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An age-structured model for a distributive channel

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CHAPTER 1

Introduction

Mathematical models may treat populations structured in many ways. A structure by age is one of the most simple, as age evolves linearly with time, which allows one to rewrite PDEs containing age and time in a simpler form.

Age-structures may be discrete or continuous: see [1], pp. 267-280, for a general presentation of the discrete ones, while the continuous ones will be the main interest of this thesis.

According to Keyfitz [27], there are four ways of modelling the evolution in time of an age-structured problem: the Lotka integral equation, the Leslie matrix, the renewal difference equation approach and the McKendrick PDE. In the recent years, mathematician have re-discovered this last one, which was used by McKendrick in epidemiology to take into account births and deaths of members of a population: denoting by $x(t,a)$ the density of a population of age a at time t, the McKendrick PDE is

$$\partial_t x(t,a) + \partial_a x(t,a) = -\mu(t,a)x(t,a)$$

where $\mu(t,a)$ is the force of mortality or instantaneous death rate of an individual of age a at time t. Such an equation may be solved through the method of characteristics, being $t-a \equiv \text{const}$ a characteristic line.

Age-structured problems are studied in a variety of situations, including harvesting

([28], [29], [30], [31]), birth control ([32], [33], [34]), epidemic disease control and optimal vaccination ([35], [36], [37]), investment economic models ([39], [40], [41], [42], [43], [44]) and a variety of models in the social area ([45], [46]).

Here, the main topic is marketing, specifically a distributive channel. There are the manufacturer and the retailer of a certain product, who are planning to introduce it into the market. They both want to maximize their profit from the sales of that product; in order to do this, the manufacturer invests his money on advertising, while the retailer on promotion. If "big" enough, the manufacturer may also decide to pay for part of the promotion.

The natural way to treat this problem is to define an appropriate differential game (see [10], page 110). That's due to the main features of marketing channels: the set of players is easy to identify, and each's payoff depends on the actions taken by the other players. So, the advertising A and the promotion P will be the manufacturer's and the retailer's controls, respectively. In chapter 3, a brief resume about differential games is given. In particular, linear state games and Open-Loop Nash Equilibria (OLNE) are recalled.

This specific model was introduced in [13]. The authors, Buratto and Grosset, simplified the situation proposed by Jørgensen, Taboubi and Zaccour in [47], by showing that the same results about coordination between manufacturer and retailer could be obtained by considering a linear-state game, instead of a linear-quadratic one.

This thesis starts from that work; it introduces an age-structure in the population to whom the product will be proposed. People will have age between 0 and a fixed ω, while time will go from 0 to $+\infty$. The choice of an infinite time horizon is because of the calculations, which are by far simpler in this context: indeed, the state equation is of McKendrick's type

$$\begin{cases} \partial_t G(t,a) + \partial_a G(t,a) = A(t,a) - \mu(a) G(t,a), \quad a \in [0,\omega] \\ \partial_t \xi(t,a) + \partial_a \xi_a(t,a) = (\mu(a)+\rho)\xi(t,a) - \pi_M(a)\gamma(a) \end{cases},$$

where $\mu(a)$ is the decay rate of the function $G(t,a)$ (which is called *Goodwill*), ρ is a positive constant called *discount rate*, and $\xi(t,a)$ is the adjoint function of $G(t,a)$. Hence, these equations may be solved via the method of characteristics, and in the infinite time-horizon case one has to take into account just the intersection of the characteristic line with $[0,+\infty) \times \{\omega\}$.

The aim is to determine whether an optimal strategy $(A^\star, P^\star)$ exists. Necessary conditions, as well as sufficient ones, for such a couple to be an OLNE are provided. Explicitly, the necessary conditions are a version of the Pontryagin's maximum principle, while the sufficient ones are of Arrow's type. That's why in Chapter 2 a brief resume about these general theorems and other concepts in Optimal Control theory is provided.

Fundamental works about the aforementioned necessary and sufficient conditions are the one by Feichtinger, Tragler and Veliov ([11]), and by Krastev ([19]), respectively. Krastev's sufficient conditions are here re-framed in terms of the current-value Hamiltonian, which is a commonly used function in the infinite-time horizon setting. The first part of Chapter 4, then, is dedicated to exposing these conditions and their adaptation; the second part, instead, explicitly presents and treats the model.
Several situations will be considered:

- calculations will be started when the functions appearing in the equations (such as $\mu(a)$) have a generic form; in particular, this will be the case for $\mu(a)$ and for the marginal profit $\pi_M(a)$ of the manufacturer on people aged a;
- secondly, computations will be deepened for two specific forms of $\pi_M(a)$ ("rectangular" and "triangular", i.e. the characteristic function of a certain interval and a modulus, respectively), when all the other parameter functions (such as $\mu(a)$) are constant;
- the results of the second part will be re-discussed when a further effect is considered. Indeed, Krastev's results may be applied also when the Hamiltonian function takes into account the following "interaction term" in the population: for every age a, people who are older than a talk about the product with people younger than a, so that they have an impact on its sales.

The results are summarized in Chapter 5.

CHAPTER 2

Age-structured Optimal Control Theory

Optimal control

In this section, a quick review of the basic notions and the useful results in the Calculus of Variation theory is given. Specifically, it's important to recall the Maximum Principle, the Mangasarian and Arrow theorems, in the general context of an infinite horizon problem. This will be important in the following chapters, where these results will be re-framed and proved for age-structured problems. See [4], [17] and [12] as references to this part.

Standard terminology

First of all, it's necessary to introduce the basic concepts in Calculus of Variations. These will be first written for the finite horizon case, and then for the infinite one. Be $[a,b]$ an interval, $n, m \in \mathbb{N}$, $f : [a,b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ the *dynamics function*, $U \subseteq \mathbb{R}^m$ a set called the *control set*. A *control* is a measurable function $u : [a,b] \to U$. The *state* corresponding to the control u is the solution x of the following initial-value problem, called the *state equation*:

$$\begin{cases} \dot{x} = f(t, x(t), u(t)), \;\; t \in [a,b] \\ x(a) = x_0 \end{cases} \tag{1}$$

where x_0 is the prescribed initial condition. One hopes that there exists - and, if so, to find - a couple (x,u), called *process*, minimizing the following *objective functional*:

$$J(x,u) = \int_a^b \Lambda(t, x(t), u(t))\,\mathrm{d}t + \lambda(x(b)) \tag{2}$$

where Λ and λ are two given functions, called the *running* and *endpoint cost*, respectively. The endpoint $x(b)$ is asked to be in a prescribed set $E \subseteq \mathbb{R}^n$, called the *target set*.

What has just been described is the *Optimal control problem* (**OC**) and, for the expression of $J(x,u)$, it is said to be in the *Bolza form*; obviously, such form won't be adapt for the infinite horizon case. An *admissible process* for OC is a couple (x,u) satisfying the constraints of the problem and for which the objective functional $J(x,u)$ is well defined.

The usual regularity conditions are the following: λ is asked to be continuously differentiable, f, Λ continuous and admit continuous derivatives with respect to x, $\partial_x f(t,x,u)$, $\partial_x \Lambda(t,x,u)$.

The *Hamiltonian function* associated to the OC problem is

$$H^\eta : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R},\ H^\eta(t,x,p,u) = \langle p, f(t,x,u)\rangle + \eta\Lambda(t,x,u) \tag{3}$$

where p is called *co-state variable*, $\eta = 1$ (*normal case*) or $\eta = 0$ (*abnormal case*). The *maximized Hamiltonian* is $M^\eta(t,x,p) = \sup_{u\in U} H^\eta(t,x,p,u)$.

Necessary conditions: the Maximum Principle

The following result is known as *Pontryagin maximum principle*. It gives a necessary condition for a process (x,u) to be a local minimizer of the objective functional. The principle will be first stated for the finite horizon case, where the following notion is needed, and then it will be generalized to the infinite horizon context.

For every $x \in E$ the closed set $N_E^L(x)$, called the *limiting normal cone* to E at the point x, is defined as such:

$$N_E^L(x) = \left\{ \xi = \lim_{i\to+\infty} \xi_i : \xi_i \in N_E^P(x_i), x_i \stackrel{i\to+\infty}{\longrightarrow} x, x_i \in E \right\}$$

and $N_E^P(x_i)$ is the *proximal cone* to E at the point x_i:

$$N_E^P(x_i) = \{\xi \in \mathbb{R}^n : \exists \sigma = \sigma(\xi, x_i) \geq 0 \text{ such that } \langle \xi, y - x_i\rangle \leq \sigma |y - x_i|^2, \forall y \in E\}$$

An example of proximal cone to a set S at a point x is given in figure (1), see [4] for an extensive presentation of their properties.

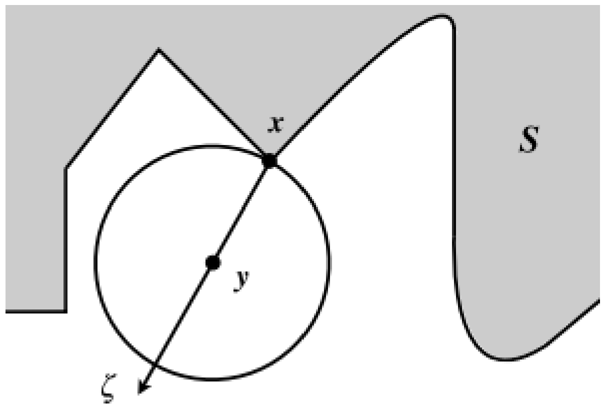

Figure 1: Proximal cone to a set S (in grey) at a point x.

Theorem 2.2.1: Pontryagin Maximum Principle

Let $(x_\star, u_\star)$ be a local minimizer for the OC problem, under the classical regularity hypotheses, where the control set U is bounded. Then there exists a co-state variable $p : [a,b] \to \mathbb{R}^n$ and $\eta \in \{0,1\}$ satisfying the nontriviality condition

$$(\eta, p(t)) \neq 0_{n+1}, \ \forall t \in [a,b], \tag{4}$$

the transversality condition

$$-p(b) \in -\eta \nabla \lambda(x_\star(b)) + N_E^L(x_\star(b)),$$

the adjoint equation

$$-\dot{p}(t) = \partial_x H^\eta(t, x_\star(t), p(t), u_\star(t)) \tag{5}$$

and the maximum condition

$$H^\eta(t, x_\star(t), p(t), u_\star(t)) = M(t, x_\star(t), p(t)) \tag{6}$$

Morever, if the problem is autonomous (i.e., f and λ do not depend on t), the Hamiltonian in equation (6) is constant in time.

The co-state variable satisfying the adjoint equation will also be called *adjoint function*. Observe that, in the normal case, the nontriviality condition is automatically satisfied.
Equivalently (see [4], Prop. 22.5), this theorem also holds if one change the transversality condition with the following: there exist $\eta \geq 0$ and a continuous piecewise $\mathscr{C}^1$ function p satisfying $\eta + \|p\| = 1$ and the other conclusions of the theorem.
More straightforwardly, as [17] introduces the problem at the beginning, the transversality condition may be re-stated as follows. Let $x_1^1, \ldots, x_1^r \in \mathbb{R}$ be $r \leq n$ fixed parame-

ters. If the components of the state x satisfy the following conditions, together with the ones in system (1):

$$\begin{cases} x^i(b) = x_1^i, & i = 1, \dots, l, \\ x^j(b) \geq x_1^j, & j = l+1, \dots, r, \\ x^k(b) \in \mathbb{R}, & k = r+1, \dots, n. \end{cases} \tag{7}$$

for some $l \leq r$, then the components of the vector p must satisfy:

$$\begin{cases} p^i(b) \in \mathbb{R}, & i = 1, \dots, l, \\ p^j(b) \geq 0 \text{ and } p^j(b)\left(x_\star^j(b) - x_1^j\right) = 0, & j = l+1, \dots, r, \\ p^k(b) = 0, & k = r+1, \dots, n. \end{cases} \tag{8}$$

Sufficient conditions: the Mangasarian and Arrow theorems

Mangasarian and Arrow's theorems give sufficient conditions for a process (x, u) to be a local minimizer of the objective functional.
Consider the OC problem in the *Lagrange form*, that is equation (2) with $\lambda = 0$.

Theorem 2.3.1: Mangasarian

Let $(x_\star(t), u_\star(t))$ be an admissible process. Suppose that the control set $U \subseteq \mathbb{R}^m$ is convex and that the dynamics function f admits derivatives with respect to the control u, and such derivatives are continuous. Let $\eta = 1$ in (3) and suppose that there exists a continuous and piecewise $\mathscr{C}^1$ $p : [a, b] \to \mathbb{R}^n$ such that it holds the following:

$$\begin{cases} \dot{p}^i(t) = -\frac{\partial H^1(t, x_\star(t), p(t), u_\star(t))}{\partial x^i}, & \text{for a.e. } t \in [a, b],\ i = 1, \dots, n, \\ \sum_{j=1}^m \frac{\partial H^1(t, x_\star(t), p(t), u_\star(t))}{\partial u^j}\left(u_\star^j(t) - u^j\right) \geq 0, & \forall u \in U,\ t \in [a, b], \\ p^i(b) \geq 0 \ \text{ and } \ p^i(b)\left(x_\star^i(b) - x_1^i\right) = 0, & i = l+1, \dots, r, \\ p^j(b) = 0, & j = r+1, \dots, n; \\ H^1(t, x, p(t), u) \text{ is convex in } (x, u), & \forall t \in [a, b]. \end{cases}$$

Notice that the first, third and fourth equations are respectively (5) and (8) and equation (4) is automatically satisfied.
Then, $(x_\star(t), u_\star(t))$ is a local minimizer of the objective functional $J(x, u)$ in (2) and both equations (1) and (7) hold.

See [17], pp. 102-103 for the proof.

Theorem 2.3.2: Arrow

Be $(x_\star(t), u_\star(t))$ an admissible process. Let $\lambda = 0$ in (2) and $\eta = 1$ in (3) and suppose that there exists a continuous and piecewise $\mathscr{C}^1$ $p : [a,b] \to \mathbb{R}^n$ such that:

$$\begin{cases} \dot{p}^i(t) = -\frac{\partial H^1(t, x_\star(t), p(t), u_\star(t))}{\partial x^i}, & \text{a.e. } t \in [a,b],\ i = 1, \dots, n; \\ H^1(t, x_\star(t), p(t), u_\star(t)) = M^1(t, x_\star(t), p(t)), & \forall\, t \in [a,b] \\ p^i(b) \geq 0 \text{ and } p^i(b)\left(x_\star^i(b) - x_1^i\right) = 0, & i = l+1, \dots, r, \\ p^j(b) = 0, & j = r+1, \dots, n, \\ M^1(t, x(t), p(t)) \text{ is well defined and concave w.r.t. } x, & \forall\, t \in [a,b]. \end{cases}$$

Then, $(u_\star(t), x_\star(t))$ is a local minimizer of the objective functional and satisfies both (1) and (7). If $M^1(t, x(t), p(t))$ is strictly concave in x, $\forall\, t \in [a,b]$, then $(x_\star(t), u_\star(t))$ is the only process for which these conclusions hold.

See again [17], pp. 106-107, for the proof.

Notice that Mangasarian and Arrow theorems differ one from the other only for the maximization assumption, and they are almost the same if U is open, by Fermat theorem.

Infinite horizon

Infinite-horizon optimal control problems are still challenging, even for systems of ordinary differential equations. This is one reason for which often optimal control problems are considered on a truncated time-horizon, although the natural formulation is in the infinite horizon. The key issue is to define appropriate transversality conditions, which allow one to select the right solution of the adjoint system for which the Pontryagin maximum principle holds. The usual notion of optimality, in which the optimal solution maximizes the objective functional, is not always appropriate, when considering infinite-horizon problems, especially for economic problems with endogenous growth. That's because the objective value can be infinite for many (even for all) admissible controls, while they may differ in their intertemporal performance. For this reason, Skritek and Veliov adapted the notions of weakly overtaking, catching up, and sporadically catching up optimality [18]. Of course, in the case of a finite objective functional, this notion coincides with the usual one. See [23] as a reference for this section.

Consider the state equation (1), where now $f : [0,+\infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. As before, suppose f continuous and that, for every couple (t,x), there exists a compact subset $U(t,x) \subset \mathbb{R}^m$ s.t. the map $(t,x) \mapsto U(t,x)$ is upper semicontinuous. Assume that there exists a finite number $M > 0$ s.t. $\|f(t,x,u)\| \le M(1+\|x\|)$, for all $(t,x,u) \in [0,+\infty) \times \mathbb{R}^n \times U(t,x)$.

A process $[0,+\infty) \to \mathbb{R}^n \times \mathbb{R}^m$, $t \mapsto (x(t),u(t))$ is called *admissible* if $t \mapsto x(t)$ is absolutely continuous and satisfies the state equation (1) a.e. on $[0,+\infty)$, $t \mapsto u(t)$ is measurable and $u(t) \in U(t,x(t))$ for a.e. $t \in [0,+\infty)$. Denote by $\mathscr{A}_\infty$ the set of all admissible process.

Now, consider the objective functional (2) with $\lambda = 0$, $a = 0$ and write it $J_b(x,u)$ to underline the dependence on b. Suppose $\Lambda : [0,+\infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ continuous. The following criteria of *optimality* can be given for a state $x_\star(\cdot)$ satisfying (1):

- *overtaking optimality* . $x_\star(\cdot)$ is overtaking optimal at x_0 if it is generated by a control $u_\star(\cdot)$ such that

$$J_\infty(x_\star,u_\star) := \lim_{b\to+\infty} J_b(x_\star,u_\star) < +\infty \tag{9}$$

  and, for any other state $x(\cdot)$ satisfying (1) generated by $u(\cdot)$, it holds

$$J_\infty(x_\star,u_\star) \ge \limsup_{b\to+\infty} J_b(x,u).$$

- *catching up optimality* . $x_\star$ is catching up optimal at x_0 if

$$\liminf_{b\to+\infty}[J_b(x_\star,u_\star) - J_b(x,u)] \ge 0 \tag{10}$$

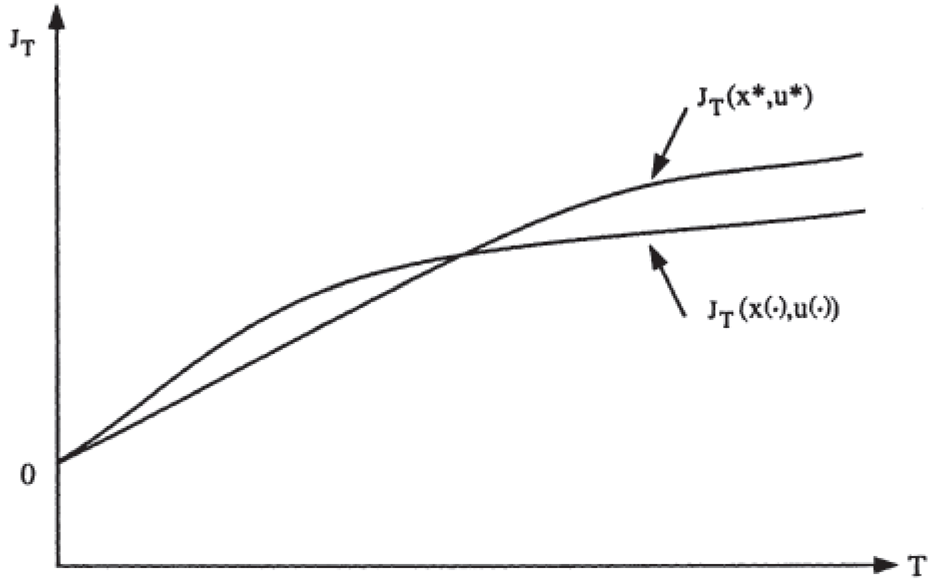

Figure 2: Catching up optimality.

for any other state $x(\cdot)$ satisfying (1) generated by $u(\cdot)$. Equivalently, $\forall \varepsilon > 0, \exists b(\varepsilon, u(\cdot))$ such that $b > b(\varepsilon, u(\cdot)) \implies J_b(x_\star, u_\star) > J_b(x, u) - \varepsilon$. See also figure (2).

- *sporadically catching up optimality* . $x_\star$ is sporadically catching up optimal at x_0 if equation (10) holds with $\limsup$ instead of $\liminf$. See also figure (3).
- *finite optimality* . $x_\star$ is finitely optimal at x_0 if, $\forall b > 0$, for every state $x(\cdot)$ solving (1) and generated by the control $u(\cdot)$ such that $x(b) = x_\star(b)$, one has $J_b(x_\star, u_\star) \geq J_b(x, u)$.

One can show that these definitions are ordered in a chain of implications, that is:

$$\text{overtaking optimality} \implies \ldots \implies \text{finite optimality}$$

Now, set $W := \sup\{J_b(x, u) : (x, u) \text{ is an admissible process}\}$. Let $x(\cdot)$ be a state solving (1) and be $\mathscr{A}(x(\cdot), \vartheta)$ the set of the processes $(y(\cdot), v(\cdot))$ such that $y(\cdot)$ satisfies equation (1) and $x(\cdot) = y(\cdot)$ on $[0, \vartheta)$. Also, set $W(b, x(\cdot), \vartheta) := \sup\{J(y(\cdot), v(\cdot)) : (y(\cdot), v(\cdot)) \in \mathscr{A}(x(\cdot), \vartheta)\}$. Then, one can give the following definitions: a state $x_\star(\cdot)$ satisfying (1) is said to be

- *decision horizon optimal* if, $\forall \vartheta > 0$, there exists $\bar{b} = \bar{b}(\vartheta) \geq 0$ such that, $\forall b \geq \bar{b}$, one has $W(b, x_\star(\cdot), \vartheta) = W$;
- *agreeable* if, $\forall \vartheta > 0$, one has $\lim_{b \to +\infty}(W - W(b, x_\star(\cdot), \vartheta)) = 0$;
- *weakly agreeable* if, $\forall \vartheta > 0$, one has $\liminf_{b \to +\infty}(W - W(b, x_\star(\cdot), \vartheta)) = 0$.

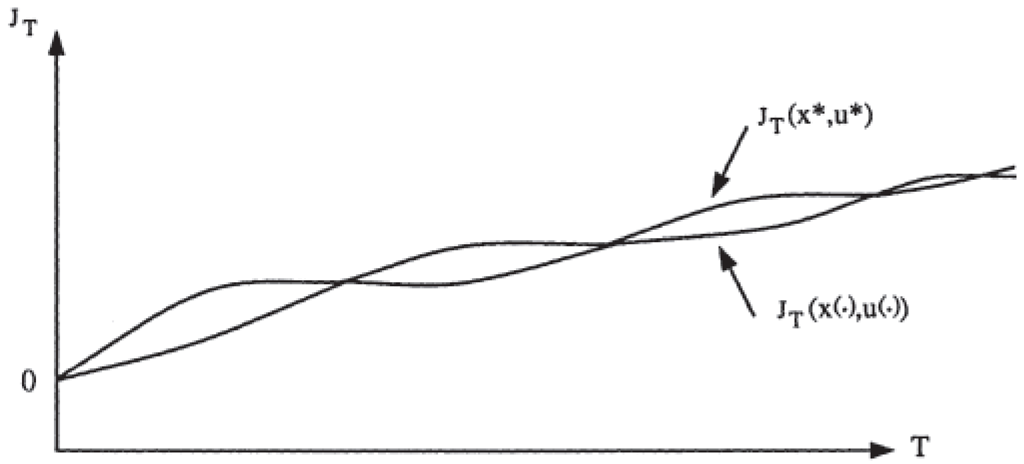

Figure 3: Sporadically catching up optimality.

As before, one has

$$\text{decision horizon optimal} \implies \dots \implies \text{weakly agreeable.}$$

It holds the following necessary optimality condition:

Theorem 2.3.3: Optimality principle

If the pair $(x_\star(\cdot), u_\star(\cdot)) \in \mathscr{A}_\infty$ is optimal, according to any one of the definitions previously given, then, for any $b \geq 0$, the restriction $x_\star^b(\cdot)$ of $x_\star(\cdot)$ (associated with the restriction $u_\star^b(\cdot)$ of $u_\star(\cdot)$) maximizes the objective functional J on the set $A_b^\star := \{(x(\cdot), u(\cdot)) : x(0) = x_0, x(b) = x_\star(b)\}$, and thus $(x_\star(\cdot), u_\star(\cdot))$ is finitely optimal.

Proof. If the result is not true for some $\hat{T} > 0$, then for some $(\hat{x}(\cdot), \hat{u}(\cdot)) \in A_{\hat{T}}^\star$ one has

$$\int_0^{\hat{T}} \Lambda(t, \hat{x}(t), \hat{u}(t))\,\mathrm{d}t > \int_0^{\hat{T}} \Lambda(t, x_\star(t), u_\star(t))\,\mathrm{d}t, \quad \hat{x}(\hat{T}) = x_\star(\hat{T}),$$

thus $\exists \varepsilon > 0$ such that

$$\int_0^{\hat{T}} \Lambda(t, \hat{x}(t), \hat{u}(t))\,\mathrm{d}t > \int_0^{\hat{T}} \Lambda(t, x_\star(t), u_\star(t))\,\mathrm{d}t + \varepsilon.$$

If one defines the process

$$(\tilde{x}(t), \tilde{u}(t)) = \begin{cases} (\hat{x}(t), \hat{u}(t)), & \text{if } t \in [0, \hat{T}) \\ (\tilde{x}(t), \tilde{u}(t)), & \text{if } t \geq \hat{T} \end{cases},$$

then it holds

$$\int_0^{T} \Lambda(t, \tilde{x}(t), \tilde{u}(t))\,\mathrm{d}t > \int_0^{T} \Lambda(t, x_\star(t), u_\star(t))\,\mathrm{d}t + \varepsilon,$$

for any $T > 0$, contradicting all the definitions of optimality given before. $\square$

From this, one can get the following:

Theorem 2.3.4: Infinite Horizon Maximum Principle

If $(x_\star, u_\star) \in \mathscr{A}_\infty$ is optimal, according to any one of the definitions previously given, then there exists a non-negative number η and a continuous piecewise $\mathscr{C}^1$ function $p : [0, +\infty) \to \mathbb{R}^n$, called adjoint function*, satisfying the* nontriviality condition

$$\|(\eta, p)\|_{\mathbb{R}^{n+1}} = 1, \tag{11}$$

the adjoint equation

$$\dot{p}(t) = -\frac{\partial}{\partial x} H^\eta(t, x_\star(t), p(t), u_\star(t)) \text{ a.e. } t \in [0, \infty), \tag{12}$$

and the maximum condition

$$H^\eta(t, x_\star(t), p(t), u_\star(t)) = M^\eta(t, x_\star(t), p(t), u(t)) \;\; \forall t \in [0, +\infty), \forall u \in U$$

Proof. Consider a strictly increasing and upperly-unbounded sequence $\{\tau_j\}_{j \in \mathbb{N}} \subset [0, +\infty)$. By the previous theorem, the restriction $(x_\star^{\tau_j}, u_\star^{\tau_j})$ maximizes the objective functional J on the set $A_\star^{\tau_j} = \{(x(\cdot), u(\cdot)) : x(0) = x_\star(0), x(\tau_j) = x_\star(\tau_j)\}$. From the Pontryagin's maximum principle for the finite case, one gets that, $\forall j \in \mathbb{N}$, there exist a scalar $\eta_j \geq 0$ and a continuous piecewise $\mathscr{C}^1$ function $p_j : [0, \tau_j]$ such that:

- $$\eta_j + \|p_j\| = 1$$

- $$\dot{p}_j(t) = -\partial_x H^\eta(t, x_\star^{\tau_j}, p_j, u_\star^{\tau_j})$$

- $$H^{\eta_j}(t, x_\star^{\tau_j}, p_j, u_\star^{\tau_j}) = M^\eta(t, x_\star^{\tau_j}, p_j)$$

Up to an appropriate subsequence, one may suppose that there exist $\eta := \lim_{j \to +\infty} \eta_j$ and $p(t) := \lim_{j \to +\infty} p_j(t)$, which in particular imply $\eta + \|p\| = 1$. By regularity, one concludes that η and p satisfy the properties of the theorem. □

Arrow Theorem can be easily generalized to the infinite horizon context:

Theorem 2.3.5: Arrow for the infinite horizon case

Suppose that:

- *the control set U is compact and that there exists a compact set X, such that the interior $\overset{\circ}{X}$ of X contains every state $x(\cdot)$ which solves (1) and is generated by an admissible control;*
- *the function $M(t,x,p,\eta) := \sup_{u\in U} H(t,x,p,u,\eta)$ is well-defined for every $x \in \overset{\circ}{X}$ and every t,p,η, and it is a concave function of x, for every t,p,η;*
- *there exists a state $x_\star(\cdot)$ generated by an admissible control $u_\star(\cdot)$ satisfying the necessary conditions of the previous theorem for an $\eta > 0$;*
- *the adjoint function $p(\cdot)$ satisfies the asymptotic transversality condition $\lim_{t\to+\infty} \|p(\cdot)\| = 0$.*

Then, the state $x_\star(\cdot)$ is catching up optimal at x_0.

Proof. Since M^η is a concave function of x,

$$M^\eta(t,x,p) \geq M^\eta(t,x_\star,p) + (x - x_\star)\partial_x M^\eta(t,x_\star,p).$$

Using equations (11) and (12), one may show that this implies

$$\eta[\Lambda(t,x_\star(t),u_\star(t)) - \Lambda(t,x(t),u(t))] \geq \frac{\mathrm{d}}{\mathrm{d}t}\left\{[x_\star(t) - x(t)]p(t)\right\}$$

for any $x(\cdot)$ emanating from x_0 and generated by a control $u(\cdot)$. Integrating the previous inequality from 0 to $T > 0$, one gets

$$\eta[J_T(x_\star,u_\star) - J(x,u)] \geq p(T)[x_\star(T) - x(T)]$$

because $x_\star(0) = x(0)$. Then, being X compact, $\eta > 0$ and using the asymptotic transversality condition, one concludes by taking the $\liminf_{T\to+\infty}$ of both the sides of the last inequality:

$$\liminf_{T\to+\infty}[J_T(x_\star,u_\star) - J(x,u)] \geq 0$$

□

The concavity requirement can be relaxed: see [24].
Now, when talking about economics, the Hamiltonian and the objective functional usually take a specific form.
In particular, one needs to introduce the *discount rate* and the *current value Hamiltonian* .

The discount rate in economics assume different meanings, depending on the context. In this field, it is defined as the interest rate to charge in order to transfer a certain amount of money "at the time 0", which will be given back in a future moment t. For some assumptions one usually make in economics, this rate is a decreasing exponential function of the time. Precisely, the objective functional takes the form

$$J(x,u) = \int_0^{+\infty} e^{-\rho t} \Lambda(t, x(t), u(t))\,\mathrm{d}t$$

(notice that, being usually Λ a polynomial in x, u and t, this guarantees the convergence of the integral). Moreover, the conditions in (7) are more appropriately replaced by the following:

$$\begin{cases} \lim_{t\to+\infty} x^i(t) = x_1^i, & i = 1,\dots,l, \\ \lim_{t\to+\infty} x^j(t) \geq x_1^j, & j = l+1,\dots,r, \\ x^k(b) \in \mathbb{R}, & k = r+1,\dots,n. \end{cases}$$

The current value Hamiltonian is defined as

$$H_c^\eta : [0,+\infty) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R},\ H^c(t,x,p,u) = \langle p, f(t,x,u)\rangle + \eta\Lambda(t,x,u)$$

Notice that it only seems like the definition in equation (3), because it neglects the discount rate.
What's important is to underline that the adjoint equation, in this context, changes its form: equation (12) becomes

$$\dot{p}(t) = -\partial_x H^c(t, x_\star(t), p(t), u_\star(t)) + \rho p(t),$$

that is, a $\rho p(t)$ term is added. The other conditions in the Infinite Horizon Maximum Principle don't change: one has just to consider the current-value Hamiltonian H^c instead of H^η.

Age-structured models

The concepts recalled in the previous pages will now be used while introducing the main notion of this thesis: *age structures*. While studying a model involving some kind of a population, one may need to take the evolution of the profile of its age into account. The reason why it is interesting for this thesis, and the works it is based on, is that the product one wants to introduce in the market may be more interesting for people of a certain age and less for another, and the age profile of the population changes with time.
An age structure in a population is far simpler to treat than another kind of structure, as one by size, for instance. That is because the age of a person increases linearly over time, while other structures may evolve in a more complicated way. In the following pages, a simple presentation of the linear discrete and linear continuous models is given. See [1] and [7] for further references.

