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A Holographic Description of Black Hole Microstates

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Introduction

General Relativity (GR) provides our deepest understanding of gravitational physics at the classical level. A hundred years after its original formulation, the gravitational waves, the last missing piece of GR predictions, have been discovered. Einstein's insight on the Equivalence Principle (in all its different formulations) and, more in general, GR have passed all the experimental tests with extremely high accuracy [1]: even the familiar technology we use, such as the GPS, is a daily confirmation of Einstein's theory.
At the present time there are no experimental evidences that GR should be modified. This is good for Einstein, but bad for us. There are, in fact, strong theoretical reasons for believing that GR is not the ultimate theory of gravity: however, because there are no experimental evidences of this, we do not have hints on which is the right way to go and we are blind to what Nature might tell us about more fundamental gravitational physics. The need of modifying GR can be justified at different levels: these can be summarized in the fact that GR is a classical theory and is mutually inconsistent with quantum mechanics.

It is generally believed that the correct, fundamental description of all physical fields should undergo the general framework of quantum mechanics. At a first stage this is an assumption we make: there cannot be two worlds, one quantum and one classical, but the quantum theory should contain the classical one as a certain limiting case. Gravitational physics should make no exception: thus, we have to look for a theory of quantum gravity that falls to GR in the infrared. As we will see, this is far from being an easy task.
It is useful to understand at which scale we expect GR to breakdown because the quantum effects become important. To make a parallel with other theories, for example, we know that Galileo's Relativity gives way to Special Relativity when the velocities are close to the speed of light c, or that classical mechanics has to be replaced with quantum mechanics when the variations of the action are of order the (reduced) Planck's constant $\hbar$, etc. In general, the breakdown of a theory is connected with the appearance of some fundamental scale which, in the above examples, is invisible in the effective theory, i.e. at scales far away from the fundamental one. By means of dimensional analysis, it is possible to combine c, $\hbar$ and the gravitational constant G_N and identify a fundamental length scale in Nature, the Planck length

$$l_P = \sqrt{\frac{\hbar G_N}{c^3}} \sim 10^{-33}\text{cm}$$

We expect the quantum gravity effects to become important at this scale[1].
The smallness of l_P is closely related to the weakness of the gravitational force: indeed, it is about 40 orders of magnitude weaker than the Electromagnetic force. The only reason why we experience it at our scales is that it is always attractive: gravity couples universally to all forms of energies and makes like charges to attract; the usual requirement from quantum mechanics that the local energy density should be positive makes the gravitational fields of a bound state of particles to add up, so that gravity can dominate over all other forces at macroscopic scales.
The fact that the l_P is so remote makes it difficult to gain any experimental evidence of quantum gravity effects: at the present time, we are able to probe physics up to a scale of 10^{-17} cm. Moreover, this makes it seem unlikely that quantum gravity may be relevant to any presently observable phenomena.

[1]Note that this does not discard the possibility that new gravitational physics could appear already at scales higher than l_P.

However, there are at least two reasons for believing that this is not the case, i.e. that a quantum theory of gravity might have consequences on the theoretical discussion on the physics at the scales we can probe. The first one has to do with the unification of the forces of Nature. No one ensures us that all the forces are different aspects of a single entity, i.e. that they should undergo a unified framework. However, this assumption would be in continuity with the history of physics of the last 150 years: in the second half of the XIX century, Maxwell has unified the classical theories of electricity and magnetism; in the sixties, Weinberg-Salam theory has done something similar with the weak and electromagnetic interactions. More recently, grand unification models suggest that the strong and electroweak interactions can be embedded into a unique simple gauge group: remarkably, the gauge coupling constants of the Standard Model, when extrapolated to high energy using the (supersymmetric) renormalization group, match each others at the grand unification scale 10^{16} GeV. The unification of the gravitational force with the strong-electroweak interactions would be the next logical step. Such an unified theory would yield new predictions of phenomena at presently observable scales in the same way as grand unification models, for example, predict the protons decay.
The second reason arises directly from GR: spacetime singularities occur in solutions of GR relevant to the gravitational collapse and cosmology. In this situations, GR predicts its own failure, and one expects the classical description of spacetime structure to break down. In particular, this has to do with the Big Bang singularity: one cannot expect that the description of our universe undergoes the equations of GR in the regime where these predict the curvature to be of order l_P^{-2}, i.e. at time comparable with the Plack time $t_P \sim 10^{-43}s$. At this regime, quantum effects would be important, and it is possible that phenomena which occur in the very early universe (and thus should be understood in the framework of quantum gravity) lead to observable predictions about the structure of the resent universe.

This issue is also related with black hole physics. Black holes are singular solutions of Einstein's equations that have an event horizon, i.e. a one-way membrane that causally divides spacetime into the external universe and the black hole interior. The analysis of black holes in GR reveals interesting (and sometimes even surprising) properties.
First of all, black holes are protected by uniqueness theorems: they are uniquely fixed in term of the conserved charges they carry, i.e. the mass M, the angular momentum J and the electric charge Q. This is quite surprising. If we consider, for instance, a star, the metric it produces is affected by its shape, its chemical composition, the distribution of the mass, the charge, etc. (you got the idea) so that, if we decompose the metric into multipole moments, an infinite number of coefficients should be specified to express the field exactly. If the star collapses into a black hole, this property no longer holds and the metric is fixed once the charges M, J and Q are given. Heuristically, we can motivate this by thinking that the formation of the event horizon causally cuts off the interior region from the external universe and thus soften all the particular characteristics of the starting condition but those that, being conserved, cannot disappear. Wheeler expressed this idea with the phrase: "black holes have no hair".
Another important feature of black holes is that they obey mechanical laws, which follow from Einstein equations, that are analogous to the 3 laws of classical thermodynamics; in this view, one can assign to a black hole a temperature T_{BH} and an entropy S_{BH} that depend only on some features of the event horizon (the surface gravity and the area respectively) and not on what lies in its interior. This behavior makes some questions rise: black holes, as the name suggest, are black so it is not clear what it means to assign them a temperature; moreover, because of the uniqueness theorems, their classical phase space is trivial and cannot account for the entropy. One could argue that, perhaps, the entropy receives contributions from the quantum mechanical states that are hidden behind the horizon (indeed, entropy is hidden information), but yet one is to face the puzzle that the entropy scales with an area instead of a volume[2]; against the typical picture for which the entropy is an extensive quantity. Therefore one would be tempted to consider the correspondence between black hole mechanics and thermodynamics as nothing more than a vague analogy.
However, taking into account quantum mechanical effects on the curved (classical) background, Hawk-

[2]As we will discuss in this thesis, this has been taken as a hint of a fundamental property of quantum gravitational theories: namely that they are holographic.

ing has shown that black holes emit a black-body radiation at a temperature T_{BH}: classically, black holes are black, but semi-classically they thermally radiate.
This fact enforces the case for taking seriously the analogy between black hole laws and thermodynamics and searching for an underlying statistical explanation of this thermodynamic behavior. This picture, as we now discuss, gives rise to puzzling situations. In statistical mechanics, a macroscopic state is viewed as a a coarse grained description of a number N of microstates, which are all compatible with the macroscopic state, in the sense that the macroscopic observables cannot distinguish between them: in this view, the entropy is the logarithm of the number of microstates. In the case of black holes, what/where are the $e^{S_{BH}}$ microstates? This is the entropy puzzle.
Furthermore, the thermal nature of Hawking's radiation has led to one of the biggest recent problems in theoretical physics: the information paradox. We can think of Hawking's radiation as a consequence of pairs creation near the event horizon: it is possible that one member of the pair falls into the hole (reducing its mass) while the other particle escapes to infinity (giving the thermal radiation). This process makes the black hole to evaporate and, eventually, to disappear: the ingoing and outgoing members of the pair are in an entangled state and, when the black hole has vanished (togheter with the infalled particle), the radiation quanta left outside are entagled with nothing, i.e. we started with a pure state and ended up with a state that must be described by a density matrix.
This violates unitarity. Because the conservation of the information, i.e. the unitary temporal evolution of quantum mechanics, is one of the pillars of theoretical physics, one may search for a failing in the above argument. In particular, since we are using GR and quantum theory in the same problem (i.e. Hawking's computation is semi-classical), one may imagine that a theory of quantum gravity should shed light upon these paradoxes. This is, however, not obvious a priori as near the horizon, in general, the curvature is small with respect to l_P^{-2} and the semi-classical approximation should be reliable. Therefore, understanding if and how information is conserved in a quantum theory of gravity turns out to be a very difficult task. In this sense, black hole physics opens a window on quantum gravity: indeed, many physicists consider black holes the analogous of the hydrogen atom for quantum mechanics.

As we have acquainted above, a consistent formulation of a theory of quantum gravity is a difficult task. The canonical quantization approach to GR fails, being it a non-renormalizable theory. The lack of observations to guide us, moreover, makes the formulation of quantum gravity a search without lighthouses and has led to many different approaches: string theory, loop quantum gravity, asymptotically safe gravity (just to name a few), driven by underlying different guiding ideas.
String theory is still a theory under construction and we have not figured out all the rules that governs it, nonetheless it seems to be the most promising approach and the aim of this thesis is to study some of the recent progresses in our understanding of black hole physics within string theory.

String theory revolutionizes what the things are made of: it pretends that the fundamental objects of Nature are not point-like particles, but one dimensional objects, namely the strings. Their oscillations give rise to a spectrum that contains a massless spin-2 field, the graviton: in this sense, it is a theory of quantum gravity.
Strings are not the only fundamental objects in string theory; there are also D-branes: multidimensional objects on which open strings can end. They are massive objects, so they source the gravitational field curving spacetime. Thus, a black hole in string theory is understood as a bound state of strings and D-branes: we call n-charge black hole the one composed of n different types of branes and strings. To be more precise, changing the gravitational strength (i.e. the coupling constant of the theory) one can interpolate between the black hole regime and the bound state description of the system.
With this point of view, for a class of supersymmetric black holes, string theory enables an exact agreement between the degeneracy of the microstates (which are essentially the different possible vibration modes of the strings and D-branes) and the thermodynamic entropy: one can compute the degeneracy in the bound state description and, thanks to supersymmetry, extrapolate this result to the black hole regime. These developments not only show that, indeed, the gravitational entropy has a statistical origin, but represent also a non trivial test on the consistency of string theory.
However, being this essentially an extrapolation from weak to strong couplig, it leaves us with some

questions: how do these microstates manifest themselves at the gravitational regime? How do they help to solve the information paradox?

The explicit solutions generated by these microstates have been constructed in some simple cases and they all show some important features: they are horizonless and non-singular geometries (well described in the supergravity limit, the low energy limit of superstring theory), which resemble the classical black hole asymptotically but deviate from it already at the horizon scale; in particular, the interior region contains informations on the particular microstate (and thus permit to distinguish one microstate from another).

This result has motivated the fuzzball proposal: according to this conjecture, associated with a black hole with entropy S_{BH}, there are $e^{S_{BH}}$ horizonless, non-singular solutions that asymptotically look like the black hole but generically differ from it up to the horizon scale. These solutions, called fuzzballs, are considered to be the microstates responsible for the entropy, while the original black hole represents the coarse grained description[3] of the system.

The absence of an horizon corroborates the idea that these are indeed microstates: if they had an horizon one could associate an entropy to them, and they would not be pure states. Moreover, the fuzzball proposal implies that quantum gravity effects are not confined at scales comparable with l_P, but extend up to the horizon scale which, in general, is a macroscopic scale.

This modification opens to the possibility that Hawking radiation does manage to carry out the information of the collapsing matter. The reason is that, while Hawking's calculation assumes that the microstates are not distinguishable near the horizon, according to this conjecture a microstate and the coarse-grained black hole differ already at the horizon scale. This means that the creation of particle pairs near the horizon is sensible to the precise form of the microstate: the modification of the black hole interior allows the emitted quanta to carry information about the microscopic configuration.

We shall stress, however, that the fuzzball proposal is a conjecture and it is not entirely accepted. To establish its validity it would be important to develop a precise map between microstates and geometries.

Moreover, we shall clarify that, by solutions, we mean in general solutions of the full string theory: only a subset of these fuzzball will solve the low energy equations of motion, i.e. will be visible (or reliably distinguishable) in the supergravity limit. The majority of the microstates, instead, are likely to be stringy fuzzballs: this imposes many technical obstacles to establish the validity of the conjecture.

Progresses have been made with the AdS/CFT duality, which is (conjectured to be) an exact equivalence between a string theory in a $(d+1)$-dimensional Anti de Sitter spacetime and a conformal field theory on the boundary. The black holes that we will consider (i.e. the 2- and 3- charge black holes) have a near-horizon region that is asymptotically $AdS_3 \times S^3$, so that AdS/CFT correspondence is applicable. The microstates, thus, can be understood as certain supersymmetric states of the dual CFT. This duality provides powerful tools to study black holes microstates, as it enables to gain insight on the gravitational physics from the CFT side of the duality.

In particular, it has been shown that, given a CFT state $|s\rangle$ (whose gravitational dual is well described in supergravity), the expectation values of some operators in $|s\rangle$ are encoded in the asymptotics of the dual AdS geometry: roughly speaking, the VeVs of these operators determine a particular deviation of the geometry of the microstate from $AdS_3 \times S^3$.

The precise dictionary VeVs/geometry expansion has been established for operators of conformal dimension 1, providing that the expectation value of such operators are controlled by the first non trivial correction around $AdS_3 \times S^3$. It is an aim of this thesis to discuss its generalization to operators of higher dimension: in particular, we will focus on operators of dimension 2, whose VeVs are controlled by corrections up to the second non trivial order around $AdS_3 \times S^3$.

It is important to work out further the CFT/gravity dictionary mainly for two reasons. The first one is that not all the geometries that have the same asymptotic charges of the black hole are microstates;

[3]The coarse-graining is a procedure borrowed from statistical mechanics: averaging microscopic degrees of freedom, the macroscopic properties of the system (for example the event horizon and, thus, the entropy) emerge. Because black hole physics undergoes a statistical mechanic description, this procedure should apply also to our systems; however, it is not clear how it analytically works in this case, i.e. it is not clear how to coarse-grain (average) geometries.

discussing their dual states in the CFT would help to confirm whether they indeed contribute to the entropy of the black hole. The second reason concernes the fact that the construction of gravity solutions is usually a difficult task (this is true also in General Relativity): developing the CFT/geometries dictionary further could provide a guide to construct more general dual geoemtries. Moreover many microstates have a vanishing expectation value of dimension 1 operators; thus, the results already known in the literature do not allow to gain any insight on the gravitational physics from the CFT side of the duality.

The strategy to extend the dictionary will be to consider microstates with known geometry and state on the dual CFT; we will compute the VeVs of the dimension 2 operators and compare them with the geometry expansion.

This generalization presents some new features compared to the case of operators of dimension 1. The extended dictionary should take into account the possible mixings between dimension 2 operators: this means that a linear combination of the coefficients of the geometry expansion around $AdS_3 \times S^3$ encode the VeVs of a linear combination of dimension 2 operators. Determining the dictionary means to provide the precise linear combinations.

In particular mixing between single- and multi-trace operators occurs. The action of the double-traces on a CFT state, however, is subtle and we will not consider it this thesis. This is done examining only states in which the double-trace operators cannot play any role, so that their contribution is invisible. Clearly, such a restriction is a limit of our discussion and it would be very interesting to extend the analysis further.

Note, moreover, that the geometry expansion gives different answers in different coordinate systems, so that, in general, the VeV/geometry dictionary depends on the gauge choice. Such a dependence is not visible in the case of dimension 1 operators, because their expectation values are controlled by the first non trivial order in the geometry expansion. This is not the case for dimension 2 operators, and it is likely that the dictionary we provide is valid only in our coordinate choice.

The work is organized as follows.

In Chapter 1 we introduce black hole solutions in General Relativity and, in particular, the analogy between black hole physics and thermodynamics. The puzzling consequences that arise when one takes this correspondence seriously are discussed, as well as the need for a quantum gravity theory in order to solve them.

In Chapter 2 we introduce string theory, supersymmetry and supergravity, reviewing the main tools and properties that will be needed in the following discussion. Moreover, we will see how black holes are described in supergravity theories.

In Chapter 3 we will treat Black holes in string theory showing how it successfully accounts for the microscopic count of states. The problem of the microstates construction will be addressed for a particular simple case (the D1-D5 black hole, i.e. a 2-charge system composed of 1-dimensional and 5-dimensional D-branes): this will lead to the discussion of the Fuzzball proposal.

As we have mentioned above, the AdS/CFT correspondence provides powerful tools for the discussion of black hole physics: the motivation and the content of the duality will be discussed in Chapter 4. The D1-D5 CFT, the conformal field theory relevant for the D1-D5 system, will be introduced.

In Chapter 5 we will state the precise dictionary between D1-D5 states and dual geometries for the 2- and 3-charge black holes. We will also review the VeVs/geometry expansion correspondence for operators of dimension 1.

In Chapter 6 we will discuss the extension of the above dictionary for operators of dimension 2, fixing some constraints on the relations between VeVs and dual geometry.

Chapter 1

General Relativity and Black Holes

1.1 General Relativity

Einstein's insight that gravity emerges from the curvature of spacetime led to the most predictive classical theory of gravitation: General Relativity. The field content of the theory is the gravitational field $g_{\mu\nu}$ (i.e. the metric of spacetime that encondes its dynamics) and all the other matter and gauge fields that, having energy, source the gravitational field. The total action is the sum of the so-called Einstein-Hilbert action S_{EH}, that governs the dynamics of spacetime, and the action for all the other fields but the metric S_M:

$$S = S_M + S_{EH} = S_M + \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}R \tag{1.1}$$

where $R = g_{\mu\nu}R^{\mu\nu}$ is the Ricci scalar and G_N is the Newton constant. The principle of stationary action yields to the following equations of motion:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_N T_{\mu\nu} \tag{1.2}$$

where the stress-energy tensor $T_{\mu\nu}$ is defined as $T_{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta S_M}{\delta g^{\mu\nu}}$.
The non-linearity of Einstein's equations reflects the back-reaction of gravity: the gravitational field, having energy, couples to itself. This is a direct consequence of the Equivalence Principle: if $g_{\mu\nu}$ did not interact with itself, a gravitational atom (two particles bound through their mutual gravitational attraction) would have a different inertial mass than gravitational mass. From a mathematical point of view, the non-linearity of the equations makes complicated the purpose of saying anything general about the properties of the solutions, therefore it is usually necessary to make some simplifying assumptions endowed with a high degree of symmetry. We will now turn to the study of these situations.

1.1.1 Schwarzschild Black Hole

The simplest case one can consider is a spherically symmetric vacuum spacetime. Birkhoff's theorem ensures that an isotropic metric is also static. The unique solution is called Schwarzschild metric, by the name of its discoverer, and in Schwarzschild coordinates (t, r, θ, ϕ) the line element reads [2]:

$$ds^2 = -\Big(1 - \frac{2MG_N}{r}\Big)dt^2 + \Big(1 - \frac{2MG_N}{r}\Big)^{-1}dr^2 + r^2 d\Omega_2^2 \tag{1.3}$$

where $d\Omega_2^2$ is the differential area element of S^2 and M is the mass of the body, in the following we will usually express formulae in terms of the reduced mass $m = MG_N$. This is the metric for spacetime outside a spherical massive object, and is relevant to describe the gravitational field created by the Earth or a star (to a good approximation). The metric coefficients of (1.3) become infinite at $r = 0$ and at the Schwarzschild radius $r = 2m \equiv r_s$.
The hypersurface $r = r_s$ is called event horizon. As the Schwarzschild solution makes sense only in vacuum, when dealing with the singularity $r = r_s$ we are interested only in the case in which the massive body has a smaller radius than r_s. Such an object is called a black hole. Being $g_{\mu\nu}$ a tensor, its components are coordinate dependent and its breakdown does not tell us whether we are considering a physical singularity or if the singularity lies in the coordinatization and the underlying spacetime manifold remains perfectly smooth. Distinguishing between these cases in generality turns out to be a difficult task, and entire books have been written about the nature of singularities in General Relativity. Analizing this issue in detail is beyond the puropuse of thesis. For the following discussion it is enought to say that if any scalar constucted from the curvature goes to infinity as we approach some point, then we are dealing with a singularity of the curvature. Considering the invariant quantity [2]

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{48G_N^2M^2}{r^6} \tag{1.4}$$

we see that at $r = 0$ the geometry is singular, while computing all the possible curvature invariants, one could check that the Schwarzschild radius is actually non singular and the breakdown of the metric has to do with a bad choice of coordinates. Physically, it reflects the fact that Schwarzschild coordinates are adapted to an observer standing still outside the hypersurface $r = r_s$ and to stand still at $r = r_s$ one should have an infinite acceleration. Choosing a coordinate system adapted to ingoing light rays, defined by the change of coordinates

$$v = t + r + 2m\log|\frac{r-2m}{2m}| \tag{1.5}$$

one gets the Schwartzshild metric in ingoing Eddington-Finkelstein coordinates (v, r, θ, ϕ)

$$ds^2 = -\Big(1 - \frac{2m}{r}\Big)dv^2 + 2dvdr + r^2d\Omega_2^2 \tag{1.6}$$

which is smooth at $r = r_s$.
Even though the singularity $r = r_s$ has to do with the coordinatization, it leads to important physics. The causal structure of (1.6) shows that the event horizon is tangent to a light cone and no future-directed timelike or null worldline can reach $r > 2m$ from $r \leq 2m$: it act as a one-way membrane separating those spacetime points that are connected to infinity by a timelike path from those that are not. It is also an infinite redshift surface: the wavelength of the radiation received by a distant observer increase as the source approaches the event horizon. Those two aspects are the essence of the invisibility of a black hole.
There is another notion of horizon in GR: if a Killing vector field ξ^μ is normal to a null hypersurface Σ, than Σ is called Killing horizon. Even thought the notion of a Killing horizon is logically independent from that of an event horizon, for spacetimes with time-traslational symmetry the two are closely related: every event horizon Σ in a stationary, asymptotically flat spacetime is a Killing horizon for some Killing vector ξ^μ, and if the spacetime is also static $\xi = \partial_t$. To every Killing horizon we can associate a quantity called surface gravity. Because ξ^μ is normal to Σ, along the Killing horizon it obeys [3]

$$\xi^\mu D_\mu \xi^\nu|_\Sigma = k\xi^\nu \tag{1.7}$$

where k is the surface gravity. We see from (1.7) that its definition is in principle arbitrary because if Σ is a Killing horizon of ξ^μ with surface gravity k, then it is also a Killing horizon of $c\xi^\mu$ with surface gravity ck, for any contstant c. There is no natural normalization of ξ_μ on Σ, as $\xi^2 = 0$ there. Yet there is a natural normalization for asymptotically flat spacetime at spatial infinity, e.g for the time-translation Killing vector field $\xi = \partial_t$ we choose

$$\xi^2(r = \infty) = -1 \tag{1.8}$$

This fixes k. The evaluation of (1.7) for the Schwarzshild metric (1.6) with $\xi^\mu = (1, 0, 0, 0)$ gives a non vanishing result only for the component $\nu = v$:

$$\xi^\rho D_\rho \xi^v = \Gamma^v_{vv} = \frac{m}{r^2}|_{r=2m} \qquad \Longrightarrow \qquad k = \frac{1}{4m} = \frac{1}{4MG_N} \tag{1.9}$$

We can interpret k as the acceleration experienced by a test particle on the horizon as measured by infinity, and we will see in the following that it is closely related to the temperature of the black hole.

1.1.2 Reissner-Nordstrom Black Hole

Let's now consider the solution to Einstein's equations for a spherically symmetric charged black hole, i.e. when $g_{\mu\nu}$ couples with the stress-energy tensor produced by an $U(1)$ gauge field. The metric reads [4]:

$$ds^2 = -\Big(1 - \frac{2mG_N}{r} + \frac{G_N Q^2}{r^2}\Big)dt^2 + \Big(1 - \frac{2mG_N}{r} + \frac{G_N Q^2}{r^2}\Big)^{-1} dr^2 + r^2 d\Omega_2^2 \tag{1.10}$$

where M is once again the mass of the body and Q is the total electric charge. Beside the geometrical singuarity $r = 0$, the metric blows up when $g^{rr} = 0$; this reflects the presence of event horizons located at

$$1 - \frac{2G_N M}{r} + \frac{G_N Q^2}{r^2} = 0 \qquad \Longrightarrow \qquad r_\pm = G_N M \pm \sqrt{G_N^2 M^2 - G_N Q^2} \tag{1.11}$$

There are 3 cases to consider:

- When $G_N M^2 < Q^2$, $g^{rr} = 0$ has no real roots and the metric is completely regular in the (t, r, θ, ϕ) coordinates all the way down to $r = 0$, which is therefore a naked singularity. This situation is however forbidden from the cosmic censorship hypothesis, which states that naked singularities cannot form in gravitational collapse. The unphysicality of this setting can be understood by the following argument based on newtonian gravity and incorporating the equivalence of inertial mass with total energy (Special Relativity) and of inertial and gravitational mass (General Relativity). The total mass is equal to the sum of the rest energy, the coulombian energy and the gravitational binding energy:

$$M = M_0 + \frac{Q^2}{r} - \frac{G_N M^2}{r} \tag{1.12}$$

  This is a quadratic equation for M, whose solution with $M(r = \infty) = M_0$ is $M(r) = \frac{1}{2G_N}(\sqrt{r^2 + 4G_N M_0 r + 4G_N Q^2} - r)$. The shell will undergo gravitational collapse if M decreases as R decreases (this allows Kinetic Energy to increase). So

$$\frac{dM}{dr} = \frac{G_N M^2 - Q^2}{2MG_N r + r^2} > 0 \qquad \Longrightarrow \qquad G_N M^2 > Q^2 \tag{1.13}$$

  and the attracting gravitational force wins the repulsive electric force.

- When $G_N M^2 > Q^2$ the metric has two coordinate singularities at $r_\pm$ corresponding to two event horizons. The one at r_+ is just like $r = 2m$ for Schwarzschild metric (the physical phenomena an observer outside the horizon witnesses, like the increasing redshift, are just like those outside an uncharged black hole): at r_+, r switches from being a spacelike coordinate to a timelike coordinate, and an infalling observer is forced to move in the direction of decreasing r untill it reaches $r = r_-$, where r switches back to a spacelike coordinate. Here one can choose whether to fall in $r = 0$ or reverse the orientation, go across the region between r_- and r_+ and re-emerge in an outside region which is actually different from the starting region: from this point of view of the Reissner–Nordstrom black hole behaves as a white hole. Both the event horizons at $r_\pm$ are Killing horizons for the Killing vector $\xi = \partial_t$. Mooving to Eddington-Finkelstein coordinates [3] and using (1.7) one gets the following result for the surface gravities:

$$k_\pm = \frac{r_+ - r_-}{2r_\pm^2} \tag{1.14}$$

  Note that in the limit $Q \to 0$, we have that $r_+ \to 2m$, $r_- \to 0$ and $k_\pm \to \frac{1}{4m}$, recovering the structure of the Schwarzschild black hole.

- In the case $G_N M^2 = Q^2$ the solution is said to be extremal. In this situation the event horizon is located at $r_+ = r_- = m$: here the r coordinate becomes null, but it is never timelike, it is spacelike on both sides. An important feature of extremal black hole is that, roughtly speaking, the mass is exactly balanced by the charge: two extremal black holes attract each other gravitationally but repell each other electrically, and the two effects precisely cancel. This allows to find exact solutions of multicentered extremal black holes in a stationary configuration [4]. In supergravity theories, as we will see later, the case of extremal solutions are of particular interest as they might[1] reflect the presence of supersymmetry, which is a considerable aid in the calculations. Let's note that (1.14) evaluated for an extremal black hole gives $k = 0$.

1.1.3 Kerr Black Hole

Let's now move to the vacuum solution of Einstein's equations generated by a stationary rotating black hole. The metric has been discovered by Kerr in 1963, about 40 years after the discovery of Schwarzshild and Reissner-Nordstrom metrics. This elapsed time gives the feeling of the mathematical difficulties related to this solution, as now the metric is stationary but no more static and presents an axial symmetry around the axis of rotation instead of a spherical symmetry. The result reads:

$$\begin{aligned} ds^2 = &-\Big(1 - \frac{2G_N M r}{\rho^2}\Big)dt^2 - \frac{2G_N M a r \sin^2\theta}{\rho^2}(dtd\phi + d\phi dt) + \frac{\rho^2}{\Delta}dr^2 \\ &+ \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\Big[(r^2 + a^2)^2 - a^2\Delta\sin^2\theta\Big]d\phi^2 \\ \Delta = r^2 &- 2G_N M r + a^2 \qquad \rho^2 = r^2 + a^2\cos^2\theta \qquad a = \frac{J}{M} \end{aligned} \tag{1.15}$$

The solutions are parametrized by the mass M and the angular momentum J of the black hole. It's possible to include the electric charge just by replacing $2G_N M r \to 2G_N M r - G_N Q^2$, however, as we are interested now in the physical phenomena due to the presence of J, we will keep Q switched off. The manifold singularity occurs at $\rho = 0$, which means $r = 0$ and $\theta = \pi/2$ (it forms a disk of radius a). The event horizons occur for $g^{rr} = 0$, and one gets

$$\Delta(r) = r^2 - 2G_N M r + a^2 = 0 \qquad \Longrightarrow \qquad r_\pm = G_N M \pm \sqrt{G_N^2 M^2 - a^2} \tag{1.16}$$

[1] Extremality is a necessary but not sufficient condition for supersymmetry.

The analysis of these null surfaces procedes analogously with Reissner-Nordstrom metric. As all the coefficients of the metric are independent of the coordinates t and ϕ, there are two manifest Killing vectors: $K = \partial_t$ and $R = \partial_\phi$. Because Kerr is stationary but not static, the event horizons are Killing horizons associated to a linear combinations of K and R: $\xi^\mu = K^\mu + \Omega_H R^\mu$. Imposing $\xi^\mu \xi_\mu = \Omega_H^2 g_{\phi\phi} + 2\Omega_H g_{t\phi} + g_{tt} = 0$ on r_+, and using $\Delta(r_+) = 0$ one gets:

$$\Omega_H = \frac{-g_{t\phi} \pm \sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}}{g_{\phi\phi}}|_{r_+} = \frac{-g_{t\phi} \pm \sqrt{\Delta \sin^2\theta}}{g_{\phi\phi}}|_{r_+} = \frac{a}{r_+^2 + a^2} \tag{1.17}$$

Mooving to Eddington-Finkelstein coordinates [3] and using using (1.7) one gets the following result for the surface gravities:

$$k_\pm = \frac{r_+ - r_-}{2(r_\pm^2 + a^2)} \tag{1.18}$$

The Killing vector associated to time translation invariance becomes null before the event horizon (except for the poles $\theta = 0, \pi$). $K^2 = g_{tt} = 0$ occurs at

$$\tilde{r}_\pm = G_N M - \sqrt{G_N^2 M^2 - a^2 cos^2\theta} \tag{1.19}$$

The region between $\tilde{r}_+$ and r_+ is known as ergoregion[2].
A static observer moves on a timelike worldline with tangent vector proportional to K^μ, so that on its worldline $dr = d\theta = d\phi = 0$. Because inside the ergosphere K^μ becomes spacelike, a static observer cannot exist there: an observer inside the ergosphere cannot stay still, but is forced to move, dragged along with the black hole rotation .

1.1.4 No-Hair Theorem

In Section 1.1.1 we have seen that the unique vacuum solution of Einstein's equation generated by a spherically symmetric charge is also static (Birkhoff's theorem) and is given by (1.3), this means that the metric is characterized by a single parameter M. Yet, this is not too surprising: an analogous statement hold also, for example, in classical electromagnetism where the only spherically symmetric field configuration in a free charge region is given by Coulomb's field. A stronger statement has been proved by Israel in 1967, who has shown that the gravitational field of a static vacuum black hole, even without further symmetry assumptions, is necessarily given by the spherically symmetric Schwarzschild metric. Holding stationarity but giving up spherical symmetry, however, one might expect that many more black hole configurations occur: if we take a non spherically-symmetric planet and decompose the metric into multipole moments, an infinite number of coefficients should be specified to express the field exactly. Surprisingly enough, black holes do not share this property: assuming that electromagnetism is the only long-range non gravitational field, black hole solutions are protected by a "no-hair theorem". It states that stationary, asymptotically flat black hole solutions to GR coupled to electromagnetism that are nonsingular outside the event horizon are fully characterized by their mass, electric charge and angular momentum (which we will call, generically, charges of the black hole). This means that it does not matter how complicated the collection of matter that collapse into a black hole is, eventually we will end up with a configuration completely specified by 3 parameters, the charges of the black hole. Not only in quantum theories, but even classically information cannot be lost; in GR, however, we can think of the information as hidden behind the horizon rather then truly been lost. We will see that in QFT, instead, black holes evaporate and eventually disappear, leaving us with a puzzling paradox.

[2] *Ergon* means work in ancient greek. The etymology of this word will become clear while studying the Penrose process in Section 1.2.2.

1.2 Black Hole thermodynamics

In GR, black holes obey laws that look analogous to the laws of thermodynamics, and follow directly from Einstein's equations. In this Section we will derive these laws and examine their physical consequences.

1.2.1 Black Hole temperature as surface gravity

In order to talk about the thermodynamical properties of a black hole, we shall firstly associate a temperature to it. We will do it examining the properties of the Euclidean black hole solution, obtained by analytically continuing the metric form Lorentzian to Euclidean signature. We will show that regularity of the metric requires time to be periodic in the imaginary direction, and this periodicity assigns a temperature to the black hole which is proportional to its surface gravity. Let's first show that QFT at a finite temperature T, to be well defined, has to be periodic in the imaginary time, with periodicity

$$t \sim t + i\beta \qquad \beta = \frac{1}{T} \tag{1.20}$$

The thermal Green's function for a field operator ϕ is

$$G_\beta(\tau, x) \equiv -\operatorname{Tr} \rho_\beta T_E[\phi(\tau, x)\phi(0,0)] = -\frac{1}{Z} \operatorname{Tr} e^{-\beta H} T_E[\phi(\tau, x)\phi(0,0)] \tag{1.21}$$

where $\tau = it$ is the euclidean time, ρ_β is the thermal density matrix and T_E denotes time ordering. Then, we have $(\tau > 0)$

$$\begin{aligned} G_\beta(\tau, x) &= -\frac{1}{Z} \operatorname{Tr} e^{-\beta H} \phi(\tau, x)\phi(0,0) = -\frac{1}{Z} \operatorname{Tr} \phi(0,0) e^{-\beta H} \phi(\tau, x) \\ &= -\frac{1}{Z} \operatorname{Tr} e^{-\beta H} \phi(\beta, 0)\phi(\tau, x) = -\frac{1}{Z} \operatorname{Tr} e^{-\beta H} T_E[\phi(\tau, x)\phi(\beta, 0)] \\ &= G_\beta(\tau - \beta, x) \end{aligned} \tag{1.22}$$

where we have used the cyclic property of the trace and the time evolution of the operators in the Heisenberg picture:

$$\phi(t + t_0) = e^{iHt}\phi(t_0)e^{-iHt} \tag{1.23}$$

Let's now examine the properties of the Euclidean Schwarzshild solution. With this purpose we shall expand Schwarzshild metric near the horizon, focusing our attention just to the (t, r) coordinates in (1.3) (the coordinates of S^2 do not play any role in this discussion and we can just ingnore them). Introducing the new coordinate $\tilde{r} = r - 2m$ that measures the coordinate distance from r_s, (1.3) reads:

$$ds^2 = -\Big(\frac{\tilde{r}}{\tilde{r} + 2m}\Big)dt^2 + \Big(\frac{\tilde{r} + 2m}{\tilde{r}}\Big)d\tilde{r}^2 \tag{1.24}$$

The near horizon expansion means $\tilde{r} << r_s = 2m$, so

$$ds^2 = -\frac{\tilde{r}}{2m}dt^2 + \frac{2m}{\tilde{r}}d\tilde{r}^2 \tag{1.25}$$

Introducing the new radial coordinate

$$d\rho^2 = \frac{2m}{\tilde{r}} d\tilde{r}^2 \Longrightarrow \rho = \sqrt{8m\tilde{r}} \tag{1.26}$$

one gets

$$ds^2 = -\frac{1}{16m^2}\rho^2 dt^2 + d\rho^2 = -k^2\rho^2 dt^2 + d\rho^2 \tag{1.27}$$

where k is the surface gravity, recovering Rindler's metric in hyperbolic coordinates. Wick rotating $\tau = it$, one finds

$$ds^2 = \rho^2 d(kt)^2 + d\rho^2 \tag{1.28}$$

If one requires this metric to be regular, this is the 2 dimentional euclidean metric in polar coordinates $ds^2 = dr^2 + r^2 d\theta^2$, where $\theta \sim \theta + 2\pi$. Hence we can assign to τ the periodicity $\beta = \frac{2\pi}{k}$. Thus, every continuous function defined on this background (and in particular every Green's function) is periodic. Using eq. (1.20), we have that to a Schwarzshild black hole with mass M we can associate a temperature

$$T = \frac{\hbar}{8\pi k_B G_N M} \tag{1.29}$$

where we have reintroduced the constants $\hbar$, G_N and the Boltzmann constant k_B. Some observation are in order. The dependence of the temperature on the surface gravity k, that in the previous computation occurs through eq. (1.27), is not a coincidence: the proportionality between the temperature and the surface gravity of a black hole

$$T = \frac{\hbar k}{2\pi k_B} \tag{1.30}$$

holds also for Reissner-Nordstrom's and Kerr's solutions. It can be shown [3] that the surface gravity is constant on the Killing horizon and that $k \geq 0$ always (where the equality sign holds for extremal black holes). The first property reminds the zeroth law of thermodynamics (that states that the temperature of a system in equilibrium is constant) and the latter reassures us that (1.30) is a consistent definition of temperature. Let's also note that if a black hole absorbs radiation its mass increases but, as eq. (1.29) shows for the Schwarzshild case, its temperature decreases, leading to a negative specific heat.
It may seem just a speculative effort to associate a non zero temperature to a black hole because GR tells us that it is an object that can only absorb and not emit particles. It turns out that this is not true once one considers QFT in the background of a stationary black hole: this is what Hawking did [5] demonstrating that black holes are unstable objects that emit black-body radiation at infinity. Let us just sketch the essential features needed to understand the Hawking radiation, without reporting the details (a complete derivation of Hawking radiation can be found, for example, in [6]). The generalization of the equations of motion of QFT in a non-Minkoskian spacetime can be obtained imposing the minimal-coupling principle. For example, the equation for a massless, non interacting scalar field ϕ reads:

$$g^{\mu\nu} D_\mu D_\nu \phi = 0 \tag{1.31}$$

where D_μ are the covariant derivatives. What one would do now is to introduce a set of positive and negative frequency modes forming a complete basis for the solutions of (1.31), expand the field ϕ in these modes, interpret the Fourier coefficients as creation and annihilation operators and quantize them imposing the commutation relations. At this point, in general, our prescription breaks down: in a spacetime with no global timelike killing vector field (and this is the case in presence of a black hole) there is no global notion of positive and negative frequency modes, and we cannot give a precise notion of particle. Still, if a spacetime is asymptotically flat, we can define positive frequency modes in the regions where it approaches Minkowski spacetime. Let us consider, for instance, a spacetime with two asymptotically flat regions: the ingoing and the outgoing one . We can define particles and vacuum states in each region, but those definitions need not to coincide. In the case of a black hole, Hawking showed that if we start with the incoming vacuum, the expectation value of the number of outgoing particles with frequency ω is

$$N_\omega \sim \frac{1}{e^{\frac{2\pi\hbar\omega}{kk_B}}-1} \tag{1.32}$$

So, starting with the vacuum in the background of a black hole, we end with a planckian distribution of outgoing particles corresponding to a black body with temperature consistent with (1.30). The energy of the emitted particles is compensated by the negative energy particles that cross the event horizon of the black hole, decreasing its mass. Thus, black holes are quantum mechanically unstable and will eventually disappear (evaporate), even thought slowly.

