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Discrete symmetries of $\mathcal{N} = 4$ String models

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"Ogni cosa che puoi immaginare, la natura l'ha già creata"
Albert Einstein

Dedicato alla vita
e a tutto ciò che ci mantiene vivi

Contents

Introduction

One of the most important problems of nowadays theoretical physics is the unification of Quantum Mechanics with General Relativity. This kind of unified description is allowed only if we are dealing with a theory of Quantum Gravity and one of the most promising candidate to this role is String Theory. For a good introduction to its formalism see [1], [8], [10] and [11].
String Theory is actually believed to be only a formally consistent theory of Quantum Gravity and that is because there are huge experimental difficulties that arise when we want to verify its validity.
The simplest type of String Theory, the Bosonic String Theory, shows some physical inconsistencies like particles with a negative squared mass (*Tachyons*) or space-time without any fermion, namely without matter, therefore we understand a priori that this kind of theory will never riproduce the reality. Those inconsistencies however may be removed by introducing a symmetry between bosons and fermions, i.e. some special transformations that allow us to link them each other. This kind of transformations are called Supersymmetries (SUSY).
By introducing SUSY in a String Theory context ([14] and [15]), we can build five different, and mathematically dual, formulations of Superstring Theory, which are: Type I, Type IIA, Type IIB, Heterotic (HO) and Heterotic (HE).
Again, also those theories present some phenomenological issues and the most evident one is the fact that they are based on a 10-dimensional space-time. Our daily experience is clearly based only on four dimensions, therefore there should exist a mechanism that allows us to "hide" the additional ones. A good way of trying to solve this problem consists in thinking the whole 10-dimensional space-time as product of a four dimensional space-time, namely the usual Minkowski one, and a set of different 6-dimensional (Ricci-flat) compact manifolds that we call *Internal Manifolds*. The mechanism used to hide the additional dimensions, called *Compactification*, consists in thinking that the internal manifolds are actually too small to be seen in today experiments, namely their typical length scale could be set between the Planck scale and an unknown experimentally verifiable scale.
There are an infinite possible consistent choices of internal manifolds and the most studied, actually, because of their properties of preserving a certain amount of *Supersymmetries*, are: $\mathbb{T}^6$, CY_3, $CY_2 \times \mathbb{T}^2$, where CY_n stands for an n-dimensional complex and Ricci-flat surface called *Calabi-Yau n-folds.* The unique CY_2 class of manifolds that there exists is the topological class of $K3$ surfaces.
K3 surfaces are very interesting objects because of their property to conserve a certain number of *Supercharges* after having compactified a particular Superstring Theory. One of the most important example of the phenomenological implications of Superstring models built on K3, is the derivation, under specifical conditions, of the Bekenstein-Hawking law describing the Entropy of Black Holes (see [24]).

The study of every possible phenomenological implication of the theory is thus of great importance and what we would like to do with this thesis is to give a contribution to this work.
A lot of different formal tools are required to study a String theory defined on these tiny structures. The most important one is with no doubt the 2-dimensional Conformal Field Theory (CFT). The great importance of this kind of theories is due to the fact that they are used to describe the *World-sheet* spanned by a closed or open string. Most of the arguments we will present will be based on [1], [6], [4] and [5]. We understand thus that CFT is the language of the String Theory and, after the introduction of supersymmetry transformations, Superconformal Field Theory (SCFT) is the correspective one for Superstring theory.
More in detail, the great importance of SCFT in our context comes from the fact that, when we compactify the Type IIA or IIB Superstring Theory on a K3 surface, 2-dimensional superconformal field theories with 8 supersymmetries arise. These, in a String Theory context, are called *Non-linear Sigma Models* (NLSMs) on K3. The main properties of K3 surfaces and the role they play in a String Theory context, are described in [28].
K3 surfaces are rather complicated objects to study and there is actually no possibility to compute explicitly some of the tipically interesting quantities of a CFT, just like, for example, the *Partition Function* of the model. One of the solution we have thus is to consider a special type of K3 that can be described as a *Torus Orbifold*, more precisely $K3 \simeq \mathbb{T}^4/\mathbb{Z}_2$. This last surface is much simpler than a generic K3 surface and, in fact, by considering that structure, we are able to compute explicitly the Partition Function of our NLSM and other similar objects like the *Elliptic Genus* and the *Twining Genera*. We will see later that the validity of both these calculations can be extended to K3 surfaces that are different from the Torus Orbifold.
More in detail the Elliptic Genus and the Twining Genus are obtained by opportunely modifying the partition function with the insertion of some interesting discrete symmetry operators of the model. There is a finite number of discrete symmetries of a NLSM on K3 and they have all been classified in [18]. The interesting thing is that, for each one of them, the corresponding Twining Genus can be potentially computed explicitly.
A lot of Twining Genera has been already computed and what we would like to do with this thesis is to continue the completion program of the classification. Thanks to those calculations, we are able to see how the fields of the NLSM behave under these special symmetries and, as previously said, this study may be relevant because symmetries of the model at very high-energy scale may have an impact on the low-energy effective field theory.

In the first chapter we present the Classical Bosonic String Theory, in particular the *Nambu-Goto* and *Polyakov* actions and the solutions of their equations of motion.

In the second chapter we introduce the canonical quantization of the bosonic string, its spectrum and the algebra of the stress-energy tensor: the *Virasoro Algebra*.

In the third chapter we present the Conformal Field Theory by introducing the fundamental concepts of *Primary Fields* and their *Operator Product Expansion* (OPE). At the end of the chapter, we make some examples of quantization of fields on the cylinder.

In the fourth chapter we introduce the perturbation approach to String Theory and we study the *Moduli Space of metrics* for manifolds with different genera g. At the end of the chapter we introduce the concept of *Non linear σ-Model* (NLSM).

In the fifth chapter we introduce the *Supersymmetry* and we apply it to both String Theory and Superconformal Field Theory.

In the sixth chapter we finally concentrate on a Conformal Field Theory defined on the Torus. We compute the *Partition Function* for some interesting systems and we present the concept of *Elliptic* and *Twining Genus*.

In the seventh and last chapter, we introduce Non linear σ-Models on $\mathbb{T}^4$ and on K3 surfaces and their Moduli Space of metrics. By choosing the proper discrete symmetries of a suitable NLSM on K3, we finally compute some interesting Twining Genera. This work may be helpful in order to prove the validity of the conjecture formulated in [19].

Chapter 1

Classical Bosonic String

In this chapter we will introduce the formalism needed to describe the dynamics of the classical bosonic string. This will be done by making an analogy between the free relativistic point particle and the classical bosonic string. We will then find the solution of the equation of motion for the fields X^μ for the case of a single free closed and open string.

1.1 The Relativistic Particle

Let us start with the description of a free relativistic particle of mass m moving in a d-dimensional Minkowski space-time. It is action is simply given by the length of its world-line:

$$S = -m \int_{s_0}^{s_1} ds = -m \int_{\tau_0}^{\tau_1} d\tau \sqrt{-\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\eta_{\mu\nu}} \tag{1.1}$$

where τ is an arbitrary parameter with which we parametrize the world-line and $\eta_{\mu\nu} = diag(-1, +1, ..., +1)$ is the usual flat Minkowski metric. We notice that the action is invariant under reparametrization of the parameter τ, in particular under a transformation $\tau \to \tilde{\tau}(\tau)$. We can write the infinitesimal form of this reparametrization taking into account also how coordinates change under this kind of transformation:

$$\tau \to \tau + \xi(\tau), \qquad \delta x^\mu(\tau) = -\xi(\tau)\partial_\tau x^\mu(\tau)$$

with the condition that $\xi(\tau_0) = 0 = \xi(\tau_1)$ and $x'^\mu(\tau') = x^\mu(\tau)$.

If we now define $\dot{x}^\mu \equiv \partial_\tau x^\mu$ and we take the functional derivative of the action S with respect to the coordinates x^μ, we obtain:

$$\frac{\delta S}{\delta x^\mu} = 0 \quad \Longrightarrow \quad 0 = \frac{d}{d\tau}\left(\frac{m\dot{x}^\mu}{\sqrt{-\dot{x}^2}}\right) = \frac{dp^\mu}{d\tau}$$

where we have defined $p^\mu = \frac{\partial \mathscr{L}}{\partial \partial_\tau x_\mu} = m\frac{\dot{x}^\mu}{\sqrt{-\dot{x}^2}}$. Using now this definition we can obtain the dynamical constraint[1] of our system: $\phi \equiv p^2 + m^2 = 0$.

In general it is possible to establish a relationship between the number of constraints of the system and the characteristics of the lagrangian (or the action), in particular it can be shown that the number of constraints of the system is equal

[1]This is the usual on-shell condition for a propagating relativistic massive particle

to the number of null eigenvalues of the Hessian matrix $\frac{\partial^2 \mathscr{L}}{\partial \dot{x}^\mu \partial \dot{x}^\nu}$.
Constraints obtained in this way are called primary constraints. When the Hessian matrix has no maximal rank, i.e. we have at least one constraint, we can express the Hamiltonian of the system as $H = H_{can} + \sum_k c_k \phi_k{}^2$, where the c_k are the Lagrangian multipliers, so it will contain every dynamical information about the system.

In order now to generalize the action of a free massive relativistic particle to the massless case, we introduce a new auxiliary variable $e(\tau)$ that should not contain any new dynamical degree of freedom. The new action is:

$$S = \frac{1}{2} \int_{\tau_0}^{\tau_1} e(e^{-2}\dot{x}^2 - m^2) d\tau \tag{1.2}$$

Since now this action has no square roots, the equation of motion appear simpler than before:

$$\begin{aligned} \frac{\delta S}{\delta e} = 0 \quad &\Longrightarrow \quad \dot{x}^2 + e^2 m^2 = 0 \\ \frac{\delta S}{\delta x^\mu} = 0 \quad &\Longrightarrow \quad \frac{d}{d\tau}\left(e^{-1}\dot{x}^\mu\right) = 0 \end{aligned}$$

We immediately notice that $e(\tau)$, being an auxiliary variable, satisfies an algebraic equation of motion, in particular its equation of motion has no derivatives in it. This means that it has no dynamical role because it can be written in term of the other true dynamical variables: the coordinates x^μ. By substituting now in the action (1.2) and in the equation of motion for x^μ, the expression of $e(\tau)$ computed by solving its equation of motion, we find again the action (1.1). This means that both actions are dinamically equivalent.
Finally we notice that, in this case, the Hessian matrix has maximal rank, so there are no primary constraints on the dynamics: the constraint ϕ can now be obtained by imposing the definition of p^μ and the equation of motion of $e(\tau)$ and x^μ. Constraints obtained in this way are called secondary constraints.
We know that the action is invariant under reparametrization of τ, therefore it is necessary to fix it. A possible choice we could do is to impose: $\dot{x}^2 = -1$. This operation of choosing a particular value of τ is properly a *Gauge Fixing*.

1.2 Polyakov and Nambu-Goto actions

Let us now generalize the arguments of the previous section about the free relativistic particle, to the string case. Strings are 1-dimensional objects that can be open or closed, depending on whether its extremal points are respectively identified or not. Now, the action of a string, analogously to the free relativistic point particle, is the area of the so called *World-Sheet*, i.e. the bidimensional analogous of the world-line. The world-sheet now, being a 2-dimensional object, has to be parametrized by two parameters that we call τ and σ.

[2] H_{can} is the canonical Hamiltonian $H_{can} = \frac{\partial L}{\partial \dot{x}^\mu}\dot{x}^\mu - L$

1.2.1 The Nambu-Goto action

In the string case the action is called *Nambu-Goto Action* and its expression is:

$$
\begin{aligned}
S_{NG} &= -T\int_\Sigma dA \\
&= -T\int_\Sigma d^2\sigma\sqrt{-\det_{\alpha\beta}\left(\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}\eta_{\mu\nu}\right)} \\
&= -T\int_\Sigma d^2\sigma\sqrt{\left(\dot{X}\cdot X'\right)^2-\dot{X}^2X'^2} \\
&= -T\int_\Sigma d^2\sigma\sqrt{-\Gamma}
\end{aligned}
\tag{1.3}
$$

where we have defined $\cdot = \frac{\partial}{\partial\tau}$, $' = \frac{\partial}{\partial\sigma}$, $\sigma^\alpha = (\tau,\sigma)$ and Σ is the world-sheet.
The coordinates $X^\mu(\sigma,\tau)$, with $\mu = 0,...,d-1$, are now d maps from the 2-dimensional world-sheet to the Minkowski d-dimensional space. The parameters σ and τ, that respectively parametrize the spatial and temporal dimension of the world-sheet, are defined in the domains: $\tau_i < \tau < \tau_f$ and $0 \leq \sigma < l$.
We have also defined for simplicity:

$$\Gamma_{\alpha\beta} \equiv \frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}\eta_{\mu\nu}$$

and $\det_{\alpha\beta}\Gamma_{\alpha\beta} \equiv \Gamma$. From the definition of $\Gamma_{\alpha\beta}$, we can interprete it as the induced metric inherited from the d-dimensional Minkowski space-time[3].
It is important to notice that we introduced a new dimensional constant, the string tension T, with $[T] = M^{-2}$. It is needed to normalize the action and make it adimensional. It is possible now, using the string tension T, to define some quantities as a scale to which our theory refers to, in particular we have:

$$
\begin{aligned}
\alpha' &= \frac{1}{2\pi T} && \textit{Regge Slope} \\
l_s &= 2\pi\sqrt{\alpha'} && \textit{String length Scale} \\
M_s &= \frac{1}{\sqrt{\alpha'}} && \textit{String Mass Scale}
\end{aligned}
$$

The NG action is invariant, just like the one of the pointlike particle, under global Poincarè transformations, $X'^\mu = X^\mu + a^\mu$, and under local reparametrizations of σ e τ.
In order now to find the equation of motion for the X^μ fields, let us vary the action keeping the extremal points of the trajectories fixed, i.e. with the condition: $\delta X^\mu(\sigma = 0, l) = 0$. The equation we obtain is thus:

$$\frac{\partial}{\partial\tau}\frac{\partial L}{\partial\dot{X}^\mu} + \frac{\partial}{\partial\sigma}\frac{\partial L}{\partial\dot{X}^\nu} = 0$$

Now the boundary conditions that we have to impose are different between open and closed strings, indeed we have that:

$$
\begin{aligned}
&\frac{\partial L}{\partial X'^\mu}\delta X^\mu = 0 \qquad \text{when} \quad \sigma = 0, l && \text{(open string)} \\
&X^\mu(\sigma + l, \tau) = X^\mu(\sigma,\tau) && \text{(closed string)}
\end{aligned}
$$

[3]This kind of metric has not to be confused with the world-sheet intrinsic metric $h_{\alpha\beta}$ that we will introduce later.

In the open string case we have that the boundary condition can be satisfied by imposing two different condition, the first is:

$$\frac{\partial L}{\partial X'^{\mu}} = 0 \quad \forall \delta X^{\mu} \qquad \text{Neumann Condition}$$

that its physical meaning is that no momentum flows off the end of the string, while the second condition we can impose is:

$$\delta X^{\mu} = 0 \qquad\qquad \text{Dirichlet Condition}$$

that means that the extremal points of the string remain fixed along the X^{μ} direction[4]. We immediately notice that keeping the extremal points fixed breaks space-time translation invariance, therefore, because of the Noether's theorem, we break the conservation of the momentum[5].
Because of the presence in the action (1.3) of a square root, the equation of motion are rather complicate. It can be shown that the Hessian matrix has two null eigenvalues that corresponds to the primary constraints:

$$\begin{cases} \Pi_{\mu} X'^{\mu} = 0 \\ \Pi^2 + T^2 X'^2 = 0 \end{cases}$$

It can be easily shown, starting from the NG action and imposing the constraints we have just found, that $H_{can} = 0$. We understand thus that the dynamics is completely determined by the primary constraints.

1.2.2 The Polyakov action

In order now to simplify the action (1.3) and extend his validity, exactly like we did previously for the point-like particle, let us introduce an auxiliary field $h_{\alpha\beta}(\sigma, \tau)$ with signature $(-, +)$. This new kind of action is called *Polyakov Action* and it is:

$$S_P = -\frac{T}{2} \int_{\Sigma} d^2\sigma \sqrt{-h} h^{\alpha\beta} \partial_{\alpha} X^{\mu} \partial_{\beta} X^{\nu} \eta_{\mu\nu} \tag{1.4}$$

where we have defined $h \equiv \det h_{\alpha\beta}$. Let us now define the stress-energy tensor of the world-sheet theory as the tensor that quantify the response of the system to the changes of the world-sheet metric:

$$T_{\alpha\beta} = \frac{4\pi}{\sqrt{-h}} \frac{\delta S_P}{\delta h^{\alpha\beta}}$$

Using the fact that $\delta h = -h_{\alpha\beta}(\delta h^{\alpha\beta}) h$, we can find its explicit expression:

$$T_{\alpha\beta} = -\frac{1}{\alpha'} \left(\partial_{\alpha} X^{\mu} \partial_{\beta} X_{\mu} - \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \partial_{\gamma} X^{\mu} \partial_{\delta} X_{\mu} \right)$$

[4] Let us notice that we can choose indipendently and differently the boundary conditions for the first or second extremal point of the open string. The notation that we will use to indicate that will be NN, ND, DN or DD.

[5] This condition should not worry us because, imposing the Dirichlet condition, we are asking that the extremal points of the string are attached to a space-time's subspace called *D-Brane*. By considering a system made of a D-brane and an open string attached to it, the momentum is conserved again.

Always from the definition of $T_{\alpha\beta}$ we notice that we can write the variation of the action as:

$$\delta S_P = \frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{-h}\; T_{\alpha\beta}\;\delta h^{\alpha\beta}$$

Computing now the equation of motion, we can obtain:

$$\begin{cases} T_{\alpha\beta} = 0 \\ \Box X^\mu = \dfrac{1}{\sqrt{-h}}\partial_\alpha(\sqrt{-h}\; h^{\alpha\beta}\partial_\beta X^\mu) = 0 \end{cases}$$

Now the boundary condition for the closed string are the same as before, while for the open string they become:

$$n^\alpha\partial_\alpha X^\mu\delta X_\mu = 0 \qquad \text{for} \quad \sigma = 0, l$$

with n_α the normal vector at the boundary. The invariance under diffeomorphisms of the Polyakov's action gives us the conservation of the stress-energy tensor, $\nabla^\alpha T_{\alpha\beta} = 0$, where the covariant derivative is $\nabla^\alpha = \partial^\alpha + \Gamma^\alpha$.[6]

Starting now from the equation of motion that we found in the case of the NG action, in particular the vanishing of the stress-energy tensor, we can obtain an useful identity:

$$\det\nolimits_{\alpha\beta}(\partial_\alpha X^\mu\partial_\beta X_\mu) = \frac{1}{4}h(h^{\gamma\delta}\partial_\gamma X_\mu\partial_\delta X^\mu)^2 \tag{1.5}$$

If we now substitute this identity into the Polyakov's action, we can check that what we recover is the NG action. For this reason the two actions are classically equivalent.

1.2.3 Symmetries of the Polyakov action

Let us now discuss the symmetries of the Polyakov's action:

1. Global Symmetries:

   - Poincarè invariance:

$$\delta X^\mu = a^\mu{}_\nu X^\nu + b^\mu \qquad\qquad (a_{\mu\nu} = -a_{\nu\mu})$$
$$\delta h_{\alpha\beta} = 0$$

2. Local Symmetries:

   - Reparametrization invariance

$$\begin{aligned} \delta X^\mu &= -\xi^\alpha\partial_\alpha X^\mu \\ \delta h_{\alpha\beta} &= -(\xi^\gamma\partial_\gamma h_{\alpha\beta} + \partial_\alpha\xi^\gamma h_{\gamma\beta} + \partial_\beta\xi^\gamma h_{\gamma\alpha}) \\ &= -(\nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha) \\ \delta\sqrt{-h} &= -\partial_\alpha(\xi^\alpha\sqrt{-h}) \end{aligned}$$

   - Weyl rescaling:

$$\delta X^\mu = 0$$
$$\delta h_{\alpha\beta} = 2\Lambda h_{\alpha\beta}$$

[6] Remember that $\Gamma^\gamma_{\alpha\beta} = \frac{1}{2}h^{\gamma\delta}(\partial_\alpha h_{\delta\beta} + \partial_\beta h_{\alpha\delta} - \partial_\delta h_{\alpha\beta})$

Let us first consider the invariance under Weyl rescaling[7]. An immediate consequence of this is that the trace of the stress-energy tensor vanishes, i.e. $T^{\alpha}{}_{\alpha} = h_{\alpha\beta}T^{\alpha\beta} = 0$. Let us indeed take a rescaling transformation for the intrinsic metric of the world-sheet $h_{\alpha\beta} \to e^{2\Lambda} h_{\alpha\beta}$. We can then write:

$$0 = \delta S = \int d^2\sigma \left(-2\frac{\delta S}{\delta h^{\alpha\beta}} h^{\alpha\beta} + \sum_i \frac{\delta S}{\delta \phi_i} d_i \phi_i \right) \delta\Lambda$$

If now the invariance holds for every $\delta\Lambda$, then we can conclude that:

$$-2 \underbrace{\frac{\delta S}{\delta h^{\alpha\beta}}}_{\propto T_{\alpha\beta}} h^{\alpha\beta} = 0 \iff h_{\alpha\beta}T^{\alpha\beta} = T^{\alpha}{}_{\alpha} = 0$$

We can now ask if there are other terms, that we can add a priori, that satisfy all the symmetries we have required. In particular, if we want also a 2-dimensional power-counting renormalizable theory, we can add to more expressions to S_P:

$$S_1 = \lambda_1 \int_\Sigma d^2\sigma \sqrt{-h} \tag{1.6}$$

$$S_2 = \frac{\lambda_2}{4\pi} \int_\Sigma d^2\sigma \sqrt{-h} R = \lambda_2 \chi(\Sigma) \tag{1.7}$$

It can be shown that the first term, called *Cosmological Term*, it is identically null because of the tracelessness condition of the stress-energy tensor, $T_\alpha{}^\alpha = -\frac{\lambda_1}{T}$, that is satisfied if and only if $\lambda_1 = 0$. We can also show that also the second term[8] does not contribute to the equation of motion because it is a total derivative.

Let us now consider the reparametrization invariance. Using the reparametrization transformations it can be shown that is always possible locally to put the world-sheet metric proportional to the flat metric, i.e.:

$$h_{\alpha\beta} = \Omega^2(\sigma, \tau)\ \eta_{\alpha\beta}$$

where the flat metric $\eta_{\alpha\beta}$ is the one for which $ds^2 = -d\tau^2 + d\sigma^2$.
Explicitly we will then have that:

$$ds^2 = -\Omega^2(d\tau^2 + d\sigma^2) = -\Omega^2 \eta_{\alpha\beta} d\sigma^\alpha d\sigma^\beta$$

Let us now consider the following change of coordinates: let us take two null vectors at each point and take their integral curves labelled by σ^+ e σ^-. If we now write this set of coordinates as $\sigma^\pm = \tau \pm \sigma$ we can write $ds^2 = -\Omega^2 d\sigma^+ d\sigma^-$. Those kind of coordinates are called *Light-Cone Coordinates* or *Conformal Coordinates.*

In general, when we act on a symmetric tensor, that has $\frac{1}{2}d(d+1)$ independent components, with some reparametrization transformations, we can remove d degrees of freedom and obtain a tensor with only $\frac{1}{2}d(d-1)$ independent parameters. When $d = 2$ we have clearly only one independent parameter that, using the Weyl

[7] It is important to notice that Weyl rescalings are not conformal transformations, indeed Weyl transformation acts only on the metric and not on the coordinates, while conformal transformations act on the coordinates causing the factorization of a certain common factor of every entry of the metric.

[8] $\chi(\Sigma)$ it is the *Euler Number* of the manifold Σ and it is a topological invariant.

invariance property of the metric, it can be eliminated. In our case we can use the Weyl rescaling in order to remove the Ω^2 factor and put the metric in the form $h_{\alpha\beta} = \eta_{\alpha\beta}$. From this arguments, we can finally put the metric in the form: $ds^2 = -d\sigma^+ d\sigma^-$.
We notice that the metric now is no more diagonal, in particular:

$$\begin{aligned} h_{\alpha\beta} = \eta_{\alpha\beta} &= \begin{pmatrix} \eta_{++} & \eta_{+-} \\ \eta_{-+} & \eta_{--} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \\ h^{\alpha\beta} &= \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} \end{aligned}$$

The derivatives with this choice of coordinates are: $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$.
In order to preserve the invariance under diffeomorphisms (i.e. reparametrizations) and Weyl rescalings locally, we require that $\delta h_{\alpha\beta} = 0$. Writing explicitly the non vanishing Christoffel's symbols in the light-cone coordinates, we obtain the following condition on the gauge: $\partial_- \xi^+ = 0 = \partial_+ \xi^-$. That obviously means that $\xi^\pm = \xi^\pm(\sigma^\pm)$. We notice that the same gauge condition would have been statisfied also taking a generic $\tilde{\sigma}^\pm = \tilde{\sigma}^\pm(\sigma^\pm)$, so we immediately see that this choice of the gauge is not complete but admit a residual gauge.
We would like now to see if the considerations hold also globally. We can explicitly see that, after a generic Weyl rescaling and reparametrization transformation, the variation of the metric becomes:

$$\begin{aligned} \delta h_{\alpha\beta} &= -\left(\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha\right) + 2\Lambda h_{\alpha\beta} \\ &\equiv (P\xi)_{\alpha\beta} + 2\tilde{\Lambda} h_{\alpha\beta} \end{aligned}$$

where we have defined:

$$\begin{aligned} (P\xi)_{\alpha\beta} &= \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha - \left(\nabla_\gamma \xi^\gamma\right) h_{\alpha\beta} \\ 2\tilde{\Lambda} &= 2\Lambda - \nabla_\gamma \xi^\gamma \end{aligned}$$

We decomposed the variation of the metric into a symmetric traceless and a trace term, but the last one can always be cancelled out by a suitable choice of Λ. We can notice that the operator P maps vectors into symmetric traceless tensors.
We ask now if the imposition of a certain condition on the metric allow us to fix completely the gauge or it admits some non-trivial transformations of the metric that leave the condition satisfied. This kind of transformations are exactly the conformal transformations. We understand therefore that the Residual Gauge are the conformal transformations globally defined on the surface.
Let us now see if it admits a kernel, i.e. it has some zero modes. As we said, the operator acts as:

$$(P\xi)_{\alpha\beta} = t_{\alpha\beta} = t_{\beta\alpha} \qquad \text{with} \quad t_\alpha{}^\alpha = 0$$

Now it may happen that, for a certain t, P does not admit a unique solution ξ. In particular, if P admit some zero modes, i.e. $(P\xi_0)_{\alpha\beta} = 0$, the most general solution could be written as $\xi + \xi_0$. The existence of a diffeomorphism corresponding to a zero mode of the P operator, does not affect the variation of the metric, therefore, it is a residual gauge. The zero modes of the P operator are called *Conformal Killing Vectors* and the equation $(P\xi)_{\alpha\beta} = 0$ is called *Conformal Killing Equation.*
We could ask now if we can always globally put the metric in a conformally flat

form and not only locally as we showed previously. Let us first define the inner product between two tensors on the world-sheet as:

$$(t_{\alpha\beta}, t_{\gamma\delta}) \equiv \int d^2\sigma\sqrt{-h}\; h^{\alpha\gamma}h^{\beta\delta}t_{\alpha\beta}t_{\gamma\delta} \tag{1.8}$$

respect to which we define the dagger operator of P:

$$\left(P^\dagger t\right)_\alpha = -\nabla^\beta t_{\alpha\beta} \tag{1.9}$$

We see that the $P^\dagger$ operator now maps symmetric traceless tensors in vectors. If it admits zero modes then we have:

$$0 = \left(\xi, P^\dagger t_0\right) = (P\xi, t_0) \qquad \forall\xi$$

We see thus that the tensor t_0, one of the zero modes of $P^\dagger$, is perpendicular to every tensor written as $(P\xi)_{\alpha\beta}$, or, equivalently, it belongs to the space perpendicular to the imagine space of the P operator. This means that it cannot be written as $(P\xi)_{\alpha\beta}$, $\forall\xi$. We can conclude therefore that if $P^\dagger$ admits a zero mode, then is not always possible globally to put the intrinsic metric in a conformally flat term, but only locally.
In conclusion, we can always, locally, put the metric in the form $h_{\alpha\beta} = e^{2\phi}\eta_{\alpha\beta}$ using the reparametrization invariance and the $e^{2\phi}$ factor can be then removed by using a Weyl rescaling. After this procedure we obtain: $h_{\alpha\beta} = \eta_{\alpha\beta}$. This gauge choice is also called *Conformal Gauge* choice.
The Polyakov action in the conformal gauge becomes:

$$\begin{aligned} S_p &= -\frac{T}{2}\int d^2\sigma\; \eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \\ &= \frac{T}{2}\int d^2\sigma\left(\dot{X}^2 - X'^2\right) \\ &= 2T\int d^2\sigma\; \partial_+ X\cdot\partial_- X \end{aligned}$$

Let us now compute the equation of motion for the X^μ fields in this particular gauge choice, with the condition $\delta X^\mu(\tau_0) = 0 = \delta X^\mu(\tau_1)$:

$$0 = \delta S = T\int d^2\sigma\delta X^\mu\left(\partial_\sigma^2 - \partial_\tau^2\right)X_\mu - T\int_{\tau_0}^{\tau_1} d\tau X'^\mu[\delta X^\mu]_{\sigma=0}^{\sigma=l} - T\int_0^{ll} d\sigma\left[\partial_\tau X^\mu \delta X_\mu\right]_{\tau_0}^{\tau_1}$$

where the last two terms vanish because of the boundary conditions for the closed and open string. We see now that the equation of motion is the same for both the open and closed string and it is:

$$\left(\partial_\sigma^2 - \partial_\tau^2\right)X^\mu = 4\cdot\partial_+\partial_- X^\mu = 0 \tag{1.10}$$

The most general solution of this equation is:

$$X^\mu(\sigma,\tau) = X_L^\mu\left(\sigma^+\right) + X_R^\mu\left(\sigma^-\right)$$

For the closed string, the left and right-moving modes are actually independent just imposing the boundary conditions, while for the open string, both modes are mixed because of the boundary conditions for the extremal point of the string.[9]

[9]This situation is exactly analogous to a wave that moves on a string and reflects in the extremal point coming back in the other direction.

Let us now come back to the equation of motion we computed for the NG action, in particular, let us impose that, in the conformal gauge, the stress-energy tensor must vanish:

$$\begin{aligned} T_{01} = T_{10} &= -2\pi T(\dot{X} \cdot X') = 0 \\ T_{00} = T_{11} &= -\pi T(\dot{X}^2 + X'^2) = 0 \end{aligned}$$

This constraints can be expressed alternatively as:

$$(\dot{X} \pm X')^2 = 0$$

In the light-cone coordinates the stress-energy tensor becomes:

$$T_{++} = -2\pi T\left(\partial_+ X \cdot \partial_+ X\right) = 0 \tag{1.11a}$$

$$T_{--} = -2\pi T\left(\partial_- X \cdot \partial_- X\right) = 0 \tag{1.11b}$$

$$T_{+-} = T_{-+} = 0 \tag{1.11c}$$

We notice that the (1.11c) expresses the traceless condition of the stress-energy tensor.[10] The condition of stress-energy tensor conservation, $\nabla^\alpha T_{\alpha\beta} = 0$, in the light-cone coordinates becomes:

$$\begin{aligned} \partial_- T_{++} &= 0 \\ \partial_+ T_{--} &= 0 \end{aligned}$$

therefore $T_{++} = T_{++}(\sigma^+)$, $T_{--} = T_{--}(\sigma^-)$. We can notice from this condition that there is an infinite number of conserved charges, indeed:

$$\partial_- f(\sigma^+) = 0 \quad \longrightarrow \quad \partial_-(f(\sigma^+)T_{++}) = 0$$

We can then associate to this expression a new conserved current with its corresponding conserved charge:

$$L_l = 2T \int_0^l d\sigma f\left(\sigma^+\right) T_{++}\left(\sigma^+\right) \tag{1.12}$$

the same is true for σ^-.
The Hamiltonian in the conformal gauge and with the light-cone coordinates becomes:

$$H = T \int_0^l d\sigma \left((\partial_+ X)^2 + (\partial_- X)^2 \right)$$

that, as we saw for the NG action, it vanishes imposing the equation of motion and the constraints. The generic Hamiltonian can be written, as we saw in the case of the free relativistic particle, as a combination of the constraints of the system when the Hessian matrix has not maximal rank, so now we have:

$$H = \int_0^l d\sigma \left[N_1(\sigma,\tau)\, \Pi \cdot X' + N_2(\sigma,\tau)\left(\Pi^2 + T^2 X^2\right)\right]$$

Using the usual expressions of the Poisson brakets at equal τ for X^μ and Π^μ:

$$\begin{aligned} \left\{X^\mu(\sigma,\tau), X^\nu\left(\sigma',\tau\right)\right\}_{PB} &= 0 = \left\{\Pi^\mu(\sigma,\tau), \Pi^\nu\left(\sigma',\tau\right)\right\}_{PB} \\ \left\{X^\mu(\sigma,\tau), \Pi^\nu\left(\sigma',\tau\right)\right\}_{PB} &= \eta^{\mu\nu}\delta(\sigma - \sigma') \end{aligned}$$

[10] Remember that, in the light-cone coordinates, the metric is made of only off diagonal components.

we can compute $\dot{X} = \{X, H\}_{PB}$, so what we obtain is:

$$\begin{cases} \dot{X}^\mu = N_1 X'^\mu + 2N_2 \Pi^\mu \\ \dot{\Pi}^\mu = \partial_\sigma (N_1 \Pi^\mu + 2T^2 N_2 X'^\mu) \end{cases}$$

Imposing now that the equation of motion (1.10) has to be satisfied, we obtain that $N_1 = 0$, $N_2 = \frac{1}{2T}$. We see that this particular choice of N_1 and N_2 equates to put ourselves in the conformal gauge. With this choice the Poisson brakets becomes:

$$\begin{aligned} \left\{X^\mu (\sigma, \tau), X^\nu (\sigma', \tau)\right\}_{PP} &= 0 = \left\{\dot{X}^\mu (\sigma, \tau), \dot{X}^\nu (\sigma', \tau)\right\}_{PP} \\ \left\{X^\mu (\sigma, \tau), \dot{X}^\nu (\sigma', \tau)\right\}_{PP} &= \frac{1}{T} \eta^{\mu\nu} \delta(\sigma - \sigma') \end{aligned}$$

Using this expression of the Poisson brakets, it can be shown that

$$-T \int \dot{X} \cdot X' d\sigma \qquad \text{and} \qquad \frac{T}{2} \int (\dot{X}^2 + X'^2) d\sigma$$

generate respectively constant σ and τ-traslations, while the conserved charges L_f generates transformations $\sigma^+ \to \sigma^+ + f(\sigma^+)$, indeed:

$$\{L_f, X(\sigma)\}_{PB} = -f(\sigma^+) \partial_+ X(\sigma)$$

We notice that this kind of transformations are exactly the ones that we expected because of the not complete gauge choice. Transformations like $\sigma^+ \to \sigma^+ + f(\sigma^-)$ are not allowed because they would introduce some diagonal terms in the intrinsic metric that obviously, in the light-cone coordinates, wouldn't leave the metric invariant.[11]

Let us finally concentrate on the global Poincaré invariance. By using the Noether's theorem for Lorentz rotations and space-time traslations, we can obtain the usual set of conserved currents:

$$P^\alpha_\mu = -T\sqrt{h} h^{\alpha\beta} \partial_\beta X_\mu \qquad J^\alpha_{\mu\nu} = X_\mu P^\alpha_\nu - P^\alpha_\mu X_\nu$$

and conserved charges (computed by integrating over a space-like section of the world-sheet at $\tau = 0$):

$$\begin{aligned} P_\mu &= \int_0^l d\sigma P^\tau_\mu = T \int_0^l d\sigma \partial_\tau X_\mu \\ J_{\mu\nu} &= \int_0^l d\sigma J^\tau_{\mu\nu} = T \int_0^l d\sigma \left(X_\mu \partial_\tau X_\nu - X_\nu \partial_\tau X_\mu\right) \end{aligned}$$

Using the Poisson brakets we can check that P_μ e $J_{\mu\nu}$ generate, as we expected, the Poincaré algebra:

$$\begin{aligned} \{P^\mu, P^\nu\}_{PB} &= 0 \\ \{P^\mu, J^{\rho\sigma}\}_{PB} &= \eta^{\mu\sigma} p^\rho - \eta^{\mu\rho} p^\sigma \\ \{J^{\mu\nu}, J^{\rho\sigma}\}_{PB} &= \eta^{\mu\sigma} J^{\nu\rho} + \eta^{\nu\sigma} J^{\mu\rho} - \eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\sigma} J^{\nu\rho} \end{aligned}$$

[11] It is easy to check by computing the ds^2 element using the $\tilde{\sigma}^+$ generic expression.

