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Bootstrap approaches in quantum mechanics

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Chapter 1

Introduction

The Schrödinger equation is a fundamental result of quantum mechanics because it governs the evolution of the wave function of any quantum system. We are interested in solving its easiest form, that is the Schrödinger equation of a free particle in two dimensions, but in a very general scenario because the space will not be the usual euclidean space $\mathbb{R}^2$ but a generic hyperbolic 2-orbifold X, that can be thought as a generalization of a 2-manifold. The best ambition would be finding an analytical form for eigenfunctions and a precise value for eigenvalues however, apart from the trivial value of the smallest eigenvalue $\lambda_0 = 0$, this problem is currently considered unsolvable. Therefore we will focus only on the lowest non zero eigenvalue λ_1, more specifically we are going to find an upper bound on λ_1 that holds for every given 2-orbifold.

We recall that a generic 2-orbifold can be seen as the quotient between the upper half complex plane $\mathbb{H}$ and a discrete subgroup Γ of $PSL(2, \mathbb{R})$ so we define $X = \mathbb{H}/\Gamma$. The main idea of the work is that the spectrum of the Laplacian on X is linked to the spectrum of the unitary irreducible representations of $G = PSL(2, \mathbb{R})$ on the Hilbert space $L^2(G/\Gamma)$ thanks to the action of the quadratic Casimir element. In fact among all the representations there are the principal and complementary series which are linear subspaces indexed by a real number Δ and we will see how the action of the Casimir on some special functions of X is both the simple multiplication by $\Delta(\Delta-1)$ and the action of the Laplacian ∇^2, from which it follows the important relation $-\lambda = \Delta(\Delta - 1)$. This implies that the problem of finding the smallest eigenvalues becomes finding, among all the irreducible representations of G on $L^2(G/\Gamma)$, all the values of Δ and choose the one that minimizes λ. In addition on $L^2(G/\Gamma)$ there is a well-defined associative and commutative product (which is the usual pointwise multiplication).

Another important element is the existence of a operator-product expansion that decomposes the product of two functions of $L^2(G/\Gamma)$ on a given basis, that in our case will be the one made by all the unitary irreducible representations. To constrain the possible values of Δ we are going to define a special class of functions on $L^2(G/\Gamma)$ that are called coherent states $\theta_n(z)$. Their peculiarity is that the product expansion of two coherent states $\theta_n(z_1)\theta_n(z_2)$ is particularly simple and it is constrained by symmetry considerations. Moreover we will define the n-function correlator $\langle ... \rangle$ which is an operator that maps the product of n functions to a complex value. We will be interested in finding the correlator of m coherent states $\langle \theta_n(z_1)...\theta_n(z_m) \rangle$, however for small m, due to the G-invariance of the function correlator, the result is trivial and it is fixed up to a multiplicative constant. The smallest m such that the result is non-trivial is $m = 4$, so we will focus on this case that is computing an expression that involves the product of 4 coherent states $\langle \theta_n(z_1)\theta_n(z_2)\theta_n(z_3)\theta_n(z_4) \rangle$. The strategy will be expanding the products before calculating the correlator and this can be done in two different ways: we can either expand the products $\theta_n(z_1)\theta_n(z_2)$ and $\theta_n(z_3)\theta_n(z_4)$ or $\theta_n(z_1)\theta_n(z_4)$ and $\theta_n(z_2)\theta_n(z_3)$. So we have two ways to compute the 4-function correlator that will apparently look different, but due to the associativity and commutativity of the product they must be equal. Finally by matching those two results we will get the so called crossing equation that after some manipulation will lead to the bound on λ_1.

Chapter 2

Setting up the problem

2.1 The problem

Let us consider the upper half complex plane $\{\mathbb{H} = x + iy \mid y > 0\}$ with the Riemann metric of constant curvature -1 :

$$ds^2 = \frac{dx^2 + dy^2}{y^2}, \tag{2.1}$$

from which we can easily see that the Laplacian on $\mathbb{H}$ is:

$$\nabla^2 = y^2(\partial_x^2 + \partial_y^2). \tag{2.2}$$

Let us also consider $G = PSL(2, \mathbb{R})$ which is the group of isometries that preserves orientation on $\mathbb{H}$ with action:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}, \qquad z \in \mathbb{H}. \tag{2.3}$$

Given a discrete subgroup Γ of G we define the space $X = \mathbb{H}/\Gamma$ which is the quotient space built on the equivalence relation:

$$x \sim y \text{ if } x = \gamma(y), \qquad \text{with } \gamma \in \Gamma \text{ and } x, y \in \mathbb{H}. \tag{2.4}$$

Our goal is to study the Schrödinger equation of a free particle on X that is finding a complete set of solution $\{(\lambda_i, f_i) \mid i \in \mathbb{N}_{\geq 0}, 0 = \lambda_0 < \lambda_1 ...\}$ of:

$$-\nabla^2 f_i(x) = \lambda_i f_i(x), \tag{2.5}$$

where f_i are smooth functions on X, that are smooth functions on $\mathbb{H}$ that satisfy:

$$f_i(x) = f_i(\gamma \cdot x), \qquad \forall \gamma \in \Gamma. \tag{2.6}$$

The main idea is that the spectrum of the Laplacian on X is related to the spectrum of unitary irreducible representations of G on the space $L^2(G/\Gamma)$.

2.2 The space $L^2(G/\Gamma)$

To define the space $L^2(G/\Gamma)$ we recall that G can be parameterized with the Iwasava NAK decomposition:

$$g(x, y, \theta) = \pm \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}, \qquad x \in \mathbb{R},\ y \in \mathbb{R}_{>0},\ \theta \in (0, 2\pi], \tag{2.7}$$

which means that every element of G can be seen as a composition of a translation of x a scaling of y and a rotation of $\frac{\theta}{2}$. On G we can also define a measure μ invariant under left and right multiplication, using the above coordinates and the normalization such that $\mu(G/\Gamma) = 1$ it takes the form:

$$d\mu(g) = \frac{1}{\mathrm{Vol}(G/\Gamma)} \frac{dx\, dy\, d\theta}{y^2} = \frac{1}{2\pi \cdot \mathrm{Vol}(\mathbb{H}/\Gamma)} \frac{dx\, dy\, d\theta}{y^2}. \tag{2.8}$$

From equation (2.7) it follows that we can see the upper half plane as $\mathbb{H} = G/K$ where

$$K = SO(2, \mathbb{R})/\{\pm\mathbb{I}\}, \tag{2.9}$$

and hence X can be seen as the double quotient $X = (G/K)/\Gamma$. This implies that the space G/Γ is a fiber bundle with base X and fiber K. We can finally define the Hilbert space $L^2(G/\Gamma)$: it is the space of functions $F : G \to C$ such that $F(\gamma g) = F(g)\ \forall\, \gamma \in \Gamma,\ \forall\, g \in G$, the norm is induced from the inner product:

$$\langle F_1, F_2\rangle = \int_{G/\Gamma} F_1(g)^* F_2(g) d\mu(g). \tag{2.10}$$

We can use the μ invariance under right multiplication to find an unitary representation of G on the space $L^2(G/\Gamma)$ with action:

$$g \in G : F(x) \to F(x \cdot g). \tag{2.11}$$

In fact let us call the image of g $\rho(g)$, it is trivial to see that equation (2.11) is a representation, to prove the unitariness let us consider:

$$\langle \rho(g)F_1, \rho(g)F_2\rangle = \int_{G/\Gamma} F_1^*(x \cdot g)F_2(x \cdot g)d\mu(x) = \int_{G/\Gamma} F_1^*(y)F_2(y)d\mu(y) = \langle F_1, F_2\rangle. \tag{2.12}$$

Where we substituted $y = x \cdot g$ and exploited the right invariance of the measure $d\mu(y \cdot g^{-1}) = d\mu(y)$.

The representation we just defined is a valid representation of every subgroup of G, in particular for $K \subset G$ and we can use it to decompose $L^2(G/\Gamma)$ in:

$$L^2(G/\Gamma) = \bigoplus_{n\in Z} V_n, \tag{2.13}$$

where $F \in V_n$ transform as $g_\theta \cdot F = e^{-in\theta}F$ with $g_\theta \in K$. Moreover it can be proven that in the NAK parametrization $F \in V_n$ takes the form:

$$F(x, y, \theta) = y^{|n|}e^{-in\theta}h(x, y). \tag{2.14}$$

From which it follows that V_0 consists of functions that don't depend on the variable θ and therefore $V_0 = L^2(X)$. The next step is finding all the unitary irreducible representations of G and using them to decompose $L^2(G/\Gamma)$.

