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Duality and the Weak Gravity Conjecture

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Introduction

The modern perspective on Quantum Field Theory is to think of every model as an *effective field theory*, describing physics consistently only up to a certain energy scale. Such description characterizes successfully those processes whose typical energy is inside the given range, but beyond it the spectrum of phenomena enlarges and eventually the theory fails to be predictive. Famous examples of effective field theories are the Fermi theory of weak interactions and the chiral Lagrangian describing the low-energy dynamics of quantum chromodynamics (QCD). In both cases we know that the underlying physics is richer: weak interactions are mediated by the Z^0 and $W^{\pm}$ bosons, though not present in the Fermi Lagrangian, and the pions, the degrees of freedom described by the chiral Lagrangian, are bound states of quarks. Nevertheless, as long as one stays in the proper energy range it is difficult to identify these fine structures and the relevant physics is equally (and more easily) described via the effective field theory. It is important to stress that in the energy range in which it holds, the effective field theory is predictive as well as the "complete" theory from which it comes.

Going beyond its purely phenomenological meaning, quantum field theories are now regarded as effective field theories in the sense that we don't expect a given theory to be descriptive of all phenomena at all energy levels; we expect instead to have to deal with different theories at different energy levels, seen all together as the various low-energy realizations of the same, unifying theory. Following this perspective, we can look at the two most relevant quantum field theories that have been established. On one side there is the Standard Model, the theory of subatomic particles and their interactions that can be seen as the effective field theory at the TeV scale. On the other there is String Theory, that is, so far, the best theory of quantum gravity we have at our disposal and lives close to the Planck scale. According to the effective field theory principles, we expect that it should be possible to connect these two infrared (IR) and ultraviolet (UV) worlds. However, how to concretely realize such connection is not at all clear. To understand this problem, we recall that String Theory is a highly-constrained theory and self-consistency requires it to be defined in ten or eleven spacetime dimensions. The contact with 4D-physics is then made via the compactification procedure, through which the extra dimensions are "wrapped" over special internal manifolds. The richness and complexity of such operation produce an extremely large spectrum of low-energy effective field theories, called "string vacua", coming from String Theory, without clear indications as to which selection mechanism to follow in order to reach, in the end, the Standard Model.

Although this connection is not clearly realizable following such top-down procedure from String theory to the 4-dimensional models, one may wonder if it is instead possible to do so following a bottom-up approach. This observation is the starting point of a research project called *Swampland program* [1–3]. The idea of the Swampland program is in fact to track back the path that leads to String Theory - or, more in general, to a theory of quantum gravity - by identifying which low-energy models are going to agree with String Theory in the UV and which won't. The set of all low-energy theories that admit a UV-completion into a theory of quantum gravity is called the *Landscape* of quantum gravity; all the other theories, well-defined in themselves but not consistent in the UV with quantum gravity, are said to belong to the *Swampland*. This distinction between theories in the Landscape and in the Swampland is made via the so called *swampland conjectures*. These are selection criteria stating the properties that a theory should have in order to belong either to the Landscape or to the Swampland. As the name suggests, these conjectures are not proven facts but rather well-established arguments, supported by vast classes of examples and motivations. The more the arguments come from different sources (e.g. String Theory, gauge/gravity correspondence, black hole physics, etc.), the better a conjecture is supported. More concretely, once a swampland conjecture has been established, it can be applied to study quantum field theories in the following way. Starting from an effective field theory, its typical higher-order extension is built as a series of irrelevant operators, which usually follows a perturbative expansion in the number of derivatives these operators contain. The resulting Lagrangian is then characterized by a set of coefficients, associated to the different higher-order operators, that at the level of this construction are completely arbitrary. The swampland conjecture acts precisely by restricting this set of parameters to the subgroup that realizes its prescriptions. Then, according to the conjecture, it follows that the theories corresponding to this subset of coefficients belong to the Landscape and all the others to the Swampland.

Among the various conjectures that have been established (see [2,3]), in this thesis work we focus on the *Weak Gravity Conjecture* (WGC) [4]. This is probably the most studied conjecture and for this reason has undergone several refinements and elaborations. In its minimal formulation, called Electric WGC, it states that a theory coupling gravity and a U(1) gauge field must describe a state for which the electric charge is (in proper units) greater than its mass. In other words, it states that it must exist a state for which the gravitational interaction is weaker than the electromagnetic one. One of the main motivation for this property to be taken as a swampland conjecture comes from black holes physics. Indeed, the Electric WGC coincides exactly with the condition that allows extremal black holes to discharge without introducing naked singularities (see [4]). Moreover, its prescription of having a state with charge-to-mass ratio greater than one can also be rephrased as the possibility for an extremal black hole to decay into smaller black holes [2,3]. The reason why a black hole should be able to discharge or decay is that otherwise we would end up with a universe filled with a very large number of stable remnants and this clearly suspect picture leads in fact to entropy inconsistencies [10].

This black hole-based arguments provide also a clear example of how the WGC can be tested following the procedure described above: starting from a theory coupling gravity to a U(1) gauge field, one can find its higher-derivative correction and study the charge-to-mass ratio of an extremal black hole solution of the extended theory; the WGC will then constrain the higher-order coefficients to the subset for which such correction results to be positive. This Swampland-based study was carried out, in the case of Einstein–Maxwell theory and its 4-derivatives extension, for example in [11] and [18]. In particular, in [18] a strong evidence supporting the WGC was pointed out. This key result is that the WGC is immediately realized in Einstein–Maxwell theory if the positivity bounds on the scattering amplitudes of the theory [20] are taken into account. This observation is very interesting because positivity bounds are a set of constraints on the theory's coefficients that come from applying the properties of locality, Lorentz symmetry and S-matrix unitarity, so that these bounds are essentially a consequence of requiring the theory to be self-consistent. The fact that the

WGC points in the same direction of such structural requirements is indeed of great support to the conjecture itself.

This equivalence, that holds exactly in Einstein–Maxwell theory, is though lost when the field content of the theory becomes richer [33]. At the same time, it seems to be restored if we make the additional requirement that the theory under examination preserves *electromagnetic duality* [33, 34]. Electromagnetic duality (EM duality) is the second main topic of this thesis work and it's the property of invariance of the set of equations of motion (EoM) and Bianchi identities (BI) of gauge fields. This is the symmetry that, for instance, allows to exchange the electric and magnetic fields in the free Maxwell equations leaving them invariant, but indeed admits a very general definition, due to M. Gaillard and B. Zumino in their famous paper [26]. The crucial characteristic of electromagnetic duality is that it is *not* a symmetry of the Lagrangian, but rather of the EoM and the BI. The idea to make use of EM duality to determine the higher-order extension of gauge theories comes from the fact it can be connected with the duality symmetries that relates the different formulations of String Theory [14–16]. From this point of view, EM duality is a manifestation of a symmetry of the UV theory and therefore it is expected to hold, in some form, at every perturbation order.

The main problem addressed by this work is precisely how to properly make use of EM duality to constrain higher-derivatives extensions of effective gauge theories. This problem has two main issues to be faced. The first one regards the nature of EM duality as a symmetry of the EoM and the BI: to implement it, it's not sufficient to construct exactly duality-invariant operators because the Lagrangian itself should not be invariant. The second one concerns instead one hypothesis of Gaillard and Zumino's paper [26]. To derive the duality group and the associated transformations of a generic Lagrangian they assume that it does not contain operators involving derivatives of the gauge fields. While in their analysis this hypothesis is essential to carry out the calculations, it results problematic when EM duality is applied to constrain higher-derivatives theories. In fact, such problematic operators indeed appear in this procedure, in greater number as the perturbative order is higher, and to exclude them a priori from the discussion seems a too strict framework.

To deal with these issues, the strategy we follow is to rely on a model-based, perturbative duality analysis, studying the duality group and the transformation of a specific Lagrangian order by order in the higher-derivatives expansion. Although the results that one obtains with this approach are limited to the model under consideration, the advantage is that there's no need to make any additional assumption on the structure of the full Lagrangian. This allows then to include in the discussion in a natural way also the problematic operators with derivatives on the gauge fields. More specifically, the theory we focus on in this work is the so called *axion-dilaton-Maxwell-Einstein theory*:

$$\mathcal{L}_2 = \frac{1}{2}R - \frac{1}{2(\mathfrak{Im}\tau)^2}\partial_\mu\bar{\tau}\partial^\mu\tau - \frac{1}{4}\mathcal{I}_{\Lambda\Sigma}(\tau,\bar{\tau})F^\Lambda_{\mu\nu}F^{\Sigma\,\mu\nu} + \frac{1}{4}\mathcal{R}_{\Lambda\Sigma}(\tau,\bar{\tau})F^\Lambda_{\mu\nu}\tilde{F}^{\Sigma\,\mu\nu}, \tag{1}$$

which couples, in a non-minimal way, gravity to two U(1) gauge fields (labelled by the capital greek indices) and a complex scalar field. Starting from this Lagrangian, by applying the procedure we outlined we were able to find its duality-preserving, 4-derivatives extension: this is the main result of this thesis work. Once the higher-derivatives extension of the theory has been found, we turn to the study of the WGC and of its claimed equivalence with the positivity bounds in the case of a duality-preserving, beyond Einstein–Maxwell theory, as our resulting theory indeed is. Following [34], we study the charge-to-mass ratio of an extremal black hole solution of the 4-derivatives theory and we determine the set of values of its coefficients that realizes the WGC. Then, we compare this result with the positivity bounds on the coefficients, showing that they indeed reproduce the WGC requirements.

The thesis is organized in the following way. In the first chapter we present in detail the Swampland Program and the WGC, focusing on the motivation supporting its Electric formulation, and we discuss how swampland conjectures constrain the coefficients of effective field theories in the context of the 4-derivatives extension of Einstein–Maxwell theory. In the second chapter we instead present the procedure through which positivity bounds on the scattering amplitudes are computed and show that they exactly reproduce the WGC constraints on Einstein–Maxwell theory. Next, the third chapter is dedicated to the description of EM duality: we first review Gaillard and Zumino analysis and then present an example of how duality can be used to determine the higher-order extension of a theory taking again Einstein–Maxwell one as benchmark. Chapters 4 and 5 are the core of the thesis work and contain the entire duality analysis we applied to theory (1): in Chapter 4 we introduce the model and then determine its duality group structure, in Chapter 5 we make use of these results to fix the 4-derivatives extension of the theory. In Chapter 6 we then present the discussion on the WGC. We conclude the thesis work by summarizing its main results and discussing the possible future developments.

Chapter 1

The Swampland Program and the Weak Gravity Conjecture

The *Swampland program* is a research project which aims to distinguish between low-energy effective field theories which admit a UV completion into a theory of quantum gravity (e.g. String Theory) and those which do not. The former are said to belong to the Landscape of quantum gravity, the latter to the Swampland. The problem of connecting the low-energy physics and the high-energy one is well known: on the former side we have the Standard Model, which successfully describes the physics of the TeV scale but needs to be extended (as an effective field theory should be when coupled to gravity); on the latter one we have String Theory, the beautiful, unifying theory that one longs to but whose low-energy realization is extremely difficult to identify. The number of consistent low-energy theories that one obtains from String Theory, the so called string vacua, is in fact very large and it is not at all clear which is the selection mechanism to follow. The Swampland program addresses this problem from a bottom-up point of view, trying to identify which effective field theories are consistent with String Theory in the UV and which are not.

The tools to make this identification are the so called *swampland conjectures*, arguments that establish some fundamental properties placing the theories that present them either in the Landscape or in the Swampland. These arguments are indeed called "conjectures" because they're not proven facts but rather well-motivated statements, usually supported by (large) classes of examples or gedankenexperiments. One of the first and most famous of these conjectures is the *Weak Gravity Conjecture* (WGC) [4], which in its minimal formulation states that a theory involving gravity and a U(1) gauge field should describe, in order to be in the Landscape, a state for which the gravitational interaction is weaker than the gauge one.

We can describe the typical Swampland-way of proceeding as follows. The first step is to formulate a conjecture: this is done by finding some recursive structures and patterns, usually in the context of String Theory or Black Hole physics, that the conjecture summarize. The following step is testing the conjecture in different models and settings, in order to see if it is truly reasonable to assume it and, in case, to make proper adjustments in its definition. The more varied are the areas from which the supporting evidence comes (e.g String Theory, black holes, AdS/CFT correspondence, etc.), the stronger is the conjecture. The final step is to understand how the conjecture constraints various low-energy models, studying both the theoretical and phenomenological outcomes.

The goal of the Swampland program is really ambitious: to guide our knowledge of fundamental physics up to the Planck scale, where quantum gravity shows up. This project is important because it would understand how quantum gravity is realized at low energy scales from both the formal and phenomenological points of view, and, in doing so, it would also establish which are the crucial properties that a theory should have in order to be meaningful. The true power of this conjectures-based procedure, which sometimes may appear in some sense arbitrary and restrictive, lies in the fact that the various established swampland conjectures, in particular the most general ones, like the Weak Gravity Conjecture, the Distance Conjecture [5] and the No Global Symmetry Conjecture [6, 7], are actually related one another: the existence of a connection between them reinforces their claims because it suggests the idea that they are different realization of a common, underlying principle, pointing towards quantum gravity along the same direction.

After describing the general settings and ideas of the Swampland Program, in this chapter we first analyze the Weak Gravity Conjecture [4] in two of its formulations, the Electric (the relevant one for this thesis work) and the Magnetic WGC, highlighting the main founding motivations.

1.1 Electric WGC

The Weak Gravity Conjecture is one of the most studied swampland conjectures. Its original formulation [4], which states that gravity should act as the weakest of the interactions described by a theory in the Landscape of quantum gravity, has undergone various developments and refinements: apart from the two of them that we describe in this section, we can mention the extension to the case of multiple U(1) gauge fields [8] and to the one in the presence of additional scalar fields [9].

The first formulation of the WGC that we present, the one that is more relevant for this work, is the so called *Electric WGC*, which in 4 spacetime dimensions states the following:

Electric Weak Gravity Conjecture (4D). *Consider a theory coupling gravity to a $U(1)$ gauge field of gauge coupling g. Such a theory must contain a state of mass M and electric charge Q such that*

$$M \leq \sqrt{2} g M_{\mathrm{P}} Q. \tag{1.1}$$

It is important to notice that the conjecture does not specify which kind of state should realize it. In [4], three possibilities are suggested:

1. the state of minimal charge;
2. the lightest charged particle;
3. the state with the minimal mass-to-charge ratio.

Depending on which of the option is taken into consideration we have a different approach to the conjecture, because different are the supporting evidence and the frameworks to test it.

First of all, we see that the conjecture can in principle be satisfied not strictly by a particle, which refers directly to the fields described by the theory under examination, but also by a more generic "state", as stated in (1.1), which can indeed by a particle but it could also be a composite object.

This characterization is necessary because in order for (1.1) to be a well posed condition the physical subject realizing the Electric WGC must be *stable*: otherwise, the meaning of M in the charge-to-mass ratio defined by (1.1) wouldn't be clear. From this observation it follows that the subject of option 1 must be a generic state and not the particle of minimal charge described by theory under examination, since the latter is not guaranteed to be stable. Instead, option 2 is well posed when its subject is a particle because the lightest charged particle described by a theory is indeed stable, as well as option 3 is well posed if its subject is a generic state.

Of the three, option 1 results to be the weakest one because there are String Theory arguments against it (see [4]). Option 2 results instead to be the most stringent because it's a direct requirement on the particles' spectrum of the theory, while according to option 3 the conjecture could instead be realized also by a heavy state with the proper charge.

Moreover, option 3 can be seen as a subcase of option 2. To see this, let's consider the case of a theory describing a spectrum of particles, of mass and charge $(m_j,\, q_j)$. Let's suppose then that the lightest of these particles satisfies the WGC according to option 2, so that in proper units we have

$$\frac{m_1}{q_1} \leq 1, \qquad m_1 < m_j \quad \forall\, j. \tag{1.2}$$

Within this framework, set out by option 2, we can show that the state with minimal mass-to-charge ratio satisfies the WGC. Indeed, among all the particles of the set different from $(m_1,\, q_1)$ we can identify the one that has the smallest mass-to-charge ratio and we call it $(m_2,\, q_2)$:

$$\frac{m_2}{q_2} \leq \frac{m_j}{q_j} \qquad \forall\, j \neq 1. \tag{1.3}$$

Comparing now the mass-to-charge ratios of particles 1 and 2, we see that we have two possibilities:

- $\frac{m_1}{q_1} \leq \frac{m_2}{q_2} \implies$ the lightest charged particle, which realizes the WGC, is also the particle with the smallest mass-to-charge ratio;
- $\frac{m_2}{q_2} \leq \frac{m_1}{q_1} \leq 1 \implies$ the smallest mass-to-charge ratio results to be such that the WGC is realized also by the corresponding particle.

Thus, in both cases we have that having the lightest charged particle realizing the WGC implies that also the particle with smallest mass-to-charge ratio realizes the WGC as well. Hence, option 3 is a subcase of option 2.

Despite its mild character, option 3 is particularly interesting because it offers the possibility for the conjecture to be realized not only from a particle but also by an extended state. The conjecture in fact does not prevent the involved state to have an arbitrary large mass, even larger than the Plank mass, as long as the associated charge guarantees that the bound (1.1) is satisfied. A good example of such states are of course black holes: we're going to see in section 1.1.2 that a relevant motivation in favour of the Electric WGC, realized via an extended state, is given by the study of the splitting process of a charged black holes into smaller black holes and the conditions under which it is allowed.

Apart from this charged black hole instability, black holes physics represents an important class of evidence supporting the Electric WGC, as discussed in section 1.1.1 for the process of discharge of a black hole. Further evidence can also be found in Holography and string compactification arguments (see [2, 3]).

1.1.1 Electric WGC and Charged Black Holes

Motivation for the Electric WGC can be found in charged black holes dynamics: the constraint (1.1) is in fact the condition that allows extremal black holes to discharge. To discuss this equivalence we consider now Einstein–Maxwell theory, i.e. the theory involving gravity and a U(1) gauge field $F_{\mu\nu}$, in 4 spacetime dimensions:

$$S_{\text{EM}} = \int d^4x \sqrt{|g|} \left[\frac{M_{\text{P}}^2}{2} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right]. \tag{1.4}$$

A charged, spherically-symmetric and static black hole solution of the Einstein equations associated to (1.4) is given by the famous Reissner–Nordström black hole [12]:

$$\begin{aligned} ds^2 =& - f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega_{S^2}^2, \\ f(r) =& 1 - \frac{2GM}{r} + \frac{Q^2 G}{4\pi r^2}, \end{aligned} \tag{1.5}$$

where $Q = gq$ is the charge of the black hole, with g being the gauge coupling constant, and M is its mass, which the Cosmic Censorship Principle constraints to be

$$M \geq \sqrt{2} Q M_{\text{P}}; \tag{1.6}$$

the solution for which we have $M = \sqrt{2} Q M_{\text{P}}$ is called *extremal black hole* because it's the black hole with minimum M for a given Q.

Let's now explore the dynamics of the discharge process of a black hole. Calling (M, Q) and (m_j, q_j) the mass and the charges of the initial black hole and of the discharge products respectively, the conservation of energy and charge read

$$M \geq \sum_j m_j, \tag{1.7}$$

$$Q = \sum_j q_j. \tag{1.8}$$

Combining these two equations we can obtain an equivalent conservation constraint on the mass-to-charge ratio:

$$\frac{M}{Q} \geq \frac{1}{Q} \sum_j m_j = \frac{1}{Q} \sum_j \frac{m_j}{q_j} q_j \geq \left(\frac{m_j}{q_j} \right)_{\text{min}} \frac{\sum_j q_j}{Q} = \left(\frac{m_j}{q_j} \right)_{\text{min}} \tag{1.9}$$

and specializing this bound to the extremal black hole case we obtain

$$\left(\frac{m_j}{q_j} \right)_{\text{min}} \leq \left(\frac{M}{Q} \right)_{\text{ext}} = \sqrt{2} M_{\text{P}}. \tag{1.10}$$

The constraint (1.10) tells us that in order for an extremal black hole to discharge there must exist a state with charge greater than its mass: this is precisely the Electric WGC as stated in (1.1).

Therefore, asking for the Electric WGC to be realized is equivalent to requiring extremal black holes to be able to discharge without creating a naked singularity. To understand why we should ask for this condition, let's make the opposite assumption and consider what would happen if they could not discharge. According to this picture, any charged black hole would evaporate until reaching the extremal condition, to which it stops. Thus, we would end up with various extremal black holes, remnants of this "truncated" discharge process, that appear to be stable; they can indeed have a mass of the planckian size, with associated charge that should respect the bound (1.6). The spectrum of the allowed charges, and consequently the (possible) number of such remnants, is as large as the gauge coupling is small. This

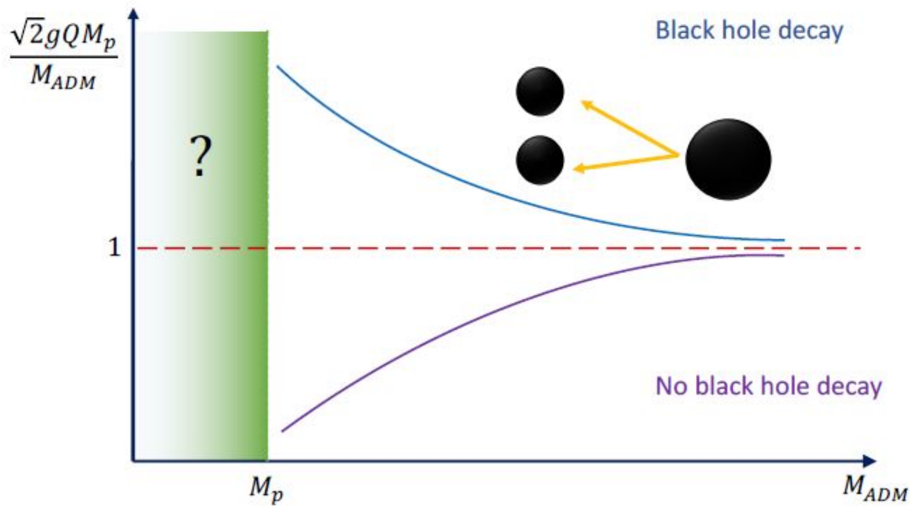


Figure 1.1: Possible higher-derivatives corrections to the charge-to-mass ratio of an extremal black hole and relation with the black hole decay process [2].

(potentially) highly-degenerate scenario, with an infinite number of stable remnants, seems immediately problematic and leads in fact to entropy problems, as discussed in [10].

However, it is important to remark that this is not a theorem but rather an argument: in fact it does not state that we necessarily end up with an infinite sets of stable remnants but only that a large number of them is expected and it is not completely clear in which terms this leads to inconsistencies (see [2]). However, it gives an effective description of what could happen - and go wrong - if we do not assume the WGC (1.1). That's why the Swampland Program deals, or has to deal, with *conjectures*: it would not be necessary to assume them if their statement were proven facts. Instead, we have arguments on the basis of a conjecture and the problem of the discharge of extremal black holes is indeed an important motivation for the WGC.

1.1.2 Black Holes as the states realizing the WGC

Proceeding further along the theme of the black hole dynamics as ground basis on which to found the WGC, we can also investigate the possibility for a charged black hole to decay into smaller black holes. In order for this process to be allowed, the charge-to-mass ratio of the decaying black hole must be greater than one: such a starting point can be reached, in the case of an extremal black hole, by taking into account higher-derivative corrections [11]. Extremal black holes appears in fact as the solution of the Einstein equations associated to Einstein–Maxwell action (1.4) with charge equal to its mass, but if one starts to include in the action also higher-order operators (e.g. $(F_{\mu\nu}F^{\mu\nu})^2$) then the charge-to-mass ratio receives corrections:

$$z \equiv \frac{\sqrt{2}M_{\rm P}Q}{M} \quad \Longrightarrow \quad z_{\rm ext} = 1 + \delta z(M). \tag{1.11}$$

If $\delta z > 0$ then the decay process into smaller black holes is possible (see Figure 1.1), with the starting black hole that represents the state satisfying the Electric WGC. We can notice that in this case the conjecture would be realized by an extended state, as described at the beginning of this section.

This realization of the Electric WGC, already suggested in [4], is particularly interesting because it provides a well defined pattern to study the conjecture: one can start with a theory involving (at least) gravity and a U(1) gauge field, find its higher-derivative extension and calculate the consequent correction to the charge-to-mass ratio for an extremal black hole solution of the theory. Asking the WGC to hold produces a constraint on this correction, which directly translates to the coefficients of the higher-derivative operator, on which the correction depends. This is a clear example of how the Swampland Program works: the models of this type with the coefficients that satisfying $\delta z > 0$ belong to the Landscape, all the others to the Swampland. A more explicit discussion of this procedure is presented in the following section.

1.2 Higher-derivative extension of Einstein–Maxwell theory

We now discuss more explicitly how we can test the WGC by looking at the higher-derivative corrections to the charge-to-mass ratio of a charged black hole. The theory we consider is the Einstein–Maxwell one (1.4) and we're going to determine its extension up to the 4-derivatives order [17, 18]. In the following, we denote with $\mathcal{L}_2$ the Einstein–Maxwell Lagrangian (1.4) and with $\mathcal{L}_4$ its 4-derivative extension.

1.2.1 4-derivatives Lagrangian

The first thing to do when trying to find the extension of a theory follwing a bottom-up approach is to list all the higher-order operators that are compatible with the symmetry of the theory considered. In our case, all possible 4-derivatives operators are:

$$\begin{aligned}
\boldsymbol{g}: &\quad R^2, \quad (R_{\mu\nu})^2, \quad (R_{\mu\nu\rho\sigma})^2; \\
\boldsymbol{g}+\boldsymbol{F}: &\quad RF^2, \quad R_{\mu\nu}F^{\mu\alpha}F^{\nu}{}_{\alpha}, \quad R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}; \\
\boldsymbol{F}: &\quad (F\cdot F)^2, \quad (F\cdot\tilde{F})^2, \quad F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu}, \\
&\quad (D_\mu F^{\mu\nu})^2, \quad (D_\mu F_{\nu\rho})^2, \quad D_\mu F_{\nu\rho}D^\nu F^{\mu\rho},
\end{aligned} \tag{1.12}$$

where $F\cdot F = F_{\mu\nu}F^{\mu\nu}$.

This set of operators can be reduced exploiting the following identities:

$$G_B = R^2 - 4(R_{\mu\nu})^2 + (R_{\mu\nu\rho\sigma})^2; \tag{1.13}$$

$$(F\cdot\tilde{F})^2 = -2(F\cdot F)^2 + 4F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu}; \tag{1.14}$$

$$(D_\mu F_{\nu\rho})^2 = 2D_\mu F_{\nu\rho}D^\nu F^{\mu\rho}; \tag{1.15}$$

$$\begin{aligned}
(D_\mu F_{\nu\rho})^2 = &-2R_{\mu\nu}F^{\mu\alpha}F^{\nu}{}_{\alpha} + R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} + 2(D_\mu F^{\mu\nu})^2 + \\
&+ 2D_\mu\left(F_{\nu\rho}D^\nu F^{\mu\rho} - F^{\mu\rho}D^\nu F_{\nu\rho}\right).
\end{aligned} \tag{1.16}$$

The first identity is the definition of the so called Gauss–Bonnet term [35], for which we can exchange the operator $(R_{\mu\nu\rho\sigma})^2$. The second one is obtained by using the Levi-Civita tensor contraction rules, while the third one is a consequence of Bianchi Identity

$$D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} + D_\rho F_{\mu\nu} = 0. \tag{1.17}$$

The last one is obtained via an integration by parts. This quantity is topological and vanish (together with the $D_\mu(\dots)$ term in (1.16) in the action. Thus, a minimal set of independent operators becomes

$$
\begin{aligned}
\boldsymbol{g}: &\quad R^2, \quad R_{\mu\nu}R^{\mu\nu};\\
\boldsymbol{g}+\boldsymbol{F}: &\quad RF^2, \quad R_{\mu\nu}F^{\mu\alpha}F^{\nu}{}_{\alpha}, \quad R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma};\\
\boldsymbol{F}: &\quad (F\cdot F)^2, \quad (F\cdot\tilde{F})^2, \quad (D_\mu F^{\mu\nu})^2.
\end{aligned}
\tag{1.18}
$$

We can further reduce this set by exploiting a field redefinition, according to the following scheme. The transformation of the fields is

$$
\begin{cases} A_\mu \longrightarrow {A_\mu}' = A_\mu + \delta A_\mu \\ g^{\mu\nu} \longrightarrow {g'}^{\mu\nu} = g^{\mu\nu} + \delta g^{\mu\nu} \end{cases} ,
\tag{1.19}
$$

with δA_μ and $\delta g_{\mu\nu}$ of order $\mathcal{O}(D^2)$, i.e. they contain two derivatives. The corresponding variation of the Lagrangian is

$$
\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 \longrightarrow \left(\sqrt{|g|}\mathcal{L}\right)' = \left(\sqrt{|g|}\mathcal{L}\right) + \delta\left(\sqrt{|g|}\mathcal{L}\right) = \sqrt{|g|}\mathcal{L} + \delta\left(\sqrt{|g|}\mathcal{L}_2\right) + \mathcal{O}(D^6),
\tag{1.20}
$$

where

$$
\frac{\delta(\sqrt{|g|}\mathcal{L}_2)}{\sqrt{|g|}} = \frac{M_{\mathrm{P}}^2}{2}\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - T_{\mu\nu}\right]\delta g^{\mu\nu} + \left[D_\mu F^{\mu\nu}\right]\delta A_\nu .
\tag{1.21}
$$

Thus, we see that, with the choice of (1.19) as

$$
\begin{cases} \delta g^{\mu\nu} = \dfrac{\alpha_1}{M_{\mathrm{P}}^4} g^{\mu\nu}F^2 + \dfrac{\alpha_2}{M_{\mathrm{P}}^4}F^{\mu\rho}F^{\nu}{}_{\rho} + \dfrac{\alpha_3}{M_{\mathrm{P}}^2}g^{\mu\nu}R + \dfrac{\alpha_4}{M_{\mathrm{P}}^2}R^{\mu\nu}, \\ \delta A_\mu = \dfrac{\beta}{M_{\mathrm{P}}^2}D^\rho F_{\rho\mu}, \end{cases}
\tag{1.22}
$$

we can remove from $\mathcal{L}_4$ the operators R^2, $(R_{\mu\nu})^2$, RF^2, $R_{\mu\nu}F^{\mu\rho}F^{\nu}{}_{\rho}$ and $(D_\mu F^{\mu\nu})^2$ by properly setting the fields redefinition coefficients:

$$
\begin{aligned}
\frac{\delta\left(\sqrt{|g|}\mathcal{L}\right)}{\sqrt{|g|}} =& -\frac{2\alpha_3+\alpha_4}{4}R^2 + \frac{\alpha_4}{2}(R_{\mu\nu})^2 - \frac{4\alpha_1+2\alpha_2-\alpha_3}{8M_{\mathrm{P}}^2}RF^2 + \\
&+ \frac{\alpha_2-\alpha_4}{2M_{\mathrm{P}}^2}R_{\mu\nu}F^{\mu\rho}F^{\nu}{}_{\rho} + \frac{\beta}{8M_{\mathrm{P}}^2}(D_\mu F^{\mu\nu})^2 + \\
&+ \frac{3\alpha_2}{8M_{\mathrm{P}}^4}(F\cdot F)^2 + \frac{\alpha_2}{8M_{\mathrm{P}}^4}(F\cdot\tilde{F})^2 .
\end{aligned}
\tag{1.23}
$$

Therefore, introducing the Weyl tensor

$$
W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - g_{\mu[\rho}R_{\sigma]\nu} + g_{\nu[\rho}R_{\sigma]\mu} + \frac{1}{6}Rg_{\mu[\rho}g_{\sigma]\nu}, .
\tag{1.24}
$$

to exchange the operator $R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$ with[1]

$$
W_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} = R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} - 2R_{\mu\nu}F^{\mu\rho}F^{\nu}{}_{\rho} + \frac{1}{3}RF^2,
\tag{1.25}
$$

[1] The extra operators in (1.25) can be re-adsorbed into a field redefinition of the type (1.22).

the resulting 4-derivatives extension of Einstein–Maxwell theory (1.4) is

$$\mathcal{L}_4 = \frac{c_1}{4M_\mathrm{P}^4}(F\cdot F)^2 + \frac{c_2}{4M_\mathrm{P}^4}(F\cdot \tilde{F})^2 + \frac{c_3}{2M_\mathrm{P}^2}\, F_{\mu\nu}F_{\rho\sigma}W^{\mu\nu\rho\sigma}. \tag{1.26}$$

1.2.2 Corrections to the charge-to-mass ratio

We now turn to the corrections to the charge-to-mass ratio that comes from the extended Lagrangian (1.26) and to the consequent comparison with the WGC and its prescriptions. The analysis that leads to the higher-order corrections to the relevant physical quantities is far from obvious and we do not reproduce it, since the goal of this section is to have a first contact with the procedure through which swampland conjectures are applied to constraint higher-derivatives theories. The reader interested in the precise calculations that here we only highlight may refer to [17].

As mentioned in Section 1.1.1, a black hole solution of the 2-derivatives Einstein–Maxwell theory is given by the Reissner–Nordström black hole (1.5), that we rewrite as:

$$\begin{aligned} ds^2 =& - f_0(r)dt^2 + \frac{1}{f_0(r)}dr^2 + r^2 d\Omega^2_{S^2}, \\ f_0(r) =& 1 - \frac{M}{4\pi M_\mathrm{P}^2 r} + \frac{Q^2}{32\pi^2 M_\mathrm{P}^2 r^2}. \end{aligned} \tag{1.27}$$

In order not to have naked singularities, the charge-to-mass ratio of the black hole is constrained to satisfy

$$z \equiv \sqrt{2}M_\mathrm{P}\frac{Q}{M} \leq 1 \tag{1.28}$$

and the solution saturating this bound is called extremal black hole ($z^{(0)}_\mathrm{ext} = 1$).

Once we correct the 2-derivative Einstein–Maxwell Lagrangian (1.4) with the higher-order terms such (1.26), the Reissner–Nordström solution (1.27) does not hold anymore since the EoM have of course changed. The solution of the new EoM can indeed be found starting from:

$$ds^2 = g_{tt}(r)\, dt^2 + g_{rr}(r)\, dr^2 + r^2 d\Omega^2_{S^2} \tag{1.29}$$

and exploiting a perturbative expansion by asking that the Reissner–Nordström solution is recovered in the limit in which the coefficient c_i of the extended Lagrangian (1.26) vanish :

$$g_{tt} = - f_0(r) + \Delta g_{tt} + \mathcal{O}(c_i^2) \qquad \text{with} \quad \Delta g_{tt} \xrightarrow{c_i\to 0} 0, \tag{1.30}$$

$$\frac{1}{g_{rr}} = f_0(r) + \Delta f + \mathcal{O}(c_i^2) \qquad \text{with} \quad \Delta f \xrightarrow{c_i\to 0} 0. \tag{1.31}$$

We're interested especially in $g_{rr}(r)$, since it's the term from which we understand the structure of the black hole horizons and, therefore, the extremality condition. Indeed, the extremal black hole horizon can be written with a more general formula as

$$r_\mathrm{H,ext} = \max_z \left\{ r \in \mathbb{R}_+ \, : \, \frac{1}{g_{rr}} = 0 \right\}. \tag{1.32}$$

Thus, we need to solve this equation in order to understand how the extremal charge-to-mass ratio gets corrected. Plugging the ansatz (1.29)–(1.31) into the corrected Einstein equations on obtains

$$\frac{1}{g_{rr}} = f_0(r) - \frac{Q^4\, c_1}{1280\pi^4 M_{\rm P}^6\, r^6} + \frac{Q^2\, c_3}{M_{\rm P}^6\, r^6}\left(\frac{5M\, r}{384\pi^3} - \frac{Q^2}{640\pi^4} - \frac{M_{\rm P}^2\, r^2}{24\pi^2}\right). \tag{1.33}$$

A perturbative solution of (1.32) with (1.33) yealds the following result for the corrected charge-to-mass ratio:

$$z_{\rm ext} = 1 + \frac{64\pi^2 M_{\rm P}^2}{5M^2}(2c_1 - c_3), \tag{1.34}$$

which is compatible also with the results of [18].

1.2.3 Weak Gravity Conjecture

Summarizing, we started from the 2-derivative Einstein–Maxwell theory (1.4) and we found its 4-derivative extension (1.26) following a bottom-up approach. This extension depends on three different coefficients c_1, c_2 and c_3, on which we have the only requirements to be subdominant with respect to $M_{\rm P}$.

It is precisely on coefficients of this type that the Swampland Program explicitly wants to act: following a given swampland conjecture, one would like to translate its prescription to some conditions on these coefficients in order for the conjecture to be realized in the theory. Then, according to the given conjecture, the set of theories with coefficients satisfying these bounds belong to the Landscape, the others to the Swampland.

As previously mentioned, when specializing this general swampland procedure to the Electric WGC (1.1), we can translate its prescription to the requirement that extremal black holes, seen as extended states realizing the conjecture, are able to decay in smaller black holes and still satisfy the Cosmic Censorship Principle. In this perspective, the Electric WGC can be phrased as the requirement that extremal black holes have a charge-to-mass ratio greater than 1:

$$\text{Electric WGC:} \quad z_{\rm ext} > 1. \tag{1.35}$$

Thus, in the case of Einstein–Maxwell theory we can immediately understand which are the higher-derivatives extensions that satisfy the Electric WGC. Indeed, from (1.34) we get

$$z_{\rm ext} = 1 + \frac{64\pi^2 M_{\rm P}^2}{5M^2}(2c_1 - c_3) > 1 \quad \Longleftrightarrow \quad 2c_1 - c_3 > 0. \tag{1.36}$$

Therefore, according to the Electric WGC (1.1) extensions of the Einstein–Maxwell theory of the type (1.26) with $2c_1 - c_3 > 0$ belong to the Landscape, all the others to the Swampland.

1.3 Magnetic WGC

We conclude this chapter presenting another version of the WGC, parallel - but not identical - to the Electric WGC, which is called *Magnetic WGC*. If we now introduce also a non-vanishing magnetic charge, we expect an statement analogous to (1.1) to hold as well. We can state it as follows:

Magnetic Weak Gravity Conjecture (4D). *Consider a theory coupling gravity to a $U(1)$ gauge field of gauge coupling g. The cutoff scale Λ associated to this effective field theory is such that*

$$\Lambda \lesssim g M_{\mathrm{P}}, \tag{1.37}$$

where g is, as before, the gauge coupling constant.