1.3 Oscillator Expansion for Closed String

Let us now solve the equation of motion of closed string in the conformal gauge. The generic solution of the equation of motion (1.10) for a closed string, with the periodic condition $X^\mu(\sigma,\tau) = X^\mu(\sigma + l, \tau)$, is:

$$X^\mu(\sigma,\tau) = X^\mu_R(\tau-\sigma) + X^\mu_L(\tau+\sigma)$$

explicitly:

$$X^\mu_R(\tau-\sigma) = \frac{1}{2}x^\mu + \frac{\pi\alpha'}{l}p^\mu(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-\frac{2\pi}{l}in(\tau-\sigma)} \tag{1.13}$$

$$X^\mu_L(\tau+\sigma) = \frac{1}{2}x^\mu + \frac{\pi\alpha'}{l}p^\mu(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\overline{\alpha}^\mu_n e^{-\frac{2\pi}{l}in(\tau+\sigma)} \tag{1.14}$$

where the α^μ_n and $\overline{\alpha}^\mu_n$ are the Fourier modes of the expansions. We define now conventionally that the Fourier modes are positive when $n < 0$ and negative when $n > 0$ and this will be very important when we will introduce the quantization of the bosonic string. Imposing also that the X^μ fields have to be real, we obtain:

$$\alpha^\mu_{-n} = (\alpha^\mu_n)^* \quad \text{and} \quad \overline{\alpha}^\mu_{-n} = (\overline{\alpha}^\mu_n)^*$$

If we now define $\alpha^\mu_0 = \overline{\alpha}^\mu_0 = \sqrt{\frac{\alpha'}{2}}p^\mu$ we can write in a compact way:

$$\partial_- X^\mu = \dot{X}^\mu_R = \frac{2\pi}{l}\sqrt{\frac{\alpha'}{2}}\sum_{n=-\infty}^{+\infty}\alpha^\mu_n e^{-\frac{2\pi}{l}in(\tau-\sigma)} \tag{1.15a}$$

$$\partial_+ X^\mu = \dot{X}^\mu_L = \frac{2\pi}{l}\sqrt{\frac{\alpha'}{2}}\sum_{n=-\infty}^{+\infty}\overline{\alpha}^\mu_n e^{-\frac{2\pi}{l}in(\tau+\sigma)} \tag{1.15b}$$

Computing now:

$$P^\mu = \int_0^l d\sigma \Pi^\mu = \frac{1}{2\pi\alpha'}\int_0^l d\sigma \dot{X}^\mu = p^\mu \tag{1.16}$$

$$q^\mu(\tau) \equiv \frac{1}{l}\int_0^l d\sigma X^\mu = x^\mu + \frac{2\pi\alpha'}{l}p^\mu \tau \tag{1.17}$$

we learn that p^μ is the total space-time momentum of the string and x^μ is the center of mass position of the string when $\tau = 0$. We can also compute the total angular momentum:

$$\begin{aligned} J^{\mu\nu} &= \int_0^l d\sigma\,(X^\mu \Pi - X^\nu \Pi^\mu) = \frac{1}{2\pi\alpha'}\int_0^l d\sigma\left(X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu\right) \\ &= l^{\mu\nu} + E^{\mu\nu} + \overline{E}^{\mu\nu} \end{aligned}$$

where

$$\begin{cases} l^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu \\ E^{\mu\nu} = -i\displaystyle\sum_{n=1}^{+\infty}\frac{1}{n}\left(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n\right) \end{cases}$$

with $\overline{E}^{\mu\nu}$ the analogous of $E^{\mu\nu}$. Using now the Poisson brakets for X^μ and $\dot{X}^\mu$ we can derive:

$$\{\alpha^\mu_m, \alpha^\nu_n\}_{PB} = \{\overline{\alpha}^\mu_m, \overline{\alpha}^\nu_n\}_{PB} = -im\delta_{m+n}\eta^{\mu\nu} \tag{1.18a}$$
$$\{\overline{\alpha}^\mu_m, \alpha^\nu_n\}_{PB} = 0 \tag{1.18b}$$
$$\{x^\mu, p^\nu\}_{PB} = \eta^{\mu\nu} \tag{1.18c}$$

where δ_{m+n} stands for $\delta_{m+n,0}$.
Using the oscillators expansion of the fields, the Hamiltonian becomes:

$$H = \frac{\pi}{l} \sum_{n=-\infty}^{+\infty} (\alpha_{-n} \cdot \alpha_n + \overline{\alpha}_{-n} \cdot \overline{\alpha}_n)$$

We can now choose the infinite functions $f(\sigma^\pm)$, that have to satisfy the correct periodicity conditions, as:

$$f_m(\sigma^\pm) = \exp(\frac{2\pi i}{l} m\sigma^\pm)$$

With this choice of the $f_m(\sigma^\pm)$ functions, we can define the *Virasoro Generators* from the equation (1.12) at $\tau = 0$:

$$L_n = -\frac{l}{4\pi^2}\int_0^l d\sigma \quad e^{-\frac{2\pi i}{l}n\sigma} T_{--} = \frac{1}{2}\sum_m \alpha_{n-m}\cdot\alpha_m$$
$$\overline{L}_n = -\frac{l}{4\pi^2}\int_0^l d\sigma \quad e^{+\frac{2\pi i}{l}n\sigma} T_{++} = \frac{1}{2}\sum_m \overline{\alpha}_{n-m}\cdot\overline{\alpha}_m$$

The Hamiltonian can be written now as:

$$H = \frac{2\pi}{l}(L_0 + \overline{L}_0)$$

The generator of constant σ-traslations, namely the momentum operator P, becomes:

$$P = -T\int \dot{X}\cdot X' d\sigma = \frac{2\pi}{l}(L_0 - \overline{L}_0)$$

and then, being no special point on a closed string, we require that $L_0 - \overline{L}_0 = 0$.[12]
Inverting the expression of L_n and expliciting the stress-energy tensor, we obtain, being it a real tensor, that:

$$L_n = L^*_{-n}, \quad \overline{L}_n = \overline{L}^*_{-n}$$

Using (1.18) we can compute the Poisson brakets for L_0 and $\overline{L}_0$:

$$\{L_m, L_n\}_{PB} = -i(m-n)L_{m+n} \tag{1.19a}$$
$$\{\overline{L}_m, \overline{L}_n\}_{PB} = -i(m-n)\overline{L}_{m+n} \tag{1.19b}$$
$$\{\overline{L}_m, L_n\}_{PB} = 0 \tag{1.19c}$$

This set of equations have to be satisfied from the Virasoro generators and represent the *Centerless Virasoro Algebra*.[13]

[12] This condition will be important, after the quantization, in building the Fock space of the bosonic string theory.

[13] In literature the equations we wrote identify actually the Witt Algebra, namely the Virasoro algebra modified by a central term.

Chapter 2

Bosonic String Quantization

In this chapter we will discuss the quantization of the bosonic string, in particular we will compute the critical dimension (that in our case will be $D = 26$) of the theory, for which the string could consistently propagate, and finally we will introduce and study the mass spectrum of the bosonic string.

2.1 Canonical Quantization

Let us quantize the bosonic string using the canonical procedure of substituting the Poisson brakets with the Lie brakets and considering the $X^\mu(\sigma,\tau)$ fields as quantum mechanical operators:

$$\{\quad,\quad\}_{P.B.} \to \frac{1}{i}\,[\quad,\quad]_{L.B.}$$

In this way we obtain the following commutators:

$$\begin{aligned}
\left[X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\right] &= 2\pi i\alpha'\eta^{\mu\nu}\delta\left(\sigma-\sigma'\right)\\
\left[X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\right] &= \left[\dot{X}^\mu(\sigma,\tau), \dot{X}^\nu(\sigma',\tau)\right] = 0
\end{aligned}$$

and consequently we obtain:

$$[x^\mu, p^\nu] = i\eta^{\mu\nu} \tag{2.1a}$$
$$[\alpha^\mu_m, \alpha^\nu_n] = [\overline{\alpha}^\mu_n, \overline{\alpha}^\nu_n] = m\delta_{m+n}\eta^{\mu\nu} \tag{2.1b}$$
$$[\overline{\alpha}^\mu_n, \alpha^\nu_n] = 0 \tag{2.1c}$$

where obviously we will have only α^μ_m for the open string and both α^μ_m, $\overline{\alpha}^\mu_m$ for the closed string[1]. The reality condition for the $X^\mu(\sigma,\tau)$ fields now becomes the hermicity condition for the operators, then we have $(\alpha^\mu_m)^\dagger = \alpha^\mu_{-m}$ e $(\overline{\alpha}^\mu_m)^\dagger = \overline{\alpha}^\mu_{-m}$. Rescaling this operators by $\alpha^\mu_m \to \frac{1}{\sqrt{m}}\alpha^\mu_m$, the equation (2.1b) becomes:

$$[\alpha^\mu_{m,}\alpha^{\nu\dagger}_n] = \delta_{m,n}\eta^{\mu\nu} \tag{2.2}$$

We can now interpret the α^μ_m operators as the creation and annihilation operators of the m oscillation mode of the string. We take conventionally α^μ_m as an annihilation operator when $m > 0$ and a creation operator when $m < 0$.

[1]Remember the different solution of the equation of motion for the open and closed string.

What we want to do now is to build the *Fock Space* of the bosonic string theory. The ground state of the Fock Space, can be defined in this way:

$$\alpha_m^\mu |0; p^\mu\rangle = 0 \qquad\qquad m > 0$$
$$\hat{p}^\mu |0; p^\mu\rangle = p^\mu |0; p^\mu\rangle$$

from which we generate all the remaining states of the space, namely the excited bosonic string states, by acting with the oscillator modes operator α_{-m} and $\bar{\alpha}_{-m}$, with $m > 0$[2].
We can now also define the number operator $\hat{N}_m \equiv: \alpha_m \cdot \alpha_{-m} := \alpha_{-m} \cdot \alpha_m$ where the ":...:" is the normal ordering prescription.
We notice now that when we take the equation (2.2) and we impose $m = n$ and $\mu = \nu = 0$, we obtain -1. This facts allow us to write:

$$\langle 0| \alpha_m^0 \alpha_{-m}^0 |0\rangle = -1 \langle 0|0\rangle < 0$$

This kind of states with negative norm are called *Ghosts* [3] and they have to be removed from our Fock space in order to make it a physical Hilbert space. It is possibile to demonstrate (no-ghost theorem) that the ghost states decouple for our physical Hilbert space if and only if the Minkowski space-time dimension is exactly $D = 26$. This value is exactly the critical dimension of the theory because, we will see later more precisely, it is exactly the value for which the theory preserve the Lorentz invariance and also Weyl invariance[4].

Let us now introduce the X^μ fields propagator with their usual definition:

$$\langle X^\mu(\sigma,\tau) X^\nu(\sigma',\tau')\rangle = T[X^\mu(\sigma,\tau) X^\nu(\sigma',\tau')] - :[X^\mu(\sigma,\tau) X^\nu(\sigma',\tau')]:$$

where T is the time ordering operator. Let us define the vacuum state of the theory as a traslationally invariant state, therefore we have $: p^\mu x^\nu := x^\nu p^\mu$, with $p^\mu |0\rangle = 0$.
Using now a new set of variables for a closed string with length l, $(z, \overline{z}) = (e^{2\pi i \frac{(\tau-\sigma)}{l}}, e^{2\pi i \frac{(\tau+\sigma)}{l}}) \in S^1 \times S^1$, we can obtain:

$$\langle X_L^\mu(\overline{z}) X_L^\nu(\overline{w})\rangle = \frac{1}{4}\alpha' \eta^{\mu\nu} \ln \overline{z} - \frac{1}{2}\alpha' \eta^{\mu\nu} \ln(\overline{z} - \overline{w}) \tag{2.3a}$$
$$\langle X_R^\mu(z) X_R^\nu(w)\rangle = \frac{1}{4}\alpha' \eta^{\mu\nu} \ln z - \frac{1}{2}\alpha' \eta^{\mu\nu} \ln(z - w) \tag{2.3b}$$
$$\langle X_R^\mu(z) X_L^\nu(\overline{w})\rangle = \frac{1}{4}\alpha' \eta^{\mu\nu} \ln z \tag{2.3c}$$
$$\langle X_L^\mu(\overline{z}) X_R^\nu(w)\rangle = \frac{1}{4}\alpha' \eta^{\mu\nu} \ln \overline{z} \tag{2.3d}$$

Considering instead the undecomposed $X^\mu(\sigma,\tau)$ fields, we have:

$$\langle X^\mu(z,\overline{z}) X^\nu(w,\overline{w})\rangle = -\frac{\alpha'}{2}\eta^{\mu\nu} \ln((z-w)(\overline{z}-\overline{w}))$$

[2] We notice that this is not the usual "second quantization procedure" because the number of particles stays the same.

[3] This is exactly the same problem of the Faddeev-Popov ghosts in a generic Yang-Mills theory.

[4] The Weyl invariance, in a generic dimension D, is broken at quantum level, i.e. it is an *Anomaly* of the theory, but it can be shown that we can recover it by imposing that $D = 26$.

We can notice that the equations (2.3d) and (2.3c) don't vanish because X_R^μ and X_L^μ have in common the same zero mode operators. If we then define:

$$X_R^\mu(z) = x_R^\mu + \frac{\pi}{l}\alpha' p_R^\mu(\tau - \sigma) + \text{oscill. terms}$$
$$X_L^\mu(\overline{z}) = x_L^\mu + \frac{\pi}{l}\alpha' p_L^\mu(\tau + \sigma) + \text{oscill. terms}$$

with also the conditions:

$$[x_R^\mu, p_R^\nu] = [x_L^\mu, p_L^\nu] = i\eta^{\mu\nu}$$
$$[x_R^\mu, p_L^\nu] = [x_L^\mu, p_R^\nu] = 0$$

we obtain that the mixed propagators vanish while the only non vanishing propagators becomes:

$$\langle X_L^\mu(\overline{z}) X_L^\nu(\overline{w})\rangle = -\frac{1}{2}\alpha'\eta^{\mu\nu}\ln(\overline{z} - \overline{w})$$
$$\langle X_R^\mu(z) X_R^\nu(w)\rangle = -\frac{1}{2}\alpha'\eta^{\mu\nu}\ln(z - w)$$

In the open string case with length l, we can analogously obtain:

$$\langle X^\mu(z,\overline{z}) X^\nu(w,\overline{w})\rangle_{NN,DD} = -\frac{\alpha'}{2}\left\{\ln|z-w|^2 \pm \ln|z-\overline{w}|^2\right\}\eta^{\mu\nu}$$
$$\langle X^\mu(z,\overline{z}) X^\nu(w,\overline{w})\rangle_{ND,DN} = -\frac{\alpha'}{2}\left\{\ln\left|\frac{\sqrt{z}-\sqrt{w}}{\sqrt{z}+\sqrt{w}}\right|^2 \pm \ln\left|\frac{\sqrt{z}-\sqrt{\overline{w}}}{\sqrt{t}+\sqrt{\overline{w}}}\right|^2\right\}\eta^{\mu\nu}$$

where with ND we denotes the Neumann boundary condition for the first extremal point and the Dirichlet boundary condition for the second extremal point of the open string.

After the introduction of the canonical quantization for the bosonic string, let us now consider the Virasoro generators and let us quantize them. We remember that at classical level we had that: $T_{++} = T_{--} = 0$. The same condition expressed with the Fourier components of the stress-energy tensor, becomes: $L_n = \overline{L}_n = 0$. At quantum level we have to promote also L_n and $\overline{L}_n$ to operators. Indeed we saw in the previous section, that they can be written as functions of the creation and annihilation operators α_n^μ. Let us now notice that the quantization procedure reveals an ambiguity for a particular operator among all the L_n operators: the L_0 operator. At classical level we define:

$$L_0 = \frac{1}{2}\sum_{n=-\infty}^{+\infty} \alpha_{-n}\cdot\alpha_n$$

but, at quantum level, L_0 has to be defined using the normal ordering prescription. We see now that the normal ordering prescription causes an ambiguity due to the fact that: $[\alpha_n^\mu, \alpha_{-n}^\mu] \propto a \neq 0$, where a is a generic constant. This can also be thought as an ambiguity in the definition of the L_0 operator itself. We are thus allowed to redefine in the following way the L_0 operator: $L_0 \to L_0 + a$.
The definition, at quantum level, of L_n thus becomes:

$$L_n = \frac{1}{2}\sum_{n=-\infty}^{+\infty} :\alpha_{n-m}\cdot\alpha_m:$$

from which we obtain:

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{+\infty} \alpha_{-m} \cdot \alpha_m$$

where the constant coming from the commutator has been absorbed by redefining, as seen above, the L_0 operator.
If we compute explicitly the commutator between L_n ed L_m, we can find that[5]:

$$[L_m, L_n] = (m-n)\, L_{m+n} + \frac{c}{12} m \left(m^2 - 1\right) \delta_{m+n}$$

where c is called *Central Charge*. This is the relation that defines the Virasoro algebra and, as we saw in (1.19), it represents the so called *Central Extension* of the Witt algebra. We can in general write a central extension of an algebra g as $\hat{g} = g \bigoplus \mathbb{C}c$, caraterized by the following commutators:

$$\begin{aligned} &[x, y]_{\hat{g}} = [x, y]_g + cp\,(x, y) \qquad x, y \in g \\ &[x, c]_{\hat{g}} = 0 \\ &[c, c]_{\hat{g}} = 0 \end{aligned}$$

with c that belongs to the center of $\hat{g}$, so it commutes with all the other generators. We can notice that, thanks to the Schur's Lemma, c is a constant in any irreducile representation of the algebra $\hat{g}$. The value of c has an important physical meaning because it can be shown that, in a theory with d free bosons, $c = \eta^{\mu}{}_{\mu} = d$.

Let us now come back to the Virasoro generators. We see that, at quantum level, the condition $L_n = 0$, $\forall n$, cannot hold for all the states $|\psi\rangle$ of our Fock Space. Indeed we see that:

$$\langle \phi \left| [L_n, L_{-n}] \right| \phi \rangle = \langle \phi \left| 2nL_0 \right| \phi \rangle + \frac{c}{12} n \left(n^2 - 1\right) \langle \phi | \phi \rangle \neq 0$$

The best we can do therefore is to require that:

$$\begin{aligned} L_n \,|phys\rangle &= 0 \qquad n > 0 \\ (L_0 + a)\,|phys\rangle &= 0 \end{aligned}$$

The conditions we have required are called *Virasoro Conditions*. The L_n operators, when $n \geq 0$, form a closed subalgebra of the Virasoro algebra. Taking the condition $L_n\,|phys\rangle = 0$ for $n > 0$ only, we effectively obtain the physical consistent constraint[6]:

$$\langle phys' \left| L_n \right| phys \rangle = 0 \qquad \forall n \neq 0$$

For closed string, the arguments about $\overline{L}_n$ are exactly analogous.
We notice that if we define a transformation like $U_\delta = e^{2\pi i \frac{\delta}{l}(L_0 - \overline{L}_0)}$, when it acts on the $X^\mu(\sigma, \tau)$ fields, it generates the rigid σ-traslations, indeed:

$$U_\delta{}^{\dagger} X^\mu\,(\sigma, \tau)\, U_\delta = X^\mu\,(\sigma + \delta, \tau)$$

so what we can impose now at quantum level, just like we did at classical level, is that:

$$(L_0 - \overline{L}_0)\,|phys\rangle = 0$$

[5] We will show the complete calculations in the dedicated appendix.

[6] This condition is analogous to the imposition of the Gupta-Bleuler condition when we build the physical QED space states, i.e. $\langle phys' \left| \partial_\mu A^\mu \right| phys \rangle = 0$

Taking now, in the closed string case, the expression of L_0 and $\overline{L}_0$:

$$L_0 \longrightarrow L_0 + a = \sum_{n=1}^{+\infty} \alpha_{-n} \cdot \alpha_n + \frac{1}{2}\alpha_0^2 = N + \frac{\alpha'}{4}p^2$$

with also:

$$m^2 = -p^\mu p_\mu = m_L^2 + m_R^2$$

where:

$$\alpha' m_L^2 = 2(\overline{N} + a)$$
$$\alpha' m_R^2 = 2(N + a)$$

and imposing the condition $L_0 - \overline{L}_0 = 0$, we obtain $m_L^2 = m_R^2$ and then $N = \overline{N}$. This condition is called *Level Matching Condition*. We notice that the mass of the string in the ground state (when $N = \overline{N} = 0$) is determined by the normal ordering constant a. The final mass relation for a closed string is then:

$$\alpha' m^2 = 4(N + a)$$

For the open string, if we define the number operator as:

$$N = \sum_{n=1}^{+\infty} (\alpha_{-n}^\mu \alpha_{\mu,n} + \alpha_{-n}^i \alpha_{i,n}) + \sum_{r \in \mathbb{N}_0 + \frac{1}{2}} \alpha_{-r}^a \alpha_{a,r}$$

where μ label the NN directions, i the DD directions and a the ND and DN directions. Using again the expression for $L_0 + a$ and imposing the Virasoro constraints, we can obtain:

$$\alpha' m^2 = N + \alpha'(T\Delta X)^2 + a$$

where $(\Delta X)^2 = \Delta X^i \Delta X_i$ is the distance between two ends of the open string in the DD directions.

2.2 Light-Cone Coordinates Quantization

Now we want to introduce, just like we did for the world-sheet, the light-cone coordinates for the Minkowski space-time. With this choice of the coordinates we will be able to fix the gauge but, on the other side, we will lose the manifest Lorentz invariance. We will show that we will recover this invariance (not manifestly) by choosing the dimension of the ambient space-time exactly as $D = 26$, the usual critical dimension of our theory. The light-cone coordinates now are defined as:

$$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^1)$$

In this set of coordinates the non-vanishing metric's entries are:

$$\eta_{+-} = \eta_{-+} = -1, \quad \eta_{ij} = \delta_{ij} \quad \text{per} \quad i, j = 2, ..., D-1$$

We saw, in the previous chapter, that the conformal gauge is a not complete gauge and it admits a residual gauge because of the existence of at least one zero mode of the P operator, therefore we had that the trasformations that put the metric in the form $h_{\alpha\beta} = e^{2\phi}\eta_{\alpha\beta}$ are not uniquely identified. This residual gauge can be seen as a freedom on the choice of the coordinates on the world-sheet, (σ^+, σ^-), indeed they can be consistently redefined as: $\sigma^\pm \to \sigma^\pm + \xi^\pm(\sigma)^\pm$. In terms of the parameters τ and σ, therefore we have:

$$\tau \to \tilde{\tau} = \frac{1}{2}[\tilde{\sigma}^+(\tau+\sigma) + \tilde{\sigma}^-(\tau-\sigma)]$$
$$\sigma \to \tilde{\sigma} = \frac{1}{2}[\tilde{\sigma}^+(\tau+\sigma) - \tilde{\sigma}^-(\tau-\sigma)]$$

We can notice that $\tilde{\tau}$ is consistently a solution of the wave equation:

$$\left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right)\tilde{\tau} = 0$$

then fixing the parameter $\tilde{\tau}$ allows us to fix automatically $\tilde{\sigma}$ and therefore the gauge becomes completely fixed. A possible choice of $\tilde{\tau}$ is to take $\tilde{\tau} \propto X^+$.[7] This is a consistent choice because also $X^\mu(\sigma,\tau)$, in on-shell condition, has to satisfy the same equation. From the equation (1.16) it is possible to determine the proportionality constant obtaining:

$$X^+ = \frac{2\pi\alpha'}{l}p^+\tau$$

From this expression it is obvious that in the X^+ directions all the oscillation modes, except the zero mode, have to vanish[8], therefore, consistently with the previous solutions, we take:

$$\alpha_0^+ = \overline{\alpha}_0^+ = \sqrt{\frac{\alpha'}{2}}p^+ \quad \text{(closed string)}, \quad \alpha_0^+ = \sqrt{2\alpha'}p^+ \quad \text{(open string)}$$

In the light-cone gauge the action becomes:

$$S = \frac{1}{4\pi\alpha'}\int d\tau d\sigma((\dot{X}^i)^2 - (X'^i)^2) - \int d\tau p^+\dot{q}^- = \int d\tau L$$

where q^μ has been defined in the (1.17). The canonical momenta now are:

$$p_- = -p^+ = \frac{\partial L}{\partial \dot{q}^-}$$
$$\Pi^i = \frac{\partial L}{\partial \dot{X}^i} = \frac{1}{2\pi\alpha'}\dot{X}^i$$

The canonical Hamiltonian becomes:

$$H_{can} = p_-\dot{q}^- + \int_0^l d\sigma \Pi_i \dot{X}^i - L = \frac{1}{4\pi\alpha'}\int_0^l d\sigma((\dot{X}^i)^2 + (X'^i)^2)$$

Using now the constraints $(\dot{X}^\mu \pm X'^\mu)^2 = 0$, we can express X^- as a function of the transverse coordinate X^i and we obtain:[9]

$$\partial_\pm X^- = \frac{l}{2\pi\alpha' p^+}(\partial_\pm X^i)^2$$

[7] It can be shown that, in this gauge, the theory becomes unitary and can be described only through physical degrees of freedom. It can be shown also that the ghosts decouple from the physical Hilbert states space.

[8] Obviously the $X^\pm$ fields have to satisfy the corresponding boundary conditions for each kind of string, in particular, for the open string, they has to satisfy the Neumann condition

[9] Let us specify that $\partial_\pm = \frac{\partial}{\partial\sigma^\pm}$ and not $\frac{\partial}{\partial X^\pm}$

By summing up now $\partial_+ X^-$ and $\partial_- X^-$, we obtain:

$$\partial_\tau X^- = \frac{l}{2\pi\alpha' p^+}((\dot{X}^i)^2 + (X'^i)^2)$$

and knowing that $p^\mu = \frac{1}{2\pi\alpha'}\int d\sigma \dot{X}^\mu$, we find:

$$p^- = \frac{l}{2\pi\alpha' p^+} H_{can} \tag{2.4}$$

For the closed string, by subtracting $\partial_+ X^-$ and $\partial_- X^-$, and integrating in $d\sigma$, we can obtain:

$$\int_0^l d\sigma X'^i \Pi^i = \frac{1}{2\pi\alpha'}\int_0^l d\sigma X'^i \dot{X}^i \overset{X(0)=X(l)}{=} 0$$

This is the usual condition for the rigid σ-traslations in the closed string case.

The dynamical variables of the theory expressed with the light-cone coordinates are finally: p_-, q^-, X^i, Π^i. Using now the oscillators expansion of the previous chapter, let us impose the canonical commutation relations:

$$\begin{aligned}
\left[q^-, p^+\right] &= -i \\
\left[q^i, p^j\right] &= i\delta^{ij} \\
\left[\alpha_n^i, \alpha_m^j\right] &= n\delta^{ij}\delta_{n+m,0} \\
\left[\overline{\alpha}_n^i, \overline{\alpha}_m^j\right] &= n\delta^{ij}\delta_{n+m,0}
\end{aligned}$$

We notice now that, writing the Hamiltonian as a function of the α_n^μ operators, we find the same ambiguity that we have already seen in the L_0 expression. So, if we define the Hamiltonian with the normal ordering prescription, we can obtain:

$$H_{can} = \frac{2\pi}{l}\left(\sum_{n>0}(\alpha_{-n}^i\alpha_n^i + \overline{\alpha}_{-n}^i\overline{\alpha}_n^i) + a + \overline{a}\right) + \frac{\pi\alpha'}{l}p^i p^i \qquad \text{(Closed Strings)}$$

$$H_{can} = \frac{\pi}{l}\sum_{n>0}\left(\alpha_{-n}^i\alpha_n^i + a\right) + \frac{\pi\alpha'}{l}\sum_{NN}p^i p^i + \frac{1}{4\pi\alpha' l}\sum_{DD}(x_1^i - x_2^i)^2 \quad \text{(Open Strings)}$$

where a and $\overline{a}$ are the usual normal ordering constant and the sum is done over the NN and DD directions for the open string [10].

Let us now concentrate on the closed string and let us compute, with a particular mathematical trick, the normal ordering constant. We can thus now write explicitly the normal ordered expression of the operators and put in evidence the normal ordering constant that comes out from the non vanishing commutator:

$$\sum_{n\neq 0}\alpha_{-n}\alpha_n = \sum_{n\neq 0} :\alpha_{-n}\alpha_n: + \sum_{n=1}^{+\infty} n = 2\left(\sum_{n=1}^{\infty}\alpha_{-n}\alpha_n + \frac{1}{2}\sum_{n=1}^{\infty} n\right) \tag{2.5}$$

We clearly see that in the last term, the second series diverges so it has to be regolarized using a cut-off parameter for which we will take the limit at infinity. But let us now for a moment consider the following series $\sum_{n=1}^{\infty} n^{-s} \equiv \zeta(s)$, i.e. the definition of the Riemann Zeta-function. The zeta-function converges when

[10] It can be shown that the number mode n for the NN and DD conditions can assume only integer values, while for the mixed conditions DN and ND it can assume only semi-integer values

$\Re(s) > 1$ but it admits a unique analytic extension at $s = -1$ and it can be shown that $\zeta(-1) = -\frac{1}{12}$. A more general expression of the analytic extension of the zeta-function given by:

$$\sum_{n=0}^{\infty} (n+q)^{-s} = \zeta(-1,q) = -\frac{1}{12}\left(6q^2 - 6q + 1\right)$$

By taking now the equation (2.4), the Hamiltonian in the light-cone gauge and the mass operator, $m^2 = 2p^+p^- - p^ip^i$, we can compute for a closed string[11]:

$$m^2 = m_L^2 + m_R^2$$

with

$$\alpha' m_L^2 = 2\left(\overline{N}_{tr} - \underbrace{\frac{1}{24}(d-2)}_{=a}\right), \quad \alpha' m_R^2 = 2\left(N_{tr} - \frac{1}{24}(d-2)\right)$$

and

$$m_L^2 = m_R^2$$

From this results we obtain again:

$$N_{tr} = \overline{N}_{tr}$$

where this time the level-matching condition take into account only the transverse directions. The final formula that we can compute for the mass operator of a closed string is:

$$\alpha' m^2 = N_{tr} - \frac{d-2}{24} \tag{2.6}$$

We can now use this general formula to compute the mass spectrum of every string physical state and that is what we are going to discuss in the next section.

2.3 String Spectrum

We have seen in the previous section that, in the light-cone gauge, the string oscillation states are obtained when the transverse creation operators act on the ground state. The excitated states must clearly remain invariant under a subgroup of the d-dimensional Lorentz group, called *Little Group*. In the case of a massive particle, it is always possible to choose a Lorentz boost in order to go to its rest frame, i.e. where $p^\mu = (m, 0, ..., 0)$, with $p^2 = -m^2$, therefore the little group for this kind of particles will be the isotropy group of the momentum p^μ, i.e. $SO(d-1)$. This means that, for a massive particle, we can label every possible excitation with the irreducible representation of the $SO(d-1)$ group. We can do the same argument for a massless particle, for which now the Lorentz boost can only take us in the frame where $p^\mu = (E, 0, ..., E)$, with $p^2 = 0$. The little group for a massless particle will be then $E(d-2)$, i.e. the $(d-2)$-dimensional Euclidean group. The real group of isostropy, in this case, will not be the entire Euclidean group, but only the subgroup that is connected to the identity transformation,

[11]We have to remember that the canonical Hamiltonian is identically null

i.e. $SO(d-2)$. Now let us concentrate on the Lorentz invariance of the theory, in particular, let us compute the normal ordering constant a and the dimension of the ambient space-time d.

Let us write the generators of the Lorentz algebra using the creation and annihilation operators α_n^μ:

$$M^{\mu\nu} = x^\mu \rho^\nu - x^\nu p^\mu - i \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu \right)$$

Those have to satisfy the commutation relation for the generators:

$$[M^{\mu\nu}, M^{\rho\sigma}] = \eta^{\nu\rho} M^{\mu\sigma} + \eta^{\mu\sigma} M^{\nu\rho} - \eta^{\mu\rho} M^{\nu\sigma} - \eta^{\nu\sigma} M^{\mu\rho}$$

Choosing now the light-cone gauge and taking the commutator betweeen M^{i-} and M^{j-}, it can be shown that the results is:

$$0 = \left[M^{-i}, M^{-j}\right] = \frac{1}{\alpha p^{+2}} \sum_{n=1}^{\infty} \left(\alpha_{-n}^i \alpha_n^j - \alpha_{-n}^j \alpha_n^i \right) \cdot \left\{ n \left[1 - \frac{(D-2)}{24} \right] + \frac{1}{n} \left[\frac{D-2}{24} + a \right] \right\}$$

and this is true $\forall n \in \mathbb{Z}^+$ if and only if:

$$D = 26, \qquad a = -1$$

Let us now consider the closed string spectrum. We see that, in this case, we have both left and right excitation states, so the final complete excited state will be the tensor product of the single left and right representation. Clearly, because of the level-matching condition, the left and right states must have the same excitation number.
The equation for the closed string mass spectrum is:

$$\alpha' m^2 = 4(N_{tr} - a)$$

When $N_{tr} = 0$, namely when we consider the ground state, we notice that a *Tachyon* appear[12], namely a particle with negative squared mass. Its value is: $\alpha' m^2 = -4a$.
As we said right above, we can write the first excited state as the action of both left and right creation operators on vacuum state, i.e. $\alpha_{-1}^i \overline{\alpha}_{-1}^j |0\rangle$. The value of the mass corresponding to this state is: $\alpha' m^2 = 4(1-a)$. If we decompose it in the different irreducible representations of the $SO(d-2)$ group, we obtain:

$$\alpha_{-1}^i \overline{\alpha}_{-1}^j |0,p\rangle = \underbrace{\left(\alpha_{-1}^{(i} \overline{\alpha}_{-1}^{j)} - \frac{1}{d-2} \delta^{ij} \alpha_{-1}^k \overline{\alpha}_{-1}^k \right)}_{\text{Traceless Symmetric Tensor}} |0,p\rangle + \underbrace{\alpha_{-1}^{[i} \overline{\alpha}_{-1}^{j]}}_{\text{Antisymmetric Tensor}} |0,p\rangle +$$

$$+ \underbrace{\frac{1}{d-2} \delta^{ij} \alpha_{-1}^k \overline{\alpha}_{-1}^k |0,p\rangle}_{\textit{Singlet}}$$

[12]This is actually a problem of the bosonic string theory, indeed, when we will consider Superstrings, Tachyons will not appear in its mass spectrum.

Taking now $D = 26$ and $a = 1$, we can identify the traceless symmetric tensor as a massless particle with spin 2, called *Graviton*, the scalar term as a scalar massless field called *Dilaton*, and finally the last part as an antisymmetric tensor field. The Regge trajectory for closed string now becomes $j_{max} = \frac{1}{2}\alpha' m^2 + 2$.

2.4 Appendix: the Virasoro Algebra

In this appendix we will explicitly compute the algebra satisfied by the Virasoro generators. We notice that we will use the euclidean metric as our conventional metric, therefore we won't have to distinguish between upper and lower indices. Let us start computing the useful commutator:

$$\left[\alpha_m^i, L_n\right] = \frac{1}{2}\sum_{p=-\infty}^{+\infty}\left[\alpha_m^i, :\alpha_p^j\alpha_{n-p}^j:\right]$$

Let us drop the normal ordering symbol because α_m^i obviously commutes with complex numbers. Use the commutators' property $[A, BC] = [A, B]\,C + B\,[A, C]$ we get:

$$\begin{aligned}\left[\alpha_m^i, L_n\right] &= \frac{1}{2}\sum_{p=-\infty}^{+\infty}\left\{\left[\alpha_m^i, \alpha_p^j\right]\alpha_{n-p}^j + \alpha_p^j[\alpha_m^i, \alpha_{n-p}^j]\right\} \\ &= \frac{1}{2}\sum_{p=-\infty}^{+\infty}\left\{\delta_{m+p}\alpha_{n-p}^j + \alpha_p^j\delta_{m+n-p}\right\}m\delta^{ij} = m\alpha_{m+n}^i\end{aligned}$$

Let us now compute:

$$\begin{aligned}[L_m, L_n] &= \frac{1}{2}\sum_{p=-\infty}^{+\infty}\left[:\alpha_p^i\alpha_{m-p}^i:, L_n\right] = \frac{1}{2}\sum_{p=-\infty}^{0}\left[\alpha_p^i\alpha_{m-p}^i, L_n\right] + \frac{1}{2}\sum_{p=1}^{+\infty}\left[\alpha_{m-p}^i\alpha_p^i, L_n\right] \\ &= \frac{1}{2}\sum_{p=-\infty}^{0}\left\{(m-p)\,\alpha_p^i\alpha_{m+n-p}^i + p\alpha_{n+p}^i\alpha_{m-p}^i\right\} \\ &\quad + \frac{1}{2}\sum_{p=1}^{+\infty}\left\{(m-p)\,\alpha_{m+n-p}^i\alpha_p^i + p\alpha_{m-p}^i\alpha_{n+p}^i\right\}\end{aligned}$$

Let us now change variable in the second and fourth term, so let us define $q = p+n$:

$$\begin{aligned}[L_m, L_n] = \frac{1}{2}\Bigg\{&\sum_{p=-\infty}^{0}(m-p)\,\alpha_p^i\alpha_{m+n-p}^i + \sum_{q=-\infty}^{n}(q-n)\,\alpha_q^i\alpha_{m+n-q}^i \\ &+ \sum_{p=1}^{+\infty}(m-p)\,\alpha_{m+n-p}^i\alpha_p^i + \sum_{q=n+1}^{+\infty}(q-n)\,\alpha_{n+m-q}^i\alpha_q^i\Bigg\}\end{aligned}$$

Considering now $n > 0$ (the $n \leq 0$ case is analogous), we can get, working a little bit on the series:

$$\begin{aligned}[L_m, L_n] = \frac{1}{2}\Bigg\{&\sum_{q=-\infty}^{0}(m-n)\,\alpha_q^i\alpha_{m+n-q}^i + \sum_{q=1}^{n}(q-n)\,\alpha_q^i\alpha_{m+n-q}^i \\ &+ \sum_{q=n+1}^{+\infty}(m-n)\,\alpha_{m+n-q}^i\alpha_q^i + \sum_{q=1}^{n}(m-q)\,\alpha_{m+n-q}^i\alpha_q^i\Bigg\}\end{aligned}$$

We notice now that all terms are already normal ordered except for the second term, so let us compute the expression of the second term only:

$$\sum_{q=1}^{n}(q-n)\,\alpha_q^i\alpha_{m+n-q}^i = \sum_{q=1}^{n}(q-n)\,\alpha_{m+n-q}^i\alpha_q^i + \sum_{q=1}^{n}(q-n)\,qd\delta_{m+n}$$

where we have used the fact that $\delta^i_i = d$. Now we can then write:

$$[L_m, L_n] = \frac{1}{2} \sum_{q=-\infty}^{+\infty} (m-n) : \alpha^i_q \alpha^i_{m+n-q} : + \frac{1}{2} d \sum_{q=1}^{n} \left(q^2 - nq\right) \delta_{n+m}$$

Using now the formulas:

$$\sum_{q=1}^{n} q^2 = \frac{1}{6} n (n+1)(2n+1) \qquad \text{and} \qquad \sum_{q=1}^{n} q = \frac{1}{2} n (n+1)$$

we finally obtain the Virasoro algebra:

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{d}{12} m \left(m^2 - 1\right) \delta_{m+n}$$

We notice, as we said previously, that the central charge is exactly $c = d$, where d is the space-time dimension. Clearly the proof would have also worked using the Minkowski metric and the central charge would have been $c = \eta^\mu{}_\mu = d$.
In order to find the exact value of the dimension of the ambient space-time, we can consider another important interpretation of the central charge and this is connected to the fact that Weyl invariance is anomalous at quantum level. When we try to quantize the theory through a Path Integral quantization and to fix the gauge of reparametrizations and the Weyl invariance, we need to introduce two anti-commuting scalar fields, called ghosts. It can be shown that the algebra of the creation and annihilation operators of the ghost fields it is exactly the Virasoro algebra with a central charge $c_{gh} = -26$. If we now consider a theory made of bosonic fields and ghost fields, and we compute the Virasoro algebra, considering the ghosts contributions to the Virasoro generators, we obtain that the central charge term (that we can for simplicity call "anomaly") cancels out if and only if $c = d = -c_{gh} = 26$, that was exactly the value we computed by requiring the Lorentz invariance of the theory. Therefore what we obtained is that ghosts contributes to the Weyl anomaly but, choosing the correct number of dimensions of our space-time, i.e. the number of our bosonic fields, we can preserve the Weyl invariance also at quantum level, making the anomaly vanish.

