used to describe the observable Universe's origin, its large-scale structures and dynamics. Some intriguing similarities between QM and GR arise in fact at a cosmological level.

We will build a cosmological model which stands upon the cosmological principle, stating that the universe must be spatially homogeneous and isotropic. We will not describe the Universe as maximally symmetric both in space and time because it would be inconsistent with the presence of a non-vacuum energy source (section 2). Vacuum energy appears in (1.1) in the form of the cosmological constant Λ, which plays a fundamental role in this thesis. These symmetry requests lead to Friedmann-Lemaître-Robertson-Walker (FLRW) metric, from which Friedmann's equations emerge. In this cosmological context GR hides an underlying linearity.

Friedmann's equations are a pair of ODEs for the scale factor $a(t)$. One is linear (equation (2.39)) while the other one is not (equation (2.36)), due to a term $\frac{\kappa}{a(t)^2}$ and a term $\left(\frac{\dot{a}(t)}{a(t)}\right)^2$, where $\kappa \in \mathbb{R}$ is the spatial curvature. We will firstly set $\kappa = 0$. In this special scenario, on account of the properties of the Schwarzian derivative and its invariance under Möbius transformations (section 3.1), it turns out that Friedmann's equations are equivalent to an infinite number of Klein-Gordon space-independent eigenvalues problems [1]

$$\begin{pmatrix} O_\beta & 0 \\ 0 & O_{-\beta} \end{pmatrix} \Psi_{\beta-\beta} = \beta^2 \frac{\Lambda}{3} \Psi_{\beta-\beta} \tag{1.2}$$

where $\beta \in \mathbb{R}$ and O_β is a Klein-Gordon space-independent operator (section 3.2). In fact, a special linear combination of the two Friedmann's equations is proportional to $\{t_\beta, t\}$[1], where $t_\beta := \int_0^t a(t')^{-2\beta} dt'$. If one selects $\beta = \frac{1}{2}$, they gets the conformal time $\eta(t)$ (section 2.3). In this case the curvature $\kappa \neq 0$ can be absorbed by exponentiation (section 3.3). We will obtain then a unique linear form for arbitrary non-vanishing curvature. Friedmann's equations are then equivalent to [1]

$$O_{1/2}\psi = \frac{\Lambda}{12}\psi \tag{1.3}$$

$$O_1 a = \frac{\Lambda}{3} a \tag{1.4}$$

These linear forms are analogous to a measurement problem, where the eigenvalue is the cosmological constant. In the $\kappa \to 0$ limit, we obtain (1.2) selecting $\beta = \frac{1}{2}$. This shows that the $\beta = \frac{1}{2}$ case is a "privileged" case and it surprisingly involves $\eta(t)$ among all the β-times.
A generic solution for (1.3) has the form [1]

$$\psi = \sqrt{a} \exp\left(\pm \frac{i}{2}\sqrt{\kappa} \int_0^t \frac{d\tau}{a(\tau)}\right) \tag{1.5}$$

[1] $\{f(z), z\}$ is the Schwarzian derivative of the holomorphic function f respect to the complex variable z (section 3.1).

Using $a \sim \frac{1}{p}$ (valid for free-falling particles in a FLRW Universe, section 2.3) we get an expression analogous to a QM WKB approximate wave function (section 4.1). We will find then the equation which leads to this approximation [2]

$$\left(\frac{d^2}{dt^2} + \frac{\kappa}{4}\frac{1}{a^2}\right)\psi = 0 \tag{1.6}$$

We will discuss the meaning of $\psi(t)\psi(t)^*$, showing its relation with Universe's evolution, and solve (1.6) for ψ, using two simple expressions for $a(t)$. We will use them as toy models to obtain exact solutions for (1.6) and discuss how singularities of the scale factor can be avoided if we express Universe's evolution through $\psi(t)\psi(t)^*$. The latter and other peculiar behaviours suggest considering the existence of physical models manifesting quantum effects that are not expressed by Friedmann's equations, which can be considered as the quasi-classical limit of (1.6). Friedmann's equations may be already characterized by quantum properties due to their linear form and the analogy with quantum WKB.

Natural Units (NU) $\hbar = c = 1$ are used throughout the whole article, except for section 4, where NU are abandoned in those contexts where the dimensionality of c and/or $\hbar$ is needed. All the information related to fundamental physical constants (such as $\hbar$ or c) has been taken from [4].

2 Friedmann-Lemaître-Robertson-Walker metric and derivation of Friedmann's equations

In this first part of the thesis, we introduce Friedmann's equations, their properties and the space-time metric they are derived from, Friedmann-Lemaître-Robertson-Walker (FLRW) metric (or simply Robertson-Walker metric), referring to the procedure followed in [3], Chapter 8. We first need to build the energy-momentum tensor of a perfect fluid to model an homogeneous and isotropic Universe (as it is done in [3], Chapters 1 and 4) and we need to justify the presence of the cosmological constant Λ in Einstein's field equation ([3], Chapter 4). The cosmological constant will play a fundamental role later on. We will treat it as the eigenvalue of two linear Klein-Gordon space-independent operators.

FLRW metric, from which Friedmann's equations are derived, is deduced from the properties of homogeneity and isotropy and from the fact that the Universe has to be described as composed by matter, and not only by vacuum energy. These symmetry requests lead to a cosmological model which hides a linear form, despite the high non-linearity of Einstein's field equation.

2.1 Energy-momentum tensor of a perfect fluid and cosmological constant

To derive Friedmann's equations from Einstein's field equations we need to use the expression of the energy-momentum tensor of a perfect fluid. We want in fact to build a cosmological model relying on the cosmological principle, stating that the Universe is homogeneous and isotropic. This is valid only for the spatial coordinates though: we will see that the entire space-time cannot be described as perfectly symmetric. This request will be then imposed only for the spatial part of the metric.
We treat the Universe as a perfect fluid described by two parameters, the rest-frame energy density ρ and the isotropic rest-frame pressure p. Isotropy implies that $T^{\mu\nu}$ is diagonal in the fluid's rest-frame, furthermore the three nonzero space-like components must all be equal. In addition, all the particles which compose the fluid are at rest with respect to each other. Using these assumptions the energy-momentum tensor in the fluid's rest-frame is

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{2.1}$$

whose expression in an arbitrary frame is

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu + pg^{\mu\nu} \tag{2.2}$$

where U^μ is the four-velocity of the fluid and $g \in T^{(0,2)}M$ is the metric of the four-dimensional pseudo-riemannian manifold M, which describes the space-time. We are using the convention in which the Lorenzian metric $g_{\mu\nu}$ has signature $(- + ++)$.

Before deriving Friedmann's equations we need to justify the presence of the cosmological constant Λ in (1.1). Writing Einstein's field equations as

$$G_{\mu\nu} = \kappa T_{\mu\nu} \tag{2.3}$$

one immediately sees a characteristic feature of General Relativity: the source for the curvature of space-time is the entire energy-momentum tensor. This sounds unfamiliar if we think of classical or quantum mechanics (or in general of non-gravitational physics), where only changes in energy from one configuration to another are measurable. This peculiar behaviour opens up the possibility of vacuum energy, a energy density characteristic of empty space. One feature that the energy-momentum tensor associated to vacuum energy needs to have is Lorentz invariance in locally inertial coordinates. When saying locally inertial coordinates we rely on the following result ([3], Chapter 2).

Lemma 2.1. Let (M, g) be a pseudo-riemannian manifold. Then at any point $p \in M$ there exists a coordinate system $x^{\hat{\mu}}$ in which $g_{\hat{\mu}\hat{\nu}}$ takes its canonical form $\mathrm{diag}(-1, -1, -1, ..., +1, +1, +1, ...)$ and the first derivatives $\partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}}$ all vanish. Such coordinates are called locally inertial coordinates and the associated basis vectors $\left\{\frac{\partial}{\partial x^{\hat{1}}}, ..., \frac{\partial}{\partial x^{\hat{m}}}\right\}$, where $m = \dim M$, constitute a local Lorentz frame.

It is also clear that if a metric takes a particular canonical form at a point $p \in M$, this will be the canonical form throughout the whole manifold. This is due to the non-degeneracy of the $\mathcal{C}^{\infty}$-bilinear map $g : TM \times TM \to \mathbb{R}$ associated to $g \in T^{(0,2)}M$[2].
We know from special relativity that Minkowski metric $\eta_{\hat{\mu}\hat{\nu}} = \mathrm{diag}(-1, +1, +1, +1)$ satisfies the following relation

$$\Lambda^{\hat{\mu}}{}_{\hat{\alpha}} \eta_{\hat{\mu}\hat{\nu}} \Lambda^{\hat{\nu}}{}_{\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}} \tag{2.4}$$

meaning Lorentz invariance in locally inertial coordinates of the metric tensor. Not only is $\eta_{\hat{\mu}\hat{\nu}}$ Lorentz invariant, but also it is the only $(0, 2)$ tensor with this characteristic. We can thus deduce an expression for the energy-momentum tensor associated to vacuum energy

$$T^{(\mathrm{vac})}_{\hat{\mu}\hat{\nu}} = -\rho_{\mathrm{vac}} \eta_{\hat{\mu}\hat{\nu}} \tag{2.5}$$

This can be easily generalized from inertial coordinates to arbitrary coordinates as

$$T^{(\mathrm{vac})}_{\mu\nu} = -\rho_{\mathrm{vac}} g_{\mu\nu} \tag{2.6}$$

$T^{(\mathrm{vac})}_{\mu\nu}$ is then proportional to the metric. Comparing this expression to the perfect-fluid energy-momentum tensor (equation (2.2)), one deduces that the vacuum looks like a perfect fluid with an isotropic pressure opposite in sign to the energy density ($p_{\mathrm{vac}} = -\rho_{\mathrm{vac}}$)[3].
Decomposing the energy-momentum tensor that appears in (1.1) into the sum of a matter energy-momentum tensor $T^{(M)}_{\mu\nu}$ and a vacuum energy-momentum tensor (equation (2.5)), Einstein's field equations become

$$\mathrm{Ric}_{\mu\nu} - \frac{1}{2}\mathcal{R} g_{\mu\nu} = 8\pi G \left(T^{(M)}_{\mu\nu} - \rho_{\mathrm{vac}} g_{\mu\nu}\right)$$

$$\mathrm{Ric}_{\mu\nu} - \frac{1}{2}\mathcal{R} g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T^{(M)}_{\mu\nu} \tag{2.7}$$

where $\Lambda = 8\pi G \rho_{\mathrm{vac}}$ is the cosmological constant. It is now clear the reason of the minus sign in the definition of the energy-momentum tensor of vacuum energy (2.5). The terms "vacuum energy" and "cosmological constant" are essentially interchangeable. Notice that the conservation of energy $(\nabla_\mu T)^{\mu\nu} = 0$ is still valid. To prove this we need to mention the second Bianchi identity ([5], Chapter 7), which can be written in coordinates as

$$(\nabla_\kappa R)^{\xi}{}_{\lambda\mu\nu} + (\nabla_\nu R)^{\xi}{}_{\lambda\kappa\mu} + (\nabla_\mu R)^{\xi}{}_{\lambda\nu\kappa} = 0 \tag{2.8}$$

[2] We are using that a tensor product vector space $(V^*)^{N\otimes}$ is always homeomorphic to $\mathrm{Hom}(V^{N\otimes} \to \mathbb{R})$. With V^* we intend the dual vector space of V [5].

[3] The vacuum energy density needs to be constant throughout spacetime.

Using the relation $(\nabla_k R)^{\mu}{}_{\lambda\mu\nu} = (\nabla_k \mathrm{Ric})_{\lambda\nu}$, one obtains the following identity by contracting the indices ξ and μ in (2.8)

$$(\nabla_k \mathrm{Ric})_{\lambda\nu} + (\nabla_\mu R)^{\mu}{}_{\lambda\nu\kappa} - (\nabla_\nu \mathrm{Ric})_{\lambda\kappa} = 0 \tag{2.9}$$

If the indices ν and λ are further contracted we have

$$\left[\nabla_\mu \left(\mathcal{R}\mathrm{Id} - 2\mathrm{Ric}\right)\right]^{\mu}{}_{\kappa} = 0 \tag{2.10}$$

Raising the κ index using the metric (remembering that ∇ is the Levi-Civita connection, which means $\nabla g = 0$), dividing both sides of equation (2.10) by two and rearranging the signs one gets

$$\left[\nabla_\mu \left(\mathrm{Ric} - \frac{1}{2}\mathcal{R}g\right)\right]^{\mu\nu} = 0 \tag{2.11}$$

At the same time, since ∇ is the Levi-Civita connection, $(\nabla_\mu g)^{\mu\nu} = 0$. This implies

$$\left[\nabla_\mu \left(\mathrm{Ric} - \frac{1}{2}\mathcal{R}g + \Lambda g\right)\right]^{\mu\nu} = 0$$

The energy-momentum tensor is then automatically conserved, $(\nabla_\mu T)^{\mu\nu} = 0$.

2.2 FLRW metric and Friedmann's equations

In this section we build the metric we use to describe our cosmological model, from which we will derive the equations which are the core of this article, Friedmann's equations.
We will use the features we introduced before, like the energy momentum tensor $T^{\mu\nu}$ of a perfect fluid and the properties of homogeneity and isotropy. The formal definition of these concepts follows, together with the definition of an isometry ([5], Chapter 7).