Linear discrete models

Suppose that the age profile of the population is divided into a finite number of classes, counted from 0 to m. Let ρ_j^n, $j \in \{0,\dots,m\}$, $n \in \mathbb{N}$, be the number of members in the j-th class at the time n.
Assume that the chance of surviving depends only on the age, it is fixed for every member of the population and it doesn't change over time: call $\sigma_j > 0$, $j = 0,\dots,m$, the chance of surviving for the members of the j-th class.
Moreover, assume that the fecundity rate has the same features as the survival rate, so denote by $\beta_j \geq 0$, $j = 0,\dots,m$, the fecundity rate of the j-th class. Hence,

$$\begin{cases} \rho_0^{n+1} = \sum_{j=0}^m \beta_j \rho_j^n \\ \rho_j^{n+1} = \sigma_{j-1}\rho_{j-1}^n, \quad j = 1,\dots,m \end{cases}$$

This model is called *Leslie matrix model*. It may be re-written using matrices: set

$$A = \begin{pmatrix} \beta_0 & \beta_1 & \dots & \beta_{m-1} & \beta_m \\ \sigma_0 & 0 & \dots & 0 & 0 \\ 0 & \sigma_1 & \dots & 0 & 0 \\ 0 & 0 & \dots & \sigma_{m-1} & 0 \end{pmatrix}, \qquad \rho^n = \begin{pmatrix} \rho_0^n \\ \dots \\ \rho_m^n \end{pmatrix},$$

so $\rho^{n+1} = A\rho^n$. By induction, one gets $\rho^n = A^n \rho^0$.
In the simplest case, where A admits $m+1$ distinct eigenvalues, $\{\lambda_j\}_{j=0,\dots,m}$, with the respective eigenvectors $\{v^j\}_{j=0,\dots,m}$, one may expand

$$\rho^n = A^n \rho^0 = \sum_{j=0}^m \langle \rho_j^0, v^j \rangle \lambda_j^n v^j.$$

Now, suppose that $\{\lambda_j\}_{j=0,\dots,m}$ is ordered in a decreasing way, so that λ_0 is *dominant* with respect to the other eigenvalues. Then, one may write $\rho^n = \lambda_0^n \langle \rho_0^0, v^0 \rangle v^0 + u^n$, where $\lambda_0^{-n} u^n \overset{n\to+\infty}{\longrightarrow} 0$.
Define $P^n := \sum_{j=0}^m \rho_j^n$ (total population at time n) and $B^n := \sum_{j=0}^m \beta_j \rho_j^n = \rho_0^{n+1}$ (births into population at time n). Then

$$P^n = \lambda_0^n \langle \rho_0^0, v^0 \rangle \sum_{j=0}^m v_j^0 + \sum_{j=0}^m u_j^n,$$

and

$$\frac{\rho^n}{P^n} = \frac{\lambda_0^n \langle \rho_0^0, v^0 \rangle v^0 + u^n}{\lambda_0^n \langle \rho_0^0, v^0 \rangle \sum_{j=0}^m v_j^0 + \sum_{j=0}^m u_j^n} \overset{n\to+\infty}{\longrightarrow} \frac{v^0}{\sum_{j=0}^m v_j^0}$$

Thus, the fraction of population within each age class tends to a limit quantity, which is proportional to the dominant eigenvector v_0. In other words, one gets a stable age distribution.
A more general analysis, which comprehends the other cases for the matrix A, may be found in [1].

Linear continuous models

Let $\rho(a,t)$ be the density of individuals of age a at time t. So, the number of individuals of age between $a - \frac{\Delta a}{2}$ and $a + \frac{\Delta a}{2}$ at time t is $\rho(a,t)\Delta a$, hence the total population is $\sum_{a=0}^{+\infty} \rho(a,t)\Delta a$. "As $\Delta a \to 0^+$", one has that the total of the population at time t is

$$P(t) = \int_0^{+\infty} \rho(a,t)\,\mathrm{d}a$$

In practice, one may assume that $\rho(a,t) = 0$ for a big enough.
Age and time are obviously related: people born at time c, at the time $t > c$ will be of age $a = t - c$. As before, suppose that there is an age-dependent death rate $\mu(a)$ (called *mortality function* or *death modulus*) which is the only way people may leave the population. This means that:

- between time t and $t + \Delta t$ a fraction $\mu(a)\Delta t$ of the people with age between a and $a + \Delta a$ at time t die.
- At time t, there are $\rho(a,t)\Delta a$ individuals in that age cohort.
- hence, the number of deaths in that age cohort at time t is $\rho(a,t)\Delta a \mu(a)\Delta t$.
- the remainder survives to the time $t + \Delta t$, being of age between $a + \Delta t$ and $a + \Delta a + \Delta t$.
- Hence, $\rho(a + \Delta a, t + \Delta t) \simeq \rho(a, \Delta a) - \rho(a,t)\Delta a \mu(a)\Delta t$.

If $\rho(a,t)$ is differentiable for any t and a, then, dividing both sides by $\Delta a \Delta t$ and taking the limit $\Delta a \to 0^+$, $\Delta t \to 0^+$, one gets the *McKendrick equation*

$$\frac{\partial \rho(a,t)}{\partial a} + \frac{\partial \rho(a,t)}{\partial t} + \mu(a)\rho(a,t) = 0$$

If $y(\alpha)$ is the number of people who survive at least until age α, then

$$y(\alpha + \Delta\alpha) - y(\alpha) = -\mu(\alpha) y(\alpha)\Delta\alpha \overset{\Delta\alpha \to 0}{\Longrightarrow} y'(\alpha) = -\mu(\alpha) y(\alpha)$$

which implies, $\forall \alpha_1 < \alpha_2$,

$$y(\alpha_2) = y(\alpha_1) e^{-\int_{\alpha_1}^{\alpha_2} \mu(\alpha)\,\mathrm{d}\alpha}$$

In particular, the probability of surviving from birth to age α is

$$\pi(\alpha) = e^{-\int_0^{\alpha} \mu(a)\,\mathrm{d}a} \tag{13}$$

From an intuitive perspective, (13) makes sense: $\mu(a)$ is the mortality rate, hence one would expect that it doesn't vanish as $a \to +\infty$; this means that $\int_0^{\alpha} \mu(a)\,\mathrm{d}a \overset{\alpha \to +\infty}{\longrightarrow} +\infty$, thus $\pi(\alpha) \to 0$ as $\alpha \to +\infty$, which makes sense, for the meaning of $\pi(\alpha)$. Now, as in the discrete case, suppose that the birth process is governed by a function $\beta = \beta(a)$, which depends only on the age, called *birth modulus*.

- The offspring for members of age between a and $a + \Delta a$ in the time interval $[t, t + \Delta t]$ is $\beta(a)\,\Delta t$.

- Thus, the total number of newborn children in the time interval $[t, t + \Delta t]$ is $\Delta t \sum \rho(a,t)\beta(a)\Delta a$, which, "as $\Delta a \to 0^+$, becomes" $\Delta t \int_0^{+\infty} \rho(a,t)\beta(a)\,\mathrm{d}a$.

- In the time interval $[t, t + \Delta t]$, such number is $\rho(0,t)\,\Delta t$, thus one gets the *renewal condition*

  $$B(t) := \rho(0,t) = \int_0^{+\infty} \beta(a)\rho(a,t)\,\mathrm{d}a$$

  In order to complete the model, one has to specify the age distribution at time 0: $\rho(a, 0) = \varphi(a)$. Then, the analogous of the Leslie model for the continuous case is

$$\begin{cases} \frac{\partial \rho(a,t)}{\partial a} + \frac{\partial \rho(a,t)}{\partial t} + \mu(a)\rho(a,t) = 0 \\ \rho(0,t) = \int_0^{+\infty} \beta(a)\rho(a,t)\,\mathrm{d}a \\ \rho(a,0) = \varphi(a) \end{cases} \tag{14}$$

See [1], pp. 275-277 for an analysis of this model through the method of the characteristics.

An alternative analysis is the following, based on the fact that, in the infinite horizon case, one is interested in a so-called *stable age distribution*, that is, a solution of (14) of the form $\rho(a,t) = A(a)T(t)$, $\forall (a,t) \in [0,+\infty) \times [0,+\infty)$, for some functions $A, T \in \mathrm{L}^1([0,+\infty))$.

Since in the discrete case there was actually a stable age distribution, one hopes to find it also in the continuous case. If so, such a solution satisfies:

$$\begin{cases} A'(a)T(t) + A(a)T'(t) + \mu(a)A(a)T(t) = 0 \\ A(0) = \int_0^{+\infty} \beta(a)A(a)\,\mathrm{d}a \\ A(a)T(0) = \varphi(a) \end{cases} \tag{15}$$

By dividing the first equation by $A(a)T(t)$, one gets

$$\frac{A'(a)}{A(a)} + \frac{T'(t)}{T(t)} + \mu(a) = 0$$

which is equivalent to the system

$$\begin{cases} \frac{A'(a)}{A(a)} + \mu(a) = c \\ \frac{T'(t)}{T(t)} = -c \end{cases}$$

for $c \in \mathbb{R}$. Of course, up to rescaling, one may assume $\int_0^{+\infty} A(a)\,\mathrm{d}a = 1$. Then, the total population at the time $t \geq 0$ is

$$P(t) = \int_0^{+\infty} \rho(a,t)\,\mathrm{d}a = T(t).$$

In particular, system (15) becomes

$$\begin{cases} \frac{A'(a)}{A(a)} + \mu(a) = c \\ \frac{P'(t)}{P(t)} = -c \\ A(0) = \int_0^{+\infty} \beta(a)A(a)\,\mathrm{d}a \\ A(a)P(0) = \varphi(a) \end{cases} \implies \begin{cases} A(a) = A(0)\pi(a)e^{ca} \\ P(t) = P(0)e^{-ct} \\ A(0) = \int_0^{+\infty} \beta(a)A(a)\,\mathrm{d}a \\ A(a)P(0) = \varphi(a) \end{cases}$$

By putting the first equation in the third one, one gets:

$$A(0) = A(0)\int_0^{+\infty} \beta(a)\pi(a)e^{ca}\,\mathrm{d}a,$$

hence

$$1 = \int_0^{+\infty} \beta(a)\pi(a)e^{ca}\,\mathrm{d}a$$

which is known as *Lotka-Sharpe equation.* One can show that this equation has a unique real root c, which is positive if $R := \int_0^{+\infty} \beta(a)\pi(a)\mathrm{d}a > 1$, null if $R = 1$ and negative if $R < 1$. Then, the stable age solution is:

$$\rho(a,t) = A(a)P(t) = A(a)P(0)e^{-ct} = \varphi(a)e^{-ct}$$

If $R = 1$, the total population is constant,

$$P(t) = \int_0^{+\infty} \varphi(a)\mathrm{d}a,$$

as well as the birth rate,

$$B(t) = \int_0^{+\infty} \beta(a)\varphi(a)\,\mathrm{d}a.$$

For a more general analysis of a general linear age-dependent model, a simple text which will be taken as reference for the rest of this section is [2].
Choose $L^1([0,+\infty))$ as the mathematical setting for the model, as it is done for many population problems.
The following assumptions will be made:

1. $\beta \in L^\infty([0,\infty))$, $\beta(a) \geq 0$, $\forall a \geq 0$, which corresponds to the idea that every age cohort has a bounded fertility rate;

2. $\mu \in L^1_{\mathrm{loc}}([0,+\infty))$, $\mu(a) \geq 0$, $\forall a \geq 0$; notice that, intuitively, one shouldn't ask the integrability on all the interval $[0,+\infty)$, as $\mu(a)$ isn't supposed to vanish as $a \to +\infty$ (see, indeed, assumption 6).

3. $\varphi \in W^{1,1}([0,+\infty))$, $\varphi(a) \geq 0$, $\forall a \geq 0$. Remember that this means that φ is an absolutely continuous function: it makes sense, as φ is a distribution of the population (in particular, the initial one), whose integral from 0 to a certain age a must be the total population with age between 0 and a at time 0.

4. $\mu\varphi \in L^1([0,+\infty))$

5. $\varphi(0) = \int_0^{+\infty} \beta(a)\varphi(a)\,\mathrm{d}a$, which is the initial renewal condition.

6. $\int_0^{+\infty} \mu(a)\,\mathrm{d}a = +\infty$ (which has already been interpreted).

By a *solution* of (14), one means a function $\rho \in L^\infty([0,+\infty); L^1([0,+\infty)))$, absolutely continuous along every characteristic line (which has equation $a - t = \text{const.}$, $a, t \in [0,+\infty)$), such that

$$\begin{cases} \frac{\partial\rho(a,t)}{\partial a} + \frac{\partial\rho(a,t)}{\partial t} + \mu(a)\rho(a,t) = 0 \\ \lim_{\varepsilon\to 0^+} \rho(\varepsilon, t+\varepsilon) = \int_0^{+\infty} \beta(a)\rho(a,t)\,\mathrm{d}a, & \text{for a.e. } t \in [0,+\infty) \\ \lim_{\varepsilon\to 0^+} \rho(a+\varepsilon,\varepsilon) = \varphi(a), & \text{for a.e. } a \in [0,+\infty) \end{cases} \tag{16}$$

The last two conditions are expressed in the limit form, because the regularity of ρ is assumed only along the characteristic lines.
One has the following:

Theorem 2.5.1: Uniqueness of solutions

Using only assumptions 1,2 and 3, the problem (16) has at most one solution. Such solution is non-negative. Actually, assumption 3 may be weakened, by just asking $\varphi \in L^1([0,+\infty))$.

Proof. Here only the idea of the proof will be given: see [2], pp. 17-20, for the details. First of all, by integration along the characteristic lines, a solution of (16) must have the form

$$\rho(a,t) = \begin{cases} B(t-a)e^{-\int_0^a \mu(\alpha)\,\mathrm{d}\alpha} = B(t-a)\pi(a), & \text{if } t \geq a \\ \varphi(a-t)e^{-\int_0^t \mu(\alpha-t+a)\,\mathrm{d}\alpha}, & t < a \end{cases} \tag{17}$$

where $B(t) = \int_0^{+\infty} \beta(a)\rho(a,t)\,\mathrm{d}a$. By inserting (17) into the expression of B, one finds

$$\begin{aligned} B(t) &= \int_0^{+\infty} \beta(a)\rho(a,t)\,\mathrm{d}a \\ &= \int_0^t \beta(a)B(t-a)\pi(a)\,\mathrm{d}a + \int_t^{+\infty} \beta(a)\varphi(a-t)e^{-\int_0^t \mu(\alpha-t+a)\,\mathrm{d}\alpha}\,\mathrm{d}a, \end{aligned} \tag{18}$$

thus, B satisfies the following Volterra equation

$$B(t) = \int_0^t K(a)B(t-a)\,\mathrm{d}a + F(t), \tag{19}$$

where $K(a) = \beta(a)\pi(a) \geq 0$ and $F(t) \geq 0$ is the second term in the last line of (18). It follows that $K, F \in \mathrm{L}^\infty([0,+\infty))$. Now, via the Banach fixed point theorem, one may prove that equation (19) has a unique solution. Indeed, it holds the estimate:

$$\begin{aligned} \|\mathscr{F}[B_1](t) - \mathscr{F}[B_2](t)\| &= \text{sup. ess.}_{t\in[0,+\infty)} \left[e^{-\lambda t} \int_0^t K(a)|B_1(t-a) - B_2(t-a)| \right] \\ &\leq \frac{1}{\lambda} \|K\|_{\mathrm{L}^\infty([0,+\infty))} \cdot \|B_1 - B_2\|, \end{aligned}$$

where $\mathscr{F}[B](t) = \int_0^t K(a)B(t-a)\,\mathrm{d}a + F(t)$, hence one has a contraction if $\lambda > \|K\|_{\mathrm{L}^\infty([0,+\infty))}$. To show that the solution is non-negative, one needs to remember that the Banach fixed-point theorem states that the solution is found through the iteration:

$$\begin{cases} B_0(t) = F(t) \\ B_{n+1}(t) = F(t) + \int_0^t K(a)B_n(t-a)\mathrm{d}a, \end{cases}$$

which converges to the solution in $\mathrm{L}^\infty([0,+\infty))$, and every B_n is non-negative. □

Theorem 2.5.2: Regularity of solutions

Under the assumption 1-6, the solution ρ of (14) is continuous and the partial derivatives $\partial_a \rho$ and $\partial_t \rho$ exist almost everywhere.

Proof. Using the same notation as in the previous theorem, one may easily verify that $F \in W^{1,\infty}([0,+\infty))$ and, by (19), that $B \in W^{1,\infty}([0,+\infty))$ and

$$B'(t) = F'(t) + K(t)B(0) + \int_0^t K(t-\alpha)B'(\alpha)\,\mathrm{d}\alpha,$$

where B' and K' are meant to be the weak derivatives of B and K respectively. Hence, equation (17) implies that $\rho \in \mathscr{C}([0,+\infty) \times [0,+\infty))$ and its partial derivatives exist almost everywhere in $[0,+\infty) \times [0,+\infty)$. □

Notice that the solution $\rho \in L^\infty((0,+\infty), L^1(0,+\infty))$ of (14) is also a weak solution, in the following sense:

$$\int_0^{+\infty}\int_0^{+\infty} \{-D\psi(a,t) + \mu(a)\psi(a,t) - \beta(a)\psi(0,t)\}\rho(a,t)\,\mathrm{d}a\,\mathrm{d}t =$$

$$= \int_0^{+\infty} \psi(a,0)\varphi(a)\,\mathrm{d}a$$

where ψ is any absolutely continuous function along almost every characteristic line and satisfies

$$\begin{cases} \psi \in L^\infty([0,+\infty)] \\ D\psi \in L^1([0,+\infty)) \\ D\psi - \mu\psi + \beta\psi(0,\cdot) \in L^\infty([0,+\infty))] \\ \lim_{a\to+\infty}\psi(a,t) = 0, \ \text{ a.e. } t \in [0,+\infty), \\ \lim_{t\to+\infty}\psi(a,t) = 0, \ \text{ a.e. } a \in [0,+\infty) \end{cases}$$

Then one has the following

Theorem 2.5.3: Uniqueness of weak solutions

Under the assumption 1-4, the system (14) has a unique weak solution.

See [2], pp. 27-29, for the proof.

3

CHAPTER

Useful concepts about Differential Games

The model treated in this work of thesis is characterised by two figures: the manufacturer and the retailer of a product, whose promotion campaign is being planned. As such, every one of them aims to maximize his earnings when the product will be introduced in the market. Also, the retailer's price towards the consumers depends on the transfer paid to the manufacturer, and it affects the quantity of the product sold as well as the one bought by the manufacturer. So, it is reasonable to define a differential game and to use the relative techniques.
This is not a case. Indeed, one has two inherent characteristics that make marketing channels meaningful to be studied via differential games theory: first, it's easy to identify the players of the game; second, each player's payoff will depend on the actions taken by the other players.
In this chapter, a brief review of the fundamental notions needed for this thesis is given. For further references about Differential Games and Economy, see [5], [10], [16], [17] and [21].

Nash equilibrium

Consider a differential game with N players over the time interval $[0,\infty)$, X a set which will be called the *space set* of the game. The *state of the game* is a vector

$x(t) \in X, t \in [0,+\infty)$, with $x(0) = x_0$ the initial state. For each player $i \in \{1,...,N\}$, write

$$u^{-i}(t) := \{u^1(t),...,u^{i-1}(t),u^{i+1}(t),...,u^N(t)\},$$

that is the set of the control variables of the other players. Choose for each player a control $u^i(t) \in U(x(t),u^{-i}(t),t) \in \mathbb{R}^{m_i}$.
Consider an $N-$tuple $\phi(t) = (\phi^1(t),...,\phi^N(t))$. The player i's decision problem is

$$\begin{aligned} &\text{Maximize } J^i_{\phi^{-i}}(u^i(\cdot)) = \int_0^{+\infty} e^{-\rho^i t}\Lambda^i_{\phi^{-i}}(t,x(t),u^i(t))\,\mathrm{d}t \\ &\text{subject to} \begin{cases} \dot{x}(t) = f^i_{\phi^{-i}(t)}(x(t),u^i(t),t) \\ x(0) = x_0 \\ u^i(t) \in U^i_{\phi^{-i}(t)}(t,x(t)) \end{cases} \end{aligned} \tag{20}$$

where the subscript ϕ^{-i} is a short form to say that each function with such subscript depends on the values $\phi^1,...,\phi^{i-1},\phi^{i+1},...,\phi^N$.
The $N-$tuple $(\phi^1,...,\phi^N)$ of functions $\phi^i : X \times [0,+\infty) \to \mathbb{R}^{m_i}$, is called *Markovian Nash equilibrium* or *feedback Nash equilibrium*, if, for each $i \in 1,...,N$, a control $u^i(\cdot)$ generating an optimal (in one of the senses described in conditions (9) and following) state for the problem (20) exists and is given by the Markovian strategy $u^i(t) = \phi^i(x(t),t)$.
The $N-$tuple $(\phi^1,...,\phi^N)$ of functions $\phi^i : [0,+\infty) \to \mathbb{R}^{m_i}$, $i \in \{1,...,n\}$, is called an *open-loop Nash equilibrium* if, for each $i \in \{1,...,N\}$, a control generating an optimal (in one of the senses described in conditions (9) and following) state for (20) exists and is given by the open-loop strategy $u^i(t) = \phi^i(t)$.
In general,

$$\{\text{open}-\text{loop Nash equilibria}\} \subseteq \{\text{Markovian Nash equilibria}\}$$

Sub-game perfectness and time consistency

Denote by $\Gamma(x_0,0)$ the game discussed in the previous section. For each pair $(x,t) \in X \times [0,+\infty)$, define a sub-game $\Gamma(x,t)$ by replacing the objective functional for the player i in equation (20) with

$$\int_t^{+\infty} e^{-\rho^i(s-t)}\Lambda^i_{\phi^{-i}}(t,x(t),u^i(t))\,\mathrm{d}t$$

and the condition $x(0) = x_0$ in the related state equation with $x(t) = x$.
Let $(\phi^1,...,\phi^N)$ be a Markovian Nash equilibrium for the game $\Gamma(x_0,0)$, and denote by $x(\cdot)$ the unique state generated by this equilibrium. The equilibrium will be said *time consistent* if, $\forall t \in [0,+\infty)$, the sub-game $\Gamma(x(t),t)$ admits a Markovian Nash

equilibrium $(\psi^1, ..., \psi^N)$ such that $\psi^i(y,s) = \phi^i(y,s)$ holds for any $i \in \{1,...,N\}$ and all $(y,s) \in X \times [t, +\infty)$.

In other words, a Markovian Nash equilibrium is time-consistent if it is a Markovian Nash equilibrium of every sub-game along the state $x(\cdot)$. Notice that this notion of time-consistency may be given for generic Nash equilibria of differential games, not only for the Markovian Nash ones.

In order to do this, one has to properly define a particular regular and non-anticipating information structure $\mathscr{H}$ (the letter stands for "History"). This implies that $\mathscr{H}(u(\cdot), t)$ depends only on the restriction of $u(\cdot)$ to the time interval $[0,t)$; call such restriction "the $t-$truncation of $u(\cdot)$", and denote it by $u_t(\cdot)$. One may define an equivalence in the set of the $t-$truncations by stating that $u_t(\cdot) \equiv v_t(\cdot)$ iif $\{s \in [0,t) : u_t(s) = v_t(s)\}$ has null Lebesgue measure. Then, the *information structure* $\mathscr{H}$ can be defined by saying that, $\forall i \in \{1,...,N\}$ and $\forall t \in [0, +\infty)$, $\mathscr{H}^i(u(\cdot), t)$ is the equivalence class to which u_t belongs. It is common to denote $H^i(u(\cdot), t)$ just by $u_t(\cdot)$, and to refer to it as the $t-$history of the game. A differential game which uses this information structure is called a *differential game with history-dependent strategies,* and in this case the subscript $\mathscr{H}$ will be used. Now one can generalize the notion of time-consistency.

Let $\Gamma_H(x_0, 0)$ be a differential game with history-dependent strategies, and let $\phi = (\phi^1, ..., \phi^N)$ be a Nash equilibrium, with correspondent $N-$tuple control paths $u(\cdot)$. The Nash equilibrium ϕ is called *time-consistent* if, $\forall t \in [0, +\infty)$, ϕ is also an equilibrium for the sub-game $\Gamma_H(u_t(\cdot), t)$.

One may show (see [5], pp. 100-101) that every Markovian Nash equilibrium of a differential game is time consistent.

Now, another important notion will be given, first for Markovian Nash equilibria and then generalized to generic Nash ones.

Let $(\phi^1, ..., \phi^N)$ be a Markovian Nash equilibrium for the game $\Gamma(x_0, 0)$. Such equilibrium is called *sub-game perfect* if, $\forall (x,t) \in X \times [0, +\infty)$, the sub-game $\Gamma(x,t)$ admits a Markovian Nash equilibrium $(\psi^1, ..., \psi^N)$ such that $\psi^i(y,s) = \phi^i(y,s)$ holds for all $i \in \{1,...,N\}$ and all $(y,s) \in X \times (t, +\infty)$. A Markovian Nash equilibrium which is sub-game perfect is also called a *Markovian perfect equilibrium* . As for time-consistency, the notion may be generalized in the following way. Denote by U_t the set of all the possible $t-$histories of the differential game $\Gamma(x_0, 0)$. Then, the Nash equilibrium ϕ is called *sub-game perfect* if, $\forall t \in [0, +\infty)$ and $\forall \tilde{u}(\cdot) \in U_t$, ϕ is also a Nash equilibrium for the sub-game $\Gamma_H(\tilde{u}_t(\cdot), t)$.

Of course, the definitions given imply that a sub-game perfect Markovian Nash equilibrium is also time-consistent. The following are sufficient conditions for a Markovian Nash equilibrium to be sub-game perfect:

Theorem 3.0.1:

et $(\phi^1,\dots,\phi^N)$ *be a given* $N-$*tuple of functions* $\phi^i : X\times[t,+\infty)\to\mathbb{R}^{m^i}$ *and make the following assumptions:*

- *for every pair* $(y,s)\in X\times[0,+\infty)$*, there exists a unique absolutely continuous solution* $x_{y,s}:[s,+\infty)\to X$ *of the initial value problem*

$$\begin{cases}\dot{x}(t)=f(x(t),\phi^1(x(t),t),\dots,\phi^N(x(t),t))\\ x(s)=y\end{cases}$$

- *for all* $i\in\{1,\dots,N\}$*, there exists a continuously differentiable function* $V^i : X\times[0,+\infty)\to\mathbb{R}$ *such that the* Hamilton-Jacobi-Bellman *equations*

$$\begin{aligned}&r^iV^i(x,t)-\partial_tV^i(x,t)=\\ &=\max\left\{\Lambda^i_{\phi^{-i}}(x,u^i,t)+\partial_xV^i(x,t)f^{-i}_{\phi^{-i}}(x,u^i,t)\,|\,u^i\in U^i_{\phi^{-i}}(x,t)\right\}\end{aligned}\tag{21}$$

  are satisfied for all $(x,t)\in X\times[0,+\infty)$*;*

- $\forall i\in\{1,\dots,N\}$*, either* V^i *is a bounded function and* $r^i>0$ *or bounded* V^i *is bounded below and not above,* $r^i>0$ *and* $\limsup_{t\to+\infty}e^{-r^it}V^i(x_{y,s}(t),t)\le 0$ *must hold* $\forall(y,s)\in X\times[0,+\infty)$.

Denote by $\Phi^i(t,x)$ *the set of all* $u^i\in U^i_{\phi^{-i}}(x,t)$ *which maximize the right-hand side of (21). If* $\phi^i(t,x)\in\Phi^i(t,x)$ *holds* $\forall i\in\{1,\dots,N\}$ *and almost all* $(x,t)\in X\times[0,+\infty)$*, then* $(\phi^1,\dots,\phi^N)$ *is a Markov perfect Nash equilibrium (where optimality is meant to be the sporadically catching up optimality).*

The following section introduces another well-known kind of equilibrium, which is not time-consistent, in general.

Stackelberg games and equilibria

The previous section dealt with differential games in which all players make their moves simultaneously. Sometimes - and this is often the case in economics - one has to deal with a situation in which some players have priority of moves over other players. For the sake of simplicity, only two players will be considered: the first will be called *leader* (L) and the latter *follower* (F).
Let $x \in \mathbb{R}^n$ denote the vector of state variables, $u^L \in \mathbb{R}^{m^L}$ the vector of the control variables of the leader, and $u^F \in \mathbb{R}^{m^F}$ the vector of control variables of the follower. The evolution of the state variables is given by

$$\dot{x}_i(t) = f_i(x(t), u^F(t), u^L(t), t), \quad x_i(0) = x_{i,0} \tag{22}$$

with $i = 1, ..., n$ and $x_{i,0}$ is given.
At time 0, the leader announces the control path $u^L(\cdot)$. The follower, taking this path as given, chooses his control path $u^F(\cdot)$ to maximize his integral of utility:

$$J^F = \int_0^{+\infty} e^{-\rho^F t} \Lambda^F(x(t), u^F(t), u^L(t), t)\, \mathrm{d}t.$$

The leader, having observed what's the follower's best response function, chooses the expression of his control u^L which maximizes his utility J^L.
What has just been described is a *Stackelberg game*. A *Stackelberg equilibrium* is a couple $(u^{L,\star}, u^{F,\star})$ of outputs, such that $u^{F^\star} = R(u^{F,\star})$ is the best response function to u^L evaluated in $u^L = u^{L,\star}$ and $u^{L,\star} \in \arg\max\{J^L(u^L, R(u^L))\}$.
Now, a procedure to find Stackelberg equilibria is going to be described.
Denote by $p(\cdot)$ the vector of co-state variables for this maximization problem, the follower's current value Hamiltonian is

$$H^F(x, u^F, u^L, p, t) = \Lambda^F(x, u^F, u^L, t) + \langle p, f(x, u^F, u^L, t)\rangle$$

In what follows, it is assumed that the controls are unconstrained. Then, assuming a sufficient regularity of the controls and given the path $u^L(\cdot)$, the optimality conditions for the follower's problem are

$$\frac{\partial \Lambda^F(x(t), u^F(t), u^L(t), t)}{\partial u_j^F} + \left\langle p, \frac{\partial f(x, u^F, u^L, t)}{\partial u_j^F} \right\rangle = 0, \tag{23}$$

for $j = 1, ..., m^F$, and

$$\dot{p}_i(t) = \rho^F p(t) - \frac{\partial \Lambda^F(x(t), u^F(t), u^L(t), t)}{\partial x^i} - \left\langle p, \frac{\partial f(x, u^F, u^L, t)}{\partial x^i} \right\rangle, \tag{24}$$

for $i = 1, ..., n$. Assume H^F is jointly concave in the variables x and u^F. Then the above conditions are sufficient for the optimality of $u^F(\cdot)$. If H^F is strictly concave in

u^F, then the condition (23) uniquely determines the value of each control variable $u^F_j(\cdot)$ as a function of $x(t)$, $p(t)$, $u^L(t)$ and t, that is

$$u^F_j = g_j(x(t), p(t), u^L(t), t), \quad j = 1, ..., m^F. \tag{25}$$

Substituting (25) in (24), one obtains

$$\begin{aligned}\dot{p}_i(t) = \rho^F p_i(t) - &\frac{\partial \Lambda^F(x(t), g(x(t), p(t), u^L(t), t), u^L(t), t}{\partial x^i} + \\ &- \left\langle p, \frac{\partial f_k(x(t), g(x(t), p(t), u^L(t), t), u^L(t), t)}{\partial x^i} \right\rangle\end{aligned} \tag{26}$$

with $i = 1, ..., n$.

These equations characterize the follower's best response to the leader's control $u^L(\cdot)$. The leader, knowing the follower's best response for each $u^L(\cdot)$, then proceeds to choose a $u^{L\star}$ to maximize the integral of his utility.

As Dockner points out in [5], pp. 115-116, the specific structure of the problem at hand determines whether the initial value $p^\star(0)$ of the adjoint function depends on the announced leader's control u^L or not. Hence, $p^\star(\cdot)$ will be said *noncontrollable* if $p^\star(0)$ doesn't depend on u^L, and vice-versa.

The leader's optimization problem is to choose a control $u^L(\cdot)$ to maximize

$$J^L = \int_0^{+\infty} e^{-\rho^L t} \lambda^L(x(t), u^F(t), u^L(t), t)\, \mathrm{d}t$$

where $u^F(t) = g(x(t), p(t), u^L(t), t)$. The maximization is subject to (22) and (26). In this optimization problem, the co-state variables p_i, $i = 1, ..., n$, of the follower's optimization problem are treated as state variables in the leader's optimization problem (in addition to the original state variables x_i, $i = 1, ..., n$). Notice that, while the initial value $x_i(0)$ is fixed at x_{i0}, the initial value $p_i(0)$ is fixed if and only if it is noncontrollable.

