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Tesi di Laurea

Il modello di Stigler: finestra competitiva, strategie di mercato e auto-organizzazione

Markovian dynamics of a limit order book: competitive windows, self-organization and market strategies in the Stigler model

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Alla mia famiglia e agli amici che mi hanno sempre sostenuto

Abstract

The evolution of prices in markets results from the interaction of traders which submit orders to buy or sell. An order book is a list that contains orders sent to a market for a particular commodity or a financial instrument. Orders enter and quit the order book following rules that depends on the particular market. Maybe the most simple and famous order book model is the one named after Stigler. It is defined by a Markovian dynamics where buy and sell orders are placed at random within a price range and a pair of buy-sell orders is cancelled any time a sell order is placed on the left of a buy order. In spite of its simple definition the Stigler model has interesting features: one of these is that the dynamics self-organise to criticality. Indeed the Stigler model is closely related to the Bak-Sneppen model and queueing models with priority which are also known to show self-organised criticality. In the Stigler model self-organisation is related to the existence of a competitive window, meaning that in the long run only orders in a restricted price range are executed. Aim of the thesis is to study through probability/complex system tools and simulations the behavior of the Stigler model.

L'andamento dei prezzi nei mercati è il risultato dell'interazione complessa tra i vari traders che inviano ordini di acquisto o di vendita. Un order book è un registro che tiene traccia di tutti gli ordini inviati a un mercato per un particolare bene o strumento finanziario. Gli ordini vengono registrati e rimossi dall'order book in base alle regole del mercato specifico che si sta considerando. Forse il modello di order book più semplice e famoso è quello che prende il nome da Stigler. È definito da una dinamica markoviana in cui gli ordini di acquisto e di vendita sono collocati in modo casuale all'interno di un intervallo di prezzi e una coppia di ordini di acquisto e di vendita viene cancellata ogni volta che un ordine di vendita viene collocato a sinistra di un ordine di acquisto. Nonostante la sua semplice definizione, il modello di Stigler presenta caratteristiche interessanti: una di queste è che la dinamica fa evolvere il sistema a uno stato stazionario dalle proprietà critiche ("self-organised criticality") . Il modello di Stigler è infatti strettamente correlato al modello di Bak-Sneppen e ai "queueing models with priority", anch'essi noti per mostrare la self-organised criticality. Nel modello di Stigler, la self-organised criticality è legata all'esistenza di una finestra competitiva, il che significa che nel lungo periodo vengono eseguiti solo gli ordini collocati in un intervallo di prezzo ristretto. L'obiettivo della tesi è studiare attraverso strumenti di indagine propri della probabilità, dei sistemi complessi e simulazioni il comportamento del modello Stigler.

Contents

Introduction

A market is a dynamic and organised environment, where buyers and sellers arrange transactions, exchanging goods, services or resources. Markets have existed since the beginning of times, long before the advent of formal currencies, when human societies initially engaged in barter, bargaining basic commodities. With the introduction of monetary systems, traders gradually transitioned to buying and selling goods under increasingly regulated circumstances. Over time, financial markets evolved, enabling the trading of specific stocks and in the modern era they have become decidedly elaborate and interdependent with a wide range of possible transactions and participants.
Considering this framework, the complexity of modern financial markets is easily noticeable and it is due to the interplay of various factors such as numerous participants, intricate trading systems and tangled macroeconomic variables. Scientists have long been interested in studying and trying to unravel this complexity in order to predict trends and develop adequate risk management strategies. Since the beginning of the last century, economists started developing several financial mathematical models in order to help investors and decision-makers optimally allocate their resources to achieve the highest possible level of profitability.
Standard economic theory was indeed initially built on one fundamental assumption: utility maximisation. However every person who expected systematic profit from trades and tried to foresee price fluctuations had to be confronted with the fundamental cornerstone of financial theory: the efficient market hypothesis. It asserts that any information one tries to utilise to gain profit from price fluctuations has already been incorporated into the existing price, making impossible to predict future prices from the observation of past ones.
In proof of this, in the 1960s and 1980s market experiments questioned the mere possibility to play the market with a delineated strategy [4]. In these models the performance of markets with human traders was confronted with the performance of markets with machine traders, that submitted random bids and offers with the only budget constraint of not buying or selling at loss. The results showed that Adam Smith's invisible hand might have been more powerful than some may have thought: in fact humans and computers obtained on average the same market results. For this reason economists began to develop simplified financial market models in which traders have no intelligence: they do not seek to optimize profits, they cannot inspect or even remember. Investors behavior can be therefore considered random and not driven by any specific rational strategy.

These are the "zero intelligence" models. The simplest of all financial markets is the limit-order market, which is characterised by the employment of an *order book*, an electronic device which records all the orders sent to a market with their features (i.e. the time of placement of the order, its execution, its type, and so on). Hence this apparatus contains fundamental and powerful market information, that offers a profound understanding of the financial market dynamics. For this very reason any model developed to simulate an order-driven market must be confronted with the empirical trend data of order books and must capture the salient features of real markets.

This dissertation will focus on the study of the characteristics of limit-order markets with the analysis of the Stigler (1964) model [9], one of the first order book zero-intelligence models, in which orders of unit size arrive according to independent Poisson processes. This simple yet eloquent model has been separately reinvented by Luckock (2003) [6], whose analysis yielded steady state distributions for the best ask and best bid prices.
My approach has been to start with the characteristics of the limit order book, assessing its fundamental properties. Hence, Chapter 1 is a basic description of stylised facts on limit order books; it will outline the experimental evidence of the analysis of price change and waiting times distributions, extracted from various real markets. Then the focus will shift to the mathematical model in consideration: Chapter 2 presents an in-depth review of the Stigler model, which is defined by a Markovian dynamics where buy and sell orders are placed at random within a price range and a pair of buy-sell orders is cancelled any time a sell order is placed on the left of a buy order. The analysis will emphasise that the model shows two interesting market features: one of these is that the dynamics self-organise to criticality. Indeed it will become apparent that the Stigler model is closely related to the Bak-Sneppen model and queueing models with priority, that are known to show self-organised criticality. In this case self-organisation is related to the existence of a competitive window, meaning that in the long run only orders in a restricted price range are executed. The existence of this competitive window will be shown both mathematically and numerically with computer simulations, that will confirm the results obtained analytically. Appendix A contains the details of the mathematical steps involved in the demonstration and Appendix B the program code utilised for the Mathematica simulations.

Chapter 1

Limit Order Books: stylised facts

In the world there are many different types of stock markets and each is different from the other in the way each one organises its trade. Typically, the most effective markets are those where both buyers and sellers engage in an active competition. In case the traded goods are completely standardised, a trader takes part in the market competition by stating the price and the quantity of the items they are prepared to purchase or sell. An operating marketplace for standardised goods, wherein both buyers and sellers can propose their offers and are able to score a trade by accepting a proposal, is known as a "continuous double action" (CDA). This mechanism by which market participants issue orders (either to buy or sell) under conditions of price and quantity provides a useful generalisation of the process of haggling, phenomenon ubiquitous in every trading society.
The continuous double action mechanism finds particular use in catering to the requirements of modern financial markets, and is the basis of almost all automated trading systems that have been implemented since 1980s with the computerisation of trades. These systems rely on electronic limit orders books, a list where all unexecuted or partially executed orders are stored and displayed while awaiting execution. Therefore, it is apparent that understanding the structure and the mechanisms of the CDA and the order books is fundamental for anyone who wants to extract a model to describe this complex system.

1.1 Order placement

What we have established is that whatever the structure of the stock market is, traders issue orders under various different conditions that are peculiar to the particular market under consideration. In general once the order is issued, it is listed in the order book and it is executed as soon as an order of the opposite type (buy/sell) can match the appropriate requirements. But what is in practice a limit order book? [1] It is a tool extensively utilised in the majority of organised electronic markets. It means that all major equity, future or derivative markets record the interests of all their market participants in a central computer system. In essence, it is a computer file in which orders to sell (ask) and buy (bid) are listed according to the following characteristics:

1. the sign of the order: this records the type of the order, whether an ask or bid
2. the price, that the trader regards as realistic or achievable
3. the quantity, i.e. the specific amount of goods the trader is interested in buying or selling
4. the time of placement, i.e. a timestamp that indicates the order's registration time
5. the time of execution, i.e. a timestamp that indicates the order's execution time

In other words the limit order book is a snapshot of all the possible transactions feasible at any given moment within a particular market. This dynamic list is updated generally (depending on the peculiar market) at a rate of seconds and thus the state of the order book evolves every time transactions occur. The process continues as shown schematically in 1.1:

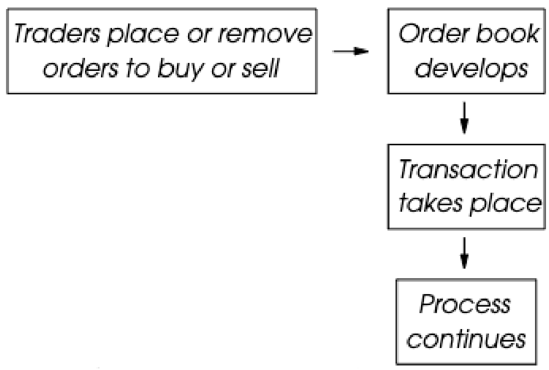

Figure 1.1: Schematic representation of a limit-order book (figure from "Econophysics and sociophysics: trends and perspectives." (2006), Chakrabarti, Bikas K., Anirban Chakraborti, and Arnab Chatterjee, eds.)

