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The Image of the Exponential Map in terms of Jordan Classes

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Chapter 0

Introduction

The purpose of this thesis is to study Jordan classes in the image of the exponential map from the Lie algebra associated to a simple linear algebraic group to the group itself. This will be done for classical complex matrix groups, for complex spin groups and for the complex simple group of type G_2.

The exponential map in Lie theory is a typical tool to understand the group operation of a Lie group by studying its associated Lie algebra. Usually computation is more easily carried out in the Lie algebra and the exponential is a standard mechanism to translate results obtained in the Lie algebra in terms of properties of the Lie group. This procedure is mainly based on the fact that the exponential function maps diffeomorphically a certain open neighborhood of 0 in the Lie algebra to an open neighborhood of the identity in the Lie group.

The surjectivity of the exponential map in the setting of classical Lie groups has been studied since the end of the 19^{th} century. In 1892, Engel and Study proved that the exponential is surjective for the projective linear groups and the general linear groups over $\mathbb{C}$. These results were rediscovered much later when Lai investigated the topic in the context of semisimple Lie groups in [5].
Starting from the seventies the problem gained popularity and, in case of non-surjectivity, the properties of the image were studied. Namely, mathematicians were interested in establishing when the image is dense and which powers of a given element belong to it.
In 1988, Djoković proved a theorem (Theorem 3.0.6) in [15] giving an explicit description of the image of the exponential map in connected complex semisimple Lie groups based on the multiplicative Jordan decomposition.

Jordan classes of a semisimple Lie algebra were first introduced in 1979

by Bohro and Kraft in [1] while studying a conjecture by Dixmier and the corresponding notion for reductive algebraic groups was presented by Lusztig in [7]. The definition of Jordan classes is based on the idea of gathering elements according to their Jordan decomposition.
As an immediate consequence of the above mentioned Djoković's theorem, it turns out that the image of the exponential map is a union of Jordan classes. It is therefore natural to ask which classes lie in the image.
As we shall see, the solution of this problem is closely related to the study of centralizers of unipotent elements and their component group.

We now briefly outline the content of this paper.
After presenting the necessary preliminary results and definitions in Chapter 1, we will introduce Jordan classes and state some of their properties in Chapter 2. We will show how to parametrize classes and point out the main differences between the classes in the Lie algebra and in the algebraic group. Then, in Chapter 3, we will proceed to define and study the exponential map. Here we will present a few general results concerning the Jordan classes in the image. In the following chapters, we will then proceed to analyze the problem case-by-case. Chapter 4 presents the solution in the special linear groups over the complex numbers. In Chapter 5, we will move on to the case of the complex symplectic groups. In Chapters 6 and 7, we will describe which classes lie in the image in the complex special orthogonal groups in odd and even dimension respectively. In Chapter 8 we will deduce a result for the complex spin groups from the study of the special orthogonal groups. Finally, in Chapter 9, the case of the simple complex group of type G_2 will be faced.

The reader is assumed to have some familiarity with linear algebraic groups and their representation theory.

Chapter 1

Preliminaries

1.1 Jordan Decomposition

We start by recalling the additive Jordan decomposition of an endomorphism. For the statements in this part we refer to Section 2.1 in [8].
Recall that an endomorphism x of a finite dimensional vector space V over a field k is semisimple if it is diagonalizable and it is nilpotent if, for some non-negative integer n, $x^n = 0$.

Proposition 1.1.1. *Let $x \in End(V)$, where V denotes a finite dimensional vector space over a field k. There exist unique $x_s, x_n \in EndV$ such that x_s is semisimple and x_n is nilpotent such that $x = x_s + x_n$ and $x_s \circ x_n = x_n \circ x_s$. Moreover, $x_s = P(x)$ and $x_n = Q(x)$, for polynomials $P(T), Q(T) \in k[T]$.*

Definition 1.1.2. Let $u \in End(V)$. If $u - Id_V$ is nilpotent, then u is called unipotent.

We can now recall the multiplicative analogue of the additive Jordan decomposition. For the following statement we refer to Proposition 2.2 in [8]

Proposition 1.1.3. *Let $x \in GL(V)$, then there exist unique $x_s, x_u \in GL(V)$ such that x_s is semisimple and x_u is unipotent and $x = x_s \circ x_u = x_u \circ x_s$.*

Definition 1.1.4. Let $x \in GL(V)$ and x_s, x_u be as in Proposition 1.1.3, then $x = x_s \circ x_u$ is called the (multiplicative) Jordan decomposition of x and x_s and x_u are called the semisimple and the unipotent part of x respectively.

We state in the following theorem the content of Theorem 2.5 in [8].

Theorem 1.1.5. *Let G be an algebraic group. Let ρ be an embedding of G into some $GL(V)$. For $g \in G$, there exist unique $g_s, g_u \in G$ such that $g = g_s g_u = g_u g_s$ and $\rho(g_s)$ is semisimple and $\rho(g_u)$ is unilpotent. Such decomposition $g = g_s g_u$ does not depend on the choice of ρ.*

Definition 1.1.6. Let $g \in G$ and $g = g_s g_u$ as in Theorem 1.1.5. Then $g = g_u g_s$ is called the Jordan decomposition of $g \in G$ and g_s, g_u are the semisimple and unipotent part respectively of g. An element $g \in G$ such that $g = g_s$ is called semisimple, while an element $g \in G$ such that $g = g_u$ is called unipotent.

1.2 Structure of Reductive Groups

Here we will present a theorem on the structure of a connected reductive group G over $\mathbb{C}$, for which we refer to Theorem 8.17 in [8].

Theorem 1.2.1. *Let G be a connected reductive group over $\mathbb{C}$. Let $\mathfrak{g} = Lie(G)$, $T \leq G$ be a maximal torus and Φ the corresponding root system of $[\mathfrak{g}, \mathfrak{g}]$. Then*

(i) $\mathfrak{g} = Lie(T) \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$.

(ii) For each $\alpha \in \Phi$ there exists a morphism of algebraic groups

$$u_\alpha : (\mathbb{C}, +) \longrightarrow G,$$

which induces an isomorphism onto $u_\alpha(\mathbb{C})$ such that $tu_\alpha(a)t^{-1} = u_\alpha(\alpha(t)c)$, for $t \in T, c \in \mathbb{C}$, If u'_α is another morphism with the same property, then for a unique $c \in \mathbb{C}^\times$, we have $u'_\alpha(z) = u_\alpha(cz), \forall z \in \mathbb{C}$.

(iii) $Im(u_\alpha)$ is the unique one-dimensional connected unipotent subgroup which is normalized by T and such that $Lie(Im(u_\alpha)) = \mathfrak{g}_\alpha$.

(iv) $G = \langle T, Im(u_\alpha) | \alpha \in \Phi \rangle$,

Definition 1.2.2. We set $U_\alpha = Im(u_\alpha), \alpha \in \Phi$. U_α is called the root subgroup of G associated to α.

1.3 Levi Subgroups

We now want to introduce Levi subgroups and parabolic subgroups of a reductive group. For this part we refer to Section 12.1 of [8]. Let G be a connected reductive group over $\mathbb{C}$. Let T be a maximal torus in G with associated root system Φ of $[\mathfrak{g}, \mathfrak{g}]$ with basis Δ. Take a subset θ of Δ and set Φ_θ to be the root subsystem of Φ generated by θ. Then

$$P_\theta = \langle T, U_\alpha | \alpha \in \Phi^+ \cup \Phi_\theta \rangle,$$

where Φ^+ is the set of positive roots, is a standard parabolic subgroup of G.

Definition 1.3.1. A parabolic subgroup of G is any subgroup containing a Borel subgroup.

Proposition 1.3.2. *Any parabolic subgroup of G is conjugate to a standard parabolic subgroup.*

We now give a result on the structure of standard parabolic subgroups that will allow us to introduce Levi subgroups.

Proposition 1.3.3. *For P_θ as above, we have that the unipotent radical of P_θ is given by $R_u(P_\theta) = \langle U_\alpha | \alpha \in \Phi^+ \setminus \Phi_\theta \rangle$ and $L_\theta = \langle T, U_\alpha | \alpha \in \Phi_\theta \rangle$ is a complement to $R_u(P_\theta)$ in P_θ.*

Definition 1.3.4. L_θ is called a standard Levi complement of P_θ. A Levi subgroup of G is a subgroup which is conjugate to a standard Levi complement.

1.4 Levi subalgebras

We now introduce the analogue of parabolic subgroups for a reductive Lie algebra $\mathfrak{g}$. Our main reference will be Section 3.8 of [3]. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and denote by Φ the associated root system of $[\mathfrak{g}, \mathfrak{g}]$ with set of simple roots Δ. Then $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$, where Φ^+ is the set of positive roots, is a Borel subalgebra of $\mathfrak{g}$, i. e. it is a maximal solvable subalgebra.

Definition 1.4.1. A parabolic subalgebra is any subalgebra containing $\mathfrak{b}$.

Let θ be a subset of Δ. Then

$$\mathfrak{p}_\theta = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_I \cup \Phi^+} \mathfrak{g}_\alpha$$

is called a standard parabolic subalgebra.

Proposition 1.4.2. *The nilradical of $\mathfrak{p}_\theta$ is $\mathfrak{n}_\theta = \oplus_{\alpha \in \Phi^+ \setminus \Phi_\theta} \mathfrak{g}_\alpha$. If we define $\mathfrak{l}_I = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_\theta} \mathfrak{g}_\alpha$, then $\mathfrak{p}_\theta = \mathfrak{l}_\theta \oplus \mathfrak{n}_\theta$.*

Definition 1.4.3. $\mathfrak{l}_\theta$ and all its conjugates is called a Levi subalgebra of $\mathfrak{g}$.

Proposition 1.4.4. *Any parabolic subalgebra is conjugate to a standard parabolic subalgebra.*

1.5 Centralizers

The purpose of this section is to introduce centralizers of elements of an algebraic group and to state their basic properties.

Let G be a linear algebraic group and $\mathfrak{g} = \mathrm{Lie}(G)$.

Definition 1.5.1. We define the centralizer of an element $g \in G$ as

$$C_G(g) = \{h \in G | hgh^{-1} = g\}.$$

We introduce the notation for the analogue of $C_G(g)$ for an element of the Lie algebra $\mathfrak{g}$.

Definition 1.5.2. Let $x \in \mathfrak{g}$. Its centralizer is given by

$$C_{\mathfrak{g}}(x) = \{y \in \mathfrak{g} | [x, y] = 0\}.$$

As we shall see, centralizers in G are not in general connected, so it is useful to present the following notation. For a subgroup H of G, H^o will denote the connected component of H containing the identity of G.

1.5.1 Centralizers of Semisimple elements

We are now interested in studying some properties of $C_G(s)$, for a semisimple $s \in G$.
Let G be a connected and reductive algebraic group over the complex numbers. Let T be a maximal torus and denote by Φ the corresponding root system of G. We will assume that Φ is irreducible.

Definition 1.5.3. A subgroup of the form $C_G(s)^o$, for $s \in G$ semisimple, is called a pseudo-Levi subgroup of G. We call a pseudo-Levi subalgebra the Lie algebra associated to a pseudo-Levi subgroup.

We refer to Theorem 14.2 in [8] for the following statement.

Theorem 1.5.4. *Let $s \in T$ and set $\Psi = \{\alpha \in \Phi | \alpha(s) = 1\}$. Then*

$$C_G(s)^o = \langle T, U_\alpha | \alpha \in \Psi \rangle.$$

We refer to Remark 3.1 in [2].

Proposition 1.5.5. *Let $s \in T$. Let Δ be a basis for Φ and let α_0 be the highest root in Φ. The root system $\Psi = \{\alpha \in \Phi | \alpha(s) = 1\}$ admits a basis which is conjugate to a subset of $\Delta \cup \{-\alpha_0\}$.*

Remark 1.5.6. When Ψ admits a basis in Δ, then $C_G(s)^o$ is actually a Levi subgroup of G.

Since we will be interested in unipotent elements in $C_G(s)$, we present the following result. We report a result from Section 2.2 in [4]

Proposition 1.5.7. *Let $g \in G$ with Jordan decomposition $g = su$. Then $u \in C_G(s)^o$.*

We now present the following result for the connectedness of $C_G(s)$ for which we refer to Section 2.11 in [4]

Proposition 1.5.8. *Let G be simply connected and let $s \in G$ be semisimple. Then $C_G(s)$ is connected.*

Definition 1.5.9. We define, for $s \in T$,

$$Z(C_G(s)^o)^{reg} = \{h \in T | C_G(h)^o = C_G(s)^o\}.$$

1.5.2 Centralizers of Semisimple elements in $\mathfrak{g}$

Let G be as in the previous section and let $\mathfrak{g} = \mathrm{Lie}(G)$. Fix a Cartan subalgebra $\mathfrak{h}$ with corresponding root system Φ and basis Δ. Let $x_s \in \mathfrak{h}$ and set $\Phi_{x_s} = \{\alpha \in \Phi | \alpha(x_s) = 0\}$.

Proposition 1.5.10. *The centralizer of x_s in $\mathfrak{g}$ is given by*

$$C_{\mathfrak{g}}(x_s) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_{x_s}} \mathfrak{g}_\alpha.$$

Up to replacing x_s by a G-conjugate, Φ_{x_s} admits a basis in Δ.

By choosing Δ_s, a basis in Δ of Φ_{x_s}, we have that $C_{\mathfrak{g}}(x_s) = \mathfrak{l}_{\Delta_s}$, hence $C_{\mathfrak{g}}(x_s)$ is a Levi subalgebra.
We now introduce a notation that will be useful later.

Definition 1.5.11. We define

$$\mathfrak{z}(C_{\mathfrak{g}}(x))^{reg} = \{y \in \mathfrak{h} | C_{\mathfrak{g}}(x) = C_{\mathfrak{g}}(y)\}.$$

1.5.3 Centralizers of Unipotent elements and Isogenies

We now turn to the study of the centralizers of unipotent elements.
Consider two groups $\bar{G}$ and G, such that there exists an isogeny $\pi : \bar{G} \longrightarrow G$. Let u be a unipotent element in G and set $\pi(u) = \bar{u}$. We are interested in studying the connection between the centralizers $C_G(u)$ and $C_{\bar{G}}(u)$ and the respective connected components of the identity.
We start by proving the following result.