We can express this version of the conjecture as a bound on the effective field theory cutoff scale because the mass of magnetic monopoles - the candidate particles to realize the conjecture - is directly proportional to the associated magnetic field, which is linearly divergent, so that

$$M_{\mathrm{mag}} \sim \frac{\Lambda}{g^2}. \tag{1.38}$$

Thus, plugging this property into the magnetic version of equation (1.1), and the fact that the magnetic coupling is the inverse of the electric one (i.e. $g_{\mathrm{mag}} = 1/g$), allows to write the Magnetic WGC in the form (1.37):

$$M_{\mathrm{mag}} \lesssim \frac{1}{g} M_{\mathrm{P}} \quad \Longrightarrow \quad \Lambda \lesssim g M_{\mathrm{P}}. \tag{1.39}$$

This conjecture can be stated also as the requirement that the magnetic monopoles described by the theory are not black holes. This condition is in fact obtained by asking that the mass of the monopole is smaller than the associated Schwarzschild radius R_{S},

$$M_{\mathrm{mag}} \leq M_{\mathrm{P}}^2 R_{\mathrm{S}}, \tag{1.40}$$

which indeed provides an estimation of the energy scale at which the effective field theory description breaks down:

$$R_{\mathrm{S}} \sim \Lambda^{-1}, \tag{1.41}$$

so that we obtain again the Magnetic WGC constraint (1.37) by putting together equations (1.40), (1.41) and (1.38).

Another argument supporting this Magnetic WGC can be found again in the problem of the stable remnants. In the previous section we stated that the degeneracy of the spectrum of charges that a black hole (that is forbidden to discharge beyond the extremal case) with mass of the planckian size is as large as the gauge coupling is small. This is clearly seen from the extremality bound (1.6):

$$Q = gq \lesssim \frac{M}{\sqrt{2} M_{\mathrm{P}}} \simeq 1 \quad \Longrightarrow \quad q \lesssim \frac{1}{g}. \tag{1.42}$$

This equation clearly tells that any black hole with charge between 0 and g^{-1} is allowed. The problem of an infinite set of stable remnants therefore shows up when the gauge coupling is taken to be small, i.e. $g \to 0$. The Magnetic WGC (1.37) represents then a solution to this problem because now taking $g \to 0$ implies also $\Lambda \to 0$ and this is clearly inconsistent since we wouldn't be able to define the starting effective field theory in any energy range.

It is precisely its connection with the effective field theory cutoff scale Λ that makes the Magnetic WGC (1.37) not just the dual counterpart of the Electric one (1.1). Assuming the Magnetic WGC corresponds in fact to have well-defined energy scale in which our effective field theory can live: such a connection with the foundation of a theory are not present in the Electric WGC (1.1).

Also, it is through the Magnetic version that we can appreciate the connection of the WGC with the other Swampland conjectures (in particular with the No Global Symmetry Conjecture, that we analyze in the following section), again because of its formulation in terms of the cutoff scale Λ.

Chapter 2

Positivity Bounds

We have introduced the Swampland program and the concept of swampland conjectures to constrain the effective field theories and their higher-order extensions. In particular, we have seen how swampland conjectures are usually formulated on the basis of examples and arguments coming from different sources, like the black hole arguments motivating the Electric WGC (1.1).

In this second chapter we discuss instead an important class of evidence for swampland conjectures, which are the so called *positivity bounds* on the scattering amplitudes described by a theory. This is a set of constraints on the coefficients of an effective field theory - usually the ones associated to higher-order operators - that are determined by the properties of locality, Lorentz invariance and unitarity of the S-matrix that a theory must possess. For example, arbitrary signs of the coefficients of higher-derivative operators may lead to the production of superluminal signals (see [20]), which is clearly inconsistent with the causal structure of a Minkowski-like spacetime.

The procedure to implement these consistency requirements, and obtain then the positivity bounds, is the following. The property of the S-matrix to be unitary is expressed via the *optical theorem* (see [21, 22]), from which we deduce that the imaginary part of the amplitude of a forward elastic scattering must be positive. This property can then be translated into constraints over the theory coefficients - the mentioned positivity bounds - because the amplitude indeed depends on them. Such connection with the imaginary part of an elastic scattering amplitude is made possible by the *locality* property of the theory. In fact, locality makes this amplitude an analytic function of the kinematic invariants, which means that it can be seen as the real boundary of an analytic function (with cuts and poles - see [20]). Locality is indeed related to the mentioned causality issues: it is in fact the requirement that commutators[1] of fields operators vanish at spacelike distance and this means precisely that there cannot be superluminal signals. Lorentz invariance is then applied by exploiting crossing symmetries to reduce the number of independent components of the amplitudes under examination.

It's interesting to compare the bounds that these requirements produce on the coefficients with the prescriptions of the swampland conjectures and see if the two are in agreement. These constraints are in fact a key test for a conjecture to be well-defined because they are expression of the structural properties that characterize a meaningful theory: they either represent an important class of evidence supporting a swampland conjecture if they match the conditions under which the conjecture is realized or, if they do not match, they point out the limits of the conjecture itself and/or the need to refine it. Indeed, in the case of Einstein–Maxwell

[1] Commutators for bosonic fields, anticommutators for fermionic ones.

theory (1.4) and its 4-derivatives extension (1.26), it has been shown in [18] that there is an exact equivalence between the positivity bounds and the Electric WGC requirements (1.35), (1.36).

After deriving, in the first section of this chapter, the optical theorem, in the second section we present the procedure to obtain positivity bounds by explicitly applying it to Euler–Heisenberg theory, which is the theory describing the interactions of a U(1) gauge field to the 4-derivatives order. Finally, in the third section we discuss the relation between the positivity bounds and the Electric WGC in the Einstein–Maxwell theory, together with the general problems that arise in the computation of such bounds when gravity is involved.

2.1 The optical theorem

We start by presenting the optical theorem, which is a direct consequence of the unitarity of the S-matrix. Calling it $\mathcal{S}$, this property states that

$$\mathcal{S}^\dagger \mathcal{S} = \mathbb{1}. \tag{2.1}$$

The S-matrix can also be written in terms of the so called transfer matrix $\mathcal{T}$ as

$$\mathcal{S} = \mathbb{1} + i\,\mathcal{T}, \tag{2.2}$$

so that the unitarity condition (2.1) becomes

$$i\left(\mathcal{T}^\dagger - \mathcal{T}\right) = \mathcal{T}^\dagger \mathcal{T}. \tag{2.3}$$

This decomposition is useful because it establishes a connection with the scattering amplitudes. Considering a generic process leading from an initial state $|A\rangle$ to a final one $|B\rangle$, the associated amplitude $\mathcal{M}(A \to B)$ is related to the transfer matrix by

$$\langle B|\, \mathcal{T}\, |A\rangle = (2\pi)^4\, \delta^{(4)}(p_A - p_B)\, \mathcal{M}(A \to B), \tag{2.4}$$

where p_A and p_B are the initial and final momenta and $\delta^{(4)}(x)$ is the four-dimensional Dirac delta function. Applying therefore $\langle B|$ and $|A\rangle$ on both sides of (2.3) we obtain

$$i\,(2\pi)^4 \delta^{(4)}(p_A - p_B)\; [\mathcal{M}^*(B \to A) - \mathcal{M}(A \to B)] = \langle B|\, \mathcal{T}^\dagger \mathcal{T}\, |A\rangle\,. \tag{2.5}$$

To write also the right-hand side of (2.5) in terms of the amplitude we insert between $\mathcal{T}^\dagger$ and $\mathcal{T}$ the completeness relation in the Hilbert space of multi-particle states, labelled by $|n\rangle$:

$$\sum_n \int d\Pi_n\; |n\rangle\, \langle n| = \mathbb{1}, \tag{2.6}$$

with

$$d\Pi_n = \prod_{j\in n} \frac{d^3 p_j}{(2\pi)^3} \frac{1}{2E_j}. \tag{2.7}$$

Thus, inserting this completeness and applying again (2.4), equation (2.5) becomes:

$$\mathcal{M}(A \to B) - \mathcal{M}^*(B \to A) = i\,(2\pi)^4 \sum_n \int d\Pi_n\, \delta^{(4)}(p_n - p_A)\, \mathcal{M}^*(B \to n)\mathcal{M}(n \to A), \tag{2.8}$$

where $\vec{p}_j$ and E_j are the three-momentum and the energy ($p_j = (E_j, \vec{p}_j)$) of the j^{th} particle of the state $|n\rangle$.

This result is known as *generalized optical theorem*. We're though interested in the subcase of the elastic scattering, i.e. the case in which the initial and final state of the process coincide: $|B\rangle = |A\rangle$. In this configuration, equation (2.8) yields what is usually called the optical theorem:

$$2\mathfrak{Im}\left[\mathcal{M}(A \to A)\right] = (2\pi)^4 \sum_n \int d\Pi_n \, \delta^{(4)}(p_n - p_A) \, |\mathcal{M}(A \to n)|^2 \,. \tag{2.9}$$

We notice that the right-hand side of this identity is positive, being a sum of positive quantities: thus, also the left-hand side, namely the imaginary part of an elastic scattering amplitude, must be positive:

$$\mathfrak{Im}\left[\mathcal{M}(A \to A)\right] > 0. \tag{2.10}$$

This is the constraint that will ultimately produce the positivity bounds on the coefficients of an effective field theory.

2.2 Positivity bounds on Euler–Heisenberg theory

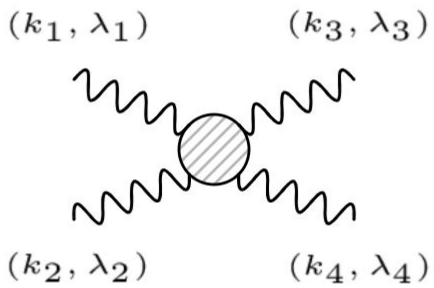
We now proceed by showing how to implement the optical theorem constraint (2.10) on the coefficients of a theory. To do so, we choose the particular setup of the Euler–Heisenberg Lagrangian

$$\mathcal{L}_{\text{EH}} = -\frac{1}{4} F \cdot F + \frac{a}{4m^2} (F \cdot F)^2 + \frac{b}{4m^2} (F \cdot \tilde{F})^2, \tag{2.11}$$

where $F \cdot F \equiv F_{\mu\nu} F^{\mu\nu}$ and m is the mass scale that drives the higher-derivative expansion. a and b are instead the coefficients of the two independent higher-order operators: while at this level they're arbitrary scalar factors, they're going to be constrained to be positive by the bounds we now compute.

2.2.1 Amplitude and crossing symmetries

The process we take under consideration is the two photons elastic scattering:



where k_1 and k_2 are the initial, incoming momenta, k_3 and k_4 the final, outgoing ones and the λ_i are the polarization indices.

Calling $\epsilon_{\lambda_i}^{\alpha_i}(k_i)$ the polarization vector associated to the i-photon, the amplitude of this process has the following general expression in terms of the kinematic invariants s, t and u:

$$\mathcal{M}_{\lambda_1\lambda_2\lambda_3\lambda_4}(s,t,u) = \epsilon^{*\,\alpha_4}_{\lambda_4}(k_4) \epsilon^{*\,\alpha_3}_{\lambda_3}(k_3) \epsilon^{\alpha_2}_{\lambda_2}(k_2) \epsilon^{\alpha_1}_{\lambda_1}(k_1) \, M_{\alpha_1\alpha_2,\alpha_3\alpha_4}(s,t,u). \tag{2.12}$$

To simplify calculations, we set this process to be a forward scattering, which corresponds to the momenta configuration such that $t = 0$. In the centre-of-mass frame we have:

$$k_1 = k_3 =(k,\, 0,\, 0,\, k), \qquad k_2 = k_4 =(k,\, 0,\, 0,\, -k). \tag{2.13}$$

The condition $t = 0$ implies that $u = -s$: thus, the dependence of the amplitude (2.12) on the kinematic invariance reduces to the only s variable.

Further, the polarization of the photons can be chosen to be the linear one:

$$\begin{aligned} \epsilon_x(k_1) =&(0,\, 1,\, 0,\, 0), \qquad & \epsilon_x(k_2) =&(0,\, -1,\, 0,\, 0), \\ \epsilon_y(k_1) =&(0,\, 0,\, 1,\, 0), \qquad & \epsilon_y(k_2) =&(0,\, 0,\, 1,\, 0). \end{aligned} \tag{2.14}$$

This is a simplifying choice because we notice that, because of Lorentz symmetry, the indices of $M_{\alpha_1\alpha_2\alpha_3\alpha_4}$ in (2.12) can be carried only by the momenta k_i or by the Minkowski metric $\eta_{\mu\nu}$. With the choice (2.14) for the polarizations, all the k-terms in $M_{\alpha_1\alpha_2\alpha_3\alpha_4}$ are then irrelevant, because each polarization vector in (2.14) is orthogonal to each momentum in (2.13).

We can then restrict the amplitude $M_{\alpha_1\alpha_2\alpha_3\alpha_4}$ in (2.12) to the only components that are proportional to the Minkowski metric:

$$M_{\alpha_1\alpha_2\alpha_3\alpha_4} = A(s)\,\eta_{\alpha_1\alpha_3}\eta_{\alpha_2\alpha_4} + B(s)\,\eta_{\alpha_1\alpha_4}\eta_{\alpha_2\alpha_3} + C(s)\,\eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4}, \tag{2.15}$$

so that the full amplitude becomes, thanks to (2.14),

$$\mathcal{M}_{\lambda_1\lambda_2\lambda_3\lambda_4}(s) = A(s)\,\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4} + B(s)\,\delta_{\lambda_1\lambda_4}\delta_{\lambda_2\lambda_3} + C(s)\,\delta_{\lambda_1\lambda_2}\delta_{\lambda_3\lambda_4}. \tag{2.16}$$

We can constrain the number of such independent amplitudes by exploiting the crossing symmetries, which requires the amplitude to be invariant under exchanges of legs in the associated diagram. If we swap the first and the third photon we have that $s \to u = -s$ and the identity the amplitude should satisfy is

$$\mathcal{M}_{\lambda_1\lambda_2\lambda_3\lambda_4}(s) = \mathcal{M}_{\lambda_3\lambda_2\lambda_1\lambda_4}(-s), \tag{2.17}$$

which yields the following relations among the functions of s in (2.15):

$$A(s) =A(-s), \qquad C(s) = B(-s). \tag{2.18}$$

The final result is that the amplitude (2.16) has only two independent components:

$$M_{xx}(s) \equiv M_{xxxx}(s) = A(s) + B(s) + B(-s), \tag{2.19}$$

$$M_{xy}(s) \equiv M_{xyxy}(s) = A(s), \tag{2.20}$$

such that

$$\mathcal{M}_{\lambda_1\lambda_2}(s) = \mathcal{M}_{\lambda_1\lambda_2}(-s). \tag{2.21}$$

Computing explicitly these amplitudes from the Euler–Heisenberg Lagrangian (2.11), one may see that

$$\begin{cases} \mathcal{M}_{xx}(s) =a\,s^2 \\ \mathcal{M}_{xy}(s) =b\,s^2 \end{cases} \iff \begin{cases} A(s) =b\,s^2 \\ B(s) =\dfrac{a-b}{2}\,s^2 \end{cases}. \tag{2.22}$$

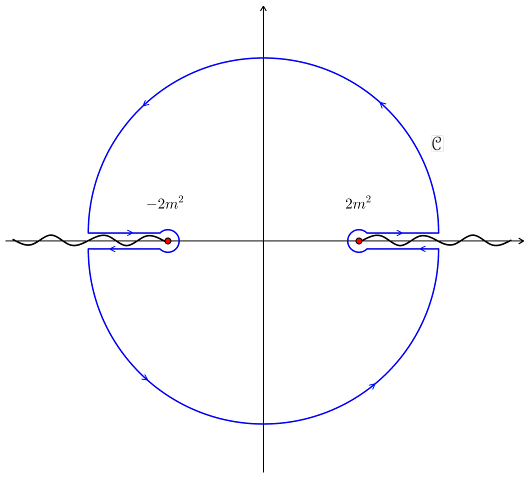


Figure 2.1: Domain of the analytic extension of $\mathcal{M}_{\lambda_1\lambda_2}$ and contour of integration.

2.2.2 Analyticity and positivity bounds

To make connection between the two amplitudes (2.19) and (2.20) and the optical theorem we now exploit the fact that, because of locality, the amplitude $\mathcal{M}_{\lambda_1\lambda_2}(s)$ should be the real boundary value of an analytic function $\mathcal{M}_{\lambda_1\lambda_2}(z)$, where the complex variable z is such that $\mathfrak{Re}z = s$. More specifically, this analytic extension of $\mathcal{M}_{\lambda_1\lambda_2}(z)$ is defined in all the complex plane $\mathbb{C}$ except for two cuts (see [20])) along the real axis, for $|s| \geq 2m^2$, as shown in Figure 2.1.

Along the contour $\mathcal{C}$ we can apply Cauchy formula and write

$$\frac{d^2\mathcal{M}_{\lambda_1\lambda_2}(z)}{d^2z}\bigg|_{z=0} = \frac{1}{i\,\pi}\oint_{\mathcal{C}} d\zeta\,\frac{\mathcal{M}_{\lambda_1\lambda_2}(\zeta)}{\zeta^3}. \tag{2.23}$$

We have now to evaluate the right-hand side of (2.23) along the different sections of the contour $\mathcal{C}$. First of all, we observe that the contributions given by the two circles at infinity are negligible because of the so called Froissart bound [23, 24] on forward scattering amplitudes:

$$\mathcal{M}(s, t=0) \lesssim \log^2 s. \tag{2.24}$$

Indeed, this bound makes the contributions of the circles vanishing at infinite radius:

$$\frac{\mathcal{M}_{\lambda_1\lambda_2}(\zeta)}{\zeta^3} = \frac{\mathcal{M}_{\lambda_1\lambda_2}(s + i\,\mathfrak{Im}(\zeta))}{(s + i\,\mathfrak{Im}(\zeta))^3} \overset{s\to+\infty}{\simeq} \frac{\log^2 s}{s^3} \longrightarrow 0. \tag{2.25}$$

Thus, we're left only with the contributions given by the four sections of $\mathcal{C}$ above and below the two branch cuts. Calling ϵ the arbitrary small, real value setting the shift from the cuts, equation (2.23) becomes

$$\begin{aligned}\frac{d^2\mathcal{M}_{\lambda_1\lambda_2}(z)}{d^2z}\bigg|_{z=0} =& \frac{1}{i\,\pi}\left[\int_{-\infty}^{-2m^2}\frac{ds}{s^3}\Big(\mathcal{M}_{\lambda_1\lambda_2}(s+i\,\epsilon) - \mathcal{M}_{\lambda_1\lambda_2}(s-i\,\epsilon)\Big)+\right.\\ &\left.+\int_{2m^2}^{+\infty}\frac{ds}{s^3}\Big(\mathcal{M}_{\lambda_1\lambda_2}(s+i\,\epsilon) - \mathcal{M}_{\lambda_1\lambda_2}(s-i\,\epsilon)\Big)\right].\end{aligned} \tag{2.26}$$

Applying Schwarz reflection principle [25]

$$\mathcal{M}(s^*) = \mathcal{M}(s)^*, \tag{2.27}$$

and taking the limit $\epsilon \to 0$ we obtain

$$\begin{aligned} \left.\frac{d^2\mathcal{M}_{\lambda_1\lambda_2}(s)}{d^2 s}\right|_{s=0} &= \frac{2}{\pi}\left[\int_{-\infty}^{-2m^2}\frac{ds}{s^3}\,\mathfrak{Im}\left[\mathcal{M}_{\lambda_1\lambda_2}(s)\right] + \int_{2m^2}^{+\infty}\frac{ds}{s^3}\,\mathfrak{Im}\left[\mathcal{M}_{\lambda_1\lambda_2}(s)\right]\right] = \\ &= \frac{2}{\pi}\int_{2m^2}^{+\infty}\frac{ds}{s^3}\,\mathfrak{Im}\left[\mathcal{M}_{\lambda_1\lambda_2}(s) + \mathcal{M}_{\lambda_1\lambda_2}(-s)\right] \overset{(2.21)}{=} \\ &= \frac{4}{\pi}\int_{2m^2}^{+\infty}\frac{ds}{s^3}\,\mathfrak{Im}\left[\mathcal{M}_{\lambda_1\lambda_2}(s)\right]. \end{aligned} \tag{2.28}$$

It is now possible to apply the optical theorem (2.9)-(2.10), from which we have

$$\left.\frac{d^2\mathcal{M}_{\lambda_1\lambda_2}(s)}{d^2 s}\right|_{s=0} > 0, \tag{2.29}$$

and making use of the explicit expression of the two amplitudes presented in (2.22) we finally obtain the positivity bounds on the two coefficients a and b of (2.11):

$$a > 0, \qquad\qquad b > 0. \tag{2.30}$$

2.3 Positivity bounds on Einstein–Maxwell theory

The procedure we explicitly worked out in the case of Euler–Heisenberg theory (2.11) is a good example of how positivity bounds on the higher-order coefficients of an effective field theory are obtained.

The positivity bounds for the 4-derivatives extension of Einstein–Maxwell theory (1.26) have been computed in [18] and result to be

$$2c_1 - c_3 > 0, \tag{2.31}$$

$$2c_2 + c_3 > 0, \tag{2.32}$$

$$c_2 > 0. \tag{2.33}$$

Thus, going back to the charge-to-mass ratio (1.34):

$$z_{\text{ext}} = 1 + \frac{64\pi^2 M_{\text{P}}^2}{5M^2}(2c_1 - c_3),$$

which characterizes an extremal black hole solution of (1.26), we immediately see that the positivity bound (2.31) set the higher-derivatives correction that it receives to be positive. Therefore, as anticipated, in the case of Einstein–Maxwell theory (1.4) there is an exact equivalence between the positivity bounds on the higher-order coefficients and the conditions (1.35), (1.36) under which the Electric WGC (1.1) is realized.

2.3.1 Subtleties with gravity

The constraints (2.31), (2.32) and (2.33) are an example of positivity bounds computed in a theory involving gravity. However, for such theories the procedure to obtain positivity bounds that we described in Section 2.2 is not effective anymore. In fact, because of the presence of gravitons, the amplitude (2.16) acquires a new term that in the forward limit becomes divergent:

$$\Delta\mathcal{M}(s,\, t \to 0) = -\frac{s^2}{M_{\mathrm{P}}^2\, t} + \mathcal{O}(s). \tag{2.34}$$

This means that when we apply the Cauchy formula (2.23) we end up with two divergences, one per side of the equation, that makes it, in this form, useless. Therefore, to be able to compute positivity bounds in a gravitational theory one needs, in general, to circumvent this divergence, known as Coulomb singularity.

In [18], the strategy to perform such an operation follows from the observation that in three spacetime dimension there is no propagating graviton and so no Coulomb divergence: the idea is then to compactify one spacetime dimension on a circle[2] and study the theory that results from the dimensional reduction. Also in this configurations the forward scattering amplitude contains contributions, that we call $\Delta\mathcal{M}_{\mathrm{div}}$, that could make useless the application of the Cauchy formula (2.23). The crucial fact is that this time the problematic terms cancel with each other from the two sides:

$$\left[\frac{d^2\mathcal{M}(s)}{d^2 s} - \frac{d^2\left(\Delta\mathcal{M}_{\mathrm{div}}\right)(s)}{d^2 s}\right]\Bigg|_{s=0} = \frac{4}{\pi}\int_0^{+\infty}\frac{ds}{s^3}\,\mathfrak{Im}\left[\tilde{\mathcal{M}}(s)\right] > 0, \tag{2.35}$$

where $\tilde{\mathcal{M}}$ denotes the "reduced" forward scattering amplitude, obtained, as said, via subtraction of the divergent contributions, whose counterpart is the term proportional to $\Delta\mathcal{M}_{\mathrm{div}}$ on the left-hand side. From this regularized amplitude it is still possible to derive meaningful positivity bounds, such as (2.31), (2.32) and (2.33).

[2]Despite this compactification indeed breaks Lorentz invariance in the 4-dimensional spacetime, in [18] it is claimed that the positivity bounds can be consistently computed exploiting the residual 3D Lorentz invariance of the non-compact dimensions.

Chapter 3

Electromagnetic Duality

This third chapter is devoted to the presentation of the second main topic of this thesis work: Electromagnetic (EM) Duality. The foundations of this peculiar symmetry, that concerns 4-dimensional theories involving gauge fields, were given by M. Gaillard and B. Zumino in their famous paper [26] and it states the invariance under rotation (on-shell) of the equations of motion (EoM) and Bianchi identities (BI) of abelian gauge fields. It is of extreme importance to remark that EM duality is *not* a symmetry of the Lagrangian but rather of the EoM and BI: we will see in fact that the Lagrangian does (and should) transform under a generic duality rotation.

The easiest example of electromagnetic duality is given by the well-known symmetry of the free Maxwell equations under the exchange of the electric and the magnetic field. This property is manifest from the explicit expression of the EoM and the BI of the free Maxwell theory:

$$\text{EoM:} \quad d \star F = 0, \tag{3.1}$$

$$\text{BI:} \quad dF = 0, \tag{3.2}$$

where $\star$ denotes the Hodge operator, acting as

$$\star F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}, \tag{3.3}$$

$$\epsilon_{\mu\nu\rho\sigma} = \sqrt{-g}\, \hat{\epsilon}_{\mu\nu\rho\sigma}, \tag{3.4}$$

with $\hat{\epsilon}$ the flat Levi-Civita tensor. To swap the electric and magnetic field is equivalent to interchange the roles of equations (3.1) and (3.2), which though keep the same expression: this is a EM duality transformation. In any case, Gaillard and Zumino's discussion is not restricted to pure gauge theories but applies to more generic ones, involving also other types of fields.

One reason that makes EM duality of great interest is it's connection with String Theory and Supergravity. Dualities are in fact a fundamental element of the String world because they allow to establish the equivalence among the different formulations of String Theory. EM duality can then be seen as a low-energy realization of the so called U-duality group of String Theory [14–16].

This property of EM duality suggests the idea to make use of it to constraint the higher-order operators of a given effective gauge theory. In the U-duality perspective, EM duality results in fact to be a symmetry of the full UV theory and therefore it is expected to hold at every perturbative order. Thus, given an independent set of higher-order operators realizing an extension of a low-energy gauge theory, one can try to further characterize, or even restrict, this set of operators by asking that the EM duality structure is preserved. An example of such an operation is given by P. Cano and A. Múrcia extension of Einstein–Maxwell theory [29].

This idea has also an immediate and very interesting connection with the Swampland Program. Similarly to the case of positivity bounds and the WGC in the context of Einstein–Maxwell theory [18], the constraints that EM duality can produce on a given theory can indeed be compared with the prescriptions of some swampland conjecture. EM duality can then be of further evidence for a conjecture (or a further requirement for the conjecture to be realized) or, on the contrary, produce some examples against it; in either cases, it provides an additional and meaningful benchmark on which test the swampland conjectures and determine the contours inside which they hold.

In this regard, an interesting example, relevant for this thesis work, is given by the fact that while positivity bounds on scattering amplitudes equivalently realize the WGC in the case of Einstein–Maxwell theory, they're not sufficient anymore when one tries to go beyond the pure Einstein–Maxwell setup. This equivalence seems to be restored if also duality requirements are included [33, 34]. This topic is the core of the main results of this thesis work and is going to be better discussed in the following chapter.

In Section 3.1 we introduce the idea of EM duality and how it works by presenting the review of Gaillard and Zumino analysis of [26]. In Section 3.2 we instead explore how EM duality can be used to constraint higher-order extensions of low-energy theories by discussing a possible duality-preserving extension of Einstein–Maxwell theory proposed by Cano and Múrcia [29].

3.1 Gaillard-Zumino duality

We now review the analysis through which Gaillard and Zumino derived in [26] the duality group of a 4-dimensional theory involving an arbitrary number of gauge fields.

Let's consider a theory coupling N abelian gauge fields F^Λ to a given set of other fields ϕ^i, described by the Lagrangian

$$\mathcal{L} = \mathcal{L}(F^\Lambda, \phi_i, \partial\phi_i), \tag{3.5}$$

which depends on the derivatives of the fields ϕ_i but is assumed to *not* depend on the derivatives of the gauge fields.

Next, we introduce the *dual fields* G_Λ as

$$\tilde{G}_{\Lambda\,\mu\nu} \equiv 2\frac{\partial\mathcal{L}}{\partial F^{\Lambda\,\mu\nu}}, \tag{3.6}$$

where

$$\tilde{F}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}. \tag{3.7}$$

With this definition, the EoM of the gauge fields F^Λ, obtained by varying (3.5) with respect to the associated vector potentials A^Λ, can be written as

$$0 = \cancel{\frac{\partial\mathcal{L}}{\partial A^\Lambda_\mu}} - \partial_\alpha\frac{\partial\mathcal{L}}{\partial\partial_\alpha A^\Lambda_\mu} = -\,\partial_\alpha\frac{\partial\mathcal{L}}{\partial F^\Sigma_{\rho\sigma}}\frac{\partial F^\Sigma_{\rho\sigma}}{\partial\partial_\alpha A^\Lambda_\mu} = -2\partial_\alpha\frac{\partial\mathcal{L}}{\partial F^\Lambda_{\alpha\mu}} = -\partial_\alpha\tilde{G}^{\alpha\mu}_\Lambda \quad \Longleftrightarrow$$

$$\Longleftrightarrow \quad \partial_\alpha\tilde{G}^{\alpha\mu}_\Lambda = 0, \tag{3.8}$$

while the BI are instead

$$\partial_\alpha \tilde{F}^{\Lambda\,\alpha\mu} = 0. \tag{3.9}$$

The system made by equations (3.8) and (3.9) is indeed invariant under a linear transformation of the gauge fields and their duals, which are called duality transformations. Calling F and G the vectors containing all the gauge fields and their duals, the infinitesimal version of such a duality transformation can be written as

$$\begin{pmatrix} \delta F \\ \delta G \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix}, \tag{3.10}$$

$$\delta\phi_i = \xi_i(\phi), \tag{3.11}$$

where $\{A, B, C, D\} \in \mathrm{GL}(N, \mathbb{R})$, while $\xi_i(\phi)$ denotes the associated duality transformation of the fields ϕ_i and it's assumed to be a non-derivative function of the various ϕ_i's.

These transformations, which clearly leave equations (3.8) and (3.9) invariant, should be consistent with the EoM of the ϕ_i and with the dual field definition (3.6). These consistency requirements translates into constraints over the duality transformations (3.10) and (3.11). The strategy is to study the F and ϕ dependence on the associated variation of the Lagrangian. This reads

$$\begin{aligned} \delta\mathcal{L} =& \frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\delta(\partial_\mu\phi_i) + \frac{\partial\mathcal{L}}{\partial F^\Lambda}\delta F^\Lambda = \\ =& \left[\xi_i\frac{\partial}{\partial\phi_i} + \partial_\mu\phi_j\frac{\partial\xi_i}{\partial\phi_j}\frac{\partial}{\partial\partial_\mu\phi_i} + \left(A^\Lambda{}_\Sigma F^\Sigma + B^{\Lambda\Sigma}G_\Sigma\right)\frac{\partial}{\partial F^\Lambda}\right]\mathcal{L}, \end{aligned} \tag{3.12}$$

where we used the property that the two operations δ and ∂_μ commute. Differentiating this equation with respect to F^Σ and making use of equations (3.6) and (3.10) we get

$$\begin{aligned} \frac{\partial}{\partial F^\Sigma}\delta\mathcal{L} =& \delta\frac{\partial\mathcal{L}}{\partial F^\Sigma} + \frac{1}{2}\left(A^\Lambda{}_\Sigma + B^{\Lambda\Omega}\frac{\partial G_\Omega}{\partial F^\Sigma}\right)\frac{\partial\mathcal{L}}{\partial F^\Lambda} = \\ =& \frac{1}{2}\delta\tilde{G}_\Sigma + \frac{1}{2}\left(A^\Lambda{}_\Sigma + B^{\Lambda\Omega}\frac{\partial G_\Omega}{\partial F^\Sigma}\right)\tilde{G}_\Lambda = \\ =& \frac{1}{2}C_{\Sigma\Omega}\tilde{F}^\Omega + \frac{1}{2}\left(A^\Lambda{}_\Sigma + D_\Sigma{}^\Lambda\right)\tilde{G}_\Lambda + \frac{1}{2}\tilde{G}_\Lambda B^{\Lambda\Omega}\frac{\partial G_\Omega}{\partial F^\Sigma} \end{aligned} \tag{3.13}$$

We can obtain some consistency condition by observing that, since the left-hand side of this equation is a derivative with respect to F^Σ, also the right-hand side must be so. To make manifest which part of the right-hand side of equation (3.13) can be seen as a derivative with respect to F^Σ and which one cannot, we observe that, since for the Levi-Civita tensor structure we have $\tilde{F}\cdot G = F\cdot\tilde{G}$, we have

$$\frac{\partial}{\partial F^\Sigma}(FC\tilde{F}) = (C_{\Sigma\Omega} + C_{\Omega\Sigma})\tilde{F}^\Omega, \tag{3.14}$$

$$\frac{\partial}{\partial G^\Sigma}(GC\tilde{G}) = \tilde{G}^\Lambda(B_{\Lambda\Omega} + B_{\Omega\Lambda})\frac{\partial G_\Omega}{\partial F^\Sigma}. \tag{3.15}$$

We can apply these identities in (3.13) by symmetrizing and anti-symmetrizing the B and C matrices, obtaining:

$$\begin{aligned}\frac{\partial}{\partial F^{\Sigma}}\delta\mathcal{L} =&\frac{1}{4}\frac{\partial}{\partial F^{\Sigma}}\left(FC\tilde{F}+GB^{T}\tilde{G}\right)+\left(A^{\Lambda}{}_{\Sigma}+D_{\Sigma}{}^{\Lambda}\right)\frac{\partial\mathcal{L}}{\partial F^{\Lambda}}+\\&+\frac{1}{4}\left(C_{\Sigma\Omega}-C_{\Omega\Sigma}\right)\tilde{F}^{\Omega}+\frac{1}{4}\tilde{G}^{\Lambda}\left(B_{\Lambda\Omega}-B_{\Omega\Lambda}\right)\frac{\partial G_{\Omega}}{\partial F^{\Sigma}}.\end{aligned} \tag{3.16}$$

The conditions that allow to write also the right-hand side of this transformation as a derivative with respect to F^{Σ} are therefore

$$B = B^{T}, \tag{3.17}$$

$$C = C^{T}, \tag{3.18}$$

$$A + D = \alpha\,\mathbb{1}, \tag{3.19}$$

with α a real constant. Equation (3.16) then becomes

$$\frac{\partial}{\partial F^{\Sigma}}\delta\mathcal{L} = \frac{\partial}{\partial F^{\Sigma}}\left(\frac{1}{4}FC\tilde{F}+\frac{1}{4}GB\tilde{G}+\alpha\mathcal{L}\right) \tag{3.20}$$

Next, a second constraint is obtained again from (3.12) in a similar fashion but this time making use of the ϕ_i EoM as a guideline. We define the EoM operator as

$$\hat{E}_i \equiv \frac{\partial}{\partial\phi^{i}} - \partial_{\mu}\frac{\partial}{\partial\partial_{\mu}\phi^{i}}, \tag{3.21}$$

so that the EoM of the field ϕ_i are

$$\hat{E}_i[\mathcal{L}] \equiv E_i = 0. \tag{3.22}$$

Under the duality transformation (3.11), the ϕ_i EoM transforms covariantly as

$$\delta E_i = -\frac{\partial\xi^{j}}{\partial\phi^{i}}E_j. \tag{3.23}$$

Now, with the same approach of equation (3.13), we apply the EoM operator (3.21) to the Lagrangian variation (3.12). After some algebraic calculation, and making use of equations (3.23), (3.6) and (3.15), we obtain

$$\begin{aligned}\hat{E}_i[\delta\mathcal{L}] =&\cancel{\delta E_i+\frac{\partial\xi^{j}}{\partial\phi^{i}}E_j}+\frac{\partial G_{\Lambda}}{\partial\phi^{i}}B^{\Lambda\Sigma}\frac{\partial\mathcal{L}}{\partial F^{\Sigma}}-\partial_{\mu}\left(\frac{\partial G_{\Lambda}}{\partial\partial_{\mu}\phi^{i}}B^{\Lambda\Sigma}\frac{\partial\mathcal{L}}{\partial F^{\Sigma}}\right)=\\=&\frac{1}{2}\frac{\partial G_{\Lambda}}{\partial\phi^{i}}B^{\Lambda\Sigma}\tilde{G}_{\Sigma}-\frac{1}{2}\partial_{\mu}\left(\frac{\partial G_{\Lambda}}{\partial\partial_{\mu}\phi^{i}}B^{\Lambda\Sigma}\tilde{G}_{\Sigma}\right)=\\=&\frac{1}{4}\left(\frac{\partial}{\partial\phi^{i}}-\partial_{\mu}\frac{\partial}{\partial\partial_{\mu}\phi^{i}}\right)\left(GB\tilde{G}\right)=\frac{1}{4}\hat{E}_i\left[GB\tilde{G}\right]\end{aligned} \tag{3.24}$$

Thus, we obtained two different transformation rules for $\delta\mathcal{L}$, equation (3.20) and (3.24):

$$\begin{cases}\dfrac{\partial}{\partial F^{\Sigma}}\delta\mathcal{L} =\dfrac{\partial}{\partial F^{\Sigma}}\left(\dfrac{1}{4}FC\tilde{F}+\dfrac{1}{4}GB\tilde{G}+\alpha\mathcal{L}\right)\\ \hat{E}_i[\delta\mathcal{L}] =\dfrac{1}{4}\hat{E}_i\left[GB\tilde{G}\right]\end{cases}, \tag{3.25}$$

describing the dependence of $\delta\mathcal{L}$ respectively on the gauge fields F^{Λ} and on the other ϕ_i.

From this two equations we may wright

$$\begin{cases} \delta\mathcal{L} = & \frac{1}{4}FC\tilde{F} + \frac{1}{4}GB\tilde{G} + \alpha\mathcal{L} + f_1(\phi) \\ \delta\mathcal{L} = & \frac{1}{4}GB\tilde{G} + f_2(F) \end{cases} \tag{3.26}$$

where $f_1(\phi)$ and $f_2(F)$ are two arbitrary real functions of the various ϕ_i and F^Λ respectively. By comparing these two equations, we see that we must have

$$f_1(\phi) = 0, \tag{3.27}$$

$$f_2(F) = \frac{1}{4}FC\tilde{F}, \tag{3.28}$$

$$\alpha = 0. \tag{3.29}$$

Therefore, the consistency conditions on the duality matrix defined in (3.10) are

$$A = -D^T, \qquad B = B^T, \qquad C = C^T; \tag{3.30}$$

these constraints (3.30) on the components of the duality matrix fix the duality group to be $\mathrm{Sp}(2N, \mathbb{R})$. The variation of the Lagrangian reads instead

$$\delta\mathcal{L} = \frac{1}{4}\left(FC\tilde{F} + GB\tilde{G}\right). \tag{3.31}$$

Equations (3.30) and (3.31) are the main result of Gaillard and Zumino's remarkable paper. Some comments are now in order. First of all, $\mathrm{Sp}(2N, \mathbb{R})$ is actually the maximal duality group that a theory can have: it may happen that the non-gauge fields further restrict it to a subgroup of $\mathrm{Sp}(2N, \mathbb{R})$ in order to have (3.31) satisfied. Further, when gravity is involved, the energy tensor of the theory results to be invariant under this actual duality group of the theory, so that Einstein equations are indeed duality-invariant (the metric does not transform under duality).