Therefore, in order-driven markets, buy and sell orders are matched and executed according to specific priority rules. Priority is always determined firstly by price, and in many markets, by the time of arrival, following the principle of "first in first out", but it always depends on the peculiarity of the specific market.

Until now, we have always taken into consideration only limit markets and limit order books. This specification is due to the order type that a trader can issue. Essentially, there are three types of orders:

- Limit orders: traders submit orders to buy a specific number of goods at a price that does not surpass a maximum one is willing to pay, or to sell a defined quantity at a price not lower than a specified minimum. If it can be matched with any opposite orders already lined up in the order book, a new limit order is promptly carried out against the most competitive. If not, the order joins the queue of unexecuted asks and bids in the order book, where it remains until it is either executed or cancelled

- Market orders: orders to buy or sell a designated quantity at the best price presently available. It has to be executed as soon and comprehensively as possible.

- Cancellation orders: orders to cancel existing limit orders previously submitted.

While some financial markets accepts many hybrids of the three basic orders described above, there is a growing trend, seen in major exchanges as those in Paris, Tokyo, Hong Kong, Sydney, Mumbai to function as pure limit order markets.

1.2 Empirical order book properties

In a limit order market, the determination of transaction prices depends on the interplay of incoming orders within the existing order book. The study of the limit order book therefore makes it possible to infer the stochastic price dynamics. Since these dynamics describe the behavior of the market, an understanding of this interaction would be of considerable value for the creation of an approachable model. For this reason the existence of the order book is of paramount importance: with the digitisation of such devices, detailed market data became easily available and this allowed to quantitatively assess the condition of a particular market at any time, extracting basic empirical findings characteristic of limit order markets. Therefore any model that aims to explain and foresee economic phenomena must show these stylised facts.

Here are the basic empirical findings emerging from the study of limit order books [8]. We will analyse both price and waiting times distributions, even if just the latter will be useful for the validation of the model under consideration. Furthermore, it is important to keep in mind that even the slightest change in the functioning of the limit market can lead to significant difference in trends between one stock market and another.

As regards price statistics, let Z_i be the price at which a transaction takes place at the instant T_i. This quantity, that is known as the 'current price', can be easily inferred from the assessment of the order book. We can then evaluate how far from the current price the limit orders are placed. Let ΔZ be this distance. The study of empirical data shows that the distribution of ΔZ is governed by the following power law:

$$P(\Delta Z) \propto (\Delta Z)^{-1-\mu}$$

The value of the exponent μ is controversial and can vary from $\backsimeq 0.5$ to $\backsimeq 1.4$. The study of the distribution of orders placed at a distance ΔZ from the current price, for France Telecom traded at Paris Bourse in 2001 shows exactly this tendency. This distribution can be seen in the following figure 1.2 where the line in the power law was fitted with a value of $\mu \backsimeq 0.5$:

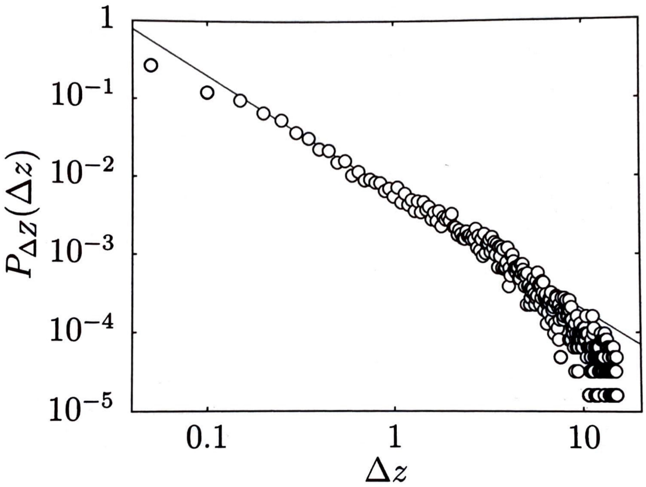

Figure 1.2: Distribution of orders placed at a distance ΔZ from the current price (in Euros), for France Telecom traded at Paris Bourse in 2001. The line in the power law was fitted with a value of $\mu \backsimeq 0.5$ (figure from [8]).

As regards the Stigler model, a significant temporal characteristic of the order placement is the lifespan of orders, the so called "waiting time" that indicates how long a specific order remains available within the order book. In the order book there are two scenarios that lead to the exit of an order from the order book: either an order is executed if matched with a complementary one, or it can be artificially removed from the order book upon request. The latter case can happen if the broker who had placed the order decides to cancel it or an order can be automatically removed upon expiration after a predefined period. In the Stigler model, we will just take into consideration limit order books, so cancellation orders fall outside the scope of this dissertation. Nonetheless, the distribution of the lifetime ΔT of both executed and removed orders is also governed by a power-law, as the following one:

$$P(\Delta T) \propto (\Delta T)^{-1-\alpha}$$

The value of the exponent varies from executed and removed orders. For limit orders empirical findings show that it is $\backsimeq 0.5$. Again, the study of the distribution of waiting times of orders (in seconds) for France Telecom traded at Paris Bourse in 2001 shows exactly this tendency. This distribution can be seen in the following figure 1.3, where the line in the power law was fitted with a value of $\alpha \backsimeq 0.5$:

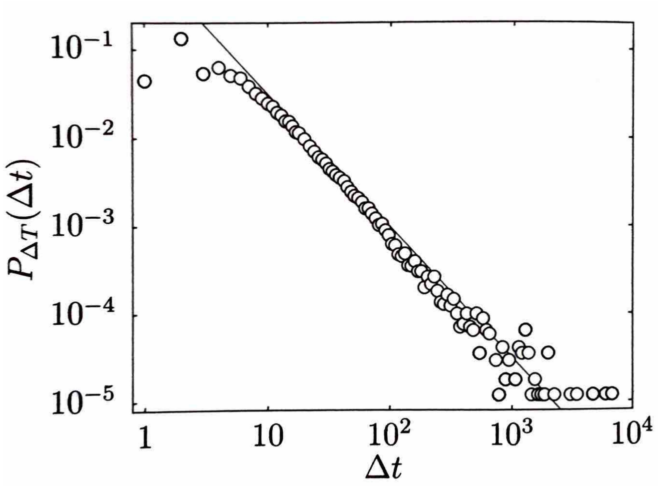

Figure 1.3: Distribution of waiting times of orders (in seconds) for France Telecom traded at Paris Bourse in 2001. The line in the power law was fitted with a value of $\alpha \backsimeq 0.5$ (figure from [8]).

As regards our model, this particular finding will be extremely helpful since the Stigler model will show the exact same behavior for the distribution of the lifetime of orders within the order book.

Chapter 2

The Stigler Model

One of the first attempts to simulate a financial market was undertaken by George J. Stigler. In 1964 [9] he proposed a simple model in which limit orders, asks or bids, were randomly submitted in the order book. Their price was uniformly distributed across the possible price interval and when an ask/bid order crossed the opposite best bid/ask order, it was executed. All orders were of unit size.

Stigler's model was then reinvented by Hugh Luckock in 2003, who was apparently unaware of Stigler's work. Luckock was able to explicitly calculate the equilibrium distribution of the bid and ask prices of his model, mathematically demonstrating the existence of a competitive window in which the vast majority of transactions are executed.

This chapter will review Luckock's mathematical findings and display the result of my simulation of the Stigler-Luckock model, which will be compared with the empirical facts presented in the previous chapter. Then it will emphasise how the simulation also shows that the system dynamics self-organise to self-criticality, a key concept in complex systems theory that will be analysed in regard to our model.

2.0.1 Stigler model results

Before delving into the specificities of the model, it proves useful to anticipate what the results of the model are in order to enable a better understanding of its theoretical and logical development. Through mathematical demonstration and with the application of a numerical computer simulation, we will show that in an order driven market with zero intelligence the system self-organises to criticality. In this particular case, this results in the formation of a competitive window that determines the possible price range in which orders can be executed. This means that all orders placed a price lower than x_{min} and higher than x_{max} are never executed and they indefinitely joins the other unexecuted orders in the order book (see Fig. 2.1).

