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Constraints on Effective Field Theories by Unitarity and Crossing Symmetry

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Introduction

The goal of fundamental physics is to describe and understand physical phenomena in terms of a small and coherent set of principles which emerge by the interplay of experimental validation and theoretical interpretation. The Standard Model (SM) of particle physics is a quantum field theory that provides such a coherent framework encompassing three of the four known fundamental interactions. i.e. the electromagnetic, weak and strong forces. It is a gauge theory based on the group $SU(3)_C \times SU(2)_W \times U(1)_Y$ under which the matter fields (leptons and the quarks) are charged. The SM, starting with the discovery of the gauge bosons $Z/W^\pm$ in 1983, passing through the years of LEP, Tevatron and the flavor factories, and finally getting to the present days of the Large Hadron Collider (LHC), has been repeatedly confirmed experimentally. Its most recent triumphs have perhaps been the discovery of the long sought Higgs boson in 2012, and the direct measurements of (some of) its couplings, in quite good agreement with the predictions of the SM. The null results, so far, of the search for physics beyond the SM may suggest that the SM is in fact the adequate description of nature in a range of energy that extends beyond the Fermi scale. As a matter of fact, the SM seems a quite good and simple description of the dynamics at the TeV scale, too.

Despite such a tremendous experimental success and its internal theoretical consistency, there are reasons of discontent with the renormalizable SM. First of all, the SM is certainly not the ultimate theory and it should be regarded instead as an Effective Field Theory (EFT) with a possibly large, yet finite, cutoff. The Planck mass provides, at very last, such a ultimate cutoff. Indeed, gravitational interactions introduce irrelevant deformations, e.g. the interactions from the Einstein-Hilbert term, which grow with energy and eventually become as important as the other forces at the Planck scale where an ultraviolet (UV) completion, such as string theory, must kick-in. Analogously, non-vanishing neutrino masses can be accommodated by deforming the SM with a dimension-5 operator, again an irrelevant operator, which introduces a new scale to be interpreted e.g. as the mass of heavy right-handed neutrinos. Moreover, most of the matter in the universe, Dark Matter, is actually not accounted by the particles of the SM. Other puzzles of the SM, such as baryogenesis, the origin of flavor, and the strong CP problem, point all toward the existence of physics beyond the SM. Furthermore, besides the irrelevant deformations, any UV threshold generates generically a dangerous relevant operator, the Higgs mass squared term $|H|^2$, which is quadratically sensitivity to those UV scales. This suggests, barring fine-tuning of the parameters of the underlying UV completion, that the cutoff of the SM should be not too far from the TeV scale.

Assuming this "new physics" is separated by a mass gap with respect to the Fermi scale, say above the reach of the LHC, one can (and should) adopt the methods of EFTs in order to study the dynamics in the infrared (IR), i.e. below the cutoff Λ. Within this approach, the leading effects in the IR are captured by an infinite towers of higher dimensional operators

$$\mathcal{L}_{EFT} = \sum \frac{c_i}{\Lambda^{\dim \mathcal{O}_i - 4}} \mathcal{O}_i \tag{0.1}$$

where the Wilson coefficient c_i parametrize the impact of new physics on the low-energy observables. The higher $\dim \mathcal{O}_i$ the more irrelevant the operator becomes at low energy $E \ll \Lambda$: only a finite number of operators is needed for any fixed accuracy, making the EFT predictive. For example, the contribution from a contact term c_i to a $2 \to 2$ scattering amplitude scales as $\sim c_i \left(E/\Lambda\right)^{\dim \mathcal{O}_i - 4}$ and can thus be discarded at sufficiently low-energy, for a given accuracy. Higher dimensional operators $\mathcal{O}_j$ with $\dim\mathcal{O}_j > \dim\mathcal{O}_i$ that contribute to the same observables can be discarded as well, (unless $c_i \ll c_j$ e.g. because of a symmetry). The paradigm of EFTs is very efficient as it retains only the relevant degrees of freedom at low-energy, capturing in a unified and simple way several UV completions at once.
In EFTs symmetries play a fundamental role because they are respected along the renormalization group (RG) flow from the UV to the IR. In other words, a (possibly approximate) symmetry of the UV is as well a (possibly approximate) symmetry of the IR described by the EFT. Symmetries can kill, or suppress by spurions insertions, the Wilson coefficients associated with operators that carry non-trivial representations of the symmetry group. Vice versa, operators which are neutral under the symmetries are expected to have sizable coefficients in the IR, as being generated along the RG flow.

But besides symmetries, the Wilson coefficients are constrained by other fundamental requirements. Indeed, the UV theory is required to be local, causal and unitary. Those conditions imply analyticity, crossing symmetry and unitarity of the scattering matrix. In turn, these UV properties survive in the IR in terms of dispersion relations that provide positivity constraints for certain Wilson coefficients of the EFT. Consider for example the EFT expressed by a Lagrangian density $\mathcal{L} = 1/2(\partial\pi)^2 + c/\Lambda^4(\partial\pi)^4 + ...$ for a Goldstone boson π arising from a spontaneously broken $U(1)$. While any value of the Wilson coefficient c is consistent with the Goldstone shift symmetry $\pi \to \pi + \epsilon$, only $c \geq 0$ is actually generated by unitarity UV completions. Indeed, by means of analyticity, unitarity, and crossing symmetry of the S-matrix, the forward scattering amplitude $\mathcal{M}(s)$ reads [1]

$$\mathcal{M}''(0) = \frac{4}{\pi} \int_0^{+\infty} ds \frac{\sigma^{tot}(s)}{s^3} \tag{0.2}$$

where each $\prime$-symbol represents a derivative with respect to s. This relation provides an IR-UV connection since the left-hand side (l.h.s) is evaluated in the deep IR, $s = t = 0$, where the EFT matches the result of the whole theory by construction, i.e. $\mathcal{M}''(0) = \mathcal{M}''|_{EFT}(0) \propto c$. On the right-hand side (r.h.s.) the integral of the total cross-section is evaluated all the way up to the UV, where the EFT is certainly no longer valid, and one should use there only the UV theory. But in fact, one does not need to know the UV theory nor calculate the integral to determine its sign: $\sigma^{tot}(s) \geq 0$ in any unitary theory, implying the claimed result:

$$\mathcal{M}''(0) \geq 0 \tag{0.3}$$

hence $c \geq 0$. Besides, the inequality is saturated only for the free theory where the total cross-section is vanishing: there is thus no interacting unitary UV theory that produces $c \leq 0$.

These kind of positivity bounds on scattering amplitudes and Wilson coefficients have been recently extended to general EFTs involving scalar particles carrying real representations of an arbitrary symmetry group[1] [2], with the applications being mostly focused on composite Higgs models and WW-scattering extending earlier results, see e.g. [3, 4, 5].

In this thesis, we extend the positivity constraints even further by including spin-1/2 particles along the lines of [6], but allowing arbitrary representations, complex or not, for arbitrary groups. This extension allows us to apply the positivity constraints to fermions of the SM. For example, we use positivity constraints to place bound on the Wilson coefficients associated to certain 4-fermion interactions that are generated within the paradigm of fermion partial compositeness that arises in composite Higgs models[2].

More specifically, we extend the action of crossing symmetry to arbitrary complex representations by studying the general structure of the crossing matrices whenever the symmetry group is non-abelian. We derive the optimal positivity bounds for the eigenamplitudes [3] and apply them to various examples. We discuss a concrete application for physics beyond the SM using data from the LHC combined with our positivities. In particular, we study the intriguing idea that (some of) the SM fermions are composite pseudo-Goldstini that emerge from an enlarged supersymmetry (SUSY) with $\mathcal{N} > 1$ that is fully broken spontaneously by some strong dynamics [9]. The pseudo-Goldstini enjoy a fermionic shift symmetry that makes them light and select only derivative interactions, in full analogy with ordinary Goldstone bosons. In turn, the lowest dimension 4-fermion operators respecting the fermionic shift symmetry have dim-8 because must contain two derivatives, i.e. they are of the schematic form $\chi^{\dagger 2}\partial^2\chi^2$. These operators produce $2 \to 2$ amplitudes that scale as $O(s^2)$, so that the associated Wilson coefficients must respect our positivity bounds. Part of the flavor and gauge groups are identified with the $SU(N)$ factors contained inside the

[1]Hereafter we refer to particles transforming under a real (complex) representation of a symmetry as *real (complex)* particles.

[2]For a review of composite Higgs models and partial compositeness see e.g. [7, 8].

[3]Eigen-amplitudes are nothing but scattering amplitudes in the basis where the S-matrix is diagonal with respect to the conserved quantum numbers. A familiar example is provided by the $SU(2)$ isospin in $\pi\pi$ scattering where the amplitudes decompose, according to $\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5}$, in eigen-amplitudes $\mathcal{M}_{I=\mathbf{1},\mathbf{3},\mathbf{5}}$ associated with isospin-0 (the singlet channel), isospin-1 (the triplet) and isospin-2 (the quintuplet).

$U(\mathcal{N})_R$ R-symmetry that acts linearly on the pseudo-Goldstini. Weakly gauging the SM group and turning-on the Yukawa couplings represents just a small explicit breaking of the non-linearly realized SUSY, again in full analogy with composite Higgs models where the Higgs shift symmetry is also approximate, being broken by gauge and Yukawa interactions. For concreteness, we apply the positivity bounds to the scenario where the three down-type right-handed quarks $\mathbf{d}_R = (d_R, s_R, b_R)$ of the SM are fully composite pseudo-Goldstini from $\mathcal{N} = 9$ SUSY. [4] They transform under R-symmetry as a $\mathbf{9}_{-1/3}$ of $U(9)_R \sim SU(9)_R \times U(1)_R$ which contains $U(1)_Y \times SU(3)_C \times SU(3)_{d_R}$ as subgroup under which the embedding reads $\mathbf{9}_{-1/3} = (\mathbf{3}, \mathbf{3})_{-1/3}$. One the $SU(3)$ factors is identified with $SU(3)_C$ while the other with the flavor $SU(3)_{d_R}$ that rotates the three down-quarks. We picked the down-type quarks just as an illustration of the general idea, the particular choice being motivated by the small bottom Yukawa, $y_b \ll 1$, which thus represents a very small breaking effect of the fermionic shift symmetry. While other choices are certainly possible, including the embedding of more types of quarks or leptons in the same Goldstino multiplet, the details of these extensions are left to future work.
By studying the dijets angular distributions measured at the LHC in run-II at $\sqrt{s} = 13$ TeV, we show that the SUSY decay constant F has to be large enough

$$\sqrt{F} \geq 2.5\text{TeV} \qquad \text{at } 95\% \text{ C.L.} \tag{0.4}$$

in order to make the effect of the derivative 4-fermion interactions compatible with present data. This corresponds roughly to a cutoff $\Lambda \gtrsim 9$ TeV $(g_*/4\pi)^{1/2}$ (for $g_* \gtrsim 3$) where $g_* = 4\pi$ is the typical size for the resonance couplings in a maximally strongly coupled model, according to the SUSY NDA [10].

The thesis is organized as follow. In chapter 1 we recall how positivity bounds are derived using analyticity, crossing symmetry and unitarity for scattering amplitudes of a single flavor real scalar particle. In chapter 2 and 3 we extend the results to several species including complex representations and spin-1/2 particles. We present as well explicit examples based on scattering amplitudes among fundamental and anti-fundamental representations of $SU(N)$. In chapter 4 we discuss the pseudo-Goldstini and put bounds on dim-8 four-fermion operators with two derivatives, using the LHC data. Four appendices with technical but useful material are also included.

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[4] We recall that $\mathcal{N} > 4$ generates no pathology for a non-linearly realized extended SUSY [9]. Indeed, there are no massless higher-spin superpartners, the supermultiplets being incomplete in a spontaneously broken SUSY. Equivalently, while supercurrents are well defined, the supercharges do not actually exist as raising/lowering operators that move from one-particle state to another one-particle state of (one of) its superpartners, when SUSY is spontaneously broken. The would-be superpartners are actually multi-particle states obtained by including Goldstino insertions that in fact raise/lower the spin [34]. Should SUSY be linearly restored at higher energy, the mass splittings in the supermultiplets would still be non-zero, although their effect on hard scattering processes above the cutoff would become smaller as the energy is increased.

Chapter 1

Positivity for scalars

In this chapter we recall how unitarity, crossing symmetry and analyticity of the S-matrix imply rigorous positivity constraints on scattering amplitudes for chargless spin-0 particles. These results represent the simplest examples of amplitudes' positivity that have been discussed in [1], [2] and references therein. Those results are extended to full generality in the next chapters.

1.1 The Unitary S-matrix

One of the central themes of quantum field theory is the study of the S-matrix, i.e. the amplitudes probabilities

$$S_{\beta\alpha} = \langle \Psi^{out}_{\beta} | \Psi^{in}_{\alpha} \rangle \tag{1.1}$$

for the transitions between the states $|\Psi^{in}_{\alpha}\rangle$ and $|\Psi^{out}_{\beta}\rangle$ whose particle content, labelled by the greek subscripts α and β, is defined in the far past (at $t \to -\infty$) and the far future ($t \to +\infty$), respectively[1]. Without interactions, $|\Psi^{in}\rangle$ and $|\Psi^{out}\rangle$ would be the same implying $S_{\beta\alpha} = \delta_{\beta\alpha}$. The rate for interactions and the differential cross-sections that are measured at colliders are thus proportional to $|S_{\beta\alpha} - \delta_{\beta\alpha}|^2$.
In the following we work with orthogonal states

$$\langle \Psi_{\beta} | \Psi_{\alpha} \rangle = N_{\alpha} \delta(\beta - \alpha) \tag{1.2}$$

where N_{α} is a normalization factor, and $\delta(\beta - \alpha)$ stands for products of delta functions and Kronecker deltas summed over all possible permutations according to the spin-statistics. We

[1]Note that we are working in the Heisenberg picture and the $|\Psi^{in(out)}_{\alpha(\beta)}\rangle$ do not represent the asymptotic limits of time-varying states. In this picture, it is in fact the time-dependence of the self-adjoint operators associated with the observables that produce time-varying expectation values. The asymptotic values of the expectations of a complete set of commuting observables, such as the set of 4-momumtum, the spins, the conserved quantum numbers,... define the label $\alpha = (p_1, \sigma_1, q_1; p_2, \sigma_2, q_2; \ldots)$ used for the *in* and *out* states $|\Psi^{in(out)}_{\alpha}\rangle$.

also adopt the relativistic normalization where one-particle states carry a $\sqrt{N} = \sqrt{2E_{\mathbf{p}}(2\pi)^3}$ factor

$$\langle p_\alpha, \sigma_\alpha | k_\beta, \sigma_\beta \rangle = \delta_{\sigma_\alpha \sigma_\beta} (2\pi)^3 2E_{\mathbf{p}} \delta^{(3)}(\mathbf{p} - \mathbf{k}) \,, \tag{1.3}$$

while multi-particle states carry products of those. With this choice, the scalar products are Lorentz invariant [2]. The sum over the states is represented by

$$\int d\alpha = \sum_{\sigma_1 n_1, \sigma_2 n_2, \dots} \int d^3 p_1 d^3 p_2 \dots \tag{1.4}$$

(but sometimes omitting the integral symbol over repeated indexes). For example, the completeness relation for states normalized as in (1.2) reads

$$\mathbb{1} = \int \frac{d\alpha}{N_\alpha} \left| \Psi_\alpha \right\rangle \left\langle \Psi_\alpha \right| \,. \tag{1.5}$$

It is often very convenient to think of the scattering amplitudes as actual matrix elements for an operator S sandwiched between free particle states $|\Phi_\alpha\rangle$

$$S_{\alpha\beta} = \langle \Phi_\alpha | S | \Phi_\beta \rangle \tag{1.6}$$

that have the same spectrum (that is the collection of possible $\{\alpha\}$) and normalization

$$H_0 |\Phi_\alpha\rangle = E_\alpha |\Phi_\alpha\rangle \,, \qquad \langle \Phi_\alpha | \Phi_\beta \rangle = N_\alpha \delta(\alpha - \beta) \,. \tag{1.7}$$

of the initial and final states $|\Psi^{in,out}\rangle$, for a suitable choice of the free hamiltonian H_0. Assuming that at $t = \pm\infty$ interactions are negligible, the *in* and *out* states are basically defined as the eigenvectors of the full Hamiltonian H,

$$H |\Psi_\alpha^{in(out)}\rangle = E_{\alpha(\beta)} |\Psi_\alpha^{in(out)}\rangle \tag{1.8}$$

with initial(final) conditions given by the $|\Phi_\alpha\rangle$, meaning

$$\int d\alpha e^{-iE_\alpha t} g(\alpha) |\Psi_\alpha^{in,out}\rangle \longrightarrow \int d\alpha e^{-iE_\alpha t} g(\alpha) |\Phi_\alpha\rangle \tag{1.9}$$

as $t \to -\infty$ or $t \to +\infty$, respectively. As this holds for any wave-packet $g(\alpha)$, one can formally write

$$|\Psi_\alpha^{in,out}\rangle = \Omega(\mp\infty) |\Phi_\alpha\rangle \,, \qquad \Omega(t) \equiv e^{iHt} e^{-iH_0 t} \,. \tag{1.10}$$

It follows that the scattering matrix (1.1) can be expressed in terms of actual "matrix elements" between free-particle states

$$S_{\beta\alpha} = \langle \Psi_\beta^{out} | \Psi_\alpha^{in} \rangle = \langle \Phi_\beta | \Omega(+\infty)^\dagger \Omega(-\infty) | \Phi_\alpha \rangle \equiv \langle \Phi_\beta | S | \Phi_\alpha \rangle \,, \tag{1.11}$$

for the S-matrix operator formally defined by

$$S = \Omega(+\infty)^\dagger \Omega(-\infty) \,. \tag{1.12}$$

[2]We work in the mostly minus signature of the Lorentz metric.

Since for free theories $S = \mathbb{1}$, it is actually convenient to define a scattering amplitude operator $\mathcal{M}$

$$S = \mathbb{1} + (2\pi)^4 \delta^{(4)} \left(\sum_i p_i \right) i\mathcal{M} \tag{1.13}$$

where the trivial evolution is removed.

1.1.1 Unitarity and the optical theorem

Crucially enough, the S-matrix is unitary

$$S^\dagger S = \mathbb{1}\,, \qquad SS^\dagger = \mathbb{1} \tag{1.14}$$

as one can directly check by means of the completeness relation[3] (1.5), i.e.

$$\langle \Phi_\alpha | S^\dagger S | \Phi_\beta \rangle = \int \frac{d\gamma}{N_\gamma} S^*_{\gamma\alpha} S_{\gamma\beta} = \int \frac{d\gamma}{N_\gamma} \langle \Psi^{in}_\alpha | \Psi^{out}_\gamma \rangle \langle \Psi^{out}_\gamma | \Psi^{in}_\beta \rangle = N_\alpha \delta(\alpha - \beta) = \langle \Phi_\alpha | \mathbb{1} | \Phi_\beta \rangle \tag{1.15}$$

and

$$\langle \Phi_\alpha | S S^\dagger | \Phi_\beta \rangle = \int \frac{d\gamma}{N_\gamma} S_{\alpha\gamma} S^*_{\beta\gamma} = \int \frac{d\gamma}{N_\gamma} \langle \Psi^{out}_\alpha | \Psi^{in}_\gamma \rangle \langle \Psi^{in}_\gamma | \Psi^{out}_\beta \rangle = N_\alpha \delta(\alpha - \beta) = \langle \Phi_\alpha | \mathbb{1} | \Phi_\beta \rangle \tag{1.16}$$

for any $|\Phi_\alpha\rangle$ and $|\Phi_\beta\rangle$.
Unitarity of the S-matrix has important consequences on the scattering amplitudes, such as the optical theorem which gives a non-perturbative relation between the imaginary part of the amplitudes and the total cross-sections. Following e.g. [11, 12] we define the $\mathcal{T}$ matrix

$$S = \mathbb{1} + i\mathcal{T}\,, \qquad \langle \Phi_\beta | \mathcal{T} | \Phi_\alpha \rangle = (2\pi)^4 \delta^4 (p_\alpha - p_\beta) \mathcal{M}_{\beta\alpha} \tag{1.17}$$

and using (1.14) we get

$$i(\mathcal{T}^\dagger - \mathcal{T}) = \mathcal{T}^\dagger \mathcal{T}. \tag{1.18}$$

The matrix elements of the r.h.s. reads

$$\begin{aligned} \langle \Phi_\beta | i(\mathcal{T}^\dagger - \mathcal{T}) | \Phi_\alpha \rangle &= i \langle \Phi_\alpha | \mathcal{T} | \Phi_\beta \rangle^* - i \langle \Phi_\beta | \mathcal{T} | \Phi_\alpha \rangle & (1.19) \\ &= i (2\pi)^4 \delta^4 (p_\alpha - p_\beta) (\mathcal{M}^*_{\alpha\beta} - \mathcal{M}_{\beta\alpha}). & (1.20) \end{aligned}$$

while the r.h.s. can be written as

$$\begin{aligned} \langle \Phi_\beta | \mathcal{T}^\dagger \mathcal{T} | \Phi_\alpha \rangle &= \int \frac{d\gamma}{N_\gamma} \langle \Phi_\beta | \mathcal{T}^\dagger | \Phi_\gamma \rangle \langle \Phi_\gamma | \mathcal{T} | \Phi_\alpha \rangle & (1.21) \\ &= \int \frac{d\gamma}{N_\gamma} (2\pi)^4 \delta^4 (p_\beta - p_\gamma) (2\pi)^4 \delta^4 (p_\alpha - p_\gamma) \mathcal{M}^*_{\gamma\beta} M_{\gamma\alpha}. & (1.22) \end{aligned}$$

[3] Trivially, the same completeness relation holds for the free fields $|\Phi_\alpha\rangle$.

having inserted a complet set of states. Therefore, we derived the generalized optical theorem

$$\mathcal{M}_{\beta\alpha} - \mathcal{M}^*_{\alpha\beta} = i \int \frac{d\gamma}{N_\gamma}(2\pi)^4 \delta^4(p_\alpha - p_\gamma)\mathcal{M}^*_{\gamma\beta} M_{\gamma\alpha}. \tag{1.23}$$

Whenever the initial and final states are equals, that is $\alpha = \beta$ (the so-called elastic forward scattering), we get

$$2\text{Im}\mathcal{M}_{\alpha\alpha} = \int \frac{d\gamma}{N_\gamma}(2\pi)^4 \delta^4(p_\alpha - p_\gamma)|M_{\gamma\alpha}|^2, \tag{1.24}$$

that is the imaginary part of the forward elastic scattering is a sum (integral) of squared matrix elements for the transition amplitudes $\alpha \to \gamma$ for any γ that is kinematically open. The most important consequence of this equation that we use in the following is the positivity of the r.h.s. that enforces the same for the l.h.s.

$$\boxed{\text{Im}\mathcal{M}_{\alpha\alpha} \geq 0}\,. \tag{1.25}$$

Notice that only the free theory, where none of the transitions in allowed, may saturate this inequality: in any interacting theory $\text{Im}\mathcal{M}_{\alpha\alpha} > 0$.
We can actually say more about the r.h.s. of (1.24). Recalling the expression for the total cross-section of the transition $\alpha \to$ *any* in the center of mass frame for an initial state α containing just two particles,

$$\sigma(2 \to \text{anything}) = \frac{1}{4E_{cm}|\mathbf{p}_i|}\int \frac{d\gamma}{N_\gamma}(2\pi)^4 \delta^4(p_\alpha - p_\gamma)|M_{\gamma\alpha}|^2\,, \tag{1.26}$$

we see that (1.24) implies

$$\boxed{\text{Im}\mathcal{M}_{2\to 2}(s)\big|_{elastic\,forward} = \sqrt{(s-m_1^2-m_2^2)^2 - 4m_1^2m_2^2}\cdot \sigma^{\text{tot}}_{2\to\text{anything}}(s)}\,. \tag{1.27}$$

The m_i are the masses of the initial particles, and $\sigma^{\text{tot}}_{\alpha\to\text{anything}}(s)$ is the total cross section for α into any final state that is kinematically open. The positivity of the imaginary part of the elastic forward scattering can be thus understood as the positivity of the total cross-section.

1.2 Crossing symmetry for spin-0 particles

Crossing symmetry is an important property of the scattering amplitudes. It is a duality that relates the amplitudes for two different scattering processes evaluated at two (mutually unphysical) set of momenta, and where the "crossed" particles are replaced by their anti-particles, swapping them from the initial to the final state. For example, consider the elastic scattering between two spin-0 particles[4]

$$\pi(p_1)\phi(p_2) \to \pi(p_3)\phi(p_4) \qquad s\text{-channel} \tag{1.28}$$

[4] Crossing symmetry for particles with spins is discussed in Section 3.1.

and its $s \leftrightarrow u$ "crossed" process

$$\pi(p_1)\overline{\phi}(p_2) \to \pi(p_3)\overline{\phi}(p_4) \qquad u\text{-channel.} \tag{1.29}$$

where particles 2 and 4 have been crossed. Crossing symmetry relates these amplitudes in the following way

$$\mathcal{M}_{\pi\phi\to\pi\phi}(p_1, p_2; p_3, p_4) = \mathcal{M}_{\pi\overline{\phi}\to\pi\overline{\phi}}(p_1, -p_4; p_3, -p_2) \qquad p_i^0 > 0 \tag{1.30}$$

or equivalently

$$\mathcal{M}_{\pi\overline{\phi}\to\pi\overline{\phi}}(p_1, p_2; p_3, p_4) = \mathcal{M}_{\pi\phi\to\pi\phi}(p_1, -p_4; p_3, -p_2) \qquad p_i^0 > 0\,. \tag{1.31}$$

Notice that the right-hand side of these crossing relations requires the evaluation of the amplitude at the unphysical kinematical point where the anti-particles have negative energy. In other words, the same function $\mathcal{M}$ defines either processes, the s-channel or u-channel scattering, depending on the sign of the energy which determines whether a particle (or anti-particle) belongs to the initial or final state.
Crossing symmetry for real scalar particles is easily understood via the Lehmann-Symanzik-Zimmerman (LSZ) reduction formula [11, 12]

$$\begin{aligned}\langle p_3, ..., p_n|S|p_1, p_2\rangle = &\left[i\int d^4x_1 e^{-ip_1\cdot x_1}\left(\Box_1 + m^2\right)\right] ... \left[i\int d^4x_1 e^{+ip_n\cdot x_n}\left(\Box_n + m^2\right)\right]\\ &\times \langle 0|T\{\phi(x_1)\phi(x_2)...\phi(x_n)\}|0\rangle\end{aligned} \tag{1.32}$$

that expresses the S-matrix elements in terms of the correlation functions stripped off their external propagators[5]. In this expression the only distinction between initial and final states is given by the sign of the 4-momentum, in agreement with Eq. (1.30) and (1.31) because we have chosen for simplicity indentical real scalars $\pi = \phi = \overline{\phi}$. The generalization to more species and complex scalars is trivial, whereas the discussion for fermions is more subtle and is presented in section 3.1 following the results of [6].
Since the scattering amplitudes among spin-0 particles are Lorentz-invariant, the $2 \to 2$ processes can be conveniently expressed in terms of the Mandelstam variables s, t, u, namely

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 \qquad t = (p_1 - p_3)^2 = (p_2 - p_4)^2 \qquad u = (p_1 - p_4)^2 = (p_2 - p_3)^2. \tag{1.33}$$

Note that only two of these variables are independent because

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2\,. \tag{1.34}$$

For elastic processes among scalars like those in (1.28) and (1.29), crossing the $s-$ and u-channel is equivalent to $p_2 \leftrightarrow -p_4$, that is

$$s \longleftrightarrow u \tag{1.35}$$

[5]The wave-functions renormalization constant have been absorbed in the definition of the fields.

where t is held fixed. (Incidentally, this transformations at the level of Mandelstam variables justifies the choice for the names of the channels). On the amplitudes it reads

$$\boxed{\mathcal{M}_{\pi\phi\to\pi\phi}(s,t,u) = \mathcal{M}_{\pi\bar{\phi}\to\pi\bar{\phi}}(u,t,s)}\,. \tag{1.36}$$

The proper meaning of this expression relies on the analytic continuation of the amplitude away from the physical configuration, as it is explained in the section 1.3. Notice that for identical real massless spin-0 particles, the forward scattering $t=0$ must be an even function in s because $u=-s$,

$$\boxed{\mathcal{M}_{\pi\pi\to\pi\pi}(s,t=0,u) = \mathcal{M}_{\pi\pi\to\pi\pi}(u,t=0,s)}\,. \tag{1.37}$$

For a general kinematics the amplitude for real identical scalars is symmetric under the inversion $s \leftrightarrow u$ which leaves invariant the point $s = u = m_1^2 + m_2^2 - t/2$.
There is another crossed channel, the t-channel, that is obtained by exchanging $s \leftrightarrow t$ at fixed u, for scalars. In the following, however, we are interested in dispersion relation in the complex s-plane for elastic scattering at fixed $t=0$ (forward scattering), where only the $s \leftrightarrow u$ crossing plays a role.

1.3 Analyticity

Another key property of the S-matrix that we use in the following is its analyticity with respect to the external particles' momenta, in particular the Mandelstam variable s at fixed $t=0$, in the forward elastic scattering $\pi(p_1)\phi(p_2) \to \pi(p_1)\phi(p_2)$. In order to investigate this property, we need to recast the LSZ reduction formula (1.32) in terms of the retarded commutators of local fields,

$$\mathcal{M}_{\pi\phi\to\pi\phi}(s,t=0) = i\int d^4y e^{+ip_2\cdot y}\left(\Box_y + m_2^2\right)^2 \theta(y^0)\langle p_1|[\phi(y),\phi(0)]|p_1\rangle\,, \tag{1.38}$$

as explained in appendix A. Here we hare reduced only the contribution from the π's. Because of the microcausality condition on the commutator (and its derivatives) which makes them vanish at spacelike distances, as well as the occurrence of the step function $\theta(y^0)$ (and its derivatives), they give vanishing contributions in the integrand outside of the forward light-cone $\{y^2 \geq 0, y^0 \geq 0\}$. In turn, such a causal structure allows to analytically continue the forward amplitude $\mathcal{M}$ in the upper complex s-plane, assuming polynomially bounded correlation functions. Conversely, the physical amplitude can be read as the upper boundary value of an analytic function in the whole upper complex s-plane:

$$\mathcal{M}_{\pi\phi\to\pi\phi}(s,t=0) = \mathcal{M}_{\pi\phi\to\pi\phi}(s+i\epsilon,t=0)\,, \tag{1.39}$$

where the $\epsilon \to 0^+$ limit is always understood, and $s \geq s_{min} = (m_1+m_2)^2$. [6]

[6] Every time s satisfies an inequality it is implicitly taken on the real axis.

This is fully analogous to the classical Kramers-Kronig relations where the retarded Green functions appear as the result of analyticity of the index of refraction with respect to the frequency in the upper complex plane, and vice versa. The analytic continuation for scattering amplitudes is however somewhat more involved because of the several complex momenta; for this reason we leave the details to appendix A.