The Hamiltonian function for the leader is

$$\begin{aligned}H^L(x, p, u^L, y, q, t) =&\Lambda^L(x, g(x, p, u^L, t), u^L, t) + \langle q, f(x, g(x, p, u^L, t), u^L, t)\rangle + \\ &+ \langle y, k(x, p, u^L, t)\rangle\end{aligned}$$

where $k(x, p, u^L, t)$ denotes the right-hand side of [26]. The variables q and y are the co-state variables associated with p and x respectively. One then has the optimality conditions:

$$\begin{cases} \frac{\partial H^L(x(t),p(t),u^L(t),\psi(t),\pi(t),t)}{\partial u^L_j} = 0 \\ \dot{y}(t) = \rho^L y(t) - \frac{\partial H^L(x(t),p(t),u^L(t),\psi(t),\pi(t),t)}{\partial x_i} \\ \dot{q}(t) = \rho^L q(t) - \frac{\partial H^L(x(t),p(t),u^L(t),\psi(t),\pi(t),t)}{\partial p_i}, \end{cases} \tag{27}$$

with $i = 1, ..., n$ and $j = 1, ..., m^L$. If the Hamiltonian H^L is jointly concave in the state variable x_i and p_i, $i = 1, ..., n$, and the control variables u^L_j, $j = 1, ..., m^L$, then the conditions (22) and (26)-(27) are sufficient for the optimality of u^L.

Linear state games

The model treated in this thesis is a linear state game, meaning something that will be explained in the following pages.
A linear state game is such if its system dynamics and the utility functions are polynomials of degree 1 on the state variables and satisfy a certain property (described below) concerning the interaction between control variables and state variables. As Dockner points out in [5], pp. 187-192, their open-loop Nash equilibria are sub-game perfect. Moreover, in the finite horizon case, if the final state is free, then their Stackelberg equilibria are sub-game perfect (see [25]). Whereas, if the final state has some constraints, then one way to deal with the time-consistency of Stackelberg equilibria is to give weaker definitions of sub-game perfectness (see [14]). As the reader may see, many things may be said about linear state games, but there are some other kinds of games that are not so difficult to discuss. For a complete review of those, see [3].
Consider a two-person differential game, with state equation

$$\dot{x}(t) = f(x(t), u^1(t), u^2(t), t),$$

where $u^1 \in \mathbb{R}^{m^1}$ and $u^2 \in \mathbb{R}^{m^2}$ are the control variables of players 1 and 2 respectively, and $x(t) \in \mathbb{R}^n$ is an $n-$dimensional vector of state variables. The objective functional of player i is given by

$$J^i = \int_0^{+\infty} e^{-\rho^i t} \Lambda^i(x(t), u^1(t), u^2(t), t) \mathrm{d}t$$

One defines the function $\tilde{H}^i : \mathbb{R}^{n+m^1+m^2} \times [0, +\infty) \to \mathbb{R}$, by

$$\tilde{H}^i(x, u^1, u^2, p^i, t) = \Lambda^i(x, u^1, u^2, t) + p^i f(x, u^1, u^2, t)$$

where $p^i \in \mathbb{R}^n$ is a vector of costate variables. A differential game is referred to as a linear state game if the conditions

$$\tilde{H}^i_{xx}(x, u^1, u^2, p^i, t) = 0$$

and

$$\tilde{H}^i_{u^i}(x, u^1, u^2, p^i, t) = 0 \implies \tilde{H}^i_{u^i x}(x, u^1, u^2, p^i, t) = 0 \tag{28}$$

hold for $i = 1, 2$ and all $(x, u^1, u^2, p^i, t) \in \mathbb{R}^{2n+m^1+m^2} \times [0, +\infty)$. Notice that (28) is automatically satisfied if

$$\tilde{H}^1_{u^i x}(x, u^1, u^2, p^1, t) = \tilde{H}^2_{u^i x}(x, u^1, u^2, p^i, t) = 0 \tag{29}$$

holds for $i = 1, 2$ and all $(x, u^1, u^2, p^1, p^2, t) \in \mathbb{R}^{3n+m^1+m^2} \times [0, +\infty)$.
Condition (29) implies that there is no multiplicative interaction at all between the

state and the control variables in game. In terms of the state equations, the objective functionals and the salvage value term this implies:

$$f(x,u^1,u^2,t) = A(t)x + g(u^1,u^2,t)$$

$$\Lambda^i(x,u^1,u^2,t) = C^i(t)x + k^i(u^1,u^2,t),$$

where $A:[0,+\infty)\to\mathbb{R}^{n\cdot n}$, $g:\mathbb{R}^{m^1+m^2}\times[0,+\infty)\to\mathbb{R}^n$, $C^i:[0,+\infty)\to\mathbb{R}^n$, $k^i:\mathbb{R}^{m^1+m^2}\times[0,+\infty)\to\mathbb{R}$ and $W^i\in\mathbb{R}^n$.
Now, this notion will be specified for age-structured models.
Consider the following maximization problem:

$$\max_{u_i\in U_i} J_i(u_1(t,a),u_2(t,a)) = \int_0^{+\infty} \mathrm{d}t \int_0^{\omega} \Lambda^i(t,a,x(t,a),p(t,a),u_1(t,a),u_2(t,a))\,\mathrm{d}a,$$

subject to the state equation

$$\begin{cases} (\partial_t+\partial_a)x(t,a) = f(t,a,x(t,a),p(t,a),u_1(t,a),u_2(t,a)) \\ x(0,a)=\varphi(a) \\ x(t,0)=0 \end{cases} \tag{30}$$

where the non local variable p is defined as follows:

$$p(t,a) = \int_0^{\omega} g(t,a,\alpha,x(t,\alpha),u_1(t,\alpha),u_2(t,\alpha))\,\mathrm{d}\alpha \tag{31}$$

Such a differential game is said to be linear state if

$$\Lambda^i(t,a,x(t,a),p(t,a),u_1(t,a),u_2(t,a)) = \bar{L}^i(t,a)x + \tilde{L}^i(t,a)p + \hat{L}^i(t,a,u_1,u_2)$$

and

$$f(t,a,x(t,a),p(t,a),u_1(t,a),u_2(t,a)) = \bar{f}(t,a)x + \tilde{f}(t,a)p + \hat{f}(t,a,u_1,u_2)$$

$$g(t,a,\alpha,x(t,\alpha),u_1(t,\alpha),u_2(t,\alpha)) = \bar{g}(t,a,\alpha)x + \hat{g}(t,a,\alpha,u_1,u_2) \tag{32}$$

Grosset and Viscolani, in [26], follow Dockner's approach ([5], p. 188) to prove the following:

Theorem 3.0.2: Subgame perfectness of OLNEs in linear state AS games

Let $(u_1^\star(t,a),u_2^\star(t,a))$ be an open-loop Nash equilibrium for the aforementioned linear age-structured differential game. If U_1 and U_2 are convex sets, then such equilibrium is sub-game perfect.

Now, sufficient conditions for the existence and uniqueness of the solution of (30) will be provided.

Suppose $f, g \in L^{\infty}([0,+\infty) \times [0,\omega])$ and that g has a little more specific form than the one in (32):

$$g(t,a,\alpha,x(t,\alpha),u_1(t,\alpha),u_2(t,\alpha)) = \chi_{[a,\omega]}(\alpha)[\bar{g}(t,a,\alpha)x + \hat{g}(t,a,\alpha,u_1,u_2)],$$

where $\chi_{[a,\omega]}(\alpha)$ is the characteristic function of the interval $[a,\omega]$. Thus, equation (31) becomes

$$p(t,a) = \int_a^{\omega} g(t,a,\alpha,x(t,\alpha),u_1(t,\alpha),u_2(t,\alpha))\,\mathrm{d}\alpha$$

and, if one can show that $\partial_a p$ exists, it holds

$$x(t,a) = -\frac{\partial_a p(t,a) + \hat{g}(t,a,u_1(t,a),u_2(t,a))}{\bar{g}(t,a)}$$

Now, given $(t,a) \in [0,+\infty) \times [0,\omega]$, by using the characteristic line $\lambda(\tau) = a - t + \tau$ (which intersects $[0,+\infty) \times \{0\}$ in $\hat{\tau} = -a + t$), one may rewrite the first equation in (30) as

$$Dx(\tau,\lambda(\tau)) - \bar{f}(\tau,\lambda(\tau))x(\tau,\lambda(\tau)) = \tilde{f}(\tau,\lambda(\tau))p(\tau,\lambda(\tau)) + \hat{f}(\tau,\lambda(\tau),u_1(\tau,\lambda(\tau)),u_2(\tau,\lambda(\tau)))$$

Then,

$$\begin{aligned} x(\tau,\lambda(\tau)) = e^{\int_0^{\tau}\bar{f}(\sigma,\lambda(\sigma))\,\mathrm{d}\sigma} \int_{t-a}^{\tau} e^{-\int_0^{\sigma}\bar{f}(\sigma_1,\lambda(\sigma_1))\,\mathrm{d}\sigma_1} [\tilde{f}(\sigma,\lambda(\sigma))p(\sigma,\lambda(\sigma)) + \\ + \hat{f}(\sigma,\lambda(\sigma),u_1(\sigma,\lambda(\sigma)),u_2(\sigma,\lambda(\sigma)))]\,\mathrm{d}\sigma \end{aligned}$$

and, by using $\lambda(t) = a$, one gets

$$\begin{aligned} x(t,a) = x(t,\lambda(t)) = & e^{\int_0^{t}\bar{f}(\tau,a-t+\tau)\,\mathrm{d}\tau} \int_{t-a}^{t} e^{-\int_0^{\tau}\bar{f}(\tau,a-t+\tau)\,\mathrm{d}\tau} [\tilde{f}(\tau,a-t+\tau)p(\tau,a-t+\tau) + \\ & + \hat{f}(\tau,a-t+\tau,u_1(\tau,a-t+\tau),u_2(\tau,a-t+\tau))]\,\mathrm{d}\tau \end{aligned} \tag{33}$$

By substituting (33) in equation (31), where g is given by (32), one gets the following Lotka-Volterra equation:

$$\begin{aligned} p(t,a) = & F(t,a) + \int_a^{\omega} \bar{g}(t,a,\alpha) e^{\int_0^{t}\bar{f}(\tau,\alpha-t+\tau)\,\mathrm{d}\tau}\,\mathrm{d}\alpha \cdot \\ & \cdot \int_{t-a}^{t} e^{-\int_0^{\tau}\bar{f}(\sigma,\alpha-t+\sigma)\,\mathrm{d}\sigma} \tilde{f}(\tau,\alpha-t+\tau)p(\tau,\alpha-t+\tau)\,\mathrm{d}\tau \end{aligned} \tag{34}$$

with

$$\begin{aligned} F(t,a) = \int_a^{\omega} \hat{g}(t,a,\alpha,u_1(t,\alpha),u_2(t,\alpha))\,\mathrm{d}\alpha \int_a^{\omega} \bar{g}(t,a,\alpha) e^{\int_0^{t}\bar{f}(\tau,\alpha-t+\tau)\,\mathrm{d}\tau}\,\mathrm{d}\alpha \cdot \\ \cdot \int_{t-\alpha}^{t} e^{-\int_0^{\tau}\bar{f}(\sigma,a-t+\sigma)\,\mathrm{d}\sigma} \hat{f}(\tau,a-t+\tau,u_1(\tau,a-t+\tau),u_2(\tau,a-t+\tau))\,\mathrm{d}\tau \end{aligned}$$

Now, consider $L^\infty([0,+\infty)\times[0,\omega])$ endowed with the following norm:

$$\|f\| = \text{ess.sup.}_{t\in[0,+\infty),a\in[0,\omega]} e^{-\lambda_1 t-\lambda_2 a}|f(t,a)|$$

Introduce on $(L^\infty([0,+\infty)\times[0,\omega]), \|\cdot\|)$ the operator

$$\mathscr{F}: L^\infty([0,+\infty)\times[0,\omega]) \longrightarrow L^\infty([0,+\infty)\times[0,\omega]),$$

defined as

$$\begin{aligned}\mathscr{F}[p](t,a) =& F(t,a)+\int_a^\omega \bar{g}(t,a,\alpha)e^{\int_0^t \bar{f}(\tau,\alpha-t+\tau)\,\mathrm{d}\tau}\,\mathrm{d}\alpha\cdot\\ &\cdot\int_{t-a}^t e^{-\int_0^\tau \bar{f}(\sigma,\alpha-t+\sigma)\,\mathrm{d}\sigma}\tilde{f}(\tau,\alpha-t+\tau)p(\tau,\alpha-t+\tau)\,\mathrm{d}\tau\end{aligned}$$

Then,

$$\begin{aligned}
&\|\mathscr{F}[p_1](t,a)-\mathscr{F}[p_2](t,a)\| \le \|\bar{g}\|_\infty\|\tilde{f}\|_\infty\cdot\\
&\cdot\text{ess.sup.}_{t\in[0,+\infty),a\in[0,\omega]} e^{-\lambda_1 t-\lambda_2 a}\int_a^\omega e^{\int_0^t \bar{f}(\tau,\alpha-t+\tau)\,\mathrm{d}\tau}\,\mathrm{d}\alpha\cdot\\
&\cdot\int_{t-a}^t e^{-\int_0^\tau \bar{f}(\sigma,\alpha-t+\sigma)\,\mathrm{d}\sigma}|p_1(\tau,\alpha-t+\tau)-p_2(\tau,\alpha-t+\tau)|\,\mathrm{d}\tau\le\\
&\le\|\bar{g}\|_\infty\|\tilde{f}\|_\infty\,\text{ess.sup.}_{t\in[0,+\infty),a\in[0,\omega]} e^{-(\lambda_1-\|\bar{f}\|_\infty)t-\lambda_2 a}\cdot\\
&\cdot\int_a^\omega\mathrm{d}\alpha\int_{t-a}^t e^{\|\bar{f}\|_\infty\tau}|p_1(\tau,\alpha-t+\tau)-p_2(\tau,\alpha-t+\tau)|\,\mathrm{d}\tau\le\\
&\le\|\bar{g}\|_\infty\|\tilde{f}\|_\infty\,\text{ess.sup.}_{t\in[0,+\infty),a\in[0,\omega]} e^{-(\lambda_1-\|\bar{f}\|_\infty)t-\lambda_2 a}\cdot\\
&\cdot\int_a^\omega\mathrm{d}\alpha\int_{t-a}^t e^{\lambda_1\tau}e^{\left(\|\bar{f}\|_\infty-\lambda_1\right)\tau}|p_1(\tau,\alpha-t+\tau)-p_2(\tau,\alpha-t+\tau)|\,\mathrm{d}\tau\le\\
&\le\frac{\|\bar{g}\|_\infty\|\tilde{f}\|_\infty}{\lambda_1}\text{ess.sup.}_{t\in[0,+\infty),a\in[0,\omega]} e^{-(\lambda_1-\|\bar{f}\|_\infty)t-\lambda_2 a}\cdot\int_a^\omega e^{\|\bar{f}\|_\infty t}|p_1(t,\alpha)-p_2(t,\alpha)|\,\mathrm{d}\alpha=\\
&=\frac{\|\bar{g}\|_\infty\|\tilde{f}\|_\infty}{\lambda_1}\text{ess.sup.}_{t\in[0,+\infty),a\in[0,\omega]} e^{-\lambda_1 t-\lambda_2 a}\cdot\int_a^\omega|p_1(t,\alpha)-p_2(t,\alpha)|\,\mathrm{d}\alpha\le\\
&\le\frac{\|\bar{g}\|_\infty\|\tilde{f}\|_\infty}{\lambda_1}\text{ess.sup.}_{t\in[0,+\infty),a\in[0,\omega]} e^{-\lambda_1 t-\lambda_2 a}\int_a^\omega e^{-\lambda_2\alpha}e^{\lambda_2\alpha}|p_1(t,\alpha)-p_2(t,\alpha)|\,\mathrm{d}\alpha\le\\
&\le\frac{\|\bar{g}\|_\infty\|\tilde{f}\|_\infty}{\lambda_1\lambda_2}\|p_1(t,a)-p_2(t,a)\|
\end{aligned}$$

Hence, if $\lambda_1\lambda_2>\|\bar{g}\|_\infty\|\tilde{f}\|_\infty$, one has that the operator $\mathscr{F}$ is a contraction. By Caccioppoli-Banach's theorem, it admits a unique fixed point, i.e.

$$\exists p(t,a)\in L^\infty([0,+\infty)\times[0,\omega])\text{ s.t. }p(t,a)=\mathscr{F}[p](t,a),$$

which is evidently a solution of (34). Moreover, that theorem states that such solution is given by the reiterative procedure:

$$\begin{cases}p_0(t,a)=F(t,a)\\ p_{n+1}(t,a)=\mathscr{F}[p_n](t,a)\end{cases}$$

Now, differentiable functions are not dense in L^∞, and one would like to prove a regularity result in order to show that the existence of the solution p implies the existence (at least, a.e. on $[0,+\infty)\times[0,\omega]$) of $x(t,a)$ by formula (32). Unfortunately, the Sobolev space L^∞ is quite a difficult one to treat as for regularity problems (see [48], p. 318); this needs further research.

CHAPTER 4

An Age-structured model for a distributive channel

As Jørgensen and Zaccour say in [10],
"*A marketing channel is formed by independent firms: a manufacturer, wholesalers, retailers and other agents who play a financial or informational facilitating role in contracting and moving the product to the final consumer. [...]*
The optimal design of marketing channel members' strategies depends on how the channel members make their marketing decisions. It is usual to distinguish two situations: the coordinated and the uncoordinated case. In game-theoretic terms, these are respectively called cooperative and noncooperative cases."
In the introduction of the previous section, a brief explanation of why it makes sense to study marketing channels through differential games theory was given. In the following pages, it will be first given a quick resume of the fundamental concepts from economics, as well as of the basic results as far as the age-structured linear-state differential games theory is concerned. Then, a model of a marketing channel with such structure is discussed.

Basic notions about distributive channels in an infinite horizon setting

What has been said until now was a rapid review of the basic mathematical notions needed to discuss this thesis. Now a brief presentation of the economic concepts is going to be given. One was already met in the first chapter: the discount rate ρ, which is strictly related to the following.

Goodwill

The returns, that the manufacturer and the retailer will have from the product, obviously depend on the public image of the product itself and of the firm. Heuristically speaking, such public image is the (*goodwill*) . Nerlove and Arrow in 1962 (see [22]) gave this definition of it:
"*One possibility of representing the temporal differences in the effects of advertising on demand [...] is to define a stock, which we shall call* ***goodwill*** *and denote by* $G(t)$, *and which we suppose summarizes the effects of current and past advertising outlays on demand.* The stock of advertising goodwill $G(t)$ evolves according to the *Nerlove-Arrow dynamics*:

$$\begin{cases} (\partial_t + \partial_a)G(t,a) = A(t,a) - \mu(a)G(t,a) \\ G(0,a) = \varphi(a),\ a \in [0,\omega] \\ G(t,0) = 0,\ t \in [0,+\infty) \end{cases}$$

where $A(t,a)$ is the advertising, i.e. the manufacturer's control, and $\mu(a)$ is a positive function that accounts for the depreciation of the goodwill stock as time goes by. Such decay may be caused by several factors, as the competition of other producers for instance, but, in any case, these factors are not discussed in the model. The derivation of the Nerlove-Arrow dynamics is similar to the one showed for the linear continuous model in the last sections of the first chapter.

Necessary and sufficient conditions

Starting from the concepts introduced in the first chapter, in the context of age-structured problems one has to formulate necessary and sufficient conditions for a control to be an equilibrium of some kind.

Necessary conditions

A set of Pontryagin-type conditions for age-structured infinite horizon problems is given in [18].
Consider the following maximization problem:

$$\max_A \int_0^{+\infty}\int_0^{\omega} \Lambda_M(t,a,G(t,a),A(t,a),P(t,a))\,\mathrm{d}a\mathrm{d}t \tag{35}$$

$$\max_P \int_0^{+\infty}\int_0^{\omega} \Lambda_R(t,a,G(t,a),A(t,a),P(t,a))\,\mathrm{d}a\mathrm{d}t \tag{36}$$

subject to:

$$\begin{cases} \partial_t G(t,a) + \partial_a G(t,a) = A(t,a) - \mu(a)G(t,a), & a \in [0,\omega] \\ G(0,a) = \varphi(a), & a \in [0,\omega] \\ G(t,0) = 0, & t \geq 0 \end{cases} \tag{37}$$

Here,

- $\omega > 0$ is fixed and it represents a sort of "maximum age that an individual can grow up to", $(t,a) \in [0,+\infty) \times [0,\omega]$;
- G is the goodwill;
- P is the promotion, i.e. the retailer’s control;
- A is the advertising, i.e. the manufacturer’s control.

The Hamiltonian functions are given by

$$\mathscr{H}_M(t,a,G,\xi_M,A,P) = \Lambda_M(t,a,G,A,P) + \xi_M(t,a)(A(t,a) - \mu(a)G(t,a)) \tag{38}$$

$$\mathscr{H}_R(t,a,G,\xi_R,A,P) = \Lambda_R(t,a,G,A,P) + \xi_R(t,a)(A(t,a) - \mu(a)G(t,a)) \tag{39}$$

The *fundamental solution* of the first equation in system (37) is the solution of

$$\begin{cases} DX(\tau,\lambda(\tau)) = -\mu(\lambda(\tau))X(\tau,\lambda(\tau)) \\ X(0,a-t) = X(\omega-a+t,\omega) = 1 \end{cases}$$

where $\lambda(\tau) = a - t + \tau$ is the characteristic line through $(t,a) \in [0,+\infty) \times [0,\omega]$. In other words,

$$X(t,a) = e^{-\int_0^{\min(t,a)} \mu(a-t+\tau)\,\mathrm{d}\tau}$$

Define the following functions

$$\hat{\xi}_M(t,a) = \left[\int_a^{\omega} \partial_G \Lambda_M(t,a,G,A,P)(t-a+\alpha)X(t-a+\alpha,\alpha)\,\mathrm{d}\alpha\right] X^{-1}(t,a)$$

$$\hat{\xi}_R(t,a) = \left[\int_a^{\omega} \partial_G \Lambda_R(t,a,G,A,P)(t-a+\alpha)X(t-a+\alpha,\alpha)\,\mathrm{d}\alpha\right] X^{-1}(t,a)$$

Make the following assumptions:

1. the functions μ, A, P and Λ, together with the partial derivatives $\partial_G\Lambda_M, \partial_G\Lambda_R$, are locally bounded, measurable in t and a and locally Lipschitz-continuous, for any fixed value of the other variables.

2. there exists two measurable functions $\sigma_1, \sigma_2 : [0,+\infty) \to [0,+\infty)$ such that

$$|\partial_G \Lambda_M(t,a,G,P,A)| \le \sigma_1(t),$$

$$|\partial_G \Lambda_R(t,a,G,P,A)| \le \sigma_2(t),$$

   $\forall (t,a) \in [0,+\infty) \times [0,\omega]$.

Denote by $\xi_M = \xi_M(t,a)$ and $\xi_R = \xi_R(t,a)$ the adjoint function of the manufacturer and of the retailer, respectively, corresponding to the goodwill G, so that they satisfy

$$-(\partial_t + \partial_a)\xi_M(t,a) = -\mu(a)\xi_M(t,a) + \partial_G\Lambda_M(t,a,G,P,A) \tag{40}$$

$$-(\partial_t + \partial_a)\xi_R(t,a) = -\mu(a)\xi_R(t,a) + \partial_G\Lambda_R(t,a,G,P,A) \tag{41}$$

and $\xi_M(t,\omega) = \xi_R(t,\omega) = 0$ for any t.

The following necessary optimality condition holds:

Theorem 4.2.1: Pontryagin's principle for infinite horizon AS problem

Suppose that the two aforementioned assumptions are satisfied. Let $(P^\star, A^\star, G^\star)$ be catching up optimal for the given problem. Then, the function $\hat{\xi}_M, \hat{\xi}_R$ are in $L^\infty_{\mathrm{loc}}([0,+\infty) \times [0,\omega])$, they are absolutely continuous along the characteristic line $t - a \equiv$ const. and they satisfy the adjoint equations. Moreover, the following maximization conditions hold:

$$\mathscr{H}_M(t,a,G^\star,A^\star,P^\star,\xi_M) = \sup_A \mathscr{H}_c(t,a,G^\star,A,P^\star,\xi_M)$$

$$\mathscr{H}_R(t,a,G^\star,A^\star,P^\star,\xi_R) = \sup_P \mathscr{H}_c(t,a,G^\star,A,P^\star,\xi_R)$$

Proof. Only the idea will be given: see [18], section 5, for the details. First, one can show that the adjoint functions $\hat{\xi}_M(t,\cdot)$ and $\hat{\xi}_R(t,\cdot)$ give the main term of the effect of a disturbance $\delta = \delta(a)$ of the state $\hat{G}(t,\cdot)$ on the objective values. Therefore, for an arbitrary $\tau \in [0,+\infty)$, one considers a disturbance $\delta = \delta(a)$ of the state $\hat{G}(\tau,\cdot)$.

The perturbation of the objective values in the interval $[\tau, T]$ $(T > \tau)$ may be then linearized as

$$\int_0^{\omega} \xi_M^T(t,a)\delta(a)\mathrm{d}a + \text{"rest terms"},$$

$$\int_0^{\omega} \xi_R^T(t,a)\delta(a)\mathrm{d}a + \text{"rest terms"},$$

for some ξ_M^T, ξ_R^T whose representations, in terms of the fundamental matrix X, may be found. Then, using the third of the aforementioned assumptions, one shows that $\xi_M^T(t,\cdot)$ and $\xi_R^T(t,\cdot)$ converge to $\hat{\xi}_M$ and $\hat{\xi}_R$, respectively.
Afterwards, one apply a needle-type variation of the controls on $[\tau-\alpha,\tau]$, which results in a specific disturbance δ of $\hat{G}(\tau,\cdot)$. One represents the direct effect of this variation on the objective value (that is, on $[\tau-\alpha,\tau]$) and the indirect effect (resulting from δ) in terms of the Hamiltonians $\mathscr{H}_M$ and $\mathscr{H}_R$. Finally, one uses the definition of catching up optimality to get the maximization conditions in the theorem. $\square$

Sufficient conditions

In [26], Grosset and Viscolani formulate the notion of age-structured and linear state games, and prove that the sufficient conditions in infinite time horizon problems, proposed by Krastev in [19], apply. In the following lines, these results are showed in the particular case of a linear state game.

Theorem 4.3.1: Arrow-type conditions for infinite-horizon AS problems

Let $(G^\star(t,a), P^\star(t,a), A^\star(t,a))$ *be a triple where the first is an admissible state and the last two are admissible controls for the age-structured control problem (35)-(37). Suppose that there exist* ξ_M *and* ξ_R *solutions of (40) for this triple. Assume that this triple satisfies the necessary conditions described in the previous section, with the same notation used there. Also, assume that the maximized Hamiltonians* $\mathscr{H}_M(t,a,G,P^\star,\xi) := \sup_A \mathscr{H}(t,a,G,A,P^\star,\xi_M)$ *and* $\mathscr{H}_R(t,a,G,P^\star,\xi) := \sup_A \mathscr{H}(t,a,G,A,P^\star,\xi_R)$ *are jointly convex with respect to* G, P, ξ_M *and* ξ_R*. Then,* $(G^\star(t,a), P^\star(t,a), A^\star(t,a))$ *is:*

- *overtaking optimal, if we add the assumption that, for each admissible triple* (G,P,A)*, there exists a finite number* τ' *such that*

$$\xi_{M,R}(\tau,a)(G(\tau,a)-G^\star(\tau,a)) \geq 0, \tag{42}$$

  for a.e. $a \in [0,\omega]$, $\tau \geq \tau'$*;*

- *catching up optimal, if one adds, instead of (42), the assumption*

$$\liminf_{t\to+\infty} \int_0^\omega \xi_{M,R}(t,a)(G(t,a)-G^\star(t,a))\,\mathrm{d}a \geq 0; \tag{43}$$

- *sporadically catching up optimal if, instead of (42) or (43), the following assumption is made:*

$$\limsup_{\tau\to+\infty} \int_0^\omega \xi_{M,R}(t,a)(G(t,a)-G^\star(t,a))\,\mathrm{d}a \geq 0 \tag{44}$$

See [19], Section 4, for the proof. Notice that equations (43) and (44) are satisfied, for example, if all the admissible G are bounded and $\lim_{t\to+\infty}\xi_{M,R}(t,a) = 0$ uniformly w.r.t. $a \in [0,\omega]$.

Here, it is useful to observe that the following holds: suppose that $G^\star, P^\star, A^\star, \mathscr{H}_M$ and $\mathscr{H}_R$ satisfy the conditions in the previous theorem, and that there exists $\xi_{M,s}(a), \xi_{R,s}(a)$ **bounded** solutions of the following system

$$\begin{cases} \partial_a \xi_{M,s}(a) = (\mu(a)+\rho)\xi_{M,s}(a) - \partial_G \Lambda_M(t,a,G,P^\star,A^\star) \\ \partial_a \xi_{R,s}(a) = (\mu(a)+\rho)\xi_{R,s}(a) - \partial_G \Lambda_R(t,a,G,P^\star,A^\star) \\ \xi_{M,s}(\omega) = \xi_{R,s}(\omega) = 0 \end{cases} \tag{45}$$

associated to the current-value Hamiltonians

$$\mathscr{H}^c_M(t,a,G,P,\xi_{M,s}) = \Lambda_M(t,a,G,P,A) + \xi_{M,s}(a)[A-\mu(a)G]$$

$$\mathscr{H}^c_R(t,a,G,P,\xi_{R,s}) = \Lambda_R(t,a,G,P,A) + \xi_{R,s}(a)[A-\mu(a)G]$$

Then, the functions $\xi_M(t,a) := e^{-\rho t}\xi_{M,s}(a)$ and $\xi_R(t,a) := e^{-\rho t}\xi_{R,s}(a)$ satisfy (40). Indeed, by equations (38),

$$\begin{aligned} e^{-\rho t}\mathcal{H}^c_{M,R}(a,G,A,P,\xi_{M,s}) =& e^{-\rho t}(\Lambda_{M,R}(t,a,G,P,A) + \xi_{M,R,s}(a)[A-\mu(a)G]) = \\ =& e^{-\rho t}\Lambda_{M,R}(t,a,G,P,A) + \xi_{M,R}(t,a)[A-\mu(a)G] = \\ =& \mathcal{H}_{M,R}(t,a,G,A,P,\xi) \end{aligned}$$

Being $e^{-\rho t}$ positive and independent on A and P, to maximize $\mathcal{H}^c_M$ with respect to A and $\mathcal{H}^c_R$ with respect to P is the same as maximizing $\mathcal{H}_M$ and $\mathcal{H}_R$ with respect to the same variables. Hence, optimal $A^\star$ and $P^\star$ are the same for the two couple of Hamiltonians. Now, assume that the triple $(G^\star, \xi_{M,s}, \xi_{R,s})$ solve (37) and (45). Then, being $\xi_{M,s}(a)$ and $\xi_{R,s}(a)$ bounded by hypothesis, one has

$$\lim_{t\to+\infty} \xi_{M,R}(t,a) = 0$$

and $\xi_{M,R}(t,a)$ satisfies (40), so one can conclude that (42) and the followings hold.

A "strategy" to find OLNEs

In [26], Section 3, one may find a nice resume of the main steps to take in order to find a OLNE, when facing a linear-state age-structured differential game. Such a scheme is structured as follows, for a two-player game:

1. find the best response of the retailer, which is assumed to be well-defined, by maximizing $\mathscr{H}_R^c(t,a,G,A,P,\xi_R)$ with respect to P:

$$P^\star(t,a,G,A,\xi_R) := \arg\max_P\{\mathscr{H}_R^c(t,A,G,P,\xi_R)\};$$

   do the same for the manufacturer, by maximizing $\mathscr{H}_M^c$ with respect to A, so as to get

$$A^\star(t,a,G,P,\xi_M) := \arg\max_A\{\mathscr{H}(t,A,G,P,\xi_M)\};$$

2. Find a solution $(P^\star, A^\star)$ of

$$\begin{cases} P = P^\star(t,a,G,A,\xi_R) \\ A = A^\star(t,a,G,P,\xi_M) \end{cases},$$

   assuming the existence and uniqueness of such solution.