Definition 2.1. Let (M, g) be a pseudo-riemannian manifold. A diffeomorphism $\phi : M \to M$ is an isometry if it preserves the metric

$$\phi^* g_{\phi(p)} = g_p$$

in other words if

$$g_{\phi(p)}\left(\phi_* X, \phi_* Y\right) = g_p(X, Y) \quad \forall X, Y \in T_p M, \quad \forall p \in M$$

Where $\phi^* : T^{(0,2)}_{\phi(p)} M \to T^{(0,2)}_{p} M$ is the pullback and $\phi_* : T_p M \to T_{\phi(p)} M$ is the pushforward. In coordinates:

$$\frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu} g_{\alpha\beta}(\phi(p)) = g_{\mu\nu}(p)$$

where x, y are the coordinates of p and $\phi(p)$ respectively.

Definition 2.2. A (pseudo-)riemannian manifold (M, g) is said to be isotropic in a neighborhood of $p \in M$ if, for any two vectors $X, Y \in T_p M$, there exists an isometry $\phi \in \mathrm{Diff}(M)$ such that $\phi_*(Y)$ is parallel to Y.

Definition 2.3. A (pseudo-)riemannian manifold (M, g) is said to be homogeneous if, for any two points $p, q \in M$, there exists an isometry $\phi \in \mathrm{Diff}(M)$ that maps p into q.

We introduce the concept of Killing vector field, in order to define the idea of a maximally symmetric space-time ([5], Chapter 7).

Definition 2.4 (Killing vector field)**.** Let (M, g) be a pseudo-riemannian manifold and $X \in \mathfrak{X}(M)$. If a displacement εX (ε infinitesimal) generates an isometry, the vector field X is called a Killing vector field. If $f : x^\mu \mapsto x^\mu + \varepsilon X^\mu$ is an isometry, it satisfies

$$\frac{\partial\left(x^\kappa+\varepsilon X^\kappa\right)}{\partial x^\mu} \frac{\partial\left(x^\lambda+\varepsilon X^\lambda\right)}{\partial x^\nu} g_{\kappa\lambda}(x+\varepsilon X)=g_{\mu\nu}(x) \tag{2.12}$$

that is equivalent to the Killing equation

$$X^\xi \partial_\xi g_{\mu\nu}+\partial_\mu X^\kappa g_{\kappa\nu}+\partial_\nu X^\lambda g_{\mu\lambda}=0 \tag{2.13}$$

which can be written in a compact way using the Lie derivative of a $(0, 2)$ tensor field

$$\left(\mathcal{L}_X g\right)_{\mu\nu}=0 \tag{2.14}$$

Killing vector fields represent the direction of the symmetry of a manifold. In the m-dimensional Minkowski space-time, there are m Killing vector fields generating translations, $m-1$ boosts and $(m-1)(m-2)/2$ space rotations, for a total of $m(m+1)/2$. Such a space-time, with the maximum number of Killing vector fields, is said to me maximally symmetric. Isotropy and homogeneity imply that a space-time is maximally symmetric.

We want to build a cosmological model which undergoes the cosmological principle, for which the Universe is homogeneous and isotropic. We will briefly analyze the case of a spacetime maximally symmetric both in space and time, in order to introduce the more general case of a spacetime maximally symmetric only in space, which is the physically relevant scenario.
It can be proved that the Riemann tensor for any maximally symmetric n-dimensional manifold can be written, at any point, in any coordinate system, as ([3], Chapter 3)

$$R_{\lambda\xi\mu\nu}=\kappa\left(g_{\lambda\mu} g_{\xi\nu}-g_{\lambda\nu} g_{\xi\mu}\right) \tag{2.15}$$

where

$$\kappa=\frac{\mathcal{R}}{n(n-1)} \tag{2.16}$$

The Ricci scalar $\mathcal{R} \in \mathcal{C}^\infty(M)$ will be a constant throughout the manifold, due to symmetry reasons. It is thus clear that a maximally symmetric manifold is well characterized by the signature of the metric (in our case $(-+++)$) and by the value of κ, since the coordinates of the Riemann tensor are determined only by the coordinates of the metric and by the Ricci scalar.
For $\kappa = 0$ we have the Minkowski space. The maximally symmetric space with positive curvature $\kappa > 0$ is called de Sitter space, whereas the maximally symmetric space with negative curvature $\kappa < 0$ is called anti-de Sitter space. The natural question is to verify if such a model can be a solution for Einstein's field equations. We start evaluating the Ricci tensor from (2.15).

$$\begin{aligned} \mathrm{Ric}_{\mu\nu}=R^\alpha{}_{\mu\alpha\nu}=g^{\alpha\beta} R_{\beta\mu\alpha\nu}&=\kappa g^{\alpha\beta}\left(g_{\beta\alpha} g_{\mu\nu}-g_{\beta\nu} g_{\mu\alpha}\right) \\ &=\kappa \delta^\alpha{}_\alpha g_{\mu\nu}-\delta^\alpha{}_\nu g_{\mu\alpha}=3\kappa g_{\mu\nu} \end{aligned} \tag{2.17}$$

and the scalar curvature $\mathcal{R} = 12\kappa$. The Einstein tensor is then

$$G_{\mu\nu}=\mathrm{Ric}_{\mu\nu}-\frac{1}{2} \mathcal{R} g_{\mu\nu}=-3\kappa g_{\mu\nu} \tag{2.18}$$

This implies that the energy-momentum tensor needs to be proportional to the metric, which is the case of vacuum energy-momentum tensor introduced in the previous section. Referring to equation (2.7) we deduce the expression for the cosmological constant, assuming that a matter energy-momentum tensor cannot be proportional to the metric ($T^{(M)}_{\mu\nu}$ needs to vanish)

$$\Lambda = 3\kappa \tag{2.19}$$

If $\Lambda > 0$ we have de Sitter while if $\Lambda < 0$ we have anti-de Sitter. It is clear that a maximally symmetric space-time cannot describe the real Universe, since we desire to take into account the presence of matter, in particular we want to use the expression of the energy-momentum tensor of a perfect fluid. We will then apply the cosmological principle only to spatial coordinates, assuming that the Universe is evolving in time. Both de Sitter and anti-de Sitter spaces have the topology of $\mathbb{R} \times \mathbb{S}^3$, where $\mathbb{R}$ represents the time coordinate. We will therefore follow the same logic, considering the space-time to be $\mathbb{R} \times \Sigma$, where Σ is a homogeneous and isotropic three-dimensional manifold. We can choose coordinates which make the metric diagonal

$$ds^2 = -dt^2 + R(t)^2 d\sigma^2 \tag{2.20}$$

where $R(t)$ is known as scale factor, a function of time which represents the idea that the spatial part of the metric (that is, at any time, maximally symmetric) is evolving. $d\sigma^2$ is the metric on Σ and we require it to be maximally symmetric, and in particular spherically symmetric. We use the coordinates (u, θ, ϕ), writing a spherically symmetric three-dimensional metric in its most general form ([3], Chapter 5)

$$d\sigma^2 = e^{2\bar{\alpha}(u)} du^2 + e^{2\bar{\beta}(u)} u^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.21}$$

where the exponentials $e^{\bar{\alpha}(u)}$ and $e^{\bar{\beta}(u)}$ have the function to preserve the signature of the metric, if we think of $d\sigma^2$ as a generalization of the Euclidean metric, that is spherically symmetric. To simplify this expression, we define a new radial coordinate $\bar{r} = e^{\bar{\beta}(u)} u$, so $d\bar{r} = \left(1 + u\frac{d\bar{\beta}(u)}{du}\right) e^{\bar{\beta}(u)} du$. Defining

$$e^{\alpha(\bar{r})} = \left(1 + u\frac{d\bar{\beta}(u)}{du}\right) e^{\bar{\alpha}(u) - \bar{\beta}(u)} \tag{2.22}$$

(notice that u is a function of $\bar{r}$) we can re-write $d\sigma^2$ as

$$d\sigma^2 = e^{2\alpha(\bar{r})} d\bar{r}^2 + \bar{r}^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.23}$$

Computing the components of the Ricci tensor associated to the Levi-Civita connection of Σ, one gets ${}^{(3)}\mathrm{Ric}_{11} = \frac{2}{\bar{r}}\frac{\partial\alpha(\bar{r})}{\partial\bar{r}}$, ${}^{(3)}\mathrm{Ric}_{22} = e^{-2\alpha(\bar{r})}\left(\bar{r}\frac{\partial\alpha(\bar{r})}{\partial\bar{r}} - 1\right) + 1$, ${}^{(3)}\mathrm{Ric}_{33} = \left[e^{-2\alpha(\bar{r})}\left(\bar{r}\frac{\partial\alpha(\bar{r})}{\partial\bar{r}} - 1\right) + 1\right]\sin^2\theta$ and ${}^{(3)}\mathrm{Ric}_{ij} = 0$ for $i \neq j$. We put the superscript ${}^{(3)}$ to specify we are referring to the Ricci tensor associated to Σ. Since we are assuming Σ to be maximally symmetric, its Riemann tensor satisfies relation (2.15). We can easily compute the Ricci tensor ${}^{(3)}\mathrm{Ric}_{ij} = 2k^{(3)}g_{ij}$, where $k = \frac{{}^{(3)}\mathcal{R}}{3\cdot(3-1)} = \frac{{}^{(3)}\mathcal{R}}{6}$. We have

$$ {}^{(3)}g_{11} = e^{2\alpha(\bar{r})}$$

$$ {}^{(3)}\mathrm{Ric}_{11} = 2k^{(3)}g_{11}$$

that is

$$\frac{2}{\bar{r}}\frac{\partial\alpha(\bar{r})}{\partial\bar{r}} = 2ke^{2\alpha(\bar{r})} \tag{2.24}$$

Solving (2.24) for $\alpha(\bar{r})$ we get

$$\alpha(\bar{r}) = -\frac{1}{2}\log\left(b - k\bar{r}^2\right) \tag{2.25}$$

where b is a real constant. We thus obtain

$$d\sigma^2 = \frac{d\bar{r}^2}{b - k\bar{r}^2} + \bar{r}^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.26}$$

For a flat space ($k = 0$) we want to reproduce the Euclidean metric. We will then set $b = 1$. The metric on our space-time maximally symmetric in space but evolving in time can be written as

$$ds^2 = -dt^2 + R(t)^2\left[\frac{d\bar{r}^2}{1 - k\bar{r}^2} + \bar{r}^2\left(d\theta^2 \sin^2\theta d\phi^2\right)\right] \tag{2.27}$$

that is the Robertson-Walker metric. It is common to normalize the curvature parameter so that k is either +1, 0 and -1 for spatially closed, flat and open Universes, respectively. In this case the physical dimension of ds^2 is carried by the scale factor ($[R(t)] =$ length) and the coordinate $\bar{r}$ is dimensionless. We will work with a dimensionless scale factor, dividing $R(t)$ by the constant $R_0 > 0$

$$a(t) := \frac{R(t)}{R_0} \tag{2.28}$$

a coordinate with the dimensions of a distance

$$r := R_0\bar{r} \tag{2.29}$$

and a curvature parameter with the dimensions of lenght^{-2}

$$\kappa := \frac{k}{R_0^2} \tag{2.30}$$

κ can take any real value, it is only important to distinguish the cases of negative, positive and vanishing curvature. In these new variables FLRW metric becomes

$$ds^2 = -dt^2 + a(t)^2\left[\frac{dr^2}{1 - \kappa r^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \tag{2.31}$$

Note that FLRW metric is invariant under the rescaling $r \mapsto \lambda r$, $a(t) \mapsto a(t)/\lambda$ and $\kappa \mapsto \kappa/\lambda^2$, with $\lambda \in \mathbb{R}$ a positive dimensionless parameter. Let us compute the components of the Ricci tensor in these coordinates. If M is the manifold decribing our spacetime and if $\nabla_X : \mathfrak{X}(M) \to \mathfrak{X}(M)$[4] is the Levi-Civita connection of M, Christoffel symbols are ([5], Chapter 7)

$$\Gamma^\kappa_{\mu\nu} = \left\{\begin{matrix}\kappa \\ \mu\nu\end{matrix}\right\} = \frac{1}{2}g^{\kappa\lambda}\left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right) \tag{2.32}$$

[4] $\mathfrak{X}(M)$ is the set of all vector fields on M.