Another important observation regards equation (3.31), which testifies that EM duality is a symmetry of the EoM and the BI and not of the Lagrangian, which indeed transform. From (3.31) we see that only a subgroup of the duality transformations, the diagonal one ($B = 0 = C$), leaves the Lagrangian invariant. Also, since the Lagrangian is function only of the "original" fields F^Λ, only lower-triangular duality transformations can be re-adsorbed via fields redefinitions. Thus, a generic duality transformation contains a component which determines a change of the so called "symplectic frame", by modifying the starting Lagrangian according to (3.31): EM duality identifies therefore a set of theories with the same EoM, i.e. the same dynamics, but which are not connected by fields redefinitions.

3.1.1 EM duality as a Legendre transform

An interesting property of Gaillard and Zumino duality is that it can be interpreted as a Legendre transform. To see this, we consider the case of one gauge field F and a Lagrangian $\mathcal{L} = \mathcal{L}[F, \dots]$ and we introduce the EM dual field G as in (3.6):

$$\tilde{G}_{\mu\nu} \equiv 2\frac{\partial\mathcal{L}[F]}{\partial F}.$$

This EM dual field can indeed be seen as the Legendre dual of F and used then to build a dual Lagrangian. Following the Legendre formalism, the dual Lagrangian is generically defined as

$$\mathcal{L}_D[F_D] = \mathcal{L}[F] - F \cdot F_D, \tag{3.32}$$

where F_D is the Legendre dual of F. The following duality relations, which link one side to the other, hold:

$$F_D^{\mu\nu} = \frac{\partial \mathcal{L}[F]}{\partial F_{\mu\nu}} \tag{3.33}$$

$$F_{\mu\nu} = - \frac{\partial \mathcal{L}_D[F_D]}{\partial F_D^{\mu\nu}} \tag{3.34}$$

By comparing equations (3.6) and (3.33), we can see that the EM dual field $\tilde{G}$ can thus be seen as Legendre dual of F, defining the following dual Lagrangian:

$$\mathcal{L}_D[\tilde{G}] = \mathcal{L}[F] - \frac{1}{2} F \cdot \tilde{G}, \tag{3.35}$$

while the duality relations become

$$\tilde{G}_{\mu\nu} = 2 \frac{\partial \mathcal{L}[F]}{\partial F^{\mu\nu}} \tag{3.36}$$

$$F^{\mu\nu} = - 2 \frac{\partial \mathcal{L}_D[\tilde{G}]}{\partial \tilde{G}_{\mu\nu}} \tag{3.37}$$

Thanks to the duality relation (3.36), to each solution of the EoM of the starting Lagrangian $\mathcal{L}[F]$ corresponds one solution of the dual Lagrangian $\mathcal{L}_D[G]$.

The Legendre formalism of EM duality was suggested in [27, 28]; for a more detailed description of how it can be used to constraint higher-order operators' coefficients, see [17].

3.1.2 EM duality and higher-derivative operators

The strength of Gaillard and Zumino's analysis is that it's carried out in a completely general setting: there is no perturbative assumption on the Lagrangian, which is kept generic, like no specifications on the non-gauge fields ϕ_i and their characteristics have been made. This fact has the important outcome of making EM a deep property of gauge theories, which are naturally equipped with this dual structure that does not depend on anything but the gauge nature of the theory, regardless in particular of the energy scale to which the theory belongs.

This general character is in agreement with the perspective for which EM duality is a manifestation of the so called U-duality of String Theory: being a symmetry of the ultimate UV theory, it is expected to hold at every perturbative order. As outlined at the beginning of this chapter, this property makes EM duality a relevant tool to constraint higher-derivatives operators of low-energy Lagrangians, which is an interesting procedure also from the Swampland Program point of view.

However, this procedure of constraining the higher-derivative extension of a gauge theory through EM duality is affected by an intrinsic problem. One of the few hypothesis of Gaillard and Zumino derivation is that the Lagrangian does *not* contain operators involving derivatives of the gauge field. This hypothesis is crucial for their analysis, as equation (3.12) clearly shows. In fact, keeping also those operators would mean that the F-contribution to the variation of the Lagrangian with respect to the duality transformation would result to be an infinite series of terms:

$$\delta_F \mathcal{L} = \frac{\partial \mathcal{L}}{\partial F^{\Lambda}_{\mu\nu}} \delta F^{\Lambda}_{\mu\nu} + \frac{\partial \mathcal{L}}{\partial \partial_\alpha F^{\Lambda}_{\mu\nu}} \partial_\alpha \delta F^{\Lambda}_{\mu\nu} + \frac{\partial \mathcal{L}}{\partial \partial_\alpha \partial_\beta F^{\Lambda}_{\mu\nu}} \partial_\alpha \partial_\beta \delta F^{\Lambda}_{\mu\nu} + \dots \, . \tag{3.38}$$

Thus, any higher-order extension that includes operator with derivatives on the gauge fields cannot be analyzed via Gaillard and Zumino duality. This issue may be ignored when the perturbative order of the extension is small because one may be able to exclude, via identities and/or fields redefinitions, the problematic operators from the independent set that is considered, but as higher is the order one wants to reach, the more difficult is to (not arbitrarily) exclude them.

This problem can be tackled in two different ways. One possible strategy is to somewhat abandon the generic setting of Gaillard and Zumino derivation and rely instead to a perturbative, model-based approach. The idea is to again start by the infinitesimal duality transformation of the type (3.10) and fix the duality group by asking the duality transformation to be self-consistent in the specific case of the (2-derivative) theory under examination. The higher-order operators are then determined by consistency with the duality group found in this way. This procedure is indeed similar to Gaillard and Zumino's one; the difference is precisely that there's no need to make assumptions on the full structure of the theory because the duality analysis is made order by order, so that also operators involving derivatives of the gauge fields are not excluded a priori. This is the procedure we followed in the main analysis of this thesis work and it's better described in the next chapter.

Another possible approach to this problem is the one developed by P. Cano and A. Múrcia in [29], in which the operators involving derivatives of the gauge fields are completely excluded from the analysis by virtue of the (claimed) consistency between the constitutive relation (3.6) and the duality transformations (3.10). In the following section we describe in more details this analysis of Cano and Múrcia, which determined a sort of algorithm to find higher-derivative extensions of Einstein–Maxwell theory (1.4) and, moreover, we discuss the criticisms of their approach, focusing in particular on the argument the two authors bring to completely exclude the operators involving derivatives of the gauge field from the higher-order Lagrangian.

3.2 Duality-preserving extension of Einstein–Maxwell theory

In [29], P. Cano and A. Múrcia want to find higher-derivative extensions of Einstein–Maxwell theory (1.4),

$$S_{\text{EM}} = \int d^4x \, \sqrt{|g|} \left[\frac{M_{\text{P}}^2}{2} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right] ,$$

that preserve EM duality, and they do so by exploiting some consistency conditions coming from the duality structure. In particular, the outcome of their analysis is a sort of algorithm to find order by order in the derivatives the different extension of (1.4).

3.2.1 Duality transformation and structure of the Lagrangian

Cano and Múrcia introduce the duality transformations in a slightly different way with respect to Gaillard and Zumino. The generic higher-order Lagrangian that the authors want to determine is denoted by

$$\mathcal{L} = \mathcal{L}(g_{\mu\nu}, R_{\mu\nu\rho\sigma}, D_\alpha R_{\mu\nu\rho\sigma}, \ldots; F_{\mu\nu}, D_\alpha F_{\mu\nu}, dots) \tag{3.39}$$

and the resulting EoM of the gauge fields are

$$0 = D_\mu \left(F^{\mu\nu} - 2\frac{\delta \mathcal{L}}{\delta F_{\mu\nu}} \right), \tag{3.40}$$

$$\frac{\delta \mathcal{L}}{\delta F_{\mu\nu}} = \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}} - D_\alpha \frac{\partial \mathcal{L}}{\partial D_\alpha F_{\mu\nu}} + \ldots . \tag{3.41}$$

The dual field is then defined as

$$\tilde{G}_{\mu\nu} \equiv -F_{\mu\nu} + 2\frac{\delta \mathcal{L}}{\delta F_{\mu\nu}}, \tag{3.42}$$

so that the EoM and the BI of the gauge fields are indeed

$$dF = 0, \tag{3.43}$$

$$dG = 0. \tag{3.44}$$

These definitions make explicit the different contributions of the starting, 2-derivative Lagrangian (1.4) and of the higher-order one $\mathcal{L}$. Also, we notice that at this level the derivatives of the gauge field are still allowed in $\mathcal{L}$.

Now, as in Gaillard and Zumino, the system (3.44) of the EoM and the BI of $F_{\mu\nu}$ is invariant, in principle, under a $\mathrm{GL}(2,\mathbb{R})$ transformation over the gauge field and its dual. Consistency with the duality definition (3.42) and with the invariance of Einstein equations (i.e. the invariance of the stress-energy tensor) reduce this group to the $SO(2,\mathbb{R})$ rotations:

$$\begin{pmatrix} F' \\ G' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} \tag{3.45}$$

At this point, Cano and Múrcia argue that internal consistency between the dual field definition (3.42) and the duality transformation (3.45) forbids $\mathcal{L}$ to contain operators with derivatives on the gauge field. The argument is the following. If $\mathcal{L}$ does contain operators with derivatives on the gauge fields, equation (3.42) is a differential relation, which makes its inverse $F = F(G)$ involving an integration. On the contrary, considering a (3.45) rotation of angle $\alpha = \pi/2$, we can write:

$$\begin{cases} F' = -G \\ G' = F \end{cases} \implies \tilde{F}' = -\tilde{G} \overset{(3.42)}{=} F - 2\frac{\partial \mathcal{L}}{\partial F} = G' - 2\frac{\partial \mathcal{L}}{\partial F}\bigg|_{F \to G'}. \tag{3.46}$$

Though analogous in form to the mentioned inverse of (3.42), this relation is differential, while the former involves integration. This is considered to be a problem: since the EoM and BI are invariant under duality rotations, the two relations should be equivalent but this is not the case if one involves integration and one differentiation. Because of this argument, operators with derivatives are excluded from the higher-order Lagrangian:

$$\mathcal{L} = \mathcal{L}(g_{\mu\nu}, R_{\mu\nu\rho\sigma}, D_\alpha R_{\mu\nu\rho\sigma}, \ldots; F_{\mu\nu}), \tag{3.47}$$

$$\frac{\delta \mathcal{L}}{\delta F_{\mu\nu}} = \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}}. \tag{3.48}$$

This is indeed a subtle point. Cano and Múrcia notice also that one may recover a differential relation also from (3.42) by a perturbative expansion. The problem of such an operation is that also the invariance under duality would hold perturbatively as well and, because of the above argument, theories with operators involving derivatives of the gauge field would lead to a divergent series instead of an exactly invariant theory (see [29, 36]).

3.2.2 Higher-derivative terms

Once the structure of the higher-order Lagrangian is restricted to (3.47), we now turn to the sort of algorithm that Cano and Múrcia derived to determine the various higher-derivative operators. This is done again by exploiting the two duality rules (3.42) and (3.45) and the associated invariance of the EoM.

Starting from (3.45), we have

$$\begin{aligned} G' &= \sin\alpha F + \cos\alpha G \\ \star G' &= \sin\alpha \star F + \cos\alpha \star G \overset{(3.42)}{=} -\cos\alpha F + 2\cos\alpha \frac{\partial \mathcal{L}}{\partial F} + \sin\alpha \star F \overset{(3.45)}{=} \\ &= -F' - \sin\alpha (G - \star F) + 2\cos\alpha \frac{\partial \mathcal{L}}{\partial F} \overset{(3.42)}{=} -F' + 2\left(\cos\alpha + \sin\alpha \star\right) \frac{\partial \mathcal{L}}{\partial F} \equiv \\ &\equiv -F' + 2\hat{R} \left.\frac{\partial \mathcal{L}}{\partial F}\right|_{F \to \cos\alpha F' + \sin\alpha G'} . \end{aligned} \tag{3.49}$$

In order for this equation to be compatible with (3.42) for F' and G', we see that we have to require

$$\hat{R} \left.\frac{\partial \mathcal{L}}{\partial F}\right|_{F \to \cos\alpha F' + \sin\alpha G'} = \left.\frac{\partial \mathcal{L}}{\partial F}\right|_{F \to F'} , \tag{3.50}$$

and since this identity should hold on-shell, we have

$$\hat{R} \left.\frac{\partial \mathcal{L}}{\partial F}\right|_{F \to \hat{R}F' - 2\sin\alpha \star\left(\frac{\partial \mathcal{L}}{\partial F}\right)} = \left.\frac{\partial \mathcal{L}}{\partial F}\right|_{F \to F'} , \tag{3.51}$$

This identity is the "master equation" that Cano and Múrcia use to determine $\mathcal{L}$ such that EM duality is preserved. To do so, although $\mathcal{L}$ is considered to be an exactly invariant theory, they rely on the following perturbative expansion:

$$\mathcal{L} = \frac{1}{m^2}\mathcal{L}_4 + \frac{1}{m^4}\mathcal{L}_6 + \dots , \tag{3.52}$$

in which m is the energy scale driving the expansion and each $\mathcal{L}_n$ term contains n derivatives of the fields. Plugging this expansion into equation (3.49) and working until the 6-derivatives order[1], we obtain

$$\begin{aligned} G'_{\mu\nu} = -F'_{\mu\nu} + \Bigg\{ &\frac{2}{m^2}\,\hat{R}\frac{\partial \mathcal{L}_4}{\partial F^{\mu\nu}} + \frac{2}{m^4}\,\hat{R}\frac{\partial \mathcal{L}_6}{\partial F^{\mu\nu}} + \\ &- \sin\alpha\,\hat{R}\left(\star\hat{R}\frac{\partial \mathcal{L}_4}{\partial F}\right)^{\alpha\beta} \frac{\partial^2 \mathcal{L}_4}{\partial F^{\alpha\beta}\partial F^{\mu\nu}} \Bigg\}\Bigg|_{F \to \hat{R}F'} + \mathcal{O}\left(\frac{1}{m^6}\right) . \end{aligned} \tag{3.53}$$

[1] In [29] the authors carry out the calculations also for the 8-derivatives operators.

Thus, in virtue of identity (3.51), we have the following identification:

$$\frac{\partial \mathcal{L}_4}{\partial F^{\mu\nu}} = \hat{R}\frac{\partial \mathcal{L}_4}{\partial F^{\mu\nu}}\bigg|_{F\to\hat{R}F}, \tag{3.54}$$

$$\frac{\partial \mathcal{L}_6}{\partial F^{\mu\nu}} = \left[\frac{\partial \mathcal{L}_6}{\partial F^{\mu\nu}} - \frac{1}{2}\left(\star\hat{R}\frac{\partial \mathcal{L}_4}{\partial F}\right)^{\alpha\beta}\frac{\partial^2 \mathcal{L}_4}{\partial F^{\alpha\beta}\partial F^{\mu\nu}}\right]\bigg|_{F\to\hat{R}F}. \tag{3.55}$$

To solve such identities, we observe that a given operator of $\mathcal{L}$ takes the general form

$$F^a D^b R^c, \tag{3.56}$$

with a, b and c that are not mixed by duality transformations, so that each of these operators independently satisfy the associated identity. From (3.56) we it follows also that

$$F^{\mu\nu}\frac{\partial \mathcal{L}}{\partial F^{\mu\nu}} = a\mathcal{L} \quad \Longrightarrow \quad F^{\mu\nu}\left(\hat{R}\frac{\partial \mathcal{L}}{\partial F^{\mu\nu}}\right)\bigg|_{F\to\hat{R}F} = a\mathcal{L}\Big|_{F\to\hat{R}F}. \tag{3.57}$$

We apply then this additional identity to determine $\mathcal{L}_4$ and $\mathcal{L}_6$.

• $\boldsymbol{\mathcal{L}_4}$ Combining equations (3.54) and (3.57) we get

$$\frac{\partial \mathcal{L}_4}{\partial F^{\mu\nu}} = \hat{R}\frac{\partial \mathcal{L}_4}{\partial F^{\mu\nu}}\bigg|_{F\to\hat{R}F} \Longrightarrow F^{\mu\nu}\frac{\partial \mathcal{L}_4}{\partial F^{\mu\nu}} = F^{\mu\nu}\hat{R}\frac{\partial \mathcal{L}_4}{\partial F^{\mu\nu}}\bigg|_{F\to\hat{R}F} \Longrightarrow \mathcal{L}_4(F) = \mathcal{L}_4(\hat{R}F). \tag{3.58}$$

This last identity (3.58) completely characterizes $\mathcal{L}_4$, that contains therefore only $\hat{R}$-invariant operators, i.e. operators invariant under a $SO(2,\mathbb{R})$-rotation of F and $\tilde{F}$. Defining the $SO(2,\mathbb{R})$ vector $\mathcal{F} \equiv (F, \tilde{F})$, we can find them by studying the invariance condition of a generic $SO(2,\mathbb{R})$-built operator:

$$\mathcal{F}_A\, M^A{}_B \mathcal{F}^B \quad \longrightarrow \quad \mathcal{F}'_A\, M^A{}_B \mathcal{F}'^B = \mathcal{F}_A\, M^A{}_B \mathcal{F}^B \quad \Longleftrightarrow \quad S^T M S = M, \tag{3.59}$$

where $S \in SO(2,\mathbb{R})$ and M is the matrix coefficient that we fix via the invariance requirement. The result of this calculation is that M must be proportional to the identity.

$$M \propto \mathbb{1}. \tag{3.60}$$

Thus, the $SO(2,\mathbb{R})$-invariant operators appearing in $\mathcal{L}_4$ are proportional to

$$\begin{aligned}\mathcal{F}_{A\,\mu\nu}\mathcal{F}^{A\,\alpha\beta} =& F_{\mu\nu}F^{\alpha\beta} + \tilde{F}_{\mu\nu}\tilde{F}^{\alpha\beta} = \\ =& \cancel{F_{\mu\nu}F^{\alpha\beta}} - \cancel{F_{\mu\nu}F^{\alpha\beta}} + 4\left[F^{[\alpha|\gamma}F_{[\mu|\gamma} - \frac{1}{4}F^2\delta^{[\alpha|}{}_{[\mu|}\right]\delta^{|\beta]}{}_{|\nu]} = T^{[\alpha}{}_{[\mu}\delta^{\beta]}{}_{\nu]},\end{aligned} \tag{3.61}$$

where $T^{\alpha}{}_{\mu}$ is precisely the stress-energy tensor associated to Einstein–Maxwell (2-derivative) action (1.4). Therefore, all the dependence of $\mathcal{L}_4$ on the gauge field must come through the stress-energy tensor (3.61) and its most general, duality-preserving expression is

$$\begin{aligned}\mathcal{L}_4 =& \alpha_1 T_{\mu\nu}T^{\mu\nu} + \alpha_2 R^{\mu\nu}T_{\mu\nu} + \alpha_3(R_{\mu\nu})^2 + \alpha_4 R^2 + \alpha_5 G_B = \\ =& \frac{\alpha_1}{4}\left[(F\cdot F)^2 + (F\cdot\tilde{F})^2\right] + \alpha_2\left[R_{\mu\nu}F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}RF^2\right] + \alpha_3(R_{\mu\nu})^2 + \alpha_4 R^2 + \alpha_5 G_B,\end{aligned} \tag{3.62}$$

where G_B is the topological Gauss–Bonnet term (1.13). We remark that while the various operators could be identified independently, the EM duality requirement that the dependence on $F_{\mu\nu}$ should occur via $T_{\mu\nu}$ fixes the relative coefficients. Indeed, this result is different from the one we obtain in (1.26): the operator $R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$ seems to violate duality.

• $\boldsymbol{\mathcal{L}_6}$ The same strategy applies now to $\mathcal{L}_6$ but this time the starting identity (3.55) is more involved. Applying (3.57) we get in fact

$$\mathcal{L}_6(\hat{R}F) = \mathcal{L}_6(F) - \frac{1}{4}\left(\sin\alpha \star \hat{R}\frac{\partial \mathcal{L}_4}{\partial F}\right)^{\mu\nu} \frac{\partial \mathcal{L}_4}{\partial F^{\mu\nu}}\bigg|_{F\to\hat{R}F}. \tag{3.63}$$

This means that $\mathcal{L}_6$ is not invariant under a $SO(2,\mathbb{R})$-rotation of F and $\tilde{F}$. Thus, $\mathcal{L}_6$ takes the form

$$\mathcal{L}_6 = \mathcal{L}_6^{\mathrm{H}} + \mathcal{L}_6^{\mathrm{IH}}, \tag{3.64}$$

where $\mathcal{L}_6^{\mathrm{H}}$ is the homogeneous term, i.e. a term that is $SO(2,\mathbb{R})$-invariant like $\mathcal{L}_4$ (see (3.58), while $\mathcal{L}_6^{\mathrm{IH}}$ is the inhomogeneous term, producing the non-trivial transformation (3.63). Cano and Múrcia show that this inhomogeneous term reads

$$\mathcal{L}_6^{\mathrm{IH}} = -\frac{1}{8}\frac{\partial \mathcal{L}_4}{\partial F^{\mu\nu}}\frac{\partial \mathcal{L}_4}{\partial F_{\mu\nu}}. \tag{3.65}$$

For more details on the explicit expression of $\mathcal{L}_6$, on the 8-derivatives order calculations and on the (crucial) invariance of the higher-order Einstein equations, see [29].

3.2.3 Remarks on Cano and Múrcia analysis

The one of Cano and Múrcia is an interesting application of Gaillard and Zumino duality and a good example of how EM duality can be used to constraint the higher-derivative extension of a Lagrangian, in this case being the Einstein–Maxwell theory (1.4).

As shown previously in this section, the two authors, similarly to Gaillard and Zumino, start from the definitions of the dual field (3.42) and of the duality transformation (3.45) under which the EoM and BI of the gauge field stays invariant and determine the higher-order, duality-preserving Lagrangian by imposing the self-consistency between the two definitions (3.42) and (3.45) (see (3.47), (3.51) and (3.57)). Although the found consistency conditions should define an exactly-duality preserving Lagrangian, to carry out explicitly the calculations they turn to a perturbative expansion (see (3.52), (3.54) and (3.55)). The result is a sort of algorithm that allows to compute order by order in the derivatives the different duality-preserving contributions to the Lagrangian (see (3.62), (3.64) and (3.65)).

The critical point, as we understood, are again the operators involving derivatives of the gauge fields and how to deal with them. Differently from Gaillard and Zumino, whose analysis is somewhat followed in parallel but not directly applied, Cano and Múrcia do not assume the absence of such operators from the Lagrangian as starting hypothesis but rather argue that their presence is entirely forbidden by, again, duality consistency.

Their argument rely on the comparison between the inverse of the dual field definition (3.42),

$$\tilde{G}_{\mu\nu} \equiv -F_{\mu\nu} + 2\frac{\delta\mathcal{L}}{\delta F_{\mu\nu}} \quad\longrightarrow\quad F = F(G),$$

and the analogous relation (3.46) that one obtains via a duality rotation (3.45) of angle $\pi/2$:

$$F' = G' - 2\frac{\partial \mathcal{L}}{\partial F}\bigg|_{F\to G'}.$$

Cano and Múrcia's claim is that these two relations should be equivalent because of the invariance of the EoM and BI under a duality rotation (3.45), but this is impossible if $\mathcal{L}$ does contain operators with derivatives on the gauge field because the former would involve integration, while the latter differentiation.

This argument is, however, somewhat vague. The presence of integration in the inverse of (3.42) is in fact not clear. In general, we can perform easily such an inversion only when the Lagrangian contains "simple" (usually, 2-derivatives) operators, but this is not the case when the Lagrangian starts to include also higher-order operators and/or a larger set of fields. Let's understand this problem by considering some examples:

1. **Einstein–Maxwell theory** in the 2-derivative Einstein Maxwell theory (1.4) the inversion is trivial:
$$\tilde{G}_{\mu\nu} = -F_{\mu\nu} \quad \longrightarrow \quad F_{\mu\nu}(G) = -\tilde{G}_{\mu\nu}. \tag{3.66}$$

2. **Non-minimally coupled theory** A more involved theory is given by
$$L = -\frac{1}{4}\mathcal{I}(\phi)F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}\mathcal{R}(\phi)F_{\mu\nu}\tilde{F}^{\mu\nu} + (\dots)\,, \tag{3.67}$$
which is an example of possible gauge sector of a theory involving also a scalar field ϕ, of which the couplings $\mathcal{I}$ and $\mathcal{R}$ are functions (the "..." denote other possible non-gauge operators, such as the ϕ kinetic term). The dual field in this case reads
$$G_{\mu\nu} = \mathcal{I}(\phi)\tilde{F}_{\mu\nu} + \mathcal{R}(\phi)F_{\mu\nu} = \left[\frac{1}{2}\mathcal{I}(\phi)\,\epsilon_{\mu\nu\rho\sigma} + \mathcal{R}(\phi)\,g_{[\mu|\rho}g_{|\nu]\sigma}\right]F^{\rho\sigma} \equiv \hat{D}(\phi)F^{\rho\sigma}, \tag{3.68}$$
so that the inverse relation yields
$$F^{\mu\nu}(G) = \hat{D}(\phi)^{-1\,\mu\nu\rho\sigma}G_{\rho\sigma} = \frac{\mathcal{I}}{\mathcal{I}^2+\mathcal{R}^2}\tilde{G}^{\mu\nu} + \frac{\mathcal{R}}{\mathcal{I}^2+\mathcal{R}^2}G^{\mu\nu}. \tag{3.69}$$

We could obtain (3.69) because we can easily invert the operator $\hat{D}(\phi)$ in (3.68), but this is far from obvious with more complicated theories. One immediate example is the Cano and Múrcia $\mathcal{L}_4$ (3.62), where combinations of the gauge filed with the Riemann and the Ricci tensor, as well as quartic gauge field operators, starts to appear. Another one can be found again in a theory such (3.67), involving a gauge field and a scalar. At the 4-derivative order operators such

$$\partial_\mu\phi\partial^\mu\phi F^2, \qquad \partial_\mu\phi\partial^\mu\phi F\tilde{F}, \qquad \partial_\mu\phi\partial_\nu\phi F^{\mu\alpha}F^{\nu}{}_{\alpha},$$

appears and complicate the task to invert (3.42). The more fields a Lagrangian describes and/or the higher is the derivative order at which is given, the less the inversion of the dual field becomes clear and with it the comparison with its counterpart (3.46) coming from a duality rotation, independently on any possible integration and/or differentiation.

In addition, since the comparison is made keeping the two relations (3.42) and ((3.46)) in their general form (i.e. not specialized to the Einstein–Maxwell Lagrangian (1.4)), this argument should prevent operators with derivatives on the gauge field to appear in any gauge theory meant to preserve EM duality, not only in Einstein–Maxwell one. Nevertheless, the comparison is difficult to understand, as discussed, not only when operators with derivatives of the gauge fields are present but also when the Lagrangian has a more involved structure (because of other fields and/or higher-order operators), so that if analogous constraints were to be derived also for this configurations of the studied Lagrangian the consistency conditions, analogous to (3.48), would be to detailed and stringent.

To deal with this comparison between (3.42) and (3.46) it seems that one needs to rely on a perturbative approach, as indeed mentioned by Cano and Múrcia themselves, regardless of whether the resulting series of operators re-add then to an exact theory or not. In this way, operators with derivatives of the gauge field are not excluded a priori, although how to include them, though perturbatively, in the above discussion is not trivial at all.

Despite this critical point, Cano and Múrcia is indeed a relevant example of how EM duality can be used to constraint higher-derivatives operators. Despite the argument they bring is not very well understood, working without operators with derivatives of the gauge field is in fact completely in agreement with Gaillard and Zumino duality framework. As it's stated, this procedure works well in the Einstein–Maxwell context, where the number higher-order operators is, after all, not too large (the authors were able to explicitly carry on the calculations to the 8-derivative order). It would be interesting then to try to apply - and possibly refine - it in more complicated theories.

Chapter 4

Beyond pure Einstein–Maxwell theory

In the first chapter we introduced the Weak Gravity Conjecture and the Swampland approach to the study of low-energy effective field theories and their higher-order extensions. We have also shown how such swampland analysis explicitly works by studying the realization of the Electric WGC (1.1) in the context of the 4-derivatives extension (1.26) of the Einstein–Maxwell theory (1.4).

The third chapter is instead dedicated to the description of Electromagnetic duality, a very deep property of gauge theories that states the invariance under rotation of the EoM and the BI of gauge fields. First we studied how M. Gaillard and B. Zumino determined in [26] how this symmetry works and the most general duality group that a theory can have (see (3.30), (3.31)). Next, we turned to discuss how EM duality, by virtue of its connection with the dualities of String Theory [14–16], can be used as a guideline to constrain higher-derivatives extension of low-energy gauge theories. Einstein–Maxwell theory (1.4) was again used as a benchmark to explicitly see how this idea can be applied and we studied its duality-preserving extension (3.62) proposed by P. Cano and A. Múrcia [29].

Following this perspective, EM duality and the Swampland Program can indeed be connected: EM duality represents a tool to determine higher-order extensions of low-energy theories, the swampland conjectures a tool to constraint the resulting coefficients. Interesting questions are then if and how EM duality works in favour of (or against) some swampland conjecture and what are its actual role and weight as a fundamental property.

In the second chapter we described how an important class of evidence supporting the WGC is represented by the positivity bounds on the theory's scattering amplitudes [19]. In the case of Einstein–Maxwell theory, the positivity bounds on the coefficients of the extended theory (1.26) automatically realize the Electric WGC condition (1.36), as shown in (2.31). However, when the WGC is studied in the context of a more involved theory the positivity bounds are not sufficient anymore to exactly reproduce its prescriptions [33]. The equivalence seems though to be restored if EM duality constraints are included [33, 34]. This picture in which EM duality plays an active and relevant role in constraining higher-derivatives corrections to low-energy models and, together with the positivity bounds on the scattering amplitudes, realizes the WGC, is the starting point for the main analysis of this thesis work.

The theory we study describes gravity, two U(1) abelian gauge fields F^Λ ($\Lambda = 1, 2$) and a complex scalar field τ, coupled in a non-minimal way. Working in $M_{\mathrm{P}} = 1$ units, the

Lagrangian is

$$\mathcal{L}_2 = \frac{1}{2}R - \frac{1}{2(\mathfrak{Im}\tau)^2}\partial_\mu\bar{\tau}\partial^\mu\tau - \frac{1}{4}\mathcal{I}_{\Lambda\Sigma}(\tau,\bar{\tau})F^\Lambda\cdot F^\Sigma + \frac{1}{4}\mathcal{R}_{\Lambda\Sigma}(\tau,\bar{\tau})F^\Lambda\cdot\tilde{F}^\Sigma,$$

where couplings $\mathcal{I}_{\Lambda\Sigma}$ and $\mathcal{R}_{\Lambda\Sigma}$ are indeed functions of the complex scalar field τ. This theory can be seen as the bosonic sector of a $\mathcal{N}=2$ Supergravity theory [37], involving the graviton and a vector multiplets.

Starting from this theory, our goal is to find its higher-derivatives extension such that EM duality is preserved, test the realization of the Electric WGC via the charge-to-mass ratio of an extremal black hole solution of the resulting theory and finally compare the resulting constraints on the higher-order coefficients with the conditions imposed by positivity bounds in order to verify that the equivalence does get restored thanks to duality.

This same model has been studied also by G. Loges, T. Noumi and G. Shiu in [34], where they indeed find a duality extension of the action and performed the Electric WGC test we described. However, the higher-order theory that they present involves a set of operators which are manifestly duality-invariant, so that the very same extension results to be exactly invariant: this is not the most precise way to deal with EM duality. Indeed, we strongly remark that *EM duality is a symmetry that concerns the EoM and the BI, not the Lagrangian*, as equation (3.31) clearly shows. It immediately follows that the higher-derivative extension proposed in [34] is not, a priori, the most general duality-preserving correction to the starting Lagrangian. The main purpose of this thesis work is precisely to fulfil this task in a more rigorous framework.

The procedure we follow to determine the higher-order Lagrangian via EM duality is one of the main aspects of this work. Differently from Cano and Múrcia, whose approach in [29] has the uncertainties described in the previous chapter, we derive the duality group of the theory again by requiring consistency between the dual field definition (3.6), and the duality transformation (3.10), but we do this in a full perturbative sense, studying the duality transformation order by order. We remark that this is compatible with the UV symmetry nature of EM duality because, being such, it is then expected to hold at every perturbative order.

More specifically, we consider an infinitesimal duality transformation of the type (3.10) on the gauge fields and their duals computed at the 2-derivatives order and we find the transformation rules that the two non-minimal couplings $\mathcal{I}(\tau,\bar{\tau})$ and $\mathcal{R}(\tau,\bar{\tau})$ should follow in order for the duality transformation to be consistent with the dual field definition. From these rules we can fix also the duality group associated to our theory by exploiting the symmetries of $\mathcal{I}(\tau,\bar{\tau})$ and $\mathcal{R}(\tau,\bar{\tau})$. Next, we determine the 4-derivatives extension by studying the duality transformations of the various operators that can provide such higher-order correction and the conditions under which they're well defined.

Similarly to Cano and Múrcia, we somewhat abandon the general approach of Gaillard and Zumino and rely on a model-based analysis of the duality group. However, Cano and Múrcia's procedure, although focused on Einstein–Maxwell theory (1.4), still looks, at the beginning, for duality constraint over the ideal full action and this leads to the (subtle) condition (3.47) about the operators with derivatives on the gauge fields. Instead, our strictly-perturbative approach is completely agnostic about the structure of the full theory and lets duality determine the higher-orders structure of the Lagrangian, without any additional requirement. This allows in particular to include in the discussion the operators with derivatives on the gauge fields in a simple way.

This chapter is dedicated to the 2-derivatives order. In the first section we present the theory we studied and describe its structure and its origin as a $\mathcal{N}=2$ Supergravity theory; in the second and third ones we discuss the duality analysis and the resulting duality group. The analysis of the 4-derivatives order is instead discussed in the next chapter.

4.1 The model

We start by presenting the theory that is the subject of this thesis work, focusing on its structure and background. As anticipated at the beginning of the chapter, the (2-derivatives) Lagrangian, written in $M_{\rm P}=1$ units, is

$$\mathcal{L}_2 = \frac{1}{2}R - \frac{1}{2(\mathfrak{Im}\tau)^2}\partial_\mu\bar{\tau}\partial^\mu\tau - \frac{1}{4}\mathcal{I}_{\Lambda\Sigma}(\tau,\bar{\tau})F^\Lambda\cdot F^\Sigma + \frac{1}{4}\mathcal{R}_{\Lambda\Sigma}(\tau,\bar{\tau})F^\Lambda\cdot\tilde{F}^\Sigma, \tag{4.1}$$

with associated action:

$$S = \int d^4x\,\sqrt{|g|}\mathcal{L}_2. \tag{4.2}$$

This model couples gravity to two U(1) abelian gauge fields F^Λ $(\Lambda=1,2)$ and a complex scalar field τ in a non-minimal way. The two gauge fields couplings $\mathcal{I}_{\Lambda\Sigma}(\tau,\bar{\tau})$ and $\mathcal{R}_{\Lambda\Sigma}(\tau,\bar{\tau})$ are functions of the scalar field τ and are symmetric in the gauge indices (Λ,Σ). Apart from these two properties, the two coupling matrices are not specified further.

The EoM of the various fields result to be:

$$\boldsymbol{F}\textbf{:}\quad \mathcal{I}_{\Lambda\Sigma}D_\mu F^{\Sigma\,\mu\nu} + (\partial_\mu\mathcal{I})_{\Lambda\Sigma}F^{\Sigma\,\mu\nu} - (\partial_\mu\mathcal{R})_{\Lambda\Sigma}\tilde{F}^{\Sigma\,\mu\nu} = 0, \tag{4.3}$$

$$\boldsymbol{\tau}\textbf{:}\quad \frac{1}{2(\mathfrak{Im}\tau)^2}\Box\tau + \frac{i}{2(\mathfrak{Im}\tau)^3}\partial_\mu\tau\partial^\mu\tau - \frac{1}{4}(\partial_{\bar{\tau}}\mathcal{I})_{\Lambda\Sigma}F^\Lambda F^\Sigma + \frac{1}{4}(\partial_{\bar{\tau}}\mathcal{R})_{\Lambda\Sigma}F^\Lambda\tilde{F}^\Sigma = 0, \tag{4.4}$$

$$\boldsymbol{g}\textbf{:}\quad G_{\mu\nu} - T_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - T_{\mu\nu} = 0, \tag{4.5}$$

where $T_{\mu\nu}$ is the stress-energy tensor, equal to

$$T_{\mu\nu} \equiv T^{(F)}_{\mu\nu} + T^{(\tau)}_{\mu\nu}, \tag{4.6}$$

$$T^{(F)}_{\mu\nu} = \mathcal{I}_{\Lambda\Sigma}\left[F^\Lambda_{\mu\rho}F^{\Sigma\,\rho}_{\nu} - \frac{1}{4}g_{\mu\nu}F^\Lambda F^\Sigma\right], \tag{4.7}$$

$$T^{(\tau)}_{\mu\nu} = -\frac{1}{2(\mathfrak{Im}\tau)^2}\left(\partial_\alpha\bar{\tau}\partial^\alpha\tau g_{\mu\nu} - 2\partial_{(\mu}\bar{\tau}\partial_{\nu)}\tau\right). \tag{4.8}$$

The Lagrangian (4.1) describes a bosonic subsector common to many Supergravity theories and coincides with that of a $\mathcal{N}=2$ Supergravity theory coupled to a vector multiplet [37]. We can in fact identify the various fields with the bosonic degrees of freedom of the $\mathcal{N}=2$ vector and graviton multiplets:

vector multiplet: 1 vector + 2 Weyl fermions + 1 complex scalar;
graviton multiplet: 1 graviton + 2 gravitini + 1 vector.

The standard bosonic sector of such theories appears like

$$\mathcal{L}_{\rm bos} = \frac{1}{2}R - g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j - \frac{1}{4}\mathcal{M}_{MN}(\phi)F^M\cdot F^N, \tag{4.9}$$

where, similarly to (4.1), we have gravity, a set of real scalars ϕ^i and a set of abelian field strengths $F^M_{\mu\nu}$. Such theories are characterized by the scalar kinetic matrix $g_{ij}(\phi)$ and the vector kinetic matrix $\mathcal{M}_{MN}(\phi)$, whose explicit expression is determined by the supersymmetric structure of the theory.