Chapter 3

Lie Groups

3.1 Lie algebra of $PSL(2,\mathbb{R})$

As defined in the previous chapter $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm\mathbb{I}\}$ where $SL(2,\mathbb{R})$ is the group of 2x2 real matrices with unitary determinant. The corresponding Lie algebra $\mathfrak{psl}(2,\mathbb{R})$ is generated by the Lie group with the exponential map and from the identity $\det(\exp\mathfrak{g}) = \exp(\mathrm{Tr}\,\mathfrak{g})$ we conclude that the Lie algebra is the vector space of 2x2 real matrices with null trace. A useful basis for the complexified Lie algebra $\mathfrak{psl}(2,\mathbb{C})$ is:

$$L_{-1} = \frac{1}{2}\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}, \quad L_0 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad L_1 = \frac{1}{2}\begin{pmatrix} -i & -1 \\ -1 & i \end{pmatrix}, \tag{3.1}$$

with the commutation relations:

$$[L_a, L_b] = (a-b)L_{a+b}. \tag{3.2}$$

In particular in the representation of equation (2.11) it can be proven, using the definition of exponential map and an infinitesimal transformation that the basis (3.1) take the form:

$$\begin{aligned} L_1 &= -e^{i\theta}\left(y(\partial_x + i\partial_y) + \partial_\theta\right); \\ L_0 &= i\partial_\theta; \\ L_{-1} &= e^{-i\theta}\left(y(\partial_x - i\partial_y) + \partial_\theta\right). \end{aligned} \tag{3.3}$$

The quadratic Casimir element will also play an important role, to calculate it we first need to compute the dual basis of the Killing form:

$$K(X,Y) = Tr(ad(X)ad(Y)), \tag{3.4}$$

where $ad(X)$ is the adjoint operator of $X \in \mathfrak{g}$:

$$ad(X)\cdot Y = [X,Y], \qquad X, Y \in \mathfrak{g}. \tag{3.5}$$

The dual basis $\{\tilde{L}_{-1}, \tilde{L}_0, \tilde{L}_1\}$ of (3.1) is defined in such a way that:

$$K(L_i, \tilde{L}_j) = \delta_{i,j}. \tag{3.6}$$

It can be easily shown that in this case the dual basis is:

$$\tilde{L}_{-1} = -\frac{1}{4}L_1, \quad \tilde{L}_0 = \frac{1}{2}L_0, \quad \tilde{L}_1 = -\frac{1}{4}L_{-1}. \tag{3.7}$$

The quadratic Casimir element C_2 is defined, up to a multiplicative constant, as:

$$C_2 = \sum_{k=-1}^{1} L_k\tilde{L}_k \tag{3.8}$$

In our case it is useful to choose 4 as multiplicative constant and the Casimir takes the explicit form:

$$C_2 = L_0^2 - \frac{L_{-1}L_1 + L_1L_{-1}}{2}. \tag{3.9}$$

In particular by direct substitution of equation: (3.3) we find that in the representation of equation (2.11) the Casimir takes the form:

$$C_2 = y^2\left(\partial_x^2 + \partial_y^2\right) + 2y\partial_x\partial_\theta. \tag{3.10}$$

3.2 Unitary irreducible representations of $PSL(2,\mathbb{R})$

It can be proven, see [3] for a proof, that the only unitary irreducible representations of $PSL(2,\mathbb{R})$ are:

- The trivial representation;
- The holomorphic discrete series D_n and anti-holomorphic discrete series $\overline{D}_n$ with $n \in \mathbb{Z}$, $n \geq 1$;
- The principal series $P^+_{i\nu}$ with $\nu \in \mathbb{R}$;
- The complementary series C_s with $s \in (0, \frac{1}{2})$.

To define them explicitly we first identify $PSL(2,\mathbb{R})$ with $SU(1,1)$ with the isomorphism $\tau:\ PSL(2,\mathbb{R}) \to SU(1,1)$:

$$\tau(X) = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}^{-1} \cdot X \cdot \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \qquad X \in PSL(2,\mathbb{R}). \tag{3.11}$$

Remember that $SU(1,1)$ is the group of complex 2x2 matrices of the form:

$$u = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}, \qquad |\alpha|^2 - |\beta|^2 = 1. \tag{3.12}$$

Moreover for $z \in \mathbb{C}$ we define the action of $SU(1,1)$ as:

$$u \cdot z = \frac{\alpha z + \beta}{\beta^* z + \alpha^*}, \qquad u \in SU(1,1), \qquad z \in \mathbb{C}. \tag{3.13}$$

This action preserves the norm hence the unit circle $\partial\mathbb{D} = \{z : |z| = 1\}$, the unit disk $\mathbb{D} = \{z : |z| < 1\}$ and $\mathbb{D}' = \{z : |z| > 1\}$ are left invariant.

Unitary irreducible representations of $PSL(2,\mathbb{R})$ can be constructed using functions on $\mathbb{D}, \mathbb{D}', \partial\mathbb{D}$:

Anti-holomorphic discrete series $\overline{D}_n$: they are realized in the space of holomorphic functions on $\mathbb{D}$ with finite norm:

$$\|f\|_{\overline{D}_n} = \int_{|z|<1} |f(z)|^2 \left(1 - |z|^2\right)^{2n-2} d^2z. \tag{3.14}$$

Using the notation of (3.12) the unitary action of G is defined as:

$$(u \cdot f)(z) \stackrel{\text{def}}{=\!=} (-\beta^* z + \alpha)^{-2n} f(u^{-1} \cdot z). \tag{3.15}$$

The action of L_{-1}, L_0, L_1 can be easily found using the identification with $SU(1,1)$, and an infinitesimal transformation:

$$\begin{aligned}
(L_{-1} \cdot f)(z) &= -\partial_z f(z); \\
(L_0 \cdot f)(z) &= -\left(\partial_z + n\right) f(z); \\
(L_1 \cdot f)(z) &= -\left(z^2\partial_z + 2nz\right) f(z).
\end{aligned} \tag{3.16}$$

We can now decompose $\overline{D}_n$ into irreducible representation of K. A good basis is $f_k = z^k,\ k \in \mathbb{Z}_{\geq 0}$ because we can see that $L_0 \cdot f_k = -(n+k)f_k$ (and L_0 generates K), this implies that the spectrum of L_0 is: $\{-n, -n-1, \ldots\}$ and L_1, L_{-1} are respectively the creation and annihilation operators, in particular there is a highest vector f_0 annihilated by L_{-1}.

Holomorphic discrete series D_n: they are realized in the space of holomorphic functions on $\mathbb{D}'$, we can define the norm in this way:

$$\|f\|^2_{D_n} = \left\|I^{-1}f\right\|^2_{\overline{D}_n}. \tag{3.17}$$

And same for the action:

$$(I(u \cdot f))\,(z) \stackrel{\text{def}}{=\!=} (-\beta^* z + \alpha)^{-2n}\,(If)\,(u^{-1} \cdot z), \tag{3.18}$$

where I is a bijection between anti-holomorphic functions on $\mathbb{D}$ and holomorphic functions on $\mathbb{D}'$:

$$(If)(z) = z^{-2n} f\left((z^*)^{-1}\right). \tag{3.19}$$

We can decompose D_n in irreducible representation of K using the basis $f_k = z^{-2n-k},\ k \in \mathbb{Z}_{\geq 0}$ because we can see that $L_0 \cdot f_k = (n+k)f_k$ and hence the spectrum of L_0 is: $\{n, n+1, ...\}$ and L_1, L_{-1} are respectively the destruction and creation operators, in particular there is a lowest vector f_0 annihilated by L_1.

Principal series $P^+_{i\nu}$: They are realized in the space $L^2(\partial\mathbb{D})$ the norm of this space is:

$$\|f\|^2_{P^+_{i\nu}} \stackrel{\text{def}}{=\!=} \int_{|z|=1} |f(z)|^2 |dz|. \tag{3.20}$$

And the group action is:

$$(u \cdot f)(z) \stackrel{\text{def}}{=\!=} |-\beta z^* + \alpha^*|^{-2\Delta} f(u^{-1} \cdot z). \tag{3.21}$$

With $\Delta = \frac{1}{2} + i\nu$, a good basis for $P_{i\nu}$ that decomposes it in irreducible representation of K is $f_k(z) = z^k,\ k \in \mathbb{Z}$ from which we find that the spectrum of L_0 is $\mathbb{Z}$. This time there is no lowest or highest weight vector but the one dimensional subspace generated by f_{-n-1} is invariant under the action of L_0. At last using equation (3.16) (of course with the substitution $n = \Delta$) it can be easily found by direct substitution in equation (3.9) that the Casimir element has the form $C_2 = \Delta(\Delta - 1)\mathbb{I}$.