They are determined only by the components of the metric. The non-zero Christoffel symbols are

$$\begin{aligned}
&\Gamma^0{}_{11} = \frac{a\dot{a}}{1-\kappa r^2} && \Gamma^1{}_{11} = \frac{\kappa r}{1-\kappa r^2} \\
&\Gamma^0{}_{22} = a\dot{a}r^2 && \Gamma^0{}_{33} = a\dot{a}r^2 \sin^2\theta \\
&\Gamma^1{}_{01} = \Gamma^2{}_{02} = \Gamma^3_{03} = \frac{\dot{a}}{a} \\
&\Gamma^1{}_{22} = -r(1-\kappa r^2) && \Gamma^1{}_{33} = -r(1-\kappa r^2)\sin^2\theta \\
&\Gamma^2{}_{12} = \Gamma^3{}_{13} = \frac{1}{r} \\
&\Gamma^2{}_{33} = -\sin\theta\cos\theta && \Gamma^3{}_{23} = \cot\theta
\end{aligned}$$

where $(x^1, x^2, x^3, x^4) = (t, r, \theta, \phi)$. The remaining Christoffel symbols can be obtained from these by symmetry (since ∇_X is torsionless, $\Gamma^\kappa{}_{\mu\nu} = \Gamma^\kappa{}_{\nu\mu}$). We can deduce the components of the Riemann tensor ([5], Chapter 7)

$$R^\kappa{}_{\lambda\mu\nu} = \partial_\mu \Gamma^\kappa{}_{\nu\lambda} - \partial_\nu \Gamma^\kappa{}_{\mu\lambda} + \Gamma^\varepsilon{}_{\nu\lambda}\Gamma^\kappa{}_{\mu\varepsilon} - \Gamma^\varepsilon{}_{\mu\lambda}\Gamma^\kappa{}_{\nu\varepsilon} \tag{2.33}$$

and consequently the components of the Ricci tensor $\mathrm{Ric}_{\lambda\nu} = R^\mu{}_{\lambda\mu\nu}$ ([5], Chapter 7), which, in these coordinates, takes a diagonal form

$$\begin{aligned}
\mathrm{Ric}_{00} &= -\frac{3\ddot{a}}{a} \\
\mathrm{Ric}_{11} &= \frac{a\ddot{a} + 2\dot{a}^2 + 2\kappa}{1-\kappa r^2} \\
\mathrm{Ric}_{22} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2\kappa) \\
\mathrm{Ric}_{33} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2\kappa)\sin^2\theta
\end{aligned}$$

The scalar curvature is therefore $\mathcal{R} = \mathrm{Ric}^\mu{}_\mu = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2}\right]$.
We now use the form of the energy-momentum tensor of a perfect fluid built in the previous section. The fluid is homogeneous and isotropic in its rest-frame. We expect that the fluid is at rest in the coordinates in which the metric generated by it is spatially isotropic. Therefore, in the coordinates discussed above $T_{\mu\nu} = \mathrm{diag}(\rho, p, p, p)$. We have, from Einstein's field equations

$$\mathrm{Ric}_{00} - \frac{1}{2}\mathcal{R} g_{00} + \Lambda g_{00} = 8\pi G T_{00} = 8\pi G\rho \tag{2.34}$$

Since $g_{00} = -1$, we obtain

$$-\frac{3\ddot{a}}{a} + 3\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2}\right] - \Lambda = 8\pi G\rho \tag{2.35}$$

Rearranging it we deduce the expression for the first Friedmann's equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\left(8\pi G\rho + \Lambda\right) - \frac{\kappa}{a^2} \tag{2.36}$$

We now take the trace on both sides of Einstein's field equations, using that $g^\mu{}_\mu = 4$ and $T^\mu{}_\mu = -\rho + 3p$

$$-\mathcal{R} + 2\Lambda = 4\pi G(-\rho + 3p) \tag{2.37}$$

$$-6\left[\frac{\ddot{a}}{a}+\left(\frac{\dot{a}}{a}\right)^2+\frac{\kappa}{a^2}\right]+2\Lambda=4\pi G(-\rho+3p) \tag{2.38}$$

Substituting the term $\left(\frac{\dot{a}}{a}\right)^2+\frac{\kappa}{a^2}$ using (2.36) we get the second Friedmann's equation

$$\frac{\ddot{a}}{a}=-\frac{4\pi G}{3}(\rho+3p)+\frac{\Lambda}{3} \tag{2.39}$$

We obtained two ordinary differential equations for the evolution of the dimensionless scale factor $a(t)$. Equation (2.39) is linear, whereas (2.36) is not. We will show in section 3 that Friedmann's equations hide an underlying linearity and can be formulated in the form of a pair of linear second-order differential equations.

2.3 Red-shift relation and Conformal Time

Before analyzing the underlying linearity of Friedmann's equations, we need to introduce some physical consequences and properties of FLRW metric.
We will follow [6, 7, 8].
Let us begin with discussing the trajectory of a photon in General Relativity. Let $x^\mu(s)$ be the worldline of a photon. The photon momentum is

$$P^\mu=\frac{dx^\mu}{ds} \tag{2.40}$$

Let us introduce the geodesic equation ([5], Chapter 7)

$$\frac{d^2x^\mu}{ds^2}+\Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}=0 \tag{2.41}$$

or equivalently

$$\frac{dP^\mu}{ds}=-\Gamma^\mu{}_{\alpha\beta}P^\alpha P^\beta \tag{2.42}$$

Photons travel along light-like geodesics, or null geodesics, such that

$$g_{\mu\nu}P^\mu P^\nu=0 \tag{2.43}$$

It is useful to consider a reparametrization of (2.31), using hyperspherical coordinates, as it is done in [7], Chapter 27

$$ds^2=-dt^2+a(t)^2\left[d\chi^2+S_\kappa(\chi)^2\left(d\theta^2+\sin^2\theta d\phi^2\right)\right] \tag{2.44}$$

where χ represent the new radial coordinate and $S_\kappa(\chi)$ is such that

$$S_\kappa(\chi)=\begin{cases}\sqrt{\kappa}^{-1}\sin\left(\chi\sqrt{\kappa}\right) & \kappa>0\\ \chi & \kappa=0\\ \sqrt{|\kappa|}^{-1}\sinh\left(\chi\sqrt{|\kappa|}\right) & \kappa<0\end{cases} \tag{2.45}$$

Now imagine to be an observer at the origin ($\chi=0$) and to be looking in some direction (θ,ϕ) at time t_0. A photon with energy E_i emitted somewhere in the Universe (t_i,χ_i,θ,ϕ) is travelling towards us. We want to determine the energy E_0 observed. Since the photon is moving radially, $P^\theta=P^\phi=0$, that is $\frac{dP^\theta}{ds}=\frac{dP^\phi}{ds}=0$. The requirement for a null trajectory can be expressed as follows

$$-E^2+a^2\left(P^\chi\right)^2=0 \tag{2.46}$$

Using (2.41), we can find an expression for $\frac{dE}{ds}$ (imposing $\mu = 0$). Among ${\Gamma^0}_{00}$, ${\Gamma^0}_{01} = {\Gamma^0}_{10}$ (the Levi-Civita connection is torsionless) and ${\Gamma^0}_{11}$, the only non-zero Christoffel symbol is ${\Gamma^0}_{11} = a\dot{a}$ (referring to (2.44)). We obtain

$$\frac{dE}{ds} = -{\Gamma^0}_{11}\left(\frac{E}{a}\right)^2 = -a\dot{a}\left(\frac{E}{a}\right)^2 = -\frac{\dot{a}}{a}E^2 \tag{2.47}$$

We also have

$$\frac{dE}{ds} = \frac{dE}{dt}\frac{dt}{ds} = \dot{E}P^0 = E\dot{E} \tag{2.48}$$

that is

$$\frac{\dot{E}}{E} = -\frac{\dot{a}}{a} \tag{2.49}$$

$$\iff \frac{d}{dt}\log(E) = -\frac{d}{dt}\log(a) \tag{2.50}$$

Solving this ODE we get

$$E(t) = \frac{\mathcal{C}}{a(t)} \tag{2.51}$$

where $\mathcal{C} \in \mathbb{R}$. For massless particles $E = |\vec{p}|$. We thus have $p \propto \frac{1}{a}$. Since the energy and the wavelength of a photon are related according to the Planck relation $E = \frac{hc}{\lambda}$, we have that, for every instant of time t, $\lambda \propto a(t)$. This leads to the red-shift relation

$$\frac{\lambda(t_0)}{a(t_0)} = \frac{\lambda(t_i)}{a(t_i)} \tag{2.52}$$

where $\lambda(t_i)$ is the wavelength of the photon emitted and $\lambda(t_0)$ is the wavelength of the photon observed. It is conventional to define the red-shift z as

$$z := \frac{\lambda(t_0) - \lambda(t_i)}{\lambda(t_i)} = \frac{a(t_0) - a(t_i)}{a(t_i)} \tag{2.53}$$

If we normalize the "today" scale factor $a(t_0) = 1$ we have

$$a(t_i) = \frac{1}{1+z} \tag{2.54}$$

The red-shift time relation $z(t)$ is simply another way to parameterize the expansion of the Universe. Red-shift gives an experimental evidence of an expanding Universe. Furthermore, we can deduce $p \propto a^{-1}$ from FLRW metric and red-shift can be observed experimentally. This leads naturally to $p \propto \lambda^{-1}$. Planck relation arises therefore also in a gravitational contest. It is immediate to generalize this result for massive particles, as it is done in [8], Chapter 1. We will use the geodesic equation imposing $\mu = 1$. We get

$$\frac{d^2\chi}{ds^2} = -\kappa\chi\left(\frac{d\chi}{ds}\right)^2 - \frac{2\dot{a}}{a}\frac{dt}{ds}\frac{d\chi}{ds} \tag{2.55}$$

Without loss of generality, suppose we adopt a spatial coordinate system in which the particle position is near the origin ($\chi = 0$). Equation (2.55) becomes

$$\frac{d}{dt}\left(\frac{d\chi}{ds}\right) = -\frac{2\dot{a}}{a}\frac{d\chi}{ds} \tag{2.56}$$

Which means $\frac{dx^i}{ds} \propto \frac{1}{a(t)^2}$. Using the expression for the spatial momentum of a free-falling particle in General Relativity

$$p_m = m\sqrt{g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}} \tag{2.57}$$

we re-obtain $p_m \propto \frac{1}{a(t)}$ (we used $g = -dt^2 + a(t)^2 d\chi^2$, since we supposed $\chi = 0$). By analogy we can associate to a massive particle a wavelength which satisfies a red-shift relation, that is the de Broglie wavelength $\lambda_m = \frac{h}{p_m}$. Therefore, already in a gravitational context, without considering quantum aspects, even massive particles are associated with a wavelength.

One more feature of FLRW metric is the relation between red-shift and distances. Let χ_i be the radial coordinate of the photon when it was emitted. Since the photon trajectory is null, $u^\mu u_\mu = 0$, that is

$$-\left(u^0\right)^2 + a^2\left(u^1\right)^2 = 0 \tag{2.58}$$

We have $u^\mu = \gamma\left(1, \frac{d\chi}{dt}, 0, 0\right)$, which implies

$$\frac{d\chi}{dt} = \frac{u^1}{u^0} = \pm\frac{1}{a} \tag{2.59}$$

Since the photon is coming towards us we choose the minus solution. We thus have

$$\chi_0 - \chi_i = -\int_{t_i}^{t_0} \frac{dt}{a(t)} \tag{2.60}$$

We set $\chi_0 = 0$ since we are observing the incoming photon standing at the origin:

$$\chi_i = \int_{t_i}^{t_0} \frac{dt}{a(t)} \tag{2.61}$$

The conformal time $\eta(t)$ is naturally defined as

$$\eta(t) := \int_0^t \frac{dt'}{a(t')} \tag{2.62}$$

where t_i was set to 0 and $t_0 = t$. Equation (2.62) implies

$$\dot{\eta} = a(t)^{-1} \tag{2.63}$$

The conformal time plays a fundamental role in finding an unique equivalent linear expression for Friedmann's equations.

3 Linear form for Friedmann's equations and Klein-Gordon space-independent eigenvalues problems

In this section we review the results in [1]. We rewrite (2.36) and (2.39) to obtain a second-order linear differential equation. We will firstly show that imposing a vanishing curvature leads to an infinite number of equivalent linear forms for Friedmann's equations. Secondly, we will show that for arbitrary non-vanishing curvature, there exists a unique linear form, which involves the conformal time introduced in section 2.3. This equivalent linear form is a space-independent Klein-Gordon eigenvalues problem and the relative eigenvalue is the cosmological constant. Among all the infinite equivalent linear forms in the $\kappa = 0$ case, only one can be interpreted as the $\kappa \to 0$ limit of the unique linear form in the $\kappa \neq 0$ case. This means that among all the equivalent linear forms, there is one that is "privileged" and it is peculiar how the conformal time is involved. To begin we need to introduce some mathematical tools that are crucial to highlight the underlying linearity of our cosmological model, such as the concepts of Schwarzian derivative and Möbius transformations, and all the properties related to them. All the information regarding Möbius transformations is taken from [9].

3.1 Möbius transformations, PSL$(2, \mathbb{C})$ and Schwarzian derivative

We start this section introducing the concept of Möbius transformation.

Definition 3.1 (Möbius transformation)**.** Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. A Möbius transformation $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a map

$$f(z) = \frac{az + b}{cz + d} \tag{3.1}$$

where $a, b, c, d \in \mathbb{C}$ and $ac - bd \neq 0$.

The requirement for $ad - bc \neq 0$ is to ensure all such transformations are invertible. If $ad - bc$ was equal to 0, f would be the trivial map to a single point and it would not be injective. It is also clear that giving four numbers a, b, c, d such that $ad - bc \neq 0$, we are not defining an unique Möbius transformation, since rescaling them as $(a, b, c, d) \mapsto \lambda(a, b, c, d)$, with $\lambda \in \mathbb{C}$, would produce the same Möbius transformation. We can avoid this non-uniqueness by requiring $ad - bc = 1$. We define then the set Möb$(\hat{\mathbb{C}})$ as

$$\text{Möb}(\hat{\mathbb{C}}) = \left\{ f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} | f \text{ is a Möbius transformation and } ad - bc = 1 \right\} \tag{3.2}$$

We still have the ambiguity given by the choice of the sign (by mapping $(a, b, c, d) \mapsto (-a, -b, -c, -d)$ we obtain the same transformation).

A list of results regarding Möbius transformations follows.

Proposition 3.1. Möb$(\hat{\mathbb{C}})$ forms a group under composition of functions.

This result highlights the analogy that stands between Möb$(\hat{\mathbb{C}})$ and SL$(2, \mathbb{C})$ (the sub-group of GL$(2, \mathbb{C})$ such that the determinant of its elements is 1). The composition between two elements of Möb$(\hat{\mathbb{C}})$ is analogous to matrix multiplication, the inverse element is associated to the inverse matrix (this is possible because $ad - bc \neq 0$) and the identity element is associated to diag$(1, 1)$. We in fact have the following result.