4.1.1 The scalar sector

Let G be the global symmetry group of the theory. The scalar field sector is given by a so called σ-model, which in general denotes a scalar field theory in which the fields take values on a given manifold $\mathscr{M}_\phi$; the scalar kinetic matrix $g_{ij}(\phi)$ introduced in (4.9) is precisely the metric tensor associated to $\mathscr{M}_\phi$. This manifold is given by the coset space G/K, which is the set of equivalence classes of elements of G connected by a transformation of K, taken in this case to be its maximal compact subgroup. The following relation among the algebras hold:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{c}, \tag{4.10}$$

where $\mathfrak{c}$ is the algebra of the coset space and has indeed dimension $\dim\mathfrak{c} = \dim\mathfrak{g} - \dim\mathfrak{k}$.

As explicit example of how such σ-models are build we consider the case, related to the Lagrangian (4.1), of the coset manifold

$$\mathscr{M}_\phi = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R}). \tag{4.11}$$

The associated algebra generators are taken to be

$$\mathfrak{sl}(2): \qquad T_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad T_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.12}$$

$$\mathfrak{so}(2): \qquad t = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{4.13}$$

so that $\mathfrak{c} = \{T_1, T_3\}$ and $\dim\mathfrak{c} = 2$.

The typical procedure to build the scalar Lagrangian starts by selecting a representative element of the coset space. There are several possibilities to do so and in general it can be built as an exponential of the generators of the coset algebra $\mathfrak{c}$. The most "economic" choice in our case is given by[1]

$$L(\varphi,\theta) = \exp\left(\theta(T_1 + T_2)\right)\exp\left(\varphi T_3\right) = \begin{pmatrix} e^{\varphi/2} & e^{-\varphi/2}\theta \\ 0 & e^{-\varphi/2} \end{pmatrix}, \tag{4.14}$$

where φ and θ are going to be the two scalar fields described by the resulting σ-model. Next, with the coset representative L one defines the matrix

$$\mathcal{M}(\varphi,\theta) = LL^T, \tag{4.15}$$

which is positive defined and invariant under K transformations. The σ-model Lagrangian can then be obtained as

$$\mathcal{L}_{\text{scalar}} = \frac{1}{8}\mathrm{Tr}\left[\partial_\mu\mathcal{M}\partial^\mu\mathcal{M}^{-1}\right], \tag{4.16}$$

and in the $\mathscr{M}_\phi = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$ case we're considering it results to be

$$\mathcal{L}_{\text{scalar}}(\varphi,\theta) = -\frac{1}{4}\left[\partial_\mu\varphi\partial^\mu\varphi + e^{-2\varphi}\partial_\mu\theta\partial^\mu\theta\right]. \tag{4.17}$$

We can recast this expression by merging the two real scalars φ and θ in a complex one

[1] This choice comes from the application of the so called "Iwasawa decomposition" [38].

τ as

$$\tau \equiv \theta + i\, e^{\varphi}, \tag{4.18}$$

so that Lagrangian (4.17) becomes

$$\mathcal{L}_{\text{scalar}}(\tau) = -\frac{1}{4(\mathfrak{Im}\tau)^2}\partial_\mu\bar{\tau}\partial^\mu\tau, \tag{4.19}$$

which is equivalent, modulo a scalar coefficient, to the scalar sector of (4.1).

We notice that the resulting σ-model (4.19) indeed enjoys the following SL$(2,\mathbb{R})$ symmetry:

$$\tau \;\longrightarrow\; \frac{a\tau+b}{c\tau+d} \qquad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}). \tag{4.20}$$

We have in fact that $\partial_\mu\bar{\tau}\partial^\mu\tau$ and $\mathfrak{Im}\tau$ transform in opposite ways:

- $\partial_\mu\tau \;\longrightarrow\; \frac{1}{(c\tau+d)^2}\partial_\mu\tau \;\Longrightarrow\; \partial_\mu\bar{\tau}\partial^\mu\tau \;\longrightarrow\; \frac{1}{|c\tau+d|^4}\,\partial_\mu\bar{\tau}\partial^\mu\tau \qquad (4.21)$
- $\mathfrak{Im}\tau \;\longrightarrow\; \frac{1}{|c\tau+d|^2}\mathfrak{Im}\tau \;\Longrightarrow\; \frac{1}{(\mathfrak{Im}\tau)^2} \;\longrightarrow\; |c\tau+d|^4\,\frac{1}{(\mathfrak{Im}\tau)^2}. \qquad (4.22)$

We highlighted here this SL$(2,\mathbb{R})$ symmetry of the scalar Lagrangian (4.19) because it's going to play a relevant role in the duality structure of Lagrangian (4.1).

4.1.2 The gauge sector

In order to go from the gauge sector of Lagrangian (4.9), expressed in terms of the gauge kinetic matrix, to the one of (4.1) in terms of the couplings $\mathcal{I}$ and $\mathcal{R}$, one needs again to make use of EM duality, which in this context corresponds to the action of the symmetry group G on the field strengths F^M. Going back to the dual field definition we gave in (3.6), we can rewrite it, in the case of (4.9), as

$$G_{M\,\mu\nu} = \epsilon_{\mu\nu\rho\sigma}\,\mathcal{M}_{MN}F^{N\,\rho\sigma}. \tag{4.23}$$

As we previously understood, EM duality identifies a set of non-equivalent theories which describe though the same dynamics, thanks to (4.23). Among this set, there is one theory in which all the fields F^M are dualized, i.e. in which the duality symmetry of the EoM and BI is manifest. Let's suppose that Lagrangian (4.9) describes precisely this picture: in even spacetime dimensions, the fields F^M splits as

$$F^M = \left(F^\Lambda,\; F_\Lambda\right), \tag{4.24}$$

in which F^Λ are the proper dynamical degrees of freedom, while F_Λ are their associated duals. For such a F^M, the dual field definition (4.23) can be rephrased in what is called twisted self-duality condition [37,39,40]:

$$F^M_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\Omega^{MJ}\mathcal{M}_{JN}F^{N\,\rho\sigma}, \tag{4.25}$$

where Ω is the symplectic matrix:

$$\Omega = \begin{pmatrix} \mathbb{0} & \mathbb{1} \\ -\mathbb{1} & \mathbb{0} \end{pmatrix}. \tag{4.26}$$

The proper Lagrangian, function only of the dynamical half of the fields F^Λ, that encodes equation (4.25) is given precisely by

$$\mathcal{L}_{\text{gauge}} = \frac{1}{4}\mathcal{I}_{\Lambda\Sigma}(\phi)F^\Lambda \cdot F^\Sigma + \frac{1}{4}\mathcal{R}_{\Lambda\Sigma}(\phi)F^\Lambda \cdot \tilde{F}^\Sigma, \tag{4.27}$$

in which the coupling matrices $\mathcal{I}(\phi)$ and $\mathcal{R}(\phi)$ are related to the starting gauge kinetic matrix $\mathcal{M}(\phi)$ by

$$\mathcal{M} = -\begin{pmatrix} \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & -\mathcal{R}\mathcal{I}^{-1} \\ -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \end{pmatrix}. \tag{4.28}$$

4.2 The duality group

We now go back to Lagrangian (4.1) and study the associated duality group. As we described in the introduction to the chapter, the strength of our duality analysis relies in its perturbative character: it allows in fact to determine the duality group of the theory already at the 2-derivatives level (4.1) and, moreover, to include in the discussion also the higher-order operators that involve derivative of the gauge fields.

In this perspective, we introduce the dual field definition as

$$\tilde{G}_{\Lambda\,\mu\nu} \equiv 2\left[\frac{\partial\mathcal{L}}{\partial F^{\Lambda\,\mu\nu}} - D_\alpha \frac{\partial\mathcal{L}}{\partial\left(D_\alpha F^{\Lambda\,\mu\nu}\right)} + \dots\right], \tag{4.29}$$

where the dots stand for the various higher-derivatives terms. Since our analysis will concern the 4-derivatives order, the relevant contributions to G_Λ are the one that are explicit in (4.29).

The infinitesimal duality transformation on the gauge fields is, similarly to (3.10),

$$\begin{pmatrix} \delta F \\ \delta G \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} F \\ G \end{pmatrix}, \tag{4.30}$$

where $F = \left(F^1,\ F^2\right)$, $G = (G_1,\ G_2)$ and $\{A,\, B,\, C,\, D\} \subset GL(2,\mathbb{R})$. For what concerns the other fields of the theory, the metric does not transform under duality, while, as anticipated in (4.20), the (full) duality transformation on the scalar field τ, under which its EoM transforms covariantly (recall (3.23)), is the $\mathrm{SL}(2,\mathbb{R})$ transformation

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d} \qquad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}). \tag{4.31}$$

The reason why the duality transformation of τ is precisely a transformation of $\mathrm{SL}(2,\mathbb{R})$ is going to be clear once the duality group of the theory is fixed.

4.2.1 Duality analysis of the gauge sector

According to definition (4.29), the dual field associated to our starting, 2-derivatives theory (4.29) is

$$G^{(2)}_{\Lambda\,\mu\nu} = \mathcal{I}_{\Lambda\Sigma}\tilde{F}^\Sigma_{\mu\nu} + \mathcal{R}_{\Lambda\Sigma}F^\Sigma_{\mu\nu}, \tag{4.32}$$

where the superscript "(2)" remarks the order at which the dual field is computed.

We can then derive the duality group of the theory by exploiting the self-consistency of the infinitesimal duality transformation (4.30). According to it, we can read the transformation of the dual field $G^{(2)}_\Lambda$ from two different points of view. On one side, it's directly involved in the duality transformation and we have

$$\delta G^{(2)}_\Lambda = C_{\Lambda\Sigma} F^\Sigma + D_\Lambda{}^\Sigma G^{(2)}_\Sigma . \tag{4.33}$$

On the other, $G^{(2)}_\Lambda$ is function of the gauge fields F^Λ and of the complex scalar τ (see (4.32)), so that its duality transformation must be proportional to δF^Λ as well as to $\delta\tau$ and $\delta\bar\tau$. At this order[2], we have

$$\delta G^{(2)}_\Lambda = \delta_\tau G^{(2)}_\Lambda + \delta_F G^{(2)}_\Lambda = \delta_\tau G^{(2)}_\Lambda + \frac{\partial G^{(2)}_\Lambda}{\partial F^\Sigma}\delta F^\Sigma = \delta_\tau G^{(2)}_\Lambda + \frac{\partial G^{(2)}_\Lambda}{\partial F^\Sigma}\left[AF + BG(F)\right]^\Sigma , \tag{4.34}$$

where $\delta_\tau G^{(2)}_\Lambda$ and $\delta_\tau G^{(2)}_\Lambda$ denote the components of the duality transformation that are proportional, respectively, to δF^Λ and to $\delta\tau$ and $\delta\bar\tau$[3].

Thus, by putting together equations (4.33) and (4.34) we obtain the following consistency identity:

$$C_{\Lambda\Sigma} F^\Sigma + D_\Lambda{}^\Sigma G^{(2)}_\Sigma = \delta_\tau G^{(2)}_\Lambda + \frac{\partial G^{(2)}_\Lambda}{\partial F^\Sigma}\left[AF + BG(F)\right]^\Sigma . \tag{4.35}$$

From this identity we can obtain the transformation rules of $\mathcal{I}$ and $\mathcal{R}$ under (4.30) and, consequently, the consistency conditions on the A, B, C and D matrices that fix the duality group.

Explicitly, the two transformations read

- G side: $$\delta G^{(2)}_\Lambda = C_{\Lambda\Sigma} F^\Sigma + D_\Lambda{}^\Sigma G^{(2)}_\Sigma = (C + D\mathcal{R})_{\Lambda\Sigma} F^\Sigma + (D\mathcal{I})_{\Lambda\Sigma} \tilde F^\Sigma ; \tag{4.36}$$
- F side: $$\begin{aligned} \delta G^{(2)}_\Lambda =& \delta\left(\mathcal{I}_{\Lambda\Sigma}\tilde F^\Sigma\right) + \delta\left(\mathcal{R}_{\Lambda\Sigma} F^\Sigma\right) = \\ =& \delta\mathcal{I}_{\Lambda\Sigma}\tilde F^\Sigma + \delta\mathcal{R}_{\Lambda\Sigma} F^\Sigma + \mathcal{I}_{\Lambda\Sigma}\delta\tilde F^\Sigma + \mathcal{R}_{\Lambda\Sigma}\delta F^\Sigma = \\ =& \left[\delta\mathcal{I} + \mathcal{I}A + \mathcal{I}B\mathcal{R} + \mathcal{R}B\mathcal{I}\right]_{\Lambda\Sigma}\tilde F^\Sigma + \\ &+ \left[\delta\mathcal{R} + \mathcal{R}A - \mathcal{I}B\mathcal{I} + \mathcal{R}B\mathcal{R}\right]_{\Lambda\Sigma} F^\Sigma . \end{aligned} \tag{4.37}$$

Identity (4.35) is satisfied by equating the coefficients of F^Σ and $\tilde F^\Sigma$ in (4.36) and (4.37). This yields the following transformation rule for $\mathcal{I}$ and $\mathcal{R}$:

$$\delta\mathcal{I} = D\mathcal{I} - \mathcal{I}A - \mathcal{I}B\mathcal{R} - \mathcal{R}B\mathcal{I}, \tag{4.38}$$

$$\delta\mathcal{R} = C + D\mathcal{R} - \mathcal{R}A + \mathcal{I}B\mathcal{I} - \mathcal{R}B\mathcal{R}. \tag{4.39}$$

We remark that these transformation rules are fixed by the consistency between the duality transformation (4.30) (the "G side") and the dual field definition (4.29) (the "F side"), expressed by identity (4.35).

Moreover, transformations (4.38) and (4.39) must also be consistent with $\mathcal{I}$ and $\mathcal{R}$ properties. As mentioned, these couplings are symmetric matrices, as it is clear from (4.1):

$$\mathcal{I}_{\Lambda\Sigma} = \mathcal{I}_{\Sigma\Lambda}, \qquad \mathcal{R}_{\Lambda\Sigma} = \mathcal{R}_{\Sigma\Lambda}. \tag{4.40}$$

Thus, plugging (4.40) into (4.38) and (4.39) we obtain consistency conditions also on the

[2] At the 2-derivatives level we have only terms proportional to δF^Λ. When higher-derivatives operators will be included, equation (4.34) is corrected by terms proportional to $D_\mu(\delta F^\Lambda)$, $D_\mu D_\nu(\delta F^\Lambda)$ and so on.

[3] Since the τ-dependence of $G^{(2)}_\Lambda$ is all in the couplings, this term corresponds to the $\delta\mathcal{I}$ and $\delta\mathcal{R}$ terms of (4.37).

duality matrices A, B, C and D. From (4.38) we get

$$\begin{aligned} 0 = \delta\mathfrak{I}_{\Lambda\Sigma} - \delta\mathfrak{I}_{\Sigma\Lambda} = &\left(D_\Lambda{}^\Omega + A^\Omega{}_\Lambda\right)\mathfrak{I}_{\Omega\Sigma} - \mathfrak{I}_{\Lambda\Omega}\left(D_\Sigma{}^\Omega + A^\Omega{}_\Sigma\right) + \\ &- \mathfrak{I}_{\Lambda A}\left(B^{AB} - B^{BA}\right)\mathfrak{R}_{B\Sigma} - \mathfrak{R}_{\Lambda A}\left(B^{AB} - B^{BA}\right)\mathfrak{I}_{B\Sigma}. \end{aligned} \quad \Longrightarrow \tag{4.41}$$

$$\Longrightarrow \quad \begin{aligned} D =& - A^T, \\ B =& B^T. \end{aligned} \tag{4.42}$$

From (4.39) we get the same conditions on A, B and D and also one on C:

$$C = C^T, \tag{4.43}$$

Conditions (4.42) and (4.43) fix the EM duality group of our theory (4.1) to be $\mathrm{Sp}(4, \mathbb{R})$.

Comparison with Gaillard and Zumino Indeed, we notice that conditions (4.42) and (4.43), and with them, of course, the duality group they determine, coincide with the Gaillard and Zumino result (3.30). The difference relies in the framework in which these constraints have been derived. Gaillard and Zumino carry out their analysis working with a generic Lagrangian, i.e. without specifying neither the fields content nor, in the effective field theory perspective, the derivatives order at which is computed. This makes their result very general and it can indeed be applied to study vast classes of theories, but at the same time they're forced to exclude (as starting assumption) the presence in the Lagrangian of operators with derivatives on the gauge fields, otherwise they would not be able to proceed with the calculations (see (3.12), (3.41)).

Instead, our derivation of the duality group is strongly model-based: we obtained the constraints (4.42) and (4.43) by directly applying the explicit expression of the dual field (4.32) and the symmetries of the $\mathfrak{I}$ and $\mathfrak{R}$ couplings (4.40). This obviously limits the results of our duality analysis to the model (4.1) we consider but at the same time we do not have to make any further assumption on the structure of Lagrangian and its higher-derivative extension: only duality fixes it, independently on the type of operators considered.

4.2.2 Duality analysis of the scalar sector

From the analysis of the gauge sector we determined the duality transformation rules of the $\mathfrak{I}$ and $\mathfrak{R}$ couplings (4.38) and (4.39): these transformations depend both on the gauge structure of the two couplings - they're matrices carrying the gauge indices (Λ, Σ) - and their dependence on the scalar field τ.

To understand how τ transform under duality we study the total variation of the Lagrangian (4.1) under (4.30). We first rewrite (4.1) as

$$\mathcal{L}_2 = \frac{1}{2}R + \mathcal{L}_{\mathrm{mat}} + \frac{1}{4}F^\Lambda \tilde{G}^{(2)}_\Lambda, \tag{4.44}$$

with

$$\mathcal{L}_{\mathrm{mat}} = -\frac{1}{2(\mathfrak{Im}\tau)^2}\partial_\mu\bar{\tau}\partial^\mu\tau, \tag{4.45}$$

and $G^{(2)}$ as in (4.32), so that under (4.30) it transforms as

$$\delta\mathcal{L}_2 = \delta\mathcal{L}_{\mathrm{mat}} + \frac{1}{4}\delta\left(F^\Lambda \tilde{G}^{(2)}_\Lambda\right). \tag{4.46}$$

The gauge term of this transformation reads

$$\begin{aligned}\delta\left(F^\Lambda\tilde{G}^{(2)}_\Lambda\right) =&\tilde{G}^{(2)}_\Lambda\delta F^\Lambda + F^\Lambda\delta\tilde{G}^{(2)}_\Lambda = \\ =&\tilde{G}^{(2)}_\Lambda\left(AF+BG^{(2)}\right)^\Lambda + F^\Lambda\left(C\tilde{F}+D\tilde{G}^{(2)}\right)_\Lambda = \\ =&FC\tilde{F}+G^{(2)}B\tilde{G}^{(2)}+\tilde{G}^{(2)}AF+FD\tilde{G}^{(2)} \overset{(4.42)}{=} \\ =&FC\tilde{F}+G^{(2)}B\tilde{G}^{(2)}+\cancel{\tilde{G}^{(2)}AF}-\cancel{\tilde{G}^{(2)}AF} = FC\tilde{F}+G^{(2)}B\tilde{G}^{(2)}.\end{aligned} \tag{4.47}$$

For what concerns $\mathcal{L}_{\text{mat}}$, a general result about the matter sector (i.e. the sector describing scalars and/or fermions) of 2-derivatives gauge theories (like (4.1)) states that if the gauge sector of the Lagrangian transforms as (4.47), the matter sector results then to be duality-invariant:

$$\delta\mathcal{L}_{\text{mat}} = 0. \tag{4.48}$$

The proof of this result is shown in Appendix A. This is a very important point of our analysis because it is the condition that fixes the duality transformations of the scalar field τ to those under which $\mathcal{L}_{\text{mat}}$ (4.45) is exactly invariant. Indeed, we have seen in (4.20), (4.21) and (4.22) that these transformations are those of $\mathrm{SL}(2,\mathbb{R})$, as stated in (4.31), which indeed is a subgroup of $\mathrm{Sp}(4,\mathbb{R})$.

Therefore, the total variation of Lagrangian (4.1) under a duality transformation is

$$\delta\mathcal{L}_2 = \frac{1}{4}\left(FC\tilde{F}+G^{(2)}B\tilde{G}^{(2)}\right), \tag{4.49}$$

which we observe is compatible with Gaillard and Zumino (3.31).

4.2.3 Structure of the duality group

The duality analysis on the gauge sector of (4.1) told us the EM duality group of the theory is $\mathrm{Sp}(4,\mathbb{R})$ (see (4.42) and (4.43)); the analysis of the scalar sector told instead that the correspondent duality transformation on τ must be in $SL(2,\mathbb{R})\subset Sp(4,\mathbb{R})$.

While $\mathrm{Sp}(4,\mathbb{R})$ is the full EM duality group, it is in fact $\mathrm{SL}(2,\mathbb{R})$ that represents the proper duality group of theory (4.1): although the EoM and the BI of the gauge fields are invariant under the larger $\mathrm{Sp}(4,\mathbb{R})$, it is only under its subgroup $\mathrm{SL}(2,\mathbb{R})$ that the EoM of the scalar field τ transforms covariantly. We notice that $\mathrm{SL}(2,\mathbb{R})$ is still a "good" duality group, in the sense that it is still a symmetry of the EoM and BI of the gauge fields and not of the Lagrangian, since the subset of generators of $\mathfrak{sp}(4,\mathbb{R})$ realizing $\mathfrak{sl}(2,\mathbb{R})$ contains also off-diagonal components (see (4.30) and (4.49)):

$$\mathfrak{sl}(2,\mathbb{R}):\qquad t_1=\frac{1}{2}\begin{pmatrix}\mathbb{0} & \sigma_3\\ \sigma_3 & \mathbb{0}\end{pmatrix},\qquad t_2=\frac{1}{2}\begin{pmatrix}\mathbb{0} & -\mathbb{1}\\ \mathbb{1} & \mathbb{0}\end{pmatrix},\qquad t_3=\frac{1}{2}\begin{pmatrix}-\sigma_3 & \mathbb{0}\\ \mathbb{0} & \sigma_3\end{pmatrix}, \tag{4.50}$$

where σ_i are the Pauli matrices.

The transformations that instead leave the Lagrangian invariant correspond indeed to the diagonal generators of $\mathrm{Sp}(4,\mathbb{R})$ (see again (4.30) and (4.49)), which identify the $\mathrm{GL}(2,\mathbb{R})$ subgroup:

$$\begin{aligned}\mathfrak{gl}(2,\mathbb{R}):\ \ v_1 &= \frac{1}{2}\begin{pmatrix}-\sigma_3 & \mathbb{0}\\ \mathbb{0} & \sigma_3\end{pmatrix},\ \ v_2=\frac{1}{2}\begin{pmatrix}\mathbb{1} & \mathbb{0}\\ \mathbb{0} & -\mathbb{1}\end{pmatrix},\\ v_3 &= \frac{1}{2}\begin{pmatrix}\sigma_1 & \mathbb{0}\\ \mathbb{0} & -\sigma_1\end{pmatrix},\ \ v_4=\frac{1}{2}\begin{pmatrix}i\,\sigma_2 & \mathbb{0}\\ \mathbb{0} & i\,\sigma_2,\end{pmatrix}\end{aligned} \tag{4.51}$$

which shares with $\mathfrak{sl}(2,\mathbb{R})$ the generator $t_3 = v_1$. These transformations can indeed be re-adsorbed via field redefinitions.

The remaining generators correspond to the group given by

$$GL(2,\mathbb{R}) \setminus Sp(4,\mathbb{R})/SL(2,\mathbb{R}). \tag{4.52}$$

Such transformations provide the change of the so called *symplectic frame*. Each one of these symplectic frames corresponds to a formulation of the Lagrangian that is inequivalent to the formulations of other frames: they in fact cannot be mapped into each other by field redefinitions because the group describing such transformations, i.e. $\mathrm{GL}(2,\mathbb{R})$, has been removed, as described in (4.52). However, since the set of EoM and BI of the gauge fields is invariant under any transformation of the full $\mathrm{Sp}(4,\mathbb{R})$, the dynamics described by all of these Lagrangian is indeed the same.

The duality group of our theory (4.1) we have now described indeed exhibits the typical structure of such duality groups. The EoM and BI of the set of gauge fields is invariant under the full EM duality group, i.e. $\mathrm{Sp}(2N_F,\mathbb{R})$, where N_F denotes the number of such gauge fields. Only a subgroup of $\mathrm{Sp}(2N_F,\mathbb{R})$ is though the proper duality group of the whole theory and this is given by the Isometry group of the scalar manifold $\mathscr{M}_\phi$ on which the scalar fields are defined (see (4.11), (4.20)): it is in fact under this group, that we call $\mathrm{Iso}(\mathscr{M}_\phi)$, that their EoM transform covariantly. The transformations between the different symplectic frames are then given by the transformations of $\mathrm{Sp}(2N_F,\mathbb{R})$ that are left once we remove the $\mathrm{Iso}(\mathscr{M}_\phi)$ and $\mathrm{GL}(N_F,\mathbb{R})$ components, the latter being the diagonal subgroup of $\mathrm{Sp}(2N_F,\mathbb{R})$ whose transformations are equivalent to fields redefinition.

Chapter 5

Duality constraints on non-minimal couplings

In the previous chapter we introduced Lagrangian (4.1), we described the Supergravity framework in which it appears and we characterized its duality group by exploiting the perturbative consistency analysis of the duality transformation (4.30) that we discussed. As a result, we found that the full EM duality group associated to (4.1) is $Sp(4, \mathbb{R})$ (see (4.42), (4.43)), while the proper duality group of the whole theory (4.1) is its $\mathrm{SL}(2, \mathbb{R})$ subgroup (see (4.48)). In doing so, we also determined the duality transformations (4.38) and (4.39) of the two $\mathcal{I}$ and $\mathcal{R}$ couplings.

The next, very important step of our analysis is the extension of the 2-derivatives Lagrangian (4.1) to higher-orders in the derivatives expansion. In particular, we derive the 4-derivatives correction to (4.1), that we call $\mathcal{L}_4$, following a bottom-up approach which exploits EM duality to fix the higher-order non-minimal coefficients.

This derivation is divided in two parts. The first one consists in finding an independent set of all the 4-derivatives operators, made out of the metric $g_{\mu\nu}$, the complex scalar field τ and the gauge fields F^Λ, that can appear in $\mathcal{L}_4$, i.e. that respects the symmetries of the model, including the $\mathrm{SL}(2, \mathbb{R})$ symmetry of the scalar sector, which we have seen to be exactly invariant under the transformations of this group (see (4.48) and Appendix A).

The second part concerns instead the determination, via duality, of the non-minimal couplings of such operators. This procedure follows exactly the one we applied in Section 3.2.1 to determine the EM duality group and the transformations (4.38) and (4.39) of the $\mathcal{I}$ and $\mathcal{R}$ couplings, only adapted to the 4-derivatives case under examination. In particular, the presence of 4-derivatives operators, a part from correcting the 2-derivatives dual field (4.32) according to (4.29), change the expression of the founding identity (4.35). In such a procedure the perturbative character of our analysis is manifest.

5.1 Higher-order operators

Following a bottom-up approach, the first thing to do when trying to find the higher-energy extension of an effective field theory is to write down all the higher-order operators that can provide such extension. In our case, we have to find all the operators composed of the metric, the scalar and the gauge fields that contain a total number of derivatives equal to 4. An independent set of such operators that can provide the higher-order extension $\mathcal{L}_4$ of Lagrangian (4.1) is given by

$$\begin{aligned}
\boldsymbol{g}: &\quad R^2\,,\ (R_{\mu\nu})^2\,,\ (R_{\mu\nu\rho\sigma})^2;\\
\boldsymbol{F}: &\quad (F^\Lambda F^\Sigma)(F^A F^B)\,,\ (F^\Lambda \tilde{F}^\Sigma)(F^A \tilde{F}^B)\,,\ (F^\Lambda F^\Sigma)(F^A \tilde{F}^B)\,,\\
&\quad \tilde{F}^{A\,\mu\nu}F^B_{\nu\rho}F^{\Lambda\,\rho\sigma}F^\Sigma_{\sigma\mu};\\
\boldsymbol{\tau}: &\quad \big[(\Box\tau)^2\,,\ (\partial_\mu\tau\partial^\mu\tau)^2\,,\ \partial_\mu\bar{\tau}\partial^\mu\tau\partial_\nu\tau\partial^\nu\tau\,,\ \Box\tau\partial_\mu\tau\partial^\mu\tau\,,\ \Box\tau\partial_\mu\bar{\tau}\partial^\mu\tau\,,\\
&\quad \Box\bar{\tau}\partial_\mu\tau\partial^\mu\tau + h.c.\big]\,,\ (\partial_\mu\bar{\tau}\partial^\mu\tau)^2\,,\ |\partial_\mu\tau\partial^\mu\tau|^2\,,\ \Box\tau\Box\bar{\tau};\\
\boldsymbol{g}+\boldsymbol{F}: &\quad RF^\Lambda F^\Sigma\,,\ RF^\Lambda\tilde{F}^\Sigma\,,\ R_{\mu\nu}F^{\Lambda\,\mu\rho}F^{\Sigma\,\nu}{}_\rho\,,\ R_{\mu\nu\rho\sigma}F^{\Lambda\,\mu\nu}F^{\Sigma\,\rho\sigma}\,,\\
&\quad R_{\mu\nu\rho\sigma}F^{\Lambda\,\mu\nu}\tilde{F}^{\Sigma\,\rho\sigma},\ D_\mu F^{\Lambda\,\mu\alpha}D^\nu F^\Sigma_{\nu\alpha};\\
\boldsymbol{\tau}+\boldsymbol{F}: &\quad \partial_\mu\bar{\tau}\partial^\mu\tau F^\Lambda F^\Sigma\,,\ \partial_\mu\bar{\tau}\partial^\mu\tau F^\Lambda\tilde{F}^\Sigma\,,\ \partial_\mu\bar{\tau}\partial_\nu\tau F^{\Lambda\,\mu\rho}F^{\Sigma\,\nu}{}_\rho\,,\\
&\quad \Big[\partial_\mu\bar{\tau}\partial_\nu\tau F^{\Lambda\,\mu\alpha}\tilde{F}^{\Sigma\,\nu}{}_\alpha,\ \partial_\mu\tau\partial^\mu\tau F^\Lambda F^\Sigma\,,\ \partial_\mu\tau\partial^\mu\tau F^\Lambda\tilde{F}^\Sigma\,,\\
&\quad \partial_\mu\partial_\nu\tau F^{\Lambda\,\mu\alpha}F^{\Sigma\,\nu}{}_\alpha\,,\ \partial_\mu\tau\partial_\nu\tau F^{\Lambda\,\mu\rho}F^{\Sigma\,\nu}{}_\rho,\ \partial_\mu\tau F^{\Lambda\,\mu\alpha}D^\nu F^\Sigma_{\nu\alpha}\,,\\
&\quad \partial_\mu\tau\tilde{F}^{\Lambda\,\mu\alpha}D^\nu F^\Sigma_{\nu\alpha}\,,\ \Box\tau F^\Lambda F^\Sigma\,,\ \Box\tau F^\Lambda\tilde{F}^\Sigma + h.c.\Big]\,;\\
\boldsymbol{g}+\boldsymbol{\tau}: &\quad [R\,\partial_\mu\tau\partial^\mu\tau\,,\ R_{\mu\nu}\partial^\mu\tau\partial^\nu\tau\,,\ \Box\tau R\,,\ \partial_\mu\partial_\nu\tau R^{\mu\nu} + h.c.]\,,\\
&\quad R\,\partial_\mu\bar{\tau}\partial^\mu\tau\,,\ R_{\mu\nu}\partial^\mu\bar{\tau}\partial^\nu\tau.
\end{aligned} \tag{5.1}$$

In principle, all these operators can be part of the 4-derivative Lagrangian $\mathcal{L}_4$ correcting (4.1). In $\mathcal{L}_4$, they appear together with a coefficient that in general is indeed function of the complex scalar τ, as the $\mathcal{I}$ and $\mathcal{R}$ couplings of (4.1). To restrict this set of operators we make use of the duality structure we derived in the previous chapter as a tool to fix these non-minimal couplings: the result will be a 4-derivatives theory that preserves EM duality.

As in the 2-derivatives case, we can divide the 4-derivatives Lagrangian $\mathcal{L}_4$ as

$$\mathcal{L}_4 = \mathcal{L}^{(4)}_{\text{mat}} + \mathcal{L}^{(4)}_{\text{gauge}}\,, \tag{5.2}$$

where $\mathcal{L}^{(4)}_{\text{gauge}}$ contains the operators involving the gauge fields, while $\mathcal{L}^{(4)}_{\text{mat}}$ contains operators made out only of the metric $g_{\mu\nu}$ and the complex scalar field τ and represents the sector of $\mathcal{L}_4$ that we expect to be exactly duality-invariant.

5.1.1 Classification of the gauge operators

The duality analysis of the exactly invariant $\mathcal{L}^{(4)}_{\text{mat}}$ is, because of it exact invariance under duality, straightforward and we discuss it in Section 4.2. This is not the case for the more challenging gauge sector $\mathcal{L}^{(4)}_{\text{gauge}}$, which requires a more detailed analysis.

In this respect, it is useful to reorganize the 4-derivatives gauge operators of (5.1) as follows. We observe that we can identify four different classes of operators involving the gauge fields:

1. operators involving *four* F^Λ's;
2. operators involving two F^Λ's and R;
3. operators involving two F^Λ's and τ derivatives;
4. operators involving *derivatives* of F^Λ.

The last two classes are the most delicate to deal with: the explicit presence of the complex scalar τ, which appears not only "inside" the coefficient but also outside and together with spacetime derivatives, complicates the duality analysis. We have in fact that this explicit $\partial_\mu \tau$ part of these operators transforms with the proper duality group $\mathrm{SL}(2,\mathbb{R})$, while the gauge fields, together with the matrix couplings that carry gauge indices, transform instead with the full EM duality group $\mathrm{Sp}(4,\mathbb{R})$. Thus, to perform the duality transformation over these operators in this explicit form one would have to restrict the analysis to the $\mathrm{SL}(2,\mathbb{R})$ group only, but this would not be correct because, as we discussed, the set EoM and BI of the gauge fields is invariant under the full EM duality group $\mathrm{Sp}(4,\mathbb{R})$, not only under $\mathrm{SL}(2,\mathbb{R})$.

To avoid such complication, we rewrite the operators belonging to classes 3 and 4 by properly including also the $\partial_\mu \tau$ component into the matrix coefficient carrying the gauge indices, so that we can study them via the correct $\mathrm{Sp}(4,\mathbb{R})$ duality group. According to this perspective, we can classify all the resulting "composed" coefficients in four different categories:

- $X_{AB} = X_{AB}(\tau,\bar{\tau},\partial\tau,\partial\bar{\tau})$, with two spacetime derivatives of τ and no free Lorentz indices (the same holds for $\tilde{X}$);
- $[Y_{AB}]_{\mu\nu} = [Y_{AB}]_{\mu\nu}(\tau,\bar{\tau},\partial\tau,\partial\bar{\tau})$, with two spacetime derivatives of τ that give the two Lorentz indices (the same holds for $\tilde{Y}$);
- $Z^{(1)}_{AB} = Z^{(1)}_{AB}(\tau,\bar{\tau})$, with no derivatives;
- $\left[Z^{(2)}_{AB}\right]_\mu = \left[Z^{(2)}_{AB}\right]_\mu(\tau,\bar{\tau},\partial\tau,\partial\bar{\tau})$, with one spacetime derivative on τ, which gives the free Lorentz index (the same holds for $\tilde{Z}^{(2)}$).

Thus, calling $\mathcal{L}^{(4)}_{\partial\tau}$ the sector of the higher-order Lagrangian that contains these operators, we can write it as

$$\begin{aligned}\mathcal{L}^{(4)}_{\partial\tau} =& X_{AB}F^AF^B + \tilde{X}_{AB}F^A\tilde{F}^B + [Y_{AB}]_{\mu\nu}F^{A\,\mu\alpha}F^{B\,\nu}{}_{\alpha} + \left[\tilde{Y}_{AB}\right]_{\mu\nu}F^{A\,\mu\alpha}\tilde{F}^{B\,\nu}{}_{\alpha} + \\ &+ Z^{(1)}_{AB}(D_\mu F^{A\,\mu\alpha})(D^\nu F^B_{\nu\alpha}) + \left[Z^{(2)}_{AB}\right]_\mu F^{A\,\mu\alpha}D^\nu F^B_{\nu\alpha} + \left[\tilde{Z}^{(2)}_{AB}\right]_\mu \tilde{F}^{A\,\mu\alpha}D^\nu F^B_{\nu\alpha},\end{aligned} \tag{5.3}$$

and the full gauge sector is therefore

$$\begin{aligned}\mathcal{L}^{(4)}_{\text{gauge}} =& + [\alpha_1]_{ABCD}(F^\Lambda F^\Sigma)(F^AF^B) + [\alpha_2]_{ABCD}(F^\Lambda\tilde{F}^\Sigma)(F^A\tilde{F}^B) + \\ &+ [\alpha_3]_{ABCD}(F^\Lambda F^\Sigma)(F^A\tilde{F}^B) + [\alpha_4]_{ABCD}\tilde{F}^{A\,\mu\nu}F^B_{\nu\rho}F^{\Lambda\,\rho\sigma}F^\Sigma_{\sigma\mu} + \\ &+ [\beta_1]_{AB}RF^\Lambda F^\Sigma + [\beta_2]_{AB}RF^\Lambda\tilde{F}^\Sigma + [\beta_3]_{AB}R_{\mu\nu}F^{\Lambda\,\mu\rho}F^{\Sigma\,\nu}{}_{\rho} + \\ &+ [\beta_4]_{AB}R_{\mu\nu\rho\sigma}F^{\Lambda\,\mu\nu}F^{\Sigma\,\rho\sigma} + [\beta_5]_{AB}R_{\mu\nu\rho\sigma}F^{\Lambda\,\mu\nu}\tilde{F}^{\Sigma\,\rho\sigma} + \\ &+ X_{AB}F^AF^B + \tilde{X}_{AB}F^A\tilde{F}^B + [Y_{AB}]_{\mu\nu}F^{A\,\mu\alpha}F^{B\,\nu}{}_{\alpha} + \\ &+ \left[\tilde{Y}_{AB}\right]_{\mu\nu}F^{A\,\mu\alpha}\tilde{F}^{B\,\nu}{}_{\alpha} + Z^{(1)}_{AB}(D_\mu F^{A\,\mu\alpha})(D^\nu F^B_{\nu\alpha}) + \\ &+ \left[Z^{(2)}_{AB}\right]_\mu F^{A\,\mu\alpha}D^\nu F^B_{\nu\alpha} + \left[\tilde{Z}^{(2)}_{AB}\right]_\mu \tilde{F}^{A\,\mu\alpha}D^\nu F^B_{\nu\alpha}.\end{aligned} \tag{5.4}$$

We now proceed with the true duality analysis of the 4-derivatives Lagrangian: first of the invariant sector $\mathcal{L}^{(4)}_{\text{mat}}$ and then of the gauge one $\mathcal{L}^{(4)}_{\text{gauge}}$.