1.2.2 First and second law of Black Holes thermodynamics

In this section two laws that remind the first and second law of classical thermodynamics will be derived for black holes. With this purpose, let's show that energy not only can flow into black holes, but there are physical processes by which one can extract energy out of them. Let's focus our attention to a massive particle (with four-momentum $p^\mu = m\frac{dx^\mu}{d\tau}$) in Kerr's metric. Two conserved quantities along the geodesics are ensured by the presence of the axial rotation Killing R^μ and the time-traslation Killing K^μ : those are the angular momentum of the particle $L = R^\mu p_\mu$ and the energy of the particle $E = -K^\mu p_\mu$ (the minus sign is required in order to have $E > 0$ at spatial infinity: there both K^μ and p^μ are timelike, so their inner product is negative). Inside the ergosphere, however, K^μ becomes spacelike, and we can immagine particles for which $E = -K^\mu p_\mu < 0$: this does not bother us, as our physical request should be that all the particle must have a positive energy ouside the static limit surface. Therefore a particle with negative energy inside the ergosphere either remains there, or needs to be accelerated until its energy becomes positive if it is to escape.
Let's start with a particle 0 outside the ergosphere (so that $E^{(0)} = -K^\mu p^{(0)}_\mu > 0$) that, moving along its geodesic, falls into the ergosphere. Suppose now that particle 0 decades into particle 1 and 2: at that istant the four-momentum is conserved, and the contraction with K^μ gives $E^{(0)} = E^{(1)} + E^{(2)}$. It is possible that the decay occurs in such a way that particle 2 has $E^{(2)} < 0$ and falls across the horizon while particle 1 (that has to have $E^{(1)} > E^{(0)}$) escapes to infinity: the particle that emerges form the ergosphere has more energy than the one that has fallen. This procedure is known as Penrose Process. Nothing is free, however, and the energy gained must come at the expense of something, that is the rotational energy of the black hole by decreasing its angular momentum. The statement that particle 2 crosses the event horizon means that

$$p^{(2)\mu}\xi_\mu = p^{(2)\mu}(K_\mu + \Omega_H\, R_\mu) = -E^{(2)} + \Omega_H L^{(2)} \leq 0 \tag{1.33}$$

Since we have arranged $E^{(2)} < 0$, then $L^{(2)} \leq \frac{E^{(2)}}{\Omega_H} < 0$ and we see that particle 2 must be emitted against the hole's rotation. Once particle 2 crosses the horizon, the black hole mass and angular momentum are changed by $\delta M = E^{(2)}$ and $\delta J = L^{(2)}$, and eq. (1.33) becomes

$$\delta J \leq \frac{\delta M}{\Omega_H} \tag{1.34}$$

The meaning of this relation is clear: the more the black hole is slowed down, the larger the energy that can be extracted out of them. Using eq. (1.17) , eq. (1.34) can be rewritten as

$$\delta\Big(M^2 + \sqrt{M^4 - \frac{J^2}{G_N^2}}\Big) = \delta\Big(\frac{r_+^2 + a^2}{2G_N^2}\Big) \geq 0 \tag{1.35}$$

We already glimpse a loose connection with thermodynamics: the decreasing of the mass and angular momentum of the black hole must occur in such a way that the quantity in (1.35) remains constant or increases. The physical meaning of this quantity can be understood computing the area of the outer event horizon: the induced metric on the horizon is

$$\begin{aligned} \gamma_{ij} dx^i dx^j &= ds^2(dt = dr = 0, r = r_+) \\ &= (r_+^2 + a^2\cos^2 a)d\theta^2 + \Big(\frac{(r_+^2 + a^2)^2 \sin^2\theta}{r_+^2 + a^2\cos^2 a}\Big)d\phi^2 \end{aligned} \tag{1.36}$$

The horizon area is then the integral of the induced volume element

$$A = \int \sqrt{|\gamma|} d\theta d\phi = 8\pi G_N^2 \Big(M^2 + \sqrt{M^4 - \frac{J^2}{G_N^2}}\Big) \tag{1.37}$$

Hence eq.s (1.35) and (1.37) give

$$\delta A \geq 0 \tag{1.38}$$

From the foregoing it follows that energy extraction from a black hole is maximally efficient when the horizon area does not change, and processes that increase the area are irreversible as it cannot be decreased: the analogy with thermodynamics is strinking, with the area playing the role of entropy. This correspondence can be formalized stating the following two laws:

- Using eqs. (1.35), (1.17) and (1.18) it follows

  $$\delta M = \frac{k}{8\pi G_N}\delta A + \Omega_H \delta J \tag{1.39}$$

  So far, we have considered only electrically neutral Kerr solutions. If a black hole is electrically charged one can extract energy from it by neutralizing it. See [7] for the description of this case, here we just state that, taking into account this process, (1.39) generalizes as

  $$\delta M = \frac{k}{8\pi G_N}\delta A + \Omega_H \delta J + \Phi \delta Q \tag{1.40}$$

  where Φ is the electrostatic potential difference between the horizon and infinity. This is the first law of black hole thermodynamics.

- The parallel between the thermodynamical quantities temperature and entropy with surface gravity and area of the black hole, as well as eq. (1.40) suggests the correspondence $\frac{k}{8\pi G_N} dA = TdS$. Using eq. (1.30) we associate to a black hole the Bekenstein-Hawking entropy

  $$S_{BH} = \frac{k_B A}{4\hbar G_N} \tag{1.41}$$

  Let's note that the entropy of a black holes grows with the area of the horizon and not with the volume, as one would expect for typical physical systems: this fact, as we will see more closely later, is one of the hints for the AdS/CFT correspondence. From (1.38) and (1.41) follows that black hole entropy is separately non decreasing, while in thermodynamics only the total entropy is non decreasing. Moreover, we have seen that a black hole radiates as a black body with temperature T given by eq. (1.30): the Hawking emission determines a decrease of the black hole mass and, thus, its area decreases (in contrast with eq. (1.38)). To solve this problem, in [8] Bekenstein conjectured the generalized second law of black hole thermodynamics, which states

$$\delta(S_{outside} + S_{BH}) \geq 0 \tag{1.42}$$

where $S_{outside}$ is the entropy outside the black hole.
In a Penrose process, however, it seems to be possible to violate (1.42): the inequality $\delta S_{BH} \geq 0$ can be saturated in this process if the particle is released from the horizon with zero radial velocity and once it crosses the horizon the entropy of the exterior has decreased. As we have learned from Hawking radiation (and as the presence of $\hbar$ in eq.s (1.30) and (1.41) suggests us), a full understanding of the thermodynamical behaviour of black holes should take into account QFT. Applying quantum mechanics, we see that in the Penrose process we cannot locate the particle (with $\dot{r} = 0$) exactly at $r = r_+$: there is an uncertainty in the position of the order of its Compton wavelenght. Therefore the process cannot be reversible and there is a minimum amount of area increasing. Its computation gives a result that is consistent with (1.42).

1.2.3 The entropy puzzle and the information paradox

We have seen that a correspondence can be established between the laws of black hole mechanics and thermodynamics. If we take this analogy seriously, and Hawking's calculations suggests we should, some problems arise: the entropy puzzle and the information paradox. In eq. (1.41) we have associated an entropy to a black hole, but what does it mean that an exact solution of GR (a classically fundamental theory), furthermore protected by a no-hair theorem, has an entropy? Statistical mechanics tells us that entropy is the logarithm of the number of microstates that have the same macroscopic properties of the macroscopical system. From a classical point of view, however the no-hair theorem tells us that there is no phase space for a black hole, as it limits the number of possible solutions one can consider; furthermore, one should not expect these microstates to be perturbations of the black hole solution because they would have an horizon, and therefore an entropy (while microstates should be pure states, i.e. states without entropy). So, what are the $e^{S_{BH}}$ microstates? This is the entropy puzzle.
The other problem is directly connected with Hawking's semiclassical computation, that showed that black holes emit radiation and therefore lose mass. The relevant property of Hawking radiation is not merely its existence, but rather the thermal nature of the radiation. If we start with a pure quantum state $|\phi\rangle_{IN}$ that collapse into the horizon, once the black hole has evoporated, we are left with a thermal radiation, a mixed state that can be described only by a density matrix. Moreover, the emitted radiation is entangled with negative energy particles that fall into the black hole and, once the black hole disappears, the emitted radiation is entagled with nothing. This picture goes against the unitary time evolution of QFT. Non-unitary evolution from pure states to mixed stated is what we call information loss: two state that are in principle distinguishable, evolve into states that are not distinguishable any more. This is the information paradox.
These problems, however, have been stated in a semiclassical contest, which means considering QFT on a classical, curved spacetime, neglecting quantum gravity corrections. It is commonly belived that a consistent theory of quantum gravity should shed light upon these paradoxes, and indeed every claimant quantum theory of gravity must confront with black hole physics.

1.2.4 Quantum gravity

It is generally believed that the fundamental description of all physical fields should be given in term of quantum physics. This confidence rises from the assumption that there cannot be two worlds, a

quantum and a classical one, but quantum mechanics should contain classical physics in the form of a certain limiting case. One of the arguments [9] relies on the postulate that quantum mechanical processes are described by probability amplitudes: it cannot be that a particle that is described by a probability amplitude interacts classically at the microscopical level.
The first attempt to give a quantum description of gravity would be to apply one of the quantization procedures to GR, starting from eq. (1.1). This approch leads to fundamental difficulties, an account of which can be found, for example, in [6]. The essential difference between GR and other field theories is that, in order to quantize the dynamical degrees of freedom of $g_{\mu\nu}$, one must also give a quantum mechanical description of the spacetime structure: this reflects the dual role played by the metric in GR, both as the quantity that describes the dynamical aspects of gravity and the quantity that describes the background spacetime. The other quantum field theories, instead, are formulated on a fixed background, which is treated classically. For example, causality leads us to expect that, at the quantum level, the metric (which becomes an operator) satisfies the commutation relation:

$$[\hat{g}_{\mu\nu}(x), \hat{g}_{\rho\sigma}(x')] = 0 \qquad (x - x')^2 > 0 \tag{1.43}$$

The problem is that this equation makes no sense because we do not know whether x and x' are spacelike related untill we know the metric, and eq. (1.43) is an operator equation that, if valid, must hold independently of the (probabilistic) value of the metric. Another difficulty arises when treating the gravitational interaction as a perturbation of the free theory (which is the typical approach to quantize interacting theories): the coupling constant is $\sqrt{G_N}$, which has dimension $[\sqrt{G_N}] = mass^{-1}$, so the theory is not renormalizable.
Non-renormalizable field theories should not be simply viewed as ill defined quantum theories, but as effective descriptions at low energy regimes. The canonical example is Fermi weak theory. From this point of view it is important to note that, attempting to canonically quantize Einstein's equations, we have done an implicit assumption, which is that GR is a fundamental theory rather then a low energy approximation. But this is not always the case: if one quantizes Maxwell theory discovers the photon, while if one tries to quantize, for instance, Navier-Stokes equations gets no hint of the microscopical physics. This justifies the search for a quantum theory of gravity, looking for a high-energy theory that flows to GR in the infrared, rather than trying to quantize GR as it is.[3] String theory seems to be one of the most promising approaches, and will be studied in the next Chapter along with its low energy limit, supergravity.

[3]This is not the only possibility, of course. There are other theory that mantain the first prospective, interpreting the non-renormalizability not as a failure of the theory but of the perturbative expantion. This is the case, for example, for loop quantum gravity that attempts to quantize gravity directly, without splitting the metric into a background and a small perturbation.

Chapter 2

String Theory, Supergravity and Black Holes

2.1 An Introduction to String Theory

String theory is a quantum theory of interacting strings: it assumes that the fundamental objects of Nature are not point-like particles, but 1-dimensional strings of length l_s. Their quantized harmonics represent particles of various masses and spin. The spectrum of the (closed) string contains a massless spin-2 particle, the graviton. In this sense, string theory is a quantum theory of gravity. Thus, it is possible to address black hole physics within string theory, and in the past decades dramatic progresses have been made in this direction. Before analyzing them, in this Section we will briefly describe, with no claim to be exhaustive, string theory and its properties that will be needed in the following.

2.1.1 Action for the Bosonic String

We can describe the classical motion of a string, through a D-dimensional spacetime, by giving its path with the coordinate $X^\mu = X^\mu(\sigma,\tau)$, with $\mu = 0, ..., D-1$. The parameters σ and τ are the coordinates for the worldsheet (the 2-dimensional manifold the string sweeps out mooving in spacetime). The action for a point particle is proportional to the lenght of its worldline

$$S_{pointparticle} = -m \int ds = -m \int d\tau \sqrt{-\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\eta_{\mu\nu}} \tag{2.1}$$

where s is the proper time and τ is a generic parametrization of the worldline of the particle. Its natural generalization for a string is the Nambu-Goto action (which is now proportional to the area of the worldsheet)

$$S_{NG} = -T \int d\tau d\sigma \sqrt{-\det(\partial_a X^\mu \partial_b X^\nu) g_{\mu\nu}} \tag{2.2}$$

Here a and b denote either σ or τ and the constant of proportionality T is the string tension, which is related to the Regge slope parameter α' or to the string lenght l_s via

$$T = \frac{1}{2\pi\alpha'} \qquad \alpha' = l_s^2 \tag{2.3}$$

The square-root in the Nambu-Goto action makes the quantization of the theory difficult; moreover eq. (2.2) is defined in term of a 2-dimensional integral, and one may want to rewrite the action to make it look more like a conventional 2-dimensional field theory. This can be done at the expense of introducing another field γ^{ab}, which acts as the dynamical metric on the worldsheet [10]:

$$S_p = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau\sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu g_{\mu\nu} \tag{2.4}$$

This is Polyakov action and it is equivalent to Nambu-Goto action, as one can check integrating away γ^{ab}. The action has the Poincaré invariance (i.e. the global symmetry $X^\mu \to \Lambda^\mu_\nu X^\nu + c^\mu$), as well as worldsheet diffeomorphism and the conformal (or Weyl) symmetries. The diffeomorphism symmetry is the invariance under reparametrization on the worldsheet

$$(\sigma,\tau) \to (\tilde{\sigma}(\sigma,\tau),\tilde{\tau}(\sigma,\tau)) \tag{2.5}$$

with X^μ transforming as a scalar and γ^{ab} as a two index tensor. This symmetry is a consequence of the string being a fundamental object: it look the same on all scales. Another consequence of this fact is that it does not have longitudinal vibrations: a longitudinal deformation means that the profile does not change, but what would change is the distance between different “parts” of the string. But there is nothing as a “part” of a string, being the string a fundamental object. Thus, the physical degrees of freedom describe only traverse oscillations of the string (this is a constraint that can be formally derived from the theory as in [10]). The Weyl symmetry is the invariance of the action under position dependent rescaling of the worldsheet metric

$$\gamma_{ab} \to e^{2\omega(\sigma,\tau)}\gamma_{ab} \tag{2.6}$$

with arbitrary $\omega(\sigma,\tau)$ and X^μ not transforming. This rescale of the metric preserves the angles between all lines, but changes the distances between distinct points in the worldsheet. It is a consequence of the theory being defined on a 2 dimensional worldsheet.
Using the above symmetries, we can choose a flat metric on the worldsheet $\gamma_{ab} = \eta_{ab}$. Imposing this gauge choice and introducing lightcone coordinates $\sigma^\pm = \tau \pm \sigma$ the equations of motion for X^μ read

$$\partial_+\partial_- X^\mu = 0 \tag{2.7}$$

The most general solution is

$$X^\mu(\sigma,\tau) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-) \tag{2.8}$$

where X^μ_L and X^μ_R describe left-moving and right-moving waves respectively.

There are essentially two types of strings, closed and open, depending on the conditions on its endpoints. A closed string is parametrized by a coordinate σ in a compact domain, which is conventionally chosen to be $[0, 2\pi]$. An open string has insted two non-coinciding ends, and can be parametrized by the coordinate $\sigma \in [0, \pi]$. The derivation of the equations of motion for the open string requires appropriate boundary conditions. We consider the evolution of the open string from some configuration at initial time τ_i to some final configuration at τ_f; the variation of the Polyakov action (in the harmonic gauge $\gamma_{ab} = \eta_{ab}$) is

$$\delta S = \delta\Big(-\frac{1}{4\pi\alpha'}\int d\sigma d\tau \partial_a X^\mu \partial^a X_\mu\Big) = -\frac{1}{2\pi\alpha'}\int d\sigma d\tau \partial_a\partial^a X_\mu \delta X^\mu + \text{total derivative} \tag{2.9}$$

The total derivative determines the boundary conditions

$$\frac{1}{2\pi\alpha'}\int d\sigma \partial_\tau X_\mu \delta X^\mu\Big|_{\tau=\tau_i}^{\tau=\tau_f} - \frac{1}{2\pi\alpha'}\int d\tau \partial_\sigma X_\mu \delta X^\mu\Big|_{\sigma=0}^{\sigma=\pi} \tag{2.10}$$

The first term is the one we are used to when dealing with the principle of stationary action (it vanishes by requiring $\delta X^\mu = 0$ at τ_i and τ_f). The second term is novel, and vanishes requiring

$$\partial_\sigma X^\mu \delta X_\mu = 0 \qquad \text{at} \qquad \sigma = 0, \pi \tag{2.11}$$

This condition can be satisfied in two different ways, corresponding to two different boundary conditions:

- Neumann boundary conditions: $\partial_\sigma X^\mu = 0$ at $\sigma = 0, \pi$. Because there is no restriction on δX^μ, with this condition the ends of the string move freely at the speed of light.
- Dirichlet boundary conditions: $\delta X^\mu = 0$ at $\sigma = 0, \pi$. This means that the ends of the string are confined at some constant position $X^\mu = c^\mu$

In order to understand the meaning of these conditions, let's consider Dirichlet boundary conditions for some coordinates, and Neumann for the others; so that, at both end-points of the string, we have

$$\begin{aligned} \partial_\sigma X^a = 0 \qquad &\text{for} \qquad a = 0, ..., p \\ X^I = c^I \qquad &\text{for} \qquad I = p+1..., D-1 \end{aligned} \tag{2.12}$$

This fixes the ends of the string to lie in a (p+1)-dimensional hypersurface in spacetime. Those hypersurfaces are called Dp-branes (where D stands for Dirichlet and p are the spatial dimensions of the brane). Those are dynamical objects of the theory on which open strings may end. The action for p-branes is the higher dimensional generalization of the Nambu-Goto action, where the tension of the brane is [11]

$$T_p = \frac{\text{mass}}{\text{p-volume}} = \frac{1}{(2\pi)^p \alpha'^{\frac{p+1}{2}} g_s} \tag{2.13}$$

We will not discuss the quantization procedure of the Polyakov action, standard references are [10] and [12]. It turns out that a consistent quantum theory of strings is possible only if the dimension of spacetime is $D = 26$.
The quantization of the theory gives rise to quantized harmonics of the string that are particles of various masses and spins. The masses are typically integral multiples of $\frac{1}{l_s}$. Since we have not yet observed stringy behaviour in any experiment, we must assume that the string length is much smaller than the smallest distances probed so far. At distances greated than l_s only the massless modes are relevant. We will generally work in this regime, focusing only on the massless excitations of the string. When one quantizes the closed string, one finds a tachyonic vacuum state (i.e. characterized by $M^2 < 0$); the presence of this tachyon is a problem of the bosonic string theory, we will see later that it can be resolved introducing supersymmetry. The first excited states are massless particles that live in the $\mathbf{24} \otimes \mathbf{24}$ of the little group $SO(24)$. Those decompose into three irreducible representations:

$$\text{symmetric traceless} \oplus \text{anti-symmetric} \oplus \text{trace}$$

We associate to each representation a massless field in spacetime such that the string oscillation can be identified with a quantum of these fields. These fields are:

- $g_{\mu\nu}(X)$, a massless spin two field, which we interpret as the metric[1].

[1] There is an argument by Feynman [9] that shows that any theory of interacting massless spin two particle must be General Relativity. Briefly, it works as follows. Gravity is a long range force, hence its mediator must be a massless particle. Moreover, it cannot have half-integer spin otherwise the resulting force would not be static. Odd spins must be descarded, as they would give rise to a theory in which like-charges generate a repulsive force [14]. A closer analysis shows that the mediator cannot be a scalar, but has to be a spin 2 particle.

- $B_{\mu\nu}(x)$, which goes by the name of Kalb-Ramond field.
- $\Phi(X)$, a scalar field called dilaton.

Regarding the open string, the first excited states are again massless, and can be classified as follows:

- Excitations polarized along the brane are described by a spin 1 gauge field A_a (with $a = 0, ..., p$) living in the Dp-brane's $(p+1)$-dimensional worldvolume. We will see later that this $U(1)$ gauge theory plays a major role in the AdS/CFT duality.
- Excitations polarized perpendicular to the brane are described by scalar fields ϕ^I (with $I = p+1, ..., D-1$). They can be interpreted as fluctuations of the brane in the transverse directions, this gives us a hint that the D-brane is a dynamical object.

2.1.2 String Interactions

So far we have discussed the free theory, we shall now consider interactions. Much of what we can presently do (as in all quantum field theories) involves treating this interaction in perturbation theory. As a Feynman diagram represents the worldlines of in- and out-going particles, a string diagram represent the worldsheet of interacting strings. Let's firstly consider closed strings: their spectrum contains massless bosonic fields which can condensate and acquire a vacuum expectation value. Let's consider string physics when just the dilaton field has a non trivial VeV. We shall extend the Polyakov action by [10]:

$$S_{string} = S_P + \frac{1}{4\pi}\int d\sigma d\tau \sqrt{\gamma} R \Phi(X) \tag{2.14}$$

If we write the dilaton as

$$\Phi(X) = \Phi_0 + \delta\Phi(X) \qquad \Phi_0 = \lim_{X\to\infty}\Phi(X) \tag{2.15}$$

eq.(2.14) becomes

$$S_{string} = S'_P + \Phi_0\chi \equiv S'_P + \frac{\Phi_0}{4\pi}\int d\sigma d\tau \sqrt{\gamma} R \tag{2.16}$$

The new term χ looks like the Einstein-Hilbert action, but we shall not deceive ourselves: we have seen that all the degrees of freedom associated to the worldsheet metric γ_{ab} can be gauged away by a conformal transformation. It turns out [10] that χ is a topological invariant of the worldsheet, known

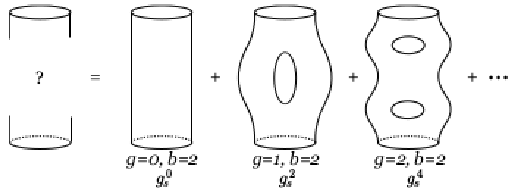

Figure 2.1: An illustration of the sum over all possible topologies.

as Euler number, and it counts the number of genus g of the surface and the number of initial and final states (boundaries of the worldsheet) b via

$$\chi = 2 - 2g - b \tag{2.17}$$

The partition function is thus obtained expanding in topologies as

$$Z = \sum_{\chi} (e^{\Phi_0})^{-\chi} \int D[X]D[\gamma]e^{-S'_P} \tag{2.18}$$

This is a series expansion in e^{Φ_0}, which suggests that $e^{\Phi_0} \equiv g_s$ plays the role of the string coupling. Thus, the string coupling is not an independent parameter of string theory: it is the asymptotic expectation value of the exponential of the dilaton, and thus it can be determined dynamically.
Every loop in a closed string diagram introduce a factor g_s^2. Suppose now that the 1 genus diagram

Figure 2.2: A 1 loop correction to the graviton propagator. For excitations well below the string scale $M_s = \frac{1}{l_s}$ strings behave like pointlike particles ans we recover Feynman diagrams.

in Figure 2.2 involves in- and out-going graviton states. In Section 1.2.4 we have seen that the coupling constant of gravitational interactions is $G_N^{\frac{1}{2}}$, thus this diagram should contribute with a factor $G_N \sim l_P^{D-2}$, where l_P is the Planck length. Computing this diagram in string theory, one finds a factor of $g_s^2 l_s^{D-2}$, where the g_s factor follows form (2.17) and the l_s dependence follows from dimensional analysis (l_s is the only length scale in string perturbation theory). Thus we have

$$g_s \propto (\frac{l_P}{l_s})^{\frac{D-2}{2}} \tag{2.19}$$

From this we see that g_s controls the hierachy of scales in string theory. The perturbation expansion is valid when $g_s << 1$, which means that stringy exitations are much less massive than the Planck scale.

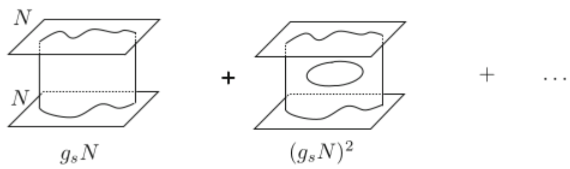

Figure 2.3: Open string perturbation theory is an expansion in $g_s N$.

Regarding open strings, we have seen that they stretch between D-branes, and their end points are labeled by the brane they end on. Thus, open string perturbation theory gains an additional factor N (the number of D-branes) from the degeneracy considered in any scattering process. The perturbative series for open string is, therefore, a power series in Ng_s (Figure 2.3).

2.1.3 Supersymmetry

Until now we have described only bosonic degrees of freedom in string theory; and yet, the world is made of fermions as well. Moreover the bosonic string theory suffers the presence of tachyonic states. Both these problems can be solved introducing supersymmetry. A supersymmetric theory is defined in terms of N fermionic generators Q^I_α (with α a spinor index and $I = 1, ..., N$), acting on the Hilbert space of the theory. The Q's are called supercharges and generate rotations that map fermionic degrees of freedom F into bosonic degrees of freedom B, and viceversa. The transformation can be schematically written as [15]

$$\delta_\epsilon B = \bar{\epsilon} F \qquad \delta_\epsilon F = \epsilon \gamma^\mu \partial_\mu B \tag{2.20}$$

where ϵ is the infinitesimal parameter (the ϵ anticommute because, for the spin-statistics theorem, fermions must be anticommuting) of the rotation and the γ^μ's are matrices satisfying the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. The set of states that transform into each other under supersymmetry transformations is called supermultiplet. The commutator of two such transformations for a bosonic field reads

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] \propto (\bar{\epsilon}_1 \gamma^\mu \epsilon_2) \partial_\mu B \tag{2.21}$$

Eq. (2.21) indicates that composing two supersymmetry transformations one gets a spacetime translation (the momentum operator $P_\mu = -i\partial_\mu$ generates the translation along the physical directions). Thus, we can deduce that a supersymmetric theory must also be invariant under spacetime translations. This suggests that there is a link between supersymmetry and Poincaré invariance, and one may think the supersymmetry algebra as an extension of the Poincaré algebra [16]. The Poincaré algebra is formed by generators of translation P_μ and boosts $M_{\mu\nu}$ satisfying the commutation relations

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] =& - i\eta_{\mu\rho} M_{\nu\sigma} - i\eta_{\nu\sigma} M_{\mu\rho} + i\eta_{\mu\sigma} M_{\nu\rho} + i\eta_{\nu\rho} M_{\mu\sigma} \\
[M_{\mu\nu}, P_\rho] =& - i\eta_{\rho\mu} P_\nu + i\eta_{\rho\nu} P_\mu \\
[P_\mu, P_\nu] =& 0
\end{aligned} \tag{2.22}$$

and, schematically, its supersymmetric extension is[2]

$$[P, Q] = 0 \qquad [M, Q] \sim Q \qquad \{Q, \bar{Q}\} \sim P \tag{2.23}$$

There are essentially two parameters characterizing a supersymmetry theory: the dimension of spacetime D (since spinorial representations get bigger the more the dimensions) and the number of supercharges N. Notice that an interacting theory of massless particles is consistent only if the particles have spin less than or equal to two[3]: this is a constraint on the number of possible supersymmetry theories one can construct. It turns out [17] that, for example, in $D = 4$ one has the bound $N \leq 8$ (more supersimmetries would lead to higher spin fields in the graviton supermultiplet) and that the biggest dimension for a supergravity theory is $D = 11$[4]. The total number of supercharges Q^I_α is N times the (real) dimension of the irreducible spinorial representation in D dimensions. A discussion of the dimensionality of spinorial representation can be found, for example, in [15]. Here, we just report the results in Table 2.1.

[2] We restrict to the case of vanishing central charge.

[3] Higher spin particles can be introduced until they do not interact; switching on the interaction requires the introducion of an infinite tower of fields, and the resulting theory would not be physical.

[4] These two statement are actually equivalent [17]: one can obtain $D = 4$ and $N = 8$ supersymmetry by dimensional reducing $D = 11$ and $N = 1$.

D	2	3	4	5	6	7	8	9	10	11
d_{irrep}	1	2	4	8	8	16	16	16	16	32

Table 2.1: The dimension (d_{irrep}) of the irreducible spinor representation in D spacetime dimensions.

2.1.4 Superstring theories

The key difference between bosonic string theory and superstring theories is the addition of fermionic modes on the worldsheet. While the bosonic string theory is unique, there are a number of discrete choices that one can make when adding fermions. The most important one is whether to add fermions in both left-moving and right-moving sectors (obtaining Type II superstring), or allow them to move in only one direction (obtaining Heterotic strings). However, later developments have shown that they are all parts of the same framework, which goes by the name of M-Theory. In this thesis we will discuss and use only Type II superstring. We introduce D Majorana spinors $\psi^\mu = (\psi^\mu_a)$ (where $\mu = 0, ..., D-1$ is the spacetime index and $a = \pm$ is a worldsheet spinor index), with action

$$S_\psi = \frac{i}{4\pi\alpha'}\int d\tau d\sigma\sqrt{-\gamma}\bar{\psi}_\mu\rho^a\partial_a\psi^\mu \tag{2.24}$$

where ρ^a are the gamma matrices in two dimensions. With the gauge choice $\gamma_{ab} = \eta_{ab}$, using null cordinates $\sigma^\pm = \tau \pm \sigma$, the fermionic equations of motion read

$$\begin{aligned} \partial_+\psi_- =&0 \qquad \rightarrow \psi_- = \psi_-(\sigma^-) \\ \partial_-\psi_+ =&0 \qquad \rightarrow \psi_+ = \psi_+(\sigma^+) \end{aligned} \tag{2.25}$$

The analysis of the boundary conditions of the fermionic fields shows that one must impose periodic ($\psi^\mu_\pm(\sigma+2\pi) = \psi^\mu_\pm(\sigma)$) or antiperiodic ($\psi^\mu_\pm(\sigma+2\pi) = -\psi^\mu_\pm(\sigma)$) boundary conditions, which correspond to the R-sector and to the NS-sector respectively.
Let's now combine the Polyakov action with eq. (2.24) and introduce the gravitino χ^α, the supersymmetric partner of γ^{ab}, in such a way that the resulting action is supersymmetric. The resulting action possesses reparametrization and conformal invariance. These symmetries can be used to fix some degrees of freedom: a useful choise is the so-called superconformal gauge, in which $\gamma_{ab} = \eta_{ab}$ and $\chi_a = 0$. With this gauge choice, the action for Type II superstring reads [18]

$$S_{II} = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma\sqrt{-\gamma}\Big[\partial_a X^\mu\partial^a X^\mu - i\bar{\psi}_\mu\rho^\alpha\partial_\alpha\psi^\mu\Big] \tag{2.26}$$

The quantization of the theory procedes analogously with the bosonic string case. One can project out of the spectrum the tachyonic state that is present in the NS sector. This can be done with the GSO projection, which keeps just the states constructed applying an odd number of fermionic creation operators to a vacuum state and projects out the others. This operation removes the tachyonic state from the Fock space, as it has an even fermionic number. It turn out that the GSO prjection has to be applied also to the R sector: in this case, whether to keep the states with even or odd fermionic number is a matter of choice, and this choice gives rise to two different theories. Consistency requires that the dimension of spacetime must be $D = 10$. The massless spectrum can be classified in 4 sectors according to the different possible boundary conditions:

- NS-NS sector: the field content is identical to the bosonic string. It consists in the dilaton Φ, the Kalb-Ramond antysimmetric tensor field $B_{\mu\nu}$ and the graviton $g_{\mu\nu}$.
- The NS-R sector contains two fermionic fields: the spin-$\frac{1}{2}$ dilatino and the spin-$\frac{3}{2}$ gravitino (supersymmetric partners of the dilaton and of the graviton respectively)

- The R-NS sector contatins the same spectrum of the NS-R sector
- R-R sector: it contains bosonic fields, but its spectrum depends on the way one makes the GSO projection. Two different theories arise: Type IIA and Type IIB supergravity teories. The former contains a 1-form and a 3-form; the latter a 0-form, a 2-form and a self dual 4-form.

2.1.5 Kaluza-Klein Mechanism

We have seen that a consistent quantization of superstring theory requires that the dimension of spacetime to be $D = 10$. And yet, our experience suggests us that we live in D=4, nor there are any experimental indications of the presence of extra-dimensions. How can such theories in $D > 4$ describe our 4-dimensional world? The point is that all of the dimensions need not to be infinitely extended: some of them can be compact. Consider, for instance, a 5-dimensional spacetime in which 4 directions are flat (with coordinates x^μ, $\mu = 0, ..., 3$) while the fifth direction is a circle of radius R (whose coordinate y is periodic: $y = y + 2\pi R$).
Consider now a massless scalar field $\phi(x, y)$; we can decompose it as

$$\phi(x,y) = \sum_n \phi_n(x) e^{\frac{iny}{R}} \tag{2.27}$$

where the integer-valued n labels the quantized momenta in the compact direction. This is a usual consequence of quantum mechanics on compact domains: it is derived by requiring that the wavefunction is single valued. The equation of motion $\partial_M \partial^M \phi(x,y) = 0$ (where $M = 0, ..., 4$ is the index of the 5-dimensional spacetime) gives:

$$\sum_n (\partial_\mu \partial^\mu - \frac{n^2}{R^2}) \phi_n(x) e^{\frac{iny}{R}} = 0 \qquad \Longrightarrow \qquad (\partial_\mu \partial^\mu - \frac{n^2}{R^2}) \phi_n(x) = 0 \qquad \forall n \tag{2.28}$$

Thus, a single field in higher dimensions becomes an infinite tower of massive fields in the noncompact world, with mass m_n given by $m_n = \frac{|n|}{R}$. At energies much lower then $\frac{1}{R}$, only the $n = 0$ mode can be excited, and at this scale $\phi(x^M) = \phi_0(x^\mu)$.
The same analysis cannot be naively applied to a 5-dimensional metric field $\tilde{g}_{MN}$: the Fourier modes of a 5-dimensional scalar field can be interpreted as scalar fields in 4 dimensions, but the Fourier modes of the 5-dimensional metric cannot be interpreted as 4-dimensional metrics because they are 5×5 matrices. What is non trivial is that, applying the same mechanism to the metric, the effective lower-dimensional world contains a metric as well as a vector gauge field and scalar matter field. In order to check this let's consider pure gravity in 5 dimensions. The action is:

$$S = \frac{1}{2k_5^2} \int d^5x \sqrt{-\tilde{g}} R_5 \tag{2.29}$$

Let's now decompose the 5-dimensional metric $\tilde{g}$ in the following ansatz[5]

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + e^{2\sigma} (dy + A_\mu dx^\mu)^2 \tag{2.30}$$

where $g_{\mu\nu}$ is the metric in the noncompact directions of spacetime, A_μ is a vector and σ a scalar field. The vector field A_μ is an honest gauge field, with gauge symmetry descending form diffeomorphisms

[5]This is the ansatz one obtains requiring that all the fields of the decomposition behaves correctly under D-dimensional diffeomorphisms [12].

in $D = 5$. Under a transformation $\delta x^M = V^M$, the metric transforms as $\tilde{g}_{MN} = D_M V_N + D_N V_M$. One can check using eq. (2.30) that diffeomorphisms on the compact direction $\delta y = \Lambda(x, y)$ turn into $\delta A_\mu = \partial_\mu \Lambda$. The action written in term of the new fields reads [10]

$$S = \frac{1}{2k_5^2} \int d^5x \sqrt{-\tilde{g}} R_5 = \frac{2\pi R}{2k_5^2} \int d^4x \sqrt{-g} e^{\sigma} (R_4 - \frac{1}{4} e^{2\sigma} F_{\mu\nu} F^{\mu\nu} + \partial_\mu \sigma \partial^\mu \sigma) \tag{2.31}$$

where $F = dA$ and R_4 is the Ricci scalar of $g_{\mu\nu}$. From eq.(2.31) one obtains a relation between Newton constants in different dimension[6]:

$$\frac{2\pi R}{k_5^2} = \frac{1}{k_4^2} \Longrightarrow G_N^{(5)} = 2\pi R G_N^{(4)} \tag{2.32}$$

where $G_N^{(D)}$ is Newton constant in D dimensions. It is worth saying that compactification preserves the supersymmetries[7].

Let's now consider the Kaluza-Klein reduction from the prespepctive of a (bosonic) string. We want to study a string moving in the background $\mathbb{R}^{1,D-2} \times S^1$. One effect of the compactification is that the momentum along the circle direction p^y is quantized in integer units

$$p^y = \frac{n}{R} \qquad n \in \mathbb{Z} \tag{2.33}$$

Another consequence of the compactification is that the boundary conditions become

$$X^y(\sigma + 2\pi) - X^y(\sigma) = 2\pi m R \qquad m \in \mathbb{Z} \tag{2.34}$$

The integer m is the winding number of the string: it is the number of times the string wraps the circle. The presence of a quantized momentum and winding number contribute to the mass of the string: beside the $\frac{1}{l_s}$ contributions, it receives the correction [10]

$$\delta M^2 = \frac{n^2}{R^2} + \frac{m^2 R^2}{l_s^4} \tag{2.35}$$

This equation tells us that a string with $n > 0$ units of momentum receives a contribution to its mass of $\frac{n}{R}$ (in agreement with eq. (2.28)); and that a string wrapping the circle picks up a contribution $2\pi m R T$ to its mass, where T is the tension of the string.