Theorem 3.2. $\phi : \text{SL}(2, \mathbb{C}) \to \text{Möb}(\hat{\mathbb{C}})$ defined by

$$\phi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to f : z \mapsto \frac{az + b}{cz + d} \tag{3.3}$$

is a group homomorphims.

We said that an element of $\text{Möb}(\hat{\mathbb{C}})$ is not uniquely defined by four coefficients a, b, c, d such that $ad - bc = 1$, because we can map a, b, c, d to $-a, -b, -c, -d$, obtaining the same transformation. It is natural then to define the equivalence relation $\sim$ which identifies two elements of $\mathrm{SL}(2, \mathbb{C})$ which differ by a minus sign. This concept can be expressed through the following theorem.

Theorem 3.3. $\text{Möb}(\hat{\mathbb{C}}) \cong \mathrm{SL}(2, \mathbb{C})/\sim$

We define $\mathrm{PSL}(2, \mathbb{C}) := \mathrm{SL}(2, \mathbb{C})/\sim$, which stands for projective special linear group.
Next step is to define the Schwarzian derivative, making immediately clear the strong relation between the latter and Möbius transformations [1].

Definition 3.2 (Schwarzian Derivative)**.** Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function such that f' does not vanish identically. The Schwarzian derivative of f at $z \in \mathbb{C}$ is defined as

$$Sf(z) = \{f, z\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 \tag{3.4}$$

Proposition 3.4. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function and let f^* its transformed via Möbius transformation. Then

$$\{f, z\} = \{f^*, z\} \tag{3.5}$$

This means that the Schwarzian derivative is invariant under Möbius transformations.

The proof follows by direct computation of the first, second and third derivatives of $f^* = \frac{af+b}{cf+d}$, together with $\{f^*, z\}$.

We end this mathematics section by enunciating the chain rule for Schwarzian derivatives. This will be useful to obtain a special unique linear form for Friedmann's equations for $\kappa \neq 0$.

Proposition 3.5 (Chain rule)**.** Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function of y such that f' does not vanish identically. Then the identity (3.6) follows

$$\{f, x\} = \left(\frac{\partial y}{\partial x}\right)^2 \{f, y\} + \{y, x\} \tag{3.6}$$

where $x \in \mathbb{C}$ is a complex variable.

The proof is trivial. One can use the chain rule for ordinary derivatives and compute directly $\{f, x\}$.

3.2 Friedmann's equations and Schwarzian derivative: an infinite number of equivalent eigenvalues problems for vanishing curvature

We use in this section the properties of the Schwarzian derivative and of $\mathrm{PSL}(2, \mathbb{C})$ in order to obtain, imposing $\kappa = 0$, an infinite number of space-independent Klein-Gordon eigenvalues problems, two for each value of a real parameter we call β, completely equivalent to Friedmann's equations, as it is discussed in [1].

Let us take the following linear combination of the left-hand side of the two Friedmann's equations

$$X_\beta(a) := \frac{\ddot{a}}{a} + (\beta - 1)\left(\frac{\dot{a}}{a}\right)^2 \tag{3.7}$$

where β is a real parameter.
Our investigation naturally leads to introduce β-time as follows

$$\dot{t}_\beta = a^{\frac{1}{\delta(\beta)}} \tag{3.8}$$

where $\delta(\beta)$ is a function of β. t_β is a generalization of the concept of conformal time. We are imposing the time derivative of t_β to be a power of the scale factor. This choice is justified by the fact that the time derivative of η is in fact a power of the scale factor. Let us rewrite (3.7) in terms of t_β. We have

$$\dot{a} = \delta(\beta) \left(\dot{t}_\beta\right)^{\delta(\beta)-1} \ddot{t}_\beta$$

and

$$\ddot{a} = \delta(\beta) \left[\left(\delta(\beta) - 1\right) \left(\dot{t}_\beta\right)^{\delta(\beta)-2} \ddot{t}_\beta^2 + \left(\dot{t}_\beta\right)^{\delta(\beta)-1} \dddot{t}_\beta \right]$$

Dividing each expression by $a = \left(\dot{t}_\beta\right)^{\delta(\beta)}$ and plugging them into (3.7) we get

$$X_\beta = \delta(\beta) \left[\frac{\dddot{t}_\beta}{\dot{t}_\beta} + \left(\frac{\ddot{t}_\beta}{\dot{t}_\beta}\right)^2 \left(\beta\delta(\beta) - 1\right) \right]$$

Now note that by setting $\beta\delta(\beta) - 1 = -\frac{3}{2}$ the expression in square brackets becomes the Schwarzian derivative of t_β respect to t. We thus have $\delta(\beta) = -\frac{1}{2\beta}$ and we can write

$$X_\beta = -\frac{1}{2\beta} \{t_\beta, t\}$$

that is

$$\frac{\ddot{a}}{a} + (\beta - 1)\left(\frac{\dot{a}}{a}\right)^2 = -\frac{1}{2\beta}\{t_\beta, t\} \tag{3.9}$$

The relation between β-time and the scale factor becomes

$$\dot{t}_\beta = a^{-2\beta} \tag{3.10}$$

Let us now consider the expression

$$\dot{t}_\beta^{\frac{1}{2}} \frac{d}{dt} \dot{t}_\beta^{-1} \frac{d}{dt} \left(\dot{t}_\beta^{\frac{1}{2}} \phi_\beta\right) \tag{3.11}$$

where ϕ_β is a smooth function of t. We have

$$\dot{t}_\beta^{\frac{1}{2}} \left(-\frac{3}{4} \dot{t}_\beta^{-\frac{5}{2}} \ddot{t}_\beta^2 + \frac{1}{2} \dot{t}_\beta^{-\frac{3}{2}} \dddot{t}_\beta + \dot{t}_\beta^{-\frac{1}{2}} \frac{d^2}{dt^2} \right) \phi_\beta = \left(\frac{1}{2} \frac{\dddot{t}_\beta}{\dot{t}_\beta} - \frac{3}{4} \left(\frac{\ddot{t}_\beta}{\dot{t}_\beta}\right)^2 + \frac{d^2}{dt^2} \right) \phi_\beta$$

Equation (3.11) is then completely equivalent to

$$\left(\frac{d^2}{dt^2} + \frac{1}{2}\{t_\beta, t\} \right) \phi_\beta \tag{3.12}$$

We define the linear operator $A_\beta := \left(\frac{d^2}{dt^2} + \frac{1}{2}\{t_\beta, t\}\right)$. It is evident that, considering the vector space $\left\langle \psi_\beta, \psi_\beta^D \right\rangle$ where $\psi_\beta = \dot{t}_\beta^{-\frac{1}{2}} = a^\beta$ and $\psi_\beta^D = \dot{t}_\beta^{-\frac{1}{2}} t_\beta = a^\beta t_\beta$, it is the kernel of the linear operator A_β (this directly follows from the equivalence between (3.11) and (3.12)). Therefore we have

$$\ker A_\beta = \left\langle \psi_\beta, \psi_\beta^D \right\rangle = \left\langle a^\beta, a^\beta t_\beta \right\rangle \tag{3.13}$$

Let us stress that solving Friedmann's equations is equivalent to finding an expression for t_β, since $t_\beta = \int_0^t a(t')^{-2\beta} dt'$. Taking a look at (3.13) we find an immediate way to obtain t_β from the two linearly independent solutions of the problem $A_\beta \phi_\beta = 0$ (where $\phi_\beta \in \mathcal{C}^\infty(\mathbb{R})$)

$$t_\beta = \frac{\psi_\beta^D}{\psi_\beta} = \frac{a^\beta t_\beta}{a^\beta} \tag{3.14}$$

We use a pair of generic linearly independent solutions of the problem $A_\beta \phi_\beta = 0$ in order to solve Friedmann's equations, and not necessarily ψ_β^D and ψ_β. An arbitrary linear combination of ψ_β^D and ψ_β is a solution of $A_\beta \phi_\beta = 0$. Let us define $\psi_\beta^{D'}$ and ψ'_β such that

$$\begin{pmatrix} \psi_\beta^{D'} \\ \psi'_\beta \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \psi_\beta^D \\ \psi_\beta \end{pmatrix} \tag{3.15}$$

assuming $AD - BC \neq 0$ (we require ψ'_β and $\psi_\beta^{D'}$ to be linearly independent). We now have

$$\frac{\psi_\beta^{D'}}{\psi'_\beta} = \frac{At_\beta + B}{Ct_\beta + D} = t'_\beta \tag{3.16}$$

which is the Möbius transformation of t_β. By computing the ratio between two arbitrary linearly independent solutions of $A_\beta \phi_\beta = 0$ one gets an expression that is the Möbius transformation of t_β with some A, B, C, D which we suppose to be real and require that $AD - BC \neq 0$.
When we introduced the concept of Schwarzian derivative, we proved that it is invariant under Möbius transformations. If we map $t_\beta \mapsto t'_\beta$ via Möbius transformation, the time derivative of t'_β is

$$\dot{t}'_\beta = \frac{A\dot{t}_\beta(Ct_\beta + D) - (At_\beta + B)C\dot{t}_\beta}{(Ct_\beta + D)^2} = (AD - BC)\frac{\dot{t}_\beta}{(Ct_\beta + D)^2} \tag{3.17}$$

Since we supposed $AD - BC \neq 0$, we can write the transformed scale factor as

$$a' = \left(\dot{t}'_\beta\right)^{-\frac{1}{2\beta}} = (AD - BC)(Ct_\beta + D)^{\frac{1}{\beta}} a \tag{3.18}$$

X_β is proportional to the Schwarzian derivative of t_β ($X_\beta = -\frac{1}{2\beta}\{t_\beta, t\}$) and the Schwarzian derivative is invariant under Möbius transformations. We thus have

$$X_\beta(a) = X_\beta(a') = \frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} + \frac{1}{3}(\beta - 1)(8\pi G\rho + \Lambda) - \frac{\kappa}{a^2} \tag{3.19}$$

The last obstacle to obtain a linear form for the two Fridemann's equations is the presence of the term $-\frac{\kappa}{a^2}$ in equation (3.19). We then impose a vanishing curvature, $\kappa = 0$. In this case, we would obtain an expression for the scale factor directly by solving $A_\beta \phi_\beta = 0$, finding two arbitrary linearly independent solutions and computing their ratio. This would lead to a generic Möbius transformation of t_β, but since X_β is invariant under Möbius transformations and the right-hand side of equation (3.19) does not contain the scale factor (because $\kappa = 0$), by computing

$$a(t) = \left[\frac{d}{dt}\left(\frac{\psi_\beta^{D'}}{\psi'_\beta}\right)\right]^{-\frac{1}{2\beta}} \tag{3.20}$$

where $\psi_\beta^{D'}$ and ψ'_β are two arbitrary linearly independent solutions for $A_\beta\phi_\beta = 0$, we would get an $a(t)$ satisfying the two Friedmann's equations.

Let us now rewrite more explicitly the reformulation of the problem in the linear form, remembering we imposed a vanishing curvature.

$$\begin{aligned} & A_\beta\phi_\beta = 0 \\ \iff \quad & \left(\frac{d^2}{dt^2} + \frac{1}{2}\{t_\beta, t\}\right)\phi_\beta = 0 \end{aligned}$$

Since $\{t_\beta, t\} = -2\beta X_\beta$ we have

$$\begin{aligned} \left(\frac{d^2}{dt^2} - \beta X_\beta\right)\phi_\beta &= \left(\frac{d^2}{dt^2} - \beta\frac{4\pi G}{3}(\rho+3p) - \beta\frac{\Lambda}{3} - \frac{\beta}{3}(\beta-1)(8\pi G\rho+\Lambda)\right)\phi_\beta = 0 \\ \iff \quad & \left(\frac{d^2}{dt^2} - \frac{4}{3}\pi G\beta(\rho+3p) + \frac{8}{3}\pi\beta G\rho(1-\beta) - \frac{\beta^2}{3}\Lambda\right)\phi_\beta = 0 \end{aligned} \tag{3.21}$$

We want to group everything that contains the cosmological constant in the right-hand side the equation (3.21). Defining

$$O_\beta = \frac{d^2}{dt^2} - \frac{4}{3}\pi G\beta(\rho+3p) + \frac{8}{3}\pi\beta G\rho(1-\beta) \tag{3.22}$$

which is a Klein-Gordon space-independent linear operator, we obtain the following eigenvalues problem, analogous to the couple of Friedmann's equations

$$O_\beta\phi_\beta = \beta^2\frac{\Lambda}{3}\phi_\beta \tag{3.23}$$

Let us now consider

$$\Psi_{\alpha\beta} := \begin{pmatrix} a^\alpha \\ a^\alpha t_\alpha \\ a^\beta \\ a^\beta t_\beta \end{pmatrix} \tag{3.24}$$

with $\alpha \neq \beta \neq 0$. We said that the couple $\left\{a^\beta, a^\beta t_\beta\right\}$ consists of two linearly independent solutions of (3.23). We can thus replace each couple of functions for both α and β in $\Psi_{\alpha\beta}$ with an arbitrary linear combination of the two. Because of this last idea, together with the fact that for every α, β ($\alpha \neq \beta \neq 0$) X_α and X_β are linearly independent, we can write a generic canonical linear form for Friedmann's equations as

$$\begin{pmatrix} O_\alpha & 0 \\ 0 & O_\beta \end{pmatrix}\Psi_{\alpha\beta} = \frac{\Lambda}{3}\begin{pmatrix} \alpha^2 & 0 \\ 0 & \beta^2 \end{pmatrix}\Psi_{\alpha\beta} \tag{3.25}$$

By taking a look at equation (3.23), we notice that for every β, there exists a solution $\phi_{-\beta}$ which has the same eigenvalue (in fact, by mapping $\beta \mapsto -\beta$, the term $\beta^2\frac{\Lambda}{3}$ remains the same). We can thus group all the set of canonical linear forms for Friedmann's equations in the following eigenvalues problem

$$\begin{pmatrix} O_\beta & 0 \\ 0 & O_{-\beta} \end{pmatrix}\Psi_{\beta-\beta} = \beta^2\frac{\Lambda}{3}\Psi_{\beta-\beta} \tag{3.26}$$

The above shows that, in case of flat space, there exists an infinite number of equivalent linear forms for Friedmann's equations. Next step is to use the chain rule for the Schwarzian derivative and select a particular value of β (selecting the conformal time) in order to absorb the curvature and find a linear form for any value of $\kappa \neq 0$.