Complementary series C_s: they are realized again in the space $L^2(\partial\mathbb{D})$ this time with norm:

$$\|f\|^2_{C_s} = \int_{|z|=1,|w|=1} \frac{f^*(z)f(w)}{|z-w|^{2-2\Delta}} |dz||dw|, \qquad \Delta = \frac{1}{2} + s. \tag{3.22}$$

And group action:

$$(u \cdot f)(z) \stackrel{\text{def}}{=\!=} |-\beta z^* + \alpha^*|^{-2\Delta} f(u^{-1} \cdot z), \qquad \Delta = \frac{1}{2} + s. \tag{3.23}$$

A good basis for C_s that decomposes it in irreducible representation of K is $f_k(z) = z^k,\ k \in \mathbb{Z}$ from which we find that the spectrum of L_0 is $\mathbb{Z}$. Again there are no lowest or highest weight vector but the one dimensional subspace generated by f_{-n-1} is invariant under the action of L_0 and the Casimir element has the form $C_2 = \Delta(\Delta - 1)\mathbb{I}$.

3.3 Decomposing $L^2(G/\Gamma)$

From the result of the previous section we know that the decomposition of $L^2(G/\Gamma)$ into unitary irreducible representations will have the form:

$$L^2(G/\Gamma) = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathcal{D}_n \oplus \bigoplus_{n=1}^{\infty} \overline{\mathcal{D}}_n \oplus \bigoplus_{k=1}^{\infty} \mathcal{C}_{\lambda_k}. \tag{3.24}$$

Notice that the decomposition is a discrete direct sum, for a proof see [2]. Moreover $\mathcal{C}_{\lambda_k}$ unify the complementary and principal series as we will see soon. Let us analyze in detail each addend:

Trivial representation $\mathbb{C}$: the trivial representation simply corresponds to constant functions which have a finite norm since G/Γ has a finite volume.

Discrete series: To build $\mathcal{D}_n$ we recall from the previous section that there is a vector F annihilated by L_1, now using the explicit form of L_1 (equation (3.3)) and the decomposition of $L^2(G/\Gamma)$ of equations (2.13),(2.14) we must have that $F \in V_n$ and by direct substitution:

$$L_1 \cdot F = 0 \to (\partial_x + i\partial_y)h(x, y) = 0, \tag{3.25}$$

and this mean that $h(x, y) = h(z)$ is holomorphic ($z = x + iy$). So we found the lowest weight vector $F(x, y, \theta) = y^n e^{-in\theta}h(z)$, to build the rest of $\mathcal{D}_n$ we can just apply iteratively the operator L_{-1}. To build $\overline{\mathcal{D}}_n$ we can just notice that $F^*(x, y, z) = y^n e^{in\theta}(h(z))^*$ is the highest weight vector for the anti holomorphic discrete series $\overline{\mathcal{D}}_n$, in fact it is annihilated by L_{-1}. Again to build the rest of the space we just apply iteratively the operator L_1.

Principal and complementary series: In the previous section we saw that each principal and complementary space contains a one dimensional subspace that is invariant under the action of K, now let F be a vector of this subspace, using the decomposition of $L^2(G/\Gamma)$ of equations (2.13),(2.14) we must have that:

$$F \in V_0 \implies F(x, y, \theta) = F(x, y). \tag{3.26}$$

To build the rest of the space we just apply iteratively the destruction and creation operators L_1 and L_{-1}. Now recall that for those series the Casimir element $C_2 = \Delta(\Delta - 1) = -\lambda$ so $\lambda \in \left[\frac{1}{4}, \infty\right)$ for the principal series and $\lambda \in \left(0, \frac{1}{4}\right)$ for the complementary series. Moreover if we apply the Casimir element on a function $F = h(x, y)$ of $P_{i\nu}$ or C_s invariant under the action of K, from equation (3.9) we have that:

$$\lambda h(x, y) = -C_2 h(x, y) = -y^2 \left(\partial_x^2 + \partial_y^2\right) h(x, y) = -\nabla^2 h(x, y). \tag{3.27}$$

hence $h(x, y)$ is an eigenfunction of the Schrödinger equation on X with eigenvalue λ and we can unify the complementary and principal series with the notation $\mathcal{C}_\lambda$. where $\lambda \in (0, \infty)$ is the corresponding eigen value of the Casimir operator.

Chapter 4

Finding the bound

4.1 Coherent states

To define coherent states we choose an orthonormal basis of lowest weight vectors in $\mathcal{D}_n$ $\{F_{n,a} \in D_n \cap V_n\}$ (the index a takes in consideration the multiplicity of the discrete series), then the coherent state $\theta_{n,a}$ in the holomorphic discrete series representation is:

$$\theta_{n,a}(z) = e^{zL_{-1}} \cdot F_{n,a}, \qquad z \in \mathbb{C},\ |z| < 1. \tag{4.1}$$

In the same way, noticing that $F^*_{n,a}$ is an orthonormal basis of highest weight vectors in $\overline{\mathcal{D}}_n$ we the coherent state $\overline{\theta}_{n,a}$ in the anti holomorphic discrete series representation is:

$$\overline{\theta}_{n,a}(z) = z^{-2n} e^{-z^{-1}L_1} \cdot F^*_{n,a}, \qquad z \in \mathbb{C},\ |z| > 1. \tag{4.2}$$

Using the definition the definition we can easily find a relation that will be useful later:

$$(\theta_{n,a}(z))^* = (z^*)^{-2n} \overline{\theta}_{n,a}((z^*)^{-1}). \tag{4.3}$$

Moreover we define:

$$\overline{\theta}_{n,a}(\infty) = \lim_{z\to\infty} z^{2n}\overline{\theta}_{n,a}(z) = \lim_{z\to\infty} e^{-z^{-1}L_1} \cdot F^*_{n,a} = F^*_{n,a}. \tag{4.4}$$

We are interested in how the complexified Lie algebra $\mathfrak{psl}(2,\mathbb{C})$ acts on coherent states, a nice way to do this is using the commutation relations:

$$\begin{aligned}
&[L_{-1}, L_{-1}] = 0;\\
&[L_0, L_{-1}] = L_{-1} \implies [L_0, (L_{-1})^k] = k(L_{-1})^k;\\
&[L_1, L_{-1}] = 2L_0 \implies [L_1, (L_{-1})^k] = 2k(L_{-1})^{k-1}L_0 + k(k-1)(L_{-1})^{k-1}.
\end{aligned} \tag{4.5}$$

So from the first commutation rule we find that:

$$L_{-1} \cdot \theta_{n,a} = L_{-1} e^{zL_{-1}} \cdot F_{n,a} = \sum_{j=0}^{\infty} \frac{z^j (L_{-1})^{j+1}}{j!} \cdot F_{n,a} = \partial_z \theta_{n,a}. \tag{4.6}$$

From the second commutation rule:

$$L_0 \cdot \theta_{n,a} = \left(\sum_{j=0}^{+\infty} \frac{z^j (L_{-1})^j L_0}{j!} + \sum_{j=0}^{+\infty} \frac{j z^j (L_{-1})^j}{j!} \right) \cdot F_{n,a} = (n + z\partial_z)\, \theta_{n,a}, \tag{4.7}$$

where we used that $L_0 \cdot F_{n,a} = nF_{n,a}$. Finally from the third commutation rule:

$$L_1 \cdot \theta_{n,a} = \sum_{j=0}^{\infty} \left(\frac{z^j}{j!} (L_{-1})^j L_1 \right) \cdot F_{n,a} + \sum_{j=1}^{\infty} \left(2\frac{z^j}{(j-1)!} (L_{-1})^{j-1} L_0 + \frac{z^j}{(j-2)!} (L_{-1})^{j-1} \right) \cdot F_{n,a}. \tag{4.8}$$

Moreover we know that $L_0 \cdot F_{n,a} = nF_{n,a}$ and $L_1 \cdot F_{n,a} = 0$ hence we conclude that:

$$L_1 \cdot \theta_{n,a} = \left(z^2\partial_z + 2nz\right)\theta_{n,a}. \tag{4.9}$$

In a similar fashion it can be proved that:

$$\begin{aligned} L_{-1} \cdot \overline{\theta}_{n,a} &= \partial_z \overline{\theta}_{n,a}; \\ L_0 \cdot \overline{\theta}_{n,a} &= (z\partial_z + n)\,\overline{\theta}_{n,a}; \\ L_1 \cdot \overline{\theta}_{n,a} &= \left(z^2\partial_z + 2nz\right)\overline{\theta}_{n,a}. \end{aligned} \tag{4.10}$$

4.2 n-function correlator

Given $F_1, F_2, ..., F_n \in C^\infty(G/\Gamma)$ we define the n-function correlator as:

$$\langle F_1 F_2 ... F_n\rangle = \int_{G/\Gamma} F_1(g)...F_n(g)d\mu(g). \tag{4.11}$$