5.2 Duality analysis of the exactly invariant sector

The sector of $\mathcal{L}_4$ that we expect to be exactly duality invariant is the one involving only the metric and the complex scalar, i.e. the one we called $\mathcal{L}^{(4)}_{\text{mat}}$ (5.2). The metric $g_{\mu\nu}$ does not transform under duality and thus all the g-operators of (5.1) are allowed in $\mathcal{L}^{(4)}_{\text{mat}}$:

$$\boldsymbol{g}\textbf{:} \quad R^2\,,\ (R_{\mu\nu})^2\,,\ (R_{\mu\nu\rho\sigma})^2. \tag{5.5}$$

Since they're already exactly invariant they appear in $\mathcal{L}^{(4)}_{\text{mat}}$ with couplings that are purely numerical factors and not functions of τ and $\bar{\tau}$. In fact the only $\mathrm{SL}(2,\mathbb{R})$ invariants that we're able to build are those resembling the τ kinetic term of Lagrangian (4.1), which indeed involves the spacetime derivatives of τ (see (4.45)). A purely algebraic invariant cannot be constructed. We also notice that, having numerical coefficients, we can exchange, as in the case of pure Einstein–Maxwell theory, the operator $(R_{\mu\nu\rho\sigma})^2$ with the topological Gauss–Bonnet term (1.13).

Instead, the complex scalar τ does transform under the proper duality group $\mathrm{SL}(2,\mathbb{R})$ (4.31), so that each corresponding operator in $\mathcal{L}^{(4)}_{\text{mat}}$ must be individually $\mathrm{SL}(2,\mathbb{R})$ invariant. Such $\mathrm{SL}(2,\mathbb{R})$ invariant operators are, as said, those constructed according to the transformation rules (4.21) and (4.22). Thus, the τ and $\tau + g$ operators described in (5.1) that can enter in $\mathcal{L}^{(4)}_{\text{mat}}$ are

$$\begin{aligned} \boldsymbol{\tau}\textbf{:} &\quad (\partial_\mu\bar{\tau}\partial^\mu\tau)^2\,,\ |\partial_\mu\tau\partial^\mu\tau|^2\,,\\ \boldsymbol{g}+\boldsymbol{\tau}\textbf{:} &\quad R\,\partial_\mu\bar{\tau}\partial^\mu\tau\,,\ R_{\mu\nu}\partial^\mu\bar{\tau}\partial^\nu\tau\,, \end{aligned} \tag{5.6}$$

with couplings proportional to some inverse power of $\mathfrak{Im}\tau$. Therefore, this sector of the 4-derivatives Lagrangian can be written as

$$\begin{aligned} \mathcal{L}^{(4)}_{\text{mat}} =&\alpha_0 G_B + \alpha_1 R^2 + \alpha_2 (R_{\mu\nu})^2 + \frac{1}{(\mathfrak{Im}\tau)^2}\left[\lambda_1 R\partial_\mu\bar{\tau}\partial^\mu\tau + \lambda_2 R_{\mu\nu}\partial^\mu\bar{\tau}\partial^\nu\tau\right] +\\ &+ \frac{1}{(\mathfrak{Im}\tau)^4}\left[\lambda_3(\partial_\mu\bar{\tau}\partial^\mu\tau)^2 + \lambda_4|\partial_\mu\tau\partial^\mu\tau|^2\right], \end{aligned} \tag{5.7}$$

where α_i and λ_i are purely numerical coefficients.

This Lagrangian can be further reduced by applying a duality-preserving fields redefinition. Making use of the same redefinition scheme introduced in (1.20) and (1.21) and the Einstein equations (4.5) of our model, we can remove some of the operators of (5.7) redefining the metric via

$$\delta g^{\mu\nu} = c_1 R g^{\mu\nu} + c_2 R^{\mu\nu} + \frac{1}{(\mathfrak{Im}\tau)^2}\left[c_3 g^{\mu\nu}\partial_\alpha\bar{\tau}\partial^\alpha\tau + c_4\partial^{(\mu}\bar{\tau}\partial^{\nu)}\tau\right], \tag{5.8}$$

where the c_i are numerical coefficients and the $(\mathfrak{Im}\tau)^{-2}$ factors ensures that the $\mathrm{SL}(2,\mathbb{R})$ symmetry is preserved by the field redefinition.

It follows that the field redefinition's coefficients can be set to remove the following operators:

$$\begin{aligned} c_1 &\longrightarrow R^2, & c_2 &\longrightarrow (R_{\mu\nu})^2,\\ c_3 &\longrightarrow R\partial_\mu\bar{\tau}\partial^\mu\tau, & c_4 &\longrightarrow R_{\mu\nu}\partial^\mu\bar{\tau}\partial^\nu\tau, \end{aligned} \tag{5.9}$$

and the resulting exactly invariant sector $\mathcal{L}^{(4)}_{\text{mat}}$ is therefore (after relabelling the coefficients)

$$\mathcal{L}^{(4)}_{\text{mat}} = \alpha_0 G_B + \frac{1}{(\mathfrak{Im}\tau)^4}\left[\lambda_1(\partial_\mu\bar{\tau}\partial^\mu\tau)^2 + \lambda_2|\partial_\mu\tau\partial^\mu\tau|^2\right]. \tag{5.10}$$

5.3 Duality analysis of the gauge sector

We now turn to the discussion of the more challenging duality analysis of the gauge sector (5.4) of the 4-derivatives Lagrangian.

The strategy to fix its non-minimal coefficients is the same we applied to in section 3.2.1: we determine the correction to the dual field according to (4.29), we apply a EM duality transformation (4.30) and constraint the coefficients via the corrected version of identity (4.35).

5.3.1 Scheme of the 4-derivatives correction duality analysis

We start by studying how the founding elements of our duality analysis, the dual field (4.32) and the consistency identity (4.35), change when we include in the Lagrangian also the 4-derivatives correction $\mathcal{L}_4$.

Starting from the dual field, its expression in terms of the perturbative expansion we're following is

$$G_\Lambda = G_\Lambda^{(2)} + G_\Lambda^{(4)}, \tag{5.11}$$

with $G_\Lambda^{(2)}$ as in (4.32) and $G_\Lambda^{(4)}$ that, according to (4.29), reads

$$G^{(4)}_{\Lambda\,\mu\nu} = 2\left[\frac{\partial \mathcal{L}_4}{\partial F^{\Lambda\,\mu\nu}} - D_\alpha \frac{\partial \mathcal{L}_4}{\partial\left(D_\alpha F^{\Lambda\,\mu\nu}\right)}\right]. \tag{5.12}$$

We remark that, given the expression (5.4) of $\mathcal{L}^{(4)}_{\text{gauge}}$, the higher-derivative term that appears in (5.12) is the only that contributes to $G^{(4)}_{\Lambda\,\mu\nu}$, as anticipated in (4.29).

To derive instead the 4-derivatives version of identity (4.35) we apply a duality transformation (4.30) to (5.11) and study it, again, from the two F and G points of view introduced in (4.37) and (4.36). This time the transformation is going to split, as G_Λ in (5.11), in the 2 and 4-derivatives components respectively, so that we're going to determine two different transformation rules, one of $G_\Lambda^{(2)}$ and one of $G_\Lambda^{(4)}$, separately, guided by the perturbative expansion in the number of derivatives. Thus, starting from (5.11), the "G-side" of the transformation reads, as before:

$$\delta G_{\Lambda\,\mu\nu} = C_{\Lambda\Sigma} F^{\Sigma}\mu\nu + D_\Lambda{}^{\Sigma} G_{\Sigma\,\mu\nu} = C_{\Lambda\Sigma} F^{\Sigma}_{\mu\nu} + D_\Lambda{}^{\Sigma} G^{(2)}_{\Sigma\,\mu\nu} + D_\Lambda{}^{\Sigma} G^{(4)}_{\Sigma\,\mu\nu}, \tag{5.13}$$

while the "F-side" now becomes

$$\begin{aligned}
\delta G_{\Lambda\,\mu\nu} =& \delta_\tau G_{\Lambda\,\mu\nu} + \frac{\partial G_{\Lambda\,\mu\nu}}{\partial F^{\Sigma}_{\alpha\beta}} \delta F^{\Sigma}_{\alpha\beta} + \frac{\partial G_{\Lambda\,\mu\nu}}{\partial D^\gamma F^{\Sigma}_{\alpha\beta}} D^\gamma\left(\delta F^{\Sigma}_{\alpha\beta}\right) = \\
=& \delta_\tau\left(G^{(2)}_{\Lambda\,\mu\nu} + G^{(4)}_{\Lambda\,\mu\nu}\right) + \left[\frac{\partial G^{(2)}_{\Lambda\,\mu\nu}}{\partial F^{\Sigma}_{\alpha\beta}} + \frac{\partial G^{(4)}_{\Lambda\,\mu\nu}}{\partial F^{\Sigma}_{\alpha\beta}}\right]\left(AF + BG^{(2)} + BG^{(4)}\right)^{\Sigma}_{\alpha\beta} + \\
&+ \frac{\partial G^{(4)}_{\Lambda\,\mu\nu}}{\partial D^\gamma F^{\Sigma}_{\alpha\beta}} D^\gamma\left(AF + BG^{(2)} + \cancel{BG^{(4)}}\right)^{\Sigma}_{\alpha\beta} + \mathcal{O}(D^6) = \\
=& \delta_\tau G^{(2)}_{\Lambda\,\mu\nu} + \frac{\partial G^{(2)}_{\Lambda\,\mu\nu}}{\partial F^{\Sigma}_{\alpha\beta}}\left(AF + BG^{(2)}\right)^{\Sigma}_{\alpha\beta} + \delta_\tau G^{(4)}_{\Lambda\,\mu\nu} + \frac{\partial G^{(2)}_{\Lambda\,\mu\nu}}{\partial F^{\Sigma}_{\alpha\beta}}\left(BG^{(4)}\right)^{\Sigma}_{\alpha\beta} + \\
&+ \frac{\partial G^{(4)}_{\Lambda\,\mu\nu}}{\partial F^{\Sigma}_{\alpha\beta}}\left(AF + BG^{(2)}\right)^{\Sigma}_{\alpha\beta} + \frac{\partial G^{(4)}_{\Lambda\,\mu\nu}}{\partial D^\gamma F^{\Sigma}_{\alpha\beta}} D^\gamma\left(AF + BG^{(2)}\right)^{\Sigma}_{\alpha\beta} + \mathcal{O}(D^6),
\end{aligned} \tag{5.14}$$

so that from (5.14) we reproduce correctly the $G^{(2)}_\Lambda$ transformation (4.37), together with its 4-derivatives version:

$$\delta G^{(2)}_{\Lambda\,\mu\nu} = \delta_\tau G^{(2)}_{\Lambda\,\mu\nu} + \frac{\partial G^{(2)}_{\Lambda\,\mu\nu}}{\partial F^\Sigma_{\alpha\beta}} \left(AF + BG^{(2)}\right)^\Sigma_{\alpha\beta}, \tag{5.15}$$

$$\begin{aligned} \delta G^{(4)}_{\Lambda\,\mu\nu} =& \delta_\tau G^{(4)}_{\Lambda\,\mu\nu} + \frac{\partial G^{(4)}_{\Lambda\,\mu\nu}}{\partial F^\Sigma_{\alpha\beta}} \left(AF + BG^{(2)}\right)^\Sigma_{\alpha\beta} + \\ &+ \frac{\partial G^{(2)}_{\Lambda\,\mu\nu}}{\partial F^\Sigma_{\alpha\beta}} \left(BG^{(4)}\right)^\Sigma_{\alpha\beta} + \frac{\partial G^{(4)}_{\Lambda\,\mu\nu}}{\partial D^\gamma F^\Sigma_{\alpha\beta}} D^\gamma \left(AF + BG^{(2)}\right)^\Sigma_{\alpha\beta}. \end{aligned} \tag{5.16}$$

Therefore, by comparison between (5.13) and (5.15) we re-obtain the 2-derivatives consistency identity (4.35), while the comparison between (5.13) and (5.16) yields its 4-derivatives corrected version:

$$\begin{aligned} D_\Lambda{}^\Sigma G^{(4)}_{\Sigma\,\mu\nu} =& \delta_\tau G^{(4)}_{\Lambda\,\mu\nu} + \frac{\partial G^{(4)}_{\Lambda\,\mu\nu}}{\partial F^\Sigma_{\alpha\beta}} \left(AF + BG^{(2)}\right)^\Sigma_{\alpha\beta} + \\ &+ \frac{\partial G^{(2)}_{\Lambda\,\mu\nu}}{\partial F^\Sigma_{\alpha\beta}} \left(BG^{(4)}\right)^\Sigma_{\alpha\beta} + \frac{\partial G^{(4)}_{\Lambda\,\mu\nu}}{\partial D^\gamma F^\Sigma_{\alpha\beta}} D^\gamma \left(AF + BG^{(2)}\right)^\Sigma_{\alpha\beta}. \end{aligned} \tag{5.17}$$

Though being expected, given that no term in the transformation containing $G^{(4)}_\Lambda$ can actually talk to the 2-derivatives ones, it's important to have re-obtained the 2-derivatives identity (4.35) because it provides further support to the results on the duality group of the theory we derived from the pure 2-derivatives order (see (4.42) and (4.43)). Also, we notice that the perturbative character of our approach is manifestly shown in the splitting of the consistency identity in (4.35) and (5.17).

Now that we derived (5.17) we can make use of it to constrain the couplings of the 4-derivatives gauge sector (5.4). As we did in the 2-derivatives case, we determine the dual field component associated to $G^{(4)}_\Lambda$ (5.12), we apply a duality transformation and derive, via (5.17), the transformation rules of the various couplings. These couplings are then going to be constrained both by the structure of their transformation rule and by the requirement that all the transformations of the various couplings are consistent with each other. In fact, a duality transformation mixes, in general, the different operators and, with them, the couplings, as we saw in the case of $\mathcal{I}$ (4.38) and $\mathcal{R}$ (4.39).

In this perspective, it is useful to identify in $\mathcal{L}^{(4)}_{\text{gauge}}$ (5.4) sets of operators that under a duality transformation get mixed between each other but not with operators of other sets. The reason is that we can perform the duality analysis on each of these sets independently on the others and this simplifies and clarifies at the same time the full discussion. This classification is in fact straightforward and yields four different sectors:

$$\mathcal{L}^{(4)}_{\text{gauge}} = \mathcal{L}^{(4F)}_4 + \mathcal{L}^{(RF)}_4 + \mathcal{L}^{(\tau)}_4 + \mathcal{L}^{(D)}_4, \tag{5.18}$$

with:

- **$4F$-sector:**

$$\begin{aligned} \mathcal{L}^{(4F)}_4 =& [\alpha_1]_{ABCD}\,(F^A F^B)(F^C F^D) + [\alpha_2]_{ABCD}\,(F^A \tilde{F}^B)(F^C \tilde{F}^D) + \\ &+ [\alpha_3]_{ABCD}\,(F^A F^B)(F^C \tilde{F}^D) + [\alpha_4]_{ABCD}\,\tilde{F}^{A\,\mu\nu} F^B_{\nu\rho} F^{C\,\rho\sigma} F^D_{\sigma\mu}, \end{aligned} \tag{5.19}$$

- ***RF*-sector:**

$$\begin{aligned}\mathcal{L}_4^{(RF)} = &[\beta_1]_{\Lambda\Sigma}\, RF^\Lambda F^\Sigma + [\beta_2]_{\Lambda\Sigma}\, RF^\Lambda \tilde{F}^\Sigma + [\beta_3]_{\Lambda\Sigma}\, R_{\mu\nu} F^{\Lambda\,\mu\rho} F^{\Sigma\,\nu}{}_\rho + \\ &+ [\beta_4]_{\Lambda\Sigma}\, R_{\mu\nu\rho\sigma} F^{\Lambda\,\mu\nu} F^{\Sigma\,\rho\sigma} + [\beta_5]_{\Lambda\Sigma}\, R_{\mu\nu\rho\sigma} F^{\Lambda\,\mu\nu} \tilde{F}^{\Sigma\,\rho\sigma},\end{aligned} \tag{5.20}$$

- ***τ*-sector:**

$$\mathcal{L}_4^{(\tau)} = X_{AB} F^A F^B + \tilde{X}_{AB} F^A \tilde{F}^B + [Y_{AB}]_{\mu\nu}\, F^{A\,\mu\alpha} F^{B\,\nu}{}_\alpha + \left[\tilde{Y}_{AB}\right]_{\mu\nu} F^{A\,\mu\alpha} \tilde{F}^{B\,\nu}{}_\alpha, \tag{5.21}$$

- ***D*-sector:**

$$\mathcal{L}_4^{(D)} = Z^{(1)}_{AB} (D_\mu F^{A\,\mu\alpha})(D^\nu F^B_{\nu\alpha}) + \left[Z^{(2)}_{AB}\right]_\mu F^{A\,\mu\alpha} D^\nu F^B_{\nu\alpha} + \left[\tilde{Z}^{(2)}_{AB}\right]_\mu \tilde{F}^{A\,\mu\alpha} D^\nu F^B_{\nu\alpha}, \tag{5.22}$$

Furthermore, a duality transformation on the 4-derivatives dual field does not mix only the operators that enter explicitly the Lagrangian but can indeed produced terms that correspond to those operators we chose to exclude from the set (5.1). These "dependent" operators are indeed related to the independent ones of (5.1) via some identities, that we exploit to properly take into account also these extra terms.

The summary of the identities we made use of is reported in Appendix B. The following sections are then dedicated to the duality analysis of the four different sectors (5.19)–(5.22) of $\mathcal{L}^{(4)}_{\text{gauge}}$ (5.4).

5.3.2 Duality analysis of the $4F$-sector

The Lagrangian of the $4F$-sector is

$$\begin{aligned}\mathcal{L}_4^{(4F)} = &[\alpha_1]_{ABCD}\, (F^A F^B)(F^C F^D) + [\alpha_2]_{ABCD}\, (F^A \tilde{F}^B)(F^C \tilde{F}^D) + \\ &+ [\alpha_3]_{ABCD}\, (F^A F^B)(F^C \tilde{F}^D) + [\alpha_4]_{ABCD}\, \tilde{F}^{A\,\mu\nu} F^B_{\nu\rho} F^{C\,\rho\sigma} F^D_{\sigma\mu},\end{aligned} \tag{5.23}$$

where the coefficients α_i have the following symmetry properites:

$$[\alpha_1]_{ABCD} = [\alpha_1]_{BACD} = [\alpha_1]_{ABDC} = [\alpha_1]_{CDAB}\,, \tag{5.24}$$

$$[\alpha_2]_{ABCD} = [\alpha_2]_{BACD} = [\alpha_2]_{ABDC} = [\alpha_2]_{CDAB}\,, \tag{5.25}$$

$$[\alpha_3]_{ABCD} = [\alpha_3]_{BACD} = [\alpha_3]_{ABDC}\,, \tag{5.26}$$

$$[\alpha_4]_{ABCD} = [\alpha_4]_{ADCB}\,, \tag{5.27}$$

$$[\alpha_4]_{AB(CD)} = 0. \tag{5.28}$$

The corresponding dual field reads, according to (5.12),

$$\begin{aligned}G^{(4F)}_{\Lambda\,\mu\nu} = &-8\,[\alpha_1]_{\Lambda BCD}\, (F^C F^D) \tilde{F}^B_{\mu\nu} + 8\,[\alpha_2]_{\Lambda BCD}\, (F^C \tilde{F}^D) F^B_{\mu\nu} + \\ &+ [\Lambda_1]_{\Lambda BCD}\, (F^C F^D) F^B_{\mu\nu} + [\Lambda_2]_{\Lambda BCD}\, F^B_{[\mu|\alpha} F^{C\,\alpha\beta} F^D_{\beta|\nu]},\end{aligned} \tag{5.29}$$

where the two coefficients Λ_1 and Λ_2 are defined as the following linear combinations of α_3

and α_4:

$$[\Lambda_1]_{\Lambda BCD} \equiv 8\,[\alpha_3]_{\Lambda(C|B|D)} + 4\,[\alpha_3]_{CD\Lambda B} + [\alpha_4]_{BC\Lambda D}\,, \tag{5.30}$$

$$[\Lambda_2]_{\Lambda BCD} \equiv 16\,[\alpha_3]_{\Lambda CBD} - 2\,[\alpha_4]_{\Lambda BCD} - 2\,[\alpha_4]_{CB\Lambda D}\,. \tag{5.31}$$

and we have $[\Lambda_1]_{\Lambda BCD} = [\Lambda_1]_{\Lambda B(CD)}$ and $[\Lambda_2]_{\Lambda BCD} = [\Lambda_2]_{\Lambda(B|C|D)}$.

We now apply a duality transformation (4.30) and exploit identity (5.17): by comparing the coefficients of the same operators on the two sides of the identity (as we did with F^Λ and $\tilde{F}^\Lambda$ in (4.36) and (4.37)) we obtain the transformation rules for the α_i coefficients of (5.19). These transformations result to be:

- $$\begin{aligned}[\delta\alpha_1]_{\Lambda\Omega CD} =& D_\Lambda{}^\Sigma\,[\alpha_1]_{\Sigma\Omega CD} - [\alpha_1]_{\Lambda\Sigma CD}\,A^\Sigma{}_\Omega - [\alpha_1]_{\Lambda\Sigma CD}\,(B\mathcal{R})^\Sigma{}_\Omega + \\ &- (\mathcal{R}B)_\Lambda{}^\Sigma\,[\alpha_1]_{\Sigma\Omega CD} + D_{(C|}{}^\Sigma\,[\alpha_1]_{\Lambda\Omega\Sigma|D)} - [\alpha_1]_{\Lambda\Omega(C|\Sigma}\,A^\Sigma{}_{|D)} + \\ &- [\alpha_1]_{\Lambda\Omega(C|\Sigma}\,(B\mathcal{R})^\Sigma{}_{|D)} - (\mathcal{R}B)_{(C|}{}^\Sigma\,[\alpha_1]_{\Lambda\Omega\Sigma|D)} + \\ &+ \frac{1}{8}\left((B\mathcal{I})^\Sigma{}_\Omega\,[\Lambda_1]_{\Lambda\Sigma CD} + (\mathcal{I}B)_\Lambda{}^\Sigma\,[\Lambda_1]_{\Sigma\Omega CD}\right) + \\ &- \frac{1}{4}\Big((B\mathcal{I})^\Sigma{}_\Lambda\,[\Lambda_2]_{\Sigma\Omega(CD)} + (B\mathcal{I})^\Sigma{}_{(C|}\,[\Lambda_2]_{\Sigma|D)\Lambda\Omega}\Big) + \\ &+ \frac{1}{8}(B\mathcal{I})^\Sigma{}_\Lambda\,[\Lambda_2]_{\Omega\Sigma(CD)}\,;\end{aligned} \tag{5.32}$$

- $$\begin{aligned}[\delta\alpha_2]_{\Lambda\Omega CD} =& D_\Lambda{}^\Sigma\,[\alpha_2]_{\Sigma\Omega CD} - [\alpha_2]_{\Lambda\Sigma CD}\,A^\Sigma{}_\Omega - [\alpha_2]_{\Lambda\Sigma CD}\,(B\mathcal{R})^\Sigma{}_\Omega + \\ &- (\mathcal{R}B)_\Lambda{}^\Sigma\,[\alpha_2]_{\Sigma\Omega CD} + D_{(C|}{}^\Sigma\,[\alpha_2]_{\Lambda\Omega\Sigma|D)} - [\alpha_2]_{\Lambda\Omega(C|\Sigma}\,A^\Sigma{}_{|D)} + \\ &- [\alpha_2]_{\Lambda\Omega(C|\Sigma}\,(B\mathcal{R})^\Sigma{}_{|D)} - (\mathcal{R}B)_{(C|}{}^\Sigma\,[\alpha_2]_{\Lambda\Omega\Sigma|D)} + \\ &+ \frac{1}{8}(B\mathcal{I})^\Sigma{}_{(C|}\,[\Lambda_2]_{\Lambda\Sigma|D)\Omega} - \frac{1}{16}(B\mathcal{I})^\Sigma{}_\Lambda\,[\Lambda_2]_{\Omega(C|\Sigma|D)} + \\ &+ \frac{1}{16}(B\mathcal{I})^\Sigma{}_{(C|}\,[\Lambda_2]_{\Lambda\Omega\Sigma|D)} + \frac{1}{32}(B\mathcal{I})^\Sigma{}_\Lambda\,[\Lambda_2]_{\Sigma\Omega(CD)} + \\ &- \frac{1}{4}(\mathcal{I}B)_{(C|}{}^\Sigma\,[\Lambda_1]_{\Lambda\Omega\Sigma|D)}\,;\end{aligned} \tag{5.33}$$

- $$\begin{aligned}[\delta\Lambda_1]_{\Lambda\Omega CD} =& D_\Lambda{}^\Sigma\,[\Lambda_1]_{\Sigma\Omega CD} - [\Lambda_1]_{\Lambda\Sigma CD}\,A^\Sigma{}_\Omega + \\ &- [\Lambda_1]_{\Lambda\Sigma CD}\,(B\mathcal{R})^\Sigma{}_\Omega - (\mathcal{R}B)_\Lambda{}^\Sigma\,[\Lambda_1]_{\Sigma\Omega CD} + \\ &+ D_{(C|}{}^\Sigma\,[\Lambda_1]_{\Lambda\Omega\Sigma|D)} - [\Lambda_1]_{\Lambda\Omega(C|\Sigma}\,A^\Sigma{}_{|D)} + \\ &- [\Lambda_1]_{\Lambda\Omega(C|\Sigma}\,(B\mathcal{R})^\Sigma{}_{|D)} - (\mathcal{R}B)_{(C|}{}^\Sigma\,[\Lambda_1]_{\Lambda\Omega\Sigma|D)} + \\ &- 16(B\mathcal{I})^\Sigma{}_{(\Lambda|}\,[\alpha_1]_{\Sigma|\Omega)CD} - 16(B\mathcal{I})^\Sigma{}_{(C|}\,[\alpha_1]_{\Lambda|D)\Sigma\Omega} + \\ &- 16(B\mathcal{I})^\Sigma{}_\Omega\,[\alpha_1]_{\Lambda(C|\Sigma|D)} + 16(B\mathcal{I})^\Sigma{}_{(C|}\,[\alpha_2]_{\Lambda\Sigma\Omega|D)} + \\ &+ 16(B\mathcal{I})^\Sigma{}_{(C|}\,[\alpha_2]_{\Lambda\Omega\Sigma|D)} + 16(B\mathcal{I})^\Sigma{}_\Lambda\,[\alpha_2]_{\Sigma(C|\Omega|D)}\,;\end{aligned} \tag{5.34}$$

- $$\begin{aligned}[\delta\Lambda_2]_{\Lambda\Omega CD} =& D_\Lambda{}^\Sigma\,[\Lambda_2]_{\Sigma\Omega CD} - 2\,[\Lambda_2]_{\Lambda\Sigma CD}\,A^\Sigma{}_\Omega - 2\,[\Lambda_2]_{\Lambda\Sigma CD}\,(B\mathcal{R})^\Sigma{}_\Omega + \\ &- (\mathcal{R}B)_\Lambda{}^\Sigma\,[\Lambda_2]_{\Sigma\Omega CD} + D_C{}^\Sigma\,[\Lambda_2]_{\Lambda\Omega\Sigma D} - (\mathcal{R}B)_C{}^\Sigma\,[\Lambda_2]_{\Lambda\Omega\Sigma D} + \\ &- 64(B\mathcal{I})^\Sigma{}_{(\Omega|}\,[\alpha_1]_{\Lambda C\Sigma|D)} + 64(B\mathcal{I})^\Sigma{}_{(\Lambda|}\,[\alpha_2]_{\Sigma|C)\Omega D}\end{aligned} \tag{5.35}$$

Solution for the coefficients Let's now look at these transformations and try to consequently fix the coefficients. In particular, a very interesting scenario is the one in which the higher-order coefficients reproduce the transformation patterns of the 2-derivatives couplings $\mathcal{I}$ and $\mathcal{R}$ or, in other words, in which it exists a duality-preserving, 4-derivatives extension of Lagrangian (4.1) whose couplings are proportional precisely to those of the previous perturbative order (4.1). This may be not the only solution but it is indeed a relevant one because first it would provide an explicit expression for the coefficients (and not only their duality transformation), then because it suggests the possibility that an exactly invariant UV theory, from which we can derive the perturbative expansion in the number of derivatives we're dealing with, could indeed exists and this is of great support to our analysis.

With this picture in mind, we see that Λ_2 transformation (5.35) is far from resembling $\mathcal{I}$ and $\mathcal{R}$ ones (4.38) and (4.39). Thus, we can start looking for a solution for the $4F$-sector's coefficients by setting

$$\Lambda_2 = 0. \tag{5.36}$$

Transformation (5.35) now is not "dynamical" anymore, but yields instead a constraint over the coefficients α_1 and α_2:

$$(B\mathcal{I})^\Sigma{}_{(\Omega|}\,[\alpha_1]_{\Lambda C\Sigma|D)} = (B\mathcal{I})^\Sigma{}_{(\Lambda|}\,[\alpha_2]_{\Sigma|C)\Omega D}\,. \tag{5.37}$$

The transformation rules of the other coefficients now become:

- $$\begin{aligned}[\delta\alpha_1]_{\Lambda\Omega CD} =& D_\Lambda{}^\Sigma\,[\alpha_1]_{\Sigma\Omega CD} - [\alpha_1]_{\Lambda\Sigma CD}\,A^\Sigma{}_\Omega - [\alpha_1]_{\Lambda\Sigma CD}\,(B\mathcal{R})^\Sigma{}_\Omega + \\ &- (\mathcal{R}B)_\Lambda{}^\Sigma\,[\alpha_1]_{\Sigma\Omega CD} + D_{(C|}{}^\Sigma\,[\alpha_1]_{\Lambda\Omega\Sigma|D)} - [\alpha_1]_{\Lambda\Omega(C|\Sigma}\,A^\Sigma{}_{|D)} + \\ &- [\alpha_1]_{\Lambda\Omega(C|\Sigma}\,(B\mathcal{R})^\Sigma{}_{|D)} - (\mathcal{R}B)_{(C|}{}^\Sigma\,[\alpha_1]_{\Lambda\Omega\Sigma|D)} + \\ &+ \frac{1}{8}\left((B\mathcal{I})^\Sigma{}_\Omega\,[\Lambda_1]_{\Lambda\Sigma CD} + (\mathcal{I}B)_\Lambda{}^\Sigma\,[\Lambda_1]_{\Sigma\Omega CD}\right);\end{aligned} \tag{5.38}$$

- $$\begin{aligned}[\delta\alpha_2]_{\Lambda\Omega CD} =& D_\Lambda{}^\Sigma\,[\alpha_2]_{\Sigma\Omega CD} - [\alpha_2]_{\Lambda\Sigma CD}\,A^\Sigma{}_\Omega - [\alpha_2]_{\Lambda\Sigma CD}\,(B\mathcal{R})^\Sigma{}_\Omega + \\ &- (\mathcal{R}B)_\Lambda{}^\Sigma\,[\alpha_2]_{\Sigma\Omega CD} + D_{(C|}{}^\Sigma\,[\alpha_2]_{\Lambda\Omega\Sigma|D)} - [\alpha_2]_{\Lambda\Omega(C|\Sigma}\,A^\Sigma{}_{|D)} + \\ &- [\alpha_2]_{\Lambda\Omega(C|\Sigma}\,(B\mathcal{R})^\Sigma{}_{|D)} - (\mathcal{R}B)_{(C|}{}^\Sigma\,[\alpha_2]_{\Lambda\Omega\Sigma|D)} + \\ &- \frac{1}{4}(\mathcal{I}B)_{(C|}{}^\Sigma\,[\Lambda_1]_{\Lambda\Omega\Sigma|D)};\end{aligned} \tag{5.39}$$

- $$\begin{aligned}[\delta\Lambda_1]_{\Lambda\Omega CD} =& D_\Lambda{}^\Sigma\,[\Lambda_1]_{\Sigma\Omega CD} - [\Lambda_1]_{\Lambda\Sigma CD}\,A^\Sigma{}_\Omega + \\ &- [\Lambda_1]_{\Lambda\Sigma CD}\,(B\mathcal{R})^\Sigma{}_\Omega - (\mathcal{R}B)_\Lambda{}^\Sigma\,[\Lambda_1]_{\Sigma\Omega CD} + \\ &+ D_{(C|}{}^\Sigma\,[\Lambda_1]_{\Lambda\Omega\Sigma|D)} - [\Lambda_1]_{\Lambda\Omega(C|\Sigma}\,A^\Sigma{}_{|D)} + \\ &- [\Lambda_1]_{\Lambda\Omega(C|\Sigma}\,(B\mathcal{R})^\Sigma{}_{|D)} - (\mathcal{R}B)_{(C|}{}^\Sigma\,[\Lambda_1]_{\Lambda\Omega\Sigma|D)}\,.\end{aligned} \tag{5.40}$$

We notice that constraint (5.37) exactly cancels all the terms proportional to α_1 and α_2 in Λ_1 transformation (5.34), which now does not depend on any other coefficients. Instead, α_1 and α_2 transformations do depend also on Λ_1.

We now look at the transformation patterns shown in (5.38), (5.39) and (5.40). We see that the transformation of Λ_1 (5.40) resembles the one of $\mathcal{I}$ (4.38) in both the (Λ, Ω) and (C, D) couples of gauge indices, namely

$$[\Lambda_1]_{\Lambda\Omega CD} \propto \mathcal{I}_{\Lambda\Omega}\mathcal{I}_{CD}, \tag{5.41}$$

or other possible configurations of indices that reproduce (5.40). This same structure is present also in the α_1 and α_2 transformations (5.38) and (5.39) but concerns only the terms that are proportional, respectively, to α_1 and α_2 themselves. Differently from (5.40), in (5.38) and (5.39) there are also the terms proportional to Λ_1, which spoil the identification with $\mathcal{I}$ of the type (5.41).

Thus, we cannot have both Λ_1 and the couple (α_1, α_2) proportional to $\mathcal{I}$ like in (5.41). In the latter case Λ_1 is forced to be vanishing by transformations (5.38) and (5.39); in the former we would not be able to determine α_1 and α_2 from their transformation rules, although very similar to (5.41). In particular, (5.38) and (5.39) excludes the possibility to set α_1 and α_2 to zero because also Λ_1 would then result to be vanishing and the entire duality transformation of the $4F$-sector would loose its meaning.

Since the latter option involves undetermined coefficients, we proceed by following the former one and we set

$$\Lambda_1 = 0. \tag{5.42}$$

At this point we can choose α_1 and α_2 proportional to $\mathcal{I}^2$ and the constraint (5.37) fixes them to be

$$[\alpha_1]_{ABCD} = [\alpha_2]_{ABCD} = \eta\, \mathcal{I}_{A(C|}\mathcal{I}_{B|D)}, \tag{5.43}$$

with η a scalar coefficient.

We recall now that Λ_1 and Λ_2 do not appear directly in the Lagrangian of the $4F$-sector (5.19), but rather are combinations of the coefficients α_3 and α_4 (5.30), (5.31). However, we now show that setting Λ_1 and Λ_2 to zero is equivalent to set to zero α_3 and α_4. In fact, (5.36) and (5.42) together give

$$\begin{cases} [\Lambda_1]_{ABCD} = 8\,[\alpha_3]_{A(C|B|D)} + 4\,[\alpha_3]_{CDAB} + [\alpha_4]_{BCAD} = 0 \\ [\Lambda_2]_{ABCD} = 16\,[\alpha_3]_{ACBD} - 2\,[\alpha_4]_{ABCD} - 2\,[\alpha_4]_{CBAD} = 0 \end{cases}, \tag{5.44}$$

so that we have

$$\begin{cases} [\alpha_3]_{A(C|B|D)} = -\frac{1}{2}\,[\alpha_3]_{CDAB} - \frac{1}{8}\,[\alpha_4]_{BCAD} \\ [\alpha_3]_{ACBD} = -\frac{1}{8}\,([\alpha_4]_{ABCD} + [\alpha_4]_{CBAD}) \end{cases}. \tag{5.45}$$

Replacing now α_3 in the first equation of system (5.45) from the second one and aking use of α_4 symmetry properties (5.27) and (5.28) we obtain

$$\begin{aligned} 0 =& \cancel{[\alpha_4]_{AB(CD)}} + [\alpha_4]_{(C|BA|D)} + \frac{1}{2}\,([\alpha_4]_{CADB} + [\alpha_4]_{DACB}) + [\alpha_4]_{BCAD} = \\ =& \frac{1}{2}\,([\alpha_4]_{CBAD} + [\alpha_4]_{DBAC} + [\alpha_4]_{CADB} + [\alpha_4]_{DACB}) + [\alpha_4]_{BCAD} = \\ =& \cancel{[\alpha_4]_{CB(AD)}} + \cancel{[\alpha_4]_{DB(AC)}} + [\alpha_4]_{BCAD} = [\alpha_4]_{BCAD}\,. \end{aligned} \tag{5.46}$$

Hence, we have

$$[\alpha_4]_{ABCD} = 0, \tag{5.47}$$

which immediately implies also

$$[\alpha_3]_{ABCD} = 0. \tag{5.48}$$

Therefore, we were able to find a solution for the coefficients of (5.19) that is compatible with the associated duality transformations (5.32)–(5.35). The resulting $4F$-sector is

$$\mathcal{L}_4^{(4F)} = \eta\, \mathcal{I}_{AC}\mathcal{I}_{BD} \left[(F^A F^B)(F^C F^D) + (F^A \tilde{F}^B)(F^C \tilde{F}^D)\right] \tag{5.49}$$

5.3.3 Duality analysis of the RF-sector

We now proceed further with the RF-sector:

$$\begin{aligned}\mathcal{L}_4^{(RF)} =& [\beta_1]_{\Lambda\Sigma}\, RF^\Lambda F^\Sigma + [\beta_2]_{\Lambda\Sigma}\, RF^\Lambda \tilde{F}^\Sigma + [\beta_3]_{\Lambda\Sigma}\, R_{\mu\nu}F^{\Lambda\,\mu\rho}F^{\Sigma\,\nu}{}_\rho + \\ &+ [\beta_4]_{\Lambda\Sigma}\, R_{\mu\nu\rho\sigma}F^{\Lambda\,\mu\nu}F^{\Sigma\,\rho\sigma} + [\beta_5]_{\Lambda\Sigma}\, R_{\mu\nu\rho\sigma}F^{\Lambda\,\mu\nu}\tilde{F}^{\Sigma\,\rho\sigma}.\end{aligned} \tag{5.50}$$

The symmetries of the coefficients are:

$$[\beta_i]_{\Lambda\Sigma} = [\beta_i]_{\Sigma\Lambda}\,, \qquad i \neq 5. \tag{5.51}$$

The associated dual field is

$$\begin{aligned}G^{(RF)}_{\Lambda\,\mu\nu} =& -4\,[\beta_1]_{\Lambda\Sigma}\, R\tilde{F}^\Sigma_{\mu\nu} + 4\left[\beta_2 + \frac{\beta_5}{2}\right]_{\Lambda\Sigma} RF^\Sigma_{\mu\nu} + 2\,[\beta_3]_{\Lambda\Sigma}\, R^{\rho\alpha}F^{\Sigma\,\sigma}{}_\alpha\epsilon_{\mu\nu\rho\sigma} + \\ &- 2\,[\beta_4]_{\Lambda\Sigma}\, R^{\alpha\beta\rho\sigma}F^\Sigma_{\rho\sigma}\epsilon_{\alpha\beta\mu\nu} + 4\,[\beta_5]_{(\Lambda\Sigma)}\, R_{\mu\nu\alpha\beta}F^{\Sigma\,\alpha\beta} + 8\,[\beta_5]_{\Lambda\Sigma}\, R_{[\mu|\alpha}F^{\Sigma\ \alpha}_{|\nu]}.\end{aligned} \tag{5.52}$$

We notice that, differently from the $4F$ case (5.29), there is a coefficient, β_5, that appears in (5.52) associated with three different operators: we immediately understand that the comparison between the transformation rules coming from these operators will determine some relevant consistency conditions on the $\mathcal{L}_4^{(RF)}$ coefficients.