3.3 Selecting the conformal time ($\beta = \frac{1}{2}$) and unique linear form for Friedmann's equations for arbitrary curvature

The goal for this section is to find an unique linear form for Fridmann's equations for arbitrary curvature, as it is discussed in [1].
Let us start from a direct consequence of the chain rule for the Schwarzian derivative (equation (3.6)), which involves the conformal time

$$\eta(t) \equiv t_{\frac{1}{2}} = \int_0^t a(t')^{-1} dt'$$

We in fact have[5]

$$\left\{e^{\pm i\sqrt{\kappa}\eta}, t\right\} = \dot{\eta}^2 \left\{e^{\pm i\sqrt{\kappa}\eta}, \eta\right\} + \{\eta, t\} = \frac{\kappa}{2}\dot{\eta}^2 + \{\eta, t\} \tag{3.27}$$

We proved in section 3.2 that $X_\beta = -\frac{1}{2\beta}\{t_\beta, t\}$. Setting $\beta = \frac{1}{2}$ one gets

$$X_{\frac{1}{2}} = \frac{\ddot{a}}{a} - \frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 = -\{\eta, t\} \tag{3.28}$$

that is, using (2.36) and (2.39)

$$-\frac{4\pi G}{3}(2\rho + 3p) + \frac{\Lambda}{6} + \frac{\kappa}{2a^2} = -\{\eta, t\} \tag{3.29}$$

Note that $a^{-2} = \dot{\eta}^2$. It is then possible to express the Schwarzian derivative of the conformal time as

$$\{\eta, t\} = \frac{4\pi G}{3}(2\rho + 3p) - \frac{\Lambda}{6} - \frac{\kappa}{2}\dot{\eta}^2 \tag{3.30}$$

We can plug this result into (3.27) obtaining

$$\left\{e^{i\sqrt{\kappa}\eta}, t\right\} = \frac{4\pi G}{3}(2\rho + 3p) - \frac{\Lambda}{6} \tag{3.31}$$

We want to find a solution for the Schwarzian equation (3.31) in order to get $e^{\pm i\sqrt{\kappa}\eta}$. To obtain the conformal time we just need to take the natural logarithm of that solution. We find then an expression for the scale factor by computing $a(t) = \dot{\eta}^{-1}$.
To solve the Schwarzian equation we will write an equivalent second-order eigenvalues problem, as we did in the previous section. Instead of considering t_β (or in this case η), we will consider $\eta \mapsto f(\eta) = e^{\pm i\sqrt{\kappa}\eta}$. Let us examine the analogous of (3.11), for which there stands the following identity

$$\dot{f}(\eta)\frac{d}{dt}\dot{f}(\eta)^{-1}\frac{d}{dt}\left(\dot{f}(\eta)^{\frac{1}{2}}\phi\right) = \left(\frac{d^2}{dt^2} + \frac{1}{2}\{f(\eta), t\}\right)\phi \tag{3.32}$$

where ϕ is a smooth function of t. We showed in section 3.2 that a basis for the kernel of the operator in round brackets is $\left\{\psi, \psi^D\right\} = \left\{\dot{f}(\eta)^{-\frac{1}{2}}, \dot{f}(\eta)^{-\frac{1}{2}}f(\eta)\right\}$, and by computing the ratio $\frac{\psi^D}{\psi}$ (or, on account of the invariance under Möbius transformations, the ratio between two arbitrary linear combinations of ψ and ψ^D that are linearly independent) one obtains $f(\eta) = e^{\pm i\sqrt{\kappa}\eta}$ or an arbitrary Möbius transformation of $f(\eta)$. Since the Schwarian derivative is invariant under Möbius transformations, if $f(\eta)$ is a solution for the

[5] $\frac{d}{d\eta}e^{\pm i\sqrt{\kappa}\eta} = \pm i\sqrt{\kappa}e^{\pm i\sqrt{\kappa}\eta}$; $\frac{d^2}{d\eta^2}e^{\pm i\sqrt{\kappa}\eta} = -\kappa e^{\pm i\sqrt{\kappa}\eta}$; $\frac{d^3}{d\eta^3}e^{\pm i\sqrt{\kappa}\eta} = \mp i\kappa\sqrt{\kappa}e^{\pm i\sqrt{\kappa}\eta}$; so $\left\{e^{\pm i\sqrt{\kappa}\eta}, \eta\right\} = \frac{\kappa}{2}$.

Schwarzian equation (3.31), also $f'(\eta) = \frac{Af(\eta)+B}{Cf(\eta)+D}$ is a solution.

The above analysis leads to the following equivalent linear form for Friedmann's equation

$$\left[\frac{d^2}{dt^2} + \frac{2\pi G}{3}(2\rho + 3p)\right]\phi = \frac{\Lambda}{12}\phi \tag{3.33}$$

which is just the rewrite of (3.32). Equation (3.33) is a Klein-Gordon space-independent eigenvalues problem, where ϕ is the eigenvector and the cosmological constant Λ (which is a synonym for vacuum energy) is the eigenvalue, with a geometric multiplicity equal to two (because (3.33) is a second order linear differential equation). There are two interesting consequences of (3.33). The first one is that, to absorb the curvature κ (which we had to impose equal to 0 to obtain a linear form for Friedmann's equations for arbitrary β-time), we needed to consider the conformal time, which is, among all β-times, the only one physically relevant. We would not have obtained a linear form for arbitrary curvature by considering an arbitrary β-time. This directly descends from the fact that $a^{-1} = \dot{\eta}$, implying that the Schwarzian derivative respect to time of $f(\eta)$ does not depend on the curvature. We thus obtained an unique linear form for Friedmann's equations for arbitrary non-vanishing curvature. It is peculiar that this unique linear form selects η among all the β-times. The second interesting consequence is that solving Friedmann's equations is completely equivalent to finding the eigenvalue for (3.33), which is Λ. Solving Friedmann's equations is then basically the same as finding a value for the vacuum energy of the Universe.

Let us now write an explicit form for ψ and ψ^D, the two linearly independent solutions for (3.33).

$$\psi = \dot{f}(\eta)^{-\frac{1}{2}} = \left[-\kappa\dot{\eta}\exp\left(\pm i\sqrt{\kappa}\eta\right)\right]^{-\frac{1}{2}}$$

$$\psi^D = \dot{f}(\eta)^{-\frac{1}{2}}f(\eta) = \left[-\kappa\dot{\eta}\exp\left(\pm i\sqrt{\kappa}\eta\right)\right]^{-\frac{1}{2}}\exp\left(\pm i\sqrt{\kappa}\eta\right)$$

Dividing both ψ and ψ^D by $(-\kappa)^{-\frac{1}{2}}$ we get (since $\dot{\eta} = a^{-1}$)

$$\psi = \sqrt{a}\exp\left(\mp\frac{i}{2}\sqrt{\kappa}\int_0^t \frac{d\tau}{a(\tau)}\right) \tag{3.34}$$

$$\psi^D = \sqrt{a}\exp\left(\pm\frac{i}{2}\sqrt{\kappa}\int_0^t \frac{d\tau}{a(\tau)}\right) \tag{3.35}$$

Let us now consider the wave function $\Psi = \sqrt{a}e^{\frac{i}{2}\sqrt{\kappa}\eta}$, that is a solution for (3.33). For $\kappa \geqslant 0$, we have $a = |\Psi|^2$.
Equation (2.39) is already written in a linear form, that is

$$\left[\frac{d^2}{dt^2} + \frac{4\pi G}{3}(\rho + 3p)\right]a = \frac{\Lambda}{3}a \tag{3.36}$$

Recalling the expression of the Klein-Gordon operator O_β (equation (3.22)) we can finally deduce the wanted unique linear form for Friedmann's equations, writing a couple of Klein-Gordon space-independent eigenvalues problems, which resemble of two measurement problems

$$O_{1/2}\psi = \frac{\Lambda}{12}\psi \tag{3.37}$$

$$O_1 a = \frac{\Lambda}{3} a \tag{3.38}$$

To conclude this section we take a look at the expression of the two linearly independent solutions for (3.33). We said in the previous chapter, when we introduced the red-shift relation, that for a free falling particle the momentum is inversely proportional to the scale factor. If we substitute $a = \frac{\mathcal{C}}{p}$ into equation (3.34) (or equivalently (3.35)) we obtain, omitting multiplicative constants

$$\psi = \frac{1}{\sqrt{p}} \exp\left(\pm \frac{i}{2} \sqrt{\kappa} \int_0^t p(\tau) d\tau \right) \tag{3.39}$$

which is interestingly analogous to the WKB approximation of 1D solutions for the time-independent Schrödinger equation. We will analyze deeply the WKB approximation in section 4, discussing this strong and interesting analogy.

Let us discuss briefly the $\kappa \to 0$ limit. We proved in the previous section that for vanishing curvature Friedmann's equations are equivalent to

$$O_\beta \psi_\beta = \beta^2 \frac{\Lambda}{3} \psi_\beta \tag{3.40}$$

and two linearly independent solutions are $\psi_\beta^D = a^\beta t_\beta$ and $\psi_\beta = a^\beta$. Imposing $\beta = \frac{1}{2}$, one selects the conformal time among all β-times and equation (3.40) becomes equation (3.37). Furthermore $\psi_{1/2} = \sqrt{a}$ and $\psi_{1/2}^D = \sqrt{a}\eta$.

Note that ψ and ψ^D in the $\kappa \neq 0$ case (equations (3.34) and (3.35)) are not linearly independent in the $\kappa \to 0$ limit, in fact

$$\lim_{\kappa \to 0} \psi = \lim_{\kappa \to 0} \psi^D = \sqrt{a}$$

which is $\psi_{1/2}$. Note that η may depend on κ. We supposed the exponent in (3.34) and (3.35) to be small for small values of κ (meaning that $\sqrt{\kappa}\eta$ becomes 0 in the $\kappa \to 0$ limit). To get the expression of $\psi_{1/2}^D$ we proceed considering $f(\eta)$ defined above. We know that $\psi = \dot{f}(\eta)^{-\frac{1}{2}}$ and $\psi^D = \dot{f}(\eta)^{-\frac{1}{2}} f(\eta)$. Expanding for small exponents we obtain

$$\psi = (-\kappa)^{-\frac{1}{2}} \sqrt{a} \pm i \frac{\eta}{2} \sqrt{a} + \mathcal{O}\left(\sqrt{\kappa}\eta\right)$$

$$\psi^D = (-\kappa)^{-\frac{1}{2}} \sqrt{a} \mp i \frac{\eta}{2} \sqrt{a} + \mathcal{O}\left(\sqrt{\kappa}\eta\right)$$

We can thus get $\psi_{1/2}^D$ as follows

$$\psi_{1/2}^D = \lim_{\kappa \to 0} (-i) \left[\psi - \psi_D\right] = \sqrt{a}\eta \tag{3.41}$$

The above shows that the linear form (3.37) is valid for arbitrary curvature (positive, negative or vanishing). In fact, if one selects $\beta = \frac{1}{2}$ in the $\kappa = 0$ case, the latter can be seen as the $\kappa \to 0$ limit of the $\kappa \neq 0$ case. We can conclude that there is a particular linear form in the $\kappa = 0$ that is "privileged" and it involves the conformal time.

4 WKB approximation and Universe as a quantum state

We will introduce the WKB method in order to proceed discussing the analogy that stands between the two linearly independent solutions of (3.37) and the semi-classical approximate solutions of the 1D stationary Schrödinger equation, as discussed in [2]. We will firstly introduce the Old Quantum Theory, the Bohr-Sommerfeld condition and the Maslov condition in order to justify the guess we will adopt for the approximate semi-classical eigenstates for the Hamiltonian $\hat{H}$. We will follow the procedure described in [10], Chapter 15, except for the last part of section 4.1 (the WKB approximation around turning points), that is taken from [11], Chapter VII.

Each of the two linearly independent solutions of (3.37) can be seen as approximate solutions. Following [2], we will heuristically find the equation this approximation come from and solve it for some simple expressions of the scale factor, finding a wave function $\psi(t)$ and discussing how it can be related to Universe's evolution. $a(t)$ can be in fact seen as a semi-classical approximation for $\psi(t)\psi(t)^*$. We will solve Friedmann's equations in some simple cases, finding $a(t)$ and then using it to determine $\psi(t)$, developing a recursive method. Although these simple cases are not physically relevant, we are using them as toy models to find exact solutions and to show how we can eliminate singularities in Universe's evolution as given by $\psi(t)\psi(t)^*$. The latter and other peculiar behaviours suggest considering the existence of physical models manifesting quantum effects that are not expressed by Friedmann's equations, which may be already characterized by quantum properties due to their linear form and the analogy with quantum WKB.