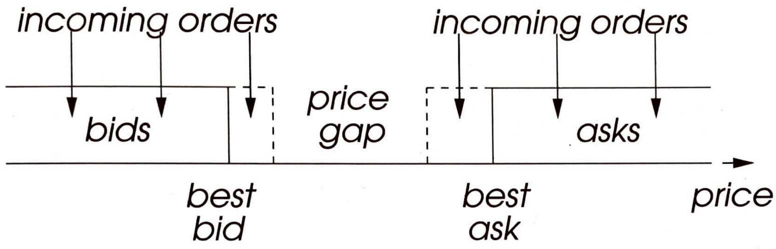

Figure 2.1: The price gap between the best bid and the best ask forms the competitive window where transactions take place. Unexecuted orders accumulate outside the competitive window (figure from "Econophysics and sociophysics: trends and perspectives." (2006), Chakrabarti, Bikas K., Anirban Chakraborti, and Arnab Chatterjee, eds.).

After demonstrating both mathematically and numerically the existence of the competitive window, we will analyse the behaviour of the lifetime of orders from placement to execution. With the aid of the computer simulation it will become apparent that both sell and buy orders follow a power law distribution. This phenomenon is strictly linked to the concept of self-organised criticality.

2.1 The characteristics of the model

The basic components of the model consist of buy and sell orders. A sell order, or "ask", submitted at a price α represents the seller's commitment to sell the considered asset at the first chance, at a price not lower than α. Contrariwise, a buy order, or "bid", submitted at a price β represents the buyer's willingness to buy the asset at the first chance, at a price not higher than β. An order is executable at a given price $x \in (0, \infty)$ if $\alpha \leq x$ and $x \leq \beta$ (so no asset can be sold or purchased at a negative price). As noted before, the model only accepts the submission of limit orders and for this reason the dynamics studied are those of limit order books.

The market has then the following properties:

1. Every order placed is for a single unit. For this reason the buyer only needs to specify the maximum price they are willing to pay, while the seller only need to indicate the minimum price there are willing to accept.

2. Buyers and sellers act independently of one another. Furthermore, they issue orders without knowing any information about the state of the order book at any time. Expected order arrival rates will be unconnected to the state of the order book. For this reason the submission of orders (i.e. the times of arrival of traders) of both types will be a Poisson process.

3. Orders cannot be cancelled once submitted and only the fulfillment of a transaction can remove a pair of orders of opposite types from the order book.

As stated before, the second property assures that this model is populated by "zero intelligence" traders. Orders are indeed submitted independently from one another: no trader has the capacity to acquire market information and infer possible trends, thus having no reason to revise his valuation of the asset in response to the state of the order book or the other traders' behavior. Traders will submit orders at prices that are random, therefore governed by a continuous uniform distribution within an interval of $[0, 1]$, that identifies a possible normalised range of prices. Despite this seemingly unrealistic assumptions, zero-intelligence models of the order book are capable to show many notable traits of real markets, as displayed by Gode and Sunder [4].

2.2 Dynamic rules of the model

At any given time t, the order book displays two queues: one for unexecuted sell orders at prices $\alpha_1(t)$, $\alpha_2(t)$, $\alpha_3(t)$, ... and one for unexecuted buy orders at prices $\beta_1(t)$, $\beta_2(t)$, $\beta_3(t)$, ... each one waiting to be paired with an incoming order. The different prices are ordered so that α_1 is the lowest ask price, followed by α_2 and so on and β_1 is the highest bid price, followed buy β_2 and so on. The quantities α_1 and β_1 are known respectively as best ask and best bid price. Once a new order is issued, the composition of the order books changes according to the following dynamic rules.

If the newly submitted order is a sell order at a price α (or respectively a buy order at a price β):

- If $\alpha \leq \beta_1$ (or respectively $\beta \geq \alpha_1$), the new sell order/buy order is matched with the best current bid/ask , resulting in the execution of a transaction at the price β_1/α_1.

- If $\alpha > \beta_1$ (or respectively $\beta < \alpha_1$), no match can occur and the new sell/buy order is added to the queue of unexecuted sell/buy orders within the order book.

As noted by Swart [7], classical economic theory predicts that a commodity will be eventually traded at its equilibrium price, determined by the market. In the Stigler-Luckock model the situation is quite different: the existence of the competitive window prevents the equilibrium price to be reached. Indeed, once the system has reached stability, bid and ask prices of feasible transactions fluctuate in a *competitive window* (x_{min}, x_{max}). We can therefore identify two "limit" prices, x_{max} and x_{min}. Our model will show that all orders submitted at a price lower than x_{min} and higher than x_{max} are never executed and they indefinitely joins the queue of unexecuted. Only orders placed at a price within the competitive window have the chance to be ultimately executed (see Fig. 2.2):

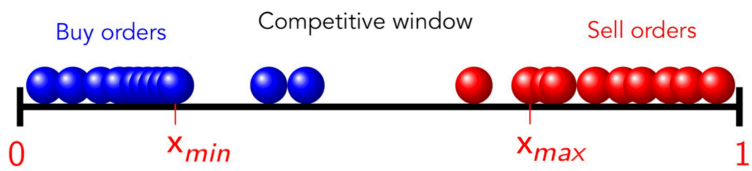

Figure 2.2: Snapshot of the order book: the competitive window is very visible and we can witness the accumulation of unexecuted orders for prices outside the competitive window. The interval $[0, 1]$ represents the possible range of prices, x_{max} and x_{min} are the endpoints of the competitive window.

2.3 Mathematical formulation of the model

The assumptions that were made to build the structure of the zero-intelligence market under consideration lead to the conclusion that in this model the submission of new sell and buy orders at a price $x \in [0, 1]$ follows a Poisson process. Therefore the probability that a sell or a buy order is issued in the possible price range $[0, 1]$ in the time interval Δt is for sell orders

$$P[\alpha \in ([0, 1], \Delta t)] = \lambda_A(x)\Delta t + o(\Delta t)^2 \tag{2.1}$$

and for buy orders

$$P[\beta \in ([0, 1], \Delta t)] = \lambda_B(x)\Delta t + o(\Delta t)^2 \tag{2.2}$$

where $\lambda_A(x)$ and $\lambda_B(x)$ is the average number of buy and sell orders issued in the time interval Δt at a price $x \in [0, 1]$. In our model, $\lambda_A(x)$ and $\lambda_B(x)$ are cumulative distribution functions (CDF) of uniform PDF in the interval $[0, 1]$. This implies that both sell and buy orders have the same probability of being issued by the trader that has arrived in the time interval Δt. But since the two options are mutually exclusive (an order will be to either buy or sell) the CDFs are $\lambda_A(x) = x$ and $\lambda_B(x) = 1 - x$. Our objective is therefore to determine the statistical characteristics, in a stable state, of an order book that follows the dynamical rules stated before with Poisson order arrivals.

To determine the endpoints of the competitive window x_{max} and x_{min}, we have to derive the steady-state distributions of the best ask α_1 and best bid β_1 price. Therefore, we define $A(x)$ to be the cumulative distribution for the best ask price α_1 and $B(x)$ to be the cumulative distribution for the best bid price β_1:

$$A(x) = P[\alpha_1 \leq x] \qquad\qquad B(x) = P[\beta_1 \geq x]$$

In other words, $A(x)$ represents the steady-state probability that the order book contains at least one executable order at the price x, similarly $B(x)$ represents the steady-state probability that the order book contains at least one buy order executable at the same

price. We have to remember that once the model has reached stability, best bid and best ask prices α_1 and β_1 fluctuate in the competitive window (x_{min}, x_{max}). Therefore only buy orders at prices exceeding x_{min} and sell orders at prices less than x_{max} have a positive probability of immediate execution against an existing order. Instead orders submitted at less competitive prices are unlikely to find matches among existing orders, leading them to accumulate within the order book.
Translating these considerations in mathematical formulas, we have:

$$A(x) = 1 \text{ for } x \in]x_{max}, 1] \qquad B(x) = 1 \text{ for } x \in [0, x_{min}[$$

The first equivalence represents the fact that no sell orders are executed at a price exceeding x_{max} and therefore the order book will certainly contain a sell order α in the price range $]x_{max}, 1]$. The second represents the fact that no buy orders are executed at a price below x_{min} and therefore the order book will certainly contain a buy order β in the price range $[0, x_{min}[$.