2.2 Supergravity

Supergravity theories are supersymmetric extensions of General Relativity and have a natural embedding in superstring theories, as supergravity corresponds to their low energy limit. The low-energy string effective action describes the low-energy dynamics of a given string theory: by low energies we mean the limit $\alpha' \to 0$ (the limit in which the string scale can be ignored and a theory of particles is recovered). Moreover, at low energies only the massless modes are relevant and their dynamics is descrived by a theory of the corresponding massless fields. The low energy theory can be obtained expanding in powers of α' the action for the massless spectrum and keeping only the lowest term (the terms of higer order in α' are also in higer order in derivatives). Actually, for some superstring theories, it is possible to arrive at the effective theory using supersymmetry arguments: one can construct a theory that contains only the massless modes of the superstring theory in such a way that

[6]We have cheated slightly here. In order to make such an identification one should first put the Einstein-Hilbert term on the r.h.s. of eq. (2.31) in its canonical form, and then rescale the metric so that it looks asymptotically flat (see [19])

[7]This is true, at least, for some compact manifolds, such as the torus. We will consider only manifolds for which this statement is true.

the resulting effective theory is supersymmetric.
Historically, however, supergravity and superstring theories were discovered independently. Before the advent of strings as a theory of quantum gravity, in fact, there was an attempt to control loop divergences in gravity by making the theory supersymmetric. The greater the number of supersymmetries, the better was the control of divergences. In four dimensions, as we have seen, the maximal number of supersymmetries is eight. Such D = 4 N = 8 supersymmetric theory appears complicated but can be obtained in a simple way from a D = 11 N = 1 theory or a D = 10 N = 2 theory via the process of dimensional reduction explained above. In this Section we will introduce $D = 11$ supergravity, as well as Type IIA and Type IIB supergravity. Since our final interest is studying black hole physics, we will mainly restrict to the bosonic content of these theories.

2.2.1 Eleven-dimensional Supergravity

This theory is the low energy limit of M-Theory; its bosonic fields are:

- The eleven-dimensional metric G_{MN}, this is a symmetric traceless tensor (44 d.o.f.)
- The antisymmetric 3-form $A_3 = A_{MNP}dx^M \wedge dx^N \wedge dx^P$ (84 d.o.f.), with field strength $F_4 = dA_3$

The fermionic content is given by:

- The gravitino ψ_M^α (128 d.o.f)

with $M, N, P = 0, ..., 10$. The bosonic part of the action is given by[8]

$$S = \frac{1}{2k_{11}^2}\int d^{11}x\sqrt{-G}\Big(R - \frac{1}{2}|F_4|^2\Big) - \frac{1}{12k_{11}^2}\int A_3 \wedge F_4 \wedge F_4 \tag{2.36}$$

The first term contains the Einstein-Hilbert action and the kinetic term for A_3. The second one is called Chern-Simons term, and is required by supersymmetry; note that it does not contain the metric: it is a topological term. The eleven-dimensional gravitational coupling constant k_{11} is related to the Planck length via $k_{11}^2 \sim G_N^{(11)} \sim l_P^9$. This theory has no free dimensionless parameters: there is only one scale, l_P.

2.2.2 Ten-dimensional Supergravity and Dualities

Type IIA Supergravity

Type IIA supergravity can be obtained from eleven-dimensional supergravity by compactifying a coordinate, say $y \equiv x^{10}$, on a circle of radius R (which is a new lenth scale of the theory). The eleven dimensional metric can be written in terms of a ten-dimensional metric $g_{\mu\nu}$, a 1 form $C_\mu^{(1)}$ and a scalar field σ (or, equivalently, the dilaton $\Phi \equiv \frac{3}{2}\sigma$, whose value at spatial infinity is related to the string coupling via $e^{\Phi_\infty} = g_s$) as

$$ds_{11}^2 = ds_{10}^2 + e^{2\sigma}(dy + C_\mu^{(1)}dx^\mu)^2 \tag{2.37}$$

where $\mu, \nu = 0, ..., 9$. The eleven-dimensional gauge fields can be decomposed into a 2-form $B^{(2)}$ and a 3-form $C^{(3)}$ via

$$A_3 = B^{(2)} \wedge dy + C^{(3)} \tag{2.38}$$

[8]The fermionic part is fixed by supersymmetry

Note that this is exactly the massless field content of Type IIA superstring: g, $B^{(2)}$ and Φ are the fields in the NS-NS sector; $C^{(1)}$ and $C^{(3)}$ are the fields in the R-R sector. The action for Type IIA supergravity can be obtained using (2.36), (2.37) and (2.38):

$$\begin{aligned} S_{IIA} = &\frac{1}{2k_{10}^2}\int d^{10}x\sqrt{-g_{10}}\Big(e^{\sigma}R_{10} + e^{\sigma}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{2}e^{3\sigma}|F^{(2)}|^2\Big) + \\ &-\frac{1}{4k_{10}^2}\int d^{10}x\sqrt{-g_{10}}\Big(e^{-\sigma}|H^{(3)}|^2 + e^{\sigma}|\tilde{F}^{(4)}|^2\Big) - \frac{1}{4k_{10}^2}\int B^{(2)}\wedge F^{(4)}\wedge F^{(4)} \end{aligned} \tag{2.39}$$

where we have introduced the field strengths $F^{(p+1)} = dC^{(p)}$, $H^{(3)} = dB^{(2)}$, and $\tilde{F}^{(4)} = dC^{(3)} - C^{(1)}\wedge F_3$. Written in this frame, (2.39) has an Einstein-Hilbert term that is not written in the canonical form $\sqrt{-g_E}R_E$. To get it in its canonical form we shall move in the so-called Einstein frame via

$$(g_E)_{\mu\nu} = e^{\frac{\Phi}{6}}(g_{10})_{\mu\nu} \tag{2.40}$$

This is the frame one shall use when derives physical results, such as the horizon area (i.e. the entropy of a black hole). Another usefull frame, the string frame, is given by

$$(g_E)_{\mu\nu} = e^{-\frac{\Phi}{2}}(g_s)_{\mu\nu} \tag{2.41}$$

With this choice the Einstein-Hilbert term reads $\sqrt{-g_s}e^{-2\Phi}R_s$; this is the frame one obtains when derives the action S_{IIA} as the low energy limit of Type IIA superstring theory.

T-Duality and Type IIB Supergravity

We have seen that Type IIA supergravity can be obtained by dimensional reducing the eleven-dimensional theory. There is another ten-dimensional supergravity theory that is the low energy limit of Type IIB superstring, Type IIB supergravity. This theory cannot be derived via compactification, but it is related to Type IIA supergravity thanks to a duality between the fields of the two theories: the T-duality. As we have seen in Section 2.1.5, if we wrap a IIA string on a circle of radius R it receives a mass contribution in units of $\frac{R}{l_s^2}$ from the winding number and in units of $\frac{1}{R}$ from the momentum modes. We can do the same for a Type IIB string wrapping a circle of radius $\tilde{R}$. It turns out that if $\tilde{R} = \frac{l_s^2}{R}$ the two theories not only have exactly the same spectra (momentum modes map to winding modes and vice versa) but they are also equivalent at the interacting level. To T-dualize the bosonic fields of Type IIA supergravity into those of Type IIB, it is convenient to rewrite the fields as

$$\begin{aligned} ds^2 &= g_{yy}(dy + A_\mu dx^\mu)^2 + \hat{g}_{\mu\nu}dx^\mu dx^\nu \\ B^{(2)} &= B_{\mu y}dx^\mu \wedge (dy + A_\mu dx^\mu) + \hat{B}^{(2)} \\ C^{(p)} &= C_y^{(p-1)} \wedge (dy + A_\mu dx^\mu) + \hat{C}^{(p)} \end{aligned} \tag{2.42}$$

The fields of the corresponding Type IIB supergravity are [21]

$$\begin{aligned} ds'^2 &= g_{yy}^{-1}(dy + B_{\mu y}dx^\mu)^2 + \hat{g}_{\mu\nu}dx^\mu dx^\nu \\ e^{2\Phi'} &= g_{yy}^{-1}e^{2\Phi} \\ B'^{(2)} &= A_\mu dx^\mu \wedge dy + \hat{B}^{(2)} \\ C'^{(p)} &= \hat{C}^{(p-1)} \wedge (dy + B_{\mu y}dx^\mu) + C_y^{(p)} \end{aligned} \tag{2.43}$$

The NS-NS sector has the same fields in Type IIA and Type IIB supergravity; the R-R sector is again made of p-forms but, in Type IIB, p takes only even values ($p = 0, 2, 4$). Thus the NS-NS term in the action (2.39) is valid also for Type IIB supergravity; while the actions, in the string frame, for the R-R fields and the Chern-Simon term become [21]

$$\begin{aligned} S_{R-R} &= -\frac{1}{4k_{10}^2}\int d^{10}x\sqrt{-g_s}\Big(|F^{(1)}|^2 + |\tilde{F}^{(3)}|^2 + \frac{1}{2}|\tilde{F}^{(5)}|^2\Big) \\ S_{CS} &= -\frac{1}{4k_{10}^2}\int C^{(4)} \wedge H^{(3)} \wedge F^{(3)} \end{aligned} \tag{2.44}$$

where we have introduced the field strengths $\tilde{F}^{(3)} = F^{(3)} - C^{(0)} \wedge H^{(3)}$, $\tilde{F}^{(4)} - C^{(1)} \wedge H^{(3)}$ and $\tilde{F}^{(5)} = F^{(5)} - \frac{1}{2}C^{(2)} \wedge H^{(3)} + \frac{1}{2}B^{(2)} \wedge F^{(5)}$.

S-Duality

An important issue is to consider the strong coupling limit of Type II theories. Regarding Type IIA, the fact that it is the dimensional reduction of D=11 supergravity leads to a KK interpretation of the dilaton as a measure of the radius R of the compact 11th dimension, as one can see from eq. (2.37). Because of the connection between g_s and the dilaton, there is a connection between the string coupling constant and R: $R \sim g_s^{\frac{2}{3}}$. In the strong coupling limit, thus, $R_{11} \to \infty$ and the theory decompactifies: the strong coupling limit of Type IIA is M-theory. Obviously, the same argument does not apply to Type IIB as it cannot be obtained through dimensional reduction. It turns out that its strong coupling limit is still a Type IIB theory, and it can be seen with S-duality. S-duality is a duality under which the coupling constant changes non-trivially, it relates different Type IIB theories. It maps the content of one theory with coupling constant g_s into a dual theory of coupling constant $\frac{1}{g_s}$: it is important to study the strong coupling limit of Type IIB theory and to generate solutions (one can S-dualize a solution, obtaining another one). The set of transformations is

$$\begin{aligned} \Phi' &= -\Phi \\ g'_{\mu\nu} &= e^{-\Phi} g_{\mu\nu} \\ B'^{(2)} &= C^{(2)} \\ C'^{(2)} &= -B^{(2)} \end{aligned} \tag{2.45}$$

2.2.3 Branes

The fields are not the only fundamental objects of supergravity theories: there are also multidimensional objects, called branes, that couple with the p-forms of the theory and play the role of electric and magnetic charges.

Let's review Maxwell theory: the 1-form $A^{(1)}$ couples to a point-particle (which is a 0-dimensional object) with worldline $x^\mu(\sigma)$ and charge q through the interaction lagrangian:

$$L_{int} = q\int d\sigma A_\mu^{(1)} \frac{dx^\mu}{d\sigma} = q\int_\gamma A^{(1)} \tag{2.46}$$

where γ is the path of the particle. The electric charge of a particle can be computed integrating the hodge dual of the field strength $\tilde{F}^{(2)} = *F^{(2)}$ over a 2-sphere surrounding the charge:

$$q_e = \int_{\mathbb{S}^2} *F^{(2)} \tag{2.47}$$

One can also introduce magnetic charges that are monopole sources for the magnetic field. They can be defined as

$$q_m = \int_{\mathbb{S}^2} F^{(2)} \tag{2.48}$$

In supergravity we have p-forms, thus we have to generalize this discussion to multidimensional objects. The interaction lagrangian of eq.s (2.46) becomes

$$L_{int} = q \int_{\gamma_p} A^{(p)} \tag{2.49}$$

where γ_p is a p-dimensional worldvolume of a $(p-1)$-dimensional object: a $(p-1)$-brane. The analog of eq.s (2.47) and (2.48) can be obtained computing the field strength of $A^{(p)}$, $F^{(p+1)} = dA^{(p)}$ and its hodge dual $\tilde{F}^{(D-p-1)} = *F^{(p+1)}$: thus, each p-form couples electrically to a $(p-1)$-brane (with electric charge Q_e) and magnetically to a $(D-p-3)$-brane (with magnetic charge Q_m). The charges can be computed as

$$Q_e = \int_{S^{D-p-1}} \tilde{F}^{(D-p-1)} \qquad Q_m = \int_{S^{p+1}} F^{(p+1)} \tag{2.50}$$

In Table 2.2 we write down all the branes in M-theory, Type IIA and Type IIB supergravity theories.

Theory	M-Theory	Type IIA			Type IIB			
Fields	A_3	$B^{(2)}$	$C^{(1)}$	$C^{(3)}$	$B^{(2)}$	$C^{(0)}$	$C^{(2)}$	$C^{(4)}$
Electric	M2	F1	D0	D2	F1		D1	D3
Magnetic	M5	NS5	D6	D4	NS5		D5	D3

Table 2.2: Coupling of branes to p-forms in M-Theory, Type IIA and TYpe IIB supergravity.

It is natural to include these multidimensional objects in supergravity theories: there must be sources the p-forms couple to. What about the interpretation of these objects in string theory? Also in superstring theories we have p-forms in the R-R sector: what do they couple to? It turns out that they are sourced by the D-branes where open strings end. Thus, we can interpret the content of Table 2.2 from a stringy point of view. The D-branes are exactly the Dirichlet-branes we have encountered when studying open string boundary conditions. The F1-brane is the fundamental string. The NS5-brane is a fundamental different object: it should be considered as the magnetic dual of the fundamental string (the analogous of a magnetic monopole).
We can see from eq.s (2.42) and (2.43) how T-duality exchanges the branes. The exchange of the NS-NS fields $g_{\mu y}$ and $B_{\mu y}$ corresponds to the transformation of the string winding number (F1) into momentum (P) along the string in the T-duality direction. From the transformation of the R-R fields $C^{(p)}$ we see that the dimension of the Dp-brane changes under T-duality depending on whether it is performed on a circle parallel or perpendicular to the brane worldvolume. To summarize we have:

$$F1 \leftrightarrow P \qquad Dp \xrightarrow{\parallel} D(p-1) \qquad Dp \xrightarrow{\perp} D(p+1) \tag{2.51}$$

The presence of a brane breaks some supersymmetries. This is not surprising: the vacuum is invariant under all the supersymmetry transformations. The important feature [13] is that, if the brane is not excited, it breaks half of the supersymmetries. If there are no branes, the left and right movers are independent; if one adds a brane boundary conditions occur and they impose constrains between left and right movers that halve the supersymmetries.
They are thus Bogomolny-Prasad-Sommerfield (BPS) states. The relevance of BPS states will be clear when we will deal with the microscopic computation of black hole entropy. For now, let's see in a simple contest what a BPS state is. Consider a theory with a simple supercharge $Q = Q^\dagger$, with Hamiltionian given by $\{Q, Q\} = H$. The energy of any state cannot be negative as

$$E = \langle\psi|H|\psi\rangle = \langle\psi|Q^2|\psi\rangle = \langle Q\psi|Q\psi\rangle \geq 0 \tag{2.52}$$

where the equality sign holds for supersymmetric states, i.e. states such that $Q|\psi\rangle = 0$. As we have seen, non supersymmetric states occur in supermultiplets containing bosonic $|B\rangle$ and fermionic states $|F\rangle$. Because $|B\rangle$ and $|F\rangle$ are connected by a supersymmetric transformation and $[Q, H] = 0$, they are states of the same energy. Supersymmetric states, instead, have $E = 0$ and need not to be paired. Consider now a theory with two real supercharges $Q_1 = Q_1^\dagger$ and $Q_2 = Q_2^\dagger$ such that:

$$Q_1^2 = H \qquad Q_2^2 = H \qquad \{Q_1, Q_2\} = Z \tag{2.53}$$

where Z is the central charge. Now we have that

$$\langle\psi|(Q_1 \pm Q_2)^2|\psi\rangle = 2E \pm 2Z \geq 0 \tag{2.54}$$

Thus, if $(Q_1 \pm Q_2)|\psi\rangle = 0$ then $E = |Z|$. To fix the ideas let's consider a state such that $(Q_1 - Q_2)|\psi\rangle = 0$ (with $Q_1|\psi\rangle$,$Q_2|\psi\rangle \neq 0$). As $Q_2|\psi\rangle = Q_1|\psi\rangle$ and $Q_2Q_1|\psi\rangle = E|\psi\rangle$, these states fall into a short multiplet described, for example, by the basis

$$\{|\psi\rangle, Q_1|\psi\rangle\} \tag{2.55}$$

The size of the multiplet is determined by the number of supersymmetries that are broken. States not annihilated by any linear combination of Q_1 and Q_2 form a long multiplet

$$\{|\psi\rangle, Q_1|\psi\rangle, Q_2|\psi\rangle, Q_2Q_1|\psi\rangle\} \tag{2.56}$$

Finally, states that are killed by both Q_1 and Q_2 are supersymmetric states (with $E = Z = 0$) and live in the singlet representation of the supersymmetry algebra. The states in the short multiplet are called BPS states, and saturate the bound $E \geq |Z|$.
Here we have analyzed BPS states as states of a quantum system. In 10-dimensional supergravity a brane in an unexcited configurations retains a half of the supersymmetries and have maximal charge for a given mass: this is the analog of the extremality condition we have encountered when studying classical black holes. In presence of different branes, each type of brane halves the number of supersymmetries of the solution: we will refer to the one charge solution (1 type of brane) as $\frac{1}{2}$-BPS, to the two charges solution as $\frac{1}{4}$-BPS, an so on.

2.3 Supergravity Solutions

In this Section we will consider the problem of finding solutions in supergravity theories for given p-brane configurations. The p-branes, as all physical object, have energy (given by eq. (2.13)); thus they couple to the metric curving spacetime. Furthermore, having an electric or magnetic charge, they source the associated q-form as in eq. (2.50). We will focus only on BPS solutions. There are two different methods to derive solutions:

- The first one consists in solving the equations of motion of the supergravity theory. As it happens for Einstein equations, this is difficult in general. However, the presence of symmetries in the brane configuration and supersymmetry simplify the task. In this contest, BPS solutions are obtained imposing constrains on the supersymmetry transformations of the fields. If we require that the configuration of the fields is supersymmetric, the fields should be invariant under a supersymmetry transformation δ_ϵ. Bosonic fields transform into fermionic ones and, when the latter are set to zero, bosonic fields are invariant. Consistency requires that also the supersymmetric variation of the fermions vanishes, leading to the BPS solutions.

- The second method starts from some trivial solution, and derives other solutions by means of T-duality and S-duality. One can add charges to the solution making boosts along a compact direction. The resulting metric is still a supergravity solution because the supergravity action is Lorentz invariant; but yet it is another solution because the boost direction is compact: the boost is not a globally defined change of coordinates and we are constructing a different solution. We will see later that, in this contest, a BPS solution can be obtained imposing the extremality condition (i.e. taking the so called BPS-limit).

Review of the first method can be found, for example, in [22] and [23]. We will use the second method to derive $\frac{1}{2}$-BPS, $\frac{1}{4}$-BPS and $\frac{1}{8}$-BPS supergravity solutions. Let's consider a 10-dimensional spacetime, with topology $\mathbb{R}^{1,4} \times S^1 \times T^4$. Let's denote with (t, x^i) the coordinates on the non-compact directions, with y the coordinate on the circle and with z^a the coordinates on the 4-dimensional torus. The starting point is the Schwarzshild metric in the $\mathbb{R}^{1,4}$ directions

$$ds^2 = -\Big(1 - \frac{2M}{r^2}\Big)dt^2 + \Big(1 - \frac{2M}{r^2}\Big)^{-1} dr^2 + r^2 d\Omega_3 + dy^2 + \sum_a (dz^a)^2 \tag{2.57}$$

where $G = 1$ and we have used polar coordinates in the 4 noncompact spatial direction:

$$x^1 = r\sin\theta\cos\phi \qquad x^2 = r\sin\theta\sin\phi \qquad x^3 = r\cos\theta\cos\psi \qquad x^4 = r\cos\theta\sin\psi \tag{2.58}$$

with $\theta \in [0, \frac{\pi}{2}]$ and $\phi, \psi \in [0, 2\pi]$. This is solution a of Einstein's equations in vacuum and, therefore, a supergravity solution when the gauge fields are switched off.

2.3.1 The 1-charge geometry

Let's now perform a boost along the circle:

$$y' = y\cosh\alpha + t\sinh\alpha \qquad t' = t\cosh\alpha + y\sinh\alpha \tag{2.59}$$

where α is the boost parameter. Renaming $y' \equiv y$ and $t' \equiv t$, the metric becomes

$$\begin{aligned} ds^2 =& \Big(1 + \frac{2M\sinh^2\alpha}{r^2}\Big)dy^2 + \Big(-1 + \frac{2M\cosh^2}{r^2}\Big)dt^2 + 2\cosh\alpha\sinh\alpha\frac{2M}{r^2}dydt + \\ &+ \Big(1 - \frac{2M}{r^2}\Big)^{-1} dr^2 + r^2 d\Omega_3 + dy^2 + (dz^a)^2 \end{aligned} \tag{2.60}$$

This is a solution of Type IIA supergravity generated not by a brane configuration, but by a wave carrying momentum. We have thus introduced a charge P_y (the momentum of the wave). Let's now apply a T-duality, the result will be a solution of Type IIB supergravity describing a fundamental string wrapping the circle, $F1_y$. Rewriting the metric in the form of eq. (2.42) and applying the correspondence of eq. (2.43) one obtains

$$\begin{aligned} ds^2 =& S_\alpha^{-1}\Big(dy^2 + \Big(-1 + \frac{2M}{r^2}\Big)dt^2\Big) + \Big(1 - \frac{2M}{r^2}\Big)^{-1} dr^2 + r^2 d\Omega_3 + dy^2 + (dz^a)^2 \\ e^{2\Phi} =& S_\alpha^{-1} \qquad B^{(2)} = \frac{2M}{r^2}\cosh\alpha\sinh\alpha S_\alpha^{-1} dt\wedge dy \\ S_\alpha \equiv& \Big(1 + \frac{2M\sinh^2\alpha}{r^2}\Big) \end{aligned} \tag{2.61}$$

This is not BPS. To obtain a BPS solution we have to impose the extremality condition, i.e. take the BPS limit

$$M \to 0 \qquad \alpha \to \infty \qquad \text{such that} \qquad Me^{2\alpha} = 2Q \tag{2.62}$$

where Q is the charge of F1. Hence, the $\frac{1}{2}$-BPS solution, in the string frame, reads

$$\begin{aligned} ds^2 =& Z_1(r)^{-1}(-dt^2 + dy^2) + dr^2 + r^2 d\Omega_3 + (dz^a)^2 \\ e^{2\Phi} =& Z(r)^{-\frac{1}{2}} \qquad B^{(2)} = -Z(r)^{-1} dt\wedge dy \qquad Z(r) = 1 + \frac{Q}{r^2} \end{aligned} \tag{2.63}$$

The metric is singular only at $r = 0$, so, if any, the horizon should be located there. In order to calculate the area of the horizon we can take two equivalent approches:

- We can find the Einstein metric in the noncompact directions via dimensional reduction, compute the area A_5 in this metric and obtain the black-hole entropy (1.41) using the 5-dimensional Newton constant.
- We can compute the area of the horizon A_{10} in the 10-dimensional Einstein metric and obtain the black-hole entropy (1.41) using the 10-dimensional Newton constant.

As we will see in Section 2.3.3, the two approches are equivalent. This should not surprise us since the entropy is a physical quantity.

Let's follow the first method: we shall evaluate the determinant of the metric in the Einstein frame, and perform the integral:

$$A_{10} = V_5 \int_0^{2\pi} d\psi \int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} d\theta \sqrt{-g_E|_{r=0}} \tag{2.64}$$

where $V_5 = V_{S^1} V_{T^4}$ is the volume of the compact space, given by the product of the volume of the circle $V_{S^1} = 2\pi R$ and the volume on the torus $V_{T^4} = (2\pi)^4 V$. Using (2.41) and (2.63) one gets $g_E|_{r=0} = 0$ and the horizon area vanishes. Thus, it is impossible to interpret this solution as a black hole with thermodynamical properties: let's add another charge.

2.3.2 The 2-charge geometry

Let's make another boost along y, with boost parameter β, proceeding from eq. (2.61) (the extremal limit should be taken once, only at the end of the computation, because a boost acts trivially on a BPS state). The result describes a string $F1_y$ wrapped in the y direction carrying momentum P_y:

$$\begin{aligned} ds^2 =& S_\alpha^{-1} S_\beta \Big(dy + \frac{2M \cosh\alpha \sinh\alpha / r^2}{1 + 2M \sinh\beta / r^2} dt \Big)^2 + S_\alpha^{-1} S_\beta^{-1} \Big(-1 + \frac{2M}{r^2} \Big) dt^2 + \\ &+ \Big(1 - \frac{2M}{r^2} \Big)^{-1} dr^2 + r^2 d\Omega_3 + dy^2 + (dz^a)^2 \\ e^{2\Phi} =& S_\alpha^{-1} \qquad B^{(2)} = \frac{2M}{r^2} \cosh\alpha \sinh\alpha S_\alpha^{-1} dt \wedge dy \end{aligned} \tag{2.65}$$

where S_β is defined with the same structure of S_α. The BPS limit gives

$$\begin{aligned} ds^2 =& Z_1(r)(-dt^2 + dy^2 + K(r)(dt + dy)^2) + dr^2 + r^2 d\Omega_3 + (dz^a)^2 \\ e^{2\Phi} =& Z(r)^{-\frac{1}{2}} \qquad B^{(2)} = -Z(r)^{-1} dt \wedge dy \\ Z_1(r) =& 1 + \frac{Q_1}{r^2} \qquad K(r) = Z_P - 1 = \frac{Q_P}{r^2} \end{aligned} \tag{2.66}$$

Again the solution is singular at $r = 0$, and the horizon area vanishes[9].
Before adding another charge, it is usefull to give the $\frac{1}{4}$-BPS solution in an other duality frame: the $D1_y D5_{yT^4}$. This configuration is summarized in Table 2.3.

	0	1	2	3	4	5	6	7	8	9
D1	−	•	•	•	•	−	•	•	•	•
D1	−	•	•	•	•	−	−	−	−	−

Table 2.3: D1D5 configuration. The • symbol indicate that the object is pointlike in the corresponding direction, the − indicates that the brane extends in the corresponding direction. The 0 direction is time; the 1-4 are the non compact spatial directions; the 5 direction is the circle and the remaining are T^4.

[9]Note that at $r = 0$ the curvature blows up, and we cannot trust anymore the supergravity answer. The supergravity action, in fact, is the low energy limit of Superstring theory: its action can be obtained keeping the first order expansion of the Type II supergravity action in $l_s^2 R$. When $R \to \infty$, higher order corrections become relevant, and we cannot trust the semiclassical limit.

Starting from the $F1_yP_y$, the $D1D5$ frame can be reached performing a chain of dualities. They are schematically:

$$\begin{aligned}(F1_yP_y) \xrightarrow{\text{S}} (D1_yP_y) \xrightarrow{\text{T along } T^4} (D5_{yT^4}P_y) \xrightarrow{\text{S}} (NS5_{yT^4}P_y)\\ \xrightarrow{\text{T along y}} (NS5_{yT^4}F1_y) \xrightarrow{\text{T along } z^1 \text{ and S}} (D5_{yT^4}D1_y)\end{aligned} \tag{2.67}$$

Note that the D5-brane has been constructed out of the fundamental string F1 and thus its charge Q_5' is related to the charge Q_1 in eq. (2.66) (and to the boost parameter α). Analogously the charge of D1 Q_1' derives from Q_P. Using eq.s (2.42), (2.43) and (2.45), the solution gets the form

$$\begin{aligned} ds^2 =& Z_1(r)^{-\frac{1}{2}} Z_5(r)^{-\frac{1}{2}}(dy^2 - dt^2) + Z_1(r)^{\frac{1}{2}} Z_5(r)^{\frac{1}{2}}(dr^2 + r^2 d\Omega_3) + Z_1(r)^{-\frac{1}{2}} Z_5(r)^{\frac{1}{2}}(dz^a)^2 \\ e^{2\Phi} =& Z_1(r)^{-1} Z_5(r) \qquad C^{(2)} = -Q_5' \sin^2\theta d\phi \wedge d\psi + (1 - Z_5(r)^{-1}) dt \wedge dy \\ Z_1(r) =& 1 + \frac{Q_1'}{r^2} \qquad Z_5(r) = 1 + \frac{Q_5'}{r^2} \end{aligned} \tag{2.68}$$

2.3.3 The 3-charge geometry

We have seen that the 1-charge and the 2-charge solutions do not give rise to a black hole with thermodynamical properties because their horizon area (and thus their entropy) vanishes. We will now add a third charge, obtaining the Strominger-Vafa black hole.
The explicit derivation of this solution is given by applying the chain of dualities (2.67) to eq. (2.65) and performing another boost along the y direction. Doing so, one arrives to a solution describing a bound state of D1, D5 branes with momentum: the D1D5P solution. The calculation is quite long (see [23]), we just state that the result, once the BPS limit is taken, is

$$\begin{aligned} ds^2 =& Z_1^{-\frac{1}{2}} Z_5^{-\frac{1}{2}}(-dt^2 + dy^2 + K(dt+dy)^2) + Z_1^{\frac{1}{2}} Z_5^{\frac{1}{2}}(dr^2 + r^2 d\Omega_3) + Z_1^{\frac{1}{2}} Z_5^{-\frac{1}{2}}(dz^a)^2 \\ e^{2\Phi} =& Z_1 Z_5^{-1} \qquad C^{(2)} = -Q_5 \sin^2\theta d\phi \wedge d\psi + (1 - Z_5(r)^{-1}) dt \wedge dy \\ Z_{1,5} =& 1 + \frac{Q_{1,5}}{r^2} \qquad K(r) = Z_P - 1 = \frac{Q_P}{r^2} \end{aligned} \tag{2.69}$$

where Q_1, Q_5 and Q_P are respectively the charges of D1, of D5 and the momentum charge. We want to derive the entropy of this black hole. The solution takes the form of (2.69) in the string frame, using eq. (2.41) one gets that the metric in the Einstein frame reads

$$ds^2 = Z_1^{-\frac{3}{4}} Z_5^{-\frac{1}{4}}(-dt^2 + dy^2 + K(dt+dy)^2) + Z_1^{\frac{1}{4}} Z_5^{\frac{3}{4}}(dr^2 + r^2 d\Omega_3) + Z_1^{\frac{1}{4}} Z_5^{-\frac{1}{4}}(dz^a)^2 \tag{2.70}$$

The solution is singular at $r = 0$ and the event horizon is located there. Let's calculate the area of the black hole in 10 dimensions, using eq. (2.64). We have that

$$\sqrt{-g_E} = r^3 \sin\theta \cos\theta \sqrt{Z_1 Z_5 Z_P} \tag{2.71}$$

and the area of the horizon is

$$A_{10} = 2\pi^2 V_5 \sqrt{Q_1 Q_5 Q_P} \tag{2.72}$$

The non-vanishing of the area gives rise to a non trivial entropy, that (in units $\hbar = k_B = 1$) is

$$S = \frac{A_{10}}{4G_{10}} = \frac{\pi^2 V_5 \sqrt{Q_1 Q_5 Q_P}}{2G_{10}} \tag{2.73}$$

where G_{10} is Newton constant in 10 dimensions.
Now, we want to compute the entropy of the 5-dimensional black hole and show that the two results

coincide. Let's follow the recipe in [19]. The first step is the dimensional reduction of the metric g_{T^4} on the torus: no off diagonal blocks are present, thus (as one can see from eq. (2.30)) the 6-dimensional metric is just the $\mathbb{R}^{1,4} \times S^1$ block in (2.70), but with a subtlety: as one can see from eq. (2.31), the ansatz (2.30) does not yield to the Einstein-Hilbert action in its canonical form. Thus, one has to rescale the 6-dimensional metric by a factor

$$|\det(g_{T^4})|^{\frac{1}{d-2}}\Big|_{d=6} = Z_1^{\frac{1}{4}} Z_5^{-\frac{1}{4}} \tag{2.74}$$

obtaining the 6-dimensional metric G_{MN}, with line element:

$$ds^2 = G_{MN}dx^M dx^N = Z_1^{-\frac{1}{2}} Z_5^{-\frac{1}{2}}(-dt^2 + dy^2 + K(dt+dy)^2) + Z_1^{\frac{1}{2}} Z_5^{\frac{1}{2}}(dr^2 + r^2 d\Omega_3) \tag{2.75}$$

We can now perform the Kaluza-Klein reduction on S^1. We define the 6-dimensional Vielbein $\hat{e}_M^{\hat{A}}$ (where $\hat{A}, \hat{B}... = 0, ..., 5$ are the 6 flat indices raised and lowered with the 6 dimensional Minkowski metric $\eta_{\hat{A}\hat{B}}$), so that $G_{MN} = \hat{e}_M^{\hat{A}} \hat{e}_N^{\hat{B}} \eta_{\hat{A}\hat{B}}$. The (t, y) block is the only one with off diagonal components. Using local Lorence invariance we can put the 6-dimensional Vielbein in triangular form. Restricting ourselves to the (t, y) block, we get

$$\left[\hat{e}_M^{\hat{A}}\right] = \begin{bmatrix} \hat{e}_t^0 & \hat{e}_t^1 \\ 0 & \hat{e}_y^1 \end{bmatrix} = \begin{bmatrix} Z_1^{-\frac{1}{4}} Z_5^{-\frac{1}{4}}(1+K)^{-\frac{1}{2}} & K Z_1^{-\frac{1}{4}} Z_5^{-\frac{1}{4}}(1+K)^{-\frac{1}{2}} \\ 0 & Z_1^{-\frac{1}{4}} Z_5^{-\frac{1}{4}}(1+K)^{\frac{1}{2}} \end{bmatrix} \tag{2.76}$$

Let's now introduce the 5-dimensional metric $g_{\mu\nu}$ with Vielbein e_μ^A ($A, B... = 0, ..., 4$ are the flat indices and $\mu, \nu = t, x^1, ..., x^4$ are the curved indices) and the metric $g_{\alpha\beta}$ on $\mathbb{S}^1$ with Vielbein E_α^a ($a, b = 1$ are the flat indices and $\alpha, \beta... = y$ are the curved indices). We can write $\hat{e}_M^{\hat{A}}$ in terms of the above as

$$\left[\hat{e}_M^{\hat{A}}\right] = \begin{bmatrix} e_t^0 & A_t^y E_y^1 \\ 0 & E_y^1 \end{bmatrix} \tag{2.77}$$

Using eq. (2.76) one gets:

$$E_y^1 = \hat{e}_y^1 = Z_1^{-\frac{1}{4}} Z_5^{-\frac{1}{4}}(1+K)^{\frac{1}{2}} \qquad e_t^0 = \hat{e}_t^0 = Z_1^{-\frac{1}{4}} Z_5^{-\frac{1}{4}}(1+K)^{-\frac{1}{2}} \tag{2.78}$$

The 5-dimensional metricm $g_{\mu\nu}$ is obtained from the Vielbein e_μ^A as $g_{\mu\nu} = e_\mu^A e_\nu^B \eta_{AB}$, with the rescaling

$$|\det(g_{yy})|^{\frac{1}{d-2}}\Big|_{d=5} = Z_1^{-\frac{1}{6}} Z_5^{-\frac{1}{6}}(1+K)^{\frac{1}{3}} \tag{2.79}$$

and the line element reads

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(Z_1 Z_5 Z_P)^{-\frac{2}{3}} dt^2 + (Z_1 Z_5 Z_P)^{\frac{1}{3}}(dr^2 + r^2 d\Omega_3) \tag{2.80}$$

It describes a 5-dimensional extremal, supersymmetric black hole with non zero horizon area. In this coordinate system, the event horizon is again located at $r = 0$, and its entropy is

$$A_5 = 2\pi^2\sqrt{Q_1 Q_5 Q_P} \Longrightarrow S = \frac{A_5}{4G_5} = \frac{\pi^2\sqrt{Q_1 Q_5 Q_P}}{2G_5} \tag{2.81}$$

Some considerations are in order. Using (2.32) one can check that this result agrees with eq. (2.73). This is no coincidence: the entropy is a physical quantity, and its value does not depend on whether we compute it in 10 dimensions or if we dimensional reduce the black hole to 5 dimensions and then make the calculation.

The fact that we have found a non vanishing area (and therefore a black hole with thermodynamical properties) depends on the number of charges, as well as on the number of non compact dimensions of spacetime. If we consider BPS black holes, in 5 non compact dimensions the minimum number of charges in order to have a non vanishing entropy is 3. If we had worked with 4 non compact dimensions, a forth charge would have been necessary: this is why, even though for phenomenological reasons we might prefer 4 dimensions, we have worked in 5 dimensions (the less the charges, the simpler the state).

Note, moreover, that we are interested in macroscopic black holes. Different (or same) kinds of branes can form bound states, and if they are not excited, the binding energy is zero. Roughly speaking, in an extremal configuration the gravitational attraction and electrical repulsion precisely cancel. Thus, we can consider black holes in supergravity that are constructed as bound states of many branes; the degeneracy N_i of each type of brane is proportional to the corresponding charge of the black hole and the macroscopic (or thermodynamic) limit occurs when one sends $N \to \infty$. Thus, there is a mapping between black hole solutions written in terms of macroscopic quantities (Q_i) and solutions written in terms of the microscopical quantities (such as the brane's degeneracy N_i). This can be seen observing that $M \sim \sum_i Q_i$ and $M \sim \sum_i N_i T_i$, where Q_i is the macroscopical charge of the brane, N_i is its degeneracy and T_i is its tension. Let's quote form [24] the precise relation between the charges of the D1D5P system and the integer charges:

$$Q_1 = \frac{g_s \alpha'^3}{V} n_1 \qquad Q_5 = g_s \alpha' n_5 \qquad Q_P = \frac{g_s^2 \alpha'^4}{V R^2} n_P \tag{2.82}$$

Using $G_{10} = 8\pi^4 g_s^2 \alpha'^4$ and eq.s (2.73), (2.82) we get the entropy of the $\frac{1}{8}$-BPS black hole in term of the integer charges:

$$S = 2\pi \sqrt{n_1 n_5 n_P} \tag{2.83}$$

From eq. (2.83) we find that the entropy does not depend on any of the continuous parameters like coupling constant or size of the internal circles. This is crucial to the possibility of reproducing this entropy by some microscopic calculation, and depends on the fact that we are considering supersymmetric black holes (as we will see in the next chapter). It is also symmetric under interchange of Q_1, Q_5 and Q_P: this is because a set of dualities can interchange the role of the charges (as emphasized in [24]).