4.1 The Old Quantum Theory and semi-classical approximation

Let us consider a particle in one-dimension (with one degree of freedom) and let C be a level set in phase space of the classical Hamiltonian $H(x,p) \in \mathcal{C}^\infty(\mathbb{R}^2)$

$$C = \left\{(x,p) \in \mathbb{R}^2 | H(x,p) = E\right\} \tag{4.1}$$

which we assume to be a closed curve. We imagine now to "draw" a wave on C. Following the de Broglie hypothesis we postulate that the local frequency $k(x)$ of the wave as a function of x is $\frac{p(x)}{\hbar}$. The wave itself can be expressed as

$$\cos\left(\frac{1}{\hbar}\int_{x_0}^{x} pdx - \delta\right) \tag{4.2}$$

where x_0 is an arbitrary starting point on the curve and δ is an arbitrary phase. In the round brackets we are integrating the 1-form $\frac{1}{\hbar}pdx$ on the differentiable manifold C. The Bohr-Sommerfeld condition is the requirement that the wave is periodic on the curve, which can be expressed as

$$\frac{1}{\hbar}\oint_C pdx = 2\pi n \qquad n \in \mathbb{Z} \tag{4.3}$$

The energy levels in the old quantum theory were taken to be those real numbers E for which the corresponding level curve C satisfies the Bohr-Sommerfeld condition. The Bohr-Sommerfeld quantization has the success to predict the energy levels of the hydrogen atom, but it fails when we desire to predict the energy levels of more complex systems. For systems with one degree of freedom we need to modify (4.3) introducing the Maslov correction

$$\frac{1}{\hbar}\oint_C pdx = 2\pi\left(n + \frac{1}{2}\right) \qquad n \in \mathbb{Z} \tag{4.4}$$

Maslov correction will be justified in a quasi-classical contest, in particular analyzing the semi-classical approximation around the turning points. Referring to Green's theorem
($\oint_C p dx = \int_B \frac{\partial p}{\partial p} dx dp = \int_B dx dp$, where $\partial B = C$), we can rewrite (4.4) as

$$\frac{1}{2\pi\hbar}\mathcal{S} = n + \frac{1}{2} \tag{4.5}$$

where $\mathcal{S}$ is the area enclosed by C. The Maslov condition does not (in most cases) give the exact energy levels, but it predicts only the leading-order in quasi-classical approximations. Notice that $\mathcal{S}$ has the dimension of an action (energy$\times$time). What we are doing is quantizing the action of a particle moving along a closed curve.

We are interested in finding approximate solutions of the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + (V(x) - E)\psi = 0 \tag{4.6}$$

for small values of $\hbar$. We analyze the behaviour of those approximate solutions in three different cases, the classical allowed region ($V(x) > E$), the classical forbidden region ($V(x) < E$) and the classical turning points (when $V(x) = E$).

Let us consider the classically allowed region. Given a potential $V(x)$ and an energy level E we can solve for the momentum as a function of x, arbitrarily choosing the plus sign

$$p(x) = \sqrt{2m(E - V(x))} \tag{4.7}$$

We are looking for approximate solutions of the form

$$\psi(x) = A(x)\exp\left(\pm i\frac{\mathcal{S}(x)}{\hbar}\right) \tag{4.8}$$

where $A(x)$ is a smooth function and $\mathcal{S}(x) := \int_{x_0}^{x} p(y)dy$, just like we did for the Old Quantum Theory. We choose $A(x)$ to be independent of $\hbar$. Our first result is to show that for any $E \in \mathbb{R}$ for which there is a classical allowed region and for any non-zero function $A(x) \in \mathcal{C}^\infty(\mathbb{R})$ we can build an approximate eigenvector of the Hamiltonian $\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$.

Proposition 4.1. For any $E_1, E_2 \in \mathbb{R}$ with $E_1 > \inf_{x\in\mathbb{R}} V(x)$ there exists a constant C and a non-zero function $A \in C^\infty(\mathbb{R})$ such that, for every $E \in [E_1, E_2]$, the support of A is contained in the classically allowed region at energy E and the function ψ given by

$$\psi(x) = A(x)\exp\left(\pm\frac{i}{\hbar}\int_{x_0}^{x} p(y)dy\right) \tag{4.9}$$

satisfies

$$\|\hat{H}\psi - E\psi\| \leqslant C\hbar\,\|\psi\| \tag{4.10}$$

Proof. For any $E \in [E_1, E_2]$, the classically allowed region for E contains the classically allowed region for E_1. We choose then $A(x)$ as a non-zero element of $\mathcal{C}^\infty(\mathbb{R})$ with support in the classically allowed region for

E_1.

We compute $\hat{H}\psi - E\psi$ by direct calculation. We have

$$\frac{d\psi}{dx} = \left(A'(x) \pm \frac{i}{\hbar}p(x)A(x)\right)\exp\left(\pm\frac{i}{\hbar}\int_{x_0}^{x} p(y)dy\right)$$

$$\frac{d^2\psi}{dx^2} = \left(A''(x) \pm \frac{2i}{\hbar}p(x)A'(x) \pm \frac{i}{\hbar}p'(x)A(x) - \frac{1}{\hbar^2}p(x)^2A(x)\right)\exp\left(\pm\frac{i}{\hbar}\int_{x_0}^{x} p(y)dy\right)$$

Since $p(x)^2 = 2m(V(x) - E)$, focusing on the term $-\frac{1}{\hbar^2}p(x)^2A(x)\exp(\dots)$, it cancels out with the term $(V(x) - E)\psi$ in the calculation of $\hat{H}\psi - E\psi$, in fact

$$-\frac{\hbar^2}{2m}\left(-\frac{1}{\hbar^2}\right)p(x)^2\psi(x) - V(x)\psi(x) + E\psi(x) = V(x)\psi(x) - E\psi(x) - V(x)\psi(x) + E\psi(x) = 0$$

We thus have

$$\hat{H}\psi - E\psi = -\frac{\hbar^2}{2m}\left(A''(x) \pm \frac{2i}{\hbar}p(x)A'(x) \pm \frac{i}{\hbar}p'(x)A(x)\right)\exp\left(\pm\frac{i}{\hbar}\int_{x_0}^{x} p(y)dy\right) \tag{4.11}$$

Recalling the triangle inequality for the L^2 norm $\|\cdot\|$ we have

$$\|\hat{H}\psi - E\psi\| \leqslant \frac{\hbar^2}{2m}\|A''\| + \frac{\hbar}{2m}\|2A'p + Ap'\| \tag{4.12}$$

Since $\|\psi\|$ is independent of $\hbar$, the right-hand side of (4.12) is of order $\hbar$. It is easy to check that $\|2A'p + Ap'\|$ is a bounded function of E in the interval $[E_1, E_2]$, so the result follows. $\square$

Using the result from proposition 4.1 it is immediate to prove that for any $E \in [E_1, E_2]$ there exists $\tilde{E}$ element of the spectrum of the Hamiltonian $\hat{H}$ such that

$$|E - \tilde{E}| \leqslant C\hbar \tag{4.13}$$

If we assume $V(x) \to +\infty$ as $x \to \pm\infty$, $\hat{H}$ will have a discrete spectrum and $\tilde{E}$ will be an eigenvalue of $\hat{H}$. The conclusion for such potentials is this: given any real number $E \in [E_1, E_2]$, there is an eigenvalue of $\hat{H}$ within $C\hbar$ of E. This is a manifestation of the classical limit: the quantum energy spectrum is approximating the classical energy spectrum as $\hbar$ goes to zero.

Let us now consider a potential $V(x)$ with the properties described above. For any $E \in [E_1, E_2]$ there are at least two points $a(E)$, $b(E)$ ($a(E) < b(E)$) for which $V(a) = V(b) = E$ (we will also assume that the derivative of V is nonzero at $a(E)$ and $b(E)$ for all $E \in [E_1, E_2]$). Let us consider $a(E)$ and $b(E)$ to be such that, for any $c(E)$ which satisfies $V(c(E)) = E$, $c(E) \notin [a(E), b(E)]$. We refer to $a(E)$ and $b(E)$ as turning points, since these are the points where a classical particle with energy E changes direction. We want to build the function $A(x)$ for both the classical allowed region and the classical forbidden region, giving it a physical interpretation.

Let us start with analyzing the approximate solution in the classical allowed region. Since we want to obtain a solution with an error smaller than $\hbar$, we require that the second and the third term in the round brackets in (4.11) cancel. This means

$$2p(x)A'(x) = -p'(x)A(x) \tag{4.14}$$

that is

$$A(x) = C\,(p(x))^{-\frac{1}{2}} \tag{4.15}$$

If $A(x)$ satisfies (4.15) we have

$$\hat{H}\psi - E\psi = -\frac{\hbar^2}{2m}\frac{A''(x)}{A(x)}\psi(x) \tag{4.16}$$

indicating that our error is of order $\hbar^2$. This applies only for the classical allowed region. Moreover, the classical momentum $p(x)$ goes to zero as $x \to a, b$, which means that $A(x)$ is unbounded around the classical turning points. We will solve this complication later on. For now, we want to conclude with the following physical analysis of the semi-classical approximate solution in the classical allowed region far from turning points.

We said that this solution takes the form

$$\psi(x) = \frac{C}{\sqrt{p(x)}}\exp\left(\pm\frac{i}{\hbar}\int_{x_0}^{x} p(y)dy\right) \tag{4.17}$$

We will refer to (4.17) as oscillating WKB function or quasi-classical solution for Schrödinger equation. The solution will be in general a linear combination of two exponentials, one with $+$ at the exponent and one with $-$. Referring to the Born postulate, the probability density function $\mathcal{P}$ related to the wave function ψ is $\mathcal{P}(x) = \frac{|\psi(x)|^2}{\|\psi(x)\|^2}$. We have that $\mathcal{P}(x)$ is such that

$$\mathcal{P}(x) \propto \frac{1}{p(x)} \tag{4.18}$$

or equivalently

$$\mathcal{P}(x) \propto \frac{1}{v(x)} \tag{4.19}$$

where $v(x)$ is the classical velocity of the particle. Let us consider $[c, d] \subset [a(E), b(E)]$. The probability to find the particle in $[c, d]$ is proportional to the following integral

$$\int_c^d \frac{dx}{v(x)} \tag{4.20}$$

Let $x(t)$ be the classical trajectory of the particle. Let t_c and t_d be such that $x(t_c) = c$ and $x(t_d) = d$. We can rewrite (4.20) as

$$\int_{t_c}^{t_d} \frac{v(x(t))}{v(x(t))}dt = \int_{t_c}^{t_d} dt = t_d - t_c = \Delta t_{cd} \tag{4.21}$$

The conclusion is that the probability of finding a particle described by the WKB approximation in the interval $[c, d]$ contained in the classical allowed region is proportional to the time that the particle spends in $[c, d]$. We will refer to this concept as quasi-classical probability.

We will briefly discuss the case of the classical forbidden region. We expect that the solution will be damped and asymptotically vanishing. Let us introduce the quantity

$$q(x) := \sqrt{2m\,(V(x) - E)} \tag{4.22}$$

We look then for an approximate solution of the time-independent 1D Schrödinger equation of the form

$$\psi(x) = A(x) \exp\left(\pm\frac{1}{\hbar}\int_{x_0}^{x} q(y)dy\right) \tag{4.23}$$

We obtain an expression that has to be satisfied by $A(x)$ similarly to what we got for the classical allowed region

$$A(x) = C\left(q(x)\right)^{-\frac{1}{2}} \tag{4.24}$$

We want a solution in $L_2(\mathbb{R})$, so we choose the minus sign at the exponent for the solution in $(b, +\infty)$ and the plus sign for the solution in $(-\infty, a)$ (we are again analyzing the solution far from turning points). We have

$$\psi(x) = \frac{C_1}{\sqrt{q(x)}} \exp\left(-\frac{1}{\hbar}\int_{x}^{a} q(y)dy\right) \qquad x \in (-\infty, a) \tag{4.25}$$

$$\psi(x) = \frac{C_2}{\sqrt{q(x)}} \exp\left(-\frac{1}{\hbar}\int_{b}^{x} q(y)dy\right) \qquad x \in (b, +\infty) \tag{4.26}$$

We refer to (4.25) and (4.26) as exponential WKB functions.
We know from general theory of ODEs that any solution for Schrödinger equation with a smooth potential is smooth, so the singularities at the turning points come from the artifact of our approximation. For small values of $\hbar$, the exact solution will track the WKB approximation until x goes close to the turning points. The exact solution will be large but finite at the turning points.

Our next goal is to discuss WKB approximation around turning points, following [11], Chapter VII. We will consider $b(E)$, defined above, such that there exists $\varepsilon \in \mathbb{R}^+$ which satisfies, for every $x \in (b, b+\varepsilon)$ and $y \in (b, b-\varepsilon)$, $V(x) > E$ and $V(y) < E$. We showed that, at a sufficient distance from the turning point, our approximation takes the form

$$\psi(x) = \frac{c}{\sqrt{q(x)}} \exp\left(-\frac{1}{\hbar}\int_{b}^{x} q(y)dy\right) \qquad x > b \tag{4.27}$$

$$\psi(x) = \frac{c_1}{\sqrt{p(x)}} \exp\left(-\frac{i}{\hbar}\int_{x_0}^{x} p(y)dy\right) + \frac{c_2}{\sqrt{p(x)}} \exp\left(+\frac{i}{\hbar}\int_{x_0}^{x} p(y)dy\right) \qquad x < b \tag{4.28}$$

It is convenient to set the starting point of the integration at the exponent of the oscillating WKB as $x_0 = b$. Since $\hat{H}$ is self-adjoint, the real and the imaginary parts of any solution is again a solution. We will therefore consider only real linear combinations, $c_1 = c_2^*$.
To determine c_1 and c_2 we will follow how $\psi(x)$ changes going from the classical forbidden region to the classical allowed region. For small $|x-b|$ we can linearize $E - V(x)$

$$E - V(x) = -\frac{dV}{dx}\,|_{x=b}\,(x-b) + \mathcal{O}\left((x-b)^2\right) = F_0(x-b) + \mathcal{O}\left((x-b)^2\right) \tag{4.29}$$

Since $b(E)$ is a right turning point with the properties described above, $F_0 := -\frac{dV}{dx}\,|_{x=b} < 0$. We can do this approximation because we assumed that the motion is quasi-classical for the entire region except for the singularities at the turning points. This means that $|x-b|$ is sufficiently small.