Chapter 3

Introduction to Conformal Field Theory

In this chapter we would like to introduce conformal field theory (CFT) which is the main tool we can use to study the perturbative string theory. Feynman diagrams in string theory are substituted by some surfaces with a suitable topology that are fundamental to compute the scattering amplitude of the string processes. That is because, in practice, the mechanism of computing scattering processes, as we will see more in detail later, consists in computing the correlation functions of a certain type of CFT defined on this particular surfaces. For this reasons we will concentrate firstly on CFTs defined on the most simple surface we can take: the complex plane.

3.1 General introduction

Under a general coordinate trasformation, the metric transforms as:

$$g_{\mu\nu} \longrightarrow g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x)$$

Let us now consider a subgroup of this kind of transformations that leaves the metric invariant up to a rescaling, namely:

$$g'_{\mu\nu}(x') = \Omega(x) g_{\alpha\beta}(x)$$

Those transformations are called *Conformal Transformations*. We notice that the Poincaré group, for flat space-time only, is a subgroup of the group of the conformal transformations, indeed we can obtain it by imposing $\Omega(x) = 1$.
In string theory we are interested in 2-dimensional conformal transformations because the 2-dimensional conformal field theory is the theory that describes the dynamic of the string world-sheet. Let us take a general infinitesimal trasformation $x'^\mu = x^\mu + \epsilon^\mu$. The metric will thus transform as:

$$\delta g_{\mu\nu} = -(\partial^\lambda \epsilon_\mu g_{\lambda\nu} + \partial^\lambda \epsilon_\nu g_{\lambda\mu}) - \epsilon^\lambda \partial_\lambda g_{\mu\nu}$$

Taking now the flat euclidean metric we obtain that the first correction in ϵ of the metric is $\delta_{\mu\nu} \to \delta_{\mu\nu} - (\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu)$. Now if we require that we are acting with conformal transformations, we expect that the variation must be proportional to the flat metric itself, namely:

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = f\,(x)\,\delta_{\mu\nu}$$

The function $f(x)$ can be found contracting both sides by $\delta_{\mu\nu}$:

$$f(x) = \frac{2}{d}(\partial \cdot \epsilon)$$

so the final equation for the metric variation becomes:

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon)\delta_{\mu\nu} \tag{3.1}$$

Acting now with ∂^μ on both sides we can obtain:

$$\Box \epsilon_\nu + \left(1 - \frac{2}{d}\right)\partial_\nu(\partial \cdot \epsilon) = 0 \tag{3.2}$$

Acting this time with $\Box$ on the equation (3.1) we can obtain another useful expression:

$$\partial_\mu \Box \epsilon_\nu + \partial_\nu \Box \epsilon_\mu = \frac{2}{d}\delta_{\mu\nu}\Box(\partial \cdot \epsilon)$$

Now, combining this equation with the previous equation (3.2), we can finally obtain the constraint equation for the parameter ϵ:

$$[\delta_{\mu\nu}\Box + (d-2)\partial_\mu\partial_\nu]\partial \cdot \epsilon = 0$$

The last equation we will need can be obtained by contracting with $\delta_{\mu\nu}$ the previous equation, namely:

$$(d-1)\Box(\partial \cdot \epsilon) = 0$$

For $d > 2$ we can see that $(\partial \cdot \epsilon)$ has to satisfy the equations:

$$\Box(\partial \cdot \epsilon) = 0$$
$$\partial_\mu\partial_\nu(\partial \cdot \epsilon) = 0$$

therefore it can be at most a linear function of x^μ, namely: $(\partial \cdot \epsilon) = A + Bx^\mu$. From this result it is clear that instead ϵ_μ will be a quadratic function in x^μ, therefore its general expression will be:

$$\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho$$

It can be shown now that the only possible solutions we can get are:

1. $\epsilon^\mu = a^\mu$, where a^μ is constant. This corresponds to constant traslations.

2. $\epsilon^\mu = \omega^\mu{}_\nu x^\nu$ where $\omega^\mu{}_\nu$ is an antisymmetric tensor. This corresponds to infinitesimal Lorentz trasformations.

3. $\epsilon^\mu = \lambda x^\mu$, $\lambda \in \mathbb{C}$. This corresponds to scale trasformations.

4. $\epsilon^\mu = b^\mu x^2 - 2x^\mu(b \cdot x)$. This kind of transformations are called *special conformal transformations*. In this case the corresponding finite trasformations are: $x^\mu \to x'^\mu = \frac{x^\mu + x^2 a^\mu}{1 + 2x \cdot a + 2x^2 a^2}$

This set of transformations forms the d-dimensional conformal group which, it can be shown, is isomorphic to the $SO(2,d)$ group.

Let us now see what happens if we take $d = 2$. By taking the the euclidean flat metric, the equation (3.1) reduces to:

$$\partial_1 \epsilon_1 = \partial_2 \epsilon_2 \tag{3.3a}$$
$$\partial_1 \epsilon_2 = -\partial_2 \epsilon_1 \tag{3.3b}$$

Defining then a new set of variables $z, \overline{z} = x_1 \pm i x_2$ and $\epsilon, \overline{\epsilon} = \epsilon_1 \pm \epsilon_2$ the equations in (3.3) becomes:

$$\partial \overline{\epsilon} = 0 \qquad\qquad \overline{\partial} \epsilon = 0$$

where we have defined $\overline{\partial} \equiv \partial_{\overline{z}}$. This means that ϵ can be an arbitrary function of z but independent from $\overline{z}$ and viceversa the $\overline{\epsilon}$. It follows that the conformal transformations for $d = 2$ are exactly the analytic coordinate transformations. We can then write:

$$\epsilon(z) = -\sum a_n z^{n+1}$$

and the generators corresponding to this kind of transformations are:

$$L_n = -z^{n+1}\partial_z$$

It can be shown that this generators satisfy the following relations:

$$\begin{aligned} [L_n, L_m] &= (m-n)L_{m+n} \\ \left[\overline{L}_n, \overline{L}_m\right] &= (m-n)\overline{L}_{m+n} \\ \left[\overline{L}_n, L_m\right] &= 0 \end{aligned}$$

but this is exactly the Virasoro algebra that we derived in Chapter 1.
What we can conclude therefore is that, in $d = 2$ and for an euclidean flat metric, the conformal transformations group is an infinite-dimensional group and the algebra of the conformal transformations group is the Virasoro algebra.

3.2 Radial Quantization

Let us now introduce the basical concepts of CFT. We found previously that, only for the Euclidean flat metric, the Poincaré group is a subgroup of the conformal group and, in $d = 2$, the algebra of the conformal group is the Virasoro algebra. It is indeed more useful to work with a flat Euclidean metric, so what we have to do is to make an analytic continuation of the world-sheet metric from Minkowskian to Euclidean. Let us therefore consider a Wick rotation, i.e. $\tau \to -i\tau$, or, for the light-cone coordinates, $\sigma^{\pm} = \tau \pm \sigma \to -i(\tau \pm i\sigma)$.
We can define complex coordinates on the cylinder of circumference l (namely the world-sheet of a free closed string):

$$\begin{aligned} w &= \tau - i\sigma \\ \overline{w} &= \tau + i\sigma \end{aligned} \qquad \text{with} \quad w \sim w + l$$

and then define the map from the cylinder to the complex plane via conformal transformation:

$$z = e^{\frac{2\pi}{l} w} = e^{\frac{2\pi}{l}(\tau - i\sigma)} \tag{3.4a}$$
$$\overline{z} = e^{\frac{2\pi}{l} \overline{w}} = e^{\frac{2\pi}{l}(\tau + i\sigma)} \tag{3.4b}$$

We see also that σ-traslations become rotations and τ-traslations become scale trasformation. An important point to stress is that now the time-ordering operator is replaced by the radial-ordering operator defined as[1]:

$$R(\phi_1(z)\phi_2(w)) = \begin{cases} \phi_1(z)\phi_2(w) & \text{for} \quad |z| > |w| \\ \phi_2(w)\phi_1(z) & \text{for} \quad |w| > |z| \end{cases}$$

We can then define the equal radius commutator as:

$$[\phi_1(z), \phi_2(w)]_{|z|=|w|} = \lim_{\delta \to 0} \{\phi_1(z)\phi_2(w)_{|z|=|w|+\delta} - \phi_2(w)\phi_1(z)_{|z|=|w|-\delta}\} \tag{3.5}$$

We notice now that lines at equal time τ are mapped into circles around the origin, for this reason the integration over σ will be now the contour integral around the origin, namely:

$$\left[\int d\sigma B, A\right] = \oint_{C_w} dz R(B(z)A(w)) \tag{3.6}$$

The main objects with which we build a conformal field theory are the conformal fields, also called *Primary Fields*, $\phi(z,\overline{z})$. Under a generic conformal trasformation $z \to z' = f(z)$, $\overline{z} \to \overline{z}' = f(\overline{z})$, primary fields transform as tensors:

$$\phi(z,\overline{z}) \to \phi'(z',\overline{z}') = \left(\frac{\partial z'}{\partial z}\right)^{-h} \left(\frac{\partial \overline{z}'}{\partial \overline{z}}\right)^{-\overline{h}} \phi(z,\overline{z})$$

Under infinitesimal transformations $z' = z + \xi(z)$, $\overline{z}' = \overline{z} + \xi(\overline{z})$ we obtain then:

$$\phi(z,\overline{z}) \to \phi'(z,\overline{z}) = \phi(z,\overline{z}) + \delta_{\xi,\overline{\xi}}\phi(z,\overline{z})$$

with:

$$\delta_{\xi,\bar{\xi}}\phi(z,\overline{z}) = -(h\partial\xi + \bar{h}\bar{\partial}\bar{\xi} + \xi\partial + \bar{\xi}\bar{\partial})\phi(z,\overline{z}) \tag{3.7}$$

where we defined $\partial \equiv \frac{\partial}{\partial z}$ and $\overline{\partial} \equiv \frac{\partial}{\partial \overline{z}}$. h and $\overline{h}$ are[2] called conformal weight of ϕ under analytic and anti-analytic transformations. Tensors with $\overline{h} = 0$ (or $h = 0$) are called holomorphic (or anti-holomorphic) tensors.
Fields that are functions purely of z or $\overline{z}$ are called *Chiral Fields*[3]. We notice also that (anti-)chiral fields have to transform necessarily with $\overline{h} = 0$ ($h = 0$). We can now link chiral fields[4] on the cylinder and chiral fields on the complex plane through the expression:

$$\phi_{plane}(z) = \left(\frac{l}{2\pi}\right)^h z^{-h}\phi_{cyl}(w) \tag{3.8}$$

If now ϕ_{cyl} admit a mode expansion, we can write:

$$\phi_{cyl}(w) = \left(\frac{2\pi}{l}\right)^h \sum_{n\in\mathbb{Z}} \phi_n e^{-\frac{2\pi}{l}nw}$$

and substituting this expression into (3.8), we obtain:

$$\phi_{plane}(z) = \sum_{n\in\mathbb{Z}} \phi_n z^{-n-h} \tag{3.9}$$

[1]For anti-commuting fields there will be a relative minus sign
[2]Note that $\overline{h}$ is not the complex-conjugate of h
[3]This is clear from (3.4) where we see that $z = f(\sigma^-)$ and $\overline{z} = f(\sigma^+)$
[4]The generalization to arbitrary primary fields is immediate

From now on we will consider all the fields as fields defined on the complex plane. The inverse of the equation (3.9) is:

$$\phi_n = \oint_{C_0} \frac{dz}{2\pi i} \phi\,(z)\, z^{n+h-1} \tag{3.10}$$

where C_0 is an arbitrary contour around the origin and the integration is counterclockwise.
Let us now have a look at the stress-energy tensor for the conformal field theory. We saw in Chapter 1 that, for a conformally invariant theory, the stress-energy tensor is traceless, i.e $T^{\alpha}{}_{\alpha} = 0$. Expressed in the light-cone coordinates $\sigma^{\pm}$ and then in the conformal coordinates, we obtain:

$$T_{z\overline{z}} = 0$$

The energy-momentum conservation becomes:

$$\partial_{\overline{z}} T_{zz} = 0 \qquad\qquad \partial_z T_{\overline{z}\overline{z}} = 0$$

so we can use the notation $T(z) \equiv -2\pi T_{zz}(z)$ and $\overline{T}(\overline{z}) \equiv -2\pi T_{\overline{z}\overline{z}}(\overline{z})$. T and $\overline{T}$ are therefore chiral and anti-chiral fields respectively.
Analogously to what we saw in the first chapter, if $T(z)$ is conserved, also $\xi(z)T(z)$ is a conserved quantity[5], therefore we have an infinite number of conserved charges and this is related to the fact that the conformal algebra in two dimensions is infinite-dimensional. The conserved charge related to the conserved current $\xi(z)T(z)$ is:

$$T_\xi = \oint_{C_0} \frac{dz}{2\pi i} \xi(z) T(z)$$

which generates the infinitesimal conformal transformation $z \to z' = z + \xi(z)$. This means that we can write:

$$\delta_\xi \phi(w) = -[T_\xi, \phi(w)]$$

Using now the definition (3.5) we obtain:

$$\begin{aligned}
\delta_\xi \phi(w) &= -\oint_{|z|>|w|} \frac{dz}{2\pi i} \xi(z) T(z) \phi(w) + \oint_{|z|<|w|} \frac{dz}{2\pi i} \xi(z) \phi(w) T(z) \\
&= -\oint_{C_w} \frac{dz}{2\pi i} \xi(z) T(z) \phi(w)
\end{aligned}$$

where C_w is a contour encircling the point w.
Comparing this result[6] with the expression in (3.7), taking $\bar{\xi} = 0$ and using the Cauchy-Riemann formula:

$$\oint_{C_z} \frac{dw}{2\pi i} \frac{f(w)}{(w-z)^n} = \frac{1}{(n-1)!} f^{(n-1)}(z)$$

we find that the R-ordered operator product with $T(z)$ is:

$$T(z)\phi(w) = \frac{h\phi(w)}{(z-w)^2} + \frac{\partial \phi}{(z-w)} + \text{finite terms} \tag{3.11}$$

[5]We should write $\xi(z)^z T_{zz}(z)$
[6]Remember the identity $[\int d\sigma B, A] = \oint dz R(B(z)A(w))$

this is the operator product expansion (OPE) between the stress-energy tensor and our conformal field. Notice that this equation can be used to define a primary field with conformal weight h.
Analogously we can use the property:

$$[\delta_{\xi_1}, \delta_{\xi_2}] = \delta_{(\xi_2 \partial \xi_1 - \xi_1 \partial \xi_2)}$$

and find:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \text{finite terms} \tag{3.12}$$

This expression can be rewritten in this way:

$$\delta_\xi T(z) = -2\partial\xi(z)T(z) - \xi(z)\partial T(z) - \frac{c}{12}\partial^3\xi(z)$$

We see that the stress-energy tensor transforms as a chiral tensor of weight two under those transformations for which $\partial^3\xi(z) = 0$. Classically we have that $c = 0$ while, as we will see later, here we have that $c \neq 0$. This fact represents the so called *Conformal Anomaly*, which is a purely quantum mechanical effect. We conclude that the stress-energy tensor is not a primary field while instead it is called a *Quasi-Primary Field*. This argument is related to the anomaly in the Virasoro algebra, indeed we will now show the equivalence between the OPE of $T(z)T(w)$ and the Virasoro algebra with central charge c. Let us expand $T(z)$ in modes:

$$T(z) = \sum_n z^{-n-2} L_n$$

The inverse relation becomes:

$$L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z) \tag{3.13}$$

where the L_n's are the Virasoro generators.
We remark that for chiral fields with conformal weight h, the hermitian conjugate is defined as:

$$[\phi(z)]^\dagger = \phi^\dagger\left(\frac{1}{\overline{z}}\right)\frac{1}{\overline{z}^{2h}}$$

and in general for the modes we have:

$$(\phi^\dagger)_{-n} = (\phi_n)^\dagger$$

We see now that, being $T(z)$ Hermitian, the Virasoro generators satisfy the Hermicity condition:

$$(L_n)^\dagger = L_{-n}$$

The Virasoro algebra can be obtained from the equation:

$$\begin{aligned}
[L_m, L_n] &= \oint_{C_0} \frac{dw}{2\pi i} \oint_{C_w} \frac{dz}{2\pi i} z^{m+1} w^{n+1} \left[\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}\right] \\
&= \frac{c}{12} m\left(m^2-1\right)\delta_{n+m,0} + (m-n) L_{m+n}
\end{aligned}$$

It can be also shown that the holomorphic and anti-holomorphic algebras commute:

$$[L_n, \overline{L}_m] = 0$$

As we see in (3.13), L_n is associated to the corresponding infinitesimal transformation $\xi(z) = -z^{n+1}$, therefore the generators L_0, L_1, L_{-1} will generate the infinitesimal transformation $\delta z = \alpha + \beta z + \gamma z^2$. Those are the generators of $sl(2,\mathbb{C})$, the maximal closed finite dimensional subalgebra of the Virasoro algebra. The corresponding finite trasformation of the three generators are:

$$\begin{aligned} L_{-1} &: z \longrightarrow z+\alpha && \text{Translations} \\ L_0 &: z \longrightarrow \lambda z && \text{Scalings} \\ L_{+1} &: z \longrightarrow \frac{z}{1-\beta z} && \text{Special Conformal} \end{aligned}$$

This $SL(2,\mathbb{C})$ group is actually called the *Special Restricted Conformal Group*. The finite transformations can be summarized in the following general expression:

$$z \longrightarrow z' = \frac{az+b}{cz+d} \tag{3.14}$$

with:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$$

We notice that the conformal transformation in (3.14) does not change if we substitute $(a,b,c,d) \to (-a,-b,-c,-d)$, so the actual conformal group we are dealing with is $PSL(2,\mathbb{C}) = SL(2,\mathbb{C})/\mathbb{Z}_2$. This group is the conformal group which we will refer to because it is the only group of globally defined invertible conformal mappings of the Riemann sphere onto itself.

3.3 The Correlation Functions

What we would like now to introduce are the correlation functions in the CFT. The correlation functions are the vacuum expectation values of the R-ordered products of field operators, so, in order to derive their explicit form, we first need to understand which are the symmetries that they have to satisfy. Let us try to justify that the global conformal group we are interested in is exactly the $SL(2,\mathbb{C})$ group. We have to require the regularity of the stress-energy tensor for the in-vacuum asymptotic state $|0\rangle$:

$$\lim_{\substack{z\to 0 \\ (\tau\to-\infty)}} \sum_{n\in\mathbb{Z}} L_n z^{-n-2} |0\rangle = \text{regular}$$

We see that this condition[7] becomes the following condition for the Virasoro generetors:

$$L_n |0\rangle = 0 \qquad \text{for} \quad n > -2$$

Analogously, for the out-vacuum asymptotic state $\langle 0|$, what we get is that:

$$L_n \langle 0| = 0 \qquad \text{for} \quad n < 2$$

We see therefore that the only generators that satisfies the conditions above are exactly the generators L_{-1}, L_0, L_{+1}, namely the $sl(2,\mathbb{C})$ generators are the only ones that annihilate both the in and out-vacuum state.

[7]When T(z) is not regular then L_n must compensate this behaviour, indeed for $n > -2$ the T(z) diverges, therefore L_n, for $n > -2$, must annihilate the in-vacuum state.

We conclude thus that the vacuum state must be invariant under the action of $SL(2,\mathbb{C})$ and the same will be for every correlation functions.
A generic n-points correlation function of n primary fields ϕ_i with conformal weight h_i and $\overline{h}_i$ can be written as:

$$G^{(n)}(z_i,\overline{z}_i) \equiv \langle R(\phi_1(z_1,\overline{z}_1)...\phi_n(z_n,\overline{z}_n))\rangle \qquad \text{with} \quad i=1,...,n$$

We immediately see from the definition that it has to satisfy the following relation for a generic conformal trasformation $z \to w(z)$:

$$\langle\phi_1(w_1,\overline{w}_1)\ldots\phi_n(w_n,\overline{w}_n)\rangle = \prod_{i=1}^{n}\left(\frac{dw}{dz}\right)^{-h_i}_{w=w_i}\left(\frac{d\overline{w}}{d\overline{z}}\right)^{-\overline{h}_i}_{\overline{w}=\overline{w}_i}\langle\phi_1(z_1,\overline{z}_1)\ldots\phi_n(z_n,\overline{z}_n)\rangle$$

Let us initially focus on the two-point correlation function. If we start from the request of translational and rotational invariance, the generic two-points function must depend on the modulus of the difference of the coordinate, namely $r_{ij} = |x_i - x_j|$. With this first consideration, we can write:

$$\langle\phi_i(z,\overline{z})\phi_j(w,\overline{w})\rangle = G_{ij}(z-w)^{-(h_i+h_j)}(\overline{z}-\overline{w})^{-(\overline{h}_i+\overline{h}_j)}$$

Invariance under special conformal transformations require that $h_i = h_j = h$ and $\overline{h}_i = \overline{h}_j = \overline{h}$. The final expression of the two-point correlation function is:

$$\langle\phi_1(z_1,\overline{z}_1)\phi_2(z_2,\overline{z}_2)\rangle = \begin{cases} \frac{C_{12}}{(z_1-z_2)^{2h}(\overline{z}_1-\overline{z}_2)^{2\overline{h}}} & \text{if} \quad h_1=h_2=h \quad \text{and} \quad \overline{h}_1=\overline{h}_2=\overline{h} \\ 0 & \text{otherwise} \end{cases}$$

where C_{12} is a constant which cannot be determined from $SL(2,\mathbb{C})$ invariance but it can be set to $\delta_{1,2}$ by choosing a suitable fields normalization. The same arguments can be applied to the three-points function case and its expression will be:

$$\begin{aligned}\langle\phi_1(z_1,\overline{z}_1)\phi_2(z_2,\overline{z}_2)\phi_3(z_3,\overline{z}_3)\rangle &= C_{123}\frac{1}{z_{12}^{h_1+h_2-h_3}z_{23}^{h_2+h_3-h_1}z_{13}^{h_3+h_1-h_2}} \\ &\times\frac{1}{\overline{z}_{12}^{\overline{h}_1+\overline{h}_2-\overline{h}_3}\overline{z}_{23}^{\overline{h}_2+\overline{h}_3-\overline{h}_1}\overline{z}_{13}^{\overline{h}_3+\overline{h}_1-\overline{h}_2}}\end{aligned}$$

where we have defined $z_{ij} \equiv (z_i - z_j)$ and $\overline{z}_{ij} = (\overline{z}_i - \overline{z}_j)$. Again C_{123} is a constant that depends on the theory we are dealing with[8].
For the four-points function things are more complicated. The explicit form of the four-points function cannot be determined from symmetry properties because we have another invariant term under special conformal transformations, that can contribute. The C_{ijkl} term will not be anymore just a constant, as we saw for the two and three-point functions, but a generic function of this invariant objects called *Anharmonic Ratios*. The anharmonic (or cross) ratio is defined as:

$$\eta = \frac{z_{ij}z_{kl}}{z_{ik}z_{jl}} \qquad \text{with} \qquad z_{ij} = (z_i - z_j)$$

[8]It can be shown that the three-points function always depends only on a single constant that here we called C_{123}.

It can be shown that in two dimensions, for n-points functions, there are $n-3$ independent cross ratios, therefore the four-points function has only one independent cross ratio. The final expression for the four-points function will be then:

$$\langle \phi_1(z_1,\overline{z}_1)...\phi_4(z_4,\overline{z}_4)\rangle = f(\eta,\overline{\eta})\prod_{i<j}^{4} z_{ij}^{h/3-h_i-h_j}\overline{z}_{ij}^{\overline{h}/3-\overline{h}_i-\overline{h}_j}$$

where $h=\sum_{i=1}^{4} h_i$ and the same for $\overline{h}$. Clearly this expression can be simplified by taking certain physical condition, in particular, taking a single field ϕ with conformal weight $h=\overline{h}$, we can obtain:

$$\langle \phi_1(z_1,\overline{z}_1)...\phi_4(z_4,\overline{z}_4)\rangle = f(\eta,\overline{\eta}) \times \left[(z_{12}z_{13}z_{14}z_{23}z_{24}z_{34})^{-2/3h} \times c.c.\right]$$

where c.c. stand for complex conjugate term.

3.3.1 Descendant States

We saw previously that the vacuum state must belong to our Hilbert space and therefore, requiring the regularity of the stress-energy tensor at the origin, it has to satisfy the condition of being annihilated by L_n when $n \geq -1$ for the asymptotic in-state and $n \leq 1$ for the asymptotic out-state.
Primary fields, when they act on the vacuum state, create asymptotic states, namely the eigenstates of the Hamiltonian. In order to see how to build the asymptotic states, let us start by computing, for a primary field $\phi(z,\overline{z})$ with conformal weight $(h,\overline{h})$, when $n \geq -1$, the following commutator:

$$\begin{aligned}
[L_n, \phi(w,\overline{w})] &= \oint_{C_w} \frac{dz}{2\pi i} z^{n+1} T(z)\,\phi(w,\overline{w}) \\
&= \oint_{C_w} \frac{dz}{2\pi i} z^{n+1}\left[\frac{h\phi(w,\overline{w})}{(z-w)^2} + \frac{\partial\phi(w,\overline{w})}{(z-w)} + \dots\right] \\
&= h(n+1)\,w^n\phi(w,\overline{w}) + w^{n+1}\partial\phi(w,\overline{w})
\end{aligned} \tag{3.15}$$

and the antiholomorphic counterpart is analogous. If we now define the in-asymptotic state as:

$$|h,\overline{h}\rangle \equiv \phi(0,0)\,|0\rangle$$

we see that:

$$L_0\,|h,\overline{h}\rangle = h\,|h,\overline{h}\rangle \qquad \text{and} \qquad \overline{L}_0\,|h,\overline{h}\rangle = \overline{h}\,|h,\overline{h}\rangle$$

therefore $|h,\overline{h}\rangle$ is consistently an eigenstate of the Hamiltonian. The asymptotic state $|h,\overline{h}\rangle$ is called *Highest Weight State*.
From (3.15), it is easy to see that:

$$\begin{aligned} L_n\,|h,\overline{h}\rangle &= 0 \\ \overline{L}_n\,|h,\overline{h}\rangle &= 0 \end{aligned} \qquad \text{if} \qquad n>0$$

Using now the expansion in (3.10), we can again compute the relation in (3.15) and obtain:

$$[L_n, \phi_m] = (n(h-1)-m)\phi_{n+m}$$

Taking $n=0$ it then becomes:

$$[L_0, \phi_m] = -m\phi_m \tag{3.16}$$

This means that the ϕ_m operators act as lowering and raising operators for the eigenstates of L_0, i.e. the asymptotic states, and increasing (or decreasing) their conformal weight by m. Also L_{-m}, for $m > 0$, acts in the same way, indeed, considering the Virasoro algebra relation:

$$[L_0, L_{-m}] = mL_{-m}$$

we see that it increase the conformal weight of the eigenstates of L_0 by m.
What we can finally conclude is that if we act with the L_{-m} operators on a generic asymptotic state $|h, \overline{h}\rangle$, we can obtain a sort of "excited states" that are usually called *Descendant States.*

The asymptotic state, created by the action of a primary field on the vacuum state, is source of an infinite set of descendant states of higher conformal dimension. We can indeed write:

$$L_{-n}\,|h\rangle = L_{-n}\phi\,(0)|\,0\rangle = \frac{1}{2\pi i}\oint dz z^{-(n-1)}T\,(z)\,\phi\,(0)\,|0\rangle$$

The natural definition of a descendant field becomes:

$$\phi^{(-n)}\,(w) \equiv (L_{-n}\phi)\,(w) = \frac{1}{2\pi i}\oint_{C_w} dz\frac{1}{(z-w)^{n-1}}T\,(z)\,\phi\,(w) \tag{3.17}$$

One simple example of descendant field is the stress-energy tensor. Indeed if we now consider the identity operator and the level $n = 2$, what we obtain is:

$$(L_{-2}\cdot 1)\,(w) = \oint_{C_w}\frac{dz}{2\pi i}\frac{1}{(z-w)}T\,(z)\cdot 1 = T\,(w)$$

We see therefore that the stress-energy tensor is a 2-level descendant field of the identity operator.
From the definition (3.17), expanding the terms in the integral with the usual OPE of T with primary fields, we find that:

$$\phi^{(0)}(w) = h\phi(w) \qquad \text{and} \qquad \phi^{(-1)}(w) = \partial\phi(w)$$

The physical properties of this fields can be then derived from the corresponding primary field, for example, let us consider the correlator:

$$\langle (L_{-n}\phi)(w)X\rangle$$

where $X = \phi_1(w_1)...\phi_N(w_N)$ is a product of N primary fields with conformal weight h_i each. It can be shown that:

$$\langle \phi^{(-n)}(w)X\rangle = \mathcal{L}_{-n}\,\langle \phi(w)X\rangle \qquad \text{for} \quad n \geq 1$$

with:

$$\mathcal{L}_{-n} \equiv \sum_i \left\{\frac{(n-1)\,h_i}{(w_i-w)^n} - \frac{1}{(w_i-w)^{n-1}}\partial_{w_i}\right\}$$

We see that for $n = 1$ we have $-\sum_i \partial_{w_i}$ and this is exactly equivalent to ∂_w because the operator $\partial_w + \sum_i \partial_{w_i}$ annihilates the correlator because of its invariance under translations. We can also define in general:

$$\begin{aligned}\phi^{(-k,-n)}\,(w) &= (L_{-k}L_{-n}\phi)\,(w)\\ &= \frac{1}{2\pi i}\oint_{C_w} dz\,(z-w)^{1-k}\,T\,(z)\,(L_{-n}\phi)\,(w)\end{aligned}$$

In particular we see that:

$$\phi^{(0,-n)}(w) = (h+n)\,\phi^{(-n)}(w) \qquad \text{and} \qquad \phi^{(-1,-n)}(w) = \partial_w \phi^{(-n)}(w)$$

so we find again the role of L_0 and L_1 as generators respectively of dilations and translations. It can be also shown that we have:

$$\langle \phi^{(-k_1,\ldots,-k_n)}(w)\, X\rangle = \mathcal{L}_{-k_1}\ldots\mathcal{L}_{-k_n}\langle \phi(w)\, X\rangle$$

so what we can finally conclude is that correlation functions of descendant fields can be reduced to correlation functions of primary fields.

3.4 Free Fields and Operator Product Expansion (OPE)

In this section we would like to find an useful method to obtain the OPE in some kinds of CFTs that are relevant in a string and superstring context. More specifically we will concentrate on the free boson and free fermion system. We have found in the previous section the OPE of the stress-energy tensor acting on a generic conformal field and on itself. What we would like now to do is to rederive those expressions in an alternative way.

3.4.1 The Free Boson System

Let us start by studying the free massless boson case with the following action:

$$S = \frac{1}{2} g \int d^2x \partial_\mu \phi \partial^\mu \phi$$

The equation of motion is:

$$\partial\bar{\partial}\phi = 0$$

Let us now compute explicitly the propagator. We can write the action in an useful way:

$$S = \frac{1}{2}\int d^2x d^2y \phi(x) A(x,y) \phi(y)$$

where $A(x,y) = -g\delta(x-y)\partial^2$, the propagator is then $K(x,y) = A^{-1}(x,y)$, or equivalently it has to satisfy the equation:

$$-g\partial_x^2 K(x,y) = \delta(x-y)$$

If we require translational and rotational invariance then the propagator will be a function of the radius $r = |x-y|$. By writing the previous equation in polar coordinate (ρ,ϕ) and integrating on both sides between 0 and r, we obtain:

$$\begin{aligned} 1 &= 2\pi g \int_0^r d\rho\rho\left(-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial K(\rho)}{\partial\rho}\right)\right) \\ &= 2\pi g\left(-r\frac{\partial K(r)}{\partial r}\right) \end{aligned}$$

The solution is then:

$$K(r) = -\frac{1}{2\pi g}\ln r + const.$$

or equivalently:

$$\langle\phi(x)\phi(y)\rangle = -\frac{1}{4\pi g}\ln{(x-y)^2} + const.$$

that in complex coordinates becomes:

$$\langle\phi(z,\overline{z})\phi(w,\overline{w})\rangle = -\frac{1}{4\pi g}[\ln{(z-w)} + \ln{(\overline{z}-\overline{w})}] + const.$$

We can now consider only the holomorphic components of the two-points function by taking the derivative respect to z and w. What we obtain is:

$$\partial\phi(z)\partial\phi(w) = -\frac{1}{4\pi g}\frac{1}{(z-w)^2}$$

This expression can be interpreted as the OPE of this fields. We can notice that if we exchange the two factors we have the same correlator, that is because the fields we are considering are bosonic fields.

The stress-energy tensor of the system is:

$$T_{\mu\nu} = \frac{\partial\mathscr{L}}{\partial\partial^\mu\phi}\partial_\nu\phi - \eta_{\mu\nu}\mathscr{L} = g(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\partial_\rho\phi\partial^\rho\phi)$$

that in its quantum version, using complex coordinates, becomes:

$$T(z) = -2\pi T_{zz} = -2\pi g : \partial\phi(z)\partial\phi(z) : \tag{3.18}$$

The normal ordering prescription is as usual taken into account in order to ensure the vanishing of its vacuum expectation value. Let us now compute:

$$\begin{aligned}
T(z)\partial\phi(w) &= -2\pi g : \partial\phi(z)\partial\phi(z) : \partial\phi(w) \\
&= -4\pi g : \partial\phi(z)\partial\phi(z) : \partial\phi(w) + ... \\
&= \frac{\partial\phi(z)}{(z-w)^2} + ... \\
&= \frac{\partial\phi(w)}{(z-w)^2} + \frac{\partial^2\phi(w)}{(z-w)} + ...
\end{aligned}$$

We notice that we considered only the singular part of the complete expression (namely the propagator) because it is the only term that contributes to the integral as we can see from the equation (3.6). If we now use the equation (3.11) as a definition of a primary field with conformal weight h, we understand that $\partial\phi$ is a primary field with weight $h = 1$.

We can use the same argument in order to find the OPE of the stress-energy tensor:

$$\begin{aligned}
T(z)\,T(w) &= 4\pi^2 g^2 : \partial\phi(z)\,\partial\phi(z) :: \partial\phi(w)\,\partial\phi(w) : \\
&= \frac{1/2}{(z-w)^4} - \frac{4\pi g : \partial\phi(z)\,\partial\phi(w) :}{(z-w)^2} + \dots \\
&= \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \dots
\end{aligned}$$

and we clearly see that the stress-energy tensor is not a primary field because of the presence of the anomalous term $\frac{1/2}{(z-w)^4}$. Making the comparison between this expression and (3.12), we understand that the central charge for a single free bosonic field system is $c = 1$.