Our goal is to calculate correlator for coherent states, sometimes the notation will be simplified by omitting the second index, i.e $\theta_{n,a} \equiv \theta_n$. Correlator is G-invariant and this follows directly from the invariance of the measure: in fact if we apply a transformation $\tilde{g}$, with the usual action defined in equation (2.11), to the product $(F_1F_2...F_n)(g)$ we have that:

$$\langle \tilde{g} \cdot (F_1 F_2 ... F_n)\rangle = \int_{G/\Gamma} (F_1...F_n)\,(g\tilde{g})d\mu(g) = \int_{G/\Gamma} (F_1...F_n)\,(g')d\mu(g') = \langle F_1F_2...F_n\rangle, \tag{4.12}$$

where we substituted $g' = g\tilde{g}$ and used the invariance of the measure $d\mu(g) = d\mu(g\tilde{g})$. In particular if we apply an infinitesimal transformation of the form $e^{i(\delta s)L_j}$ (with $j \in \{-1, 0, 1\}$) we have that:

$$\langle e^{i(\delta s)L_j} \cdot (F_1F_2...F_n)\rangle = \langle (\mathbb{I} + i(\delta s)L_j)\cdot(F_1F_2...F_n)\rangle = \langle F_1F_2...F_n\rangle + i(\delta s)\langle L_j \cdot (F_1F_2...F_n)\rangle = \langle F_1F_2...F_n\rangle. \tag{4.13}$$

Where in the last equality we used equation (4.12), this in particular implies that:

$$\langle L_j \cdot (F_1F_2...F_n)\rangle = 0. \tag{4.14}$$

G-invariance fixes the form of n-function correlator for small n up to some constants:

1-function correlator: By applying equation (4.14) with $j = -1$ and $j = 0$ on $F_1 = \theta_n(z)$ we find that:

$$\begin{gathered} \int_{G/\Gamma} \partial_z \theta_n(z)d\mu(g) = 0 \implies \partial_z\langle\theta_n(z)\rangle = 0; \\ \int_{G/\Gamma} (\partial_z + n)\theta_n(z)d\mu(g) = 0 \implies (\partial_z + n)\langle\theta_n(z)\rangle = 0. \end{gathered} \tag{4.15}$$

By combining these two equations we conclude that $\langle\theta_n(z)\rangle = 0$ and in the same way it can be found that $\langle\overline{\theta}_n(z)\rangle = 0$.

2-function correlator: We choose $F_1 = \theta_{m,a}(z_1)$,$F_2 = \theta_{n,b}(z_2)$ and let us define

$$f(z_1, z_2) = \langle\theta_{m,a}(z_1)\theta_{n,b}(z_2)\rangle, \tag{4.16}$$

if we apply equation (4.14) with $j = -1$ we find that:

$$\partial_{z_1} f(z_1, z_2) + \partial_{z_2} f(z_1, z_2) = 0, \tag{4.17}$$

and this is solved solved if

$$f(z_1, z_2) \equiv f(z_1 - z_2), \tag{4.18}$$

which means that f depends only on the difference of the two variables. Now if we apply equation (4.14) with $j=0$ we find that:

$$z_1\partial_{z_1}f(z_1,z_2)+z_2\partial_{z_2}f(z_1,z_2)+2nf(z_1,z_2)=0=\left[(z_1-z_2)\partial_{z_1}+2n\right]f(z_1-z_2). \tag{4.19}$$

Which is solved for:

$$f(z_1-z_2)=\frac{C_{n,m,a,b}}{(z_1-z_2)^{2n}}. \tag{4.20}$$

We can see that in the limit $z_1\to z_2$ we have a singularity and hence $C_{n,m,a,b}=0$ and we conclude that:

$$\langle\theta_{m,a}(z_1)\theta_{n,b}(z_2)\rangle=0. \tag{4.21}$$

In the exact same way it can be proven that equation (4.20) holds with $F_1=\theta_{m,a}(z_1)$,$F_2=\overline{\theta}_{n,b}(z_2)$ but in this case there is no singularity because $|z_1|<1$ and $|z_2|>1$. The constant $C_{n,m,a,b}$ however is fixed from orthogonality and we have that $C_{n,m,a,b}=\delta_{n,m}\delta_{a,b}$ and hence

$$\langle\theta_{m,a}(z_1)\overline{\theta_{n,b}}(z_2)\rangle=\frac{\delta_{n,m}\delta_{a,b}}{(z_1-z_2)^{2n}}. \tag{4.22}$$

3-function correlator: We choose $F_1=\theta_n(z_1)$,$F_2=\theta_n(z_2)$, $F_3=\overline{\theta}_{p,a}(z_3)$ and let us define

$$f(z_1,z_2,z_3)=\langle\theta_n(z_1)\theta_n(z_2)\overline{\theta}_{p,a}(z_3)\rangle, \tag{4.23}$$

if we apply equation (4.14) with $j=-1$ we find that:

$$\partial_{z_1}f(z_1,z_2,z_3)+\partial_{z_2}f(z_1,z_2,z_3)+\partial_{z_3}f(z_1,z_2,z_3)=0. \tag{4.24}$$

We can generalize the case of 2-function correlator and this equation is solved if

$$f(z_1,z_2,z_3)\equiv f(z_1-z_2,z_1-z_3,z_2-z_3), \tag{4.25}$$

now if we apply (4.14) with $j=0$ we find that:

$$z_1\partial_{z_1}f(z_1,z_2,z_3)+z_2\partial_{z_2}f(z_1,z_2,z_3)+z_3\partial_{z_3}f(z_1,z_2,z_3)+(2n+p)f(z_1,z_2,z_3)=0. \tag{4.26}$$

Now let us define $z_{ij}\equiv z_i-z_j$ with $i,j\in\{1,2,3\}$, using the chain rule we can rewrite the equation as:

$$\begin{aligned}&z_1\left(\partial_{z_{12}}f(z_{12},z_{13},z_{23})+\partial_{z_{13}}f(z_{12},z_{13},z_{23})\right)+z_2\left(-\partial_{z_{12}}f(z_{12},z_{13},z_{23})+\partial_{z_{23}}f(z_{12},z_{13},z_{23})\right)+\\&+z_3\left(-\partial_{z_{13}}f(z_{12},z_{13},z_{23})-\partial_{z_{23}}f(z_{12},z_{13},z_{23})\right)+(2n+p)f(z_{12},z_{13},z_{23})=0.\end{aligned} \tag{4.27}$$

Which can be rearranged as:

$$z_{12}\partial_{z_{12}}f(z_{12},z_{13},z_{23})+z_{13}\partial_{z_{13}}f(z_{12},z_{13},z_{23})+z_{23}\partial_{z_{23}}f(z_{12},z_{13},z_{23})=-(2n+p)f(z_{12},z_{13},z_{23}). \tag{4.28}$$

This last equation is in general solved for:

$$f(z_1,z_2,z_3)=\frac{f_{n,p,a}}{z_{12}^{\alpha}z_{13}^{\beta}z_{23}^{\gamma}},\qquad \alpha+\beta+\gamma=2n+p. \tag{4.29}$$

We can constrain it even more by applying (4.14) with $j=1$:

$$\begin{aligned}&z_1^2\left(\partial_{z_{12}}f(z_{12},z_{13},z_{23})+\partial_{z_{13}}f(z_{12},z_{13},z_{23})\right)+z_2^2\left(-\partial_{z_{12}}f(z_{12},z_{13},z_{23})+\partial_{z_{23}}f(z_{12},z_{13},z_{23})\right)+\\&+z_3^2\left(-\partial_{z_{13}}f(z_{12},z_{13},z_{23})-\partial_{z_{13}}f(z_{12},z_{13},z_{23})\right)=-\left(2nz_1+2nz_2+2pz_3\right)f(z_{12},z_{13},z_{23}).\end{aligned} \tag{4.30}$$

Using equation (4.29) we can rewrite it as:

$$\left(\alpha(z_1+z_2)+\beta(z_1+z_3)+\gamma(z_2+z_3)\right)f(z_{12},z_{13},z_{23})=\left(2nz_1+2nz_2+2pz_3\right)f(z_{12},z_{13},z_{23}). \tag{4.31}$$

Which is solved if and only if

$$\beta=p, \tag{4.32}$$
$$\gamma=p, \tag{4.33}$$
$$\alpha=2n-p, \tag{4.34}$$

hence we conclude that the 3-function correlator is fixed up to a constant:

$$\langle\theta_n(z_1)\theta_n(z_2)\overline{\theta}_{p,a}(z_3)\rangle=\frac{f_{n,p,a}}{z_{12}^{2n-p}z_{13}^{p}z_{23}^{p}}. \tag{4.35}$$

4-function correlator: This is the smallest n such that the n-function correlator is not uniquely determined up to a constant, in fact it can be proven that, see [5]:

$$\langle \theta_n(z_1)\theta_n(z_2)\overline{\theta}_n(z_3)\overline{\theta}_n(z_4)\rangle = \frac{g(z)}{z_{12}^{2n} z_{34}^{2n}}. \tag{4.36}$$

Where $g(z)$ is an unfixed single variable function and

$$z = \frac{z_{12}z_{34}}{z_{13}z_{24}} \tag{4.37}$$

is the cross ratio of the four points.