3. Solve the following equation:

$$\begin{cases} \partial_t G(t,a) + \partial_a G(t,a) = A^\star(t,A) - \mu(a)G(t,a) \\ \partial_t \xi_M(t,a) + \partial_a \xi_M(t,a) = \mu(a)\xi_M(t,a) - \partial_G \Lambda_M(t,a,G,P^\star,A^\star) \\ \partial_t \xi_R(t,a) + \partial_a \xi_R(t,a) = \mu(a)\xi_R(t,a) - \partial_G \Lambda_R(t,a,G,P^\star,A^\star) \\ G(t,\omega) = G_\omega \\ \xi_M(t,\omega) = \xi_R(t,\omega) = 0 \end{cases}$$

If one can find a unique solution to the last systems, then the sufficient conditions imply that the couple $(A^\star, P^\star)$ is an open-loop sub-game perfect Nash equilibrium.

The model

The considered model is a marketing channel where the retailer promotes the manufacturer's product, while the manufacturer spends on advertising to build a stock of goodwill. Sales will depend on goodwill and promotion.
The problem will be formalized both *à la* Nash and *à la* Stackelberg.
The starting point is the finite horizon model discussed in [13]. An age structure is introduced, as both the advertising and the promotion strategy are assumed to be dependent on the age of the consumers.
Let $\omega \in \mathbb{R}^{>0}$ be fixed. The goodwill is controlled by the manufacturer's advertising effort $A = A(t,a)$ at time t for people of age a, and it follows the dynamics described in equation (37), which is here recalled:

$$\begin{cases} \partial_t G(t,a) + \partial_a G(t,a) = A(t,a) - \mu(a)G(t,a), & a \in [0,\omega] \\ G(t,\omega) = G_\omega, & a \in [0,\omega] \end{cases}$$

Assume that the sales function is linear with respect to the goodwill, i.e.

$$Q(a,P,G) = \beta(a)P + \gamma(a)G.$$

Here, $P = P(t,a)$ is the promotion effort and represents the retailer's control ($\beta = \beta(a) > 0$, $\forall a \in [0,\omega]$, stands for the marginal sales with respect to promotion). Suppose that both the advertising and promotion cost are quadratic:

$$C_M(t,a) = k_M(a)\frac{A^2(t,a)}{2}$$

$$C_R(t,a) = k_R(a)\frac{P^2(t,a)}{2}$$

The advertising costs are sustained by the manufacturer, while the promotion costs are sustained by the manufacturer for a fraction $r \in (0,1)$ and by the retailer for the remaining $1-r$ part.
Under these assumptions, the manufacturer's profit is

$$\begin{aligned} J_M(A,r) = \int_0^{+\infty}\int_0^{\omega} e^{-\rho t}\Big\{\pi_M(a)[\beta(a)P(t,a)+\gamma(a)G(t,a)]+ \\ -\frac{k_M(a)}{2}A^2(t,a) - r\frac{k_R(a)}{2}P^2(t,a)\Big\}\,\mathrm{d}a\mathrm{d}t \end{aligned} \tag{46}$$

while the retailer's one is

$$\begin{aligned} J_R(P) = \int_0^{+\infty}\int_0^{\omega} e^{-\rho t}\{\pi_R(a)[\beta(a)P(t,a)+\gamma(a)G(t,a)]+ \\ -(1-r)\frac{k_R(a)}{2}P^2(t,a)\}\,\mathrm{d}a\mathrm{d}t, \end{aligned}$$

where

- π_M is the manufacturer's marginal profit, gross to the market expenditure;
- π_R is the retailer's marginal profit, gross to the market expenditure;
- $k_M(a)$ is the advertising cost parameter;
- $k_r(a)$ is the promotion cost parameter.

The current-value Hamiltonian for the manufacturer is

$$\begin{aligned}\mathcal{H}^c_M(t,a,G,A,P,\xi) =&\pi_M(a)[\beta(a)P+\gamma(a)G]-\frac{k_M(a)}{2}A^2+\\ &-r\frac{k_R(a)}{2}P^2+\xi_M[A-\mu(a)G]\end{aligned}$$

while the one of the retailer is

$$\begin{aligned}\mathcal{H}^c_R(t,a,G,A,P,\xi) =&\pi_R(a)[\beta(a)P+\gamma(a)G]+\\ &-(1-r)\frac{k_R(a)}{2}P^2+\xi_R(A-\mu(a)G)\end{aligned}$$

So, the best response function for the manufacturer is:

$$A^\star(r,a,G,P,\xi_M) = \arg\max_A\{\mathcal{H}^c_M(t,a,G,A,P,\xi_M)\} = \frac{\xi_M}{k_M(a)} \tag{47}$$

while the one for the retailer is

$$P^\star(t,a,G,A,\xi_R) = \arg\max_P\{\mathcal{H}^c_R(t,a,G,A,P,\xi)\} = \frac{\pi_R(a)\beta(a)}{(1-r)k_R(a)} \tag{48}$$

This shows that the goodwill and the objective functional don't depend on ξ_R. Being ξ_M the only adjoint function to have an impact in the following calculation, for the sake of simplicity the subscript M for ξ_M will be just omitted:

$$\xi_M \Longrightarrow \xi$$

Hence, one needs to solve:

$$\begin{cases}\partial_t G(t,a)+\partial_a G(t,a) = \frac{\xi(t,a)}{k_M(a)}-\mu(a)G(t,a)\\ \partial_t\xi(t,a)+\partial_a\xi(t,a) = (\mu(a)+\rho)\xi(t,a)-\pi_M(a)\gamma(a)\\ G(t,\omega)=G_\omega\\ \xi(t,\omega)=0\end{cases} \tag{49}$$

Suppose that μ is integrable along any characteristic lines $t-a\equiv cost$. Given $(t,a)\in[0,+\infty)\times[0,\omega]$, the characteristic line $\lambda(\tau)=a-t+\tau$ through (t,a) intersects $[0,+\infty)\times\{\omega\}$ for $\hat{\tau}=\omega-a+t$. Then, the second equation in (49) may be re-written as the ODE:

$$\mathrm{D}\xi(\tau,\lambda(\tau)) = (\mu(\lambda(\tau))+\rho)\xi(\tau,\lambda(\tau))-\pi_M(\lambda(\tau))\gamma(\lambda(\tau))$$

Hence

$$\xi(\tau,\lambda(\tau)) = -e^{\int_0^{\lambda(\tau)}\mu(\sigma)\,\mathrm{d}\sigma+\rho\tau}\int_{\hat{\tau}}^{\tau}\pi_M(\lambda(\sigma_1))\gamma(\lambda(\sigma_1))e^{-\int_0^{\lambda(\sigma_1)}\mu(\sigma_2)\,\mathrm{d}\sigma_2-\rho\sigma_2)}\,\mathrm{d}\sigma_1$$

As the characteristic line passes through (t,a), one eventually finds

$$\begin{aligned}\xi(t,a) = \xi(t,\lambda(t)) = & e^{\int_0^{a}\mu(\alpha)\,\mathrm{d}\alpha+\rho t}\cdot \\ & \cdot\int_t^{\omega+t-a}\pi_M(a-t+\tau)\gamma(a-t+\tau)e^{-\int_0^{a-t+\tau}\mu(\sigma)\,\mathrm{d}\sigma-\rho\tau}\,\mathrm{d}\tau\end{aligned} \tag{50}$$

From equation (47), one gets:

$$A^\star(t,a) = \frac{e^{\int_0^{a}\mu(\alpha)\,\mathrm{d}\alpha+\rho t}}{k_M(a)}\int_t^{\omega+t-a}\pi_M(a-t+\tau)\gamma(a-t+\tau)e^{-\int_0^{a-t+\tau}\mu(\sigma)\,\mathrm{d}\sigma-\rho\tau}\,\mathrm{d}\tau \tag{51}$$

Thus, the first equation in (49) becomes

$$\begin{aligned}&\partial_t G(t,a)+\partial_a G(t,a) = -\mu(a)G(t,a)+ \\ &+\frac{e^{\int_0^{a}\mu(\alpha)\,\mathrm{d}\alpha+\rho t}}{k_M(a)}\int_t^{\omega+t-a}\pi_M(a-t+\tau)\gamma(a-t+\tau)e^{-\int_0^{a-t+\tau}\mu(\sigma)\,\mathrm{d}\sigma-\rho\tau}\,\mathrm{d}\tau\end{aligned}$$

Using again the aforementioned characteristic $\lambda(\tau)$, one may re-write this equation as

$$\begin{aligned}&\mathrm{D}G(\tau,\lambda(\tau))+\mu(\lambda(\tau))G(\tau,\lambda(\tau)) = \\ &=\frac{e^{\int_0^{\lambda(\tau)}\mu(\sigma)\,\mathrm{d}\sigma+\rho\tau}}{k_M(\lambda(\tau))}\int_\tau^{\hat{\tau}}\pi_M(\lambda(\sigma))\gamma(\lambda(\sigma))e^{-\int_0^{\lambda(\sigma)}\mu(\sigma_1)\,\mathrm{d}\sigma_1-\rho\sigma}\,\mathrm{d}\sigma\end{aligned}$$

Thus

$$\begin{aligned}G(\tau,\lambda(\tau)) = & e^{-\int_0^{\lambda(\tau)}\mu(\sigma)\,\mathrm{d}\sigma}\int_{\hat{\tau}}^{\tau}\frac{e^{2\int_0^{\lambda(\sigma)}\mu(\sigma_1)\,\mathrm{d}\sigma_1+\rho\sigma}}{k_M(\lambda(\sigma))}\cdot \\ & \cdot\int_\sigma^{\hat{\tau}}\pi_M(\lambda(\sigma_1))\gamma(\lambda(\sigma_1))e^{-\int_0^{\lambda(\sigma_1)}\mu(\sigma_2)\,\mathrm{d}\sigma_2-\rho\sigma_1}\,\mathrm{d}\sigma_1\,\mathrm{d}\sigma\end{aligned}$$

so

$$\begin{aligned}&G^\star(t,a) = G(t,\lambda(t)) = e^{-\int_0^{a}\mu(\sigma)\,\mathrm{d}\sigma}\int_{\omega+t-a}^{t}\frac{e^{2\int_0^{a-t+\tau}\mu(\sigma)\,\mathrm{d}\sigma+\rho\tau}}{k_M(a-t+\tau)}\cdot \\ &\cdot\int_\tau^{\omega+t-a}\pi_M(a-t+\sigma)\gamma(a-t+\sigma)e^{-\int_0^{a-t+\sigma}\mu(\sigma_1)\,\mathrm{d}\sigma_1-\rho\sigma}\,\mathrm{d}\sigma\,\mathrm{d}\tau\end{aligned} \tag{52}$$

By equation (46), the manufacturer's profit is

$$\begin{aligned}J_M(r) = & \int_0^{+\infty}\int_0^{\omega}e^{-\rho t}\Big\{\pi_M(a)\frac{\beta^2(a)\pi_R(a)}{(1-r)k_R(a)}-\frac{r\pi_R^2(a)\beta^2(a)}{2k_R(a)(1-r)^2}+ \\ & +\pi_M(a)\gamma(a)e^{-\int_0^{a}\mu(\sigma)\,\mathrm{d}\sigma}\int_{\omega+t-a}^{t}\frac{e^{2\int_0^{a-t+\tau}\mu(\sigma)\,\mathrm{d}\sigma+\rho\tau}}{k_M(a-t+\tau)}\cdot \\ & \cdot\int_\tau^{\omega+t-a}\pi_M(a-t+\sigma)\gamma(a-t+\sigma)e^{-\int_0^{a-t+\sigma}\mu(\sigma_1)\,\mathrm{d}\sigma_1-\rho\sigma}\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ & -\frac{e^{2\int_0^{a}\mu(\alpha)\,\mathrm{d}\alpha+2\rho t}}{2k_M(a)}\left[\int_t^{\omega+t-a}\pi_M(a-t+\tau)\gamma(a-t+\tau)e^{-\int_0^{a-t+\tau}\mu(\sigma)\,\mathrm{d}\sigma-\rho\tau}\,\mathrm{d}\tau\right]^2\Big\}\,\mathrm{d}a\,\mathrm{d}t\end{aligned}$$

Now, in order to maximize this quantity, one derives it with respect to r and equals the result to 0:

$$\frac{1}{(1-r)^2}\int_0^\omega \pi_M(a)\frac{\beta^2(a)\pi_R(a)}{k_R(a)}\,\mathrm{d}a - \frac{1+r}{(1-r)^3}\int_0^\omega \frac{\pi_R^2(a)\beta^2(a)}{2k_R(a)}\,\mathrm{d}a = 0$$

and finds this way the optimal r:

$$r = \frac{\int_0^\omega \pi_M(a)\frac{\beta^2(a)\pi_R(a)}{k_R(a)}\,\mathrm{d}a - \int_0^\omega \frac{\pi_R^2(a)\beta^2(a)}{2k_R(a)}\,\mathrm{d}a}{\int_0^\omega \pi_M(a)\frac{\beta^2(a)\pi_R(a)}{k_R(a)}\,\mathrm{d}a + \int_0^\omega \frac{\pi_R^2(a)\beta^2(a)}{2k_R(a)}\,\mathrm{d}a} = \frac{\int_0^\omega \frac{\pi_R(a)\beta^2(a)}{k_R(a)}\left(\pi_M(a) - \frac{\pi_R(a)}{2}\right)\mathrm{d}a}{\int_0^\omega \frac{\pi_R(a)\beta^2(a)}{k_R(a)}\left(\pi_M(a) + \frac{\pi_R(a)}{2}\right)\mathrm{d}a}$$

Finally, one should show that the triple $(G^\star, A^\star, P^\star)$, given by equations (52), (51) and (48), is a catching-up optimal solution of the given differential game. As seen in Section "Sufficient conditions" in this chapter, it is enough to show that all the admissible G's are bounded and that $\lim_{t\to+\infty}\xi(t,a) = 0$ uniformly w.r.t. $a \in [0,\omega]$. As for the second condition, it is useful to notice that, through a change of variable $a - t + \tau \mapsto \alpha_1$, equation (50) may be rewritten as

$$\xi(t,a) = e^{\int_0^a \mu(\alpha)\,\mathrm{d}\alpha}\int_a^\omega \pi_M(\alpha_1)\gamma(\alpha_1)e^{-\int_0^{\alpha_1}\mu(\alpha_2)\,\mathrm{d}\alpha_2 - \rho(\alpha_1 - a)}\,\mathrm{d}\alpha_1 \tag{53}$$

Equation (53) shows that, actually, the adjoint function doesn't depend on the time t, but only on the age a. For this reason, for the rest of the thesis, the adjoint function in (53) will be denoted by $\xi_s(a)$, the subscript s standing for "*stable age*". As seen in section "Sufficient conditions", it is enough to prove that $\xi_s(a)$ is bounded. But this is indeed the case: ξ_s is a continuous function on the compact interval $[0,\omega]$. So, in order to prove that the triple $(G^\star, A^\star, P^\star)$ is catching up optimal, it is enough to show that any admissible state G is bounded. By re-writing equation (52) with the same change of variables used before, one gets:

$$G^\star(t,a) = -e^{-\int_0^a \mu(\alpha)\,\mathrm{d}\alpha}\int_a^\omega \frac{\xi_s(\alpha)}{k_M(\alpha)}e^{\int_0^\alpha \mu(\alpha_1)\,\mathrm{d}\alpha_1}\,\mathrm{d}\alpha + G_\omega e^{\int_a^\omega \mu(\alpha)\,\mathrm{d}\alpha} \tag{54}$$

Equation (54) shows that also $G^\star$ doesn't depend on time. As it was said for the co-state function, for the rest of the thesis the goodwill will be denoted with $G_s^\star(a)$, where the subscript "s" stands for "steady state" and remarks the aforementioned dependence.

So, G_s and ξ_s just satisfy the system:

$$\begin{cases} \partial_a G_s(a) = \frac{\xi_s(a)}{k_M(a)} - \mu(a)G_s(a) \\ \partial_a \xi_s(a) = (\mu(a)+\rho)\xi_s(a) - \pi_M(a)\gamma(a) \\ \xi_s(\omega) = 0 \\ G_s(\omega) = G_\omega \end{cases} \tag{55}$$

Notice that, for any strategy (A,P), G_s is a continuous function with compact support $[0,\omega]$, thus it is bounded. The conclusion is that $(G_s^\star, A^\star, P^\star)$ is a catching-up optimal solution of the differential game.

These were the results if the game is treated *à la Nash*. Now, it's immediate to see that the analysis is the same when looking for a Stackelberg equilibrium.

Stackelberg equilibrium

Suppose that at time t the manufacturer announces his control $A(t,a)$ for the individuals of age a. Then, the retailer best response function is

$$P^\star(t,a) = \arg\max_P \mathcal{H}_R^c(t,a,G,A,P,\xi),$$

that is

$$P^\star(t,a) = \frac{\pi_R(a)\beta(a)}{(1-r)k_R(a)},$$

The adjoint function ξ satisfies

$$(\partial_t + \partial_a)\xi(t,a) = -\pi_R(a)\gamma(a) + (\mu(a)+\rho)\xi(t,a)$$

Hence, the manufacturer has to maximize

$$\begin{aligned} J_M(A,r) = \int_0^{+\infty}\int_0^\omega e^{-\rho t}\Big\{\pi_M(a)\left[\beta(a)\frac{\pi_R(a)\beta(a)}{(1-r)k_R(a)} + \gamma(a)G(t,a)\right] + \\ -\frac{k_M(a)}{2}A^2(t,a) - r\frac{\pi_R^2(a)\beta^2(a)}{2(1-r)^2k_R(a)}\Big\}\,\mathrm{d}a\mathrm{d}t \end{aligned}$$

The current-value Hamiltonian function for the manufacturer is

$$\begin{aligned} \mathcal{H}_M^c(t,a,G,\xi,\Gamma,\zeta) =& \pi_M(a)\left[\beta(a)\frac{\pi_R(a)\beta(a)}{(1-r)k_R(a)} + \gamma(a)G(t,a)\right] + \\ & -\frac{k_M(a)}{2}A^2(t,a) - r\frac{\pi_R^2(a)\beta^2(a)}{2(1-r)^2k_R(a)} + \\ & +\Gamma(A(t,a) - (\mu(a)+\rho)G(t,a)) + \\ & +\zeta[-\pi_R(a)\gamma(a) + \xi(t,a)(\mu(a)+\rho)], \end{aligned}$$

where Γ and ζ are the co-state variables associated to G and ξ respectively. Hence, the manufacturer will choose the control

$$A^\star(a,t) = \arg.\max._A \mathcal{H}_M^c(t,a,G,\xi,A,\Gamma,\zeta) = \frac{\Gamma(t,a)}{k_M(a)}, \tag{56}$$

where Γ satisfies

$$(\partial_t + \partial_a)\Gamma(t,a) = -\frac{\partial \mathcal{H}_M^c}{\partial G} = -\pi_M(a)\gamma(a) + (\mu(a)+\rho)\Gamma(t,a)$$

This shows that ξ and Γ satisfy the same PDE, so they are equal. Equation (56), then, proves that $A^\star$ is the same as the one in the open-loop equilibrium. Being the same consideration evident for $P^\star$, one concludes that the Stackelberg equilibrium is the same as the open-loop Nash one.

Simulations

Constant marginal profits

Now, consider again the system of state and adjoint equations in (55). The manufacturer's marginal profit (as well as the one of the retailer) is often non-null only for a certain segment of the population: for the sake of simplicity, one considers

$$\pi_M(a) = \chi_{[\omega_{1,M},\omega_{2,M}]}(a)$$

$$\pi_R(a) = R_\pi \chi_{[\omega_{1,R},\omega_{2,R}]}(a)$$

where $\chi_{[\omega_{1,i},\omega_{2,i}]}$ is the characteristic function on the interval $[\omega_{1,i},\omega_{2,i}]$ and $0 < \omega_{1,i} < \omega_{2,i} < \omega$, $i = M, R$, and R_π is a positive constant (which is needed to take into account that the two profits are generally different in amplitude). These are non-continuous functions, so one in general cannot use the stable age solution found before. Thus, one considers (55) and treats separately the cases $a \in [0, \omega_1)$, $a \in [\omega_{1,M}, \omega_{2,M}]$ and $a \in (\omega_{2,M}, \omega]$. Hence:

$$\begin{cases} \partial_a G_s(a) = \frac{\xi_s(a)}{k_M(a)} - \mu(a) G_s(a) \\ \partial_a \xi_s(a) = (\mu(a)+\rho)\xi_s(a) \\ \lim_{a\to\omega_{1,M}^-} \xi_s(a) = \xi_s(\omega_{1,M}) \\ \lim_{a\to\omega_{1,M}^-} G_s(a) = G_s(\omega_{1,M}) \end{cases} \qquad a \in [0, \omega_{1,M}),$$

$$\begin{cases} \partial_a G_s(a) = \frac{\xi_s(a)}{k_M(a)} - \mu(a) G_s(a) \\ \partial_a \xi_s(a) = (\mu(a)+\rho)\xi_s(a) - \gamma(a) \\ \xi_s(\omega_{2,M}) = \lim_{a\to\omega_{2,M}^+} \xi_s(a) \\ G_s(\omega_{2,M}) = \lim_{a\to\omega_{2,M}^+} G_s(a) \end{cases} \qquad a \in [\omega_{1,M}, \omega_{2,M}]$$

and

$$\begin{cases} \partial_a G_s(a) = \frac{\xi_s(a)}{k_M(a)} - \mu(a) G_s(a) \\ \partial_a \xi_s(a) = (\mu(a)+\rho)\xi_s(a) \\ \xi_s(\omega) = 0 \\ G_s(\omega) = G_\omega \end{cases} \qquad a \in [\omega_{2,M}, \omega].$$

Then,

$$\xi_s(a) = \begin{cases} e^{\int_0^a \mu(\alpha)\,\mathrm{d}\alpha + \rho a} \int_{\omega_{1,M}}^{\omega_{2,M}} e^{-(\int_0^\alpha \mu(\alpha_1)\,\mathrm{d}\alpha_1 + \rho\alpha)} \gamma(\alpha)\,\mathrm{d}\alpha, & a \in [0, \omega_{1,M}), \\ e^{\int_0^a \mu(\alpha)\,\mathrm{d}\alpha + \rho a} \int_{a}^{\omega_{2,M}} e^{-(\int_0^\alpha \mu(\alpha_1)\,\mathrm{d}\alpha_1 + \rho\alpha)} \gamma(\alpha)\,\mathrm{d}\alpha, & a \in [\omega_{1,M}, \omega_{2,M}], \\ 0, & a \in (\omega_{2,M}, \omega] \end{cases}$$

and

$$
\begin{aligned}
G_s(a) = &-e^{\int_0^a \mu(\alpha)\,\mathrm{d}\alpha} \int_a^{\omega_{1,M}} \frac{e^{2\int_0^\alpha \mu(\alpha_1)\,\mathrm{d}\alpha_1 + \rho\alpha}}{k_M(\alpha)} \int_{\omega_{1,M}}^{\omega_{2,M}} e^{-\left(\int_0^{\alpha_1} \mu(\alpha_2)\,\mathrm{d}\alpha_2 + \rho\alpha_1\right)} \gamma(\alpha_1)\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha + \\
&- e^{-\int_0^a \mu(\alpha)\,\mathrm{d}\alpha} \int_{\omega_{1,M}}^{\omega_{2,M}} \frac{e^{2\int_0^\alpha \mu(\alpha_1)\,\mathrm{d}\alpha_1 + \rho\alpha}}{k_M(\alpha)} \int_\alpha^{\omega_{2,M}} e^{-\left(\int_0^{\alpha_1} \mu(\alpha_2)\,\mathrm{d}\alpha_2 + \rho\alpha_1\right)} \gamma(\alpha_1)\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha + \\
&+ G_\omega e^{\int_a^\omega \mu(\alpha)\,\mathrm{d}\alpha},
\end{aligned}
$$

if $a \in [0, \omega_{1,M})$,

$$
\begin{aligned}
G_s(a) = {} & G_\omega e^{\int_a^\omega \mu(\alpha)\,\mathrm{d}\alpha} + \\
&- e^{-\int_0^a \mu(\alpha)\,\mathrm{d}\alpha} \int_a^{\omega_{2,M}} \frac{e^{2\int_0^\alpha \mu(\alpha_1)\,\mathrm{d}\alpha_1 + \rho\alpha}}{k_M(\alpha)} \int_\alpha^{\omega_{2,M}} e^{-\left(\int_0^{\alpha_1} \mu(\alpha_2)\,\mathrm{d}\alpha_2 + \rho\alpha_1\right)} \gamma(\alpha_1)\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha,
\end{aligned}
$$

if $a \in [\omega_{1,M}, \omega_{2,M}]$, and

$$
G_s(a) = G_\omega e^{\int_a^\omega \mu(\alpha)\,\mathrm{d}\alpha},
$$

if $a \in (\omega_{2,M}, \omega]$.

The best response functions for the retailer and for the manufacturer are given by (48) and (47), respectively:

$$
P^\star(t,a) = \frac{\beta(a) R_\pi \chi_{[\omega_{1,R},\omega_{2,R}]}(a)}{(1-r)k_R(a)}
$$

and

$$
A^\star(t,a) = \frac{\xi_s(t,a)}{k_M(a)} =
$$

$$
= \begin{cases}
\frac{e^{\int_0^a \mu(\alpha)\,\mathrm{d}\alpha + \rho a}}{k_M(a)} \int_{\omega_{1,M}}^{\omega_{2,M}} e^{-(\int_0^\alpha \mu(\alpha_1)\,\mathrm{d}\alpha_1 + \rho\alpha)} \gamma(\alpha)\,\mathrm{d}\alpha, & a \in [0, \omega_{1,M}), \\
\frac{e^{\int_0^a \mu(\alpha)\,\mathrm{d}\alpha + \rho a}}{k_M(a)} \int_a^{\omega_{2,M}} e^{-(\int_0^\alpha \mu(\alpha_1)\,\mathrm{d}\alpha_1 + \rho\alpha)} \gamma(\alpha)\,\mathrm{d}\alpha, & a \in [\omega_{1,M}, \omega_{2,M}], \\
0, & a \in (\omega_{2,M}, \omega]
\end{cases}
$$

Thus, the manufacturer's profit is maximized by $r^\star$, satisfying:

$$
\frac{1}{(1-r^\star)^2} \int_{\omega_{1,M}}^{\omega_{2,M}} \frac{e^{-\rho t} R_\pi \beta^2(a) \chi_{[\omega_{1,R},\omega_{2,R}]}(a)}{k_R(a)}\,\mathrm{d}a = \frac{1+r^\star}{(1-r^\star)^3} \int_{\omega_{1,R}}^{\omega_{2,R}} \frac{R_\pi^2 \beta^2(a)}{2k_R(a)}\,\mathrm{d}a,
$$

that is,

$$
r^\star = \max\left\{0, \frac{\int_{\omega_{1,M}}^{\omega_{2,M}} \frac{\beta^2(a)\chi_{[\omega_{1,R},\omega_{2,R}]}(a)}{k_R(a)}\,\mathrm{d}a - \int_{\omega_{1,R}}^{\omega_{2,R}} \frac{R_\pi \beta^2(a)}{2k_R(a)}\,\mathrm{d}a}{\int_{\omega_{1,M}}^{\omega_{2,M}} \frac{\beta^2(a)\chi_{[\omega_{1,R},\omega_{2,R}]}(a)}{k_R(a)}\,\mathrm{d}a + \int_{\omega_{1,R}}^{\omega_{2,R}} \frac{R_\pi \beta^2(a)}{2k_R(a)}\,\mathrm{d}a}\right\}
$$

Notice that, if the manufacturer and the retailer are interested into the same age segment, that is, if $\omega_{1,M} = \omega_{1,R}$ and $\omega_{2,M} = \omega_{2,R}$, then $r^\star = \max\left\{0, \frac{2-R_\pi}{2+R_\pi}\right\}$. This shows that the optimal $r^\star$ is a strictly decreasing function of R_π (see figure (4)): the higher is the marginal profit of the retailer with respect to the manufacturer one, the lower is the participation fraction that is convenient for the manufacturer to spend. Moreover, by asking $r^\star > 0$, one has the upper bound for the value of R_π:

$$R_\pi < \frac{\int_{\omega_{1,M}}^{\omega_{2,M}} \frac{\beta^2(a)\chi_{[\omega_{1,R},\omega_{2,R}]}(a)}{k_R(a)}\,\mathrm{d}a}{\int_{\omega_{1,R}}^{\omega_{2,R}} \frac{\beta^2(a)}{2k_R(a)}\,\mathrm{d}a} = 2,$$

if the retailer and the manufacturer are interested into the same age segment.

Now, in order to make further considerations about the results, suppose that μ, β, γ, k_M

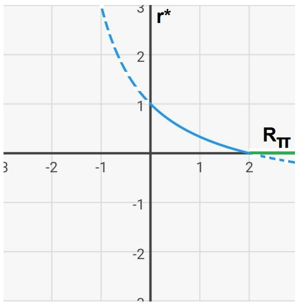

Figure 4: $r^\star$ as a function of R_π. Notice that R_π and $r^\star$ must be positive by definition.

and k_R are constant with respect to the age a. As for k_M and k_R, this might be reasonable: these quantities are the cost for the manufacturer and the retailer, so they might actually be independent of age, even non-null for an age outside the ranges $[\omega_{1,M}, \omega_{2,M}]$ and $[\omega_{1,R}, \omega_{2,R}]$.
The adjoint function becomes

$$\xi_s(a) = \begin{cases} \frac{\gamma}{\mu+\rho}\left[e^{(\mu+\rho)(a-\omega_{1,M})} - e^{(\mu+\rho)(a-\omega_{2,M})}\right], & a \in [0, \omega_{1,M}], \\ \frac{\gamma}{\mu+\rho}\left[1 - e^{(\mu+\rho)(a-\omega_{2,M})}\right], & a \in (\omega_{1,M}, \omega_{2,M}), \\ 0, & a \in [\omega_{2,M}, \omega] \end{cases}$$

and the stable-age goodwill is:

$$G_s(a) = \frac{\gamma}{k_M(\mu+\rho)(2\mu+\rho)}\left[e^{(\mu+\rho)(a-\omega_{1,M})} - e^{(\mu+\rho)(a-\omega_{2,M})} - e^{\mu(\omega_{1,M}-a)}\right] + \\ + \frac{\gamma}{k_M(\mu+\rho)}\left[\frac{e^{\mu(\omega_{1,M}-a)} - e^{\mu(\omega_{2,M}-a)}}{\mu} + \frac{e^{\mu(\omega_{2,M}-a)}}{2\mu+\rho}\right] + G_\omega e^{\mu(\omega-a)} \tag{57}$$

if $a \in [0, \omega_{1,M})$,

$$G_s(a) = \frac{\gamma}{k_M(\mu+\rho)}\left[\frac{1 - e^{\mu(\omega_{2,M}-a)}}{\mu} - \frac{e^{(\mu+\rho)(a-\omega_{2,M})} - e^{\mu(\omega_{2,M}-a)}}{2\mu+\rho}\right] + G_\omega e^{\mu(\omega-a)},$$

if $a \in [\omega_{1,M}, \omega_{2,M}]$, and

$$G_s(a) = G_\omega e^{\mu(\omega-a)},$$

if $a \in (\omega_{2,M}, \omega]$. Notice that $G_s(a)$ and $\xi_s(a)$ are both bounded; one may analytically compute the maximum value of $\xi_s(a)$:

$$0 \le \xi_s(a) \le \frac{\gamma}{\mu+\rho}\left(1 - e^{(\mu+\rho)(\omega_{1,M}-\omega_{2,M})}\right)$$

The behaviour of the goodwill G_s on the age segment $[0, \omega_{1,M}]$ strongly depends on G_ω, i.e. on the desired goodwill at $a = \omega$: if G_ω is "high", then $G_s(a)$ has a minimum in $[0, \omega_{1,M}]$; instead, for low values of G_ω, G_s is strictly increasing on $[0, \omega_{1,M}]$. See figures (5), (6) and (7), where $\omega_{1,M} = 25, \omega_{2,M} = 35$, $\omega = 50$, $\mu = 0.3$, $\rho = 0.05$ and $\gamma = k_M$. Now, the best response functions are

Figure 5: $G_s(a)$ for low values of G_ω.

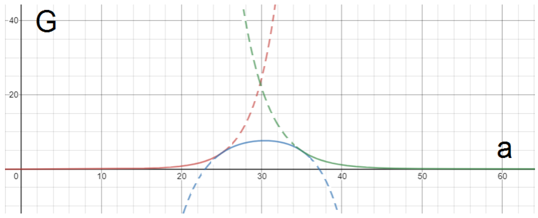

Figure 6: $G_s(a)$ for an almost threshold value of G_ω.