Much can be deduced about the functions A and B from simple arguments. We start by defining mathematically the endpoints of the competitive window:

$$x_{min} \equiv inf\{x \in [0,1] : B(x) < 1\} \qquad x_{max} \equiv sup\{x \in [0,1] : B(x) < 1\}$$

The function $A : [0,1] \to [0,1]$ is non-decreasing and right-continuous and similarly $B : [0,1] \to [0,1]$ is non-decreasing and left-continuous. The left-continuity of B and the right-continuity of A then imply:

$$B(x_{min}) = 1 \qquad A(x_{max}) = 1$$

Since $A(x) + B(x)$ is the probability that either $\alpha_1 \leq x$ or $\beta_1 \geq x$ and since these alternatives are mutually exclusive, $0 \leq A(x) + B(x) \leq 1$ everywhere. Therefore we also have

$$A(x_{min}) = 0 \qquad B(x_{max}) = 0$$

and, since A is a non-decreasing function and $A(x_{min}) < A(x_{max})$,

$$x_{min} < x_{max}$$

The identities $A(x_{min}) = 0$ and $A(x_{max}) = 1$ display the fact that the best ask price α_1 will almost always surpass x_{min} but hardly ever surpass x_{max}, while the boundary conditions for B indicate that the best bid price β_1 will almost always be lower than x_{max} but almost never less than x_{min}:

$$x_{min} < \alpha_1 \leq x_{max} \qquad x_{min} \leq \beta_1 < x_{max}$$

Up to now, we have therefore successfully characterise the mathematical qualitative conditions of the competitive window. In order to obtain the values we are searching for, a more thorough analysis is in order.

Let $M(t,x)$ be the number of sell orders α in the order book at the time t that are executable at the price x and $N(t,x)$ be the number of buy orders β in the order book at the time t that are executable at the same price x. The evolution of the functions M and N is influenced by the stochastic submission of new orders, so for any fixed price $x > 0$ and future time t, $M(t,x)$ and $N(t,x)$ are random variables. Therefore the quantities $\mathbb{E}[M(t,x_2) - M(t,x_1)]$ and $\mathbb{E}[N(t,x_1) - N(t,x_2)]$ represent respectively the expected number of unexecuted sell orders in the price interval $(x_1, x_2]$ and of unexecuted buy orders in the price interval $[x_1, x_2)$.
We are interested in their time evolution. For all prices x_1 and x_2 with $0 < x_1 < x_2 < 1$ we can infer the following identities (the detailed mathematical steps are given in *Appendix A, Proof I*).
For $M(t,x)$:

$$\frac{\partial \mathbb{E}[M(t,x_2) - M(t,x_1)]}{\partial t} = \int_{x_1}^{x_2} [(1-B)\, d\lambda_A - \lambda_B dA] \tag{2.3}$$

And for $N(t,x)$:

$$\frac{\partial \mathbb{E}[N(t,x_1) - N(t,x_2)]}{\partial t} = \int_{x_1}^{x_2} [\lambda_A dB - (1-A)\, d\lambda_B] \tag{2.4}$$

The consequences of 2.3 and 2.4 are very useful. These integral must be non negative in the steady state, since otherwise the quantity of unexecuted orders in the specified intervals would be indefinitely decreasing and would eventually become negative, which would make no sense. Therefore, for any $[x_1, x_2] \subset]0, \infty[$

$$\int_{x_1}^{x_2} (1-B)\, d\lambda_A \geq \int_{x_1}^{x_2} \lambda_B dA \geq 0 \tag{2.5}$$

and

$$-\int_{x_1}^{x_2} (1-A)\, d\lambda_B \geq -\int_{x_1}^{x_2} \lambda_A dB \geq 0 \tag{2.6}$$

Now let's suppose that $[x_1, x_2] \subset (x_{min}, \infty)$ (it's also obviously valid in $]x_{min}, x_{max}[$). Then $B(x_1) < 1$, this implies the existence of a non zero probability $1 - B(x_1) > 0$ that at some remote point in time t in the future, the order book will not contain any buy orders at a price of x_1 or higher, i.e. $N(t,x_1) = 0$. Since this probability remains constant and positive as $t \to \infty$, it can be deduced that $N(t,x_1)$ and therefore the difference $N(t,x_1) - N(t,x_2)$ must cyclically return to zero from time to time. A similar argument can be developed for $M(t,x_1)$.
Therefore, the two expected values $\mathbb{E}[N(t,x_2) - N(t,x_1)]$ and $\mathbb{E}[M(t,x_2) - M(t,x_1)]$ cannot be continually growing. This implies that for $N(t,x)$

$$\int_{x_1}^{x_2} [\lambda_A dB - (1-A) d\lambda_B] = 0 \qquad \text{for } [x_1, x_2] \subset (x_{min}, \infty) \tag{2.7}$$

and for $M(t,x)$

$$\int_{x_1}^{x_2} [(1-B) d\lambda_A - \lambda_B dA] = 0 \qquad \text{for } [x_1, x_2] \subset (0, x_{max}) \tag{2.8}$$

These equations shows that in the competitive window the average number of orders placed is constant and allow to find an explicit expression for the functions $A(x)$ and $B(x)$.

Indeed, if 2.8 and 2.7 are valid in the specified intervals, then, for any $x \in (x_{min}, x_{max})$ (for proof see *Appendix A, Proof II*),

$$[1 - A(x)]\lambda_B(x) + [1 - B(x)]\lambda_A(x) = \kappa \tag{2.9}$$

where κ is a constant equal to the overall frequency of the market. Now we can finally determine $A(x)$ and $B(x)$ from the supply and demand functions $\lambda_A(x)$ and $\lambda_B(x)$. For any $x_0 \in (x_{min}, x_{max})$, the general solution to equations 2.8 and 2.7 on (x_{min}, x_{max}) has the form

$$A(x) = 1 - \frac{[1 - A(x_0))]\lambda_A(x)}{\lambda_A(x_0)} - \kappa\lambda_A(x)\int_{x_0}^{x}\left(\frac{1}{\lambda_B}\right) d\left(\frac{1}{\lambda_A}\right) \tag{2.10}$$

$$B(x) = 1 - \frac{[1 - B(x_0))]\lambda_B(x)}{\lambda_B(x_0)} - \kappa\lambda_B(x)\int_{x_0}^{x}\left(\frac{1}{\lambda_A}\right) d\left(\frac{1}{\lambda_B}\right) \tag{2.11}$$

We can finally consider our uniform model and impose that $\lambda_A(x) = x$ and $\lambda_B(x) = 1-x$. The previous equations become

$$A(x) = 1 - \frac{[1 - A(x_0)]x}{x_0} + \kappa x\int_{x_0}^{x}\frac{1}{x^2(1-x)}dx \tag{2.12}$$

$$B(x) = 1 - \frac{[1 - B(x_0)](1-x)}{1-x_0} - \kappa(1-x)\int_{x_0}^{x}\frac{1}{x(1-x)^2}dx \tag{2.13}$$

We have now to determine the value of κ. In order to do so, we consider the previous expressions for $A(x)$ and $B(x)$ and we take the limit $x_0 \to x_{min}$ for the equation of $A(x)$ and the limit $x_0 \to x_{max}$ for the equation of $B(x)$. Remembering the boundary conditions $A(x_{min}) = 0$ and $B(x_{max}) = 0$ we obtain (we do not immediately specify the expression of λ_A and λ_B for the sake of generality):

$$A(x) = 1 - \frac{\lambda_A(x)}{\lambda_A(x_{min})} - \kappa\lambda_A(x)\int_{x_{min}}^{x}\left(\frac{1}{\lambda_B}\right) d\left(\frac{1}{\lambda_A}\right) \tag{2.14}$$

$$B(x) = 1 - \frac{\lambda_B(x)}{\lambda_B(x_{max})} + \kappa\lambda_B(x)\int_{x}^{x_{max}}\left(\frac{1}{\lambda_A}\right) d\left(\frac{1}{\lambda_B}\right) \tag{2.15}$$

We then use the two previous equations and the equation 2.7. We write the latter as $\lambda_A dB = (1-A)d\lambda_B$. We then find the following equation. Remembering that for the symmetry of the system $x_{min} + x_{max} = 1$, we find an explicit expression of κ as (proof of this is shown in *Appendix A, Proof III*):

$$\kappa = \frac{1}{1 - x_{min} \cdot \ln\left(\frac{x_{min}}{1-x_{min}}\right)} \tag{2.16}$$

Finally, in order to determine the endpoints of the competitive window, we consider $A(x_{max})$ utilising the expression of κ just found. The general solution for x_{max} is therefore (proof is shown in *Appendix A, Proof IV*):

$$\frac{1}{x_{max}} = \ln\left(\frac{x_{max}}{1 - x_{max}}\right) \tag{2.17}$$

Considering $\frac{1}{x_{max}} = z$, the previous equation becomes $e^{-z} = z - 1$. This is an exponential equation that can only be resolved numerically $z \approx 1.2785$. We can finally calculate the values for x_{min} and x_{max}:

$$x_{min} \approx 0.2178 \qquad\qquad x_{max} \approx 0.7822 \tag{2.18}$$

2.4 Computer simulation

We will now illustrate the details of the computer simulation of the Stigler-Luckock model. In the simulation we impose that the arrival of traders in the market and therefore the submission of a buy or sell order is governed by a Poisson process. This is equivalent to impose that the probability that an order (whether buy or sell) will be placed in the possible normalised price range $[0, 1]$ is $\lambda \cdot \Delta T$, where λ is the average number of orders issued in the time interval ΔT and ΔT represents the minimum time interval in which the market evolves and the state of the order book is updated. We have set $\lambda = 1$ both for buy and sell orders, so that they have the same probability of occurrence and $\Delta T = 0.001$.