The duality transformations of the coefficients of each term of the dual field (5.52) are the following:

- $$\begin{aligned}[\delta\beta_1]_{\Lambda\Omega} =& D_\Lambda{}^\Sigma\,[\beta_1]_{\Sigma\Omega} - A^\Sigma{}_\Omega\,[\beta_1]_{\Lambda\Sigma} - (B\mathcal{R})^\Sigma{}_\Omega\,[\beta_1]_{\Lambda\Sigma} - (\mathcal{R}B)_\Lambda{}^\Sigma\,[\beta_1]_{\Sigma\Omega} + \\ &+ (B\mathcal{I})^\Sigma{}_\Omega\left(\beta_2 + \frac{3}{2}\beta_5\right)_{\Lambda\Sigma} + (\mathcal{I}B)_\Lambda{}^\Sigma\left(\beta_2 + \frac{\beta_5}{2}\right)_{\Sigma\Omega} + \\ &+ \frac{1}{2}(B\mathcal{I})^\Sigma{}_\Omega\,[\beta_5]_{\Sigma\Lambda}\,;\end{aligned} \tag{5.53}$$

- $$\begin{aligned}[\delta\beta_2]_{\Lambda\Omega} + \frac{1}{2}\,[\delta\beta_5]_{\Lambda\Omega} =& D_\Lambda{}^\Sigma\left(\beta_2 + \frac{\beta_5}{2}\right)_{\Sigma\Omega} - A^\Sigma{}_\Omega\left(\beta_2 + \frac{\beta_5}{2}\right)_{\Lambda\Sigma} + \\ &- (B\mathcal{R})^\Sigma{}_\Omega\left(\beta_2 + \frac{\beta_5}{2}\right)_{\Lambda\Sigma} - (\mathcal{R}B)_\Lambda{}^\Sigma\left(\beta_2 + \frac{1}{2}\beta_5\right)_{\Sigma\Omega} + \\ &- (B\mathcal{I})^\Sigma{}_\Omega\left(\beta_1 + \frac{\beta_3}{2} - \beta_4\right)_{\Lambda\Sigma} - (\mathcal{I}B)_\Lambda{}^\Sigma\,[\beta_1]_{\Sigma\Omega}\,;\end{aligned} \tag{5.54}$$

- $$\begin{aligned}[\delta\beta_3]_{\Lambda\Omega} =& D_\Lambda{}^\Sigma\,[\beta_3]_{\Sigma\Omega} - A^\Sigma{}_\Omega\,[\beta_3]_{\Lambda\Sigma} - (B\mathcal{R})^\Sigma{}_\Omega\,[\beta_3]_{\Lambda\Sigma} - (\mathcal{R}B)_\Lambda{}^\Sigma\,[\beta_3]_{\Sigma\Omega} + \\ &- 4(B\mathcal{I})^\Sigma{}_\Omega\,[\beta_5]_{(\Lambda\Sigma)} - 2(B\mathcal{I})^\Sigma{}_\Omega\,[\beta_5]_{\Lambda\Sigma} - 2(\mathcal{I}B)_\Lambda{}^\Sigma\,[\beta_5]_{\Sigma\Omega}\,;\end{aligned} \tag{5.55}$$

- $$\begin{aligned}[\delta\beta_4]_{\Lambda\Omega} =& D_\Lambda{}^\Sigma\,[\beta_4]_{\Sigma\Omega} - A^\Sigma{}_\Omega\,[\beta_4]_{\Lambda\Sigma} - (B\mathcal{R})^\Sigma{}_\Omega\,[\beta_4]_{\Lambda\Sigma} - (\mathcal{R}B)_\Lambda{}^\Sigma\,[\beta_4]_{\Sigma\Omega} + \\ &+ (B\mathcal{I})^\Sigma{}_\Omega\,[\beta_5]_{(\Lambda\Sigma)} + (\mathcal{I}B)_\Lambda{}^\Sigma\,[\beta_5]_{(\Sigma\Omega)}\,;\end{aligned} \tag{5.56}$$

- $$\begin{aligned}[\delta\beta_5]_{\Lambda\Omega} + [\delta\beta_5]_{\Omega\Lambda} =& D_\Lambda{}^\Sigma\,([\beta_5]_{\Lambda\Sigma} + [\beta_5]_{\Sigma\Lambda}) - A^\Sigma{}_\Omega\,([\beta_5]_{\Lambda\Sigma} + [\beta_5]_{\Sigma\Lambda}) + \\ &- (B\mathcal{R})^\Sigma{}_\Omega\,([\beta_5]_{\Lambda\Sigma} + [\beta_5]_{\Sigma\Lambda}) + 2(B\mathcal{I})^\Sigma{}_\Omega\,[\beta_4]_{\Lambda\Sigma} + \\ &- (\mathcal{R}B)_\Lambda{}^\Sigma\,([\beta_5]_{\Sigma\Omega} + [\beta_5]_{\Omega\Sigma}) + 2(\mathcal{I}B)_\Lambda{}^\Sigma\,[\beta_4]_{\Sigma\Omega}\,;\end{aligned} \tag{5.57}$$

- $$\begin{aligned}[\delta\beta_5]_{\Lambda\Omega} =& D_\Lambda{}^\Sigma\,[\beta_5]_{\Sigma\Omega} - A^\Sigma{}_\Omega[\beta_5]_{\Lambda\Sigma} - (B\mathcal{R})^\Sigma{}_\Omega[\beta_5]_{\Lambda\Sigma} - \mathcal{R}B)_\Lambda{}^\Sigma\,[\beta_5]_{\Sigma\Omega} + \\ &+ 2(B\mathcal{I})^\Sigma{}_\Omega\,[\beta_4]_{\Lambda\Sigma} - \frac{1}{2}(B\mathcal{I})^\Sigma{}_\Omega\,[\beta_3]_{\Lambda\Sigma} + \frac{1}{2}(\mathcal{I}B)_\Lambda{}^\Sigma\,[\beta_3]_{\Sigma\Omega}\,.\end{aligned} \tag{5.58}$$

Solution for the coefficients As previously mentioned, the coefficient β_5 enters three different transformations: (5.54), together with β_2, (5.57) and (5.58). Let's focus on the last two of them. The first one (5.57) is symmetrized in the gauge indices (Λ, Ω) but split consistently in the two components, yielding

$$\begin{aligned}[\delta\beta_5]_{\Lambda\Omega} =& D_\Lambda{}^\Sigma\,[\beta_5]_{\Sigma\Omega} - [\beta_5]_{\Lambda\Sigma}\,A^\Sigma{}_\Omega - (\mathcal{R}B)_\Lambda{}^\Sigma\,[\beta_5]_{\Sigma\Omega} + \\ &- [\beta_5]_{\Lambda\Sigma}\,(B\mathcal{R})^\Sigma{}_\Omega + 2(B\mathcal{I})^\Sigma{}_\Omega\,[\beta_4]_{\Lambda\Sigma}\,.\end{aligned} \tag{5.59}$$

In order for this transformation to agree with (5.58) we see that we must have that β_3 is such that

$$(\mathcal{I}B[\beta_3])_{[\Lambda\Omega]} = 0. \tag{5.60}$$

This requirement can be satisfied by setting $\beta_3 = 0$ or $\beta_3 = \xi\,\mathcal{I}$, with ξ a numerical coefficient. We can test the latter option by looking at the transformation of β_3 (5.55) and check what additional conditions we should impose in order to reproduce $\mathcal{I}$ transformation rule (4.38). From (5.55) we see that this additional constraint is

$$2(\mathcal{I}B)_\Lambda{}^\Sigma\,[\beta_5]_{\Sigma\Omega} + 2(\mathcal{I}B)_\Omega{}^\Sigma\,[\beta_5]_{\Sigma\Lambda} + 4\,[\beta_5]_{\Lambda\Sigma}\,(B\mathcal{I})^\Sigma{}_\Omega = \xi\,(\mathcal{I}B\mathcal{R})_{\Lambda\Omega}\,. \tag{5.61}$$

However, the way indices are contracted in the left-hand side does not allow this constraint to be solved. Thus, the only option we're left with is

$$\beta_3 = 0. \tag{5.62}$$

The transformation (5.55) now becomes the following constraint over β_5:

$$2(\beta_5\,B\mathcal{I})_{\Lambda\Omega} + (\mathcal{I}B\,\beta_5)_{\Omega\Lambda} + (\mathcal{I}B\,\beta_5)_{\Lambda\Omega} = 0, \tag{5.63}$$

which cannot be solved but via

$$\beta_5 = 0, \tag{5.64}$$

and from β_5 transformation (5.59) we get also

$$\beta_4 = 0. \tag{5.65}$$

Therefore, consistency among the duality transformations reduce the RF-sector Lagrangian (5.20) to

$$\mathcal{L}_4^{(RF)} = [\beta_1]_{\Lambda\Sigma}\, RF^\Lambda F^\Sigma + [\beta_2]_{\Lambda\Sigma}\, RF^\Lambda \tilde{F}^\Sigma, \tag{5.66}$$

with the couplings β_1 and β_2 that transform under duality as

$$[\delta\beta_1]_{\Lambda\Omega} = D_\Lambda{}^\Sigma\, [\beta_1]_{\Sigma\Omega} - [\beta_1]_{\Lambda\Sigma}\, A^\Sigma{}_\Omega - ([\beta_1]B\mathcal{R})_{\Lambda\Omega} + ([\beta_2]B\mathcal{I})_{\Lambda\Omega} + 2\,(\mathcal{I}B[\beta_2])_{(\Lambda\Omega)}\,, \tag{5.67}$$

$$[\delta\beta_2]_{\Lambda\Omega} = D_\Lambda{}^\Sigma\, [\beta_2]_{\Sigma\Omega} - [\beta_2]_{\Lambda\Sigma}\, A^\Sigma{}_\Omega - ([\beta_2]B\mathcal{R})_{\Lambda\Omega} - ([\beta_1]B\mathcal{I})_{\Lambda\Omega} - 2\,(\mathcal{I}B[\beta_1])_{(\Lambda\Omega)}\,. \tag{5.68}$$

Weyl transformation At this point, it is not necessary to find an explicit expression for β_1 and β_2 because we can re-adsorb these coefficients, and with them the entire RF-sector (5.66), via a field redefinition of the metric $g_{\mu\nu}$ called *Weyl transformation*:

$$g_{\mu\nu} \quad\longrightarrow\quad g'_{\mu\nu} = e^{2\Lambda} g_{\mu\nu}, \tag{5.69}$$

with Λ an arbitrary real function. This transformation acts on the Ricci scalar R as

$$R \quad\longrightarrow\quad R' = e^{-2\Lambda}\,(R - 6\Box\Lambda + 6\partial_\mu\Lambda\partial^\mu\Lambda)\,. \tag{5.70}$$

Assuming that Λ is a 2-derivatives function and exploiting a perturbative expansion to the 4-derivative order, the Weyl transformation on the Ricci scalar becomes

$$\begin{aligned} R \quad\longrightarrow\quad R' =& e^{-2\Lambda}\,(R - 6\Box\Lambda + 6\partial_\mu\Lambda\partial^\mu\Lambda) = \\ =& (1-2\Lambda)\,(R - 6\Box\Lambda + \cancel{6\partial_\mu\Lambda\partial^\mu\Lambda}) + \mathcal{O}(D^6) = \\ =& R - 2\Lambda R + \Box\Lambda + \mathcal{O}(D^6). \end{aligned} \tag{5.71}$$

The $\Box\Lambda$ term is irrelevant because it is a total derivative and vanishes in the total action, but we can indeed get rid of the entire RF-sector (5.66) the $2\Lambda R$ term, with the choice

$$\Lambda = \frac{1}{2}\left([\beta_1]_{AB}\, F^\Lambda F^\Sigma + [\beta_2]_{AB}\, F^\Lambda \tilde{F}^\Sigma\right). \tag{5.72}$$

Thus, no operator of the RF-sector appears in the resulting 4-derivatives, duality-invariant Lagrangian (5.2).

5.3.4 Duality analysis of the τ-sector

Now our analysis goes on with the τ-sector, which, together with the D-sector, is one of the most subtle sectors because of the presence of the derivatives of the complex scalar field τ, which we have placed inside the couplings, as described in (5.3).

The τ-sector Lagrangian is

$$\mathcal{L}_4^{(\tau)} = X_{AB}F^A F^B + \tilde{X}_{AB}F^A\tilde{F}^B + [Y_{AB}]_{\mu\nu}\, F^{A\,\mu\alpha}F^{B\,\nu}{}_\alpha + \left[\tilde{Y}_{AB}\right]_{\mu\nu} F^{A\,\mu\alpha}\tilde{F}^{B\,\nu}{}_\alpha, \tag{5.73}$$

with the coefficients having the following symmetries:

$$X_{AB} = X_{(AB)}, \qquad [Y_{AB}]_{\mu\nu} = [Y_{BA}]_{\nu\mu}, \tag{5.74}$$

$$\tilde{X}_{AB} = \tilde{X}_{(AB)}, \qquad \left[\tilde{Y}_{AB}\right]_{\mu\nu} = \left[\tilde{Y}_{AB}\right]_{[\mu\nu]}. \tag{5.75}$$

The corresponding dual field reads:

$$G^{(\tau)}_{\Lambda\,\mu\nu} = -4X_{\Lambda\Sigma}\tilde{F}^{\Sigma}_{\mu\nu} + 4\tilde{X}_{\Lambda\Sigma}F^{\Sigma}_{\mu\nu} + 2\left[(Y_{\Lambda\Sigma})\right]^{\rho\alpha} F^{\Sigma\,\sigma}{}_{\alpha}\epsilon_{\mu\nu\rho\sigma} - 4\left[\tilde{Y}_{[\Lambda\Sigma]}\right]^{\rho}{}_{[\mu|}F^{\Sigma}_{\rho|\nu]} \tag{5.76}$$

and the duality transformations of the various coefficients are:

- $$\begin{aligned}\delta X_{\Lambda\Omega} =& D_{\Lambda}{}^{\Sigma}X_{\Sigma\Omega} - A^{\Sigma}{}_{\Omega}X_{\Lambda\Sigma} - (B\mathcal{R})^{\Sigma}{}_{\Omega}X_{\Lambda\Sigma} - (\mathcal{R}BX)_{\Lambda\Omega} + (\tilde{X}B\mathcal{I})_{\Lambda\Omega} +\\ &+ (\mathcal{I}B\tilde{X})_{\Lambda\Omega};\end{aligned} \tag{5.77}$$

- $$\begin{aligned}\delta \tilde{X}_{\Lambda\Omega} =& D_{\Lambda}{}^{\Sigma}\tilde{X}_{\Sigma\Omega} - A^{\Sigma}{}_{\Omega}\tilde{X}_{\Lambda\Sigma} - (B\mathcal{R})^{\Sigma}{}_{\Omega}\tilde{X}_{\Lambda\Sigma} - (\mathcal{R}B\tilde{X})_{\Lambda\Omega} +\\ &- (XB\mathcal{I})_{\Lambda\Omega} - (\mathcal{I}BX)_{\Lambda\Omega} - \frac{1}{2}(B\mathcal{I})^{\Sigma}{}_{\Omega}\,(Y_{\Lambda\Sigma})^{\alpha}{}_{\alpha};\end{aligned} \tag{5.78}$$

- $$\begin{aligned}(\delta Y_{\Lambda\Omega})^{\rho\alpha} =& D_{\Lambda}{}^{\Sigma}\,(\delta Y_{\Sigma\Omega})^{\rho\alpha} - A^{\Sigma}{}_{\Omega}(Y_{\Lambda\Sigma})^{\rho\alpha} - (B\mathcal{R})^{\Sigma}{}_{\Omega}(Y_{\Lambda\Sigma})^{\rho\alpha} +\\ &- (\mathcal{R}B)_{\Lambda}{}^{\Sigma}\,(Y_{\Sigma\Omega})^{\rho\alpha} - 2(\mathcal{I}B)_{\Lambda}{}^{\Sigma}\left(\tilde{Y}_{[\Omega\Sigma]}\right)^{\rho\alpha};\end{aligned} \tag{5.79}$$

- $$\begin{aligned}\left(\delta\tilde{Y}_{[\Lambda\Omega]}\right)^{\alpha}{}_{[\mu|} =& D_{\Lambda}{}^{\Sigma}\left(\delta\tilde{Y}_{[\Sigma\Omega]}\right) - A^{\Sigma}{}_{\Omega}\left(\tilde{Y}_{[\Lambda\Sigma]}\right)^{\alpha}{}_{[\mu|} +\\ &- (B\mathcal{R})^{\Sigma}{}_{\Omega}\left(\tilde{Y}_{[\Lambda\Sigma]}\right)^{\alpha}{}_{[\mu|} - (\mathcal{R}B)_{\Lambda}{}^{\Sigma}\left(\tilde{Y}_{[\Sigma\Omega]}\right)^{\alpha}{}_{[\mu|} +\\ &- (B\mathcal{I})^{\Sigma}{}_{\Omega}\,(Y_{\Lambda\Sigma})^{\alpha}{}_{[\mu|} + (\mathcal{I}B)_{\Lambda}{}^{\Sigma}\,(Y_{\Omega\Sigma})^{\alpha}{}_{[\mu|}.\end{aligned} \tag{5.80}$$

Solution for the coefficients We notice that the transformations of the various coefficients do not seem to be influenced by the presence of the τ spacetime derivatives: the transformation patterns are in fact similar to the ones of $\mathcal{I}$ (4.38) and $\mathcal{R}$ (4.39), as well as the transformations of the other sectors. This fact suggests the idea that the derivative component of these coefficients appears always together with a (non-derivative) function of the complex scalar field in such a way that their combination is duality invariant. We know that such a term is given by

$$\frac{1}{(\mathfrak{Im}\tau)^2}\partial_\mu\bar{\tau}\partial_\nu\tau, \tag{5.81}$$

as we saw for example in (4.48).

Working then with this assumption, we first focus on the Y and $\tilde{Y}$ transformations (5.79) and (5.80). $\tilde{Y}$ transformation (5.80) resembles the transformation of $\mathcal{I}$ (4.38) in the terms proportional to $\tilde{Y}$ itself, so that a possible solution is obtained by asking that the terms in (5.80) that are proportional to Y cancel out:

$$(B\mathcal{I})^{\Sigma}{}_{\Omega}\,(Y_{\Lambda\Sigma})^{\alpha}{}_{\beta} = (\mathcal{I}B)_{\Lambda}{}^{\Sigma}\,(Y_{\Omega\Sigma})^{\alpha}{}_{\beta}. \tag{5.82}$$

This constraint is satisfied by setting Y to

$$(Y_{\Lambda\Sigma})_{\mu\nu} = \frac{\alpha}{(\mathfrak{Im}\tau)^2}\partial_{(\mu}\bar{\tau}\partial_{\nu)}\tau\,\mathcal{I}_{\Lambda\Sigma}, \tag{5.83}$$

where α is a scalar factor and the symmetrization in the (μ,ν) indices makes it compatible with Y symmetry properties (5.74). This solution is compatible with the transformation of Y in (5.79) if we require $\tilde{Y}$ to be symmetric in the gauge indices and this is indeed the case, since (5.83) was meant to make the $\tilde{Y}$ transformation (5.80) compatible with the $\mathcal{I}$ one (4.38).

Thus, a solution that sees $\tilde{Y}$ proportional to $\mathcal{I}$ and antisymmetric in the (μ,ν) indices, as prescribed by its symmetry (5.75), is given by

$$\left(\tilde{Y}_{\Lambda\Sigma}\right)_{\mu\nu} = \frac{\beta}{(\mathfrak{Im}\tau)^2}\left[i\,\partial_\mu\bar{\tau}\partial_\nu\tau + h.c.\right]\mathcal{I}_{\Lambda\Sigma}, \tag{5.84}$$

where β is, again, a numerical coefficient.

Let's now turn to the discussion of X and $\tilde{X}$ transformations (5.77) and (5.78). Again, the part of these transformations that is proportional, respectively, to X and $\tilde{X}$ follows the same pattern of $\mathcal{I}$ transformation (4.38). To have such a solution we need then to set to zero the other parts of the transformations:

$$(\tilde{X}B\mathcal{I})_{\Lambda\Omega} + (\mathcal{I}B\tilde{X})_{\Lambda\Omega} = 0, \tag{5.85}$$

$$(XB\mathcal{I})_{\Lambda\Omega} + (\mathcal{I}BX)_{\Lambda\Omega} + \frac{1}{2}(B\mathcal{I})^{\Sigma}{}_{\Omega}\,(Y_{\Lambda\Sigma})^{\alpha}{}_{\alpha} = 0. \tag{5.86}$$

Equation (5.85) immediately sets

$$\tilde{X} = 0, \tag{5.87}$$

while (5.86) can be rewritten, via (5.83), as

$$(XB\mathcal{I})_{(\Lambda\Omega)} = -\frac{\alpha}{4}\partial_\mu\bar{\tau}\partial^\mu\tau(\mathcal{I}B\mathcal{I})_{\Lambda\Omega}, \tag{5.88}$$

so that X is set to be

$$X_{\Lambda\Omega} = \frac{\alpha}{4}\partial_\mu\bar{\tau}\partial^\mu\tau\,\mathcal{I}_{\Lambda\Omega}, \tag{5.89}$$

which is in agreement with the X transformation (5.77) after imposing (5.87).

Thus, the resulting τ-sector is

$$\begin{aligned}\mathcal{L}_4^{(\tau)} =& -\frac{\alpha}{4}\frac{1}{(\mathfrak{Im}\tau)^2}\partial_\mu\bar{\tau}\partial^\mu\tau\,\mathcal{I}_{\Lambda\Sigma}F^\Lambda F^\Sigma + \frac{\alpha}{(\mathfrak{Im}\tau)^2}\partial_\mu\bar{\tau}\partial_\nu\tau\,\mathcal{I}_{\Lambda\Sigma}F^{\Lambda\,\mu\alpha}F^{\Sigma\,\nu}{}_{\alpha} + \\ &+ \frac{\beta}{(\mathfrak{Im}\tau)^2}\mathcal{I}_{\Lambda\Sigma}\left(i\,\partial_\mu\bar{\tau}\partial_\nu\tau F^{\Lambda\,\mu\alpha}\tilde{F}^{\Sigma\,\nu}{}_{\alpha} + h.c.\right).\end{aligned} \tag{5.90}$$

A remark is now in order. Back to the solutions (5.83) and (5.84) for the Y and $\tilde{Y}$ coefficients, their derivation started by imposing (5.82) in order to remove the extra terms from transformation (5.80) and identify then $\tilde{Y}$ with $\mathcal{I}$. Equivalently, we could have considered the Y transformation (5.79) first and tried to reproduce some transformation pattern by acting on the extra $\tilde{Y}$ term. In this case we can indeed try to identify, as in (5.83), Y with $\mathcal{I}$ and to do so is sufficient to require $\tilde{Y}$ to be symmetric in the gauge indices. However, this $(Y,\tilde{Y})$ solution exactly solve also the $\tilde{Y}$ transformation (5.80) without the need to specify further the explicit expression of $\tilde{Y}$.

Although this is indeed a more general solution for the τ-sector coefficients, of which

(5.84) can be seen as a subcase, it also features an undetermined $\tilde{Y}$ which is not fully understood and also spoils the final result. Thus, we decide to proceed with the more specific solution (5.84).

5.3.5 Duality analysis of the D-sector

The last sector to be analyzed is the one involving the operators containing derivatives of the gauge fields (as well as of the complex scalar). The D-sector Lagrangian is

$$\mathcal{L}_4^{(D)} = Z^{(1)}_{AB}(D_\mu F^{A\,\mu\alpha})(D^\nu F^B_{\nu\alpha}) + \left[Z^{(2)}_\mu\right]_{AB} F^{A\,\mu\alpha} D^\nu F^B_{\nu\alpha} + \left[\tilde{Z}^{(2)}_\mu\right]_{AB} \tilde{F}^{A\,\mu\alpha} D^\nu F^B_{\nu\alpha}, \quad (5.91)$$

where the only coefficient having a symmetry property is $Z^(1)$, which is symmetric in the gauge indices:

$$Z^{(1)}_{AB} = Z^{(1)}_{BA}. \quad (5.92)$$

The dual field results to be

$$\begin{aligned}
G^{(D)}_{\Lambda\,\mu\nu} =& \left[2\partial^\rho \left(Z^{(1)} - \left(Z^{(2)}\right)^\rho\right)\right]_{\Lambda\Sigma} D_\alpha F^{\Sigma\,\alpha\sigma}\epsilon_{\mu\nu\rho\sigma} + 4\left[\tilde{Z}^{(2)}_{(\Lambda\Sigma)}\right]_{[\mu|} D^\alpha F^\Sigma_{\alpha|\nu]} + \\
&+ 2\left[Z^{(1)}\right]_{\Lambda\Sigma} D^\rho D_\beta F^{\Sigma\,\beta\sigma}\epsilon_{\mu\nu\rho\sigma} + \left[Z^{(2)}_{\Sigma\Lambda}\right]_\alpha D^\rho F^{\Sigma\,\alpha\sigma}\epsilon_{\mu\nu\rho\sigma} - \left[\tilde{Z}^{(2)}_{\Sigma\Lambda}\right]_\alpha D^\alpha F^\Sigma_{\mu\nu} + \\
&+ \left[\partial^\rho\left(Z^{(2)}_{\Sigma\Lambda}\right)_\alpha\right] F^{\Sigma\,\alpha\sigma}\epsilon_{\mu\nu\rho\sigma} - \left[\partial^\alpha\left(\tilde{Z}^{(2)}_{\Sigma\Lambda}\right)_\alpha\right] F^\Sigma_{\mu\nu} + 2\left[\partial^\alpha\left(\tilde{Z}^{(2)}_{\Sigma\Lambda}\right)_{[\mu|}\right] F^\Sigma_{\alpha|\nu]}
\end{aligned} \quad (5.93)$$

and because of the presence of operators with derivatives of the gauge fields the duality transformation is now more involved. Making this calculation explicitly, we obtain:

$$\begin{aligned}
\delta G^{(D)}_{\Lambda\,\mu\nu} =& D_\Lambda{}^\Sigma G^{(D)}_{\Sigma\,\mu\nu} \\
& | \\
=& \Big[\left(2\partial^\rho(\delta Z^{(1)\Lambda\Omega}) - (\delta Z^{(2)}_{\Lambda\Omega})^\rho\right) + (A+B\mathcal{R})^\Sigma{}_\Omega\left(2\partial^\rho(Z^{(1)}_{\Lambda\Sigma}) - (Z^{(2)}_{\Lambda\Sigma})^\rho\right) + \\
&+(\mathcal{R}B)_\Lambda{}^\Sigma\left(2\partial^\rho(Z^{(1)}_{\Sigma\Omega}) - (Z^{(2)}_{\Sigma\Omega})^\rho\right) + 2Z^{(1)}_{\Lambda\Sigma}(B\partial^\rho\mathcal{R})^\Sigma{}_\Omega + \\
&+2(\mathcal{I}B)_\Lambda{}^\Sigma\left(\tilde{Z}^{(2)}_{(\Sigma\Omega)}\right)^\rho\Big] D_\alpha F^{\Omega\,\alpha\sigma}\epsilon_{\mu\nu\rho\sigma} + \\
&+ 4\Big[\left(\delta\tilde{Z}^{(2)}_{(\Lambda\Omega)}\right)_{[\mu|} + (A+B\mathcal{R})^\Sigma{}_\Sigma\left(\tilde{Z}^{(2)}_{(\Lambda\Sigma)}\right)_{[\mu|} + (\mathcal{R}B)_\Lambda{}^\Sigma\left(\tilde{Z}^{(2)}_{(\Sigma\Omega)}\right)_{[\mu|} + \\
&+\frac{1}{2}Z^{(1)}_{\Lambda\Sigma}(B\partial_{[\mu|}\mathcal{I})^\Sigma{}_\Omega + \frac{1}{2}(B\mathcal{I})^\Sigma{}_\Omega\left(Z^{(2)}_{\Sigma\Lambda}\right)_{[\mu|} + \\
&-\frac{1}{2}(\mathcal{I}B)_\Lambda{}^\Sigma\left(2\partial_{[\mu|}(Z^{(1)}) - (Z^{(2)})_{[\mu|}\right)_{\Sigma\Omega}\Big] D^\alpha F^\Omega_{\alpha|\nu]} + \\
&+ 2\left[\delta Z^{(1)}_{\Lambda\Omega} + (A+B\mathcal{R})^\Sigma{}_\Omega Z^{(1)}_{\Lambda\Sigma} + (\mathcal{R}B)_\Lambda{}^\Sigma Z^{(1)}_{\Sigma\Omega}\right] D^\rho D_\alpha F^{\Omega\,\alpha\sigma}\epsilon_{\mu\nu\rho\sigma} + \\
&+ \Big[\left(\delta Z^{(2)}_{\Omega\Lambda}\right)_\alpha + (A+B\mathcal{R})^\Sigma{}_\Omega\left(Z^{(2)}_{\Sigma\Lambda}\right)_\alpha + (\mathcal{R}B)_\Lambda{}^\Sigma\left(Z^{(2)}_{\Omega\Sigma}\right)_\alpha + \\
&+2Z^{(1)}_{\Lambda\Sigma}(B\partial_\alpha\mathcal{R})^\Sigma{}_\Omega\Big] D^\rho F^{\Omega\,\alpha\sigma}\epsilon_{\mu\nu\rho\sigma} + \\
&- \Big[\left(\delta\tilde{Z}^{(2)}_{\Omega\Lambda}\right)_\alpha + (A+B\mathcal{R})^\Sigma{}_\Omega\left(\tilde{Z}^{(2)}_{\Sigma\Lambda}\right)_\alpha + (\mathcal{R}B)^\Sigma_\Lambda\left(\tilde{Z}^{(2)}_{\Omega\Sigma}\right)_\alpha + (B\mathcal{I})^\Sigma{}_\Omega\left(Z^{(2)}_{\Sigma\Lambda}\right)_\alpha + \\
&+2Z^{(1)}_{\Lambda\Sigma}(B\partial_\alpha\mathcal{I})^\Sigma{}_\Omega\Big] D^\alpha F^\Omega_{\mu\nu} +
\end{aligned}$$

$$
\begin{aligned}
&+\Big[\partial^\rho\left(\delta Z^{(2)}_{\Omega\Lambda}\right)_\alpha+(A+B\mathcal{R})^\Sigma{}_\Omega\partial^\rho\left(Z^{(2)}_{\Sigma\Lambda}\right)_\alpha+(\mathcal{R}B)_\Lambda{}^\Sigma\partial^\rho\left(Z^{(2)}_{\Omega\Sigma}\right)_\alpha+\\
&\quad+(B\partial_\alpha\mathcal{R})\left(2\partial^\rho(Z^{(1)}_{\Lambda\Sigma})-(Z^{(2)}_{\Lambda\Sigma})^\rho\right)+2(B\partial^\rho\partial^\alpha\mathcal{R})^\Sigma{}_\Omega Z^{(1)}_{\Lambda\Sigma}+\\
&\quad+(B\partial^\rho\mathcal{R})^\Sigma{}_\Omega\left(Z^{(2)}_{\Lambda\Sigma}\right)_\alpha+(\mathcal{J}B)_\Lambda{}^\Sigma\left(\partial_\alpha(\tilde{Z}^{(2)}_{\Omega\Sigma})^\rho\right)\Big]F^{\Omega\,\alpha\sigma}\epsilon_{\mu\nu\rho\sigma}+\\
&-\Big[\partial^\alpha\left(\delta\tilde{Z}^{(2)}_{\Omega\Lambda}\right)_\alpha+(A+B\mathcal{R})^\Sigma{}_\Omega\partial^\alpha\left(\tilde{Z}^{(2)}_{\Sigma\Lambda}\right)_\alpha+(\mathcal{R}B)_\Lambda{}^\Sigma\partial^\alpha\left(\tilde{Z}^{(2)}_{\Omega\Sigma}\right)_\alpha+\\
&\quad+(B\mathcal{J})^\Sigma{}_\Omega\partial^\alpha\left(Z^{(2)}_{\Sigma\Lambda}\right)_\alpha+(B\partial^\alpha\mathcal{R})^\Sigma{}_\Omega\left(\tilde{Z}^{(2)}_{\Sigma\Lambda}\right)_\alpha+2(B\Box\mathcal{J})^\Sigma{}_\Omega Z^{(1)}_{\Lambda\Sigma}+\\
&\quad+\left(2\partial^\alpha(Z^{(1)}_{\Lambda\Sigma})-(Z^{(2)}_{\Lambda\Sigma})^\alpha\right)(B\partial_\alpha\mathcal{J})^\Sigma{}_\Omega+(B\partial^\alpha\mathcal{J})^\Sigma{}_\Omega\left(Z^{(2)}_{\Sigma\Lambda}\right)_\alpha\Big]F^\Omega_{\mu\nu}+\\
&+2\Big[\partial^\alpha\left(\delta\tilde{Z}^{(2)}_{\Omega\Lambda}\right)_{[\mu|}+(A+B\mathcal{R})^\Sigma{}_\Omega\partial^\alpha\left(\tilde{Z}^{(2)}_{\Sigma\Lambda}\right)_{[\mu|}+(\mathcal{R}B)_\Lambda{}^\Sigma\partial^\alpha\left(\tilde{Z}^{(2)}_{\Omega\Sigma}\right)_{[\mu|}+\\
&\quad+(B\mathcal{J})^\Sigma{}_\Omega\partial^\alpha\left(Z^{(2)}_{\Sigma\Lambda}\right)_{[\mu|}+2(B\partial^\alpha\mathcal{R})^\Sigma{}_\Omega\left(\tilde{Z}^{(2)}_{(\Lambda\Sigma)}\right)_{[\mu|}+\\
&\quad+\left(2\partial^\alpha(Z^{(1)}_{\Lambda\Sigma})-(Z^{(2)}_{\Lambda\Sigma})^\alpha\right)(B\partial_{[\mu|}\mathcal{J})^\Sigma{}_\Omega+\frac{1}{2}(B\partial^\alpha\mathcal{J})^\Sigma{}_\Omega\left(Z^{(2)}_{\Sigma\Lambda}\right)_{[\mu|}+\\
&\quad+Z^{(1)}_{\Lambda\Sigma}(B\partial^\alpha\partial_{[\mu|}\mathcal{J})^\Sigma{}_\Omega-(\mathcal{J}B)_\Lambda{}^\Sigma\left(\partial_{[\mu|}(Z^{(2)}_{\Omega\Sigma})^\alpha\Big]\right)F^\Omega_{\alpha|\nu]}+\\
&+2\Big[(B\mathcal{J})^\Sigma{}_\Omega\partial^\alpha\left(\tilde{Z}^{(2)}_{\Sigma\Lambda}\right)_{[\mu|}+2(B\partial^\alpha\mathcal{J})^\Sigma{}_\Omega\left(\tilde{Z}^{(2)}_{(\Lambda\Sigma)}\right)_{[\mu|}\Big]\tilde{F}^\Omega_{\alpha|\nu]}+\\
&-\Big[(B\partial^\alpha\mathcal{J})^\Sigma{}_\Omega\left(\tilde{Z}^{(2)}_{\Sigma\Lambda}\right)_\alpha+(B\mathcal{J})^\Sigma{}_\Omega\partial^\alpha\left(\tilde{Z}^{(2)}_{\Sigma\Lambda}\right)_\alpha+(\mathcal{J}B)_\Lambda{}^\Sigma\left(\partial^\alpha(\tilde{Z}^{(2)}_{\Omega\Sigma})_\alpha\right)\Big]\tilde{F}^\Omega_{\mu\nu}+\\
&-\Big[(B\mathcal{J})^\Sigma{}_\Omega\left(\tilde{Z}^{(2)}_{\Sigma\Lambda}\right)_\alpha\Big]D^\alpha\tilde{F}^\Omega_{\mu\nu}-4\Big[(\mathcal{J}B)_\Lambda{}^\Sigma Z^{(1)}_{\Sigma\Omega}\Big]D_{[\mu|}D^\alpha F^\Omega_{\alpha|\nu]}+\\
&-2\Big[(\mathcal{J}B)_\Lambda{}^\Sigma\left(Z^{(2)}_{\Omega\Sigma}\right)^\alpha\Big]D_{[\mu|}F^\Omega_{\alpha|\nu]}-\frac{1}{2}\Big[(\mathcal{J}B)_\Lambda{}^\Sigma\left(\tilde{Z}^{(2)}_{\Omega\Sigma}\right)^\alpha\Big]D_\alpha F^{\Omega\,\rho\sigma}\epsilon_{\mu\nu\rho\sigma}.
\end{aligned}
\tag{5.94}
$$

Solution for the coefficients We focus on the last four lines of this very complicated transformation:

$$
\begin{aligned}
&2\Big[(B\mathcal{J})^\Sigma{}_\Omega\partial^\alpha\left(\tilde{Z}^{(2)}_{\Sigma\Lambda}\right)_{[\mu|}+2(B\partial^\alpha\mathcal{J})^\Sigma{}_\Omega\left(\tilde{Z}^{(2)}_{(\Lambda\Sigma)}\right)_{[\mu|}\Big]\tilde{F}^\Omega_{\alpha|\nu]}+\\
&-\Big[(B\partial^\alpha\mathcal{J})^\Sigma{}_\Omega\left(\tilde{Z}^{(2)}_{\Sigma\Lambda}\right)_\alpha+(B\mathcal{J})^\Sigma{}_\Omega\partial^\alpha\left(\tilde{Z}^{(2)}_{\Sigma\Lambda}\right)_\alpha+(\mathcal{J}B)_\Lambda{}^\Sigma\left(\partial^\alpha(\tilde{Z}^{(2)}_{\Omega\Sigma})_\alpha\right)\Big]\tilde{F}^\Omega_{\mu\nu}+\\
&-\Big[(B\mathcal{J})^\Sigma{}_\Omega\left(\tilde{Z}^{(2)}_{\Sigma\Lambda}\right)_\alpha\Big]D^\alpha\tilde{F}^\Omega_{\mu\nu}-4\Big[(\mathcal{J}B)_\Lambda{}^\Sigma Z^{(1)}_{\Sigma\Omega}\Big]D_{[\mu|}D^\alpha F^\Omega_{\alpha|\nu]}+\\
&-2\Big[(\mathcal{J}B)_\Lambda{}^\Sigma\left(Z^{(2)}_{\Omega\Sigma}\right)^\alpha\Big]D_{[\mu|}F^\Omega_{\alpha|\nu]}-\frac{1}{2}\Big[(\mathcal{J}B)_\Lambda{}^\Sigma\left(\tilde{Z}^{(2)}_{\Omega\Sigma}\right)^\alpha\Big]D_\alpha F^{\Omega\,\rho\sigma}\epsilon_{\mu\nu\rho\sigma}.
\end{aligned}
\tag{5.95}
$$

We can see in fact that these operators do not appear in the dual field of the D-sector (5.93). They're indeed an example of those operators which are not independent with respect to the considered set (5.1). Such operators appeared also in the other sectors' transformations and we properly took them into account by exploiting the identities that connect them to the operators of (5.1), as described in Appendix B.