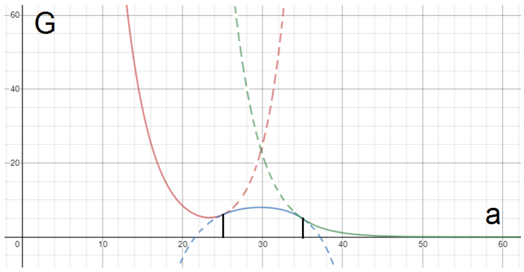

Figure 7: $G_s(a)$ for high values of G_ω.

$$P^\star(a) = \frac{\beta R_\pi}{(1-r)k_R}\chi_{[\omega_{1,R},\omega_{2,R}]}(a) \tag{58}$$

and

$$A^\star(a) = \begin{cases} \frac{\gamma}{k_M(\mu+\rho)}\left[e^{(\mu+\rho)(a-\omega_{1,M})} - e^{(\mu+\rho)(a-\omega_{2,M})}\right], & a \in [0, \omega_{1,M}], \\ \frac{\gamma}{k_M(\mu+\rho)}\left[1 - e^{(\mu+\rho)(a-\omega_{2,M})}\right], & a \in (\omega_{1,M}, \omega_{2,M}), \\ 0, & a \in [\omega_{2,M}, \omega] \end{cases}$$

Notice that $A^\star$ is non-null on $[0, \omega_{1,M}]$: this means that, even if the manufacturer has no marginal profit in that segment age, it is also (increasingly) convenient to spend on advertising there. This is obvious: people will grow older, as time goes by, and progressively get into the segment age $[\omega_{1,M}, \omega_{2,M}]$. Instead, $A^\star$ decreases in the interval $[\omega_{1,M},\omega_{2,M}]$ and vanishes on $[\omega_{2,M},\omega]$: this can be explained with the same arguments used before. See figure (8) for the graph of $A^\star = A^\star(a)$.

Then, if one defines

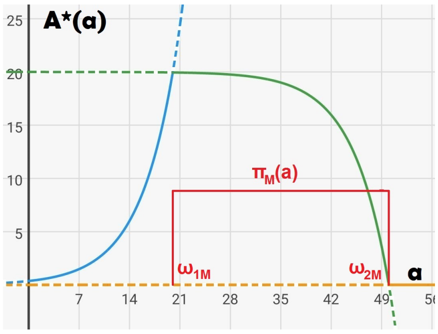

Figure 8: Optimal advertising as a functions of the age. Here $\mu+\rho = 0.2$, $k_M = 0.5$, $\gamma = 1$, $\omega_{1,M} = 20$ and $\omega_{2,M=50}$.

$$l([\omega_{1,R},\omega_{2,R}]\cap[\omega_{1,M},\omega_{2,M}]) = \begin{cases} 0, & \text{if } \omega_{2,R} < \omega_{1,M} \text{ or } \omega_{1,R} > \omega_{2,M} \\ \min\{\omega_{2,M},\omega_{2,R}\} - \max\{\omega_{1,M},\omega_{1,R}\}, & \text{otherwise.} \end{cases}$$

as the length of the overlap between the two age segments $[\omega_{1,M},\omega_{2,M}]$ and $[\omega_{1,R},\omega_{2,R}]$, one has that the optimal $r^\star$ satisfies

$$\frac{\beta^2 R_\pi}{k_R(1-r)^2} l([\omega_{1,R},\omega_{2,R}]\cap[\omega_{1,M},\omega_{2,M}]) = \frac{1+r}{(1-r)^3}\frac{\beta^2 R_\pi^2(\omega_{2,R}-\omega_{1,R})}{2k_R},$$

that is

$$\begin{aligned} r^\star =& \max\left\{0, \frac{\frac{\beta^2 R_\pi}{k_R} l([\omega_{1,R},\omega_{2,R}]\cap[\omega_{1,M},\omega_{2,M}]) - \frac{\beta^2 R_\pi^2(\omega_{2,R}-\omega_{1,R})}{2k_R}}{\frac{\beta^2 R_\pi}{k_R} l([\omega_{1,R},\omega_{2,R}]\cap[\omega_{1,M},\omega_{2,M}]) + \frac{\beta^2 R_\pi^2(\omega_{2,R}-\omega_{1,R})}{2k_R}}\right\} = \\ =& \max\left\{0, \frac{2l([\omega_{1,R},\omega_{2,R}]\cap[\omega_{1,M},\omega_{2,M}]) - R_\pi(\omega_{2,R}-\omega_{1,R})}{2l([\omega_{1,R},\omega_{2,R}]\cap[\omega_{1,M},\omega_{2,M}]) + R_\pi(\omega_{2,R}-\omega_{1,R})}\right\} \end{aligned} \tag{59}$$

Notice that $r^\star > 0$, if and only if

$$\frac{l([\omega_{1,R},\omega_{2,R}]\cap[\omega_{1,M},\omega_{2,M}])}{\omega_{2,R}-\omega_{1,R}} > \frac{R_\pi}{2},$$

This means that it will be convenient for the manufacturer to bear part of the cost of the promotion, only if he and the retailer focus at least partially on the same age segment. Moreover, one notices that the higher is R_π, the higher must be the overlap, and that (as seen above) R_π cannot be greater than 2: if the retailer's marginal profit is much greater than the one of the manufacturer, it won't be convenient for this last one to cover part of the retailer's expenditures on promotion. Equation (59) also shows that, in the constant case here examined, the optimal $r^\star$ depends only on R_π and on the overlap ratio

$$s := \frac{l([\omega_{1,R}, \omega_{2,R}] \cap [\omega_{1,M}, \omega_{2,M}])}{\omega_{2,R} - \omega_{1,R}},$$

while it doesn't depend neither on k_R, nor on β. Obviously, in the more general case ($\beta = \beta(a)$ and $k_R = k_R(a)$), this is false, while there's still a certain dependence on the overlap ratio (as already pointed out, indeed). In the end, notice that the optimal participation rate $r^\star$ is an increasing function of the overlap ratio s (see figure (9)): in particular, if the common age segment of interest is small, then the manufacturer won't find convenient to take on part of the retailer's expenditures on promotion.

By inserting equation (59) in the expression of the promotion function $P^\star$ (equa-

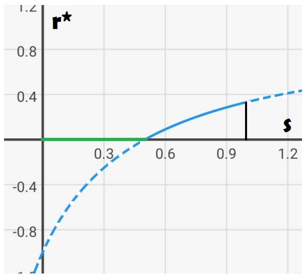

Figure 9: $r^\star$ as a function of the overlap ratio s (remember that $s \in [0,1]$), with $R_\pi = 1$.

tion (58)), one gets

$$P_s^\star(a) = \frac{\beta \chi_{[\omega_{1,R}, \omega_{2,R}]}(a)}{k_R} \cdot \begin{cases} R_\pi, & \text{if } 0 \le s \le \frac{R_\pi}{2}, \\ \left(s + \frac{R_\pi}{2}\right), & \text{if } \frac{R_\pi}{2} < s \le 1 \end{cases}$$

See figure (10) for the dependence of $P_s^\star$ on s, when $a \in [\omega_{1,R}, \omega_{2,R}]$ and $\omega_{1,R}$ and $\omega_{2,R}$ are fixed (hence the variation of the overlap ratio s depends only on $\omega_{1,M}$ and $\omega_{2,M}$).

As pointed out earlier, $A^\star$ is non-null on the interval $[0, \omega_{1,M}]$, which is outside of the support of $\pi_M(a) = \xi_{[\omega_{1,M}, \omega_{2,M}]}(a)$. This is reasonable: as people grow old, the new-born children get into the age segment on which the manufacturer is focused, hence the manufacturer has to invest also on the younger people. But the optimal advertising on the young ages may be much smaller than the one on the age segment $[\omega_{1,M}, \omega_{2,M}]$. So, it makes sense to conclude this section by computing and comparing the average value of the advertising A in the segment $[0, \omega_{1,M})$, the one on $[\omega_{1,M}, \omega_{2,M}]$ and the one $(\omega_{2,M}, \omega]$ (the last one being obviously null).

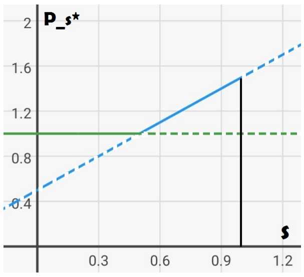

Figure 10: The optimal promotion $P_s^\star$ as a function of $s \in [0,1]$. Here, $R_\pi = 1$ and $\beta = k_R$. Notice that $P^\star$ is constant exactly in the region where $r^\star = 0$ $(0 < s < \frac{R_\pi}{2})$, while it increases with the overlap ratio s in the same region where $r^\star$ is strictly positive. Thus, the promotion is maximum when $[\omega_{1,R}, \omega_{2,R}]$ and $[\omega_{1,M}, \omega_{2,M}]$ are fully overlapped (i.e. one contained into the other, no matter which one).

One has

$$\begin{aligned}\bar{A}_1^\star :=& \frac{1}{\omega_{1,M}} \int_0^{\omega_{1,M}} \frac{e^{\int_0^a \mu(\alpha)\,\mathrm{d}\alpha + \rho a}}{k_M(a)} \int_{\omega_{1,M}}^{\omega_{2,M}} e^{-\left(\int_0^\alpha \mu(\alpha_1)\,\mathrm{d}\alpha_1 + \rho\alpha\right)} \gamma(\alpha)\,\mathrm{d}\alpha = \\ =& \frac{\gamma}{k_M(\mu+\rho)^2} \left[1 - e^{-(\mu+\rho)\omega_{1,M}} - e^{(\mu+\rho)(\omega_{1,M}-\omega_{2,M})} + e^{-(\mu+\rho)\omega_{2,M}}\right]\end{aligned}$$

$$\begin{aligned}\bar{A}_2^\star =& \frac{1}{\omega_{2,M} - \omega_{1,M}} \int_{\omega_{1,M}}^{\omega_{2,M}} \frac{e^{\int_0^a \mu(\alpha)\,\mathrm{d}\alpha + \rho a}}{k_M(a)} \int_a^{\omega_{2,M}} e^{-\left(\int_0^\alpha \mu(\alpha_1)\,\mathrm{d}\alpha_1 + \rho\alpha\right)} \gamma(\alpha)\,\mathrm{d}\alpha = \\ =& \frac{\gamma}{k_M(\mu+\rho)^2(\omega_{2,M}-\omega_{1,M})} \left[e^{(\mu+\rho)(\omega_{1,M}-\omega_{2,M})} - 1 - (\mu+\rho)(\omega_{1,M}-\omega_{2,M})\right]\end{aligned}$$

Thus, the wanted ratio is

$$\begin{aligned}\frac{\bar{A}_2^\star}{\bar{A}_1^\star} =& \frac{\omega_{1,M}}{\omega_{2,M}-\omega_{1,M}} \frac{\int_{\omega_{1,M}}^{\omega_{2,M}} \frac{e^{\int_0^a \mu(\alpha)\,\mathrm{d}\alpha + \rho a}}{k_M(a)} \int_a^{\omega_{2,M}} e^{-\left(\int_0^\alpha \mu(\alpha_1)\,\mathrm{d}\alpha_1 + \rho\alpha\right)} \gamma(\alpha)\,\mathrm{d}\alpha}{\int_0^{\omega_{1,M}} \frac{e^{\int_0^a \mu(\alpha)\,\mathrm{d}\alpha + \rho a}}{k_M(a)} \int_{\omega_{1,M}}^{\omega_{2,M}} e^{-\left(\int_0^\alpha \mu(\alpha_1)\,\mathrm{d}\alpha_1 + \rho\alpha\right)} \gamma(\alpha)\,\mathrm{d}\alpha} = \\ =& \frac{\omega_{1,M}}{\omega_{2,M}-\omega_{1,M}} \frac{e^{(\mu+\rho)(\omega_{1,M}-\omega_{2,M})} - 1 - (\mu+\rho)(\omega_{1,M}-\omega_{2,M})}{\left[1 - e^{(\mu+\rho)(\omega_{1,M}-\omega_{2,M})}\right]\left[1 - e^{-(\mu+\rho)\omega_{1,M}}\right]}\end{aligned}$$

Then, figure (11) shows that the ratio is an increasing function of $\mu+\rho$ and of the width of the interval $[\omega_{1,M}, \omega_{2,M}]$. Notice that the ratio does not diverge neither as $\omega_{2,M} \to \omega_{1,M}^+$, nor as $\mu+\rho \to 0$: indeed, one has respectively

$$\lim_{\omega_{2,M}\to\omega_{1,M}^+} \frac{\bar{A}_2^\star}{\bar{A}_1^\star} = \frac{(\mu+\rho)\omega_{1,M}}{2\left[1 - e^{-(\mu+\rho)\omega_{1,M}}\right]}$$

and

$$\lim_{\mu+\rho\to 0} \frac{\bar{A}_2^\star}{\bar{A}_1^\star} = \frac{1}{2}$$

This last result is important: it shows that, independently of the values $\omega_{1,M}$ and $\omega_{2,M}$, the ratio is smaller than 1, for $\mu+\rho$ small enough. That is, if the goodwill decays at a small enough rate μ, and if the discount rate ρ is small enough, then the

average optimal advertising on the age segment $[0, \omega_{1,M})$ will be higher than the one on $[\omega_{1,M}, \omega_{2,M}]$, even though the manufacturer doesn't profit on the first interval. Shortly said, if people remember about the product for quite a long time, then the aforementioned anticipating effect makes convenient for the manufacturer to invest just on young people.
Also, notice that the ratio does not diverge even when $\omega_{2,M} \to +\infty$ (though, in this model one must have $\omega_{2,M} \leq \omega$):

$$\lim_{\omega_{2,M} \to +\infty} \frac{\bar{A}_2^\star}{\bar{A}_1^\star} = \frac{(\mu+\rho)\omega_{1,M}}{1 - e^{-(\mu+\rho)\omega_{1,M}}}$$

Figure (11) and (12) sum up these results.

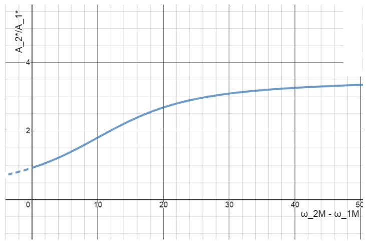

Figure 11: The ratio $\frac{\bar{A}_2^\star}{\bar{A}_1^\star}$ as a function of the amplitude $\omega_{2,M} - \omega_{1,M}$. Remember that this quantity must be smaller than $\omega - \omega_{1,M}$. Here, $\omega_{1,M} = 10$ and $\mu + \rho = 0.35$.

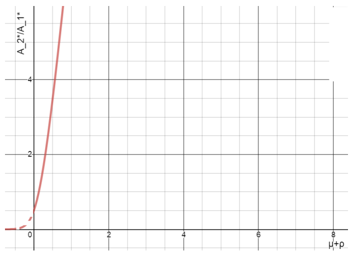

Figure 12: The ratio $\frac{\bar{A}_2^\star}{\bar{A}_1^\star}$ as a function of $\mu + \rho$. Here, $\omega_{1,M} = 10$ and $\omega_{2,M} = 15$.

Triangular marginal profits

Now, the same computations will be made for a different kind of π_M and π_R. Suppose that the manufacturer and the retailer aim at two (generally different) specific ages, a_M and a_R, and their interest for the other ones linearly decreases to zero:

$$\pi_M(a) = \max\{0, -f_M|a - a_M| + 1\} = (1 - f_M|a - a_M|)\chi_{\left]a_M - \frac{1}{f_M}, a_M + \frac{1}{f_M}\right[}(a)$$

$$\pi_R(a) = \max\{0, (-f_R|a - a_R| + 1)R_\pi\} = R_\pi(1 - f_R|a - a_R|)\chi_{\left]a_R - \frac{1}{f_R}, a_R + \frac{1}{f_R}\right[}(a)$$

with $f_M, f_R > 0$ positive constants that express how much the manufacturer and the retailer are focused on addressing people aged a_M and a_R, respectively, and R_π

positive constant that express how big is the retailer's marginal profit with respect to the manufacturer's one. Notice that π_M and π_R have compact support in $[0,\omega]$ if and only if

$$0 < a_i - \frac{1}{f_i} < a_i + \frac{1}{f_i} < \omega,$$

with $i = M, R$.
One has to solve

$$\begin{cases} \partial_a G_s(a) = \frac{\xi_s(a)}{k_M(a)} - \mu(a) G_s(a) \\ \partial_a \xi_s(a) = (\mu(a)+\rho)\xi_s(a) + (f_M|a-a_M|-1)\gamma(a)\chi_{\left]a_M-\frac{1}{f_M},a_M+\frac{1}{f_M}\right[}(a) \\ \xi_s(\omega) = 0 \\ G_s(\omega) = G_\omega \end{cases}$$

Then, one gets

$$\begin{aligned} \xi_s(a) =& e^{\int_0^a \mu(\alpha)\,\mathrm{d}\alpha+\rho a} \int_a^{a_M-\frac{1}{f_M}} e^{-\left(\int_0^\alpha \mu(\alpha_1)+\rho\alpha\right)} (1+f_M\alpha - f_M a_M)\gamma(\alpha)\,\mathrm{d}\alpha + \\ &+ e^{\int_0^a \mu(\alpha)\,\mathrm{d}\alpha+\rho a} \int_{a_M-\frac{1}{f_M}}^{a_M} e^{-\left(\int_0^\alpha \mu(\alpha_1)\,\mathrm{d}\alpha_1+\rho\alpha\right)} (1+f_M\alpha - f_M a_M)\gamma(\alpha)\,\mathrm{d}\alpha + \\ &+ e^{\int_0^a \mu(\alpha)\,\mathrm{d}\alpha+\rho a} \int_{a_M}^{a_M+\frac{1}{f_M}} e^{-\left(\int_0^\alpha \mu(\alpha_1)\,\mathrm{d}\alpha_1+\rho\alpha\right)} (1+f_M a_M - f_M\alpha)\gamma(\alpha)\,\mathrm{d}\alpha, \end{aligned} \tag{60}$$

if $a \in \left[0, a_M - \frac{1}{f_M}\right]$,

$$\begin{aligned} \xi_s(a) =& e^{\int_0^a \mu(\alpha)\,\mathrm{d}\alpha+\rho a} \int_a^{a_M} e^{-\left(\int_0^\alpha \mu(\alpha_1)\,\mathrm{d}\alpha_1+\rho\alpha\right)} (1+f_M\alpha - f_M a_M)\gamma(\alpha)\,\mathrm{d}\alpha + \\ &+ e^{\int_0^a \mu(\alpha)\,\mathrm{d}\alpha+\rho a} \int_{a_M}^{a_M+\frac{1}{f_M}} e^{-\left(\int_0^\alpha \mu(\alpha_1)\,\mathrm{d}\alpha_1+\rho\alpha\right)} (1-f_M\alpha + f_M a_M)\gamma(\alpha)\,\mathrm{d}\alpha, \end{aligned}$$

if $a \in \left(a_M - \frac{1}{f_M}, a_M\right)$,

$$\xi_s(a) = e^{\int_0^a \mu(\alpha)\,\mathrm{d}\alpha+\rho a} \int_a^{a_M+\frac{1}{f_M}} e^{-\left(\int_0^\alpha \mu(\alpha_1)\,\mathrm{d}\alpha_1+\rho\alpha\right)} (1-f_M\alpha + f_M a_M)\gamma(\alpha)\,\mathrm{d}\alpha$$

if $a \in \left[a_M, a_M + \frac{1}{f_M}\right)$, and

$$\xi_s(a) = 0$$

if $a \in \left[a_M + \frac{1}{f_M}, \omega\right]$.
Thus, the stable-age goodwill is

$$G_s(a) = G_\omega e^{\int_a^\omega \mu(\alpha)\,\mathrm{d}\alpha} +$$
$$+ e^{-\int_0^a \mu(\alpha)\,\mathrm{d}\alpha} \int_a^{a_M - \frac{1}{f_M}} \frac{e^{2\int_0^\alpha \mu(\alpha_1)\,\mathrm{d}\alpha_1 + \rho\alpha}}{k_M(\alpha)} \int_{a_M - \frac{1}{f_M}}^{\alpha} e^{-\left(\int_0^{\alpha_1} \mu(\alpha_2)\,\mathrm{d}\alpha_2 + \rho\alpha_1\right)} (f_M a_M - 1 - f_M \alpha_1)\gamma(\alpha_1)\,\mathrm{d}\alpha_1\, \alpha +$$
$$e^{-\int_0^a \mu(\alpha)\,\mathrm{d}\alpha} \int_a^{a_M - \frac{1}{f_M}} \frac{e^{2\int_0^\alpha \mu(\alpha_1)\,\mathrm{d}\alpha_1 + \rho\alpha}}{k_M(\alpha)} \int_{a_M - \frac{1}{f_M}}^{a_M} e^{-\left(\int_0^{\alpha_1} \mu(\alpha_2)\,\mathrm{d}\alpha_2 + \rho\alpha_1\right)} (1 - f_M a_M + f_M \alpha_1)\gamma(\alpha_1)\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha +$$
$$+ e^{-\int_0^a \mu(\alpha)\,\mathrm{d}\alpha} \int_a^{a_M - \frac{1}{f_M}} \frac{e^{2\int_0^\alpha \mu(\alpha_1)\,\mathrm{d}\alpha_1 + \rho\alpha}}{k_M(\alpha)} \int_{a_M}^{a_M + \frac{1}{f_M}} e^{-\left(\int_0^{\alpha_1} \mu(\alpha_2)\,\mathrm{d}\alpha_2 + \rho\alpha_1\right)} (-f_M \alpha_1 + f_M a_M + 1)\gamma(\alpha_1)\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha +$$
$$+ e^{-\int_0^a \mu(\alpha)\,\mathrm{d}\alpha} \int_{a_M - \frac{1}{f_M}}^{a_M} \frac{e^{2\int_0^\alpha} \mu(\alpha_1)\,\mathrm{d}\alpha_1 + \rho\alpha}{k_M(\alpha)} \int_\alpha^{a_M} e^{-\left(\int_0^{\alpha_1} \mu(\alpha_2)\,\mathrm{d}\alpha_2 + \rho\alpha_1\right)} (-f_M \alpha_1 + f_M a_M - 1)\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha +$$
$$+ e^{-\int_0^a \mu(\alpha)\,\mathrm{d}\alpha} \int_{a_M - \frac{1}{f_M}}^{a_M} \frac{e^{2\int_0^\alpha \mu(\alpha_1)\,\mathrm{d}\alpha_1 + \rho\alpha}}{k_M(\alpha)} \int_{a_M}^{a_M + \frac{1}{f_M}} e^{-\left(\int_0^{\alpha_1} \mu(\alpha_2)\,\mathrm{d}\alpha_2 + \rho\alpha_1\right)} (f_M \alpha_1 - f_M a_M - 1)\gamma(\alpha_1)\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha +$$
$$+ e^{-\int_0^a \mu(\alpha)\,\mathrm{d}\alpha} \int_{a_M}^{a_M + \frac{1}{f_M}} \frac{e^{2\int_0^\alpha \mu(\alpha_1)\,\mathrm{d}\alpha_1 + \rho\alpha}}{k_M(\alpha)} \int_\alpha^{a_M + \frac{1}{f_M}} e^{-\left(\int_0^{\alpha_1} \mu(\alpha_2)\,\mathrm{d}\alpha_2 + \rho\alpha_1\right)} (f_M \alpha_1 - f_M a_M - 1)\gamma(\alpha_1)\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha$$

if $a \in \left[0, a_M - \frac{1}{f_M}\right]$,

$$G_s(a) = G_\omega e^{\int_a^\omega \mu(\alpha)\,\mathrm{d}\alpha} +$$
$$+ e^{-\int_0^a \mu(\alpha)\,\mathrm{d}\alpha} \int_a^{a_M} \frac{e^{2\int_0^\alpha \mu(\alpha_1)\,\mathrm{d}\alpha_1 + \rho\alpha}}{k_M(\alpha)} \int_\alpha^{a_M} e^{-\left(\int_0^{\alpha_1} \mu(\alpha_2)\,\mathrm{d}\alpha_2 + \rho\alpha_1\right)} (-1 - f_M \alpha_1 + f_M a_M)\gamma(\alpha_1)\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha +$$
$$+ e^{-\int_0^a \mu(\alpha)\,\mathrm{d}\alpha} \int_a^{a_M} \frac{e^{2\int_0^\alpha \mu(\alpha_1)\,\mathrm{d}\alpha_1 + \rho\alpha}}{k_M(\alpha)} \int_{a_M}^{a_M + \frac{1}{f_M}} e^{-\left(\int_0^{\alpha_1} \mu(\alpha_2)\,\mathrm{d}\alpha_2 + \rho\alpha_1\right)} (f_M \alpha_1 - 1 - f_M a_M)\gamma(\alpha_1)\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha +$$
$$+ e^{-\int_0^a \mu(\alpha)\,\mathrm{d}\alpha} \int_{a_M}^{a_M + \frac{1}{f_M}} \frac{e^{2\int_0^\alpha \mu(\alpha_1)\,\mathrm{d}\alpha_1 + \rho\alpha}}{k_M(\alpha)} \int_\alpha^{a_M + \frac{1}{f_M}} e^{-\left(\int_0^{\alpha_1} \mu(\alpha_2)\,\mathrm{d}\alpha_2 + \rho\alpha_1\right)} (f_M \alpha_1 - f_M a_M - 1)\gamma(\alpha_1)\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha$$

if $a \in \left]a_M - \frac{1}{f_M}, a_M\right[$,

$$\begin{aligned} G_s(a) = & G_\omega e^{\int_a^\omega \mu(\alpha)\,\mathrm{d}\alpha} + e^{-\int_0^a \mu(\alpha)\,\mathrm{d}\alpha} \int_a^{a_M + \frac{1}{f_M}} \frac{e^{2\int_0^\alpha \mu(\alpha_1)\,\mathrm{d}\alpha_1 + \rho\alpha}}{k_M(\alpha)} \cdot \\ & \cdot \int_\alpha^{a_M + \frac{1}{f_M}} e^{-\left(\int_0^{\alpha_1} \mu(\alpha_2)\,\mathrm{d}\alpha_2 + \rho\alpha_1\right)} (f_M \alpha_1 - f_M a_M - 1)\gamma(\alpha_1)\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha \end{aligned}$$

if $a \in \left[a_M, a_M + \frac{1}{f_M}\right)$, and

$$G_s(a) = e^{\int_a^\omega \mu(\alpha)\,\mathrm{d}\alpha} G_\omega,$$

if $a \in \left[a_M + \frac{1}{f_M}, \omega\right]$.
The best response functions are

$$P^\star(a) = \frac{\beta(a)}{(1-r)k_R(a)} R_\pi (1 - f_R |a - a_R|) \chi_{\left]a_R - \frac{1}{f_R}, a_R + \frac{1}{f_R}\right[}(a) \qquad (61)$$

and

$$\begin{aligned} A^{\star}(a) = & \frac{e^{\int_0^a \mu(\alpha)\,\mathrm{d}\alpha + \rho a}}{k_M(a)} \int_a^{a_M - \frac{1}{f_M}} e^{-\left(\int_0^{\alpha} \mu(\alpha_1) + \rho\alpha\right)} (1 + f_M\alpha - f_M a_M)\gamma(\alpha)\,\mathrm{d}\alpha + \\ & + \frac{e^{\int_0^a \mu(\alpha)\,\mathrm{d}\alpha + \rho a}}{k_M(a)} \int_{a_M - \frac{1}{f_M}}^{a_M} e^{-\left(\int_0^{\alpha} \mu(\alpha_1)\,\mathrm{d}\alpha_1 + \rho\alpha\right)} (1 + f_M\alpha - f_M a_M)\gamma(\alpha)\,\mathrm{d}\alpha + \\ & + \frac{e^{\int_0^a \mu(\alpha)\,\mathrm{d}\alpha + \rho a}}{k_M(a)} \int_{a_M}^{a_M + \frac{1}{f_M}} e^{-\left(\int_0^{\alpha} \mu(\alpha_1)\,\mathrm{d}\alpha_1 + \rho\alpha\right)} (1 + f_M a_M - f_M\alpha)\gamma(\alpha)\,\mathrm{d}\alpha, \end{aligned} \tag{62}$$

if $a \in \left[0, a_M - \frac{1}{f_M}\right]$,

$$\begin{aligned} A^{\star}(a) = & \frac{e^{\int_0^a \mu(\alpha)\,\mathrm{d}\alpha + \rho a}}{k_M(a)} \int_a^{a_M} e^{-\left(\int_0^{\alpha} \mu(\alpha_1)\,\mathrm{d}\alpha_1 + \rho\alpha\right)} (1 + f_M\alpha - f_M a_M)\gamma(\alpha)\,\mathrm{d}\alpha + \\ & + e^{\int_0^a \mu(\alpha)\,\mathrm{d}\alpha + \rho a} \int_{a_M}^{a_M + \frac{1}{f_M}} e^{-\left(\int_0^{\alpha} \mu(\alpha_1)\,\mathrm{d}\alpha_1 + \rho\alpha\right)} (1 - f_M\alpha + f_M a_M)\gamma(\alpha)\,\mathrm{d}\alpha, \end{aligned} \tag{63}$$

if $a \in \left(a_M - \frac{1}{f_M}, a_M\right)$,

$$A^{\star}(a) = \frac{e^{\int_0^a \mu(\alpha)\,\mathrm{d}\alpha + \rho a}}{k_M(a)} \int_a^{a_M + \frac{1}{f_M}} e^{-\left(\int_0^{\alpha} \mu(\alpha_1)\,\mathrm{d}\alpha_1 + \rho\alpha\right)} (1 - f_M\alpha + f_M a_M)\gamma(\alpha)\,\mathrm{d}\alpha \tag{64}$$

if $a \in \left[a_M, a_M + \frac{1}{f_M}\right)$, and

$$A^{\star}(a) = 0,$$

if $a \in \left[a_M + \frac{1}{f_M}, \omega\right]$.

The manufacturer's profit is, then, maximized by the following optimal $r^{\star}$:

$$r^{\star} = \frac{\int_{a_R - \frac{1}{f_R}}^{a_R + \frac{1}{f_R}} \frac{\beta^2(a)}{k_R(a)} (1 - f_R|a - a_R|)[(1 - f_M|a - a_M|)\chi_M(a) - R_{\pi}(1 - f_R|a - a_R|)]\,\mathrm{d}a}{\int_{a_R - \frac{1}{f_R}}^{a_R + \frac{1}{f_R}} \frac{\beta^2(a)}{k_R(a)} (1 - f_R|a - a_R|)[(1 - f_M|a - a_M|)\chi_M(a) + R_{\pi}(1 - f_R|a - a_R|)]\,\mathrm{d}a} \tag{65}$$

where $\chi_M(a) := \chi_{\left]a_M - \frac{1}{f_M}, a_M + \frac{1}{f_M}\right[}(a)$.

Now, as done before, suppose that k_M, k_R, β, γ and μ are constant, in order to make further considerations about the results.

Then, equation (60) becomes:

$$\xi_s(a) = f_M\gamma e^{(\mu+\rho)\left(a - a_M + \frac{1}{f_M}\right)} \left(\frac{1 - e^{-\frac{\mu+\rho}{f_M}}}{\mu + \rho}\right)^2 \tag{66}$$

if $a \in \left[0, a_M - \frac{1}{f_M}\right]$,

$$\xi_s(a) = \frac{\gamma}{\mu + \rho}\left(1 - f_M a_M + f_M a + \frac{f_M}{\mu + \rho}\right) + \frac{f_M\gamma e^{(\mu+\rho)(a - a_M)}}{(\mu + \rho)^2}\left(e^{-\frac{\mu+\rho}{f_M}} - 2\right),$$

if $a \in \left(a_M - \frac{1}{f_M}, a_M\right]$,

$$\xi_s(a) = \frac{\gamma(1 + f_M a_M - f_M a)}{\mu+\rho} + \frac{f_M\gamma\left(e^{(\mu+\rho)\left(a-a_M-\frac{1}{f_M}\right)} - 1\right)}{(\mu+\rho)^2},$$

if $a \in \left(a_M, a_M + \frac{1}{f_M}\right)$ and

$$\xi_s(a) = 0$$

if $a \in \left[a_M + \frac{1}{f_M}, \omega\right]$. Observe that ξ_s is a bounded function, just as for the rectangular case:

$$0 \le \xi_s(a) \le \frac{\gamma}{\mu+\rho}\left[1 - \frac{f_M}{\mu+\rho}\log\left(2 - e^{-\frac{\mu+\rho}{f_M}}\right)\right]$$

Also, it is interesting to point out that

$$\lim_{\mu+\rho\to 0^+}\xi_s(a) = \frac{\gamma}{f_M},$$

i.e., as $\mu+\rho \to 0^+$, this maximum tends to the positive value $\frac{\gamma}{f_M}$, and decreases with $\mu+\rho$ (see figure (13)), while the maximum point increases to a_M as $\mu+\rho$ increases.