Vertex Operator There is now another bosonic primary field, besides the $\partial\phi$ field, that can be constructed by exponentiating the scalar field ϕ. This kind of field is usually called *Vertex Operator*. The definition of vertex operator thus becomes:

$$\mathcal{V}_k(z,\overline{z}) =: e^{ik\phi(z,\overline{z})} :$$

We can now easily find the OPE of the vertex operator with the bosonic field $\partial\phi$:

$$\partial\phi(z)\mathcal{V}_k(w,\overline{w}) = -\frac{ik}{4\pi g}\frac{\mathcal{V}_k(w,\overline{w})}{(z-w)} + ...$$

Using this result, it can be shown that it is effectively a primary field by computing the OPE of the stress-energy tensor acting on it. What we can found namely is that:

$$T(z)\mathcal{V}_k(w,\overline{w}) = \frac{k^2}{8\pi g}\frac{\mathcal{V}_k(w,\overline{w})}{(z-w)^2} + \frac{\partial\mathcal{V}_k(w,\overline{w})}{(z-w)} + ...$$

so we can conclude that the vertex operator is a primary field with conformal weight:

$$h(k) = \overline{h}(k) = \frac{k^2}{8\pi g} \tag{3.19}$$

The OPE with $\overline{T}$ is exactly analogous. Using the Wick's theorem we can also write another useful identity:

$$: e^{a\phi_1} :: e^{b\phi_2} :=: e^{a\phi_1 + b\phi_2} : e^{ab\langle\phi_1\phi_2\rangle}$$

and applying it to the vertex operator we can obtain:

$$\mathcal{V}_\alpha(z,\overline{z})\mathcal{V}_\beta(w,\overline{w}) = |z-w|^{2\alpha\beta/4\pi g}\mathcal{V}_{\alpha+\beta}(w,\overline{w}) + ... \tag{3.20}$$

We remember now that the two-points correlation function of primary fields does not vanish if and only if the two fields have the same conformal weight. This means that:

$$h(\alpha) = h(\beta) \quad \Longrightarrow \quad \alpha^2 = \beta^2$$

Furthermore we need to impose that the correlator does not grow with distance and this leads to the condition $\alpha = -\beta$[9]. This means that the equation (3.20) becomes:

$$\mathcal{V}_\alpha(z,\overline{z})\mathcal{V}_{-\alpha}(w,\overline{w}) = |z-w|^{-2\alpha^2/4\pi g} + ...$$

3.4.2 The Free Fermion System

Let us now make the same exercise for the free fermion case. The action of a free Majorana fermion[10] in two dimensions is:

$$S = \frac{1}{2}g\int d^2x \Psi^\dagger\gamma^0\gamma_\mu\partial^\mu\Psi$$

where the Dirac gamma matrices[11]now are:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^1 = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

[9]It can be shown that, for the correlation function of several vertex operators, the more general condition is $\sum_i \alpha_i = 0$

[10]It can be shown that in $d = 2$, spinors can be real, i.e. there may exist Majorana spinors in $d = 2$.

[11]Choosing $\eta_{\mu\nu} = diag(1,1)$, they have to satisfy the fundamental relation $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$

Therefore we have that:

$$\gamma^0\gamma^\mu\partial_\mu = \gamma^0\left(\gamma^0\partial_0 + \gamma^1\partial_1\right) = 2\begin{pmatrix}\partial_{\overline{z}} & 0\\ 0 & \partial_z\end{pmatrix}$$

If we now define $\Psi = (\psi, \overline{\psi})$, the action becomes:

$$S = g\int d^2x\left(\psi\overline{\partial}\psi + \overline{\psi}\partial\overline{\psi}\right)$$

In this case the equations of motion are:

$$\begin{aligned}\overline{\partial}\psi &= 0\\ \partial\overline{\psi} &= 0\end{aligned}$$

Again we want to compute the propagator, therefore, just like the case of the free boson, we can write:

$$S = \frac{1}{2}\int d^2xd^2y\Psi_i^\dagger(x)A_{ij}(x,y)\Psi_j(y)$$

where $A(x,y) = g\delta(x-y)\left(\gamma^0\gamma^\mu\right)_{ij}\partial_\mu$ and $K(x,y) = A^{-1}(x,y)$. The equation for $K(x,y)$ now becomes:

$$g\left(\gamma^0\gamma^\mu\right)_{ik}\partial_\mu K_{kj}(x,y) = \delta(x-y)\delta_{ij}$$

In complex coordinates it becomes:

$$2g\begin{pmatrix}\partial_{\overline{z}} & 0\\ 0 & \partial_z\end{pmatrix}\begin{pmatrix}\langle\psi(z,\overline{z})\psi(w,\overline{w})\rangle & \langle\psi(z,\overline{z})\overline{\psi}(w,\overline{w})\rangle\\ \langle\overline{\psi}(z,\overline{z})\psi(w,\overline{w})\rangle & \langle\overline{\psi}(z,\overline{z})\overline{\psi}(w,\overline{w})\rangle\end{pmatrix} = \frac{1}{\pi}\begin{pmatrix}\partial_{\overline{z}}\frac{1}{z-w} & 0\\ 0 & \partial_z\frac{1}{\overline{z}-\overline{w}}\end{pmatrix}$$

where we have used the identity for the delta $\partial_{\overline{z}}\frac{1}{z} = \overline{\partial}\frac{\overline{z}}{|z|^2} = \lim_{\varepsilon\to 0}\overline{\partial}\frac{\overline{z}}{|z|^2+\varepsilon^2} = 2\pi\delta$. The solution of the equation above can be immediately written:

$$\begin{aligned}\langle\psi(z,\overline{z})\psi(w,\overline{w})\rangle &= \frac{1}{2\pi g}\frac{1}{(z-w)}\\ \langle\overline{\psi}(z,\overline{z})\overline{\psi}(w,\overline{w})\rangle &= \frac{1}{2\pi g}\frac{1}{(\overline{z}-\overline{w})}\\ \langle\overline{\psi}(z,\overline{z})\psi(w,\overline{w})\rangle &= 0\end{aligned}$$

From the correlation functions we just found above, we can write the OPE of the fermionic field with itself:

$$\psi(z)\psi(w) = \frac{1}{2\pi g}\frac{1}{(z-w)}$$

We can notice that again the propagator reflects the fermionic nature of the fields, indeed by exchanging the two factors we get a minus sign.
Let us now again compute the stress-energy tensor of the system. From the definition:

$$T(z) = -2\pi T_{zz} = -\frac{1}{2}\pi T^{\overline{z}\overline{z}} = -\pi\frac{\partial\mathscr{L}}{\partial\overline{\partial}\psi}\partial\psi = -\pi g :\psi(z)\partial\psi(z): \tag{3.21}$$

Let us then compute the OPE of the stress energy tensor acting on the fermion ψ:

$$\begin{aligned} T(z)\psi(w) &= -\pi g : \psi(z)\partial\psi(z) : \psi(w) \\ &= \frac{1}{2}\frac{\psi(z)}{(z-w)^2} + \frac{1}{2}\frac{\partial\psi(z)}{(z-w)} + ... \\ &= \frac{1}{2}\frac{\psi(w)}{(z-w)^2} + \frac{1}{2}\frac{\partial\psi(w)}{(z-w)} + \frac{1}{2}\frac{\partial\psi(w)}{(z-w)} + ... \\ &= \frac{1}{2}\frac{\psi(w)}{(z-w)^2} + \frac{\partial\psi(w)}{(z-w)} + ... \end{aligned}$$

Therefore we see that the fermion ψ is a field with conformal dimension $h = 1/2$. Let us compute again the OPE of the stress-energy tensor acting on itself:

$$\begin{aligned} T(z)T(w) &= \pi^2 g^2 : \psi(z)\partial\psi(z) :: \psi(w)\partial\psi(w) : \\ &= ... = \frac{1/4}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + ... \end{aligned}$$

What we obtain is again that the stress-energy tensor is not a primary field beacuse of the presence of the anomalous term (that is different from the one in the bosonic case). We see that the central charge for the single free Majorana fermion system now is $c = 1/2$.

3.4.3 The Ghost System

Another simple system that we can study is the ghost system. We are interested in this kind of system because this is what we obtain when we transform the Faddeev-Popov determinant (namely the Jacobian coming from the change of variables in the metric integration measure) into the exponential of the action of two anticommuting bosonic fields. The ghost action is:

$$S = \frac{1}{2}g\int d^2x b_{\mu\nu}\partial^\mu c^\nu$$

where $b_{\mu\nu}$ is a symmetric traceless tensor and both $b_{\mu\nu}$ and c^μ are anticommuting fields. Their equation of motions are:

$$\partial^\alpha b_{\alpha\mu} = 0 \qquad\qquad \partial^\alpha c^\mu + \partial^\mu c^\alpha = 0$$

that in holomorphic coordinates become:

$$\begin{aligned} \overline{\partial} b &= 0 \\ \partial\overline{b} &= 0 \end{aligned} \qquad\qquad \begin{aligned} \overline{\partial} c &= 0 \\ \partial\overline{c} &= 0 \\ \partial c &= -\overline{\partial}\overline{c} \end{aligned}$$

Again we can compute the ghost propagator by writing the action as:

$$S = \frac{1}{2}\int d^2x d^2y b_{\mu\nu}(x) A^{\mu\nu}_\alpha(x,y) c^\alpha(y)$$

with:

$$A^{\mu\nu}_\alpha(x,y) = \frac{1}{2}g\delta^\nu_\alpha\delta(x-y)\partial^\mu$$

where the $\frac{1}{2}$ factor compensate the double counting in the sum of the indices (μ, ν) because of the fact that $b_{\mu\nu}$ is a symmetric tensor. The propagator now has to satisfy the equation:

$$\frac{1}{2} g \delta^{\mu}_{\alpha} \partial^{\nu} K^{\beta}_{\mu\nu}(x, y) = \delta(x - y) \delta_{\alpha\beta}$$

that in complex coordinate[12] becomes:

$$g \partial_{\overline{z}} K^{\beta}_{zz} = \frac{1}{\pi} \partial_{\overline{z}} \frac{1}{z - w} \delta_{\beta z}$$

The solution of the equation is:

$$\langle b(z) c(w) \rangle = K^{z}_{zz}(z, w) = \frac{1}{\pi g} \frac{1}{z - w} = \langle c(z) b(w) \rangle$$

where last equality is due to the fact that also $K^{\alpha}_{\mu\nu}(x, y)$ must be a symmetric tensor in the (μ, ν) indices.
The stress-energy tensor of the ghost system is:

$$T^{\mu\nu}_{(c)} = \frac{1}{2} g (b^{\mu\alpha} \partial^{\nu} c_{\alpha} - \eta^{\mu\nu} b^{\alpha\beta} \partial_{\alpha} c_{\beta})$$

and we see that it is clearly not symmetric. In order to put it in a completely symmetric form we can use the Belinfante procedure by adding a term like $\partial_{\rho} B^{\rho\mu\nu}$. With some calculations what we can obtain is the completely symmetric stress-energy tensor, called also Belinfante tensor:

$$T^{\mu\nu}_{B} = \frac{1}{2} g \left\{ b^{\mu\alpha} \partial^{\nu} c_{\alpha} + b^{\nu\alpha} \partial^{\mu} c_{\alpha} + \partial^{\alpha} b^{\mu\nu} c_{\alpha} - \eta^{\mu\nu} b^{\alpha\beta} \partial_{\alpha} c_{\beta} \right\}$$

If we take $\mu = \nu = \overline{z}$ and remembering that $T^{\overline{z}\overline{z}} = 4 T_{zz}$ we obtain that:

$$T(z) = \pi g : (2 b \partial c + c \partial b) :$$

We can now compute, just like we did previously, the OPE for the stress-energy tensor acting on the ghost fields and on itself. After the usual calculations, remembering that the c and b ghost fields anticommute, we can obtain:

$$\begin{aligned} T(z) c(w) &= -\frac{c(w)}{(z - w)^2} + \frac{\partial c(w)}{(z - w)} + ... \\ T(z) b(w) &= 2 \frac{b(w)}{(z - w)^2} + \frac{\partial b(w)}{(z - w)} + ... \\ T(z) T(w) &= \frac{-13}{(z - w)^4} + 2 \frac{T(w)}{(z - w)^2} + \frac{\partial T(w)}{(z - w)} + ... \end{aligned}$$

We understand from this results that c and b are primary fields with conformal weight $h = -1$ and $h = 2$ respectively. We also notice that the central charge for a ghost system is $c = -26$, exactly the opposite value of the dimension of space-time that we chose in Chapter 2 and this is not a coincidence. If we consider a

[12] Remember that the metric expressed with holomorphic coordinates has only out-diagonal not null entries, namely $g_{zz} = g_{\overline{z}\overline{z}} = 0$.

system made by 26 bosonic fields, namely in string theory the coordinates[13] X^μ with $\mu = 0, ..., 25$, we find that the stress-energy tensor of the whole system is

$$T_{tot}(z) = T_{gh}(z) + 26 \cdot T_{bos}(z) = 2\frac{T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + ...$$

so it is finally a primary field because the anomaly has been removed. What we can conclude is therefore that a string theory based on $d = 26$ is a consistent theory because the conformal anomaly has been cancelled out and therefore all the simmetries are preserved from classical to quantum level.
Let us now see another more physical interpretation of the central charge c. Let us recall for a moment the infinitesimal transformation of the stress-energy tensor under the transformation $z'^\mu = z^\mu + \epsilon$:

$$\delta_\xi T(z) = -2\partial\xi(z)T(z) - \xi(z)\partial T(z) - \frac{c}{12}\partial^3\xi(z)$$

it can be shown that the corresponding finite transformation[14] is:

$$T'(w) = \left(\frac{dw}{dz}\right)^{-2}\left[T(z) - \frac{c}{12}\{w; z\}\right]$$

where { ; } is called *Schwarzian Derivative* and it is defined as:

$$\{w; z\} = \frac{d^3w/dz^3}{dw/dz} - \frac{3}{2}\left(\frac{d^2w/dz^2}{dw/dz}\right)^2$$

We see that if we apply this transformation law to the conformal transformation $w = \frac{L}{2\pi}\ln z$ we find that:

$$T_{cyl}(w) = \left(\frac{2\pi}{L}\right)^2\left[T_{plane}(z)z^2 - \frac{c}{24}\right] \tag{3.22}$$

If we assume that the vacuum expectiation value of the stress-energy tensor must vanish on the plane, then we obtain that the vacuum expectation value of the stress-energy tensor on the cylinder in not zero, in particular:

$$\langle T_{cyl}(w)\rangle = -\frac{\pi^2 c}{6L^2}$$

The central charge is therefore proportional to the *Casimir Energy.*

3.5 Free Boson Quantization on the Cylinder

In Section 3.4 we analyzed different kinds of systems, in particular we focused on the free Boson, Fermion and Ghosts systems. What we would like now to introduce is the canonical quantization of the free boson system on a different kind of surface respect to the plane: the cylinder. We know that the cylinder and the plane are linked each other through conformal transformations and we will use this correspondance to define the conformal generators and the vacuum energies.

[13] But this is exactly the dimension of space-time

[14] It can be shown that this finite transformation is preserved in form from any transformation of the $SL(2, \mathbb{C})$ group

Let us introduce a bosonic field $\phi(x,t)$ defined on a cylinder of circumference L and flat Minkowski metric. What we initially require is that $\phi(x+L,t) = \phi(x,t)$. The Fourier representation of the field then becomes:

$$\phi(x,t) = \sum_n e^{2\pi inx/L}\phi_n(t)$$
$$\phi_n(t) = \frac{1}{L}\int dx\ e^{-2\pi inx/L}\phi(x,t)$$

The Lagrangian of the system becomes:

$$L = \frac{1}{2}g\int dx\ \partial_\mu\phi(x,t)\partial^\mu\phi(x,t) = \frac{1}{2}g\int dx\ [(\partial_t\phi)^2 - (\partial_x\phi)^2]$$

In terms of the Fourier coefficients becomes:

$$L = \frac{1}{2}gL\sum_n\left[\dot\phi_n\dot\phi_{-n} - \left(\frac{2\pi n}{L}\right)^2\phi_n\phi_{-n}\right]$$

The conjugate momentum of ϕ_n is $\pi_n = gL\dot\phi_{-n}$ and it satysfies the canonical relation $[\phi_n,\pi_m] = i\delta_{nm}$. The Hamiltonian is:

$$H = \frac{1}{2gL}\sum_n\left[\pi_n\pi_{-n} + (2\pi ng)^2\phi_n\phi_{-n}\right]$$

We notice immediately that $\pi_n^\dagger = \pi_{-n}$ and similarly $\phi_n^\dagger = \phi_{-n}$. The Hamiltonian represent a sum of n decoupled harmonic oscillators with frequencies $\omega_n = \frac{2\pi|n|}{L}$. We know also that it can be as usual written in terms of the annihilation and creation operators $\tilde a_n$ and $\tilde a_n^\dagger$:

$$\tilde a_n = \frac{1}{\sqrt{4\pi g\,|n|}}\left(2\pi g\,|n|\,\phi_n + i\pi_{-n}\right)$$

but this expression of the operators does not work when $n = 0$. The problem can be solved by introducing a new set of operators:

$$a_n = \begin{cases} -i\sqrt{n}\ \tilde a_n & (n>0)\\ i\sqrt{-n}\ \tilde a_{-n}^\dagger & (n<0)\end{cases} \qquad\qquad \bar a_n = \begin{cases} -i\sqrt{n}\ \tilde a_{-n} & (n>0)\\ i\sqrt{-n}\ \tilde a_n^\dagger & (n<0)\end{cases}$$

with the commutation relations:

$$[a_n,a_m] = n\delta_{n+m} \qquad [a_n,\overline{a}_m] = 0 \qquad [\overline{a}_n,\overline{a}_m] = n\delta_{n+m}$$

The Hamiltonian therefore becomes:

$$H = \frac{1}{2gL}\pi_0^2 + \frac{2\pi}{L}\sum_{n\neq0}\left(a_{-n}a_n + \overline{a}_{-n}\overline{a}_n\right)$$

Computing the commutator:

$$[H,a_{-m}] = \frac{2\pi}{L}ma_{-m}$$

we understand that applying a_{-m} to an eigenstate of the Hamiltonian, we obtain another eigenstate with shifted energy $E+2\pi m/L$. In terms of constant operators, namely the operators a and $\bar{a}$ computed at $t = 0$, the mode expansion field becomes:

$$\phi(x,t) = \phi_0 + \frac{1}{gL}\pi_0 t + \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}\left(a_n e^{2\pi in(x-t)/L} - \overline{a}_{-n}e^{2\pi in(x+t)/L}\right)$$

If we now perform a Wick rotation, i.e. $t \to -i\tau$, and use the conformal coordinates $z = e^{2\pi(\tau-ix)/L}$ and $\overline{z} = e^{2\pi(\tau+ix)/L}$ we can finally obtain:

$$\phi(z,\overline{z}) = \phi_0 - \frac{i}{4\pi g}\pi_0 \ln(z\overline{z}) + \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}\left(a_n z^{-n} + \overline{a}_n\overline{z}^{-n}\right)$$

We know that in conformal field theory only $\partial\phi$ is a primary field so let us write the holomorphic primary field we need:

$$i\partial\phi(z) = \frac{1}{4\pi g}\frac{\pi_0}{z} + \frac{1}{\sqrt{4\pi g}}\sum_{n\neq 0} a_n\ z^{-n-1}$$

We can now define the operators:

$$a_0 = \overline{a}_0 = \frac{\pi_0}{\sqrt{4\pi g}} \tag{3.23}$$

and write in a compact way our primary field:

$$i\partial\phi(z) = \frac{1}{\sqrt{4\pi g}}\sum_{n} a_n\ z^{-n-1} \tag{3.24}$$

Analogously to what we saw in the closed string case, the operator a_n can be interpreted as the creation or annihilation operator of the right-moving excitations while the $\overline{a}_n$ operator, the counterpart for the left-moving excitations.

3.5.1 Compactified Boson

We can notice that the free boson Lagrangian is invariant under a shift of a constant quantity of the bosonic field, namely $\phi \to \phi + const.$, so we can impose consistently a new boundary condition for the ϕ field:

$$\phi(x+L,t) = \phi(x,t) + 2\pi mR$$

with m the *winding number* of the field configuration. We notice that here the bosonic field has assumed the role of an angular variable because we restricted the domain of variation of the field to a circle of radius R. After this considerations, the new expression of the mode expansion of the field becomes:

$$\phi(x,t) = \phi_0 + \frac{n}{gRL}t + \frac{2\pi mR}{L}x + \frac{i}{\sqrt{4\pi g}}\sum_{k\neq 0}\frac{1}{k}\left(a_k e^{2\pi ik(x-t)/L} - \overline{a}_{-k}e^{2\pi ik(x+t)/L}\right)$$

Performing now the Wick rotation and expressing everything in complex coordinates what we obtain is:

$$\begin{aligned}\phi(z,\overline{z}) = \phi_0 &- i\left(\frac{n}{4\pi gR} + \frac{1}{2}mR\right)\ln z + \frac{i}{\sqrt{4\pi g}}\sum_{k\neq 0}\frac{1}{k}a_k z^{-k} \\ &- i\left(\frac{n}{4\pi gR} - \frac{1}{2}mR\right)\ln \overline{z} + \frac{i}{\sqrt{4\pi g}}\sum_{k\neq 0}\frac{1}{k}\overline{a}_k\overline{z}^{-k}\end{aligned}$$

We can also write:

$$i\partial\phi(z) = \left(\frac{n}{4\pi gR} + \frac{1}{2}mR\right)\frac{1}{z} + \frac{1}{\sqrt{4\pi g}}\sum_{k\neq 0} a_k z^{-k-1}$$

We can also write the expression of the L_0 and $\overline{L}_0$ operators:

$$L_0 = \sum_{n>0} a_{-n}a_n + \frac{1}{2}a_0^2 = \sum_{n>0} a_{-n}a_n + 2\pi g\left(\frac{n}{4\pi gR} + \frac{1}{2}mR\right)^2 \tag{3.25}$$

$$\overline{L}_0 = \sum_{n>0} \overline{a}_{-n}\overline{a}_n + \frac{1}{2}\overline{a}_0^2 = \sum_{n>0} \overline{a}_{-n}\overline{a}_n + 2\pi g\left(\frac{n}{4\pi gR} - \frac{1}{2}mR\right)^2 \tag{3.26}$$

We notice that the second term represent now the new conformal dimension of the vacuum state[15]. We can therefore now label the vacuum state with the state $|n, m\rangle$, with its conformal dimension:

$$h_{n,m} = 2\pi g\left(\frac{n}{4\pi gR} + \frac{1}{2}mR\right)^2$$

annihilated by all the a_{-n} operators with $n > 0$. We should notice that the state $|k, n, m\rangle$ is the state generated by acting with the vertex operator $\mathcal{V}_k(z, \bar{z})$ on $|0\rangle$, where the momentum k is quantized[16].

3.6 Free Fermion Quantization on the Cylinder

Let us now quantize, analogously to what we have done in the free boson case, a free fermion system on the cylinder. In this case the action will be:

$$S = \frac{1}{2}g\int d^2x\ \Psi^\dagger\gamma^0\gamma^\mu\partial_\mu\Psi$$

where $\Psi = (\psi, \bar{\psi})$. Now, just like we did for the boson case, let us write the Fourier mode expansion of the fermionic field that lives on a cylinder of circumference L:

$$\psi(x, t) = \sqrt{\frac{2\pi}{L}}\sum_k b_k\ e^{2\pi ik(x-t)/L}$$

with the operators of creation and annihilation b_k that obey to the anticommutation relation:

$$\{b_k, b_q\} = \delta_{q+k,0} \tag{3.27}$$

Making the usual Wick rotation of the time variable and defining $w = \tau - ix$, we get the holomorphic fermionic field expression:

$$\psi(w) = \sqrt{\frac{2\pi}{L}}\sum_k b_k\ e^{-2\pi kw/L}$$

We can now notice that there are two possible choices of boundary conditions for the fermionic field, that are compatible with the Action:

$$\begin{aligned} \psi(x+L, t) &\equiv \psi(x, t) && \text{Ramond Sector (R)} \\ \psi(x+L, t) &\equiv -\psi(x, t) && \text{Neveau-Schwarz Sector (NS)} \end{aligned} \tag{3.28}$$

[15] Indeed this is the eigenvalue of the L_0 operator.

[16] This can be easily verified by looking at (3.19) and noting that h is quantized, since $n, m \in \mathbb{Z}$

In order to take into account the antiperiodicity of the fermionic field, in the NS sector, k takes half-integer values, i.e. $k \in \mathbb{Z} + \frac{1}{2}$, while in the R sector, as usual, it takes integer value, i.e. $k \in \mathbb{Z}$.
The Hamiltonian could be written in general as:

$$H = \sum_{k>0} \omega_k \; b_{-k} b_k + E_0 \qquad\qquad \omega_k = \frac{2\pi |k|}{L}$$

where E_0 is a constant having the meaning of a vacuum energy.
We can now see that in the R sector there exists a zero-mode operator b_0 that does not contribute to the Hamiltonian and therefore leads to a degeneracy of the vacuum state. If we consider indeed the $b_0 \left|0\right\rangle$ state, we clearly see that it has the same energy of the vacuum state and it is annihilated from all the annihilation operator because of the relation (3.27). Using again the relation (3.27), we can find that $b_0^2 = \frac{1}{2}$.

Let us now map the fermionic field from the cylinder to the plane. This map is given by the relation $z = e^{2\pi w/L}$. Remembering now that the conformal weight of the fermionic field is $h = \frac{1}{2}$, what we obtain is that:

$$\psi_{plane}(z) = \left(\frac{dw}{dz}\right)^{1/2} \psi_{cyl}(w) = \sqrt{\frac{L}{2\pi z}} \; \psi_{cyl}(w)$$

The final mode expansion of the field on the plane becomes then:

$$\psi(z) = \sum_k b_k \; z^{-k-1/2} \tag{3.29}$$

From this expression we can notice that the meanings of the two types of boundary conditions has been interchanged, indeed:

$$\begin{aligned} \psi(e^{2\pi i} z) &= \;\;\psi(z) \qquad \text{NS Sector} \\ \psi(e^{2\pi i} z) &= -\psi(z) \qquad \text{R Sector} \end{aligned}$$

Let us compute now the two-point function on the complex plane for each kind of sector. If we consider first the NS sector, we have:

$$\begin{aligned} \langle \psi(z)\psi(w) \rangle &= \sum_{q,k \in \mathbb{Z}+\frac{1}{2}} \langle b_k b_q \rangle \, z^{-k-1/2} w^{-q-1/2} \\ &= \sum_{k \in \mathbb{Z}+\frac{1}{2},\; k>0} z^{-k-1/2} w^{k-1/2} \\ &= \sum_{n=0}^{\infty} \frac{1}{z} \left(\frac{w}{z}\right)^n \\ &= \frac{1}{z-w} \end{aligned}$$

and that is consistent to what we found in Section (3.4.2). If we consider secondly the R sector, we have:

$$\begin{aligned}\langle\psi(z)\psi(w)\rangle &= \sum_{q,k\in\mathbb{Z}} \langle b_k b_q\rangle\, z^{-k-1/2} w^{-q-1/2}\\ &= \frac{1}{2\sqrt{zw}} + \sum_{k=1}^{\infty} z^{-k-1/2} w^{k-1/2}\\ &= \frac{1}{\sqrt{zw}}\left\{\frac{1}{2} + \sum_{k=1}^{\infty}\left(\frac{w}{z}\right)^k\right\}\\ &= \frac{1}{\sqrt{zw}}\,\frac{z+w}{2(z-w)}\\ &= \frac{1}{2}\frac{\sqrt{z/w}+\sqrt{w/z}}{z-w}\end{aligned}$$

We notice that this result coincide with the previous one by taking the $z \to w$ limit. This means that the behaviour of the theory at short distances is independent of the boundary conditions.

We can now compute the stress-energy tensor in the two types of sectors. Starting from the NS sector, we can write the expression of the strees-energy tensor:

$$T(z) = -\pi g : \psi(z)\,\partial\psi(z) :$$

that trivially gives, usign the normalization $g = 1/2\pi$:

$$\langle T(z)\rangle = \frac{1}{2}\lim_{\varepsilon\to 0}\left(-\langle\psi(z+\varepsilon)\,\partial\psi(z)\rangle + \frac{1}{\varepsilon^2}\right) = 0 \tag{3.30}$$

In the R sector we have instead:

$$\begin{aligned}\langle T(z)\rangle &= \frac{1}{2}\lim_{w\to z}\left[-\frac{1}{2}\partial_w\left(\frac{\sqrt{z/w}+\sqrt{w/z}}{z-w}\right) + \frac{1}{(z-w)^2}\right]\\ &= \frac{1}{16z^2}\end{aligned} \tag{3.31}$$

We can now turn back to the cylinder and compute the vacuum expectation value of the stress-energy tensor. It can be easily shown, using the equation (3.22), that:

$$\langle T_{cyl}(z)\rangle = \begin{cases} -\dfrac{1}{48}\left(\dfrac{2\pi}{L}\right)^2 & \text{NS Sector}\\ \dfrac{1}{24}\left(\dfrac{2\pi}{L}\right)^2 & \text{R Sector}\end{cases}$$

3.7 Affine Current Algebras

In Sections (3.4.1) and (3.4.2) we discussed the free boson and fermion systems. In both cases we computed explicitly the stress-energy tensor but there may be also conserved currents that we didn't compute directly. These currents thus satisfy the following equations:

$$\partial^\mu J_\mu = 0 \qquad \epsilon^{\mu\nu}\partial_\mu J_\nu = 0$$

Using complex coordinates, the equations above give us the (anti-)holomorphicity conditions:

$$\partial_z J_{\bar{z}} = 0 \qquad \partial_{\bar{z}} J_z = 0$$

If we now consider the whole set of holomorphic conserved currents of the theory, we can write the most general OPE of them compatible with their properties, namely:

$$J^a(z)J^b(w) = \frac{G^{ab}}{(z-w)^2} + \frac{i f^{ab}{}_c J^c(w)}{z-w} + \text{finite}$$

where G^{ab} and $f^{ab}{}_c$ are respectively symmetric and antisymmetric in the a and b indices. By writing the OPE product expansion of $J^a(z)J^b(w)J^c(y)$ and using the associativity property, it can be shown that $f^{abc} = f^{ab}{}_d\, G^{dc}$ is completely antisymmetric and satisfy the Jacobi identity. We understand therefore that f^{abc} are the structure constants of a Lie algebra g with invariant Killing metric G_{ab}. If we now expand the currents:

$$J^a(z) = \sum_n J^a_n \, z^{-n-1} \tag{3.32}$$

$$J^a_n = \oint_{C_0} \frac{dz}{2\pi i} \, z^n J^a(z) \tag{3.33}$$

where C_0 is an arbitrary contour taken anticlockwise around $z = 0$, we can compute the following commutator:

$$[J^a_n, J^b_m] = m \, G^{ab}\delta_{n+m,0} + i f^{ab}{}_c \, J^c_{n+m} \tag{3.34}$$

We immediately see that this algebra $\hat{g}$ is an infinite-dimensional generalization of the Lie algebra g and it is called *Affine* or *Current Algebra*. We can also notice that the algebra of the zero modes J^a_0 is exactly the algebra g with $f^{ab}{}_c$ structure constants.

The currents J^a, having conformal weight $h = 1$, are necessary primary field in a positive theory. The OPE with the stress-energy tensor therefore will be:

$$T(z)J^a(w) = \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w} + \text{finite}$$

Let us now consider a simple group G. If we perform an opportune change of basis, we can set the Killing metric as $G^{ab} = k\delta^{ab}$, where k is called the level of the affine algebra $\hat{g}_k$. We can notice now that, writing the operator:

$$T_G(z) = \frac{1}{2(k+\tilde{h})} : J^a(z)J^a(z) :$$

it satisfies the Virasoro algebra with central charge $c_G = \frac{kD_G}{k+\tilde{h}}$, where $\tilde{h}$ is called *Dual Coxeter Number*[17] of G and D_G is the dimension of the G group. The $T_G(z)$ operator is called *Sugawara* affine stress-energy tensor..
In order to build the representations of the affine algebra $\hat{g}$, we can take a set of states $|R_i\rangle$ that transforms in the representation R of the zero-mode current subalgebra and is annihilated by all the positive modes of the currents, namely:

$$J^a_{m>0} \, |R_i\rangle = 0 \qquad J^a_0 \, |R_i\rangle = (T^a_R)_{ij} \, |R_j\rangle \tag{3.35}$$

[17]For $SU(N)$ it is $\tilde{h} = N$, while for $SO(N)$ it is $\tilde{h} = N - 2$.

The states $|R_i\rangle$ are generated by local operators $R_i(z,\bar{z})$ acting on the vacuum state. We notice that there is a parallelism between the Virasoro algebra and the affine current algebra. Indeed we can see that currents play the role of the Virasoro generators L_n and the $R_i(z,\bar{z})$ operators play the role of the primary field ϕ_n modes operators.
Using now the expression of the current modes in (3.33), the (3.35) relation becomes:

$$J^a(z)R_i(w,\bar{w}) = \frac{(T_R^a)_{ij}}{z-w}\, R_j(w,\bar{w}) + \text{finite} \tag{3.36}$$

This can be seen as the definition of *Affine Primary Fields*, exactly analogously to what we did in the case of the Virasoro algebra. From the (3.36) expression, we can compute the conformal weight of the $R_i(z,\bar{z})$ operators from the Sugawara affine stress-energy tensor expression:

$$\Delta_R = \frac{C_R}{k+\tilde{h}}$$

where C_R is the quadratic Casimir of the representation R of the G group.

Chapter 4

String Perturbation Theory

In this chapter we will present how scattering amplitudes in String Theory are defined and we will study some of the possible surfaces that describe the World-sheet of such processes. The surfaces on which we will focus will be, in particular, the Sphere and the Torus. We will introduce also the concepts of Low-Energy Effective Action and Non-Linear $\sigma-$Model.

4.1 String Perturbation Expansion

In QFT we are used to compute the physical quantities of our interest by expanding them in powers of a small parameter and by computing order by order every contribution to the series. In string theory there are no interaction terms but, as we will show later, we have coupling constant that allow us to define a perturbative expansion that is analogous to the one we have usually in QFT.
Let us therefore start considering the most simple scattering process: the freely propagating closed string. The Feynman diagram in this case is clearly a cylinder, where its two ends are the asymptotic incoming and outcoming closed string.
The cylinder can be described by the Euclidean flat metric $ds^2 = d\tau^2 + d\sigma^2$ with $-\infty < \tau < +\infty$, $0 \leq \sigma < 2\pi$. Through the conformal transformation $\tau = \ln r$, the cylinder can be mapped in a puntured plane with metric:

$$ds^2 = r^{-2}(dr^2 + r^2 d\sigma^2)$$

where now $0 < r < \infty$.
Again, by rescaling the metric with the factor $4r^2(1+r^2)^{-2}$ and taking $z = r\ e^{i\sigma}$, we can map the plane to the sphere. The metric we can obtain is then:

$$ds^2 = \frac{4\ dzd\bar{z}}{(1+|z|^2)^2}$$

namely the metric of the sphere stereographically projected onto the plane.
Choosing $z = \cot\left(\frac{\theta}{2}\right)e^{i\phi}$, we can get the usual metric of the sphere with radius $r = 1$, $ds^2 = d\theta^2 + \sin^2\theta\ d\phi^2$.
Let us now consider a tree-level scattering between four closed strings. The Feynman diagram, namely the world-sheet, that we obtain is four cylinders that all meet on a sphere.
Through the usual conformal transformations, the world-sheet can be mapped into a sphere with four punctures which correspond to the four external legs of

the scattering process. This way to describe string diagrams remains also useful for loop corrections of the process, in particular the external strings will be mapped to points on a sphere with g handles, where g indicates the number of loops. The amplitude of a generic process will be therefore a sum over all possible topologies of the spheres with g handles.
It can be shown that two-dimensional compact oriented surface without boundaries are topologically described as spheres with g handles. This means that the total amplitude of the process has to be defined by the following path-integral:

$$\begin{aligned} A_n &= \sum_{g=0}^{\infty} A_n^{(g)} \\ &= \sum_{g=0}^{\infty} e^{-\lambda \cdot \chi(\Sigma)} \int \mathcal{D}h\mathcal{D}X^{\mu} \int \prod_{i=1}^{n} d^2 z_i \; V_i(z_i, \overline{z}_i) \; e^{-S[X,\, h]} \end{aligned} \tag{4.1}$$

where $V(z, \overline{z})$ are the conformal vertex operator for the massless closed string excitation. We have now to notice that the coefficient $e^{-\lambda\cdot\chi(\Sigma)}$ contains all the informations about the topology of the world-sheet, indeed it holds the following relation:

$$-\chi(\Sigma) \cdot \lambda = \frac{\lambda}{4\pi} \int d^2\sigma \sqrt{h} \; R = 2\,(g-1)\,\lambda$$

where g is also called the *Genus* of the surface and $\chi(\Sigma)$ is its Euler number.
The complete action will be therefore made by the sum of two terms:

$$S = S_p + \chi\lambda$$

If we now define $g_s \equiv e^{\lambda}$, we see that the perturbative expansion in surfaces with increasing genus g, becomes a perturbative expansion in powers of g_s. For this reason we are allowed to interpret it as the string coupling constant. Considering for example the scattering amplitude at two-loops order, we see that it receives contributions from the sphere ($g = 0$), the torus ($g = 1$) and a $g = 2$ surface. This becomes an expansion in powers of the g_s coupling: g_s^{-2} for the sphere, g_s^0 for the torus and g_s^2 for the last surface.