4.3 Product expansion of coherent states

In section (2.2) we described the decomposition of $L^2(G/\Gamma)$ in unitary irreducible representations of G from which we derive that every function $F \in L^2(G/\Gamma)$ can be decomposed as:

$$F = P_{\mathbb{C}}(F) + \sum_{n=0}^{\infty}\left(P_{\mathcal{D}_n}(F) + P_{\overline{\mathcal{D}}_n}(F)\right) + \sum_{k=1}^{\infty} P_{\mathcal{C}_{\lambda_k}}(F). \tag{4.38}$$

Where P_H is the projector onto the subspace H. In this section we want to constrain, using the G-invariance, the decomposition of the product of two coherent states.

Expansion of $\theta_n(z_1)\theta_n(z_2)$: We will prove that the $\theta_n(z_1)\theta_n(z_2)$ has a non-trivial projection only on $\mathcal{D}_n$:

$$P_H\left(\theta_n(z_1)\theta_n(z_2)\right) = 0, \qquad \text{unless } H = \mathcal{D}_p,\ p \text{ even},\ p \geq 2n. \tag{4.39}$$

The proof is straightforward, we just compute the projection using G-invariance and all the results of the previous section. We will also use, without giving a proof, that there are some well-defined projectors that maps the irreducible unitary representations on $L^2(G/\Gamma)$ to their corresponding irreducible unitary representations defined on $\mathbb{H}$ in section (3.2):

$$\begin{aligned}
&\pi_{D_n} : \mathcal{D}_n \to D_n \\
&\pi_{\overline{D}_n} : \overline{\mathcal{D}}_n \to \overline{D}_n \\
&\pi_{\mathcal{C}_\lambda} : \mathcal{C}_\lambda \to P^+_{i\lambda} \text{ or } C_\lambda
\end{aligned} \tag{4.40}$$

Let us begin from the projection on the trivial representation $P_{\mathbb{C}}(\theta_n(z_1)\theta_n(z_2))$: it is a G-invariance function of $z_1, z_2 \in \mathbb{D}$ and we proved in the previous section that the only possible form is:

$$P_{\mathbb{C}}(\theta_n(z_1)\theta_n(z_2)) = \frac{A}{(z_1 - z_2)^{2n}} \tag{4.41}$$

But for $z_1 = z_2$ the projector is singular, hence $A = 0$ and

$$P_{\mathbb{C}}(\theta_n(z_1)\theta_n(z_2)) = 0. \tag{4.42}$$

Now consider $P_{\overline{\mathcal{D}}_p}(\theta_n(z_1)\theta_n(z_2))$ and its projection on $\overline{D}_p$ evaluated on a point z_3:

$$P_{\overline{\mathcal{D}}_p}(\theta_n(z_1)\theta_n(z_2)) \to \delta_{z_3}\left(\pi_{\overline{D}_p}\left(P_{\overline{\mathcal{D}}_p}(\theta_n(z_1)\theta_n(z_2))\right)\right), \tag{4.43}$$

it is a G-invariant holomorphic function of z_1, z_2, z_3 and we have already seen in the previous section that the only possible form is:

$$\delta_{z_3}\left(\pi_{\overline{D}_p}\left(P_{\overline{\mathcal{D}}_p}(\theta_n(z_1)\theta_n(z_2))\right)\right) = \frac{A}{z_{12}^{2n-p} z_{13}^{p} z_{23}^{p}} \tag{4.44}$$

but for the anti-holomorphic discrete representation we have that $z_3 \in \mathbb{D}$, hence have a singularity for $z_1 = z_3$ and this implies that $A = 0$ and

$$P_{\overline{\mathcal{D}}_p}(\theta_n(z_1)\theta_n(z_2)) = 0. \tag{4.45}$$

For the holomorphic discrete series it can be proven the exact same result:

$$\delta_{z_3}\left(\pi_{D_p}\left(P_{\mathcal{D}_p}(\theta_n(z_1)\theta_n(z_2))\right)\right) = \frac{A}{z_{12}^{2n-p} z_{13}^p z_{23}^p} \tag{4.46}$$

and if $p \geq 2n$ there are no singularities since $z_3 \in \mathbb{D}'$. However for p odd the left hand side is even under the permutation $z_1 \leftrightarrow z_2$ while the right hand side is odd, hence we conclude that in general

$$P_{\mathcal{D}_p}(\theta_n(z_1)\theta_n(z_2)) \neq 0 \qquad \text{only for } p \geq 2n,\ p \text{ even}. \tag{4.47}$$

Finally let us consider $P_{\mathcal{C}_{\lambda_k}}(\theta_n(z_1)\theta_n(z_2))$, using again G-invariance and considering its projection onto $P^+_{i\lambda_k}$ or C_{λ_k} evaluated in a point z_3 it can be proven that:

$$\delta_{z_3}\left(\pi_{\mathcal{C}_{\lambda_k}}\left(P_{\mathcal{C}_{\lambda_k}}(\theta_n(z_1)\theta_n(z_2))\right)\right) = \frac{A z_3^{\Delta_k}}{z_{12}^{2n-\Delta_k} z_{13}^{\Delta_k} z_{23}^{\Delta_k}} \tag{4.48}$$

Notice that the result is slightly different because in this case $z_3 \in \partial\mathbb{D}$ and hence the function is not holomorphic in z_3. Again the function is singular for $z_1 = z_3$ and hence we conclude that:

$$P_{\mathcal{C}_{\lambda_k}}(\theta_n(z_1)\theta_n(z_2)) = 0 \tag{4.49}$$

The explicit form of the expansion can also be calculated we have that:

$$\theta_n(z_1)\theta_n(z_2) = \sum_{\substack{p=2n \\ p \text{ even}}}^{\infty} \sum_{a=1}^{l_p} f_{p,a} \tau_p(e_a \otimes C_p(z_1, z_2)). \tag{4.50}$$

Where:

$$C_p(z_1, z_2) = \sqrt{\frac{2n-1}{\pi}} \frac{1}{z_{12}^{2n-p} z_{13}^p z_{23}^p}. \tag{4.51}$$

This explicit form of the expansion will not be formally proven, for a rigorous proof see [4], here we will just give an idea: we already saw in section (3.3) that the space $\mathcal{D}_p$ contains many independent copies of D_p, one for each linearly independent lowest weight vector. Hence if we call l_p the number of independent copies we have that $\mathcal{D}_p$ is isomorphic to $\mathbb{C}^{l_p} \otimes D_p$. Moreover it can be proven that there exists an unitary isomorphism $\tau_p : \mathbb{C}^{l_p} \otimes D_p \to \mathcal{D}_p$, so if we define an orthonormal basis $\{e_a,\ 0 < a \leq l_p\}$ for $\mathbb{C}^{l_p}$ we can reverse the projector π_{D_p} of equation (4.46):

$$P_{\mathcal{D}_p}(\theta_n(z_1)\theta_n(z_2)) = \sum_{a=1}^{l_p} A_{p,a} \tau_p\left(e_a \otimes \frac{1}{z_{12}^{2n-p} z_{13}^p z_{23}^p}\right), \qquad p \geq 2n,\ p \text{ even}. \tag{4.52}$$

Where $A_{p,a}$ are just some unknown constants, the expansion follows directly from the substitution:

$$A_{p,a} = f_{p,a}\sqrt{\frac{2n-1}{\pi}}. \tag{4.53}$$

Expansion of $\theta_n(z_1)\overline{\theta}_n(z_2)$: We will prove that the $\theta_n(z_1)\overline{\theta}_n(z_2)$ has a non-trivial projection only on $\mathbb{C}$ and $\mathcal{C}_{\lambda_k}$:

$$P_H\left(\theta_n(z_1)\overline{\theta}_n(z_2)\right) = 0, \qquad \text{unless } H = \mathcal{C}_{\lambda_k} \text{ or } H = \mathbb{C}. \tag{4.54}$$

The proof is almost identical to the one used for the product expansion of $\theta_n(z_1)\theta_n(z_2)$: for the trivial representation we find that

$$P_{\mathbb{C}}(\theta_n(z_1)\overline{\theta}_n(z_2)) = \frac{A}{(z_1 - z_2)^{2n}}. \tag{4.55}$$

And this time there are no singularity since $z_1 \in D'$, $z_2 \in \overline{D}'$.