In conclusion, we have derived supergravity solutions corresponding to some configurations of branes. As supergravity is the low energy limit of superstring theory, one might think that these solutions as superstring solutions (at least in the low energy limit). This prospective will be discussed in the next Chapter.

Chapter 3

Black Holes microstates and the Fuzzball proposal

In the previous Chapter we have found supergravity solutions and we have interpreted some of them as black holes with thermodynamical properties. It is important to note that we have derived such solutions using the symmetries of supergravity, which are different from those of superstring theory; therefore, it is not obvious that the corresponding singularity is allowed in string theory: it is possible that there is no microscopic source that can generate these solutions. We will analyze this issue when discussing the 2-charge system. The aim of this Section is to give a stringy interpretation of these solutions: this approach has been suggested by Susskind in [25], where he proposed that there is a correspondence between excited string states and black holes. This prospective will allow us to turn to a microscopical computation of the entropy. Bekenstein-Hawking entropy of eq. (1.41) is a "macroscopic" entropy: it arises from the thermodynamical behaviour of the black hole and can be computed by means of the macroscopic variables (like mass and charges) that enter in the equation of state. From the statistical mechanics point of view, the thermodynamic properties of a system emerge as a coarse graining of a more microscopic description, so that states with similar macroscopic behavior are lumped into a single thermodynamic state. With this prospective, Boltzmann has thought us that the entropy of a system is the logarithm of the number N of microscopic states that have been lumped into one thermal state

$$S_{micro} = \ln(N) \tag{3.1}$$

This is the "microscopic" entropy (in unity of $k_B = 1$). It is a task of string theory, as a candidate theory of quantum gravity, to reproduce the Bekenstein-Hawking entropy with a microscopic count of states so that $S_{micro} = S_{BH}$.

3.1 Why BPS?

The goal is to give a stringy description of black hole solutions, compute the microstates degeneracy, calculate the microscopical entropy through eq. (3.1) and compare it with the macroscopic one. However, we are able to count states only in the free theory, i.e when $g_s \to 0$ and the theory reduces to a manageable field theory. In this regime, however, gravity is switched off because $G_N \sim g_s^2$ and our system is not a black hole. On the other hand, S_{BH} is the entropy of a true black hole. Because we are comparing systems at different values of g_s, a priori, there is no reason for which the microscopic and macroscopic computation should match each other. And, indeed, in general we obtain different answers. To see this, let's recall that S_{micro} can be obtained out of the partition function

$$Z = e^{-\beta F} = \mathrm{Tr}_{\mathcal{H}}\, e^{-\beta H} \tag{3.2}$$

where $\mathcal{H}$ is the Hilbert space, via

$$S = -\frac{\partial F}{\partial T} \tag{3.3}$$

Because energy levels shift as we change g_s, one can see from eq. (3.2) and (3.3) that the partition function and, thus, the entropy change as we move in the moduli space. This makes it incorrect to directly compare degeneracies at different g_s.

In the theory there are quantities, called indices, that are protected by supersymmetry and do not change as we change the coupling constant from small to large values. The simplest example of a supersymmetric index is Witten index [26]

$$I_{Wit} = \mathrm{Tr}_{\mathcal{H}}[(-1)^F e^{-\beta H}] \tag{3.4}$$

where the operator $(-1)^F$ is defined as follows

$$(-1)^F|\text{Boson}\rangle = |\text{Boson}\rangle \qquad (-1)^F|\text{Fermion}\rangle = -|\text{Fermion}\rangle \tag{3.5}$$

We have seen in Section 2.2.3 that if we consider a theory with a single real supercharge Q and hamiltionian $H = Q^2$, non supersymmetric states are organized into doublets containing a bosonic and a fermionic state that have the same energy, while supersymmetric states need not to be paired and have zero energy. As we vary the coupling constant, the non supersymmetric states move around in the energy space; but they move in Bose-Fermi pairs. It is possible that some $E \neq 0$ pairs move down to $E = 0$ or zero energy states gain a non zero energy. What is not possible is that a supersymmetric state acquires a non-zero energy alone: as soon as it has a non-zero energy it must have a supersymmetric partner. This means that the difference

$$I_{Wit} = (\#\text{bosonic zero energy states}) - (\#\text{fermionic zero energy states}) \tag{3.6}$$

does not change as we changes g_s. It is possible to apply this idea to BPS states: one can construct a generalization of Witten index so that it receives a non vanishing contribution only from the short multiplets (BPS states):

$$I_{BPS} = (\#\text{bosonic BPS states}) - (\#\text{fermionic BPS states}) \tag{3.7}$$

This index, again, will be protected from corrections as we vary the moduli.

One may wonder our interest in indices since Boltzmann relations of (3.1) counts the absolute degeneracy of states, and

$$N_{BPS} = (\#\text{bosonic BPS states}) + (\#\text{fermionic BPS states}) > I_{BPS} \tag{3.8}$$

In other words I_{BPS} can only give us a lower bound on the number of BPS states: a priori index and absolute degeneracy are not the same. It can be shown, however, that in the interacting theory and large charge limit the two coincide at the leading order. This justifies our interest in BPS solutions: we can compute their microscopic degeneracy in the free theory, calculate the entropy through eq. (3.1) and compare it with the macroscopic entropy. We cannot do the same for non-BPS solution because their microscopic degeneracy is not protected by supersymmetry: this is why we have no analytic control of the microscopic counting of, say, the Schwarzshild black hole.

3.2 The microscopic count of states

In this Section we will give a stringy interpretation of the $\frac{1}{2}$-BPS, $\frac{1}{4}$-BPS and $\frac{1}{8}$-BPS solutions and count the number of microstates of the system in the limit in which the coupling constant g_s is very small.

Entropy for the 1-charge solution

Let's consider Type IIA superstring theory, where we label the compact direction S^1 (with radius R) with the coordinate y that has periodicity $0 \leq y < 2\pi R$. The 1-charge solution corresponds to a fundamental string F1 wrapped n_1 times around the circle. From eq. (2.63) we read the expression of the metric for this solution:

$$
\begin{aligned}
ds^2 =& Z_1(r)^{-1}(-dt^2 + dy^2) + dr^2 + r^2 d\Omega_3 + (dz^a)^2 \\
Z(r) =& 1 + \frac{Q_1}{r^2}
\end{aligned}
\tag{3.9}
$$

As $n_1 \propto Q_1$, with n_1 large this will give a BPS state with large mass. The singularity is located at $r = 0$; from the stringy point of view this mirrors the fact that, because every brane has a tension along its worldvolume directions, the string, wrapping the circle, causes it to shrink, collapsing at $r = 0$. Let's recall that the horizon area vanishes and, thus, $S_{BH} = 0$. This result agrees with the microscopic count. BPS solution means that the string is in an oscillator ground state: the degeneracy comes from the zero modes of the string, which give 128 bosonic and 128 fermionic states. Thus, $S_{micro} = \ln(256)$ does not grow with n_1. In the thermodynamical limit $n_1 \to \infty$, we write $S_{micro} = S_{BH} = 0$.

Entropy for the 2-charge solution

Let's now consider the 2-charge solution in the duality frame in which it arises as an F1-P bound state: a string wrapping the circle n_1 times and carring n_p units of momentum. The degeneracy is due to the fact that we can partition the momentum along different harmonics of vibration of the string, where each harmonic describes a Fourier mode. The total lenght of the F1 is $L_T = 2\pi R n_1$. The total momentum of the string can be written as

$$
P = \frac{n_p}{R} = \frac{2\pi n_1 n_p}{L_T} \tag{3.10}
$$

where each excitation of the Fourier mode contributes with momentum

$$
p_k = \frac{2\pi k}{L_T} \tag{3.11}
$$

Let's now focus just on one direction of vibration. The state will be described by the quanta (m_i, k_i), where m_i are the units of the Fourier mode k_i, satisfying

$$
\sum_i m_i k_i = n_1 n_p \tag{3.12}
$$

This equation shows that the degeneracy is given by counting the partitions of the integer $n_1 n_p$. The number of such partitions is given asymptotically by $\sim e^{2\pi\sqrt{\frac{n_1 n_p}{6}}}$.

So far we have just considered one direction of vibration: we must take into account that the momentum is partitioned among 8 bosonic and 8 fermionic vibrations, since we are in $D = 10$ and the longitudinal vibrations are forbidden. Because, statistically, a fermionic degree of freedom counts a half a bosonic one, there are a total of $8 + 4 = 12$ bosonic vibrations. Thus we are interested in the partitions of $\frac{n_1 n_p}{12}$ and, as all the directions are independent, we have to multiply the deceneracies of each mode. Thus we get

$$
S_{micro} = 2\sqrt{2}\pi\sqrt{n_1 n_p} \tag{3.13}
$$

Note that we have obtained a non vanishing microscopical entropy and this is not in agreement with the macroscopic entropy: the macroscopic computation, in this case, is $S_{BH} = 0$ (the area of the horizon vanishes and we have a naked singularity). It is believed that this inconsistence is due to the fact that the supergravity regime is not reliable at $r = 0$: the curvature blows up there and higher derivative corrections become important. It can be shown that, if one starts from a spacetime topology $\mathbb{R}^{1,4} \times S^1 \times K3$ (i.e. replaces the T^4 with a $K3$) and includes the higher derivative corrections in the action, the corresponding solutions develops an horizon, and the area of this horizon correctly reproduces the microscopic entropy. We will further discuss this issue when dealing with the construction of microstates in Section 3.3.

Entropy for the 3-charge solution

Let's now consider the 3-charge solutions in the duality frame of $NS5_{yT^4}F1_yP_y$. The microscopical entropy for this system[1] was first obtained by Strominger and Vafa in [27]. Let's first consider the system with $NS5 - F1$: the two branes are bound if the string lies along the NS5, so that it can vibrate only in the direction parallel to the NS5 and transverse to the F1. This frame can be reached from the $F1 - P$ using the chain of dualities in eq. (2.67) and the charges are mapped as

$$F1(n_1)P(n_p) \to NS5(n_1)F1(n_p) \tag{3.14}$$

As the two systems are mapped into each other by dualities, their microscopical entropy must coincide and it is given by eq. (3.13). We have seen in the previous Section that, in the $F1 - P$ frame, the total lenght of the string wrapping the circle n_1 times is $2\pi R n_1$ and, thus, the momentum comes in units of $\Delta p = \frac{1}{n_1 R}$. The total momentum is $\frac{n_p}{R} = n_1 n_p \Delta p$: this means that we have $n_1 n_p$ units of momentum that can be partitioned in different ways, giving rise to the entropy of (3.13). Let's now give an explanation of the degeneracy for the $NS5(n_1) - F1(n_p)$ bound state. If we have just one NS5 ($n_1 = 1$) we expect the degeneracy to be the number of different state that can occur partitioning the number n_p. Pictorially, we can say that the F1 lives in the NS5, but can be joined up to make multiwond strings in different ways: we can have m_i multiwound strings, each one with winding k_i, with the constrain

$$\sum_i m_i k_i = n_p \tag{3.15}$$

If $n_1 > 1$, we have $n_1 n_p$ strands in all and they can be joined in various ways, so that the total number of states one can construct is

$$\sum_i m_i k_i = n_1 n_p \tag{3.16}$$

Let's now return to the 3-charge system. We have seen that a bound state of n_1 F1 branes and n_5 NS5 branes gives rise to states composed of m_i multiwound strings with winding k_i, according to eq. (3.16). If we now add the momentum charge P , we can take the F1-NS5 bound state in any of the configuration in eq. (3.16) and distribute the n_p units of momentum among the strings. The leading contribution comes from the configurations $m = 1$, $k = n_1 n_5$, i.e. when we consider the maximally wounded effective string. The degeneracy can be computed as in the previous case, with the difference that now the momentum comes in units of $\Delta p \sim \frac{1}{n_1 n_5}$ and the string can oscillate only in 4 transverse directions. Therefore, the total number of states goes as $\sim e^{2\pi\sqrt{n_1 n_5 n_p}}$, and we get that the microscopical entropy is

$$S_{micro} = 2\pi\sqrt{n_1 n_5 n_p} \tag{3.17}$$

Remarkably, the microscopic entropy eq. (3.17) agrees with the macroscopic entropy (2.83).

[1] The system, actually, was slightly different: they took the compactification $\mathbb{R}^{1,4} \times S^1 \times K^3$. The case of K3 repaced by T^4 (that is our compactification choice) was done soon after in [28].

3.3 Microstates construction

String theory allows for a microscopical count of states which agrees with the macroscopic entropy. However the counting we have described leaves us with the question of how the microstates manifest themselves at the gravitational regime, i.e. when the coupling is strong and the system forms a black hole. Before dealing with the structure of the microstates, let's pause on the problem we have encountered in Section 3.2: for the 2 charge system the microscopic computation gives a non zero microscopic entropy, while the supergravity solution (2.66) has a vanishing entropy. This is strange, as the entropy (3.13) arose in a similar way to (3.17).
Note, however, that the metric (2.66) is a solution of supergravity (the low energy approximation of string theory) and is not reliable near $r = 0$ because the curvature blows up there: the fact that such a solution exist in supergravity does not mean that it is allowed in string theory. The F1-P system is generated by a fundamental string wrapping S^1 and the momentum P is carried as traveling waves along the string. Because there are no longitudinal vibrations on the fundamental string, the momentum must be carried by transverse vibrations. This makes the string bend away from its central axis, and will not be confined at $r = 0$. We thus conclude that the metric (2.66) cannot be generated in string theory. We will refer to this metric as the "naive" metric for the F1-P system.
We may ask what solution is produced by a fundamental string carrying momentum. The F1 has many strands, as it is multiwound; when a strand carries a wave described by the transverse displacement profile $\vec{F}(t-y)$, the metric produced is [24] ($u = t + y$ $v = t - y$)

$$
\begin{aligned}
&ds^2 = H(-dudv + Kdv^2 + 2A_i dx^i dv) + dx^i dx^i + dz^a dz^a \\
&H^{-1}(\vec{x}, y, t) = 1 + \frac{Q}{|\vec{x} - \vec{F}(t-y)|^2} \qquad K(\vec{x}, y, t) = \frac{Q|\dot{\vec{F}}(t-y)|^2}{|\vec{x} - \vec{F}(t-y)|^2} \\
&A_i(\vec{x}, y, t) = -\frac{Q_1 \dot{F}_i(t-y)}{|\vec{x} - \vec{F}(t-y)|^2} \qquad e^{2\Phi} = H
\end{aligned}
\tag{3.18}
$$

The metric is singular along the curve $x_i = F_i(t-y)$. We are interested in multiwound strings ($n_1 > 1$) where each strand carries a vibration profile[2] $\vec{F}(t-y)$. In this way the strands are mutually BPS, and the solution for $n_1 > 1$ can be obtained superposing the harmonic functions H^{-1}, K and A_i. Moreover we are interested in the thermodynamic limit $n_1, n_p \to \infty$ and, when this limit is performed, we can approximate the sums with integrals. The structure of the metric is still the one in (3.18), while the functions are

$$
\begin{aligned}
&H^{-1} = 1 + \frac{Q_1}{L}\int_0^L \frac{dv}{|\vec{x} - \vec{F}(t-y)|^2} \qquad K = \frac{Q_1}{L}\int_0^L \frac{dv|\dot{\vec{F}}(t-y)|^2}{|\vec{x} - \vec{F}(t-y)|^2} \\
&A_i = -\frac{Q_1}{L}\int_0^L \frac{dv \dot{F}_i(t-y)}{|\vec{x} - \vec{F}(t-y)|^2}
\end{aligned}
\tag{3.19}
$$

where $L = 2\pi R n_1$ is the total range of the y coordinate on the multiwound string.
For the following, it is useful to express this solution in D1-D5 duality frame. This can be done applying the chain of dualities given in eq. (2.67) to the solution in eq.s (3.19) and (3.18). The D1 and D5 charges are denoted, respectively, by Q_1 and Q_5, their values in terms of the integer charges are given in eq. (2.82). The metric in the D1D5 frame is [23]

$$
\begin{aligned}
&ds^2 = (Z_1 Z_2)^{-\frac{1}{2}}(-(dt - A_i dx^i)^2 + (dy + B_i dx^i)^2) + (Z_1 Z_2)^{\frac{1}{2}} dx^i dx^i + (Z_1)^{\frac{1}{2}}(Z_2)^{-\frac{1}{2}} dz^a dz^a \\
&Z_1 = 1 + \frac{Q_5}{L}\int_0^L \frac{dv|\dot{\vec{F}}(t-y)|^2}{|\vec{x} - \vec{F}(t-y)|^2} \qquad Z_2 = 1 + \frac{Q_5}{L}\int_0^L \frac{dv}{|\vec{x} - \vec{F}(t-y)|^2} \\
&A_i = -\frac{Q_5}{L}\int_0^L \frac{dv \dot{F}_i(t-y)}{|\vec{x} - \vec{F}(t-y)|^2} \qquad dB = -\star_4 dA
\end{aligned}
\tag{3.20}
$$

[2] All the strands must carry momentum in the same direction of y for the state to be supersymmetric: the profiles can depend on $t + y$ or $t - y$, but there must be no mixing.

where $\star_4$ denotes the Hodge dual operator in the non-compact space, $L = \frac{2\pi Q_5}{R}$ and the total D1 charge is given by $Q_1 = \frac{Q_5}{L}\int_0^L |\dot{F}|^2 dv$.

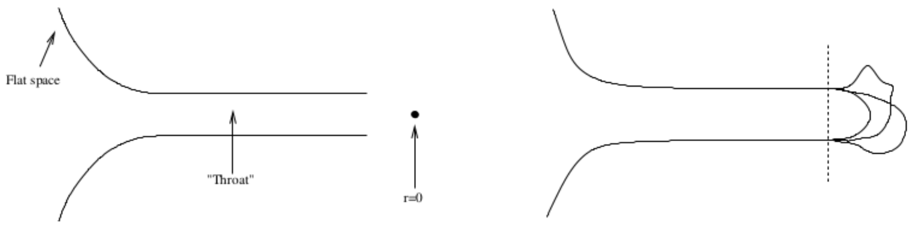

Figure 3.1: On the left, there is the naive geometry of extremal D1D5 black hole. On the right, the geometry of the microstates is represented.

Let's discuss the geometrical structure of this solution, and compare it with the naive geometry (2.68); a pictorial representation is given in Figure 3.1.

- For $r^2 >> Q_{1,5}$ we recover the asymptotically flat regime: the constant term in the Z_1 and Z_2 functions cause the geometries of both the naive black hole and the microstates to be Minkowski spacetime at infinity.
- For $r^2 \sim Q_{1,5} >> |\vec{F}|^2$, where $|\vec{F}|$ is the characteristic length of the profile, we have that $Z_1 \sim 1 + \frac{Q_1}{r^2}$, $Z_2 \sim 1 + \frac{Q_5}{r^2}$ and the contribution of the functions A and B can be neglected: the microstate and the naive geometry are still indistinguishable.
- When $|\vec{F}|^2 << r << Q_{1,5}$ we are in the decoupling (or near horizon) limit. Here the functions Z_1, Z_2 etc. receive contributions in the form of higher negative powers of r: the shape of $\vec{F}$ gives the deviation of the microstate geometry from that of the black hole. The important fact is that the constant term in Z_1 and Z_2 can be neglected: because they cause the geometry to be asymptotically flat, in the decoupling limit asymptotic flatness is lost. In fact, if we take the $r \to \infty$ limit we get

$$ds^2 = \frac{r^2}{\sqrt{Q_1 Q_5}}(-dt^2 + dy^2) + \frac{\sqrt{Q_1 Q_5}}{r^2}dr^2 + \sqrt{Q_1 Q_5}d\Omega_3 + \frac{\sqrt{Q_1}}{\sqrt{Q_5}}dz^a dz^a \qquad (3.21)$$

Introducing the new coordinate $\tilde{r} = \frac{r}{\sqrt{Q_1 Q_5}}$, the metric becomes

$$ds^2 = \sqrt{Q_1 Q_5}\Big[\frac{d\tilde{r}^2}{\tilde{r}^2} + \tilde{r}^2(-dt^2 + dy^2)\Big] + \sqrt{Q_1 Q_5}d\Omega_3 + \frac{\sqrt{Q_1}}{\sqrt{Q_5}}dz^a dz^a \qquad (3.22)$$

This is $AdS_3 \times S^3 \times T^4$, with the AdS term written in Poincar coordinates and with radii $R_{AdS} = R_{S^3} = \sqrt{Q_1 Q_5}$. Note that the asymptotic limit is the same for both the naive geometry and the microstate, as terms with $O(r^{-3})$ in the microstate expansion are subleading as $r \to \infty$. As we will see later, the AdS_3 factor is crucial for the holographic construction to hold.
- As r decreases we do not encounter any singularity: this is remarkable as, at first sight it seems that the metric is singular on the profiles (the functions in (3.20) blow up at $\vec{x} = \vec{F}(v)$). However, it turns out [24] that this is just a coordinate singularity and the geometries are completely smooth. This is a peculiarity of the D1D5 frame. To summarize, the metric is horizonless, it is regular on the profile F and as $r \to 0$ the geometry remains smooth, and ends with a cap.

3.4 The Fuzzball proposal

In the previous Section we have constructed the microstates for the D1D5 black hole: they are horizonless, non singular solutions of string theory that look like the naive black hole in the asymptotic region, while deviate from it already at the horizon scale. The fuzzball proposal [24] [29] states that, for a given black hole with entropy S, there are e^S microstates that behave like those we have constructed. In this picture, the original black hole emerges as an averaged description of the system. This proposal would not only account for the entropy of the black hole, but could also solve the information paradox. The reason is that, while Hawking's calculation assumes that the microstates are not distinguishable near the horizon, according to this conjecture two microstates (or a microstate and the naive geometry) differ already at the horizon scale. This means that the creation of particle pairs near the horizon is sensible to the precise form of the microstate: the modification of the black hole interior allows the emitted quanta to carry information about the microscopic configuration.

Another important feature of these microstates is that they are horizonless. For a long time people believed that if one takes a free string and increases g_s an horizon forms around the string (as its size becomes smaller than its Schwarzschild radius) and it ends up being a black hole [25]. It is generally believed that if we find any horizon in GR we should associated an entropy to it. Does it make sense to associate an entropy to a microstate? A microstate is any one of the possible states in an ensemble that share the same macroscopic parameters of the black hole: if we consider the macroscopic system the entropy is a measure of the hidden information (i.e. the microscopic configuration of the system), if we consider a microstate we have a full description of the state and we should not associate an entropy to it. In our picture, indeed a microstate has no horizon and for this to happen the geometry must change all through the interior region of the horizon. It is commonly accepted that quantum gravity effects should become important at some size comparable with some microscopical fixed length scale (e.g. the planck scale l_p or the string length l_s). With this proposal, the classical solution should be modified long before: corrections occur already at the horizon scale which, in general, is a macroscopic scale.

Note, however, that the fuzzball picture is a conjecture and it is not entirely accepted. So far we have a good understanding of all the microstates responsible for the entropy of the 2-charge extremal black hole and all off them have the properties required by the fuzzball proposal. However we do not have the complete family of microstates for the 3-charge case and, so, it has not been proved that all the microstates are horizonless smooth solution. Moreover, general microstates might not be well described in supergravity: this is a property of the 2-charge solutions, but to describe general 3-charge microstates string theory might be needed. Another point that is not well understood is what happens when we give up extremality. Thus, to establish the validity of the fuzzball conjecture it would be important to provide a full gravitational description of the microstates. In the next Chapter, we will introduce the AdS/CFT correspondence, which provides powerful tools to extend our understanding on the geometry of microstates.

Chapter 4

The AdS/CFT Conjecture and the D1D5 CFT

The AdS/CFT correspondence states that string theory in a $d+1$-dimensional anti-de Sitter (AdS) spacetime is dual to a d-dimensional conformal field theory. This is one of the most important recent ideas that arose in theoretical physics. The power of the correspondence lies in the fact that when one theory is strongly coupled (and, thus, it is difficult to treat) the dual one is weakly coupled, and vice versa. In this Chapter we will introduce the conjecture and we will study the dual theory of the D1D5 geometries: the D1D5 CFT.

4.1 Motivating the conjecture

Although the AdS/CFT correspondence was originally discovered [30] [31] by studying D-branes and black holes in string theory, the fact that such an equivalence may exist can be heuristically motivated from certain aspects of gauge theories and gravity.

We have seen that the entropy of a black hole scales with the area of the event horizon. Eq. (1.41) is not only the entropy of a black hole but it is also a bound on the maximal entropy that can be stored in a region of space (that contains gravity).
The argument works as follows: consider a region of space Γ that contains a thermodynamic system with more entropy than $\frac{A}{4G_N}$, where A is the area of $\partial\Gamma$. The total mass of the system cannot exceed the mass of a black hole of area A, otherwise it would be bigger than the region Γ. Now imagine we throw in some extra matter so that we form a black hole which just fills the region, i.e. the area of the event horizon is A. The entropy in the exterior of Γ has obviously decreased, as well as the entropy inside the region, because now its entropy is just $\frac{A}{4G_N}$ which is smaller than the initial entropy by our assumption. So, the second law has been violated. Therefore, unless the second law is untrue, the entropy of any system that includes gravity is limited by

$$S_{\max} = \frac{A}{4G_N} \tag{4.1}$$

This bound implies that the number of degrees of freedom inside some region that contains gravity grows as the area of the boundary and not like the volume of the region. In standard quantum field theories this is certainly not possible. Attempting to understand this behavior led Susskind to the "holographic principle" [32], which states that in a quantum gravity theory all physics within some volume can be described in terms of some non gravitational theory on the boundary of the volume.

A second indication that a gauge theory could be dual to a string theory[1] comes from the 't Hooft large N limit. A pure 4-dimensional $SU(N)$ gauge theory, beside the dimensionless coupling g_{YM}

[1] The relation between gauge theories and string theories may not surprise: after all, string theory was first invented to describe strong interactions. Different vibration modes of a string provided an economical way to explain many

(which is dimensionally transmuted in the scale Λ_{QCD}), has another dimensionless parameter: the number of colors N. 't Hooft's idea was to treat the number of colors N of a non-Abelian gauge theory as a parameter, take it to be large, and expand physical quantities in $\frac{1}{N}$. With this aim, we have to impose the scaling of g_{YM} as we take $N \to \infty$: a natural choice is to impose a scaling such that Λ_{QCD} remains constant. The β-function for pure $SU(N)$ Yang-Mills theory is [33]

$$\mu \frac{dg_{YM}}{d\mu} = \frac{11}{2} N \frac{g_{YM}^3}{(4\pi)^2} + O(g_{YM}^4) \tag{4.2}$$

and we find that the leading terms are of the same order if we take the combination $N \to \infty$ while keeping 't Hooft coupling $\lambda = g_{YM}^2 N$ fixed. Now, consider the partition function:

$$Z = \int DA_\mu \exp\Big(- \frac{1}{4g_{YM}^2} \int dx^4 \operatorname{Tr} F^2 \Big) \tag{4.3}$$

It turns out that the generating functional can be written as an expansion in $\frac{1}{N}$ as [33]

$$\log Z = \sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda) \tag{4.4}$$

where $f_g(\lambda)$ are functions of the 't Hooft coupling only and they include the contributions of all the diagrams that can be drawn on a 2-dimensional surface with g genus without crossing any line. Because the topology of a two-dimensional compact surface is classified by its number of holes, the expansion (4.4) can be considered as an expansion in term of the topology of Feynman diagrams. In the large N limit we see that any computation will be dominated by the diagrams with minimal genus: these "planar diagrams" give a contribution of order N^2, while all other diagrams will be suppressed by powers of $\frac{1}{N^2}$.
This is in remarkable parallel with the perturbative expansion of a closed string theory. Comparing eq.s (2.18) and (4.4) one could identify the string coupling constant as something proportional to $\frac{1}{N}$. This analogy suggests that, indeed, gauge and string theories may be related in such a way that, in the large N limit, the string theory is weakly coupled.

One of the strongest motivations for believing the AdS/CFT correspondence is to consider it as a realization of the open/closed string duality. We have seen that superstring theories contain multidimensional objects: D-branes. D-branes are non perturbative objects: since they have mass proportional to the inverse of the string coupling $\frac{1}{g_s}$ they do not arise in the perturbative expansion (one cannot scatter two strings and get a D-brane, at least for small g_s). However, if a D-brane is present in spacetime one can do a perturbative expansion around this background.
We have seen that D-branes play a double role in the theory. In the closed string description they are charged objects with respect to the Ramond-Ramond fields and, since they are massive objects, they curve the spacetime geometry in which closed string propagate. In the open string description D-branes are multidimensional objects where an opens string can end.

Let's first consider D-branes from the open string prospective. As we have seen in Section 2.1.1, the quantization of the theory gives an open string spectrum that can be identified with fluctuations of the brane. For a single D-brane, the massless spectrum consists in scalar field ϕ_i that describe fluctuations of the brane in the transverse direction and a U(1) gauge field A_μ that lives on the brane. The remarkable feature is the possible appearance of a non-Abelian gauge theory when one considers multiple parallel D-branes. If one has N D-branes, open strings that have both endpoints on the same brane form a $U(1)$ gauge field as before, so that we have an overall gauge group $U(1)^N$; we will

resonances discovered in the sixties which obey the so-called Regge behavior (i.e. the relation $M \propto J$ between the mass and the angular momentum of a particle). Later it was discovered that there is another description of strong interactions: QCD (an $SU(3)$ gauge theory).

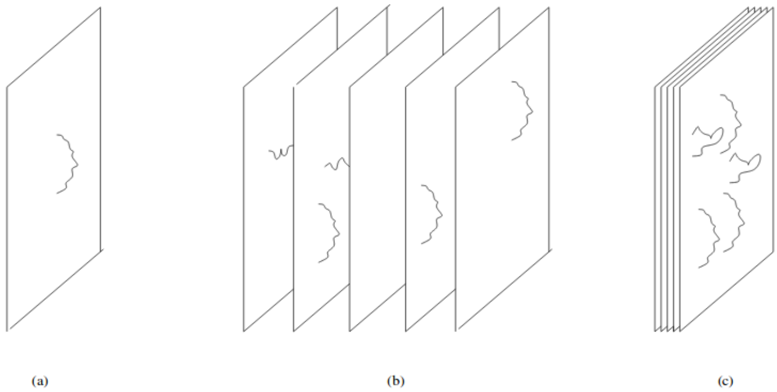

Figure 4.1: Open strings stretching between (a) single, (b) separated, (c) coincident D-branes.

denote the gauge fields with $(A_\mu)^a{}_a$, where the upper (lower) index labels the brane on which the string starts (ends). We can also have strings that have endpoints on different branes (as in Figure 4.1): if they are separated by a distance r, the corresponding field $(A_\mu)^a{}_b$ (with $a \neq b$) have a mass given by the tension of the string times the separation of the branes $m = \frac{r}{2\pi\alpha'}$. If the branes are on top of each other ($r = 0$) all the $(A_\mu)^a{}_b$ are massless and the resulting theory is a non-Abelian gauge theory with gauge group $U(N)$. Similarly one finds that the massless scalars become $N \times N$ matrices $(\phi_i)^a{}_b$, which transform in the adjoint representations of the gauge group.

To be more specific, let's treat the most studied case: consider N D3-branes in Type IIB superstring. In the low energy limit, the effective action of the massless modes will be

$$S = S_{\text{bulk}} + S_{\text{branes}} + S_{\text{bulk-branes}} \tag{4.5}$$

where S_{bulk} is the ten-dimensional supergravity action (plus some higher derivative corrections), S_{branes} is the brane action and $S_{\text{bulk-branes}}$ describes the interactions between the brane modes and the bulk modes. Let's consider each term separately.
S_{branes} describes the theory on the brane: in the low energy limit it contains the gauge fields $(A_\mu)^a{}_b$ ($\mu = 0, ..., 3$) and the scalar fields $(\phi_i)^a{}_b$ ($i = 1, ..., 6$) which all live in the adjoint representation of $U(N)$. As a D-brane breaks one half of the 32 supersymmetries of the $D = 10$ $N = 2$ superstring theory, S_{branes} describes an $N = 4$ super-Yang-Mills theory with gauge group $U(N)$ in $(3 + 1)$-dimensions. The bosonic lagrangian can be written as [34]

$$L = -\frac{1}{g_{YM}^2}\,\text{Tr}(\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}D_\mu\phi^i D^\mu\phi^i + \sum_{i,j}[\phi^i, \phi^j]) \tag{4.6}$$

with the Yang-Mills coupling is given by

$$g_{YM}^2 = 4\pi g_s \tag{4.7}$$

The presence of supersymmetry makes the theory conformally invariant. At the classical level, Yang-Mills theories in 4-dimensions are scale invariant: the coupling constant is dimensionless so that there are no mass scales in the theory. This invariance does not extend at the quantum level: the renormalization group introduces a scale dependence and g_{YM} runs with the energy scale. Supersymmetry makes the β-function to vanish exactly, restoring conformal invariance.
The $U(N)$ gauge group is equivalent to $U(1) \times SU(N)$ (up to some Z_N identification that affects only global issues and not the Lie algebra). The diagonal $U(1)$ degree of freedom describes the motion of the branes' center of mass (i.e. rigid motion of the entire system of branes); we are not interested in this trivial type of motion and we will focus only on the $SU(N)$ gauge group left after removing the $U(1)$.

The full action (4.5) contains also closed string modes which can interact among themselves (S_{bulk}) and with open strings ($S_{\text{bulk-brane}}$): since gravity couples universally to all form of matter, the interaction will always be proportional to some power of $\sqrt{G_N} \sim g_s \alpha'^2$. Let's take the low energy limit sending $l_s \to 0$ while keeping the energy and all the dimensionless parameters (e.g. g_s and N) fixed. In this limit the coupling $\sqrt{G_N} \to 0$ and all the interactions vanish (gravity is free at large distances). Thus, we have two decoupled systems: free gravity in the bulk and a 4-dimensional gauge theory on the brane.

Let's now change prospective (closed string description), considering the fact that the D3-branes are massive and charged objects which act as sources for the various supergravity fields. Denoting with x_i the three spatial directions in which the branes are extended, the D3-brane supergravity metric takes the form [33]

$$\begin{aligned} d_s^2 &= H^{-\frac{1}{2}}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + H^{\frac{1}{2}}(dr^2 + r^2 d\Omega_5^2) \\ H &= 1 + \frac{R^4}{r^4} \qquad R^4 = 4\pi g_s \alpha'^2 N \end{aligned} \tag{4.8}$$

For $r >> R$ we recover Minkowsky spacetime. In the near horizon limit, $r << R$, we have $H \sim \frac{R^4}{r^4}$ and the geometry becomes $AdS_5 \times S^5$, with AdS radius equal to R.
Because g_{tt} is not constant, the energy E_p of an object as measured by an observer at a constant position r and the energy E measured by an observer at infinity are related by $E = (H)^{-\frac{1}{4}} E_p$. This means that the closer the object to $r = 0$ the lower the energy measured at infinity.
Let's now consider the low energy limit, there are to kinds of low energy excitations from the point of view of an observer at infinity: we can have massless particles in the bulk region with large wavelengths or we can have a generic excitation very close to $r = 0$. In the low energy limit these two types of excitations decouple. The bulk particle decouples from the near horizon region because, roughly speaking, if its wavelength is large compared to the typical gravitational size of the brane ($\sim R$) it will not be absorbed. Similarly, the excitations that live near $r = 0$ find it hard to climb the gravitational potential and escape to the asymptotic region. Note, moreover, that the strength of the interactions between closed string is proportional to the Newton constant G_N, which has dimension $[G_N] = \text{lenght}^{d-2}$, so the dimensionless coupling constant at energy E is $G_N E^8$: in the low energy limit the interactions vanish. Thus, the low energy modes in the Minkowski region decouple from each other; this does not happen in the throat, because massive string excitations survive there. Therefore, in the closed string description the interacting sector at low energies reduces to closed string in $AdS_5 \times S^5$.

We have seen two different descriptions for treating the D3-brane: the open string and the closed string description. However, superstring theories are invariant under worldsheet reparametrizations (2.5): this allows to exchange the proper time τ and the proper length σ on the worldsheet. With this

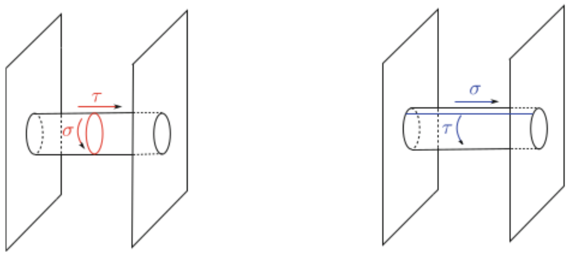

Figure 4.2: Exchanging τ and σ we can interpret the process as an exchange of a closed string between D-branes (left), or as an open string loop diagram (right).

prospective, tree level closed string processes are equivalent to open string loop diagrams, as shown in Figure 4.2. This is the so-called open/close string duality. How does this duality manifest itself in the effective theory? Are S_{bulk} and S_{brane} dual to each other?

4.2 AdS/CFT correspondence

In the previous Section, we have discussed the system composed of N D3-branes. We have seen that both from the point of view of the field theory of open strings living on the brane, and from the point of view of the supergravity description, we have two decoupled systems in the low-energy limit. In both cases one of them is supergravity in flat space. In [30] Maldacena has identified the other systems in the two descriptions. This led to the conjecture that

$$\{N = 4\ SU(N) \text{ super-YM in 3+1-dimensions}\} = \{\text{Type IIB superstring in } AdS_5 \times S^5\} \tag{4.9}$$

The relation between the parameters of the two sides of (4.9) can be obtained from eq.s (4.7) and (4.8)

$$\left(\frac{R}{l_s}\right)^4 = \lambda = N g_{YM}^2 \qquad \left(\frac{R}{l_p}\right)^4 = \frac{\sqrt{2}}{\pi^2} N \tag{4.10}$$

The above identifications are part of what is called the dictionary of the correspondence. From the relations in eq. (4.10) it follows that the two descriptions are tractable in different parameters regimes. When the t'Hooft coupling $g_{YM}^2 N \sim g_s N << 1$ we have $R << l_s$: this regime consists in the weakly coupled Yang-Mills theory, where the perturbative expansion is reliable; however, because the radius charactering the gravitational effect becomes small in string units, the closed string description becomes intractable since one has to deal with highly stringy behaviors.
On the other hand, when $g_{YM}^2 N >> 1$ we have $R >> l_s$, the geometry becomes weakly curved and the supergravity description is reliable; instead, the Yang-Mills theory is strongly coupled and one cannot control the loop expansion. Thus, we see that the supergravity regime and the gauge theory are perfectly incompatible: this is the reason why this correspondence is called a "duality". The two theories are conjectured to be exactly the same, but when one is weakly coupled the other is strongly coupled and vice versa. This makes the correspondence useful (we can treat a strongly coupled gauge theory via classical supergravity or a stringy system using the tools provided by weakly coupled Yang-Mills theories) but, at the same time, hard to prove.
In this sense, it remains a conjecture even though supported by a large number of evidences [35]. So far, we have implicitly assumed $N \to \infty$: this is the case in which we are most interested, because it corresponds to the thermodynamical limit for macroscopic black holes[2].