From this simulation we can indeed easily calculate estimates for the endpoint of the competitive window (x_{min}, x_{max}), that we will compare with the theoretical one. Furthermore, we will study another temporal property very important to our scope: we will calculate the lifetime of the various orders, in other words how long a specific buy or sell order lasts in the order book. This will allow us to confirm that the lifetime ΔT of executed orders is power-law distributed and to therefore verify that our model successfully simulate the behavior of a real market. Both of these results are strictly related to the phenomenon of *self organised criticality*: the simulation will therefore confirm that the system under consideration self-organises to criticality. The Mathematica code utilised for the simulation can be found on *Appendix B*.

2.4.1 Results of the simulation: competitive window

With the simulation just described, considering a number of iteration equal to 10^7, we can construct the following histogram. It represents the state of the order book at the end of the process, as if we analysed the remaining orders in the order book at the end of the day after the market had closed. In other words the Figure 2.3 is a plot of the unexecuted buy and sell orders (the bins of the histogram have a width of 0.01):

Figure 2.3: Histogram of the order book: the competitive windows is very visible and we can witness the accumulation of unexecuted orders for prices outside the competitive window. The interval $[0, 1]$ represents the possible range of prices.

We can observe the competitive window, whose endpoints of the competitive window are virtually equal to the theoretical ones listed in 2.18.

2.4.2 Results of the simulation: waiting times distribution

With the simulation it is also possible to inspect another property of the system: the lifetime of orders from placement to execution. If our model effectively reproduces the behavior of a real market, we should be able to obtain a graph very similar to 1.3 in *Chapter 1*, a log-log plot where the horizontal axis represents the interval of the time Δt an order has to wait before being executed and the vertical axis represents its probability, i.e. the normalised frequency of the waiting times of executed orders in the order book.

Both sell and buy orders are clearly power law distributed. If we take into account the empirical findings listed in *Chapter 1*, the power law that governs the waiting times distribution depends on the parameter α as $P(\Delta T) \propto (\Delta T)^{-1-\alpha}$ where $\alpha \approx 0.5$ from empirical findings. If we plot this power law with the simulation data with $\alpha \approx 0.5$ for buy orders and for sell orders we obtain the two graphs in the following page. As we can observe, the simulation data are distributed according the empirical results. This means that our model once again faithfully reproduces the behaviour of a real market and therefore exhibits power-law distributions for the lifetime of orders.

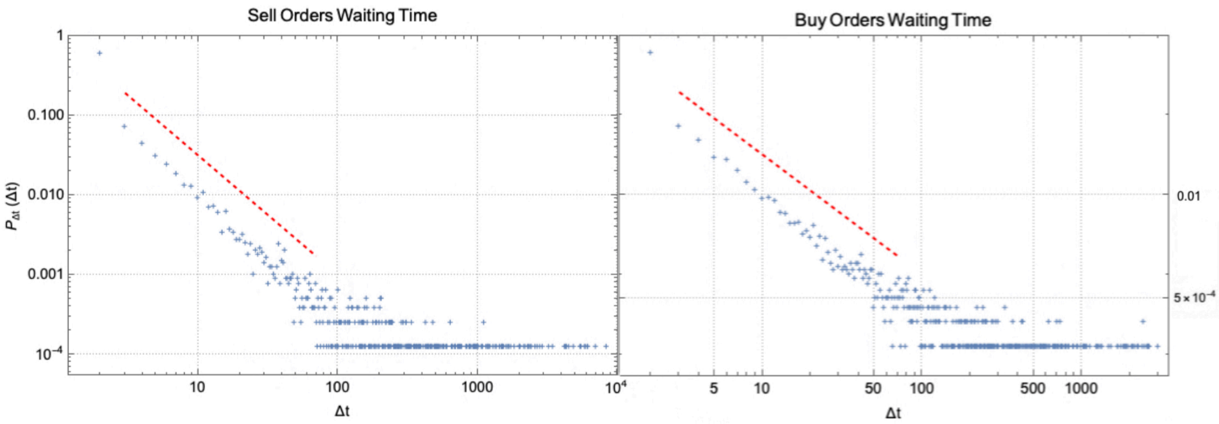

Figure 2.4: Graph of the log-log distribution of the lifetime of buy and sell orders from placement to execution in the simulation. The line is the power law $\propto (\Delta t)^{-1.5}$

This phenomenon, as we will see in the next section, is strictly linked to the concept of *self-organised criticality*. Indeed, physical systems at the phase transition point (also called the critical point) present power law decay of quantities of interest. In analogy with physics, systems or models that are driven to criticality by their own dynamics are said to show self-organised criticality.

Our "temporal" analysis does not stop there. It is of particular interest to study the evolution of the best buy and the best bid once the system has self-organised itself as traders arrive on the market. As predicted by theory, the simulation data will show that the equilibrium price is never attained therefore entering in contrast with classical 1900s economic theory.

If we inspect how the best bid and best ask prices evolve as orders are submitted by new traders in the market, we clearly see from the image below that these prices do not settle at an equilibrium price but keep fluctuating in the competitive window. In this graph the horizontal axis represents the $n-th$ trader that arrives in the market submitting either a sell or a buy order and the vertical axis represents the best ask and best buy price at the time t the $n-th$ trader has entered the market. In a *zero-intelligence* model therefore, the equilibrium price is never attained, and the best bid and best ask prices are contained within the respective endpoint of the competitive window. In fact, see Fig.2.5, the best ask price never goes below 0.8 and the best buy price never goes below 0.2 (i.e. the approximations of x_{max} and x_{min} obtained theoretically).

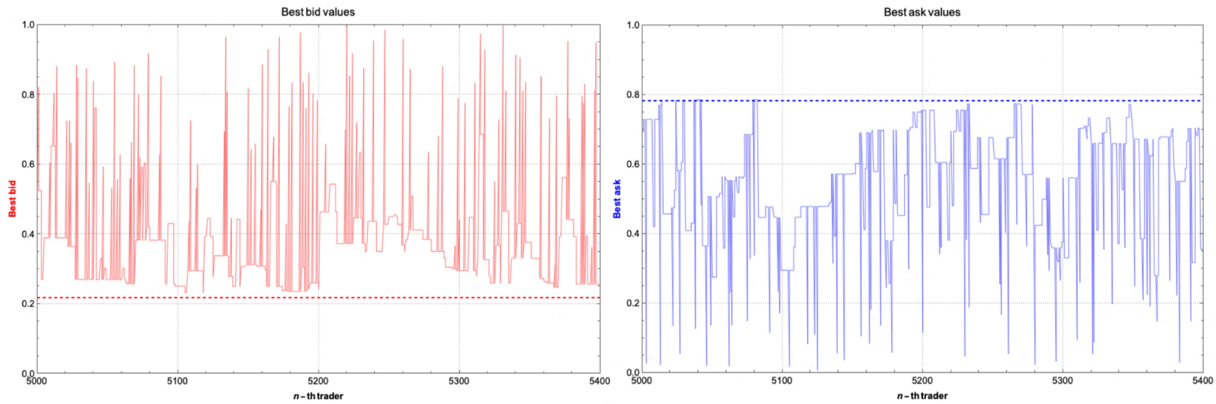

Figure 2.5: Evolution of the best ask and bid price between the arrival of the 5000-th and the 5400-th trader. As we can observe, the best ask price never goes above x_{max} and the best bid price never goes below x_{min}. The dashed lines represent the endpoint x_{max} and x_{min}, calculated mathematically before.

2.5 The Stigler model and Self-organised criticality

Financial markets can be compared to common microscopical physical systems: both of them consist of many body with out-of-equilibrium stochastic dynamics. In the limit order market we have considered so far (i.e. the Stigler model), the device that allows the evolution of the system is the limit order book which records the placement and removal of the buy and the sell orders and their settlement. If we wanted to juxtapose these two entities, we could consider the various orders as "particles" that interact with each other within the system depositing and evaporating at prescribed rates. But how can we model such composite and complicated systems? Indeed, a financial market is a good example of a complex system: the market participants are various and they each act with an agenda, the dynamical rules (i.e. the norms that regulate possible transactions) are often complicated and depend on the considered asset, every second an enormous number of transaction is carried out... What patterns a system like this shows? Does it showcase a behavior similar to other complex economic systems or even physical ones? These considerations do not just apply to economical ensembles but are of great importance to a vast class of systems [5].

Let us take into consideration the following systems: a group of electrons, a pile of sand, an ecosystem or the community of stock-market traders. What do they all have in common? Each one of these is composed of many components that interact with each other exchanging energy or information with some kind of action. In addition to these "internal" forces, the overall dynamic of the system can be also driven by external interactions (such as an electric field or environmental alterations). The following questions arise: what happens during this evolution? Is there a simplifying mechanism that leads to typical behaviors shared among many systems, or does each system always depend on the specific details of the dynamical rules?