We stress here, however, that the amplitude can be analytically extended in the lower complex s-plane too. Indeed, it is enough that the amplitude takes real values $\mathcal{M}(s,t=0)=\mathcal{M}^*(s,t=0)$ over an open interval of the real axis (e.g. below threshold) so that the Schwarz reflection principle extends it to an analytical and real function of complex variable everywhere in the cut s-plane

$$\boxed{\mathcal{M}_{\pi\phi\to\pi\phi}(s^*,t=0)=\mathcal{M}^*_{\pi\phi\to\pi\phi}(s,t=0)}\,, \tag{1.40}$$

except for discontinuities located on the real axis. Those are branch-cuts associated to multiparticle production, or simple poles associated to stable one-particle states, see Fig.1 . As a matter of fact, crossing symmetry gives physical meaning to the boundary amplitude on the real axis approached from below

$$\boxed{\mathcal{M}_{\pi\phi\to\pi\phi}(s-i\epsilon,t=0)=\mathcal{M}^*_{\pi\overline{\phi}\to\pi\overline{\phi}}(-s+i\epsilon+2m_1^2+2m_2^2,t=0)}\,, \tag{1.41}$$

$$\boxed{\mathcal{M}_{\pi\overline{\phi}\to\pi\overline{\phi}}(-s-i\epsilon+2m_1^2+2m_2^2,t=0)=\mathcal{M}_{\pi\phi\to\pi\phi}(s+i\epsilon,t=0)}\,. \tag{1.42}$$

On the other hand, the optical theorem (1.27) provides the discontinuities across the real axis in the physical regions, e.g.

$$\mathrm{Disc}\,\mathcal{M}_{\pi\phi\to\pi\phi}(s+i\epsilon,t=0)=2i\sqrt{(s-m_1^2-m_2^2)^2-4m_1^2m_2^2}\cdot\sigma^{\mathrm{tot}}_{\pi\phi\to\mathrm{anything}}(s) \tag{1.43}$$

for $s\geq s_{min}$, and

$$\mathrm{Disc}\,\mathcal{M}_{\pi\overline{\phi}\to\pi\overline{\phi}}(s+i\epsilon,t=0)=-2i\sqrt{(s-m_1^2-m_2^2)^2-4m_1^2m_2^2}\cdot\sigma^{\mathrm{tot}}_{\pi\overline{\phi}\to\mathrm{anything}}(u)\,. \tag{1.44}$$

for $s<u_{min}=-s_{min}+2m_1^2+2m_2^2$.

1.4 Positivity for flavourless scalar particles

In this section we discuss the simplest example of positivity contraint that arises from unitarity, analyticity and crossing symmetry of the S-matrix that we have studied in the previous sections. We consider a theory of a single massive real scalar particles π and study the two-body scattering. For simplicity, we drop the particle label, and leave the $t=0$ in the argument understood, that is

$$\mathcal{M}_{\pi\pi\to\pi\pi}(s,t=0)\equiv\mathcal{M}(s)\,. \tag{1.45}$$

Moreover, we demand there are no lighter states than π which could be exchanged the scattering. As an example of interesting EFT that satisfies there requirements one could

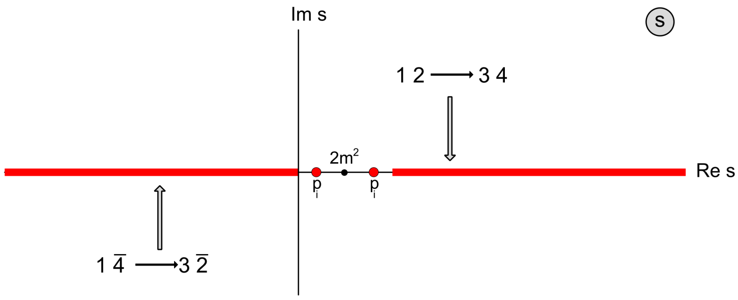

Fig. 1: Analytic structure in the $s-$ plane for the scalars scattering amplitude $\pi(p_1)\phi(p_2) \to \pi(p_3)\phi(p_4)$ in the forward limit $t = 0$ and with $m_1 = m_2 = m$. There may exist poles between the branch-cuts shown in the figure with the points p_i. We identify the amplitude of the $s-$channel by approaching the right cut from above, whereas the physical amplitude of the crossed reaction in the s plane can be obtaining by approaching the left cut from below.

keep in mind the lagrangian for a Goldstone Boson (GB) π from a spontaneously broken $U(1)$

$$\mathcal{L} = \frac{1}{2}(\partial\pi)^2 + \frac{c}{\Lambda^4}(\partial\pi)^4 + \dots \tag{1.46}$$

perturbed by a small mass term $-m^2\pi^2/2$, with m arbitrarily small[7]. As it should be clear from the general derivation presented below, the resulting positivity conditions are not specific of this example only but apply to any theory with this analytic structure.
Because of analyticity away from the real axis or below the elastic threshold, we can expand the amplitude around a point μ^2 in the complex cut s-plane as

$$\mathcal{M}(s) = \mathcal{M}(\mu^2) + \mathcal{M}'(\mu^2)(s-\mu^2) + \frac{1}{2!}\mathcal{M}''(\mu^2)(s-\mu^2)^2 + \dots \tag{1.47}$$

The coefficient $\mathcal{M}''(\mu^2)$ can be obtained by means of the Cauchy integral formula

$$\mathcal{M}''(\mu^2) = \frac{2!}{2\pi i}\oint_{\mathcal{C}} \frac{\mathcal{M}(s)}{(s-\mu^2)^3}, \tag{1.48}$$

where $\mathcal{C}$ is a contour shown in Fig.2 .
There are no singularities between $s = 0$ and $s = 4m^2$ because of the absence of light intermediate states. The contour integral can be smoothly deformed into the integral over the curve Γ, see Fig.2 , as long as we do not cross singularities. We have seen in the previous sections that there are none in the upper and lower complex cut s-plane. Thanks to the Schwartz reflection principle we have $\mathcal{M}(s^*) = \mathcal{M}^*(s)$ and in particular

$$\mathcal{M}^*(s+i\epsilon) = \mathcal{M}(s-i\epsilon) \tag{1.49}$$

[7]When we will apply the general positivity constraint (1.55) to this particular example we will eventually send the mass to zero at the end of the calculation.

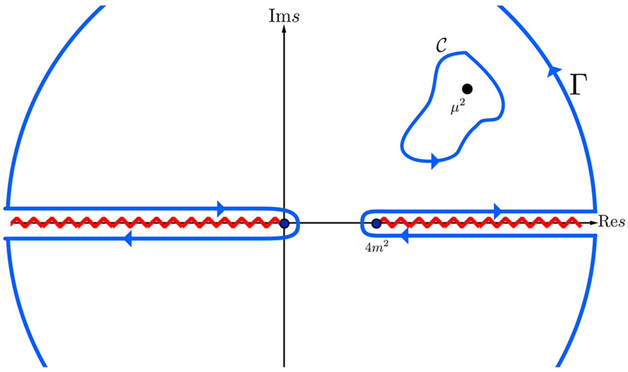

Fig. 2: Analytic structure of a forward scattering amplitude of massive scalars in the configuration $s+u=4m^2$ without lighter intermediate states. We show the contours along which we perform the integrals.

across the branch-cuts, implying that the discontinuities are the imaginary parts of the amplitude. Sending to $+\infty$ the radius of the big circle in Γ, the contour integral can thus be organized into the sum of three pieces

$$\mathcal{M}''(\mu^2) = C_\infty + \left(\int_{-\infty}^{0} \frac{ds}{(s-\mu^2)^3} + \int_{4m^2}^{+\infty} \frac{ds}{(s-\mu^2)^3}\right) \mathrm{Im}\mathcal{M}(s+i\epsilon) \tag{1.50}$$

where C_∞ is the contribution of the integral along the big circle whose radius is sent to infinity. This latter term can actually be discarded because the Froissart bound [13] ensures that $|\mathcal{M}(s)| \leq s\log^2 s$ for $s \to \infty$, and therefore $C_\infty \to 0$.
We can further simplify (1.50) by changing variable on the second integral $s \to u = -s+4m^2$

$$\int_{-\infty}^{0} ds \frac{\mathrm{Im}\mathcal{M}(s+i\epsilon)}{(s-\mu^2)^3} = -\int_{4m^2}^{\infty} ds \frac{\big(\mathcal{M}(-s+4m^2+i\epsilon) - \mathcal{M}(-s+4m^2-i\epsilon)\big)}{(-s+4m^2-\mu^2)^3}. \tag{1.51}$$

By crossing symmetry

$$\mathcal{M}(u) = \mathcal{M}(s) \tag{1.52}$$

and therefore

$$\int_{-\infty}^{0} ds \frac{\mathrm{Im}\mathcal{M}(s+i\epsilon)}{(s-\mu^2)^3} = \int_{4m^2}^{\infty} ds \frac{\mathrm{Im}\mathcal{M}(s+i\epsilon)}{(s-4m^2+\mu^2)^3} \tag{1.53}$$

which in turns gives the dispersion relation

$$\boxed{\mathcal{M}''(\mu^2) = \int_{4m^2}^{+\infty} ds \left(\frac{1}{(s-\mu^2)^3} + \frac{1}{(s-4m^2+\mu^2)^3}\right) \mathrm{Im}\mathcal{M}(s+i\epsilon)} \tag{1.54}$$

where the second derivative of the amplitude is expressed in terms of an integral along the imaginary parts. The imaginary parts of the forward elastic amplitude in the r.h.s. are positive in any interacting theory, see the inequalities (1.25) and (1.27), because of unitarity: $\mathrm{Im}\mathcal{M}(s+i\epsilon) = s\sqrt{1-4m^2/s}\,\sigma_{\pi\pi\to\text{anything}} > 0$. By choosing any real scale μ^2 between $s=0$ and $s=4m^2$ such that the denominators in (1.53) are positive as well, we get the positivity condition

$$\boxed{\mathcal{M}''(\mu^2) > 0} \tag{1.55}$$

in any interacting theory that admits an unitary UV completion. Notice that it is convenient to choose the scale μ^2 just below $s=4m^2$ which is a physical threshold where the scattering opens kinematically. For smaller values of μ^2, the amplitude is still positive but it should be run to another scale $\tilde{\mu}^2$, above or at threshold, where the experiments are performed and the Wilson coefficients measured in principle.
Let's see what the positivity constraint (1.55) implies on the EFT (1.46), e.g. for $m\to 0$. First, we can use (1.46) to calculate the l.h.s. of (1.55) because $s=t=0$ is a kinematical point in the deep IR: we have thus $\mathcal{M}(s) = 2cs^2/\Lambda^4$ and therefore

$$\boxed{c > 0}\,. \tag{1.56}$$

This condition means that there are no non-trivial UV completions of this EFT for the GB that produce $c \leq 0$. This positivity conditions is clearly beyond the normal constraints provided by the symmetry as any value of c would actually be allowed by the GB's shift symmetry.

We can easily extend these results to scattering amplitudes of theories with different analytic structure, i.e. with singularities on the real axis between the elastic branch-cuts that start at $s=4m^2$ and $s=0$, that is to theories with light intermediate states. For example, let us assume there are stable intermediates states that show up as poles on the real axis as in Fig.3 .
Now, the relation (1.54) becomes

$$\mathcal{M}''(\mu^2) = \int_{4m^2}^{+\infty} ds \left(\frac{1}{(s-\mu^2)^3} + \frac{1}{(s-4m^2+\mu^2)^3}\right)\mathrm{Im}\mathcal{M}(s+i\epsilon) - \sum_i \frac{\mathrm{Res}\left[\mathcal{M}(s)\right]\Big|_{s=p_i}}{(p_i-\mu^2)^3} \tag{1.57}$$

where p_i are the locations of extra poles. Following the same steps as above, and moving the residues of the simple poles on the left-hand side, we end up with the positivity condition

$$\mathcal{M}''(\mu^2) + \sum_i \frac{\mathrm{Res}\left[\mathcal{M}(s)\right]\Big|_{s=p_i}}{(p_i-\mu^2)^3} > 0. \tag{1.58}$$

We can calculate the left-hand side of the inequality within the EFT, given that $t=0$ and $\mu^2 \ll \Lambda^2$. The tree-level EFT has no branch-cut and the sum of the residues in the IR is the same as (minus) the residue at infinity calculated with the EFT lagrangian, meaning that

$$\boxed{\mathcal{M}''(p_i \ll \mu^2 \ll \Lambda^2)\big|_{EFT} > 0\,.}\,. \tag{1.59}$$

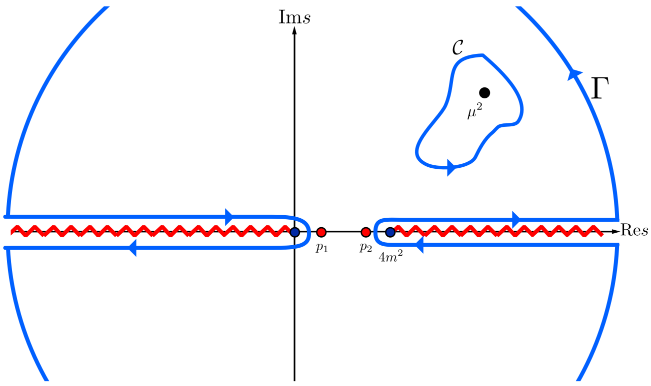

Fig. 3: General analytic structure of a forward scattering amplitude of massive scalars in the configuration $s + u = 4m^2$. The points p_i stand for poles coming from intermediate one-particle states, i.e. propagators.

This relation expresses the fact that the dispersion relations allow one to place positivity constraints on the leading $O(s^2)$ coefficients produced by the EFT, for s larger than any other IR scale (e.g. the small masses of the light intermediate states) but still below the cutoff of the EFT.

Finally, It is instructive to give an example of explicit UV completion and see where the positivity constraint seen in the IR has originated from in the UV. Let's consider again the example of a spontaneously broken $U(1)$ with its GB described in the IR by the lagrangian (1.46), where we have seen that the Wilson coefficient c must satisfy the constraint (1.56). At the leading order, the effective amplitude of a generic process $\pi\pi \to \pi\pi$ is

$$\mathcal{M}(s,t) = \frac{c}{\Lambda^4}\left(s^2 + t^2 + u^2\right), \qquad \mathcal{M}(s) \equiv \mathcal{M}(s, t=0) = \frac{2c}{\Lambda^4}\left(s^2\right). \tag{1.60}$$

An exampe of calculable UV completion for this theory is a linear sigma model from which the π arise as the only states in the spectrum below the mass of the radial (or Higgs-like) mode. The lagrangian reads

$$\mathcal{L} = \partial_\mu \Phi^* \partial^\mu \Phi - \lambda\left(|\Phi|^2 - \frac{v^2}{2}\right)^2 \tag{1.61}$$

where we reparametrize the fields as $\Phi(x) = \frac{1}{\sqrt{2}}(v + h(x))e^{i\pi(x)/v}$, and

$$\lambda > 0 \tag{1.62}$$

because of vacuum stability and to ensure the desired IR spectrum with a massless particle. Expanding the terms around the vev v, the lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\left(1 + \frac{h}{v}\right)^2 (\partial\pi)^2 + (\partial h)^2 - \frac{1}{2}M_h^2 h^2 - \lambda v h^3 - \frac{\lambda}{4}h^4\,, \tag{1.63}$$

where $M_h^2 = 2\lambda v^2$. The forward amplitude (1.60) is now extended to values of s above the cutoff, that is above the mass M_h

$$\mathcal{M}(s) = \frac{\lambda}{M_h^2}\left[\frac{s^2}{s+M_h^2} - \frac{s^2}{s-M_h^2}\right] . \tag{1.64}$$

Expanding for small values of s with respect to M_h^2, i.e. integrating out the field h, we get the matching condition between the IR and UV

$$\frac{c}{\Lambda^4} = \frac{\lambda}{M_h^4} , \tag{1.65}$$

where the positivity of c has been tracked back to the positivity of the UV parameter λ. More generally, the $\mathcal{M}''(\mu^2)$ can be expressed in principle in terms of the parameters of the UV theory which are responsible for its positivity.

Chapter 2

Positivity constraints for complex scalars

In the previous chapter we showed how crossing symmetry works for real scalar particles, and how it can be used together with unitarity and analyticity of the S-matrix to derive the positivity of the second derivative with respect to s of the forward elastic amplitudes. Now we want to extend these results to complex scalars which may transform under some internal symmetry group. We first discuss how crossing symmetry works when particles carry representations of non-abelian symmetry groups. In particular we study the properties of the crossing matrices associated to $s \leftrightarrow u$ crossing between the irreducible representations found in the decomposition of the initial and final two-particle states, in the 2-to-2 scattering. This allows us to obtain sum rules and positivity constraints involving the Wilson coefficients of the EFTs, generalizing the results of [2] to arbitrary complex irreps. As explicit example we discuss in detail the positivity constraints for the scattering amplitudes of particles carrying fundamental and anti-fundamental representations of $SU(N)$. This example is relevant for the model building involving the pseugo-Goldstini presented in chapter 4.

2.1 Internal symmetries and eigen-amplitudes

A special role in the classification of fundamental interactions is played by internal symmetries, that is symmetries that commute with the generators of Poincaré. Examples are the nuclear isospin, flavor symmetries as well as the gauge groups of the SM. These transformations act on the Hilbert space of the states as unitary operators $U(g)$ which act on the internal labels without changing the momenta and spins of the particles. For example, considering a generic element g in the symmetry group G g, a state with two scalar particles transforms as

$$U(g)\,|\Psi_{p_1,n_1;p_2,n_2}\rangle = \sum_{\tilde{n}_1,\tilde{n}_2} \mathcal{D}^{\mathbf{r}_1}(g)_{\tilde{n}_1,n_1}\mathcal{D}^{\mathbf{r}_2}(g)_{\tilde{n}_2,n_2}\,|\Psi_{p_1,\tilde{n}_1;p_2,\tilde{n}_2}\rangle \tag{2.1}$$

where $\mathcal{D}^{\mathbf{r}_i}(g)$ are the matrix elements of the representations $\mathbf{r}_i$ of the symmetry group G carried by the two particles.
A symmetry of the dynamics is a symmetry of the S-matrix elements [11, 12], meaning

$$\langle\Phi_\beta|S|\Phi_\alpha\rangle = \langle\Phi_\beta|U(g)^\dagger SU(g)|\Phi_\alpha\rangle \qquad \forall|\Phi_{\alpha,\beta}\rangle \tag{2.2}$$

that is $S = U(g)^\dagger SU(g)$. In other words, the S-matrix is invariant, that is a scalar (or singlet), under the action of the symmetry. Lie groups are continuous groups where the elements around the identity can be reach by an expansion in the generators Y^n

$$U(g) \simeq \mathbb{1} + i\alpha_n Y^n + ... \tag{2.3}$$

Applying this expansion to (2.2) it follows the commutation rule

$$[Y^n, S] = 0. \tag{2.4}$$

It is often very convenient to decompose the initial and final multiparticle states $|\Phi_\alpha\rangle$ that transform as the tensor product of single-particle states (that is as in (2.1)) in irreducible representations (irreps) $\mathbf{r}_{I(\xi)}$ of the symmetry group. For example, for a two-particle state

$$\mathbf{r}_1 \otimes \mathbf{r}_2 = \bigoplus_{I(\xi)} \mathbf{r}_{I(\xi)}\,, \tag{2.5}$$

where I labels the inequivalent irrep (e.g. the collection of its Casimirs) whereas ξ counts how several times the (equivalent) irrep I appears in the decomposition[1]. Because the S-matrix is a singlet under a symmetry, $S|\Phi_\alpha\rangle$ has the same decomposition in irreps than $|\Phi_\alpha\rangle$. Thanks to the Wigner-Eckart theorem, there are no transitions between states $|I(\xi), i\rangle$ carrying inequivalent irreps

$$\langle J(\eta), j|S|\, I(\xi), i\rangle = \delta_{IJ}\delta_{ij} S_{I(\xi\eta)}\,, \tag{2.6}$$

where the indexes i and j label the particular states inside the multiplets $I(\xi)$ and $J(\eta)$ respectively. The transitions do not depend on the particular representative state inside the irrep one is picking. The $S_{I(\xi\eta)}$ are known as reduced-matrix elements and the associated amplitudes $\mathcal{M}_{I(\xi\eta)}$ are called *eigen-amplitudes*. The latter name is very appropriate especially for non-degenerate irreps (those that do not appear more than once in the decomposition such that the labels ξ and η are not needed), since the amplitudes are diagonal in this basis for the internal space, $\mathcal{M}_I$ being the associated "eigenvalues". For example, in the $\pi\pi$-scattering of pions the $SU(2)$-isospin decomposition reads $\mathbf{3}\otimes\mathbf{3} = \mathbf{1}\oplus\mathbf{3}\oplus\mathbf{5}$ where we have labeled the irreps by their dimensions. There is no equivalent irrep that appears more than once in the decomposition, hence $\langle J, j|\mathcal{M}|\, I, i\rangle = \delta_{ij}\delta_{IJ}\mathcal{M}_I$ where $I = \mathbf{1}, \mathbf{3}, \mathbf{5}$, and $i, j = 1, \ldots, \mathrm{dim}I$.

[1] For example, two particles transforming in the adjoint of $SU(3)$ give the following decomposition $\mathbf{8}\otimes\mathbf{8} = \mathbf{1}+\mathbf{8}_1+\mathbf{8}_2+\mathbf{10}+\overline{\mathbf{10}}+\mathbf{27}$ where the counting label ξ has been put as subscript to avoid clutter of notation. The adjoint irrep appears twice in the decomposition.

2.2 Clebsch-Gordan coefficients and eigen-amplitudes

In the following we need to express the eigen-amplitudes in terms of the ordinary amplitudes of multi-particle states (that transform as the tensor product (2.1) of one-particle states). This is achieved by means of the so-called Clebsch-Gordan coefficients.
Since crossing symmetry exchanges particles with anti-particles, we will need to consider the following two-particle decompositions

$$\mathbf{N} \otimes \mathbf{M} = \bigoplus_{I(\xi)} Y_{I(\xi)} \tag{2.7}$$

$$\mathbf{N} \otimes \overline{\mathbf{M}} = \bigoplus_{I(\xi)} Z_{I(\xi)} \tag{2.8}$$

$$\overline{\mathbf{N}} \otimes \mathbf{M} = \bigoplus_{I(\xi)} \overline{\mathrm{Z}}_{I(\xi)} \tag{2.9}$$

$$\overline{\mathbf{N}} \otimes \overline{\mathbf{M}} = \bigoplus_{I(\xi)} \overline{\mathrm{Y}}_{I(\xi)} \tag{2.10}$$

where the capital letters (Y, Z) label the sets of irreps, I is a collective index labeling nonequivalent irreps, and ξ labels possible degenerate irreps i.e. those that appear more than once. In the following, we shall choose $\mathbf{M}$=$\mathbf{N}$ for simplicity.
Any multi-particle quantum state can be decomposed into irreps of G using its Clebsch-Gordan (CG) coefficients. We define $|a\rangle$ and $|\bar{a}\rangle$ the one-particle states and its complex-conjugated respectively, where a is a collective index which groups all the quantum numbers. Under a generic representation $U(g)$ of the group G they transform as

$$|a\rangle \equiv \pi^a \to U(g)^a{}_b \pi^b \tag{2.11}$$

$$|\bar{a}\rangle \equiv \bar{\pi}^a \equiv \pi_a \to \overline{\mathrm{U}}(g)_a{}^b \pi_b, \tag{2.12}$$

where we defined $\overline{\mathrm{U}}(g)_a{}^b = U(g)^{*a}{}_b$. We are considering the general case of states transforming under complex representations. A two-particle state obtained by the product of two single-particle states transforms as

$$|a\rangle \otimes |\bar{b}\rangle \equiv \pi^a \pi_b \to U^a_c \overline{\mathrm{U}}(g)_b{}^d \pi^c \pi_d\,. \tag{2.13}$$

Equations (2.7)÷(2.10) mean that we can decompose the two-particles state in terms of irreps as follow

$$\pi^a \pi^b = \sum_{I(\xi)i}^{Y} C^{ab}_{I(\xi)i} \, |I(\xi), i\rangle \tag{2.14}$$

$$\pi_a \pi^b = \sum_{I(\xi)i}^{Z} C^{\bar{a}b}_{I(\xi)i} \, |I(\xi), i\rangle \equiv \sum_{I(\xi)i}^{Z} C^{b}_{aI(\xi)i} \, |I(\xi), i\rangle \tag{2.15}$$

$$\pi_a \pi_b = \sum_{I(\xi)i}^{\bar{Y}} C_{ab}^{I(\xi)i} \, |I(\xi), i\rangle\,. \tag{2.16}$$

where the $C^{ab}_{I(\xi)i}, C^{\bar{a}b}_{I(\xi)i}, C^{I(\xi)i}_{ab}$ are the CG coefficients associated to the irreps under scrutiny. We use the following notation for the complex conjugate

$$(C^{ab}_{I(\xi)i})^* \equiv \bar{C}^{ab}_{I(\xi)i} \equiv C^{I(\xi)i}_{ab}. \tag{2.17}$$

Crucially, since the CG coefficients map one basis into another one, they are unitary matrices, i.e.

$$\sum_{I,\xi\, i} C^{I(\xi)i}_{ab} C^{cd}_{I(\xi)i} = \delta^c_a \delta^d_b \qquad \sum_{ab} C^{J(\xi)j}_{ab} C^{ab}_{I(\chi)i} = \delta_{IJ}\delta_{ij}\delta_{\chi\xi}\,. \tag{2.18}$$

With these definitions and using the Wigner-Eckart theorem

$$\langle J(\xi'), j|\mathcal{M}|\, I(\xi), i\rangle = \delta_{IJ}\delta_{ij}\widehat{\mathcal{M}}_{I(\xi\xi')}\,, \qquad I, J \in Y \tag{2.19}$$

the scattering amplitude of $|a\rangle\,|b\rangle \to |c\rangle\,|d\rangle$ can be written in terms of the eigen-amplitudes $\widehat{\mathcal{M}}$ among the irreps $I \in Y$

$$\boxed{\mathcal{M}_{ab\to cd}(s,t) = \sum_{I(\xi),i}^{Y}\sum_{J(\xi'),j}^{Y} C^{J(\xi')j}_{cd} C^{ab}_{I(\xi)i}\,\langle J(\xi'), j|\mathcal{M}|I(\xi), i\rangle = \sum_{I,i,\xi\xi'}^{Y} C^{I(\xi')i}_{cd} C^{ab}_{I(\xi)i}\widehat{\mathcal{M}}_{I(\xi\xi')}(s,t)}\,. \tag{2.20}$$

Compared to notation of the previous section we are inserting a hat over $\mathcal{M}$ to distinguish the eigen-amplitudes associated to the irreps in Y, i.e. found in $\mathbf{N}\otimes\mathbf{N}$, from the eigen-amplitudes with a tilde, $\widetilde{\mathcal{M}}$, which are for the transitions of the irreps $J \in Z$, i.e. those associated to irreps found in $\mathbf{N}\otimes\overline{\mathbf{N}}$:

$$\langle J(\xi'), j|\mathcal{M}|\, I(\xi), i\rangle = \delta_{IJ}\delta_{ij}\widetilde{\mathcal{M}}_{I(\xi\xi')}\,, \qquad I, J \in Z \tag{2.21}$$

and

$$\boxed{\mathcal{M}_{\bar{c}b\to\bar{a}d}(s,t) = \sum_{I(\xi),i}^{Z}\sum_{J(\xi'),j}^{Z} C^{J(\xi')j}_{\bar{a}d} C^{\bar{c}b}_{I(\xi)i}\,\langle J(\xi'), j|\mathcal{M}|I(\xi), i\rangle = \sum_{I,i,\xi\xi'}^{Z} C^{I(\xi')i}_{\bar{a}d} C^{\bar{c}b}_{I(\xi)i}\widetilde{\mathcal{M}}_{I(\xi\xi')}(s,t)}\,. \tag{2.22}$$

These relations along with unitarity of the CG coefficients will allow us to express the action of crossing symmetry directly on the eigen-amplitudes.

2.3 The crossing matrix

Crossing symmetry relates different eigen-amplitudes because the decomposition in irreps is different in $\mathbf{N}\otimes\mathbf{N}$, $\overline{\mathbf{N}}\otimes\mathbf{N}$ and their complex conjugate. We will show that the crossed amplitudes are related by a constant involutory matrix $\mathbb{X}$ called *crossing matrix*, built out of two smaller blocks of crossing matrices that send the irreps found in $\mathbf{N}\otimes\mathbf{N}$ in those of $\overline{\mathbf{N}}\otimes\mathbf{N}$ and vice versa.

We restrict ourselves to the case of elastic scattering at $t = 0$ displaying only the dependence on s, e.g. $\widetilde{\mathcal{M}}(s) \equiv \widetilde{\mathcal{M}}(s, t = 0)$. The s-channel $|ab\rangle \to |cd\rangle$ and its crossed process $|\bar{c}b\rangle \to |\bar{a}d\rangle$, the u-channel, are related by crossing symmetry

$$\mathcal{M}_{ab\to cd}(s) = \mathcal{M}_{\bar{c}b\to\bar{a}d}(u) \tag{2.23}$$

as in the previous chapter. However, we find now convenient to decompose the amplitudes into eigen-amplitudes using (2.20) and (2.22)

$$\sum_{I\xi i\xi'}^{Y} C^{ab}_{I(\xi),i}\bar{C}^{cd}_{I(\xi')i}\widehat{\mathcal{M}}_{I(\xi\xi')}(s) = \sum_{I\xi i\xi'}^{Z} C^{\bar{c}b}_{I(\xi),i}C^{I(\xi')i}_{\bar{a}d}\widetilde{\mathcal{M}}_{I(\xi\xi')}(u). \tag{2.24}$$

Defining the projection operators[2]

$$[\widehat{P}_{J(\xi'\xi)}]^{cd}_{ab} = \sum_j \left[C_{J(\xi')j}\right]^{cd}\left[C^{J(\xi)j}\right]_{ab} \tag{2.25}$$

$$[\widetilde{P}_{I(\chi\chi')}]^{ba}_{cd} = \sum_j \left[C_{I(\chi)j}\right]^{b}_{c}\left[C^{I(\chi')j}\right]^{a}_{d}, \tag{2.26}$$

we can write the amplitudes as

$$\mathcal{M}_{ab\to cd}(s) = \sum_{I\xi i\xi'}^{Y} [\widehat{P}_{I(\xi'\xi)}]^{ab}_{cd}\widehat{\mathcal{M}}_{I(\xi\xi')}(s) \tag{2.27}$$

$$\mathcal{M}_{\bar{c}b\to\bar{a}d}(u) = \sum_{I\xi i\xi'}^{Z} [\widetilde{P}_{I(\xi'\xi)}]^{ab}_{cd}\widetilde{\mathcal{M}}_{I(\xi\xi')}(u) \tag{2.28}$$

multiplying both sides of (2.24) by the expression in (2.25), and summing over the index a, b, c, d we get

$$\mathrm{dim}Y_J\widehat{\mathcal{M}}_{J(\xi\xi')}(s) = \sum_{I\chi\chi'}^{Z}\sum_{abcd}\left[\widehat{P}_{J(\xi'\xi)}\right]^{cd}_{ab}\left[\widetilde{P}_{I(\chi\chi')}\right]^{ba}_{cd}\widetilde{\mathcal{M}}_{I(\chi\chi')}(u)\,. \tag{2.29}$$

This expression defines the crossing matrix $\mathbb{X}_1$

$$\boxed{\widehat{\mathcal{M}}(s) = \mathbb{X}_1\widetilde{\mathcal{M}}(u)} \tag{2.30}$$

relating the s-channel eigen-amplitudes $\widehat{\mathcal{M}}(s)$ with the u-channel eigen-amplitudes $\widetilde{\mathcal{M}}(u)$:

$$\boxed{[\mathbb{X}_1]_{J(\xi\xi')I(\chi\chi')} = \frac{1}{\mathrm{dim}Y_J}\sum_{abcd}\left[\widehat{P}_{J(\xi'\xi)}\right]^{ab}_{cd}\left[\widetilde{P}_{I(\chi\chi')}\right]^{dc}_{ab}}. \tag{2.31}$$

We recall that the indices I and J run over the different sets Z and Y which in general contain a different number of irreps. Therefore, the matrix $\mathbb{X}_1$ is in general rectangular.