Proposition 1.5.12. *For $G, \bar{G}, u, \bar{u}$, we have*

$$C_G(u) = \pi^{-1}(C_{\bar{G}}(\bar{u})).$$

Proof. It is clear that $x \in C_G(u)$ implies $\pi(x) = \bar{x} \in C_{\bar{G}}(\bar{u})$. We now show that if $\bar{x} \in C_{\bar{G}}(\bar{u})$, then $x \in C_G(u), \forall x \in \pi^{-1}(\bar{x})$ We have $\bar{x}\bar{u}\bar{x}^{-1} = \bar{u}$, so for any $x \in \pi^{-1}(\bar{x})$, we get $xux^{-1} = z_0 u$ for some $z_0 \in \ker(\pi)$. Since xux^{-1} is unipotent and z_0 is semisimple and central, we must have $z_0 = 1$. □

Moreover, we can prove the following proposition for $C_G(u)^o$ and $C_{\bar{G}}(u)^o$.

Proposition 1.5.13. *Let $G, \bar{G}, u, \bar{u}$ be as above, then*

$$\pi^{-1}(C_{\bar{G}}(\bar{u})^o) = \cup_{z \in \ker(\pi)} z C_G(u)^o.$$

Proof. We certainly have $\pi(zC_G(u)^o) \subseteq C_{\bar{G}}(\bar{u})^o, \forall z \in \ker(\pi)$ and the opposite inclusion follows from the fact that $\pi(C_G(u)^o)$ is a normal subgroup of finite index in $C_{\bar{G}}(\bar{u})$. □

Consider now the mapping

$$\varphi : C_G(u) \longrightarrow C_{\bar{G}}(\bar{u}) \longrightarrow C_{\bar{G}}(\bar{u})/C_{\bar{G}}(\bar{u})^o,$$

given by the composition of π and the projection to the quotient. Then

$$\ker(\varphi) = \pi^{-1}(C_{\bar{G}}(\bar{u})^o) = \cup_{z\in\ker(\pi)} zC_G(u)^o.$$

Hence, we have proved

Proposition 1.5.14. *For $G, \bar{G}, u, \bar{u}$ as above,*

$$(C_G(u)/C_G(u)^o)\big/(\pi^{-1}(C_{\bar{G}}(\bar{u})^o)/C_G(u)^o) \simeq C_{\bar{G}}(\bar{u})/C_{\bar{G}}(\bar{u})^o.$$

We deduce what follows.

Lemma 1.5.15. $|C_G(u)/C_G(u)^o| = |C_{\bar{G}}(\bar{u})/C_{\bar{G}}(\bar{u})^o|$ *holds if and only if $zC_G(u)^o = C_G(u)^o$ for any $z \in \ker(\pi)$ or, equivalently, if and only if $\ker(\pi) \subseteq C_G(u)^o$.*

1.5.4 The Component Group $C_G(u)/C_G(u)^o$

We now present part of the content of [12] that will be useful to deduce further information on unipotent centralizers in case G is adjoint.
We start by introducing some notations. Let G be a connected simple algebraic group over $\mathbb{C}$ of adjoint type. Let Φ be its root system with respect to a fixed maximal torus T of dimension n and set $\Delta = \{\alpha_i | 1 \le i \le n\}$ to be a basis for Φ. Denote by α_0 the highest root and let $c_i \in \mathbb{C}$ be such that $\alpha_0 = \sum_{i=1}^n c_i\alpha_i$ and define $c_0 = 1$.
Let $s \in T$. Recall that, up to replacing s by a conjugate, we can find $\Delta_s \subseteq \Delta \cup \{-\alpha_0\}$ such that

$$C_G(s)^o = \langle T, U_\alpha | \alpha \in \Psi_s\rangle,$$

where Ψ_s is the root subsystem generated by Δ_s. We define $d_s = g.c.d.\{c_i|\alpha_i \notin \Delta_s\}$.

Definition 1.5.16. A unipotent element in G is said to be distinguished if it is not centralized by a nontrivial torus.

Theorem 13 in [12] establishes a correspondence between $G-$conjugacy classes of pairs $(\mathfrak{l}, N)$, where $\mathfrak{l}$ is a pseudo-Levi subalgebra of the Lie algebra $\mathfrak{g}$ of G and N is a distinguished nilpotent in $\mathfrak{l}$, and $G-$conjugacy classes of pairs (N, C), where N is a nilpotent element in $\mathfrak{g}$ and C is a conjugacy class in $C_{\mathfrak{g}}(N)/C_{\mathfrak{g}}(N)^o$. More precisely, it is shown that, given a pair (N, C) as above, there exists a pseudo-Levi subalgebra $\mathfrak{l}$ such that N is distinguished in it and the class C in $C_{\mathfrak{g}}(N)/C_{\mathfrak{g}}(N)^o$ is represented by an element of $\mathfrak{z}(\mathfrak{l})^{reg}$. We will not discuss in detail this result, but we will be interested in some of its consequences. We start by stating a result from page 10 in [12].

Proposition 1.5.17. *Let $u \in G$ be unipotent and $s \in G$ a semisimple element such that $s \in C_G(u)^o$ and u is distinguished in $C_G(s)^o$. Then $C_G(s)^o$ is a Levi subgroup.*

As a consequence, we also have the following result from Corollary 15 in [12] and the comments immediately following it.

Proposition 1.5.18. *Let $u \in G$ be unipotent and let $s \in G$ semisimple such that $su = us$. Assume $C_G(s)^o$ is not a Levi subgroup of G and u is distinguished in it. Then the projection of s to the quotient $C_G(u)/C_G(u)^o$ is nontrivial. Moreover, the class of s has order dividing d_s.*

Remark 1.5.19. Recall that $C_G(s)^o$ is in general a pseudo-Levi subgroup. If it is not a Levi subgroup, then Ψ_s does not admit a basis conjugate to a subset of Δ, i.e. Δ_s needs to contain $-\alpha_0$.

We also have the following consequence also from page 10 of [12].

Corollary 1.5.20. *Let $u \in G$ be unipotent and let $s \in G$ semisimple such that $su = us$. Assume $C_G(s)^o$ is not a Levi subgroup of G and u is distinguished in it. If d_s is prime, then the class of s in $C_G(u)/C_G(u)^o$ has order exactly d_s.*

Chapter 2

Jordan Classes

2.1 Jordan Classes in the Lie Algebra of a Linear Algebraic Group

The aim of this section is to classify the elements of a Lie algebra associated to a linear algebraic group according to their Jordan decomposition. Let G be a reductive linear algebraic group over the complex numbers with simply connected semisimple part and $\mathfrak{g} = \mathrm{Lie}(G)$. We will assume $\mathfrak{g}$ is reductive and G is reductive with simply connected semisimple part. We recall that an element $x \in \mathfrak{g}$ admits a decomposition $x = x_s + x_n$, where $x_s \in \mathfrak{g}$ is semisimple and $x_n \in \mathfrak{g}$ is nilpotent.

Definition 2.1.1. Let $x, y \in \mathfrak{g}$ and $x = x_s + x_n$ and $y = y_s + y_n$ be their Jordan decomposition. We say x and y lie in the same Jordan class if there exists a $g \in G$ such that

1. $C_{\mathfrak{g}}(y_s) = \mathrm{Ad}(g) \cdot (C_{\mathfrak{g}}(x_s))$;
2. $y_n = \mathrm{Ad}(g) \cdot x_n$.

We denote by $\mathfrak{J}(x)$ the class of an element x. We see that the Jordan class of $x = x_s + x_n \in \mathfrak{g}$ is the set

$$\mathfrak{J}(x) = \mathrm{Ad}(G) \cdot (\mathfrak{z}(C_{\mathfrak{g}}(x_s))^{\mathrm{reg}} + x_n).$$

Example 1. The matrices

$$\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{bmatrix}, \begin{bmatrix} -\mu & 0 & 0 & 1 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & -\mu \end{bmatrix} \in \mathfrak{sl}_4(\mathbb{C}),$$

with $\lambda \neq \mu \neq 0$, belong to the same Jordan class.

Remark 2.1.2. The regularity condition is necessary. Indeed, take for example any element $z \in \mathfrak{z}(\mathfrak{g})$, then certainly $z \in \mathfrak{z}(C_{\mathfrak{g}}(x_s))$, for any semisimple $x_s \in \mathfrak{g}$. Clearly the centralizer $C_{\mathfrak{g}}(z)$ is the whole $\mathfrak{g}$, which is not equal to $C_{\mathfrak{g}}(x_s)$ for any semisimple x_s.

It is easy to verify that the classes we have just defined are equivalence classes and therefore determine a partition of $\mathfrak{g}$. We now want to show that Jordan classes come in finite number. First we fix a Cartan subalgebra $\mathfrak{h} \in \mathfrak{g}$. Let $x = x_s + x_n \in \mathfrak{g}$. Up to replacing x by one of its conjugates- hence an element of the same class- we may assume $x_s \in \mathfrak{h}$. We know $\mathfrak{g}$ admits a root space decomposition, so that we can write

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha,$$

where Φ is a root system. Now we set $\Phi_{x_s} = \{\alpha \in \Phi | \alpha(x_s) = 0\}$. Then we can write

$$C_{\mathfrak{g}}(x_s) = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi_{x_s}} \mathfrak{g}_\alpha.$$

Up to acting on x_s by an element of G, Φ_{x_s} admits a basis which is a subset of a given basis Δ of Φ. Therefore $C_{\mathfrak{g}}(x_s)$ is determined by a subset of Δ and there are only finitely many of them. Finally the existence of finitely many nilpotent orbits in any reductive Lie algebras (see Chapter 3 of [3]) allows us to conclude that there are finitely many Jordan classes.
We now summarize some properties of Jordan classes, stated and proved in Section 39.1 of [14], in the following proposition.

Proposition 2.1.3. *Let $x \in \mathfrak{g}$ and $\mathfrak{J}(x)$ be its Jordan class, then*

1. *$\mathfrak{J}(x)$ is a union of adjoint orbits of the same dimension;*
2. *$\mathfrak{J}(x)$ is locally closed;*
3. *$\mathfrak{J}(x)$ is smooth;*
4. *$\mathfrak{J}(x)$ is irreducible;*
5. *the closure of $\mathfrak{J}(x)$ is a union of Jordan classes.*

2.2 Jordan Classes in a Reductive Linear Algebraic Group

We now want to define a sort of analogue of the classes we presented in the previous section for a connected and reductive linear algebraic group G over $\mathbb{C}$. By Theorem 1.1.5, an element $g \in G$ admits a unique multiplicative Jordan decomposition $g = su$, where $s \in G$ is semisimple and $u \in G$ is unipotent.

Definition 2.2.1. Let $g, h \in G$ and let $g = su$ and $h = rv$ be their Jordan decomposition. We say g and h lie in the same Jordan class if there exists an $f \in G$ such that

1. $fvf^{-1} = u$;
2. $C_G(s)^o = C_G(frf^{-1})^o$;
3. $sZ(C_G(s)^o)^o = (frf^{-1})Z(C_G(s)^o)^o$;

We notice that the class of $g = su$ is given by the set

$$J(su) = G \cdot ((sZ(C_G(s)^o)^o)^{reg}u).$$

The class of an element $g = su$ can be described up to simultaneous conjugation by an element of G by the triple

$$(C_G(s)^o, sZ(C_G(s)^o)^o, \mathcal{O}_u^{C_G(s)^o}), \tag{2.1}$$

where $\mathcal{O}_u^{C_G(s)^o}$ is the orbit of u in $C_G(s)^o$.

Remark 2.2.2. Assume G is simple. Let s be an element of a maximal torus T. Denote by Φ the set of roots of G and set $\Psi_s = \{\alpha \in \Phi | \alpha(s) = 1\}$. Let α_0 be the highest positive root of G and Δ be a basis for Φ, then, as we saw in Proposition 1.5.5, up to conjugation, Ψ_s admits a basis Δ_s contained in $\Delta \cup \{-\alpha_0\}$.

Proposition 2.2.3. *Jordan classes in G are finitely many.*

Proof. $C_G(s)^o$ is determined by a subsystem of Φ, so there are only finitely many choices for this parameter. The group $Z(C_G(s)^o)$ has finitely many connected components. The unipotent orbits in $C_G(s)^o$ are finitely many. As a class is determined by the parameters in (2.1), the result follows. □

Exaclty as in $\mathfrak{g}$, Jordan classes in G are irreducible.
We have just seen a few analogies between Jordan classes in the Lie algebra $\mathfrak{g}$ and in the group G, now we want to point out some differences between the two.
We observe that Remark 2.2.2 already presents one. Indeed if we consider a semisimple $x_s \in \mathfrak{h}$, up to an action by an element of G, the root system $\Phi_{x_s} = \{\alpha \in \Phi | \alpha(x_s) = 0\}$ admits a basis which is a subset of a given basis for the root system of $\mathfrak{g}$. As we have observed in Remark 2.2.2, for $s \in G$ semisimple, the root system $\Psi_s = \{\alpha \in \Phi | \alpha(s) = 1\}$ may not have a basis which is conjugate to a subset of a fixed basis of the root system of G.
Moreover, under our assumptions on $\mathfrak{g}$, the vector space $\mathfrak{z}(C_{\mathfrak{g}}(x_s))$ is connected, for any semisimple $x_s \in \mathfrak{g}$. We see through an example that this is not always the case for $Z(C_G(s)^o)$, where $s \in G$ is semisimple.

Example 2. Consider the matrix

$$s = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & -\lambda^{-1} & 0 \\ 0 & 0 & 0 & -\lambda^{-1} \end{bmatrix} \in SL_4,$$

where SL_4 is the special linear group over $\mathbb{C}$. Then

$$Z(C_G(s)^o) = Z(C_G(s)) = \left\{ \begin{bmatrix} t & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & v & 0 \\ 0 & 0 & 0 & v \end{bmatrix} | t, v \in \mathbb{C}, t^2v^2 = 1 \right\}.$$

As the polynomial $t^2v^2 - 1$ splits into $(tv-1)(tv+1)$, $Z(C_G(s))$ is not connected and has two connected components. The component of the identity is given by

$$Z(C_G(s))^o = \left\{ \begin{bmatrix} t & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & t^{-1} & 0 \\ 0 & 0 & 0 & t^{-1} \end{bmatrix} | t \in \mathbb{C}^\times \right\}$$

Notice that in this case $s \notin Z(C_G(s))^o$ and

$$sZ(C_G(s))^o = \left\{ \begin{bmatrix} t & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & -t^{-1} & 0 \\ 0 & 0 & 0 & -t^{-1} \end{bmatrix} | t \in \mathbb{C}^\times \right\}$$

is the other connected component.