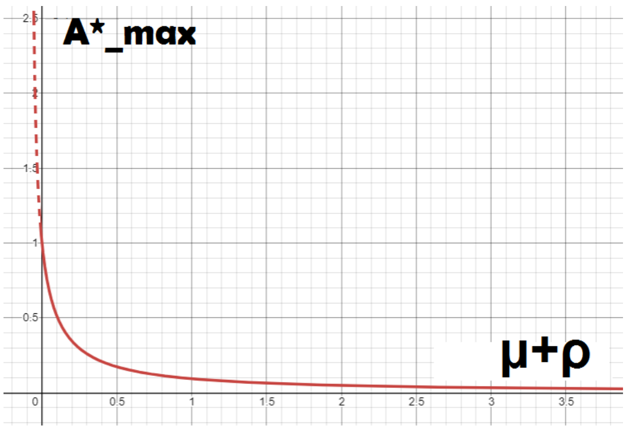

Figure 13: The maximum advertising as a function of $\mu+\rho$. Notice that it is positive for $\mu+\rho \to 0^+$, and that it is decreasing with $\mu+\rho$.

The stable-age goodwill is

$$G_s(a) = \frac{f_M\gamma\left(1 - e^{-\frac{\mu+\rho}{f_M}}\right)^2}{(\mu+\rho)^2}\frac{e^{(\mu+\rho)\left(a-a_M+\frac{1}{f_M}\right)} - e^{\mu\left(a_M-\frac{1}{f_M}-a\right)}}{2\mu+\rho} - \frac{f_M\gamma\rho e^{\mu\left(a_M-\frac{1}{f_M}-a\right)}}{\mu^2(\mu+\rho)^2}\left[1 - e^{\frac{\mu}{f_M}}\right] +$$
$$+G_\omega e^{\mu(\omega-a)} + e^{\mu\left(a_M-\frac{1}{f_M}-a\right)}\frac{f_M\gamma}{(\mu+\rho)^2}\left(e^{-\frac{\mu+\rho}{f_M}} - 2\right)\frac{e^{-\frac{\mu+\rho}{f_M}} - e^{\frac{\mu}{f_M}}}{2\mu+\rho} +$$
$$+\frac{f_M\gamma}{\mu^2(\mu+\rho)}e^{\mu\left(a_M-\frac{1}{f_M}-a\right)}\left(e^{\frac{\mu}{f_M}} - e^{2\frac{\mu}{f_M}}\right) + \frac{f_M\gamma}{(\mu+\rho)^2}e^{\mu(a_M-a)}\left[\frac{e^{-\frac{\mu+\rho}{f_M}} - e^{\frac{\mu}{f_M}}}{2\mu+\rho} - \frac{1-e^{\frac{\mu}{f_M}}}{\mu}\right]$$

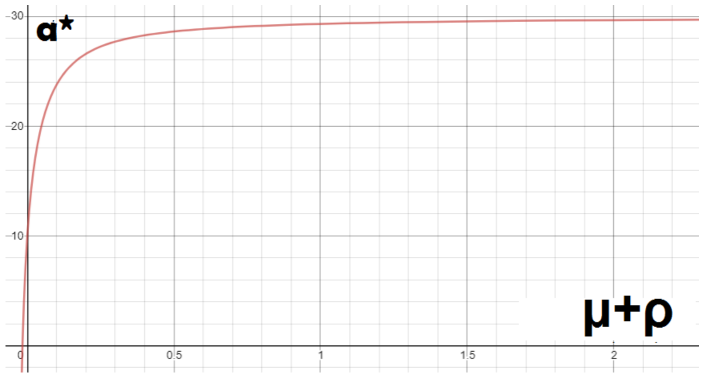

Figure 14: The maximum point of the function $A^\star(a)$, shown with respect to $\mu+\rho$. Here, $a_M = 30$ and $f_M = 0.05$.

if $a \in \left(0, a_M - \frac{1}{f_M}\right)$,

$$G_s(a) = \frac{\gamma}{\mu(\mu+\rho)}\left[1 - f_M a_M + f_M a + \frac{f_M}{\mu+\rho}\left(1 - e^{\mu(a_M-a)}\right)\right] - \frac{f_M\gamma}{\mu^2(\mu+\rho)}\left[1 - e^{\mu(a_M-a)}\right] + \\ + G_\omega e^{\mu(\omega-a)} + \frac{f_M\gamma\left(e^{-\frac{\mu+\rho}{f_M}} - 2\right)}{(\mu+\rho)^2}\frac{e^{(\mu+\rho)(a-a_M)} - e^{\mu(a_M-a)}}{2\mu+\rho} + \\ + \frac{f_M\gamma}{\mu^2(\mu+\rho)}\left[e^{\mu(a_M-a)} - e^{\mu\left(a_M+\frac{1}{f_M}-a\right)}\right] + \frac{f_M\gamma e^{\mu(a_M-a)}}{(\mu+\rho)^2}\left[\frac{e^{-\frac{\mu+\rho}{f_M}} - e^{\frac{\mu}{f_M}}}{2\mu+\rho} - \frac{1 - e^{\frac{\mu}{f_M}}}{\mu}\right]$$

if $a \in \left(a_M - \frac{1}{f_M}, a_M\right)$,

$$G_s(a) = \frac{\gamma}{\mu(\mu+\rho)}\left(1 + f_M a_M - f_M a\right) + G_\omega e^{\mu(\omega-a)} + \frac{f_M\gamma}{\mu^2(\mu+\rho)}\left[1 - e^{\mu\left(a_M+\frac{1}{f_M}-a\right)}\right] + \\ + \frac{f_M\gamma}{(\mu+\rho)^2}\left[\frac{e^{(\mu+\rho)\left(a-a_M-\frac{1}{f_M}\right)} - e^{\mu\left(a_M+\frac{1}{f_M}-a\right)}}{2\mu+\rho} - \frac{1 - e^{\mu\left(a_M+\frac{1}{f_M}-a\right)}}{\mu}\right]$$

if $a \in \left[a_M, a_M + \frac{1}{f_M}\right)$, and

$$G_s(a) = e^{\mu(\omega-a)}G_\omega,$$

if $a \in \left[a_M + \frac{1}{f_M}, \omega\right]$.

First, notice that, contrary to what happens for the adjoint function ξ_s, here the goodwill depends effectively on the value of μ and ρ, not only on the sum $\mu+\rho$ (that is, the age structure that introduces the function μ hasn't only the effect of "reparameterizing" ρ).

As it happened in the rectangular case, the behaviour of the goodwill strongly depends on the value of G_ω: for "high" values of G_ω, G_s is strictly decreasing (see fig. (17)); instead, for "low" values of G_ω, G_s vanishes on an interval of the form $[0, a_1)$, it is increasing on an interval of the form $[a_1, a_2]$, and then decreases on $[a_2, \omega]$ (see fig. (15)). The manufacturer's best response function is given by

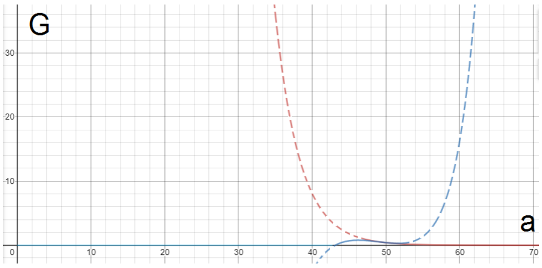

Figure 15: The goodwill for "low" values of G_ω.

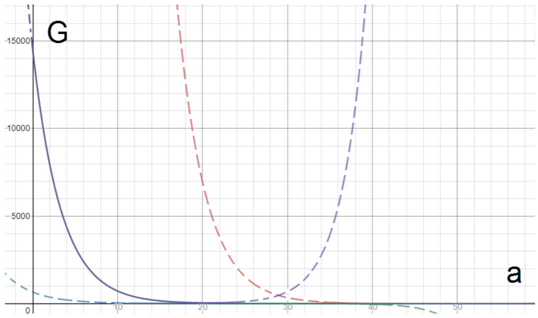

Figure 16: The goodwill for G_ω nearly equal to the "critical" value.

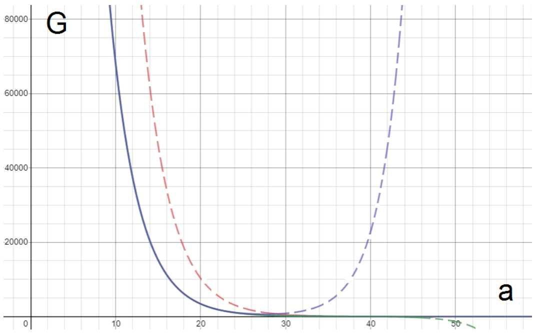

Figure 17: The goodwill for "high" values of G_ω.

$$A^\star(a) = \frac{f_M \gamma e^{(\mu+\rho)\left(a - a_M + \frac{1}{f_M}\right)}}{k_M} \left(\frac{1 - e^{-\frac{\mu+\rho}{f_M}}}{\mu+\rho} \right)^2$$

if $a \in \left[0, a_M - \frac{1}{f_M}\right]$,

$$A^\star(a) = \frac{\gamma}{k_M(\mu+\rho)} \left(1 - f_M a_M + f_M a + \frac{f_M}{\mu+\rho}\right) + \frac{f_M \gamma e^{(\mu+\rho)(a-a_M)}}{k_M(\mu+\rho)^2} \left(e^{-\frac{\mu+\rho}{f_M}} - 2\right),$$

if $a \in \left(a_M - \frac{1}{f_M}, a_M\right]$,

$$A^\star(a) = \frac{\gamma(1 + f_M a_M - f_M a)}{k_M(\mu+\rho)} + \frac{f_M \gamma \left(e^{(\mu+\rho)\left(a - a_M - \frac{1}{f_M}\right)} - 1\right)}{k_M(\mu+\rho)^2},$$

if $a \in \left(a_M, a_M + \frac{1}{f_M}\right)$ and

$$A^\star(a) = 0$$

if $a \in \left[a_M + \frac{1}{f_M}, \omega\right]$.

Notice that $A^\star(a)$ is increasing in the interval $\left[0, a_M - \frac{1}{f_M}\right]$, decreasing in the interval $\left[a_M, a_M + \frac{1}{f_M}\right]$; it has a maximum for a certain $a^\star \in \left(a_M - \frac{1}{f_M}, a_M\right)$ and that it is non-zero even outside the interval $\left]a_M - \frac{1}{f_M}, a_M + \frac{1}{f_M}\right[$ (that is, it is observed the same "anticipating effect" as the one in the constant marginal profit case).

As done in the previous section, it may be interesting to calculate the average op-

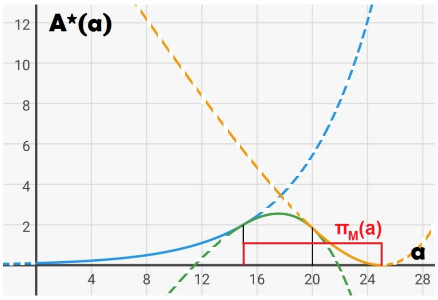

Figure 18: The advertising $A^\star$ as a function of the age a. Here, $\mu+\rho = 0.2$, $\gamma = k_M$, $f_M = 0.2$ and $a_M = 20$: the blue line holds for $a \in [0,15]$, the green one for $a \in (15,20]$, the orange one for $a \in (20,25)$. $A^\star$ vanishes on $[25,\omega]$.

timal advertising on the intervals $\left[0, a_M - \frac{1}{f_M}\right]$, $\left(a_M - \frac{1}{f_M}, a_M\right)$ and $\left[a_M, a_M + \frac{1}{f_M}\right)$. One gets, respectively:

$$A_1^\star = \frac{f_M \gamma}{k_M\left(a_M - \frac{1}{f_M}\right)}\left(\frac{1-e^{-\frac{\mu+\rho}{f_M}}}{\mu+\rho}\right)^2 \frac{1-e^{-(\mu+\rho)\left(a_M - \frac{1}{f_M}\right)}}{\mu+\rho},$$

$$\begin{aligned} A_2^\star =& \frac{\gamma}{k_M(\mu+\rho)}\left(1 - f_M a_M + \frac{f_M}{\mu+\rho}\right) + \frac{f_M^2\gamma}{k_M(\mu+\rho)}\frac{a_M^2 - \left(a_M - \frac{1}{f_M}\right)^2}{2} + \\ &+ \frac{f_M^2 \gamma}{k_M(\mu+\rho)^3}\left(e^{-\frac{\mu+\rho}{f_M}} - 2\right)\left(1 - e^{-\frac{\mu+\rho}{f_M}}\right), \end{aligned}$$

$$\begin{aligned} A_3^\star =& \frac{\gamma}{k_M(\mu+\rho)}(1 + f_M a_M) - \frac{f_M^2\gamma}{k_M(\mu+\rho)}\frac{\left(a_M + \frac{1}{f_M}\right)^2 - a_M^2}{2} + \\ &- \frac{f_M\gamma}{k_M(\mu+\rho)^2} + \frac{f_M^2\gamma}{k_M(\mu+\rho)^3}\left(1 - e^{-\frac{\mu+\rho}{f_M}}\right) \end{aligned}$$

The dependence of the two ratios $\frac{A_2^\star}{A_1^\star}$ and $\frac{A_2^\star}{A_3^\star}$ on the quantity $\mu+\rho$ is shown in figures (19) and (20), respectively.

The retailer's best response function is

$$P^\star(a) = \frac{\beta}{(1-r)k_R} R_\pi (1 - f_R|a - a_R|)\chi_{\left]a_R - \frac{1}{f_R}, a_R + \frac{1}{f_R}\right[}(a)$$

By using equation (65), one finds that the manufacturer's profit J_M is maximized by the participation rate

$$r^\star = \max\left\{\frac{\int_{a_R - \frac{1}{f_R}}^{a_R + \frac{1}{f_R}} (1 - f_R|a - a_R|)[\chi_M(a)(1 - f_M|a - a_M|) - R_\pi(1 - f_R|a - a_R|)]\,\mathrm{d}a}{\int_{a_R - \frac{1}{f_R}}^{a_R + \frac{1}{f_R}} (1 - f_R|a - a_R|)[\chi_M(a)(1 - f_M|a - a_M|) + R_\pi(1 - f_R|a - a_R|)]\,\mathrm{d}a}, 0\right\},$$

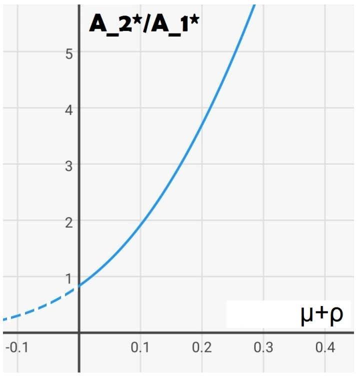

Figure 19: The graph of the function $\mu+\rho \mapsto \frac{A_2^\star}{A_1^\star}(\mu+\rho)$. Notice that, as in the case of constant marginal profits, one has that this ratio can be lower than 1, for $\mu+\rho$ small enough. Here $f_M = 0.2$, $\gamma = k_M$ and $a_M = 20$.

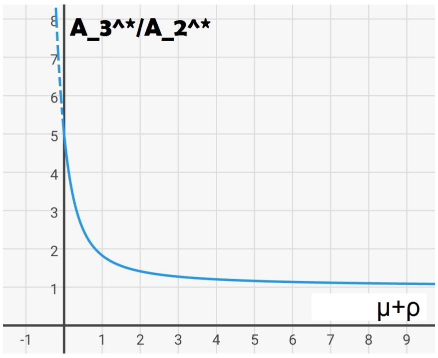

Figure 20: The graph of the function $\mu+\rho \mapsto \frac{A_2^\star}{A_3^\star}(\mu+\rho)$. Notice that this quantity is always bigger than 1, and asymptotically tends to 1, as $\mu+\rho \to +\infty$. This, again, is a consequence of the fact that, as time goes by, people aged a_M will grow older and will be less interesting for the manufacturer.

where $\chi_M(a) = \chi_{\left[a_M - \frac{1}{f_M}, a_M + \frac{1}{f_M}\right]}(a)$.
Notice that the intersection between $\left[a_M - \frac{1}{f_M}, a_M + \frac{1}{f_M}\right]$ and $\left[a_R - \frac{1}{f_R}, a_R + \frac{1}{f_R}\right]$ must be non-empty, otherwise $r = 0$.
Set $M := \left[a_M - \frac{1}{f_M}, a_M + \frac{1}{f_M}\right]$, $R := \left[a_R - \frac{1}{f_R}, a_R + \frac{1}{f_R}\right]$, $M^- := \left[a_M - \frac{1}{f_M}, a_M\right]$, $M^+ := \left(a_M, a_M + \frac{1}{f_M}\right]$, $R^- := \left[a_R - \frac{1}{f_R}, a_R\right]$, $R^+ = \left]a_R, a_R + \frac{1}{f_R}\right]$. Depending on the values of a_M, a_R, f_M and f_R, these sets change their expression and they may even be void. The simplest case is when $a_M = a_R =: a_C$, where the previous expression may be simplified, so as to get:

$$r^\star = \max,\left\{0, \frac{3f_R - 2R_\pi f_R + f_M}{3f_R + 2R_\pi f_R - f_M}\right\},$$

which doesn't depend neither on a_M, nor on a_R, as one might expect. In order to have $0 < r^\star < 1$, one has to ask

$$\frac{f_M}{2} < R_\pi < \frac{3}{2} + \frac{f_M}{2 f_R}$$

that is: the more narrow is the age segment on which the manufacturer focus, the higher must be the maximum marginal profit of the retailer with respect to the one of the manufacturer. Moreover, notice that $r^\star$ is a decreasing function of f_R, when f_M is fixed, and

$$\lim_{f_R \to +\infty} r^\star(f_R) = \frac{3-2R_\pi}{3+2R_\pi} > 0 \iff R_\pi < \frac{3}{2}$$

Instead, for a fixed value of f_R, $r^\star$ is an increasing function of f_M (see figure (22)). Now, consider the case $a_M < a_R$. The values of f_M and f_R give eight different situations, illustrated in figure (23).

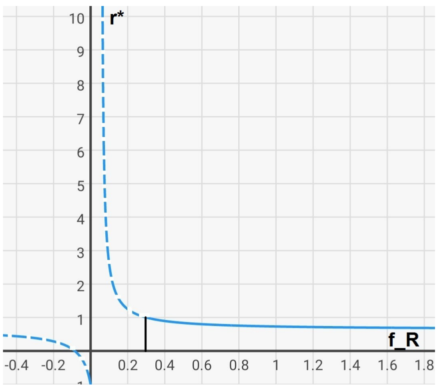

Figure 21: The optimal participation rate $r^\star$ as a function of f_R, for $a_M = a_R$. Here $R_\pi = \frac{2}{3}$ and $f_M = 0.2$.

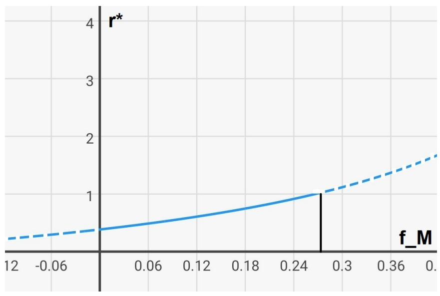

Figure 22: The optimal participation rate $r^\star$ as a function of f_M, for $a_M = a_R$. Here $R_\pi = \frac{2}{3}$ and $f_R = 0.2$.

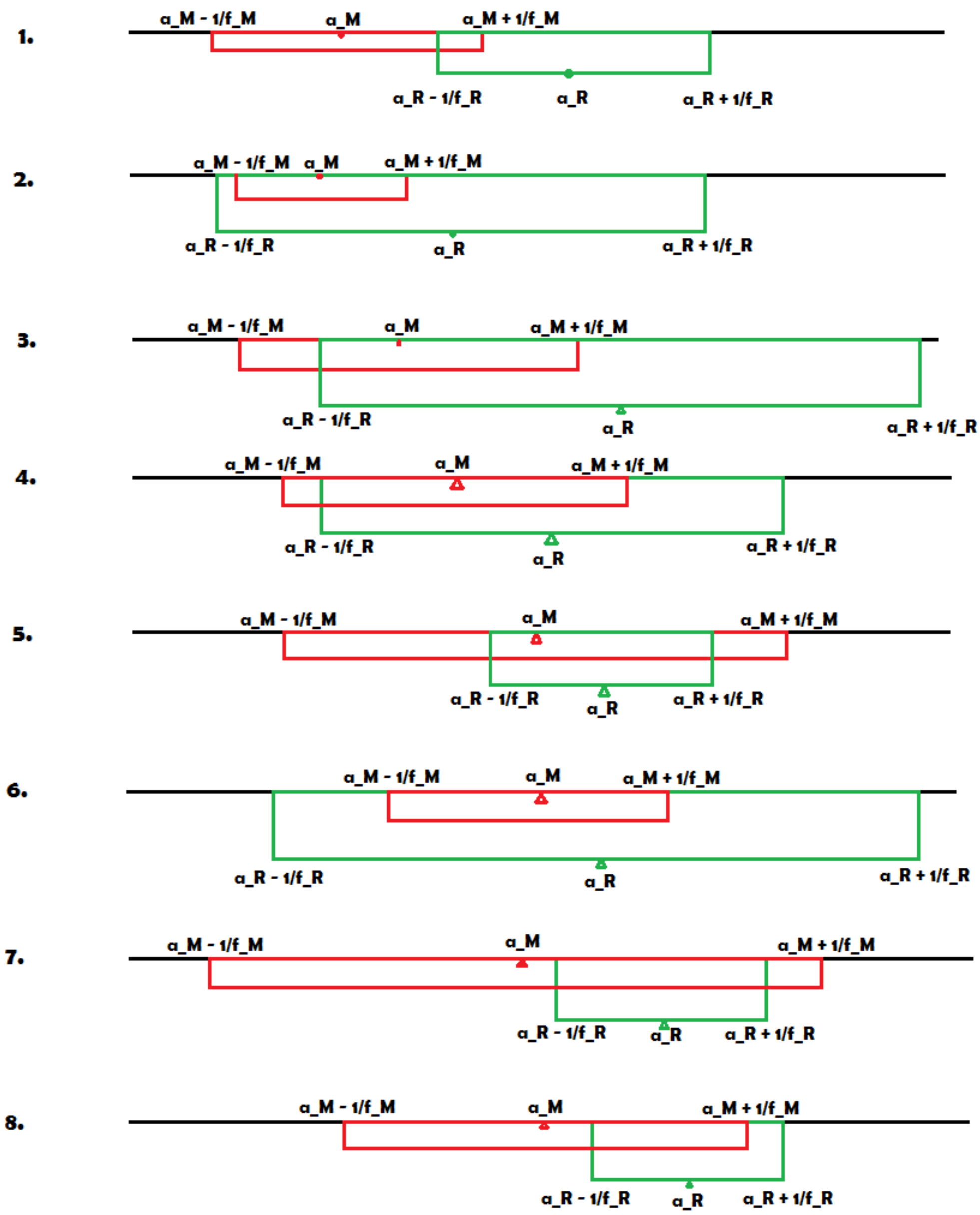

Figure 23: Possible superpositions between the sets $M = \left[a_M - \frac{1}{f_M}, a_M + \frac{1}{f_M}\right]$ (in red) and $R = \left[a_R - \frac{1}{f_R}, a_R + \frac{1}{f_R}\right]$ (in green), when $a_M < a_R$.

Then, referring to the enumeration in figure (23), one has respectively:

1. $R^- \setminus M = \left(a_M + \frac{1}{f_M}, a_R\right)$, $R^+ \setminus M = \left[a_R, a_R + \frac{1}{f_R}\right]$, $M^- \cap R = \emptyset$, $M^+ \cap R^- = \left[a_R - \frac{1}{f_R}, a_M + \frac{1}{f_M}\right]$, $M \cap R^+ = \emptyset$;

2. $R^- \setminus M = \left[a_R - \frac{1}{f_R}, a_M - \frac{1}{f_M}\right) \cup \left(a_M + \frac{1}{f_M}, a_R\right)$, $R^+ \setminus M = \left[a_R, a_R + \frac{1}{f_R}\right]$, $M^- \cap R = \left[a_M - \frac{1}{f_M}, a_M\right]$, $M^+ \cap R^- = \left[a_M, a_M + \frac{1}{f_M}\right]$, $M \cap R^+ = \emptyset$;

3. $R^- \setminus M = \left(a_M + \frac{1}{f_M}, a_R\right]$, $R^+ \setminus M = \left[a_R, a_R + \frac{1}{f_R}\right]$, $M^- \cap R = \left[a_R - \frac{1}{f_R}, a_M\right]$, $M^+ \cap R^- = \left[a_M, a_M + \frac{1}{f_M}\right]$, $M \cap R^+ = \emptyset$;

4. $R^- \setminus M = \emptyset$, $R^+ \setminus M = \left(a_M + \frac{1}{f_M}, a_R + \frac{1}{f_R}\right]$, $M^- \cap R = \left[a_R - \frac{1}{f_R}, a_M\right]$, $M^+ \cap R^- = [a_M, a_R]$, $M \cap R^+ = \left[a_R, a_M + \frac{1}{f_M}\right]$;

5. $R^- \setminus M = \emptyset$, $R^+ \setminus M = \emptyset$, $M^- \cap R = \left[a_R - \frac{1}{f_R}, a_M\right]$, $M^+ \cap R^- = [a_M, a_R]$, $M \cap R^+ = \left[a_R, a_R + \frac{1}{f_R}\right]$;

6. $R^- \setminus M = \left[a_R - \frac{1}{f_R}, a_M - \frac{1}{f_M}\right)$, $R^+ \setminus M = \left[a_M + \frac{1}{f_M}, a_R + \frac{1}{f_R}\right]$, $M^- \cap R = \left[a_M - \frac{1}{f_M}, a_M\right]$, $M^+ \cap R^- = [a_M, a_R]$, $M \cap R^+ = \left[a_R, a_M + \frac{1}{f_M}\right]$;

7. $R^- \setminus M = \emptyset$, $R^+ \setminus M = \emptyset$, $M^- \cap R = \emptyset$, $M^+ \cap R^- = \left[a_R - \frac{1}{f_R}, a_R\right]$, $M \cap R^+ = \left[a_R + \frac{1}{f_R}\right]$;

8. $R^- \setminus M = \emptyset$, $R^+ \setminus M = \left(a_M + \frac{1}{f_M}, a_R + \frac{1}{f_R}\right]$, $M^\setminus \cap R = \emptyset$, $M^+ \cap R^- = \left[a_R - \frac{1}{f_R}, a_R\right]$, $M \cap R^+ = \left[a_R, a_M + \frac{1}{f_M}\right]$.

Hence, for instance, in case 7 one has

$$r^\star = \max\left\{0, 1 - \frac{4R_\pi}{1 + f_M a_M + \frac{2}{3}R_\pi - f_M a_R}\right\} \tag{67}$$

In order to have $r^\star \in (0,1)$, one has the condition

$$\frac{3}{2}\left[(a_R - a_M)f_M - 1\right] < R_\pi < \frac{3}{10}\left[1 + f_M(a_M - a_R)\right]$$

Notice that, by equation (67), $r^\star$ is a decreasing function of R_π (see figure (24)): the greater is the retailer's marginal profit with respect to the manufacturer's one, the lesser it is necessary to the manufacturer to take on part of the retailer's expenditures. Moreover, $r^\star$ is a decreasing function of f_M (see figure (25)): the more narrow is the age segment on which the manufacturer focuses, the lower it is convenient to be the participation rate of the manufacturer in the retailer's expenditures. Also, notice that the participation rate doesn't depend on how narrow is the age segment on which the retailer focuses (i.e., $r^\star$ doesn't depend on f_R): the only constraint is that f_R has to be such that the situation in case 7 (fig. (23)) holds. Now, in order to see the dependence of $r^\star$ with respect to a_M and a_R, it may be interesting to compare the case $a_M = a_R =: a_C$ with the 5$^{\text{th}}$ or the 6$^{\text{th}}$ case in figure (23).

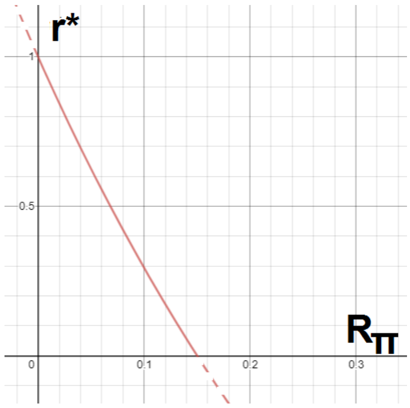

Figure 24: The optimal participation rate $r^\star$ as a function of R_π for the 7th case, with $a_M < a_R$. Here, $a_M = 30$, $a_R = 40$ and $f_M = 0.05$.

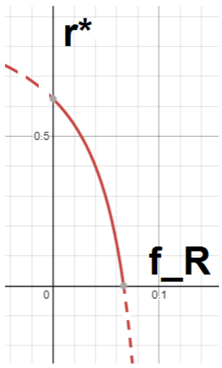

Figure 25: The optimal participation rate $r^\star$ as a function of f_R. Here $a_R = 30$, $a_M = 40$ and $R_\pi = 0.1$.

Consider, for instance, the fifth case, where $r^\star$ results:

$$r^\star = \max\left\{0, 1 - \frac{4R_\pi}{3f_R}\left[2f_M a_M a_R - f_M a_M^2 - f_M f_R a_M a_R^2 + f_M f_R a_M^2 a_R + \frac{1}{f_R} + \frac{2R_\pi}{3f_R} + 2R_\pi f_R a_R - \frac{1}{3} f_M f_R a_M^3 + - f_M a_R^2 + \frac{f_M f_R}{3} a_R^3 - 2R_\pi f_R a_R^2 - \frac{f_M}{3f_R^2}\right]^{-1}\right\}$$

In order to have $r^\star \in (0,1)$, it has to be $R_\pi \in (\lambda_1, \lambda_2)$, where

$$\lambda_1 := \frac{-2f_M a_M a_R + f_M a_M^2 + f_M f_R a_M a_R^2 - f_M f_R a_M^2 a_R - \frac{1}{f_R} + \frac{f_M f_R}{3} a_M^3 + f_M a_R^2 - \frac{f_M f_R}{3} a_R^3 + \frac{f_M}{3f_R^2}}{\frac{2}{3f_R} + 2f_R a_R - 2f_R a_R^2}$$

and

$$\lambda_2 := \frac{2f_M a_M a_R - f_M a_M^2 - f_M f_R a_M a_R^2 + f_M f_R a_M^2 a_R + \frac{1}{f_R} - \frac{f_M f_R}{3} a_M^3 - f_M a_R^2 + \frac{f_M f_R}{3} a_R^3 - \frac{f_M}{3f_R^2}}{\frac{2}{3f_R} - 2f_R a_R + 2f_R a_R^2}$$

As already pointed out in case 7, $r^\star$ is a decreasing function of R_π (see figure (26)). See figure (27) for the graph of $r^\star$ as a function of a_M: it is a decreasing function of a_M, which means that $r^\star$ is maximized when $a_M = a_R =: a_C$, i.e., when the manufacturer and the retailer have people of the same age as their main aim. Obviously,

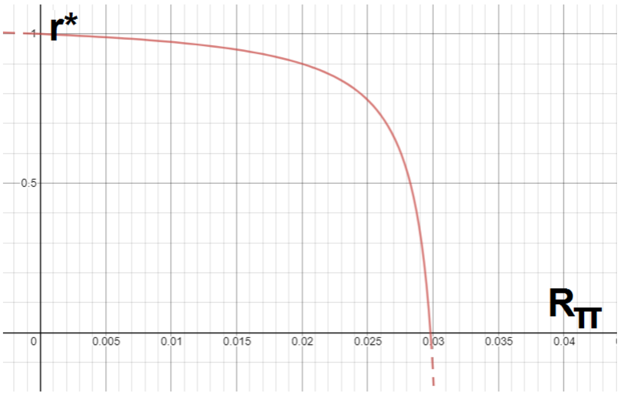

Figure 26: The optimal participation rate $r^\star$ is a decreasing function of R_π in case 5 (as well as it was in case 7). Here $a_M = 30$, $a_R = 35$, $f_M = 0.05$ and $f_R = 0.1$.