[2] They are actual orthogonal projectors only for $\xi = \xi'$.

Multiplying and summing both sides of (2.24) by $\left[P_{J(\chi\chi')}\right]_{ab}^{dc}$, we obtain instead

$$\widetilde{\mathcal{M}}_{J(\chi\chi')}(u) = \sum_{I\xi\xi'}^{Y} \frac{1}{\dim Z_J} \sum_{abcd} \left[\widetilde{P}_{J(\chi'\chi)}\right]_{ab}^{dc} \left[\widehat{P}_{I(\xi\xi')}\right]_{cd}^{ab} \widehat{\mathcal{M}}_{I(\xi\xi')}(s) \tag{2.32}$$

which defines the other crossing matrix $\mathbb{X}_2$

$$\boxed{[\mathbb{X}_2]_{J(\chi\chi')I(\xi\xi')} = \frac{1}{\dim Z_J} \sum_{abcd} \left[\widetilde{P}_{J(\chi'\chi)}\right]_{ab}^{dc} \left[\widehat{P}_{I(\xi\xi')}\right]_{cd}^{ab}} \tag{2.33}$$

that relates back the u-channel to the s-channel:

$$\boxed{\widetilde{\mathcal{M}}(u) = \mathbb{X}_2 \widehat{\mathcal{M}}(s)}\,. \tag{2.34}$$

In (2.30) and (2.34) we are using an index free notation where $\widehat{\mathcal{M}}$ and $\widetilde{\mathcal{M}}$ are vector with components $\widehat{\mathcal{M}}_{I(\xi\xi')}$ and $\widetilde{\mathcal{M}}_{J(\xi\xi')}$ respectively. Notice that in general $\mathbb{X}_1$, $\mathbb{X}_2$ are not the same matrices, see Fig.1 , although they are the left- or right-inverse of each other

$$\mathbb{X}_2\mathbb{X}_1 = \mathbb{1}_{z\times z}, \tag{2.35}$$

$$\mathbb{X}_1\mathbb{X}_2 = \mathbb{1}_{y\times y} \tag{2.36}$$

where z (y) is the number of irreps in Z (Y).

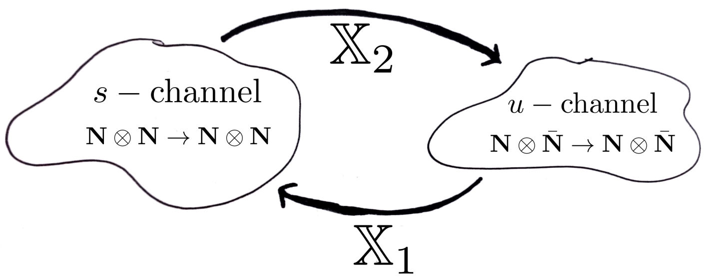

Fig. 1

By definition, see (2.31) and (2.33), we have

$$\left[\mathbb{X}_1^\dagger\right]_{H(\theta\theta')J(\xi\xi')} = [\mathbb{X}_1]^*_{J(\xi\xi')H(\theta\theta')} = [\mathbb{X}_1]_{J(\xi'\xi)H(\theta'\theta)} \tag{2.37}$$

$$\dim Y_J\,[\mathbb{X}_1]_{J(\xi\xi')I(\chi\chi')} = \dim Z_I\,[\mathbb{X}_2]_{I(\chi'\chi)J(\xi'\xi)}\,. \tag{2.38}$$

Moreover, defining the diagonal matrices

$$\widehat{\Delta}_{I(\chi\chi')J(\xi\xi')} \equiv \dim Y_I \delta_{IJ}\delta_{\chi\xi}\delta_{\chi'\xi'} \tag{2.39}$$

$$\widetilde{\Delta}_{I(\chi\chi')J(\xi\xi')} \equiv \dim Z_I \delta_{IJ}\delta_{\chi\xi}\delta_{\chi'\xi'}\,, \tag{2.40}$$

we get

$$\mathbb{X}_1^\dagger \widehat{\Delta} \mathbb{X}_1 = \widetilde{\Delta} \tag{2.41}$$

$$\mathbb{X}_2^\dagger \widetilde{\Delta} \mathbb{X}_2 = \widehat{\Delta}\,. \tag{2.42}$$

All these relations are better expressed in terms of a single crossing matrix that acts on the whole set of eigen-amplitudes. Indeed, collecting the various eigen-amplitudes associated to the irreps in the two decompositions $\mathbf{N} \otimes \mathbf{N}$ and $\mathbf{N} \otimes \overline{\mathbf{N}}$ in a single master eigen-amplitude

$$\boxed{\mathcal{M} = \begin{pmatrix} \widehat{\mathcal{M}} \\ \widetilde{\mathcal{M}} \end{pmatrix}} \tag{2.43}$$

whose components are $\mathcal{M} = (\cdots \widehat{\mathcal{M}}_{J(\xi\xi')} \cdots, \cdots \widetilde{\mathcal{M}}_{I(\chi\chi')} \cdots)^T$, we see that the $s \leftrightarrow u$ crossing takes the form

$$\boxed{\mathcal{M}(s) = \mathbb{X}\mathcal{M}(u) \qquad \mathcal{M}(u) = \mathbb{X}\mathcal{M}(s)}\,. \tag{2.44}$$

where

$$\mathbb{X} = \begin{bmatrix} 0 & \mathbb{X}_1 \\ \mathbb{X}_2 & 0 \end{bmatrix}. \tag{2.45}$$

The matrix $\mathbb{X}$ represents the *crossing matrix* of the entire set of eigen-amplitudes, it is thus the matrix we were after. Its entries are defined in terms of the CG coefficients in (2.31) and (2.33). They are purely geometric objects that depend on the group and the irreps, but that know nothing about the dynamics. The crossing matrix $\mathbb{X}$ is an involutory matrix and satisfies the following properties

$$\boxed{\mathbb{X}^\dagger \mathcal{G} \mathbb{X} = \mathcal{G}, \qquad \mathbb{X}^2 = \mathbb{1}_{n\times n}} \tag{2.46}$$

where $n = y + z$ and

$$\mathcal{G} = \begin{bmatrix} \widehat{\Delta} & 0 \\ 0 & \widetilde{\Delta} \end{bmatrix}. \tag{2.47}$$

In other words, the crossing matrix is not only involutory but also unitary with respect to the positive definite metric $\mathcal{G}$ built out of the dimensions of the irreps that appear in (2.39) and (2.40). These relations follow directly from (2.41), (2.42), (2.35) and (2.36). Being involutory, the eigenvalues of $\mathbb{X}$ are ± 1. Actually, the $(+1)$-eigenspace always contains the vector $v^{(+)}$ whose entries are irrep-independent and diagonal with respect to the degenerate irreps' labels:

$$v^{(+)} = \begin{pmatrix} \delta_{\xi\xi'} \\ \delta_{\chi\chi'} \end{pmatrix}, \qquad \sum_I [\mathbb{X}_1]_{J(\xi\xi')I(\chi\chi')}\, \delta_{\chi\chi'} = \delta_{\xi\xi'}, \qquad \sum_I [\mathbb{X}_2]_{J(\xi\xi')I(\chi\chi')}\, \delta_{\chi\chi'} = \delta_{\xi\xi'}. \tag{2.48}$$

For theories with no degenerate irreps appearing in the CG decomposition this (+1)-eigenvector reduces just to vector of identical entries,

$$v^{(+)} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{2.49}$$

2.4 Sum rules

Now that we understood the general properties of the crossing matrices and how they can be constructed, we are able to obtain dispersion relations that provide sum rules for the eigen-amplitudes and the Wilson coefficients. As in the previous chapter we assume:

- *Analyticity*, in the cut s-plane as shown in Fig.2 . Since we assume that particles have all the same mass, $m_i = m$, the IR branch-points given by the threshold for elastic scattering are at $s_{IR} = (m_1+m_2)^2 = 4m^2$ in the s-channel, and $u_{IR} = (m_1-m_2)^2 = 0$ in the u-channel. Moreover, a generalization of the Schwarz reflection principle (1.40) follows from (2.20)
$$\mathcal{M}_{I(\xi\xi')}(s)^* = \mathcal{M}_{I(\xi'\xi)}(s^*), \tag{2.50}$$
for the eigen-amplitudes. It relates the discontinuity between the upper and lower complex plane to the imaginary parts of the eigen-amplitudes between the same irreps $\xi = \xi'$.

- *Unitarity*, which implies the optical theorem
$$\text{Im}\mathcal{M}_{I(\xi\xi)}(s) = s\sqrt{1-\frac{4m^2}{s}}\sigma^{\text{tot}}_{I(\xi\xi)}(s) > 0 \tag{2.51}$$
for $s \geq s_{IR}$.

- *Crossing symmetry*, which acts on the entire set of eigen-amplitudes as $\mathcal{M}(s) = \mathbb{X}\mathcal{M}(u)$ where the crossing matrix $\mathbb{X}$ is defined in section 2.3 in terms of the relevant CG coefficients.

By analyticity, we can Taylor expand the amplitude around a scale μ^2 in the upper complex plane away from the singularities

$$\mathcal{M}(s) = \sum_n \frac{1}{n!}\mathcal{M}^{(n)}(\mu^2)(s-\mu^2)^n. \tag{2.52}$$

The coefficient $\mathcal{M}^{(n)}(\mu^2)$ can be computed by means of the Cauchy integral formula,

$$\mathcal{M}^{(n)}(\mu^2) = \frac{n!}{2\pi i}\oint_{\mathcal{C}} ds \frac{\mathcal{M}(s)}{(s-\mu^2)^{n+1}} \tag{2.53}$$

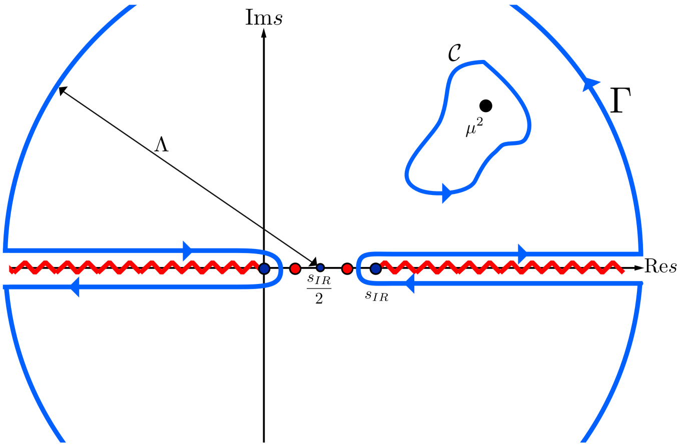

Fig. 2: General analytic structure of a forward scattering amplitude of massive scalars in the configuration $s+u=4m^2$. The red points stand for poles coming from intermediate one-particle states, i.e. propagators, and they always come in pairs because of crossing symmetry. The black dot marked μ^2 is the point around where we are Taylor expanding the amplitude. Λ is the radius of the big circle, while $s_{IR}=4m^2$ is the branch-point associated to the elastic threshold in the s-channel. The other branch-point at $s=0$ comes from the u-channel, that is by crossing symmetry. In the following, we refer to the energies between $s=0$ and s_{IR} as "IR masses".

where the contour $\mathcal{C}$ does not cross any singular point as shown in Fig.2 .
The integral on the r.h.s. can be performed along the curve Γ by subtracting the residue of poles on the real axis, if any. Just for simplicity, we assume in the following that there are no lighter stable particles so that these residues are actually not present. The more general case with lighter particles in the spectrum and their residues included in the dispersion relations can be obtained straightforwardly, as it was done for a single flavor in section 1.4.
The contribution c_n^Λ coming from the integration over the big circle of radius Λ^2 (see Fig.2)

$$c_n^\Lambda = n! \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{|s_\Lambda| e^{i\theta} \mathcal{M}(|s_\Lambda| e^{i\theta})}{(|s_\Lambda| e^{i\theta} - \mu^2)^{n+1}}, \qquad |s_\Lambda| = s_{IR}/2 + \Lambda^2 \tag{2.54}$$

can be discarded for $n \geq 2$ since $c_2^\Lambda \to 0$ as one take the limit $\Lambda \to \infty$. This follows from the Froissart bound [13] which ensures $|\mathcal{M}(s)| \leq s \log^2 s$ for $s \to \infty$. The contribution from the integrals along the branch-cuts reads

$$n!\int_{s_{IR}}^{\Lambda^2+s_{IR}/2}\frac{ds}{2\pi i}\left[\frac{\mathcal{M}(s+i\epsilon)-\mathcal{M}(s-i\epsilon)}{(s-\mu^2)^{n+1}}+(-1)^n\frac{\mathcal{M}(-s+s_{IR}-i\epsilon)-\mathcal{M}(-s+s_{IR}+i\epsilon)}{(s-s_{IR}+\mu^2)^{n+1}}\right] \tag{2.55}$$

$$= n!\int_{s_{IR}}^{\Lambda^2+s_{IR}/2}\frac{ds}{2\pi i}\left[\frac{1}{(s-\mu^2)^{n+1}}+(-1)^n\frac{\mathbb{X}}{(s-s_{IR}+\mu^2)^{n+1}}\right][\mathcal{M}(s+i\epsilon)-\mathcal{M}(s-i\epsilon)] \tag{2.56}$$

where in the second equality we used crossing symmetry. By (2.50) we have

$$\mathcal{M}(s+i\epsilon)-\mathcal{M}(s-i\epsilon) = 2\mathrm{Re}\mathcal{M}^-(s+i\epsilon)+2i\mathrm{Im}\mathcal{M}^+(s+i\epsilon) \tag{2.57}$$

where, in components,

$$\mathcal{M}^{\pm}_{I(\xi\xi')}(s) \equiv \frac{1}{2}\left[\mathcal{M}_{I(\xi\xi')}(s)\pm\mathcal{M}_{I(\xi'\xi)}(s)\right]. \tag{2.58}$$

With no degenerate irreps in the CG decomposition, or in presence of particular selection rules between the degenerate irreps [2] , $\mathcal{M}^-=0$ and $\mathcal{M}^+=\mathcal{M}$. Hereafter we restrict to the case of no degenerate irreps, since all examples we will discuss later fall in this category. Therefore, using also the optical theorem to write the imaginary parts in terms of the total cross-section, the dispersion relation for $n\geq 2$ and $\Lambda\to\infty$ reads

$$\boxed{\mathcal{M}^{(n)}(\mu^2) = n!\int_{s_{IR}}^{\infty}\frac{ds}{\pi}\left[\frac{s}{(s-\mu^2)^{n+1}}+(-1)^n\frac{s\,\mathbb{X}}{(s-s_{IR}+\mu^2)^{n+1}}\right]\sqrt{1-\frac{4m^2}{s}}\sigma^{\mathrm{tot}}(s)}. \tag{2.59}$$

where σ^{tot} is the vector with components σ^{tot}_I, and $s_{IR}=4m^2$.
As we have seen in the previous sections, the crossing matrix $\mathbb{X}$ is involutory and it has therefore eigenvalues ±1. We can thus define the projectors over the positive and negative subspaces

$$P_\pm = \frac{1}{2}\left(\mathbb{1}\pm\mathbb{X}\right) \tag{2.60}$$

which satisfy the properties

$$P_\pm^2=\mathbb{1}\,,\qquad [P_\pm,\mathbb{X}]=0,,\qquad P_\pm X=\pm P_\pm\,. \tag{2.61}$$

Projecting (2.59) onto the ±1-eigenspaces, we get the sum rules

$$P_\pm\mathcal{M}^{(n)}(\mu^2) = n!\int_{s_{IR}}^{+\infty}\frac{ds}{\pi}\left[\frac{s}{(s-\mu^2)^{n+1}}\pm(-1)^n\frac{s}{(s-s_{IR}+\mu^2)^{n+1}}\right]\sqrt{1-\frac{4m^2}{s}}P_\pm\sigma^{tot}(s)\,. \tag{2.62}$$

which for $n = 2$ read

$$\boxed{P_\pm \mathcal{M}^{(2)}(\mu^2) = 2\int_{s_{IR}}^{+\infty} \frac{ds}{\pi}\left[\frac{s}{(s-\mu^2)^3} \pm \frac{s}{(s-s_{IR}+\mu^2)^3}\right]\sqrt{1-\frac{4m^2}{s}}P_\pm \sigma^{tot}(s)}\,. \tag{2.63}$$

Notice that at the crossing symmetric point $\mu^2 = (s_{IR} + u_{IR})/2 = 2m^2$ the projection with P_- kills the r.h.s. and therefore it makes vanishing the l.h.s. of the dispersion relation as well

$$\boxed{P_- \mathcal{M}''(2m^2) = 0}\,, \tag{2.64}$$

meaning that $\mathcal{M}''(2m^2)$ must belong to the $(+1)$-eigenspace of the crossing matrix. This provides an useful linear constraint among the amplitudes in the IR which are not all independent in that kinematical point.
In the massless limit and choosing $\mu^2 = 0$ the sum rules become even more neat

$$P_\pm \mathcal{M}^{(n)}(0) = 2\left[1 \pm (-1)^n\right]\int_0^{+\infty} \frac{ds}{\pi s^n} P_\pm \sigma^{tot}(s)\,. \tag{2.65}$$

2.5 Positivity for flavorfull scalar particles

Using the sum rules obtained by the dispersion relations, we can derive positivity constraints for scattering amplitudes as it was done in section (1.4), except that now the particles transform non-trivially under a symmetry group of the theory. This allows us in general to obtain stronger positivity conditions, as it was obtained in the special case of real irreps in [2].
Let us focus then on the elastic forward scattering

$$\pi^a(p_1)\pi^b(p_2) \to \pi^a(p_1)\pi^b(p_2) \tag{2.66}$$

where π^i stands for an element inside the representation of the group G. Should we follow the same steps as in section (1.4), that is without taking advantage of the symmetry, we would simply get

$$\boxed{\mathcal{M}''_{ab\to ab}(\mu^2) \geq 0}\,. \tag{2.67}$$

No symmetry has been invoked to obtain this positivity, the labels a and b playing no role in the derivation above. In order to get (possibly) stronger bounds on $\mathcal{M}''$, we instead use the sum rules (2.63). For μ^2 chosen between $s = 0$ and $s = s_{IR}$, the terms in the square brackets in (2.63) are positive, and so is the entries of the vector σ^{tot} made of total cross sections associated to the transitions $I \to$ *anything*. The only possibly negative sources in (2.63) are some of the entries of the projectors $P_\pm$. However, we have seen in Eq. (2.46) that the crossing matrix $\mathbb{X}$ that defines the projectors $P_\pm = (1 \pm \mathbb{X})/2$ is unitary with respect to a positive definite metric $\mathcal{G}$, i.e. $\mathbb{X}^\dagger \mathcal{G} \mathbb{X} = \mathcal{G}$, which is diagonal and whose entries are the dimensions of the irreps

$$\mathcal{G}_{IJ} = \delta_{IJ}\mathrm{dim}\mathbf{r}_I\,, \qquad I, J \in Z, Y\,. \tag{2.68}$$

This means that we can project on the (± 1)-eigenspaces with the scalar product defined by $\mathcal{G}$, i.e.

$$w_\pm^\dagger \mathcal{G} P_\pm \mathcal{M}''(\mu^2) = 2\int_{s_{IR}}^{+\infty} \frac{ds}{\pi}\left[\frac{s}{(s-\mu^2)^3} \pm \frac{s}{(s-s_{IR}+\mu^2)^3}\right]\sqrt{1-\frac{4m^2}{s}}\, w_\pm^\dagger \mathcal{G} P_\pm \sigma^{tot}(s) \tag{2.69}$$

that implies

$$\boxed{w_\pm^\dagger \mathcal{G} \mathcal{M}''(\mu^2) = 2\int_{s_{IR}}^{+\infty} \frac{ds}{\pi}\left[\frac{s}{(s-\mu^2)^3} \pm \frac{s}{(s-s_{IR}+\mu^2)^3}\right]\sqrt{1-\frac{4m^2}{s}}\, w_\pm^\dagger \mathcal{G} \sigma^{tot}(s)} \tag{2.70}$$

where $w_\pm$ is any (± 1)-eigenvector of $\mathbb{X}$, $\mathbb{X} w_\pm = \pm w_\pm$. But here is the catch: among the various eigenvectors w_+, we have seen in the previous section that there always exists a $v^{(+)} = (1, \ldots, 1)^T$ which has all positive entries. Projecting on such a eigenvector

$$v^{(+)\dagger} \mathcal{G} \mathcal{M}''(\mu^2) = 2\int_{s_{IR}}^{+\infty} \frac{ds}{\pi}\left[\frac{s}{(s-\mu^2)^3} \pm \frac{s}{(s-s_{IR}+\mu^2)^3}\right]\sqrt{1-\frac{4m^2}{s}}\, v^{(+)\dagger} \mathcal{G} \sigma^{tot}(s) \tag{2.71}$$

we see that the integrand on the r.h.s is strictly positive, so that the following positivity must hold

$$v^{(+)\dagger} \mathcal{G} \mathcal{M}''(\mu^2) > 0\,, \tag{2.72}$$

or equivalently

$$\boxed{\sum_I \mathrm{dim}\mathbf{r}_I\, \mathcal{M}''_I(\mu^2) > 0}\,. \tag{2.73}$$

In fact, since the $(+1)$-eigenspace of the crossing matrix $\mathbb{X}$ is linear, there are other $m_+ - 1$ positivity conditions that are obtained by adding $(m_+ - 1)$ sufficiently small and linearly independent $(+1)$-eigenvectors to $v^{(+)}$, where m_+ is the dimension of the $(+1)$-eigen-space. As long as the $(+1)$-eigenvectors $V^{(+)}$ obtained in this way have non-negative entries the positivity conditions

$$\boxed{V^{(+)\dagger} \mathcal{G} \mathcal{M}''(\mu^2) > 0} \tag{2.74}$$

hold too.

The strongest, i.e. optimal, bounds are those that imply the others (e.g. by taking linear combinations of the inequality with just positive coeffients), that is those that are obtained by intersecting the linear $(+1)$-eigenspace with the positive quadrant where the entries of the vectors are all non-negative. This intersection defines a convex polyhedric cone whose edges $V_i^{(+)}$, by constructions, define the strongest positivity conditions $V_i^{(+)\dagger} \mathcal{G} \mathcal{M}''(\mu^2) > 0$ (see Fig.3). The edges can be determined explicitly, given the crossing matrix $\mathbb{X}$, by using the algorithm described in [2], that is by looking for the $(+1)$-eigenvectors of $\mathbb{X}$ with $(m_+ - 1)$ vanishing entries (since we need to find a one-dimensional subspace living on the boundary of the quadrant). Notice that the $(+1)-$eigenspace can be easily identified by the algebric condition $P_- w_+ = 0$. In the next section we provide a detailed and explicit example of these positivity constraints.

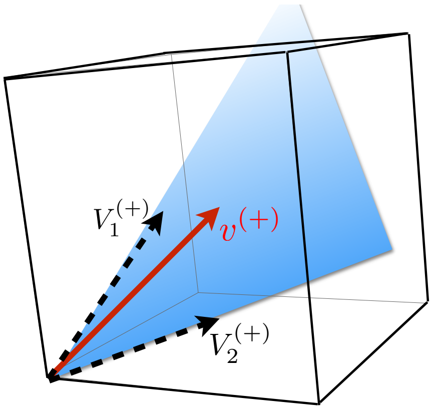

Fig. 3: In sky blue the convex polyhedric cone obtained by intersecting the positive quadrant of amplitudes with the the $(+1)$-eigenspace (a m_+-dimensional hyperplane) of the crossing matrix. (We are of course able to draw explicitly only a two-dimensional polyhedric cone with two edges, i.e. a triangle, embedded into a three-dimensional space of amplitudes). The boundary of the cone corresponds to $(+1)$-eigenvectors $V_i^{(+)}$ which have all positive entries but $m_+ - 1$ which are vanishing. While projecting $\mathcal{M}''(\mu^2)$ on any vector inside the cone, as e.g. on $v^{(+)} = (1, \dots, 1)^T$ as in (2.72), gives a positivity constraint, the strongest bounds are obtained by projecting on the boundary eigenvectors of the cone $V_i^{(+)}$, as any other vector inside the cone can be reached by linear combinations with positive coefficients.

2.6 An example: $SU(N)$

Let us consider a $2 \to 2$ scattering of identical scalar particles transforming under the fundamental (anti-fundamental) $\mathbf{N}$ ($\overline{\mathbf{N}}$) of $SU(N)$. The decompositions (2.7) and (2.8) reads

$$\mathbf{N} \otimes \mathbf{N} = \frac{\mathbf{N(N\text{-}1)}}{\mathbf{2}} \oplus \frac{\mathbf{N(N{+}1)}}{\mathbf{2}} \equiv \mathbf{A} \oplus \mathbf{S} \tag{2.75}$$

$$\mathbf{N} \otimes \overline{\mathbf{N}} = \mathbf{1} \oplus (\mathbf{N}^2 - 1) \equiv \mathbf{1} \oplus \mathbf{Adj}. \tag{2.76}$$

We cluster all eigen-amplitudes of $\mathbf{N} \otimes \mathbf{N}$ and $\mathbf{N} \otimes \overline{\mathbf{N}}$ in a single master eigen-amplitude:

$$\mathcal{M}(s) = \begin{pmatrix} \widehat{\mathcal{M}} \\ \widetilde{\mathcal{M}} \end{pmatrix} = \begin{pmatrix} \widehat{\mathcal{M}}_{\mathbf{A}}(s) \\ \widehat{\mathcal{M}}_{\mathbf{S}}(s) \\ \widetilde{\mathcal{M}}_{\mathbf{1}}(s) \\ \widetilde{\mathcal{M}}_{\mathbf{Adj}}(s) \end{pmatrix}. \tag{2.77}$$

In the following we will omit the tilde and the hat over the amplitudes. It should not be ambiguous since the irrep subscript tell us which decomposition of the tensor product we are considering.

In order to determine explicitly the crossing matrix, we use the projectors over the invariant subspaces in (2.75) and (2.76), which are

$$\left[\widehat{P}_{\mathbf{S}}\right]^{ab}_{cd} = \frac{1}{2}\left(\delta^a_c\,\delta^b_d + \delta^b_c\,\delta^a_d\right) \tag{2.78}$$

$$\left[\widehat{P}_{\mathbf{A}}\right]^{ab}_{cd} = \frac{1}{2}\left(\delta^a_c\,\delta^b_d - \delta^b_c\,\delta^a_d\right) \tag{2.79}$$

$$\left[\widetilde{P}_{\mathbf{1}}\right]^{dc}_{ab} = \frac{1}{N}\delta^d_a\,\delta^c_b \tag{2.80}$$

$$\left[\widetilde{P}_{\mathbf{Adj}}\right]^{dc}_{ab} = \delta^c_a\,\delta^d_b - \frac{1}{N}\delta^d_a\,\delta^c_b\,. \tag{2.81}$$

Therefore, the crossing matrix $\mathbb{X}$ is given by

$$\mathbb{X} = \begin{pmatrix} 0 & 0 & -\frac{1}{N} & \frac{N+1}{N} \\ 0 & 0 & \frac{1}{N} & \frac{N-1}{N} \\ \frac{1-N}{2} & \frac{N+1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}, \tag{2.82}$$

which is diagonalized

$$\mathbb{X}_{diag} = M\mathbb{X}M^{-1} = \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} \tag{2.83}$$

by

$$M = \begin{pmatrix} -\frac{1}{4} & -\frac{1}{4} & 0 & \frac{1}{2} \\ \frac{N-1}{4} & \frac{-N-1}{4} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \\ \frac{1-N}{4} & \frac{N+1}{4} & \frac{1}{2} & 0 \end{pmatrix}. \tag{2.84}$$

A basis for the two-dimensional (+1)-eigenspace is given by

$$V_1^{(+)} = (0, 2, N+1, 1)^T \qquad V_2^{(+)} = (N+1, N-1, 0, N)^T\,. \tag{2.85}$$

Notice that the vector $(1,1,1,1)^T = (V_1^{(+)} + V_2^{(+)})/(N+1)$ is indeed an element of the positive eigenspace as expect by our general arguments. Since all the entries of $V_i^{(+)}$ are non-negative, one immediately gets the positivity conditions from (2.74)

$$\boxed{\mathcal{M}''_{\mathbf{1}}(\mu^2) + N\mathcal{M}''_{\mathbf{S}}(\mu^2) + (N-1)\mathcal{M}''_{\mathbf{Adj}}(\mu^2) > 0} \tag{2.86}$$

$$\boxed{\mathcal{M}''_{\mathbf{A}}(\mu^2) + \mathcal{M}''_{\mathbf{S}}(\mu^2) + 2\mathcal{M}''_{\mathbf{Adj}}(\mu^2) > 0} \tag{2.87}$$

where we used

$$\mathcal{G} = \mathrm{diag}\left(N\frac{N-1}{2}, N\frac{N+1}{2}, 1, N^2-1\right) \tag{2.88}$$

and $0 \leq \mu^2 \leq 4m^2$. These positivity conditions are actually optimal because there is no other (+1)-eigenvector with one vanishing entry while its other entries are kept all strictly positive.

Further simplifications occur by choosing to work at the crossing symmetric point $\mu^2 = 2m^2$ because of the crossing relations $P_-\mathcal{M}''(2m^2) = 0$ in (2.64). In our $SU(N)$ example, it gives rise to the constraints

$$\mathcal{M}''_{\mathbf{A}}(2m^2) + \mathcal{M}''_{\mathbf{S}}(2m^2) - 2\mathcal{M}''_{\mathbf{Adj}}(2m^2) = 0\,, \tag{2.89}$$

$$(N-1)\mathcal{M}''_{\mathbf{A}} - (N+1)\mathcal{M}''_{\mathbf{S}}(2m^2) + 2\mathcal{M}''_{\mathbf{1}}(2m^2) = 0. \tag{2.90}$$

The first equation immediately implies $\mathcal{M}''_{\mathbf{Adj}}(2m^2) > 0$ and $\mathcal{M}''_{\mathbf{A}}(2m^2) + \mathcal{M}''_{\mathbf{S}}(2m^2) > 0$. Solving the system (2.89) and (2.90) for $\mathcal{M}''_{\mathbf{S},\mathbf{A}}(2m^2)$, and plugging them back into (2.86) and (2.87) we get

$$\boxed{\mathcal{M}''_{\mathbf{A}}(2m^2) + \mathcal{M}''_{\mathbf{S}}(2m^2) > 0\,, \quad \mathcal{M}''_{\mathbf{Adj}}(2m^2) > 0\,, \quad \mathcal{M}''_{\mathbf{1}}(2m^2) + (N-1)\mathbf{M}''_{\mathbf{Adj}}(2m^2) > 0}\,. \tag{2.91}$$

These positivity constraints on the scattering amplitudes will be used in the next chapter to derive positivity constraints on the Wilson coefficients of a theory that respects $SU(N)$ and where the states transform like the fundamental and anti-fundamental representation.

Chapter 3

Positivity constraints for fermions

In this chapter we extend the previous results about the positivity of scattering amplitudes and Wilson coefficients to the case of particles with spin. This task is non-trivial because crossing symmetry is generically not just exchanging $s \leftrightarrow u$ as it is for scalars in (2.23), but it involves instead transformations on the polarizations of the external states and extra overall signs for fermions. Despite these extra features for spinning particles, we show how to obtain positivity constraints on scattering amplitudes, in the elastic forward limit, for generic spins. Moreover, exteanding the work of [6], we apply these results to the specific case of spin-1/2 fermions that carry fundamental and anti-fundamental representations of $SU(N)$ that we use in the next chapter when discussing composite pseudo-Goldstini that are charged under the R-symmetry from an extended SUSY.