Chapter 3

The Exponential Map

We now introduce the exponential map defined over a Lie group. For this part we will refer to Section 3 of Chapter 2 of [10]. Here G will denote a Lie group and $\mathfrak{g}$ will be its Lie algebra, which in this setting can be thought as the tangent space T_1G at the identity of G.
For $g \in G$ and $\xi \in \mathfrak{g}$ we define

$$\xi g = d(r_g)(\xi) \in T_gG,$$

where $r_g : G \to G$ is the right multiplication by g and $d(r_g)$ is the differential of r_g computed in 1. We recall that a one paramenter subgroup of G is a differentiable path $g(t)$ defined for $t \in \mathbb{R}$ such that

$$g(s+t) = g(s)g(t),$$

i.e. g is homomorphism of Lie groups from $(\mathbb{R}, +)$ into G. For any differentiable path $g(t)$ one can define a path $\xi(t)$ in $\mathfrak{g}$ given by the equation

$$\frac{dg(t)}{dt} = \xi(t)g(t).$$

Such a ξ is called the velocity of the path g. A path $g(t)$ as above is a one parameter subgroup if and only if its velocity is constant and $g(0) = e$. As a result, to any $\xi \in \mathfrak{g}$ we can associate the one parameter subgroup $g_\xi(t)$ with constant velocity $\xi(t) = \xi$.

Definition 3.0.1. The exponential map $\exp : \mathfrak{g} \to G$ is defined by

$$\exp(\xi) = g_\xi(1).$$

Remark 3.0.2. Consider $G = GL_n$, the general linear group over $\mathbb{C}$. Then G is the group of the invertible elements of the algebra of $n \times n$ matrices over $\mathbb{C}$. For each $n \times n$ matrix a, we have

$$g_a(t) = \exp(ta) = \sum_{n=0}^{\infty} \frac{(ta)^n}{n!}.$$

In other words, $\exp : \mathfrak{gl}_n \longrightarrow GL_n$ is the usual matrix exponential. This is also true for other classical matrix groups and can be shown by taking the restriction of the exponential and using the correspondence between Lie subalgebras of $\mathfrak{gl}_n$ and connected Lie subgroups of GL_n.
Furthermore, we have $g \cdot \exp(\xi) g^{-1} = \exp(\mathrm{Ad}g \cdot \xi)$, for all $g \in G$ and $\xi \in \mathfrak{g}$.

We present a few general properties of the exponential map for which we refer to Section 5 of Chapter 4 of [11].

Proposition 3.0.3. *There exists a neighborhood of zero in* $\mathfrak{g}$ *such that the exponential* $\exp : \mathfrak{g} \to G$ *maps it diffeomorphically onto a neighborhood of the identity in* G.

Proposition 3.0.4. *Let* $\xi, \eta \in \mathfrak{g}$. *If* $[\xi, \eta] = 0$, *then* $\exp(\xi+\eta) = \exp(\xi)\exp(\eta)$.

The exponential is not in general surjective. We will denote its image by E_G and its complement in G by E'_G.

Example 3. The exponential map in GL_n is surjective.
It is not surjective in SL_n, the special linear group over $\mathbb{C}$. For example, as we willl verify later, a matrix made of a unique Jordan block of dimension n with eigenvalue different from 1 is not contained in E_{SL_n}.

We have the following result

Theorem 3.0.5. *Let* G *be a connected complex Lie group.* E_G *is dense in* G.

The following theorem gives a description of its image for a connected complex semisimple Lie group.

Theorem 3.0.6 (Djoković)**.** *Let* G *be a connected complex semisimple Lie group. Let* $g \in G$ *and* $g = su$ *be its Jordan decomposition. Then the following are equivalent*

(i) $g \in E_G$;

(ii) $s \in C_G(g)^o$;

(iii) $s \in C_G(u)^o$.

Example 4. Consider a matrix in SL_3 as in Example 3:

$$g := \begin{bmatrix} \zeta & 1 & 0 \\ 0 & \zeta & 1 \\ 0 & 0 & \zeta \end{bmatrix},$$

where ζ is a primitive $3-$rd root of 1. Its unipotent part is

$$u = \begin{bmatrix} 1 & \frac{1}{\zeta} & 0 \\ 0 & 1 & \frac{1}{\zeta} \\ 0 & 0 & 1 \end{bmatrix}$$

The centralizer of u in SL_3 is made of the matrices of the form

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ 0 & a_{1,1} & a_{1,2} \\ 0 & 0 & a_{1,1} \end{bmatrix},$$

where $a_{i,j} \in \mathrm{C}$, with determinant equal to 1. The equation for the determinant is $a_{1,1}^3 - 1 = 0$. The polynomial $a_{1,1}^3 - 1$ is not irreducible and splits in $a_{1,1}^3 - 1 = (a_{1,1} - 1)(a_{1,1} - \zeta)(a_{1,1} - \zeta^2)$. Thus the connected component of the identity is the one satisfying $a_{1,1} - 1 = 0$. Clearly s does not belong to $C_G(u)^o$, so $g \notin E_{SL_3}$. In general, consider a matrix g with a Jordan block of dimension n in SL_n. The centralizer of the unipotent part will be made of upper triangular matrices whose elements on the diagonal are all equal. As a consequence, the polynomial for the determinant will be given by $p(x) = x^n - 1$ which splits into n distinct factors of the form $x - \eta$, where η is a $n-$th root of 1. Clearly the identity component is that corresponding to the factor $x - 1$. Thus, in order to obtain $g \in E_{SL_n}$, the only choice for the eigenvalue of g is 1.

Example 5. Consider now the 4×4 matrix in SL_4

$$g = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & -\lambda^{-1} & 1 \\ 0 & 0 & 0 & -\lambda^{-1} \end{bmatrix}$$

Here the centralizer of the unipotent part is given by matrices of the form

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ 0 & a_{1,1} & 0 & a_{1,3} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ 0 & a_{3,1} & 0 & a_{3,3} \end{bmatrix},$$

with components in $\mathbb{C}$ and determinant equal to 1. The condition on the determinant means that the components must satisfy $(a_{1,1}a_{3,3} - a_{1,3}a_{3,1})^2 - 1 = 0$. Also in this case the polynomial can be reduced in two irreducible factors. Thus the centralizer has the two components, that of identity, where $a_{1,1}a_{3,3} - a_{1,3}a_{3,1} - 1 = 0$, and that of g, where $a_{1,1}a_{3,3} - a_{1,3}a_{3,1} + 1 = 0$ holds. Applying Theorem 3.0.6, we have that $g \notin E_G$.

Remark 3.0.7. We know that in GL_n, the centralizer of any matrix is connected. This in particular holds for a unipotent element u. The centralizer of u in SL_n is $C_{SL_n}(u) = C_{GL_n}(u) \cap SL_n$. $C_{SL_n}(u)$ is not connected when the polynomial function $p(z_{i,j}) = \det(z) - 1$ restricted to $C_{GL_n}(u)$ is reducible. Indeed, whenever p splits, we have a connected component for each of its irreducible factors.

In the remaining of this section G will be a connected semisimple algebraic group over $\mathbb{C}$. Since we are interested in describing the image of the exponential map in terms of the Jordan classes of G, we present the following

Proposition 3.0.8. *E_G is a union of Jordan classes.*

Proof. Let $g \in E_G$ and $g = su$ be its Jordan decomposition. By Theorem 3.0.6 we have $s \in C_G(u)^o$. Recall $u \in C_G(s)^o$ and therefore u commutes with all elements of $Z(C_G(s)^o)$. It follows that $Z(C_G(s)^o) \subseteq C_G(u)$, so $Z(C_G(s)^o)^o \subseteq C_G(u)^o$. Clearly $s \in Z(C_G(s)^o)^o s$. As a consequence we get $Z(C_G(s)^o)^o s \subseteq C_G(u)^o$. If $h \in J(g)$ and $h = rv$, then r is conjugate to an element of $Z(C_G(s)^o)^o s$ and is therefore contained in $C_G(v)^o$. $\square$

Remark 3.0.9. In general $J(x) \in E_G$ does not imply $\overline{J(x)} \in E_G$. Consider for example SL_n. We have shown that the exponential map is not surjective and we know that E_{SL_n} is dense in SL_n. Assume now $\overline{J(x)} \in E_{SL_n}$ whenever $J(x)$ is in E_{SL_n}. Then E_{SL_n} is a union of closed sets and such union is finite as Jordan classes are finitely many. As a consequence E_{SL_n} is closed and therefore coincides with its closure. But E_{SL_n} is dense in SL_n, so we obtain $E_{SL_n} = SL_n$, a contradiction.

Proposition 3.0.10. *Let $g \in E'_G$ with Jordan decomposition su. Let $r \in C_G(u)$ such that $J(r) \subseteq \overline{J(s)}$. Then $J(ru) \nsubseteq E_G$.*

Proof. By Theorem 3.0.6 $s \notin C_G(u)^o$. By the properties of Jordan decomposition $u \in C_G(s)$, so $Z(C_G(s)) \subseteq C_G(u)$, hence $Z(C_G(s)^o)^o s \subseteq C_G(u)^o s \neq C_G(u)^o$. If $J(r) \subseteq \overline{J(s)}$, then, as we can see in the proof of Proposition 4.9 in [2], up to conjugation, $C_G(s)^o \subseteq C_G(r)^o$ and $Z(C_G(r)^o)^o \subseteq Z(C_G(s)^o)^o$ and $Z(C_G(s)^o)^o r \subseteq Z(C_G(r)^o)^o s$. Hence $r \in C_G(u)^o s$.

$\square$

Proposition 3.0.11. *Assume $E_G \neq G$, then the set E'_G is not closed.*

Proof. Since the exponential map is not surjective, there exists an element $g \notin E_G$, so $\mathcal{O}_g \subseteq E'_G$. Let $g = su$ be its Jordan decomposition. Then the identity is contained in the closure of $\mathcal{O}_u^{C_G(s)^o}$, hence there exists a path joining u and 1 entirely contained in $C_G(s)^o$. We surely have $s \in E_G$, as it is semisimple, but also $s \in \overline{\mathcal{O}}_{su}$, so $\overline{\mathcal{O}}_{su} \nsubseteq E'_G$. $\square$

We present here a result which applies in case G is adjoint.

Proposition 3.0.12. *Assume G is adjoint. Let $s \in G$ be semisimple and such that $C_G(s)^o$ is a Levi subgroup. Then for any unipotent $u \in C_G(s)$, we have $s \in C_G(u)^o$.*

Proof. Since $C_G(s)^o$ is a Levi subgroup and G is adjoint, we have, by Lemma 33 in [9], $Z(C_G(s)^o)/Z(C_G(s)^o)^o = \{1\}$. So $s \in Z(C_G(s)^o)^o \subseteq C_G(u)^o$. $\square$

Corollary 3.0.13. *Let G be adjoint. If s is a semisimple element of G such that $C_G(s)^o$ is a Levi subgroup, then $J(su) \subseteq E_G$, for any unipotent $u \in C_G(s)$.*

Proof. By Proposition 3.0.12 we know $s \in C_G(u)^o$, which implies by Theorem 3.0.6 that $su \in E_G$. We finish the proof applying Proposition 3.0.8. □

Proposition 3.0.14. *Let G be adjoint. Let $u \in G$ be a unipotent element such that $C_G(u)/C_G(u)^o$ is nontrivial. Then for each $s \in C_G(u)$ semisimple, such that $C_G(s)^o$ is not a Levi subgroup and u is distinguished in $C_G(s)^o$, we have $J(su) \notin E_G$.*

Proof. By Proposition 1.5.18, we know that the projection of s in the quotient $C_G(u)/C_G(u)^o$ is nontrivial, hence $s \notin C_G(u)^o$, which means, by Theorem 3.0.6, that $su \notin E_G$, hence the class $J(su)$ is not in the image. □

Not only does Proposition 1.5.18 tell us that $su \notin E_G$, it also give us information on the index of su in G.

Definition 3.0.15. The index of an element $g \in G$ is the minimal non-negative integer k for which $g^k \in E_G$ if such minimum exists, otherwise we say that the index of g is $+\infty$.

Defining d_s as in Section 1.5.4, we have that the index of su, under the hypothesis of Proposition 1.5.18, is a divisor of d_s. Indeed, Theorem 1.3 in [16] states that the index of su in G is the order of the class of s in $C_G(u)/C_G(u)^o$, which is a divisor of d_s by Proposition 1.5.18.

Chapter 4

The Classical Group SL_n

We now focus our attention on the special linear group over the complex numbers SL_n. We recall a few facts that will be useful to treat this case. The Lie algebra associated to SL_n is

$$\mathfrak{sl}_n = \{A \in \mathfrak{gl}_n | Tr(A) = 0\}.$$

We set $\mathfrak{h} \subseteq \mathfrak{sl}_n$ to be the toral subalgebra of traceless diagonal matrices. A root system for $\mathfrak{h}$ in $\mathfrak{sl}_n$ is given by

$$\Phi = \{\varepsilon_i - \varepsilon_j | 1 \leq i, j \leq n, i \neq j\}.$$

A basis for Φ is

$$\Delta = \{\varepsilon_i - \varepsilon_{i+1} | 1 \leq i \leq n-1\}.$$

The root space $\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{C}e_{i,j}$, where $e_{i,j}$ is the matrix whose components are all zeroes except for a 1 in position i, j. The Lie algebra $\mathfrak{sl}_n$ can be decomposed into

$$\mathfrak{sl}_n = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha = \mathfrak{h} \oplus \bigoplus_{1 \leq i,j \leq n : i \neq j} \mathbb{C}e_{i,j}.$$

For all $h \in \mathfrak{h}$, we have $[h, e_{i,j}] = (\varepsilon_i - \varepsilon_j)(h)e_{i,j}$.

We are now ready to state the theorem which describes the classes in E_{SL_n}.

Theorem 4.0.1. *Let $s \in SL_n$ be a diagonal matrix with eigenvalues $s_i, 1 \leq i \leq k$. Set n_i to be the multiplicity of the eigenvalue s_i. Suppose all equal eigenvalues appear in consecutive positions along the diagonal. Let $u \in SL_n$ be a unipotent element such that $su = us$ is in Jordan normal form and let M be the g.c.d. of the sizes of the (multiples of the) Jordan blocks in u. Then $J(su) \subseteq E_{SL_n}$ if and only if $\prod_i s_i^{n_i/M} = 1$.*

Remark 4.0.2. Observe that M is always a devisor of each n_i, hence all the exponents appearing in the condition $\prod_i s_i^{n_i/M} = 1$ are positive integers.

Remark 4.0.3. Any matrix in SL_n is conjugated to a matrix in Jordan normal form, therefore we may choose a representative in Jordan normal form to verify that a class in contained in E_{SL_n}. Recall that by Proposition 3.0.8 if $g \in E_{SL_n}$, then the whole class $J(g)$ lies in E_{SL_n}. Therefore Theorem 4.0.1 covers all the Jordan classes in SL_n.