For the holomorphic/anti-holomorphic discrete series representations we find that:

$$\begin{aligned} \delta_{z_3}\left(\pi_{\overline{D}_p}\left(P_{\overline{\mathcal{D}}_p}(\theta_n(z_1)\overline{\theta}_n(z_2))\right)\right) &= \frac{A}{z_{12}^{2n-p}z_{13}^{p}z_{23}^{p}}, \\ \delta_{z_3}\left(\pi_{D_p}\left(P_{\mathcal{D}_p}(\theta_n(z_1)\overline{\theta}_n(z_2))\right)\right) &= \frac{A}{z_{12}^{2n-p}z_{13}^{p}z_{23}^{p}}. \end{aligned} \tag{4.56}$$

But this time they are both singular: the first one is singular for $z_1 = z_3$, the second one for $z_2 = z_3$ and hence we conclude that:

$$P_{\mathcal{D}_p}(\theta_n(z_1)\overline{\theta}_n(z_2)) = P_{\overline{\mathcal{D}}_p}(\theta_n(z_1)\overline{\theta}_n(z_2)) = 0. \tag{4.57}$$

Finally for the projection on $\mathcal{C}_{\lambda_k}$ it can be shown that:

$$\delta_{z_3}\left(\pi_{\mathcal{C}_{\lambda_k}}\left(P_{\mathcal{C}_{\lambda_k}}(\theta_n(z_1)\theta_n(z_2))\right)\right) = \frac{A z_3^{\Delta_k}}{z_{12}^{2n-\Delta_k}z_{13}^{\Delta_k}z_{23}^{\Delta_k}}, \tag{4.58}$$

and the projection is in general non-zero.

The expansion can be calculated explicitly:

$$\begin{aligned} &(1)\ \ P_{\mathbb{C}}(\theta_n(z_1)\overline{\theta}_n(z_2)) = \frac{A}{(z_1-z_2)^{2n}}, \\ &(2)\ \ P_{\mathcal{C}_{\lambda_k}}(\theta_n(z_1)\overline{\theta}_n(z_2)) = \sum_{a=1}^{d_k} c_{k,a}\kappa_k(e_a \otimes \tilde{C}_k(z_1,z_2)). \end{aligned} \tag{4.59}$$

Where $\tilde{C}_k(z_1,z_2)(z_0)$:

$$\tilde{C}_k(z_1,z_2)(z_0) = \frac{N_{\Delta_k} z_3^{\Delta_k}}{z_{12}^{2n-\Delta_k}z_{13}^{\Delta_k}z_{23}^{\Delta_k}}, \tag{4.60}$$

and N_{Δ_k} is defined in such a way that $\tilde{C}_k(z_1,z_2)(z_0)$ has unit norm in $P_{i\lambda_k}$ or C_{λ_k}. The explicit form of the projection on $\mathbb{C}$ has already been proved, for the projection on $\mathcal{C}_{\lambda_k}$ as in the previous case we will just give an idea of the proof, a rigorous one can be found in [4]. Let $H = P^+_{i\lambda_k}$ or $H = C_{\lambda_k}$ be the appropriate irreducible representation for the value λ_k. We saw in section (3.3) that in general in $\mathcal{C}_{\lambda_k}$ there will be many copies of H, one for each linearly independent vector defined in (3.26). Hence if we call d_k the number of independent copies we have that $\mathcal{C}_{\lambda_k}$ is isomorphic to $\mathbb{C}^{d_k} \otimes H$. Moreover it can be proven that there exists an unitary isomorphism $\kappa_k : \mathbb{C}^{d_k} \otimes H \to \mathcal{C}_{\lambda_k}$, so by defining an orthonormal basis $\{e_a\}$ for $\mathbb{C}^n$ we can find the inverse of the projector $\pi_{\mathcal{C}_{\lambda_k}}$ of equation (4.58) up to some unknown constants:

$$P_{\mathcal{C}_{\lambda_k}}(\theta_n(z_1)\overline{\theta}_n(z_2)) = \sum_{a=1}^{d_k} A_{k,a}\kappa_k\left(e_a \otimes \frac{z_3^{\Delta_k}}{z_{12}^{2n-\Delta_k}z_{13}^{\Delta_k}z_{23}^{\Delta_k}}\right). \tag{4.61}$$

Where $A_{k,a}$ are some unknown constants, the expansion follows directly from the substitution:

$$A_{k,a} = c_{k,a}N_{\Delta_k}. \tag{4.62}$$

4.4 Bootstrap

In section (4.2) we saw that the 4-function correlator $\langle\theta_n(z_1)\theta_n(z_2)\overline{\theta}_n(z_3)\overline{\theta}_n(z_4)\rangle$ is determined by a single variable function $g(z)$. In this section we want to use the product expansion and the crossing symmetry to find two explicit forms for $g(z)$: the first will be found by expanding the products $\theta_n(z_1)\theta_n(z_2)$ and $\overline{\theta}_n(z_3)\overline{\theta}_n(z_4)$, while the second by expanding $\theta_n(z_1)\overline{\theta}_n(z_4)$ and $\theta_n(z_2)\overline{\theta}_n(z_3)$. Ultimately by matching those two expressions we will find the bound on the lowest eigenvalue.

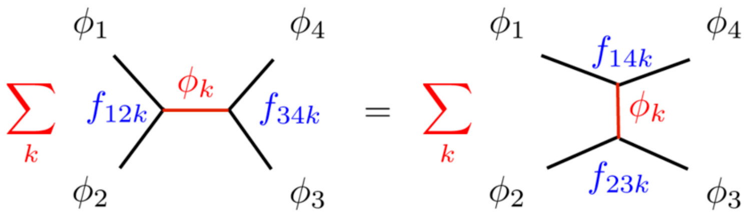

Figure 4.1: Graphical interpretation of the bootstrap, $\phi_i = \theta_n(z_i)$ or $\overline{\theta}_n(z_i)$

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Using the constants $f_{p,a}$ and $c_{k,a}$ defined in the previous section (4.3) we have that the function $g(z)$ in equation (4.36) can be written as:

(1) s-channel expansion:

$$g(z) = \sum_{\substack{p=2n \\ p \text{ even}}}^{\infty} \sum_{a=1}^{l_p} |f_{p,a}|^2 \mathcal{G}_p(z). \tag{4.63}$$

With:

$$\mathcal{G}_\Delta(z) = z^\Delta {}_2F_1(\Delta; \Delta; 2\Delta; z), \tag{4.64}$$

where ${}_2F_1$ is the Gauss hypergeometric function.

(2) t-channel expansion:

$$g(z) = \left(\frac{z}{1-z}\right)^{2n} \left(1 + \sum_{k=1}^{\infty} \sum_{a=1}^{d_k} c_{k,a}^2 \mathcal{H}_{\Delta_k}(z)\right). \tag{4.65}$$

With:

$$\mathcal{H}_\Delta(z) = {}_2F_1\left(\Delta; 1-\Delta; 1; \frac{z}{z-1}\right). \tag{4.66}$$

Proof of part (1): As we anticipated we want to expand the products $\theta_n(z_1)\theta_n(z_2)$ and $\overline{\theta}_n(z_3)\overline{\theta}_n(z_4)$, we don't have a formula for expanding two coherent states in the anti holomorphic discrete series representation, however we can exploit equation (4.1):

$$\left(\overline{\theta}_n(z_3)\overline{\theta}_n(z_4)\right)^* = (z_3^*)^{-2n}(z_4^*)^{-2n}\theta_n((z_3^*)^{-1})\theta_n((z_4^*)^{-1}). \tag{4.67}$$

Moreover given two functions $F_1, F_2 \in C^\infty(G/\Gamma)$ we can transform their correlator in a scalar product:

$$\langle F_1 F_2 \rangle = \langle F_1^*, F_2 \rangle, \tag{4.68}$$

with the usual inner product defined in (2.10). So using equation (4.67) and the relation between correlator and scalar product we can rewrite the 4-function correlator as:

$$\langle \theta_n(z_1)\theta_n(z_2)\overline{\theta}_n(z_3)\overline{\theta}_n(z_4)\rangle = \langle (z_3^*)^{-2n}(z_4^*)^{-2n}\theta_n((z_3^*)^{-1})\theta_n((z_4^*)^{-1}), \theta_n(z_1)\theta_n(z_2)\rangle. \tag{4.69}$$

Finally we can expand the two products $\theta_n((z_3^*)^{-1})\theta_n((z_4^*)^{-1})$ and $\theta_n(z_1)\theta_n(z_2)$ using equation (4.50):

$$\langle (z_3^*)^{-2n}(z_4^*)^{-2n}\theta_n((z_3^*)^{-1})\theta_n((z_4^*)^{-1}), \theta_n(z_1)\theta_n(z_2)\rangle = \sum_{\substack{p=2n \\ p \text{ even}}}^{\infty} \sum_{a=1}^{l_p} |f_{p,a}|^2 z_{12}^{-2n} z_{34}^{-2n} \mathcal{G}_p(z). \tag{4.70}$$