3.1 Crossing symmetry for spinning particles

3.1.1 Crossing one particle

Let us consider a multi-particle scattering process

$$\psi_{p,\sigma,b} X_{\{k_i,\sigma_i,a_i\}} \to Y_{\{k_o,\sigma_o,a_o\}} \tag{3.1}$$

represented in Fig. 1, where we single out a particle ψ that we are going to cross from the initial to the final state. Here

- $\psi_{p,\sigma,b}$ is a certain particle with four-momentum p, Lorentz little-group index[1] σ (either spin or helicity depending whether it is massive or massless), and internal index b that labels the state inside an irrep of an internal symmetry group carried by the particle;
- X and Y stand for generic initial (apart from ψ) and final states respectively that contain other spectator particles.

[1]We focus on pure states with definite quantum numbers. A more general discussion involving mixed states with unpolarized particles can be found in [6].

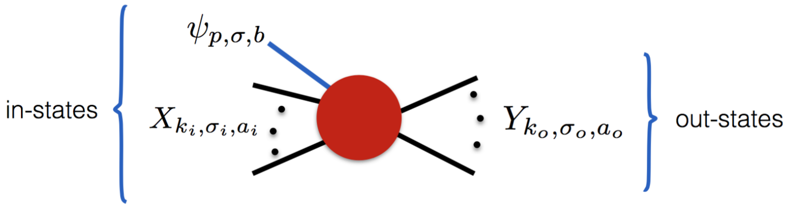

Fig. 1: Generic scattering process from an initial state $in = \{\psi + X\}$ to a final state $out = \{Y\}$.

Following the general LSZ reduction formula [11], the scattering amplitudes for the process (3.1) are obtained by dotting the amputated Green functions (i.e. the Green functions $\langle 0|\mathrm{T}\psi_{\beta\, a}(y)\ldots\psi^\dagger_{\alpha\, b}(x)|0\rangle$ contracted with the inverse of the propagators of the external particles) with the wave-functions polarizations u^σ_α of the incoming particles, as represented in the upper part of Fig. 2. The resulting amplitude can be expressed in the form

$$\mathcal{M}(\psi_{p,\sigma,b}X_{\{k_i,\sigma_i,a_i\}} \to Y_{\{k_o,\sigma_o,a_o\}}) = \mathcal{O}^\alpha(\{k_i,\sigma_i,a_i\},\{k_o,\sigma_o,a_o\},p,b)\cdot u^\sigma_\alpha(\mathbf{p}) \tag{3.2}$$

where we have singled out for convenience the wave-function polarization $u^\sigma_\alpha(\mathbf{p})$ for the particle ψ. We use the usual definition for the wave-function polarizations: ψ annihilates a particle and creates an anti-particle, whereas $\psi^\dagger$ creates a particle and annihilates an anti-particle

$$\langle 0|\psi_{\alpha\, b}(0)|p^\sigma_a\rangle \propto \delta_{ab}u^\sigma(\mathbf{p})\,, \qquad \langle p^\sigma_{\bar a}|\psi_{\alpha\, b}|0\rangle \propto \delta_{ab}v^\sigma_\alpha\,. \tag{3.3}$$

Notice that we have not committed to any particular spin yet; the index α in u^σ_α and v^σ_α is spinorial or Lorentzian depending on the Lorentz representation carried by the field ψ: the trivial one for scalars, a two-component spinor index for Weyl fermions, a 4-component spinor index for Dirac fermions, and a four-vector Lorentz index $\alpha=\mu$ for vectors, ...
The crossed process [2]

$$X_{\{k_i,\sigma_i,a_i\}} \to \bar\psi_{\bar p,\bar\sigma,\bar b}Y_{\{k_o,\sigma_o,a_o\}} \tag{3.4}$$

is obtained again by the LSZ reduction formula by dotting the amputated Green function in momentum space with the wave-function polarization $v^{\bar\sigma}_\alpha(\bar{\mathbf{p}})$ of the outgoing anti-particle

$$\mathcal{M}(X\to Y\bar\psi_{\bar p,\bar\sigma,\bar a}) = \pm\mathcal{O}^\alpha(\{k_i,\sigma_i,a_i\},\{k_o,\sigma_o,a_o\},-\bar p,b)v^{\bar\sigma}_\alpha(\bar{\mathbf{p}}) \tag{3.5}$$

as represented in Fig. 2. Crossing symmetry is the statement that $\mathcal{O}^\alpha$ in (3.5) is the same fuction that appears in (3.2) but evaluated at the unphysical momentum $-\bar p$ (since $-\bar p$ has negative energy) given that the (anti-)particle now belongs to the final state. In practice, the amputated Green function $\mathcal{O}^\alpha$ has the same functional dependence on the kinematical and internal variables except for the sign of the momentum for the crossed particles, since

[2] We denote with a bar over the internal indexes the states of anti-particles transforming under the complex conjugate representation of the internal symmetry group. The bar over the spin and the momentum means instead that we are provisionally considering generic spin and momentum for the anti-particle.

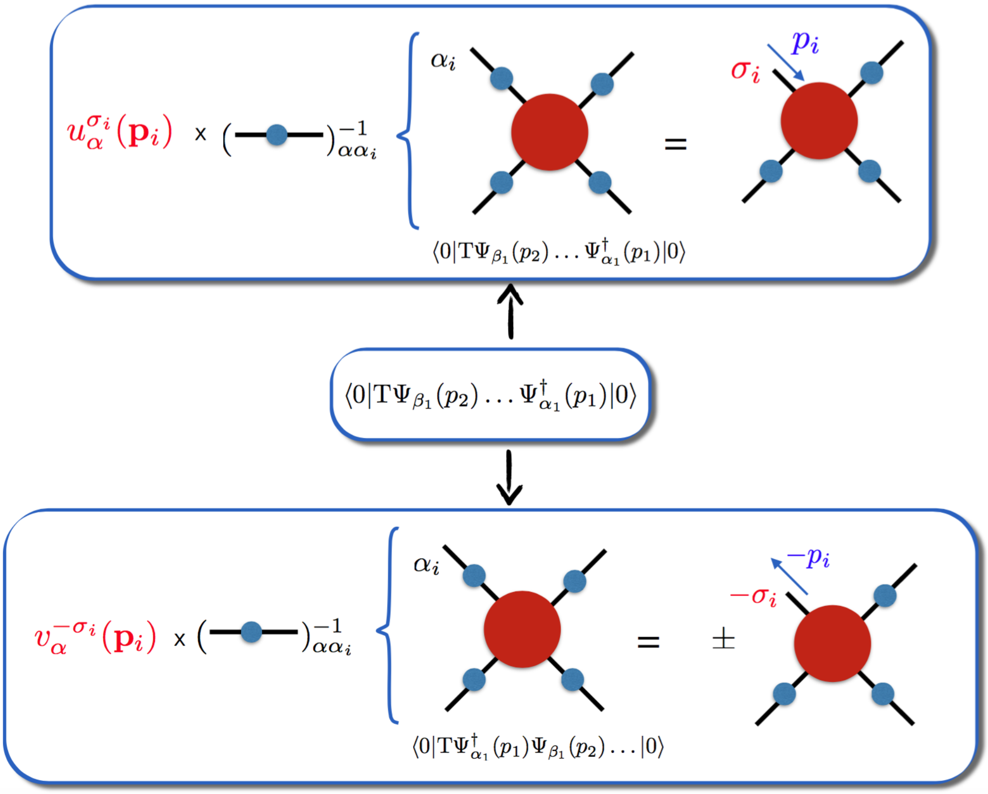

Fig. 2: A single Green-function gives rise to several scattering processes where particles/anti-particles move from/to the initial and final state, according to the LSZ reduction formula. In the upper part we show schematically the scattering amplitudes for $\psi_{p,\sigma,b}X \to Y$, obtained by dotting the amputated Green function with the on-shell wave-function polarization $u^\sigma(p)$ of the particle. In the lower part we show that the amplitude for the crossed process, where the anti-particle carries opposite polarization $\bar{\sigma} = -\sigma$, arises from the the same amputated Green-function evaluated in $-p$ and contracted with $v^{-\sigma} \sim u^\sigma$, see Eq. (3.6). The overall sign is determined by the statistics of ψ. The lines with the blue dot in the middle represent the external propagators.

in the LSZ reduction formula one has $+k$ for incoming and $-k$ for outgoing. Notice that, besides the momentum flip in $\mathcal{O}$ and the different external wave-function polarizations, there is also an overall sign which depends on the statistics of ψ: it is $+1$ for bosons and $(-1)^n$ for fermions, where n is number of fermionic pair exchanges performed from the starting Green function.

Notice that the particle/anti-particle wave-function polarizations dotted in the amputated matrix elements are actually related via CPT invariance. For vector, Dirac, and left- or right-handed (massless) Weyl representation we have respectively [3]

$$\epsilon_\mu^{\sigma\,*}(\mathbf{p}) = (-1)^\sigma \epsilon_\mu^{-\sigma}(\mathbf{p})\,, \quad v^\pm(\mathbf{p}) = \mp\gamma^5 u^\mp(\mathbf{p})\,, \quad v_L^+(\mathbf{p}) = u_L^-(\mathbf{p})\,, \quad v_R^-(\mathbf{p}) = u_R^+(\mathbf{p}). \tag{3.6}$$

We find thus simple relations between the amplitudes by simply considering opposite helicities in the crossed scattering, i.e. $\bar\sigma = -\sigma$ as represented in the lower part of Fig. 2.

3.1.2 Crossing two particles

We are interested in a $2 \to 2$ scattering

$$1_{b_1}^{\sigma_1} 2_{a_2}^{\sigma_2} \to 3_{b_3}^{\sigma_3} 4_{a_4}^{\sigma_4} \tag{3.7}$$

where the b's and a's are internal indexes and $1_{b_1}^{\sigma_1} \equiv \psi_{p_1,\sigma_1,b_1}$ with the same notation as before (analogously for the other particles). We are going to cross two particles: particle 1 from the initial to the final state, and particle 3 from the final to the initial state

$$\bar 3_{\bar b_3}^{\bar\sigma_3} 2_{a_2}^{\sigma_2} \to \bar 1_{\bar b_1}^{\bar\sigma_1} 4_{a_4}^{\sigma_4}\,, \tag{3.8}$$

looking provisionally to a generic kinematics with momenta $\bar p$ and spin $\bar\sigma$ for the crossed particles. Using the results of the previous subsection for each crossed particles, we get the following amplitudes

$$\mathcal{M}(1_{b_1}^{\sigma_1} 2_{a_2}^{\sigma_2} \to 3_{b_3}^{\sigma_3} 4_{a_4}^{\sigma_4}) = \left[u_{\alpha_4}^{\sigma_4\,\dagger}(\mathbf{p}_4) u_{\alpha_3}^{\sigma_3\,\dagger}(\mathbf{p}_3)\right] \mathcal{O}^{\alpha_1\alpha_2}_{\alpha_3\alpha_4}(p_1, p_3, p_2, p_4, \{b\}, \{a\}) \left[u_{\alpha_1}^{\sigma_1}(\mathbf{p}_1) u_{\alpha_2}^{\sigma_2}(\mathbf{p}_2)\right] \tag{3.9}$$

$$\mathcal{M}(\bar 3_{\bar b_3}^{\bar\sigma_3} 2_{a_2}^{\sigma_2} \to \bar 1_{\bar b_1}^{\bar\sigma_1} 4_{a_4}^{\sigma_4}) = \pm \left[u_{\alpha_4}^{\sigma_4\,\dagger}(\mathbf{p}_4) v_{\alpha_3}^{\bar\sigma_3\,\dagger}(\bar{\mathbf{p}}_3)\right] \mathcal{O}^{\alpha_1\alpha_2}_{\alpha_3\alpha_4}(-\bar p_1, -\bar p_3, p_2, p_4, \{b\}, \{a\}) \left[v_{\alpha_1}^{\bar\sigma_1}(\bar{\mathbf{p}}_1) u_{\alpha_2}^{\sigma_2}(\mathbf{p}_2)\right]\,. \tag{3.10}$$

These expression can be further simplified in the forward scattering where the kinematics of the initial and final state are the same

$$p_1 = p_3\,, \qquad k_2 = k_4\,, \qquad \sigma_1 = \sigma_3\,, \qquad \sigma_2 = \sigma_4\,, \tag{3.11}$$

and analogous for the barred quantities. In this special kinematics the wave-function polarizations in (3.9) and (3.10) pair to actually form the matrix-elements of *density matrices* (also called *spin projectors*, the same that appear in matrix elements squared when calculating cross-sections)

$$u_\alpha^\sigma(\mathbf{k}) u_{\alpha'}^{\sigma\,\dagger}(\mathbf{k}) \equiv \rho^\sigma_{\alpha\alpha'}(\mathbf{k}), \qquad v_\alpha^\sigma(\mathbf{k}) v_{\alpha'}^{\sigma\,\dagger}(\mathbf{k}) \equiv \tilde\rho^\sigma_{\alpha\alpha'}(\mathbf{k})\,, \tag{3.12}$$

and (3.9), (3.10) become

$$\mathcal{M}(1_{b_1}^{\sigma_1} 2_{a_2}^{\sigma_2} \to 1_{b_3}^{\sigma_1} 2_{a_4}^{\sigma_2}) = \rho^{\sigma_1}_{\alpha_1\alpha_3}(\mathbf{p}_1) \mathcal{O}^{\alpha_1\alpha_2}_{\alpha_3\alpha_4}(p_1, k_2, \{b\}, \{a\}) \rho^{\sigma_2}_{\alpha_2\alpha_4}(\mathbf{p}_2) \tag{3.13}$$

$$\mathcal{M}(\bar 1_{\bar b_3}^{\bar\sigma_1} 2_{a_2}^{\sigma_2} \to \bar 1_{\bar b_1}^{\bar\sigma_1} 2_{a_4}^{\sigma_2}) = (-1)^{2S}\, \tilde\rho^{\bar\sigma_1}_{\alpha_1\alpha_3}(\bar{\mathbf{p}}_1) \mathcal{O}^{\alpha_1\alpha_2}_{\alpha_3\alpha_4}(-\bar p_1, k_2, \{b\}, \{a\}) \rho^{\sigma_2}_{\alpha_2\alpha_4}(\mathbf{p}_2) \tag{3.14}$$

[3] For a generic irreducible representation (A, B) of the Lorentz group $SO(3,1) \times SU(2) \times SU(2)$ where A and B are positive half-integer numbers that label the irreps of $(2A+1)(2B+1)$ dimension, the relation (3.6) becomes $u_\alpha^\sigma(\mathbf{p}) = (-1)^{2B+S+\sigma} v_\alpha^{-\sigma}(\mathbf{p})$ where S is the spin of the particle, see [6].

respectively, see Fig. 3. The overall sign of the crossed process is determined by the spin S of the crossed particles 1 and 3. Since the number of fermions in the amplitude must be even, for a given ordering of the spectator states 2 and 4 there are necessarily an odd number of particles exchanges, hence the overall $(-1)^{2S}$ factor from the statistics of the particles.

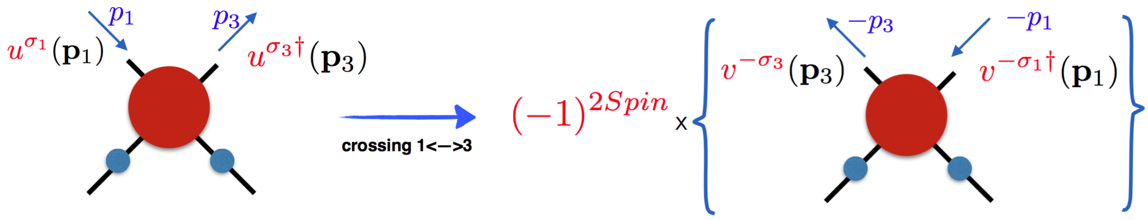

Fig. 3: We show schematically the action of crossing symmetry (3.10) that in the forward limit reduces to (3.14) and then to simply $s \leftrightarrow u$ exchange, thanks to locality, as explained in the main text.

The expressions (3.13) and (3.14) become more closely related to each other by choosing[4] $\bar{\sigma} = -\sigma$ because of the relations (3.6). We are also going to take the same physical 4-momentum for the crossed particles: $\bar{p}_i = p_i$. For simplicity let's consider first the massless case:

$$\mathcal{M}(1^{\sigma_1}_{b_1} 2^{\sigma_2}_{a_2} \to 1^{\sigma_1}_{b_3} 2^{\sigma_2}_{a_4}) = \rho^{\sigma_1}_{\alpha_1\alpha_3}(\mathbf{p}_1) \mathcal{O}^{\alpha_1\alpha_2}_{\alpha_3\alpha_4}(p_1, k_2, \{b\}, \{a\}) \rho^{\sigma_2}_{\alpha_2\alpha_4}(\mathbf{p}_2) \tag{3.15}$$

$$\boxed{\mathcal{M}(\bar{1}^{-\sigma_1}_{\bar{b}_3} 2^{\sigma_2}_{a_2} \to \bar{1}^{-\sigma_1}_{\bar{b}_1} 2^{\sigma_2}_{a_4}) = (-1)^{2S}\, \rho^{\sigma_1}_{\alpha_1\alpha_3}(\mathbf{p}_1) \mathcal{O}^{\alpha_1\alpha_2}_{\alpha_3\alpha_4}(-p_1, k_2, \{b\}, \{a\}) \rho^{\sigma_2}_{\alpha_2\alpha_4}(\mathbf{p}_2)}\,. \tag{3.16}$$

Compared to the scalar forward scattering we are facing here a difficulty since the 4-momentum is reversed in the crossed process only inside the amputated Green-function but not in the external wave-function polarizations which are a priori function of three-momentum $\mathbf{p}$ only, and not the 4-momentum $p = (p^0, \mathbf{p})$. Moreover, there is an extra minus sign for fermions that are exchanged under crossing. In other words, crossing symmetry does not look, at first sight, as simple as exchanging $p_1 \leftrightarrow -p_3 = -p_1$ nor $s \leftrightarrow u$. But in fact there is more than meet the eyes. The density matrix can be indeed uniquely extended analytically to a function $\rho(p)$ of the whole 4-momentum p with definite parity under reflection

$$\boxed{\rho(p) = (-1)^{2S} \rho(-p)}\,. \tag{3.17}$$

This can be directly checked on the generic expression of $\rho(p)$ for arbitrary massless spins that have been calculated long ago by Weinberg [14, 15, 11]. For example, for a massless spin-1/2 particle we have

$$\rho^{-}(\mathbf{p}) = u^{-}(\mathbf{p}) u^{-\dagger}(\mathbf{p}) = v^{+}(\mathbf{p}) v^{+\dagger}(\mathbf{p}) = \tilde{\rho}^{+}(\mathbf{p}) = p_\mu \sigma^\mu\,. \tag{3.18}$$

[4] Actually, for massless particles this is not a choice: the anti-particle can carry only the opposite helicity than its particle.

which is a linear, odd, monomial of the 4-momentum, whereas for a massless spin-1 one can always choose the gauge with $\rho^{\pm}_{\mu\nu} = -\eta_{\mu\nu}$ which is constant, hence trivially even in the 4-momentum. More generally, it is locality that requires (3.17) to hold true. One quick way to see this is realizing that the analytically continued density matrices with definite parity for massless particles have to exist since they are nothing but the numerators of the Lorentz covariant version of the propagators [11, 12]

$$\langle 0|\mathrm{T}\psi_{\alpha_1}(x_1)\psi^{\dagger}_{\alpha_2}(x_2)|0\rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x_1-x_2)} \frac{i\rho_{\alpha_1\alpha_2}(k)}{k^2 - i\epsilon} = (-1)^{2S}\langle 0|\mathrm{T}\psi^{\dagger}_{\alpha_2}(x_2)\psi_{\alpha_1}(x_1)|0\rangle \tag{3.19}$$

$$= (-1)^{2S} \int \frac{d^4k}{(2\pi)^4} e^{-ik(x_1-x_2)} \frac{i\tilde{\rho}_{\alpha_1\alpha_2}(-k)}{k^2 - i\epsilon}\,. \tag{3.20}$$

Locality implies the spin-statistic theorem for which fermions have half-integer spin and anti-commute, hence the second equality in (3.19), which can be expressed as in (3.20) implying therefore the relation (3.17). Another way to reach the same conclusion is by looking at the commutator or anti-commutator of two fields at equal times and requires that it vanishes for $\mathbf{x}_1 \neq \mathbf{x}_2$, as required by locality.
The case of massive integer spins works similarly, e.g. $\rho_{\mu\nu} = -g_{\mu\nu} + p_\mu p_\nu/m^2$ for a spin-1, which is an even function of the 4-momentum as it is claimed. The massive Dirac fermions are slightly more complicated because the representation is reducible, $(1/2, 0) \oplus (0, 1/2)$, which is reflected in the presence of a γ^5 in (3.6). Nevertheless, by direct inspection of the Dirac density matrices

$$\rho^{\sigma}(k) = u^{\sigma}(k)u^{\sigma\dagger}(k) = (\not{k} + m)\,\frac{1 + \gamma^5 \not{a}^{\sigma}(k)}{2}\gamma^0 \tag{3.21}$$

$$\tilde{\rho}^{\sigma}(k) = v^{\sigma}(k)v^{\sigma\dagger}(k) = (\not{k} - m)\,\frac{1 - \gamma^5 \not{a}^{-\sigma}(k)}{2}\gamma^0 \tag{3.22}$$

where $a^{\sigma}_{\mu}(k) = -a^{\sigma}_{\mu}(-k)$ is the (analytically continued) polarization 4-vector [6, 31], one sees that indeed $\tilde{\rho}^{-\sigma}(k) = \gamma^5\rho^{\sigma}(k)\gamma^5 = -\rho^{\sigma}(-k)$.

All in all, the crossed amplitudes can be obtained for spinning particles, in the forward limit, simply by flipping the spin, taking the complex conjugate representation of the internal quantum numbers, and reversing the 4-momentum not only in the amputated Green-functions but also in the (analytically continued) wave-function polarizations. Expressing the amplitudes in terms of Mandelstam variables at $t = 0$, crossing symmetry corresponds thus to the following statement:

$$\boxed{\mathcal{M}(1^{\sigma_1}_{b_1} 2^{\sigma_2}_{a_2} \to 1^{\sigma_1}_{b_3} 2^{\sigma_2}_{a_4})(s) = \mathcal{M}(1^{-\sigma_1}_{\bar{b}_3} 2^{\sigma_2}_{a_2} \to 1^{-\sigma_1}_{\bar{b}_1} 2^{\sigma_2}_{a_4})(u)} \tag{3.23}$$

which is almost identical to expression (2.23) that holds for particles without spin. We remark once more that this relation on the amplitudes is true only for the special kinematical configuration of the forward scattering.

3.2 The crossing matrix for spinning particles

Given that the crossing relation (3.23) in the complex s-plane at $t = 0$ differs from the one for scalars (2.23) just because of the spin flip, the crossing matrix relations are also very similar. Indeed, we are considering only internal symmetries which do not act on the spin indexes that are inert under the internal transformations. The decomposition in irreps inside $\mathbf{N}\otimes\mathbf{N}$ and $\mathbf{N}\otimes\overline{\mathbf{N}}$ works exactly as for scalars, producing i.e. eigen-amplitudes $\widehat{\mathcal{M}}^{\sigma_1\sigma_2}$ and $\widetilde{\mathcal{M}}^{-\sigma_1\sigma_2}$ respectively, up to flipping the spin in the crossed channel. Therefore, Eq. (2.30) and (2.34) read now

$$\widehat{\mathcal{M}}^{\sigma_1\sigma_2}(s) = \mathbb{X}_1\widetilde{\mathcal{M}}^{-\sigma_1\sigma_2}(u) \qquad \widetilde{\mathcal{M}}^{-\sigma_1\sigma_2}(u) = \mathbb{X}_2\widehat{\mathcal{M}}^{\sigma_1\sigma_2}(s)\,, \tag{3.24}$$

where the crossing matrices $\mathbb{X}_i$ are given by the same expression (2.31) and (2.33), enjoying thus all their geometric properties.
Again as in (2.45), it is convenient to define the general crossing involutory matrix $\mathbb{X}$

$$\mathbb{X} = \begin{pmatrix} 0 & \mathbb{X}_1 \\ \mathbb{X}_2 & 0 \end{pmatrix} \tag{3.25}$$

that acts on the master eigen-amplitude $\mathcal{M}$

$$\mathcal{M} \equiv \begin{pmatrix} \widehat{\mathcal{M}}^{\sigma_1\sigma_2} \\ \widetilde{\mathcal{M}}^{-\sigma_1\sigma_2} \end{pmatrix} \tag{3.26}$$

as

$$\mathcal{M}(s) = \mathbb{X}\mathcal{M}(u) \qquad \mathcal{M}(u) = \mathbb{X}\mathcal{M}(s)\,. \tag{3.27}$$

3.3 Positivity bounds for fermions

Positivity bounds can be obtained now for fermions following the same steps that we discussed in the previous chapter for scalars, yielding again the twice subtracted dispersion relation (2.63), the crossing constraint (2.64), the positivity (2.67), and the optimal bounds (2.74). We restrict for definiteness and simplicity to the case of scattering amplitudes for massless fermions which always have the same analytic structure of the amplitudes for scalars[5], i.e. as in Fig.2 with no light poles and $s_{IR} \to 0$ in the massless limit.
To derive positivity bounds, we make use of the dispersion relations (2.63) and consider a theory that starts with dimensional-8 operators for four massless fermions[6].

[5]There may exist some extra IR branch-cuts of finite extension of the type $\sqrt{s(s-4m^2)}$ that is coming from the discontinuities of the wave-function polarizations, but only in certain parity-violating theories for massive fermions as discussed in [6]. Being finite in extension and residing in the IR, they pose actually no real problem and positivity conditions can be derived as well [6], although the discussion becomes more involved. The effect of these extra IR branch-cuts is effectively very small and disappears in the massless limit.

[6]Should we start with lower dimensional operators the dispersion relation would not be IR convergent,

3.3.1 Positivity for fermions in the fundamental of $SU(N)$

More specifically, we consider the $SU(N)$ invariant effective theory that besides the kinetic terms for the massless fermions starts with

$$\mathcal{L} = c_1\mathcal{O}_1 + c_2\mathcal{O}_2 + c_3\mathcal{O}_3 + c_4\mathcal{O}_4 \tag{3.28}$$

where

$$\mathcal{O}_1 = \bar\psi_a \partial_\mu \bar\chi_b \partial^\mu \chi^a \psi^b\,, \tag{3.29}$$
$$\mathcal{O}_2 = \bar\psi_a \partial_\mu \bar\chi_b \partial^\mu \chi^b \psi^a\,, \tag{3.30}$$
$$\mathcal{O}_3 = \bar\psi_a \bar\chi_b \partial_\mu \chi^a \partial^\mu \psi^b + \text{h.c.}\,, \tag{3.31}$$
$$\mathcal{O}_4 = \bar\psi_a \bar\chi_b \partial_\mu \chi^b \partial^\mu \psi^a + \text{h.c.}\,, \tag{3.32}$$

The spinor contractions with dotted and undotted greek indexes in the 2-component notation of [17], e.g. $\mathcal{O}_1 = \bar\psi_{a\,\dot\alpha}\partial\bar\chi_b^{\dot\alpha}\partial\chi^{a\,\beta}\psi^b_\beta$, are left understood whenever clear. The lagrangian $\mathcal{L}$ in (3.28) is the most general EFT, up to Fierz identities and field redefinitions, that involves two Weyl fermion species χ and ψ which transform under the fundamental representation $\mathbf{N}$ of $SU(N)$, and that contains two derivatives. We can compute the forward scattering amplitude

$$\mathcal{M}(\psi^a_{\sigma_1}(p_1)\chi^b_{\sigma_2}(p_2) \to \psi^c_{\sigma_1}(p_1)\chi^d_{\sigma_2}(p_2)) = \mathcal{M}_3 + \mathcal{M}_4 \tag{3.33}$$

where we wrote explicitly the contributions to the amplitude which come from the operators $\mathcal{O}_3$ and $\mathcal{O}_4$ (the contributions of $\mathcal{O}_1$ and $\mathcal{O}_2$ vanish in the forward limit). Using the expression of the density matrices (3.12) we obtain the amplitudes for polarized Weyl fermions

$$i\mathcal{M}3^{(\sigma_1\sigma_2)}_{ab\to cd} = -is\, c_3 \delta^a_d \delta^b_c \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}\rho^{\sigma_1}_{\alpha\dot\alpha}(p_1)\rho^{\sigma_2}_{\beta\dot\beta}(p_2) \tag{3.34}$$
$$i\mathcal{M}4^{(\sigma_1\sigma_2)}_{ab\to cd} = -is\, c_4 \delta^a_c \delta^b_d \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}\rho^{\sigma_1}_{\alpha\dot\alpha}(p_1)\rho^{\sigma_2}_{\beta\dot\beta}(p_2). \tag{3.35}$$

Since the particles are massless they carry definite helicity, say $\sigma_{1,2} = -$, and therefore $\rho^-_{\alpha\dot\alpha}(p) = p_\mu \sigma^\mu_{\alpha\dot\alpha}$ and $\epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}\rho^-_{\beta\dot\beta}(p) = p_\mu(\bar\sigma^\mu)^{\dot\alpha\alpha}$. Using $\mathrm{Tr}[\sigma^\mu\bar\sigma^\nu] = 2\eta^{\mu\nu}$ we get

$$\mathcal{M}3^{(--)}_{ab\to cd} = -c_3\delta^a_d\delta^b_c s^2 \tag{3.36}$$
$$\mathcal{M}4^{(--)}_{ab\to cd} = -c_4\delta^a_c\delta^b_d s^2 \tag{3.37}$$

and therefore

$$\boxed{\mathcal{M}^{(--)}_{ab\to cd} = -s^2(c_3\delta^a_d\delta^b_c + c_4\delta^a_c\delta^b_d)}\,. \tag{3.38}$$

in the strict massless limit. For example, the interaction $(\bar\psi\psi)^2/\Lambda^2$ produces an amplitude which behaves as $\mathrm{Im}\mathcal{M}(s\to 0) \sim s^2/\Lambda^4$ which is not enough to grant the IR convergence of (2.62) for $m^2 = \mu^2 = 0$ with $n = 2$. We recall that $n = 1$, which would yields an IR convergent dispersion relation, is no longer necessarily UV convergent; moreover, the integrand for $n = 1$ (or odd in generality) is not necessarily positive definite even for a UV convergent theory [2].