The paper by Bak, Tang and Wiesenfeld *Self-organized criticality: an explanation of 1/f noise* (1987) presented the hypothesis that in fact complex many-bodies systems exhibit common characteristic behaviors and dynamical patterns. The claim they made was that, under very general conditions, some dynamical systems tend to organise themselves into a state that displays a complex yet general structure. The interesting implication is that although the dynamical response of the system is intricate, the simplifying aspect is that the statistical properties are governed by power laws.
Bak, Tang and Wiesenfeld proposed that this common behavior arises without any external fine "tuning" of the system and this self-organised state has the same properties of equilibrium systems at the critical point. They described this behavior as *self-organised criticality*. In such systems, small events or perturbations have the capacity to trigger larger cascades and avalanches, causing events of various sizes. This behavior is reflected in the word "criticality". Indeed in equilibrium thermodynamics criticality refers to phase transitions: when a thermodynamic system reaches exactly its transition temperature, something unexpected happens. While for every other temperature, one can locally perturb the system and the effect of this disturbance is only noticed by the local region where the perturbation was applied; at the transition temperature, the local distortion will propagate throughout the entire system. These systems are therefore critical because all members of the system influence each other.

Self-organised critical systems evolve to the complex critical state without the interference from any outside agent and the process of self-organisation takes place after a transient period [2]. For our purposes, one of the most important features of self-organised critical systems is that in the theory of self-organised criticality the power-law distributions emerge spontaneously without any parameter tuning. Essentially, all the phenomena subjected to self-organisation can be expressed in terms of power laws. Thus, the problem of explaining the observed statistical features of complex systems can be translated mathematically into the the problem of explaining the underlying power laws and more specifically the values of the exponents.
These properties are showed by a large number of complex systems and phenomena like earthquakes, solar flares, avalanches, ecosystems... In particular a model that exhibits self-organisation criticality is indeed the Stigler model under consideration. This is confirmed by the fact that after a transient period, the order book self-organises itself showcasing the insurgence of the competitive window, meaning that in the long run only orders in a restricted price range are executed. Another fundamental proof of its self-organised critical behavior is its power-law distributions of waiting times: the lifetime of orders within the order book presents a power law decay distribution, an hard evidence of its self-organised properties.

2.5.1 The Stigler model and the Bak Sneppen evolution model: how self-organised criticality is transversely present

In 1993 Per Bak and Kim Sneppen in their paper "*Punctuated equilibrium and criticality in a simple model of evolution*" [3] presented a simple but robust model of the biological evolution of an ecosystem of interacting species. This model is considered a cornerstone in the validation of self-organised criticality since the simple dynamics that regulate the model automatically lead the system to self-organise itself into a steady critical state with intermittent evolutionary "avalanches" of all sizes. In order to be able to proficiently compare these two models, let us delve into the details of its structure and the assumptions of its theorizing.
With their article Bak and Sneppen tried to find a mathematical answer to the following questions. Does evolution exhibit a consistent and steady pattern, characterised by the continuous flow of old species disappearing while new species emerge and better adapt? Or does it exhibit a more erratic behavior, involving long periods of peaceful co-existence among species, punctuated by brief but intense period of extinction, after which new species can rapidly proliferate? From studies of the fossil record, the paleontological evidence suggest that the latter scenario is more reflective of reality: species tend to survive for long periods and the vanish relatively quickly over a relative short span of time. Moreover, it is often observed that the extinction of one species coincides with the extinction of several others. Therefore we can summarise the evolutionary phenomenon as a "process of long time spans of equilibrium separated by short periods of activity".

Bak and Sneppen managed to mathematically translate these speculations in a very simple probabilistic model, astoundingly similar to the Stigler one. Consider a fixed number of species N, each labeled by x. The fitness of a specific species is defined as the ability of an individual to survive in a given environment/ecosystem and it is represented by a number $B(x)$ ranging from 0 to 1. If $B(x)$ is close to zero, it means that within the given environment the species x has a phenotype poorly suited for competing with other co-existing species. If the fitness $B(x)$ is close to 1, the species has great possibilities of surviving in the ecosystem. It is important to note that this model only considers co-evolution: the fitness of a certain species is determined by random peculiarities of co-existing species and not by the physical environment, whether harsh or favorable.
In order to determine how the species interact with each other, and consequently how fitness changes, Bak and Sneppen followed a principle often used in statistical mechanics: they choose the simplest possible representation of the phenomenon. In the Bak-Sneppen model fitness is a relative quantity and it can therefore increase or decrease when another species disappears or appears. Typically, species with low fitness are more likely to become extinct. Hence, the dynamics of the model consist of identifying the species x_S with the lowest fitness B and then removing it. In order to maintain constant the number N of total species in the system, the extinct species x_S is immediately replaced by a new one. Since mutations are by definition random, we expect that the fitness of the newly arrived species will be random too and chosen at random within the range $[0, 1]$. The replacement of the species x_S by a new species of a different phenotype will

affect all the species that interacted with that species. For this reason the species that interact with the new species in the environment are given new randomly chosen fitness. The process is then repeated iteratively. Once a new species is introduced, we will the re-locate the species x_S with the smallest fitness and we will update the fitness of the two neighboring species.

In order to identify extinction events in the model and try to highlight the punctuated equilibrium dynamics, we will analyse the temporal evolution of $B_S(t)$, the smallest fitness value at the time t. This function fluctuates up and down as the model progresses. An extinction is defined as "the set of consecutive updates for which $B_S(t)$ is below a chosen value B_0". So, the extinction begins at the time t_0 when $B_S(t)$ passes from above to below B_0 and it ends when it rises above B_0 again. The size s of an extinction event is determined by the number of time steps that B_S spends below B_0. The dynamics of the Bak-Sneppen model self-organises leading to a state in which nearly all species has a value B uniformly distributed between the value $B_c = 0.6670$ and 1. Another proof of its self-organisation behavior is that the the sizes s of the extinctions are power-law distributed, as $P(s) \propto s^{-\tau}$ with $\tau \approx 1.09$. The following figure shows the probability density of the fitness B after a transient period; we can easily observe how the B-values are uniformly distributed between B_c and 1.

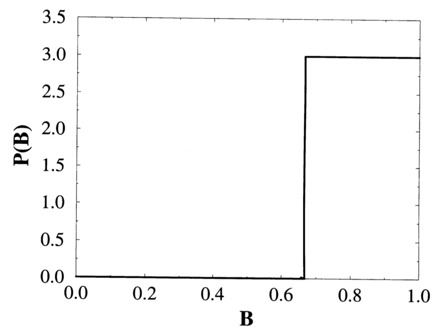

Figure 2.6: The line shows the probability density of the fitness B in the Bak-Sneppen model after a transient period. The B-values are uniformly distributed between $B_c = 0.6670$ and 1 (figure from [3]).

The similarities between this model and the Stigler one are apparent: both systems self-organises to a steady critical state in which the only surviving subjects are relegated within restricted ranges. In other words, for the Stigler model we can observe how all unexecuted orders are uniformly distributed in restricted intervals, entirely similar to those in the Bak-Snappen model. In this sense we can also identify a competitive window in the Bak-Sneppen model, within which species may be susceptible to extinction. As for the distribution analysis, we can observe that both models displays power law distribution for the phenomena of interest, as expected by a system governed by self-organise criticality.

Conclusions

In this thesis, my approach has been to start with the characteristics of the limit order book, assessing its fundamental properties. Hence, in Chapter 1 we outlined a basic description of stylised facts on limit order books. We have focused our attention on the experimental evidence and on the analysis of price change and waiting times distributions extracted from various real markets. We studied the main characteristics of limit and market orders and how they are placed in the markets. We then showed how the study of limit order book makes it possible to infer stochastic price dynamics information and to asses the performance of a market. We analysed the basic empirical findings regarding both price and waiting times distribution; in particular we have used these empirical trends to find matches in the Stigler model in order to verify its feasibility.

Then the focus shifted to the mathematical model in consideration. In Chapter 2 we presented an in-depth review of the Stigler-Luckock model, analysing its dynamics. We discussed the process of orders' placement: buy and sell orders are placed at random within a price range $[0, 1]$ and a pair of buy-sell orders is cancelled any time a sell order is placed on the left of a buy order. The assumption of this random behavior is based on the concept of zero-intelligence market, often used in simplified financial models. The analysis has emphasised that the model shows two interesting market features: one of these is that the dynamics self-organise to criticality. Indeed it has become apparent that the Stigler model is closely related to the Bak-Sneppen model that is known to show self-organised criticality. In this case self-organisation is related to the existence of a competitive window, meaning that in the long run only orders in a restricted price range are executed. The existence of this competitive window has been shown both mathematically and numerically with a computer simulation, that has confirmed the results obtained analytically. The values (x_{min}, x_{max}) of the competitive window obtained mathematically and numerically match and are $x_{min} \approx 0.218$ and $x_{max} \approx 0.782$. The results of our simulation have confirmed all mathematical predictions and empirical market trends: the existence of the competitive window is clearly visible in the histogram of the order book and as for the lifetime of orders from placement to execution we can see that for both buy and sell orders they are clearly power-law distributed in accordance with the empirical distribution $P(\Delta T) \propto (\Delta T)^{-1-\alpha}$ where $\alpha \approx 0.5$.