Where $\mathcal{G}_p(z)$ is defined as:

$$z_{12}^{-2n} z_{34}^{-2n} \mathcal{G}_p(z) = z_3^{-2n} z_4^{-2n} \langle C_p((z_3^*)^{-1}, (z_4^*)^{-1}), C_p(z_1, z_2)\rangle. \tag{4.71}$$

Notice that we exploited that τ_p is a unitary isomorphism:

$$\langle \tau_p(e_{a_1} \otimes C_p(z_1, z_2)), \tau_p(e_{a_2} \otimes C_p((z_3^*)^{-1}, (z_4^*)^{-1})) \rangle = \delta_{a_1,a_2} \langle C_p(z_1, z_2), C_p((z_3^*)^{-1}, (z_4^*)^{-1}) \rangle, \quad (4.72)$$

and that the subspaces $\mathcal{D}_p$ are orthogonal:

$$\langle \tau_{p_1}(...), \tau_{p_2}(...) \rangle = 0, \qquad \text{for } p_1 \neq p_2. \quad (4.73)$$

To conclude the proof we have to show that $\mathcal{G}_\Delta(z) = z^\Delta {}_2F_1(\Delta; \Delta; 2\Delta; z)$, the easiest way is to apply the quadratic Casimir operator of $D_p \otimes D_p$ on both side of the equation: in fact it can be proven, see [1], that the right hand side of the equation is an eigenfunction of that Casimir operator with eigenvalue $p(p-1)$. Moreover remember that given two representations of a Lie Algebra $\mathfrak{g}$: (V_1, π_1) and (V_2, π_2) the representation induced on the tensor product $(V_1 \otimes V_2, \pi)$ is:

$$\pi(X) = \pi_1(X) \otimes \mathbb{I} + \mathbb{I} \otimes \pi_2(X), \qquad X \in \mathfrak{g}. \quad (4.74)$$

In our case $V_1 = D_p$ and $V_2 = D_p$ and to calculate the quadratic Casimir equation (3.9) we only need the images of L_{-1}, L_0, L_1 which can be found by combining equation (3.16) and equation (4.74). The result is a differential operator in the variables z_1, z_2 which applied to the left hand side of equation (4.71) leads to:

$$z^2(1-z)\partial_z^2 \mathcal{G}_p(z) - z^2 \partial_z \mathcal{G}_p(z) = p(p-1)\mathcal{G}_p(z). \quad (4.75)$$

This is a second order ordinary differential equation which for $p \geq 2n > 1$ has general solution:

$$\mathcal{G}_p^z = Az^p {}_2F_1(p; p; 2p; z) + Bz^{1-p} {}_2F_1(1-p; 1-p; 1-2p; z). \quad (4.76)$$

For $z \to 0$ the second term is singular hence $B = 0$, to find the value of A we set $z_1 = z_2 = 0$ and $z_3 = z_4 = \infty$ such that $z \to 0$ and ${}_2F_1(p; p; 2p; z) \approx 1$ which leads to $\mathcal{G}_p(z) \approx Az^p$. With the same substitution equation (4.71) becomes:

$$\mathcal{G}_p(z) = z^p \langle C_p(0,0), C_p(0,0) \rangle. \quad (4.77)$$

With $C_p(0,0)(x) = \sqrt{\frac{2n-1}{\pi}} x^{-2p}$, it can be proven that the scalar product $\langle C_p(0,0), C_p(0,0) \rangle = 1$ which implies that $A = 1$ and this ends the proof of part (1).

Proof of part (2): It follows the same idea of part (1) but this time we expand the products $\theta_n(z_2)\overline{\theta_n}(z_3)$ and $\theta_n(z_1)\overline{\theta_n}(z_4)$. First of all we transform the 4-function correlator in a scalar product:

$$\langle \theta_n(z_1)\theta_n(z_2)\overline{\theta}_n(z_3)\overline{\theta}_n(z_4) \rangle = \langle (\theta_n(z_2)\overline{\theta}_n(z_3))^*, \theta_n(z_1)\overline{\theta}_n(z_4) \rangle. \quad (4.78)$$

Then using (4.1) we rewrite $\theta_n(z_2)\overline{\theta_n}(z_3)$ as:

$$\left(\theta_n(z_2)\overline{\theta}_n(z_3)\right)^* = (z_2^*)^{-2n}(z_3^*)^{-2n}\overline{\theta}_n((z_2^*)^{-1})\theta_n((z_3^*)^{-1}). \quad (4.79)$$

So we substitute equation (4.79) into equation (4.78) and we use equations (4.54) and (4.59) to expand the products:

$$\langle \theta_n(z_1)\theta_n(z_2)\overline{\theta}_n(z_3)\overline{\theta}_n(z_4) \rangle = z_{14}^{-2n} z_{23}^{-2n} + \sum_{k=1}^{\infty} \sum_{a=1}^{d_k} c_{k,a}^2 z_2^{-2n} z_3^{-2n} \langle \tilde{C}_k((z_3^*)^{-1}, (z_2^*)^{-1}), \tilde{C}_k(z_1, z_4) \rangle, \quad (4.80)$$

which can be rewritten as:

$$\langle \theta_n(z_1)\theta_n(z_2)\overline{\theta}_n(z_3)\overline{\theta}_n(z_4) \rangle = z_{14}^{-2n} z_{23}^{-2n} + \sum_{k=1}^{\infty} \sum_{a=1}^{d_k} c_{k,a}^2 z_{12}^{-2n} z_{34}^{-2n} \left(\frac{z}{1-z}\right)^{2n} \mathcal{H}_{\Delta_k}(z). \quad (4.81)$$

Where $\mathcal{H}_{\Delta_k}(z)$ is defined as:

$$z_{12}^{-2n} z_{34}^{-2n} \left(\frac{z}{1-z}\right)^{2n} \mathcal{H}_{\Delta_k}(z) = z_2^{-2n} z_3^{-2n} \langle \tilde{C}_k((z_3^*)^{-1}, (z_2^*)^{-1}), \tilde{C}_k(z_1, z_4) \rangle. \quad (4.82)$$

Notice that we used that κ_k is a unitary isomorphism:

$$\langle \kappa_k(e_{a_1} \otimes \tilde{C}_k((z_3^*)^{-1}, (z_2^*)^{-1})), \kappa_k(e_{a_2} \otimes \tilde{C}_k(z_1, z_4))\rangle = \delta_{a_1,a_2}\langle \tilde{C}_k((z_3^*)^{-1}, (z_2^*)^{-1}), \tilde{C}_k(z_1, z_4)\rangle, \quad (4.83)$$

and that the subspaces $\mathcal{C}_\lambda$ are orthogonal:

$$\langle \kappa_{k_1}(...), \kappa_{k_2}(...)\rangle = 0, \qquad \text{for } k_1 \neq k_2. \quad (4.84)$$

Finally applying the Casimir operator to both side of (4.82) and using the same strategy of part (1) it can be proven that:

$$\mathcal{H}_\Delta(z) = {}_2F_1\left(\Delta, 1-\Delta, 1, \frac{z}{1-z}\right). \quad (4.85)$$

4.5 Crossing equation

In the previous section we found two equivalent expressions for the function $g(z)$ that can be matched to get the crossing equation:

$$\sum_{\substack{p=2n \\ p \text{ even}}}^{\infty} \sum_{a=1}^{l_p} |f_{p,a}|^2 \mathcal{G}_p(z) = \left(\frac{z}{1-z}\right)^{2n} \left(1 + \sum_{k=1}^{\infty}\sum_{a=1}^{d_k} c_{k,a}^2 \mathcal{H}_{\Delta_k}(z)\right). \quad (4.86)$$

From this equation we want to find a bound on the lowest non-zero eigenvalue of the Schrödinger equation on X (2.5). We begin by rewriting (4.86) in the following way:

$$\sum_{\substack{p=2n \\ p \text{ even}}}^{\infty} S_p \mathcal{G}_p(z) = \left(\frac{z}{1-z}\right)^{2n} \left(1 + \sum_{k=1}^{\infty} T_k \mathcal{H}_{\Delta_k}(z)\right). \quad (4.87)$$

where we substituted

$$S_p = \sum_{a=1}^{l_p} |f_{p,a}|^2, \quad (4.88)$$

$$T_k = \sum_{a=1}^{d_k} c_{k,a}^2. \quad (4.89)$$

Now we can extract information by expanding both side of the crossing equation order by order around $z = 0$, in such a way to write each S_p in function of T_k and eigenvalues data (which are stored in $\mathcal{H}_{\Delta_k}$, remember that $\Delta_k(1-\Delta_k) = \lambda_k$ with λ_k eigenvalue). In particular we know that $S_p = 0$ for p odd so we can constrain λ_k and T_k by simply considering the Taylor expansion for odd powers of z. Moreover notice that the each addend of the left hand side of the crossing equation is symmetric under the substitution $z \to \frac{z}{z-1}$ while each term of the right hand doesn't have this symmetry, so we can find more constraints on λ_k and T_k by antisymmetrizing the crossing equation. An elegant way to both antisymmetrize and find the Taylor expansion is to exploit the orthogonality of hypergeometric functions:

$$\frac{1}{2\pi i}\oint z^{-2}\mathcal{G}_{1-p}(z)\mathcal{G}_q(z) = \delta_{p,q}. \quad (4.90)$$