4.1.1 Vertex Operators in String Theory

Let us now discuss the physical meaning of vertex operators in string theory. Let us define the vertex operator of our interest as:

$$\mathcal{V}_k(z, \overline{z}) \equiv : e^{ikX(z,\overline{z})} :$$

We can now apply this operator to the vacuum state in order to create a state with momentum k:

$$|k\rangle = \lim_{z,\overline{z}\to 0} \mathcal{V}_k(z, \overline{z})\,|0\rangle$$

Let us show that k is the momentum associated to the state $|k\rangle$. From the definition of the momentum operator $P^{\mu} = \frac{2}{\alpha'} \oint \frac{dz}{2\pi i} i\partial X^{\mu}(z)$, we can compute:

$$P^{\mu}\,|k\rangle = \frac{2}{\alpha'} \oint \frac{dz}{2\pi i} i\partial X^{\mu}(z) : e^{ik\cdot X(z,\overline{z})} : |0\rangle \overset{OPE}{=} k^{\mu}\,|k\rangle$$

Imposing now the physical state condition on $|k\rangle$, $h = \overline{h} = 1$, we obtain that $m^2 = -k^2 = -8\pi g < 0$, therefore a state with negative mass. What we understand

now is that the state $\mathcal{V}_k(z,\overline{z})\,|0\rangle$ has to be interpreted as a tachyon, i.e. the ground state of the bosonic closed string spectrum. It can also be shown that the annihilation operators a_n and $\overline{a}_n$, for $n > 0$, annihilate the state $|k\rangle$, therefore we can refer to this state as the *Conformal Vacuum.* From this consideration, the Fock Space associated to $|k\rangle$ can be constructed by acting with the creation operators on it. This is straightforward because we can increase or decrease the conformal dimension[1] of the state using the a_n and $\overline{a}_n$ operators and therefore modify the energy and consequently the mass of the state.
Using this procedure, we can introduce the vertex operator also for the next level of excitation of the closed bosonic string. Indeed we may have:

$$\begin{aligned}|k,\epsilon\rangle &= -\frac{2}{\alpha'}\epsilon_{\mu\nu}(k)\lim_{z,\overline{z}\to 0} :\partial X^\mu(z)\,\overline{\partial} X^\nu(\overline{z})\,e^{ik\cdot X(z,\overline{z})}: |0\rangle \\ &= \alpha^\mu_{-1}\overline{\alpha}^\nu_{-1}\,|k\rangle\,\epsilon_{\mu\nu}\end{aligned}$$

where $\epsilon_{\mu\nu}$ is the polarization tensor. By computing the OPE of the stress-energy tensor acting on this vertex, we can find that:

$$\begin{aligned}T(z) :\epsilon_{\mu\nu}\partial X^\mu(w)\,\overline{\partial}X^\nu(\overline{w})\,e^{ik\cdot X(w,\overline{w})} :&= -i\frac{\alpha'}{2}\frac{k^\mu\epsilon_{\mu\nu}}{(z-w)^3} : \overline{\partial}X^\nu(\overline{w})\,e^{ik\cdot X(w,\overline{w})} : \\ &+ \left[\frac{\frac{\alpha'}{4}k^2+1}{(z-w)^2} + \frac{\partial_w}{(z-w)}\right]\epsilon_{\mu\nu} : \partial X^\mu(w)\,\overline{\partial}X^\nu(\overline{w})\,e^{ik\cdot X(w,\overline{w})} : +...\end{aligned}$$

We have now to impose that the vertex operator transforms as a physical primary field operator[2], therefore we need to require that:

$$\begin{aligned}k^\mu\epsilon_{\mu\nu} = 0 = \epsilon_{\mu\nu}k^\nu \\ k^2 = 0\end{aligned}$$

This last condition is exactly the on-shell condition for a massless tensor particle which we can identify with the graviton, antisymmetric tensor and dilaton, depending on whether $\epsilon_{\mu\nu}$ is respectively symmetric traceless, antisymmetric or pure trace.

4.2 The Moduli Space

We recall that Polyakov's action is invariant under Weyl transformations and diffeomorphisms, therefore the direct computation of the amplitude in (4.1) is highly divergent because we are integrating over an infinite number of equivalent metrics. In order to remove the infinite factors that would come out of the computation, we need to change the functional integration measure:

$$\int \mathcal{D}h\mathcal{D}X^\mu \quad \longrightarrow \quad \int \frac{\mathcal{D}h\mathcal{D}X^\mu}{Vol(\text{diff})\times Vol(\text{Weyl})} = \int_{M_g}\mathcal{D}h\mathcal{D}X^\mu \tag{4.2}$$

The meaning of the new functional integration measure is that we are now integrating over a manifold M_g, called *Moduli Space*, defined in the following way:

$$M_g = \frac{\{\text{metrics}\}}{\{\text{Weyl}\}\times\{\text{diff}\}}$$

[1]Remember that it is the eigenvalue of the L_0 operator and then of the Hamiltonian
[2]Namely we have to require that $h = 1$ and we have to remove the cubic singularity.

A point of M_g is an equivalence class of metrics on a surface of genus g, where two metrics are equivalent if they can be transformed into each other by Weyl transformations and diffeomorphisms. The moduli space M_g can then be parametrized by a certain number of parameters τ_i, called *moduli.* Fixing thus the value of the moduli τ_i corresponds to choose a representative of a specific equivalence class of metrics, namely a so called *Fiducial Metric*, $h_{\alpha\beta}$.
The space $\mathcal{M}_g$ has a natural structure of a fiber bundle with a finite dimensional base space and infinite dimensional fibers. We can thus think this space as composed by gauge slides labelled by a finite number of moduli τ_i and, over this gauge slides, an infinite numbers of fibers labelled by every possible diffeomorphism and Weyl transformation that we can apply to the metric at that point[3].
The integration measure of the metric in (4.2) therefore becomes:

$$\mathcal{D}h = J\ \mathcal{D}\Lambda\mathcal{D}'\xi\ \prod_i \tau_i$$

where the J matrix is the Jacobian arising from the change of coordinates and the prime index means that we do not have to integrate over the zero modes of P because we already counted them in the integration over the Weyl factor.
We notice also that $\mathcal{M}_g$ parametrizes every possible complex structure, namely every possible choice of local complex coordinates, on a surface of a given topology. If we have indeed, for example, a metric with Minkowski signature, $ds^2 = 2e^{2\phi}\ ((d\sigma^1)^2 - (d\sigma^0)^2)$, we have showed that it is always possible, locally, by performing a Wick rotation $\sigma^0 = -i\sigma^2$ and defining the complex coordinate $z = \sigma^1 + i\sigma^2$, to put the metric in the form:

$$ds^2 = 2e^{2\phi}\ dzd\overline{z} \equiv 2h_{z\overline{z}}\ dzd\overline{z}$$

This means that an equivalence class of metrics admits a representative that can be always locally put in the form above. We understand thus that there is a one-to-one correspondence between complex structures and equivalence classes of metrics of a surface of a given topology.
Metrics that are linked to each other through conformal transformations are called conformally related. We already know that if $P^\dagger$ admits zero modes, then there exist metrics that are not conformally related. It can be actually shown that the number of moduli is the number of zero modes of the $P^\dagger$ operator defined in (1.9)[4].

We now ask ourselves how many moduli parameters τ_i exist for a two-dimensional compact Riemann surface of genus g without boundaries. The answer comes from the Riemann-Roch theorem that states that:

$$\dim_{\mathbb{C}} \ker P - \dim_{\mathbb{C}} \ker P^\dagger = 3 - 3g = \frac{3}{2}\chi$$

We see that, thanks to this theorem, if we are able to find the number of Killing vectors for our surface, we are also immediately able to find the number of moduli.

Let us now apply the theorem to the case of a surface with genus $g = 0$, namely the Sphere. Recalling now the metric of the sphere stereographically projected

[3]In fact we apply Weyl transformations and diffeomorphisms to the representative of the equivalence class specified by the modulus τ_i.

[4]This is clear since that zero modes of the $P^\dagger$ operator are the variations of the metric that cannot be reached by diffeomorphisms and Weyl transformations.

onto the plane and solving for that the equation (3.1), what we can obtain is that the only globally defined analytic vector fields are:

$$\partial_z, \qquad z\partial_z, \qquad z^2\partial_z$$

That is not a coincidence because we already know that those fields are the one generated by the $L_0, L_{\pm 1}$ Virasoro generators, which generate the Conformal Killing Group $SL(2, \mathbb{C})$ on the sphere.
We find therefore that there exist three Killing vectors for the sphere, so, through the Riemann-Roch theorem, we know that the number of moduli is exactly zero. This statement is really very important because it says that on the sphere all the metrics are conformally equivalent.

4.2.1 The Torus

We can now extend those arguments to the torus, namely a surface with genus $g = 1$. The torus admits a global flat metric $ds^2 = dzd\overline{z}$ and then the Ricci scalar will be $R = 0$. The conformal Killing equations in this case are very easy to solve, indeed:

$$\partial_z V^{(n)} = 0 = \partial_{\overline{z}} V^{(n)} \qquad \Longrightarrow \qquad V^{(n)} = \text{const.}$$

This means that there is only one globally defined Killing vector and therefore, thanks to the Riemann-Roch theorem, the number of moduli is exactly one. The Killing group in this case is $U(1) \times U(1)$, namely the group of translations along the two main cycles of the torus.
What we learned from this analysis is that there exists only one parameter that characterize two conformally inequivalent tori. We know that taking two complex numbers on the complex plane $\lambda_1, \lambda_2 \in \mathbb{C}$ with $\text{Im}(\lambda_2/\lambda_1) \neq 0$, they generates a lattice $\Lambda = \{n\lambda_1 + m\lambda_2 | n, m \in \mathbb{Z}\}$. The definition of the torus follows immediately:

$$z \sim z + n\lambda_1 + m\lambda_2, \qquad n, m \in \mathbb{Z}, \qquad \lambda_1, \lambda_2 \in \mathbb{C}, \qquad \text{Im}(\lambda_1/\lambda_2) \neq 0$$

namely the torus is defined as $\mathbb{T} \sim \mathbb{C}/\Lambda$ with $\mathbb{C}$ its covering space. Since now the modulus has necessarily to be invariant under Weyl transformations and diffeomorphisms, namely a conformal invariant parameter, the only one complex parameter that we can build that satisfies those requests is:

$$\tau = \frac{\lambda_1}{\lambda_2} \equiv \tau_1 + i\tau_2$$

Without loss of generalities we can restrict the domain of τ in $\text{Im}\,\tau > 0$. The new domain of the τ parameter is called *Teichmüller Space* and it is indicated with $\mathbb{H}_+$.
We have now to notice that there are some residual transformations not connected to the identity of the global group of diffeomorphisms, i.e. those transformations that cannot be obtained by exponentiating the infinitesimal corresponding ones, that changes the parameters τ without changing the torus itself. Those transformations are called *Dehn Twists*. Those kind of transformations consists in the rotation of 2π of two different points around respectively the "cycle a" and the "cycle b" of the torus.
The twist around the "cycle a" practically becomes the following transformation on our parameters:

$$\begin{cases} \lambda_1 \to \lambda_1 \\ \lambda_2 \to \lambda_2 + \lambda_1 \end{cases} \qquad \Longrightarrow \qquad \tau \to \tau + 1$$

while the twist around the "cycle b" becomes:

$$\begin{cases} \lambda_2 \to \lambda_2 \\ \lambda_1 \to \lambda_1 + \lambda_2 \end{cases} \implies \tau \to \frac{\tau}{\tau + 1}$$

We can summarize a generic transformations that changes the τ parameter without changing the torus, in the following expression:

$$\begin{cases} \lambda_1 \to c\lambda_2 + d\lambda_1 \\ \lambda_2 \to a\lambda_2 + b\lambda_1 \end{cases} \implies \tau \to \frac{a\tau + b}{c\tau + d} \qquad \text{with} \qquad ad - bc = 1$$

where the condition on the coefficients a, b, c, d means that we want to take into account only invertible trnasformations of λ_1, λ_2, that also preserve the orientation of the surface. We also notice that performing the transformation $(a, b, c, d) \to (-a, -b, -c, -d)$, we generates the same transformation of τ, therefore we need to remove this other residual condition. What we can therefore say is that those residual transformations generates the so called *Modular Group* of the torus: $SL(2, \mathbb{Z})/\mathbb{Z}_2 \sim PSL(2, \mathbb{Z})$.
Clearly the real domain of τ in which we are interested in will be:

$$M_1 = \frac{\text{Teichmüller space}}{\text{Modular group}} \sim \mathbb{H}_+/PSL(2, \mathbb{Z})$$

We can also alternatively see that the Modular group can be generated from other two transformations:

$$\begin{aligned} T: &\quad \tau \to \tau + 1 \\ S: &\quad \tau \to -\frac{1}{\tau} \end{aligned} \tag{4.3}$$

so any element of $PSL(2, \mathbb{Z})$ can be composed of S and T transformations. It can be easily seen that, through the T and S transformations, we can reach any point in the Teichmüller space by starting from a small region of it. This particular domain is called *Fundamental Region* and, in the case of the torus, it can be identified as:

$$\mathscr{F} = \left\{ -\frac{1}{2} \leq \text{Re}\ \tau \leq 0, |\tau|^2 \geq 1 \quad \cup \quad 0 < \text{Re}\ \tau < \frac{1}{2}, |\tau|^2 > 1 \right\}$$

We can notice also that $\tau = i$ is a fixed point of the S transformation with $S^2(\tau = i) = 1$, while $\tau = e^{2/3\pi i}$ is a fixed point of the ST transformation with $(ST)^3(\tau = e^{2/3\pi i}) = 1$. This means that the moduli space of the torus M_1 is not a smooth manifold but admits singularities at the fixed points. These kind of structures are called *Orbifolds*.

4.3 Non-Linear Sigma Models

In discussing the Polyakov action we focused on the properties of the world-sheet metric only, assuming that the space-time metric was simply flat. What does it happen then if we consider a background given by a curved space-time? Let us answer to this question starting from the Polyakov action in which we consider a curved space-time metric:

$$S_\sigma = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{g}\ g^{ab} G_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu \tag{4.4}$$

Let us now assume that the metric $G_{\mu\nu}(X)$ is a small perturbation of the flat metric, namely:

$$G_{\mu\nu}(X) \sim \eta_{\mu\nu} + \chi_{\mu\nu}(X)$$

we can now see that, in this approximation, we have:

$$e^{-S_\sigma} = e^{-S_P}\left(1 - \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{g}\; g^{ab}\chi_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu\right) \tag{4.5}$$

Assuming now that the small perturbation is a plane wave:

$$\chi_{\mu\nu}(X) = -4\pi g_c\; e^{ik\cdot X} s_{\mu\nu}$$

with $s_{\mu\nu}$ a symmetric tensor, we immediately recognize the graviton's vertex operator. We are therefore allowed to interpret, consistently with General Relativity, the curved space-time background as a coherent state of gravitons.
Let us now try to go beyond this arguments and include also the other massless excitation states of the closed bosonic string as our background fields. The immediate generalization of the previous action is:

$$S_\sigma = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{g}\left[\left(g^{ab}\; G_{\mu\nu}(X) + i\epsilon^{ab}B_{\mu\nu}(X)\right)\partial_a X^\mu \partial_b X^\nu + \alpha' R\; \Phi(X)\right] \tag{4.6}$$

where R is the scalar curvature of the world-sheet metric, $B_{\mu\nu}(X)$ is the antisymmetric tensor field and $\Phi(X)$ is a scalar field that, with the diagonal part of the $G_{\mu\nu}(X)$ tensor, contributes to the dilaton background.
We notice that we can interprete the $B_{\mu\nu}$ tensor as the corresponding rank-2 gauge field of the A_μ gauge field that we have in QED. Indeed gauge invariance on a 1-dimensional world-sheet requires a gauge vector fields, while on a 2-dimensional world-sheet it requires a gauge tensor field of rank 2.
Analogously to the QED gauge trasformation $A_\mu \to A_\mu + \partial\alpha$, we have that the gauge trasformation $B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu\zeta_\nu - \partial_\nu\zeta_\mu$ leaves the string action invariant. Again, exactly like we usually do in QED, we could write a gauge invariant object also in this case:

$$H = dB \quad \longrightarrow \quad H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$$

called also field strength.

Let us now turn back to the S_σ action. We can immediately notice that the kinetic term is a non-quadratic function of the fields X^μ. Models with this kind of kinetic terms are called *Non-Linear Sigma Models*.
In order to simplify the kinetic term expression, we can expand the metric around the classical field solution x_0, namely $X^\mu(\sigma) = x_0^\mu + \sqrt{\alpha'}\, Y^\mu(\sigma)$, so what we obtain expanding the metric is:

$$G_{\mu\nu}(X)\,\partial_a X^\mu \partial_b X^\nu = \left[G_{\mu\nu}(x_0) + \sqrt{\alpha'}\,\partial_\omega G_{\mu\nu}(x_0)Y^\omega + \frac{1}{2}\alpha'\,\partial_\omega\partial_\rho G_{\mu\nu}Y^\omega Y^\rho + ...\right]\partial_a Y^\mu \partial_b Y^\nu$$

In this expression we can see the standard quadratic term in the Y^μ fields but also an infinite number of always higher order interacting terms. From a dimensional point of view, the first derivative term of the metric has one mass dimension, i.e. the dimension of the inverse of a length. Indeed we can write:

$$\frac{\partial G}{\partial X} \sim \frac{1}{R_c} \qquad \text{with} \quad [R_c] = -1$$

with R_c the radius of curvature of space-time[5]. The effective dimensionless coupling of the theory becomes therefore:

$$\frac{\sqrt{\alpha'}}{R_c} \qquad \text{with} \quad \left[\frac{\sqrt{\alpha'}}{R_c}\right] = 0$$

The previous expansion is then perfectly allowed if and only if $\sqrt{\alpha'}/R_c \ll 1$. This means that we are working at length scales that are much larger than the typical string length . This means clearly that we are completely ignoring the internal strings structure, therefore what we are doing is to work with a *Low-Energy Effective Field Theory.* Always from a dimensional point of view, we can notice that $[X] = -1$ and $[Y] = 0$, therefore the interactions terms in Y^μ have all dimension two. We can conclude then that the non-linear sigma model (4.4) is a renormalizable theory.

4.3.1 Low Energy Effective Action

Let us now occupy of the Weyl invariance of the action (4.6). Following the perturbative expansion of the action in (4.5), we are allowed to write:

$$G_{\mu\nu}(X) = \eta_{\mu\nu} - 4\pi g_c \; s_{\mu\nu} \; e^{ik\cdot X} \tag{4.7a}$$

$$B_{\mu\nu}(X) = -4\pi g_c \; a_{\mu\nu} \; e^{ik\cdot X} \tag{4.7b}$$

$$\Phi(X) = -4\pi g_c \; \phi \; e^{ik\cdot X} \tag{4.7c}$$

By substituting the expressions (4.7) in the action (4.6) and expanding its exponential to the second order in $\chi_{\mu\nu}$, $B_{\mu\nu}$ and Φ, we can compute the stress-energy tensor of the theory. Computing its trace what we obtain is:

$$T^a{}_a = -\frac{1}{2\alpha'}\beta^G_{\mu\nu} g^{ab}\partial_a X^\mu \partial_b X^\nu - \frac{i}{2\alpha'}\beta^B_{\mu\nu}\varepsilon^{ab}\partial_a X^\mu \partial_b X^\nu - \frac{1}{2}\beta^\Phi R$$

where:

$$\beta^G_{\mu\nu} = \alpha' R_{\mu\nu} + 2\alpha'\nabla_\mu\nabla_\nu\Phi - \frac{\alpha'}{4}H_{\mu\lambda w}H_\nu{}^{\lambda\omega} + O\left(\alpha'^2\right) \tag{4.8a}$$

$$\beta^B_{\mu\nu} = -\frac{\alpha'}{2}\nabla^\omega \; H_{\omega\mu\nu} + \alpha'\nabla^\omega\Phi H_{\omega\mu\nu} + O\left(\alpha'^2\right) \tag{4.8b}$$

$$\beta^\Phi = \frac{D-26}{6} - \frac{\alpha'}{2}\nabla^2\Phi + \alpha'\nabla_\omega\Phi\nabla^\omega\Phi - \frac{\alpha'}{24}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + O\left(\alpha'^2\right) \tag{4.8c}$$

where in β^Φ we took into account the ghost contribution to the Weyl anomaly. We notice also that in the expression of $\beta^G_{\mu\nu}$, the $R_{\mu\nu}$ tensor is the space-time, and not the world-sheet, Ricci tensor.
Requiring now Weyl invariance means to require that the stress-energy tensor is traceless. This means to have:

$$\beta^G_{\mu\nu} = \beta^B_{\mu\nu} = \beta^\Phi = 0 \tag{4.9}$$

Those set of conditions are also called *Background Field Equations.*
We can notice that those equations physically consistent, indeed we can interpret the first condition as the Einstein's equation with a source term given by the

[5]We can alternatively call it the *Target Space* of the X^μ fields.

antisymmetric tensor field and dilaton, while the second condition as the generalization of the Maxwell's equation for the divergence of the $H_{\mu\nu\rho}$ field strenght. A consistent choice (and not the only possible one) of the background fields is:

$$
\begin{aligned}
G_{\mu\nu}(X) &= \eta_{\mu\nu} \\
B_{\mu\nu}(X) &= 0 \\
\Phi(X) &= \Phi_0 = \text{const.} = \lambda
\end{aligned}
$$

with necessarily, the already seen condition, $D = 26$. It may seems that, changing the background fields, we are giving birth to a new theory but this is not true: we have always the same theory but with different background choices.
Interpreting now the conditions (4.9) as the physical equations of motion, it can be shown that they can be derived from the following space-time action:

$$
S = \frac{1}{2\kappa_0^2}\int d^D X\sqrt{-G}\, e^{-2\Phi}\left[-\frac{2\,(D-26)}{3\alpha'} + R - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + 4\partial_\mu\Phi\partial^\mu\Phi + O\left(\alpha'\right)\right] \tag{4.10}
$$

where κ_0 is a normalization constant that can be absorbed from the redefinition of the fields, therefore it has no physical meaning. The action (4.10) is called *Low-Energy Effective Action* because it is the one that governs the low-energy behaviour of space-time fields.
Making now the useful redefinition of the metric:

$$
\tilde{G}_{\mu\nu}(X) = e^{2\omega(X)}G_{\mu\nu}(X)
$$

where $\omega\,(X) = \frac{2\Phi}{D-2} \equiv \frac{2(\Phi-\Phi_0)}{D-2}$. The Ricci scalar becomes:

$$
\tilde{R} = e^{-2\omega(X)}\left[R - 2\,(D-1)\,\nabla^2\omega - (D-2)\,(D-1)\,\partial_\mu\omega\partial^\mu\,\omega\right]
$$

Redefining also the normalization constant as $\kappa = \kappa_0\; e^{\Phi_0}$, the action (4.10) becomes:

$$
S = \frac{1}{2\kappa}\int d^D X\sqrt{-\tilde{G}}\left[-\frac{2\,(D-26)}{3\alpha'}e^{\frac{4\tilde{\phi}}{D-2}} + \tilde{R} - \frac{1}{12}e^{-\frac{8\tilde{\phi}}{D-2}}H_{\mu\nu\lambda}\tilde{H}^{\mu\nu\lambda} - \frac{4}{D-2}\partial_\mu\tilde{\Phi}\tilde{\partial}^\mu\tilde{\Phi} + O\left(\alpha'\right)\right] \tag{4.11}
$$

where the $\sim$ symbol means that the indices have been raised with the $\tilde{G}_{\mu\nu}$ metric. This new expression of the action is very important because we put in the canonical form the kinetic term of the dilaton and also recovered the Hilbert action term. Indeed we have:

$$
S_H = \frac{1}{2\kappa}\int d^D x\sqrt{-\tilde{G}}\;\tilde{R}
$$

from which we can find that:

$$
\kappa \underset{D=4}{=} \sqrt{8\pi G_N} = \frac{8\pi}{M_P} = (2,43\cdot 10^{-8}\;GeV)^{-1}
$$

In the action (4.11) we can also notice that the dilaton has no potential or mass term. This means that it gives rise to long-range forces that mix with gravitational interactions and therefore cause the violation of the equivalence principle. The equivalence principle holds actually very well, so we expect that there will be a certain mechanism that would give mass to the dilaton[6].

[6] We have to remember that forces carried by massive particles has a decreasing range proportionally to the mass of the carrier.

Chapter 5

Supersymmetry and Superstrings

In this chapter we would like to introduce the concept of Supersymmetry[1] and, without going into details, see its application to String Theory and Conformal Field Theory.

5.1 Brief Introduction

String Theory is one of the best candidates for describing gravitational interaction at quantum level and give a unified vision of all fundamental interactions. However, the only kind of strings we have seen so far are bosonic strings and this leads to two great problems. The first problem is the fact that we do not have space-time fermions and this is clearly not accettable if we want to reproduce, under a certain limit, the Standard Model. The second problem is that the closed bosonic string spectrum contains tachyons which are clearly unphysical states.
In order to solve the first problem, we could introduce a transformation that takes bosons and gives us fermions, in other words we would like to introduce what is usually called a *Supersymmetry*.
Supersymmetry is realized through generators, that we call Q, that behave like space-time fermions with $1/2$ spin and satisfy the anticommutation relation:

$$\{Q, Q^\dagger\} \sim p^\mu$$

Just like any other symmetry, supersymmetry can be local or global. This implies that we can have a parameter ϵ that may depend on space-time coordinates, namely, $\epsilon(x)Q$ or simply ϵQ. A locally supersymmetric theory is necessarily diffeomorphism invariant, moreover, it can be shown that a theory with local supersymmetry is consistent if and only if it is a theory of gravity, namely a *Supergravity* theory. This fact allow us to think that if local supersymmetry is realized in nature, then an elegant extension of Standard Model that includes also gravity is permitted.
There are other advantages of considering a supersymmetric Standard Model[2]. The first is that the so called *Hierarchy Problem* is automatically solved, i.e.

[1] A detailed discussion about the role of Supersymmetry in String Theory and SM can be found in [12].

[2] In this contest, one of the theories that is actually studied is the Minimal Supersymmetric Standard Model. "Minimal" here means that we consider the Standard Model with the minimum number of supersymmetric partners, consistently with the actual particle fenomenology, because, in fact, we still haven't seen any supersymmetric particle.

supersymmetry allow us to fix the electroweak energy scale and thus predict the correct scale of the mass of the Higgs boson. The second is that the gauge coupling constants of the weak, strong and electromagnetic interactions, α_s, α_w and α_{em}, collapse to a unique gauge coupling constant α_g at the energy scale of $\Lambda \sim 10^{15}$ Gev[3]. This means that it may exist a unified theory describing these three fundamental interactions as a unique more fundamental one, called *Great Unification Theory* (GUT).
We just saw that there are strong motivations to introduce supersymmetry into SM, therefore there are good formal and phenomenological reasons to introduce supersymmetry also in String Theory.
Now, there are two ways of introducing supersymmetry in string theory: impose supersymmetry on the world-sheet (Ramond-Neveau-Schwarz procedure), or impose supersymmetry in the target-space (Green-Schwarz procedure). It can be shown that those procedures are perfectly equivalent. For simplicity, we will discuss only the RNS procedure.
We will see later that the introduction of the supersymmetry will luckily solve also the second problem, indeed, through a particular procedure called *Gliozzi-Scherk-Olive projection* or GSO projection, we will remove all the unphysical states from the closed string spectrum, tachyons included.
The world-sheet of a Closed Bosonic String is described by a 2-dimensional Conformal Field Theory while the world-sheet of a Closed Superstring is described by a 2-dimensional Superconformal Field Theory (SCFT). It is therefore useful to introduce the so called *Superconformal Symmetry* in the Conformal Field Theory context.

5.2 Superconformal Symmetry

We have seen previously, in the CFT context, many chiral operators: the stress-energy tensor $T(z)$, the fermionic and bosonic field $\Psi = (\psi(z), \bar{\psi}(\bar{z}))$ and $\phi(z)$, and finally the currents $J^a(z)$. The currents, in particular, are present when a system shows a certain symmetry and they are conserved thanks to the Noether's theorem. What we would like to do now is to introduce the conserved currents related to the supersymmetry trasformations and study their properties.

5.2.1 The $\mathcal{N} = 1$ case

Let us consider the action of a free bosonic field and a free Majorana fermion:

$$S = \frac{1}{2\pi l_s^2} \int d^2z \; (\partial X \bar{\partial} X - (\psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi})) \tag{5.1}$$

The action is invariant under a particular symmetry that exchange bosons and fermions, called, as written above, supersymmetry. The left-moving (or chiral) transformations are:

$$\delta X = \epsilon(z)\psi, \qquad \delta\psi = \epsilon(z)\partial X, \qquad \delta\bar{\psi} = 0$$

while the right-moving (or anti-chiral) transformations are:

$$\delta X = \bar{\epsilon}(\bar{z})\bar{\psi}, \qquad \delta\bar{\psi} = \bar{\epsilon}(\bar{z})\bar{\partial} X, \qquad \delta\psi = 0$$

[3]Actually into Standard Model, the three gauge couplings only approximately meet, while, considering also supersymmetry, they meet exactly in a single point.

with $\epsilon(z)$ and $\bar{\epsilon}(\bar{z})$ anticommuting objects. The invariance of the action under left-moving transformations is easy to prove:

$$\begin{aligned}
\delta S &= \frac{1}{2\pi l_s^2}\int d^2z(\partial\delta X\bar{\partial}X + \partial X\bar{\partial}\delta X - (\delta\psi\bar{\partial}\psi + \psi\bar{\partial}\delta\psi + \underbrace{\delta\bar{\psi}\partial\bar{\psi} + \bar{\psi}\partial\delta\bar{\psi}}_{=0}))\\
&= \frac{1}{2\pi l_s^2}\int d^2z(\partial(\epsilon\psi)\bar{\partial}X + \partial X\epsilon\bar{\partial}\psi - \epsilon\partial X\bar{\partial}\psi - \psi\epsilon\bar{\partial}\partial X)\\
&= \frac{1}{2\pi l_s^2}\int d^2z(-\epsilon\psi\partial\bar{\partial}X - \psi\epsilon\bar{\partial}\partial X)\\
&= 0
\end{aligned}$$

where in the last step we used the fact that $\{\epsilon, \psi\} = 0$. The proof for the right-moving transformations is exactly analogous. In our conventions, if the action is invariant under both left and right-moving transformations, we say that we have $\mathcal{N} = (1,1)_2$ supersymmetry.
The conserved currents, also called *Supercurrents*, associated to this kind of supersymmetry are:

$$G(z) = i\frac{\sqrt{2}}{l_s^2}\psi\partial X, \qquad \bar{G}(\bar{z}) = i\frac{\sqrt{2}}{l_s^2}\bar{\psi}\bar{\partial}X$$

with clearly $\bar{\partial}G = 0 = \partial\bar{G}$. Remembering now the two-points functions:

$$\begin{aligned}
\langle\partial X(z)\partial X(w)\rangle &= -\frac{l_s^2}{2}\frac{1}{(z-w)^2}\\
\langle\psi(z)\psi(w)\rangle &= l_s^2\,\frac{1}{z-w}\\
\langle\bar{\psi}(\bar{z})\bar{\psi}(\bar{w})\rangle &= l_s^2\,\frac{1}{\bar{z}-\bar{w}}
\end{aligned}$$

we can find the OPE of the chiral current $G(z)$:

$$\begin{aligned}
G(z)G(w) &= \frac{1}{(z-w)^3} + 2\frac{T(w)}{z-w} + \text{finite}\\
T(z)G(w) &= \frac{3}{2}\frac{G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w} + \text{finite}
\end{aligned}$$

where we used the following expression for the stress-energy tensor:

$$T(z) = -\frac{1}{l^s} : \partial X\partial X : -\frac{1}{2l_s^2} : \psi\partial\psi :$$

We see therefore that the chiral current $G(z)$ is a primary field with conformal weight $h = 3/2$. Introducing now the following expression for the currents:

$$\begin{aligned}
G(z) &= \sum_r G_r\ z^{-r-3/2}\\
G_r &= \oint_{C_0}\frac{dz}{2\pi i}\ z^{r+1/2}\ G(z)
\end{aligned}$$

and using the usual mode expansion for the stress-energy tensor, we obtain the relations:

$$\begin{aligned}
\{G_r, G_s\} &= \frac{\hat{c}}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0} + 2L_{r+s}\\
[L_m, G_r] &= \left(\frac{m}{2} - r\right)G_{m+r}
\end{aligned} \tag{5.2}$$

where we have defined $\hat{c} = 2c/3$. We notice that the algebra in (5.2) shows the symmetry $G \to -G$ and $T \to T$, therefore NS and R boundary conditions are possible also for supercurrents. In the NS sector we have that $r \in \mathbb{Z} + 1/2$ and $G_r |0\rangle = 0$ for $r > 0$. The R sector is analogous but the modes now take integer values, namely $r \in \mathbb{Z}$.
The zero mode clearly exists only in the R sector and, according to (5.2), it satisfies:

$$\{G_0, G_0\} = 2G_0^2 = 2L_0 - \frac{\hat{c}}{8}$$

A unitary theory requires that $h \geq \hat{c}/16$. We see that when $h > \hat{c}/16$ the state to which we apply the G_0 operator is doubly degenerate and G_0 links the two degenerate states. On the other hand, when $h = \hat{c}/16$, we have that the eigenvalues are $G_0^2 = 0$ that implies $G_0 = 0$. Just like the fermion case, we can insert the operator $(-1)^F$, with F the usual fermion number, that anticommutes with the supercurrent G. This is useful because, when we compute the trace of $(-1)^F$ in the R sector, the only contributions will come from the ground states with $h = \hat{c}/16$. This kind of trace is called the *Witten Index* of the supersymmetric theory.

5.3 Superstring Theories

Let us now turn back to String Theory. The simplest way to present the RNS procedure is to make a parellelism with the previous bosonic theory. In the bosonic string case we have the coordinates $X^\mu(\sigma,\tau)$ that are world-sheet scalars but target-space vectors, while in the supersymmetric case, we add a set of fields $\psi^\mu(\sigma,\tau)$ that behave like world-sheet spinors and target-space vectors. Analogously, we should also add to the action a fermionic superpartner of the world-sheet metric $h^{\alpha\beta}$, that we call *Gravitino*, χ^α. The gravitino, just like the world-sheet metric, behaves like a target-space scalar but like a spin 3/2 world-sheet fermion.
Now, it can be shown that the local supersymmetric action becomes:

$$\begin{aligned} S = -\frac{1}{8\pi}\int d^2\sigma\, \sqrt{-h}\Big[&\frac{2}{\alpha'}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu + 2i\overline{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \\ &- i\overline{\chi}_\alpha \rho^\beta \rho^\alpha \psi^\mu \left(\sqrt{\frac{2}{\alpha'}}\partial_\beta X_\mu - \frac{i}{4}\overline{\chi}_\beta \psi_\mu\right)\Big] \end{aligned} \tag{5.3}$$

where ψ^μ, χ^α are the chiral (left-moving) spinors, while $\bar{\psi}^\mu$, $\bar{\chi}^\alpha$ are the antichiral (right-moving) spinors. We have also ρ^α that is:

$$\rho^\alpha = (\rho^0, \rho^1) \qquad \text{with} \qquad \rho^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \rho^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

the two-dimensional Gamma matrices.
The action (5.3) is invariant under the supersymmetric version of the groups of the bosonic string action, namely (sDiff)×(sWeyl)×(Poincarè), where the first two are local trasformations of the world-sheet.
By choosing, analogously to the bosonic case, a gauge fixing for the metric, namely by setting locally $h^{\alpha\beta} = \eta^{\alpha\beta}$ and the gravitino $\chi^\alpha = 0$, we find that the residual

gauge trasformations are the superconformal symmetries. The action, with this gauge choice, thus reduces to:

$$S = -\frac{1}{8\pi}\int d^2\sigma \left(\frac{2}{\alpha'}\partial_\alpha X^\mu \partial_\alpha X_\mu + 2i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\right) \tag{5.4}$$

Proceeding then with the Faddeev-Popov quantization, because of the presence of the gravitino, we find the usual chiral and antichiral ghosts $b, c, \bar{b}, \bar{c}$ but also the so called *Ghostini* $\beta, \gamma, \bar{\beta}, \bar{\gamma}$.[4]
As we can expect now, ghostini give a not null contribution to the central charge of the theory, therefore we can compute the new dimension for which the central charge vanishes:

$$0 = c = \underbrace{d + \frac{d}{2}}_{bos.+ferm.} \quad \underbrace{-26}_{ghosts} \quad \underbrace{+11}_{ghostini} \quad \Longrightarrow \quad \frac{3}{2}d = 15$$

Thus the dimension for which the theory does not show anomalies is $d = 10$, differently from the bosonic theory case where we had $d = 26$.

5.3.1 Physical states and GSO projection

With the introduction of world-sheet fermions, we are allowed to choose the periodicity conditions for ψ^μ and $\bar{\psi}^\mu$ separately. Again, just like we saw previously, periodic fermions belong to the R sector while the antiperiodic ones to the NS sector[5]. Combining all the possibilities, we give rise to four different sectors of the theory, namely: (NS-NS), (R-NS), (NS-R), (R-R).

Now the physical superstring spectrum can be found by imposing some physical state conditions, similarly to what has been done for the Bosonic String case. It is useful thus to define two independent left and right-moving worldsheet fermion numbers, F_L and F_R, and then define two left and right-moving operators, called *G-Parity* operators, as $G_{L,R} \equiv (-1)^{F_{L,R}+1}$. We can thus associate to each sector four different combinations of the eigenvalues of the operators $G_{L,R}$, namely: $(+,+)$, $(+,-)$, $(-,+)$ and $(-,-)$.
We can now notice that, because of the level-matching condition, we have to act necessarily with an equal number of fermion operators b^μ_{-n} and $\bar{b}^\mu_{-n}$, therefore the possible NS sectors reduces to $(\text{NS}_+, \text{NS}_+)$ or $(\text{NS}_-, \text{NS}_-)$.
In the NS sector we do not have zero modes, therefore we will have only a unique ground state, i.e. we can act only with a vertex operator with charge k and build the state $|k\rangle$ in the $(\text{NS}_-, \text{NS}_-)$ sector. If we impose the physical state condition on this state, exactly like we saw at the end of Section (2.2) for the bosonic theory, we obtain $k^2 = -1/2$. Therefore this state represents a tachyon and it should be eliminated from the spectrum.
The first fermionic excited state we can build is $b^\mu_{-1/2}\, \bar{b}^\nu_{-1/2}\, |k\rangle$ and it belongs to the $(\text{NS}_+, \text{NS}_+)$ sector. By imposing again the physical state condition on it, what we obtain is $k^2 = 0$. We thus find again the graviton, the dilaton and the B-field, just like in the bosonic theory.

[4] It can be shown that ghostini behave like commuting fermions, exactly the opposite of the usual ghosts that behave like anticommuting bosons.

[5] The choice of the sectors is consistent with the one in Section (3.7), because our world-sheet is still a cylinder.