4.1 Proof of Theorem 4.0.1

We start by introducing the method we want to use, which is inspired by that used in [5] to compute the index of the exponential map. Consider $g \in SL_n$ in Jordan normal form with unipotent and semisimple part u and s respectively. Certainly $s, u \in E_{SL_n}$ by Theorem 3.0.6. Therefore we can write

$$g = \exp(h)\exp(N)$$

for a semisimple $h \in \mathfrak{g}$ and a nilpotent $N \in \mathfrak{g}$. Clearly, our assumptions on g implies that h lies in $\mathfrak{h}$. Notice that the choice of h is not unique. Indeed, we may add integer multiples of $2\pi i$ to the elements of the diagonal of h without changing its image through the map exp(.). On the contrary, N is fixed since the exponential map is injective on the nilpotent part of $\mathfrak{sl}_n$.

Our purpose is to find such an $h \in \mathfrak{g}$ satisfying $[h, N] = 0$, so that, by Proposition 3.0.4, $g = \exp(h)\exp(N) = \exp(h + N)$, which means g belongs to E_{SL_n}. The nilpotent part N is a strictly upper triangular matrix, so it is a linear combination

$$N = \sum_{i<j} n_{i,j} e_{i,j}.$$

More precisely, N presents blocks of strictly upper triangular matrices along the diagonal that have the same dimension as the Jordan blocks of u. Set $\Delta_N = \{\varepsilon_i - \varepsilon_{i+1} | n_{i,i+1} \neq 0\}$ and let I be the set of indeces i for which $\varepsilon_i - \varepsilon_{i+1} \in \Delta_N$.

Observe that the coefficients $n_{i,j}$ which may be non-zero are those for which the corresponding root $\varepsilon_i - \varepsilon_j$ can be obtained as a sum of roots in Δ_N. Therefore, requiring that h commutes with N is the same as asking that $(\varepsilon_i - \varepsilon_{i+1})(h) = 0$, for all $\varepsilon_i - \varepsilon_{i+1} \in \Delta_N$.

The above arguement implies that $[h, N] = 0$ if and only if $h_{i,i} = h_{i+1,i+1}$

for all $i \in I$. So h is a matrix of the form

$$\begin{bmatrix} h_1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & h_1 & & & & & & \\ & & & h_2 & & & & & \\ & & & & \ddots & & & & \\ & & & & & h_2 & & & \\ & & & & & & \ddots & & \\ & & & & & & & \ddots & \\ & & & & & & & & h_k \end{bmatrix}$$

Let h_l be the eigenvalues of h as in (4.1) and let m_l be the number of repetitions of the value h_l on the diagonal. By assumption $s = \exp(h)$, so s is diagonal with elements on the diagonal $s_{i,i} = e^{h_{i,i}}$. Thus s must satisfy $s_{i,i} = s_{i+1,i+1}$, for all $i \in I$. We set $s_l = e^{h_l}$.

Clearly h must also satisfy the condition $Tr(h) = 0$ since $h \in \mathfrak{sl}_n$. This condition gives us

$$\sum_{l=1}^{k} m_l h_l = 0.$$

If $M := g.c.d.\{m_l\} \neq 1$, we can divide both sides by M to get

$$\sum_{l=1}^{k} m'_l h_l = 0,$$

where $m'_l = m_l/M$. Now we can certainly write

$$e^{\sum_{l=1}^{k} m'_l h_l} = \prod_{l=1}^{k} (e^{h_l})^{m'_l} = 1.$$

Therefore s must satisfy

$$\prod_{l=1}^{k} s_l^{m'_l} = 1.$$

Remark 4.1.1. Notice that in principle we knew that the product on the left hand side had to be an $M-$th root of 1. Indeed, since $s \in SL_n$, its determinant must be equal to 1, so $\prod_{l=1}^{k} s_l^{m_l} = 1$. Hence the $M-$th root of the left hand side is an $M-$th root of 1.

Example 6. Consider the matrix

$$g = \begin{bmatrix} \mu & 1 & 0 \\ 0 & \mu & 1 \\ 0 & 0 & \mu \end{bmatrix},$$

with $\mu^3 = 1$. In this case we have $\Delta_N = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3\}$. The semisimple elements $h \in$ for which $[h, N] = 0$ are scalar matrices satisfying the further condition $3h_1 = 0$, where h_1 denotes the only eigenvalue of h. This means $h_1 = 0$, so clearly $e^{h_1} = 1$. Therefore the only diagonal element commuting with u such that $su \in E_{SL_3}$ is the identity, i.e. $g \in E_{SL_3}$ if and only if $\mu = 1$. As a result we can see that $g \notin E_{SL_3}$.

Example 7. Consider a 4×4 matrix g in SL_4 with two 2×2 Jordan blocks with eigenvalues $\lambda, \mu \neq 0$. As $g \in SL_4$, λ and μ satisfy $\lambda^2\mu^2 = 1$. So we have either $\mu = \lambda^{-1}$ or $\mu = -\lambda^{-1}$. The unipotent part is given by

$$\begin{bmatrix} 1 & 1/\lambda & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/\mu \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

In our notation, we have $\Delta_N = \{\varepsilon_1 - \varepsilon_2, \varepsilon_3 - \varepsilon_4\}$. An element $h \in \mathfrak{h}$ such that $\alpha(h) = 0$ for all $\alpha \in \Delta_N$ is of the form

$$\begin{bmatrix} h_1 & 0 & 0 & 0 \\ 0 & h_1 & 0 & 0 \\ 0 & 0 & h_2 & 0 \\ 0 & 0 & 0 & h_2 \end{bmatrix}$$

The condition on the trace tells us that $2h_1 + 2h_2 = 0$, which means $h_1 + h_2 = 0$. This means that $\lambda\mu = 1$, hence $\mu = \lambda^{-1}$. As we have already seen in Example 4, if $\mu = -\lambda^{-1}$, then $g \notin E_{SL_4}$.

4.2 Parametrization of the Classes in E_G

Now we want to translate Theorem 4.0.1 in terms of the parametrization of Jordan classes. Recall that the class of an element $g = su$ can be parametrized by the triple

$$(C_G(s)^o, sZ(C_G(s)^o)^o, \mathcal{O}_u^{C_G(s)^o}).$$

Let T be the torus of diagonal matrices. We saw that, for $s \in T$, $C_G(s) = C_G(s)^o$ is fixed once we choose a subset Δ_s of Δ. We can attach to Δ_s a partition (n_i) of n in the following way. Set n_1 so that $\alpha_1, \ldots, \alpha_{n_1-1} \in \Delta_s$ and $\alpha_{n_1} \notin \Delta_s$. Let n_2 be such that $\alpha_{n_1+1}, \ldots, \alpha_{n_1+n_2-1} \in \Delta_s$ and $\alpha_{n_1+n_2} \notin \Delta_s$. Then we define inductively n_i such that it satifsfies $\alpha_{\sum_{l=1}^{i-1}+1n_l+1}, \ldots, \alpha_{\sum_{l=1}^{i} n_l - 1} \in \Delta_s$ and $\alpha_{\sum_{l=1}^{i} n_l} \notin \Delta_s$. Now that we have defined (n_i), it is immediate to verify that we can chose a representative with semisimple part

$$s = \begin{bmatrix} S_{n_1} & & & & \\ & S_{n_2} & & & \\ & & \ddots & & \\ & & & S_{n_{k-1}} & \\ & & & & S_{n_k} \end{bmatrix},$$

where S_{n_i} is a scalar matrix, with eigenvalue s_i and the s_i's are all distinct. A matrix in the centralizer of s will have the form

$$\begin{bmatrix} A_{n_1} & & & & \\ & A_{n_2} & & & \\ & & \ddots & & \\ & & & A_{n_{k-1}} & \\ & & & & A_{n_k} \end{bmatrix}, \tag{4.1}$$

where $A_{n_i} \in GL_{n_i}$ and $\prod_i \det(A_{n_i}) = 1$. Since $u \in C_{SL_n}(s)$, then it must have the form

$$u = \begin{bmatrix} U_{n_1} & & & & \\ & U_{n_2} & & & \\ & & \ddots & & \\ & & & U_{n_{k-1}} & \\ & & & & U_{n_k} \end{bmatrix},$$

where each U_{n_i} is a unipotent in GL_{n_i}. So we can parametrize the unipotent classes by taking refinements $(n_{i,j})$ of (n_i), so that $\sum_j n_{i,j} = n_i$. Indeed, for each fixed i, $(n_{i,j})$ is a partition of n_i and determines therefore a unipotent class in GL_{n_i}. By Theorem 4.0.1, if $g.c.d.(n_{i,j}) = 1$, then the class is always in the image. Otherwise, if $g.c.d.(n_{i,j}) = M \neq 1$, then s must satisfy the condition

$$\prod_i s_i^{n_i/M} = 1,$$

which determines the admissible cosets $sZ(C_{SL_n}(s))$. Indeed, the matrices in $Z(C_{SL_n}(s))$ will be of the form (4.1) with the extra condition that each A_{n_i} a scalar matrix. The connected components will be determined by the irreducible factors of the polynomial

$$P(z_i) = \prod_i z_i^{n_i} - 1.$$

Clearly

$$F(z_i) = \prod_i z_i^{n_i/M} - 1$$

is always a factor of $P(z_i)$. Therefore the connected components that can be chosen as second parameter of the class are those corresponding to the irreducible factors of $F(z_i)$.

Remark 4.2.1. The same result can be achieved by analyzing the structure of the centralizer unipotent part u and by determining which semisimple elements lie in $C_{SL_n}(u)^o$ to conclude by applying Theorem 3.0.6.

Example 8. Consider SL_3, then a unipotent element is conjugated to one of these matrices

$$Id = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, u_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, u_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Let $g = su \in SL_3$. If u is conjugated to the Id, then $g = s$, so $J(g) \in E_{SL_n}$. If the eigenvalues of g are all distinct, then it lies in the class with representative

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix},$$

with $abc = 1, a \neq b \neq c$. If g has two disctinc eigenvalues, then it is in the class of

$$\begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & q \end{bmatrix},$$

with $r^2q = 1, r \neq q$. Finally if g is a scalar matrix then its eigenvalue is a $3-$rd root of unity and g is in a different class for each choice of such root. Assume now u is conjugated to u_1. Then s is of the form

$$\begin{bmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & v \end{bmatrix}.$$

The condition $t^2v = 1$ simply means $s \in SL_3$, so $g \in E_{SL_3}$. In this case g may belong to the class of

$$\begin{bmatrix} t & 1 & 0 \\ 0 & t & 0 \\ 0 & 0 & v \end{bmatrix}, t \neq v,$$

if its eigenvalues are distinct. If this is not the case, all the eigenvalues of g are $3-$rd roots of one and again for each choice of such root, say w, we have a different class with representative of the form

$$\begin{bmatrix} w & 1 & 0 \\ 0 & w & 0 \\ 0 & 0 & w \end{bmatrix}.$$

Finally if u is conjugate to u_2, then necessarily s is a scalar matrix with elements on the diagonal that are $3-$rd roots of 1. In this case g may lie in one of three different class with representatives

$$y_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, y_2 = \begin{bmatrix} \zeta & 1 & 0 \\ 0 & \zeta & 1 \\ 0 & 0 & \zeta \end{bmatrix}, y_3 \begin{bmatrix} \zeta^2 & 1 & 0 \\ 0 & \zeta^2 & 1 \\ 0 & 0 & \zeta^2 \end{bmatrix}.$$

Here we have $J(y_1) \in E_{SL_3}$, while $J(y_2), J(y_3) \notin E_{SL_3}$.

Chapter 5

The Symplectic Group Sp_{2n}

Here we consider the symplectic group Sp_{2n} over the complex numbers. For us the form that Sp_{2n} leaves invariant is given by the matrix

$$K = \begin{bmatrix} & -\omega_n \\ \omega_n & \end{bmatrix},$$

where ω_n is $n \times n$ is the antidiagonal form given by

$$\omega_n = \begin{bmatrix} & & & 1 \\ & & \cdot & \\ & \cdot & & \\ 1 & & & \end{bmatrix}. \tag{5.1}$$

The Lie algebra associated to Sp_{2n} is

$$\mathfrak{sp}_{2n} = \{A \in \mathfrak{gl}_{2n} | A^T K + KA = 0\}.$$

A toral subalgebra $\mathfrak{h} \subseteq \mathfrak{sp}_{2n}$ is that of the diagonal matrices in $\mathfrak{sp}_{2n}$. A root system for $\mathfrak{h}$ is

$$\Phi = \{\pm(\varepsilon_i + \varepsilon_j), 1 \leq i \leq j \leq n, \pm(\varepsilon_i - \varepsilon_j), 1 \leq i < j \leq n\}$$

and a basis for it is

$$\Delta = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1}, 1 \leq i < n; \alpha_n = 2\varepsilon_n\}.$$

The root spaces are $\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{C}(e_{i,j} - e_{2n+1-j,2n+1-i})$, for $1 \leq i, j \leq n, i \neq j$, $\mathfrak{g}_{2\varepsilon_i} = \mathbb{C}e_{i,2n+1-i}$, for $1 \leq i \leq n$, $\mathfrak{g}_{\varepsilon_i + \varepsilon_j} = \mathbb{C}(e_{i,2n+1-j} + e_{j,2n+1-i})$, and $\mathfrak{g}_{-\varepsilon_i - \varepsilon_j} = \mathbb{C}(e_{2n+1-j,i} - e_{2n+1-i,j})$, $1 \leq i < j \leq n$.

Now we briefly describe the unipotent classes in Sp_{2n}.