4.3 D1D5 CFT

In this Section we will apply the above correspondence to the D1D5 system: the basic setup is a spacetime with topology $\mathbb{R}^{1,4} \times T^4 \times S^1$ with N_5 D5-branes wrapping the hole compact space and N_1 D1-branes wrapping S^1. We want to consider the region of parameters in which the radius of S^1 is much bigger than $(V_4)^{\frac{1}{4}} \sim l_s$.
In the low energy limit $E << \frac{1}{l_s}$, because the T^4 has a size of order the string length we conclude, using eq. (2.35), that the masses of the winding and the momentum modes on the T^4 can be neglected. On the other hand, because the radius of the circle is much larger than the string scale, we have to retain all the momentum modes in this direction.
We have seen in Section 3.3 that the supergravity description, in the near horizon limit, becomes $AdS_3 \times S^3 \times T^4$. We want to find the description that arise from the low energy behavior of open string modes. We expect the dual theory to be a two-dimensional conformal field theory with 8 supercharges (as the D1D5 breaks $\frac{1}{4}$ supersymmetries).

Such a description can be obtained with two different approaches, which we now summarize (for a more exhaustive treatment see, for example, [11] and [36]). The first one consists in explicitly consider

[2]There are, however, different forms of the conjecture: the strong form is to assume that the two theories are exactly the same for all values of g_s and N.

the gauge theory that arise from open strings: because they can begin and end on either a D1 or a D5, there are three sectors we have to consider:

- The 5-5 strings have both endpoints on D5 and give rise to a $U(N_5)$ gauge theory in $5+1$-dimensions with 16 supercharges.
- The 1-1 strings have both endpoints on D1 and give rise to a $U(N_1)$ gauge theory in $1+1$-dimensions, again with 16 supercharges.
- The 5-1 (and 1-5) strings are fundamental under $U(N_5)$ ($U(N_1)$) and antifundamental under $U(N_1)$ ($U(N_5)$) and break the supersymmetries of the theory down to 8.

Because we have taken the size of the T^4 to be on the string scale, we can dimensional reduce the theory down to $1+1$ dimensions, parametrized by the time and the S^1 coordinate y.
Because we are discussing the low energy limit, we are interested in the supersymmetric minima of the theory. It turns out that there are two classes of minima, that select two different regions of the moduli space of the theory.

- In the first one, the so-called Coulomb branch, the state of the $(1,1)$ and $(5,5)$ string along the transverse directions acquire a non vanishing expectation value; this makes the brane to separate from each other, breaking the gauge group down.
- In the second one, the so-called Higgs branch, the states of the $(1,5)$ and $(5,1)$ strings acquire a non zero Vev, and all the branes remain on top of each other. We are interested in this branch, which truly describes a bound state. Studying the theory in this way, however, is complicated.

Nonetheless, there is an alternative but equivalent approach to the problem. It consists in considering the D1 branes as instantonic solutions of the 6-dimensional $U(N_5)$ gauge theory on the D5 branes: these are strings wrapping S^1 but localized in T^4. From the above discussion, we conclude that we are interested in N_1 instantons in the D5 theory: these form a family of solution whose corresponding parameters form the instanton moduli space. The effective description of the Higgs branch, thus, is a $1+1$-dimensional sigma model with target space the moduli space of N_1 $U(N_5)$ instantons on T^4. In general, this space is complicated; however, for particular values of the closed string moduli, it reduces to

$$\frac{(T^4)^{N_1 N_5}}{S_{N_1 N_5}} \tag{4.11}$$

where S_n is the symmetric group of degree n. This is the so-called orbifold point.
To summarize, the proposed correspondence is between Type IIB superstring theory defined on an asymptotically $AdS_3 \times S^3$ space and the D1D5 conformal fields theory, a $1+1$-dimensional $N=(4,4)$ sigma-model with target space given by (4.11) at the orbifold point. Throughout all the discussion, we will work at the orbifold point (the free point), in which all the couplings vanish and the CFT reduces to a collection of free bosons and fermions. At the orbifold point the target space is the symmetric product of N_1N_5 copies of the manifold T^4: thus, we can visualize the CFT as a collection of N_1N_5 strings wrapping the circle with target space T^4; the S_n identification is required as there is no physical distinction between permutations of the strings.
However, different configurations can occur. We can have that all the strings wrap the circle once, i.e. if we cirle S^1 once we end up on the same string: this is the untwisted sector of the theory. We can join up k different strings to form an effective string wrapping the circle k times, we will call such objects strands: this is the twisted sector of the theory (see Figure 4.3). Thus, we can have a generic configuration formed by m_i strands with winding k_i; with the constrain $\sum_i k_i m_i = N_1 N_5$.

Let's now discuss the agreement of the symmetries of the two theories. On the gravity side we have $AdS_3 \times S^3 \times T^4$ that enjoys an $SO(2,2) \simeq SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ isometry group of AdS_3, an $SO(4)_E$ isometry group of S^3 and an $SO(4)_I$ of the torus that is broken by compactification. When we consider the matching with the CFT side we become quite confused: the conformal group in 2-dimensions is infinite-dimensional, with Virasoro generators L_n, $\bar{L}_n$ ($n \in \mathbb{Z}$), while we would like to identify the

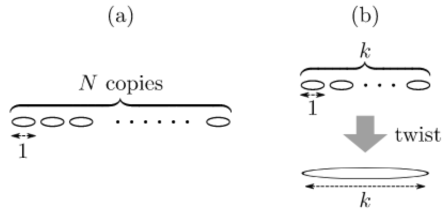

Figure 4.3: (a) We have N copies of singly wound strands $k = 1$. (b) As we will see, a twist operator can glue k singly wound strands into a single multy wound strand with length k.

conformal group with the isometry group of AdS_3. However, only the subalgebra spanned by $n = 0, \pm 1$ is well-defined globally [33]: this subalgebra generates an $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ group of transformations which we identify with the isometries of AdS_3. The CFT also has an $SO(4)$ R-symmetry which we identify with $SO(4)_E$ and another $SO(4)$ symmetry which we identify with $SO(4)_I$. We write the non simple group $SO(4)$ as the product of its simple subgroups as: $SO(4)_E \simeq SU(2)_L \times SU(2)_R$ and $SO(4)_I \simeq SU(2)_1 \times SU(2)_2$.

4.3.1 Field content

In this Section we will state the main ingredients of the CFT relevant to the D1D5 microstates and we will follow the conventions of [38]. Appendix A contains a brief review on conformal field theories in 2 dimensions. The D1D5 CFT can be visualized, at the free orbifold point, as a collection of N_1N_5 strands, each one with 4 bosons and 4 doublets of fermions

$$\left(X^{A\dot{A}}_{(r)}(\tau,\sigma), \psi^{\alpha\dot{A}}_{(r)}(\tau+\sigma), \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)}(\tau-\sigma)\right) \tag{4.12}$$

where $r = 1, ..., N_1N_5$ labels the different copies, (τ,σ) are the timelike and spacelike directions on the cylinder base space (which corresponde to the (t,y) coordinates on the bulk side), and the spinorial indices $A, \dot{A}, \alpha, \dot{\alpha} = 1, 2$[3]live in the following internal spaces:

$$\begin{array}{llll} \alpha,\beta & \text{fundamental of } SU(2)_L & \dot{\alpha},\dot{\beta} & \text{fundamental of } SU(2)_R \\ A,B & \text{fundamental of } SU(2)_1 & \dot{A},\dot{B} & \text{fundamental of } SU(2)_2 \end{array} \tag{4.13}$$

We can Wick rotate to Euclidean space by taking $\tau_E = i\tau$ and we can then map the theory from the cylinder to the complex plane via

$$z = e^{\tau_E + i\sigma} \qquad \bar{z} = e^{\tau_E - i\sigma} \tag{4.14}$$

Writing the fields in the complex plane, we have that the bosons' derivatives with respect to z ($\partial \equiv \partial_z$) and $\bar{z}$ ($\bar{\partial} \equiv \partial_{\bar{z}}$) and the fermions break into holomorphic (left-movers) and antiholomorfic (right-movers) functions:

$$\partial X^{A\dot{A}}_{(r)}(z), \quad \bar{\partial} X^{A\dot{A}}_{(r)}(\bar{z}), \quad \psi^{\alpha\dot{A}}_{(r)}(z), \quad \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)}(\bar{z}) \tag{4.15}$$

Each of the N_1N_5 copies of the CFT contribute with $c_{\text{copy}} = 6$ to the central charge (corresponding to 4 free bosons and 4 free fermions); the total central charge is the sum of c_{copy} over all the N_1N_5 copies, so that $c = 6N_1N_5$. From eq. (4.11) we see that all the states must be invariant under the permutations of the N_1N_5 strands.

[3]We will sometimes use the notation $\alpha, \dot{\alpha} = \pm$, with the identifications $1 \equiv +$ and $2 \equiv -$.

4.3.2 The untwisted ($k = 1$) sector

The untwisted sector is composed of singly wound strands: thus, we have a collection of $N_1 N_5$ independent strands with length (winding) $k = 1$. From now on, we will mainly focus our attention on the holomorphic (left) sector, as the discussion for the antiholomorfic sector works analogously. The OPE for the fermions and the bosons are respectively

$$\psi^{1\dot{A}}_{(r)}(z)\psi^{2\dot{A}}_{(s)}(w) = \frac{\epsilon^{\dot{A}\dot{B}}\delta_{rs}}{z-w} + [\text{reg.}] \qquad \partial X^{A\dot{A}}_{(r)}(z)\partial X^{B\dot{B}}_{(s)}(w) = \frac{\epsilon^{AB}\epsilon^{\dot{A}\dot{B}}\delta_{rs}}{(z-w)^2} + [\text{reg.}] \tag{4.16}$$

where, in our conventions $\epsilon_{12} = \epsilon_{\dot{1}\dot{2}} = -\epsilon^{12} = -\epsilon^{\dot{1}\dot{2}} = 1$.
Since we are dealing with singly wound strands, the boundary conditions have to be imposed so that, taking $\sigma \to \sigma + 2\pi$ on the cylinder (or $z \to e^{2\pi i}z$ on the complex plane), we return back on the same copy. For the bosons we have

$$\partial X^{A\dot{A}}_{(r)}(e^{2\pi i}z) = \partial X^{A\dot{A}}_{(r)}(z) \qquad \partial X^{A\dot{A}}_{(r)}(z) = \sum_{n\in\mathbb{Z}} \alpha^{A\dot{A}}_{(r)n} z^{-n-1} \tag{4.17}$$

For fermions we can have either Ramond (R) or Neveu-Schwartz (NS) bounadry conditions, which correspond, respectively, to periodic and antiperiodic boundary conditions on the cylinder. When one maps to the complex plane, there is a Jacobian (-1) factor that switches the periodicity: periodic fermions on the cylinder correspond to antiperiodic fermions on the complex plane and vice versa. The boundary conditions are reflected in the mode expansions of the fields. In the z-plane for fermions in the R sector we have

$$\psi^{\alpha\dot{A}}_{(r)}(e^{2\pi i}z) = -\psi^{\alpha\dot{A}}_{(r)}(z) \qquad \psi^{\alpha\dot{A}}_{(r)}(z) = \sum_{n\in\mathbb{Z}} \psi^{\alpha\dot{A}}_{(r)n} z^{-n-\frac{1}{2}} \tag{4.18}$$

while for the NS sector

$$\psi^{\alpha\dot{A}}_{(r)}(e^{2\pi i}z) = \psi^{\alpha\dot{A}}_{(r)}(z) \qquad \psi^{\alpha\dot{A}}_{(r)}(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \psi^{\alpha\dot{A}}_{(r)n} z^{-n-\frac{1}{2}} \tag{4.19}$$

The OPEs (4.16), as well as the mode expansions of the fields, yield to the following (non zero) commutation relations

$$[\alpha^{A\dot{A}}_{(r)n}, \alpha^{B\dot{B}}_{(s)m}] = n\epsilon^{AB}\epsilon^{\dot{A}\dot{B}}\delta_{n+m,0}\delta_{rs} \qquad \{\psi^{\alpha\dot{A}}_{(r)n}, \psi^{\beta\dot{B}}_{(s)m}\} = -\epsilon^{\alpha\beta}\epsilon^{\dot{A}\dot{B}}\delta_{n+m,0}\delta_{rs} \tag{4.20}$$

where the relations hold for both the fermionic sectors. Note that the boundary conditions (4.18) and (4.19) select respectively integer and half integer modes and, thus, only the R-sector has zero modes in the expansion.

Vacuum states

In each CFT copy the vacuum state is the tensor product of a vacuum state for the bosons and for the fermions. To be precise, we should say that each one of them is also the tensor product of the vacuum state for the holomorphic and antiholomorphic sector; however, the right-mover and left-mover modes commute with each other and we will not distinguish between left and right vacua.
The bosonic vacuum state corresponding to the (r) copy, $|0\rangle_{(r)}$, is annihilated by all the non negative bosonic modes and we assume the following normalization

$$\begin{aligned} &\alpha^{A\dot{A}}_{(r)n}|0\rangle_{(r)} = 0 \qquad \forall n \geq 0, A, \dot{A} \\ &{}_{(r)}\langle 0|0\rangle_{(s)} = \delta_{rs} \end{aligned} \tag{4.21}$$

Analogous rules hold for the NS sector: the NS vacuum state, denoted with $|0\rangle_{(r)NS}$, is annihilated by all the positive modes

$$\begin{aligned} &\psi^{\alpha\dot{A}}_{(r)n}|0\rangle_{(r),NS} = 0 \qquad \forall n > 0, A, \alpha \\ &{}_{NS,(r)}\langle 0|0\rangle_{(s),NS} = \delta_{rs} \end{aligned} \tag{4.22}$$

In the R sector, instead, things work differently: we still have that the vacuum state is annihilated by the positive modes, but the presence of the zero modes in the expansion (4.18) give rise to a degeneracy of vacuum states. Considering the zero modes, the anticommutation relation (4.20) becomes $\{\psi^{\alpha\dot{A}}_{(r)0}, \psi^{\beta\dot{B}}_{(s)0}\} = -\epsilon^{\alpha\beta}\epsilon^{\dot{A}\dot{B}}\delta_{rs}$. This is the Clifford algebra (with reference to $SU(2)_L$) in 2-dimensions, and its irriducible representation has dimension 2. Thus, in this representation , there should be two vacuum states that can be expressed (using the rising and lowering generators) as a "spin-up" and a "spin-down" state. We will come back on this issue when discussing the currents operators.
We can define a Ramond vacuum state $|{+}{+}\rangle_r$ such that:

$$\begin{aligned} &\psi^{\alpha\dot{A}}_{(r)n}|{+}{+}\rangle_r = 0, \qquad \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)n}|{+}{+}\rangle_r = 0 \qquad \forall n > 0, A, \alpha, \dot{\alpha} \\ &\psi^{1\dot{A}}_{(r)0}|{+}{+}\rangle_r = 0, \qquad \tilde{\psi}^{\dot{1}\dot{A}}_{(r)0}|{+}{+}\rangle_r = 0 \\ &\psi^{2\dot{A}}_{(r)0}|{+}{+}\rangle_r \neq 0, \qquad \tilde{\psi}^{\dot{2}\dot{A}}_{(r)0}|{+}{+}\rangle_r \neq 0 \end{aligned} \tag{4.23}$$

where $\tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)n}$ are antiholomorphic R sector modes. Again, we choose a normalization such that:

$${}_{(r)}\langle + + | + +\rangle_{(s)} = \delta_{rs} \tag{4.24}$$

Note that the degenerate vacua can be obtained from the vacuum state $|{+}{+}\rangle_r$ only acting with zero modes: acting with negative modes would rise the energy of the state.
The topology of the gravitational theory selects the periodicity of the fermions in the CFT. If we have global AdS_3 space, the conformal field theory has fermions with antiperiodic boundary conditions around the circle[4]. The reason is that the circle is contractible in AdS_3 and going around the S^1 at the boundary looks like a 2π rotation close to the center of AdS. Because fermions are invariant under 4π, a 2π rotation gives a minus sign and the D1D5 CFT is in the NS-NS sector. The D1D5 geometries, instead, must have an asymptotically flat extension, and this is possible only if the vacuum energy is zero. This selects the R sector in the CFT: in this way fermions and bosons have the same periodicity and supersymmetry makes their contribution to the vacuum energy precisely to cancel. Therefore, the R sector will be our building block for the discussions in the next Chapter. The total vacuum for the R sector of the CFT is the tensor product of the vacua of each copy.

Other operators

The theory contains other operators beside the bosonic and fermionic fields. For each copy of the CFT, we have two current operators associated to the $SU(2)_L \times SU(2)_R$ symmetry[5]:

$$\begin{aligned} J^{\alpha\beta}_{(r)}(z) &= \frac{1}{2}\psi^{\alpha\dot{A}}_{(r)}\epsilon_{\dot{A}\dot{B}}\psi^{\beta\dot{B}}_{(r)} \\ \tilde{J}^{\alpha\beta}_{(r)}(\bar{z}) &= \frac{1}{2}\tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)}\epsilon_{\dot{A}\dot{B}}\tilde{\psi}^{\dot{\beta}\dot{B}}_{(r)} \end{aligned} \tag{4.26}$$

[4]The boundary of AdS_3 is a cylinder, so the conformal field theory is naturally defined there: its topology constrains the periodicity of fermions on the cylinder.

[5]By operator on the r-th copy $O_{(r)}$, we mean that it acts non trivially in the r-th copy while it is just the identity on the other copies:

$$O_{(r)} \equiv \mathbb{I}_{(1)} \otimes ... \otimes \mathbb{I}_{(r-1)} \otimes O_{(r)} \otimes \mathbb{I}_{(r+1)} \otimes ... \otimes \mathbb{I}_{(N_1 N_5)} \tag{4.25}$$

where, from now on, normal ordering with respect to the $|{+}{+}\rangle_r$ vacuum is understood. The conformal dimensions of the operators are $(h,\bar{h}) = (1,0)$ and $(h,\bar{h}) = (0,1)$ respectively. The total currents are just the sum of the currents (4.26) on the N_1N_5 copies

$$J^{\alpha\beta}(z) = \sum_{r=1}^{N_1N_5} J^{\alpha\beta}_{(r)}(z) \qquad \tilde{J}^{\alpha\beta}(\bar{z}) = \sum_{r=1}^{N_1N_5} \tilde{J}^{\dot{\alpha}\dot{\beta}}_{(r)}(\bar{z}) \tag{4.27}$$

In terms of the standard $SU(2)$ generators we have

$$\begin{aligned} J^{+}_{(r)} &= \frac{1}{2}\psi^{1\dot{A}}_{(r)}\epsilon_{\dot{A}\dot{B}}\psi^{1\dot{B}}_{(r)} \equiv J^{1}_{(r)} + iJ^{2}_{(r)} \\ J^{-}_{(r)} &= -\frac{1}{2}\psi^{2\dot{A}}_{(r)}\epsilon_{\dot{A}\dot{B}}\psi^{2\dot{B}}_{(r)} \equiv J^{1}_{(r)} - iJ^{2}_{(r)} \\ J^{3}_{(r)} &= -\frac{1}{2}(\psi^{1\dot{A}}_{(r)}\epsilon_{\dot{A}\dot{B}}\psi^{2\dot{B}}_{(r)} - 1) \end{aligned} \tag{4.28}$$

Analogous definitions hold for the antiholomorphic currents. The constant term in $J^3_{(r)}$ has been fixed in such a way that the state $|{+}{+}\rangle_r$ has quantum number $(\frac{1}{2},\frac{1}{2})$ under $(J^3_{(r)},\tilde{J}^3_{(r)})$.
The mode expansion, the OPE rules and the mode algebra for the currents are:

$$\begin{aligned} J^{a}_{(r)}(z) &= \sum_{n\in\mathbb{Z}} J^{a}_{(r)n} z^{-n-1} \\ J^{a}_{(r)}(z)J^{b}_{(s)}(w) &= \frac{\delta_{rs}}{z-w} i\epsilon^{abc}J^{c}_{(r)}(w) + [\text{reg.}] \\ [J^{a}_{(r)n}, J^{b}_{(s)m}] &= i\epsilon^{abc}J^{c}_{(r)n+m}\delta_{r,s} + \frac{c_{\text{copy}}}{12} n\delta^{ab}\delta_{r,s}\delta_{m+n,0} \end{aligned} \tag{4.29}$$

Where $a = 1,2,3$ or $a = \pm, 3$ is an $SU(2)$ triplet index. We use the zero modes of the current operators to define the R vacua with different spin

$$J^{-}_{(r)0}|{+}{+}\rangle_{(r)} \equiv |{-}{+}\rangle_{(r)} \qquad \tilde{J}^{-}_{(r)0}|{+}{+}\rangle_{(r)} \equiv |{+}{-}\rangle_{(r)} \qquad \tilde{J}^{-}_{(r)0}J^{-}_{(r)0}|{+}{+}\rangle_{(r)} \equiv |{-}{-}\rangle_{(r)} \tag{4.30}$$

Another important family of operators is

$$O^{\alpha\dot{\alpha}}_{(r)}(z,\bar{z}) \equiv \frac{-i}{2}\psi^{\alpha\dot{A}}_{(r)}\epsilon_{\dot{A}\dot{B}}\tilde{\psi}^{\dot{\alpha}\dot{B}}_{(r)} = \sum_{n,m\in\mathbb{Z}} O^{\alpha\dot{\alpha}}_{(r)nm} z^{-n-\frac{1}{2}}\bar{z}^{-m-\frac{1}{2}} \tag{4.31}$$

They have conformal dimension $(h,\bar{h}) = (\frac{1}{2},\frac{1}{2})$ and the conjugation relations are $(O^{1\dot{1}})^\dagger = O^{2\dot{2}}$ and $(O^{1\dot{2}})^\dagger = -O^{2\dot{1}}$.
The action of this operator on the state $|{+}{+}\rangle_{(r)}$ produces another R vacuum that carries spin $(0,0)$ under $(J^3_{(r)},\tilde{J}^3_{(r)})$.

$$|00\rangle_{(r)} \equiv O^{2\dot{2}}_{(r)00}|{+}{+}\rangle_{(r)} \tag{4.32}$$

The normalizations of the R vacua are derived from the normalizations of $|{+}{+}\rangle_{(r)}$ and from the commutation relations of the operators; one gets

$${}_{(r)}\langle s|s'\rangle_{(s)} = \delta_{s,s'}\delta_{r,s} \tag{4.33}$$

where $s, s' = (\pm\pm)$ or (00).

Supersymmetry generates the (holomorphic) supersymmetry currents

$$G^{\alpha A}_{(r)} = \psi^{\alpha\dot{A}}_{(r)}\epsilon_{\dot{A}\dot{B}}\partial X^{A\dot{B}}_{(r)} = \sum_{n} G^{\alpha a}_{(r)n} z^{-n-\frac{3}{2}} \tag{4.34}$$

where $n \in \mathbb{Z}$ ($n \in \mathbb{Z} + \frac{1}{2}$) in the R sector (NS sector). These operators have conformal dimension $(h, \bar{h}) = (\frac{3}{2}, 0)$, and the modes form the algebra:

$$\begin{aligned} \{G^{\alpha A}_{(r)m}, G^{\beta B}_{(s)n}\} = & -\frac{c_{\text{copy}}}{6}(m^2 - \frac{1}{2})\epsilon^{AB}\epsilon^{\alpha\beta}\delta_{m+n,0}\delta_{r,s} + \\ & + (m-n)\epsilon^{AB}\epsilon^{\beta\gamma}(\sigma^{*a})^{\alpha}_{\gamma} J^a_{m+n}\delta_{r,s} - \epsilon^{AB}\epsilon^{\alpha\beta}L_{m+n}\delta_{r,s} \end{aligned} \tag{4.35}$$

The right sector has the corresponding anti-holomorphic supercurrents. The stress energy operator of the theory receives a contribution from the bosons T^B and from the fermions T^F as

$$\begin{aligned} T_{(r)}(z) &= T^B_{(r)}(z) + T^F_{(r)}(z) = \sum_{n\in\mathbb{Z}} L_{(r)n} z^{-n-2} \\ T^B_{(r)}(z) &= \frac{1}{2}\epsilon_{AB}\epsilon_{\dot{A}\dot{B}}\partial X^{A\dot{A}}_{(r)}\partial X^{B\dot{B}}_{(r)} \\ T^F_{(r)}(z) &= \frac{1}{2}\epsilon_{\alpha\beta}\epsilon_{\dot{A}\dot{B}}\psi^{\alpha\dot{A}}_{(r)}\partial\psi^{\beta\dot{B}}_{(r)} \end{aligned} \tag{4.36}$$

This is the current associated to conformal invariance, and its modes generate the Virasoro algebra on each copy of the CFT

$$[L_{(r)n}, L_{(s)m}] = (n-m)L_{(r)n+m}\delta_{r,s} - \frac{c_{\text{copy}}}{12}n(n^2-1)\delta_{n+m,0}\delta_{r,s} \tag{4.37}$$

4.3.3 The twisted ($k > 1$) sector

In the previous section we have discussed the untwisted sector: a collection of N_1N_5 independent strands of winding $k = 1$. This was obtained imposing boundary condition such that circling the point z around the origin (or transforming $\sigma \to \sigma + 2\pi$ on the cylinder) we return on the same CFT copy.
As we have seen, this is not the only possibility: we may have m_i strands of winding k_i so that $\sum_i m_i k_i = N_1N_5$. In this case the boundary conditions are non trivial. Consider a strand of length k: as we circle the origin one copy of the CFT gets mapped into the adjacent

$$\begin{aligned} \partial X^{A\dot{A}}_{(r)}(e^{2\pi i}z) &= \partial X^{A\dot{A}}_{(r+1)}(z) \qquad \bar{\partial} X^{A\dot{A}}_{(r)}(e^{-2\pi i}\bar{z}) = \bar{\partial} X^{A\dot{A}}_{(r+1)}(\bar{z}) \\ \psi^{\alpha\dot{A}}_{(r)}(e^{2\pi i}z) &= \psi^{\alpha\dot{A}}_{(r+1)}(z) \qquad \bar{\psi}^{\dot{\alpha}\dot{A}}_{(r)}(e^{-2\pi i}\bar{z}) = \bar{\psi}^{\dot{\alpha}\dot{A}}_{(r+1)}(\bar{z}) \end{aligned} \tag{4.38}$$

where $r = 1, ..., k$. Note that, because $\bar{z}$ is the complex conjugate of z, if the boundary conditions for the holomorphic fields are imposed circling the point z anticlockwise ($z \to e^{2\pi i}z$), the boundary conditions for $\bar{z}$ must be taken clockwise ($\bar{z} \to e^{-2\pi i}\bar{z}$).
This conditions hold both for the R and the NS sector. The distinction between the two fermionic sector comes from the identifications

$$\begin{aligned} &\text{Bosons:} \quad \partial X^{A\dot{A}}_{(k+1)} = \partial X^{A\dot{A}}_{(1)} \qquad \bar{\partial} X^{A\dot{A}}_{(k+1)} = \bar{\partial} X^{A\dot{A}}_{(1)} \\ &\text{R sector:} \quad \psi^{\alpha\dot{A}}_{(k+1)} = \psi^{\alpha\dot{A}}_{(1)} \qquad \bar{\psi}^{\dot{\alpha}\dot{A}}_{(k+1)} = \bar{\psi}^{\dot{\alpha}\dot{A}}_{(1)} \\ &\text{NS sector:} \quad \psi^{\alpha\dot{A}}_{(k+1)} = (-1)^{k+1}\psi^{\alpha\dot{A}}_{(1)} \qquad \bar{\psi}^{\dot{\alpha}\dot{A}}_{(k+1)} = (-1)^{k+1}\bar{\psi}^{\dot{\alpha}\dot{A}}_{(1)} \end{aligned} \tag{4.39}$$

From eq. (4.38) we see that the boundary conditions expressed in terms of the copies labeled by (r) are non trivial. A natural choice is to diagonalize these conditions: this can be done performing a change of basis $(r) = 1, ..., k \to \rho = 0, ..., k-1$. For the bosons this is

$$
\begin{aligned}
\partial X^{1\dot{1}}_{\rho}(z) &= \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{-2\pi i\frac{r\rho}{k}}\partial X^{1\dot{1}}_{(r)}(z) = \sum_{n\in\mathbb{Z}} \alpha^{1\dot{1}}_{\rho,n-\frac{\rho}{k}} z^{-n-1+\frac{\rho}{k}} \\
\partial X^{2\dot{2}}_{\rho}(z) &= \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{2\pi i\frac{r\rho}{k}}\partial X^{2\dot{2}}_{(r)}(z) = \sum_{n\in\mathbb{Z}} \alpha^{2\dot{2}}_{\rho,n+\frac{\rho}{k}} z^{-n-1-\frac{\rho}{k}} \\
\partial X^{1\dot{2}}_{\rho}(z) &= \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{2\pi i\frac{r\rho}{k}}\partial X^{1\dot{2}}_{(r)}(z) = \sum_{n\in\mathbb{Z}} \alpha^{1\dot{2}}_{\rho,n+\frac{\rho}{k}} z^{-n-1-\frac{\rho}{k}} \\
\partial X^{2\dot{1}}_{\rho}(z) &= \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{-2\pi i\frac{r\rho}{k}}\partial X^{2\dot{1}}_{(r)}(z) = \sum_{n\in\mathbb{Z}} \alpha^{2\dot{1}}_{\rho,n-\frac{\rho}{k}} z^{-n-1+\frac{\rho}{k}}
\end{aligned}
\tag{4.40}
$$

The antiholomorphic sector works analogously, upon taking $(z,i) \to (\bar{z},-i)$. The bosonic modes in the ρ-basis satisfy the following commutation relations

$$
[\alpha^{A\dot{A}}_{\rho_1,n}, \alpha^{B\dot{B}}_{\rho_2,m}] = \epsilon^{AB}\epsilon^{\dot{A}\dot{B}} n\delta_{n+m,0}\delta_{\rho_1,\rho_2} \tag{4.41}
$$

For the R sector, the change of basis $(r) \to \rho$ gives

$$
\begin{aligned}
\psi^{1\dot{A}}_{\rho}(z) &= \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{2\pi i\frac{r\rho}{k}}\psi^{1\dot{A}}_{(r)}(z) = \sum_{n\in\mathbb{Z}} \psi^{1\dot{A}}_{\rho,n+\frac{\rho}{k}} z^{-n-1-\frac{\rho}{k}} \\
\psi^{2\dot{A}}_{\rho}(z) &= \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{-2\pi i\frac{r\rho}{k}}\psi^{2\dot{A}}_{(r)}(z) = \sum_{n\in\mathbb{Z}} \psi^{1\dot{A}}_{\rho,n-\frac{\rho}{k}} z^{-n-1+\frac{\rho}{k}}
\end{aligned}
\tag{4.42}
$$

and the modes satisfy the following nonzero anticommutation relations

$$
\{\psi^{1\dot{A}}_{\rho_1,n}, \psi^{2\dot{A}}_{\rho_2,m}\} = \epsilon^{\dot{A}\dot{B}}\delta_{n+m,0}\delta_{\rho_1,\rho_2} \tag{4.43}
$$

Note from eq.s (4.40) and (4.42) that only the $\rho = 0$ terms contain the zero modes. Analogous relations hold for the right sector. In the NS sector things work differently because of the identifications in (4.39), see [38] for an exhaustive treatment.

Vacuum states

The discussion of the vacua and the operators in the twisted sector is analogous to the discussion in the untwisted sector, with the difference that now the natural basis is the one that diagonalizes the boundary conditions, i.e. the ρ-basis.
The bosonic vacuum for a strand of length k, $|0\rangle_k$, is annihilated by all the non negative bosonic modes in the ρ-basis given in eq. (4.40). Analogously one generalizes eq. (4.22) for the NS sector. In the R sector we have vacua $|\pm\pm\rangle_k$ and $|00\rangle_k$, and

$$
\begin{aligned}
&\psi^{\alpha\dot{A}}_{\rho,n}|{+}{+}\rangle_r = 0, \qquad \tilde{\psi}^{\dot{\alpha}\dot{A}}_{\rho,n}|{+}{+}\rangle_r = 0 \qquad \forall n > 0, A, \alpha, \dot{\alpha} \\
&\psi^{1\dot{A}}_{\rho,0}|{+}{+}\rangle_r = 0, \qquad \tilde{\psi}^{\dot{1}\dot{A}}_{\rho,0}|{+}{+}\rangle_r = 0
\end{aligned}
\tag{4.44}
$$

and the other vacua are obtained acting on $|{+}{+}\rangle_k$ with the generalization of the operators J^-, $\tilde{J}^-$ and O^{--} in the twisted sector.

Other operators

We want now to discuss operators in the twisted sector. To fix the ideas, consider the case we have $\frac{N}{k}$ strands of length k. In the untwisted sector, an operator $\mathcal{O}$ is the sum of the operators $\mathcal{O}_{(r)}$ defined on each copy of length $k=1$. We can defined it on a strand of length k splitting the summation on the different copies and performing a change of basis $(r) \to \rho$ as

$$\mathcal{O} = \sum_{I=1}^{\frac{N}{k}} \sum_{r=1}^{k} \mathcal{O}^I_{(r)} = \sum_{I=1}^{\frac{N}{k}} \sum_{\rho=0}^{k-1} \mathcal{O}^I_\rho \tag{4.45}$$

where the index $I = 1, ..., \frac{N}{k}$ runs on the different strands. Restricting our attention on a single strand of winding k (i.e. neglecting the sum over I), the current operators become

$$\begin{aligned} J^+ &= \frac{1}{2}\sum_{r=1}^{k} \psi^{1\dot{A}}_{(r)} \epsilon_{\dot{A}\dot{B}} \psi^{1\dot{B}}_{(r)} = \frac{1}{2}\Big(\psi^{1\dot{A}}_{\rho=0}\epsilon_{\dot{A}\dot{B}}\psi^{1\dot{B}}_{\rho=0} + \sum_{\rho=1}^{k-1}\psi^{1\dot{A}}_{\rho}\epsilon_{\dot{A}\dot{B}}\psi^{1\dot{B}}_{k-\rho}\Big) \\ J^- &= -\frac{1}{2}\sum_{r=1}^{k} \psi^{2\dot{A}}_{(r)} \epsilon_{\dot{A}\dot{B}} \psi^{2\dot{B}}_{(r)} = -\frac{1}{2}\Big(\psi^{2\dot{A}}_{\rho=0}\epsilon_{\dot{A}\dot{B}}\psi^{2\dot{B}}_{\rho=0} + \sum_{\rho=1}^{k-1}\psi^{2\dot{A}}_{\rho}\epsilon_{\dot{A}\dot{B}}\psi^{2\dot{B}}_{k-\rho}\Big) \\ J^3 &= -\frac{1}{2}\sum_{r=1}^{k} (\psi^{1\dot{A}}_{(r)} \epsilon_{\dot{A}\dot{B}} \psi^{2\dot{B}}_{(r)} - 1) = -\frac{1}{2}\sum_{\rho=0}^{k-1}(\psi^{1\dot{A}}_{\rho}\epsilon_{\dot{A}\dot{B}}\psi^{2\dot{B}}_{\rho} - 1) \end{aligned} \tag{4.46}$$

where we have used the inverse relations of eq. (4.42) (which are obtained just inverting the sign of the phases) and the orthonormality condition

$$\sum_{r=1}^{k} e^{2\pi i \frac{r}{k}(\rho_1+\rho_2)} = k\delta_{\rho_1+\rho_2,0} \tag{4.47}$$

Analogously, one can define the operator $O^{\alpha\dot{\alpha}}$ on a strand of winding k: for instance, the $(+,+)$ spin component becomes:

$$O^{++} = \frac{-i}{\sqrt{2}}\sum_{\rho=0}^{k-1}\psi^{+\dot{A}}_{\rho}\tilde{\psi}^{+\dot{B}}_{\rho}\epsilon_{\dot{A}\dot{B}} \tag{4.48}$$

As in the case $k=1$ (4.30) and (4.32) we can define different R vacua acting with the zero modes as

$$\begin{aligned} J^-_{0\rho=0}|{+}{+}\rangle_k &= |{-}{+}\rangle_k \qquad \tilde{J}^-_{0\rho=0}|{+}{+}\rangle_k = |{+}{-}\rangle_k \\ J^-_{0\rho=0}\tilde{J}^-_{0\rho=0}|{+}{+}\rangle_k &= |{+}{-}\rangle_k \qquad O^{--}|{+}{+}\rangle_k = \frac{-i}{\sqrt{2}}\psi^{-\dot{A}}_{0\rho=0}\tilde{\psi}^{-\dot{B}}_{0\rho=0}\epsilon_{\dot{A}\dot{B}}|{+}{+}\rangle_k = |00\rangle_k \end{aligned} \tag{4.49}$$

4.3.4 The Twist operator

The untwisted and twisted sector of the CFT are not disconnected: there are operators, the twist operators, that merge k untwisted vacua to give a single twisted one with length k. There are twist operators for the bosonic vacua as well as for the fermionic vacua in the NS and in the R sectors. We will focus on the latter, discussions about the formers can be found in [38].