We can decompose the amplitude in eigen-amplitudes inside $\mathbf{N}\otimes\mathbf{N}$ using (2.20) and (2.25)

$$\mathcal{M}^{(--)}_{ab\to cd} = \sum_{\mathbf{I}\in Y} \left[\widehat{P}_{\mathbf{I}}\right]^{ab}_{cd} \widehat{\mathcal{M}}^{(--)}_{\mathbf{I}} \tag{3.39}$$

whereas the crossed eigen-amplitudes in $\overline{\mathbf{N}}\otimes\mathbf{N}$ are obtained by the exchanging $a \leftrightarrow \bar{c}$ and using (2.26)

$$\mathcal{M}^{(+-)}_{\bar{c}b\to\bar{a}d} = \sum_{\mathbf{I}\in Z} \left[\widetilde{P}_{\mathbf{I}}\right]^{ab}_{cd} \widetilde{\mathcal{M}}^{(+-)}_{\mathbf{I}}. \tag{3.40}$$

We collect the the eigen-amplitudes in the vector

$$\mathcal{M}(s) \equiv \begin{pmatrix} \widehat{\mathcal{M}}^{(--)}_{\mathbf{A}}(s) \\ \widehat{\mathcal{M}}^{(--)}_{\mathbf{S}}(s) \\ \widetilde{\mathcal{M}}^{(+-)}_{\mathbf{1}}(s) \\ \widetilde{\mathcal{M}}^{(+-)}_{\mathbf{Adj}}(s) \end{pmatrix} \tag{3.41}$$

Because of crossing symmetry

$$\mathcal{M}^{(+-)}_{\bar{c}b\to\bar{a}d}(-s) = \mathcal{M}^{(--)}_{ab\to cd}(s) \tag{3.42}$$

we see that $\mathcal{M}^{(--)}_{ab\to cd}(s)$ is enough to derive the positivity bounds. In order to do so, we use the projectors (2.78)÷(2.81) together with

$$\frac{1}{\dim\mathbf{I}} \sum_{abcd} \left[\widehat{P}_{\mathbf{I}}\right]^{ab}_{cd} \left[\widehat{P}_{\mathbf{J}}\right]^{cd}_{ab} = \delta_{\mathbf{IJ}}\,, \qquad \frac{1}{\dim\mathbf{I}} \sum_{abcd} \left[\widetilde{P}_{\mathbf{I}}\right]^{ab}_{cd} \left[\widetilde{P}_{\mathbf{J}}\right]^{cd}_{ab} = \delta_{\mathbf{IJ}} \tag{3.43}$$

In this way, we get the eigen-amplitudes (the sum over the SU(N) indices is understood)

$$\widehat{\mathcal{M}}^{(--)}_{\mathbf{A}}(s) = \frac{2}{N(N-1)} \left[\widehat{P}_{\mathbf{A}}\right]^{ab}_{cd} \mathcal{M}_{ab\to cd}(s) = s^2(c_3 - c_4) \tag{3.44}$$

$$\widehat{\mathcal{M}}^{(--)}_{\mathbf{S}}(s) = \frac{2}{N(N+1)} \left[\widehat{P}_{\mathbf{S}}\right]^{ab}_{cd} \mathcal{M}_{ab\to cd}(s) = -s^2(c_3 + c_4) \tag{3.45}$$

$$\widetilde{\mathcal{M}}^{(+-)}_{\mathbf{1}}(s) = \left[\widetilde{P}_{\mathbf{1}}\right]^{ab}_{cd} \mathcal{M}_{\bar{c}b\to\bar{a}d}(s) = -s^2(Nc_3 + c_4) \tag{3.46}$$

$$\widetilde{\mathcal{M}}^{(+-)}_{\mathbf{Adj}}(s) = \frac{1}{N^2-1} \left[\widetilde{P}_{\mathbf{Adj}}\right]^{ab}_{cd} \mathcal{M}_{\bar{c}b\to\bar{a}d}(s) = -s^2 c_4 \tag{3.47}$$

and hence, using the inequality (2.91), we get from (3.44) and (3.45)

$$\boxed{c_4 \leq 0, \qquad c_3 + c_4 \leq 0}\,. \tag{3.48}$$

Incidentally, these bounds happen to be the same than those that one would obtain by using the positivity (2.67) for each pair of flavor $a = b$ or $a \neq b$ in the ordinary amplitude, namely

$$\mathcal{M}^{(--)\prime\prime}_{ab\to cd}(0) = -2(c_3\delta^a_b + c_4) \geq 0\,. \tag{3.49}$$

In conclusion, there is no UV completion that gives rise to $\mathcal{O}_1$ and $\mathcal{O}_3$ with negative Wilson coefficients. Moreover, the inequality is saturated only for the free theory.

Chapter 4

Composite quarks and pseudo-Goldstini at the LHC

An EFT is a theory with a limited range validity but which is very effective in capturing the low-energy features of phenomena below a certain energy cutoff Λ. [1] The associated effective Lagrangian is an infinite tower of operators $\mathcal{O}_i$ of increasing dimension $\Delta_i = \dim \mathcal{O}_i$,

$$\mathcal{L}_{EFT} = \sum_i \frac{c_i}{\Lambda^{\Delta_i - 4}} \mathcal{O}_i \,, \tag{4.1}$$

where c_i are the so-called Wilson coefficients. Despite the infinite number of operators and Wilson coefficients, EFT's are predictive for $E \ll \Lambda$ because only a finite number of operators affect appreciably (that is above the fixed experimental resolution) the value of low-energy observables. For $E \sim \Lambda$, infinitely many terms become important and the theory should be superseded by a UV completion or another EFT that includes the new degrees of freedom that start propagating at around Λ.

In a generic setup where all Wilson coefficients are sizeable, one expects by dimensional analysis that higher dimensional operators contributing to a certain process become quickly less important in the IR than the lower dimensional operators that contribute to the same process for $E \ll \Lambda$. Generically, one can thus truncate the infinite tower of operators to the lowest dimensional ones in the IR. However, symmetries can forbid or suppress certain Wilson coefficients so that the would-be leading lower dimensional operators may actually be dominated by some higher dimensional operator at intermediate energy, i.e. still below the cutoff, consistency with the EFT expansion. In this chapter we are going to see precisely an example of such a sort, where marginal and dim-6 operators that enter in 2-to-2 scattering of fermions have suppressed Wilson coefficients, so that the amplitude is actually dominated by dim-8 operators, hence scaling as E^4 for E larger than any IR scale but still well below the cutoff Λ. For those dim-8 operators we can apply the positivity conditions that we have derived on a firm theoretical ground in the previous chapters. The set of rules, symmetries,

[1] We are focusing on relativistic EFTs for particles only, for an introduction see e.g. [19]. The general ideas of EFT have actually found applications in a much wider range of subjects and fields.

and spurions that determine which operator enters with a sizeable Wilson coefficient versus those which are suppressed are known as "power counting". For definiteness in the following we adopt the one-coupling one-scale power counting of composite Higgs models, although several of the results that we present actually extend beyond that framework.

4.1 Power counting and composite dynamics

Strongly-coupled models of the electroweak symmetry breaking and the Higgs sector provide a solution to the hierarchy problem of the SM. Inspired by the power counting of the chiral Lagrangian in QCD and by holographic dual models, the dynamics of the states that emerge from these strong sectors is usually assumed to be broadly described by a simple one-coupling (g_*) one-scale (Λ) power counting [20]

$$\mathcal{L}_C = \frac{\Lambda^4}{g_*^2}\hat{\mathcal{L}}_C\left[\frac{\partial}{\Lambda}, \frac{g_*\sigma}{\Lambda}, \frac{g_*\chi}{\Lambda^{3/2}}\right]. \tag{4.2}$$

where $\hat{\mathcal{L}}_C$ is a dimensional function that can be taylor expanded in its arguments with $O(1)$ coefficients. This power counting encompasses the naive dimensional analysis (NDA) of fully strongly coupled sectors with $g_* \sim 4\pi$ like in QCD [19, 21], and the moderate coupling limit $g_* \sim O(1)$. The σ and χ fields in (4.2) represent generic composite spin-0 and spin-1/2 resonances, respectively. However, particular resonances can enjoy extra selections rules dictated by symmetries that forbid certain interactions.
For example, the $2 \to 2$ scattering among generic scalar resonances would give $\mathcal{M} \sim g_*^2$, e.g. from a marginal operator $g_*^2\sigma^4$. But should the scalar resonances be composite Goldstone Bosons (GB) π emerging from a symmetry broken spontaneously by the strong sector, we know that the amplitudes should actually be dependent on the momentum, as the GBs are derivatively coupled. This means that GBs interactions come from higher dimensional operators that requires at least two extra derivatives, schematically of the type $g_*^2\partial^2\pi^4/\Lambda^2$, giving rise to the desired scaling at leading order $\mathcal{M} \sim g_*^2E^2/\Lambda^2$. Should the spontaneosly broken symmetry be approximate, the π would actually be pseudo-GBs and admit a potential schematically of the type $\epsilon^2\left(g_*^2\pi^4 + \Lambda^2\pi^2 + \ldots\right)$. The spurion $\epsilon \ll 1$ that slightly breaks the symmetry generates thus lower dimensional operators which, being less irrelevant that the symmetry preserving interactions, change the amplitude in the deep IR,

$$\mathcal{M} \sim g_*^2\epsilon^2 + g_*^2\frac{E^2}{\Lambda^2}. \tag{4.3}$$

But for intermediate energy above the IR and yet below the cutoff

$$\epsilon\Lambda \ll E \ll \Lambda \tag{4.4}$$

the amplitude is still dominated by the irrelevant higher dimensional operator,

$$\mathcal{M}(\epsilon\Lambda \ll E \ll \Lambda) \sim g_*^2E^2/\Lambda^2, \tag{4.5}$$

within the validity of the EFT. In Composite Higgs model where the Higgs boson is one of the GB's of the strong sector, the spurion ϵ is provided by the ratio of couplings $\epsilon = g_{SM}/g_*$ where g_{SM} is a gauge coupling or a Yukawa coupling. [2]
The lesson to be drawn from this specific example is in fact quite general: higher dimensional operators may dominate lower dimensional ones because of selection rules that require to go high enough in operator dimension to construct a singlet under the symmetry. In the case of the GBs one had to add at least two extra derivatives because of the GB shift symmetry; see e.g. [9, 22] for other explicit examples. In the next sections we show that the same may happen for other resonances of the strong sector, and we discuss in detail the case of the Goldstone fermion of SUSY, the Goldstino. The leading interactions for a Goldstino do no start at dim-6 with $(\chi^\dagger\chi)^2$-type of 4-fermion interaction as one could naively expect for a generic spin-1/2 resonance, but rather at dim-8 as the fermionic shift symmetry from SUSY breaking requires two extra derivatives to be inserted, schematically $\chi^\dagger\chi^\dagger\partial^2\chi^2$.

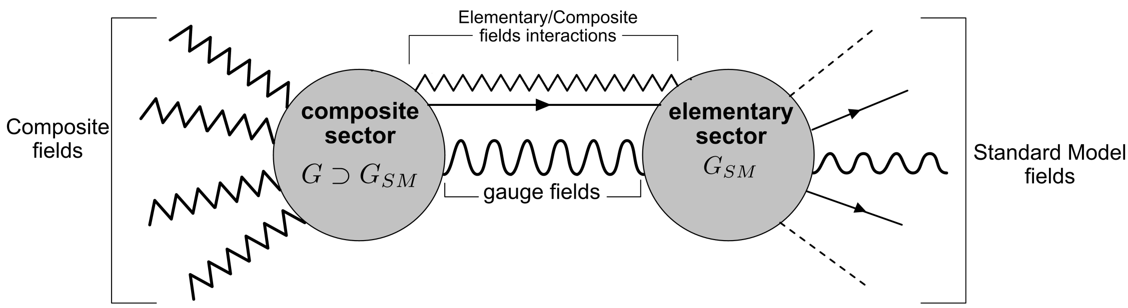

Fig. 1: Cartoon of the partial compositeness framework where states from a weakly coupled elementary sector mix linearly with composite resonances of a strongly coupled sector.

Beside composite particles of the strong sector there are also elementary particles, for example the transverse gauge bosons, that we need to include in the EFT in the IR. In order to couple them consistently to the strong sector, the latter must contain currents J_i^μ associated to the SM symmetry group. Essentially, the strong sector must have a sufficiently large symmetry group G to include $SU(3) \times SU(2) \times U(1)$; the associated currents are thus weakly gauged by coupling the currents to the elementary gauge fields

$$\mathcal{L}_{g,mix} = g_i A_\mu^i J_i^\mu \,. \tag{4.6}$$

Since the currents of the strong sector will generically produce spin-1 resonances acting on the vacuum, $\langle 0|J_\mu(0)|p,\sigma\rangle \propto \epsilon_\mu^\sigma(p)$, the Lagrangian (4.6) represents a mixing between the

[2]In composite Higgs models, in contrast to pions of QCD that get a tree-level potential from the insertion of Yukawas, the potential is actually further suppressed because it is generated at one-loop.

spin-1 resonances and the elementary ones, the angle of the mixing being controlled by $\epsilon_i = g_i/g_*$. Analogously, a spin-1/2 field ψ of the elementary sector can source a spin-1/2 operator $\mathcal{O}_F$ of the strong sector

$$\mathcal{L}_{y,mix} = \lambda\bar{\psi}\mathcal{O}_F \tag{4.7}$$

which generates spin-1/2 resonances χ when acting on the vacuum, $\langle 0|\mathcal{O}_F(0)|p,\sigma\rangle \propto u^\sigma(p)$. The Lagrangian (4.7) represents again a linear mixing controlled by $\epsilon = \lambda/g_*$ between elementary and composite states ψ and χ respectively. Since the physical states are a mixture of the particles in the two sectors, this scenario is known as "partial compositeness". The physical states inherit couplings from both sector upon insertions of the ϵ's. For example, 4-fermi interactions that are present for the resonance χ of the strong sector generate in turn 4-fermion interactions for the physical state, although containing 4 insertions of ϵ's. In the regime where $\lambda/g_* \ll 1$, the fermion ψ couples very weakly to the strong sector and it is thus mostly elementary, the 4-fermion interaction being strongly suppressed. On the other hand, for $\lambda/g_* \sim 1$, the fermion as $O(1)$ mixing and becomes part of the strong sector, it is i.e. (almost) *fully composite* and it enjoys unsuppressed 4-fermion interactions. When this happens we can drop the distinction from ψ and χ and use just a single letter χ.
Since the couplings of the elementary gauge fields A_μ and matter fields ψ to the strong sector pass only through g_i and y_i, the effective Lagrangian is summarized, apart from extra selection rules, by the following scaling

$$\mathcal{L}_{\text{EFT}} = \frac{\Lambda^4}{g_*^2}\hat{\mathcal{L}}_{\text{EFT}}\left[\frac{\partial}{\Lambda}, \frac{g_*\sigma}{\Lambda}, \frac{g_*\chi}{\Lambda^{3/2}}, \frac{g}{\Lambda}A_\mu, \frac{\lambda_{L,R}}{\Lambda^{3/2}}\psi_{L,R}\right] + \mathcal{L}_{elem.}(A_\mu, \psi)\,, \tag{4.8}$$

where $\mathcal{L}_{elem.}$ contains the kinetic term and the weak interactions of the elementary sector. Notice that the strong sector produces corrections to the elementary kinetic terms of $O(\lambda^2/g_*^2)$ or $O(y^2/g_*^2)$. We should also mention that, given a leading operator to a certain process, the insertion of extra derivatives or fields sitting at their VEVs, produces small corrections suppressed by

$$k_v^2 = \left(\frac{g_* v}{\Lambda}\right)^2, \qquad k_E^2 = \left(\frac{E}{\Lambda}\right)^2. \tag{4.9}$$

4.2 Constraints on dim-8 four-fermion operators

Let us specialize now to the $2 \to 2$ scattering of fermion fields. The amplitude, due to SM interactions and dim-6 four-fermion operators, goes approximately as

$$\mathcal{M}(\chi\chi \to \chi\chi) \sim g_{SM}^2 + \frac{g_*^2 E^2}{\Lambda^2} \tag{4.10}$$

at energy well above the fermion and gauge boson masses. [3] We are assuming that the fermion mix almost maximally with the strong sector, i.e. $\psi \sim \chi$ is essentially fully composite, so that the insertions of $\epsilon = y/g_*$ can be omitted in the 4-fermion vertex. We are

[3] This scaling is simply understood by dimensional analysis: each fermion wave-function for $E \gg m$ goes like $\sqrt{E}$, while the gauge boson propagators go as $1/E^2$ above its mass.

also provisionally assuming that these dim-6 operators are unsuppressed by symmetries of the strong sector; we will relax this assumption later.
Since $g_* \ll g_{SM}$, in the validity range of the effective theory $E < \Lambda$, the contributions of the higher-dimensional operator can beat the SM gauge coupling and dominate the amplitude for

$$\left(\frac{g_{SM}}{g_*}\right) \times \Lambda < E < \Lambda \,. \tag{4.11}$$

This is no surprise since irrelevant operators grow with energy and can eventually beat marginal operators (such as the gauge interactions) within the validity of the EFT because the gauge coupling is small compared to g_*. Adding the contribution from $dim-8$ operators (with two more derivatives) does not change anything in this picture

$$\mathcal{M}(\chi\chi \to \chi\chi) \sim g_{SM}^2 + \frac{g_*^2 E^2}{\Lambda^2} + \frac{g_*^2 E^4}{\Lambda^4} \tag{4.12}$$

since they are always subdominant to the dim-6 for $E < \Lambda$, given that both dim-6 and dim-8 Wilson coefficients are unsuppressed.
But let's suppose now that for because of a symmetry, or some dynamical reasons, the dim-6 four-fermion operators from the strong sector are actually suppressed by a small spurion ϵ

$$\mathcal{M}(\chi\chi \to \chi\chi) \sim g_{SM}^2 + \epsilon^2 \frac{g_*^2 E^2}{\Lambda^2} + \frac{g_*^2 E^4}{\Lambda^4} + \ldots \tag{4.13}$$

In this case the dim-8 operators dominate the amplitude within the range of the EFT

$$\max\{\epsilon\Lambda \,, \sqrt{g_{SM}/g_*}\Lambda\} < E < \Lambda \tag{4.14}$$

while even higher dimensional operators are still suppressed.
Through the remainder of this chapter we will assume precisely this latter scenario where (some or all) dim-6 operators of four-fermion interactions are suppressed, while dim-8 four-fermion operators with two extra derivatives dominate the amplitude at high enough energy, but still below the cutoff such that our EFT approach is valid. We will assume that some of the quarks of the SM are fully composite, i.e. they mix by an $O(1)$ factor with the fermions of the strong sector, inheriting the unsuppressed 4-fermion interactions with two derivatives. We will be agnostic about the precise dynamics or symmetry that forbids the dim-6 operators, although we provide an explicit example of symmetry of the strong sector that does so. It is a spontaneously broken extended $\mathcal{N}$-SUSY where the composite fermions are (pseudo-)Goldstini protected by a fermionic shift symmetry

$$\chi(x) \to \chi'(x) = \chi(x) + \xi - v^\mu(\xi, \chi)\partial_\mu \chi(x) + \ldots \tag{4.15}$$

where $v^\mu(\xi, \chi) = i\left(\xi\sigma^\mu\chi^\dagger - \chi\sigma^\mu\xi^\dagger\right)$, as discussed in detail in the appendix D. The SM as composite pseudo-Goldstini is actually a hold idea by Bardeen and Visnjic [23] that was recently invoked in [9] in the context of composite Higgs models. We revive this idea and make it more concrete by comparing a simple realization against the LHC data.

In Tab. 4.1, we list the expressions of dimension-8 operators involving four Weyl fermions based on the symmetry they respect[4]. The $SU(N_C) \times SU(N_F)$ is eventually identified with the color and flavor group carried by χ.

$U(1)$	$U(1) \times SU(N)$	$U(1) \times SU(N_C) \times SU(N_F)$
$(\partial\chi^\dagger)\chi^\dagger(\partial\chi)\chi$	$\partial_\mu\bar{\chi}_a\bar{\chi}_b\partial^\mu\chi^a\chi^b$	$\partial\chi^{\dagger\alpha}_{\ a}\chi^{\dagger\beta}_{\ b}\partial\chi^a_\alpha\chi^b_\beta$
	$\partial_\mu\bar{\chi}_a\bar{\chi}_b\partial^\mu\chi^b\chi^a$	$\partial\chi^{\dagger\alpha}_{\ a}\chi^{\dagger\beta}_{\ b}\partial\chi^a_\beta\chi^b_\alpha$
		$\partial\chi^{\dagger\alpha}_{\ a}\chi^{\dagger\beta}_{\ b}\partial\chi^b_\alpha\chi^a_\beta$
		$\partial\chi^{\dagger\alpha}_{\ a}\chi^{\dagger\beta}_{\ b}\partial\chi^b_\beta\chi^a_\alpha$

Table 4.1: Dim-8 operators involving massless right-handed Weyl fermions charged under the fundamental or bi-fundamental representation of the symmetries listed in the headlines. We use the two-component spinor notation with spinor indexes not displayed, $\chi^\dagger\chi^\dagger = \chi^\dagger_{\dot\alpha}\chi^{\dagger\dot\alpha}$, $\chi\chi = \chi^\alpha\chi_\alpha$. The Lorentz indexes of the derivatives are contracted between each other and not displayed. We use upper(lower) indexes $\chi \equiv \chi^a$ ($\chi^\dagger = \chi_a$) for the (anti-)fundamental of $SU(N)$. Greek and Latin letters label $SU(N_C)$ and $SU(N_F)$ indexes respectively.

4.2.1 Positivity bounds

As a concrete example, we identify χ with a fully composite right-handed down-type quark d_R which carries two $SU(3)$ indexes, flavor and color, on top of the hypercharge. As we stressed above, we assume that dim-6 four-d_R operators are suppressed, either by dynamics or symmetry, (e.g. an extended SUSY, see appendix D and Eq. D.66).
The independent hermitian dim-8 operators are

$$\mathcal{O}_1^{(8)} = \partial\bar{d}_{R_a}^{\alpha}\bar{d}_{R_b}^{\beta}\partial d_{R\alpha}^{a}d_{R\beta}^{b} \tag{4.16}$$
$$\mathcal{O}_2^{(8)} = \partial\bar{d}_{R_a}^{\alpha}\bar{d}_{R_b}^{\beta}\partial d_{R\beta}^{a}d_{R\alpha}^{b} \tag{4.17}$$
$$\mathcal{O}_3^{(8)} = \partial\bar{d}_{R_a}^{\alpha}\bar{d}_{R_b}^{\beta}\partial d_{R\alpha}^{b}d_{R\beta}^{a} \tag{4.18}$$
$$\mathcal{O}_4^{(8)} = \partial\bar{d}_{R_a}^{\alpha}\bar{d}_{R_b}^{\beta}\partial d_{R\beta}^{b}d_{R\alpha}^{a} \tag{4.19}$$

where Greek and Latin letters stand for color and flavor indexes respectively, and the effective Lagrangian takes the form

$$\mathcal{L}_{eff} = \mathcal{L}_{SM} + \sum_i \frac{g_i}{\Lambda^4}\mathcal{O}_i^{(8)} . \tag{4.20}$$

The amplitude of a generic scattering is is[5]

$$\mathcal{M}^{(--)}{}^{ab,\alpha\beta}_{cd,\gamma\delta} = \langle 4^{-\,(d\,\delta)}3^{-\,(c\,\gamma)}|\mathcal{M}|1^{-\,(a\,\alpha)}2^{-\,(b\,\beta)}\rangle \tag{4.21}$$

[4] A sketched derivation of this list is given in the Appendix B.

[5] We recall the notation (2.12) which gives the position of the indexes in the amplitude. For instance $\langle 2^{-\,(d\,\delta)}|$ transforms as a tensor T of $SU(3)_C \times SU(3)_F$ with components $T_{cd,\gamma\delta}$ whereas $\langle 2^{-\,(d\,\delta)}\bar{1}^{+\,(\bar{a}\,\bar{\alpha})}|$ as $T^{a\ \ ,\alpha}_{\ d,\ \ \delta}$.

and it reads

$$\mathcal{M}^{(--)}{}^{ab,\alpha\beta}_{cd,\gamma\delta} = u^{\dagger\dot{\alpha}}(p_4,\sigma)u^{\dagger}{}_{\dot{\alpha}}(p_3,\sigma)u_{\sigma}(p_1,\sigma)u^{\sigma}(p_2,\sigma)\left[\mathcal{M}_1+\mathcal{M}_2+\mathcal{M}_3+\mathcal{M}_4\right] \tag{4.22}$$

where we are omitting the indexes in the r.h.s. and the polarization is fixed, say $\sigma = -$ (for d^c_R), giving rise to

$$\mathcal{M}_1 = -g_1\left(t\cdot\delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta}\delta^{a}_{c}\delta^{b}_{d} + u\cdot\delta^{\alpha}_{\delta}\delta^{\beta}_{\gamma}\delta^{b}_{c}\delta^{a}_{d}\right) \tag{4.23}$$

$$\mathcal{M}_2 = -g_2\left(t\cdot\delta^{\alpha}_{\delta}\delta^{\beta}_{\gamma}\delta^{a}_{c}\delta^{b}_{d} + u\cdot\delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta}\delta^{b}_{c}\delta^{a}_{d}\right) \tag{4.24}$$

$$\mathcal{M}_3 = -g_3\left(t\cdot\delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta}\delta^{a}_{d}\delta^{b}_{c} + u\cdot\delta^{\alpha}_{\delta}\delta^{\beta}_{\gamma}\delta^{b}_{d}\delta^{a}_{c}\right) \tag{4.25}$$

$$\mathcal{M}_4 = -g_4\left(t\cdot\delta^{\alpha}_{\delta}\delta^{\beta}_{\gamma}\delta^{a}_{d}\delta^{b}_{c} + u\cdot\delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta}\delta^{b}_{d}\delta^{a}_{c}\right)\,. \tag{4.26}$$

In the elastic forward limit $p_1 = p_3 = p$, $p_2 = p_4 = k$, we can apply the positivity bounds we derived for fermions transforming in the fundamental of $SU(N)$ in section (2.6) In this limit, the wave-function polarization can be written in terms of the density matrix (3.12)

$$u^{\dagger\dot{\alpha}}(k,\sigma)u^{\dagger}{}_{\dot{\alpha}}(p,\sigma)u_{\sigma}(p,\sigma)u^{\sigma}(k,\sigma) = \rho^{-}(p)_{\sigma\dot{\alpha}}\epsilon^{\sigma\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\rho^{-}(k)_{\beta\dot{\beta}}. \tag{4.27}$$

Then, setting $t = 0$ and replacing (4.27) in (4.22), the amplitude becomes

$$\mathcal{M} = s^2\left[\delta^{a}_{c}\delta^{b}_{d}a_1 + \delta^{a}_{d}\delta^{b}_{c}a_2\right] \tag{4.28}$$

where

$$a_1 = +g_4\delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta} + g_3\delta^{\alpha}_{\delta}\delta^{\beta}_{\gamma} \tag{4.29}$$

$$a_2 = +g_2\delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta} + g_1\delta^{\alpha}_{\delta}\delta^{\beta}_{\gamma}. \tag{4.30}$$

We can decompose the amplitude in eigenamplitudes respect to the flavor group $SU(N_F)$ using the projectors (2.78)÷(2.81). Indeed, the amplitude of the process can be written as

$$\mathcal{M}^{(--)}{}^{ab,\alpha\beta}_{cd,\gamma\delta} = \sum_{\mathbf{I}\in Y}\left[\widehat{P}_{\mathbf{I}}\right]^{ab}_{cd}\left[\widehat{M}^{(--)}_{\mathbf{I}}\right]^{\alpha\beta}_{\gamma\delta} \tag{4.31}$$

whereas, exchanging $1 \leftrightarrow 3$ in the forward limit we get the crossed amplitude $\mathcal{M}^{(+-)}{}^{ab,\alpha\beta}_{cd,\gamma\delta} = \langle 2^{-\,(d\,\delta)}\bar{1}^{+\,(\bar{a}\,\bar{\alpha})}|\mathcal{M}|\bar{3}1\rangle^{+\,(\bar{c}\,\bar{\gamma})}\,2^{-\,(b\,\beta)}$ which can be decomposed as

$$\mathcal{M}^{(+-)}{}^{ab,\alpha\beta}_{cd,\gamma\delta} = \sum_{\mathbf{I}\in Z}\left[\widetilde{P}_{\mathbf{I}}\right]^{ab}_{cd}\left[\widetilde{M}^{(+-)}_{\mathbf{I}}\right]^{\alpha\beta}_{\gamma\delta}\,. \tag{4.32}$$

By crossing symmetry, $\mathcal{M}^{(+-)}{}^{ab,\alpha\beta}_{cd,\gamma\delta}(-s) = \mathcal{M}^{(--)}{}^{ab,\alpha\beta}_{cd,\gamma\delta}(s)$ in the forward limit, and since (4.2.1) is an even function of s, we can extract the eigen-amplitudes from $\mathcal{M}^{(--)}{}^{ab,\alpha\beta}_{cd,\gamma\delta}(s)$. The completeness relations in the flavour space

$$\frac{1}{\dim\mathbf{I}}\sum_{abcd}\left[\widehat{P}_{\mathbf{I}}\right]^{ab}_{cd}\left[\widehat{P}_{\mathbf{J}}\right]^{cd}_{ab} = \delta_{\mathbf{IJ}}\,,\qquad \frac{1}{\dim\mathbf{I}}\sum_{abcd}\left[\widetilde{P}_{\mathbf{I}}\right]^{ab}_{cd}\left[\widetilde{P}_{\mathbf{J}}\right]^{cd}_{ab} = \delta_{\mathbf{IJ}}\,. \tag{4.33}$$

allow us to extract the eigen-amplitudes[6]. To avoid clutter of notation, in the following we omit the tilde and the hat over the amplitudes (and then the projectors) since the irrep index labels uniquely the components of the master eigen-amplitude

$$\mathcal{M}(s) \equiv \begin{pmatrix} \widehat{\mathcal{M}}^{(--)}_{\mathbf{A}}(s) \\ \widehat{\mathcal{M}}^{(--)}_{\mathbf{S}}(s) \\ \widetilde{\mathcal{M}}^{(+-)}_{\mathbf{1}}(s) \\ \widetilde{\mathcal{M}}^{(+-)}_{\mathbf{Adj}}(s) \end{pmatrix} \tag{4.34}$$

where we are again omitting the color indexes.

$$\mathcal{M}_{\mathbf{A}} = \frac{2}{N(N-1)} [P_{\mathbf{A}}]^{ab}_{cd} \mathcal{M}^{cd}_{ab} = s^2(a_1 - a_2) \tag{4.35}$$

$$\mathcal{M}_{\mathbf{S}} = \frac{2}{N(N+1)} [P_{\mathbf{S}}]^{ab}_{cd} \mathcal{M}^{cd}_{ab} = s^2(a_2 + a_1) \tag{4.36}$$

$$\mathcal{M}_{\mathbf{1}} = [P_{\mathbf{1}}]^{ab}_{cd} \mathcal{M}^{cd}_{ab} = s^2(N a_1 + a_2) \tag{4.37}$$

$$\mathcal{M}_{\mathbf{Adj}} = \frac{1}{N^2-1} [P_{\mathbf{Adj}}]^{ab}_{cd} \mathcal{M}^{cd}_{ab} = s^2 a_2. \tag{4.38}$$

Applying now the general positivity bound (2.91) we get the constraints

$$a_1 \geq 0 \qquad a_2 \geq 0\,. \tag{4.39}$$

Setting $\alpha = \gamma \neq \beta = \delta$ we extract the positivity bounds for g_4 and g_2 whereas setting $\alpha = \delta \neq \beta = \gamma$ we constraint g_1 and g_3

$$\boxed{g_1 \geq 0, \quad g_2 \geq 0, \quad g_3 \geq 0, \quad g_4 \geq 0}\,. \tag{4.40}$$

This result matches the positivity condition found in [6] for $a = b = c = d$ and $\alpha = \beta = \gamma = \delta$ with the identification $\chi = d_R$.

4.3 Dijets analysis

The goal of this section is twofold. First, we want first derive the exclusion region on the parameter space of dim-8 four-fermion operators using the data from the experiments at the LHC. Second, we want to determine the quantitative impact of the positivity constraints on those experimental bounds. This will be done by studying the dijets angular distribution at $\sqrt{s} = 13$ TeV measured at LHC by ATLAS [24]. We thus we need to say something about the interactions of the other quarks that can significantly affect the LHC analysis for $pp \to jj$ that we use. We impose a flavor symmetry

$$G_F = U(3)_{q_L} \times U(3)_{d_R} \times U(3)_{u_R} \tag{4.41}$$

[6]Notice that the sum over the SU(N) indices is understood and we do not write the color indexes which are included in the definition of a_1 and a_2.

which rotates the three left-handed q_L doublets, the three right-handed down-type and up-type quarks. This symmetry is obviously broken by the Yukawa couplings of the SM, that is their masses, but we assume it is respected by the strong sector. The bounds will be placed using events with large momentum transferred compared to the masses of the quarks that can thus be neglected. Notice that, while d_R is taken fully composite, we haven't committed yet to a particular choice of degree of compositeness of the q_L and u_R. For a sizable fraction of compositeness of the quarks other than d_R, we will need to include also dim-6 four-fermion operators involving them.