However, in this case we cannot exploit the same procedure because some of the operators of (5.95) are connected to those appearing in the dual field (5.93) via an integration by parts. This means that in the identity we should exploit to account for them it appears a total derivative that completely spoils the duality transformation, since there are no counterparts of such terms in the G-side of the transformation.

Thus, the only way to avoid this problem is to set to zero the coefficients of these operators, which though means that all the coefficients of the D-sector result to be vanishing:

$$Z^{(1)} = 0 = Z^{(2)}_\mu = \tilde{Z}^{(2)}_\mu . \tag{5.96}$$

5.4 Resulting duality-preserving Lagrangian

The duality analysis of the different sectors (5.18) of the gauge part of the 4-derivatives Lagrangian (5.2) allows to determine which operators of the set (5.1) give rise to a duality-invariant extension of the starting model (4.1) and, with them, an explicit expression for the associated non-minimal couplings. The resulting duality-preserving 4-derivatives Lagrangian is

$$\begin{aligned}\mathcal{L}_4 =& \frac{1}{(\mathfrak{Im}\tau)^4}\left[\lambda_1(\partial_\mu\bar{\tau}\partial^\mu\tau)^2 + \lambda_2|\partial_\mu\tau\partial^\mu\tau|^2\right] - \frac{\alpha}{4}\frac{1}{(\mathfrak{Im}\tau)^2}\partial_\mu\bar{\tau}\partial^\mu\tau\,\mathfrak{I}_{\Lambda\Sigma}F^\Lambda F^\Sigma + \\ &+ \frac{\alpha}{(\mathfrak{Im}\tau)^2}\partial_\mu\bar{\tau}\partial_\nu\tau\,\mathfrak{I}_{\Lambda\Sigma}F^{\Lambda\,\mu\alpha}F^{\Sigma\,\nu}{}_\alpha + \frac{\beta}{(\mathfrak{Im}\tau)^2}\mathfrak{I}_{\Lambda\Sigma}\Big(i\,\partial_\mu\bar{\tau}\partial_\nu\tau F^{\Lambda\,\mu\alpha}\tilde{F}^{\Sigma\,\nu}{}_\alpha + h.c.\Big) + \\ &+ \eta\,\mathfrak{I}_{AC}\mathfrak{I}_{BD}\Big[(F^AF^B)(F^CF^D) + (F^A\tilde{F}^B)(F^C\tilde{F}^D)\Big]. \end{aligned} \tag{5.97}$$

Let's describe now the properties characterizing our solution. First of all, duality reduced significantly the set of allowed operators (see (5.1)). In particular, the operators with derivatives of the gauge fields are excluded from this set dynamically, i.e. not by assumption, and this was possible thanks to the perturbative character of our duality analysis.

Moreover, also the allowed couplings result to be strongly constrained. Focusing on the 4-derivatives gauge sector (5.4), we have seen that the components carrying the gauge indices have been fixed for all the couplings to be proportional to $\mathfrak{I}$. This fact is quite remarkable. First, because it establish a direct connection between the couplings of the 2 and 4-derivatives orders, indeed suggesting the idea that the one we're dealing with is the perturbative expansion of an exactly duality-invariant theory, which supports the meaning of the perturbative expansion itself. Then, because it makes manifest that the duality group characterizing the invariance of the gauge fields' EoM and BI is the full EM duality group $\mathrm{Sp}(4,\mathbb{R})$ presented in (4.42), (4.43). A very important property of (5.97) is in fact that we were able to derive it working with the full $\mathrm{Sp}(4,\mathbb{R})$, namely without any $\mathrm{SL}(2,\mathbb{R})$ additional and simplifying assumption. This means that the 4-derivatives Lagrangian (5.97) holds in every symplectic frame we can identify via a duality transformation.

It's important to highlight also that the 4-derivatives Lagrangian (5.97) we found may not be the only possible solution for the couplings' duality transformations. To derive it we followed the strategy to search for the $\mathfrak{I}$ transformation pattern (4.38) in the duality transformations of the various coefficients and then impose constraints over them in order to realize such a solution. This though does not exclude the possibility that other solutions may be allowed[1]. However, the only way we have to identify an explicit expression for the higher-order coefficients is to rely on those of the 2-derivatives Lagrangian (4.1) and their duality transformations (4.38) and (4.39): although possible, the alternative solutions for the 4-derivatives Lagrangian would lack of a precise determination of its couplings, of which we would know only the duality transformation patterns. This does not make them good candidates for applications such as the test of the WGC we had in mind at the beginning of

[1] This does not hold properly for the RF-sector (5.20), in which first the constraints (5.62), (5.65) and (5.64) followed strictly from formal consistency among the duality transformations, second the β_1 and β_2 coefficients were removed via the Weyl transformation (5.69), (5.72) without the need to solve their transformation.

this discussion.

On the contrary, the Lagrangian (5.97) is perfect to perform such test. In fact, it depends on the five undetermined scalar coefficients $\{\eta,\, \alpha,\, \beta,\, \lambda_1,\, \lambda_2\}$ that are precisely the subject of the WGC constraints.

Chapter 6

Non-minimal couplings and the Weak Gravity Conjecture

In chapters 4 and 5 we discussed in detail the duality group of Lagrangian (4.1) and how it can be used to determine its 4-derivatives correction, which was found to be Lagrangian (5.97) and is the main result of this thesis work. In particular, it is the perturbative analysis we applied to obtained it that strongly support our result. As we described, the strategy we have adopted was to require the self consistency of the duality transformation (4.30) order by order in the higher-derivative expansion of the Lagrangian under examination. This means that no further assumption was needed on the general structure of the resulting theory, which was the subtle point of Cano and Múrcia approach in [29]. The analysis of the 2-derivatives order (4.1) provided the EM duality group of the theory (4.42), (4.43) and the transformation rules (4.38) and (4.39) of the $\mathcal{I}$ and $\mathcal{R}$ couplings. These results have then be applied to the analysis of the 4-derivatives order (5.2), (5.4), (5.7) and allowed to fix the higher-order couplings in a duality-preserving way, yielding eventually Lagrangian (5.97).

Now that we indeed determined a duality-preserving extension of the starting Lagrangian (4.1) we can turn to the discussion of its connection with the Weak Gravity Conjecture (1.1). In the context of pure Einstein–Maxwell theory (1.4), we have seen in Chapter 2 how positivity bounds on the scattering amplitudes [20–22] equivalently realize the Electric WGC (1.1) [18, 19]. Instead, in the context of a theory beyond Einstein–Maxwell, like the model (4.1) we studied, to realize this equivalence one needs to add to the positivity bounds also EM duality constraints [33, 34].

Especially [34] represents an important basis of comparison for this analysis. In fact, also in [34] a duality-preserving, 4-derivatives extension of Lagrangian (4.1) is proposed:

$$\begin{aligned}\mathcal{L}_{\mathrm{S}}^{(4)} =&(\mathfrak{Im}\tau)^2\alpha_{ABCD}\left[(F^AF^B)(F^CF^D)+(F^A\tilde{F}^B)(F^C\tilde{F}^D)\right]+\\ &+\frac{1}{\mathfrak{Im}\tau}\alpha_{AB}\left[-\frac{1}{8}\partial_\mu\bar{\tau}\partial^\mu\tau F^AF^B+\frac{1}{2}\partial_\mu\bar{\tau}\partial_\nu\tau F^{A\,\mu\alpha}F^{B\,\nu}{}_{\alpha}+\right]+\\ &+\frac{1}{\mathfrak{Im}\tau}\alpha_{AB}\left[\frac{1}{4}\left(i\,\partial_\mu\bar{\tau}\partial_\nu\tau F^{A\,\mu\alpha}\tilde{F}^{B\,\nu}{}_{\alpha}+h.c.\right)\right]+\\ &+\frac{1}{(\mathfrak{Im}\tau)^4}\left[\alpha_1(\partial_\mu\bar{\tau}\partial^\mu\tau)^2+\alpha_2|\partial_\mu\tau\partial^\mu\tau|^2\right],\end{aligned} \tag{6.1}$$

where the α-coefficients are scalars and have the following symmetries:

$$\alpha_{abcd} = \alpha_{bacd} = \alpha_{abdc} = \alpha_{cdab}, \tag{6.2}$$

$$\alpha_{ab} = \alpha_{ba}. \tag{6.3}$$

However, as anticipated at the beginning of Chapter 4, this extension of (4.1) seems to have been constructed just by adding operators that are exactly invariant under a duality transformation but this is not an accurate way of deal with duality. Indeed, EM duality is a symmetry of the EoM and BI of the gauge fields and *not* of the Lagrangian that describes them. Therefore, the higher-order theory (6.1) proposed in [34] is not, in principle, the most general duality-invariant, 4-derivatives correction to (4.1). Also, it is defined only in one specific symplectic frame, the one in which the $\mathcal{I}$ and $\mathcal{R}$ couplings of (4.1) become proportional to identity matrix according to

$$\mathcal{I} = \mathfrak{Im}\tau\, \mathbb{1}, \tag{6.4}$$

$$\mathcal{R} = \mathfrak{Re}\tau\, \mathbb{1}. \tag{6.5}$$

Indeed, the 4-derivatives extension (5.97) we derived represents an improvement to (6.1) in both its issues: first, it was determined with a well-defined procedure, in full agreement with the definition of EM duality as a symmetry of the EoM and BI of the gauge fields; second, it's definition is independent of the choice of symplectic frame.

Despite the structural uncertainties of (6.1), we see that its expression and the one of (5.97) are indeed similar. To properly compare them, we make use of its freedom in the symplectic frame choice and write our 4-derivatives Lagrangian in the frame (6.4)-(6.5) in which (6.1) is defined. In this way we can make an identification between the coefficients $\{\eta, \alpha, \beta, \lambda_1, \lambda_2\}$ of (5.97) and the α-ones of (6.1) and this allows to re-read the results of [34] - and understand how they change - from the more rigorous point of view of our result (5.97).

In particular, after introducing (6.1), in [34] the authors proceed by computing the correspondent correction to the charge-to-mass ratio of an extremal black hole solution of the theory and test then the Electric WGC as described in (1.11). To do so, they exploit a thermodynamic formulation of the black hole and carry out the calculations by following the procedure described in [41]. Further, they compute the positivity bounds associated to the coefficients of (6.1) that enter the correction to the charge-to-mass ratio and show that they are sufficient conditions to realize the Electric WGC (1.1) because they make this correction positive. What we do is to start back from the charge-to-mass ratio computed in [34] rewritten in terms of the coefficients of (5.97) and we study in detail which values of these coefficients realize the Electric WGC. Similarly, we then translate to the coefficients of (5.97) also the positivity bounds reported in [34] and show that they set them to a configuration in agreement with the WGC.

This chapter is organized in the following way. In the first section we discuss the black hole solution and the higher-order corrections to its charge-to-mass ratio presented in [34]. Next, in the second section we describe the symplectic frame in which (6.1) is defined (see (6.4), (6.5)), we specify (5.97) in this frame and we make the identification between the coefficients of the two theories. This allows to make use of the black hole solution of [34] to study the WGC constraints on the coefficients of (5.97): this analysis is the topic of the third section.

6.1 Higher-order corrections to the charge-to-mass ratio

In this first section we focus on the black hole solution of Lagrangian (4.1) that is presented in [34] and on the higher-order correction to the associated charge-to-mass ratio that come from the higher-derivative Lagrangian (6.1). To compute such corrections, the authors exploit a thermodynamic description of the black hole and the physical quantities associated to it, following the procedure they describe in [41].

6.1.1 The black hole solution

The Einstein equations (4.5) associated with the 2-derivatives Lagrangian (4.1) admit the following dyonic black hole solution:

$$\begin{aligned} ds^2 =& -f(r)dt^2 + \frac{1}{f(r)}dr^2 + (r+\kappa_1)(r+\kappa_2)\left(d\theta^2 + \sin^2\theta\varphi^2\right), \\ f(r) =& \frac{r(r-2\xi)}{(r+\kappa_1)(r+\kappa_2)}, \end{aligned} \tag{6.6}$$

together with

$$\mathfrak{Re}\tau = 0, \qquad \mathfrak{Im}\tau = \frac{r+\kappa_1}{r+\kappa_2}, \tag{6.7}$$

as solution for the τ EoM (4.4) and with

$$A_1 = -\frac{q}{r+\kappa_1}\,dt, \qquad A_2 = -\,p\cos\theta\,d\varphi, \tag{6.8}$$

for what concerns instead the vector potentials associated to F^1 and F^2 and their EoM (4.3). q and p are the reduced electric and magnetic charges, namely

$$Q = 4\pi q, \qquad P = 4\pi p, \tag{6.9}$$

and are related to the constants ξ, $\kappa_1 > 0$ and $\kappa_2 > 0$ that appear in the metric as

$$q^2 = \kappa_1(\kappa_1 + 2\xi), \qquad p^2 = \kappa_2(\kappa_2 + 2\xi). \tag{6.10}$$

From the metric (6.6) we see that the two black hole horizons corresponds to $r = 0$ and $r = 2\xi$, so that the extremal configuration is achieved via $\xi \to 0$.

6.1.2 Thermodynamic description

This black hole can be characterized via a thermodynamic description, presented in [41]. The reason why such an approach is applied is that it allows to compute the higher-order corrections to the physical quantities associated to the black hole in a simple way.

The building block of this thermodynamic picture is the fact that we can identify the Euclidean version I_{E} of the action under examination with the free energy H in the grand canonical ensemble:

$$H = TI_{\mathrm{E}}, \tag{6.11}$$

where T is the temperature of the black hole. The free energy H acts as a bridge between the (Euclidean) action and the thermodynamic quantities characterizing the black hole. First of all, the mass M and the entropy S of the black hole are connected by the First Law of

Thermodynamics as

$$dM = TdS + \Phi dQ + \Psi dP, \tag{6.12}$$

where Φ is the electric potential evaluated at the outer horizon and Ψ its magnetic counterpart (see A_1 and A_2 in (6.8) and [34, 41]). Then, from the definition of free energy and from the First Law (6.12) we have

$$H \equiv M - TS - Q\Phi, \tag{6.13}$$

$$dH = -\, SdT - Qd\Phi + \Psi dP. \tag{6.14}$$

Equation (6.14) tells that the grand canonical ensemble we have introduced is described in terms of the three variables T, Φ and P and, moreover, provides, together with (6.13), an explicit procedure to compute the physical quantities characterizing the black hole, such as

$$S = -\left(\frac{\partial H}{\partial T}\right)\bigg|_{\Phi,\,P}, \qquad Q = -\left(\frac{\partial H}{\partial \Phi}\right)\bigg|_{T,\,P}, \qquad \Psi = \left(\frac{\partial H}{\partial Q}\right)\bigg|_{T,\,\Phi}. \tag{6.15}$$

The crucial point is that the free energy H is connected to the action under examination, as stated in (6.11). Indeed, in all this description we have not specified the perturbative order at which the action should be computed and in fact the fundamental relation (6.11) does hold also when higher-derivative corrections to the leading order action are included[1].

Let's then explore further the case of an action including also higher-order terms. Following the perturbative expansion, we can write its Euclidean version as

$$I_{\mathrm{E}} = I_2 + \Delta I + I_\partial, \tag{6.16}$$

where I_2 is the 2-derivatives, starting action, ΔI denotes all the included higher-derivatives correction and I_∂ is a boundary term (taken to be the Gibbons-Hawking-York term [42]) that is necessary to have a well-defined Euclidean action, free of divergences.

From (6.16) we immediately see that this thermodynamic approach allows to compute the higher-derivative corrections in a very simple way because they're already included into equations (6.14) and (6.15). For instance, in the case of the entropy we have

$$\begin{aligned} S = &-\left(\frac{\partial H}{\partial T}\right)\bigg|_{\Phi,\,P} = -\left(\frac{\partial (T I_{\mathrm{E}})}{\partial T}\right)\bigg|_{\Phi,\,P} = \\ = &-\left(\frac{\partial\,(T(I_2 + I_\partial))}{\partial T}\right)\bigg|_{\Phi,\,P} - \left(\frac{\partial T \Delta I}{\partial T}\right)\bigg|_{\Phi,\,P} = \\ = &-\left(\frac{\partial\,(T(I_2 + I_\partial))}{\partial T}\right)\bigg|_{\Phi,\,P} - \left(\frac{\partial T I_4}{\partial T}\right)\bigg|_{\Phi,\,P} + \cdots \equiv S_2 + \Delta S. \end{aligned} \tag{6.17}$$

This computation naturally splits into the different higher-derivative contributions, so that all the corrections to the physical quantities characterizing the black hole can be derived in the same way from I_{E}.

The 4-derivatives order is particularly interesting because the full Euclidean action, describing a set of fields that we call $\mathcal{F}$, is such that (see [44]):

$$I_{\mathrm{E}}\left[\mathcal{F}\right] = I_{\mathrm{E}}\left[\mathcal{F}_2\right] + \mathcal{O}(\lambda^2), \tag{6.18}$$

[1] This statement is true if the entropy S is the Wald entropy [43].

where $\mathcal{F}_2$ denotes the solution of the EoM of the 2-derivatives action and λ the generic higher-order coefficient driving the perturbative expansion. Thus, from this property it follows that the corrections computed at the 4-derivatives order along the solution of the leading order EoM - in the case of theory (4.1), given by the Euclidean version of (6.6), (6.7) and (6.8) - are, at this order, exact. In other words, the 4-derivatives corrections are fully characterized by evaluating I_4 along $\mathcal{F}_2$.

Let's now outline the results of applying such thermodynamic procedure to theory (4.1) and its 4-derivatives extension (6.1) proposed in [34].

2-derivatives order The free energy associated to Lagrangian (4.1) results

$$H(T,\Phi,P) = \frac{1-\Phi^2}{2T} + \frac{P^2 T}{2(1-\Phi^2)}, \tag{6.19}$$

so that from equation (6.14) we obtain the electric charge and the entropy of the black hole, while from (6.13) we then obtain its mass:

$$Q_2(T,\Phi,P) = \frac{\Phi}{T}\left(1 - \frac{P^2T^2}{(1-\Phi^2)^2}\right), \tag{6.20}$$

$$S_2(T,\Phi,P) = \frac{1-\Phi^2}{2T^2}\left(1 - \frac{P^2T^2}{(1-\Phi^2)^2}\right), \tag{6.21}$$

$$M_2(T,\Phi,P) = \frac{1}{T}\left(1 - \frac{\Phi^2P^2T^2}{(1-\Phi^2)^2}\right). \tag{6.22}$$

One can indeed check that these expression of S, Q and M match the ones that can be read from the metric (6.6) (see [34]).

4-derivatives order Applying the strategy described by equations (6.16), (6.17) and (6.18), the 4-derivatives correction to Q and M that comes from the α_{1111}-operator are

$$\begin{aligned} Q(T,\Phi,P) =& Q_2(T,\Phi,P) + \frac{64\pi^2\,\alpha_{1111} T\Phi^2}{5(1-\Phi^2)^2}\Bigg[2(2-\Phi^2)\,{}_2F_1\left(1,1,6,y(T,\Phi,P)\right) + \\ &+ \left(\frac{\Phi^2[3P^2T^2-(1-\Phi^2)^2]}{3(1-\Phi^2)^3}\right){}_2F_1\left(2,2,7,y(T,\Phi,P)\right)\Bigg], \end{aligned} \tag{6.23}$$

$$\begin{aligned} M(T,\Phi,P) =& M_2(T,\Phi,P) + \frac{64\pi^2\,\alpha_{1111} T\Phi^4}{5(1-\Phi^2)^2}\Bigg[2(2-\Phi^2)\,{}_2F_1\left(1,1,6,y(T,\Phi,P)\right) + \\ &+ \left(-\frac{\Phi^2}{3(1-\Phi^2)^2} + \frac{P^2T^2(1+2\Phi^2)}{3(1-\Phi^2)^3}\right){}_2F_1\left(2,2,7,y(T,\Phi,P)\right)\Bigg], \end{aligned} \tag{6.24}$$

where the function $y(T,\Phi,P)$ is equal to

$$y(T,\Phi,P) = \frac{P^2T^2 - \Phi^2(1-\Phi^2)^2}{(1-\Phi^2)^3}, \tag{6.25}$$

and the functions ${}_2F_1(a,b,c,d)$ are the hypergeometric functions.

The other contributions, proportional to the remaining coefficients of (6.1), are obtained in the same way and have a similar form. Once the full set of corrections has been computed, one can indeed derive the charge-to-mass ratio of an extremal black hole configuration of (6.6). To do so, one more step is in order. In fact, to carry on this calculation it is convenient to exchange the roles of Φ and Q and make use of the latter as a "coordinate" to describe the ensemble: this operation corresponds to the transition from the grand canonical ensemble to the canonical one and is achieved by inverting, at first order in the α-coefficients, equation (6.23).

With this substitution, equation (6.24) describes now the mass of the black hole as function of its electric charge. The corrected charge-to-mass ratio of an extremal black hole can indeed be computed from this relation and (after taking the limit $T \to 0$) results to be

$$\begin{aligned} z_{\text{ext}} = 1 + \frac{1}{5p(p+q)} \Bigg[& (\alpha_{1111})\ {}_2F_1\left(1,1;6;1-Q/P\right) + (\alpha_{2222})\ {}_2F_1\left(1,5;6;1-Q/P\right) + \\ & + (4\alpha_{1212} - 2\alpha_{1122})\ {}_2F_1\left(1,3;6;1-Q/P\right) + \\ & - \left(1-\frac{Q}{P}\right)^2 \left(\frac{\alpha_{11}}{84}\ {}_2F_1\left(3,3;8;1-Q/P\right) + \frac{\alpha_{22}}{84}\ {}_2F_1\left(3,5;8;1-Q/P\right)\right) + \\ & + \left(1-\frac{Q}{P}\right)^4 \frac{\alpha_1+\alpha_2}{126}\ {}_2F_1\left(5,5;10;1-Q/P\right) \Bigg]. \end{aligned} \tag{6.26}$$

We notice how this result is written, as seen in (6.17), as a perturbative series of corrections to the leading order value, in terms of the higher-order coefficients, which makes manifest the contributions to z_{ext} of the different higher-derivatives operators of (6.1). Moreover, this result holds in the case of positive charges: $Q > 0$, $P > 0$. The reason is that, otherwise, the two "coordinates" of the (now canonical) ensemble could give rise to a negative value of the mass M.

6.2 4-derivatives Lagrangian in the diagonal frame

The charge-to-mass ratio (6.26) is is the result of [34] that we want to re-read in terms of the coefficients of (5.97) in order to test the WGC and its requirements over them. To do so, we first need to write it in the proper symplectic frame to make the comparison with (6.1).

6.2.1 The diagonal frame

The frame in which (6.1) is defined is the one in which the $\mathcal{I}$ and $\mathcal{R}$ couplings take the form presented in (6.4) and (6.5) and we call it "diagonal frame". To see that such frame can indeed be reached via duality transformation, we need to introduce the following complex combination of $\mathcal{I}$ and $\mathcal{R}$:

$$\mathcal{N} \equiv \mathcal{R} - i\,\mathcal{I}. \tag{6.27}$$

From the rules (4.38) and (4.39), we see that the infinitesimal duality transformation of this new quantity is

$$\delta\mathcal{N} = C + D\mathcal{N} - \mathcal{N}A - \mathcal{N}B\mathcal{N}, \tag{6.28}$$

From this infinitesimal transformation rule we can actually track back the correspondent finite transformation. Going back to (4.30), the finite duality transformation on the gauge fields and their duals is written as

$$\begin{pmatrix} F' \\ G' \end{pmatrix} = \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} = \left[\mathbb{1}_4 + \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right] \begin{pmatrix} F \\ G \end{pmatrix} + \mathcal{O}(2), \tag{6.29}$$

where

$$\begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{pmatrix} \in \mathrm{Sp}(4, \mathbb{R}), \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{sp}(4, \mathbb{R}), \tag{6.30}$$

$\mathbb{1}_4$ is the 4×4 identity matrix and $\mathcal{O}(2)$ denotes the higher-order terms. Then, the correspondent finite transformation on $\mathcal{N}$ can be understood from (6.28) as follows:

$$\begin{aligned} \mathcal{N}' = \mathcal{N} + \delta\mathcal{N} + \mathcal{O}(2) =& \mathcal{N} + C + D\mathcal{N} - \mathcal{N}A - \mathcal{N}B\mathcal{N} + \mathcal{O}(2) = \\ =& (C + (\mathbb{1} + D)\mathcal{N}) - \mathcal{N}(A + B\mathcal{N}) + \mathcal{O}(2) = \\ =& [C + (\mathbb{1} + D)\mathcal{N}] [\mathbb{1} - A - B\mathcal{N}] + \mathcal{O}(2) = \\ =& [C + (\mathbb{1} + D)\mathcal{N}] [(\mathbb{1} + A) + B\mathcal{N}]^{-1} + \mathcal{O}(2), \end{aligned} \tag{6.31}$$

so that, looking at (6.29), we find

$$\mathcal{N}' = \left(\hat{C} + \hat{D}\mathcal{N}\right) \left(\hat{A} + \hat{B}\mathcal{N}\right)^{-1}. \tag{6.32}$$

This fractional, finite duality transformation is the rule that we can use to set the $\mathcal{I}$ and $\mathcal{R}$ in the diagonal frame given by (4.38) and (4.39). To understand which is the duality transformation that we have to apply to reach such a frame we have to better specify the explicit expression of the matrix $\mathcal{N}$. To do so, we need to go back to the construction of theory (4.1). In Section 4.1 we introduced the concept of coset manifolds, specifying that the scalar sector of (4.1) is given by the $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2, \mathbb{R})$ coset manifold (4.11). This manifold can be seen also as a so called Kähler manifold (see [32], [31]), which is the typical structure of the scalar manifolds of supersymmetric models. Then, one can show (see [31]) that the Kähler geometry fixes the matrix $\mathcal{N}$ to be diagonal and of the form

$$\mathcal{N} = \begin{pmatrix} \bar{\tau} & 0 \\ 0 & -\frac{1}{\bar{\tau}} \end{pmatrix}, \tag{6.33}$$

and we see that it holds

$$\mathcal{N}_{11} = -\frac{1}{\mathcal{N}_{22}}. \tag{6.34}$$

Now that the matrix $\mathcal{N}$ has been fixed to (6.33), we can look for the finite duality transformation that sets the $\mathcal{I}$ and $\mathcal{R}$ couplings to be as in (6.4) and (6.5). This transformation is given by the following $\mathrm{Sp}(4, \mathbb{R})$ matrix (see [31]):

$$\hat{\mathcal{D}} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right). \tag{6.35}$$

The correspondent transformation is in fact (6.32) on $\mathcal{N}$ is

$$\mathcal{N}' = \begin{pmatrix} \mathcal{N}_{11} & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\mathcal{N}_{22}} \end{pmatrix} = \begin{pmatrix} \mathcal{N}_{11} & 0 \\ 0 & -\frac{1}{\mathcal{N}_{22}} \end{pmatrix} \overset{(6.34)}{=} \mathcal{N}_{11}\,\mathbb{1}_4, \tag{6.36}$$

which recalling (6.27) and (6.33) indeed reproduces (6.4) and (6.5).

We finally remark that the duality transformation (6.35) is indeed a symplectic frame transformation: the correspondent transformation of the algebra $\mathfrak{sp}(4,\mathbb{R})$ is in fact given by

$$\mathcal{D} = -i\,\frac{\pi}{2}\left[t_2 + \frac{1}{2}\begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}\right], \tag{6.37}$$

where t_2 is one of the $\mathfrak{sl}(2,\mathbb{R})$ generators (4.50) and the second one belongs to the symplectic subgroup (4.52).

6.2.2 Identification between the coefficients

We showed that the symplectic frame we have called diagonal indeed exists and how it can be reached via duality transformations. It is in this frame that we can properly compare the 4-derivatives correction we determined and the one proposed in [34].

As said, Lagrangian (6.1) is essentially a guess, in the diagonal frame, based on the minimal operator content that is expected to enter the 4-derivatives, duality-invariant extension of (4.1). It appears in fact in terms of generic matrix coefficients α_i, which still carry the gauge indices (A, B) although the couplings' dependence on the complex scalar field τ appears explicitly. Moreover, as highlighted in the previous chapters, such a construction is not fully respectful of EM duality, which truly is a symmetry of the EoM and BI of the gauge fields and not of the Lagrangian. The result of such vague structure is precisely the fact that (6.1) is not in clear connection with all the EM duality group but it is instead defined in one particular symplectic frame.

Looking now at Lagrangian (5.97), we see that though the operators it contains are the same of (6.1), they appear together with proper combinations of the coupling $\mathcal{I}$ of (4.1), without the need to specify it further, neither in the Lagrangian nor in the analysis that lead to (5.97). It is this dependence on $\mathcal{I}$ that makes (5.97) frame-independent, a property that, together with its perturbative, duality based derivation, supports (5.97).

In fact, Lagrangian (5.97) can indeed be seen as a generalization of (6.1) which fills all the gaps the latter leaves open. Specifying then (5.97) in the diagonal frame, which means substituting to the $\mathcal{I}$ terms of (5.97) its diagonal frame expression (6.4), we obtain

$$\begin{aligned} \mathcal{L}^{(4)}_{\text{diag}} =& \eta(\mathfrak{Im}\tau)^2\left[(F^AF^B)(F_AF_B) + (F^A\tilde{F}^B)(F_A\tilde{F}_B)\right] - \frac{\alpha}{4}\frac{1}{\mathfrak{Im}\tau}\partial_\mu\bar{\tau}\partial^\mu\tau F^\Lambda F_\Lambda + \\ &+ \frac{\alpha}{\mathfrak{Im}\tau}\partial_\mu\bar{\tau}\partial_\nu\tau F^{\Lambda\,\mu\alpha}F^\nu_{\Lambda\,\alpha} + \frac{\beta}{\mathfrak{Im}\tau}\left(i\,\partial_\mu\bar{\tau}\partial_\nu\tau F^{\Lambda\,\mu\alpha}\tilde{F}^\nu_{\Lambda\,\alpha} + h.c.\right) + \\ &+ \frac{1}{(\mathfrak{Im}\tau)^4}\left[\lambda_1(\partial_\mu\bar{\tau}\partial^\mu\tau)^2 + \lambda_2|\partial_\mu\tau\partial^\mu\tau|^2\right]. \end{aligned} \tag{6.38}$$

We immediately see that Lagrangian (6.38) not only reproduces the dependence on the complex scalar field, in terms of $\mathfrak{Im}\tau$, of the non-minimal couplings shown in (6.1), but it also better specifies the expression of the α-coefficients of (6.1), where they were left generic, and with them the structure of the associated operators.

Therefore it's now possible to make an identification between the α-coefficients of (6.1) and the set $\{\eta, \alpha, \beta, \lambda_1, \lambda_2\}$ of (6.38). Following the Lagrangian subdivision seen in (5.2) and (5.18), we have:

- **$4F$-sector** we start with the operators of (6.1) and (6.38) that we can track back to the $4F$-sector (5.19), so that the coefficients to be compared are α_{ABCD} and η. The symmetries of α_{ABCD} (6.2) tell that out of its 16 components the independent ones are

$$\begin{aligned} &\alpha_{1111}\ (\times 1), &&\alpha_{1112}\ (\times 4), &&\alpha_{1212}\ (\times 4) \\ &\alpha_{1122}\ (\times 2), &&\alpha_{1222}\ (\times 4), &&\alpha_{2222}\ (\times 1). \end{aligned} \tag{6.39}$$

Instead, the $4F$-term in Lagrangian (6.38) reads

$$\eta(F^A F^B)(F_A F_B) = \eta\left[(F^1F^1)^2 + (F^2F^2)^2 + 2(F^1F^2)^2\right]. \tag{6.40}$$

Thus, by comparing (6.39) and (6.40) we obtain the following identification between the coefficients:

$$\begin{aligned} &\alpha_{1111} = \eta, &&\alpha_{1112} = 0, &&\alpha_{1212} = \frac{\eta}{2}, \\ &\alpha_{1122} = 0, &&\alpha_{1222} = 0, &&\alpha_{2222} = \eta. \end{aligned} \tag{6.41}$$

- **τ-sector** For what regards the operators of the τ-sector, we immediately see that there's a relevant difference between (6.38) and (6.1): in the former the coefficient of the operator

$$i\,\partial_\mu\bar{\tau}\partial_\nu\tau F^{A\,\mu\alpha}\tilde{F}^{B\,\nu}{}_{\alpha} + h.c. = i\left(\partial_\mu\bar{\tau}\partial_\nu\tau - \partial_\mu\tau\partial_\nu\bar{\tau}\right)F^{A\,\mu\alpha}\tilde{F}^{B\,\nu}{}_{\alpha} \tag{6.42}$$

is independent of the other two of this sector (i.e. there is no relation between α and β), while in the latter such relation does exits and would correspond, in (6.38), to the additional constraint

$$\beta = \frac{1}{2}\alpha. \tag{6.43}$$

This means that a one-to-one correspondence between the coefficients of (6.38) and (6.1) is not possible. The reason is that the former, thanks to the rigorous derivation of Chapter 5, has a more specified structure that the latter, more guessed than derived, necessary lacks.
However, we saw that the black hole solution presented in [34] with which we want to make contact is characterized by (see (6.7))

$$\mathfrak{Re}\tau = 0,$$

and along this solution for the complex scalar field τ the operator (6.42) results to be identically vanishing, independently on the coefficient β:

$$\begin{aligned} &\mathfrak{Re}\tau = 0 \quad \Longrightarrow \quad \tau = i\,\mathfrak{Im}\tau = -\bar{\tau} \quad \Longrightarrow \\ &\qquad \Longrightarrow \quad \partial_\mu\bar{\tau}\partial_\nu\tau - \partial_\mu\tau\partial_\nu\bar{\tau} = -\partial_\mu\tau\partial_\nu\tau + \partial_\mu\tau\partial_\nu\tau = 0. \end{aligned} \tag{6.44}$$

Thus, since the purpose is to study, from the (6.38) point of view, the higher-derivatives corrections to the charge-to-mass ratio (6.26) of the black hole solution determined in [34] and, according to this solution, the operator (6.42) is identically vanishing, it is sufficient to perform the identification between the coefficients in absence of this operator. In this case, it results to be

$$\begin{aligned}\alpha_{11} =&\alpha_{22} = 2\alpha,\\ \alpha_{12} =&\alpha_{21} = 0.\end{aligned} \tag{6.45}$$

- **scalar sector** For the remaining operators, the ones involving the complex scalar field only, the identification is trivial and reads

$$\alpha_1 = \lambda_1, \qquad \alpha_2 = \lambda_2. \tag{6.46}$$

Thanks to this identification, we can now read the results of [34] in terms of the coefficients of (6.38) and study then which conditions they should satisfy to realize the Electric WGC (1.1).

6.3 Weak Gravity Conjecture

We're now able, thanks to (6.41), (6.45) and (6.46), to write the charge-to-mass ratio (6.26) of [34] in terms of the coefficients of the 4-derivatives Lagrangian (5.97) we derived. Defining

$$x \equiv \frac{Q}{P}, \tag{6.47}$$

$$\lambda \equiv \lambda_1 + \lambda_2, \tag{6.48}$$

we can write the resulting charge-to-mass ratio as

$$z_{\text{ext}} = 1 + \Delta z_{\text{ext}}, \tag{6.49}$$

$$\Delta z_{\text{ext}} = \frac{1}{QP}\left[c_\eta(x)\,\eta - c_\alpha(x)\,\alpha + c_\lambda(x)\lambda\right], \tag{6.50}$$

where we have:

$$c_\eta(x) = \frac{16\pi^2}{5}\frac{x}{1+x}\left({}_2F_1(1,1;6;1-x) + {}_2F_1(1,5;6;1-x) + 2\,{}_2F_1(1,3;6;1-x)\right), \tag{6.51}$$

$$c_\alpha(x) = \frac{4\pi^2}{105}\frac{x(1-x)^2}{1+x}\left({}_2F_1(3,3;8;1-x) + {}_2F_1(3,5;8;1-x)\right), \tag{6.52}$$

$$c_\lambda(x) = \frac{8\pi^2}{315}\frac{x(1-x)^4}{1+x}\,{}_2F_1(5,5;10;1-x), \tag{6.53}$$

and all these functions are non-negative for all values of x. We can get an idea of the relative contribution of each of these pieces to z_{ext} by plotting

$$QP\frac{\Delta z_{\text{ext}}}{\lambda_i},$$

where $\lambda_i = \{\eta,\,\alpha,\,\lambda\}$, as function of x: this is shown in Figure 6.1. The contributions are relative because their true weight in z_{ext} depends on the values of the η, α and λ coefficients.