5.1 Unipotent conjugacy classes in Sp_{2n}

We will parametrize unipotent classes through suitable partitions of $2n$. A partition $(n_i)_i$ of $2n$ determines a unipotent conjugacy class in Sp_{2n} if and only if odd values of n_i appear an even number of times. Given a partition $(n_i)_i$ satisfying this condition, we see how to construct a representative of the corresponding orbit. If n_i is odd, denote by $2q_i$ the number of times it appears; if n_i is even, let $2q_i + \epsilon_i$ be number of its repetitions, where $\epsilon_i \in \{0, 1\}$. Clearly

$$n = \sum_{n_i\ odd} q_i n_i + \sum_{n_i\ even} (q_i n_i + \epsilon_i \frac{n_i}{2}).$$

There exists an embedding

$$\prod_{n_i\ odd} (Sp_{2n_i})_i^q \times \prod_{n_i\ even;\, q_i \neq 0} (Sp_{2n_i})^{q_i} \times \prod_{\epsilon_i \neq 0} Sp_{n_i} \longrightarrow Sp_{2n}.$$

Indeed, anytime we write $n = \sum_{l=1}^m k_l$, we can embed $\prod_l Sp_{2k_l}$ into Sp_{2n} as follows. The matrix

$$\begin{bmatrix} A_{k_1} & B_{k_1} \\ C_{k_1} & D_{k_1} \end{bmatrix} \times \begin{bmatrix} A_{k_2} & B_{k_2} \\ C_{k_2} & D_{k_2} \end{bmatrix} \times \cdots \times \begin{bmatrix} A_{k_m} & B_{k_m} \\ C_{k_m} & D_{k_m} \end{bmatrix} \in \prod_l Sp_{2k_l}$$

is sent to

$$\begin{bmatrix} A_{k_1} & & & & & & & B_{k_1} \\ & A_{k_2} & & & & & B_{k_2} & \\ & & \cdot & & & \cdot & & \\ & & & A_{k_m} & B_{k_m} & & & \\ & & & C_{k_m} & D_{k_m} & & & \\ & & \cdot & & & \cdot & & \\ & C_{k_2} & & & & & D_{k_2} & \\ C_{k_1} & & & & & & & D_{k_1} \end{bmatrix} \in Sp_{2n}.$$

For the factors of the type Sp_{2n_i} we may take an upper triangular representative of the class corresponding to the partition (n_i, n_i) of $2n_i$, with A_{k_1} having nonzero elements only on the the diagonal and just above it. For the factors Sp_{n_i} we consider a representative of the regular orbit , i.e. of the class corresponding to the partition of n_i given by (n_i), which is upper triangular. Then we embed the chosen matrices as above.

Remark 5.1.1. The upper triangular matrices in the regular orbit are those for which the positions corresponding to simple roots are nonzero as we can deduce from Theorem 1 in Section 3.7 of [13].

Remark 5.1.2. A completely analogue factor decomposition can be carried out in $\mathfrak{sp}_{2n}$. Observe that the exponential map preserves this factor decomposition. This fact follows by the definition of the exponential for matrix groups.

Remark 5.1.3. By a similar argument one can show that the nilpotent conjugacy classes in $\mathfrak{sp}_{2n}$ correspond to partitions of $2n$ satisfying the same condition that odd values must occur an even number of times. It will be useful to notice that the factor decomposition determined by the partition is preserved by the exponential map.

5.2 The Image of the Exponential map

5.2.1 Statement of result

Theorem 5.2.1. *Let $\Delta_s \subseteq \Delta \cup \{-\alpha_0\}$. Choose a diagonal matrix s such that Δ_s is a basis for $\Psi_s = \{\alpha \in \Phi | \alpha(s) = 1\}$. Let $u \in C_{Sp_{2n}}(s)$ be unipotent. Then*

(i) If $\Delta_s \subseteq \Delta \setminus \{\alpha_n\}$, then $J(su) \subseteq E_{Sp_{2n}}$.

(ii) If (up to conjugation) $\Delta_s \subseteq \Delta$ and $\alpha_n \in \Delta_s$, then, if all the values of the partition of $2n$ associated to u have even multiplicity, $J(su) \subseteq E_{Sp_{2n}}$. Otherwise, either $J(su)$ or $J(-su)$ is in $E_{Sp_{2n}}$.

(iii) If $-\alpha_0, \alpha_n \in \Delta_s$, then $C_{Sp_{2n}}(s) \cong Sp_{2n_1} \times \prod_{i=2}^{m-1} GL_{n_i} \times Sp_{2n_m}$, for some (n_i) such that $\sum_i n_i = n$. Set v_1, v_m to be the projection of u to the factor Sp_{2n_1}, Sp_{2n_m} respectively. Then

(a) If the partitions associated to the class of v_1, v_m in Sp_{2n_1}, Sp_{2n_m} present only values with even multiplicity, $J(su) \subseteq E_{Sp_{2n}}$.

(b) If only one of the partitions as in (a) has a value with odd multiplicity, then either $J(su)$ or $J(-su)$ is in $E_{Sp_{2n}}$.

(c) If both partitions as in (a) have a value with odd multiplicity, then $J(su) \nsubseteq E_{Sp_{2n}}$.

Remark 5.2.2. Observe that Theorem 5.2.1 covers all possible cases, i.e. for each Jordan class we can find a representative satisfying the hypothesis of one of the cases in the statement.

5.2.2 Proof of Theorem 5.2.1

Suppose we have fixed Δ_s and we have picked a diagonal matrix s such that Δ_s is a basis for $\Psi_s = \{\alpha \in \Phi | \alpha(s) = 1\}$. Let $u \in C_G(s)$. Both s and u are in the image of the exponential map and are the image of a semisimple and a nilpotent element respectively. Since s is diagonal, it can be written as

$s = \exp(h)$ for $h \in \mathfrak{h}$. As we observed in the case of SL_n, the choice of h is not unique. We now see how $C_{Sp_{2n}}(s)$ varies according to the choice of Δ_s. We start by constructing a partition of n depending on Δ_s.

Define n_1 so that $\alpha_1, \ldots, \alpha_{n_1-1} \in \Delta_s$ and $\alpha_{n_1} \notin \Delta_s$ or $n_1 = n$. Let n_2 be such that $\alpha_{n_1+1}, \ldots, \alpha_{n_1+n_2-1} \in \Delta_s$ and $\alpha_{n_1+n_2} \notin \Delta_s$ or $n_1 + n_2 = n$. Inductively we define n_i to be the integer for which $\alpha_{\sum_{k=1}^{i-1} n_k+1}, \ldots, \alpha_{\sum_{k=1}^{i-1} n_k+n_i-1} \in \Delta_s$ and $\alpha_{\sum_{k=1}^{i} n_k} \notin \Delta_s$ or $\sum_{k=1}^{i} n_k = n$. Since s is assumed to be diagonal, we have

$$s = \begin{bmatrix} S_{n_1} & & & & & & & & \\ & S_{n_2} & & & & & & & \\ & & \cdot & & & & & & \\ & & & \cdot & & & & & \\ & & & & S_{n_m} & & & & \\ & & & & & S_{n_m}^{-1} & & & \\ & & & & & & \cdot & & \\ & & & & & & & \cdot & \\ & & & & & & & & S_{n_2}^{-1} & \\ & & & & & & & & & S_{n_1}^{-1} \end{bmatrix},$$

where S_{n_i} is a scalar matrix of dimension $n_i \times n_i$ and the eigenvalues appearing in the S_{n_i}'s are all distinct. Under our assumptions, a matrix in $C_{Sp_{2n}}(s)$ has block form

$$\begin{bmatrix} A_{n_1} & & & & & & & B_{n_1} \\ & A_{n_2} & & & & & 0 & \\ & & \cdot & & & \cdot & & \\ & & & \cdot & 0 & & & \\ & & & A_{n_m} & B_{n_m} & & & \\ & & & C_{n_m} & D_{n_m} & & & \\ & & 0 & & & \cdot & & \\ & \cdot & & & & & \cdot & \\ 0 & & & & & & D_{n_2} & \\ C_{n_1} & & & & & & & D_{n_1} \end{bmatrix},$$

where $A_{n_i} \in GL_{n_i}$ and $D_{n_i} = {\omega_{n_i}}^T A_{n_i}^{-1} \omega_{n_i}$, for $i \neq 1, m$. We now treat separately the case $i = 1$ and $i = m$. If $-\alpha_0 \in \Delta_s$, then the matrix $\begin{bmatrix} A_{n_1} & B_{n_1} \\ C_{n_1} & D_{n_1} \end{bmatrix} \in Sp_{2n_1}$ This follows from the fact that $-\alpha_0 \in \Delta_s$ implies that $S_{n_1} = \pm Id$. If $-\alpha_0 \notin \Delta_s$, then $A_{n_1} \in GL_{n_1}$, $D_{n_1} = {\omega_{n_1}}^T A_{n_1}^{-1} \omega_{n_1}$ and the blocks B_{n_1}, C_{n_1} are zeroes. Similarly, if $\alpha_n \in \Delta_s$, then $S_{n_m} = \pm 1$, so $\begin{bmatrix} A_{n_m} & B_{n_m} \\ Cn_m & D_{n_m} \end{bmatrix} \in Sp_{2n_m}$. Otherwise, once again $A_{n_m} \in GL_{n_m}$, $D = {\omega_{n_m}}^T A_{n_m}^{-1} \omega_{n_m}$ and B_{n_m}, C_{n_m} are the $n_m \times n_m$ zero matrix.

Remark 5.2.3. Under our assumptions, when the eigenvalues of S_{n_1} and S_{n_m} are ± 1, they must differ, otherwise $C_G(s)$ would be a larger subgroup.

Now we look at the unipotent elements that lie in $C_{Sp_{2n}}(s)$. To see how to parametrize the orbits, we distinguish into three cases depending on the presence of the roots $-\alpha_0$ and α_n in the set Δ_s, as in the statement of Theorem 5.2.1.
If $-\alpha_0, \alpha_n \notin \Delta_s$, the unipotent conjugacy classes are parametrized by refinements $(n_{i,j})$ of (n_i), i.e. $(n_{i,j})$ satisfies $\sum_j n_{i,j} = n_i$. This holds because in this case A_{n_i} can be any unipotent matrix of GL_{n_i}. When $-\alpha_0 \notin \Delta_s$ and $\alpha_n \in \Delta_s$, we parametrize the classes through $(n_{i,j})$, with $\sum_j n_{i,j} = n_i$, for $i \neq m$, and $\sum n_{m,j} = 2n_m$, with even multiplicity for odd values of $(n_{m,j})$. This is due to the fact that now the block $\begin{bmatrix} A_{n_m} & B_{n_m} \\ C_{n_m} & D_{n_m} \end{bmatrix}$ lies in Sp_{2n_m} and a unipotent in Sp_{2n_m} corresponds to a partition of $2n_m$ in which odd values appear an even number of times. Finally, when both $-\alpha_0$ and α_n belong to Δ_s, we need partitions $(n_{i,j})$, where $\sum_j n_{i,j} = n_i$, for $i \neq 1, m$, $\sum_j n_{i,j} = 2n_i$, for $i = 1, m$, and in $(n_{1,j})$ and $(n_{m,j})$ odd values have even multiplicity.
Assume we have chosen the partition $(n_{i,j})$ that determines the conjugacy class of u. Let $N \in \mathfrak{sp}_{2n}$ be such that $u = \exp(N)$. We want to define a suitable set of roots Δ_N such that $[h, N] = 0$ if and only if $\alpha(h) = 0, \forall \alpha \in \Delta_N$.

To do so, we first consider case (i), in which neither $-\alpha_0$ nor α_n are in Δ_s. In this case we can assume that u is of the form

$$\begin{bmatrix} U_{n_1} & & & & & & & & \\ & U_{n_2} & & & & & & & \\ & & \cdot & & & & & & \\ & & & \cdot & & & & & \\ & & & & U_{n_m} & & & & \\ & & & & & V_{n_m} & & & \\ & & & & & & \cdot & & \\ & & & & & & & \cdot & \\ & & & & & & & & V_{n_2} & \\ & & & & & & & & & V_{n_1} \end{bmatrix},$$

with $V_{n_i} = {\omega_{n_i}}^T U_{n_i}^{-1} \omega_{n_i}, \forall i$ and U_{n_i} is such that $S_{n_i} U_{n_i}$ is in Jordan normal form, with blocks of dimensions $n_{i,j}$. As we have already observed, the exponential preserves the blocks along the diagonal, therefore Δ_N can be defined as

$$\Delta_N := \{\varepsilon_i - \varepsilon_{i+1} | u_{i,i+1} \neq 0, 1 \leq i < n\}.$$

Now observe that $\alpha(h) = 0$ for all $\alpha \in \Delta_N$ simply means $h_i = h_{i+1}$ when $\varepsilon_i - \varepsilon_{i+1} \in \Delta_N$, which results in the condition

$$s_{i,i} = e^{h_{i,i}} = e^{h_{i+1,i+1}} = s_{i+1,i+1}.$$

This is always satisfied as $\Delta_N \subseteq \Delta_s$ and therefore we knew already that $s_{i,i} = s_{i+1,i+1}$. As a result, we can find h, N such that $[h, N] = 0$, from which $g = \exp(h)\exp(N) = \exp(h + N)$.

Consider now case (ii), for which $\alpha_n \in \Delta_s$ and $-\alpha_0 \notin \Delta_s$. This time u has a representative

$$\begin{bmatrix} U_{n_1} &&&&&&&& \\ & U_{n_2} &&&&&&& \\ && \cdot &&&&&& \\ &&& \cdot &&&&& \\ &&&& U_{n_m} & X_{n_m} &&& \\ &&&& Y_{n_m} & V_{n_m} &&& \\ &&&&&& \cdot && \\ &&&&&&& \cdot & \\ &&&&&&&& V_{n_2} & \\ &&&&&&&&& V_{n_1} \end{bmatrix},$$

with U_{n_i}, V_{n_i} as above for $i \neq m$ and $\begin{bmatrix} U_{n_m} & X_{n_m} \\ Y_{n_m} & V_{n_m} \end{bmatrix} \in Sp_{2n_m}$ is a representative of the partition $(n_{m,j})$ of $2n_1$.

Define as in the previous section $2q_{m,j}$ or $2q_{m,j} + \epsilon_{m,j}$, $\epsilon_{m,j} \in \{\pm 1\}$, to be the number of repetitions of the value $n_{m,j}$ if it is odd or even respectively. Recall that a representative of the orbit of $\begin{bmatrix} U_{n_m} & X_{n_m} \\ Y_{n_m} & V_{n_m} \end{bmatrix}$ can be obtained through the immersion

$$\prod_{n_{m,j}\ odd} (Sp_{2n_{m,j}})^{q_j} \times \prod_{n_{m,j}\ even;\ q_{m,j} \neq 0} (Sp_{2n_{m,j}})^{q_{m,j}} \times \prod_{\epsilon_{m,j} \neq 0} Sp_{n_{m,j}} \longrightarrow Sp_{2n_1},$$

taking a representatives of $(n_{m,j}, n_{m,j})$ for the factors $Sp_{2n_{m,j}}$ and of $(n_{m,j})$ for the factors $Sp_{n_{m,j}}$ as in the previous section. If each $n_{m,j}$ is repeated an even number of times, i.e. $\epsilon_{m,j} = 0$ for all even $n_{m,j}$, then, by our choice of the representatives, $\begin{bmatrix} U_{n_m} & X_{n_m} \\ Y_{n_m} & V_{n_m} \end{bmatrix}$ will have X_{n_m} and Y_{n_m} equal to zero, since no factor of the form $Sp_{n_{m,j}}$ appear. Therefore Δ_N can be defined as above

$$\Delta_N := \{\varepsilon_i - \varepsilon_{i+1} | u_{i,i+1} \neq 0, 1 \leq i < n\}.$$

Hence Δ_N contains only roots of the form $\varepsilon_i - \varepsilon_{i+1}$ and therefore the condition on h that follows can always be satisfied by what we observed in the previous case. So $g = \exp(h + N) \in E_{Sp_{2n}}$.