Where we are integrating on a loop that encloses $z = 0$ as only point of singularity. By applying this to both sides of the crossing equation we find that:

$$-S_p + F_p^n(0) + \sum_{k=1}^{\infty} T_k F_p^n(\lambda_k) = 0, \qquad p \geq 2n. \quad (4.91)$$

Where we defined $S_p = 0$ for p odd and

$$F_p^n(\lambda) = \frac{1}{2\pi i}\oint z^{-2}\mathcal{G}_{1-p}(z)\left(\frac{z}{1-z}\right)^{2n}\mathcal{H}_\Delta(z). \quad (4.92)$$

In particular notice that using the residue theorem we only have to calculate the residue at $z = 0$ of the function under the integral.

The easiest way to calculate a bound on the lowest eigenvalue is to compute equation (4.91) for $p = 2n + 1$ and $p = 2n + 3$:

Case $p = 2n + 1$: we want to find the residue at $z = 0$ of:

$$f(z) = z^{-2}\mathcal{G}_{-2n}(z)\left(\frac{z}{1-z}\right)^{2n}\mathcal{H}_\Delta(z). \tag{4.93}$$

Expanding every function at the first order around $z = 0$ is enough:

$$\mathcal{G}_{-2n}(z) \approx z^{-2n}(1 - nz), \tag{4.94}$$
$$\mathcal{H}_\Delta(z) \approx 1 + \Delta(\Delta - 1)z = 1 - \lambda z, \tag{4.95}$$
$$\left(\frac{z}{1-z}\right)^{2n} \approx z^{2n}(1 + 2nz), \tag{4.96}$$

by direct substitution in (4.93) we find that the residue is

$$Res(f, 0) = F^n_{2n+1}(\lambda) = n - \lambda \tag{4.97}$$

and hence we find the identity:

$$n - \sum_{k=1}^{\infty}(\lambda_k - n)T_k = 0. \tag{4.98}$$

Case $p = 2n + 3$: this time we want to calculate the residue at $z = 0$ of the function:

$$f(z) = z^{-2}\mathcal{G}_{-2n-2}(z)\left(\frac{z}{1-z}\right)^{2n}\mathcal{H}_\Delta(z) \tag{4.99}$$

the method is the same of the former case but this time we have to expand to the third order around $z = 0$:

$$\mathcal{G}_{-2n-2}(z) \approx z^{-2n-2}\left(1 - (n+1)z + \frac{(n+1)(2n+1)^2}{8n+6}z^2 + \frac{n^2(n+1)(2n+1)}{12n+9}z^3\right), \tag{4.100}$$
$$\mathcal{H}_\Delta(z) \approx 1 - \lambda z - \frac{1}{4}\lambda(-\lambda + 2)z^2 - \frac{1}{36}\lambda(\lambda^2 - 10\lambda + 12)z^3, \tag{4.101}$$
$$\left(\frac{z}{1-z}\right)^{2n} \approx z^{2n}\left(1 + 2nz + n(2n+1)z^2 + \frac{2}{3}n(2n^2 + 3n + 1)z^3\right). \tag{4.102}$$

From which we can easily compute the residue $F^n_{2n+3}(\lambda)$ and $F^n_{2n+3}(0)$ and get an equation of the form:

$$F^n_{2n+3}(0) + \sum_{k=1}^{\infty} T_k F^n_{2n+3}(\lambda_k) = 0. \tag{4.103}$$

We want to find an equation in which the contribution of $\lambda_0 = 0$ drops out and this can be easily done by subtracting equation (4.103) multiplied by n and equation (4.98) multiplied by $F^n_{2n+3}(0)$ the result is:

$$\sum_{k=1}^{\infty}\lambda_k\left(\lambda_k^2 - (9n+1)\lambda_k + 12n^2\right)T_k = 0. \tag{4.104}$$

The polynomial $\lambda_k^2 - (9n + 1)\lambda_k + 12n^2$ has roots:

$$\lambda_{k,\pm} = \frac{9n + 1 \pm \sqrt{33n^2 + 18n + 1}}{2}, \tag{4.105}$$

which are both positive for each $n > 0$ because

$$(9n+1)^2 = 81n^2 + 18n + 1 > 33n^2 + 18n + 1, \qquad \forall n > 0. \tag{4.106}$$

Since $0 = \lambda_0 < \lambda_1 < ... < \lambda_n < ...$ and $T_k \geq 0$, see equation (4.88), we conclude that in order to satisfy (4.104) λ_1 can be at most the greatest root:

$$\lambda_1 \leq \frac{9n + 1 + \sqrt{33n^2 + 18n + 1}}{2}. \tag{4.107}$$

Actually we just need the assumption that there is a $\beta > 0$ such that $T_\beta > 0$ because in that case equation (4.107) holds for the smallest λ_k such that $T_k > 0$ and we know that $\lambda_k > \lambda_1$.

This bound holds for each orbifold X such that the subspace V_n defined in (2.13) is not empty. In a more mathematical language it means that there must be a non zero modular form of weight $2n$. In particular it can be proven, see [6], that every hyperbolic 2-orbifold has a modular form of weight at most 12. So by substituting $n = 6$ in equation (4.107) we find that:

$$\lambda_1 \leq \frac{55 + \sqrt{1297}}{2} \approx 45.50694. \tag{4.108}$$

This bound is valid for every hyperbolic 2-orbifold and in particular is close to the λ_1 of the $(2, 3, 7)$ triangle orbifold which has the largest known eigenvalue $\lambda_1 \approx 44.88835$. However we exploited only two of the infinite addends of the crossing equation (4.86), with a more sophisticated method that includes the contribution of more terms of the crossing equation the bound can be improved and it can be found, see [4] for a proof, that:

$$\lambda_1 \leq 44.8883537, \tag{4.109}$$

which is up to many decimal digits indistinguishable from the λ_1 on the triangle orbifold.

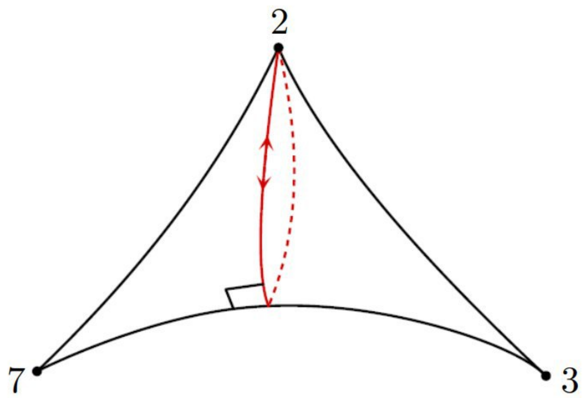

Figure 4.2: Drawing of $(2, 3, 7)$, the hyperbolic 2-orbifold with smallest area

Bibliography

[1] F. A. Dolan and H. Osborn. "Conformal partial waves and the operator product expansion". In: *Nucl. Phys. B* 678 (2004), pp. 491–507. DOI: 10.1016/j.nuclphysb.2003.11.016. arXiv: hep-th/0309180.

[2] I. M. Gelfand, M. I. Graev, and I. I. Pyatetskii-Shapiro. *Generalized functions*. Vol. 6. Providence: AMS Chelsea Publishing, 2016.

[3] A. W. Knapp. *Representation theory of semisimple groups*. Princeton: Princeton University Press, 2001.

[4] Petr Kravchuk, Dalimil Mazac, and Sridip Pal. *Automorphic Spectra and the Conformal Bootstrap*. Nov. 2021. arXiv: 2111.12716 [hep-th].

[5] Aritra Pal and Koushik Ray. "Conformal Correlation functions in four dimensions from Quaternionic Lauricella system". In: *Nucl. Phys. B* 968 (2021), p. 115433. DOI: 10.1016/j.nuclphysb.2021.115433. arXiv: 2005.12523 [hep-th].

[6] John Voight and David Zureick-Brown. *The canonical ring of a stacky curve*. Mar. 2020. arXiv: 1501.04657 [math.AG].