Since our approximation fails at turning points, we will consider $\psi(x)$ as a function of $x \in \mathbb{C}$, passing from $x - b > 0$ to $x - b < 0$ going "around" the turning point $b(E)$ along a semicircle of radius ρ in the superior complex half-plane. At the end of this modified path the exponential WKB function will gain an imaginary exponent, which has to match with the exponent of the oscillating WKB function. Using this technique we can determine the coefficients c_1 and c_2. We get [11]

$$c_1 = \frac{c}{2} e^{i\frac{\pi}{4}} \tag{4.30}$$

$$c_2 = \frac{c}{2} e^{-i\frac{\pi}{4}} \tag{4.31}$$

This leads to an expression for the WKB approximation in the left neighbourhood of $b(E)$

$$\psi(x) = \frac{c}{\sqrt{p(x)}} \cos\left(\frac{1}{\hbar}\int_b^x p(y)dy - \frac{\pi}{4}\right) \tag{4.32}$$

From this last expression one can deduce the origin of the term $\frac{1}{2}$ in the Maslov condition (4.4). The boundary condition (4.32) needs to be valid both for $a(E)$ and $b(E)$. These two expressions need to coincide for the entire classical allowed region, meaning the the sum of the phases (which is a constant) has to be equal to $n\pi$ ($n \in \mathbb{Z}$).

$$\frac{1}{\hbar}\int_a^b p(x)dx - \frac{\pi}{2} = n\pi \tag{4.33}$$

$$\frac{1}{2\pi\hbar}\oint p(x)dx = n + \frac{1}{2} \tag{4.34}$$

that is the Maslov condition ($\oint p(x)dx := 2\int_a^b p(x)dx$).

To conclude this section, we stress the idea that from a WKB function we can determine the equation which the approximate solution come from, determining at first the potential. This will be useful later on. Let $\psi = \frac{1}{\sqrt{p}} e^{\pm\frac{i}{\hbar}\mathcal{S}}$ be an oscillating WKB function. We have

$$\frac{1}{2m\,|\psi(x)|^4} = E - V(x) \tag{4.35}$$

To get the Hamiltonian we just need the expression of the kinetic energy operator.

4.2 Quantum Friedmann's equations and Universe's evolution

Here, following [2], we analyze deeply the intriguing analogy between QM WKB and the solutions of the linear form of Friedmann's equations, as it was said in the introduction of this Chapter. We proved in section 3.3 that Friedmann's equations admit an equivalent linear form

$$\left[\frac{d^2}{dt^2} + \frac{2\pi G}{3}(2\rho + 3p)\right]\psi = \frac{\Lambda}{12}\psi \tag{4.36}$$

for arbitrary curvature. Two linearly independent solutions for (4.36) are

$$\psi = \sqrt{a}\exp\left(\pm\frac{i}{2}\sqrt{\kappa}\int_0^t \frac{d\tau}{a(\tau)}\right) \tag{4.37}$$

Using the relation that stands between the momentum of a free-falling particle and the scale factor $p \propto a^{-1}$ in FLRW Universe (as seen in section 2.3) we get, omitting multiplicative coefficients

$$\psi = \frac{1}{\sqrt{p}} \exp\left(\pm \frac{i}{2}\sqrt{\kappa}\int_0^t p(\tau)d\tau\right) \tag{4.38}$$

which is strongly reminiscent of a WKB function, where $\frac{\sqrt{\kappa}}{2}$ plays the role of the inverse of $\hbar$. We can thus see ψ as a WKB approximation. The obvious question is to find the equation whose WKB approximation leads to (4.37).

We said in section 4.1 that we can get the potential from oscillating WKB functions by computing $\frac{1}{|\psi|^4} = 2m(E-V)$, if we choose one of the two linearly independent approximate solutions. The analogy between quantum WKB and cosmological WKB suggests $x \mapsto t$, $\hbar^2 \mapsto \frac{4}{\kappa}$ and $2m(E-V) \mapsto \frac{1}{a^2}$. Stationary Schrödinger equation $\left[\frac{d^2}{dx^2} + \frac{2m}{\hbar^2}(E-V)\right]\psi = 0$ becomes

$$\left(\frac{d^2}{dt^2} + \frac{\kappa}{4}\frac{1}{a^2}\right)\psi = 0 \tag{4.39}$$

The cosmological WKB wave function is oscillating only for $\kappa > 0$. We assume (4.39) to be valid also for negative curvature and for vanishing curvature. We are not analyzing the $\kappa = 0$ case for the sake of simplicity. We will only see that for a special solution for Friedmann's equations $a_\kappa(t)$ depends on the curvature and the ratio $\frac{\kappa}{a_\kappa^2}$ in equation (4.39) is well defined (non-zero) in the $\kappa \to 0$ limit.

Friedmann's equations can be seen then as approximate equations and the scale factor $a(t)$ emerges quasi-classically as $a(t) = |\psi_{\mathrm{WKB}}|^2$ if $\kappa > 0$.

For $\kappa > 0$ we have oscillating solutions and for $\kappa < 0$ we have exponential solutions. Note that one can get quasi-classical solutions for $\kappa = 0$ by imposing $\beta = \frac{1}{2}$ in (3.13), obtaining $\psi^D_{1/2} = \sqrt{a}\eta$ and $\psi_{1/2} = \sqrt{a}$ as discussed in section 3.27. The scale factor emerges as $a = \left|\psi_{1/2}\right|^2$.

We expect ψ to be related to the scale factor via $|\psi|^2$. The cosmological ψ_{WKB} is an approximate solution for (4.39) and the error is of the order of $\frac{4}{\kappa}$. We have

$$\psi = \psi_{\mathrm{WKB}} + \mathcal{O}\left(\frac{4}{\kappa}\right) \tag{4.40}$$

which implies, if $\kappa > 0$ and if we treat ψ_{WKB} as one of the two linearly independent solutions in (4.37)

$$|\psi|^2 = |\psi_{\mathrm{WKB}}|^2 + \mathcal{O}\left(\frac{4}{\kappa}\right) = a(t) + \mathcal{O}\left(\frac{4}{\kappa}\right) \tag{4.41}$$

It seems that Friedmann's equations emerge as an approximation for large values of the spatial curvature. The spatial curvature is not a well defined physical observable though, since it depends on the choice of the coordinates. We said in fact in section 2.2 that FLRW metric is invariant under the rescaling $r \mapsto \lambda r$, $\kappa \mapsto \frac{\kappa}{\lambda^2}$, $a(t) \mapsto \frac{a(t)}{\lambda}$, where $\lambda \in \mathbb{R}^+$, and so are Friedmann's equations. Note that measurable quantities, like the Hubble parameter $H := \frac{\dot{a}}{a}$ or the density parameter $\Omega := \frac{8\pi G}{3H^2}\rho$ ([3], Chapter 8), are invariant under this rescaling. The density parameter is such that $\Omega - 1 = \frac{\kappa}{H^2 a^2}$. One can then deduce the sign of κ by measuring Ω.

The fact that one can choose a scale for κ and a arbitrarily makes unclear the meaning of the error $\mathcal{O}\left(\frac{4}{\kappa}\right)$.

Furthermore, one could work adopting the convention of a dimensionless normalized curvature $k \in \{0, \pm 1\}$ and a scale factor $a(t)$ with the dimensions of a length (see section 2.2). In this scenario it is not clear what $\mathcal{O}\left(\frac{4}{k}\right)$ means.

A natural consequence of the analogy between the linear form for Friedmann's equations and QM WKB approximation is to consider FLRW model at a quantum scale. For this reason we parametrize the curvature (which we assumed to be positive) as

$$\kappa = \frac{\mu^2 c^2}{\hbar^2} \tag{4.42}$$

The choice of $\mu \in \mathbb{R}$ is arbitrary and sets the curvature. Note that μ has the dimensions of a mass. If one calls $\mu \equiv m$ they gets $\frac{m^2 c^2}{\hbar^2}$, which is analogous to the inverse squared of the reduced Compton wavelength. Let us now consider a constant $a(t)$, which can be a valid approximation for short periods of time (this means we are expanding $a(t)$ at order zero). This is the analogous of a free particle in QM, since the scale factor plays the role of the potential in (4.39). Let us set $a = \frac{1}{2}$. Equation (4.39) becomes, if one restores the c factor that was set to 1 because of NU

$$\left(\frac{d^2}{dt^2} + \frac{m^2 c^4}{\hbar^2}\right)\psi = 0 \tag{4.43}$$

which resemble a Klein-Gordon equation for a free particle. This strong and intriguing analogy convinces us that equation (4.39) is valid at all scales (microscopic and macroscopic) and it is characterized by quantum properties. This analogy may lead to quantum behaviours that are not expressed by Friedmann's equations. Note that by this reparametrization ($\kappa = \frac{m^2 c^2}{\hbar^2}$) the exponent of (4.37) becomes

$$\pm \frac{i}{2} \frac{mc}{\hbar} \int_0^t \frac{d\tau}{a(\tau)} \tag{4.44}$$

which is reminiscent of QM WKB approximation.

Following [13, 14, 15], we write a generic solution to the time-dependent 1D Schrödinger equation $i\hbar\frac{\partial \psi}{\partial t} = \hat{H}\psi$ using the polar decomposition

$$\psi(x,t) = R \exp\left(i\frac{\mathcal{S}_q}{\hbar}\right) \tag{4.45}$$

We put the subscript q to distinguish the "quantum action" from the classical action $\mathcal{S}$, which appears in the Hamilton-Jacobi equation and which can be seen as the classical limit of $\mathcal{S}_q$. R and $\mathcal{S}_q$ are functions of x and t and they satisfy the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \frac{p_q}{m}\right) = 0 \tag{4.46}$$

where $\rho = |\psi|^2 = R^2$ is the probability density associated to the wave function ψ and $p_q := \frac{\partial \mathcal{S}_q}{\partial x}$. For stationary states $\frac{\partial \rho}{\partial t} = 0$, then $R = \frac{1}{\sqrt{\mathcal{S}_q'}}$ satisfies the continuity equation. One can write, omitting a potential normalization coefficient

$$\psi(x) = \frac{1}{\sqrt{\mathcal{S}_q'(x)}} \exp\left(i\frac{\mathcal{S}_q(x)}{\hbar}\right) \tag{4.47}$$

which is an exact solution for the 1D Schrödinger equation. It looks like we replaced the classical momentum $p(x)$ that appears in the WKB function with the "quantum momentum" $p_q(x) = \mathcal{S}'_q(x)$. Our investigation leads to construct by analogy the cosmological wave function (solution of (4.39)) as

$$\psi(t) = \sqrt{a_q(t)} \exp\left(i\frac{\sqrt{\kappa}}{2}\int_0^t \frac{d\tau}{a_q(\tau)}\right) \tag{4.48}$$

It is manifestly clear how we may express Universe's evolution by $\psi(t)\psi(t)^*$. The scale factor emerges quasi-classically through $|\psi_{\text{WKB}}|^2$ and the "quantum scale factor" emerges as $a_q(t) = |\psi|^2$. This means that Friedmann's equations can be seen as a quasi-classical approximation of (4.39). We will solve equation (4.39) for two simple examples and show some peculiar behaviours of the squared module of the solution $\psi(t)$, which may be imputed to quantum effects that are not expressed by Friedmann's equations. Note that Friedmann's equations may be already characterized by quantum properties due to their linear form, from which "cosmological" WKB (equation (4.37)) emerges.

We will consider two examples of simple solutions for Friedmann's equations in order to get an expression for $a(t)$. We can consequently find an expression for ψ, solving (4.39). We are using a recursive method to get, for each step, a more accurate result. The goal for this last part of the article is to show how this recursive method works and how it could be useful to avoid singularities of the scale factor. We will use simple examples as toy models to find exact solutions for (4.39). The recursive method consists of using an expression for $a(t)$, plugging it in (4.39) and solve it. One could see (4.39) as an approximation of some other equation, going further with the recursive method. We will proceed with two steps and treat $\psi(t)\psi(t)^*$ as related to the evolution rate of the Universe, as it was said above.