Assume now that in $(n_{m,j})$ there are some even values which appear an odd number of times, hence for some j we have $\epsilon_j = 1$. Now for the factors of the type $Sp_{n_{m,j}}$ we are taking the upper triangular representative of the regular orbit, so we can still assume that Y_{n_m} is zero, but X_{n_m} contains blocks on the antidiagonal which are nonzero. By Remarks 5.1.1 and 5.1.2 this implies that Δ_N must contain at least a root of the form $2\varepsilon_k$, where k ranges between $n-(n_m-1)$ and n.

The condition $2\varepsilon_k(h)=0$ means that $h_{k,k}=0$ and thus the $k-$th component of s along the diagonal has to be equal to $e^{h_{k,k}}=1$. As we have already pointed out $S_{n_m}=\pm Id$, hence this condition tells us that s is in the image of the exponential map only when $S_{n_m}=Id$. This can be stated equivalently that either $su\in E_G$ or $-su\in E_G$.

Finally, we turn to case (iii), in which both $-\alpha_0$ and α_n are in Δ_s. Here we need to look at both $(n_{1,j})$ and $(n_{m,j})$. If all values have an even multiplicity, then we have no further condition on s. Assume that only in $(n_{1,j})$ at least an even value is repeated an odd number of times, then to obtain $g\in E_{Sp_{2n}}$ the condition $S_{n_1}=Id$ must hold, since there exists at least an index k_1, $1\le k_1\le n_1$, for which $2\varepsilon_{k_1}\in\Delta_N$. If this happens instead only for $(n_{m,j})$, there exists a k_2, $n-(n_m-1)\le k_2\le n$, such that $2\varepsilon_{k_2}\in\Delta_N$. We deduce the condition $S_{n_m}=Id$. Assume finally that this is true for both partitions. Now this clearly means that there are at least two indeces k_1 and k_2, $1\le k_1\le n_1$, $n-(n_m-1)\le k_2\le n$, such that $2\varepsilon_{k_1},2\varepsilon_{k_2}\in\Delta_N$. As a consequence, to get $g\in E_{Sp_{2n}}$, the blocks S_{n_1} and S_{n_m} should be the identity matrix of dimension n_1 and n_m respectively, but, since we are now assuming that the eigenvalues of S_{n_1} and S_{n_m} are distinct, this cannot happen, hence such g is never in the image.

Example 9. Consider for example $g\in Sp_4$ with

$$s=\begin{bmatrix} s_1 & & & \\ & -1 & & \\ & & -1 & \\ & & & s_1^{-1}\end{bmatrix}, u=\begin{bmatrix} 1 & & & \\ & 1 & -1 & \\ & & 1 & \\ & & & 1\end{bmatrix}.$$

Then u is the exponential of

$$N=\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},$$

so $\Delta_N=\{2\varepsilon_2\}$. We are looking for a semisimple $h\in\mathfrak{sp}_4$ such that $2\varepsilon_2(h)=0$ and $s=\exp(h)$. But $2\varepsilon_2(h)=0$ implies h has the form

$$h=\begin{bmatrix} h_1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -h_1\end{bmatrix},$$

hence its image must have 1 in the second and third position along the diagonal. Therefore $g \notin E_{Sp_4}$. Observe that we are in case (ii) of Theorem 5.2.1 and $-g$ is indeed in E_G.

5.2.3 Parametrization of the Classes in E_G

Now we see how we can interpret what we have just seen in terms the usual parameters for Jordan classes. Recall once again that a class is determined (up to conjugation) by the triple

$$(C_G(s), sZ(C_G(s))^o, \mathcal{O}_u^{C_G(s)}).$$

We know that fixing $C_G(s)$ means choosing the set Δ_s. If Δ_s contains neither $-\alpha_0$ nor α_n, for all choices of the second and third component the Jordan class lies in the image of the exponential map.

Assume $\alpha_n \in \Delta_s$ and $-\alpha_0 \notin \Delta_s$. As above we construct the partition (n_i). We check if all values of $(n_{m,j})$ appear an even number of times. If this is the case, the class is in the image. Otherwise, only for $sZ(C_G(s))^o = Z(C_G(s))^o$ the class lies in $E_{Sp_{2n}}$. If $-\alpha_0 \in \Delta_s$ and $\alpha_n \notin \Delta_s$, we can replace s by a conjugate to obtain again $-\alpha_0 \notin \Delta_s$ and $\alpha_n \in \Delta_s$, so this class is actually included in the previous case. The last possible case is that both α_n and $-\alpha_0$ are in Δ_s. Now we need to look at both $(n_{1,j})$ and $(n_{m,j})$. Once again if no values have an odd multiplicity, the class is in the image. If in both partitions an even value appears an odd number of times, then the class cannot be in $E_{Sp_{2n}}$. If this happens just fo the first partition, the condition $S_{n_1} = Id$ determines the second component for which the class is in the image. Lastly, if it is true only for $(n_{m,j})$, it's the condition $S_{n_m} = Id$ which determines the coset $sZ(C_G(s))^o$ such that the class is in $E_{Sp_{2n}}$.

Chapter 6

The Special Orthogonal Group SO_{2n+1}

We consider now the special orthogonal group over the complex numbers SO_{2n+1}. The form that it leaves invariant will be

$$J = \begin{bmatrix} & & \omega_n \\ & 1 & \\ \omega_n & & \end{bmatrix} = \omega_{2n+1},$$

with ω_n as in (5.1).
Its associated Lie algebra is

$$\mathfrak{so}_{2n+1} = \{A \in \mathfrak{gl}_{2n+1} | A^T J + JA = 0\},$$

with toral subalgebra $\mathfrak{h}$ given by the diagonal matrices. A root system for $\mathfrak{h}$ is given by

$$\Phi = \{\pm\epsilon_i \pm \epsilon_j, 1 \leq i < j \leq n; \pm\epsilon_i, 1 \leq i \leq n\}$$

with basis

$$\Delta = \{\alpha_i = \epsilon_i - \epsilon_{i+1}, 1 \leq i < n; \alpha_n = \epsilon_n\}.$$

The highest root is $\alpha_0 = \epsilon_1 + \epsilon_2$.

Once again we will use partitions to parametrize the unipotent classes in the centralizer of a semisimple element. To do so, we need to make a few remarks concerning the classes in special orthogonal groups. A partition (l_i) of l corresponds to at least a class in SO_l if each even value of (l_i) occurs an even number of times. Moreover (l_i) corresponds to a single class if there exists an odd value l_i, otherwise the partition corresponds to exactly two classes. This phenomenon appears since the two classes are conjugated by a matrix in O_l with negative determinant, i.e. one can be obtain from the other by applying an external automorphism. Clearly this can occur only when l is even.

We are now ready to state the theorem which tells us which classes are in the image of the exponential map.

Theorem 6.0.1. *Let $\Delta_s \subseteq \Delta \cup \{-\alpha_0\}$. Choose a diagonal $s \in SO_{2n+1}$ such that Δ_s is a basis of $\Psi_s = \{\alpha \in \Phi | \alpha(s) = 1\}$. Let $u \in C_{SO_{2n+1}}(s)$. Then*

(i) If (up to conjugation) $\Delta_s \subseteq \Delta$, then $J(su) \subseteq E_{SO_{2n+1}}$.

(ii) If $-\alpha_0, \alpha_1 \in \Delta_s$, then, if $\alpha_n \in \Delta_s$, for some (n_i) such that $\sum_i n_i = n$, we have $C_{SO_{2n+1}}(s)^o \cong SO_{2n_1} \times \prod_{i=2}^{m-1} GL_{n_i} \times SO_{2n_m+1}$, otherwise, for some (n_i') such that $\sum_i n_i' = n$, $C_{SO_{2n+1}}(s)^o \cong SO_{2n_1'} \times \prod_{i=2}^{m} GL_{n_i'}$. By setting v_1 to be the projection of u to the first factor (in both cases), $J(su) \subseteq E_{SO_{2n+1}}$ if and only if the partition associated to the class of v_1 has no value with odd multiplicity.

Remark 6.0.2. For each class there exists a representative $g = su$ such that it falls in one of the cases listed in the statement of Theorem 6.0.1.

6.1 Proof of Theorem 6.0.1

Suppose we have fixed $\Delta_s \subseteq \Delta \cup \{-\alpha_0\}$ and a diagonal s such that Δ_s is a basis of $\Psi_s = \{\alpha \in \Phi | \alpha(s) = 1\}$. As we did in the previous chapter, we construct a partition (n_i) of n associated to Δ_s so that

$$s = \begin{bmatrix} S_{n_1} & & & & & & \\ & \ddots & & & & & \\ & & S_{n_m} & & & & \\ & & & 1 & & & \\ & & & & S_{n_m}^{-1} & & \\ & & & & & \ddots & \\ & & & & & & S_{n_1}^{-1} \end{bmatrix},$$

where each block S_{n_i} is a scalar matrix of size $n_i \times n_i$. When $\alpha_n \in \Delta_s$, the block S_{n_m} will be the identity matrix of dimension n_m. When α_1 and $-\alpha_0$ are contemporarily in Δ_s, we are assuming $S_{n_1} = -Id_{n_1}$. Indeed, $S_{n_m} = Id$ would mean that $C_G(s)^o$ is a larger subgroup, i.e. Δ_s would not be a basis for Ψ_s.

Statement (i) can be proved by applying 3.0.13 as SO_{2n+1} is adjoint. As for the computation of the centralizer, the result follows from the block form of s.

We now turn to prove statement (ii), in which we are assuming $\alpha_1, -\alpha_0 \in \Delta_s$, which means, under our assumptions, $S_{n_1} = -Id_{n_1}$. This time a matrix in $C_{SO_{2n+1}}(s)^o$ is

$$\begin{bmatrix} A_{n_1} & & & & & & & B_{n_1} \\ & A_{n_2} & & & & & 0 & \\ & & \cdot & & & \cdot & & \\ & & & A_{n_m} & a & B_{n_m} & & \\ & & & b & 1 & c & & \\ & & & C_{n_m} & d & D_{n_m} & & \\ & & \cdot & & & \cdot & & \\ & 0 & & & & & D_{n_2} & \\ C_{n_1} & & & & & & & D_{n_1} \end{bmatrix},$$

with A_{n_i}, D_{n_i} as above for $i \neq 1, m$, while $\begin{bmatrix} A_{n_1} & B_{n_1} \\ C_{n-1} & D_{n_1} \end{bmatrix} \in SO_{2n_1}$ and $\begin{bmatrix} A_{n_m} & a & B_{n_m} \\ b & 1 & c \\ C_{n_m} & d & D_{n_m} \end{bmatrix} \in SO_{2n_m+1}$ if $\alpha_n \in \Delta_s$, otherwise $a, b, c, d, B_{n_m}, C_{n_m}$ are all zero and $A_{n_m} \in GL_{n_m}, D_{n_m} = \omega_{n_m}{}^T A_{n_i}^{-1} \omega_{n_m}$. Now the unipotent classes correspond to $(n_{i,j})$, with $\sum_j n_{i,j} = n_i$ when $i \neq 1, m$, $\sum_j n_{1,j} = 2n_1$, with even $n_{1,j}{}'$s occurring an even number of times and $\sum_j n_{m,j} = 2n_m + 1$, with even values of $(n_{m,j})$ having even multiplicity if $\alpha_n \in \Delta_s$, otherwise $(n_{m,j})$ is any partition of n_m. Again we examine the blocks separately. By what we have seen above, any choice of $(n_{i,j})$ does not imply any condition to be satisfied by the blocks $S_{n_i}, i \neq 1$. Since $S_{n_1} = S_{n_1}^{-1} = -Id_{n_1}$, we now need to understand when $-Id_{2n_1} v \in E_{SO_{2n_1}}$, where v is a representative for $(n_{1,j})$. We would like to apply Theorem 3.0.6, hence we need to verify when $-Id_{2n_1} \in C_{SO_{2n_1}}(v)^o$. To do so, we refer to [6] to study the centralizer of a unipotent element in an orthogonal group O_l.

6.1.1 Centralizer of a unipotent element in O_l

First we fix a partition of l in which even values occur with even multiplicity and we denote by r_i the number of times the value i appears in the partition. We will consider the unipotent v, having Jordan blocks of size i repeated r_i times along the diagonal. There exists an orthogonal form for which such v lies in an orthogonal group SO_l. Here we may need to replace our usual bilinear form by one for which we can take a representative of the chosen partition as the v above. Observe that this procedure is harmless for our purpose. Indeed, changing the bilinear form does not affect whether or not $-Id_l$ lies in $C_{O_l}(v)^o$, as $-Id_l$ is always left invariant by conjugation.

Choosing another form means performing a change of basis, hence two special orthogonal groups relative to different equivalent forms are conjugate. We construct a group denoted by R' as follows. Consider first the case in which $r_i \neq 0$ for only one value of i. An element of R' is a $r_i i \times r_i i$ matrix of the form

$$\begin{bmatrix} A_{1,1} & \dots & A_{1,r_i} \\ \vdots & \dots & \vdots \\ A_{r_i,1} & \dots & A_{r_i,r_i} \end{bmatrix}, \tag{6.1}$$

where each $A_{j,j}$ is a scalar matrix of dimension $i \times i$. If $r_i \neq 0$ for different values of i, a matrix in R' will have blocks of size ir_i on the diagonal of the form (6.1). Now we define R to be the set of matrices in R' satisfying the orthogonality condition, i.e. for which each block of dimension $ir_i \times ir_i$ lies in O_{ir_i}. By Theorem 3.1 in [6], we have $C_{O_l}(v) = QR$, where Q is the unipotent radical of $C_{O_l}(v)$. Moreover, by Lemma 3.11 in [6], $C_{SO_l}(v) = Q(SO_l \cap R)$. Since Q is connected, we study R to understand what conditions are required to have $-Id_l \in C_{SO_l}(v)^o$.
A connected subgroup of O_k lies in SO_k, therefore an element of R is in the connected component of the identity if and only if each block of size $ir_i \times ir_i$ is contained in SO_{ir_i}. For $-Id_l$, we have that the determinant of each block is $(-1)^{ir_i}$, thus $-Id_l$ is in the connected component of the identity if and only if ir_i is even for each i.