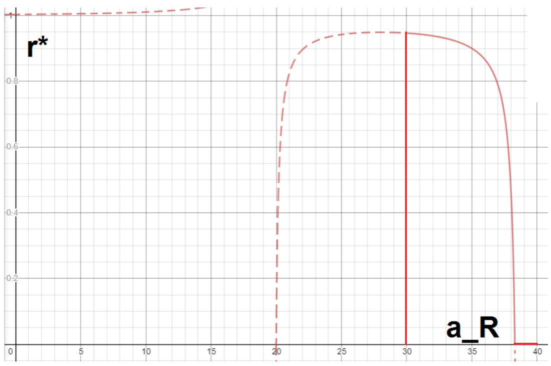

Figure 27: The participation rate $r^\star$ as a function of a_R, in the fifth case of figure (23). Here $a_M = 30$, $f_R = 0.1$, $f_M = 0.05$ and $R_\pi = 0.02$. Notice that, in order to actually be in such a case, one has to ask $a_M \in (30, 40)$.

one may consider the expression of $r^\star$ when $a_M > a_R$, and the possible situations as for the overlap between the intervals M and R become the ones in figure (28).

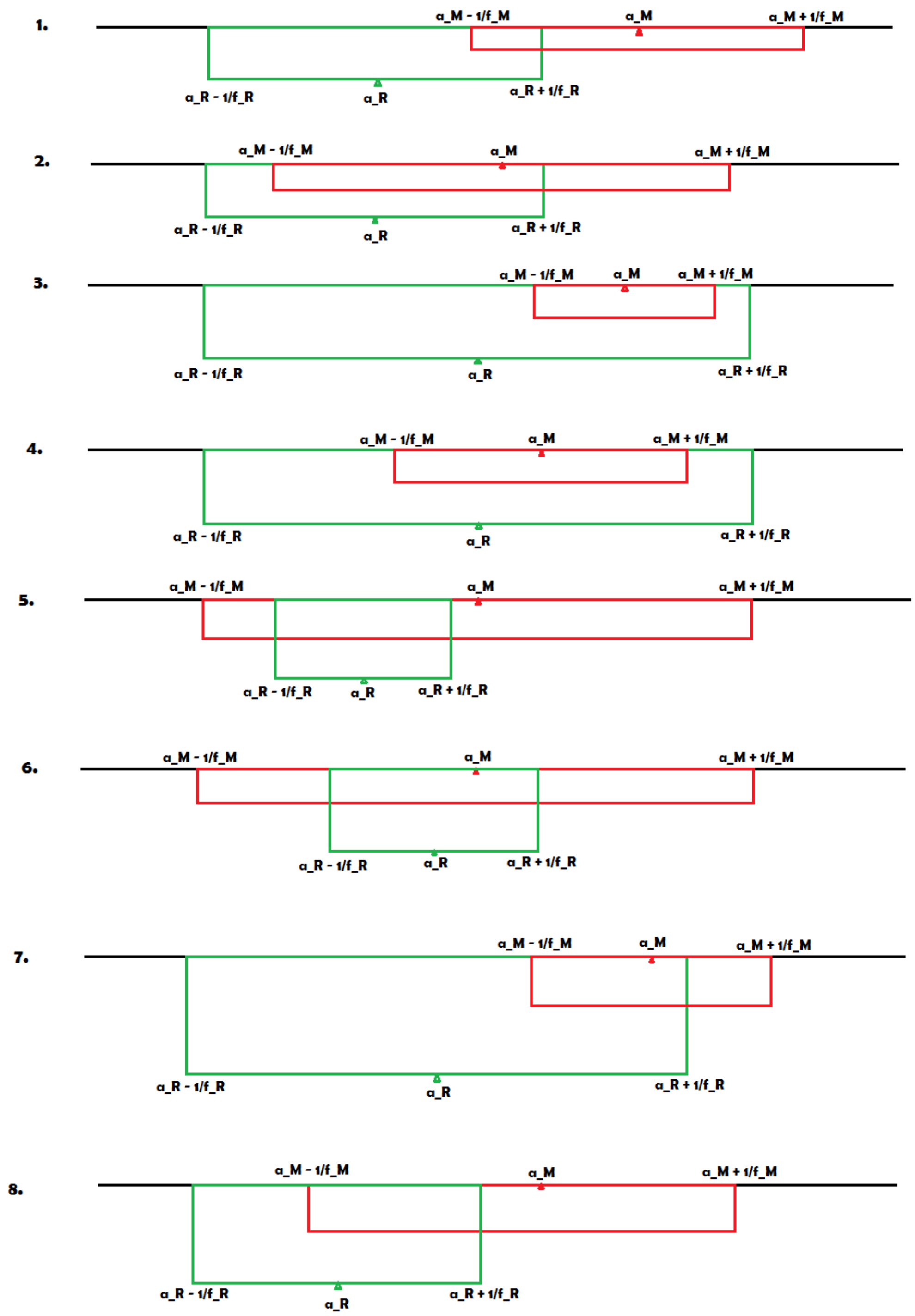

Figure 28: Possible overlaps between $M = \left[a_M - \frac{1}{f_M}, a_M + \frac{1}{f_M}\right]$ and $R = \left[a_R - \frac{1}{f_R}, a_R + \frac{1}{f_R}\right]$, when $a_M > a_R$. They are, obviously, specular with respect to the ones in the case $a_M < a_R$.

Introducing an interaction term in the model

In the population, it generally happens the following: people talk about the product, whose introduction on the market is being planned, so that there's an "interaction" term as for the state equation describing the goodwill.
Models with such an interaction term were analysed by Feichtinger, Tragler and Veliov in [11] as for the necessary conditions, and by Krastev in [19] as for the sufficient conditions for the existence of an OLNE.
In the following pages, the model is going to be modified, so that, for any age a, the goodwill among people younger than a will be affected by how people older than a talk about the product. In particular, at any time t and for any age $\alpha \in [a, \omega]$, the impact of people aged α on people younger than a is quantified as $c_{\text{old}} G_s(\alpha)$, for a certain real constant c_{old}; this means that the higher (in modulus) is c_{old}, the higher is the impact. Moreover, if $c_{\text{old}} > 0$, one is assuming that the "old" people are positively influencing the Goodwill of the firm, and vice-versa.
The state equation for the goodwill $G(t, a)$, then, will be an integro-differential equation:

$$\frac{\partial G(t,a)}{\partial t} + \frac{\partial G(t,a)}{\partial a} = -\mu(a) G(t,a) + A(t,a) + \int_a^{\omega} c_{\text{old}} G(t,a')\,\mathrm{d}a' \tag{68}$$

If ξ is the adjoint function of the stable-age goodwill G and

$$p(t,a) := \int_a^{\omega} c_{\text{old}} G(t,a')\,\mathrm{d}a',$$

then the current-value Hamiltonian is given by

$$\begin{aligned} H_c(a, A, P, G, \xi) =& \pi_M(a)[\beta(a)P + \gamma(a)G] - \frac{k_M}{2}A^2 - \frac{r k_R}{2}P^2 + \\ & + \xi\left(-\mu(a)G + A + p(t,a)\right) + \int_a^{\omega} \eta(t,a')G(t,a')\,\mathrm{d}a' \end{aligned}$$

for a certain function η.
Thus

$$\frac{\partial}{\partial t}\xi(t,a) + \frac{\partial}{\partial a}\xi(t,a) = -\pi_M(a)\gamma(a) + (\mu(a) + \rho)\xi(t,a)$$

which shows that the adjoint function is the same as the one in the no-interaction case, and

$$\eta(t,a) = \frac{\partial H_c}{\partial p}(a) = \xi(t,a)$$

Notice that in the current-value Hamiltonian no new terms containing A, P, or r appeared. This means that the optimal strategy $(A^{\star}, P^{\star})$ is the same as the one in the "no interaction" case. Moreover, in the current-value Hamiltonian, no new terms depending on t appeared; this implies that the Goodwill, as well as all the other aforementioned functions, will depend just on a and not on t. For this reason, the same subscript "s" will be used in this chapter: $G(t,a) \to G_s(a), \xi(t,a) \to \xi_s(a)$, and so on. Equation (68) becomes

$$\frac{\partial G_s(a)}{\partial a} = -\mu(a) G_s(a) + A(a) + \int_a^\omega c_{\text{old}} G_s(a')\, \mathrm{d}a' \tag{69}$$

Now, consider equation (69): if one asks G_s and A to be $\mathscr{C}^1$, then the member on the right is a $\mathscr{C}^1$ function: this implies that $\partial_a G_s(a)$ (the member on the left of the same equation) is a $\mathscr{C}^1$ function, hence G_s is $\mathscr{C}^2$.
Notice that, in the triangular case of the previous section, A was indeed $\mathscr{C}^1$, while in the "rectangular" case A was $\mathscr{C}^1$ everywhere but in $a = \omega_{1,M}, \omega_{2,M}$.

Triangular marginal profits

For the triangular case, one may just derive equation (68) with respect to the age a. If so, one gets

$$\begin{aligned}\frac{\partial^2}{\partial a^2} G_s(a) + \mu(a) \frac{\partial}{\partial a} G_s(a) + \left(\frac{\partial}{\partial a}\mu(a) + c_{\text{old}}\right) G_s(a) = -\frac{\pi_M(a)\gamma(a)}{k_M(a)} + \\ +\xi_s(a) \frac{(\mu(a)+\rho) k_M(a) - \partial_a k_M(a)}{k_M^2(a)}\end{aligned} \tag{70}$$

where it was used the relationship $A^\star(a) = \frac{\xi_s(a)}{k_M(a)}$.
Notice that, if $\mu = \mu(a)$ constant, then (70) is a second-order ODE with constant coefficients. In particular, if the member on the right has the form

$$e^{c_1 a}(P_1(a)\cos(c_2 a) + P_2(a)\sin(c_2 a)) \tag{71}$$

for some constant c_1, c_2 and polynomials $P_1(a), P_2(a)$, then one may analytically solve the equation.
In the previous section, calculations were deepened in the hypothesis $\mu(a) \equiv \mu$ (constant), and, if also $\gamma(a) \equiv \gamma$ and $k_M(a) \equiv k_M$, the adjoint function ξ_s was given by equations (66) and following:

$$\xi_s(a) = f_M \gamma e^{(\mu+\rho)\left(a - a_M + \frac{1}{f_M}\right)} \left(\frac{1 - e^{-\frac{\mu+\rho}{f_M}}}{\mu+\rho}\right)^2$$

if $a \in \left[0, a_M - \frac{1}{f_M}\right]$,

$$\xi_s(a) = \frac{\gamma}{\mu+\rho}\left(1 - f_M a_M + f_M a + \frac{f_M}{\mu+\rho}\right) + \frac{f_M \gamma e^{(\mu+\rho)(a-a_M)}}{(\mu+\rho)^2}\left(e^{-\frac{\mu+\rho}{f_M}} - 2\right),$$

if $a \in \left(a_M - \frac{1}{f_M}, a_M\right]$,

$$\xi_s(a) = \frac{\gamma(1 + f_M a_M - f_M a)}{\mu+\rho} + \frac{f_M \gamma \left(e^{(\mu+\rho)\left(a - a_M - \frac{1}{f_M}\right)} - 1\right)}{(\mu+\rho)^2},$$

if $a \in \left(a_M, a_M + \frac{1}{f_M}\right)$ and

$$\xi_s(a) = 0$$

if $a \in \left[a_M + \frac{1}{f_M}, \omega\right]$.
Thus, equation (70) becomes

$$\begin{aligned}&\frac{\partial^2}{\partial a^2} G_s(a) + \mu \frac{\partial}{\partial a} G_s(a) + c_{\text{old}} G_s(a) = \\ &= \frac{f_M \gamma}{k_M} e^{(\mu+\rho)\left(a - a_M + \frac{1}{f_M}\right)} \frac{\left(1 - e^{-\frac{\mu+\rho}{f_M}}\right)^2}{\mu+\rho}\end{aligned}$$

if $a \in \left[0, a_M - \frac{1}{f_M}\right]$,

$$\begin{aligned}&\frac{\partial^2}{\partial a^2} G_s(a) + \mu \frac{\partial}{\partial a} G_s(a) + c_{\text{old}} G_s(a) = \\ &= \frac{f_M \gamma}{k_M(\mu+\rho)} \left[\left(e^{-\frac{\mu+\rho}{f_M}} - 2\right) e^{(\mu+\rho)\left(a - a_M - \frac{1}{f_M}\right)} + 1\right]\end{aligned}$$

if $a \in \left(a_M - \frac{1}{f_M}, a_M\right)$,

$$\frac{\partial^2}{\partial a^2} G_s(a) + \mu \frac{\partial}{\partial a} G_s(a) + c_{\text{old}} G_s(a) = \frac{f_M \gamma \left(e^{(\mu+\rho)\left(a - a_M - \frac{1}{f_M}\right)} - 1\right)}{k_M(\mu+\rho)},$$

if $a \in \left[a_M, a_M + \frac{1}{f_M}\right)$, and

$$\frac{\partial^2}{\partial a^2} G_s(a) + \mu \frac{\partial}{\partial a} G_s(a) + c_{\text{old}} G_s(a) = 0 \tag{72}$$

if $a \in \left[a_M + \frac{1}{f_M}, \omega\right]$.
The homogeneous equation associated to these second-order ODEs is the same for every $a \in [0, \omega]$, and it coincides with (72). The characteristic polynomial relative to it is

$$z^2 + \mu z + c_{\text{old}} = 0,$$

hence one has three different possibilities:

- if $c_{\text{old}} < \frac{\mu^2}{4}$, then the solution of the homogeneous equation is

$$G_{s,0} = c_1 e^{-\frac{\left(\mu + \sqrt{\mu^2 - 4c_{\text{old}}}\right)a}{2}} + c_2 e^{-\frac{\left(\mu - \sqrt{\mu^2 - 4c_{\text{old}}}\right)a}{2}}, \tag{73}$$

  for some real constants c_1, c_2;

- if $c_{\text{old}} = \frac{\mu^2}{4}$, then the solution of the homogeneous equation is

$$G_{s,0} = e^{-\frac{\mu a}{2}} (c_1 + c_2 a), \tag{74}$$

  for some real constants c_1, c_2;

- if $c_{\text{old}} > \frac{\mu^2}{4}$, then the solution to the homogeneous equation is

$$G_{s,0} = e^{-\frac{\mu a}{2}} \left(c_1 \cos\left(a\sqrt{c_{\text{old}} - \frac{\mu^2}{4}} \right) + c_2 \sin\left(a\sqrt{c_{\text{old}} - \frac{\mu^2}{4}} \right) \right) \tag{75}$$

for some real constants c_1, c_2.

Denote by $G_\omega := G_s(\omega)$, so that, by equation (68), one has $\partial_a G_s(\omega) = -\mu G_\omega$. Notice that the manufacturer's payoff J_M depends just on what happens in the interval $\left[a_M - \frac{1}{f_M}, a_M + \frac{1}{f_M}\right]$: thus, it is interesting to compute the solution G_s just on the interval. It results:

- for $c_{\text{old}} < \frac{\mu^2}{4}$,

$$
\begin{aligned}
G_s(a) = {} & G_\omega e^{\frac{\mu}{2}(\omega-a)} \left[\cosh\left(\sqrt{\frac{\mu^2}{4} - c_{\text{old}}}\omega - a \right) + \frac{\mu}{2\sqrt{\frac{\mu^2}{4} - c_{\text{old}}}} \left(\sqrt{\frac{\mu^2}{4} - c_{\text{old}}}(\omega - a) \right) \right] + \\
& - \frac{2 f_M \gamma}{k_M(\mu+\rho) c_{\text{old}}} \cosh\left(\sqrt{\frac{\mu^2}{4} - c_{\text{old}}}(a_M - a) \right) + \\
& + \frac{f_M \gamma e^{-\frac{\mu+\rho}{f_M}} \left(3 - e^{-\frac{\mu+\rho}{f_M}} \right)}{k_M \sqrt{\frac{\mu^2}{4} - c_{\text{old}}}(\mu+\rho) \left[\left(\frac{3}{2}\mu + \rho\right)^2 + c_{\text{old}} - \frac{\mu^2}{4} \right]} \left[\cosh\left(\sqrt{\frac{\mu^2}{4} - c_{\text{old}}}(a_M - a) \right) + \right. \\
& \left. - \left(\frac{3}{2}\mu + \rho \right) \sinh\left(\sqrt{\frac{\mu^2}{4} - c_{\text{old}}}(a_M - a) \right) \right] + \\
& + \frac{f_M \gamma e^{\frac{\mu}{2}\left(a_M + \frac{1}{f_M}\right)}}{k_M \left(\left(\frac{3}{2}\mu + \rho\right)^2 + c_{\text{old}} - \frac{\mu^2}{4} \right)} \left[\cosh\left(\frac{\sqrt{\frac{\mu^2}{4} - c_{\text{old}}}}{f_M} \right) \left(\frac{\frac{3}{2}\mu + \rho}{c_{\text{old}}} \cosh\left(\sqrt{\frac{\mu^2}{4} - c_{\text{old}}}(a_M - a) \right) + \right. \right. \\
& \left. - \frac{\frac{3}{4}\mu^2 + \frac{\rho\mu}{2} - c_{\text{old}}}{\sqrt{\frac{\mu^2}{4} - c_{\text{old}}} c_{\text{old}}} \sinh\left(\sqrt{\frac{\mu^2}{4} - c_{\text{old}}}(a_M - a) \right) \right) + \\
& + \sinh\left(\frac{\sqrt{\frac{\mu^2}{4} - c_{\text{old}}}}{f_M} \right) \left(\frac{\cosh\left(\sqrt{\frac{\mu^2}{4} - c_{\text{old}}}(a_M - a) \right)}{\sqrt{\frac{\mu^2}{4} - c_{\text{old}}}} + \right. \\
& \left. \left. - \left(\frac{\mu}{\mu^2 - 4c_{\text{old}}} - \frac{\frac{3}{2}\mu + \rho}{c_{\text{old}}} \right) \sinh\left(\sqrt{\frac{\mu^2}{4} - c_{\text{old}}}(a_M - a) \right) \right) \right]
\end{aligned}
$$

if $a \in \left(a_M - \frac{1}{f_M}, a_M\right)$;

$$G_s(a) = e^{\frac{\mu}{2}(\omega-a)} G_\omega \left[\cosh\left(\sqrt{\frac{\mu^2}{4} - c_{\text{old}}}(\omega-a)\right) + \frac{\mu}{2\sqrt{\frac{\mu^2}{4}-c_{\text{old}}}}\sinh\left(\sqrt{\frac{\mu^2}{4} - c_{\text{old}}}(\omega-a)\right)\right] +$$
$$+ \frac{f_M\gamma e^{\frac{\mu}{2}\left(a_M+\frac{1}{f_M}\right)}}{k_M\left[\left(\frac{3}{2}\mu+\rho\right)^2 + c_{\text{old}} - \frac{\mu^2}{4}\right]}\left[\frac{\frac{3}{2}\mu+\rho}{c_{\text{old}}}\cosh\left(\sqrt{\frac{\mu^2}{4}-c_{\text{old}}}\left(a_M+\frac{1}{f_M}-a\right)\right)+\right.$$
$$\left. + \frac{1}{\sqrt{\frac{\mu^2}{4}-c_{\text{old}}}}\sinh\left(\sqrt{\frac{\mu^2}{4}-c_{\text{old}}}\left(a_M+\frac{1}{f_M}-a\right)\right)\right] + \frac{f_M\gamma}{k_M(\mu+\rho)}\left[\frac{e^{(\mu+\rho)\left(a-a_M-\frac{1}{f_M}\right)}}{\left(\frac{3}{2}\mu+\rho\right)^2 + c_{\text{old}} - \frac{\mu^2}{4}} - \frac{1}{c_{\text{old}}}\right]$$

for $a \in \left[a_M, a_M + \frac{1}{f_M}\right)$,

- for $c_{\text{old}} = \frac{\mu^2}{4}$,

$$G_s(a) = G_\omega e^{\frac{\mu}{2}(\omega-a)}\left[1 + \frac{\mu}{2}(\omega-a_M) + \frac{\mu^2}{2}(a-a_M)\left(\frac{\mu}{2}-1\right)(\omega-a_M)\right] +$$
$$+ \frac{\gamma e^{\frac{\mu}{2}\left(a_M+\frac{1}{f_M}-a\right)}}{k_M(\mu+\rho)}\left[-\frac{2}{\mu}+\mu(a-a_M)\right] + \frac{f_M\gamma e^{\frac{\mu}{2}(a_M-a)}}{k_M\left(\frac{3}{2}\mu+\rho\right)^2}\left\{e^{\frac{\mu}{2f_M}}\left(a_M+\frac{1}{f_M}\right)+\right.$$
$$\left. + \frac{\mu^2}{2}(a-a_M)\left[e^{\frac{\mu+\rho}{f_M}}\left(e^{-\frac{\mu+\rho}{f_M}}-2\right)\left(\frac{1}{\mu+\rho}-\frac{2}{\mu}\right) - \frac{e^{\frac{\mu}{2f_M}}\left(\frac{\mu}{2f_M}-1\right)}{\mu+\rho}\right]\right\}$$

if $a \in \left(a_M - \frac{1}{f_M}, a_M\right)$,

$$G_s(a) = e^{\frac{\mu}{2}(\omega-a)} G_\omega\left[1 + \frac{\mu}{2}(\omega-a)\right] +$$
$$+ \frac{f_M\gamma e^{\frac{\mu}{2}\left(a_M+\frac{1}{f_M}-a\right)}}{k_M}\left[\frac{-1+\frac{\mu}{2}\left(a_M+\frac{1}{f_M}-a\right)}{\left(\frac{3}{2}\mu+\rho\right)^2(\mu+\rho)} + \frac{4\left(1+\frac{\mu}{2}\left(a-a_M-\frac{1}{f_M}\right)\right)}{\mu^2(\mu+\rho)} + \frac{a_M+\frac{1}{f_M}}{\left(\frac{3}{2}\mu+\rho\right)^2}\right]$$

if $a \in \left[a_M, a_M + \frac{1}{f_M}\right)$;

- for $c_{\text{old}} > \frac{\mu^2}{4}$,

$$
\begin{aligned}
G_s(a) = & e^{\frac{\mu}{2}(\omega-a)}\left[-\cos\left(\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}(a_M-a)\right)-\frac{\mu}{2\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}}\sin\left(\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}(a_M-a)\right)\right]\cdot\\
& \cdot\left[\cos\left(\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}(\omega-a_M)\right)+\frac{\mu}{2\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}}\sin\left(\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}(\omega-a_M)\right)\right]+\\
& e^{\frac{\mu}{2}(a_M+\frac{1}{f_M}-a)}\frac{f_M\gamma\sin\left(\frac{\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}}{f_M}\right)}{k_M(\mu+\rho)\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}}\left[\frac{\frac{3}{2}\mu+\rho}{\left(\frac{3}{2}\mu+\rho\right)^2+c_{\text{old}}-\frac{\mu^2}{4}}-\frac{\mu}{2c_{\text{old}}}\right]\cdot\\
& \cdot\left[\cos\left(\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}(a_M-a)\right)-\frac{\mu}{2\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}}\sin\left(\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}(a_M-a)\right)\right]+\\
& +\frac{f_M\gamma\left(e^{-\frac{\mu+\rho}{f_M}}-2\right)e^{-\frac{\mu+\rho}{f_M}}}{k_M(\mu+\rho)}\cdot\left[-\cos\left(\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}(a_M-a)\right)+\right.\\
& \left.+\frac{\frac{3}{2}\mu+\rho}{\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}}\sin\left(\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}(a_M-a)\right)+e^{(\mu+\rho)(a-a_M)}\right]+\\
& +\frac{f_M\gamma}{k_M c_{\text{old}}(\mu+\rho)}\left[-\cos\left(\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}(a_M-a)\right)+\right.\\
& \left.+\frac{\mu}{2\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}}\sin\left(\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}(a_M-a)\right)+1\right]+\\
& -e^{\frac{\mu}{2}(\omega-a_M)}\sin\left(\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}(a_M-a)\right)\left[\sin\left(\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}(\omega-a_M)\right)+\right.\\
& \left.-\frac{\mu}{2\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}}\cos\left(\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}(\omega-a_M)\right)\right]+\\
& +\frac{\mu}{2}e^{\frac{\mu}{2}(\omega-a_M)}\sin\left(\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}(a_M-a)\right)\left[\cos\left(\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}(\omega-a_M)\right)+\right.\\
& \left.+\frac{\mu}{2\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}}\sin\left(\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}(\omega-a_M)\right)\right]+\\
& +\frac{f_M\gamma e^{\frac{\mu}{2f_M}}}{k_M(\mu+\rho)\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}}\left[\frac{\frac{3}{2}\mu+\rho}{\left(\frac{3}{2}\mu+\rho\right)^2+c_{\text{old}}-\frac{\mu^2}{4}}-\frac{\mu}{2c_{\text{old}}}\right]\cdot\\
& \cdot\left[-\frac{\mu}{2}\sin\left(\frac{\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}}{f_M}\right)-\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}\cos\left(\frac{\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}}{f_M}\right)\right]
\end{aligned}
$$

if $a \in \left(a_M - \frac{1}{f_M}, a_M\right)$;

$$G_s(a) = e^{\frac{\mu}{2}(\omega-a)}\left[\cos\left(\sqrt{c_{\text{old}} - \frac{\mu^2}{4}}(\omega-a)\right) + \frac{\mu}{2\sqrt{c_{\text{old}} - \frac{\mu^2}{4}}}\sin\left(\sqrt{c_{\text{old}} - \frac{\mu^2}{4}}(\omega-a)\right)\right] +$$
$$+\frac{f_M\gamma\left(\frac{3}{2}\mu+\rho\right)e^{\frac{\mu}{2}\left(a_M+\frac{1}{f_M}-a\right)}\sin\left(\sqrt{c_{\text{old}} - \frac{\mu^2}{4}}\left(a_M+\frac{1}{f_M}-a\right)\right)}{k_M\sqrt{c_{\text{old}} - \frac{\mu^2}{4}}(\mu+\rho)\left[\left(\frac{3}{2}\mu+\rho\right)^2 + c_{\text{old}} - \frac{\mu^2}{4}\right]} +$$
$$-\frac{f_M\gamma\mu e^{\frac{\mu}{2}\left(a_M+\frac{1}{f_M}-a\right)}\sin\left(\sqrt{c_{\text{old}} - \frac{\mu^2}{4}}\left(a_M+\frac{1}{f_M}-a\right)\right)}{2k_M(\mu+\rho)c_{\text{old}}\sqrt{c_{\text{old}} - \frac{\mu^2}{4}}}$$

if $a \in \left[a_M, a_M + \frac{1}{f_M}\right)$

Notice that, as it should be, the stable-age goodwill G_s tends to the "no-interaction" case, as $c_{\text{old}} \to 0^+$. Moreover, the solutions in the two cases are very similar one another if $c_{\text{old}} < \frac{\mu^2}{4}$: for instance, for $a \in \left[a_M + \frac{1}{f_M}, \omega\right]$, see figure (29).

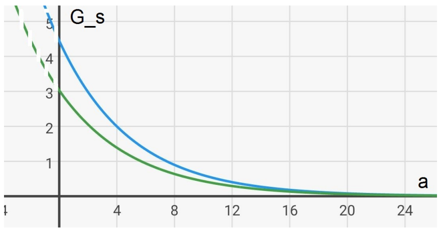

Figure 29: The stable-age goodwill G_s in the no-interaction case (blue line) and in the with-interaction case (green line), for $a \in \left[a_M + \frac{1}{f_M}, \omega\right]$. Here $G_\omega = 5 \cdot 10^{-7}$, $\omega = 80$, $\mu = 0.2$ and $c_{\text{old}} = 0.001$.

The same holds for $c_{\text{old}} = \frac{\mu^2}{4}$ (see figure (30)), where one has the ratio between the two expressions of the goodwill which is

$$\frac{G_{s,\text{no int}}}{G_{s,\text{int}}} = \frac{e^{\frac{\mu}{2}(\omega-a)}}{1+\frac{\mu}{2}(\omega-a)} \simeq 1 + \frac{\mu^2}{8}(\omega-a)^2$$

for $a \to \omega^-$. For $a \to \left(a_M + \frac{1}{f_M}\right)^+$, that ratio is just

$$\frac{G_{s,\text{no int}}}{G_{s,\text{int}}} = \frac{e^{\frac{\mu}{2}\left(\omega - a_M - \frac{1}{f_M}\right)}}{1+\frac{\mu}{2}\left(\omega - a_M - \frac{1}{f_M}\right)}$$

which is a strictly increasing function of $\frac{\mu}{2}\left(\omega - a_M - \frac{1}{f_M}\right)$; in particular, the higher is μ, the higher is the goodwill in the no-interaction case with respect to the with-interaction case.

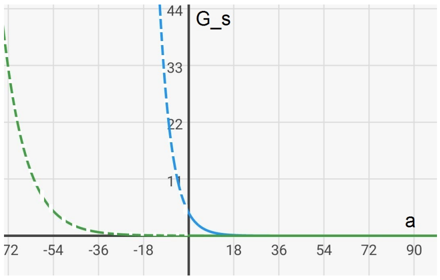

Figure 30: The stable-age goodwill G_s in the no-interaction case (blue line), vs. the one in the with-interaction case (green line), for $a \in \left[a_M + \frac{1}{f_M}, \omega\right]$ and $c_{\text{old}} = \frac{\mu^2}{4}$.

Instead, if $c_{\text{old}} > \frac{\mu^2}{4}$, then the two solutions $G_{s,\text{no int}}$ and $G_{s,\text{int}}$ may significantly differ in their behaviour (see figure (31)), while they keep being similar for $a \to \omega^-$:

$$\frac{G_{s,\text{no int}}}{G_{s,\text{int}}} = \frac{e^{\frac{\mu}{2}(\omega-a)}}{\cos\left((\omega-a)\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}\right) + \frac{\mu}{2\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}}\sin\left((\omega-a)\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}\right)}$$

$$\simeq 1 + \frac{\left(c_{\text{old}}-\frac{\mu^2}{4}\right)(\omega-a)^2}{2}$$

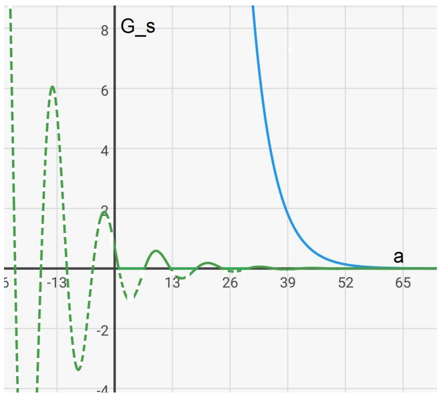

Figure 31: The stable-age goodwill G_s in the no-interaction case (blue line), vs. the one in the with-interaction case (green line), for $a \in \left[a_M + \frac{1}{f_M}, \omega\right]$ and $c_{\text{old}} > \frac{\mu^2}{4}$.