Then, we analysed how the Stigler model is related to the phenomenon of self-organised criticality, after a thorough general description of what it is about. Lastly we analysed another famous model that shows self-organised criticality behaviour: the Bak-Sneppen evolution model. We highlighted how even such a different model about biological evolution which self-organises to criticality shows many properties similar to the Stigler model.

Appendix A

Appendix A: Mathematical demonstrations

A.1 Mathematical proofs

A.1.1 Proof I

Suppose a new sell order enters the market during the time interval $[t, t + \Delta t]$. If the price α of this new sell order is either less or equal to the best bid price β_1, these two orders are matched and executed. Therefore, for $x \leq \beta_1$ the number $N(t, x)$ of remaining buy orders that can be executed at the price x will decrease by one. (If the order book does not contain any buy orders $\beta_1 = 0$). Conversely, if $\alpha > \beta_1$, no match can occur and the new sell order is placed in the queue, increasing by one the number $M(t, x)$ of sell orders that can be executed at a price $x \geq \alpha$.
Therefore, for a given $x > 0$, the changes of $M(t, x)$ and $N(t, x)$ due to the submission of a new sell order are:

$$\Delta M_\alpha(x) = I(x \geq \alpha)I(\alpha > \beta_1) \qquad \Delta N_\alpha(x) = -I(x \leq \beta_1)I(\alpha \leq \beta_1)$$

Similarly, the submission of a new buy order at a price β results in the following changes:

$$\Delta M_\beta(x) = -I(x \geq \alpha_1)I(\beta \geq \alpha_1) \qquad \Delta N_\beta(x) = I(x \leq \beta)I(\beta < \alpha_1)$$

where I denotes the indicator function.

During the time interval Δt, there is a probability $\lambda_A(x)\Delta t$ of receiving a new sell order at a price $\alpha \leq x$ and a probability $\lambda_B(x)\Delta t$ of receiving a new buy order at a price $\beta \geq x$. The expected increments in $M(t, x)$ and $N(t, x)$ over this time interval are:

$$\mathbb{E}[\Delta M(x)] = \{I(x > \beta_1)[\lambda_A(x) - \lambda_A(\beta_1)] - I(x \geq \alpha_1)\lambda_B(\alpha_1)\}\Delta t \qquad (A.1)$$

$$\mathbb{E}[\Delta N(x)] = \{I(x < \alpha_1)[\lambda_B(x) - \lambda_B(\alpha_1)] - I(x \leq \beta_1)\lambda_A(\beta_1)\}\Delta t \qquad (A.2)$$

For $x_1 > 0$ and $x_2 > x_1$, the rate of change of the expectation value of the number of unexecuted sell orders in the price interval $(x_1, x_2]$ at time t is:

$$\begin{aligned}\frac{\partial \mathbb{E}[M(t,x_2) - M(t,x_1)]}{\partial t} &= \lim_{\Delta t \to 0} \frac{\mathbb{E}[\Delta M(x_2)] - \mathbb{E}[\Delta M(x_1)]}{\Delta t} = \lambda_A(x_2)\mathbb{E}[I(x_2 > \beta_1)] \\ &- \lambda_A(x_1)\mathbb{E}[I(x_1 > \beta_1)] - \mathbb{E}[I(x_2 > \beta_1 \geq x_1)\lambda_A(\beta_1)] \\ &- \mathbb{E}[I(x_2 \geq \alpha_1 > x_1)\lambda_B(\alpha_1)] \\ &= \lambda_A(x_2)[1 - B(x_2)] - \lambda_A(x_1)[1 - B(x_1)] + \int_{x_1}^{x_2} \lambda_A(\beta) dB(\beta) \\ &- \int_{x_1}^{x_2} \lambda_B(\alpha) dA(\alpha) \\ &= \int_{x_1}^{x_2} [(1 - B) d\lambda_A - \lambda_B dA]\end{aligned} \tag{A.3}$$

where we integrated by parts:

$$\int_{x_1}^{x_2} \lambda_A(\beta) dB(\beta) = [\lambda_A(\beta)B(\beta)]_{x_1}^{x_2} + \int_{x_1}^{x_2} B(\beta) d\lambda_A(\beta) \tag{A.4}$$

A similar procedure for $N(t,x)$ gives

$$\frac{\partial \mathbb{E}[N(t,x_1) - N(t,x_2)]}{\partial t} = \int_{x_1}^{x_2} [\lambda_A dB - (1 - A) d\lambda_B] \tag{A.5}$$

A.1.2 Proof II

Let $x \in (x_{min}, x_{max})$. Then, equation 2.8 implies that

$$\begin{aligned}\int_{x_{min}}^{x} [\lambda_B dA - \lambda_A dB] &= \int_{x_{min}}^{x} [(1 - B) d\lambda_A + \lambda_A d(1 - B)] = [(1 - B)\lambda_A]_{x_{min}}^{x} \\ &= [1 - B(x)]\lambda_A(x)\end{aligned} \tag{A.6}$$

where we used $B(x_{min}) = 1$. A similar argument from equation 2.7 gives

$$\int_{x}^{x_{max}} [\lambda_B dA - \lambda_A dB] = [1 - A(x)]\lambda_B(x) \tag{A.7}$$

We can then sum the two equations:

$$\int_{x_{min}}^{x_{max}} [\lambda_B dA - \lambda_A dB] = [1 - B(x)]\lambda_A(x) + [1 - A(x)]\lambda_B(x) \tag{A.8}$$

On account of boundary condition, the integral in the first member of the previous equation exists and it is a constant equal to the overall frequency of the market κ. In order to find explicit expression of $A(x)$ and $B(x)$ we consider the forms of the supply and demand function of the model under consideration, i.e. $\lambda_A(x) = x$ and $\lambda_B(x) = 1 - x$. For any $x_0 \in (x_{min}, x_{max})$, we have to find the general solution to equations 2.7 and 2.8 on (x_{min}, x_{max}):

$$\begin{cases} \int_{x_0}^{x}[xdB + (1-A)dx] = 0 \\ \int_{x}^{x_0}[(1-B)dx - (1-x)dA] = 0 \end{cases} \tag{A.9}$$

We integrate by parts:

$$\begin{cases} [xB(x)]_{x_0}^{x} - \int_{x_0}^{x} Bdx + x - x_0 - \int_{x_0}^{x} Adx = 0 \\ x_0 - x - \int_{x}^{x_0} Bdx - A(x_0) + A(x) + \int_{x}^{x_0} xdA = 0 \end{cases} \tag{A.10}$$

Now using the substition method:

$$\begin{cases} -\int_{x}^{x_0} Bdx = xB(x) - x_0B(x_0) + x - x_0 - \int_{x_0}^{x} Adx \\ xB(x) - x_0B(x_0) - A(x_0) + A(x) - xA(x) + x_0A(x_0) = 0 \end{cases} \tag{A.11}$$

Considering the second equation, by factoring out the expression we obtain:

$$A(x)(1-x) + xB(x) + A(x_0)(x_0 - 1) - x_0B(x_0) = 0 \tag{A.12}$$

Now, considering the expression of κ in the model under consideration (i.e. $\kappa = A(x)(x - 1) + 1 - xB(x)$,

$$\kappa - 1 + A(x)(1-x) + xB(x) = 0$$

We thus find an expression of $A(x)$:

$$A(x) = \frac{1 - \kappa - xB(x)}{1 - x} \tag{A.13}$$

Substituting the previous expression of $A(x)$ in equation 2.8 we obtain:

$$xdB + \left(\frac{\kappa + xB(x) - x}{1 - x}\right) dx = 0 \tag{A.14}$$

We can thus highlight and resolve the first-order linear ordinary differential equation:

$$\frac{dB(x)}{dx} + \frac{B(x)}{1-x} = \frac{1}{1-x} - \frac{\kappa}{x(1-x)} \tag{A.15}$$

The solution is:

$$B(x) = 1 - \frac{(1-x)(1-B(x_0))}{1-x_0} - \kappa \cdot (1-x) \int_{x_0}^{x} \frac{1}{x(1-x)^2} dx \tag{A.16}$$