6.1.2 Conclusion of the Proof

Going back to our original problem, this gives the condition that each odd value of $(n_{1,j})$ has to occur an even number of times to have $g = su \in E_{SO_{2n+1}}$. Since we know the multiplicity of even values is necessarily even, we can simply say that every value of the partition must appear an even number of times.

6.2 Parametrization of the Classes in E_G

We briefly see what the result means in terms of the parameters

$$(C_G(s)^o, sZ(C_G(s)^o)^o, \mathcal{O}_u^{C_G(s)}).$$

Suppose we have fixed Δ_s, hence $C_{SO_{2n+1}}(s)^o$. As we have seen, when Δ_s can be taken to be a subset of Δ_s, the class is in the image. Assume now $\alpha_1, -\alpha_0 \in \Delta_s$. We parametrize the unipotent orbits through suitable partitions $(n_{i,j})$. We consider $(n_{1,j})$ and check if all odd values appear an even number of times. If this is the case, then the class is in the image. Indeed this implies that each product of the values of $(n_{1,j})$ and their multiplicity is an even number, therefore by the discussion in the previous section we can conclude. If at least an odd value occurs with odd multiplicity, the class is not contained in $E_{SO_{2n+1}}$ because for that value the product with its multiplicity will be odd.

Chapter 7

The Special Orthogonal Group SO_{2n}

We now treat the special orthogonal group over the complex numbers in even dimension. Now the form left invariant by SO_{2n} will be

$$J = \begin{bmatrix} & \omega_n \\ \omega_n & \end{bmatrix} = \omega_{2n},$$

with ω_n as in (5.1).
Its associated Lie algebra is

$$\mathfrak{so}_{2n} = \{A \in \mathfrak{gl}_{2n} | A^T J + JA = 0\}.$$

As usual we consider the toral subalgebra $\mathfrak{h}$ given by the diagonal matrices. The root system is

$$\Phi = \{\pm\varepsilon_i \pm \varepsilon_j, 1 \leq i < j \leq n\}$$

with basis

$$\Delta = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1}, 1 \leq i < n, \alpha_n = \varepsilon_{n-1} + \varepsilon_n\}.$$

The highest root is $\alpha_0 = \varepsilon_1 + \varepsilon_2$.

We now present the theorem describing the classes in $E_{SO_{2n}}$.

Theorem 7.0.1. *Let $\Delta_s \subseteq \Delta \cup \{-\alpha_0\}$. Choose a diagonal matrix s such that Δ_s is a basis of $\Psi_s = \{\alpha \in \Phi | \alpha(s) = 1\}$. Let $u \in C_{SO_{2n}}(s)$. Then*

(i) If $\{-\alpha_0, \alpha_1\}, \{\alpha_{n-1}, \alpha_n\} \not\subseteq \Delta_s$, then $J(su) \subseteq E_{SO_{2n}}$.

(ii) If (up to conjugation) $\{\alpha_{n-1}, \alpha_n\} \subseteq \Delta_s$ and $\{-\alpha_0, \alpha_1\} \not\subseteq \Delta_s$, then, if the partition of $2n$ corresponding to the class of u has only values with even multiplicity, $J(su) \subseteq E_{SO_{2n}}$. Otherwise, either $J(su)$ or $J(-su)$ is in $E_{SO_{2n}}$.

(iii) *If* $\{-\alpha_0, \alpha_1\}, \{\alpha_{n-1}, \alpha_n\} \subseteq \Delta_s$, *then, for some* (n_i) *such that* $\sum_i n_i = n$, $C_{SO_{2n}}(s)^o \cong SO_{2n_1} \times \prod_{i=2}^{m-1} GL_{n_i} \times SO_{2n_m}$. *Let* v_1, v_m *be the projection of* u *on* SO_{2n_1}, SO_{2n_m} *respectively. Then*

(a) *If the partitions corresponding to the classes of* v_1 *and* v_m *in* SO_{2n_1} *and* SO_{2n_m} *present no value with odd multiplicity, then* $J(su) \subseteq E_{SO_{2n}}$.

(b) *If one of the two partitions as in* (a) *has a value with odd multiplicity, then either* $J(su)$ *or* $J(-su)$ *is in* $E_{SO_{2n}}$.

(c) *If both partitions as in* (a) *have at least a value with odd multiplicity, then* $J(su) \nsubseteq E_{SO_{2n}}$.

Remark 7.0.2. The result covers all possible cases. Indeed the only possibility not covered by the statement is that in which Δ_s contains $-\alpha_0$, but not α_n, but it suffices to replace s by a conjugate to obtain $\alpha_n \in \Delta_s$ and $-\alpha_0 \notin \Delta_s$.

7.1 Proof of Theorem 7.0.1

Once we have fixed a subset $\Delta_s \subseteq \Delta \cup \{-\alpha_0\}$ and a diagonal s such that Δ_s is a basis of $\Psi_s = \{\alpha \in \Phi | \alpha(s) = 1\}$, we define a partition (n_i) of n such that

$$s = \begin{bmatrix} S_{n_1} & & & & & & \\ & \cdot & & & & & \\ & & \cdot & & & & \\ & & & S_{n_m} & & & \\ & & & & S_{n_m}^{-1} & & \\ & & & & & \cdot & \\ & & & & & & \cdot \\ & & & & & & & S_{n_1}^{-1} \end{bmatrix},$$

with S_{n_i} a scalar matrix. When $\alpha_{n-1}, \alpha_n \in \Delta_s$ the block S_{n_m} is $\pm Id_{n_m}$. Similarly, when $-\alpha_0, \alpha_1 \in \Delta_s$, then $S_{n_1} = \pm Id_{n_1}$. If $-\alpha_0, \alpha_1, \alpha_{n-1}, \alpha_n \in \Delta_s$, then, by assumption, the blocks S_{n_1} and S_{n_m} have opposite sign. Moreover, when only one between 1 and -1 appears as an eigenvalue we may suppose $\Delta_s \subseteq \Delta$.
Assume now we are in case (i), so $\{-\alpha_0, \alpha_1\}$ and $\{\alpha_{n-1}, \alpha_n\}$ are not subsets

of Δ_s, then a matrix in $C_{SO_{2n}}(s)$ will have the form

$$\begin{bmatrix} A_{n_1} & & & & & & \\ & \cdot & & & & & \\ & & \cdot & & & & \\ & & & A_{n_m} & & & \\ & & & & D_{n_m} & & \\ & & & & & \cdot & \\ & & & & & & \cdot \\ & & & & & & & D_{n_1} \end{bmatrix},$$

where $A_{n_i} \in GL_{n_i}$ and $D_{n_i} = {\omega_{n_i}}^T A_{n_i}^{-1} \omega_{n_i}$. We may parametrize the unipotent classes in $C_{SO_{2n}}(s)$ through partitions $(n_{i,j})$ such that $\sum_j n_{i,j} = n_i$. We now want to prove that for any choice of the partition $g = su \in E_{SO_{2n}}$. To do so, we can argue as we did for the symplectic group when $-\alpha_0, \alpha_n \notin \Delta_s$.

Remark 7.1.1. Here we cannot suppose we have gathered all the eigenvalues of s that are equal in a unique block S_{n_i}. For example, the matrices

$$\begin{bmatrix} \lambda & & & \\ & \lambda & & \\ & & \lambda^{-1} & \\ & & & \lambda^{-1} \end{bmatrix}, \begin{bmatrix} \lambda & & & \\ & \lambda^{-1} & & \\ & & \lambda & \\ & & & \lambda^{-1} \end{bmatrix},$$

with $\lambda \neq \pm 1$, are not conjugate in SO_4. Indeed, they are conjugate in O_4 through a matrix of negative determinant. This actually tells us that one is the image of the other through an external automorphism (given precisely by the conjugation by the matrix in O_4 through which they are conjugates). Such automorphism, being given by conjugation by a matrix, certainly commutes with the exponential map, hence we do not really need to treat these two cases separately. Notice that this situation cannot occur when ± 1 appears as an eigenvalue.

We proceed to study case (ii), in which $\alpha_{n-1}, \alpha_n \in \Delta_s$ and $-\alpha_0, \alpha_1$ are not both in Δ_s. The elements of $C_{SO_{2n}}(s)$ have block form

$$\begin{bmatrix} A_{n_1} & & & & & & 0 \\ & \cdot & & & & \cdot & \\ & & \cdot & & \cdot & & \\ & & & A_{n_m} \quad B_{n_m} & & & \\ & & & C_{n_m} \quad D_{n_m} & & & \\ & & \cdot & & \cdot & & \\ & \cdot & & & & \cdot & \\ 0 & & & & & & D_{n_1} \end{bmatrix},$$

with A_{n_i}, D_{n_i} as above for $i \neq m$ and $\begin{bmatrix} A_{n_m} & B_{n_m} \\ C_{n_m} & D_{n_m} \end{bmatrix} \in SO_{2n_m}$. Now unipotent classes can be parametrized through $(n_{i,j})$ with $\sum_j n_{i,j} = n_i$, for $i \neq m$, and

$\sum_j n_{m,j} = 2n_m$ with even values of $(n_{m,j})$ occurring with even multiplicity. If $S_{n_m} = Id_{n_m}$, then, for any choice of $(n_{i,j})$, g lies in $E_{SO_{2n_m}}$, while when $S_{n_m} = -Id_{n_m}$, then $g \in E_{SO_{2n}}$ if and only if all values of $(n_{m,j})$ occur with even multiplicity. Indeed, we know that g is in the image if and only if $S_{n_m} \in C_{SO_{2n_m}}(v)^o$, where v is a representative in SO_{2n_m} of the partition $(n_{m,j})$. When $S_{n_m} = Id_{n_m}$, this clearly holds. When $S_{n_m} = -Id_{n_m}$, as seen in Section 6.1.1, the condition is satisfied if and only if all values of $(n_{m,j})$ have even multiplicity.

Suppose now $-\alpha_0, \alpha_1, \alpha_{n-1}, \alpha_n \in \Delta_s$, so we have to prove statement (iii). A matrix in $C_{SO_{2n}}(s)^o$ has block form

$$\begin{bmatrix} A_{n_1} & & & & & & B_{n_1} \\ & A_{n_2} & & & & 0 & \\ & & \cdot & & \cdot & & \\ & & & A_{n_m} \quad B_{n_m} & & & \\ & & & C_{n_m} \quad D_{n_m} & & & \\ & & \cdot & & \cdot & & \\ & 0 & & & & D_{n_2} & \\ C_{n_1} & & & & & & D_{n_1} \end{bmatrix},$$

where A_{n_i} and D_{n_i} are as above, for $i \neq 1, m$, and $\begin{bmatrix} A_{n_1} & B_{n_1} \\ C_{n_1} & D_{n_1} \end{bmatrix} \in SO_{2n_1}$, $\begin{bmatrix} A_{n_m} & B_{n_m} \\ C_{n_m} & D_{n_m} \end{bmatrix} \in SO_{2n_m}$. To parametrize unipotent classes we take $(n_{i,j})$ with $\sum_j n_{i,j} = n_i$, for $i \neq 1, m$, and $\sum_j n_{i,j} = 2n_i$ and even values of $(n_{i,j})$ repeated an even number of times, for $i = 1, m$. To check if $g \in E_{SO_{2n}}$ we consider the blocks separately as we did in the previous chapter. For $i \neq 1, m$, the choice of $(n_{i,j})$ does not imply any condition on s. We now look at $(n_{1,j})$ and $(n_{m,j})$. If in both partitions all values appear with even multiplicity, then $g \in E_{SO_{2n}}$. Indeed, if this is the case, $\pm Id_{n_i} \in C_{SO_{2n_i}}(v_i)^o, i = 1, m$, where v_i is a representative of the unipotent class in SO_{2n_i} corresponding to $(n_{i,j})$. Therefore, $S_{n_i} \in C_{SO_{2n_i}}(v_i)^o, i = 1, m$, so $g \in E_{SO_{2n}}$. If this happens just for $(n_{1,j})$, then to have $g \in E_{SO_{2n}}$ the block S_{n_m} must be the identity block and consequently $S_{n_1} = -Id_{n_1}$ by our assumptions. Similarly, if it holds only for $(n_{m,j})$, then we have the condition $S_{n_1} = Id_{n_1}$, so $S_{n_m} = -Id_{n_m}$. Finally if in both partitions at least a value has odd multiplicity, then g cannot be in $E_{SO_{2n}}$.

7.2 Parametrization of the Classes in E_G

We can now check which Jordan classes are in $E_{SO_{2n}}$. We start by fixing Δ_s, hence $C_{SO_{2n}}(s)^o$. If $\{-\alpha_0, \alpha_1\}, \{\alpha_{n-1}, \alpha_n\}$ are not subsets of Δ_s, then the class is in $E_{SO_{2n}}$. Assume $\alpha_{n-1}, \alpha_n \in \Delta_s$ and at least one between $-\alpha_0$ and α_1 is not in Δ_s. We construct (n_i) and choose $(n_{i,j})$ to fix $\mathcal{O}_u^{C_{SO_{2n}}(s)^o}$.

If in $(n_{m,j})$ all values have even multiplicity, the class is in the image of the exponential. If this does not hold, then the condition for the representative that $S_{n_m} = Id_{n_m}$ determines the choice of the second parameter $sZ(C_{SO_{2n}}(s)^o)^o$ for which the class is in $E_{SO_{2n}}$. Finally, consider the case $-\alpha_0, \alpha_1, \alpha_{n-1}, \alpha_n \in \Delta_s$. Construct (n_i) and fix $(n_{i,j})$. If all values of $(n_{1,j})$ and $(n_{m,j})$ have even multiplicity, then the class is in the image. If this does not hold for one of the two partitions, say $(n_{k,j})$, then the condition $S_{n_k} = Id_{n_k}$ determines the coset $sZ(C_{SO_{2n}}(s)^o)^o$ such that the class is in $E_{SO_{2n}}$. When in both partitions at least a value has odd multiplicity, the class is not in $E_{SO_{2n}}$.

Chapter 8

The Spin Group $Spin_n$

We now want to deduce some information on the image of the exponential map in the spin group $Spin_n$ over $\mathbb{C}$ from our study on the special orthogonal groups.
$Spin_n$ is the universal cover of SO_n, i.e. it is the simply connected group in the isogeny class of SO_n. We define $\pi : Spin_n \longrightarrow SO_n$ to be the isogeny between the groups. We may observe that, in case n is odd, the kernel $\ker(\pi)$ is precisely $Z(Spin_n)$ and SO_n is the adjoint group of $Spin_n$. When n is even this is no longer true, although $|\ker(\pi)| = 2$ also in this case.
In the following we set $G = Spin_n$, $\overline{G} = SO_n$ and $\ker(\pi) = \{1, z\}$.