We will firstly consider a simple cosmological model, for which the only energy source is the vacuum energy. This implies $T^{\mu\nu} = 0$, that is $\rho = p = 0$. Friedmann's equations become

$$\ddot{a}(t) = \frac{\Lambda}{3}a(t) \tag{4.49}$$

$$\dot{a}(t)^2 = \frac{\Lambda}{3}a(t)^2 - \kappa \tag{4.50}$$

The most general solution for (4.49) is

$$a(t) = Ae^{\sqrt{\frac{\Lambda}{3}}t} + Be^{-\sqrt{\frac{\Lambda}{3}}t} \tag{4.51}$$

where A, B are real coefficients. We will refer to this solution as classical non-singular bouncing solution. Note that according to this expression, Universe's expansion is accelerating for $t > 0$. We can rewrite (4.51) as

$$a(t) = C_1 \cosh\left(\sqrt{\frac{\Lambda}{3}}t + C_2\right) \qquad A = \frac{C_1}{2}e^{C_2},\ B = \frac{C_1}{2}e^{-C_2} \tag{4.52}$$

Plugging (4.52) into (4.50) we get an expression for the coefficient C_1

$$C_1^2 = \frac{3\kappa}{4\Lambda} \tag{4.53}$$

Choosing the plus solution (we require the scale factor to be positive) we get

$$a(t) = \sqrt{\frac{3\kappa}{\Lambda}} \cosh\left(\sqrt{\frac{\Lambda}{3}}t + C_2\right) \tag{4.54}$$

Using the bouncing solution (4.54), equation (4.39) becomes

$$\frac{d^2\psi}{dt^2} + \frac{\Lambda}{12}\left[\cosh\left(\sqrt{\frac{\Lambda}{3}}t + C_2\right)\right]^{-2}\psi = 0 \tag{4.55}$$

Note that this expression does not depend on the spatial curvature (and so does the solution $\psi(t)$). This behaviour was anticipated above and it shows how equation (4.39) could be valid also for vanishing κ. With the change of variable $t = \sqrt{\frac{3}{\Lambda}}(\tau - C_2)$ we get

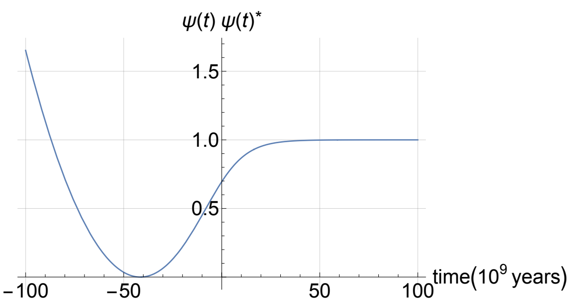
$$\frac{d^2\psi}{d\tau^2} + \frac{1}{4}\frac{1}{\cosh^2\tau}\psi = 0 \tag{4.56}$$

The most general solution of (4.56) is a linear combination of a Legendre function of first kind and a Legendre function of second kind, where $\lambda = -\frac{1}{2} + \frac{1}{\sqrt{2}}$ (see section 5.1)

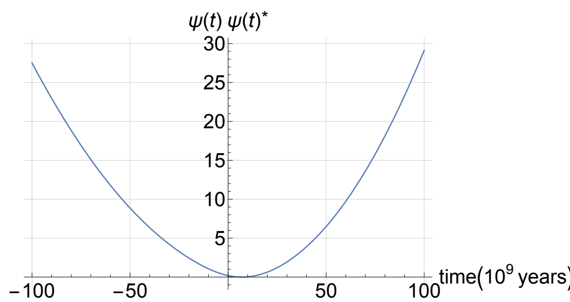
$$\psi(\tau) = \mathcal{A}P_{-\frac{1}{2}+\frac{1}{\sqrt{2}}}(\tanh\tau) + \mathcal{B}Q_{-\frac{1}{2}+\frac{1}{\sqrt{2}}}(\tanh\tau) \qquad \mathcal{A}, \mathcal{B} \in \mathbb{R} \tag{4.57}$$

We obtain $\psi(t)$ just by setting $\tau = \sqrt{\frac{\Lambda}{3}}t + C_2$. We will plot and discuss $\psi^*(t)\psi(t)$ for $\mathcal{A} = 1, \mathcal{B} = 0$ and for $\mathcal{A} = 0, \mathcal{B} = 1$ respectively. We use $\Lambda = (4.24 \pm 0.11) \times 10^{-66}\mathrm{eV}^2$ [12] (note that the cosmological constant has the units of a length^{-2} [6]) and we set $C_2 = 0$.

Figures 1a and 1b show the cases ($\mathcal{A} = 1, \mathcal{B} = 0$) and ($\mathcal{A} = 0, \mathcal{B} = 1$) respectively.



(a) $P_{-\frac{1}{2}+\frac{1}{\sqrt{2}}}(\tanh\tau)$



(b) $Q_{-\frac{1}{2}+\frac{1}{\sqrt{2}}}(\tanh\tau)$

Figure 1a shows an asymptotic behaviour of $|\psi|^2$ in the $\mathcal{B} = 0$ case. Note that we obtained an asymptotic behaviour from a non-asymptotic scale factor. Figure 1b ($\mathcal{A} = 0$ case) shows an accelerating expansion.

To conclude this work, let us consider an expression for $a(t)$ which solves Friedmann's equations for $p = \frac{\rho}{3}$ and $\Lambda = 0$

$$a(t) = \left(\frac{8\pi G\rho_0}{3\kappa} - \kappa t^2\right)^{\frac{1}{2}} \tag{4.58}$$

[6] We are adopting in this case the convention for which $\hbar = c = 1$. This means that $1\mathrm{eV} = 5.067730716... \times 10^6\mathrm{m}^{-1}$

where $\rho = \frac{\rho_0}{a^4}$ and $\rho_0 \in \mathbb{R}$ is a constant with the dimensions of an energy density. Note that (4.58) is singular for $t_0 = \pm\frac{1}{\kappa}\sqrt{\frac{8\pi G\rho_0}{3}}$. Plugging this solution into (4.37) we get

$$\left(\frac{d^2}{dt^2} + \frac{\kappa}{4}\left(\frac{8\pi G\rho_0}{3\kappa} - \kappa t^2\right)^{-1}\right)\psi(t) = 0 \tag{4.59}$$

Defining $\xi := \frac{8\pi G\rho_0}{3}$, a generic solution for (4.59) is

$$\psi(t) = \alpha\, {}_2F_1\left(\frac{1}{4}(-1-\sqrt{5}), \frac{1}{4}(-1+\sqrt{5}); \frac{1}{2}; \frac{\kappa^2 t^2}{4\xi}\right) + i\beta\frac{\kappa}{2\sqrt{\xi}}t\, {}_2F_1\left(\frac{1}{4}(1-\sqrt{5}), \frac{1}{4}(1+\sqrt{5}); \frac{3}{2}; \frac{\kappa^2 t^2}{4\xi}\right) \tag{4.60}$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric function (see section 5.1) and $\alpha, \beta \in \mathbb{R}$. This example shows how we can avoid singularities using this recursive method, expressing the evolution of the Universe through $\psi(t)\psi(t)^*$. This is clear if taking a look at figure 2, which shows $\psi(t)\psi(t)^*$ as given by equation (4.60) (in blue) and $a(t)$ as given by equation (4.58) (in orange).

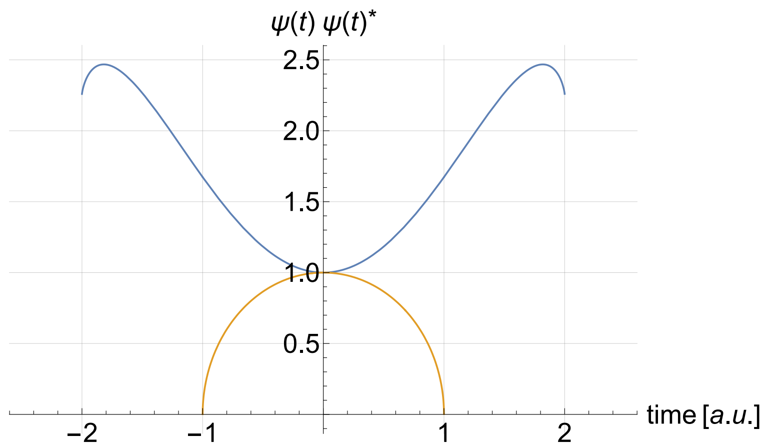


Figure 2. Comparison between the singular solution for Friedmann's equations (equation (4.58), orange) and $\psi(t)\psi(t)^*$ (equation (4.60), blue). α, β were chosen to be such that $\alpha = \beta = 1$ and we set $\xi = 1$, $\kappa = 1$. Time is just an dimensionless parameter (a.u. stands for arbitrary units). This figure's sake is only to compare the two solutions.

Figure 2 shows that $\psi(t)\psi(t)^*$ does not have the singularities at t_0, defined above, and it tracks $a(t)$ for small values of t. Although the expression for the scale factor in equation (4.58) does not have any significant physical meaning, it is interesting how we obtained an evolution without singularities via $\psi(t)$ starting from a singular $a(t)$. We also obtained an expansion starting from a contraction.

All peculiar behaviours of $|\psi(t)|^2$ showed above may be imputed to quantum effects which do not emerge in the context of FLRW Universe, as it was said in the previous paragraphs.

5 Appendix

5.1 Legendre functions of first and second kind

For this last section we are referring to [16]. Let us consider the general Legendre equation

$$(1-x^2)y'' - 2xy' + \left[\lambda(\lambda+1) - \frac{\mu^2}{1-x^2}\right] y = 0 \tag{5.1}$$

where $\lambda, \mu \in \mathbb{C}$. If $\mu = 0$ and $\lambda \in \mathbb{N}$ the solutions for (5.1) are the Legendre polynomials. If $\lambda \in \mathbb{N}$ and $\mu \in \mathbb{Z}$ such that $|\mu| \leqslant \lambda$ the solutions are the associated Legendre polynomials.

Equation (5.1) has two linearly independent solutions

$$P_\lambda^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left[\frac{1+z}{1-z}\right]^{\frac{\mu}{2}} {}_2F_1\left(-\lambda, \lambda+1; 1-\mu; \frac{1-z}{2}\right) \qquad |1-z| < 2 \tag{5.2}$$

$$Q_\lambda^\mu(z) = \frac{\sqrt{\pi}\Gamma(\lambda+\mu+1)}{2^{\lambda+1} + \Gamma\left(\lambda+\frac{3}{2}\right)} \frac{e^{i\mu\pi}(z^2-1)^{\frac{\mu}{2}}}{z^{\lambda+\mu+1}} {}_2F_1\left(\frac{\lambda+\mu+1}{2}, \frac{\lambda+\mu+2}{2}; \lambda+\frac{3}{2}; \frac{1}{z^2}\right) \quad |z| > 1 \tag{5.3}$$

where $P_\lambda^\mu(z)$ and $Q_\lambda^\mu(z)$ are denoted respectively as Legendre functions of first and second kind. Γ is the Euler-gamma function while ${}_2F_1$ is the hypergeometric function defined for $|z| < 1$

$${}_2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \tag{5.4}$$

where $(q)_n$ is the falling factorial

$$(q)_n := \begin{cases} 1 & n = 0 \\ q \cdot (q+1) \cdot \ldots \cdot (q+n-1) & n > 0 \end{cases} \tag{5.5}$$

${}_2F_1(a, b; c; z)$ is undefined (or infinite) if c equals a non-positive integer. When we write P_λ or Q_λ we intend P_λ^0 and Q_λ^0 respectively.

References

[1] Marco Matone, Universe as Klein–Gordon eigenstates, Eur. Phys. J. C **81** (2021) no.12, 1052 doi:10.1140/epjc/s10052-021-09865-4 [arXiv:2110.01557 [hep-th]].

[2] Marco Matone and Nikolaos Dimakis, in preparation.

[3] Sean Carroll, *Spacetime and Geometry: An Introduction to General Relativity*, Cambridge University Press, 2019.

[4] National Institute of Standards and Technology, NIST, The NIST Reference on Constants, Units, and Uncertainty, 2018, `https://physics.nist.gov/cuu/Constants/index.html`.

[5] Mikio Nakahara, *Geometry, Topology and Physics*, Second Edition, CRC Press, 2003.

[6] Christopher M. Hirata, Lecture Notes, Physics 217bc: The Standard Model - Cosmology, TAPIR (Theoretical AstroPhysics Including Relativity and Cosmology), Caltech (California Institute of Technology), Lecture 2, `http://www.tapir.caltech.edu/~chirata/ph217/lec02.pdf`.

[7] John A. Wheeler, Charles W. Misner, Kip S. Thorne, *Gravitation*, W. H. Freeman and Company, 1973.

[8] Steven Weinberg, *Cosmology*, Oxford University Press, 2008.

[9] John Olsen, The Geometry of Möbius Transformations, University of Rochester, 2010, `https://johno.dk/mathematics/moebius.pdf`.

[10] Brian C. Hall, *Quantum Theory for Mathematicians*, Springer Link, 2013.

[11] Lev D. Landau, Evgenij M. Lifsits, *Fisica Teorica 3. Meccanica quantistica. Teoria non relativistica*, Editori Riuniti, University Press, 2010.

[12] Nabila Aghanim *et al.* [Planck], Planck 2018 results. VI. Cosmological parameters, Astron. Astrophys. **641** (2020), A6 [erratum: Astron. Astrophys. **652** (2021), C4] doi:10.1051/0004-6361/201833910 [arXiv:1807.06209 [astro-ph.CO]].

[13] David Bohm, A Suggested Interpretation of the Quantum Theory in Terms of "Hidden Variables" I, Physical Review. **85**, 166–179 (1952). Archived text.

[14] A. E. Faraggi and M. Matone, "Quantum mechanics from an equivalence principle", Phys. Lett. B **450**, 34-40 (1999) doi:10.1016/S0370-2693(99)00113-6 [arXiv:hep-th/9705108 [hep-th]]. 108 citations counted in INSPIRE as of 05 Sep 2023.

[15] A. E. Faraggi and M. Matone, "The Equivalence postulate of quantum mechanics", Int. J. Mod. Phys. A **15**, 1869-2017 (2000) doi:10.1142/S0217751X00000811 [arXiv:hep-th/9809127 [hep-th]]. 158 citations counted in INSPIRE as of 05 Sep 2023.

[16] Timothy M. Dunster, Frank W. J. Olver, Daniel M. Lozier, Ronald F. Boisvert, Charles W. Clark, NIST Handbook of Mathematical Functions, Cambridge University Press, 2010, `https://dlmf.nist.gov/14`.