We study the angular distribution of dijets processes $pp \to jj$ at LHC, which allows to probe the size of four-quark contact interaction [25],[26],[27],[28]. We will closely follow the strategy put forward by [29] but using the analysis of [26].

The process is dominated by QCD interactions which may interfere with the higher-dimension operators that are generated by the strong sector. Since we are after operators that grow with energy, we look at the events with pretty high dijets invariant mass[7], in the range $[3.4\text{TeV}, 4.0\text{TeV}]$. In this range of energy, the SM gives

$$\left(\frac{\sigma(u\bar{u} \to u\bar{u})}{\sigma(uu \to uu)}\right)_{SM} \simeq 0.04 \qquad \left(\frac{\sigma(uc \to uc)}{\sigma(uu \to uu)}\right)_{SM} \simeq 0.02\,, \qquad \left(\frac{\sigma(gg \to gg)}{\sigma(uu \to uu)}\right)_{SM} \simeq 0.35\,, \tag{4.42}$$

so that the the dominant contributions in $pp \to jj$ come from uu, dd, du, gg initial states. Initial states involving other quark families are suppressed as well. As example, we show in Fig.2 the MSTW2008NNLO PDFs for each quark (anti-quark) flavour at the factorization scale 3.4 TeV. The contribution from gg does not receive contribution from the operators of the strong sector that we want to bound, it is purely QCD and does not interfere at tree-level with the BSM sector.

Processes with different quark families in final states such as $uu \to ss$ do not arise by four-fermion interaction thanks to the flavour symmetry.

The dimension-6 operators involving only the relevant quarks from the first family are [29][8]

$$\mathcal{O}^{(6)}_{uu} = (\bar{u}_R\bar{\sigma}^\mu u_R)(\bar{u}_R\bar{\sigma}_\mu u_R) \tag{4.43}$$
$$\mathcal{O}^{(6)}_{qq} = (\bar{q}_L\bar{\sigma}^\mu q_L)(\bar{q}_L\bar{\sigma}_\mu q_L) \tag{4.44}$$
$$\mathcal{O}^{(6)}_{qu} = (\bar{q}_L\bar{\sigma}^\mu q_L)(\bar{u}_R\bar{\sigma}_\mu u_R) \tag{4.45}$$
$$\mathcal{Q}^{(6)}_{qq} = (\bar{q}_L\bar{\sigma}^\mu T^A q_L)(\bar{q}_L\bar{\sigma}_\mu T^A q_L) \tag{4.46}$$
$$\mathcal{Q}^{(6)}_{qu} = (\bar{q}_L\bar{\sigma}^\mu T^A q_L)(\bar{u}_R\bar{\sigma}_\mu T^A u_R) \tag{4.47}$$

where $q_L = (u_L, d_L)$ and T^A are the generators of $SU(3)_C$.

The relevant dimension-8 operators involving right-handed down-type quarks are those in

[7] The dijets invariant mass m_{jj} is the characteristic energy of the partonic process, $m_{jj} = \sqrt{\hat{s}}$ where $\hat{s}$ is the partonic Mandelstam variable. These kinematical variables will be defined later.

[8] Contrary to [29], we construct the operators using Weyl spinors.

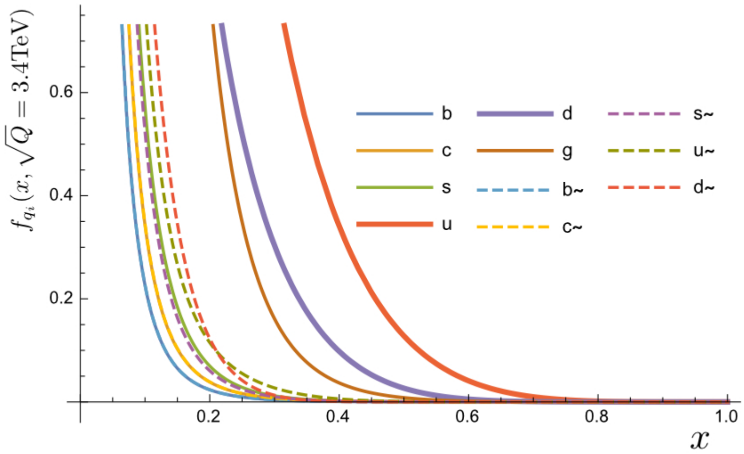

Fig. 2: Parton distribution functions $f_{q_i}(x)$ plotted respect to the momentum fraction carried by the parton q_i at the factorization scale $\sqrt{Q} = 3.4\text{TeV}$.

(4.16)÷(4.19) that we write again

$$\mathcal{O}_1^{(8)} = \partial\bar{d}_{Ra}^{\alpha}\bar{d}_{Rb}^{\beta}\partial d_{R\alpha}^{a}d_{R\beta}^{b} \tag{4.48}$$
$$\mathcal{O}_2^{(8)} = \partial\bar{d}_{Ra}^{\alpha}\bar{d}_{Rb}^{\beta}\partial d_{R\beta}^{a}d_{R\alpha}^{b} \tag{4.49}$$
$$\mathcal{O}_3^{(8)} = \partial\bar{d}_{Ra}^{\alpha}\bar{d}_{Rb}^{\beta}\partial d_{R\alpha}^{b}d_{R\beta}^{a} \tag{4.50}$$
$$\mathcal{O}_4^{(8)} = \partial\bar{d}_{Ra}^{\alpha}\bar{d}_{Rb}^{\beta}\partial d_{R\beta}^{b}d_{R\alpha}^{a}. \tag{4.51}$$

In Fig.3 we show an example of the partonic processes we are interested in. All physical quantities at partonic level are given in terms of the partonic Mandelstam variables which are defined by the momenta carried by partons inside the protons

$$\hat{s} = (k_1 + k_2)^2 = (k_3 + k_4)^2, \tag{4.52}$$
$$\hat{t} = (k_1 - k_3)^2 = (k_2 - k_4)^2 \tag{4.53}$$
$$\hat{u} = (k_1 - k_4)^2 = (k_2 - k_3)^2. \tag{4.54}$$

In pp collisions, QCD contributes mainly with a t-channel gluon exchange, giving rise to a differential cross-section which grows approximately as

$$\left(\frac{d\hat{\sigma}}{d\hat{t}}\right)_{QCD} \propto \frac{\alpha_s^2}{\hat{t}^2} \tag{4.55}$$

where $\hat{t} = -\hat{s}/2(1 - \cos\hat{\theta})$, $\hat{\theta}$ being the scattering angle in the collision center of mass frame.

Since in general the center of mass frame of the parton-parton scattering is boosted respect the collision one, it is convenient to describe the processes using kinematical variables which transform fairly well under longitudinal boosts. For massless particles, one of these variables

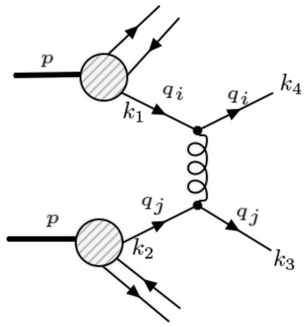

Fig. 3

is the rapidity y which is addictive under boosts along the beam axis. The definition of rapidity is

$$y = \frac{1}{2}\ln\left(\frac{E+p_z}{E-p_z}\right) \tag{4.56}$$

where the p_z is the particle momentum component along the beam line. The rapidity is used to define another variable χ which is useful to study the angular distribution of dijets, $\chi = e^{|\eta_1-\eta_2|} = e^{2|\eta^*|}$ where $\eta_{1,2}$ are the pseudorapidities (or rapidity in the massless limit) of the two leading jets and $\eta^* = (\eta_1 - \eta_2)/2$. The variable χ can be expressed also in terms of the scattering angle $\hat{\theta}$ as

$$\chi = \frac{1+|\cos\hat{\theta}|}{1-|\cos\hat{\theta}|} \simeq \frac{1}{1-|\cos\hat{\theta}|} \propto \frac{\hat{s}}{\hat{t}} \tag{4.57}$$

Under this approximation the differential cross-section (4.55) behaves as

$$\left(\frac{d\hat{\sigma}}{d\chi}\right)_{QCD} \propto \frac{\alpha_s^2}{\hat{s}} \tag{4.58}$$

for fixed $\hat{s}$ which means that we always produce dijets with the same invariant mass. At the hadronic level, this means also that the product of the PDFs is approximately fixed (up to logarithmic scaling variations with the factorization scale) and then the cross-section is approximately constant as well. We therefore expect a flat distribution for $d\hat{\sigma}/d\chi$ in the SM.

Taking into account now the contributions of higher-dimension operators, the distribution of the partonic differential cross section becomes peaked for small values of χ, see Fig. 4. New physics signals should thus emerge as deviation in the distributions for small values of χ. In order to observe significant deviations, we will use an useful measurable variable F_χ which is defined as the ratio of events

$$F_\chi = \frac{N(\chi < \chi_c, m_{jj}^{min} < m_{jj} < m_{jj}^{max})}{N(\chi < \chi_{max}, m_{jj}^{min} < m_{jj} < m_{jj}^{max})} \tag{4.59}$$

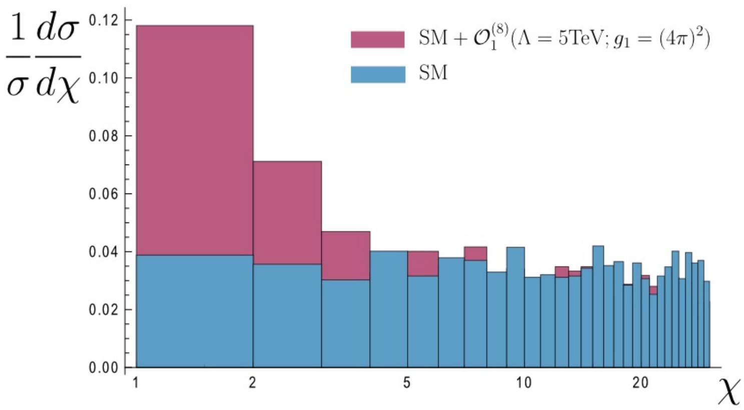

Fig. 4: Distribution of the normalized differential dijets cross-section respect to the variable χ at $\sqrt{s} = 13$TeV and with 3.4TeV $< m_{jj} <$ 4TeV. We show two distributions: Standard Model and BSM. In the latter, we turned on only $\mathcal{O}_1^{(8)}$ in addition to the SM.

for some values of χ_c and χ_{max}. The interval $|y^*| < 0.6$ defines the region where the variable is most sensitive to new physics effect and corresponds to the angular region $\chi < \chi_c \equiv 3.32$ whereas $|y^*| < 1.7$ corresponds to $\chi < \chi_{max} \equiv 30$ i.e. the region where QCD contribution dominate. We will follow the same analysis approach of [26].
To obtain the total dijets cross-section, we need to weight each partonic process with the parton density functions (PDF's). The general expression of the total cross-section is

$$\sigma(pp \to jj) = \sum_{i_1,i_2}^{\text{initial states}} \sum_{f_1,f_2}^{\text{final states}} \frac{1}{1+\delta_{i_1,i_2}} \int_0^1 d\tau \int_\tau^1 \frac{dx}{x} f_{i_1}(x,\sqrt{\hat{s}}) f_{i_2}(\frac{\tau}{x},\sqrt{\hat{s}}) \hat{\sigma}(i_1 i_2 \to f_1 f_2) + (1 \leftrightarrow 2) \tag{4.60}$$

where $\hat{\sigma}$ is the partonic cross-section computed analytically for different quark states, and $f_{q_j}(x,\sqrt{\hat{s}})$ is the PDF corresponding to the quark q_j carrying x fraction of the proton momentum at the factorization scale $\sqrt{\hat{s}}$.
The variable F_χ can be also written in terms of the total dijets cross section (at hadronic-level)

$$F_\chi = \frac{\sigma(pp \to jj)^{\chi<3.32}}{\sigma(pp \to jj)^{\chi<30}}\Big|_{m_{jj}^{min}<m_{jj}<m_{jj}^{max}} \tag{4.61}$$

and then it can be computed analytically using the effective lagrangian

$$\mathcal{L} = \mathcal{L}_{SM} + \sum_{i,j} \frac{c^{(6)}_{q_i,q_j}}{\Lambda^2}\mathcal{O}^{(6)}_{q_i,q_j} + \sum_{i,j} \frac{w^{(6)}_{q_i,q_j}}{\Lambda^2}\mathcal{Q}^{(6)}_{q_i,q_j} + \sum_i \frac{g_i^{(8)}}{\Lambda^4}\mathcal{O}_i^{(8)}. \tag{4.62}$$

where $c^{(6)}_{q_i,q_j}$ and $w^{(6)}_{q_i,q_j}$ are respectively the Wilson coefficients of the dim-6 operators $\mathcal{O}^{(6)}_{ij}$ and $\mathcal{Q}^{(6)}_{ij}$ in Eq. (4.43), (4.44), (4.45), (4.46), and (4.47). We can calculate partonic and hadronic level cross sections with (4.62) and the QCD Lagrangian. The relevent BSM contributions in the massless quarks limit are reported in the Appendix C. Within this approach, the partonic cross-section of a specific process can be written schematically as (see Appendix C)

$$\hat{\sigma} = \frac{\hat{s}^3}{\Lambda^8}\hat{\sigma}_{3,0} + \frac{\hat{s}^2}{\Lambda^6}\hat{\sigma}_{2,0} + \frac{\hat{s}}{\Lambda^4}\hat{\sigma}_{1,0} + \frac{\hat{s}\alpha_s(\hat{s})}{\Lambda^4}\hat{\sigma}_{1,1} + \frac{\alpha_s(\hat{s})}{\Lambda^2}\hat{\sigma}_{0,1} + \frac{\alpha_s^2(\hat{s})}{\hat{s}}\hat{\sigma}_{-1,2} \tag{4.63}$$

where the $\hat{\sigma}_{i,j}$ is the coefficients of the term that scales as $\hat{s}^i\alpha_s^j$ which are obtained by integrating over χ

$$\int d\chi \frac{d\hat{\sigma}}{d\chi}\,. \tag{4.64}$$

Notice that $\hat{\sigma}_{-1,2}$ comes purely from QCD contributions and represent the SM contribution in isolation.

As we stated at the beginning of this section, we can neglect different initial states from uu, dd, ud, gg. Within this approximation, and taking the expasion (4.63) into account, the dijets cross-section can be written schematically as

$$\sigma(pp \to jj) \simeq \sum_{j,k} \frac{1}{\Lambda^{2(j+1)}}\vec{P}_{j,k}\cdot\vec{\hat{\sigma}}_{j,k} + \sigma_{gg\to gg}(pp \to jj) \tag{4.65}$$

where

$$\vec{P}_{m,n} = \left(P^{uu}_{m,n}, P^{dd}_{m,n}, P^{ud}_{m,n}\right) \tag{4.66}$$

$$P^{q_j q_k}_{m,n} = \frac{1}{1+\delta_{j,k}}\int_0^1 d\tau \int_\tau^1 dx f_{q_j}(x,\sqrt{\hat{s}}) f_{q_k}(\frac{\tau}{x},\sqrt{\hat{s}})\frac{\hat{s}^m\alpha_s(\hat{s})^n}{x} + (j \leftrightarrow k) \tag{4.67}$$

$$\vec{\hat{\sigma}}_{j,k} = \left(\hat{\sigma}_{j,k}(uu \to uu), \hat{\sigma}_{j,k}(dd \to dd), \hat{\sigma}_{j,k}(ud \to ud)\right). \tag{4.68}$$

We have factored out the gluon initiated contribution because it does not interfere at leading order with the 4-fermion operators. Using the expression (4.65) and implementing the lagrangian (4.62) in MadGraph5, we can compute the coefficients $P^{q_j,q_k}_{m,n}$ by fitting the formula for the hadron-level process

$$\sigma(q_j q_k \to q_j q_k) = \frac{P^{q_j,q_k}_{m,n}}{\Lambda^{2(1+m)}}\hat{\sigma}_{m,n}(q_j q_k \to q_j q_k). \tag{4.69}$$

The values of $P^{q_j,q_k}_{m,n}$ and $\hat{\sigma}_{m,n}$ depend on the dijets momentum cuts which translate on the invariant mass cut $m_{jj}^{min} < m_{jj} < m_{jj}^{max}$. We will consider the energy window 3.4 TeV $< m_{jj} <$ 4 TeV and we will take the data from [24]. All the simulations are performed fixing both the factorization and the renormalization scale to 3.4 TeV and imposing the cuts as described in [24], i.e. we require the p_T[9] of the leading and subleading jets greater than 440 GeV and 50 Gev respectively and $|y^*| < 1.7$. The radius parameter is 0.4 and the PDF's selected are the NNPDF2.3 grid.
The parameters that we obtain in this way are reported in Table 4.2. These values have

[9] The transverse momentum is the component perpendicular to the beam direction.

	$P_{3,0}(pb^{-3})$	$P_{2,0}(pb^{-2})$	$P_{1,0}(pb^{-1})$	$P_{1,1}(pb^{-1})$	$P_{0,1}$
uu	–	–	7.4E-4	–	1.4E-3
dd	1.4E-7	–	1.2E-4	8.2E-6	2.6E-4
du	–	–	3.0E-4	–	5.8E-4

Table 4.2: Partonic coefficients $P_{m,n}$ computed using MadGraph and checked analytically with the NNPDF2.3 set. The values which are not useful for our purposes are labelled with the dash.

been checked with the analytic expression (4.67) using the NNPDF2.3 partonic distribution functions and imposing the cuts on the invariant mass which translates into the integration intervals $x \in [\tau, 1]$ and $\tau \in [m_{jj}^{min2}/s, m_{jj}^{max2}/s]$.
The variable F_χ can be decomposed as

$$F_\chi = \frac{\sigma_{SM}^{\chi<3.32} + \sigma_{BSM}^{\chi<3.32}}{\sigma_{SM}^{\chi<30} + \sigma_{BSM}^{\chi<30}} \equiv \frac{F_\chi^{SM} + \sigma_{BSM}^{\chi<3.32}/\sigma_{SM}^{\chi<30}}{1 + \sigma_{BSM}^{\chi<30}/\sigma_{SM}^{\chi<30}} \tag{4.70}$$

where $F_\chi^{SM} = \sigma_{SM}^{\chi<3.32}/\sigma_{SM}^{\chi<30} = 0.08$. From the data reported in [24], we extract $F_\chi = 0.084 \pm 0.0039$ and therefore the 2σ confidence level bound

$$0.076 < F_\chi < 0.092. \tag{4.71}$$

4.3.1 Bounds on composite quarks

Bounds on the Wilson coefficients (over a positive power of the BSM scale Λ) follow from (4.71) and the positivity constraints (4.40). Example of excluded regions obtained by turning on just two Wilson coefficients are shown in Fig.5 , Fig.6 as illustration of the bounds. These figures are obtained by drawing the contour plot of the F_χ function for values at 1σ and 2σ away from the central value. In these figures, the regions allowed experimentally are colored in green (1σ) and yellow (2σ). The dashed parts are those that do not respect the positivity conditions (4.40), and are thus theoretically excluded. As we see, the positivity conditions (4.40) strongly improve the impact of the experimental bounds. This effect is dramatic for the plot on the right-hand side in Fig.5 where the positivity conditions resolve a flat direction unbounded by the data at LHC.

Range of validity

Since we are working in an EFT, we should make sure that the bounds are obtained consistently within its range of validity, namely $E < \Lambda$. To illustrate this general point, let us focus for example on g_1/Λ^4 and write the bound as

$$\delta_-^{exp} \leq \frac{g_1}{\Lambda^4} \leq \delta_+^{exp} \tag{4.72}$$

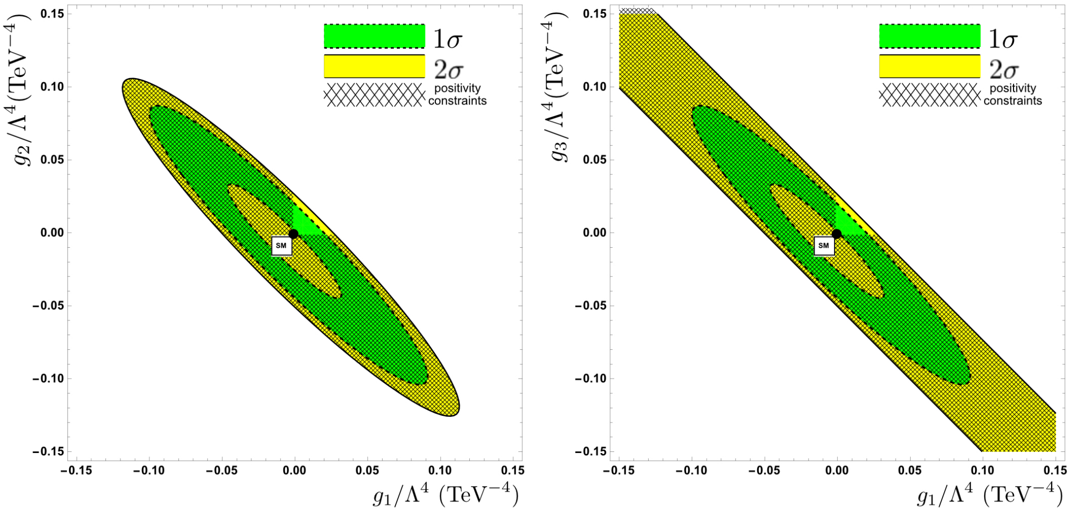

Fig. 5: Allowed regions at 1σ (green) and 2σ (yellow) in the g_1/Λ^4 – g_2/Λ^4 (left) and g_1/Λ^4 – g_3/Λ^4 (right) plane, obtained by setting the other Wilson coefficients to zero. The dashed darker regions correspond to values of Wilson coefficients that do not satisfy the positivity conditions (4.40), and then are excluded theoretically.

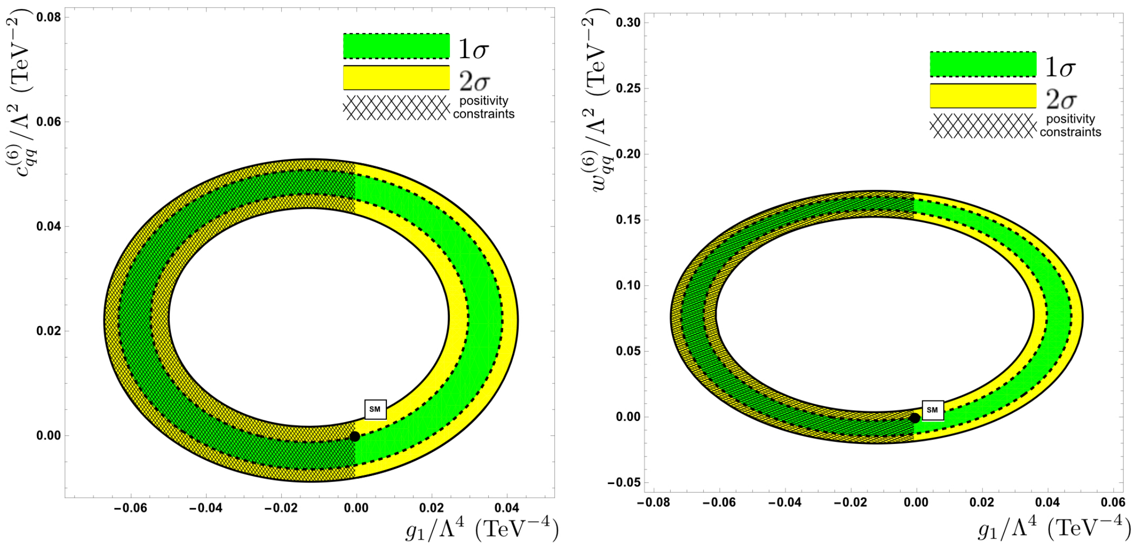

Fig. 6: Allowed regions at 1σ (green) and 2σ (yellow) in the g_1/Λ^4 – $c_{qq}^{(6)}/\Lambda^2$ (left) and g_1/Λ^4 –$w_{qq}^{(6)}/\Lambda^2$ (right) plane, obtained by setting the other couplings to zero. The dashed darker regions correspond to values Wilson coefficients that do not satisfy the positivity conditions (4.40), and are thus excluded. The $w^{(6)}$ and $c_{qq}^{(6)}$ can take either signs because are associated to dim-6 operators which are not subject to the positivity constraints.

where $\delta_{\pm}^{exp}$ are the values at 2σ obtained by the plots taking the positivity constraints into account. In particular, $\delta_{-}^{exp} = 0$. Since $g_1/\Lambda^4 < g_1/m_{jj,max}^4$, our bounds add non-trivial information only for a coupling g_1 such that $\delta_{+}^{exp} < g_1/m_{jj,max}^4$, i.e. only for sufficiently large values of g_1

$$g_1 \geq m_{jj,max}^4 \delta_{+}^{exp}. \tag{4.73}$$

This is why strongly coupled models are better suited for these type of analysis. One can extend the region of validity to smaller values of couplings by working with a sliding energy window for the cuts where $m_{jj,max}$ is taken smaller, as advocated e.g. in [22].

Bounds on One-Coupling-One-Scale models

In a One-Coupling-One-Scale model where we also take fully composite d_R quarks we have $g_i = a_i g_*^2$ with $a_i = O(1)$ and g_* can be taken as large as 4π. We can thus translate the experimental constraints on lower bound on the scale Λ

$$\Lambda \geq \sqrt[4]{\frac{(4\pi)^2}{\delta_{+}^{exp}}} \left(\frac{g_*}{4\pi}\right)^{1/2} \tag{4.74}$$

where δ_{+}^{exp} is taken in the direction $g_1/\Lambda^4 = g_2/\Lambda^4$ and we choosen for definiteness $a_1 = a_2 = 1$ (the scaling of the bound being obvious for other values). In this example the plot on the left in Fig.5 gives $\delta_{+}^{exp} \simeq 0.013\text{TeV}^{-4}$ which corresponds to

$$\boxed{\Lambda \geq 10.5 \left(\frac{g_*}{4\pi}\right)^{1/2} \text{TeV}}. \tag{4.75}$$

This bound can be consistently applied only for $g_* \geq 1.83$, such that our EFT expansion didn't break down, see 4.73. We group in Table 4.3 the lower bounds on the composite scale Λ depending on the coupling g_*.

g_*	Λ_{min} (TeV)
2	4.2
3	5.1
5	6.6
10	9.4
4π	10.5

Table 4.3: Compositeness scale Λ depending on the value of the coupling g_*. The value $g_* = 4\pi$ corresponds to a maximally strongly coupled theory.

Bounds on the SUSY breaking scale

Let's assume now that the dim-8 operators actually arise from a spontaneously broken extended SUSY of the strong sector, the d_R being identified with the Goldstini that interact

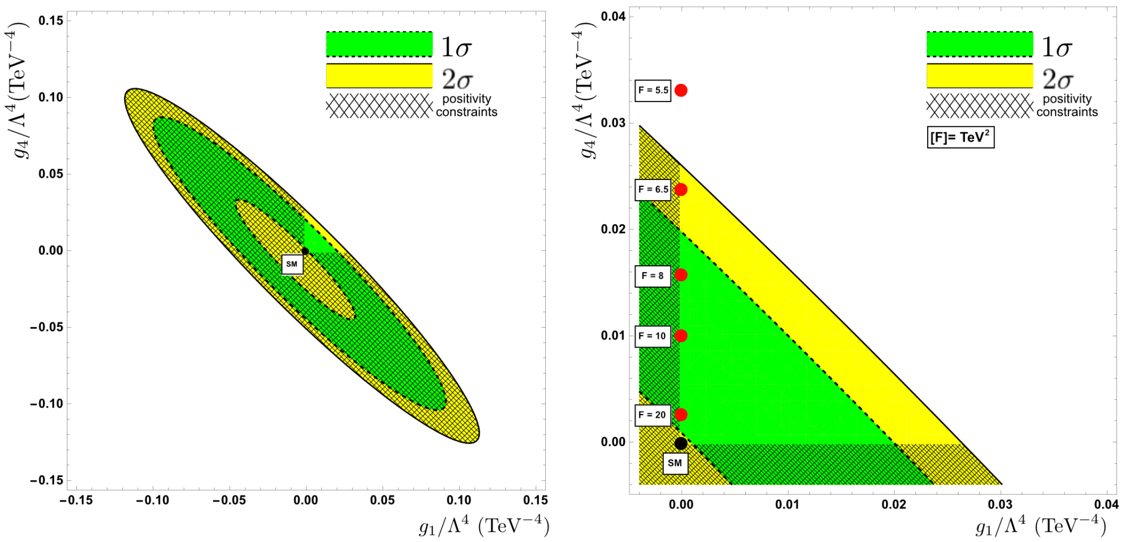

Fig. 7: Left: Allowed regions in the g_1/Λ^4 – g_4/Λ^4 plane at 1σ (green) and 2σ (yellow). The shaded region is excluded by our positivity bounds. Right: zoom of the same regions where the predictions from spontaneously broken SUSY are the red dots, for different values of the decay constant F.

with a strength set by the decay constant F (see the Appendix D). We recall that $[F] = 2$, since F^2 is proportional to the SUSY breaking contribution to the vacuum energy.
We take in particular $\mathcal{N} = 9$ and embed the d_R quarks (with color and flavor indexes) in the same multiplet of the R-symmetry

$$U(9)_R \supset SU(3) \times SU(3) \times U(1)\,, \tag{4.76}$$

identifying the various subgroup factors with color, flavor and hypercharge. The defining representation of $SU(9)$ has one index i that can take $3 \cdot 3$ values; it can be replaced by a pair of indexes $i = (\alpha, a)$ where $\alpha, a = 1, 2, 3$, realizing the embedding

$$\mathbf{9}_y = (\mathbf{3}, \mathbf{3})_y \tag{4.77}$$

as one can promptly check, see e.g. [30]. Within this embedding, the dimension-8 Goldstini interactions (D.66)

$$\frac{1}{F^2}\chi^{\dagger\, i}\partial\chi^{\dagger\, j}\partial\chi_i\chi_j \tag{4.78}$$

can be identified with $\mathcal{O}_4^{(8)}$ in (4.51). In fact, i and j are pairs of indexes, i.e. $i = (\alpha, a)$ and $j = (\beta, b)$ where Greek and latin letters stand for color and flavor indexes respectively. In practice, only one dim-8 operator is allowed by the non-linearly realized SUSY. Moreover, its coefficient is precisely the (inverse squared) decay constant F that we want to bound

$$\frac{g_4}{\Lambda^4} = \frac{1}{F^2}, \qquad g_1 = g_2 = g_3 = 0\,, \tag{4.79}$$

The experimental bounds on g_4/Λ^4 can be interpreted as lower bound on the SUSY decay constant F, see Fig.7 . The experimental bounds imply

$$\frac{g_4}{\Lambda^4} = \frac{1}{F^2} \leq \delta_+^{exp} \Rightarrow F \geq \sqrt{\frac{1}{\delta_+^{exp}}}. \tag{4.80}$$

From the plot, $\delta_+^{exp} = 0.026\text{TeV}^{-4}$ and then

$$\boxed{F \geq 6.2\text{TeV}^2} \tag{4.81}$$

that is $\sqrt{F} \gtrsim 2.5\text{TeV}$.