Let us now take into account also the R sector. In the R sector we have that the zero modes of the ψ^μ and $\bar{\psi}^\mu$ fields satisfy the Dirac algebra:

$$\{\psi_0^\mu, \psi_0^\nu\} = 2\eta^{\mu\nu}, \qquad \{\bar{\psi}_0^\mu, \bar{\psi}_0^\nu\} = 2\eta^{\mu\nu}, \qquad \{\psi_0^\mu, \bar{\psi}_0^\nu\} = 0$$

therefore they can be expressed in term of $2d \times 2d$ matrices that acts on the ground state of the Fock space. The dimension of the Fock space generated is $2^{d/2}$, with d the dimension of the target-space. Choosing now $d = 10$ we generate 32 ground states[6] that we can generically write as $|\alpha\rangle$. Let us notice that the zero-modes behave like target-space vectors but the 32 ground states form a 10-dimensional spinor representation of the Dirac algebra.
In order to see this fact, let us take a transformation $g \in SO(1,9)$, and $\rho(g)$ its representation on the ground states $|\alpha\rangle$. This representation will act on the generic ground state $|\alpha\rangle$ as $\rho(g)\,|\alpha\rangle$. In order to fix which representation $\rho(g)$ we are using, let us take another ground state $\psi_0^\mu\,|\alpha\rangle$. The action of the symmetry g on the fields ψ_0^μ is $\psi_0^\mu \to g^\mu{}_\nu \psi_0^\nu$.
We can now act in two different ways on the ground state $\psi_0^\mu\,|\alpha\rangle$: the first consist in acting separately on both ψ_0^μ and $|\alpha\rangle$, namely

$$\psi_0^\mu\,|\alpha\rangle \longmapsto g^\mu{}_\nu\ \psi_0^\nu\ \rho(g)\,|\alpha\rangle\,,$$

while the second consist in acting directly with $\rho(g)$ on the whole ground state $\psi_0^\mu\,|\alpha\rangle$, namely

$$\psi_0^\mu\,|\alpha\rangle \longmapsto \rho(g)\ \psi_0^\mu\,|\alpha\rangle$$

Imposing now that the two transformations have to be the same, since they hold $\forall\,|\alpha\rangle$, what we get is:

$$\rho(g)\ \psi_0^\mu = g^\mu{}_\nu\ \psi_0^\nu\ \rho(g) \implies \rho(g)\ \psi_0^\mu\ \rho^{-1}(g) = g^\mu{}_\nu\ \psi_0^\nu$$

This relation means that $\rho(g)$ is the spinorial representation of the $SO(1,9)$ and therefore that the generic state $|\alpha\rangle$ transforms as a 10-dimensional spinor, namely a space-time spinor. That is exactly what we asked for when we initially introduced supersymmetry, indeed we have now a target-space made of bosons and fermions.
Now, a generic massless state belonging to the ($R_\pm$, NS_+) sector can be written as $|\alpha\rangle \otimes \bar{\psi}^\mu_{-1/2}\,|k\rangle$, with clearly $k^2 = 0$. The lowest states in this sector can be interpreted as Gravitinos or *Dilatinos*, depending on the properties of the tensorial decomposition.
Finally, if we consider the ($R_\pm$, $R_\pm$) sector, we can write all the possible states (32×32 possibilities) as $|\alpha\rangle \otimes |\beta\rangle$. Being respectively $|\alpha\rangle$ and $|\beta\rangle$ target-space spinors, their tensor product is a target-space boson.

We see now that, keeping all the possible states that can be generated by acting with all kinds of creation operators, tachyons are still present in our spectrum. In order to eliminate them, we should apply a procedure that allow us to select in a consistent way the physical sectors of the theory. This procedure is called *GSO projection* and it consists in selecting the sectors for which the following requests are satisfied:

[6]When we act with a zero mode on the ground state $|0\rangle$, we create a degenerate state of the fundamental ground state because we are not changing its energy but only its fermonic number.

1. The OPEs of fields belonging to selected sectors must be closed[7]: the fields appearing in the OPE must not be elements of projected-out sectors.

2. The OPEs of fields belonging to selected sectors must be consistent (or local): the OPEs must not show branch cuts.

3. The partition function on the two-dimensional torus must be modular invariant.

Let us start by removing the NS$_-$ sector, for which the spectrum contains tachyons[8]. The remaining sectors can be chosen by imposing the previous conditions and by looking at the possible results of the OPEs:

$$\begin{cases} NS_+ \times NS_+ = NS_+ \\ R_\pm \times NS_+ = R_\pm \\ R_\pm \times R_\pm = NS_+ \end{cases}$$

A consistent choice of physical sectors is thus (NS$_+$, NS$_+$), (R$_\alpha$, R$_\beta$), (NS$_+$, R$_\alpha$) and (R$_\alpha$, NS$_+$), where $\alpha, \beta \in \{+, -\}$. It can be also shown that (R$_+$, R$_+$) and (R$_-$, R$_-$) sectors give rise to perfectly equivalent theories and the same is true also for (R$_+$, R$_-$) and (R$_-$, R$_+$) sectors. The possible inequivalent choices thus reduces to (R$_+$, R$_+$) and (R$_+$, R$_-$) sectors.
We see therefore that there are two consistent possibilities of choosing the R sectors and this arbitrariness give rise to two possible superstring theories:

$$\begin{array}{ll} (NS_+, NS_+), (R_+, R_+), (R_+, NS_+), (NS_+, R_+) & \text{Type II B} \\ (NS_+, NS_+), (R_+, R_-), (R_-, NS_+), (NS_+, R_-) & \text{Type II A} \end{array}$$

The two theories we built are called "Type II" because, it can be seen, they present two space-time supersymmetries, namely they are $\mathcal{N} = 2$ theories. It is also important to notice that Type II A and Type II B superstring theories are made exclusively of closed bosonic and fermionic strings.

It may seem that Type II theories are the only two possible consistent superstring theories. It can be seen instead that there are other possible constructions of a superstring theory, in particular there are three more theories that can be formulated by including also open strings or mixing the superstring formulation with the bosonic one. There exists indeed a theory called Type I superstring theory that includes also unoriented open strings. In this theory we do not have left and right-moving operators, therefore there are less possible choices of the physically consistent sectors. This theory, it can be seen, has $\mathcal{N} = 1$ space-time supersymmetry. There exists also another theory that presents $\mathcal{N} = 1$ space-time supersymmetry and it is called Heterotic superstring theory. This theory is built by constructing chiral fields with the usual superstring theory prescription, while anti-chiral fields with the bosonic string prescripton. Here, similarly to Type II theories, ghostini arise from the Faddeev-Popov quantization procedure of the chiral gravitino. By introducing then by hand sixteen anti-chiral bosons, it can

[7] Notice that the OPEs are defined on the plane, therefore the role of the two sectors is reversed, namely we have antiperiodic fermions in the R sector and periodic fermions in the NS sector.

[8] Clearly the NS$_-$ sector can be eliminated also by requiring that the R sectors are present in the theory. That is because the OPEs will not be local anymore as we can easily see right below.

be seen that the central charge vanishes if and only if the dimension of the target-space is $d = 10$ and therefore the theory becomes physically consistent because free from anomalies.
We can notice that the set of sixteen anti-chiral bosons forms an anti-holomorphic conformal field theory with a central charge $c = 16$. It can be seen that the only possible conformal field theories with this properties admit only two kinds of algebra that are $SO(32)$ and $E_8 \times E_8$. There are therefore in reality two types of Heterotic superstring theories and those are called respectively HO and HE superstring theories.
It is finally believed that there exists a 11-dimensional theory, called *M-Theory*, that contains the all known five superstring theories. In particular, it can be seen that the five superstring theories can be obtained by taking some limit cases of the M-theory itself.

5.3.2 Compactifications

As we saw in the previous section, a consistent formulation of superstring theory requires a 10-dimensional target-space. The main question we may ask therefore is, we live in a 4-dimensional space-time, where are the remaining six dimensions? The standard mechanism to explain this discrepancy is to think that the other six dimensions extend along Riemannian compact manifolds that are too small to be investigated from nowadays experiments. Therefore these six more dimensions are said to be *compactified*. From this point of view, the whole 10-dimensional space-time can be thought as the product of two different terms:

$$\mathcal{M}_{10} = \mathcal{M}_{1,3} \times K_6$$

where $\mathcal{M}_{1,3}$ is the usual 4-dimensional Minkowski space-time and K_6 is a 6-dimensional Riemannian Ricci-flat compact manifold. The procedure of going from a theory defined on $\mathcal{M}_{10}$ to a theory defined on $\mathcal{M}_{1,3}$ is properly called *Compactification*.
There is a huge number of K_6 manifolds that could a priori be chosen and the choice depends on the theory we would like to study. The choice is strictly related to the number of conserved supercharges as we will immediately show[9].
If we call Q the supercharge related to a supersymmetry and ϵ the supersymmetry trasformation parameter[10], then the condition for the supersymmetry to be unbroken is:

$$\epsilon Q \left|0\right\rangle = 0$$

If we now take a generic field ϕ, taking its supersymmetric variation, we obtain:

$$\langle \delta_\epsilon \phi \rangle \equiv \left\langle 0\right| \delta_\epsilon \phi \left|0\right\rangle = \left\langle 0\right| [\epsilon Q, \phi] \left|0\right\rangle = 0 \tag{5.5}$$

Since in a supergravity theory there exists always the gravitino χ, it can be shown that:

$$\delta_\epsilon \chi = \nabla_\mu \epsilon + ...$$

[9]Supersymmetric extensions of the Standard Model require $d = 4$ and $\mathcal{N} = 1$, therefore, in general, one should pay attention on the choice of the K_6 manifold and verify that the number of conserved supercharges after the compactification is compatible with the required properties of the SM extension.

[10]Notice that both ϵ and Q are $SO(1, 9)$ spinors.

where "..." stands for other fields contributions (B-field, dilaton and so on). Now the condition (5.5) becomes a condition on the supersymmetry transformation parameter ϵ:

$$\nabla_\mu \epsilon = 0 \quad \Longrightarrow \quad \nabla_m \epsilon = 0 \tag{5.6}$$

where $\mu = 0, ..., 9$ is the whole target-space coordinates index, while $m = 4, ..., 9$ is the index of the coordinates on K_6. The condition of unbroken supersymmetry thus reduces to compute the covariantly constant spinors on K_6. It can be shown that a necessary condition for the existence of at least a solution of the (5.6) on a Riemannian manifold, is the vanishing of its Ricci tensor. This therefore impose a constraint on the possible choices of the K_6 manifold.

The equation above admits solutions depending on the *Holonomy* of the manifold[11]. In general, if we take an oriented manifold with six real dimensions, the Holonomy group will be $SO(6)$ or one of its proper subgroups. It can be shown that the equation (5.6) admits solutions if and only if the Holonomy group is a proper subgroup of $SO(6)$. Those kind of manifolds are called *Restricted Holonomy Manifolds.*
The torus case is the easiest one because its holonomy group is trivial and the equation (5.6) always admits a solution. A more complicated case is when we take $H = SU(3)$ as possible subgroup of $H \subseteq SO(6)$. Three-dimensional complex manifolds whose holonomy group is $SU(3)$ are called *Calabi-Yau 3-folds.* Another possibility is $H = SU(2)$ that is the holonomy group of the class of 2-dimensional complex manifolds called *Calabi-Yau 2-folds.* Actually the only class of CY_2 that there exist is the K3 class of manifolds[12] and for our purpose this is the most important. If we want instead to take a 3-dimensional complex manifold whose holonomy group is $H = SU(2)$, the only possible choice we could make is to take $K_6 \simeq K3 \times \mathbb{T}^2$.
We are interested in K3 manifolds because the equation (5.6) admits only two solutions and, with some work, it can be seen that, compactifying on this kind of manifolds, are preserved exactly 1/2 of the initial amount of supercharges. This may be relevant for a phenomenological point of view if we want, for example, compute the entropy of a black hole that preserve a certain number of supercharges. This kind of computations has been done by Strominger and Vafa in [24].

5.4 Extended Superconformal Symmetry

In order now to describe supersymmetric systems with two or more supercurrents, we need to extend the concept of superconformal symmetry previously introduced. This extension will be relevant when we will consider NLSM whose target-space is $\mathbb{T}^4$ or, for example, a K3 surface.

5.4.1 The $\mathcal{N} = 2$ case

The simplest case of extended superconformal algebra is the $\mathcal{N} = (2, 0)_2$ case. This kind of algebra contains now two supercurrents, that we can call $G^{\pm}$, and a

[11] If we perform the parallel transport of a vector on a closed path on a manifold, what we obtain is not always the same initial vector $\vec{v}_1$ but a different one $\vec{v}_2$. Writing $\vec{v}_2 = R\,\vec{v}_1$, we can define the group of all possible transformations R as the *Holonomy Group* of the manifold.

[12] All K3 surfaces share the same topology but may have different metrics. This will be the key point for our purposes.

$U(1)$ current that we call J. The OPEs of this new set of currents are:

$$G^+(z)G^-(w) = \frac{2c}{3}\frac{1}{(z-w)^3} + \left(\frac{2J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w}\right) + \frac{2}{z-w}T(w) + ... \quad (5.7)$$

$$T(z)G^\pm(w) = \frac{3}{2}\frac{G^\pm}{(z-w)^2} + \frac{\partial G^\pm}{z-w} + ... \quad (5.8)$$

$$J(z)G^\pm(w) = \pm\frac{G^\pm}{z-w} + ... \quad (5.9)$$

$$T(z)J(w) = \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w} + ... \quad (5.10)$$

$$J(z)J(w) = \frac{c/3}{(z-w)^2} + ... \quad (5.11)$$

and

$$G^+(z)G^+(w) = \text{finite} \qquad G^-(z)G^-(w) = \text{finite} \quad (5.12)$$

In general we can introduce a superconformal algebra by choosing the number of supercurrents that generate the algebra and writing their consistent OPEs[13], or simply writing the conserved supercurrents for a particular action that has two supersymmetries, namely for example:

$$S = \frac{1}{2\pi l_s^2}\int d^2z \left(\sum_i \partial X^i \bar{\partial} X^i - \sum_j \left(\psi^j \bar{\partial}\psi^j + \bar{\psi}^j \partial \bar{\psi}^j\right)\right)$$

with $i, j = 1, 2$.
We notice that if we now take the real combinations $G^1 = G^+ + G^-$ and $G^2 = i(G^+ - G^-)$, the algebra shows a $SO(2) \sim U(1)$ symmetry. Moreover we have also a $\mathbb{Z}_2$ symmetry due to the transformations $G^1 \to G^1$, $G^2 \to -G^2$. The global symmetry group of the $\mathcal{N} = (2,0)_2$ algebra is thus $O(2)$.
We can use now the $SO(2) \sim U(1)$ symmetry in order to impose different boundary coinditions to the supercurrents $G^\pm(z)$. We can therefore write:

$$G^\pm(e^{2\pi i}z) = e^{\mp 2\pi i(\alpha+1/2)}G^\pm(z)$$

When $\alpha = 0$ we have the NS sector where the mode expansion index takes half-integer values. When $\alpha = \pm 1/2$ we have the R sector where instead the index of the mode expansion takes integer values.
Being the $U(1)$ symmetry an automorphism, then the algebras of the two sectors are isomorphic. This isomorphism is called *Spectral Flow* and it gives a continuous link between the two sectors. It can be written explicitly as:

$$\begin{aligned} J_n^\alpha &= J_n - \alpha\frac{c}{3}\delta_{n,0} \\ L_n^\alpha &= L_n - \alpha J_n + \alpha^2\frac{c}{6}\delta_{n,0} \\ G_{r+\alpha}^{\alpha,+} &= G_r^+ \\ G_{r-\alpha}^{\alpha,-} &= G_r^- \end{aligned} \quad (5.13)$$

where $n \in \mathbb{Z}$ and $r \in \mathbb{Z} + 1/2$.
As usual, in the NS sector, every state is generated by acting with the negative

[13] We have to look at the total conformal weight of the radial ordered supercurrents and write every possible singular term with any possible consistent combination of primary fields. One simple example is exactly the equation (5.7).

modes of T, J and $G^{\pm}$ on the ground state $|h, q\rangle$, where h is the eigenvalue of L_0 and q the eigenvalue of J_0. In the R sector we have also to take into account the zero mode of the two supercurrents, in particular the fact that:

$$(G^{\pm})^2 = 0, \qquad \{G_0^+, G_0^-\} = 2\left(L_0 - \frac{c}{24}\right)$$

Just like the $\mathcal{N} = (1,0)_2$ case, the unitarity of the theory requires that $h \geq c/24$. When $h > c/24$, the G_0^+ and G_0^- operators do not anticommute therefore their algebra admits, it can be shown, a 2-dimensional representation. This means that the ground states space is 2-dimensional and thus there is degeneracy. instead when $h = c/24$, we have that the $G_0^{\pm}$ operators anticommute and therefore their algebra admits a 1-dimensional representation. This means again that the ground states space is 1-dimensional and therefore there is no degeneracy.
Inserting, just like in the $\mathcal{N} = 1$ case, the $(-1)^F$ operator in the trace computed in the R sector, we see that the only contributions that do not cancel each others are the ground states ones. We will see later how this fact will affect our final calculations.
From the relations in (5.13), we deduce that:

$$J_0^{R\pm} = J_0^{NS} \mp \frac{c}{6}, \qquad L_0^{R\pm} - \frac{c}{24} = L_0^{NS} \mp J_0^{NS} \tag{5.14}$$

From this relation we find that the non degenerate ground state in the R sector, namely the ones that have $h = c/24$, correspond to NS states with $2h = q$. This kind of states are called *Chiral States* and they are generated from the chiral field operators acting on vacuum state. We notice also that the states in NS sector with $h = 0 = q$ correspond to R states with $h = c/24$, $q = \pm c/6$[14]. Applying again the spectral flow to this R states we obtain the NS states with $h = c/6$, $q = \pm c/3$, therefore we understand that there must exist a chiral operator with this particular parameters.

5.4.2 The $\mathcal{N} = 4$ case

The last case of superconformal algebra that is relevant to our interests, and in general to superstring theory, is the $\mathcal{N} = (4,0)_2$ superconformal algebra. It contains the usual stress-energy tensor, the four supercurrents G^α and $\bar{G}^\alpha$ with $\alpha = 1, 2$ and three current that generate the current algebra of $SU(2)_k$ where k is called the *level* of the $SU(2)$ algebra. The Virasoro central charge is related to k as $c = 6k$. The OPEs of the objects we introuced above are:

$$J^a(z)J^b(w) = \frac{k}{2}\frac{\delta^{ab}}{(z-w)^2} + i\epsilon^{abc}\frac{J^c(w)}{z-w} + \text{finite} \tag{5.15}$$

$$J^a(z)G^\alpha(w) = \frac{1}{2}\sigma^a_{\alpha\beta}\frac{G^\beta}{z-w} + \text{finite} \tag{5.16}$$

$$J^a(z)\bar{G}^\alpha(w) = -\frac{1}{2}\sigma^a_{\alpha\beta}\frac{\bar{G}^\beta}{z-w} + \text{finite} \tag{5.17}$$

$$G^\alpha(z)\bar{G}^\beta(w) = \frac{4k\delta^{\alpha\beta}}{(z-w)^3} + 2\sigma^a_{\alpha\beta}\left[\frac{2J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w}\right] + 2\delta^{\alpha\beta}\frac{T^{\alpha\beta}}{z-w}\text{finite} \tag{5.18}$$

[14]Notice that they are the maximal "charged" ground states in the R sector.

and:

$$G^{\alpha}(z)G^{\beta}(w) = \text{finite} \qquad \bar{G}^{\alpha}(z)\bar{G}^{\beta}(w) = \text{finite}$$

where $\sigma^{a}_{\alpha\beta}$, $a = 1, 2, 3$, are the Pauli matrices.
Analogously to the $\mathcal{N} = (2, 0)_2$ there are different boundary conditions that can be imposed to the supercurrents G^{α} and $\bar{G}^{\alpha}$. Again we have an NS and R sector and, as usual, the index of the mode expansion of the supercurrents will take half-integer values (NS) or integer values (R). We have also another spectral flow, similar to the one in the $\mathcal{N} = (2, 0)_2$ case, that link the NS and R sectors.
Again primary states are annihilated by the positive modes of the expansion and are labelled by the conformal weight h and the $SU(2)_k$ spin j. A unitary theory requires that $j \leq k/2$.
The case of $k = 1$ is relevant for superstring compactification, in particular for non-linear sigma models built on K3 surfaces.

Chapter 6

CFT on a Torus

In this chapter we would like to introduce the Conformal Field Theory on a torus worldsheet, in particular we will compute the partition function of the theory and study the properties of Bosons and Fermions fields defined on it.

6.1 The Partition Function

As we already saw in Section (4.2.1), the torus is a surface that can be defined as $\mathbb{T} \sim \mathbb{C}/\Lambda$, where Λ is the lattice generated by two complex numbers ω_1, ω_2 which we can call the periods of the lattice. The parameter that univocally identifies the conformal structure of the torus is what we called the modular parameter (or modulus) $\tau \equiv \frac{\omega_2}{\omega_1} = \tau_1 + i\tau_2$.
Now, before proceeding to the direct calculation of the partition function, we can notice that a torus is simply a cylinder with the boundary circles glued together. We can therefore extend the operator formalism defined on the cylinder to the torus, by imposing periodic boundary conditions along it.
By taking the the equation (3.22) we can see that:

$$T_{cyl}(w) = \sum_{n\in\mathbb{Z}} L_n \; z^{-n} - \frac{c}{24} = \sum_{n\in\mathbb{Z}} \left(L_n - \frac{c}{24}\delta_{n,0} \right) \; z^{-n}$$

and therefore:

$$L_{0,cyl} = L_{0,pl} - \frac{c}{24}$$

and analogously we can obtain the relation for $\overline{L}_{0,cyl}$. [1]
We can now define the partition function on the torus by just choosing the spatial and time directions respectively as the real and imaginary axis of $\mathbb{C}/\Lambda$. We can now define the partition function as:

$$Z(\tau_1, \tau_2) \equiv \mathrm{Tr}_{\mathcal{H}} e^{-2\pi(\tau_2 H - i\tau_1 P)}$$

where the trace is taken over all states in the Hilbert space $\mathcal{H}$. We have also:

$$H = \frac{2\pi}{l}\left(L_0 + \overline{L}_0 - \frac{c}{24} - \frac{\overline{c}}{24} \right)$$
$$P = \frac{2\pi}{l}\left(L_0 - \overline{L}_0 - \frac{c}{24} + \frac{\overline{c}}{24} \right)$$

[1] From now we will call $L_{0,pl} \equiv L_0$.

that are respectively the Hamiltonian and the momentum operator defined on the cylinder. If we now consider to have a theory in which $c = \overline{c}$ and the length of the circumference of the cylinder as $l = 2\pi$, the partition function becomes:

$$\begin{aligned} Z(\tau, \bar{\tau}) &= \mathrm{Tr}_{\mathcal{H}}\ e^{\pi i[(\tau-\bar{\tau})(L_0+\overline{L}_0-\frac{c}{24}-\frac{c}{24})+(\tau+\bar{\tau})(L_0-\overline{L}_0)]} \\ &= \mathrm{Tr}_{\mathcal{H}}\ e^{2\pi i[\tau(L_0-\frac{c}{24})+\bar{\tau}(\overline{L}_0-\frac{c}{24})]} \end{aligned}$$

If we now define:

$$q \equiv e^{2\pi i\tau} \qquad \text{and} \qquad \overline{q} \equiv e^{-2\pi i\bar{\tau}}$$

what we obtain is:

$$Z(\tau, \bar{\tau}) = \mathrm{Tr}_{\mathcal{H}}\bigg(q^{L_0-\frac{c}{24}}\ \overline{q}^{\bar{L}_0-\frac{c}{24}}\bigg) = (q\overline{q})^{-c/24}\ \mathrm{Tr}_{\mathcal{H}}\bigg(q^{L_0}\ \bar{q}^{\bar{L}_0}\bigg) \tag{6.1}$$

We have now to remember that, as we saw in Section 4.2.1, the domain of the parameter τ is the moduli space of the torus, namely $M_1 \sim \mathbb{H}_+/PSL(2,\mathbb{Z})$.
We know also that different tori, whose parameters τ can be linked by a certain modular transformation, have the same conformal structure[2]. This means that also their partition functions can be linked with a modular transformation. Since now we are working with a theory that is invariant under conformal transformations, also the partition function must be invariant under modular transformations. The partition function thus will be a function $Z(\tau, \bar{\tau})$ that are invariant under the transformations:

$$\tau \to \frac{a\tau + b}{c\tau + d}, \qquad \bar{\tau} \to \frac{a\bar{\tau} + b}{c\bar{\tau} + d}$$

with $ad - bc = 1$ and $a, b, c, d \in \mathbb{Z}$.

6.1.1 The Free Boson System

What we would like to do now is the partition function in (6.1) for a single free boson system. We can now anticipate that what we will find is:

$$Z_{bos}(\tau) = \frac{1}{\sqrt{\tau_2}|\eta(\tau)|^2} = \frac{1}{\sqrt{\mathrm{Im}\tau}|\eta(\tau)|^2} \tag{6.2}$$

We can verify that the partition function is modular invariant. Since every modular transformation can be written as the composition of T and S transformations, using the definition we saw in (4.3), we need only to check how those two act on the Dedekind function:

$$\begin{aligned} \eta^T(\tau) &= \eta(\tau+1) = e^{\frac{2\pi i}{24}}\ \eta(\tau) \\ \eta^S(\tau) &= \eta\bigg(-\frac{1}{\tau}\bigg) = \sqrt{-i\tau}\ \eta(\tau) \end{aligned}$$

It is easy now to check that the partition function is actually invariant under a generic modular transformation.

Let us now show how to obtain the result in (6.2). We have initially the following expression for the partition function:

$$Z(\tau, \bar{\tau}) = (q\overline{q})^{-c/24}\ \mathrm{Tr}_{\mathcal{H}}\bigg(q^{L_0}\ \bar{q}^{\bar{L}_0}\bigg)$$

[2]This means that the metrics of both tori are linked with a conformal transformation.

Using now the stress-energy tensor expression obtained in (3.18) and substituting the expansion of the bosonic field in (3.24), we can obtain the full expression of the L_0 and $\bar{L}_0$ operators. The trace therefore breaks up into a sum over the occupation numbers $N_{\mu n}$ and $\tilde{N}_{\mu n}$ for each μ and n and an integral over momentum k_μ. The dependence from the momentum k_μ of the Virasoro generators is straightforward by looking at the expression of a_0 in (3.23). What we can finally obtain is:

$$Z(\tau,\overline{\tau}) = V_d\,(q\overline{q})^{-d/24}\int\frac{d^dk}{(2\pi)^d}\;e^{-\pi\tau_2\alpha' k^2}\prod_{\mu=1}^{d}\prod_{n=1}^{\infty}\sum_{N_{\mu n}\tilde{N}_{\mu n}=0}^{\infty} q^{nN_{\mu n}}\overline{q}^{n\tilde{N}_{\mu n}}$$

where we substituted:

$$\sum_k \quad\longrightarrow\quad (V_d)\int\frac{d^dk}{(2\pi)^d}$$

We know now that:

$$\sum_{N=0}^{\infty} q^{nN} = \frac{1}{1-q^n}$$

therefore the partition function becomes:

$$Z(\tau,\overline{\tau}) = iV_d Z_X(\tau,\bar{\tau})^d$$

with:

$$Z_X(\tau,\bar{\tau}) = \frac{1}{\sqrt{4\pi^2\alpha'\tau_2}}\frac{1}{|\eta(\tau)|^2}$$

where the i factor in the previous expression comes from the Wick rotation needed to have a well defined integral over momenta. By fixing the constant $\alpha' = \frac{1}{4\pi^2}$, we can obtain exactly the expression in (6.2).

6.1.2 The Free Fermion System

Analogously to the free boson system, we can start doing the same calculations also for the free fermion case and find the partition function of this system. The action in this case is:

$$S = \frac{1}{2\pi}\int d^2x\;(\bar{\psi}\partial\bar{\psi} + \psi\bar{\partial}\psi)$$

The most general periodicity conditions for a fermionic field defined on a torus are:

$$\psi(z+\omega_1) = e^{2\pi i v}\psi(z) \qquad\qquad \psi(z+\omega_2) = e^{2\pi i u}\psi(z)$$

but, as we saw in (3.28), the only ones that are compatible with the action are:

$$\begin{aligned}
(v,u) &= (0,0) &&\text{or}\quad (R,R)\\
(v,u) &= (0,\tfrac{1}{2}) &&\text{or}\quad (R,NS)\\
(v,u) &= (\tfrac{1}{2},0) &&\text{or}\quad (NS,R)\\
(v,u) &= (\tfrac{1}{2},\tfrac{1}{2}) &&\text{or}\quad (NS,NS)
\end{aligned} \tag{6.3}$$

where "NS" stands for Neveau-Schwarz sector and "R" for Ramond sector. We shall assume now that the same periodicity conditions are satisfied by the antiholomorphic fermionic field $\bar{\psi}$. We can now denote the partition function computed with the (v, u)[3] periodicity conditions, with $Z_{v,u}$. Because of the decoupling between ψ and $\bar{\psi}$, denoting by $d_{v,u}$ the partition function computed only for the holomorphic component, the complete partition function will assume the following structure:

$$Z = \sum_{v,u} |d_{v,u}|^2$$

What we would like to do now is to compute the partition function, analogously to what we did for the free boson case, by using the operator formalism. In order to do this, we need to implement the periodicity conditions in time direction. As we saw in Section 3.7, we have to remember that, when we chose a periodicity condition for the fermionic field on the plane, it would get exchanged on the cylinder[4]. In order then to have the natural antiperiodicity condition in time direction, since we are working with the L_0 operator defined on the plane and with the partition function defined on the cylinder, we need to insert in the case of periodic condition $u = 0$, an operator that anticommutes with $\psi(z)$, whatever the value of z is. This operator is $(-1)^F$, with:

$$F = \sum_{k\geq 0} F_k \qquad \text{with} \qquad F_k = b_{-k}b_k \quad (k > 0)$$

where F_0 is defined in the space-periodic case, equal to 0 when it acts on $|0\rangle$ and 1 when it acts in $b_0 |0\rangle$. F is called *Fermion Number*. It can be defined analogously a fermion number $\bar{F}$ for the antiholomorphic component $\bar{\psi}$

With this kind of procedure, the holomorphic partition functions $d_{v,u}$ become:

$$\begin{aligned}
d_{0,0} &= \frac{1}{\sqrt{2}}\mathrm{Tr}\,(-1)^F\, q^{L_0 - 1/48} &&= \frac{1}{\sqrt{2}}\mathrm{Tr}\,(-1)^F\, q^{\sum_k kb_{-k}b_k + 1/24} \\
d_{0,\frac{1}{2}} &= \frac{1}{\sqrt{2}}\mathrm{Tr}\; q^{L_0 - 1/48} &&= \frac{1}{\sqrt{2}}\mathrm{Tr}\; q^{\sum_k kb_{-k}b_k + 1/24} \\
d_{\frac{1}{2},0} &= \mathrm{Tr}\,(-1)^F\, q^{L_0 - 1/48} &&= \mathrm{Tr}\,(-1)^F\, q^{\sum_k kb_{-k}b_k - 1/48} \\
d_{\frac{1}{2},\frac{1}{2}} &= \mathrm{Tr}\; q^{L_0 - 1/48} &&= \mathrm{Tr}\; q^{\sum_k kb_{-k}b_k - 1/48}
\end{aligned}$$

where the factor $1/\sqrt{2}$ for the first two lines is conventional. We notice that we used the expressions of L_0 obtained by substituting the fermionic field expansion (3.29), in the expression (3.21) of the stress-energy tensor, namely:

$$\begin{aligned}
L_0 &= \sum_{k>0} kb_{-k}b_k \qquad (NS : k \in \mathbb{Z} + \frac{1}{2}) \\
L_0 &= \sum_{k>0} kb_{-k}b_k + \frac{1}{16} \qquad (R : k \in \mathbb{Z})
\end{aligned}$$

with the adding constant fixed from the relations (3.30) and (3.31).

Let us now explicitly compute:

$$\begin{aligned}
d_{\frac{1}{2},0} &= q^{-1/48}\mathrm{Tr}_{\mathcal{H}} \prod_{k>0} q^{kb_{-k}b_k}\,(-1)^{F_k} \\
&= q^{-1/48} \prod_{k>0} \left(\mathrm{Tr}_{\mathcal{H}_k}\; q^{kb_{-k}b_k}\,(-1)^{F_k}\right)
\end{aligned}$$

[3]They are also called *spin structure* for the fermion.

[4]A torus is simply a cylinder with its two ends glued together.

where we have used the fact that Tr(AB) with A and B operators acting on different factors of a tensor product, is simply Tr(A)Tr(B). Indeed what we did in this case is to write our Hilbert space as $\mathcal{H} \simeq \otimes_{k>0}^{\infty} \mathcal{H}_k$ with $\mathcal{H}_k$ the two-dimensional Hilbert subspace[5] generated by $|0\rangle$ and $b_k|0\rangle$, with $k \in \mathbb{Z}+1/2$.
Trivially the fermion number can assume only the values 0 or 1, therefore we can compute the previous traces:

$$\begin{aligned} \text{Tr } q^{kb_{-k}b_k} &= 1+q^k \\ \text{Tr } q^{kb_{-k}b_k}(-1)^{F_k} &= 1-q^k \end{aligned}$$

Doing these computations for all the $d_{v,u}$ we need, what we can obtain is:

$$\begin{aligned} d_{0,0} &= \frac{1}{\sqrt{2}}\, q^{1/24} \prod_{n=0}^{\infty} (1-q^n) && = 0 \\ d_{0,\frac{1}{2}} &= \frac{1}{\sqrt{2}}\, q^{1/24} \prod_{n=0}^{\infty} (1+q^n) && = \sqrt{\frac{\theta_2(\tau)}{\eta(\tau)}} \\ d_{\frac{1}{2},0} &= q^{-1/48} \prod_{r=1/2}^{\infty} (1-q^r) && = \sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}} \\ d_{\frac{1}{2},\frac{1}{2}} &= q^{-1/48} \prod_{r=1/2}^{\infty} (1+q^r) && = \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}} \end{aligned}$$

where we have defined the Theta functions as:

$$\begin{aligned} \theta_2(\tau) &= 2q^{1/8} \prod_{n=1}^{\infty} (1-q^n)(1+q^n)^2 \\ \theta_3(\tau) &= \prod_{n=1}^{\infty} (1-q^n)\left(1+q^{n-1/2}\right)^2 \\ \theta_4(\tau) &= \prod_{n=1}^{\infty} (1-q^n)\left(1-q^{n-1/2}\right)^2 \end{aligned}$$

We can notice that the vanishing of $d_{0,0}$ is a consequence of the fact that the b_0 operator creates, when applied to a generic state with a certain energy and fermion number, a state with the same energy but with opposite fermion number[6]. Because of this effect, every state $b_0|n\rangle$ cancels every contribution coming from the $|n\rangle$ state and therefore $d_{0,0}$ is identically null.
We are now ready to write the final expression of the fermionic partition function:

$$\begin{aligned} Z &= Z_{0,0} + Z_{\frac{1}{2},0} + Z_{0,\frac{1}{2}} + Z_{\frac{1}{2},\frac{1}{2}} \\ &= |d_{\frac{1}{2},0}|^2 + |d_{0,\frac{1}{2}}|^2 + |d_{\frac{1}{2},\frac{1}{2}}|^2 \\ &= \left|\frac{\theta_2(\tau)}{\eta(\tau)}\right| + \left|\frac{\theta_3(\tau)}{\eta(\tau)}\right| + \left|\frac{\theta_4(\tau)}{\eta(\tau)}\right| \end{aligned}$$

[5]This is clear since every combination of the creation operators like $(b_k)^l$ with $l > 1$ is identically null because of the anticommutation relation.

[6]We saw explicitly this fact in Section 3.7.

It can be now shown that the partition function is invariant under modular transformations, that is exactly what we need. It can be seen that, for the modular transformation $\tau \to -1/\tau$ we have:

$$d_{0,1/2}(-1/\tau) = d_{1/2,0}(\tau)$$
$$d_{1/2,0}(-1/\tau) = d_{0,1/2}(\tau)$$
$$d_{1/2,1/2}(-1/\tau) = d_{1/2,1/2}(\tau)$$

while for the modular transformation $\tau \to \tau + 1$:

$$d_{0,1/2}(\tau+1) = e^{i\pi/8}\, d_{0,1/2}(\tau)$$
$$d_{1/2,0}(\tau+1) = e^{-i\pi/24}\, d_{1/2,1/2}(\tau)$$
$$d_{1/2,1/2}(\tau+1) = e^{-i\pi/24}\, d_{1/2,0}(\tau)$$

so what we can conclude is that the partition function is invariant under modular transformations.