A similar argument for $A(x)$ gives the equivalent differential equation:

$$\frac{dA}{dx} - \frac{A(x)}{x} = \frac{\kappa}{x(1-x)} - \frac{1}{x} \tag{A.17}$$

whose solution is

$$A(x) = 1 + \frac{x(A(x_0) - 1)}{x_0} + \kappa \cdot x \int_{x_0}^{x} \frac{1}{x^2(1-x)} dx \tag{A.18}$$

A.1.3 Proof III

In terms of the supply and demand functions $\lambda_A(x)$ and $\lambda_B(x)$, given any $x_0 \in (x_{min}, x_{max})$ the general solutions to equations 2.7 and 2.8 is:

$$A(x) = 1 - \frac{[1 - A(x_0))]\lambda_A(x)}{\lambda_A(x_0)} - \kappa\lambda_A(x) \int_{x_0}^{x} \left(\frac{1}{\lambda_B}\right) d\left(\frac{1}{\lambda_A}\right) \tag{A.19}$$

$$B(x) = 1 - \frac{[1 - B(x_0))]\lambda_B(x)}{\lambda_B(x_0)} - \kappa\lambda_B(x) \int_{x_0}^{x} \left(\frac{1}{\lambda_A}\right) d\left(\frac{1}{\lambda_B}\right) \tag{A.20}$$

We take the limit $x_0 \to x_{min}$ for the equation of $A(x)$ and the limit $x_0 \to x_{max}$ for the equation of $B(x)$. Remembering the boundary conditions $A(x_{min}) = 0$ and $B(x_{max}) = 0$ we obtain (we do not immediately specify the expression of λ_A and λ_B for the sake of generality):

$$A(x) = 1 - \frac{\lambda_A(x)}{\lambda_A(x_{min})} - \kappa\lambda_A(x) \int_{x_{min}}^{x} \left(\frac{1}{\lambda_B}\right) d\left(\frac{1}{\lambda_A}\right) \tag{A.21}$$

$$B(x) = 1 - \frac{\lambda_B(x)}{\lambda_B(x_{max})} + \kappa\lambda_B(x) \int_{x}^{x_{max}} \left(\frac{1}{\lambda_A}\right) d\left(\frac{1}{\lambda_B}\right) \tag{A.22}$$

We then use the two previous equations and the equation 2.7. We write the latter as $\lambda_A dB = (1 - A)d\lambda_B$. It is readily shown by direct substitution that the two previous equations represents a solution if and only if:

$$\frac{1}{\kappa}\left[\frac{1}{\lambda_A(x_{min})} + \frac{1}{\lambda_B(x_{max})}\right] - \frac{1}{\lambda_A(x_{min})\lambda_B(x_{min})} - \int_{x_{min}}^{x_{max}} \left(\frac{1}{\lambda_A}\right) d\left(\frac{1}{\lambda_B}\right) = 0 \tag{A.23}$$

or equivalently after integrating by parts:

$$\frac{1}{\kappa}\left[\frac{1}{\lambda_A(x_{min})} + \frac{1}{\lambda_B(x_{max})}\right] - \frac{1}{\lambda_A(x_{max})\lambda_B(x_{max})} + \int_{x_{min}}^{x_{max}} \left(\frac{1}{\lambda_B}\right) d\left(\frac{1}{\lambda_A}\right) = 0 \tag{A.24}$$

We now consider the explicit expression of the supply and demand function of the model under consideration, i.e. $\lambda_A(x) = x$ and $\lambda_B(x) = 1 - x$. By substituting in the latter equation:

$$\frac{1}{\kappa}\left[\frac{1}{x_{min}} + \frac{1}{1 - x_{max}}\right] - \frac{1}{x_{max}(1 - x_{max})} + \int_{x_{min}}^{x_{max}} \frac{1}{1 - x} d\left(\frac{1}{x}\right) = 0 \tag{A.25}$$

We integrate:

$$\int_{x_{min}}^{x_{max}} \frac{1}{1 - x} d\left(\frac{1}{x}\right) = \left[\frac{1}{x} + \ln\left(\frac{1 - x}{x}\right)\right]_{x_{min}}^{x_{max}} \tag{A.26}$$

Substituting this result in the previous equation we obtain

$$\frac{1}{\kappa}\left[\frac{1}{x_{min}} + \frac{1}{1 - x_{max}}\right] - \frac{1}{x_{max}(1 - x_{max})} + \frac{1}{x_{max}} - \frac{1}{x_{min}} + \ln\left(\frac{1 - x_{max}}{1 - x_{min}} \cdot \frac{x_{min}}{x_{max}}\right) = 0$$

Remembering that for the symmetry of the system $x_{min} + x_{max} = 1$, we find:

$$\frac{1}{\kappa} \cdot \frac{1}{x_{min}} + \frac{1}{x_{min}} + \ln\left(\frac{x_{min}}{1 - x_{min}}\right) = 0 \tag{A.27}$$

We can therefore find an explicit expression for κ:

$$\kappa = \frac{1}{1 - x_{min} \cdot \ln\left(\frac{x_{min}}{1-x_{min}}\right)} \tag{A.28}$$

A.1.4 Proof IV

Considering the following equation for $A(x)$, previoulsy derived:

$$A(x) = 1 + \frac{x(A(x_0) - 1)}{x_0} + \kappa \cdot x \int_{x_0}^{x} \frac{1}{x^2(1-x)} dx \tag{A.29}$$

We want to evaluate $A(x_{max})$ with $x_0 = x_{min}$. Integrating we obtain (for the boundary conditions $A(x_{min}) = 0$ and $A(x_{max}) = 1$):

$$A(x_{max}) = 1 \stackrel{!}{=} 1 - \frac{x_{max}}{x_{min}} - \kappa \cdot x_{max} \left[\frac{1}{x_{max}} - \frac{1}{x_{min}} + \ln\left(\frac{1 - x_{max}}{1 - x_{min}} \cdot \frac{x_{min}}{x_{max}}\right)\right] \tag{A.30}$$

Remembering that for the symmetry of the system $x_{min} + x_{max} = 1$, we find:

$$0 = -\frac{1 - x_{min}}{x_{min}} - \frac{1 - x_{min}}{1 - x_{min} \cdot \ln\left(\frac{x_{min}}{1-x_{min}}\right)} \left[\frac{1}{1 - x_{min}} - \frac{1}{x_{min}} + 2\ln\left(\frac{x_{min}}{1 - x_{min}}\right)\right] \tag{A.31}$$

By simplifying both members, we obtain:

$$\frac{1}{1 - x_{min}} = \ln\left(\frac{1 - x_{min}}{x_{min}}\right) \tag{A.32}$$

Remembering that for the symmetry of the system $x_{min} + x_{max} = 1$, we find:

$$\frac{1}{x_{max}} = \ln\left(\frac{x_{max}}{1 - x_{max}}\right) \tag{A.33}$$

Appendix B

Appendix B: Simulation Mathematica code

B.1 Mathematica Code

```
Lambda1=1;
Lambda2=1;
Niter=10000000;

dT=0.001;

SellOrders=Table[3,{i,1,1}];
BuyOrders=Table[0,{i,1,1}];
BuyTime=Table[0,{i,1,1}];
SellTime=Table[0,{i,1,1}];
SellWaiting=Table[0,{i,1,1}];
BuyWaiting=Table[0,{i,1,1}];

BestAsks=Table[0,{i,1,1}];
BestBids=Table[0,{i,1,1}];

Lambda=Lambda1+Lambda2;

Nrtraders=0;

For[j=1,j<=Niter,j++,
  Z=RandomReal[];
    If[Z<=Lambda*dT,
      W =RandomReal[];
      If[W<=1/2,
          NewSell=RandomReal[];
          AppendTo[SellOrders,NewSell];
          AppendTo[SellTime,j*dT];
          Nrtraders+=1,
          NewBuy=RandomReal[];
          AppendTo[BuyOrders,NewBuy];
          AppendTo[BuyTime,j*dT];
```

```
            Nrtraders+=1];];
     BestBidPosition=Ordering[BuyOrders,-1];
     Bestbid=BuyOrders[[BestBidPosition]].{1};
     BestAskPosition=Ordering[SellOrders,1];
     Bestask=SellOrders[[BestAskPosition]].{1};
     If[Nrtraders>=5000&&Nrtraders<=5500,AppendTo[BestAsks, {Nrtraders,
    Bestask}];
     AppendTo[BestBids,{Nrtraders,Bestbid}];];
   If[Bestbid>=Bestask,
       AppendTo[SellWaiting,j*dT-SellTime[[BestAskPosition]]];
       AppendTo[BuyWaiting,j*dT-BuyTime[[BestBidPosition]]];
       SellOrders=Delete[SellOrders,BestAskPosition];
       BuyOrders=Delete[BuyOrders,BestBidPosition];
       BuyTime=Delete[BuyTime,BestBidPosition];
       SellTime=Delete[SellTime,BestAskPosition];
         ];

     ];
```

Listing B.1: Mathematica code used for the simulation

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