Rectangular marginal profits

For the rectangular case, one may find the Goodwill in the same way as the one adopted for the triangular case, by asking $a \neq \omega_{1,M}, \omega_{2,M}$; otherwise, one may actually apply the Fourier (or the Laplace) transform to every member of equation (68), with no problems about regularity. If one follows the first way, the equation to solve is:

$$\partial_a^2 G_s(a) + \mu \partial_a G_s(a) + c_{\text{old}} G_s(a) = \frac{\gamma}{k_M}\left(e^{(\mu+\rho)(a-\omega_{1,M})} - e^{(\mu+\rho)(a-\omega_{2,M})}\right),$$

if $a \in [0, \omega_{1,M})$;

$$\partial_a^2 G_s(a) + \mu \partial_a G_s(a) + c_{\text{old}} G_s(a) = -\frac{\gamma}{k_M} e^{(\mu+\rho)(a-\omega_{2,M})},$$

if $a \in (\omega_{1,M}, \omega_{2,M})$;

$$\partial_a^2 G_s(a) + \mu \partial_a G_s(a) + c_{\text{old}} G_s(a) = 0,$$

if $a \in (\omega_{2,M}, \omega)$. As before, one has different solutions depending on c_{old}. For reasons that will be clear in the following pages, it is enough to compute $G_s(a)$ only in the interval $[\omega_{1,M}, \omega]$:

- for $c_{\text{old}} < \frac{\mu^2}{4}$, by equation (73), one has

$$\begin{aligned} G_s(a) =& G_\omega e^{\frac{\mu}{2}(\omega-a)}\left[\cosh\left(\sqrt{\frac{\mu^2}{4} - c_{\text{old}}}(\omega-a)\right) + \frac{\mu}{2\sqrt{\frac{\mu^2}{4}-c_{\text{old}}}}\text{senh}\left(\sqrt{\frac{\mu^2}{4} - c_{\text{old}}}(\omega-a)\right)\right] + \\ &+\frac{\gamma e^{\frac{\mu}{2}(\omega_{2,M}-a)}\left[\cosh\left(\sqrt{\frac{\mu^2}{4}-c_{\text{old}}}(\omega_{2,M}-a)\right) - \frac{\frac{3}{2}\mu+\rho}{\sqrt{\frac{\mu^2}{4}-c_{\text{old}}}}\text{senh}\left(\sqrt{\frac{\mu^2}{4}-c_{\text{old}}}(\omega_{2,M}-a)\right)\right]}{k_M\left[\left(\frac{3\mu}{2}+\rho\right)^2 + c_{\text{old}} - \frac{\mu^2}{4}\right]} + \\ &-\frac{\gamma e^{(\mu+\rho)(a-\omega_{2,M})}}{k_M\left[\left(\frac{3\mu}{2}+\rho\right)^2 + c_{\text{old}} - \frac{\mu^2}{4}\right]} \end{aligned}$$

  if $a \in (\omega_{1,M}, \omega_{2,M})$, and

$$G_s(a) = G_\omega e^{\frac{\mu}{2}(\omega-a)}\left[\cosh\left(\sqrt{\frac{\mu^2}{4} - c_{\text{old}}}(\omega-a)\right) + \frac{\mu}{2\sqrt{\frac{\mu^2}{4}-c_{\text{old}}}}\sinh\left(\sqrt{\frac{\mu^2}{4} - c_{\text{old}}}(\omega-a)\right)\right]$$

  if $a \in [\omega_{2,M}, \omega]$;

- for $c_{\text{old}} = \frac{\mu^2}{4}$, by equation (74),

$$\begin{aligned} G_s(a) =& e^{\frac{\mu}{2}(\omega-a)} G_\omega \left(1 + \frac{\mu}{2}(\omega-a)\right) + \\ &+\frac{\gamma\left(\frac{1}{\frac{3}{2}\mu+\rho} + a - \omega_{2,M}\right)}{k_M\left(\frac{3}{2}\mu+\rho\right)} - \frac{\gamma e^{(\mu+\rho)(a-\omega_{2,M})}}{k_M\left(\frac{3}{2}\mu+\rho\right)^2} \end{aligned}$$

  if $a \in (\omega_{1,M}, \omega_{2,M})$ and

$$G_s(a) = e^{\frac{\mu}{2}(\omega-a)} G_\omega \left[1 + \frac{\mu}{2}(\omega-a)\right]$$

  if $a \in [\omega_{2,M}, \omega]$;

- for $c_{\text{old}} > \frac{\mu^2}{4}$, by equation (75),

$$G_s(a) = \max\left\{0, G_\omega e^{\frac{\mu}{2}(\omega-a)}\left[\cos\left(\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}(\omega-a)\right)+ \right.\right.$$
$$+\frac{\mu}{2\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}}\sin\left(\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}(\omega-a)\right)\Bigg]-\frac{\gamma e^{(\mu+\rho)(a-\omega_{2,M})}}{k_M\left[\left(\frac{3}{2}\mu+\rho\right)^2+c_{\text{old}}-\frac{\mu^2}{4}\right]}+$$
$$+\frac{\gamma e^{\frac{\mu}{2}(\omega_{2,M}-a)}}{k_M\left[\left(\frac{3}{2}\mu+\rho\right)^2+c_{\text{old}}-\frac{\mu^2}{4}\right]}\left[\cos\left(\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}(\omega_{2,M}-a)\right)+\right.$$
$$\left.\left.-\frac{\frac{3}{2}\mu+\rho}{\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}}\sin\left(\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}(\omega_{2,M}-a)\right)\right]\right\} \tag{76}$$

if $a \in \left(\omega_{1,M}, \omega_{2,M}\right)$ and

$$G_s(a) = \max\left\{0, G_\omega e^{\frac{\mu}{2}(\omega-a)}\left[\cos\left(\sqrt{-\frac{\mu^2}{4}+c_{\text{old}}}(\omega-a)\right)+\right.\right.$$
$$\left.\left.+\frac{\mu}{2\sqrt{-\frac{\mu^2}{4}+c_{\text{old}}}}\sin\left(\sqrt{-\frac{\mu^2}{4}+c_{\text{old}}}(\omega-a)\right)\right]\right\} \tag{77}$$

if $a \in \left[\omega_{2,M}, \omega\right]$.

As in the triangular case, one may observe that the behaviour of $G_{s,\text{int}}$ is very similar to the one of $G_{s,\text{no int}}$ (given by equation (57) and following), when $c_{\text{old}} < \frac{\mu^2}{4}$:

$$\frac{G_{s,\text{no int}}}{G_{s,\text{int}}} = \frac{e^{\frac{\mu}{2}(\omega-a)}}{\cosh\left((\omega-a)\sqrt{-c_{\text{old}}+\frac{\mu^2}{4}}\right)+\frac{\mu}{\sqrt{-4c_{\text{old}}+\mu^2}}\sinh\left((\omega-a)\sqrt{-c_{\text{old}}+\frac{\mu^2}{4}}\right)}$$
$$\simeq 1-\left(\frac{\mu^2}{4}-c_{\text{old}}\right)(\omega-a)^2$$

when $a \to \omega^-$, and $G_{s,\text{int}} \to G_{s,\text{no int}}$ as $c_{\text{old}} \to 0^+$.
As for the cases $c_{\text{old}} = \frac{\mu^2}{4}$ and $c_{\text{old}} > \frac{\mu^2}{4}$, the discussion is the same as the one done for the triangular marginal profits.
Further details in the comparison will be provided below, but first one needs to be more precise about the goodwill for high values of c_{old}.
Indeed, one has to see whether $G_s(a)$ is strictly positive. Analytically, this is far simpler for the expression (77): one gets $G_s(a) > 0$, $\forall a \in [\omega_{2,M}, \omega]$, if and only if

$$\tan\left(\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}(\omega-a)\right) \geq -\frac{2\sqrt{c_{\text{old}}-\frac{\mu^2}{4}}}{\mu} \tag{78}$$

Notice that the member on the left always vanishes at $a=\omega$; while, if c_{old} is high enough, the argument of the tangent becomes greater than $\frac{\pi}{2}$, thus one has to check (78) for many branches of graph of the tangent. Precisely, the threshold value is

$$c_{\text{old},t} = \frac{\mu^2}{4} + \frac{\pi^2}{4\omega^2} \tag{79}$$

Now, if c_{old} is high enough, equation (76) tends to (77); that is, if the impact of old people is high enough, then the advertising has a negligible effect on the goodwill, and for the manufacturer it is unnecessary to invest on it.

Hence, the condition (78) can be studied also for $a \in [\omega_{1,M}, \omega_{2,M}]$. Set:

$$k_1 := \max\left\{k \in \mathbb{N} : -\frac{\pi}{2} + k\pi < \omega_{1,M}\right\} = \left\lfloor \frac{\omega_{1,M}}{\pi} + \frac{1}{2} \right\rfloor$$

and, if

$$\omega_{2,M} - \omega_{1,M} > \pi, \tag{80}$$

set also

$$k_2 := \max\left\{k \in \mathbb{N} : -\frac{\pi}{2} + k\pi < \omega_{2,M}\right\} = \left\lfloor \frac{\omega_{2,M}}{\pi} + \frac{1}{2} \right\rfloor$$

Then, the solution of equation (78) is

$$\omega - \frac{\frac{\pi}{2} + k\pi}{\sqrt{c_{\text{old}} - \frac{\mu^2}{4}}} < a < \omega + \frac{\arctan\left(\frac{2\sqrt{c_{\text{old}} - \frac{\mu^2}{4}}}{\mu}\right) - k\pi}{\sqrt{c_{\text{old}} - \frac{\mu^2}{4}}} \tag{81}$$

for all $k \in \{k_1, \dots, k_2\}$, and, if

$$\omega_{2,M} > \omega + \frac{\arctan\left(\frac{2\sqrt{c_{\text{old}} - \frac{\mu^2}{4}}}{\mu}\right) - k_2\pi}{\sqrt{c_{\text{old}} - \frac{\mu^2}{4}}},$$

then (78) is satisfied also for

$$a \in \left(\omega + \frac{\arctan\left(\frac{2\sqrt{c_{\text{old}} - \frac{\mu^2}{4}}}{\mu}\right) - k_2\pi}{\sqrt{c_{\text{old}} - \frac{\mu^2}{4}}}, \omega_{2,M}\right).$$

Instead, if (80) is not satisfied (i.e., $\omega_{2,M} - \omega_{1,M} < \pi$), then (78) is satisfied for

$$a \in \left(\omega + \frac{\arctan\left(\frac{2\sqrt{c_{\text{old}} - \frac{\mu^2}{4}}}{\mu}\right) - k_1\pi}{\sqrt{c_{\text{old}} - \frac{\mu^2}{4}}}, \omega_{2,M}\right)$$

These results will be used in a few lines.
In order to compare $G_{s,\mathrm{int}}$ and $G_{s,\mathrm{no\,int}}$ more effectively, it is meaningful to compute the difference in the profit J_M of the manufacturer in the two situations. By looking at the expression of J_M (equation (46)), one defines

$$\Delta_{J_M} := \gamma \int_{\omega_{1,M}}^{\omega_{2,M}} \left(G_{s,\mathrm{int}}(a) - G_{s,\mathrm{no\,int}}(a) \right) \mathrm{d}a \tag{82}$$

Then, one gets

- for $c_{\mathrm{old}} < \frac{\mu^2}{4}$,

$$\begin{aligned}
\frac{\Delta_{J_M}}{\gamma} =& \frac{G_\omega}{\sqrt{\frac{\mu^2}{4} - c_{\mathrm{old}}}} \left[e^{\frac{\mu}{2}(\omega - \omega_{1,M})} \sinh\left(\sqrt{\frac{\mu^2}{4} - c_{\mathrm{old}}} (\omega - \omega_{1,M}) \right) + \right. \\
& \left. - e^{\frac{\mu}{2}(\omega - \omega_{2,M})} \sinh\left(\sqrt{\frac{\mu^2}{4} - c_{\mathrm{old}}} (\omega - \omega_{2,M}) \right) \right] + \\
& + \frac{\gamma}{k_M \left[\left(\frac{3}{2}\mu + \rho \right)^2 + c_{\mathrm{old}} - \frac{\mu^2}{4} \right]} \left[- \frac{2\mu + \rho}{c_{\mathrm{old}}} + \right. \\
& + \frac{2\mu + \rho}{c_{\mathrm{old}}} \cosh\left(\sqrt{\frac{\mu^2}{4} - c_{\mathrm{old}}} (\omega_{2,M} - \omega_{1,M}) \right) e^{\frac{\mu}{2}(\omega_{2,M} - \omega_{1,M})} + \\
& - \frac{\mu^2 + \frac{\rho\mu}{2} - c_{\mathrm{old}}}{c_{\mathrm{old}} \sqrt{\frac{\mu^2}{4} - c_{\mathrm{old}}}} \sinh\left(\sqrt{\frac{\mu^2}{4} - c_{\mathrm{old}}} (\omega_{2,M} - \omega_{1,M}) \right) e^{\frac{\mu}{2}(\omega_{2,M} - \omega_{1,M})} + \\
& \left. + \frac{e^{(\mu+\rho)(\omega_{1,M} - \omega_{2,M})} - 1}{\mu + \rho} \right] - \frac{\gamma}{k_M(\mu + \rho)} \left[\frac{\omega_{2,M} - \omega_{1,M}}{\mu} + \right. \\
& + \frac{1 - e^{\mu(\omega_{2,M} - \omega_{1,M})}}{\mu^2} - \frac{1 - e^{(\mu+\rho)(\omega_{1,M} - \omega_{2,M})}}{(\mu + \rho)(2\mu + \rho)} + \\
& \left. - \frac{1 - e^{\mu(\omega_{2,M} - \omega_{1,M})}}{\mu(2\mu + \rho)} \right] + G_\omega \frac{e^{\mu(\omega - \omega_{2,M})} - e^{\mu(\omega - \omega_{1,M})}}{\mu}
\end{aligned}$$

- for $c_{\mathrm{old}} = \frac{\mu^2}{4}$,

$$\begin{aligned}
\frac{\Delta_{J_M}}{\gamma} =& G_\omega \left[e^{\frac{\mu}{2}(\omega - \omega_{1,M})} (\omega - \omega_{1,M}) - e^{\frac{\mu}{2}(\omega - \omega_{2,M})} (\omega - \omega_{2,M}) \right] + \\
& + \frac{2\gamma}{k_M \mu \left(\frac{3}{2}\mu + \rho \right)} \left[-\frac{2}{\mu} + e^{\frac{\mu}{2}(\omega - \omega_{1,M})} \left(\omega_{1,M} - \omega_{2,M} + \frac{2}{\mu} \right) \right] + \\
& + \frac{4\gamma}{k_M \mu \left(\frac{3}{2}\mu + \rho \right)^2} \left[\cosh\left(\frac{\mu}{2} (\omega_{2,M} - \omega_{1,M}) \right) - 1 \right] - G_\omega \frac{e^{\mu(\omega - \omega_{2,M})} - e^{\mu(\omega - \omega_{1,M})}}{\mu} + \\
& - \frac{\gamma}{k_M(\mu + \rho)} \left[\frac{\omega_{2,M} - \omega_{1,M}}{\mu} + \frac{1 - e^{\mu(\omega_{2,M} - \omega_{1,M})}}{\mu^2} + \frac{e^{(\mu+\rho)(\omega_{1,M} - \omega_{2,M})} - 1}{(2\mu + \rho)(\mu + \rho)} + \frac{e^{\mu(\omega_{2,M} - \omega_{1,M})} - 1}{\mu(2\mu + \rho)} \right]
\end{aligned}$$

- for $c_{\text{old}} > \frac{\mu^2}{4}$, calculations must take into account when $G_s(a)$ vanishes. As said before, this always happens when c_{old} is strictly greater than the threshold value $c_{\text{old},t}$ in equation (79). Thus, if $\frac{\mu^2}{4} < c_{\text{old}} \le c_{\text{old},t}$, then

$$\begin{aligned}\frac{\Delta_{J_M}}{\gamma} =& \frac{2}{\sqrt{-\frac{\mu^2}{4}+c_{\text{old}}}}\left[e^{\frac{\mu}{2}(\omega-\omega_{2,M})}\sin\left(\sqrt{-\frac{\mu^2}{4}+c_{\text{old}}}(\omega-\omega_{2,M})\right)+\right.\\ &\left.-e^{\frac{\mu}{2}(\omega-\omega_{1,M})}\sin\left(\sqrt{-\frac{\mu^2}{4}+c_{\text{old}}}(\omega-\omega_{1,M})\right)\right]-G_\omega\frac{e^{\mu(\omega-\omega_{2,M})}-e^{\mu(\omega-\omega_{1,M})}}{\mu}+\\ &-\frac{\gamma}{k_M(\mu+\rho)}\left[\frac{\omega_{2,M}-\omega_{1,M}}{\mu}+\frac{1-e^{\mu(\omega_{2,M}-\omega_{1,M})}}{\mu^2}+\right.\\ &\left.+\frac{e^{(\mu+\rho)(\omega_{1,M}-\omega_{2,M})}-1}{(2\mu+\rho)(\mu+\rho)}+\frac{e^{\mu(\omega_{2,M}-\omega_{1,M})}-1}{\mu(2\mu+\rho)}\right]\end{aligned}$$

Instead, if $c_{\text{old}} > c_{\text{old},t}$, one has to count the (finitely many) branches of the tangent (see equation (78)) in which the goodwill is non-null. This was done in equation (81), which may be used to find:

$$\begin{aligned}\frac{\Delta_{J_M}}{\gamma} =& \frac{2}{\sqrt{-\frac{\mu^2}{4}+c_{\text{old}}}}\sum_{k=k_1}^{k_2}\left[e^{\frac{\mu}{2}(\omega-a)}\sin\left(\sqrt{-\frac{\mu^2}{4}+c_{\text{old}}}(\omega-a)\right)\right|_{a=\omega-\frac{\frac{\pi}{2}+k\pi}{\sqrt{-\frac{\mu^2}{4}+c_{\text{old}}}}}^{a=\omega+\frac{k\pi+\arctan\left(\frac{2\sqrt{-\frac{\mu^2}{4}+c_{\text{old}}}}{\mu}\right)}{\sqrt{-\frac{\mu^2}{4}+c_{\text{old}}}}} +\\ & \frac{2}{\sqrt{-\frac{\mu^2}{4}+c_{\text{old}}}}\left[e^{\frac{\mu}{2}(\omega-a)}\sin\left(\sqrt{-\frac{\mu^2}{4}+c_{\text{old}}}(\omega-a)\right)\right|_{\omega+\frac{k_2\pi+\arctan\left(\frac{2\sqrt{-\frac{\mu^2}{4}+c_{\text{old}}}}{\mu}\right)}{\sqrt{-\frac{\mu^2}{4}+c_{\text{old}}}}}^{\omega_{2,M}}\cdot\\ &\cdot\max\left\{0,\omega_{2,M}-\omega-\frac{k_2\pi+\arctan\left(\frac{2\sqrt{-\frac{\mu^2}{4}+c_{\text{old}}}}{\mu}\right)}{\sqrt{-\frac{\mu^2}{4}+c_{\text{old}}}}\right\}-G_\omega\frac{e^{\mu(\omega-\omega_{2,M})}-e^{\mu(\omega-\omega_{1,M})}}{\mu}+\\ &-\frac{\gamma}{k_M(\mu+\rho)}\left[\frac{\omega_{2,M}-\omega_{1,M}}{\mu}+\frac{1-e^{\mu(\omega_{2,M}-\omega_{1,M})}}{\mu^2}+\frac{e^{(\mu+\rho)(\omega_{1,M}-\omega_{2,M})}-1}{(2\mu+\rho)(\mu+\rho)}+\frac{e^{\mu(\omega_{2,M}-\omega_{1,M})}-1}{\mu(2\mu+\rho)}\right]\end{aligned}$$

Notice that, in everyone of the previous cases, Δ_{J_M} tends to a positive quantity even if $k_M \to +\infty$. This is equivalent to say that, no matter how much is the marginal cost for the manufacturer, he profits from the effect of the people's talking to each other about the product.

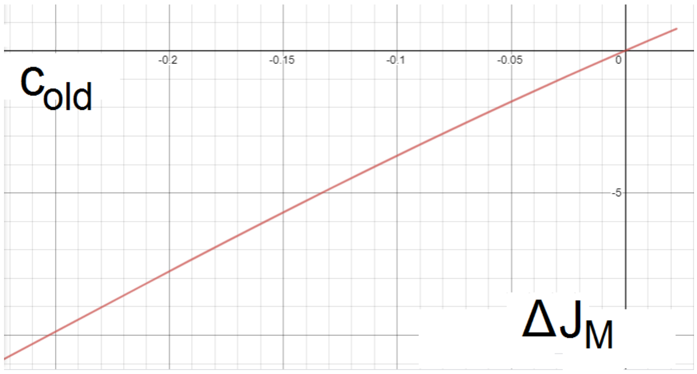

Figure 32: Δ_{J_M} as a function of c_{old}, for $c_{\text{old}} < \frac{\mu^2}{4}$. Here $G_\omega = 2 \cdot 10^{-10}$, $\mu = 0.3$, $\omega = 25$, $\omega_{1,M} = 10$, $\omega_{2,M} = 15$, $\rho = 0.05$, $\frac{\gamma}{k_M} = 1$. Then, the domain of the function is $-0.2725 = -(\mu+\rho)(2\mu+\rho) < c_{\text{old}} < \frac{\mu^2}{4} = 1$.

Also, as expected, $\Delta_{J_M} \to 0$ as $c_{\text{old}} \to 0$ (see fig. (32)), being this situation exactly the same as the no-interaction one.
Also, it's interesting to point out that Δ_{J_M} tends to a finite quantity as $c_{\text{old}} \to +\infty$ (see figure (33)): this quantity is

$$\begin{aligned}\lim_{c_{\text{old}} \to +\infty} \Delta_{J_M}(c_{\text{old}}) = & G_\omega \frac{e^{\mu(\omega-\omega_{1,M})} - e^{\mu(\omega-\omega_{2,M})}}{\mu} + \\ & - \frac{\gamma}{k_M(\mu+\rho)} \left[\frac{\omega_{2,M}-\omega_{1,M}}{\mu} + \frac{1-e^{\mu(\omega_{2,M}-\omega_{1,M})}}{\mu^2} + \right. \\ & \left. + \frac{1-e^{(\mu+\rho)(\omega_{1,M}-\omega_{2,M})}}{(\mu+\rho)(2\mu+\rho)} + \frac{1-e^{\mu(\omega_{2,M}-\omega_{1,M})}}{\mu(2\mu+\rho)} \right]\end{aligned}$$

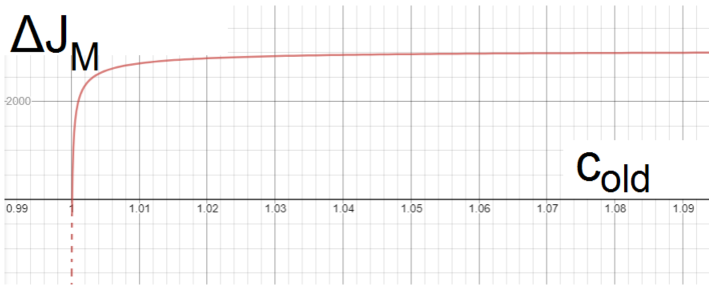

Figure 33: The variation in the profit Δ_{j_M} as a function of c_{old}, for $c_{\text{old}} > \frac{\mu^2}{4}$. It is a strictly increasing function, with an horizontal asymptote as $c_{\text{old}} \to +\infty$. Here $G_\omega = 2 \cdot 10^{-10}$, $\mu = 2$, $\omega = 25$, $\omega_{1,M} = 10$, $\omega_{2,M} = 15$, $\rho = 1$, $\frac{\gamma}{k_M} = 1$. Then, the domain of the function is $c_{\text{old}} > \frac{\mu^2}{4} = 1$.

CHAPTER 5

Conclusions

In this thesis, the distributive channel proposed by Buratto and Grosset in [13] was equipped with an age structure. Following the line of thought presented in the paragraph "Linear continuous models" in Chapter 2, the state equation about the Goodwill G and the adjoint equation have the form of a McKendrick PDE. By using the method of the characteristics, these equations were solved; the solutions assumed explicit shapes, when the decay rate $\mu(a)$ and the marginal costs $k_M(a)$ and $k_R(a)$ were supposed to be constant, for the two chosen expressions of the marginal profit $\pi_M(a)$ (i.e. the rectangular and the triangular ones). For any strategy (P, A), $G_s(a)$ is a bounded function: in particular, such is in the rectangular and triangular cases. Moreover, the adjoint function $\xi_s(a)$ is a bounded function. By the necessary and the sufficient conditions introduced in Chapter 4, this means that $(G_s, A^\star, P^\star)$ is catching up optimal for the considered problem.
When the manufacturer's and the retailer's marginal profits, $\pi_M(a)$ and $\pi_R(a)$, are assumed to be constant on two intervals (specifically, $\pi_M(a) = \chi_{]\omega_{1,M},\omega_{2,M}[}(a)$ and $\pi_R(a) = R_\pi \chi_{]\omega_{1,M},\omega_{2,M}[}(a)$, for a positive constant R_π), one observes the following:

- the optimal advertising $A^\star$ is non-null and increasing on the interval $[0, \omega_{1,M}]$ (see figure (8)), though $\pi_M(a)$ vanishes on such interval. In other words, there's an anticipating effect, due to the age structure of the population: it is convenient for the manufacturer to invest on advertising towards people younger than $\omega_{1,M}$, even though his product isn't thought for that age, just because these people will eventually grow up and get into the segment $]\omega_{1,M},\omega_{2,M}[$. For the same reason, $A^\star(a)$ is decreasing on $]\omega_{1,M},\omega_{2,M}[$, even though the marginal profit is constant on that age segment.

- $A^\star(a)$ is bounded: it assumes its maximum in $\omega_{1,M}$, where it is $A^\star(\omega_{1,M}) = \frac{\gamma}{\mu+\rho}\left[1-e^{(\mu+\rho)(\omega_{1,M}-\omega_{2,M})}\right]$.

- In order to better understand how much it is convenient for the manufacturer to invest on the age segment $[0,\omega_{1,M}]$, with respect to how much he should do on $]\omega_{1,M},\omega_{2,M}[$, the average value of $A^\star$ on the age segments $[0,\omega_{1,M}]$ (called $A_1^\star$), $]\omega_{1,M},\omega_{2,M}[$ ($A_2^\star$) and $[\omega_{2,M},\omega]$ was computed. The ratio $\frac{A_2^\star}{A_1^\star}$ is represented in figure (11) as a function of the width of the interval $]\omega_{1,M},\omega_{2,M}[$, and in figure (12) as a function of $\mu+\rho$. In particular, the greater are the decay rate μ of the goodwill or the discount rate ρ, the greater is $A_2^\star$ with respect to $A_1^\star$; for fixed value of $\omega_{1,M}$, the same holds when $\omega_{2,M}$ increases. So, it is true that there's an anticipating effect on the optimal advertising $A^\star$, but it may be negligible for big values of $\mu+\rho$ or $\omega_{2,M}-\omega_{1,M}$.

- As for the participation rate $r \in [0,1)$, i.e. the part of the retailer expenditures on promotion that it is convenient for the manufacturer to take on, it was shown that it is non-null if and only if the ratio R_π isn't too high (that is, iif the retailer's marginal profits aren't too high with respect to the manufacturer's ones) and there's at least a partial overlap between the intervals $]\omega_{1,M},\omega_{2,M}[$ and $]\omega_{1,R},\omega_{2,R}[$ (see equation (59)). Furthermore, r is an increasing function of such overlap, for any fixed value of R_π (see figure (9)).

When π_M and π_R are assumed to have triangular shape, i.e.

$$\pi_M(a) = (1-f_M|a-a_M|)\chi_{\left]a_M-\frac{1}{f_M},a_M+\frac{1}{f_M}\right[}(a)$$
$$\pi_R(a) = R_\pi(1-f_R|a-a_R|)\chi_{\left]a_R-\frac{1}{f_R},a_R+\frac{1}{f_R}\right[}(a)$$

one gets the following:

- the optimal advertising $A^\star(a)$ is bounded, just as it was for the rectangular case. It's interesting to point out, though, that in the rectangular case the maximum point for $A^\star(a)$ was $\omega_{1,M}$, i.e. where the marginal profit $\pi_M(a)$ started to be non-null. Instead, in the triangular case, $A^\star(a)$ assumes its maximum value on a value of a smaller than a_M (which is the maximum point of $\pi_M(a)$). Precisely, $A^\star$ has its maximum point for

$$a = a_M - \frac{1}{\mu+\rho}\log\left(2-e^{-\frac{\mu+\rho}{f_M}}\right),$$

which is an increasing function of $\mu+\rho$ (see figure (14)). The maximum value of $A^\star$, instead, is a decreasing function of $\mu+\rho$ (see figure (13)), and

$$\lim_{\mu+\rho\to 0^+} A^\star(a) = \frac{\gamma}{k_M f_M}.$$

In other words, the lower are the discount rate ρ or the decay rate μ, the "more anticipated" is the maximum point and the higher is the maximum optimal advertising $A^\star$.

- the advertising is non-null and increasing on $\left[0, a_M - \frac{1}{f_M}\right]$, even though $\pi_M(a)$ vanishes on such interval, and rapidly decreases on $\left[a_M, a_M + \frac{1}{f_M}\right[$, until it vanishes on $\left[a_M + \frac{1}{f_M}, \omega\right]$.

- as for the rectangular case, the average value of $A^\star$ on the intervals $\left[0, a_M - \frac{1}{f_M}\right]$, $\left]a_M - \frac{1}{f_M}, a_M\right[$ and $\left[a_M, a_M + \frac{1}{f_M}\right[$ was computed, and called respectively $A_1^\star$, $A_2^\star$ and $A_3^\star$. It turns out that $A_2^\star$ may be lower than $A_1^\star$, for $\mu+\rho$ small enough (see figure (19)), while $A_2^\star$ is always bigger than $A_3^\star$ (see figure (20)). This means that, if $\mu+\rho$ is small enough, it may be more convenient for the manufacturer to invest on people younger than $a_M - \frac{1}{f_M}$, then on the one aged between $a_M - \frac{1}{f_M}$ and a_M.

- the analysis of the optimal participation rate $r^\star$ requires a bit more attention than in the rectangular case. First, one notices that $r^\star = 0$ if the manufacturer and the retailer focus on two completely different intervals. Now, depending on a_M, a_R, f_M and f_R, the intervals $\left]a_M - \frac{1}{f_M}, a_M + \frac{1}{f_M}\right[$ and $\left]a_R - \frac{1}{f_R}, a_R + \frac{1}{f_R}\right[$ are overlapped in a different way. Figures (23) and (28) show all the possible superpositions: among them, three interesting configurations were further analyzed. In particular, when $a_M = a_R$, $r \in (0,1)$ if and only if $R_\pi \in \left(\frac{f_M}{2}, \frac{3}{2} + \frac{f_M}{2f_R}\right)$. That is, one has a lower and an upper bound for R_π, and not just an upper one, as it was in the rectangular case. Moreover, $r^\star$ is an increasing function of f_M and a decreasing one of f_R, and

$$\lim_{f_R\to+\infty} r^\star(f_R) = \frac{3-2R_\pi}{3+2R_\pi}.$$

Analogous observations are made for two other possible configurations, and the dependence of $r^\star$ on R_π is shown in figures (26) and (27).

In the end, an interaction term in the population was introduced. Specifically, for any age a, it was considered the effect produced by people older than a talking about the product with people younger than a. This makes the state equation an integro-differential one (see formula (68)). The impact of "old" people is expressed through a constant c_{old}: the higher it is, the higher is such impact. If $c_{\text{old}} > 0$, older

people interact with younger people in a favourable way for the firm, and vice-versa.
The Hamiltonian function modifies accordingly, but no new terms containing A or P were introduced. This means that the optimal advertising and promotion are the same as the ones in the no-interaction case; and they are still an OLNE, being the adjoint function $\xi_s(a)$ the same. So, the new term contributes only to modify the goodwill $G_s(a)$.
Equation (68) was solved just by deriving it with respect to the age a. This is reasonable in the triangular case: the regularity of $A^\star(a)$ ensures that $G_s(a)$ is twice differentiable. Instead, in the rectangular case, $A^\star$ is just continuous, hence $G_s(a)$ is just $\mathscr{C}^1$: this means that either one repeats the previous way, and asks $a \neq \omega_{1,M}, \omega_{2,M}$, or one applies the Laplace transform to (68), with no stronger assumptions on the regularity of $G_s(a)$. Calculations are just easier by using the first method, especially because the Laplace's anti-transform is not so simple to compute, hence this way was followed.
Thus, equation (68) becomes a linear second-order ODE (equation (70)), with constant coefficients, one of which is c_{old}. Thus, the explicit form of $G_s(a)$ depends on c_{old}, both in the triangular and the rectangular case: specifically, the threshold value is $c_{\text{old}} = \frac{\mu^2}{4}$.
Again, $G_{s,\text{int}}(a)$ is bounded, as well as it is for any strategy (P, A); being ξ_s the same as the "no interaction" case, it follows that $(G_{s,\text{int}}, A^\star, P^\star)$ is catching up optimal.
The solutions $G_{s,\text{int}}(a)$ and $G_{s,\text{no int}}$ - when the interaction term in the Hamiltonian is present or not, respectively - were then compared, by computing the difference Δ_{J_M} in the profit of the manufacturer (see equation (82)). Notice that such difference depends only on the behaviour of the two solutions on the interval $[\omega_{1,M}, \omega_{2,M}]$. Then, Δ_{J_M} was plotted as a function of c_{old} (see figures (32) and (33)). In particular, Δ_{J_M} results ever increasing and positive for $c_{\text{old}} > 0$; it is obviously null for $c_{\text{old}} = 0$, and it is negative for $c_{\text{old}} < 0$. It's also interesting to point out that Δ_{J_M} tends to a finite quantity as $c_{\text{old}} \to +\infty$.
These calculations may be re-done for different shapes of the marginal profit $\pi_M(a)$. Also, they are easily applicable for a non-constant marginal cost $k_M(a)$ and $\gamma(a)$, if they produce a term as (71) in equation (70). In such a case, obviously, one has to re-compute the adjoint function $\xi_s(a)$ and to check if it keeps being a bounded function of a.
In the end, it may be a good starting point for further research the following observation. $\Delta_{J_M} < 0$ is a rather undesirable condition for the manufacturer. So, one may imagine that he has a "security plan": to intervene in the promotion, so as to raise the sales $Q = \gamma A + \beta P$ and take back Δ_{J_M} on a positive value. This may be done by modifying the expression of β, for example, writing it as a function of c_{old}: this

way, it keeps being constant with respect to the age a, and calculations don't get too complicated.

CHAPTER 5

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