6.2 Compactified Boson

We would like now to extend the discussion we did in Subsection (3.6.2) to a free boson compactified on a circle. The main difference is that now the bosonic field has two possible winding numbers on the target space (m, m') because of the two possible cycles that the field describe on the worldsheet, that is, analogously to the previous discussion, a torus. We can therefore consider:

$$\varphi\left(z + k\omega_1 + k'\omega_2\right) = \varphi(z) + 2\pi R\left(km + k'm'\right) \qquad k, k' \in \mathbb{Z}$$

The doublet of integers (m, m') specify the topological class of configurations of field that obey to the previous periodicity conditions. We can then define a partition function $Z_{m,n}$ obtained by integrating over the configurations of the class we are considering. What we do thus is to perform the path-integral after we decomposed ϕ in a classical term $\phi^{cl}_{m,m'}$ that satisfies the classical equation of motion, namely $\partial\bar{\partial}\phi^{cl}_{m,m'} = 0$, and in a periodic field $\tilde{\phi}$. It can be shown that this procedure consists in writing:

$$\phi = \phi^{cl}_{m,m'} + \tilde{\phi}$$

with:

$$\phi^{cl}_{m,m'} = 2\pi R\left\{\frac{z}{\omega_1}\frac{m\overline{\tau} - m'}{\overline{\tau} - \tau} - \frac{z}{\omega_1^*}\frac{m\tau - m'}{\overline{\tau} - \tau}\right\}$$

It can be checked that the field ϕ decomposed in this way has the right periodicity condition and it is real.
The action of the ϕ field becomes now:

$$S[\phi] = S[\phi^{cl}_{m,m'}] + S[\tilde{\phi}]$$

Since now $\partial\bar{\partial}\phi^{cl}_{m,m'} = \nabla^2\phi^{cl}_{m,m'} = 0$, the crossed term in the action are null after integrating by parts, namely:

$$\int d^2x\ \nabla\varphi^{cl}_{m,m}\nabla\tilde{\varphi} = -\int d^2x\ \tilde{\varphi}\nabla^2\varphi^{cl}_{m,m} = 0$$

We can now compute easily:

$$\begin{aligned}
S[\varphi^{cl}_{m,m'}] &= \frac{1}{8\pi}\int d^2x \left(\nabla\varphi^{cl}_{m,m'}\right)^2 \\
&= \frac{1}{2\pi}\int dz d\bar{z}\ \partial\varphi^{cl}_{m,m'}\bar{\partial}\varphi^{cl}_{m,m'} \\
&= 2\pi R^2 A \frac{1}{|\omega_1|^2}\left|\frac{m\bar{\tau}-m'}{\bar{\tau}-\tau}\right|^2 \\
&= \pi R^2 \frac{|m\tau - m'|^2}{2\ \mathrm{Im}\tau}
\end{aligned}$$

with $A = \mathrm{Im}(\omega_2\omega_1^*)$ the area of the torus. The partition function therefore is:

$$Z_{m,m'}(\tau) = Z_{bos}(\tau)\ e^{-\frac{\pi R^2|m\tau - m'|^2}{2\ \mathrm{Im}\tau}}$$

where $Z_{bos}(\tau)$ is the partition function obtained by integrating over the periodic field $\tilde{\phi}$. We computed its expression in (6.2). We have now to verify if the total partition function is invariant under modular transformations. It can be easily checked that:

$$\frac{|m\tau - m'|^2}{2\ \mathrm{Im}\tau} \quad \longrightarrow \quad \frac{|ma\tau + bm - m'c\tau - m'd|^2}{\mathrm{Im}\tau}$$

therefore the doublet (m, m') transforms like:

$$\begin{pmatrix} m \\ m' \end{pmatrix} \longrightarrow \underbrace{\begin{pmatrix} a & -c \\ -b & d \end{pmatrix}}_{\equiv R} \begin{pmatrix} m \\ m' \end{pmatrix}$$

We can notice that the R matrix is the inverse matrix of $SL(2,\mathbb{Z})$ with which we transform the module τ. This means that the doublet transforms like the periods (k_1, k_2) of the reciprocal lattice.
By taking now $a = b = d = 1$, $c = 0$ and $a = d = 0$, $c = -1 = -b$, namely taking the T and S transformations respectively, we see that:

$$\begin{aligned}
Z_{m,m'}(\tau+1) &= Z_{m,m'-m}(\tau) \\
Z_{m,m'}\left(-\frac{1}{\tau}\right) &= Z_{-m',m}(\tau)
\end{aligned}$$

What we can do now is, in order to obtain a modular invariant partition function, to sum over all the doublets $(m, m') \in \mathbb{Z}^2$ with equal weights $Z_{bos}(\tau)$, namely we can write:

$$Z(\tau, R) = \frac{R}{\sqrt{2}} Z_{bos}(\tau) \sum_{m,m'} e^{-\frac{\pi R^2|m\tau - m'|^2}{2\ \mathrm{Im}\tau}} \tag{6.4}$$

which is clearly invariant under modular transformations. Notice that the factor $\frac{R}{\sqrt{2}}$ can be derived from a zero-mode integration of ϕ.
Let us now introduce an important formula that we can use to rewrite in a useful way the previous partition function. The formula is called *Poisson's resummation*

formula and it is[7]:

$$\sum_{n\in\mathbb{Z}} e^{-\pi a n^2+bn} = \frac{1}{\sqrt{a}}\sum_{k\in\mathbb{Z}} e^{-\frac{\pi}{a}(k+b/2\pi i)^2}$$

Setting now $a = R^2/2\tau_2$, $b = \pi m R^2\tau_1/\tau_2$, with $\tau = \tau_1 + i\tau_2$, the partition function in (6.4) becomes:

$$Z(\tau, R) = \frac{1}{|\eta(\tau)|^2}\sum_{e,m\in\mathbb{Z}} q^{(e/R+mR/2)^2/2}\ \bar{q}^{(e/R-mR/2)^2/2} \tag{6.5}$$

where we notice that the $\sqrt{\mathrm{Im}\tau}$ factor, contained in the $Z_{bos}(\tau)$ pre-factor, has been reabsorbed in the resummation formula. The partition function is compatible with the expression of L_0 and $\bar{L}_0$ in the (3.25) and (3.26). The compactification of the free boson give rise to an adding term in the conformal dimension expression, namely:

$$h_{e,m} = \frac{1}{2}\left(\frac{e}{R}+\frac{mR}{2}\right)^2 \qquad \bar{h}_{e,m} = \frac{1}{2}\left(\frac{e}{R}-\frac{mR}{2}\right)^2$$

We can now see that fields with $e \neq 0$ correspond to vertex operator $e^{ie\phi/R}$ with charge e/R, namely we can consistently interprete, as we saw in the (3.6.1) Subsection, the e/R charge as the momentum of the created state. Instead when $m \neq 0$ we can interprete the contribution coming from the value of m, as some vortex configurations of the field ϕ that correspond to lines on which the fields have discontinuity of $2\pi m R$. A field with $e, m \neq 0$ is a superposition of these two possibilities.
We can at last notice that this model, by performing the interchanges $R \leftrightarrow 2/R$ and $e \leftrightarrow m$, shows an interesting e-m duality that is usually called *T-Duality*.

6.2.1 Multi-component Compactified Bosons

We would like now to compute the modular invariant partition functions of a set of n compactified free bosons, namely a multi-component boson system. However, before proceeding, we need to introduce the concept of multidimensional lattice. A n-dimensional lattice Γ is a set of points in $\mathbb{R}^n$ with the property that its elements can be written in the following way:

$$\Gamma = \left\{ x = \sum_i x_i \epsilon_i \;\middle|\; x_i \in \mathbb{Z} \right\}$$

with ϵ_i a set of n basis vectors. A lattice is said to be Lorentzian if the signature of its inner product is:

$$(\underbrace{+, ..., +}_{=s}, \underbrace{-, ..., -}_{=\bar{s}})$$

[7] The demonstration is easy by starting from the identity:

$$\sum_{n\in\mathbb{Z}} \delta(x-n) = \sum_{k\in\mathbb{Z}} e^{2\pi i k x}$$

and integrating it over $e^{-\pi a x^2+bx}$.

with s the number of $+$ signs and $\bar{s}$ the number of $-$ signs. We define then the volume of the lattice to be:

$$\mathrm{Vol}(\Gamma) = \det[\epsilon_i \cdot \epsilon_j]$$

We can analogously define the dual lattice Γ^* as the set of points p that satisfy the relation $x \cdot p \in \mathbb{Z}$. A natural set of basis vectors for Γ^* are ϵ_i^*, satisying the relation $\epsilon_i \cdot \epsilon_j^* = \delta_{ij}$. The volume of the dual lattice is $\mathrm{Vol}(\Gamma^*) = 1/\mathrm{Vol}(\Gamma)$.
A lattice is called *self-dual* if $\Gamma^* = \Gamma$, from which it follows that $\mathrm{Vol}(\Gamma) = 1$.
Moreover we call *integer* a lattice whose elements satisfy the property $x \cdot y \in \mathbb{Z}$ and *even-integer* if, for all its elements, $x^2 \in 2\mathbb{Z}$.
Considering now the partition function in (6.5), with the useful ridefinition $e \to m$ (momentum) and $m \to w$ (winding), and defining:

$$p = \frac{m}{R} + \frac{wR}{2} \qquad \bar{p} = \frac{m}{R} - \frac{wR}{2}$$

we can write:

$$Z(\tau) = \frac{1}{|\eta(\tau)|^2} \sum_{p,\bar{p}} e^{i\pi\tau p^2 - i\pi\bar{\tau}\bar{p}^2}$$

where the sum is taken over all possible integer values of e and m. We notice now that we can write $(p, \bar{p}) = me_1 + we_2$, with $e_1 = (1/R, 1/R)$ and $e_2 = (R/2, -R/2)$. If we now take the Lorentzian product as:

$$(x, y) \cdot (x', y') = xx' - yy'$$

we see that the points $(p, \bar{p})$ forms an even-integer, self-dual and Lorentzian lattice $\tilde{\Gamma}$ generated by e_1 and e_2 basis vectors[8]. Let us now see why this fact is closely related to the modular invariance of the partition function.
Let us consider n bosons with only holomorphic modes and analogously $\bar{n}$ bosons with only antiholomorphic modes. The Virasoro generators are:

$$L_0 = \frac{1}{2}p^2 + \sum_{i=1}^{n} \sum_{k>0} a_{-k}^{(i)} a_k^{(i)}$$
$$\bar{L}_0 = \frac{1}{2}\bar{p}^2 + \sum_{i=1}^{\bar{n}} \sum_{k>0} \bar{a}_{-k}^{(i)} \bar{a}_k^{(i)}$$

where $(p, \bar{p})$ belongs to a lattice $\bar{\Gamma}$ with signature $(n, \bar{n})$, analogous to the lattice $\tilde{\Gamma}$ we described above. The partition function of the system is:

$$Z_{n,\bar{n}}(\tau) = \frac{1}{\eta(\tau)^n \bar{\eta}(\tau)^{\bar{n}}} \sum_{p,\bar{p}} e^{i\pi\tau p^2 - i\pi\bar{\tau}\bar{p}^2}$$

Let us now see how modular transformations act on $Z_{n,\bar{n}}(\tau)$. It can be easily checked that under the transformation $\tau \to \tau + 1$ we have:

$$Z_{n,\bar{n}}(\tau + 1) = Z_{n,\bar{n}}(\tau) e^{2\pi i \frac{(n-\bar{n})}{24}} \tag{6.6}$$

[8]It can be immediately checked that $e_1 \cdot e_1 = e_2 \cdot e_2 = 0$ and $e_1 \cdot e_2 = 1$.

where we have used the fact that $p^2 - \bar{p}^2 \in 2\mathbb{Z}$.
In order now to see how the transformation $\tau \to -1/\tau$ acts on $Z_{n,\bar{n}}(\tau)$, we need to use an extended version of the Poisson's resummation formula, that is:

$$\sum_{q\in\Gamma} e^{-\pi a q^2 + q\cdot b} = \frac{1}{\mathrm{Vol}(\Gamma)} \frac{1}{a^{n/2}} \sum_{p\in\Gamma^*} e^{-\frac{\pi}{a}\left(p+\frac{b}{2\pi i}\right)} \tag{6.7}$$

where a is a constant with Re $a > 0$ and b an n-dimensional constant vector. Applying now the (6.7) formula to the (6.6) partition function, it can be shown that, remembering that $\tilde{\Gamma}$ is self-dual, the partition function $Z_{n,\bar{n}}(\tau)$ is invariant under the transformation $\tau \to -1/\tau$.
We can conclude now that $Z_{n,\bar{n}}(\tau)$ is a modular invariant partition function if and only if the lattice of the momentum vectors $\tilde{\Gamma}$ is even-integer and self-dual. Moreover we require that the number of holomorphic and antiholomorphic bosons satisfy the constraint:

$$\frac{(n-\bar{n})}{24} \in \mathbb{Z}$$

or alternatively $n - \bar{n} = 0 \mod 24$.

6.3 The Elliptic and Twining Genus

We have computed untill now only partition functions in the case of boson and fermion systems. As we saw in Subsection (6.1.2), chiral and anti-chiral fermions may have different periodicity conditions, in particular they may be periodic or anti-periodic in space and time direction respectively. Let us now specify the notation we are going to use: we will write "(Chiral fermionic sector, Anti-chiral fermionic sector)" to specify the periodicity conditions for the corresponding type of fermionic field, for example, the compact notation (R,NS) means that we take a periodic chiral fermion and an anti-periodic anti-chiral fermion[9].
It is in general very complicated to compute explicitly the partition function for some NLSMs defined over some kinds of manifolds, like the Calabi-Yau manifolds. In particular it can be seen that the partition function will depend on the metric and the B-field of the target-space manifold. It is thus useful to introduce, for a superconformal field theory, a new quantity that is independent of the metric and the B-field choice but depends only on the topology of the target-space manifold. This quantity is called *Elliptic Genus*. By opportunely modifying the Elliptic Genus, one can also obtain another interesting quantity: the *Twining Genus*.
In order now to give a precise definition of Elliptic and Twining Genus, let us recall the partition function for a generic $\mathcal{N} = 2$ or $\mathcal{N} = 4$ superconformal field theory. Let us take, for simplicity, the partition function computed in the R-R sector, namely where $\psi^i(z)$ and $\bar{\psi}^i(\bar{z})$ are periodic in space direction. What we have thus is:

$$Z_{RR}(\tau) = \mathrm{Tr}_{RR}\left[q^{L_0 - c/24}\ \bar{q}^{\bar{L}_0 - c/24}(-1)^{F+\bar{F}}\right]$$

where the F and $\bar{F}$ operators are respectively the world-sheet fermion and the anti-fermion numbers. Their insertion in the partition function, as we have already discussed, is necessary in order to ensure anti-periodicity in time direction. We can now insert another operator which weighs every state contribution with different coefficients. The operator we insert, following the definition given by Witten in

[9]Clearly all the periodicity conditions are taken on the cylinder.

[25], is the zero mode of the J^3 current, J_0^3. The J^3 current is an operator that belongs to the superconformal algebra, as we can see in (5.15).
The partition function modified in this way is usually called *Elliptic Genus* and becomes:

$$Z(\tau, z)_{RR} = \mathrm{Tr}_{RR}\big[q^{L_0-c/24}\ \bar{q}^{\bar{L}_0-c/24}(-1)^{F+\bar{F}}y^{J_0^3}\big] \tag{6.8}$$

where $y \equiv e^{2\pi i z}$, with z a real parameter.
The Elliptic Genus a very interesting quantity because shows two interesting properties: it is an holomorphic function[10] of τ and z, and it is invariant under deformations of the model, in particular by taking deformations of the metric $G_{\mu\nu}$ and the B-field $B_{\mu\nu}$, as we will see later.
Our study is focused on how fields behave under a certain set of discrete symmetries that fix the $\mathcal{N} = 4$ superconformal algebra, therefore we can try to insert a symmetry operator $g \in G$, where G is the discrete symmetry group of our superconformal field theory, and see how the result depend on g. The Elliptic Genus modified in this way is called *Twining Genus* and it is:

$$Z_g(\tau, z)_{RR} = \mathrm{Tr}_{RR}\big[q^{L_0-c/24}\ \bar{q}^{\bar{L}_0-c/24}(-1)^{F+\bar{F}}y^{J_0^3}\ g\big] \tag{6.9}$$

Also the Twining Genus is an interesting object because, it can be shown, it is invariant under deformations of the model as long as g is a symmetry. We will actually exploit this property to compute the Twining Genus when it is easy to do it.

[10] This is discussed by Witten in [25]

Chapter 7

The $\mathbb{T}^4/\mathbb{Z}_2$ Orbifold

In this last chapter we would like to introduce non linear sigma-models on $\mathbb{T}^4$ and K3, and compute the Twining Genus for some discrete symmetries g. More specifically, the object we are going to compute will be the Twining Genus for a symmetry g of a $\mathcal{N} = (4, 4)$ superconformal field theory with target-space the orbifold $\mathbb{T}^4/\mathbb{Z}_2$. The motivation is that the computation of the Twining Genus of the theory is a good way of studying how fermionic and bosonic fields of this superconformal field theory, that arises from Type IIA and Type IIB superstring compactifications on K3 surfaces, behave under some discrete symmetries of the orbifold[1], in particular, the ones that preserve the OPEs of the model and fix the $\mathcal{N} = 4$ superconformal algebra. This could be relevant because symmetries of the fields at very high-energy level, may have an impact on the low-energy physics, namely on the properties of the compactified superstring theory.
As we said at the end of Section (5.3), we are interested in K3 surfaces, so it is natural to ask ourselves why we are focusing on the $\mathbb{T}^4/\mathbb{Z}_2$ orbifold. The fact is that, as we will see in Section (7.3), this orbifold can be obtained by taking an opportunely deformation of the metric of a K3 surface. An interesting property of the Twining Genus we are going to compute, is that it should not depend on small deformations of the metric as long as the symmetry g remains a symmetry of the corresponding NLSM.

7.1 Orbifold, a simple example: $S^1/\mathbb{Z}_2$

In order to present our final calculations, we need now to introduce the concept of a particular topologic structure called *Orbifold* and to make a simple example of how a partition function on this structure has to be computed.
If we consider a manifold $\mathcal{M}$ and discrete group G acting on a generic point $x \in \mathcal{M}$ with $x \to gx$, we can define the corresponding orbifold $\mathcal{M}/G$ in the following way:

$$\mathcal{M}/G = \{x \in \mathcal{M} \mid x \sim gx, \ \forall g \in G\}$$

An example of orbifold is $S^1/\mathbb{Z}_2$. The structure of this orbifold can be understood by considering the following relations:

$$x \sim x + 2\pi R, \qquad x \sim -x$$

[1]The complete classification of all discrete symmetries of the orbifold $\mathbb{T}^4/\mathbb{Z}_2$ can be found in [16].

where $x \in \mathbb{R}$ and R is the radius of the circle. In this case the orbifold is simply a segment of length π. Notice that the boundaries, namely $x = 0$ and $x = \pi$, are the two fixed points under the action of $\mathbb{Z}_2$ on the circle.

Let us now consider a free compactified boson with the assumption that the bosonic field ϕ can be identified with $-\phi$, namely the field now take its values on the space where we have perfomed the quotient by the action of the Z_2 group. In order to identify the class of bosonic fields we are dealing with, we label the lowest weight states generated by the vertex operator with winding number n and momentum m, with $|m, n\rangle$. Taking now the non-trivial transformation $h \in \mathbb{Z}_2$, namely the one that maps a generic field of our theory $\phi \to -\phi$, we have the following transformation property $h\,|m, n\rangle = |-m, -n\rangle$ with h anticommuting with the mode creation and annihilation operators α_{-n} and $\bar{\alpha}_{-n}$. In order now to compute the partition function, we need to project on the states of the Hilbert space that are invariant under the action of h. We need therefore to introduce the projector $(1 + h)/2$ in the trace:

$$Z^{\text{invar}}(R) = \frac{1}{2}\text{Tr}[(1+h)\ q^{L_0-1/24}\ \bar{q}^{L_0-1/24}]$$

where:

$$L_0 = \sum_{k\in\mathbb{Z}} \alpha_{-k}\alpha_k$$

The previous trace can be splitted in two contributions:

$$Z^{\text{invar}}(R) = \frac{1}{2}Z(R) + \frac{1}{2}\text{Tr}[h\ q^{L_0-1/24}\ \bar{q}^{L_0-1/24}]$$

where the first term is exactly the partition function of a compactified boson. In order now to compute the h trace term, we notice that we can build two vacua states $|a\rangle = |m, n\rangle + |-m, -n\rangle$ and $|b\rangle = |m, n\rangle - |-m, -n\rangle$ respectively symmetric and antisymmetric under the action of h. We see that the contributions coming from the excited states built upon $|a\rangle$ and $|b\rangle$ cancel out because both have the same L_0 eigenvalue but, because of the presence of the h transformation, have opposite signs. We conclude then that the only states that contribute to the h trace are the ones built upon the vacuum $|0, 0\rangle$.
The direct computation now gives us:

$$\begin{aligned}
\text{Tr}\big[h\ q^{\sum_n \alpha_{-n}\alpha_n - 1/24}\big] &= q^{-1/24}\prod_{n=1}^{\infty}\text{Tr}_{\mathcal{H}_n} h\ q^{\alpha_{-n}\alpha_n} \\
&= q^{-1/24}\prod_{n=1}^{\infty}(1 - q^n + q^{2n} - q^{3n} + ...) \\
&= q^{-1/24}\prod_{n=1}^{\infty}\sum_{k=0}^{\infty}(-q^n)^k \\
&= q^{-1/24}\prod_{n=1}^{\infty}\frac{1}{1+q^n} \\
&= \sqrt{\frac{2\eta(\tau)}{\theta_2(\tau)}}
\end{aligned}$$

where we decomposed the Hilbert space $\mathcal{H} \simeq \otimes_{n=1}^{\infty}\mathcal{H}_n$, where $\mathcal{H}_n$ is the infinite-dimensional Hilbert subspace built upon the vacuum $|0, 0\rangle$ and generated by $|0, 0\rangle$

and $(\alpha_{-n})^k |0,0\rangle$ with $k = 1, ..., \infty$.
The result thus becomes:

$$Z^{\text{invar}}(R) = \frac{1}{2}Z(R) + \left|\frac{\eta(\tau)}{\theta_2(\tau)}\right|$$

We notice that we clearly obtained a partition function that is not modular invariant. That is because we considered the contributions of the h invariant states coming from the so called *Untwisted Sector*, namely the periodic boson fields $\phi(\sigma + 2\pi, \tau) \sim \phi(\sigma, \tau)$. Let us now thus consider the so called *Twisted Sector* in which we have instead $\phi(\sigma + 2\pi, \tau) \sim -\phi(\sigma, \tau)$. It can be shown that, in this sector, the mode expansion of the bosonic field becomes:

$$\phi(\sigma, \tau) = \phi_0 + \frac{i}{\sqrt{4\pi g}} \sum_{n\in\mathbb{Z}} \left(\frac{\alpha_{n+1/2}}{n+1/2} e^{i(n+1/2)(\sigma+\tau)} + \frac{\bar{\alpha}_{n+1/2}}{n+1/2} e^{-i(n+1/2)(\sigma-\tau)} \right)$$

The ground state ϕ_0 now, imposing the twisted sector condition, has necessarily to assume the values $\phi_0 = 0$ and $\phi_0 = \pi$. Differently from the untwisted sector case therefore now we have two ground states $|0,0\rangle_0$ and $|0,0\rangle_\pi$ on which we can act with the creation operators α_{-n} and $\bar{\alpha}_{-n}$ with $n > 0$, and build all the possible bosonic excited states.
Again, just like we did for the untwisted sector, we need to compute the trace on the invariant states under the action of the h transformation. Thus we need to insert again the projector $(1 + h)/2$ into the trace:

$$Z^{\text{tw}}(R) = \frac{1}{2}\text{Tr}[(1+h)\ q^{L_0 - 1/24}\ \bar{q}^{L_0 + 1/24}]$$

where we used the expression of L_0 obtained by substituting the bosonic field expansion (3.24), setting the coupling constant as $g = 1/4\pi$, in the expression of the stress-energy tensor in (3.18), namely:

$$L_0 = \sum_{k\in\mathbb{Z}+\frac{1}{2}} \alpha_{-k}\alpha_k + \frac{1}{16}$$

with the additive constants fixed by computing the vacuum expectation value of the stress-energy tensor considering a bosonic field with respectively periodic and antiperiodic conditions on the plane[2].
The direct computation leads to:

$$\begin{aligned}
Z^{\text{tw}}(R) &= \frac{1}{2}\text{Tr}[(1+h)\ q^{L_0-1/24}\ \bar{q}^{L_0-1/24}] \\
&= \frac{1}{2}\text{Tr}[(1+h)\ q^{\sum_n \alpha_{-n}\alpha_n + 1/48}\ \bar{q}^{\sum_n \alpha_{-n}\alpha_n + 1/48}] \\
&= \frac{1}{2}(q\bar{q})^{1/48}\left[\prod_{n=1}^{\infty} \frac{1}{(1-q^{n-1/2})(1-\bar{q}^{n-1/2})} + \prod_{n=1}^{\infty} \frac{1}{(1+q^{n-1/2})(1+\bar{q}^{n-1/2})}\right] \\
&= \left|\frac{\eta(\tau)}{\theta_4(\tau)}\right| + \left|\frac{\eta(\tau)}{\theta_3(\tau)}\right|
\end{aligned}$$

[2]Since the conformal dimension of the bosonic field is $h = 1$, the boundary conditions defined on the cylinder, unlike what happens in the fermionic case, stay the same when mapping the cylinder onto the plane.

The complete partition function therefore becomes:

$$\begin{aligned}
Z^{\text{orb}}(R) &= Z^{\text{untw}}(R) + Z^{\text{tw}}(R) \\
&= \frac{1}{2}Z(R) + \left|\frac{\eta(\tau)}{\theta_2(\tau)}\right| + \left|\frac{\eta(\tau)}{\theta_4(\tau)}\right| + \left|\frac{\eta(\tau)}{\theta_3(\tau)}\right| \\
&= \frac{1}{2}\left(Z(R) + \frac{|\theta_2\theta_3|}{|\eta|^2} + \frac{|\theta_2\theta_4|}{|\eta|^2} + \frac{|\theta_3\theta_4|}{|\eta|^2}\right)
\end{aligned}$$

where we use the identity $\theta_2\theta_3\theta_4 = 2\eta^3$. We notice now that, by adding the missing piece, namely the twisted sector trace term, the total partition function is modular invariant as expected.
This construction can be generalized to $\mathbb{T}^d/\mathbb{Z}_2$ orbifolds, with $d > 1$, where the symmetry h maps the fields $\phi^i \to -\phi^i$ where $i = 1, ..., d$. In the next sections we will consider the $d = 4$ supersymmetric case: the $\mathbb{T}^4/\mathbb{Z}_2$ orbifold.

7.2 Non linear σ-Models on $\mathbb{T}^4$ and its Symmetries

As previously introduced, we need to study a $\mathcal{N} = (4, 4)$ superconformal field theory and see its possible application for superstring compactification. Let us thus recall the action:

$$S = \frac{1}{2\pi}\int d^2z \left(\sum_a \partial X^a \bar{\partial} X^a - \left(\sum_b \psi^b \bar{\partial}\psi^b + \bar{\psi}^b \partial \bar{\psi}^b\right)\right)$$

with $a, b = 1, ..., 4$. Introducing now the currents $j^a(z) = i\partial X^a(z)$ and $\tilde{j}^a(\bar{z}) = i\bar{\partial}X^a(\bar{z})$, the OPEs of our objects become:

$$j^a(z)j^b(w) \sim \frac{\delta^{ab}}{(z-w)^2}, \qquad \psi^a(z)\psi^b(w) \sim \frac{\delta^{ab}}{z-w}$$

where the anti-holomorphic counterparts satisfy analogous relations and all other possible combinations vanish. Their mode expansion follows immediately:

$$j^a(z) = \sum_{n\in\mathbb{Z}} \alpha^a_n \, z^{-n-1}, \qquad \psi^a(z) = \sum_{r\in\mathbb{Z}+\nu} \psi^a_r \, z^{-r-1/2}$$

with $\nu = 0$ if we are in the R sector or $\nu = 1/2$ if we are in the NS sector. The mode operators satisfy the following commutation and anticommutation relations:

$$[\alpha^a_m, \alpha^b_n] = m\delta^{ab}\delta_{m+n}, \qquad \{\psi^a_r, \psi^b_s\} = \delta^{ab}\delta_{r+s}$$

Let us now define a useful set of chiral bosonic fields:

$$\begin{aligned}
\partial Z^{(1)}(z) &\equiv \frac{1}{\sqrt{2}}(j^1(z) + ij^3(z)), \qquad & \partial Z^{(1)*}(z) &\equiv \frac{1}{\sqrt{2}}(j^1(z) - ij^3(z)) \\
\partial Z^{(2)}(z) &\equiv \frac{1}{\sqrt{2}}(j^2(z) + ij^4(z)), \qquad & \partial Z^{(2)*}(z) &\equiv \frac{1}{\sqrt{2}}(j^2(z) - ij^4(z))
\end{aligned}$$

chiral fermionic fields:

$$\begin{aligned}
\chi^{(1)}(z) &\equiv \frac{1}{\sqrt{2}}(\psi^1(z) + i\psi^3(z)), \qquad & \chi^{(1)*}(z) &\equiv \frac{1}{\sqrt{2}}(\psi^1(z) - i\psi^3(z)) \\
\chi^{(2)}(z) &\equiv \frac{1}{\sqrt{2}}(\psi^2(z) + i\psi^4(z)), \qquad & \chi^{(2)*}(z) &\equiv \frac{1}{\sqrt{2}}(\psi^2(z) - i\psi^4(z))
\end{aligned}$$

and, analogously, anti-chiral bosonic and fermionic fields $\partial\tilde{Z}^{(1)}$, $\partial\tilde{Z}^{(2)}$, $\partial\tilde{Z}^{(1)*}$, $\partial\tilde{Z}^{(2)*}$, $\tilde{\chi}^{(1)}$, $\tilde{\chi}^{(2)}$, $\tilde{\chi}^{(1)*}$, $\tilde{\chi}^{(2)*}$.
We can now build the four supercurrents using the definitions above:

$$\begin{aligned}
G^{+}(z) &\equiv i\sqrt{2}(:\chi^{(1)*}(z)\partial Z^{(1)}(z):+:\chi^{(2)*}(z)\partial Z^{(2)}(z):)\\
G^{-}(z) &\equiv i\sqrt{2}(:\chi^{(1)}(z)\partial Z^{(1)*}(z):+:\chi^{(2)}(z)\partial Z^{(2)*}(z):)\\
G'^{+}(z) &\equiv \sqrt{2}(-:\chi^{(1)*}(z)\partial Z^{(2)*}(z):+:\chi^{(2)*}(z)\partial Z^{(1)*}(z):)\\
G'^{-}(z) &\equiv \sqrt{2}(:\chi^{(1)}(z)\partial Z^{(2)}(z):-:\chi^{(2)}(z)\partial Z^{(1)}(z):)
\end{aligned} \tag{7.1}$$

We notice now that our torus model contains four chiral fermions from which, it can be shown, we can build an $so(4)$ algebra. By using now the property that $so(4) \simeq su(2) \oplus su(2)$, we can define two inequivalent commuting $su(2)$ algebras that are generated respectively from the following set of currents:

$$\begin{aligned}
J^{3}(z) &\equiv \frac{1}{2}\big(:\chi^{(1)*}(z)\chi^{(1)}(z):+:\chi^{(2)*}(z)\chi^{(2)}(z):\big)\\
J^{+}(z) &\equiv i:\chi^{(1)*}(z)\chi^{(2)*}(z):\\
J^{-}(z) &\equiv i:\chi^{(1)}(z)\chi^{(2)}(z):
\end{aligned} \tag{7.2}$$

and:

$$\begin{aligned}
A^{3}(z) &\equiv \frac{1}{2}\big(:\chi^{(1)*}(z)\chi^{(1)}(z):-:\chi^{(2)*}(z)\chi^{(2)}(z):\big)\\
A^{+}(z) &\equiv i:\chi^{(1)*}(z)\chi^{(2)}(z):\\
A^{-}(z) &\equiv i:\chi^{(1)}(z)\chi^{(2)*}(z):
\end{aligned} \tag{7.3}$$

We can now notice that the first set of affine currents, $J^3(z)$ and $J^{\pm}(z)$, are exactly the ones that appear in the definition of the $\mathcal{N}=4$ superconformal algebra, in particular in the OPEs of the four supercurrents, as we can see from the equation (5.18).

After having introduced all the fields we will need for our calculations, let us now give a qualitative description of all possible discrete symmetries of the $\mathbb{T}^4$ manifold that preserves all the OPEs and fixes the $\mathcal{N}=4$ superconformal algebra, namely that act trivially on the supercurrents in (7.1) and the currents in (7.2). In a $\mathcal{N}=(4,4)$ superconformal field theory, the fields we are dealing with are: the bosons $\partial X^i(z)$, the fermions $\psi^i(z)$, their anti-holomorphic counterparts and the vertex operators $V_\lambda(z,\bar{z})$.
We have now to remember that:

$$\lim_{z,\bar{z}\to 0} V_\lambda(z,\bar{z})\,|0\rangle = |\lambda\rangle$$

with $\vec{\lambda}=(\vec{\lambda}_L,\vec{\lambda}_R)$, and that $|\lambda\rangle$ satisfy the following conditions:

$$\begin{aligned}
\alpha_0^i\,|\lambda\rangle &= \lambda_L^i\,|\lambda\rangle\\
\bar{\alpha}_0^i\,|\lambda\rangle &= \lambda_R^i\,|\lambda\rangle
\end{aligned} \tag{7.4}$$

As we saw in Subsection (6.2.1), $\vec{\lambda}$ forms a 8-dimensional lattice $\Gamma^{4,4}$ that must be, in order to have a modular invariant partition function, even-integer and self-dual (or *unimodular*). We have thus that $\lambda_L^2-\lambda_R^2\in 2\mathbb{Z}$, therefore the lattice has signature $(4,4)$ and the metric is clearly $g_{ab}=\mathrm{diag}(1,...,1,-1,...,-1)$.

In order to find the symmetries we need, let us recall the OPEs of the fields we are considering:

$$\partial X^i(z)\partial X^j(w) \sim -\frac{\delta^{ij}}{(z-w)^2} \tag{7.5}$$

$$\psi^i(z)\psi^j(w) \sim \frac{\delta^{ij}}{z-w} \tag{7.6}$$

$$\partial X^i(z)V_\lambda(w,\bar{w}) \sim -i\frac{\lambda_L^i}{z-w}V_\lambda(w,\bar{w}) \tag{7.7}$$

$$V_\lambda(z,\bar{z})V_\mu(w,\bar{w}) \sim \xi(\lambda,\mu)\ (z-w)^{\vec{\lambda}_L\cdot\vec{\mu}_L}\ (\bar{z}-\bar{w})^{\vec{\lambda}_R\cdot\vec{\mu}_R}\ V_{\lambda+\mu}(w,\bar{w}) \tag{7.8}$$

with $\xi(\lambda,\mu)$ a phase that satisfies the condition $\xi(\lambda,\mu)=(-1)^{\lambda_L\mu_L-\lambda_R\mu_R}\ \xi(\mu,\lambda)$.

The first transformation we could do in order to preserve the equation (7.5), is $\partial X'^j(z)=R^{ij}\partial X^i(z)$, with $R^{ij}\in O(4)_L$. A necessary condition for this transformations to act trivially on the four supercurrents is to transform exactly in the inverse way the set of four fermionic fields of the model. This implies that the fermionic fields must transform with $(R^{-1})^{ij}=R^{ji}$. Focusing now on the equation (7.7), we notice that, in order to preserve the OPE, the same $O(4)_L$ transformation that acts on ∂X^i, has also to act on $\vec{\lambda}_L^i$. Taking into account now also the anti-chiral components and making analogous considerations, the allowed transformations have to belong necessarily to the group $O(4)_L\times O(4)_R$[3].
As we saw before, the $o(4)_L$ algebra can be divided in two commuting algebras: $su(2)_J$ and $su(2)_A$. Now the group $SU(2)_A$ acts trivially on the three currents J^3 and $J^\pm$ while transforms in a non-trivial way, up to a $\mathbb{Z}_2$ transformation, the other three currents A^3 and $A^\pm$. Therefore, in order to fix the three currents $J^\pm$ and J^3, we have to require that the allowed transformations belong only to a subgroup of $O(4)_L$, in particular to $SU(2)_A\equiv SU(2)_L$. Analogous considerations hold for the right-moving terms.
We need also to impose another very important constraint: the transformed vector $\vec{\lambda}'_{L/R}$ has still to belong to the lattice $\Gamma^{4,4}$. We know that the metric of the lattice is left invariant from a $SO(4,4)$ transformation, but only a group of discrete transformations can map a point of the lattice to another point of the same lattice. We can call for simplicity this group $SO(\Gamma^{4,4})$.
We conclude thus that $\vec{\lambda}\in\Lambda$ has to transform with a transformation belonging to $(SU(2)_L\times SU(2)_R)\cap SO(\Gamma^{4,4})$.
We notice that we can still act on $V_\lambda(z,\bar{z})$ and preserve the equations (7.7) and (7.8), with the following transformations:

$$V_\lambda(z,\bar{z})\longrightarrow V_\lambda(z,\bar{z})\cdot e^{2\pi i(\vec{\epsilon}_L\cdot\vec{\lambda}_L-\vec{\epsilon}_R\cdot\vec{\lambda}_R)}$$

where $\vec{\epsilon}_{L/R}\in\Gamma^{4,4^*}\otimes\mathbb{R}$.[4] The group generated by the transformations above is $U(1)^4_L\times U(1)^4_R$.
We can now finally conclude that the symmetry group we are looking for is:

$$G=(U(1)^4_L\times U(1)^4_R)\rtimes(SU(2)_L\times SU(2)_R)\cap SO(\Gamma^{4,4})$$

Since the transformation $\lambda_{L/R}\to-\lambda_{L/R}$ is always allowed, we can rewrite the

[3]Notice that L and R components have both to transform independently from each other.
[4]This notation means that $\vec{\epsilon}\cdot\vec{\lambda}\in\mathbb{R}$, $\vec{\lambda}\in\Gamma^{4,4}$.

previous group decomposition as[5]:

$$G = (U(1)^4_L \times U(1)^4_R) : \mathbb{Z}_2 \,.\, G_0 \tag{7.9}$$

with:

$$G_0 = ((SU(2)_L \times SU(2)_R) \cap SO(\Gamma^{4,4}))/\mathbb{Z}_2$$

The complete classification of all possible symmetries $g \in G_0$ and their eigenvalues can be found in [16], Table 2, pag.20.

7.2.1 The Moduli Space

We have already seen that, in order to classify all the possible metrics over a certain manifold, we can build the so called Moduli Space of (pseudo-)Riemannian metrics of the corresponding manifold. In general, the Moduli Space is a space that parametrize every possible property of a class of mathematical objects. In our context, it is very important to study this space for $\mathbb{T}^4$ and K3 surfaces cases because we can parametrize all possible choices of the metric and, possibly, the B-field on those manifolds. Indeed the metric $G_{\mu\nu}$ and the B-field $B_{\mu\nu}$ are the quantities that determine the NLSM on that manifold. We can thus build a Moduli Space whose points correspond to a fixed metric and B-field, namely, every point corresponds to different NLSMs defined over the manifold itself.