Theorem 8.0.1. *Let $g \in G$ and $\bar{g} = \pi(g)$, then*

(i) If $J(\bar{g}) \not\subseteq E_{\overline{G}}$, then $J(g) \not\subseteq E_G$.

(ii) If $J(\bar{g}) \subseteq E_{\overline{G}}$, then, if $\bar{g}$ has Jordan decomposition $\bar{g} = \bar{s}\bar{u}$,

  (a) If the partition associated to the $\overline{G}-$class of $\bar{u}$ has an odd value with multiplicity ≥ 2, then $J(g) \subseteq E_G$.

  (b) Otherwise, either $J(g)$ or $J(zg)$ is in E_G.

Before proving Theorem 8.0.1, we state Proposition 3.19 is [6].

Proposition 8.0.2. *Let $u \in G$ be unipotent and set $\pi(u) = \bar{u}$. Then we have $|C_G(u)/C_G(u)^o| = |C_{\overline{G}}(\bar{u})/C_{\overline{G}}(\bar{u})^o|$ if the partition associated to the class of $\bar{u}$ in $\overline{G}$ has an odd value occurring with multiplicity ≥ 2.*
Otherwise, $|C_G(u)/C_G(u)^o| = 2|C_{\overline{G}}(\bar{u})/C_{\overline{G}}(\bar{u})^o|$.

Proof of Theorem 8.0.1. We start by considering statement (i). If $\bar{g}$ is not in $E_{\overline{G}}$, then certainly $g \notin E_{\overline{G}}$. Indeed, $\bar{g} \notin E_G$ means that its semisimple part $\pi(s) = \bar{s}$ is not in $C_{\overline{G}}(\bar{u})^o$ and so s cannot be in $C_G(u)^o$. If it was the case then clearly $\bar{s} = \pi(s) \in C_{\overline{G}}(\bar{u})^o$, contradicting the assumption, so (i) is proved.
We proceed to case (ii), so we assume $\bar{g} \in E_{\overline{G}}$.

Recall that by Lemma 1.5.15, we have $|C_G(u)/C_G(u)^o| = |C_{\overline{G}}(\bar{u})/C_{\overline{G}}(\bar{u})^o|$ if and only if $yC_G(u)^o = C_G(u)^o$ for any $y \in \ker(\pi)$ or, equivalently, if and only if $\ker(\pi) \subseteq C_G(u)^o$.

Suppose the partition associated to u has an odd value which appears at least two times, then, by Proposition 8.0.2 and Lemma 1.5.15, we have $\ker(\pi) \subseteq C_G(u)^o$, so $ys \in C_G(u)^o$ for all $y \in \ker(\pi)$. We can deduce that in this case any lift of $\bar{g}$ is in E_G, in particular this will hold for g, so the whole class $J(g) \subseteq E_G$. Consider now the case in which no odd value of the partition has multiplicity ≥ 2. Now $|\pi^{-1}(C_{\overline{G}}(\bar{u})^o)/C_G(u)^o| = 2$, so there will be a lift s_1 of $\bar{s}$ in $C_G(u)^o$ and the other out of $C_G(u)^o$, since we will have $zs_1 \notin C_G(u)^o$. This means a lift of $\bar{g}$ will lie in E_G and the other will not be in E_G and the same holds for their respective Jordan classes.

□

Remark 8.0.3. When we studied the special orthogonal groups, we did not use partitions of n to parametrize unipotent classes in $C_G(s)^o$. We now see how to recover the partition of n associated to the class of a unipotent element from those we used.
We start by considering the case in which n is odd, so we write $n = 2k+1$. For a fixed Δ_s we constructed a partition (k_i) of k and we parametrized the unipotent classes through suitable partitions $(k_{i,j})$ as in Chapter 6. If $\alpha_k \notin \Delta_s$ and $\{-\alpha_0, \alpha_1\} \nsubseteq \Delta_s$, we simply need to count twice each value in $(k_{i,j})$ and add a 1 correponding to the isolated block in the middle. If $-\alpha_0, \alpha_1 \in \Delta_s$ then the values in $(k_{1,j})$ has to be counted only once. Similarly, when $\alpha_k \in \Delta_s$ the values $(k_{m,j})$ will also be counted just once and we will not need to add the value 1 for the block in the middle.
We now pass to the case in which $n = 2k$ is even. Again we associeted Δ_s a partition (k_i) and parametrized the classes through partitions $(k_{i,j})$ as explained in Chapter 7. If $\{-\alpha_0, \alpha_1\}, \{\alpha_k, \alpha_{k-1}\}$ are not contained in Δ_s, then simply double each value in $(k_{i,j})$. If $\{-\alpha_0, \alpha_1\} \subseteq \Delta_s$, then count just once the values in $(k_{1,j})$ and when $\{\alpha_{k-1}, \alpha_k\} \subseteq \Delta_s$, count once those in $(k_{m,j})$.

Example 10. In SO_{12}, assume $\Delta_s = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_5 - \varepsilon_6, \varepsilon_5 + \varepsilon_6\}$, then the associated partition is $(3, 1, 2)$, so $n_1 = 3$, $n_2 = 1$, $n_3 = 2$. We choose for example the unipotent class associated to $n_{1,1} = 2$, $n_{1,2} = 1$, $n_{2,1} = 1$, $n_{3,1} = 1$, $n_{3,2} = 3$ (recall that here $n_{3,1}, n_{3,2}$ must satisfy $n_{3,1} + n_{3,2} = 2n_3$). According to Remark 8.0.3, the corresponding total partition of 12 is $(1^5, 2^2, 3)$, where a^b means that the value a appears in the partition $b-$times.

Remark 8.0.4. Observe that, when $\bar{g} \notin E_{\bar{G}}$, so both its lifts are not in E_G, or when both lifts are in E_G, from our discussion in the proof of Theorem 8.0.1 we cannot deduce whether the lifts belong to the same class or to two different classes. On the contrary, if one lift is in the image of the exponential and the other is not, we know that they belong to different classes as an immediate consequence of Proposition 3.0.8.

Chapter 9

The Exceptional Group G_2

In this chapter G will be the complex simple algebraic group of type G_2. As usual, $\mathfrak{g}$ will denote its Lie algebra. We fix T a maximal torus of G. Recall that the root system of G will be given by

$$\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2), \pm(2\alpha_1+\alpha_2), \pm(3\alpha_1+\alpha_2), \pm(3\alpha_1+2\alpha_2)\},$$

where α_1 and α_2 are the short and the long root respectively of the basis Δ. The highest root is $\alpha_0 = 3\alpha_1 + 2\alpha_2$.
The extended Dynkin diagram is given by

$$\underset{-\alpha_0}{\circ} \;\frac{\qquad}{}\; \underset{\alpha_2}{\circ} \Rrightarrow \underset{\alpha_1}{\circ}$$

For $s \in T$ semisimple, we know that, up to conjugation, we can find a basis for $\Psi_s = \{\alpha \in \Phi | \alpha(s) = 1\}$ in $\Delta \cup \{-\alpha_0\}$ and such basis will be denoted as usual by Δ_s.

Theorem 9.0.1. *Let $\Delta_s \subseteq \Delta \cup \{-\alpha_0\}$ and choose a semisimple $s \in T$ such that Δ_s is a basis for $\Psi_s = \{\alpha \in \Phi | \alpha(s) = 1\}$. Let $u \in G$ be unipotent such that $su = us$. Then*

(i) If (up to taking a conjugate) $\Delta_s \subseteq \Delta$, then $J(su) \subseteq E_G$.

(ii) If $\Delta_s = \{-\alpha_0, \alpha_1\}$, then

- *(a) If u is in the class of $u_{\alpha_0}(1)u_{\alpha_1}(1)$, then $J(su) \not\subseteq E_G$.*
- *(b) Otherwise $J(su) \subseteq E_G$.*

(iii) If $\Delta_s = \{-\alpha_0, \alpha_2\}$, then

- *(a) If u is conjugate to $u_{\alpha_0}(1)u_{\alpha_2}(1)$, then $J(su) \not\subseteq E_G$.*
- *(b) Otherwise $J(su) \subseteq E_G$.*

Remark 9.0.2. Observe that case (ii) and (iii) of Theorem 9.0.1 are all the possible cases for which $C_G(s)^o$ is not a Levi subgroup, hence the result covers all the Jordan classes in G.

Proof. Since G is adjoint, we can apply Corollary 3.0.13 to prove (i).
We now consider statement (ii). Here $\Delta_s = \{-\alpha_0, \alpha_1\}$ so Ψ_s is of type $A_1 \times \tilde{A}_1$ and the unipotent classes in $C_G(s)^o$ can be represented by $1, u_{\alpha_1}(1)$ or $u_{\alpha_0}(1)u_{\alpha_1}(1)$, where u_{α_i} is defined as in Theorem 1.2.1.
Assume u is in the conjugacy class of $u_{\alpha_0}(1)u_{\alpha_1}(1)$. This means that u is distinguished in $C_G(s)^o$, as it is regular. We know by Proposition 1.5.18 that the image of s in $C_G(u)/C_G(u)^o$ is nontrivial, since $C_G(s)^o$ is not a Levi subgroup. As a result, we obtain that $J(su)$ is not in the image of the exponential map.
As shown in Section 4 of [12] and as we reported in Table 9.1, there exists only one unipotent $G-$orbit for which $C_G(u)/C_G(u)^o$ is nontrivial, hence such orbit must be that of $u_{\alpha_0}(1)u_{\alpha_1}(1)$. When u is not in that $G-$conjugacy class, then $C_G(u)/C_G(u)^o$ is trivial. So when u is conjugate to 1 or $u_{\alpha_1}(1)$ we can apply Theorem 3.0.6 to conclude that $J(su) \subseteq E_G$.
We finally turn to (iii) in which Ψ_s is of type A_2. Again we have that a unipotent u conjugate to $u_{\alpha_0}(1)u_{\alpha_2}(1)$ is distinguished in $C_G(s)^o$. We can apply again Proposition 1.5.18 because $C_G(s)^o$ is not a Levi subgroup. So $J(su) \not\subseteq E_G$. We deduce that the only class for which $C_G(u)/C_G(u)^o$ is nontrivial is that of $u_{\alpha_0}(1)u_{\alpha_2}(1)$.
For the other conjugacy classes, which may be represented by 1 and $u_{\alpha_2}(1)$, we can ague as in (i) to conclude.

□

Remark 9.0.3. In the proof, we actually saw that the elements $u_{\alpha_0}(1)u_{\alpha_1}(1)$ and $u_{\alpha_0}(1)u_{\alpha_2}(1)$ belong to the same conjugacy class of G. Indeed, they belong to the only orbit in G, for which $C_G(u)/C_G(u)^o$ is nontrivial, where u denotes any element of the class, i.e. the subregular orbit. We know from Section 4 of [12] that $C_G(u)/C_G(u)^o$ is isomorphic to the group of permutations of 3 elements S_3.
In case $(ii), (a)$ the projection of s in $C_G(u)/C_G(u)^o$ must belong to the conjugacy class of order 2 of S_3, while in case $(iii), (a)$ the projection of s in $C_G(u)/C_G(u)^o$ is in the conjugacy class of order 3. Indeed, to prove this we just need to compute d_s defined as in Section 1.5.4 in the two cases and apply Corollary 1.5.20 to conclude.

In Table 9.1 we list the unipotent classes in G_2 with relative component group $C_G(u)/C_G(u)^o$. We use the notation as in Section 8.4 in [3], so A_1 is the minimal orbit, $\tilde{A}_1$ is the class associated (in the Bala-Carter classification) to a Levi sugroup of type $\tilde{A}_1$ (i.e. of type A_1, but corresponding (up to conjugation) to the short root α_1), $G_2(a_1)$ is the subregular orbit and G_2 is the regular one.

Table 9.1: Unipotent classes and component groups

Class of u	$C_G(u)/C_G(u)^o$
1	1
A_1	1
$\tilde{A}_1$	1
$G_2(a_1)$	S_3
G_2	1

Bibliography

[1] W. Borho and H. Kraft. Über bahnen und deren deformationen bei linearen aktionen reduktiver gruppen. *Comment. Math. Helv.*, 54:61–104, 1979.

[2] G. Carnovale and F. Esposito. On sheets of conjugacy classes in good characteristic. *International Mathematics Research Notices*, 2012(4):810–828, 2012.

[3] D. H. Collingwood and W. M. McGovern. *Nilpotent orbits in semisimple Lie algebras*. van Nostrand-Reinhold, 1993.

[4] J. E. Humphreys. *Conjugacy classes in semisimple algebraic groups*. American Mathematical Society, 1995.

[5] H.-L. Lai. Surjectivity of the exponential map on semisimple lie groups. *J. Math. Soc. Japan*, 29(2):303–325, 1977.

[6] M. W. Liebeck and G. M. Seitz. *Unipotent and Nilpotent Classes in Simple Algebraic Groups and Lie Algebras*. American Mathematical Society, 2012.

[7] G. Lusztig. Intersection cohomology complexes on a reductive group. *Invent. Math.*, 75(2):205–272, 1984.

[8] G. Malle and D. Testerman. *Linear Algebraic Groups and Finite Groups of Lie Type*. Cambridge University Press, 2011.

[9] G. J. McNinch and E. Sommers. Component groups of unipotent centralizers in good characteristic. *Journal of Algebra*, (260):323–337, 2003.

[10] A. L. Onishchik and E. B. Vinberg. *Lie Groups and Lie Algebras I: Foundations of Lie Theory Lie Transformation Groups*. Encyclopaedia of Mathematical Sciences. Springer, 1993.

[11] A. L. Onishchik and E. B. Vinberg. *Lie Groups and Lie Algebras III: Structure of Lie Groups and Lie Algebras*. Encyclopaedia of Mathematical Sciences. Springer, 1994.

[12] E. Sommers. A generalization of the bala-carter theorem for nilpotent orbits. *Internat. Math. Res. Notices*, (11):539–562, 1998.

[13] R. Steinberg. *Conjugacy Classes in Algebraic Groups.* Lecture Notes in Mathematics. Springer-Verlag, 1974.

[14] P. Tauvel and R. W. T. Yu. *Lie Algebras and Algebraic Groups.* Springer, 2005.

[15] D. Ž. Djoković. The exponential image of simple complex lie groups of exceptional type. *Geom. Dedicata*, 27:101–111, 1988.

[16] D. Ž. Djoković and K. H. Hofmann. The surjectivity question for the exponential function of real lie groups: A status report. *Journal of Lie Theory*, 7:171–199, 1997.