Conclusions

In this thesis we discussed the fundamental consistency conditions that Wilson coefficients must respect in order for an EFT to admit a UV completion that is Lorentz invariant, unitarity and crossing symmetric. These consistency conditions take the form of sum rules and positivity constraints for the low-energy scattering amplitudes that are expressed in terms of the Wilson coefficients. We extended significantly the results from earlier literature by treating in full generality particles carrying arbitrary spin and arbitrary representations of internal symmetry groups. We applied these positivity constraints to certain EFTs for physics beyond of the SM, and probe them with data from the current run of the LHC.

Several of the results presented in this work are based on crossing symmetry of scattering amplitudes. In chapter 2 we encapsulated these properties in a general crossing matrix for particles with arbitrary spin, and that carry a (generically complex) representation $\mathbf{r}$ of an internal symmetry group. We found that the crossing matrix $\mathbb{X}$ is an involutory matrix which is also unitary with respect to the diagonal and positive definite metric made of the dimensions of all the irreps exchanged in the elastic scattering where the initial state transforms as $\mathbf{r} \otimes \mathbf{r}$ (s-channel) or $\bar{\mathbf{r}} \otimes \mathbf{r}$ (u-channel). The crossing matrix takes an anti-diagonal form and it is built out of two sub-matrices $\mathbb{X}_{1,2}$ which relate the s- and u-channel to each other. For real representations these two matrices become equal and one recovers the results of [2] that we have generalized to the case of arbitrary complex irreps. In Chapter 2, we have also obtained rigorous positivity constraints that follow from the analyticity, crossing symmetry and unitarity of the S-matrix. These principles allow us to obtain dispersion relations for the scattering amplitudes: projecting those dispersion relations on the positive eigenspace of the general crossing matrix we obtain positivity bounds that must be respected by the scattering amplitudes evaluated in the IR. In chapter 3 we extended these results to particles with arbitrary spin, and discussed the physically relevant example of positivity constraints for particles carrying fundamental and anti-fundamental representations of an internal $SU(N)$ symmetry group. These low-energy eigen-amplitudes must satisfy the following positivity constraints

$$\mathcal{M}''_{\mathbf{A}}(2m^2) + \mathcal{M}''_{\mathbf{S}}(2m^2) > 0\,, \quad \mathcal{M}''_{\mathbf{Adj}}(2m^2) > 0\,, \quad \mathcal{M}''_{\mathbf{1}}(2m^2) + (N-1)\mathbf{M}''_{\mathbf{Adj}}(2m^2) > 0\,.$$

where m is the mass of the particles and $\mathbf{1}$, $\mathbf{A}$, $\mathbf{S}$ and $\mathbf{Adj}$ are the singlet, symmetric, anti-symmetric and adjoint irreps either found in the decomposition of $\mathbf{N}\otimes\mathbf{N} = \mathbf{A}\oplus\mathbf{S}$ or $\overline{\mathbf{N}}\otimes\mathbf{N} =$

$\mathbf{1} \oplus \mathbf{Adj}$ of $SU(N)$. In turn, these conditions translate on certain positivity conditions of the Wilson coefficients of the EFT that are used to calculate such an amplitudes.
In chapter 4 we discussed an application of the positivity bounds to the scenario of fermion compositeness. We showed that the positivity constraints have a tremendous impact on the experimental bounds that one can put on dimension-8 four-fermion operators generated by a strongly coupled sector that is responsible for the electroweak symmetry breaking. Indeed, we first obtained the experimental bounds on dimension-8 four-quark interactions of the schematic type $\mathcal{O}^{(8)} = \psi^\dagger \partial \psi^\dagger \psi \partial \psi$ by studying the distributions of dijets at the LHC. Then we imposed our theoretical positivity constraints that removed most of parameter space that was otherwise experimentally viable. As illustration of these bounds, in Fig. 8 we show the allowed parameter space at 1σ and 2σ C.L. for the Wilson coefficients associated to two dim-8 operators involving the down-type quarks. The dashed darker region is theoretically excluded by our positivity bounds based on crossing symmetry and unitarity, while the lighter green or yellow regions are experimentally and theoretically allowed.

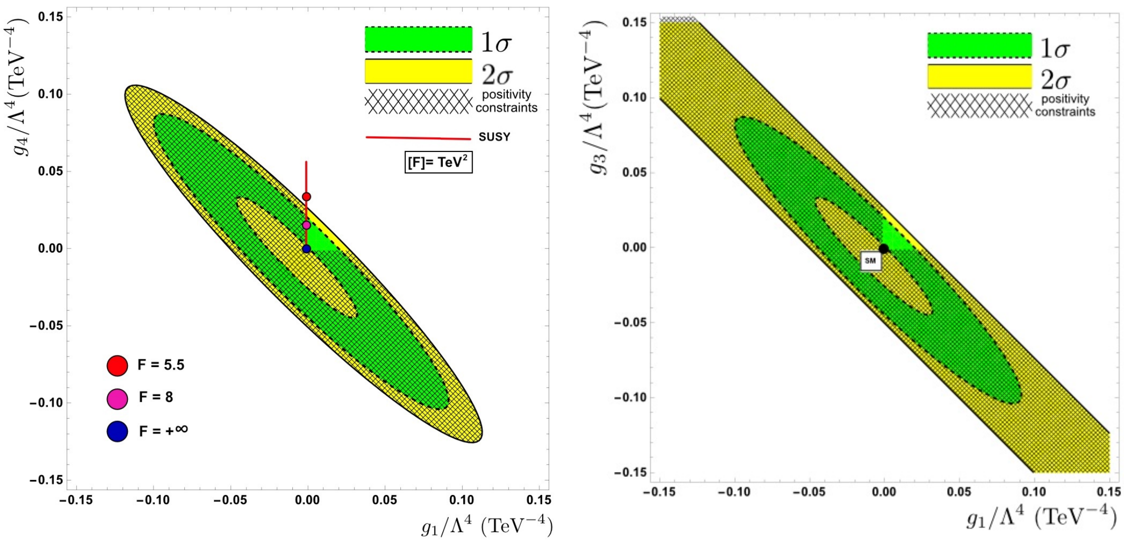

Fig. 8: Allowed region of parameter space in the $g_1 - g_4$ and $g_3 - g_4$ planes, where g_i are the Wilson coefficients of four-d_R dim-8 operators $\mathcal{O}_1^{(8)} = \partial \bar{d}_{Ra}^{\,\alpha} \bar{d}_{Rb}^{\,\beta} \partial d_{R\alpha}^{a} d_{R\beta}^{b}$, $\mathcal{O}_2^{(8)} = \partial \bar{d}_{Ra}^{\,\alpha} \bar{d}_{Rb}^{\,\beta} \partial d_{R\beta}^{a} d_{R\alpha}^{b}$, $\mathcal{O}_3^{(8)} = \partial \bar{d}_{Ra}^{\,\alpha} \bar{d}_{Rb}^{\,\beta} \partial d_{R\alpha}^{b} d_{R\beta}^{a}$, and $\mathcal{O}_4^{(8)} = \partial \bar{d}_{Ra}^{\,\alpha} \bar{d}_{Rb}^{\,\beta} \partial d_{R\beta}^{b} d_{R\alpha}^{a}$. Darker dashed regions are theoretically excluded by the positivity constraints. The red line in the plot on the left hand side represents the prediction from a model of spontaneously broken extended-SUSY where its $\mathcal{N} = 9$ Goldstini are identified with the three down-type right-handed quarks, and the R-symmetry is the maximal $U_R(9)$ group. The dots on the red line correspond to different choices of the SUSY decay constant F in units of TeV^2.

There could be dynamical reasons or a symmetry that explain why dim-8 four-fermion operators are the leading operators of the EFT in the IR for this process, rather than the familiar dim-6 four-fermion operators with no derivatives. We actually provided an example where it is a spontaneously broken extended $\mathcal{N}$-SUSY of the strong sector that suppresses indeed all dim-6 four-fermion operators for the $\mathcal{N}$ Goldstini that transform non linearly, with a fermionic shift symmetry. Within this setup we identified the down-type right-handed quarks with fully composite $\mathcal{N} = 9$ (pseudo-)Goldstini, and interpreted the experimental constraints on the Wilson coefficients as lower bounds the SUSY decay constant

$$\sqrt{F} > 2.5\text{TeV} \qquad \text{at } 95\% \text{ C.L.}\,,$$

see Fig. 8. It would be very interesting to push this idea of SM fermions as pseudo-Goldstini even further, and try to embed all quarks inside the same R-symmetry multiplet of pseudo-Goldstini, alhtough probably giving up maximal R-symmetry.

In conclusion, not all EFTs are born equal: some live in the "swampland" i.e. in the space of EFTs that do not admit sensible UV completions. [10] In this work we proved rigorously certain necessary conditions that an EFT must satisfy not to live in such a swampland. They take the form of positivity constraints for the scattering amplitudes in the IR, and hence for the Wilson coefficients of the EFTs. Should these positivity conditions be violated, the EFT at hand would thus live in such a swampland as its UV completion does not have an unitary, crossing symmetric, and analytic S-matrix. Finally, we showed with a concrete example that these positivity constraints are relevant also phenomenologically for the searches of physics beyond the SM that are done at the LHC.

[10] The concept of swampland in the space of EFTs was originally introduced in the context of string theory and its landscape by Cumrun Vafa [32], with the weak gravity conjecture [33] being perhaps the most concrete example of condition that an EFT with gravity and electric type of forces should satisfy.

Appendix A

Analyticity and Causality

In this appendix we briefly review the relation between causality and the analytic structure of the forward elastic amplitude $\mathcal{M}_{\pi\phi\to\pi\phi}(s,t=0)$ that was initiated in the classic works [42] and [43]. For the sake of simplicity, we restrict to the case where particles 2 and 4 are actually massless, $m_2 = m_4 = 0$ as in ref. [11]. The general case can be found in several textbooks, e.g. [44, 45, 46], as well an in more specialized monographs [47, 48].
Let us use the LSZ reduction formula (1.32) only for particles 2 and 4 which move in the background of 1 and 3,

$$\langle p_3, p_4|S-\mathbb{1}|p_1,p_2\rangle = \left[i\int d^4x e^{-ip_2\cdot x}\left(\Box_x\right)\right]\left[i\int d^4y e^{+ip_4\cdot y}\left(\Box_y\right)\right]\langle p_3|T\phi(y)\phi(x)|p_1\rangle \quad \text{(A.1)}$$

Since we are interested in the forward elastic scattering we go in kinematics $p_1\to p_3$ (and hence $p_2\to p_4$ is enforced by momentum conservation). Using the following identity

$$T\phi(y)\phi(x) = \theta(y^0-x^0)[\phi(y),\phi(x)] + \phi(x)\phi(y) \quad \text{(A.2)}$$

we can rewrite the amplitude as

$$\begin{aligned}\langle p_1, p_4|S-\mathbb{1}|p_1,p_2\rangle = &\left[i\int d^4x e^{-ip_2\cdot x}\left(\Box_x\right)\right]\left[i\int d^4y e^{+ip_4\cdot y}\left(\Box_y\right)\right] \\ &\left[\theta(y^0-x^0)\langle 1|[\phi(y),\phi(x)]|1\rangle + \langle p_1|\phi(x)\phi(y)|p_1\rangle\right] .\end{aligned} \quad \text{(A.3)}$$

The last term without the retarded commutator does not actually contribute to the amplitude for physical values of the momenta. Indeed, inserting a complete set of states in in (A.3) and using the invariance under spacetime translations i.e. $\phi(x) = e^{iPx}\phi(0)e^{-iPx}$, we get

$$\begin{aligned}&\left[i\int d^4x e^{-ip_2\cdot x}\left(\Box_x\right)\right]\left[i\int d^4y e^{+ip_4\cdot y}\left(\Box_y\right)\right]\langle p_1|\phi(x)\phi(y)]|p_1\rangle \\ &= -(2\pi)^4\delta^4(p_2-p_4)\int\frac{d\beta}{N_\beta}(2\pi)^4\left(-p_2^2\right)^2|\langle p_1|\phi(0)|\beta\rangle|^2\Big|_{p_\beta=p_1-p_2}\end{aligned} \quad \text{(A.4)}$$

This expression is almost a total cross-section except that would require the non-physical negative energy p_2^0 to appear in the initial state[1]. Actually, by crossing symmetry it is proportional to the decay width of particle 1 $\to$ anything, which would be a violation of our initial assumption that it was a stable asymptotic one-particle state. Therefore, for this physical kinematics such a matrix elements must vanish. In turn, the scattering amplitude for elastic forward scattering evaluated for physical momenta can be written in terms of an integral over the forward lightcone alone

$$\langle p_1, p_4|S - \mathbb{1}|p_1, p_2\rangle = \left[i\int d^4x e^{-ip_2\cdot x}\left(\Box_x\right)\right]\left[i\int d^4y e^{+ip_4\cdot y}\left(\Box_y\right)\right] \theta(y^0 - x^0)\langle p_1|[\phi(y), \phi(x)]|p_1\rangle\,. \tag{A.5}$$

Using again invariance under translations and recalling the definition of the scattering amplitude $\mathcal{M}$ in (1.13) where a $i(2\pi)^4\delta^4(p_2 - p_4)$ factor is removed, we get the actual expression that we wish to analytically continue to complex momenta

$$\mathcal{M}(s, t = 0) = i\int d^4y e^{+ip_2\cdot y}\Box_y^2\left\{\theta(y^0)\langle p_1|[\phi(y), \phi(0)]|p_1\rangle\right\}\,. \tag{A.6}$$

By Lorentz invariance, it is a function of the Mandelstam variable s. As long as we are concerned about the analytic properties of $\mathcal{M}$, we can safely move the $\theta(y^0)$ to the left of the $\Box_y^2$-operator, the mismatch between the two expressions being only a polynomial in p_2, hence analytic, because of the microcausality condition for $[\phi(y), \phi(0)]$: its time derivatives vanish at equal times, $y_0 = 0$, except at coincidence points $\mathbf{y} = 0$ where a delta function may occur. The Fourier transform of a differential operator acting on such a delta functions returns the claimed analytic polynomial.

Microcausality and the presence of the step function imply that the integrand vanishes outside the forward lightcone $\{y^2 > 0\,, y^0 > 0\}$. This allows us to analytically continue $\mathcal{M}$ in the upper complex s-plane, assuming polynomially bounded correlation functions. One simple way to see this is by working in the rest frame of particle 1, $p_1 = (m_1, \mathbf{0})^T$. In this frame s is just a linear combination of the energy of particle 2

$$s = 2m_1p_2^0 + m_1^2\,, \tag{A.7}$$

so that analyticity with respect to p_2^0 trivially implies the analytic structure with respect to s. In this frame the correlation function inside (A.6) is actually a function of $|\mathbf{y}|^2$ and y^0 only, given that any rotation leaves p_1 invariant. Therefore, the integral over the angular variables $d^3\Omega_{\mathbf{y}}$ in spherical coordinates can be carried out explicitly[2]

$$\mathcal{M}(s, t = 0) \sim 2\pi^2\int_0^\infty d|\mathbf{y}|\mathbf{y}^2\int_0^{+\infty} dy^0 e^{ip_2^0y^0}\left(\frac{\sin|\mathbf{y}||\mathbf{p}_2|}{|\mathbf{p}_2|}\right)\langle p_1|\Box_y^2[\phi(y), \phi(0)]|p_1\rangle \tag{A.8}$$

[1] Moreover, for identical particles in the center of mass frame, it would require $p_\beta = 0$ which corresponds only to the vacuum state. The stability of the vacuum forbids such a process.

[2] The $\sim$ symbol means that we are omitting the analytic polynomial contribution from the Fourier transform of the delta functions that arise from the time derivatives in $\Box$ that hit the step function.

where $|\mathbf{p}_2| = p_2^0$ because $m_2 = 0$, and we restricted the integration to $y^0 > 0$ thanks to the step function of the retarded commutator. The analyticity in the upper complex p_2^0-plane (hence upper s-plane) of $\mathcal{M}$ is now established because of the exponential damping that $e^{ip_2^0 y^0}$ provides for $\mathrm{Im}\, p_2^0 > 0$, given that $y_0 > 0$ too in the integration region. Even for $p_2^0 \to \infty$ in the complex upper plane the integral over $|\mathbf{y}|$ can be carried out. The would-be dangerous terms from the sin function in (A.8) have the following behavior $\mathrm{Exp}\, i\left(p_2^0 y_0 \pm |\mathbf{p}_2||\mathbf{y}|\right) = \mathrm{Exp}\, ip_2^0\,(y_0 \pm |\mathbf{y}|)$ which is integrable for $\mathrm{Im}\, p_2^0 > 0$ and $y_0 \geq |\mathbf{y}|$, i.e. inside the forward lightcone selected by the retarded commutator.
In practice, the integral representation (A.8) provides an analytic extension of the amplitude for $\mathrm{Im}\, s > 0$, while the physical amplitude is recovered as the boundary value on the real axis approached from above

$$\mathcal{M}_{\pi\phi\to\pi\phi}(s, t = 0) = \mathcal{M}_{\pi\phi\to\pi\phi}(s + i\epsilon, t = 0)\,, \qquad s \geq s_{min} = (m_1 + m_2)^2\,, \tag{A.9}$$

and $\epsilon \to 0^+$ limit is always understood.
Assuming there exists an open interval on the real axis where the amplitude is real, meaning $\mathcal{M}(s, t = 0) = M^*(s^*, t = 0)$ on such an interval, one can actually extend it as a real function to the lower complex s-plane too. In fact, using the Schwarz reflection principle the amplitude is analytically extended everywhere in the complex s-plane:

$$\mathcal{M}(s^*, t = 0) = \mathcal{M}^*(s, t = 0)\,, \tag{A.10}$$

except for some discontinuities on the real axis that come from stable particles and branch-cuts of multiparticle states. Crossing symmetry relates the discontinuities between the s- and u-channel that are boundary value of the analytic function approached either from above or below, see Eq. (1.41) and (1.42). Remarkably, crossing symmetry, unitarity and the Schwarz reflection principle are all consistent with each other as one can check re-running for $\mathcal{M}^\dagger$ the analysis that have presented above for $\mathcal{M}$. In particular, the time-ordered product can be replaced by the advanced commutator

$$T\phi(y)\phi(x) = -\,\theta(x^0 - y^0)[\phi(y), \phi(x)] + \phi(y)\phi(x) \tag{A.11}$$

which effectively allows one to extend $\mathcal{M}^*_{\alpha\alpha}$ in the lower complex plane, consistently with the extension presented above in terms of the Schwarz reflection principle because of unitarity (1.18).

Appendix B

Dim-8 four-fermion operators

In this appendix, we want to show a sketch of the derivation of the most general dimension-8 operators involving a right-handed Weyl fermion field as those listed in Table 4.1.

B.1 Spinor notation

Since we will work with an big number of operators, it is convenient to define a notation to make easier the lecture of this appendix. A generic fermion bilinear takes the form

$$\bar{\psi}\Gamma^A\psi \tag{B.1}$$

where Γ^A is an arbitrary combination of gamma matrices which do not annihilate the operator. The choice of the gamma matrices basis will be performed in the next section. In the following, we use the notation

$$\bar{\psi}\Gamma^A\psi = \left(\Gamma^A\right) \tag{B.2}$$

where the brackets stands for the fermions field attached to Γ^A. In presence of more bilinears involving different species of fermion field, we use a different kind of brackets. For instance,

$$\bar{\psi}\Gamma^A\psi\bar{\chi}\Gamma_B\chi = \left(\Gamma^A\right)\left[\Gamma_B\right]. \tag{B.3}$$

If some derivative acts on the fields, we write its Lorentz index on the top of the bracket. For example,

$$\partial_\mu\bar{\psi}\Gamma^A\partial_\nu\psi\partial_\rho\bar{\chi}\Gamma_B\partial_\sigma\chi = \overset{\mu}{(}\ \Gamma^A\ \overset{\nu}{)}\ \overset{\rho}{[}\ \Gamma_B\ \overset{\sigma}{]}\,. \tag{B.4}$$

B.2 Chiral basis and Fierz identities

Since we assumed to work with massless fermions, the most convenient choice of the gamma matrices basis is the chiral one which is defined as

$$\{\Gamma^A\} = \{R, L, \gamma^\mu L, \gamma^\mu R, \Sigma^{\mu\nu}\}, \qquad \{\Gamma_A\} = \{R, L, \gamma_\mu R, \gamma_\mu L, \frac{1}{2}\Sigma_{\mu\nu}\}, \tag{B.5}$$

where

$$R = \frac{1+\gamma^5}{2}, \; L = \frac{1-\gamma^5}{2}, \; \Sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right], \tag{B.6}$$

and we use the Weyl representation of the gamma matrices, i.e.

$$\gamma^\mu = \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \tag{B.7}$$

where $\sigma^\mu = (1, \sigma^i)$ and $\bar{\sigma}^\mu = (1, -\sigma^i)$ with σ^i the Pauli matrices.
The orthogonality property of the chiral basis is

$$\mathrm{Tr}\left[\Gamma_A \Gamma^B\right] = 2\delta_A^B \tag{B.8}$$

and the completeness relation, written in terms of the spinor notation defined above, is [16]

$$(\;)[\;] = -\frac{1}{2}\left(\Gamma_A\right]\left[\Gamma^A\right) \tag{B.9}$$

where the sum over A is understood and the minus is due to the anticommuting nature of the fermionic fields. From this relation we obtain the chiral Fierz identites

$$\left(\Gamma^A\right)\left[\Gamma^B\right] = -\frac{1}{4}\mathrm{Tr}\left[\Gamma^A \Gamma_C \Gamma^B \Gamma_D\right]\left(\Gamma^D\right]\left[\Gamma^C\right). \tag{B.10}$$

Some useful relations follow from (B.10)

$$(\gamma^\mu R)[\gamma_\mu R] = (\gamma^\mu R][\gamma_\mu R) \tag{B.11}$$

$$(\gamma^\mu R)[\gamma_\nu R] = -\frac{1}{2}\left\{(\gamma^\mu R][\gamma_\nu R) + (\gamma^\nu R][\gamma_\mu R) - \eta^{\mu\nu}(\gamma^\lambda R][\gamma_\lambda R) + i\epsilon^{\mu\nu\rho\sigma}(\gamma_\rho R][\gamma_\sigma R)\right\}. \tag{B.12}$$

The following relations will be useful

$$\sigma^{\mu\nu} = \frac{i}{4}(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu) \tag{B.13}$$

$$\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = 2g^{\mu\nu} \tag{B.14}$$

$$\chi^\dagger{}_{\dot{\alpha}} \chi^\dagger{}_{\dot{\beta}} = -\frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\left(\chi^\dagger\right)^2 \tag{B.15}$$

$$\partial_\nu \chi^\dagger{}_{\dot{\beta}} \left(\bar{\sigma}^\mu\right)^{\dot{\beta}\beta} \left(\bar{\sigma}^\nu\right)^{\dot{\alpha}\alpha} \partial_\mu \chi_\alpha = -2\partial_\mu \chi^{\dagger\dot{\alpha}} \partial^\mu \chi^\beta \tag{B.16}$$

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}(\bar{\sigma}_\mu)_{\dot{\beta}\beta} = 2\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}} \tag{B.17}$$

B.3 Lorentz structure for dimension-8 operators

Let us consider a massless right-handend Weyl spinor χ. The dimension-8 operators we can construct with *only* this fermion species involve two derivatives ∂'s and four fields. We can group these operators in three classes.

- Class I) Operators involving the Levi-Civita tensor

$$C_1 = \epsilon^{\mu\nu\rho\sigma}(\bar{\sigma}^{\mu}\ {}^{\rho}\ )\ {}^{\sigma}[\ \bar{\sigma}^{\nu}] \qquad C_2 = \epsilon^{\mu\nu\rho\sigma}(\bar{\sigma}^{\mu}\ {}^{\rho}\ )\ [\bar{\sigma}^{\nu}\ {}^{\sigma}\ ] \tag{B.18}$$

$$C_3 = \epsilon^{\mu\nu\rho\sigma}(\bar{\sigma}^{\mu})\ {}^{\rho}[\ \bar{\sigma}^{\nu}\ {}^{\sigma}\ ]\ . \tag{B.19}$$

Other arrangements of the derivatives can be neglected up to integration by parts.

- Class II) Operators involving derivatives with same Lorentz indices (*i.e.* contracted derivatives)

$$A_1 = \partial\chi^{\dagger}\bar{\sigma}^{\mu}\partial\chi\chi^{\dagger}\bar{\sigma}_{\mu}\chi \qquad A_2 = \partial\chi^{\dagger}\bar{\sigma}^{\mu}\chi\partial\chi^{\dagger}\bar{\sigma}_{\mu}\chi \tag{B.20}$$

$$A_3 = \partial\chi^{\dagger}\bar{\sigma}^{\mu}\chi\chi^{\dagger}\bar{\sigma}_{\mu}\partial\chi \qquad A_4 = \chi^{\dagger}\bar{\sigma}^{\mu}\partial\chi\partial\chi^{\dagger}\bar{\sigma}_{\mu}\chi \tag{B.21}$$

$$A_5 = \chi^{\dagger}\bar{\sigma}^{\mu}\partial\chi\chi^{\dagger}\bar{\sigma}_{\mu}\partial\chi \qquad A_6 = \chi^{\dagger}\bar{\sigma}^{\mu}\chi\partial\chi^{\dagger}\bar{\sigma}_{\mu}\partial\chi \tag{B.22}$$

Using the identity (B.17), we can simplify this list

$$A_1 = A_3 = A_6 = A_4 = 2(\partial\bar{\chi})\chi^{\dagger}(\partial\chi)\chi \tag{B.23}$$

$$A_2 = A_5^{\dagger} \tag{B.24}$$

$$A_5 = (\chi^{\dagger})^2\Box(\chi^2). \tag{B.25}$$

Integrating by parts, we end up with only one operator. We choose A_5.

- Type III) Derivatives with different Lorentz indices (*i.e.* derivatives contracted with Γ^A and Γ^D). The possible operators for a massless fermion field are

$$\mathcal{O}_1 = (\bar{\sigma}^{\nu}\ {}^{\mu}\ )\ {}^{\nu}[\ \bar{\sigma}^{\mu}] \qquad \mathcal{O}_2 = (\bar{\sigma}^{\nu}\ {}^{\mu}\ )\ [\bar{\sigma}^{\mu}\ {}^{\nu}\ ] \tag{B.26}$$

$$\mathcal{O}_3 = ({}^{\mu}\ \bar{\sigma}^{\nu})\ {}^{\nu}[\ \bar{\sigma}^{\mu}] \qquad \mathcal{O}_4 = ({}^{\mu}\ \bar{\sigma}^{\nu})[\bar{\sigma}^{\mu}\ {}^{\nu}\ ]\ . \tag{B.27}$$

Integrating by parts (IBP) and using Fierz identities, we can reduce this sets of operators. The calculus are tedious and not so illuminating. What we learn is that the operators of the first class can be expressed in terms of the third class ones, and so we can neglect C_1, C_2, C_3. Integrating by parts, we can neglect also $\mathcal{O}_3$,$\mathcal{O}_4$ and $\mathcal{O}_2$

$$\mathcal{O}_3 \stackrel{IBP}{=} -\mathcal{O}_1 \tag{B.28}$$

$$\mathcal{O}_4 \stackrel{IBP}{=} -\mathcal{O}_2 \tag{B.29}$$

$$\mathcal{O}_2 \stackrel{IBP}{=} -\mathcal{O}_1. \tag{B.30}$$

Then, we end up with $\mathcal{O}_1$. Using the identity (B.16), we can write $\mathcal{O}_1 = -2(\partial_{\nu}\chi^{\dagger})\chi^{\dagger}(\partial^{\nu}\chi)\chi$. Integrating by parts, $A_5 = 2\mathcal{O}_1$ and then only one operators remains, that is $\mathcal{O}_1$.