We have seen that a configuration in the CFT can be labeled by the positive integers (m_i, k_i), where m_i is the number of strands of length k_i, with the constrain $\sum_i m_i k_i = N_1 N_5$. The untwisted sector corresponds to take $m = N_1N_5$ and $k=1$. If we forget about the degeneracy of the R ground states, (m_i, k_i) characterizes completely the configuration because of the symmetrization on the copies (i.e. the fact that all the copies are identical). Thus, there is a bijection between the configurations and the representations in terms of cycles of the permutation group $S_{N_1N_5}$. The untwisted sector can be represented as the product of N_1N_5 1-cycles (the identity permutation):

$$(1)(2)...(r)...(s)...(N_1N_5) \tag{4.50}$$

while a configuration in the twisted sector will be something like

$$(12...)...(r...s)...(...N_1N_5) \tag{4.51}$$

Acting with a k-cycle (or a product of k-cycles) one can pass from one equivalence class of $S_{N_1N_5}$ to another, and this is exactly what a twist operator does.
Suppose we start from the untwisted configuration and we have a set of fields $X^i_{(a)}$ defined on each copy (where $a = 1...N_1N_5$ is the copy index); if we act with the n-cycle $(1...n)$, the fields $X^i_{(a)}$ are mapped as

$$X^i_{(1)} \to X^i_{(2)} \to ... \to X^i_{(n)} \to X^i_{(1)} \tag{4.52}$$

while the other copies of $X^i_{(a)}$ are left unchanged. These are precisely the boundary conditions we have imposed when defining the twisted sector in eq.s (4.38) and (4.39).
The twist operator for the R sector will be denoted with $\Sigma_k^{s_1\dot{s}_2}(z,\bar{z})$: the index k is the length of the k-cycle, the indices s_1 and $\dot{s}_2$ transform in the representation of $(\frac{k-1}{2}, \frac{k-1}{2})$ under $SU(2)_L \times SU(2)_R$. This operator has conformal dimension $(\frac{k-1}{2}, \frac{k-1}{2})$[6].
Because there are many vacuum states in the R-sector and $\Sigma_k^{s_1\dot{s}_2}(z,\bar{z})$ carries spin $(j,\tilde{j}) = (\frac{k-1}{2}, \frac{k-1}{2})$, we have to be careful to the spin conservation: not all the actions of $\Sigma_k^{s_1\dot{s}_2}(z,\bar{z})$ give a non vanishing result on a certain configuration.
To fix the ideas, and because we will primarily use this special case, let's set $k = 2$. The spin indices transform in the fundamental representation of $SU(2)_L \times SU(2)_R$ and we have $\Sigma_2^{\alpha\dot{\alpha}}$, with $\alpha, \dot{\alpha} = \pm$: this operator carries spin $(j,\tilde{j}) = (\frac{1}{2}, \frac{1}{2})$. The conjugacy relations are

$$\left(\Sigma_2^{1\dot{1}}\right)^\dagger = \Sigma_2^{2\dot{2}} \qquad \left(\Sigma_2^{1\dot{2}}\right)^\dagger = -\Sigma_2^{2\dot{1}} \tag{4.53}$$

Let's consider the following example: we have N strands of length $k = 1$, in the vacuum state $|{+}{+}\rangle_{(r)}$, so that the total state is just the tensor product of the N vacua

$$|s\rangle = \bigotimes_{r=1}^{N} |{+}{+}\rangle_{(r)} = |{+}{+}\rangle_{(1)} \otimes ... \otimes |{+}{+}\rangle_{(r)} \otimes ... \otimes |{+}{+}\rangle_{(N)} \tag{4.54}$$

and we act with the twist field $\Sigma_2^{\pm\pm}$. Because of the symmetrization on the copies of the CFT, all the states (or dually the operators) must be invariant under permutations of the copy subscript: thus, the action of $\Sigma_2^{\pm\pm}$ generates the sum af all possible states obtained from (4.54) gluing any two singly wound strands into a strand of length $k = 2$. So, we shall write

$$\Sigma_2^{\pm\pm} = \sum_{r=1}^{N}\sum_{r<s} \Sigma_{(rs)}^{\pm\pm} \tag{4.55}$$

where the operators $\Sigma_{(rs)}^{\pm\pm}$, acting on the state (4.54), can merge together the copies (r) and (s).
In order to understand for which value of $(\alpha,\dot{\alpha}) = (\pm,\pm)$ this process can be carried out, we must apply the angular momentum conservation. Let's focus on any two copies in (4.54), to fix the ideas

[6] We have cheated slightly here. This is the conformal dimension when we consider the twist operator as the product of the bosonic and fermionic twist operators. The vacuum state is the product of the vacua of the fermionic and bosonic sectors of the CFT: because the fields live on strands of length k, it would not make sense to have different lengths for the bosons and the fermions. In this view, the twist operator has to merge both the bosonic and fermionic vacua to produce a strand of length k. The sum of the conformal dimensions of the bosonic and fermionic twist operators gives the above result. For more details on this, see [38]

let's take $|s'\rangle = |{+}{+}\rangle_{(1)} \otimes |{+}{+}\rangle_{(2)}$. Because $|{+}{+}\rangle$ is an eigenstate of $(J^3, \tilde{J}^3)$ with eigenvalues $(\frac{1}{2}, \frac{1}{2})$, both J^3 and $\tilde{J}^3$ are diagonal on $|s'\rangle$, and its total left (right) spin is the sum of the left (right) spin of the two $k = 1$ vacuum states

$$\begin{aligned} J^3\Big(|{+}{+}\rangle_{(1)} \otimes |{+}{+}\rangle_{(2)}\Big) &= |{+}{+}\rangle_{(1)} \otimes |{+}{+}\rangle_{(2)} \\ \tilde{J}^3\Big(|{+}{+}\rangle_{(1)} \otimes |{+}{+}\rangle_{(2)}\Big) &= |{+}{+}\rangle_{(1)} \otimes |{+}{+}\rangle_{(2)} \end{aligned} \tag{4.56}$$

If we act with $\Sigma^{\pm\pm}_{(12)}$ on $|s'\rangle$ the left (right) spin of the state produced will be the sum of the left (right) spin of $|s'\rangle$ and $\Sigma^{\pm\pm}_{(12)}$. We have seen in Section 4.3.3 that the $k = 2$ vacuum state cannot have eigenvalues greater than $\frac{1}{2}$ under $(J^3, \tilde{J}^3)$. The only non zero possibility is to act with $\Sigma^{--}_{(12)}$, which gives a state with spin $(2 \times \frac{1}{2} - \frac{1}{2}, 2 \times \frac{1}{2} - \frac{1}{2}) = (\frac{1}{2}, \frac{1}{2})$

$$\Sigma^{--}_{(12)}\Big(|{+}{+}\rangle_{(1)} \otimes |{+}{+}\rangle_{(2)}\Big) = |{+}{+}\rangle_{(12)} \tag{4.57}$$

From the permutations point of view, we can interpret the action of the operator $\Sigma^{--}_{(rs)}$ on the state $|s\rangle$ as the action of the k-cycle (rs) on the untwisted configuration (4.50)

$$\Sigma^{--}_{(rs)}|s\rangle \to \qquad (rs) \quad (1)...(r)...(s)...(N) = (1)...(rs)...(N) \tag{4.58}$$

The action of Σ^{--}_2 will thus be the sum of all the possible permutations on the r.h.s. of (4.58): because Σ^{--}_2 can form a strand of winding $k = 2$ gluing any two strands out of the N, the result will be the sum of $\binom{N}{2}$ states

$$\Sigma^{--}_2|s\rangle \to (12)(3)...(N) + [\text{permutations}] \tag{4.59}$$

This is just a pictorial representation that does not naturally takes into account whether the action of the twist operator on a state is allowed or not (one has always to impose the spin conservation to obtain a physically consisten result); however, it is useful to compute the combinatorial factors in the CFT.

The operator $\Sigma^{\pm\pm}_2$ does not only glue strands, but it can also split a strand of winding k into (the symmetrization of) two strands of winding k_1 and $k_2 = k - k_1$. In the language of permutations, we can have for instance

$$\begin{aligned} &(rs) \quad (12...)...(rs)...(...N) = (12...)...(r)(s)...(...N) \\ &(rs) \quad (12...)...(...rs...)(...N) = (12...)...(r)...(s...)(...N) \end{aligned} \tag{4.60}$$

As in the previous example, when one considers the splitting process due to the action of the twist operator, the spin conservation gives constrains on the allowed values of $\alpha, \dot{\alpha}$.

4.3.5 Spectral Flow

The D1D5 superconformal algebra is generated by (the modes of) the local operators $\{J^a(z), G^{\alpha A}(z), T(z)\}$: these are the currents associated to the R-symmetry, supersymmetry and conformal invariance respectively. There are, however, different inequivalent ways to satisfy the superconformal algebra: in particular one can make the following z-dependent, 1-parameter (ν) transformation on the operators [11]:

$$\begin{aligned} T(z) \to T_\nu(z) &= T(z) - \frac{2\nu}{z} J^3(z) + \frac{c\nu^2}{6z^2} \\ J^3(z) \to J^3_\nu(z) &= J^3(z) - \frac{c\nu}{6z} \\ J^\pm(z) \to J^\pm_\nu(z) &= z^{\mp 2\nu} J^\pm(z) \\ G^{\pm A}(z) \to G^{\pm A}_\nu(z) &= z^{\mp\nu} G^{\pm A}(z) \end{aligned} \tag{4.61}$$

which gives rise to the modes transformation

$$\begin{aligned} L_m &\to L_m - 2\nu J^3_m + \frac{c\nu^2}{6}\delta_{m,0} \\ J^3_m &\to J^3_m - \frac{c\nu}{6}\delta_{m,0} \\ J^\pm_m &\to J^\pm_{m\mp 2\nu} \\ G^{\pm A}_m &\to G^{\pm A}_{m\mp\nu} \end{aligned} \tag{4.62}$$

This set of transformations is the so-called spectral flow.
From the transformation rule of the supercharges we see that, depending on the value of ν, spectral flow provides an interpolation between the NS and R sector. If we start from the NS sector (where the fermions, and thus the supercurrents, are half-integrally moded), for $\nu = 0$ we are left in the NS sector while for $\nu = \pm\frac{1}{2}$ spectral flow maps the theory in the R sector, where the supercurrents are integrally moded. Spectral flow also acts on states: suppose we start from the NS sector and let's set $\nu = -\frac{1}{2}$. A state with (left) conformal dimensions h^{NS} and spin j_3^{NS} is mapped, under spectral flow, into a state in the R sector with dimension and spin given by

$$h^R = h^{NS} + j_3^{NS} + \frac{c}{24} \qquad j_3^R = j_3^{NS} + \frac{c}{12} \tag{4.63}$$

Spectral flow acts independently on the left and right sector; for the antiholomorphic sector anologous rules hold.

A representation of the spectral flow is naturally defined in the context of bosonized fermions: this is a technique, called bosonization, that enables to realize the properties of the fermions in terms of a set of bosons.
We will introduce this technique in the untwisted sector, and use it to provide a representation of the spectral flow; the generalization to the twisted sector can be found in [38]. We introduce the bosons $H_{(r)}(z)$, $K_{(r)}(z)$ in the holomorphic sector and $\tilde{H}_{(r)}(\bar{z})$, $\tilde{K}_{(r)}(\bar{z})$ in the anitholomorphic sector: they are defined on strands of length 1 and obey to the OPE rules

$$\begin{aligned} H_{(r)}(z)H_{(s)}(\omega) = K_{(r)}(z)K_{(s)}(\omega) = -\delta_{r,s}\log(z-\omega) + [\text{reg.}] \\ \tilde{H}_{(r)}(\bar{z})\tilde{H}_{(s)}(\bar{\omega}) = \tilde{K}_{(r)}(\bar{z})\tilde{K}_{(s)}(\bar{\omega}) = -\delta_{r,s}\log(\bar{z}-\bar{\omega}) + [\text{reg.}] \end{aligned} \tag{4.64}$$

The fermions can be written in terms of the bosons as

$$\begin{aligned} \psi^{1\dot{1}}_{(r)} = i :e^{iH_r}: \quad \psi^{2\dot{2}}_{(r)} = i :e^{-iH_r}: \quad \psi^{1\dot{2}}_{(r)} =: e^{iK_r}: \quad \psi^{2\dot{1}}_{(r)} =: e^{-iK_r}: \\ \tilde{\psi}^{\dot{1}\dot{1}}_{(r)} = i :e^{i\tilde{H}_r}: \quad \tilde{\psi}^{\dot{2}\dot{2}}_{(r)} = i :e^{-i\tilde{H}_r}: \quad \tilde{\psi}^{\dot{1}\dot{2}}_{(r)} =: e^{i\tilde{K}_r}: \quad \tilde{\psi}^{\dot{2}\dot{1}}_{(r)} =: e^{-i\tilde{K}_r}: \end{aligned} \tag{4.65}$$

It turns out that the operator $e^{i\alpha X(z)}$ (where $X(z)$ is a bosonic operator that satisfies the OPE rules (4.64)) has conformal dimension $(h, \bar{h}) = (\frac{\alpha^2}{2}, 0)$ and spin $(j_3, \tilde{j}_3) = (\frac{\alpha}{2}, 0)$ (if X had been an antiholomorphic boson the left and right dimension and spin would have been exchanged). Knowing this, one can check that the definitions (4.65) satisfy (4.16) as a consequence of (4.64).

The operators H and K and their antiholomorphic partners can be used to define the operator that maps the NS vacuum $|0\rangle_{(r)NS}$ to the R ground state $|{+}{+}\rangle_{(r)}$:

$$|{+}{+}\rangle^N = \bigotimes_{r=1}^{N} |{+}{+}\rangle_{(r)} = \bigotimes_{r=1}^{N}\left[\lim_{z,\bar{z}\to 0} e^{\frac{i}{2}(H_{(r)}(z)+K_{(r)}(z)+\tilde{H}_r(\bar{z})+\tilde{K}_r(\bar{z}))}|0\rangle_{(r)NS}\right] \tag{4.66}$$

Because the left and right conformal dimension (spin) of the spectral flow operator (4.66) is the sum of the left and right conformal dimension (spin) of each term in the tensor product, and since the NS vacuum has zero dimension and spin, we conclude that the R ground state $|{+}{+}\rangle^N$ has dimension and spin

$$\begin{aligned} h = \bar{h} &= \sum_{r=1}^{N} \frac{1}{2}\left(\frac{1}{2}\right)^2 2 = \frac{c}{24} \\ j_3 = \bar{j}_3 &= \sum_{r=1}^{N} \frac{1}{2}\left(\frac{1}{2}\right) 2 = \frac{c}{12} \end{aligned} \tag{4.67}$$

Note that this result is consistent with (4.63). Because R ground states can be obtained acting on $|{+}{+}\rangle^N$ with fermion zero modes, they all have conformal dimension $h = \bar{h} = \frac{c}{24}$.

Chapter 5

Holographic dictionary for D1D5 states

In the previous Section we have introduced the AdS/CFT duality and we have discussed the D1D5 CFT: the CFT relevant for the D1D5 microstates. The duality implies that, given a state $|s\rangle$ in the CFT, there is a corresponding asymptotically AdS spacetime; moreover, it can be shown that the VeVs $\langle s|\mathcal{O}^i|s\rangle$ of some operators $\mathcal{O}^i$ computed in the CFT state capture the coefficient of the dual geometry expansion around $AdS_3 \times S^3$. In this Section we will discuss this correspondence in detail: this point of view was pioneered by [40] [41] for the $\frac{1}{4}$-BPS geometries and then extended in [42] for the $\frac{1}{8}$-BPS geometries constructed in [43].

5.1 Chiral Primaries

We have introduce the D1D5 CFT at the free orbifold point, i.e. at a special point in the moduli space where the theory is made up of free bosons and fermions. We have also seen that the AdS/CFT duality relates the moduli spaces of the two sides of the correspondence in a non trivial way. It turns out that the region of moduli space of the CFT dual to the low-energy supergravity regime in the bulk is distant from the solvable free orbifold point, and would require to work in a strongly interacting point in the CFT.

In general, if we compute quantities (such as VeVs) at the free orbifold point and move in the moduli space of the CFT, these quantities will receive corrections: this makes it seem difficult to gain any information about the gravitational physics from the CFT side of the duality. Once again, however, supersymmetry simplifies the task: if we focus on quantities that are protected by supersymmetry (i.e. which do not vary as we change the couplings of the theory) we can reliably compare the computation at the free point of the CFT with the bulk physics in the supergravity regime.

With this aim, let's introduce some representation theory of the NS sector of the D1D5 CFT[1]. We choose a representation in which the generator L_0 is diagonal; and we will call Virasoro primary a state $|\psi\rangle$ such that

$$L_0|\psi\rangle = h|\psi\rangle \qquad L_n|\psi\rangle = 0 \quad \forall n > 0 \tag{5.1}$$

This is the highest weight state of the representation space, and it is the asymptotic state created by applying a primary field operator $\psi(0)$ of dimension h on the NS vacuum [39]. From eq. (4.35), we have that the D1D5 superconformal algebra includes the anticommutators

$$\begin{aligned} \{G^{-A}_{+\frac{1}{2}}, G^{+B}_{-\frac{1}{2}}\} &= \epsilon^{AB}(J^3_0 - L_0) \\ \{G^{+A}_{+\frac{1}{2}}, G^{-B}_{-\frac{1}{2}}\} &= \epsilon^{AB}(J^3_0 + L_0) \end{aligned} \tag{5.2}$$

[1] Here we just restrict our attention to the left sector, as the right sector works analogously.

Let's consider a generic state state $|\psi\rangle$ with quantum numbers (h, j^2, m) under $(L_0, (J_0^a)^2, J_0^3)$; taking the expectation value of the anticommutators (5.2) on $|\psi\rangle$ one gets

$$\begin{aligned} &\sum_B \left|G^{+B}_{-\frac{1}{2}}|\psi\rangle\right|^2 + \sum_B \left|G^{-B}_{\frac{1}{2}}|\psi\rangle\right|^2 = 2(h-m) \\ &\sum_B \left|G^{-B}_{-\frac{1}{2}}|\psi\rangle\right|^2 + \sum_B \left|G^{+B}_{\frac{1}{2}}|\psi\rangle\right|^2 = 2(h+m) \end{aligned} \tag{5.3}$$

Because in any unitary theory the left hand sides of (5.3) are the sum of two non negative quantities, we obtain the following bound on any physical state of the theory

$$h \geq |m| \quad \Rightarrow \quad h \geq j \tag{5.4}$$

Let's now restrict our attention to a state $|\phi\rangle$ that saturates the bound $h = m$. This state has the following properties

$$\begin{aligned} &G^{\pm A}_n|\phi\rangle = L_n|\phi\rangle = 0 \qquad \forall n > 0 \\ &G^{+A}_{-\frac{1}{2}}|\phi\rangle = 0 \end{aligned} \tag{5.5}$$

The first row can be derived noting that, if these operators did not annihilate the state, the inequality (5.4) would be violated; the second row saturated the bound $h = m$. We call chiral primary state a state that satisfies (5.5) and chiral primary operator the operator dual to a chiral primary state. We call antichiral primary state (or, dually, operator) a state that satisfies $h = -m$. From the bound $h = m = j$, we see that chiral primaries are highest weight state with respect to both the Virasoro algebra and the $SU(2)_L$ algebra.
It turns out that the VeVs of chiral primary operators (and their descendants, obtained acting with L_{-1} and J_0^- on the chiral primary) computed on Ramond ground states are protected when one moves in moduli space. To be more precise, the three point functions of chiral primaries (and descendants) are protected by a non-renormalization theorem, which ensures that they do not receive corrections as one changes the couplings of the theory. From the spectral flow eq. (4.63) one can check that antichiral primary operators are in one to one correspondence with Ramond vacua. Therefore, the VeVs of chiral primary operators computed on Ramond ground states are protected. Thus, we can compute the VeVs of these operators on states that are dual to the $\frac{1}{4}$-BPS and $\frac{1}{8}$-BPS microstates and reliably use the results to extract informations about the bulk physics in the supergravity regime.

5.2 Holographic map between geometries and CFT states

The aim of this Section is to describe the mapping between microstate geometries and CFT states. We first discuss the geometric setting of the gravitational theory for $\frac{1}{4}$-BPS microstates in order to introduce the notations that will be followed throughout this Chapter. We then identify the CFT dual states. We also briefly outline the recipe to construct (some) $\frac{1}{8}$-BPS microstates starting from 2-charge geometries and identify their CFT duals. We will just report the results that will be used in the following discussion, with reference to [43] and [44].

5.2.1 $\frac{1}{4}$-BPS Sector

The geometric side

The general BPS solution of type IIB supergravity on a spacetime with topology $\mathbb{R}^{1,4} \times S^1 \times T^4$, assuming invariance under T^4 rotations, that preserves the same supercharges as the D1-D5-P system is described, in the string frame, by the 10-dimensional metric

$$
\begin{aligned}
ds^2_{(10)} &= -\frac{2\sqrt{\alpha}}{\sqrt{\mathcal{P}}}(dv+\beta)\big[du+\omega+\frac{\mathcal{F}}{2}(dv+\beta)\big]+\sqrt{\mathcal{P}}\sqrt{\alpha}ds^2_4+\sqrt{\frac{Z_1}{Z_2}}d\hat{s}^2_4 \\
e^{2\Phi} &= \frac{Z_1^2}{\mathcal{P}} \qquad \alpha\equiv\frac{Z_1Z_2}{Z_1Z_2-Z_4^2} \qquad \mathcal{P}\equiv Z_1Z_2-Z_4^2
\end{aligned}
\tag{5.6}
$$

where ds^2_4 corresponds to a (generically non trivial) Euclidean metric in the 4 spatial non compact directions that reduces asymptotically to flat $\mathbb{R}^4$; $d\hat{s}^2_4$ denotes the flat metric on T^4 and Φ is the dilaton. We have introduced light-cone coordinates

$$
u=\frac{t-y}{\sqrt{2}} \qquad v=\frac{t+y}{\sqrt{2}} \tag{5.7}
$$

where t is the time coordinate and y is the coordinate on S^1. The metric $ds^2_{(10)}$ depends on the following objects: four scalar functions Z_1, Z_2, Z_4 and $\mathcal{F}$; two 1-forms on $\mathbb{R}^4$ β and ω. The naive 3-charge geometry (2.69) has the form of (5.6) with

$$
\begin{aligned}
&Z_1=1+\frac{Q_1}{r^2} \qquad Z_2=1+\frac{Q_5}{r^2} \qquad \mathcal{F}=-\frac{2Q_P}{r^2} \\
&Z_4=0 \qquad \beta=\omega=0 \qquad ds^2_4=dx^idx^i
\end{aligned}
\tag{5.8}
$$

where the relations between the macroscopic charges Q_1, Q_5 and Q_P and the integer charges are given by (2.82). Note that in the 2 charge geometry $Q_P=0$ and, thus, $\mathcal{F}=0$.

It is useful to think of the 3-charge solution as obtained by adding momentum to some 2-charge solutions, so let's start discussing the latter. In Section 3.3 we have constructed microstates corresponding to a D1D5 bound state: we started with a duality frame where the system is described in terms of a fundamental string carrying momentum; we have parametrized the profile of the string by a curve $F^I(v)$ in the 4 non compact directions $I=1,...,4$ and then we applied a chain of dualities on the known solution in the F1-P frame to rewrite the solution for the D1D5 configuration (i.e. in a duality frame where the profiles do not have a direct geometric meaning).
These are not the more general 2 charge microstates. From a geometrical point of view, to discuss the most general case, we should have started giving 8 functions $g_A(v)$ transverse to the fundamental string in order to describe its profile; these functions can be split into four $\mathbb{R}^4$ components ($A=1,...,4$) and four T^4 components ($A=5,...,8$). When the latter are non-vanishing, invariance under rotation in the T^4 directions is broken. However, when one applies the chain of dualities (2.67), one of the T^4 direction (which we take to be x^5) plays a special role: it turns out that, in the D1-D5 frame, geometries that have non trivial values of the profiles $g_A(v)$ (for $A=1,...,5$) preserve rotational symmetry in the T^4 directions. Thus, these solutions fall into the class of eq. (5.6). Generalizing eq. (3.20), we will now give the expressions of the functions and 1-forms in (5.6) in terms of the profiles $g_A(v)$. Defining

$$
h_1(v)\equiv g_1(v)+ig_2(v) \qquad h_2(v)\equiv g_3(v)+ig_4(v) \tag{5.9}
$$

we have

$$
\begin{aligned}
&Z_1=1+\frac{Q_5}{L}\int_0^L dv\frac{|\dot{h}_1|^2+|\dot{h}_2|^2+|\dot{g}_5|^2}{|x_i-g_i|^2} \qquad Z_2=1+\frac{Q_5}{L}\int_0^L dv\frac{1}{|x_i-g_i|^2} \\
&Z_4=-\frac{Q_5}{L}\int_0^L dv\frac{\dot{g}_5}{|x_i-g_i|^2} \qquad A=-\frac{Q_5}{L}\int_0^L dv\frac{\dot{g}_j dx^j}{|x_i-g_i|^2} \qquad dB=-\star_4 dA \\
&\beta\equiv\frac{-A+B}{\sqrt{2}} \qquad \omega\equiv\frac{-A-B}{\sqrt{2}} \qquad \mathcal{F}=0 \qquad ds^2_4=dx^idx^i
\end{aligned}
\tag{5.10}
$$

where the dot on the profiles denote a derivative with respect to v, $\star_4$ is the hodge dual with respect to the flat metric ds^2_4 and the denominator $|x_i-g_i|^2$ is a short hand for

$$
|x_i-g_i|^2\equiv\sum_{i=1}^4(x_i-g_i)^2=|(x_1+ix_2)-h_1|^2+|(x_3+ix_4)-h_2|^2 \tag{5.11}
$$

The length of the curve L and the D1 charge are given by

$$L = \frac{2\pi Q_5}{R} \qquad Q_1 = \frac{Q_5}{L}\int_0^L dv(|\dot{h}_1|^2 + |\dot{h}_2|^2 + |\dot{g}_5|^2) \tag{5.12}$$

The CFT side

In the previous Chapter we have seen that the relevant Sector of the CFT for the $\frac{1}{4}$-BPS microstates is the Ramond Sector. We will denote the RR ground states of a strand of length k with $|s\rangle_k \equiv |j_3\bar{j}_3\rangle_k$, where $j_3, \bar{j}_3 = 0, \pm$, so that $(j_3\bar{j}_3)$ denote the left and right spin carried by the state. A generic ground state configuration can be labeled by the integers $\{N_k^{(s)}\}$, where $N_k^{(s)}$ is the number of strands $|s\rangle_k$, with the constraint

$$\sum_{s,k} kN_k^{(s)} = N_1 N_5 \equiv N \tag{5.13}$$

and we will denote it as

$$\psi_{\{N_k^{(s)}\}} = \prod_{k,s}(|s\rangle_k)^{N_k^{(s)}} \tag{5.14}$$

We now identify the state/geometry dictionary for the $\frac{1}{4}$-BPS microstates (5.10). The basic idea is that the Fourier expansion of the profiles entering in the geometry encodes the informations about the dual CFT state: in particular, the 5 components of profiles $g_A(v)$ determine the quantum numbers $(j_3\bar{j}_3)$ of the strands under the $SU(2)_L \times SU(2)_R$ $\mathcal{R}$-symmetry generators; the Fourier mode numbers correspond to the winding of the strands and the Fourier coefficients are related to the number of strands of each type.
To be more precise, let's consider the Fourier expansion of the profiles:

$$\begin{aligned} g_1 + ig_2 &= \sum_{k>0}\Big(\frac{a_k^{++}}{k}e^{\frac{2\pi ik}{L}v} + \frac{a_k^{--}}{k}e^{-\frac{2\pi ik}{L}v}\Big) \\ g_3 + ig_4 &= \sum_{k>0}\Big(\frac{a_k^{+-}}{k}e^{\frac{2\pi ik}{L}v} + \frac{a_k^{-+}}{k}e^{-\frac{2\pi ik}{L}v}\Big) \\ g_5 &= -\mathrm{Im}\Big(\sum_{k>0}\frac{a_k^{00}}{k}e^{\frac{2\pi ik}{L}v}\Big) \end{aligned} \tag{5.15}$$

The presence of a non vanishing coefficient $a_k^{(s)}$ signals the presence of strands $|s\rangle_k$ in the CFT dual state. Note that eq. (5.12) imposes the following constraint on the Fourier coefficients

$$\sum_k \Big[|a_k^{++}|^2 + |a_k^{+-}|^2 + |a_k^{-+}|^2 + |a_k^{--}|^2 + \frac{1}{2}|a_k^{++}|^2\Big] = \frac{Q_1 Q_5}{R^2} \tag{5.16}$$

We introduce new dimensionless coefficients $A_k^{(s)}$ that are related to $a_k^{(s)}$ through

$$A_k^{\pm\pm} \equiv R\sqrt{\frac{N}{Q_1 Q_5}}a_k^{\pm\pm} \qquad A_k^{00} \equiv R\sqrt{\frac{N}{2Q_1 Q_5}}a_k^{00} \tag{5.17}$$

in terms of which eq. (5.16) becomes

$$\sum_{k,s}|A_k^{(s)}|^2 = N \tag{5.18}$$

The profiles $g_A(v)$ determine a geometry and a set of coefficients $\{A_k^{(s)}\}$ and the CFT state dual to this geometry is given in terms of $\{A_k^{(s)}\}$ by

$$\psi(\{A_k^s\}) = \sum_{\{N_k^{(s)}\}}{}' \Big(\prod_{k,s} A_k^{(s)}\Big)^{N_k^{(s)}} \psi_{\{N_k^{(s)}\}} = \sum_{\{N_k^{(s)}\}}{}' \prod_{k,s} \Big(A_k^{(s)}|s\rangle_k\Big)^{N_k^{(s)}} \tag{5.19}$$

where the superscript over $\sum'_{\{N_k^{(s)}\}}$ denotes that the sum is constrained by eq. (5.13). Note that the dual CFT states are coherent states: microstate geometries do not correspond to a state like (5.14), where the degeneracies of the strands are fixed.

In the supergravity description of black holes we take the thermodynamic limit $N \to \infty$; the supergravity side is well described only by states in which the average number of each type of strand ($\bar{N}_k^{(s)}$) is large: $\bar{N}_k^{(s)} >> 1$ (in the thermodynamic limit, variations of order one in $\bar{N}_k^{(s)}$ are invisible on the gravity side). Eq. (5.19) shows that the state $\psi(\{A_k^s\})$ receives contribution from all the possible configurations $\{N_k^{(s)}\}$ compatible with eq. (5.13); however, in the above limit, the sum is peaked over the average number $\bar{N}_k^{(s)}$.

Let's now describe how the magnitude of $A_k^{(s)}$ determines the average degeneracy of the corresponding strand. With this aim, we have to introduce the norm of the state $\psi(\{A_k^s\})$ and show that its (constrained) maximum is determined by $A_k^{(s)}$. By convention we define the norm of the state $\psi_{\{N_k^{(s)}\}}$ in terms of the number of ways, $\mathcal{N}(\{N_k^{(s)}\})$, the configuration $\{N_k^{(s)}\}$ can be reached starting from $|{+}{+}\rangle_N$:

$$\langle\psi_{\{N_k^{(s)}\}}|\psi_{\{N_k^{(s)}\}}\rangle = \delta_{\{N_k^{(s)}\},\{N_k^{(s)}\}}\mathcal{N}(\{N_k^{(s)}\}) \tag{5.20}$$

In order to compute the factor $\mathcal{N}(\{N_k^{(s)}\})$ we have to generalize the discussion on the combinatorics that arises when one considers the action of the twist operator Σ_k on a state. We start from $|{+}{+}\rangle_N$, i.e. from the untwisted Sector of the CFT; to produce a single strand with winding k, the twist operator can choose k singly wound strands among N, giving rise to the combinatorial factor $\binom{N}{k}$. However, this does not uniquely determines a k-cycle: we can still permute $k-1$ copies out of the k forming the multy wound strand and obtain a different k-cycle. So we obtain the combinatorics $\binom{N}{k}(k-1)!$. The configuration $\{N_k^{(s)}\}$ can be reached acting repeatedly with the twist operator; iterating the combinatorial computation one gets that the total number of terms produced is

$$\frac{N!}{(N-k_1)!k_1}\frac{(N-k_1)!}{(N-k_1-k_2)!k_2}... = \frac{N!}{\prod_{k,s} k^{N_k^{(s)}}} \tag{5.21}$$

Because all the copies are identical, for strand with degeneracy $N_k^{(s)} > 1$, the order by which the twist operator acts is immaterial: thus, one should divide by $N_k^{(s)}!$. The final result is

$$\mathcal{N}(\{N_k^{(s)}\}) = \frac{N!}{\prod_{k,s} N_k^{(s)}! k^{N_k^{(s)}}} \tag{5.22}$$

The norm of the CFT states dual the $\frac{1}{4}$-BPS geometries, then, is

$$|\psi(\{A_k^s\})|^2 = \sum_{\{N_k^{(s)}\}}{}' \mathcal{N}(\{N_k^{(s)}\}) \prod_{k,s} |A_k^{(s)}|^{2N_k^{(s)}} \equiv \sum_{\{N_k^{(s)}\}}{}' e^{S(\{N_k^{(s)}\})} \tag{5.23}$$

We now consider the maxima $\bar{N}_k^{(s)}$ of $|\psi(\{A_k^s\})|^2$. Because we are dealing with the large charge limit, we can expand $S(\{N_k^{(s)}\})$ using Stirling approximation $\log N_k^{(s)}! \sim (N_k^{(s)} + \frac{1}{2})\log N_k^{(s)} - N_k^{(s)}$. The result is

$$S(\{N_k^{(s)}\}) = \sum_{k,s} N_k^{(s)} \log |A_k^{(s)}|^2 - N_k^{(s)} \log N_k^{(s)} + N_k^{(s)} - N_k^{(s)} \log k \tag{5.24}$$

Using the method of Lagrange multipliers, we can evaluate the stationary points of $S(\{N_k^{(s)}\})$, with the constrain (5.13). We obtain that the sum in (5.19) is peaked over the average degeneracy

$$\bar{N}_k^{(s)} = \frac{|A_k^{(s)}|^2}{k} \tag{5.25}$$

5.2.2 $\frac{1}{8}$-BPS Sector

The geometric side

In the $\frac{1}{8}$-BPS Sector, a complete construction on the gravity side of the microstates has not been achieved yet. However, there are solution generating techniques that enable to construct some classes of 3-charge geometries starting from particular 2-charge solutions (the so-called "seed solutions"). Roughly speaking, this procedure adds momentum to the D1D5 geometry. We will now briefly review these techniques, with no aim of being exhaustive, trying to highlight the underlying idea that will help us to identify the CFT states that are dual to these 3-charge geometries.

Consider the circular profile in the (x_1, x_2) plane:

$$g_1(v) + ig_2(v) = ae^{\frac{2\pi iv}{L}} \qquad g_3(v) = g_4(v) = g_5(v) = 0 \tag{5.26}$$

Parametrizing the non compact space with coordinates

$$x_1 + ix_2 = \sqrt{r^2 + a^2} \sin\theta e^{i\phi} \qquad x_3 + ix_4 = r\cos\theta e^{i\psi} \tag{5.27}$$

and using eq. (5.10), one obtains the following 2-charge geometry

$$\begin{aligned}
ds_4^2 &= \Sigma\Big(\frac{dr^2}{r^2+a^2} + d\theta^2\Big) + (r^2+a^2)\sin^2\theta d\psi^2 + r^2\cos^2\theta d\psi^2 \\
Z_1 &= 1 + \frac{Q_1}{\Sigma} \qquad Z_2 = 1 + \frac{Q_5}{\Sigma} \qquad Z_4 = \mathcal{F} = 0 \qquad \Sigma \equiv r^2 + a^2\cos^2\theta \\
\beta &= \frac{Ra^2}{\sqrt{2}\Sigma}(\sin^2\theta d\phi - \cos^2\theta d\psi) \qquad \omega = \frac{Ra^2}{\sqrt{2}\Sigma}(\sin^2\theta d\phi + \cos^2\theta d\psi)
\end{aligned} \tag{5.28}$$

Eq. (5.16) links the parameter a with the radius R of S^1 and the charges by $a = \frac{\sqrt{Q_1Q_5}}{R}$. Let's now take the decoupling limit so that we can use the tools that the AdS/CFT duality provides: the geometry reduces to pure $AdS_3 \times S^3 \times T^4$, and this can be made explicit performing the coordinate redefinition

$$\phi \to \phi + \frac{t}{R} \qquad \psi \to \psi + \frac{y}{R} \tag{5.29}$$

We can interpret this result from the CFT point of view: from the discussion in Section 5.2.1, we conclude that the profile (5.26) generates a CFT state that is proportional to $|{+}{+}\rangle^N$. On the other hand, pure AdS_3 is just the NS-NS vacuum. The change of coordinates (5.29), thus, maps $|{+}{+}\rangle^N \to |0\rangle_{NS}^N$: using (4.66) we conclude that this is the geometrical representation of the spectral flow transformation that maps the R-R sector to the NS-NS sector. The advantage of working in the NS sector is that pure $AdS_3 \times S^3$ enjoys an $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R \times SU(2)_L \times SU(2)_R$ isometry group and one can use the corresponding generators to produce 3-charge solutions.
With this aim, let's consider a perturbation of the system, obtained adding a non vanishing $g_5(v)$ profile to (5.26); this will be our seed solution

$$g_1(v) + ig_2(v) = ae^{\frac{2\pi iv}{L}} \qquad g_5(v) = -\frac{b}{k}\sin(\frac{2\pi kv}{L}) \qquad g_3(v) = g_4(v) = 0 \tag{5.30}$$

The corrensponding geometry can be found in [43] (eq. (3.11)). If one takes the decoupling limit and performs the change of coordinate (5.29), one obtains a 2-charge geometry that is a fluctuation around the $AdS_3 \times S^3 \times T^4$ background geometry. Because the background is rotationally invariant, acting with the $SU(2)_L$ generators[2] one produces a new fluctuation: the resulting geometry, written in the original R-R coordinates, has a non-vanishing momentum charge.
The CFT state generated by (5.26) represents a R-R ground state (composed of strands of type $|{+}{+}\rangle_1$ and $|00\rangle_k$) so, when one performs the change of coordinates (5.29), it is mapped to an antichiral primary in the NS sector. Because an antichiral primary is annihilated by the lowering operator J_0^-, new states can be generated only acting with J_0^+. In particular we can act m_k times with J_0^+, with $m_k \le k$: the reason is that the action of $(J_0^+)^k$ generates a chiral primary, which is annihilated by any further action of J_0^+. From eq. (4.62) we see that the operator J_0^+ in the NS sector is mapped to J_{-1}^+ via spectral flow; thus, we conclude the state dual to the freshly produced 3-charge geometry is composed of strands $|{+}{+}\rangle_1$ and $\left(J_{-1}^+\right)^{m_k}|00\rangle_k$.
As it will be used in the following, here we just quote from [43] the structure of the function Z_4 entering in the 3-charge geometry

$$\begin{aligned} Z_4^{(k,m_k)} &= Rb\frac{\Delta_{k,m_k}}{\Sigma}\cos\Big(m_k\frac{\sqrt{2}v}{R} + (k-m_k)\phi - m_k\psi\Big) \\ \Delta_{k,m_k} &\equiv \Big(\frac{a}{\sqrt{r^2+a^2}}\Big)^k \sin^{k-m_k}\theta\cos^{m_k}\theta \end{aligned} \tag{5.31}$$

These $\frac{1}{8}$-BPS states are, by construction, descendants of $\frac{1}{4}$-BPS states. Exploiting the linearity of the BPS equations (the supergravity equations of motion when the BPS condition has been imposed), one can construct 3-charge geometries that are not descendants of a 2-charge microstate taking arbitrary linear combinations of the functions $Z_4^{(k,m_k)}$. Doing so, the function Z_4 reads

$$Z_4 = R\sum_{k=1}^{\infty}\sum_{m_k=0}^{k} b_{k,m_k}\frac{\Delta_{k,m_k}}{\Sigma}\cos\Big(m\frac{\sqrt{2}v}{R} + (k-m_k)\phi - m_k\psi + \eta_{k,m_k}\Big) \tag{5.32}$$

The CFT side

Let's now discuss the CFT states dual to the $\frac{1}{8}$-BPS geometries. A generic configuration of the theory can be labeled by the integers $\{N_{k,m_k}^{(s)}\} \equiv \{N_k^{(\pm\pm)}, N_{k,m_k}^{(00)}\}$, where $N_{k,m_k}^{(s)}$ denotes the number of strands $|s\rangle_k$, and we allow the ground state $|00\rangle_k$ to carry (left-moving) momentum number $m_k \le k$. Again, $\{N_{k,m_k}^{(s)}\}$ must satisfy the constrain

$$\sum_{s=1}^{4}\sum_k kN_k^{(s)} + \sum_{k,m_k} kN_{k,m_k}^{(00)} = N \tag{5.33}$$