As we discussed in Section 1.1.2, the Electric WGC (1.1) can be formulated as the request that extremal black holes are able to decay into smaller black holes. This process is possible

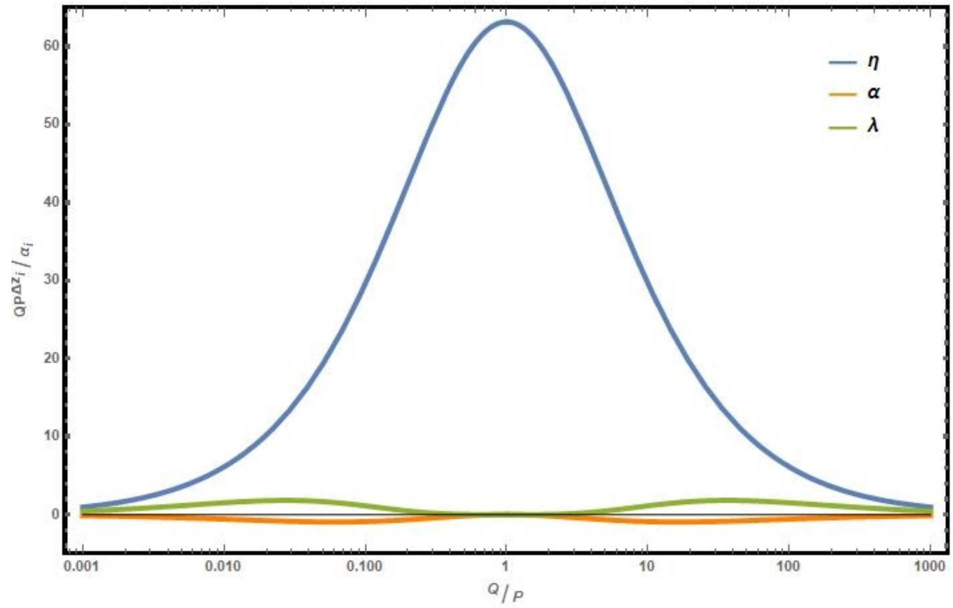


Figure 6.1: Relative contributions to the 4-derivatives correction to z_{ext}.

if the higher-order correction that the charge-to-mass ratio of such black hole receives are positive, as shown in (1.11). Therefore, the Electric WGC is realized in our theory if the coefficients η, α and λ are such that

$$\mathcal{Z}(x) \equiv c_\eta(x)\,\eta - c_\alpha(x)\,\alpha + c_\lambda(x)\,\lambda \geq 0 \qquad \forall\, x. \tag{6.54}$$

Let's then study this condition in order to extract the WGC constraints over the coefficients of (5.97). First of all, from Figure 6.1 we can see that for the value $x = 1$ we have

$$c_\eta(1) > 0, \qquad c_\alpha(1) = 0, \qquad c_\lambda(1) = 0, \tag{6.55}$$

which means

$$\mathcal{Z}(1) = c_\eta(1)\eta = \frac{32\pi^2}{5}\eta. \tag{6.56}$$

Since for the WGC to be realized the condition (6.54) must be satisfied for every values of x, $\mathcal{Z}(1)$ yields a bound over η, which is

$$\eta \geq 0. \tag{6.57}$$

This is the first WGC constraint we obtain. To deal also with α and λ, it is convenient to rewrite (6.54) dividing everything by η:

$$\tilde{\mathcal{Z}}(x) \equiv c_\eta(x) - c_\alpha(x)\,\tilde{\alpha} + c_\lambda(x)\,\tilde{\lambda} \geq 0, \tag{6.58}$$

where

$$\tilde{\alpha} \equiv \frac{\alpha}{\eta}\,, \qquad \tilde{\lambda} \equiv \frac{\lambda}{\eta}\,. \tag{6.59}$$

From (6.58) we get (for $x \neq 1$)

$$\tilde{\lambda} \geq \frac{c_\alpha(x)}{c_\lambda(x)}\,\tilde{\alpha} - \frac{c_\eta(x)}{c_\lambda(x)}. \tag{6.60}$$

To study this bound (6.60) we need to consider separately the four possible combinations of signs of $\tilde{\alpha}$ and $\tilde{\lambda}$.

(I) $\boldsymbol{\alpha \leq 0}$, $\boldsymbol{\lambda \geq 0}$. This case is the simplest one because for $\alpha \leq 0$ the right-hand side of (6.60) is negative, so that for $\lambda \geq 0$ the bound is trivially satisfied for all values of x.

(II) $\boldsymbol{\alpha \geq 0}$, $\boldsymbol{\lambda \geq 0}$. for this combination we have to distinguish two cases, depending on whether the right-hand side (RHS) of (6.60) is positive or negative.

1. For a positive RHS we have a non-trivial condition over both α and λ:

$$\tilde{\alpha} \geq \frac{c_\eta(x)}{c_\alpha(x)} \quad \longrightarrow \quad \tilde{\lambda} \geq \frac{c_\alpha(x)}{c_\lambda(x)}\,\tilde{\alpha} - \frac{c_\eta(x)}{c_\lambda(x)}. \tag{6.61}$$

   Since, as said, (6.60) should be satisfied for all values of x, these conditions translate to the maximum values of the functions of x:

$$\tilde{\alpha} \geq \max_x \left(\frac{c_\eta(x)}{c_\alpha(x)}\right), \qquad \tilde{\lambda} \geq \max_x \left(\frac{c_\alpha(x)}{c_\lambda(x)}\,\tilde{\alpha} - \frac{c_\eta(x)}{c_\lambda(x)}\right). \tag{6.62}$$

   However, studying the $\tilde{\alpha}$ bound we immediately find that

$$\max_x \left(\frac{c_\eta(x)}{c_\alpha(x)}\right) = +\infty, \tag{6.63}$$

   and therefore a positive RHS of (6.60) with both α and λ positive is not an allowed configuration.
2. If the RHS of (6.60) is negative then $\lambda \geq 0$ is sufficient to satisfy the bound, as happened in case (I). We have a negative RHS for

$$\tilde{\alpha} \leq \frac{c_\eta(x)}{c_\alpha(x)}, \tag{6.64}$$

   and this time this condition translates to the minimum of the function of x, which is now finite:

$$\tilde{\alpha} \leq \min_x \left(\frac{c_\eta(x)}{c_\alpha(x)}\right) = 4. \tag{6.65}$$

   Therefore we can conclude that for both α and λ positive the WGC is realized if

$$0 \leq \alpha \leq 4\eta, \qquad \lambda \geq 0. \tag{6.66}$$

(III) $\boldsymbol{\alpha \leq 0}$, $\boldsymbol{\lambda \leq 0}$. In this case the RHS of (6.60) is negative and we have

$$\frac{c_\alpha(x)}{c_\lambda(x)}\,\tilde{\alpha} - \frac{c_\eta(x)}{c_\lambda(x)} \leq \tilde{\lambda} \leq 0. \tag{6.67}$$

In order to extremes this bound and obtain the WGC constraint, it is convenient to rewrite it in terms of the absolute values of $\tilde{\alpha}$ and $\tilde{\lambda}$:

$$\begin{aligned} \tilde{\lambda} \geq \frac{c_\alpha(x)}{c_\lambda(x)}\,\tilde{\alpha} - \frac{c_\eta(x)}{c_\lambda(x)} \quad &\Longrightarrow \quad -|\tilde{\lambda}| \geq -\left[\frac{c_\alpha(x)}{c_\lambda(x)}\,|\tilde{\alpha}| + \frac{c_\eta(x)}{c_\lambda(x)}\right] \quad \Longrightarrow \\ &\Longrightarrow \quad |\tilde{\lambda}| \leq \frac{c_\alpha(x)}{c_\lambda(x)}\,|\tilde{\alpha}| + \frac{c_\eta(x)}{c_\lambda(x)}. \end{aligned} \tag{6.68}$$

We can then can minimize this expression with respect to x, obtaining

$$|\tilde{\lambda}| \leq \min_x \left(\frac{c_\alpha(x)}{c_\lambda(x)} |\tilde{\alpha}| + \frac{c_\eta(x)}{c_\lambda(x)} \right) = \min_x \left(\frac{c_\alpha(x)}{c_\lambda(x)} \right) |\tilde{\alpha}| + \min_x \left(\frac{c_\eta(x)}{c_\lambda(x)} \right) = 1 + \frac{|\tilde{\alpha}|}{4}, \tag{6.69}$$

so that

$$|\tilde{\lambda}| = -\tilde{\lambda} \leq 1 + \frac{|\tilde{\alpha}|}{4} = 1 - \frac{\tilde{\alpha}}{4} \quad \Longrightarrow \quad \tilde{\lambda} \geq -1 + \frac{\tilde{\alpha}}{4}. \tag{6.70}$$

The resulting WGC bounds on the negative α and λ are therefore

$$\alpha \leq 0, \qquad \frac{\alpha}{4} - \eta \leq \lambda \leq 0. \tag{6.71}$$

(IV) $\boldsymbol{\alpha \geq 0}$, $\boldsymbol{\lambda \leq 0}$. In this last configuration we have a negative λ, so that also the RHS of (6.60) must be so. This yields the following condition on α:

$$\tilde{\alpha} \leq \frac{c_\eta(x)}{c_\alpha(x)} \quad \Longrightarrow \quad \tilde{\alpha} \leq \min_x \left(\frac{c_\eta(x)}{c_\alpha(x)} \right) =4 \quad \Longrightarrow \quad 0 \leq \alpha \leq 4\eta. \tag{6.72}$$

The associated bound on λ is instead

$$\frac{c_\alpha(x)}{c_\lambda(x)} \tilde{\alpha} - \frac{c_\eta(x)}{c_\lambda(x)} \leq \tilde{\lambda} \leq 0, \tag{6.73}$$

and we can study it again by analyzing the correspondent condition on its absolute value:

$$\begin{aligned} \tilde{\lambda} \geq \frac{c_\alpha(x)}{c_\lambda(x)} \tilde{\alpha} - \frac{c_\eta(x)}{c_\lambda(x)} \quad &\Longrightarrow \quad -|\tilde{\lambda}| \geq \frac{c_\alpha(x)}{c_\lambda(x)} \tilde{\alpha} - \frac{c_\eta(x)}{c_\lambda(x)} \quad \Longrightarrow \\ &\Longrightarrow \quad |\tilde{\lambda}| \leq \frac{c_\eta(x)}{c_\lambda(x)} - \frac{c_\alpha(x)}{c_\lambda(x)} \tilde{\alpha}. \end{aligned} \tag{6.74}$$

It follows that

$$|\tilde{\lambda}| \leq \min_x \left(\frac{c_\eta(x)}{c_\lambda(x)} - \frac{c_\alpha(x)}{c_\lambda(x)} \tilde{\alpha} \right) = 1 - \frac{\tilde{\alpha}}{4} \quad \Longrightarrow \quad \tilde{\lambda} \geq \frac{\tilde{\alpha}}{4} - 1. \tag{6.75}$$

We conclude that the WGC is realized by positive α and negative λ if they're such that

$$0 \leq \alpha \leq 4\eta, \qquad \frac{\alpha}{4} - \eta \leq \lambda \leq 0. \tag{6.76}$$

Summarizing, the coefficients α and λ agree with the WGC for the following subsets of their values:

$$\begin{aligned} &\text{(I)} \quad \begin{cases} \alpha \leq 0 \\ \lambda \geq 0 \end{cases}, \qquad &&\text{(II)} \quad \begin{cases} 0 \leq \alpha \leq 4\eta \\ \lambda \geq 0 \end{cases}, \\ &\text{(III)} \quad \begin{cases} \alpha \leq 0 \\ \frac{\alpha}{4} - \eta \leq \lambda \leq 0 \end{cases}, \qquad &&\text{(IV)} \quad \begin{cases} 0 \leq \alpha \leq 4\eta \\ \frac{\alpha}{4} - \eta \leq \lambda \leq 0 \end{cases}. \end{aligned} \tag{6.77}$$

Looking at the different combinations, we notice that we can indeed merge the different bounds over one coefficients keeping fixed the sign of the other one. For instance, at fixed α

we have

$$\alpha \leq 0 \quad \longrightarrow \quad \lambda \geq \frac{\alpha}{4} - \eta, \tag{6.78}$$

$$0 \leq \alpha \leq 4\eta \quad \longrightarrow \quad \lambda \geq \frac{\alpha}{4} - \eta. \tag{6.79}$$

Since the resulting bound on λ does not change when the sign of α does, we can merge also the bounds on α and obtain, as final bounds on the two coefficients,

$$\alpha \leq 4\eta \tag{6.80}$$

$$\lambda \geq \frac{\alpha}{4} - \eta. \tag{6.81}$$

Equations (6.57), (6.80) and (6.81) are therefore the conditions over the coefficients of (5.97) in order for the Electric WGC (1.1) to be realized in (5.97). Following the Swampland prescriptions, the set of theories (5.97) with coefficients satisfying (6.57), (6.80) and (6.81) belong to the Landscape, all the others to the Swampland.

6.3.1 Positivity bounds

Once the conditions under which the WGC is realized have been found, one can look for supporting evidence in the positivity bounds on the scattering amplitudes that the theory under examination describes. We have seen that in the pure Einstein–Maxwell theory (1.4) there is an exact equivalence between the WGC conditions and the positivity bounds [18] (see Chapter 2). However, as discussed in [33, 34], they're not sufficient, alone, when the field content of the theory is more involved: the equivalence seems to be restored if the theory satisfies the additional requirement to be duality-preserving. Therefore, our resulting 4-derivatives Lagrangian (5.97) is a good benchmark to make such a test.

We rely again on [34] for the computation of such amplitudes[2]. The resulting positivity bounds over the coefficients η, α and $\lambda = \lambda_1 + \lambda_2$ of (5.97) are the following:

$$\eta \geq 0, \tag{6.82}$$

$$\alpha \leq 0, \tag{6.83}$$

$$\lambda \geq 0. \tag{6.84}$$

By comparing these positivity bounds with the WGC requirements we found in the previous section, we immediately see that while the bound (6.57) on the coefficient η exactly match the WGC condition (6.57), the bounds (6.83) and (6.84) on α and λ correspond only to a subclass of the WGC constraints (6.80) and (6.81), the one we denoted with (I). There are indeed configurations of the coefficients - the ones we named (II), (III) and (IV) - that are in agreement with the WGC but are instead excluded from the positivity bounds, so that the latter result to be a stronger requirement over the coefficients than the former.

We can then conclude that in the context of the duality-preserving theory (5.97) an exact equivalence between the positivity bounds on the scattering amplitudes and the Electric WGC (1.1) was not found: the positivity bounds represent only a sufficient condition to realize Electric the WGC, not a necessary one.

[2] This computation is performed in [34] assuming that the graviton exchange amplitude is subdominant, so that problems with the Coulomb singularity (2.34) are avoided.

Summary and Outlook

The first and main goal of this thesis work was to find a 4-derivatives extension of action (4.1) that was duality preserving. To do so, the issues to overcome were two: the first was to understand how to properly implement EM duality as a symmetry of the set of EoM and BI (it's definitely not a symmetry of the Lagrangian); the second was how to not exclude a priori from the discussion the operators with derivatives of the gauge fields. The strategy we applied was to abandon the general approach of Gaillard and Zumino in [26] and rely instead to a model-based, perturbative duality analysis that exploits the self-consistency of the duality transformations (4.29), (4.30). This indeed restricts our results to the specific model (4.1) we considered, but at the same time it does not require - unlike in [26] - any further assumption on the structure of the full theory, which is determined only in virtue of EM duality. We also remark that this perturbative approach is consistent with the perspective under which EM duality is a manifestation of the duality symmetries of String Theory [14–16]: being a symmetry of the full UV theory is then expected to hold at every perturbation order.

More specifically, we determined the duality group (4.42)-(4.43) of the theory and the transformations (4.38) and (4.39) of the $\mathcal{I}$ and $\mathcal{R}$ couplings from the duality analysis of the 2-derivatives, starting Lagrangian (4.1). We then used these results of the 2-derivatives order to determine which operators and couplings give rise to a duality-preserving 4-derivatives correction to (4.1). This analysis led to Lagrangian (5.97), which is the main result of the thesis work. The main feature of this Lagrangian is that it does not depend on the choice of the particular symplectic frame: this shows that our result was indeed derived in full respect of EM duality as symmetry of the EoM and BI. Also, it supports our perturbative duality analysis as a valuable analytic tool to determine duality-preserving extensions of effective gauge theories.

After presenting this duality analysis in chapters 4 and 5, in Chapter 6 we discussed the relationship, in the context of the resulting theory (5.97), between the Electric WGC (1.1) and the duality-preserving higher-order theories, which is the second topic addressed in this work. Following [34], we studied the charge-to-mass ratio (6.50) characterizing a black hole solution of (5.97) in terms of the higher-order coefficients η, α and $\lambda = \lambda_1 + \lambda_2$ and we determined which subset of these parameters satisfy the Electric WGC prescription (1.11). This subset is given by conditions (6.57), (6.80) and (6.81). Then, we compared these results with the positivity bounds (6.82)–(6.84) presented in [34] and showed that they're indeed a sufficient condition for the Electric WGC (1.1) to be realized but not a necessary one, similarly to what was found [34]. Therefore, while our resulting Lagrangian (5.97) is indeed an improvement with respect to [34] because of the rigorous duality framework in which it was derived and because it allows to explicitly determine the WGC requirements on its coefficients (which are not determined in [34]), it's not able to overcome it on the question about the relationship between the WGC and the positivity bounds, since, despite the improvements, the exact equivalence between the two was not found.

In this respect, one question that must be explored is whether this equivalence between the WGC and the positivity bounds may follow by including in the analysis also Supersymmetry. Indeed, Supersimmetry imposes additional constraints over the couplings of the theory: it is then possible that the higher-order coefficients result to be able to now realize the equivalence between the two requirements in an exact way.

More in general, this relationship among EM duality, the WGC and the positivity bounds on the scattering amplitudes is a topic that surely needs to be discussed further. In fact, although the claimed equivalence between the Electric WGC (1.1) and the positivity bounds has not been found, neither in [34] nor in this work, to hold in the duality-preserving extension of axion-dilaton-Maxwell-Einstein theory (4.1), it was indeed shown in [33] to be exactly realized in the context of the axion-dilaton-Einstein theory. Thus, to better understand which is the true connection between the WGC and the EM duality, and if they truly point in the same direction in the theories' space, is then crucial to test the WGC in different realizations of EM duality. To do so, the perturbative duality analysis we exploited to derive Lagrangian (5.97) is indeed a valuable tool that can be applied to many different models.

Another question the duality analysis presented in this work can be useful to address regards the relation between EM duality and the operators with derivatives on the gauge fields. As said, they're excluded from the general discussion of Gaillard and Zumino in [26] but when EM duality is used to determine the higher-derivatives extensions of effective gauge theories it results to be too restrictive a hypothesis. Indeed, the perturbative approach we exploited was able to overcome this issue and treat those operators in the same way as the others. However, it does not give, in itself, any further clues about the true role of these operators in duality-preserving theories. Thus, studying many different models with this approach may point out the conditions under which such higher-order extensions do include also these operators or if instead, as happened for (4.1), they're always excluded.

Appendix A

Duality transformation on L_{mat}

In this section we show the result (4.48) on the duality transformation of the matter sector of a 2-derivatives gauge theory.

We consider the case of a Lagrangian density L describing a set of scalar fields ϕ^i and one U(1) gauge field F (and not on its derivatives):

$$L = L_{\text{mat}}(\phi) + L_{\text{g}}(\phi, F), \tag{A.1}$$

such that under a duality transformation we have

$$\delta L_{\text{g}} = \frac{1}{4}\left(FC\tilde{F} + GB\tilde{G}\right), \tag{A.2}$$

where G is the dual field as defined in (4.29). Let's denote the associated duality transformation on the scalars ϕ^i as

$$\delta\phi^i \equiv \xi^i(\phi), \tag{A.3}$$

$$\xi^i_\mu \equiv \partial_\mu \xi^i = \frac{\partial \xi^i}{\partial \phi^j}\partial_\mu \phi^j \equiv \frac{\partial \xi^i}{\partial \phi^j}\phi^j_\mu. \tag{A.4}$$

Recalling the definitions (3.22) and (3.21) of the ϕ_i EoM and the associated operator, the variation $\delta\mathcal{L}_{\text{mat}}$ can be written in terms of the total duality transformation of the Lagrangian L in the following way:

$$\begin{aligned}\delta L =& \delta L_{\text{mat}} + \delta L_{\text{g}} = \delta L_{\text{mat}} + \frac{1}{4}\left(FC\tilde{F} + GB\tilde{G}\right)\\ &| \\ =&\xi^k \frac{\partial L}{\partial \phi^k} + \xi^k_\alpha \frac{\partial L}{\partial \phi^k_\alpha} + \delta F \frac{\partial L}{\partial F} = \xi^k E_k + \partial_\alpha\left[\xi^k \frac{\partial L}{\partial \phi^k_\alpha}\right] + \frac{1}{2}(AF + BG)\tilde{G},\end{aligned} \tag{A.5}$$

so that

$$\delta L_{\text{mat}} = \xi^k E_k + \partial_\alpha\left[\xi^k \frac{\partial L}{\partial \phi^k_\alpha}\right] + \frac{1}{2}FA\tilde{G} + \frac{1}{4}GB\tilde{G} - \frac{1}{4}FC\tilde{F}. \tag{A.6}$$

Similarly to what we have seen in Gaillard and Zumino analysis (see (3.24)), we can study the ϕ-dependence of δL_{mat} by applying to it the EoM operator (3.21):

$$\hat{E}_i[\delta L_{\text{mat}}] = \hat{E}_i[\xi^k E_k] + \hat{E}_i\left[\partial_\alpha\left(\xi^k \frac{\partial L}{\partial \phi^k_\alpha}\right)\right] + \hat{E}_i\left[\frac{1}{2}FA\tilde{G} + \frac{1}{4}GB\tilde{G} \cancel{- \frac{1}{4}FC\tilde{F}}\right], \tag{A.7}$$

where the last term cancels because F and ϕ_i are of course independent. After some algebraic computations, each of the pieces of (A.7) reads

- $$\hat{E}_i\left[\frac{1}{2}FA\tilde{G}+\frac{1}{4}GB\tilde{G}\right]=\delta_F E_i-\partial_\mu(\delta F)\frac{\partial}{\partial F}\frac{\partial L}{\partial\phi^i_\mu},\tag{A.8}$$
- $$\hat{E}_i\left[\xi^k E_k\right]=\delta_\phi E_i+\delta_{\partial\phi}E_i+\frac{\partial\xi^k}{\partial\phi^i}E_k-\xi^k_\mu\frac{\partial E_k}{\partial\phi^i_\mu}-\partial_\mu\left[\xi^k\frac{\partial E_i}{\partial\phi^k_\mu}\right],\tag{A.9}$$
- $$\hat{E}_i\left[\partial_\alpha\left(\xi^k\frac{\partial L}{\partial\phi^k_\alpha}\right)\right]=\partial_\alpha\left[\xi^k\frac{\partial E_i}{\partial\phi^k_\alpha}-\xi^k_\mu\frac{\partial^2 L}{\partial\phi^k_\alpha\partial\phi^i_\mu}\right],\tag{A.10}$$

in which we have introduced the quantities $\delta_F E_i$, $\delta_\phi E_i$ and $\delta_{\partial\phi}E_i$ according to (A.5):

$$\delta E_i=\delta_F E_i+\delta_\phi E_i+\delta_{\partial\phi}E_i\equiv\frac{\partial E_i}{\partial F}\delta F+\frac{\partial E_i}{\partial\phi^j}\delta\phi^j+\frac{\partial E_i}{\partial\phi^j_\alpha}\partial_\alpha(\delta\phi^j).\tag{A.11}$$

Putting (A.8), (A.9) and (A.10) together and exploiting the covariance of the scalar fields' EoM (3.23), we obtain

$$\hat{E}_i[\delta L_{\text{mat}}]=\cancel{\delta E_i+\frac{\partial\xi^k}{\partial\phi^i}E_k}-\left[\partial_\mu(\delta F)\frac{\partial}{\partial F}+\partial_\mu(\delta\phi^k)\frac{\partial}{\partial\phi^k}+\partial_\mu(\delta\phi^k_\alpha)\frac{\partial}{\partial\phi^k_\alpha}\right]\frac{\partial L}{\partial\phi^i_\mu}.\tag{A.12}$$

This last equation can be further manipulated by exploiting first the fact that spacetime derivatives and field derivatives are commutative operations (since the fields and their spacetime derivatives are considered as independent degrees of freedom in the Lagrangian):

$$\begin{aligned}\frac{\partial}{\partial\phi^i}(\partial_\mu L)=&\frac{\partial}{\partial\phi^i}\left(\frac{\partial L}{\partial\phi^k}(\partial_\mu\phi^k)+\frac{\partial L}{\partial\partial_\alpha\phi^k}(\partial_\mu\partial_\alpha\phi^k)\right)=\\=&\frac{\partial^2 L}{\partial\phi^k}(\partial_\mu\phi^k\partial\phi^i)+\frac{\partial^2 L}{\partial\partial_\alpha\phi^k\partial\phi^i}(\partial_\mu\partial_\alpha\phi^k)=\partial_\mu\frac{\partial L}{\partial\phi^i},\end{aligned}\tag{A.13}$$

then the commutativity also between ∂_μ and δ. The right-hand side of equation (A.12) thus becomes

$$\begin{aligned}&\partial_\mu(\delta\phi^k)\frac{\partial^2 L}{\partial\phi^k\partial\phi^i_\mu}+\partial_\mu(\delta\phi^k_\alpha)\frac{\partial^2 L}{\partial\phi^k_\alpha\partial\phi^i_\mu}+(\partial_\mu\delta F)\frac{\partial^2 L}{\partial F\partial\phi^i_\mu}=\\&=\partial_\mu\left[\delta\phi^k\frac{\partial}{\partial\phi^k}+\delta\phi^k_\alpha\frac{\partial}{\partial\phi^k_\alpha}+\delta F\frac{\partial}{\partial F}\right]\frac{\partial L}{\partial\phi^i_\mu}-\left[\delta\phi^k\partial_\mu\frac{\partial}{\partial\phi^k}+\delta\phi^k_\alpha\partial_\mu\frac{\partial}{\partial\phi^k_\alpha}+\delta F\partial_\mu\frac{\partial}{\partial F}\right]\frac{\partial L}{\partial\phi^i_\mu}\overset{(A.13)}{=}\\&=\partial_\mu\left(\delta\frac{\partial L}{\partial\phi^i_\mu}\right)-\left[\delta\phi^k\frac{\partial}{\partial\phi^k}+\delta\phi^k_\alpha\frac{\partial}{\partial\phi^k_\alpha}+\delta F\frac{\partial}{\partial F}\right]\partial_\mu\frac{\partial L}{\partial\phi^i_\mu}=\\&=\partial_\mu\left(\delta\frac{\partial L}{\partial\phi^i_\mu}\right)-\delta\left(\partial_\mu\frac{\partial L}{\partial\phi^i_\mu}\right)=0.\end{aligned}\tag{A.14}$$

Thus, the final result of this computation is

$$\hat{E}_i[L_{\text{mat}}]=0\tag{A.15}$$

and this is not an equation but rather an identity. The only way to satisfy it consistently with the fact that L_{mat} indeed carries a dependence on the scalar fields ϕ_i is therefore

$$\delta L_{\text{mat}}=0.\tag{A.16}$$

Appendix B

Identities among higher-order operators

In this Section we describe the relevant identities among the 4-derivatives gauge operators of (5.1) and how to use them to properly deal with the duality transformations of (5.29), (5.52) and (5.76).

B.1 $4F$-sector

The extra operator that appear in the duality transformation of the $4F$-sector dual field (5.29) are

$$F^A_{\alpha\beta}F^{B\,\beta\gamma}F^C_{\gamma\delta}F^{D\,\delta\alpha}, \quad \tilde{F}^A_{\alpha\beta}F^{B\,\beta\gamma}F^C_{\gamma\delta}\tilde{F}^{D\,\delta\alpha}, \quad \tilde{F}^A_{\alpha\beta}F^{B\,\beta\gamma}\tilde{F}^C_{\gamma\delta}F^{D\,\delta\alpha}, \tag{B.1}$$

which can be related to those appearing in (5.19) by exploiting the Levi-Civita tensor contraction rule, which in D spacetime dimension reads

$$\epsilon^{\mu_1\ldots\mu_p\nu_{p+1}\ldots\nu_D}\epsilon_{\mu_1\ldots\mu_p\rho_{p+1}\ldots\rho_D} = \frac{g}{|g|}(D-p)!\;p!\;\delta^{\nu_{p+1}}_{\;[\rho_{p+1}}\ldots\delta^{\nu_D}_{\;\;\rho_D]}, \tag{B.2}$$

and the identity

$$F^A_{\alpha\beta}F^{B\,\beta\gamma}F^C_{\gamma\delta}F^{D\,\delta\alpha} = \frac{1}{4}(F^AF^D)(F^CF^B) + \frac{1}{4}(F^AF^B)(F^CF^D) + \frac{1}{4}(F^A\tilde{F}^C)(F^B\tilde{F}^D). \tag{B.3}$$

The resulting relations between the operators of (B.1) and those of (5.19) are:

- $$\begin{aligned}[\beta_1]_{ABCD}\,F^A_{\alpha\beta}F^{B\,\beta\gamma}F^C_{\gamma\delta}F^{D\,\delta\alpha} =& \frac{1}{2}\,[\beta_1]_{ABCD}\,(F^AF^B)(F^CF^D)+\\ &+\frac{1}{4}\,[\beta_1]_{ACBD}\,(F^A\tilde{F}^B)(F^C\tilde{F}^D),\end{aligned} \tag{B.4}$$

- $$\begin{aligned}[\beta_2]_{ABCD}\,\tilde{F}^A_{\alpha\beta}F^{B\,\beta\gamma}F^C_{\gamma\delta}\tilde{F}^{D\,\delta\alpha} =& \frac{1}{4}\left([\beta_2]_{ACDB}-[\beta_2]_{ACBD}\right)(F^AF^B)(F^CF^D)+\\ &+\frac{1}{4}\,[\beta_2]_{ABCD}\,(F^A\tilde{F}^B)(F^C\tilde{F}^D),\end{aligned} \tag{B.5}$$

- $$[\beta_3]_{ABCD}\,\tilde{F}^A_{\alpha\beta}F^{B\,\beta\gamma}\tilde{F}^C_{\gamma\delta}F^{D\,\delta\alpha} = \left(\frac{1}{2}\,[\beta_3]_{A(B|C|D)} - \frac{1}{4}\,[\beta_3]_{ADBC}\right)(F^A\tilde{F}^B)(F^C\tilde{F}^D), \tag{B.6}$$

where β_1, β_2 and β_3 are couplings function of τ and $\bar{\tau}$ and possess the following symmetries:

$$[\beta_1]_{ABCD} = [\beta_1]_{ADCB} = [\beta_1]_{CBAD}\,, \tag{B.7}$$
$$[\beta_2]_{ABCD} = [\beta_2]_{DCBA}\,, \tag{B.8}$$
$$[\beta_3]_{ABCD} = [\beta_3]_{CDAB}\,. \tag{B.9}$$

As we described in Chapter 5, when acting the duality transformation of the dual field (5.29) we obtain also terms that can be linked to the dependent operators (B.1) as if they were explicitly present in $G^{(4F)}_\Lambda$, while they are instead "hidden" inside the independent ones via the identities (B.4), (B.5) and (B.6). We can find the dual field analogue of such identities by acting, following the dual field definition (4.29), on both sides with the operator

$$\epsilon_{\mu\nu\rho\sigma}\frac{\partial}{\partial F^\Lambda_{\mu\nu}}. \tag{B.10}$$

Exploiting the symmetries (B.7)–(B.9) and redefining the coefficients as

$$[\Omega_1]_{\Lambda BCD} \equiv [\beta_1]_{\Lambda BCD} + [\beta_1]_{B\Lambda DC}\,, \tag{B.11}$$
$$[\Omega_2]_{\Lambda DCB} \equiv [\beta_2]_{\Lambda DCB} - [\beta_2]_{D\Lambda CB}\,, \tag{B.12}$$
$$[\Omega_3]_{\Lambda BCD} \equiv [\beta_3]_{\Lambda BCD} - [\beta_3]_{B\Lambda CD}\,, \tag{B.13}$$

the identities we obtain by applying (B.10) to (B.4)–(B.6) are:

- $$[\Omega_1]_{\Lambda BCD}\, F^{B\,\rho\alpha} F^C_{\alpha\beta} F^{D\,\beta\sigma}\epsilon_{\mu\nu\rho\sigma} = -\frac{1}{2}\Big([\Omega_1]_{\Lambda B(CD)} + [\Omega_1]_{(CD)\Lambda B}\Big)\,(F^C F^D)\tilde{F}^B_{\mu\nu} + \\ -\frac{1}{2}\,[\Omega_1]_{\Lambda(C|B|D)}\,(F^C\tilde{F}^D)F^B_{\mu\nu}; \tag{B.14}$$

- $$[\Omega_2]_{\Lambda DCB}\,\tilde{F}^B_{[\mu|\alpha}F^{C\,\alpha\beta}F^D_{\beta|\nu]} = \frac{1}{2}\,[\Omega_2]_{BDC\Lambda}\,(F^C F^D)\tilde{F}^B_{\mu\nu} + \\ -\frac{1}{2}\,[\Omega_2]_{\Lambda BCD}\,(F^C\tilde{F}^D)F^B_{\mu\nu}; \tag{B.15}$$

- $$[\Omega_3]_{\Lambda BCD}\,F^B_{[\mu|\alpha}\tilde{F}^{C\,\alpha\beta}F^D_{\beta|\nu]} = \frac{1}{2}\,([\Omega_3]_{BD\Lambda C} - [\Omega_3]_{\Lambda BCD})\,(F^C\tilde{F}^D)F^B_{\mu\nu}. \tag{B.16}$$

These identities are then plugged into the duality transformation of the dual field (5.29) and allow to re-adsorb the extra terms into those that have a corresponding term in (5.29). The final result are the transformations (5.32)–(5.35).

B.2 RF-sector

Next, we discuss the identities used in the RF-sector. The extra operators that appear in $\delta G^{(RF)}_\Lambda$ are

$$R_{\alpha\beta}\tilde{F}^{\Lambda\,\alpha\gamma}\tilde{F}^{\Sigma\,\beta}{}_{\gamma}, \quad R_{\alpha\beta\gamma\delta}\tilde{F}^{\Lambda\,\alpha\beta}\tilde{F}^{\Sigma\,\gamma\delta}, \tag{B.17}$$

which are related to (5.20) by the identities

- $$R_{\alpha\beta}\tilde{F}^{\Lambda\,\alpha\gamma}\tilde{F}^{\Sigma\,\beta}{}_{\gamma} = -\frac{1}{2}RF^\Lambda F^\Sigma + R_{\alpha\beta}F^{\Lambda\,\alpha\gamma}F^{\Sigma\,\beta}{}_{\gamma}, \tag{B.18}$$
- $$R_{\alpha\beta\gamma\delta}\tilde{F}^{\Lambda\,\alpha\beta}\tilde{F}^{\Sigma\,\gamma\delta} = RF^\Lambda F^\Sigma + R_{\alpha\beta\gamma\delta}F^{\Lambda\,\alpha\beta}F^{\Sigma\,\gamma\delta} - 4R_{\alpha\beta}F^{\Lambda\,\alpha\gamma}F^{\Sigma\,\beta}{}_{\gamma}. \tag{B.19}$$

Proceeding as we did for the $4F$-sector, we apply on both sides of these identities the dual field operator (B.10) and we obtain, in terms of two generic non-minimal couplings $[\beta_1]_{\Lambda\Sigma}$ and $[\beta_2]_{\Lambda\Sigma}$, symmetric in the gauge indices, their analogue for the duality transformation terms:

- $$[\beta]_{\Lambda\Sigma}\, R_{[\mu|\alpha}\tilde{F}^{\Sigma}_{|\nu]}{}_{\alpha} = \frac{1}{4}\,[\beta]_{\Lambda\Sigma}\, R\tilde{F}^{\Sigma}_{\mu\nu} + \frac{1}{2}\,[\beta]_{\Lambda\Sigma}\, R^{\rho\alpha}F^{\Sigma\,\sigma}{}_{\alpha}\,\epsilon_{\mu\nu\rho\sigma}; \tag{B.20}$$

- $$\begin{aligned}[\beta]_{\Lambda\Sigma}\, R_{\mu\nu\rho\sigma}\tilde{F}^{\Sigma\,\rho\sigma} = [\beta]_{\Lambda\Sigma}\, R\tilde{F}^{\Sigma}_{\mu\nu} + \frac{1}{2}\,[\beta]_{\Lambda\Sigma}\, R^{\rho\sigma\alpha\beta}F^{\Sigma}_{\alpha\beta}\,\epsilon_{\rho\sigma\mu\nu} + \\ + 2\,[\beta]_{\Lambda\Sigma}\, R^{\rho\alpha}F^{\Sigma\,\sigma}{}_{\alpha}\,\epsilon_{\mu\nu\rho\sigma}.\end{aligned} \tag{B.21}$$

These identities yields then the transformations (5.53)–(5.58) of the RF-sector couplings.

B.3 τ-sector

In the case of the τ-sector (5.21) there is only extra operator in the dual field transformation,

$$\tilde{F}^{\Lambda\,\alpha\gamma}\tilde{F}^{\Sigma\,\beta}{}_{\gamma}, \tag{B.22}$$

which is related to (5.21) by the identity

$$[\beta_{\Lambda\Sigma}]_{\alpha\beta}\,\tilde{F}^{\Lambda\,\alpha\gamma}\tilde{F}^{\Sigma\,\beta}{}_{\gamma} = -\frac{1}{2}\,[\beta_{\Lambda\Sigma}]^{\alpha}{}_{\alpha} + [\beta_{\Sigma\Lambda}]_{\alpha\beta}\,F^{\Lambda\,\alpha\gamma}F^{\Sigma\,\beta}{}_{\gamma}, \tag{B.23}$$

where $[\beta]_{\mu\nu}$ is a generic non-minimal coupling such that

$$[\beta_{\Lambda\Sigma}]_{\mu\nu} = [\beta_{\Sigma\Lambda}]_{\nu\mu}\,. \tag{B.24}$$

The analogue of such identity, obtained via (B.10), to be applied in the duality transformation results to be

$$[\beta_{\Lambda\Sigma}]_{[\mu|\alpha}\,\tilde{F}^{\Sigma}_{|\nu]}{}^{\alpha} = -\frac{1}{4}\,[\beta_{\Lambda\Sigma}]^{\alpha}{}_{\alpha}\tilde{F}^{\Sigma}_{\mu\nu} - \frac{1}{2}\,[\beta_{\Sigma\Lambda}]^{\rho\alpha}\,F^{\Sigma\,\sigma}{}_{\alpha}\,\epsilon_{\mu\nu\rho\sigma}, \tag{B.25}$$

yielding then transformations (5.77)–(5.80).

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