B.4 SU(N)-invariant operators

Now, we demand that χ transforms under the fundamental representation of $SU(N)$, i.e. $\chi \equiv \chi^a$ where a is a $SU(N)$ fundamental index. We normalize the generators of the $SU(N)$ algebra as

$$Tr[T^A T^B] = \frac{1}{2}\delta^{AB}. \tag{B.31}$$

Expressions involving T^A's can be further simplified using the Fierz Identity for $SU(N)$

$$\sum_A T^A_{ab} T^A_{cd} = \frac{1}{2}\left(\delta_{ad}\delta_{bc} - \frac{1}{N}\delta_{ab}\delta_{cd}\right). \tag{B.32}$$

Now, a little remark is indispensable. Let us consider for example the *not* Lorentz invariant operator of the form

$$\partial_\rho \chi^{\dagger a} \Gamma^A_{ab} \bar{\sigma}_\mu \partial_\sigma \chi^b \chi^{\dagger c} \Gamma^A_{cd} \bar{\sigma}_\nu \chi^d \tag{B.33}$$

where Γ^A_{ij} can be δ_{ij} or a generator of $SU(N)$. Using the Fierz Identity (B.32), we can express it as the linear combination

$$\frac{1}{2}\partial_\rho \chi^{\dagger a} \bar{\sigma}_\mu \partial_\sigma \chi^b \chi^{\dagger b} \bar{\sigma}_\nu \chi^a - \frac{1}{2N}\partial_\rho \chi^{\dagger a} \bar{\sigma}_\mu \partial_\sigma \chi^a \chi^{\dagger b} \bar{\sigma}_\nu \chi^b. \tag{B.34}$$

Then, in order to classify the dimension-8 four-fermion operators, each of them carrying an $SU(N_C)$ index, we can restrict ourselves to those where the color indexes are contracted in the same bilinear or in different bilinears, because otherwise the operator which contains a combination of T^A's can be simplified.
If the fields carry an additional $SU(N)_F$ index (for example a flavor index) the same argument holds. Indeed, let us consider

$$\partial_\rho \chi^{\dagger a}_{\ \alpha} \Gamma^C_{ab} \Gamma^F_{\alpha\beta} \bar{\sigma}_\mu \partial_\sigma \chi^b_\beta \chi^{\dagger c}_{\ \gamma} \Gamma^C_{cd} \Gamma^F_{\gamma\delta} \bar{\sigma}_\nu \chi^d_\delta. \tag{B.35}$$

The only relevant case to study is when the Γ's are the generators of the groups. Using the Fierz Identity (B.32) for both kinds of generators

$$\Gamma^C_{ab}\Gamma^F_{\alpha\beta}\Gamma^C_{cd}\Gamma^F_{\gamma\delta} = \frac{1}{4}\left(\delta_{ad}\delta_{bc} - \frac{1}{N_C}\delta_{ab}\delta_{cd}\right)\left(\delta_{\alpha\delta}\delta_{\beta\gamma} - \frac{1}{N_F}\delta_{\alpha\beta}\delta_{\gamma\delta}\right). \tag{B.36}$$

Then we can write (B.35) as

$$\frac{1}{4}\Big(\partial_\rho \chi^{\dagger a}_{\ \alpha} \bar{\sigma}_\mu \partial_\sigma \chi^b_\beta \chi^{\dagger b}_{\ \beta} \bar{\sigma}_\nu \chi^a_\alpha - \frac{1}{N_F}\partial_\rho \chi^{\dagger a}_{\ \alpha} \bar{\sigma}_\mu \partial_\sigma \chi^b_\alpha \chi^{\dagger b}_{\ \delta} \bar{\sigma}_\nu \chi^a_\delta \tag{B.37}$$

$$-\frac{1}{N_C}\partial_\rho \chi^{\dagger a}_{\ \alpha} \bar{\sigma}_\mu \partial_\sigma \chi^a_\beta \chi^{\dagger b}_{\ \beta} \bar{\sigma}_\nu \chi^b_\alpha - \frac{1}{N_F N_C}\partial_\rho \chi^{\dagger a}_{\ \alpha} \bar{\sigma}_\mu \partial_\sigma \chi^a_\alpha \chi^{\dagger b}_{\ \beta} \bar{\sigma}_\nu \chi^b_\beta\Big). \tag{B.38}$$

Then, also in this case we can restrict ourselves to the cases with all possible $SU(N)\otimes SU(N)$-indices contracted with the deltas.
The procedure to derive the list of independent operators is the same as above. The independent $SU(N)$-invariant operators involving the massless right-handed field χ^a are

$$\mathcal{O}_1 = \partial\chi^\dagger{}_a \chi^\dagger{}_b \partial\chi^a \chi^b \tag{B.39}$$

$$\mathcal{O}_2 = \partial\chi^\dagger{}_a \chi^\dagger{}_b \partial\chi^b \chi^a. \tag{B.40}$$

B.5 Implementing four-fermion operators in MadGraph

In this thesis we used MadGraph5_aMCNLO [18] to compute the partonic coefficients in the table 4.2 and to quantify the effects of higher-dimension operators (dim-6 and dim-8). However, when identical particles are involved in the same vertex, MadGraph5 does not handle directly with four-fermion interactions since there is an ambiguity on deriving the correct fermionic flow. The splitting of the vertex by means of an heavy auxiliary field is necessary and here we give an example of how to generate dim-6 operators.
To reproduce the dim-6 operators in (4.43)÷(4.47), it is enough to use a spin-1 field which couples with the current

$$J^\mu(x) = c_{1q}\bar{q}_L\gamma^\mu q_L + c_{1d}\bar{d}_R\gamma^\mu d_R + c_{1u}\bar{u}_R\gamma^\mu u_R. \tag{B.41}$$

Notice that we want to reproduce also dim-6 operators for d_R quarks.
Integrating an heavy field at tree-level means solving the equations of motion for this field evaluating the lagrangian at zero momentum. Then, we get

$$\mathcal{L} = \frac{1}{2}M^2_{aux}V^\mu V_\mu + V^\mu J_\mu \Longrightarrow V^\mu = -J^\mu/M^2_{aux} \tag{B.42}$$

and the lagrangian becomes

$$\mathcal{L} = -\frac{1}{2M^2_{aux}}J^\mu J_\mu = -\frac{1}{2M^2_{aux}}\big(c^2_{1q}\bar{q}_L\gamma^\mu q_L\bar{q}_L\gamma_\mu q_L + c^2_{1d}\bar{d}_R\gamma^\mu d_R\bar{d}_R\gamma_\mu d_R \tag{B.43}$$

$$+ c^2_{1u}\bar{u}_R\gamma^\mu u_R\bar{u}_R\gamma_\mu u_R + 2c_{1q}c_{1d}\bar{q}_L\gamma_\mu q_L\bar{d}_R\gamma^\mu d_R \tag{B.44}$$

$$+ 2c_{1q}c_{1u}\bar{q}_L\gamma^\mu q_L\bar{u}_R\gamma_\mu u_R + 2c_{1d}c_{1u}\bar{d}_R\gamma^\mu d_R\bar{u}_R\gamma_\mu u_R\big). \tag{B.45}$$

We have generated some of the desired operators and by matching the coefficients we identify

$$\frac{-c^2_{1q,d,u}}{2M^2_{aux}} = \frac{c^{(1)}_{qq,dd,uu}}{\Lambda^2} \tag{B.46}$$

$$\frac{-c_{1(u,q)}c_{1d}}{M^2_{aux}} = \frac{c^{(1)}_{(u,q)d}}{\Lambda^2} \tag{B.47}$$

$$\frac{-c_{1q}c_{1u}}{M^2_{aux}} = \frac{c^{(1)}_{qu}}{\Lambda^2}. \tag{B.48}$$

We can generate the remaining operators (those with the $SU(N)$ generators) by adding a spin-1 field transforming in the adjoint of $SU(N)$. The lagrangian becomes

$$\mathcal{L}_{tot} = \mathcal{L} + \frac{1}{2}M^2_{aux}V^A_\mu V^A_\mu + J^A_\mu V^\mu_A$$

where the currents are given by

$$J^A_\mu(x) = c_{8q}\bar{q}_L\gamma^\mu T^A q_L + c_{8d}\bar{d}_R\gamma^\mu T^A d_R + c_{8u}\bar{u}_R\gamma^\mu T^A u_R.$$

Performing the same integration as before gives the same matching conditions. Note that, using the Fierz identities both for spinors and $SU(N)$ generators, we can write

$$c_{8d}^2 \bar{d}_R \gamma^\mu T^A d_R \bar{d}_R \gamma^\mu T^A d_R = \frac{1}{3} c_{8d}^2 \bar{d}_R \gamma^\mu d_R \bar{d}_R \gamma^\mu d_R \tag{B.49}$$

then, we can redefine

$$c_{1d}^2 + \frac{1}{3} c_{8d}^2 \to c_{1d}^2. \tag{B.50}$$

Appendix C

Partonic BSM cross-sections

We give the list of the at partonic-level cross-sections due to higher-dimensional operators. We recall that we assume the flavor symmetry $G_F = U(3)_{q_L} \times U(3)_{d_R} \times U(3)_{u_R}$ together with color $SU(3)_C$.

$$\frac{\hat{\sigma}(dd \to dd)_{BSM}}{d\hat{t}} = \frac{2\alpha_s(\hat{s})\hat{s}}{9\Lambda^2\hat{t}\hat{u}}A_1^{dd} + \frac{\alpha_s(\hat{s})}{9\Lambda^4}\left(\frac{\hat{t}}{2\hat{u}}B_1^{dd} + \frac{\hat{u}}{2\hat{t}}B_1^{dd} + B_2^{dd}\right) + \frac{1}{3\pi\Lambda^4}C^{dd} \tag{C.1}$$

$$+ \frac{1}{64\pi\Lambda^8}\left[t^2 D_1^{dd} + tuD_2^{dd} + u^2 D_1^{dd}\right] \tag{C.2}$$

$$\frac{\hat{\sigma}(uu \to uu)_{BSM}}{d\hat{t}} = \frac{\alpha_s(\hat{s})}{\Lambda^2}\left[\frac{\hat{s}}{9\hat{t}\hat{u}}A_1^{uu} + \frac{1}{9\hat{s}}A_2^{uu}\left(1 - \frac{\hat{t}}{\hat{u}} - \frac{\hat{u}}{\hat{t}}\right)\right] + \frac{1}{9\pi\Lambda^4}\left[\left(\frac{\hat{t}^2 + \hat{u}^2}{2\hat{s}^2}\right)C_1^{uu} + C_2^{uu}\right] \tag{C.3}$$

$$\frac{\hat{\sigma}(ud \to ud)_{BSM}}{d\hat{t}} = \frac{\alpha_s(\hat{s})}{9\hat{t}\Lambda^2}\left[A_1^{ud} + \frac{\hat{u}^2}{\hat{s}^2}A_2^{ud}\right] + \frac{1}{4\pi\Lambda^4}\left[C_1^{ud} + \frac{\hat{u}^2}{\hat{s}^2}C_2^{ud}\right] \tag{C.4}$$

where

$$A_1^{dd} = c_{qq}^{(6)} + \frac{1}{3} w_{qq}^{(6)} \tag{C.5}$$

$$B_1^{dd} = g_1 + g_3 \tag{C.6}$$

$$B_2^{dd} = g_2 + g_4 \tag{C.7}$$

$$C^{dd} = -{c_{qq}^{(6)}}^2 - \frac{2}{3} c_{qq}^{(6)} w_{qq}^{(6)} - \frac{1}{9} {w_{qq}^{(6)}}^2 \tag{C.8}$$

$$D_1^{dd} = \frac{1}{6} \left(3{g_1}^2 + 2g_1(g_2 + 3g_3 + g_4) + 3{g_2}^2 + 2g_2(g_3 + 3g_4) + 3{g_3}^2 + 2g_3g_4 + 3{g_4}^2\right) \tag{C.9}$$

$$D_2^{dd} = \frac{1}{3} \left({g_1}^2 + 2g_1(3g_2 + g_3 + 3g_4) + {g_2}^2 + 2g_2(3g_3 + g_4) + {g_3}^2 + 6g_3g_4 + {g_4}^2\right) \tag{C.10}$$

$$A_1^{uu} = -2c_{qq}^{(6)} - \frac{2}{3} w_{qq}^{(6)} - 2c_{uu}^{(6)} - \frac{2}{3} w_{uu}^{(6)} \tag{C.11}$$

$$A_2^{uu} = w_{qu}^{(6)} \tag{C.12}$$

$$C_1^{uu} = \frac{9}{2} c_{qq}^{(6)} c_{uu}^{(6)} + w_{qq}^{(6)} w_{uu}^{(6)} \tag{C.13}$$

$$C_2^{uu} = 3{c_{qq}^{(6)}}^2 + 2c_{qq}^{(6)} w_{qq}^{(6)} + \frac{1}{3} {w_{qq}^{(6)}}^2 + 3{c_{uu}^{(6)}}^2 + 2c_{uu}^{(6)} w_{uu}^{(6)} + \frac{1}{3} {w_{uu}^{(6)}}^2 \tag{C.14}$$

$$A_1^{ud} = 2w_{qq}^{(6)} \tag{C.15}$$

$$A_2^{ud} = {2_{qu}^{(6)}}^2 \tag{C.16}$$

$$C_1^{ud} = {c_{qq}^{(6)}}^2 + \frac{2}{9} {w_{qq}^{(6)}}^2 \tag{C.17}$$

$$C_2^{ud} = c_{qq}^{(6)} c_{uu}^{(6)} + \frac{2}{9} w_{qq}^{(6)} w_{uu}^{(6)} \tag{C.18}$$

Appendix D

Extendend $\mathcal{N}$-SUSY and effective action for Goldstini

In this appendix we derive the effective action for $\mathcal{N}$ Goldstini which emerge from a spontaneously broken $\mathcal{N}$-SUSY, extending the original work of Akulov and Volkov [41] for $\mathcal{N} = 1$. We do not investigate the details of the mechanism that breaks the symmetry, focusing ourselves instead on the impact of such symmetry breaking in the IR. We will follow a CCWZ-like construction, see e.g. [36, 37, 38, 39, 40], although we believe to have improved in clarity earlier derivations. The CCWZ formalism is also better suited than alternative approaches, such as the constraint superfield formalism [34],[35] for what concerns the coupling to matter and gauge fields in an extended SUSY.

In addition to the Poincaré generators, the $\mathcal{N}$-SUSY algebra includes $\mathcal{N}$ spinor supercharges. The relevant part of this algebra is

$$[P_\mu, Q^i_\alpha] = [P_\mu, Q^\dagger_{\dot\alpha\, j}] = 0 \tag{D.1}$$

$$\{Q^i_\alpha, Q^j_\beta\} = \{Q^\dagger_{\dot\alpha\, i}, Q^\dagger_{\dot\beta\, j}\} = 0 \tag{D.2}$$

$$\{Q^i_\alpha, Q^\dagger_{\dot\beta\, j}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu \delta^i_j \tag{D.3}$$

where $i, j = 1, ..., \mathcal{N}$. These (anti-)commutation relations are invariant under a $U(\mathcal{N})_R$ symmetry which transforms the Q^i_α among themselves and we assume thus such symmetry. The elements of $U(\mathcal{N})_R = SU(\mathcal{N})_R \times U(1)_R$ act by means of the $\mathcal{N}^2$ generators R^a which transform the supercharges as

$$[R^a, Q^i_\alpha] = (U^a)^i_j\, Q^j_\alpha\,, \qquad [R^a, Q^\dagger_{\dot\alpha\, i}] = -\,(U^{*\,a})^i_j\, Q^\dagger_{\dot\alpha j}\,, \qquad a = 1, ...\mathcal{N}^2\,. \tag{D.4}$$

where $\mathcal{N}^2 - 1$ matrices are the generators of $SU(\mathcal{N})_R$ and the last one is just a phase due to $U(1)_R$.
The symmetry breaking pattern that we assume is

$$\text{SUSY and maximal R-symmetry} \to U(\mathcal{N})_R \times \text{Poincaré}. \tag{D.5}$$

Thus, let us consider the generic group element made of unbroken and broken generators

$$U = e^{i\chi(x)Q+i\chi(x)^\dagger Q^\dagger} e^{ix_\mu P^\mu} \tag{D.6}$$

where the adopted notation is $\chi Q = \chi_i^\alpha Q_\alpha^i$ and $\chi^\dagger Q^\dagger = \chi_{\dot\alpha}^{\dagger\, i} Q_i^{\dagger\,\dot\alpha}$. From (D.3) we see that the dimension of the supercharges is $[Q] = 1/2$ and the fields $\chi(x)$ are not canonically normalized, i.e. $[\chi] = -1/2$.
We want to derive the Goldstino transformation in the particular case of Poincaré (unbroken) and broken transformations. To this purpose, we will use the fact that, in general, the commutators among broken ($\hat{T}^{\hat a}$) and unbroken generators (T^a) can be written schematically as

$$\left[T^a, T^b\right] = i f_c^{ab} T^c \tag{D.7}$$

$$\left[T^a, \hat{T}^{\hat b}\right] = i f_{\hat c}^{a\hat b} T^{\hat c} \tag{D.8}$$

$$\left[\hat{T}^{\hat a}, \hat{T}^{\hat b}\right] = i f_c^{\hat a\hat b} T^c + i f_{\hat c}^{\hat a\hat b} T^{\hat c} \tag{D.9}$$

where f_c^{ab} are the structure constants of the algebra. In this way, we can always write the action of a general transformation g on U as

$$gU = U' \cdot (\text{unbroken group element}). \tag{D.10}$$

From (D.10), we see that we can derive the Goldstino transformation from U'.
Let us begin with an unbroken transformation, for example a Lorentz transformation L. It acts as

$$LU(x,\chi(x)) = Le^{i\chi(x)Q+i\chi(x)^\dagger Q^\dagger} L^{-1} L e^{ix^\mu P_\mu} L \tag{D.11}$$

and this transformation defines $U'(x',\chi'(x'))$ where

$$x^\mu \longrightarrow x'^{\,\mu} = \Lambda^\mu{}_\nu x^\nu \tag{D.12}$$

$$\chi^\alpha(x) \longrightarrow \chi'^{\,\alpha}(x') = \chi^\beta(x(x'))\, \widetilde{\Lambda}_\beta{}^\alpha = \chi^\beta(\Lambda^{-1}x')\, \widetilde{\Lambda}_\beta{}^\alpha \tag{D.13}$$

$$\chi^\dagger_{\dot\alpha}(x) \longrightarrow \chi^{\dagger\prime}_{\dot\alpha}(x') = \chi^\dagger_{\dot\beta}(x(x'))\, \widehat{\Lambda}^{\dot\beta}{}_{\dot\alpha} = \chi^\dagger_{\dot\beta}(\Lambda^{-1}x')\, \widehat{\Lambda}^{\dot\beta}{}_{\dot\alpha} \tag{D.14}$$

because the supercharges Q_α,$Q^\dagger_{\dot\alpha}$ and P_μ carry irreducible representations of the Lorentz group which are $(\frac{1}{2},0) = \tilde\Lambda$,$(0,\frac{1}{2}) = \tilde\Lambda^*$ and $(\frac{1}{2},\frac{1}{2}) = \Lambda$ respectively, i.e.

$$LP_\mu L^{-1} = P_\nu \Lambda^\nu{}_\mu\,, \qquad LQ^i_\alpha L^{-1} = \widetilde{\Lambda}_\alpha{}^\beta Q^i_\beta\,, \qquad LQ_i^{\dagger\,\dot\alpha} L^{-1} = \widehat{\Lambda}^{\dot\alpha}{}_{\dot\beta} Q_i^{\dot\beta} \tag{D.15}$$

where $\widehat{\Lambda} = \widetilde{\Lambda}^{-1\,\dagger}$. Analogously, under space-time translations $T = e^{ia^\mu P_\mu}$ we get

$$TU = U' = e^{i\chi'(x')Q+i\chi^{\dagger\prime}(x')Q^\dagger} e^{ix'^{\,\mu} P_\mu} \tag{D.16}$$

where

$$x'^{\,\mu} = x^\mu + a^\mu, \qquad \chi'(x') = \chi(x(x')) = \chi(x'-a). \tag{D.17}$$

We now want to derive how a general SUSY transformation generated by the supercharges acts on the Goldstini. We write such transformation as

$$g_\xi = e^{i\xi Q + i\xi^\dagger Q^\dagger} \tag{D.18}$$

where ξ is an anticommuting global variable. Acting with g_ξ on U, we get the *non-linear* transformation of the Goldstini under a general broken transformation

$$\begin{aligned} g_\xi U(x,\chi(x)) &= e^{i\xi Q+i\xi^\dagger Q^\dagger} e^{i\chi(x)Q+i\chi(x)^\dagger Q^\dagger} e^{ix_\mu P^\mu} &\text{(D.19)}\\ &= e^{i\chi Q+i\xi Q+i\chi^\dagger Q^\dagger+i\xi^\dagger Q^\dagger-\frac{1}{2}\left[\xi^\dagger Q^\dagger,\chi Q\right]-\frac{1}{2}\left[\xi Q,\chi^\dagger Q^\dagger\right]} e^{ix_\mu P^\mu} &\text{(D.20)}\\ &= e^{i(\chi+\xi)Q+i(\chi^\dagger+\xi^\dagger)Q^\dagger+\chi\sigma^\mu\xi^\dagger P_\mu-\xi\sigma^\mu\chi^\dagger P_\mu+ix_\mu P^\mu} &\text{(D.21)}\\ &\equiv e^{i\chi'(x')Q+i\chi'^{,\dagger}(x')Q^\dagger+ix'^\mu P_\mu} &\text{(D.22)}\end{aligned}$$

where we used the BHC formula[1] $e^{iA}e^{iB} = e^{iA+iB-\frac{1}{2}[A,B]}$ and $\left[\xi^\dagger Q^\dagger, \chi Q\right] = -2\chi\sigma^\mu\xi^\dagger P_\mu$ (see SUSY algebra (D.3)). From (D.22), we get the non-linear SUSY representation on $\chi(x)$

$$\begin{aligned} v^\mu(\xi,\chi(x)) &\equiv i\left(\xi\sigma^\mu\chi(x)^\dagger - \chi(x)\sigma^\mu\xi^\dagger\right) &\text{(D.23)}\\ x \to x'^\mu &= x^\mu + v^\mu(\xi,\chi(x)) &\text{(D.24)}\\ \chi(x) \to \chi'(x') &= \chi(x(x')) + \xi &\text{(D.25)}\\ \chi^\dagger(x) \to \chi'^\dagger(x') &= \chi^\dagger(x(x')) + \xi^\dagger. &\text{(D.26)}\end{aligned}$$

One can invert (D.24) by expanding in ξ

$$x^\mu = x'^\mu - v^\mu(\xi,\chi(x')) + ... \tag{D.27}$$

from which we obtain the variation of the field χ at a given point

$$\begin{aligned} \chi(x) \to \chi'(x) &= \chi(x'(x)) + \xi = \chi(x) + \xi - v^\mu(\xi,\chi)\partial_\mu\chi(x) + \ldots &\text{(D.28)}\\ \chi^\dagger(x) \to \chi'^\dagger(x) &= \chi^\dagger(x'(x)) + \xi^\dagger = \chi^\dagger(x) + \xi^\dagger - v^\mu(\xi,\chi(x))\partial_\mu\chi^\dagger(x) + \ldots &\text{(D.29)}\end{aligned}$$

For *matter fields* $\Phi(x)$, i.e matter (non-gauge) fields that aren't Goldstini, a representation is obtained simply by omitting the non-linear shift, that is

$$\Phi(x) \to \Phi'(x) = \Phi(x'(x)) = \Phi(x) - v^\mu(\xi,\chi(x))\partial_\mu\Phi(x) + \ldots \tag{D.30}$$

In order to construct an effective action for Goldstini, we want to build all the objects which transform covariantly under SUSY transformations. First of all, let us focus on the derivative terms. We would like derivative-like terms transforming covariantly, that is

$$\nabla_a\chi(x) \to (\nabla_a\chi)'(x) = \nabla_a\chi(x'(x)). \tag{D.31}$$

[1] Notice that all higher commutators vanish because of the SUSY algebra (D.1), (D.2) and (D.3) and P_μ commutes with everybody.

for some operator ∇_a. It is straightforward to see that the standard derivative do not transform like (D.31). At a given point

$$\partial_\mu \chi(x) \to \partial_\mu \chi'(x) = \partial_\mu \chi(x'(x)) = (\partial_\nu \chi)(x'(x)) \frac{\partial x'^\nu}{\partial x^\mu} = \partial_\mu \chi(x'(x)) - \partial_\nu \chi(x'(x))\, v^\nu(\xi, \partial_\mu \chi(x)) + ... \tag{D.32}$$

In order to obtain objects transforming covariantely, it is useful to introduce the Maurer-Cartan 1-form built with the group element U

$$(U^{-1}dU)(x) = U^{-1}(x, \chi(x)) dU(x, \chi(x)) \qquad d = dx^\mu \partial_\mu = dx'^\mu \partial'_\mu . \tag{D.33}$$

Notice that the Maurer-Cartan 1-form is invariant under SUSY transformations. Since the ξ spinor in (D.18) is coordinate-independent

$$(U^{-1}dU)(x) \longrightarrow (U^{-1}dU)'(x') = U(x, \chi(x)^{-1} g_\xi^{-1} dx^\mu \partial_\mu \left(g_\xi U(x, \chi(x))\right) = (U^{-1}dU)(x). \tag{D.34}$$

We can write $(U^{-1}dU)$ in terms of the SUSY generators

$$(U^{-1}dU)(x) = i dx^\mu E_\mu{}^a \left(P_a + \nabla_a \chi\, Q + \nabla_a \chi^\dagger\, Q^\dagger\right) \tag{D.35}$$

where for future convenience we have factored out the coefficient $E_\mu{}^a$ of the momentum

$$E_\mu{}^a = \delta_\mu{}^a + i\partial_\mu \chi \sigma^a \chi^\dagger - i\chi \sigma^a \partial_\mu \chi^\dagger = E_\mu{}^{a\,\dagger}\,, \tag{D.36}$$

$$\nabla_a \chi = (E^{-1})_a{}^\mu \partial_\mu \chi\,, \tag{D.37}$$

$$\nabla_a \chi^\dagger = (E^{-1})_a{}^\mu \partial_\mu \chi^\dagger\,. \tag{D.38}$$

We define the inverse of $E_\mu{}^a$ which satisfies

$$(E^{-1})_a{}^\mu \equiv E_a{}^\mu\,, \qquad E_a{}^\mu E_\mu{}^b = \delta_a^b\,, \qquad E_\mu{}^a E_a{}^\nu = \delta_\mu^\nu. \tag{D.39}$$

The transformation of the Maurer-Cartan form at a *given point*

$$(U^{-1}\partial_\mu U)(x) \to (U^{-1}\partial_\mu U)'(x) = \frac{\partial x'^\nu}{\partial x^\mu}(U^{-1}\partial'_\nu U)(x'(x)) \tag{D.40}$$

suggests that $E_\mu{}^a$ and its inverse transform with the Jacobian (anti-Jacobian)

$$E_\mu{}^a(x) \to E'_\mu{}^a(x) = \frac{\partial x'^\nu}{\partial x^\mu} E_\nu{}^a(x'(x))\,, \tag{D.41}$$

$$E_a{}^\mu(x) \to E'_a{}^\mu(x) = \frac{\partial x^\mu}{\partial x'^\nu} E_a{}^\nu(x'(x))\,. \tag{D.42}$$

It is clear now that $\nabla_a \chi$ transforms covariantly

$$\nabla_a \chi(x) \to (\nabla_a \chi)'(x) = E'_a{}^\mu(x) \partial_\mu \chi'(x) \tag{D.43}$$

$$= \frac{\partial x^\mu}{\partial x'^\nu} E_a{}^\nu(x'(x)) \partial_\mu \chi(x'(x)) \tag{D.44}$$

$$= \frac{\partial x^\mu}{\partial x'^\nu} \frac{\partial x'^\rho}{\partial x^\mu} E_a{}^\nu(x'(x)) \partial'_\rho \chi(x'(x)) \tag{D.45}$$

$$= \nabla_a \chi(x'(x)). \tag{D.46}$$

We can now render covariant the derivatives of any fields by acting

$$\nabla_b \nabla_a \Psi(x) = E_b^\mu \partial_\mu \left(E_a^\nu \partial_\nu \Psi(x) \right). \tag{D.47}$$

There is another tensor built out of the Goldstini,

$$F_{bc}{}^a(x) \equiv E_b{}^\mu(x) E_c{}^\nu(x) \left(\partial_\mu E_\nu{}^a(x) - \partial_\nu E_\nu{}^a(x) \right) \tag{D.48}$$

which transforms covariantly

$$F_{bc}{}^a(x) \to F'_{bc}{}^a(x'(x)) \tag{D.49}$$

because the term $\partial^2 x'^\rho / \partial x^\mu \partial x^\nu E_\rho^a$ cancels out in the difference $\mu \leftrightarrow \nu$. Moreover, it is easy to see that

$$\left[\nabla_b, \nabla_a \right] \Psi(x) = -F_{ba}{}^c \nabla_c \Psi(x). \tag{D.50}$$

Gauge fields A_μ associated to local internal symmetry groups behave just as the ordinary derivatives (they are 1-forms) and should thus be compensated by Goldstino insertions as well

$$\mathbb{A}_a \equiv E_a{}^\mu A_\mu \tag{D.51}$$

so that gauge covariant derivatives

$$D_a \equiv \nabla_a - ig\mathbb{A}_a \tag{D.52}$$

transform covariantly under the SUSY and gauge transformations. The gauge field strength is defined analogously by compensating the two lorentz indexes with the Vielbien $E_a{}^\mu$. Now we have all the useful object to construct the effective lagrangian. We want an invariant measure in the action. Thus let's define $\mathcal{E}(x) \equiv \det E_\mu{}^a(x)$. We have

$$\mathcal{E}(x) \to \mathcal{E}'(x) = |\frac{\partial x'}{\partial x}| \mathcal{E}(x'(x)) \tag{D.53}$$

and therefore the invariant measure we were looking for is

$$d^4x \mathcal{E}(x) \to d^4x |\frac{\partial x'}{\partial x}| \mathcal{E}(x'(x)) = d^4x' \mathcal{E}(x') \tag{D.54}$$

that we use to build actions[2]

$$S[\chi, \Phi] = \int d^4x \mathcal{E}(x) \mathcal{L}(\nabla_a \chi(x), \Phi(x), \nabla_a \Phi(x), F_{bc}{}^a(x), \ldots) \tag{D.55}$$

where the ellipses denote other possible covariant object we can build. For our purposes, the Goldstini and matter fields are enough. The actions (D.55) are invariant under the transformations of the field at a given point

$$S[\chi, \Phi] \to S[\chi', \Phi'] = S[\chi, \Phi]\,. \tag{D.56}$$

[2]To avoid clutter of notation we are showing only neutral matter fields Φ. Including charged fields and gauge fields is straightforwardly done by promoting the $\nabla_a \to D_a$ and by adding the fields strengths $E_a{}^\mu E_b{}^\nu(\partial_\mu A_\nu - \partial_\nu A_\mu)$.

because

$$S[\chi', \Phi'] = \int d^4x \mathcal{E}'(x) \mathcal{L}((\nabla_a \chi)'(x), \Phi'(x), (\nabla_a \Phi)'(x), F'_{bc}{}^a(x), \ldots) \tag{D.57}$$

$$= \int d^4x |\frac{\partial x'}{\partial x}| \mathcal{E}(x'(x)) \mathcal{L}(\nabla_a \chi(x'(x)), \Phi(x'(x)), \nabla_a \Phi(x'(x)), F_{bc}{}^a(x'(x)), \ldots) \tag{D.58}$$

$$= \int d^4x' \mathcal{E}(x') \mathcal{L}(\nabla_a \chi(x'), \Phi(x'), \nabla_a \Phi(x'), F_{bc}{}^a(x'), \ldots) = S[\chi, \Phi]\,. \tag{D.59}$$

We can build the action by expanding the lagrangian with respect to the fields and their derivatives. The first term may be just a constant, $\mathcal{L} = -F^2 + ...$ where $[F] = 2$. The sign is determined by the correct sign of the Goldstino kinetic term which arises from $\mathcal{E}(x)$

$$\mathcal{E} = 1 + \left(i\partial_\mu \chi \sigma^\mu \chi^\dagger + h.c.\right) + \frac{1}{2}\left[\left(i\partial_\mu \chi \sigma^\mu \chi^\dagger + h.c.\right)^2 - \left(i\partial_\mu \chi \sigma^a \chi^\dagger + h.c.\right)\left(i\partial_a \chi \sigma^\mu \chi^\dagger + h.c.\right)\right] + \ldots \tag{D.60}$$

and it generates a positive vacuum energy $E_{vac} = F^2 > 0$, as it must be for a spontaneously broken SUSY. Notice that $\sqrt{F}$ has the physical meaning of the SUSY breaking scale. The sign and the coefficient of 4-fermion interactions are fixed in terms of F only

$$-F^2 \mathcal{E}(x) \to -2F^2 \left(\chi^\dagger \bar{\sigma}^a \partial_\mu \chi\right)\left(\chi^\dagger \bar{\sigma}^\mu \partial_a \chi\right) + (\text{vanish on-shell or total der.}) \tag{D.61}$$

If we canonically normalize the Goldstino fields,$\chi \to \chi/(\sqrt{2}F)$, the dimension-8 operator (D.61) enters in the effective lagrangian as

$$-\frac{1}{2F^2}\left(\chi^\dagger \bar{\sigma}^a \partial_\mu \chi\right)\left(\chi^\dagger \bar{\sigma}^\mu \partial_a \chi\right) \tag{D.62}$$

The overall square-two factor is due to the presence of the hermitian conjugate of the kinetic term, since the latter is hermitian up to integration by parts.
Integrating by parts we can rewrite (D.62) as

$$\frac{1}{2F^2}\left(\chi^\dagger \bar{\sigma}^a \partial_\mu \chi\right)\left(\partial_a \chi^\dagger \bar{\sigma}^\mu \chi\right) = \frac{1}{2F^2}\left(\chi^{\dagger\,i}_{\dot{\alpha}} \bar{\sigma}^{\nu\,\dot{\alpha}\alpha} \partial_\mu \chi_{i\,\alpha}\right)\left(\partial_\nu \chi^{\dagger\,j}_{\dot{\beta}} \bar{\sigma}^{\mu\,\dot{\beta}\beta} \chi_{j\,\beta}\right) \tag{D.63}$$

$$= -\frac{1}{2F^2} \chi^{\dagger\,i}_{\dot{\alpha}} \partial_\nu \chi^{\dagger\,j}_{\dot{\beta}} \bar{\sigma}^{\mu\,\dot{\beta}\beta} \bar{\sigma}^{\nu\,\dot{\alpha}\alpha} \partial_\mu \chi_{i\,\alpha} \chi_{j\,\beta} \tag{D.64}$$

$$= \frac{1}{F^2} \chi^{\dagger\,i}_{\dot{\alpha}} \partial_\mu \chi^{\dagger\,j\,\dot{\alpha}} \partial^\mu \chi_i^\beta \chi_{j\,\beta} \tag{D.65}$$

$$= \frac{1}{F^2} \chi^{\dagger\,i} \partial \chi^{\dagger\,j} \partial \chi_i \chi_j \tag{D.66}$$

where we used (B.16). When we choose $\mathcal{N} = 1$, this operator reduces to the quartic term that appears in the Goldstino lagrangian in [34]

$$\frac{1}{F^2} \chi^\dagger \partial \chi^\dagger \partial \chi \chi = \frac{1}{4F^2} \partial\left(\chi^\dagger \chi^\dagger\right) \partial\left(\chi\chi\right) = -\frac{1}{4F^2} \chi^\dagger \chi^\dagger \Box\left(\chi\chi\right). \tag{D.67}$$

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