Generalizing (5.14), we will denote the state in the configuration $\{N_{k,m_k}^{(s)}\}$ as:

$$\psi_{\{N_{k,m_k}^{(s)}\}} = \prod_{s=1}^{4}\prod_k (|s\rangle_k)^{N_k^{(s)}} \prod_{k,m_k}\Big(\frac{(J_{-1}^+)^{m_k}}{m_k!}|00\rangle_k\Big)^{N_{k,m_k}^{(00)}} \tag{5.34}$$

where $s = 1, ..., 4$ denotes the ground state with spin $(\pm,\pm)$.
The norm of the state is a generalization of (5.20): in this case, however, $\mathcal{N}(\{N_{k,m_k}^{(s)}\})$ takes into account not only the combinatoric factor (5.22) that arises from the action of the twist operators, but

[2] In order to preserve supersymmetry, one can act either with the right $SL(2,\mathbb{R})_R \times SU(2)_R$ or with the left $SL(2,\mathbb{R})_L \times SU(2)_L$ generators, not both at the same time. Here, we just restrict our attention to the (left) $\mathcal{R}$-symmetry generators.

also an extra factor due to the presence of the operators $(J^+_{-1})^{m_k}$. Using the commutation relation for k twisted CFT copies

$$[J^a_m, J^b_n] = i\epsilon^{abc} J^c_{m+n} + \frac{k}{2} m \delta_{m,-n} \delta^{ab} \tag{5.35}$$

which can be derived form (4.29), one can compute the norm of the state $(J^+_{-1})^{m_k}|00\rangle_k$. Consider the state

$$\begin{aligned} J^-_1 (J^+_{-1})^{m_k}|00\rangle_k &= \Big(-2J^3_0 + k + J^+_{-1}J^-_1 \Big)(J^+_{-1})^{m_k-1}|00\rangle_k = ... = \\ &= \Big(-2\sum_{j=0}^{m_k-1} j + mk \Big)(J^+_{-1})^{m_k-1}|00\rangle_k = m(k-(k-1))(J^+_{-1})^{m_k-1}|00\rangle_k \end{aligned} \tag{5.36}$$

where we have used the fact that $(J^+_{-1})^j|00\rangle_k$ has eigenvalue j under J^3_0. Iterating (5.36), one finds the following norm

$$\begin{aligned} {}_k\langle 00|(J^-_1)^{m_k}(J^+_{-1})^{m_k}|00\rangle_k &= m_k!(k-(m_k-1))(k-(m_k-2))...k = \\ &= (m_k!)^2 \binom{k}{m_k} \end{aligned} \tag{5.37}$$

Taking into account the normalization factor $\frac{1}{m_k!}$ in (5.34), one gets

$$\mathcal{N}(\{N^{(s)}_{k,m_k}\}) = \Big(\prod_{s=1}^{4} \prod_k \frac{N!}{N^s_k! k^{N^s_k}} \Big) \Big(\prod_{k,m_k} \frac{1}{N^{00}_{k,m_k}! k^{N^{00}_{k,m_k}}} \Big) \prod_{k,m_k} \binom{k}{m_k}^{N^{00}_{k,m_k}} \tag{5.38}$$

For 3-charge systems the coefficient b_{k,m_k} in (5.31) plays the same role as a^{00}_k for the 2-charge microstates (indeed, if the momentum charge is switched off, $b_{k,0} = a^{00}_k$). The dual CFT states are

$$\psi(\{A^{(s)}_k, B_{k,m_k}\}) = \sum_{\{N^{(s)}_{k,m_k}\}}{}' \prod_{s=1}^{4} \prod_k \Big(A^{(s)}_k |s\rangle_k \Big)^{N^{(s)}_k} \prod_{k,m_k} \Big(B_{k,m_k} \frac{(J^+_{-1})^{m_k}}{m_k!} |00\rangle_k \Big)^{N^{(00)}_{k,m_k}} \tag{5.39}$$

where the dimensionless coefficients $A^{(s)}_k$ (with $s = (\pm\pm)$) are related to the Fourier coefficients $a^{(s)}_k$ as in (5.17) and B_{k,m_k} is defined as

$$B_{k,m_k} \equiv R\sqrt{\frac{N}{2Q_1Q_5}} \binom{k}{m_k}^{-1} b_{k,m_k} e^{i\eta_{k,m_k}} \tag{5.40}$$

The norm of $\psi(\{A^{(s)}_k, B_{k,m_k}\})$

$$|\psi(\{A^{(s)}_k, B_{k,m_k}\})|^2 = \sum_{\{N^{(s)}_{k,m_k}\}}{}' \mathcal{N}(\{N^{(s)}_{k,m_k}\}) \prod_{s=1}^{4} \prod_k |A^{(s)}_k|^{2N^{(s)}_k} \prod_{k,m_k} |B_{k,m_k}|^{2N^{(00)}_{k,m_k}} \tag{5.41}$$

is peaked over the average degeneracy $\bar{N}^{(s)}_k$ and $\bar{N}^{(00)}_{k,m_k}$ of the strands $(\pm,\pm)$ and $(0,0)$ respectively. The computation is analogous to the one for the 2-charge system, and the result reads:

$$k\bar{N}^{(s)}_k = |A^{(s)}_k|^2 \qquad k\bar{N}^{(00)}_{k,m_k} = \binom{k}{m_k} |B_{k,m_k}|^2 \tag{5.42}$$

5.3 Holography for operators of dimension 1

In the previous Section we have discussed the dictionary between geometries and dual CFT states. It has been shown in [40] [41] [42] that the geometries of $\frac{1}{4}$- BPS and $\frac{1}{8}$- BPS microstates encode informations about the VeVs of chiral primary operators in the dual state: one can extract these 3-point functions from the fluctuations around $AdS_3 \times S^3$ of the 6D metric in the Einstein frame. One can take advantage of the fact that 3-point functions of chiral primaries are protected as one moves in the moduli space, to argue that the VeVs extracted from the geometry match those computed at the free orbifold point.
The 6D metric in the Einstein frame for D1-D5-P microstates can be obtained from eq. (5.6) by dimensional reduction; it reads

$$ds^2_{(6)} = -\frac{2}{\sqrt{\mathcal{P}}}(dv+\beta)\big[du+\omega+\frac{\mathcal{F}}{2}(dv+\beta)\big]+\sqrt{\mathcal{P}}ds^2_4 \tag{5.43}$$

The expansion around $AdS_3 \times S^3$ has the form:

$$\begin{aligned}
Z_1 &= \frac{Q_1}{r^2}\Big(1+\sum_k \sum_{m_k,\tilde{m}_k=-\frac{k}{2}}^{\frac{k}{2}} \frac{f^1_{k(m_k,\tilde{m}_k)}}{r^k} Y_k^{m_k,\tilde{m}_k}\Big)\\
Z_2 &= \frac{Q_5}{r^2}\Big(1+\sum_k \sum_{m_k,\tilde{m}_k=-\frac{k}{2}}^{\frac{k}{2}} \frac{f^5_{k(m_k,\tilde{m}_k)}}{r^k} Y_k^{m_k,\tilde{m}_k}\Big)\\
Z_4 &= \frac{\sqrt{Q_1Q_5}}{r^2}\Big(\sum_k \sum_{m_k,\tilde{m}_k=-\frac{k}{2}}^{\frac{k}{2}} \frac{\mathcal{A}_{k(m_k,\tilde{m}_k)}}{r^k} Y_k^{m_k,\tilde{m}_k}\Big)\\
\mathcal{F} &= -\frac{2Q_P}{r^2}+O(r^{-3}) \qquad ds^2_4 = dx^i dx^i + O(r^{-4})\\
A &= -\frac{Q_5}{r^2}\Big(\sum_{j=1}^{4}\frac{(A_{1j})_i Y_1^j}{r}+O(r^{-2})\Big)dx^i
\end{aligned} \tag{5.44}$$

where x^i are the non compact space directions ($i, j = 1, ..., 4$). We then define

$$a^{\alpha\pm} = \frac{\sqrt{Q_5}}{\sqrt{Q_1}}\sum_{i>j} e^{\pm}_{\alpha ij}(A_{1j})_i \tag{5.45}$$

where $\alpha = 1, 2, 3$ is an adjoint index of $SU(2)$. The relevant properties the of spherical harmonics for the following discussion and the definition of the coefficients $e^{\pm}_{\alpha ij}$ are reviewed in Appendix B.
It is always possible to choose coordinates such that

$$f^1_{1(m_1,\tilde{m}_1)} + f^5_{1(m_1,\tilde{m}_1)} = 0 \tag{5.46}$$

and, from now on, we will work in this gauge.
The VeVs of operators of dimension 1 are encoded in the first non-trivial corrections around the asymptotic background, i.e. in the functions $f^1_{1(m_1,\tilde{m}_1)}, \mathcal{A}_{1(m_1,\tilde{m}_1)}, a^{\alpha\pm}$ (note that $f^5_{1(m_1,\tilde{m}_1)}$ is not an independent degree of freedom, because of the gauge choice (5.46)). We will see that higher order corrections control the VeVs of higher dimension operators. In Section 4.3 we have introduced chiral primary operators of dimension 1: they are the $SU(2)_L \times SU(2)_R$ currents J^α and $\tilde{J}^\alpha$ (with $(h,\bar{h}) = (j,\bar{j})$ equal to $(1,0)$ and $(0,1)$ respectively); the twist field $\Sigma_2^{\alpha\dot{\alpha}}$ and the operator $O^{\alpha\dot{\alpha}}$ (both with $(h,\bar{h}) = (j,\bar{j}) = (\frac{1}{2},\frac{1}{2})$).

The relation between the VeVs of these operators on a state $|s\rangle$ and the dual geometry expansion is

$$\begin{aligned} \langle s|O^{++}|s\rangle &= -\sqrt{2}C_{O_1}\mathcal{A}_{1(--)} & \langle s|O^{+-}|s\rangle &= -\sqrt{2}C_{O_1}\mathcal{A}_{1(-+)} \\ \langle s|O^{-+}|s\rangle &= \sqrt{2}C_{O_1}\mathcal{A}_{1(+-)} & \langle s|O^{--}|s\rangle &= \sqrt{2}C_{O_1}\mathcal{A}_{1(++)} \\ \langle s|\Sigma_2^{++}|s\rangle &= -\sqrt{2}C_{\Sigma_2}f^1_{1(--)} & \langle s|\Sigma_2^{+-}|s\rangle &= -\sqrt{2}C_{\Sigma_2}f^1_{1(-+)} \\ \langle s|\Sigma_2^{-+}|s\rangle &= \sqrt{2}C_{\Sigma_2}f^1_{1(+-)} & \langle s|\Sigma_2^{--}|s\rangle &= \sqrt{2}C_{\Sigma_2}f^1_{1(++)} \\ \langle s|J^{\alpha}|s\rangle &= C_J a^{\alpha+} & \langle s|\tilde{J}^{\alpha}|s\rangle &= C_{\tilde{J}} a^{\alpha-} \end{aligned} \tag{5.47}$$

Note that the coefficient coupled to the spherical harmonic $Y_1^{m_1,\tilde{m}_1}$ controls the component $(-m_1, -\tilde{m}_1)$ of the corresponding operators. The coefficients entering in the dictionary are given by:

$$C_{O_1} = \frac{\sqrt{2}NR}{\sqrt{Q_1Q_5}} \qquad C_{\Sigma_2} = \frac{N^{\frac{3}{2}}R}{\sqrt{Q_1Q_5}} \qquad C_J = -C_{\tilde{J}} = \frac{NR}{\sqrt{Q_1Q_5}} \tag{5.48}$$

They depend only on the moduli of the theory, but not on the particular state $|s\rangle$ taken in consideration. It is difficult to determine the values of these coefficients a priori, what one usually does is to compute the VeVs at the orbifold point, take advantage of the fact that it is protected, and compare it with the coefficients of the asymptotic expansion of the dual geometry.
Before discussing the generalization of the above dictionary to operators of dimension 2, let's clarify the meaning of the correspondence with an example.

5.3.1 Example on a 2-charge microstate

Let's consider a 2-charge state composed of $N_1^{(++)} \equiv p$ strands $|{+}{+}\rangle_{k=1}$ and $N_1^{00} = N - p$ strands $|00\rangle_{k=1}$ (where we have used the constrain (5.13)). We have seen in Section 5.2.1 that a microstate geometry is determined by a set of coefficients $\{A_k^{(s)}\}$ (related to the profiles $g_A(v)$), which, in general, identify a multiplicity of configurations $\{N_k^{(s)}\}$ compatible with the constrain (5.13) (the only exception being when only a single $A_k^{(s)}$ does not vanish). The dual geometry is a coherent state obtained superposing all the possible $\psi(\{N_k^{(s)}\})$. Renaming $A_1^{(++)} \equiv A$ and $A_1^{(00)} \equiv B$, we consider the coherent state

$$\psi(A,B) = \sum_{p=1}^{N} (A|{+}{+}\rangle_1)^p (B|00\rangle_1)^{N-p} \tag{5.49}$$

Eq. (5.18) fixes

$$|A|^2 + |B|^2 = N \tag{5.50}$$

and from the general result (5.25) we conclude that the average degeneracy of the strands is given by

$$\bar{N}_1^{(++)} = \bar{p} = |A|^2 \qquad \bar{N}_1^{(00)} = N - \bar{p} = |B|^2 \tag{5.51}$$

which is consistent with (5.50).
Among the dimension 1 operators, $\Sigma_2^{\alpha\dot{\alpha}}$ is the only one that has a vanishing VeV on the state (5.49) because it is composed only by singly wound strands.
The operator O^{--} transforms a strand $|{+}{+}\rangle$ into $|00\rangle$ (and O^{++} viceversa), thus it acts non trivially on this state; the VeVs of O^{+-} and O^{-+}, instead, have a vanishing VeV for angular momentum conservation. Let's consider the VeV of O^{++}: its action on the state $\psi(A,B)$ gives the relevant contribution

$$O^{++}((|{+}{+}\rangle_1)^p(|00\rangle_1)^{N-p}) = (p+1)((|{+}{+}\rangle_1)^{p+1}(|00\rangle_1)^{N-p-1}) \tag{5.52}$$

The factor $p+1$ on the r.h.s. is a combinatorial factor that arises when one considers the different possible actions of the operator on the state and requires that the states on the l.h.s. and on the r.h.s.

are composed of the same number of terms. In this case, O^{++} can act on any of the $N-p$ copies of $|00\rangle_1$; $p+1$ is the factor that satisfies the equality

$$(N-p)\mathcal{N}(p) = (p+1)\mathcal{N}(p+1) \tag{5.53}$$

Thus, the VeV of O^{++} on the state $\psi(A,B)$ is

$$\begin{aligned}\langle O^{++}\rangle =&|\psi(A,B)|^{-2}\langle s|O^{++}|s\rangle \\ =&|\psi(A,B)|^{-2}\sum_{p=1}^{N}\bar{A}^p\bar{B}^{N-p}A^{p-1}B^{N-p-1}p = \frac{B}{A}\bar{p} = \bar{A}B\end{aligned} \tag{5.54}$$

where in the last line we have used eq. (5.51). The VeV of O^{--} can be computed analogously, or simply noticing that $\left(O^{--}\right)^\dagger = O^{++}$ and thus

$$\langle O^{--}\rangle = \langle O^{++}\rangle^* = A\bar{B} \tag{5.55}$$

Let's consider the $\mathcal{R}$-currents: for angular momentum conservation the only operators that have a non vanishing VeV on $\psi(A,B)$ are J^3 and $\tilde{J}^3$. These operators are diagonal on the state: each strand $|{+}{+}\rangle_1$ contributes to the total left and right spin by $(\frac{1}{2},\frac{1}{2})$ while $|00\rangle_1$ by $(0,0)$. Thus, the VeVs are only sensitive to the average number of strands $\bar{N}_1^{(++)}$:

$$\langle J^3\rangle = \langle\tilde{J}^3\rangle = \frac{1}{2}\bar{p} = \frac{|A|^2}{2} \tag{5.56}$$

Let's now consider the gravitational side. From the dictionary between geometries and CFT state, we conclude that the dual geometry is generated by the profile in (5.26), with $k=1$; the relation between the coefficients a and A and between b and B are given by eq. (5.17) and read

$$A = R\sqrt{\frac{N}{Q_1Q_5}}a \qquad B = R\sqrt{\frac{N}{2Q_1Q_5}}b \tag{5.57}$$

We are interested in the function Z_4 and in the 1-form A as they encode the VeVs of $O^{\alpha\dot{\alpha}}$ and J^a, $\tilde{J}^{\dot{a}}$ respectively. Using eq. (5.10), and expanding the geometry up to the first non trivial correction around $AdS_3\times S^3$ we get

$$\begin{aligned}&Z_4 = \frac{Rab\sin\theta\cos\theta}{\sqrt{r^2+a^2}(r^2+a^2cos^2\theta)} \sim \frac{\sqrt{Q_1Q_5}}{r^3}\frac{Rab}{2\sqrt{2}\sqrt{Q_1Q_5}}(Y_1^{++}-Y_1^{--}) \\ &A_1 \sim -\frac{Q_5}{r^2}\Big(\frac{Ra^2Y_1^2}{2Q_5r}\Big) \qquad A_2 \sim \frac{Q_5}{r^2}\Big(\frac{Ra^2Y_1^1}{2Q_5r}\Big)\end{aligned} \tag{5.58}$$

the coefficients a and b are taken to be real. Using the formalism in (5.44) and (5.45) we recognize:

$$\mathcal{A}_{1(++)} = -\mathcal{A}_{1(--)} = \frac{Rab}{2\sqrt{2}\sqrt{Q_1Q_5}} \qquad a^{3\pm} = \pm\frac{R^2}{\sqrt{Q_1Q_5}}\frac{a^2}{2} \tag{5.59}$$

Using the dictionary in (5.47) and the coefficients C_{O_1}, C_J and $C_{\tilde{J}}$ in (5.48) one verifies that the VeVs (5.54), (5.55) and (5.56) computed in the CFT agree with the holographically derived values (5.59).

Chapter 6

Holography for operators of dimension 2

The aim of this Section is to discuss the generalization of the holographic dictionary VeVs/geometry for operators of higher dimension: in particular we will focus on operators of dimension 2, whose VeVs are controlled by the coefficients of the geometry expansion up to the second non trivial correction.
Following [40], we will consider an ansatz that generalizes the dictionary (5.47) and, as it has been suggested in [45], we will assume a mixing between single- and multi-trace operators. We will compute their VeVs on particular states at the free orbifold point and compare them with the dual geometry expansion: this will allow to fix constrains on the analogous to the linear coefficients in (5.48).

6.1 Single-trace and Multi-trace operators

There are two types of dimension two operators: single-trace and multi-trace operators. All the operators we have encountered so far are single-trace operators: their action on the D1D5 CFT can be written just as the sum of the corresponding operator defined on each copy. A multi-trace operator is the off-diagonal product of single-trace operators.
Let's clarify this point with an example: consider an operator $\mathcal{O}_{(r)}$ defined on the (r)-th copy of the CFT; the total operator is given by $\mathcal{O} = \sum_r \mathcal{O}_{(r)}$, this is a single-trace operator. We can also consider the product of two $\mathcal{O}_{(r)}$, in this case we have

$$\begin{aligned} &\text{single-trace:} \sum_{r,s=1}^{N} \delta_{rs} : \mathcal{O}_{(r)}\mathcal{O}_{(s)} := \sum_{r=1}^{N} : \mathcal{O}_{(r)}\mathcal{O}_{(r)} :\equiv \sum_{r=1}^{N} \tilde{\mathcal{O}}_{(r)} \\ &\text{multi-trace:} \sum_{r\neq s} : \mathcal{O}_{(r)}\mathcal{O}_{(s)} : \end{aligned} \tag{6.1}$$

where we have defined $\tilde{\mathcal{O}}_{(r)} \equiv: \mathcal{O}_{(r)}\mathcal{O}_{(r)} :$. Note, that when one considers the VeV

$$\langle s| \sum_{r,s=1}^{N} \mathcal{O}_{(r)}\mathcal{O}_{(s)}|s\rangle \tag{6.2}$$

it receives contribution form both the single- and the multi-trace: however, the contribution of the former scales with N while the contribution of the latter with N^2.
For the following discussion we are interested only in dimension two operators whose VeV on a R-R state is protected as we move in the moduli space, so that we can compute it explicitly at the free orbifold point and argue that it matches the computation in a region of the moduli dual to the supergravity regime. Therefore, for what concerns the single-trace, we are interested only in chiral

primary operator and descendants (in particular, those obtained acting with the generators $J_0^{\pm}$ or $\tilde{J}_0^{\pm}$). Multi-trace operators, instead, are protected if the "single trace" constituents are both chiral or anti-chiral primary operators; if not their anomalous dimension scales as $\frac{1}{N}$ and thus vanish in the large charge limit $N \to \infty$ (and we are working in this regime).

6.2 The operator O_2

The first operator of dimension two we are going to consider is $O_2^{\alpha\dot{\alpha}\beta\dot{\beta}} \equiv: O^{\alpha\dot{\alpha}}\Sigma_2^{\beta\dot{\beta}}$:, it has dimension $(h, \bar{h}) = (1, 1)$. This is the product of two fundamental representations of $SU(2)_L$ and two of $SU(2)_R$, so both split into two irreducible representations: a symmetric triplet and an antisymmetric singlet. As explained above, we are interested in chiral primary operators and descendants, i.e. in the triplet representation which is characterized by $(h, \bar{h}) = (j, \bar{j})$. We will denote them with $O_2^{a\dot{a}}$, where $a = -1, 0, 1$ and $\dot{a} = -1, 0, 1$ are indices in the triplets of $SU(2)_L$ and $SU(2)_R$ respectively.
The highest weight state is

$$O_2^{11} = O^{++}\Sigma_2^{++} \tag{6.3}$$

and its descendants can be obtained acting upon it with the lowering operators J_0^- and $\tilde{J}_0^-$.
In the following we will consider CFT states in which both $O^{\alpha\dot{\alpha}}$ and $\Sigma_2^{\alpha\dot{\alpha}}$ have a vanishing VeV, this will allow to consider only the single-trace operator

$$\sum_{r<s} \left(O_{(r)}\Sigma_{(rs)}\right)^{a\dot{a}} \tag{6.4}$$

neglecting the mixing with the double-trace. Following the formalism introduced in Section 5.3, we consider the ansatz:

$$\langle s|O_2^{a\dot{a}}|s\rangle = C_{O_2}\mathcal{A}_{2(-a,-\dot{a})} \tag{6.5}$$

6.2.1 Switching on the O_2's VeVs

In order to fix the coefficient C_{O_2} we will consider $\frac{1}{4}$-BPS and $\frac{1}{8}$-BPS microstates whose dual geometry is given by (5.31). In particular, we will focus our attention on states characterized by $k = 2$ and $m = 0, 1, 2$:

$$\begin{aligned} Z_4^{(m)} &= Rb\frac{\Delta_m}{\Sigma}\cos\left(m\frac{\sqrt{2}v}{R} + (2-m)\phi - m\psi\right) \\ \Delta_m &\equiv \frac{a^2}{r^2+a^2}\sin^{2-m}\theta\cos^m\theta \end{aligned} \tag{6.6}$$

where the coefficients a and b are taken to be real. Note that, since the coefficient of the Z_4 expansion at order r^{-3} vanishes, we can conclude, using eq. (5.47), that this geometry describes a state where the VeV of the operator $O^{\alpha\dot{\alpha}}$ is zero.

The case $k = 2$, $m = 0$

The CFT state contains two types of strands: there are $N_2^{(00)} \equiv p$ strands $|00\rangle_2$ and, using the constrain (5.13), $N_1^{(++)} \equiv N - 2p$ strands $|{+}{+}\rangle_1$. Renaming the only non vanishing coefficients entering in the coherent state as $A_1^{(++)} \equiv A$ and $A_2^{(00)} \equiv B$, we get

$$\psi(A, B) = \sum_{p=1}^{\frac{N}{2}} (A|{+}{+}\rangle_1)^{N-2p}(B|00\rangle_2)^p \tag{6.7}$$

The average degeneracy of the two types of strand follows from (5.25)

$$\bar{p} = \frac{|B|^2}{2} \qquad N - 2\bar{p} = |A|^2 \tag{6.8}$$

which is consistent with the constraint given by (5.18). By conservation of angular momentum, we conclude that the only componets of $O_2^{a\dot{a}}$ that have a non-vanishing VeVs on the state $\psi(A, B)$ are $O_2^{-1,-1}$ and, obviously, its conjugate $O_2^{11} = \left(O_2^{-1,-1}\right)^\dagger$. Let's consider the former: its action on the state can be split into the following two relevant processes:

- The operator Σ_2^{--} joins two singly wound strands $|{+}{+}\rangle_1$ into the strand of winding 2 $|{+}{+}\rangle_2$. This process gives

$$\Sigma_2^{--}((|{+}{+}\rangle_1)^{N-2p}(|00\rangle_2)^p) = ((|{+}{+}\rangle_1)^{N-2(p+1)}(|00\rangle_2)^p|{+}{+}\rangle_2) \tag{6.9}$$

  The operator Σ_2^{--} can choose any two among the $N - 2p$ strands $|{+}{+}\rangle_1$ contributing to the l.h.s. number of terms with a factor $\binom{N-2p}{2}$: no further combinatorial factor is required for the l.h.s and r.h.s. number of terms to match.

- Next, the operator O^{--} converts the freshly produced strand $|{+}{+}\rangle_2$ into a $|00\rangle_2$ strand, this contribution is represented by:

$$O^{--}((|{+}{+}\rangle_1)^{N-2(p+1)}(|00\rangle_2)^p|{+}{+}\rangle_2) = (p+1)((|{+}{+}\rangle_1)^{N-2(p+1)}(|00\rangle_2)^{p+1}) \tag{6.10}$$

  The operator O^{--} can act only on the freshly produced $|{+}{+}\rangle_2$ strand so it carries no combinatorial factor: imposing the matching of the total number of terms on the two sides of the equation the factor $p + 1$ arises.

Proceding as in eq. (5.54), one finds the VeV of $O_2^{-1,-1}$ is

$$\langle O_2^{-1,-1}\rangle = \frac{|A|^2}{B}\bar{p} = \frac{|A|^2\bar{B}}{2} = \frac{R^3 N^{\frac{3}{2}}}{2\sqrt{2}(Q_1Q_5)^{\frac{3}{2}}} a^2 b \tag{6.11}$$

where we have used (6.8) and the relations (5.17) between the dimensionless coefficients A and B and the coefficients a and b entering in the geometry. The dual geometry (6.6) expansion is

$$Z_4^{(m=0)} \sim \frac{\sqrt{Q_1Q_5}}{r^4}\frac{Ra^2b}{2\sqrt{3}\sqrt{Q_1Q_5}}(Y_2^{1,1} + Y_2^{-1,-1}) \tag{6.12}$$

and we read the relevent coefficients

$$\mathcal{A}_{2(\pm1,\pm1)} = \frac{Ra^2b}{2\sqrt{3}\sqrt{Q_1Q_5}} \tag{6.13}$$

Inserting the VeV (6.11) and the geometric coefficient (6.13) into the ansatz (6.5), we obtain:

$$\langle s|O_2^{-1,-1}|s\rangle = C_{O_2}\mathcal{A}_{2(1,1)} \qquad C_{O_2} = \frac{\sqrt{3}R^2N^{3/2}}{\sqrt{2}Q_1Q_5} \tag{6.14}$$

Note that, as in the case of dimension one operators, the coefficient depends on the moduli of the theory but carries no information about the precise state considered.

The case $k = 2$, $m = 1$

A consistency check of the result in (6.14) can be obtained considering other states. With this purpose, let's switch on the third charge, and discuss a $\frac{1}{8}$-BPS state. In particular, we consider a state with

$N^{(00)}_{2,1} \equiv p$ strands of type $|00\rangle_p$ acted upon by the operator J^+_{-1} once and $N-2p$ strands $|{+}{+}\rangle_1$. Renaming the dimensionless coefficients $A^{(++)} \equiv A$ and $B_{2,1} \equiv B$, we consider the state

$$\psi(A,B) = \sum_{p=1}^{\frac{N}{2}} (A|{+}{+}\rangle_1)^{N-2p} (BJ^+_{-1}|00\rangle_2)^p \tag{6.15}$$

We can borrow the general result (5.42) to claim that the average numbers of strands in the state $\psi(A,B)$ is

$$\bar{p} = |B|^2 \qquad N - 2\bar{p} = |A|^2 \tag{6.16}$$

Because the strands $|{+}{+}\rangle_1$ and $J^+_{-1}|00\rangle_2$ carry spin $(\frac{1}{2},\frac{1}{2})$ and $(1,0)$ respectively, by angular momentum conservation we conclude that only $O^{0,-1}_2$ (and its hermitian conjugate) has non zero VeV on the state. This descendant is obtained acting with $J^-_0(\tilde{J}^-_0)^2$ on the highest weight state (6.3): imposing the proper normalization one gets

$$O^{0,-1}_2 = -\frac{1}{\sqrt{2}}\Big(O^{+-}\Sigma^{--}_2 + O^{--}\Sigma^{+-}_2\Big) = -\frac{1}{\sqrt{2}}\Big(O^{+---}_2 + O^{--+-}_2\Big) \tag{6.17}$$

where the minus sign has been introduced for later convenience.
Because $|00\rangle_2$ is acted upon by J^+_{-1}, to determine the VeV of $O^{a\dot{a}}_2$ on the state $\psi(A,B)$ we will need the commutation relations of $O^{\alpha\dot{\alpha}}$ and the twist operator with the $SU(2)_L$ current algebra. These are:

$$\begin{aligned}\Big[(J^i_n)^{\alpha\beta}, O^{\beta\dot{\alpha}}(u,v)\Big] &= \frac{1}{2}e^{in\frac{\sqrt{2}v}{R}}(\sigma^i)^{\alpha\beta}O^{\beta\dot{\alpha}}(u,v)\\ \Big[(J^i_n)^{\alpha\beta}, \Sigma^{\beta\dot{\alpha}}_2(u,v)\Big] &= \frac{1}{2}e^{in\frac{\sqrt{2}v}{R}}(\sigma^i)^{\alpha\beta}\Sigma^{\beta\dot{\alpha}}_2(u,v)\end{aligned} \tag{6.18}$$

Both O^{+---}_2 and O^{--+-}_2 contribute to the VeV of the operator $O^{0,-1}_2$: let's first consider the former. As in the previous case, it is useful to split the action of O^{+---}_2 into the following two processes

- The operator Σ^{--}_2 joins two $|{+}{+}\rangle_1$ strands into $|{+}{+}\rangle_2$; as it can choose any two strands among the $N-2p$ this process carries a $\binom{N-2p}{2}$ factor. Requiring that both sides of the equality contain the same number of terms on gets

$$\Sigma^{--}_2((|{+}{+}\rangle_1)^{N-2p}(J^+_{-1}|00\rangle_2)^p) = ((|{+}{+}\rangle_1)^{N-2(p+1)}(J^+_{-1}|00\rangle_2)^p|{+}{+}\rangle_2) \tag{6.19}$$

  Note that in this case the total number of terms on the two sides of the equation is given by (5.38).

- Let's now consider the action of $O^{+-}(v,u)$ on (6.19). Note that we have to insert it at a generic worldsheet point (v,u), otherwise the action of O^{+-} on the strand $|{+}{+}\rangle_2$ is zero. In particular, using the commutation relations in eq. (6.18), as well as the fact that the ground state is annihilated the positive modes of the current operator, we obtain the following VeV for an individual strand

$${}_2\langle 00|J^-_{+1}O^{+-}(v,u)|{+}{+}\rangle_2 = e^{i\frac{\sqrt{2}v}{R}} \tag{6.20}$$

  Taking into account the proper combinatorial factor, one gets

$$\begin{aligned}&O^{+-}(v,u)((|{+}{+}\rangle_1)^{N-2(p+1)}(J^+_{-1}|00\rangle_2)^p|{+}{+}\rangle_2) =\\ &\qquad = e^{i\frac{\sqrt{2}v}{R}}\frac{p+1}{2}((|{+}{+}\rangle_1)^{N-2(p+1)}(J^+_{-1}|00\rangle_2)^{p+1})\end{aligned} \tag{6.21}$$

Because the state $J^+_{-1}|00\rangle_2$ has norm 2 as a consequence of eq. (5.37), the VeV of $O^{+---}_2(v,u)$ on $\psi(A,B)$ is

$$\langle O^{+---}_2(v,u)\rangle = e^{i\frac{\sqrt{2}v}{R}}\frac{|A|^2}{2B}\bar{p} = e^{i\frac{\sqrt{2}v}{R}}\frac{|A|^2\bar{B}}{2} = e^{i\frac{\sqrt{2}v}{R}}\frac{R^3N^{\frac{3}{2}}}{4\sqrt{2}(Q_1Q_5)^{\frac{3}{2}}}a^2b \tag{6.22}$$

where we have used eq.s (5.17) and (5.40) to obtain the relations between the dimensionless parameters A, B and the coefficients a, b that enter in the geometry:

$$A = R\sqrt{\frac{N}{Q_1 Q_5}}a \qquad B = \frac{1}{2}R\sqrt{\frac{N}{2Q_1 Q_5}}b \tag{6.23}$$

The computation of $\langle O_2^{--+-}(v,u)\rangle$ proceeds analogously to the previous case, here we just report the result:

$$\langle O_2^{--+-}(v,u)\rangle = \langle O_2^{+---}(v,u)\rangle = e^{i\frac{\sqrt{2}v}{R}} \frac{R^3 N^{\frac{3}{2}}}{4\sqrt{2}(Q_1Q_5)^{\frac{3}{2}}} a^2 b \tag{6.24}$$

The dual geometry expansion up to the first non trivial correction is

$$Z_4 \sim \frac{\sqrt{Q_1Q_5}}{r^4} \frac{Ra^2 b}{2\sqrt{6}\sqrt{Q_1Q_5}} (-e^{i\frac{\sqrt{2}v}{R}} Y_2^{0,1} + e^{-i\frac{\sqrt{2}v}{R}} Y_2^{0,-1}) \tag{6.25}$$

Inserting the results (6.17), (6.22), (6.24) and (6.25) into the ansatz (6.5) one obtains

$$\langle s|O_2^{0,-1}|s\rangle = C_{O_2}\mathcal{A}_{2(0,1)} \qquad C_{O_2} = \frac{\sqrt{3}R^2 N^{3/2}}{\sqrt{2}Q_1Q_5} \tag{6.26}$$

in agreement with eq. (6.14).

The case $k = 2$, $m = 2$

As a last consistency check, we consider a state made up of strands of type $|{++}\rangle_2$ and of type $|00\rangle_2$ acted upon by J^+_{-1} twice, so that $|00\rangle_2$ carries the maxima allowed units of momentum. Thus, we consider the coherent state

$$\psi(A,B) = \sum_{p=1}^{\frac{N}{2}} (A|{++}\rangle_1)^{N-2p}(B(J^+_{-1})^2|00\rangle_2)^p \tag{6.27}$$

where we have renamed $A_1^{(++)} \equiv A$, $B_{2,2} \equiv B$ and $N_{2,2}^{(00)} \equiv p$ and we have used the constraint (5.33) to claim that $N_1^{(++)} = N - 2p$. The average degeneracy of the strands is

$$\bar{p} = \frac{|B|^2}{2} \qquad N - 2\bar{p} = |A|^2 \tag{6.28}$$

The strand $(J^+_{-1})^2|00\rangle_2$ carries spin $(2,0)$, thus, by conservation of angular momentum, we conclude that the only non trivial processes on individual strands are

$${}_2\langle 00|(J^-_{+1})^2 O_2^{1,-1}(v,u)|{++}\rangle_1|{++}\rangle_1 = {}_2\langle 00|(J^-_{+1})^2 O^{+-}\Sigma_2^{+-}(v,u)|{++}\rangle_1|{++}\rangle_1 \tag{6.29}$$

and its inverse, which is given by $O_2^{-1,1}$. The expression of $O_2^{1,-1}$ in terms of $O^{\alpha\dot{\alpha}}$ and the twist operator is obtained acting twice on the highest weight state (6.3) with $\tilde{J}_0^-$. Note that, as in the previous case, the operator $O_2^{a,\dot{a}}(v,u)$ needs to be inserted at a generic worldsheet point, otherwise it would kill the state. Using the commutators (6.18), one obtaines that (6.29) is

$$\begin{aligned} &{}_2\langle 00|(J^-_{+1})^2 O^{+-}\Sigma_2^{+-}(v,u)|{++}\rangle_1|{++}\rangle_1 = \\ &\qquad = 2e^{i\frac{2\sqrt{2}v}{R}} {}_2\langle 00|O^{--}\Sigma_2^{--}(v,u)|{++}\rangle_1|{++}\rangle_1 \end{aligned} \tag{6.30}$$

The complete action of the operator O_2^{+-+-} on the state is obtained implementing the appropriate combinatorial factor:

$$
\begin{aligned}
&O^{+-}\Sigma^{+-}(v,u)((|{+}{+}\rangle_1)^{N-2p}((J^+_{-1})^2|00\rangle_2)^p) = \\
&\qquad = e^{i\frac{2\sqrt{2}v}{R}}(p+1)((|{+}{+}\rangle_1)^{N-2(p+1)}((J^+_{-1})^2|00\rangle_2)^{p+1})
\end{aligned}
\tag{6.31}
$$

This gives rise to the VeV

$$
\langle O_2^{1,-1}(v,u)\rangle = e^{i\frac{2\sqrt{2}v}{R}}\bar{p}\frac{|A|^2}{B} = e^{i\frac{2\sqrt{2}v}{R}}\frac{|A|^2\bar{B}}{2} = e^{i\frac{2\sqrt{2}v}{R}}\frac{R^3N^{\frac{3}{2}}}{2\sqrt{2}(Q_1Q_5)^{\frac{3}{2}}}a^2b \tag{6.32}
$$

Expanding the dual geometry (6.6) around $AdS_3\times S^3$ up to the first non trivial order, one gets

$$
Z_4 \sim \frac{\sqrt{Q_1Q_5}}{r^4}\frac{Ra^2b}{2\sqrt{3}\sqrt{Q_1Q_5}}(e^{i\frac{2\sqrt{2}v}{R}}Y_2^{-1,1}+e^{-i\frac{2\sqrt{2}v}{R}}Y_2^{+1,-1}) \tag{6.33}
$$

Using the holographic dictionary (6.5), along with (6.32) and (6.33) we obtain

$$
\langle s|O_2^{1,-1}|s\rangle = C_{O_2}\mathcal{A}_{2(-1,1)} \qquad C_{O_2} = \frac{\sqrt{3}R^2N^{3/2}}{\sqrt{2}Q_1Q_5} \tag{6.34}
$$

which is consistent with (6.14) and (6.26).

6.3 The operators Ω, Σ_3 and the associated multi-traces

The aim of this Section is to consider more dimension 2 operators and fix constrains on the corresponding holographic dictionary.

We first consider the operator $\Omega^{\alpha\beta\dot{\alpha}\dot{\beta}}_{(r)} = J^{\alpha\beta}_{(r)}\tilde{J}^{\dot{\alpha}\dot{\beta}}_{(r)}$. The highest weight state

$$
\Omega^{++} = \sum_r^N J^+_{(r)}\tilde{J}^+_{(r)} = \frac{1}{4}\sum_r^N \psi^{+1}_{(r)}\psi^{+2}_{(r)}\tilde{\psi}^{+1}_{(r)}\tilde{\psi}^{+2}_{(r)} \tag{6.35}
$$

and descendants we will be denoted with $\Omega^{a\dot{a}}$. The associated multi-trace operator is

$$
\sum_{r\neq s}^N J^a_{(r)}\tilde{J}^{\dot{a}}_{(s)} \tag{6.36}
$$

These operators have $(h,\bar{h}) = (j,\bar{j}) = (1,1)$.
Consider now the twist fields. The operator $\Sigma_3^{a,\dot{a}}$ has dimension $(h,\bar{h}) = (1,1)$; in terms of the permutation group, this field acts as a 3-cycle on a strand configuration: it can glue into an effective string of length k three strands of winding k_1, k_2 and $k_3 = k-k_1-k_2$.
We should also consider the associated multi-trace: this is given by the product of two $k=2$ twist operators:

$$
\sum_{r\neq s\neq p\neq q}(\Sigma_{(rp)}\Sigma_{(sq)})^{a,\dot{a}} \tag{6.37}
$$

The constraints $r<p$ and $s<q$, as in eq (4.56), are understood in the summation. The summation restrictions can be understood recalling that the operator $\Sigma_{(rp)}$ act as a 2-cycle from the point of view of the permutation group: the above summation means that, when dealing with the multi-trace operator, we have to consider disjoint 2-cycles. If the 2-cycles had a copy index in common, then they would represent a 3-cycle and the operator would fall into the category of Σ_3; if the 2-cycles had

both copy indices in common, instead, they would just be the identity permutation. Again, we should restrict our attention on the highest weight state

$$\sum_{r\neq s\neq p\neq q}(\Sigma_{(rp)}\Sigma_{(sq)})^{11}=\sum_{r\neq s\neq p\neq q}\Sigma^{++}_{(rp)}\Sigma^{++}_{(sq)} \tag{6.38}$$

and descendants.

We now discuss the holographic dictionary for these operators. In [40], with reference to eq. (6.4), it has been proposed the following ansatz

$$\begin{pmatrix}\langle\Sigma_3^I\rangle\\ \langle\Omega^I\rangle\end{pmatrix}=\frac{n_1n_5}{4\pi}\begin{pmatrix}\sqrt{6}(f^1_{2I}-f^5_{2I})\\ \sqrt{2}(-(f^1_{2I}+f^5_{2I})+8a^{\alpha-}a^{\beta+}f_{I\alpha\beta})\end{pmatrix} \tag{6.39}$$

In [45], with reference to eq.s (1.6) and (1.7), the above dictionary has been modified, taking into account the mixing between the single-trace and multi-trace operators and claiming that the r.h.s. of (6.39) should be rotated by some $SO(2)$ matrix.
Following the above suggestions, we impose the ansatz:

$$\begin{aligned}&\begin{pmatrix}\langle\alpha\frac{1}{N}\Sigma_3^{a,\dot a}+\beta\frac{1}{N^2}\sum\limits_{r\neq s\neq p\neq q}(\Sigma_{(rp)}\Sigma_{(sq)})^{a,\dot a}+\gamma\Omega^{a,\dot a}+\delta\frac{1}{N}\sum\limits_{r\neq s}(J_{(r)}\tilde J_{(s)})^{a,\dot a}\rangle\\ \langle\tilde\alpha\frac{1}{N}\Sigma_3^{a,\dot a}+\tilde\beta\frac{1}{N^2}\sum\limits_{r\neq s\neq p\neq q}(\Sigma_{(rp)}\Sigma_{(sq)})^{a,\dot a}+\tilde\gamma\Omega^{a,\dot a}+\tilde\delta\frac{1}{N}\sum\limits_{r\neq s}(J_{(r)}\tilde J_{(s)})^{a,\dot a}\rangle\end{pmatrix}=\\ &=\begin{pmatrix}\sqrt{6}(f^1_{2(-a,-\dot a)}-f^5_{2(-a,-\dot a)})\\ \sqrt{2}(-(f^1_{2(-a,-\dot a)}+f^5_{2(-a,-\dot a)})+8a^{\alpha-}a^{\beta+}f^{(2)}_{(-a,-\dot a)\alpha\beta})\end{pmatrix}\equiv\begin{pmatrix}g_{(-a,-\dot a)}\\ \tilde g_{(-a,-\dot a)}\end{pmatrix}\end{aligned} \tag{6.40}$$

where α, β, γ,δ, $\tilde\alpha$, $\tilde\beta$, $\tilde\gamma$, $\tilde\delta$ are constants (that we expect to depend only the moduli of the theory) to be determined and $f^{(2)}_{(a\dot a)\alpha\beta}$ is defined in (B.12). The $\frac{1}{N}$ factors in front of the operators makes them to have the same large N scaling behavior.
This is the most general ansatz that generalize [40] and includes the suggestions in [45]. The linear combination on the l.h.s. not only takes into account all the possible mixing among the single- and multi-trace operators, but, at the same time, includes all the possible rotated versions of the r.h.s.. In fact, rotating the r.h.s. by some $SO(2)$ matrix $\mathcal{M}$ is equivalent to rotating the l.h.s. by the inverse matrix $\mathcal{M}^{-1}$ which just changes the coefficients of the linear combination.
Note from eq.s (6.5) and (6.40) that the operator O_2 does not mix with the twist operator Σ_3 and Ω (and the associated multi-trace operators). The reason is that the operator O_2 is associated with the profile $g_5(v)$ which selects a privileged direction on the T^4: the only one that does not break invariance under rotations in the T^4 direction.

The strategy to fix constrains on the unknown coefficients is analogous to the one to fix the dictionary (6.5): we will compute the VeVs of the dimension two operators and compare them with the geometry expansion. The action of the double-traces, however, is subtle and we will not consider it in this thesis: we will examine only states and angular momentum components in which the double-trace operators cannot play any role, as we have done in Section 6.2.1. This will be enough in order to fix constrains on the coefficients α, γ, $\tilde\alpha$ and $\tilde\gamma$ in (6.40).

6.3.1 The state composed of strands of type $|{++}\rangle_1$ and $|{--}\rangle_1$

We consider a state that allows for a fixing of γ and $\tilde\gamma$: the simples case is obtained considering strands of type $|{++}\rangle_1$ and $|{--}\rangle_1$: the coherent state is

$$\psi(A,B)=\sum_{p=1}^{N}(A_+|{++}\rangle_1)^{N-p}(A_-|{--}\rangle_1)^p \tag{6.41}$$

where we have renamed $A_1^{(++)} \equiv A_+$, $A_1^{(--)} \equiv A_-$ and $N_1^{--} = p$ so that $N_1^{++} = N - p$. The average degeneracy of the strands in terms of the dimensionless coefficients A_+ and A_- is

$$\bar{p} = |A_-|^2 \qquad N - \bar{p} = |A_+|^2 \tag{6.42}$$

The general discussion in Section 5.2.1 implies that the geometry dual to the state $\psi(A, B)$ is generated by the profiles

$$h_1(v) \equiv g_1(v) + i g_2(v) = a_+ e^{\frac{2\pi i v}{L}} + a_- e^{-\frac{2\pi i v}{L}} \qquad h_2(v) = g_5(v) = 0 \tag{6.43}$$

where

$$A_+ = R\sqrt{\frac{N}{Q_1 Q_5}}\, a_+ \qquad A_- = R\sqrt{\frac{N}{Q_1 Q_5}}\, a_- \tag{6.44}$$

Using eq. (5.10) one gets that at the first non trivial order in the expansion

$$\begin{aligned}
Z_1 &\sim \frac{Q_1}{r^2}\Big(1 - \frac{R^2}{Q_1 Q_5}\frac{(a_+^4 + a_-^4)\cos 2\theta}{r^2} + \frac{2a_+ a_- \cos 2\phi \sin^2\theta}{r^2}\Big)\\
Z_2 &\sim \frac{Q_5}{r^2}\Big(1 - \frac{R^2}{Q_1 Q_5}\frac{(a_+^4 + 2a_+^2 a_-^2 + a_-^4)\cos 2\theta}{r^2} + \frac{4a_+ a_- \cos 2\phi \sin^2\theta}{r^2}\Big)\\
A_1 &\sim -\frac{Q_5}{r^2}\Big(\frac{R(a_+^2 - a_-^2) Y_1^2}{2Q_5 r}\Big) \qquad A_2 \sim \frac{Q_5}{r^2}\Big(\frac{R(a_+^2 - a_-^2) Y_1^1}{2Q_5 r}\Big) \qquad A_3 = A_4 = 0
\end{aligned} \tag{6.45}$$

The non-vanishing coefficients extracted from this geometry and encoding the VeVs of the operators are

$$\begin{aligned}
f^1_{2,00} &= -\frac{R^2}{Q_1 Q_5}\frac{(a_+^4 + a_-^4)}{\sqrt{3}} \qquad & f^5_{2,00} &= -\frac{R^2}{Q_1 Q_5}\frac{(a_+^4 + 2a_+^2 a_-^2 + a_-^4)}{\sqrt{3}}\\
f^1_{2,+1+1} &= f^1_{2,-1-1} = \frac{a_+ a_-}{\sqrt{3}} \qquad & f^5_{2,+1+1} &= f^5_{2,-1-1} = \frac{2a_+ a_-}{\sqrt{3}}\\
a^{3\pm} &= \pm\frac{R}{\sqrt{Q_1 Q_5}}\frac{a_+^2 - a_-^2}{2}
\end{aligned} \tag{6.46}$$

and it is straightforward to compute

$$\begin{aligned}
g_{(00)} &= 2\sqrt{2}\frac{R^2}{Q_1 Q_5} a_+^2 a_-^2 \qquad & \tilde{g}_{(00)} &= 2\sqrt{6}\frac{R^2}{Q_1 Q_5} a_+^2 a_-^2\\
g_{(\pm1\pm1)} &= -\sqrt{2} a_+ a_- \qquad & \tilde{g}_{(\pm1\pm1)} &= -\sqrt{6} a_+ a_-
\end{aligned} \tag{6.47}$$

Let's now discuss the CFT side. The expectation value of Σ_3 is zero because all the strands have winding $k = 1$. The VeV of $\Sigma_2\Sigma_2$ is zero as well: the process that glues two singly wound strands into a strand of winding 2 and then splits it back into two singly wound strands is just the identity permutation, and we do not consider it.

Among the operators $\Omega^{a,\dot{a}}$ and $\frac{1}{N}\sum\limits_{r\neq s}(J_{(r)}\tilde{J}_{(s)})^{a,\dot{a}}$, for angular momentum conservation, the $a, \dot{a} = 0$ component have non trivial VeV. Moreover, the double-trace $\sum_{r\neq s}\big(J_{(r)}\tilde{J}_{(s)}\big)^{\pm1,\pm1}$ has a vanishing expectation value: the reason is that the two current operators have to act on different copies of the CFT, and there are no strands of type $|+-\rangle_1$ or $|-+\rangle_1$ in $\psi(A, B)$. We shall thus focus on the angular momentum components $(a, \dot{a} = \pm1)$: in this case, only Ω has a non trivial VeV and this will allow us to fix γ and $\tilde{\gamma}$.

The operator $\Omega^{+1,+1}$ transforms a $|--\rangle_1$ strand into $|++\rangle_1$, giving the main contribution:

$$\Omega^{+1,+1}\big((|++\rangle_1)^{N-p}(|--\rangle_1)^p\big) = (N - p + 1)\big((|++\rangle_1)^{N-p+1}(|--\rangle_1)^{p-1}\big) \tag{6.48}$$

The combinatorial factor rises observing that $\Omega^{+1,+1}$ can act on any of the p copies of $|--\rangle_1$ in the state $((|++\rangle_1)^{N-p}(|--\rangle_1)^p)$, which is made of $\mathcal{N}(p)$ terms, and imposing the matching of the total number of terms on the l.h.s and on the r.h.s.. This process gives rise to the VeV

$$\langle\Omega^{1,1}\rangle = \frac{(N-\bar{p})A_-}{A_+} = A_-\bar{A_+} = \frac{R^2 N}{Q_1 Q_5} a_+ a_- \tag{6.49}$$

Using the dictionary (6.40), along with eq.s (6.47) and (6.49) one obtains

$$\gamma = -\sqrt{2}\frac{Q_1 Q_5}{R^2 N} \qquad \tilde{\gamma} = -\sqrt{6}\frac{Q_1 Q_5}{R^2 N} \tag{6.50}$$

6.3.2 Switching on the VeV of Σ_3

The operator $\Sigma_3^{a\dot{a}}$ glues three strands of windings k_1, k_2 and k_3 into a strand of winding $k = k_1+k_2+k_3$; if, for instance, we focus on the spin component $(a,\dot{a}) = (-1,-1)$, the following process can occur

$$\Sigma_3^{-1,-1}|++\rangle_{k_1}|++\rangle_{k_2}|++\rangle_{k_3} = c_{k_1k_2k_3}|++\rangle_{k_1+k_2+k_3} \tag{6.51}$$

where $c_{k_1k_2k_3}$ is some unknown factor that depends on the windings of the initial strands. In the case of the operator Σ_2, the analogous coefficient $c_{k_1k_2}$ is given by $c_{k_1k_2} = \frac{k_1+k_2}{2k_1k_2}$; note that, in the case of singly wound strands, $c_{11} = 1$ and thus its effect was invisible in the computations in Section 6.2.1.

We now analyze the simplest 2-charge state in which the VeV of $\Sigma_3^{a\dot{a}}$ is non-zero: the one composed of strands of type $|++\rangle_1$ and $|++\rangle_3$. This will allow us to fix constrains on α and $\tilde{\alpha}$ in the dictionary (6.40). Because the value of $c_{k_1k_2k_3}$ is for us unknown, we will just keep track of it. Renaming $A_1^{(++)} \equiv A$, $A_3^{(++)} \equiv B$ and $N_3^{(++)} \equiv p$, we consider the coherent state

$$|s\rangle = \sum_{p=1}^{\frac{N}{3}} (A|++\rangle_1)^{N-3p}(B|++\rangle_3)^p \tag{6.52}$$

where we have used the constrain (5.13) to fix $N_1^{(++)} \equiv N-3p$. The average number of strands of each type in this case reads

$$\bar{N}_1^{(++)} = N - 3\bar{p} = |A|^2 \qquad \bar{N}_3^{(++)} = \bar{p} = \frac{|B|^2}{3} \tag{6.53}$$

Let's first consider the r.h.s. of eq. (6.40). The geometry dual to the state (6.52) can be constructed using the following profiles as seeds:

$$h_1(v) \equiv g_1(v) + i g_2(v) = a e^{\frac{2\pi i v}{L}} + \frac{b}{3} e^{\frac{6\pi i v}{L}} \qquad h_2(v) = g_5(v) = 0 \tag{6.54}$$

where the coefficients a and b are taken to be real and are related to the coefficients A and B through (5.17). Using the general result in (5.10), we obtain that the relevant functions to the r.h.s. of the ansatz (6.40) are given by

$$\begin{aligned}
Z_1 \sim& \frac{Q_1}{r^2}\Big(1 - \frac{R^2}{Q_1Q_5}\big(\frac{(9a^4+16a^2b^2+b^4)\cos(2\theta) + 18a^3b\cos(2\phi)\sin^2\theta}{9r^4}\big)\Big) \\
Z_2 \sim& \frac{Q_5}{r^2}\Big(1 - \frac{R^2}{Q_1Q_5}\big(\frac{(9a^4+10a^2b^2+b^4)\cos(2\theta)}{9r^4}\big)\Big) \\
A_1 \sim& -\frac{Q_5}{r^2}\Big(\frac{R(3a^2+b^2)Y_1^2}{6Q_5 r}\Big) \qquad A_2 \sim \frac{Q_5}{r^2}\Big(\frac{R(3a^2+b^2)Y_1^1}{6Q_5 r}\Big) \qquad A_3 = A_4 = 0
\end{aligned} \tag{6.55}$$

Hence, one reads out:

$$\begin{aligned} f^1_{2,00} &= -\frac{R^2}{Q_1Q_5}\frac{(9a^4+16a^2b^2+b^4)}{9\sqrt{3}} \qquad f^5_{2,00} = -\frac{R^2}{Q_1Q_5}\frac{(9a^4+10a^2b^2+b^4)}{9\sqrt{3}} \\ f^1_{2,+1+1} &= f^1_{2,-1-1} = -\frac{R^2}{Q_1Q_5}\frac{a^3b}{\sqrt{3}} \qquad a^{3\pm} = \pm\frac{R}{\sqrt{Q_1Q_5}}\frac{3a^2+b^2}{6} \end{aligned} \tag{6.56}$$

The coefficients that encodes the VeVs of the operators are

$$\begin{aligned} g_{(00)} &= -6\sqrt{2}\frac{R^2}{Q_1Q_5}a^2b^2 \qquad \tilde{g}_{(00)} = \frac{14\sqrt{6}}{27}\frac{R^2}{Q_1Q_5}a^2b^2 \\ g_{(11)} &= g_{(-1-1)} = -\sqrt{2}\frac{R^2}{Q_1Q_5}a^3b \qquad \tilde{g}_{(11)} = \tilde{g}_{(-1-1)} = \frac{\sqrt{2}}{\sqrt{3}}\frac{R^2}{Q_1Q_5}a^3b \end{aligned} \tag{6.57}$$

By angular momentum conservation, we conclude that the only processes that contribute to the VeVs of $\Sigma_3^{a\dot{a}}$ are the one in eq. (6.51) and its inverse. Considering the $(-1,-1)$ component one gets

$$\Sigma_3^{-1,-1}\big((|{+}{+}\rangle_1)^{N-3p}(|{+}{+}\rangle_3)^p\big) = c_{111}\frac{p+1}{2}\big((|{+}{+}\rangle_1)^{N-3p-3}(|{+}{+}\rangle_3)^{p+1}\big) \tag{6.58}$$

The operator $\Sigma_3^{-1,-1}$ can act on any group of 3 strands of length 1 out of $N-3p$ and, thus, carries a $\binom{N-3p}{3}$ combinatorial factor; as usual, the factor $\frac{p+1}{2}$ arises requiring that the total number of terms on the two sides of (6.58) match each other. This gives rise to the VeV

$$\langle\Sigma_3^{-1,-1}\rangle = c_{111}\frac{A^3}{B}\frac{\bar{p}}{2} = c_{111}\frac{A^3\bar{B}}{6} = c_{111}\frac{R^4N^2}{6(Q_1Q_5)^2}a^3b \tag{6.59}$$

Using the ansatz (6.40), along with eq.s (6.57) and (6.59), one obtains the following constrains on the coefficients α and $\tilde{\alpha}$

$$\alpha c_{111} = -6\sqrt{2}\frac{Q_1Q_5}{R^2N} \qquad \tilde{\alpha}c_{111} = 2\sqrt{6}\frac{Q_1Q_5}{R^2N} \tag{6.60}$$

Conclusions

In this work we have studied some of the recent progresses that have been made in understanding black hole physics in string theory. We have devoted particular attention to the holographic tools that the AdS/CFT conjecture provides in this sense, discussing the dictionary between $\frac{1}{4}$-BPS and $\frac{1}{8}$-BPS gravitational solutions and states of the dual CFT, the dictionary between VeVs and geometry expansion for dimension 1 operators and its extension to operators of dimension 2.

We first discussed the basic features of black holes in General Relativity, their thermodynamic behavior and the puzzles that arise when one takes this behavior seriously (and Hawking's computation suggests that, indeed, we should): the entropy puzzle and the informations paradox. We have also discussed why their solution requires a formulation of a consistent theory of quantum gravity.
After reviewing some of the main properties of string theory, one of the leading candidates as a theory of quantum gravity, and supergravity, its low energy limit, we discussed how to deal with black hole physics in these theories. With this point of view, black holes are bound states of branes and strings, possibly carrying momentum. Restricting our attention to supersymmetric solutions, we have shown how string theory allows an exact matching between Bekenstein-Hawking entropy and the microscopic degeneracy of the microstates. The discussion of the properties of the microstates led us to the fuzzball proposal, a promising idea which has the potential to resolve the longstanding puzzles related to black hole physics.
We have discussed some of the recent results that support the interpretation of the fuzzball geometries as black holes microstates and the holographic technology provided by the AdS/CFT conjecture that could further corroborate the fuzzball proposal. With this aim, we have introduced the conformal field theory relevant for the D1-D5 microstates, the D1D5 CFT, at a point in the moduli space, the free orbifold point, in which it is just a collection of free bosons and fermions. Even thought this solvable point is distant from the regions of the moduli space admitting a low-energy supergravity description (so that in general one cannot relate states at the free orbifold point with supergravity solutions), one can take advantage of the fact that certain quantities are protected by supersymmetry: this allows to gain informations about the gravitational physics from the CFT side of the duality. We have thus provided the dictionary between microstate geometries and particular CFT states in the RR sector of the CFT (both for $\frac{1}{4}$-BPS and some $\frac{1}{8}$-BPS microstates) and we have discussed how the geometry expansion around $AdS_3 \times S^3$ of these microstates encode the VeVs of dimension 1 operators in the dual CFT state.
Finally, we have analyzed what predictions can be made for VeVs of higher dimension operators. Having an analytic control of higher dimension operators would be important because they give informations on higher order corrections around $AdS_3 \times S^3$: indeed, many microstates have trivial VeVs of operators of dimension 1. In particular, we have restricted our attention to operators of dimension 2. This case involves some technical difficulties compared to the case of operators of dimension 1, because of the mixing between dimension 2 operators. We have avoided part of these difficulties by considering particular CFT states in which the VeVs of the double-trace operators vanish: this has allowed us to give constrains on the coeffients in front of the single-trace operators in the ansatz (6.40). Nonetheless, this restriction is an evident limit of our results: it would be very interesting to extend this discussion in order to prove the full linear combination in (6.40).

To conclude, the fuzzball proposal is a concrete idea that has the potential to shed light upon black hole issues; however, substantial work and technical progress is still needed to show its validity. For example, it would we important to understand how the black hole thermal properties emerge from the microstates, i.e. how to coarse-grain geometries. Moreover, we shall note that the microstates we have considered are well described in supergravity: however, as we have already emphasized, we expect that they are only a subset of the fuzzball solutions. To develop the fuzzball proposal further one should also understand more deeply the stringy fuzzballs.

Appendix A

Conformal Field Theories

Symmetries play a major role in our understanding of physics. So far we have encountered field theories that are invariant under the scale transformation $x^\mu \to \lambda x^\mu$. Examples are string theory and classical Yang-Mills theories in 4 dimensions. The importance of scale transformations is evident in the study of critical phenomena and it leads to the renormalization group equations. Generally, this invariance does not extend to the quantum theory because renormalization introduces a scale dependence and the parameters of the theory (such as the coupling constant) run with the energy scale; however there are exceptions, such as $D = 4$ $N = 4$ super-Yang-Mills theories.

Under a scale transformation, the metric rescales as $g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = |\lambda|^2 g_{\mu\nu}(x)$. The generalization of this transformation leads to the conformal group. In this Appendix we will review some basic aspects of Conformal Field Theories (CFTs), i.e. theories that are invariant under the action of the conformal group. In particular, we will focus on theories in 2-dimensions. The content of this Appendix is far from being exhaustive as we will just report some of the features used in the thesis; a standard reference is [39].

A.1 CFT in 2 dimensions

The conformal group is the group of transformations that leave invariant the metric up to some local rescaling

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega^2(x) g_{\mu\nu}(x) \tag{A.1}$$

It can be viewed as the minimal extension of the Poincaré group containing the inversion symmetry $x^\mu \to \frac{x^\mu}{x^2}$. In D dimensions, the conformal group is generated by the scale transformation and the special conformal transformations

$$x^\mu \to \lambda x^\mu \qquad x^\mu \to \frac{x^\mu + a^\mu x^2}{1 + 2a_\nu x^\nu + a^2 x^2} \tag{A.2}$$

and the Poincaré transformations. The generators of the conformal group form an algebra isomorphic to $SO(D, 2)$.

The case D=2 is special: if we start with euclidean coordinates (x^1, x^2) and define the complex coordinates $z, \bar{z} = x^1 \pm ix^2$, the two dimensional conformal transformations coincide with the analytic coordinate transformations

$$z \to f(z) \qquad \bar{z} \to \bar{f}(\bar{z}) \tag{A.3}$$

In fact, under (A.3) the metric transforms as

$$ds^2 = (dx^1)^2 + (dx^2)^2 = dz d\bar{z} \to \left|\frac{\partial f}{\partial z}\right|^2 dz d\bar{z} \tag{A.4}$$

and we recognize the scale factor in $\Omega^2 = \left|\frac{\partial f}{\partial z}\right|^2$. The local algebra that generates (A.3) is infinite dimensional, with generators

$$L_n = -z^{z+1}\partial_z \qquad \bar{L}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \tag{A.5}$$

They satisfy the algebra

$$[L_n, L_m] = (m-n)L_{m+n} \qquad [\bar{L}_n, \bar{L}_m] = (m-n)\bar{L}_{m+n} \qquad [L_n, \bar{L}_m] = 0 \tag{A.6}$$

The generators of the local conformal algebra (A.5) are not well defined globally: in order to form a group, the mappings (A.3) must be invertible and must map the hole complex plane (or more precisly, its compactification obtained adding the point at infinity) into itself. Because the conformal transformations are generated by the vector fields $v(z) = -\sum_n a_n L_n = \sum_n a_n z^{n+1}\partial_z$, the non singularity of $v(z)$ at $z = 0$ requires $a_n = 0$ for $n < -1$. Analogously, one can investigate the behaviour of $v(z)$ as $z \to \infty$ performing the transformation $z = -\frac{1}{w}$, and one obtains $a_n = 0$ for $n > -1$. The same restrictions apply to antiholomorphic transformations. The globally defined algebra (restricted conformal algebra) is thus generated by $\{L_{0,\pm}, \bar{L}_{0,\pm}\}$. We can identify L_{-1}, $\bar{L}_{-1}$ as the generators of translations; $L_0 + \bar{L}_0$ and $i(L_0 - \bar{L}_0)$ as the generators of dilatations and rotations respectively and L_1, $\bar{L}_1$ as the generators of special conformal transformations. The finite form of these transformations is

$$z \to \frac{az+b}{cz+d} \qquad ad - bc = 1 \quad a, b, c, d \in \mathbb{C} \tag{A.7}$$

These are called projective transformations, and are in one to one correspondence with invertible 2×2 complex matrices with unit determinant. Thus, the global conformal transformation is isomorphic to $SL(2, \mathbb{C})$.

The global conformal algebra is useful to characterize the physical properties of the states. A conformal field theory contains an infinite set of fields, among which there are also derivatives of other fields. A physically interesting representation of the conformal group involve fields (or operators) that are eigenfunctions of the operators L_0 and $\bar{L}_0$, with eigenvalues h and $\bar{h}$ respectively: they are the so-called conformal dimensions (or conformal weight). We call "quasi-primary field", a field that under a global conformal transformation (A.7) transforms as

$$\phi(z, \bar{z}) \to \left(\frac{\partial z'}{\partial z}\right)^h \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{\bar{h}} \phi(z', \bar{z}') \tag{A.8}$$

A field whose variation under any local conformal transformation is given by (A.8) is called "primary field".
It can be shown that this transformation property gives constrains on the structure of the n-point functions of the fields. For example, the 2-point function for a scalar field with conformal dimensions h and $\bar{h}$ is

$$\langle \phi(z, \bar{z})\phi(0, 0)\rangle = \frac{C_{12}}{z^{2h}\bar{z}^{2\bar{h}}} \tag{A.9}$$

A.2 Quantization

The quantization procedure usually starts with Euclidean "space" and "time" coordinates[1] (τ, σ). We compactify the space direction $\sigma = \sigma + 2\pi$: this defines a cylinder. Next we consider the conformal map

$$z = e^{(\tau + i\sigma)} \qquad \bar{z} = e^{(\tau - i\sigma)} \tag{A.10}$$

[1]The distinction between time and space is natural in Minkowsky space-time, but somewhat arbitrary in Euclidean space-time.

which maps the cylinder to the compactified complex plane. Surfaces of equal time τ on the cylinder become circles of equal radii on the complex plane; in particular the infinite past ($\tau = -\infty$) gets mapped into the origin on the plane ($z = 0$) while the infinite future becomes $z = \infty$; time reversal on the complex plane becomes $z \to \frac{1}{z^*} (= e^{(-\tau + i\sigma)})$.
We have seen that the operator $L_0 + \bar{L}_0$ generates dilatations on the complex plane, which correspond to time translation on the cylinder, and that $i(L_0 - \bar{L}_0)$ generates rotations on the plane, which correspond to spatial translation on the cylinder. Thus, we can identify the latter with the momentum operator and the former can be regarded as the Hamiltonian for the system, and the Hilbert space is built up on surfaces of constant radius. This procedure for defining a quantum theory on the plane is known as radial quantization. Quantization assumes the existence of a vacuum state $|0\rangle$ upon which the Hilbert space is constructed by application of creation operators.

For an interacting quasi primary field ϕ, the Hilbert space of the CFT can be described with the standard formalism of in- and out-states of QFT: we assume that the interaction is attenuated as $\tau \to \pm\infty$, so that the asymptotic field $\phi_{\text{in}} \propto \lim_{\tau \to \infty} \phi(\tau, \sigma)$ is free. With radial quantization, $\tau \to -\infty$ corresponds to $z \to 0$, so we define a in-state as

$$|\phi_{\text{in}}\rangle = \lim_{z, \bar{z} \to 0} \phi_{z, \bar{z}} |0\rangle \tag{A.11}$$

In order to compute amplitudes, we must define a scalar product on the Hilbert space: this can be done defining asymptotic out-states, together with hermitian conjugation. As time reversal on the complex plane is $z \to \frac{1}{z^*}$, we define the hermitian conjugation as

$$\phi(z, \bar{z})^\dagger = \bar{z}^{-2h} z^{-2\bar{h}} \phi(\frac{1}{\bar{z}}, \frac{1}{z}) \tag{A.12}$$

With this choice the out-state $\langle\phi_{\text{out}}| = |\phi_{\text{in}}\rangle^\dagger$ has a well defined inner product with $|\phi_{\text{in}}\rangle$, consistent with eq. (A.9).

Appendix B

Spherical harmonics S^3

The spherical harmonics on S^3 are a representation of the isometry group of the three-sphere $SO(4) \simeq SU(2)_L \times SU(2)_R$. We will use spherical coordinates on the noncompact space, that are related to the Cartesian coordinates via

$$\begin{aligned} x^1 &= r \sin\theta \cos\phi \qquad x^2 = r \sin\theta \sin\phi \\ x^3 &= r \cos\theta \cos\psi \qquad x^4 = r \cos\theta \sin\psi \end{aligned} \tag{B.1}$$

where $\theta \in [0, \frac{\pi}{2}]$ and $\psi, \phi \in [0, 2\pi]$. With this coordinate choice, the line element ds_3^2 and the volume form η_3 on S^3 are

$$ds_3^2 = d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2 \qquad \eta_3 = \sin\theta\cos\theta d\theta \wedge d\phi \wedge d\psi \tag{B.2}$$

A degree k scalar harmonic transforms in the $(\frac{k}{2}, \frac{k}{2})$ representation of $SU(2)_L \times SU(2)_R$ and we will denote it with $Y_k^{m,\tilde{m}}$, where $(m, \tilde{m})$ are the spin charges under $(J^3, \tilde{J}^3)$. They satisfy the following equation

$$\Box_{S^3} Y_k^{m,\tilde{m}} = -k(k+2) Y_k^{m,\tilde{m}} \tag{B.3}$$

We use normalized spherical harmonics such that

$$\int Y_{k_1}^{*m_1,\tilde{m}_1} Y_{k_2}^{m_2,\tilde{m}_2} = \Omega_3 \delta_{k_1,k_2} \delta^{m_1,m_1} \delta^{\tilde{m}_1,\tilde{m}_2} \tag{B.4}$$

where $\Omega_3 = 2\pi^2$ is the volume of the three sphere.
The generators of the isometry group of S^3, written in terms of the standard $SU(2)$ generators, are

$$\begin{aligned} J^\pm &= \frac{1}{2} e^{\pm i(\phi+\psi)} (\pm\partial_\theta + i\cot\theta\partial_\phi - i\tan\theta\partial_\psi) \qquad J^3 = -\frac{i}{2}(\partial_\phi + \partial_\psi) \\ \tilde{J}^\pm &= \frac{1}{2} e^{\pm i(\phi-\psi)} (\pm\partial_\theta + i\cot\theta\partial_\phi + i\tan\theta\partial_\psi) \qquad \tilde{J}^3 = -\frac{i}{2}(\partial_\phi - \partial_\psi) \end{aligned} \tag{B.5}$$

One can generate the degree k spherical harmonics acting with the raising and lowering operators in (B.5) on the higher spin, degree k spherical harmonic which is

$$Y_k^{\pm\frac{k}{2},\pm\frac{k}{2}} = \sqrt{k+1} \sin^k\theta e^{\pm ik\phi} \tag{B.6}$$

In this thesis we make use of the degree $k = 1, 2$, normalized spherical harmonics, that are:

$$\begin{aligned} Y_1^{+\frac{1}{2},+\frac{1}{2}} &= \sqrt{2}\sin\theta e^{i\phi} \qquad Y_1^{+\frac{1}{2},-\frac{1}{2}} = -\sqrt{2}\cos\theta e^{i\psi} \\ Y_1^{-\frac{1}{2},+\frac{1}{2}} &= -\sqrt{2}\cos\theta e^{-i\psi} \qquad Y_1^{-\frac{1}{2},-\frac{1}{2}} = -\sqrt{2}\sin\theta e^{-i\phi} \end{aligned} \tag{B.7}$$

$$
\begin{aligned}
Y_2^{+1,+1} &= \sqrt{3}\sin^2\theta e^{2i\phi} & Y_2^{+1,0} &= -\sqrt{6}\sin\theta\cos\theta e^{i(\phi+\psi)} \\
Y_2^{+1,-1} &= \sqrt{3}\cos^2\theta e^{2i\psi} & Y_2^{0,+1} &= -\sqrt{6}\sin\theta\cos\theta e^{i(\phi-\psi)} \\
Y_2^{0,0} &= \sqrt{3}\cos 2\theta & Y_2^{0,-1} &= \sqrt{6}\sin\theta\cos\theta e^{-i(\phi-\psi)} \\
Y_2^{-1,+1} &= \sqrt{3}\cos^2\theta e^{-2i\psi} & Y_2^{-1,0} &= \sqrt{6}\sin\theta\cos\theta e^{-i(\phi+\psi)} \\
Y_2^{-1,-1} &= \sqrt{3}\sin^2\theta e^{-2i\phi}
\end{aligned}
\tag{B.8}
$$

When useful, we use the cartesian representation of the spherical harmonics instead of the standard $SU(2)$ representation:

$$
Y_k^{m_k,\tilde{m}_k} \leftrightarrow Y_k^{I_k} \tag{B.9}
$$

where $Y_k^{I_k}$ is a (complex) linear combination of $\frac{x^{(i_1}\dots x^{i_k)}}{r^k}$, where the indices are not only symmetric but also traceless (in the sense that tracing over any pair of indices gives zero.) The normalized, degree 1 spherical harmonics in the cartesian representation are given by $Y_1^i = 2\frac{x^i}{r}$, and the change of basis reads:

$$
\begin{aligned}
Y_1^{+\frac{1}{2},+\frac{1}{2}} &= \frac{Y^1 + iY^2}{\sqrt{2}} & Y_1^{+\frac{1}{2},-\frac{1}{2}} &= -\frac{(Y^3 + iY^4)}{\sqrt{2}} \\
Y_1^{-\frac{1}{2},+\frac{1}{2}} &= -\frac{(Y^3 - iY^4)}{\sqrt{2}} & Y_1^{-\frac{1}{2},-\frac{1}{2}} &= -\frac{Y^1 - iY^2}{\sqrt{2}}
\end{aligned}
\tag{B.10}
$$

We also introduce degree 1 vector spherical harmonics $Y_1^{\alpha\pm}$

$$
\begin{aligned}
Y_1^{\alpha+} &= \frac{\eta^\alpha_{ij} dx^i x^j}{r^2} & Y_1^{\alpha-} &= \frac{\bar{\eta}^\alpha_{ij} dx^i x^j}{r^2} \\
\eta^\alpha_{ij} &= \delta_{\alpha i}\delta_{4j} - \delta_{\alpha j}\delta_{4i} + \epsilon_{\alpha ij4} & \bar{\eta}^\alpha_{ij} &= \delta_{\alpha i}\delta_{4j} - \delta_{\alpha j}\delta_{4i} - \epsilon_{\alpha ij4}
\end{aligned}
\tag{B.11}
$$

where $\alpha = 1, 2, 3$ is an index in the adjoint of $SU(2)$.
We define the following triple integrals:

$$
\int \left(Y_1^{\alpha\pm}\right)^a Y_1^j D_a Y_1^i = \Omega_3 e^{\pm}_{\alpha ij} \qquad \int Y_k^{(m_k,\tilde{m_k})}\left(Y_1^{\alpha-}\right)_a \left(Y_1^{\beta+}\right)^a = \Omega_3 f^{(k)}_{(m_k,\tilde{m_k})\alpha\beta} \tag{B.12}
$$

where a is an S^3 index and D_a is the covariant derivative with respect to the metric (B.2). The explicit values for the overlaps $e^{\pm}_{\alpha ij}$ are

$$
\begin{aligned}
&e^+_{312} = -1 \quad e^-_{312} = 1 \quad e^+_{334} = -1 \quad e^-_{334} = -1 \quad e^+_{113} = 1 \quad e^-_{113} = 1 \\
&e^+_{124} = -1 \quad e^-_{124} = 1 \quad e^+_{214} = -1 \quad e^-_{214} = -1 \quad e^+_{223} = 1 \quad e^-_{223} = -1
\end{aligned}
\tag{B.13}
$$

Note that $e^{\pm}_{\alpha ij} = -e^{\pm}_{\alpha ji}$. The explicit value of the components of $f^{(k)}_{(m_k,\tilde{m_k})\alpha\beta}$, defined in (B.12), that have been used in this thesis are

$$
f^{(2)}_{(00)33} = \frac{1}{\sqrt{3}} \qquad f^{(2)}_{(\pm1,\pm1)33} = 0 \tag{B.14}
$$

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