If we now consider the $\mathbb{T}^4$ case, we know that the winding-momentum lattice is $\Gamma^{4,4} \subseteq \mathbb{R}^{4,4}$ and, because of modular invariance properties of the partition function, it must be even-integer and self-dual[6]. This lattice is also called *Narain lattice*. The signature of the metric of the lattice is $(4,4)$, therefore we can separate the lattice in two 4-dimensional subspaces: a definite positive one denoted by Π and a definite negative one denoted by $\Pi^\perp$. A generic vector of the lattice can be written as $\vec{p} = (\vec{p}_L, \vec{p}_R)$, with $p_{L,R}$ the eigenvalues respectively of the α_0^i and $\bar{\alpha}_0^i$ operators, with $i = 1, ..., 4$, and the relation $p_L^2 - p_R^2 \in 2\mathbb{Z}$. We notice now that, since the momentum depends on the geometric properties of the manifold, fixing the lattice $\Gamma^{4,4}$, means to fix the geometry of the manifold, i.e. the metric, and the B-field.

In order now to build the Moduli Space of the metrics and B-fields for the torus $\mathbb{T}^4$, we can start considering the generic group that preserves the metric on $\mathbb{R}^{4,4}$, namely $O(4,4)$. Since we divided the lattice into two subspaces Π and $\Pi^\perp$, we should remove all the possible transformations that rotate every vector of a subspace into another vector of the same subspace, i.e. the group $O(4)_\Pi \times O(4)_{\Pi^\perp}$. At this point what we obtained is:

$$\mathcal{M}_1 = O(4,4)/(O(4) \times O(4))$$

and the meaning of $\mathcal{M}_1$ is to be the space of all possible couples $(G_{\mu\nu}, B_{\mu\nu})$. We have now to notice that, since the B-field can be deduced from the momenta of the Narain lattice, we should remove all the possible transformations that give us different B-fields but corresponding to equivalent NLSMs. We have therefore to take the quotient of $\mathcal{M}_1$ by the so called *Duality Group* of the Lattice, namely the group of all possible basis changing transformations, B-field shifting with

[5]Since the product of the groups in (7.9) is neither direct nor semi-direct, we indicate with "." and ":" their kind of group product.

[6]Notice that self-dual lattices can exist if and only if, by taking a genirc lattice $\Gamma^{m,n}$, we have that $m - n = 8l$, with $l \in \mathbb{Z}$.

an anti-simmetric integer matrix N_{ij}, and exchanging between winding numbers and momenta (called also *T-Duality* transformations). The final structure of the Moduli Space thus becomes:

$$\mathcal{M} = O(\Gamma^{4,4})\backslash O(4,4)/(O(4)\times O(4))$$

This is finally the space of the couples $(G_{\mu\nu}, B_{\mu\nu})$ corresponding to all possible inequivalent NLSMs on the torus $\mathbb{T}^4$.

7.3 Non-linear σ-Models on K3

As discussed in Subsection (5.3.2), K3 surfaces are two-dimensional complex and Ricci-flat manifolds whose holonomy group is $H = SU(2)$. This allows us to find non trivial solutions to the equation (5.6) that fixes the number of conserved supercharges after the compactification on them of a superstring theory. That is why they are so important in this context, so let us now consider non-linear σ models on K3.

NLSM defined on K3 surfaces are two-dimensional superconformal field theories with $\mathcal{N} = (4,4)$ superconformal symmetry and central charge $c = 6$. Unfortunately, for a generic NLSM on K3, we are not able to compute directly even basic quantities such as the partition function of the theory or, for example, the Elliptic Genus.

K3 surfaces are a class of manifolds that share the same topology but not the same metric (and B-field) therefore, as we saw right above, we can consider the Moduli Space parametrizing all the possible choices of metric and B-field. It can be shown that the Moduli Space of NLSMs built on K3 has the following structure[7]:

$$\mathcal{M}_{K3} = O(\Gamma^{4,20})\backslash O(4,20)/(O(4)\times O(20)) \tag{7.10}$$

with $\Gamma^{4,20}$ the unique, up to isomorphisms, even, self-dual lattice, of signature $(4,20)$ and $O(\Gamma^{4,20})$ its infinite discrete group of symmetry. As usual, we will focus in symmetries preserving the $\mathcal{N} = 4$ superconformal algebra, just like we did for the $\mathbb{T}^4$ case. In [18] and [22] it is shown that, in each point of $\mathcal{M}_{K3}$, those kind of symmetries form a finite proper subgroup of $O(\Gamma^{4,20})$.

It can be now shown that there are some special points of (7.10) for which the corresponding NLSM on K3 is exactly equivalent to a NLSM defined on the orbifold $\mathbb{T}^4/\mathbb{Z}_2$, namely, with a proper choice of the couple $(G_{\mu\nu}, B_{\mu\nu})$, we have that $K3 \simeq \mathbb{T}^4/\mathbb{Z}_2$. This kind of orbifold has a much simpler structure than a generic K3 surface and, in particular, we are able to compute explicitly a lot of interesting quantities such as the partition function, the Elliptic Genus and many of its Twining Genera.

As discussed in Section (6.3), the Elliptic Genus is invariant under deformations of the metric and B-field. Since now the Moduli Space $\mathcal{M}_{K3}$ is connected, this implies that the Elliptic Genus is the same for every point of $\mathcal{M}_{K3}$, namely the same for all NLSMs on K3. It is therefore sufficient to compute it in a special point of the Moduli Space, i.e. where $K3 \simeq T^4/\mathbb{Z}_2$. The same holds for the Twining Genus of a discrete symmetry g, at least for the subspace of $\mathcal{M}_{K3}$ where g remains a good symmetry of the K3 model. We will now finally show how to compute these quantities.

[7] The construction of the Moduli Space in this case is very complicated and it is accurately presented in [17].

7.4 The Elliptic Genus of K3

As we said right above, the Elliptic Genus computed for K3 surfaces and the $\mathbb{T}^4/\mathbb{Z}_2$ orbifold is exactly the same. That is because it has the property of being invariant under deformations of K3 models, namely it stays the same for every point of $\mathcal{M}_{K3}$, therefore it can be computed for the orbifold case. Following the definition given in (6.8) and inserting in the trace the projector $(1+h)/2$, with $h \in \mathbb{Z}_2$ the transformation that maps the generic fields $\phi^i \to -\phi^i$, the Elliptic Genus defined on $\mathbb{T}^4/\mathbb{Z}_2$ orbifold becomes:

$$\begin{aligned} Z^{orb}(\tau,z) = \mathrm{Tr}_{(h-untw)}&\left[\frac{(1+h)}{2}\; q^{L_0-c/24}\;\bar{q}^{\bar{L}_0-c/24}(-1)^{F+\bar{F}}y^{J_0^3}\right] \\ &+ \mathrm{Tr}_{(h-tw)}\left[\frac{(1+h)}{2}\; q^{L_0-c/24}\;\bar{q}^{\bar{L}_0-c/24}(-1)^{F+\bar{F}}y^{J_0^3}\right] \\ &\equiv Z^{orb}_{untw}(\tau,z)+Z^{orb}_{tw}(\tau,z) \end{aligned} \tag{7.11}$$

As we already know, in order to compute explicitly the trace, we need to specify how the Hilbert space $\mathcal{H}$ has to be factorized in this case. This is useful because of the well known property of traces $\mathrm{Tr}(AB)$ =$\mathrm{Tr}(A)\mathrm{Tr}(B)$ when A and B are operators that act on different factors of the Hilbert space.
As usual, we can start from a generic ground state $|\Omega\rangle$ and build, acting with the mode creation operators of the fields of the theory, the corresponding Hilbert subspace $\mathcal{H}_{f_i}$, with f_i the index that specify the kind of field we are considering, for example, $\partial Z^{(1)}$ or $\tilde{\chi}^{(2)*}$.
We have also to remember that in the R-R sector there exist fermionic zero modes forming a non-trivial algebra. This implies that the ground states must be degenerate, with the zero modes mapping the ground states into each other. We can call states created in this way as ground states. The chiral and anti-chiral zero modes satisfy the following non-vanishing anticommutation relations:

$$\{\chi_0^a,\chi_0^{b*}\}=\delta^{ab}, \qquad \{\tilde{\chi}_0^a,\tilde{\chi}_0^{b*}\}=\delta^{ab}$$

therefore they can be represented as matrices living in a $2^{d/2}$-dimensional space, with d the total number of fermionic operator. Being $d=8$, we have that the zero modes generate a 16-dimensional Hilbert space. Since there is no canonical choice of creation and annihilation operators by starting from the zero modes, we conventionally assign to $\chi^{(1)*}$, $\chi^{(2)*}$, $\tilde{\chi}^{(1)*}$, $\tilde{\chi}^{(2)*}$ the role of creation operator and to $\chi^{(1)}$, $\chi^{(2)}$, $\tilde{\chi}^{(1)}$, $\tilde{\chi}^{(2)}$ the role of annihilation operators.
In order to explicitly describe the 16-dimensonal Hilbert space of ground states $\mathcal{H}_{gs}$, we can conveniently choose a basis of eigenstates for the current operators J_0^3, $\tilde{J}_0^3$, A_0^3, $\tilde{A}_0^3$, namely $|s_1,s_2,\tilde{s}_1,\tilde{s}_2\rangle$, with $s_{1,2}=\pm1/2$ and $\tilde{s}_{1,2}=\pm1/2$.
In order to generate the whole 16-dimensional space, we can choose a state $|s_1,s_2,\tilde{s}_1,\tilde{s}_2\rangle$, that is annihilated from all the annihilation operators, and act in all possible ways with the creation operators. The consistent relations that

have to be satisfied in order to obtain 16 states are:

$$\chi_0^{(1)*}\,|-\frac{1}{2},s_2,\tilde{s}_1,\tilde{s}_2\rangle = |\frac{1}{2},s_2,\tilde{s}_1,\tilde{s}_2\rangle \qquad \tilde{\chi}_0^{(1)*}\,|s_1,s_2,-\frac{1}{2},\tilde{s}_2\rangle = |s_1,s_2,\frac{1}{2},\tilde{s}_2\rangle$$
$$\chi_0^{(2)*}\,|s_1,-\frac{1}{2},\tilde{s}_1,\tilde{s}_2\rangle = |s_1,\frac{1}{2},\tilde{s}_1,\tilde{s}_2\rangle \qquad \tilde{\chi}_0^{(2)*}\,|s_1,s_2,\tilde{s}_1,-\frac{1}{2}\rangle = |s_1,s_2,\tilde{s}_1,\frac{1}{2}\rangle$$
$$\chi_0^{(1)*}\,|\frac{1}{2},s_2,\tilde{s}_1,\tilde{s}_2\rangle = 0 \qquad \tilde{\chi}_0^{(1)*}\,|s_1,s_2,\frac{1}{2},\tilde{s}_2\rangle = 0$$
$$\chi_0^{(2)*}\,|s_1,\frac{1}{2},\tilde{s}_1,\tilde{s}_2\rangle = 0 \qquad \tilde{\chi}_0^{(2)*}\,|s_1,s_2,\tilde{s}_1,\frac{1}{2}\rangle = 0$$

and the same holds, with opposite signs, for the corresponding annihilation operators. The action of the current operators on a generic state of $\mathcal{H}_{gs}$ is described by the following relations:

$$J_0^3\,|s_1,s_2,\tilde{s}_1,\tilde{s}_2\rangle = (s_1+s_2)\,|s_1,s_2,\tilde{s}_1,\tilde{s}_2\rangle \qquad A_0^3\,|s_1,s_2,\tilde{s}_1,\tilde{s}_2\rangle = (s_1-s_2)\,|s_1,s_2,\tilde{s}_1,\tilde{s}_2\rangle$$
$$\tilde{J}_0^3\,|s_1,s_2,\tilde{s}_1,\tilde{s}_2\rangle = (\tilde{s}_1+\tilde{s}_2)\,|s_1,s_2,\tilde{s}_1,\tilde{s}_2\rangle \qquad \tilde{A}_0^3\,|s_1,s_2,\tilde{s}_1,\tilde{s}_2\rangle = (\tilde{s}_1-\tilde{s}_2)\,|s_1,s_2,\tilde{s}_1,\tilde{s}_2\rangle$$

The last factor of the Hilbert space factorization we need to consider is the Hilbert space generated by vertex operators. It can be written as $\mathcal{H}_{V_\lambda} = \bigoplus_{\lambda\in\Gamma^{4,4}}\mathcal{H}_\lambda$, with $\mathcal{H}_\lambda$ the 1-dimensional Hilbert subspace generated by the vector $|\lambda\rangle$.
Since now $\vec{\lambda}$ has to satisfy the conditions in (7.4), we understand that the eigenstate $|\lambda\rangle$ may exist if and only if the bosonic zero mode operators are present. This is not true if we consider the h-twisted sector, indeed, in that sector, the bosonic and fermionic zero modes are labelled with $r\in\mathbb{Z}+1/2$.
The role of the vertex operators thus is different in the h-untwisted and the h-twisted sector: in the first they give rise to the Hilbert space defined above as $\mathcal{H}_{V_\lambda}$, while in the second, as we will see more precisely later, they are the operators that map a twisted ground state into another twisted ground states, namely they generate $\mathcal{H}_{gs}^{tw}$.
The final decomposition of our Hilbert space for the h-untwisted sector thus finally becomes:

$$\mathcal{H} = \left(\bigoplus_{\lambda\in\Lambda}|\lambda\rangle\right)\otimes\mathcal{H}_{gs}\otimes\mathcal{H}_{\partial Z^{(1)}}\otimes\mathcal{H}_{\partial Z^{(2)}}\otimes\mathcal{H}_{\chi^{(1)}}\otimes\mathcal{H}_{\chi^{(2)}}$$
$$\otimes\,\mathcal{H}_{\partial Z^{(1)*}}\otimes\mathcal{H}_{\partial Z^{(2)*}}\otimes\mathcal{H}_{\chi^{(1)*}}\otimes\mathcal{H}_{\chi^{(2)*}}\otimes\ldots$$

where "..." stand for the Hilbert spaces generated with right-moving operators.

Before starting with the calculations, we can notice now that it is useful to rewrite the trace in (7.11) into four different terms:

$$Z^{(1)}(\tau,z) = \frac{1}{2}\mathrm{Tr}_{(h-untw)}\left[q^{L_0-c/24}\ \bar{q}^{\bar{L}_0-c/24}(-1)^{F+\bar{F}}y^{J_0^3}\right] \tag{7.12}$$
$$Z^{(2)}(\tau,z) = \frac{1}{2}\mathrm{Tr}_{(h-untw)}\left[q^{L_0-c/24}\ \bar{q}^{\bar{L}_0-c/24}(-1)^{F+\bar{F}}y^{J_0^3}\ h\right] \tag{7.13}$$
$$Z^{(3)}(\tau,z) = \frac{1}{2}\mathrm{Tr}_{(h-tw)}\left[q^{L_0-c/24}\ \bar{q}^{\bar{L}_0-c/24}(-1)^{F+\bar{F}}y^{J_0^3}\right] \tag{7.14}$$
$$Z^{(4)}(\tau,z) = \frac{1}{2}\mathrm{Tr}_{(h-tw)}\left[q^{L_0-c/24}\ \bar{q}^{\bar{L}_0-c/24}(-1)^{F+\bar{F}}y^{J_0^3}\ h\right] \tag{7.15}$$

with clearly:

$$Z^{orb}(\tau,z) = \sum_{i=1}^{4}Z^{(i)}(\tau,z)$$

We can now notice that every $Z^{(i)}(\tau, z)$ can be decomposed as the product of three different terms. One of each term take into account contributions coming from, respectively, the ground states, the oscillator terms (mode operators with index $-n$, $n \neq 0$) and the vertex operators. We can thus write:

$$\begin{aligned} Z^{(i)}_{untw}(\tau, z) &= \frac{1}{2}\, Z^{(i)}_{gs}(\tau, z) \cdot Z^{(i)}_{osc}(\tau, z) \cdot Z^{(i)}_{\lambda}(\tau, z) \\ Z^{(i)}_{tw}(\tau, z) &= \frac{1}{2}\, Z^{(i)}_{gs}(\tau, z) \cdot Z^{(i)}_{osc}(\tau, z) \end{aligned} \tag{7.16}$$

We are now ready to compute the ground states contribution to $Z^{(1)}(\tau, z)$. In the R-R sector the ground states has conformal weight $h = \bar{h} = c/24$, therefore we don't get any contribution to the trace from the operators $q^{L_0 - c/24}$ and $\bar{q}^{\bar{L}_0 - c/24}$. Let us take the state $|-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\rangle \equiv |gs\rangle$ as the state from which we can generate $\mathcal{H}_{gs}$. Applying the operators in the trace to the $|gs\rangle$ state we obtain:

$$(-1)^F \; y^{J_0^3} \;\; |gs\rangle = (+1) \cdot (y^{-1})\, |gs\rangle = y^{-1}\, |gs\rangle$$

Acting now for example with the creation operator $\chi_0^{(1)*}$ on $|gs\rangle$, calling it for simplicity $|\chi_0^{(1)*}\rangle$, we can find:

$$\begin{aligned} (-1)^F \; y^{J_0^3} \;\; |\chi_0^{(1)*}\rangle &= (-1)^F \; y^{J_0^3} \; \left| +\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right\rangle = \\ &= (-1) \cdot (+1)\, |\chi_0^{(1)*}\rangle = -\, |\chi_0^{(1)*}\rangle \end{aligned}$$

using now $|\chi_0^{(2)*}\rangle$ we find:

$$\begin{aligned} (-1)^F \; y^{J_0^3} \;\; |\chi_0^{(2)*}\rangle &= (-1)^F \; y^{J_0^3} \; \left| -\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right\rangle = \\ &= (-1) \cdot (+1)\, |\chi_0^{(2)*}\rangle = -\, |\chi_0^{(2)*}\rangle \end{aligned}$$

and analogously with $|\chi_0^{(2)*}\chi_0^{(1)*}\rangle$ we get:

$$(-1)^F \; y^{J_0^3} \;\; |\chi_0^{(2)*}\chi_0^{(1)*}\rangle = y\, |\chi_0^{(2)*}\chi_0^{(1)*}\rangle$$

Acting now also with all the possible combinations of right-moving operators, we can obtain the remaining contributions. By computing then the trace what we can obtain is:

$$Z^{(1)gs}(\tau, z) = y^{-1}(1 - y)^2 \cdot 0 = 0 \tag{7.17}$$

From this result is immediate to understand that

$$Z^{(1)}(\tau, z) = 0 \tag{7.18}$$

Let us now focus on the $Z^{(2)}(\tau, z)$ term. Since the effect of the h operators is to give us a -1 every time it acts on a single bosonic or fermionic field, we can compute $Z^{(2)gs}(\tau, z)$ following the previous considerations. This time what we can indeed obtain is:

$$Z^{(2)gs}(\tau, z) = y^{-1}(1 + y)^2 \cdot 4 \tag{7.19}$$

where the factor four comes from the right-moving operators contribution.
In order now to compute the $Z^{(2)osc}$ term, we need first to understand how the

operator $y^{J_0^3}$ acts on fermionic fields. As we can see from (7.2), being $y^{J_0^3}$ defined with only fermionic operators, it will not act on bosonic fields and therefore their eigenvalues will be simply 1. Using thus the definition of J_0^3, for fermionic fields, we can find:

	$\chi^{(1)*}$	$\chi^{(2)*}$	$\chi^{(1)}$	$\chi^{(2)}$	$\tilde{\chi}^{(1)*}$	$\tilde{\chi}^{(2)*}$	$\tilde{\chi}^{(1)}$	$\tilde{\chi}^{(2)}$
$y^{J_0^3}$	y	y	y^{-1}	y^{-1}	1	1	1	1

We are now ready to proceed with the computation of $Z^{(2)osc}(\tau, z)$. Let us start with the factor of the Hilbert space generated by the $\partial Z^{(1)}_{-n}$ operators. Analogously to the previous case, we can start computing, using the table above:

$$\begin{aligned}(-1)^F\ q^{L_0}\ \bar{q}^{\bar{L}_0}\ y^{J_0^3}\ h\,|\Omega\rangle &= (+1)\cdot|\Omega\rangle = |\Omega\rangle\\ (-1)^F\ q^{L_0}\ \bar{q}^{\bar{L}_0}\ y^{J_0^3}\ h\ \partial Z^{(1)}_{-n}\,|\Omega\rangle &= -q^n\ \partial Z^{(1)}_{-n}\,|\Omega\rangle\\ (-1)^F\ q^{L_0}\ \bar{q}^{\bar{L}_0}\ y^{J_0^3}\ h\ (\partial Z^{(1)}_{-n})^2\,|\Omega\rangle &= q^{2n}\ (\partial Z^{(1)}_{-n})^2\,|\Omega\rangle\\ &\ldots\end{aligned}$$

The contribution to the trace then can be expressed in a compact way:

$$1 - q^n + q^{2n} + \ldots = \sum_{l=0}^{+\infty}(-q^n)^l = \frac{1}{1+q^n}$$

Repeating the same calculations for the $\partial Z^{(1)*}$ case, we get:

$$\begin{aligned}(-1)^F\ q^{L_0}\ \bar{q}^{\bar{L}_0}\ y^{J_0^3}\ h\ \ |\Omega\rangle &= (+1)\cdot|\Omega\rangle = |\Omega\rangle\\ (-1)^F\ q^{L_0}\ \bar{q}^{\bar{L}_0}\ h\ \partial Z^{(1)*}_{-n}\,|\Omega\rangle &= -q^n\ \partial Z^{(1)*}_{-n}\,|\Omega\rangle\\ (-1)^F\ q^{L_0}\ \bar{q}^{\bar{L}_0}\ y^{J_0^3}\ h\ (\partial Z^{(1)*}_{-n})^2\,|\Omega\rangle &= q^{2n}\ (\partial Z^{(1)*}_{-n})^2\,|\Omega\rangle\\ &\ldots\end{aligned}$$

Again we can use the geometric series to express the previous results:

$$1 - q^n + q^{2n} + \ldots = \sum_{l=0}^{+\infty}(-q^n)^l = \frac{1}{1+q^n}$$

The same can be done for the $\partial Z^{(2)}$ and $\partial Z^{(2)*}$ fields.
Let us now consider the fermionic contributions. We can start by considering the $\chi^{(1)}$ contribution, therefore what we obtain is:

$$\begin{aligned}(-1)^F\ q^{L_0}\ \bar{q}^{\bar{L}_0}\ y^{J_0^3}\ h\ \ |\Omega\rangle &= |\Omega\rangle\\ (-1)^F\ q^{L_0}\ \bar{q}^{\bar{L}_0}\ y^{J_0^3}\ h\ \chi^{(1)}_{-n}\,|\Omega\rangle &= y^{-1}q^n\,|\Omega\rangle\end{aligned}$$

The contribution to the trace therefore becomes: $1 + y^{-1}q^n$. The same obviously has to be done with the remaining chiral fields. We can notice that we are not considering the anti-chiral field terms and that is because, it can be shown, the only contributions coming from the right-moving field operators are the ground states contributions. The motivation, that we are not going to present here, is the fact that the anti-chiral bosonic and fermionic contributions exactly cancel each other because of supersymmetry, except for ground states contributions. After this statement, we have finally that:

$$Z^{(2)osc}(\tau, z) = \prod_{n=1}^{\infty} \frac{(1 + y^{-1}q^n)^2(1 + yq^n)^2}{(1 + q^n)^4} \tag{7.20}$$

In order now to find the final expression of $Z^{(2)}(\tau, z)$, we still need to compute the contribution to the trace coming from the vertex operators.
It can be now shown that vertex operators contribute only under specifical condition that, for the $Z^{(2)}(\tau, z)$ case, are not satisfied. This will become more clear when we will compute directly the Twining Genus on K3, but the key point is that the eigenvalues we are dealing with in this case are $\zeta_L = \zeta_R = -1$, where the minus sign arises from the action of the h symmetry. In conclusion what we obtain is that $Z^{(2)\lambda}(\tau, z) = 1$, therefore there is no contribution from the vertex operators.
We can now use the special functions defined in the Appendix (7.3), in particular the relation (7.51), to write in a useful way the expression in (7.20). What we can finally obtain thus is:

$$Z^{(2)}(\tau, z) = \underbrace{4y^{-1}(1 + y)^2}_{gs} \cdot \underbrace{\frac{\theta_2(z|\tau)^2}{\theta_2(0|\tau)^2} \frac{4}{y^{-1}(1 + y)^2}}_{osc} \cdot \underbrace{1}_{vertex} = 16 \cdot \frac{\theta_2(z|\tau)^2}{\theta_2(0|\tau)^2} \tag{7.21}$$

Let us now focus on the h-twisted sector terms, namely on $Z^{(3)}(\tau, z)$ and $Z^{(4)}(\tau, z)$. In the h-twisted sector, the oscillation modes of the fields have an half-integer index, namely $r \in \mathbb{Z} + 1/2$. As we said previously, we cannot have zero modes now, so we will see later how to define the twisted ground states Hilbert space and compute their contribution to the trace in (7.11).
We can start computing the $Z^{(3)osc}(\tau, z)$ term first. Just like we did for the untwisted sector, we can start computing the $\partial Z^{(1)}$ contribution to the trace:

$$\begin{gathered}
(-1)^F \; q^{L_0} \; \bar{q}^{\bar{L}_0} \; y^{J_0^3} \; |\Omega\rangle = |\Omega\rangle \\
(-1)^F \; q^{L_0} \; \bar{q}^{\bar{L}_0} \; y^{J_0^3} \; \partial Z_r^{(1)} \, |\Omega\rangle = q^r \; \partial Z_r^{(1)} \, |\Omega\rangle \\
(-1)^F \; q^{L_0} \; \bar{q}^{\bar{L}_0} \; y^{J_0^3} \; (\partial Z_r^{(1)})^2 \, |\Omega\rangle = q^{2r} \; (\partial Z_r^{(1)})^2 \, |\Omega\rangle \\
\ldots
\end{gathered}$$

The contribution to the trace can be written again as:

$$1 + q^r + q^{2r} + \ldots = \sum_{l=0}^{\infty} (q^r)^l = \frac{1}{1 - q^r}$$

Proceeding now analogously with the remaining bosonic and fermionic chiral fields, the oscillator term assume a very similar form to the term in (7.20), namely:

$$Z^{(3)osc}(\tau, z) = \prod_{r \in \mathbb{N}+1/2} \frac{(1 - y^{-1}q^r)^2(1 - yq^r)^2}{(1 - q^r)^4}$$

Using the functions defined in the Appendix (7.3), in particular the relation (7.49), we can rewrite the previous term as:

$$Z^{(3)osc}(\tau, z) = \frac{\theta_4(z|\tau)^2}{\theta_4(0|\tau)^2} \tag{7.22}$$

The $Z^{(4)osc}(\tau, z)$ term can now be immediately obtained by considering, like we did previously, the action of the h operator. What we can finally obtain thus, using the relation (7.48) in the Appendix, is:

$$Z^{(4)osc}(\tau, z) = \frac{\theta_3(z|\tau)^2}{\theta_3(0|\tau)^2} \tag{7.23}$$

Let us now compute the $Z^{(3)gs}(\tau, z)$ and $Z^{(4)gs}(\tau, z)$ terms. We can start by noting that in the h-twisted R-R sector there are no zero mode operators for the bosonic and fermionic fields, therefore we can't build the Hilbert subspace of ground states $\mathcal{H}_{gs}^{tw}$ analogously to the previous case. In this case the construction of $\mathcal{H}_{gs}^{tw}$ involves the vertex operators. The vertex operator in the h-twisted sector is:

$$V_\lambda(z, \bar{z}) = E_\lambda^-(z, \bar{z}) E_\lambda^+(z, \bar{z}) e_\lambda$$

with:

$$E_\lambda^\pm = \exp\left(- \sum_{n \in \mathbb{Z}+1/2} \lambda_L \cdot \alpha_n \frac{z^{-n}}{n} - \sum_{n \in \mathbb{Z}+1/2} \lambda_R \cdot \bar{\alpha}_n \frac{\bar{z}^{-n}}{n} \right)$$

The OPE of vertex operators defined in this way must be the same as (7.8), namely:

$$V_\lambda(z, \bar{z}) V_\mu(w, \bar{w}) = \xi(\lambda, \mu)(z - w)^{\lambda_L \cdot \mu_L} (\bar{z} - \bar{w})^{\lambda_R \cdot \mu_R} V_{\lambda+\mu}(w, \bar{w})$$

where the operators e_λ commute with the oscillators $E_\lambda^\pm$ and have to satisfy the following consistency condition:

$$e_\lambda e_\mu = \xi(\lambda, \mu) e_{\lambda+\mu} = (-1)^{\lambda_L \mu_L - \lambda_R \mu_R}\ \xi(\mu, \lambda) e_{\lambda+\mu} \tag{7.24}$$

Now, as usual, we define the twisted ground states through the following conditions:

$$\alpha_n^i \left| gs \right\rangle = 0, \qquad \psi_n^i \left| gs \right\rangle = 0 \qquad n \in \mathbb{N} + 1/2$$

and we can thus notice that:

$$\lim_{z, \bar{z} \to 0} V_\lambda(z, \bar{z}) \left| gs \right\rangle = e_\lambda \left| gs \right\rangle$$

Since now the e_λ operators commute with the oscillator terms, we deduce that also $e_\lambda \left| gs \right\rangle$ is a twisted ground state. What we obtain thus is that e_λ maps a twisted ground state to another twisted ground state, therefore, in other words, $\mathcal{H}_{gs}^{tw}$ is the representation of the algebra of the e_λ operators. Let us now find the irreducible representaion of this algebra. If we now consider a basis $\{w_1, ..., w_4, m_1, ..., m_4\}$ of the winding-momentum lattice $\Gamma^{4,4}$ satisfying the following conditions:

$$m_i \circ m_j = 0 = w_i \circ w_j, \qquad m_i \circ w_j = \delta_{ij}$$

where "$\circ$" is the bilinear form of signature $(4, 4)$ of $\Gamma^{4,4}$, the relation (7.24) for the e_{m_i}, e_{w_i} operators becomes:

$$[e_{m_i}, e_{m_j}] = 0 = [e_{w_i}, e_{w_j}], \qquad e_{m_i} e_{w_j} = (-1)^{\delta_{ij}}\ e_{w_j} e_{m_i} \tag{7.25}$$

For each $\lambda \in \Gamma^{4,4}$, i.e. $\lambda = \sum_i a_i w_i + b_i m_i$, we can now define:

$$e_\lambda \equiv e_{w_1}^{a_1} \cdots e_{w_4}^{a_4}\ e_{m_1}^{b_1} \cdots e_{m_4}^{b_4}$$

that satisfies the relation (7.24), as we can easily check. We can now see that the operators $e_{2w_i} = e^2_{w_i}$ and $e_{2m_i} = e^2_{m_i}$ commute with every e_λ, therefore, in an irreducible representation, they must be proportional to the identity. Since now we are free to rescale by a phase e_λ without modifying its algebra relations and the OPE of the vertex operators, we can set:

$$e_{2w_i} = e^2_{w_i} = 1, \qquad e_{2m_i} = e^2_{m_i} = 1$$

We notice now that the four operators $e_{m_1}, ..., e_{m_4}$, thanks to the relations in (7.25), commute with each other, therefore we can find a simultaneous eigenvector of them that we can call $|r_1, r_2, r_3, r_4\rangle$. Since $e^2_{m_i} = 1$ we have then that $r_i = \pm 1$, with $i = 1, ..., 4$. Using again the relations in (7.25), the vector $e_{w_i}|r_1, r_2, r_3, r_4\rangle$ is still an eigenvector of e_{m_i} with eigenvalues $(-1)^{\delta_{ij}} r_j$. We can thus use the operators e_{w_i} to obtain all 16 possible eigenvectors of e_{m_1}, indeed, starting for example from the eigenvector $|1, 1, 1, 1\rangle$, what we can obtain is:

$$\begin{aligned}
&|1,1,1,1\rangle && e_{w_4}|1,1,1,1\rangle = |1,1,1,-1\rangle \\
&e_{w_1}|1,1,1,1\rangle = |-1,1,1,1\rangle && e_{w_1}e_{w_2}|1,1,1,1\rangle = |-1,-1,1,1\rangle \\
&e_{w_2}|1,1,1,1\rangle = |1,-1,1,1\rangle && e_{w_1}e_{w_3}|1,1,1,1\rangle = |-1,1,-1,1\rangle \\
&e_{w_3}|1,1,1,1\rangle = |1,1,-1,1\rangle && e_{w_1}e_{w_4}|1,1,1,1\rangle = |-1,1,1,-1\rangle
\end{aligned}$$

$$\begin{aligned}
&e_{w_2}e_{w_3}|1,1,1,1\rangle = |1,-1,-1,1\rangle && e_{w_1}e_{w_2}e_{w_4}|1,1,1,1\rangle = |-1,-1,1,-1\rangle \\
&e_{w_2}e_{w_4}|1,1,1,1\rangle = |1,-1,1,-1\rangle && e_{w_1}e_{w_3}e_{w_4}|1,1,1,1\rangle = |-1,1,-1,-1\rangle \\
&e_{w_3}e_{w_4}|1,1,1,1\rangle = |1,1,-1,-1\rangle && e_{w_2}e_{w_3}e_{w_4}|1,1,1,1\rangle = |1,-1,-1,-1\rangle \\
&e_{w_1}e_{w_2}e_{w_3}|1,1,1,1\rangle = |-1,-1,-1,1\rangle && e_{w_1}e_{w_2}e_{w_3}e_{w_4}|1,1,1,1\rangle = |-1,-1,-1,-1\rangle
\end{aligned} \tag{7.26}$$

Those vectors form in fact a basis for $\mathcal{H}^{tw}_{gs}$, namely they are the 16 twisted ground states of our model. Since now all the operators inserted in the trace act trivially on those states, namely their eigenvalues are identically 1, what we obtain is simply:

$$Z^{(3)gs}(\tau, z) = Z^{(4)gs}(\tau, z) = \mathrm{Tr}_{\mathcal{H}_{gs}}[1] = 16 \tag{7.27}$$

We are now finally able to write the final expressions for $Z^{(3)}(\tau, z)$ and $Z^{(4)}(\tau, z)$. By putting together every contribute what we obtain is:

$$Z^{(3)}(\tau, z) = 16\, \frac{\theta_4(z|\tau)^2}{\theta_4(0|\tau)^2} \tag{7.28}$$

$$Z^{(4)}(\tau, z) = 16\, \frac{\theta_3(z|\tau)^2}{\theta_3(0|\tau)^2} \tag{7.29}$$

The Elliptic Genus for K3 thus results:

$$Z^{orb}(\tau, z) = 8 \cdot \left[\frac{\theta_2(z|\tau)^2}{\theta_2(0|\tau)^2} + \frac{\theta_4(z|\tau)^2}{\theta_4(0|\tau)^2} + \frac{\theta_3(z|\tau)^2}{\theta_3(0|\tau)^2} \right] \tag{7.30}$$

7.5 Some Twining Genera of K3

We are now ready to present our goal: the computation in the R-R sector of some Twining Genera of the $\mathcal{N} = (4, 4)$ superconformal field theory whose target-space

is the $\mathbb{T}^4/\mathbb{Z}_2$ orbifold. Notice that, although the Twining Genus is defined on the R-R sector, we can get informations also about the NS sectors using the spectral flow that we defined in Subsection (5.4.1).
In order now to obtain the Twining Genus, we need to take the expression seen in (7.11) and insert a generic discrete symmetry $g \in G$ of our NLSM. A possible choice of the operator representation of g acting on fermionic fields is $g \equiv \zeta_L^{A_0^3} \cdot \zeta_R^{\tilde{A}_0^3}$. The final object we would like to compute thus is:

$$\begin{aligned} Z_g^{orb}(\tau, z) = \mathrm{Tr}_{(h-untw)} \left[\frac{(1+h)}{2} q^{L_0 - c/24} \; \bar{q}^{\bar{L}_0 - c/24} (-1)^{F+\bar{F}} y^{J_0^3} \; g \right] \\ + \mathrm{Tr}_{(h-tw)} \left[\frac{(1+h)}{2} q^{L_0 - c/24} \; \bar{q}^{\bar{L}_0 - c/24} (-1)^{F+\bar{F}} y^{J_0^3} \; g \right] \end{aligned} \tag{7.31}$$

Again, it is useful to rewrite the trace above into four different terms:

$$Z_g^{(1)}(\tau, z) = \frac{1}{2} \mathrm{Tr}_{(h-untw)} \left[q^{L_0 - c/24} \; \bar{q}^{\bar{L}_0 - c/24} (-1)^{F+\bar{F}} y^{J_0^3} \; g \right] \tag{7.32}$$

$$Z_g^{(2)}(\tau, z) = \frac{1}{2} \mathrm{Tr}_{(h-untw)} \left[q^{L_0 - c/24} \; \bar{q}^{\bar{L}_0 - c/24} (-1)^{F+\bar{F}} y^{J_0^3} \; g \; h \right] \tag{7.33}$$

$$Z_g^{(3)}(\tau, z) = \frac{1}{2} \mathrm{Tr}_{(h-tw)} \left[q^{L_0 - c/24} \; \bar{q}^{\bar{L}_0 - c/24} (-1)^{F+\bar{F}} y^{J_0^3} \; g \right] \tag{7.34}$$

$$Z_g^{(4)}(\tau, z) = \frac{1}{2} \mathrm{Tr}_{(h-tw)} \left[q^{L_0 - c/24} \; \bar{q}^{\bar{L}_0 - c/24} (-1)^{F+\bar{F}} y^{J_0^3} \; g \; h \right] \tag{7.35}$$

with:

$$Z_g^{orb}(\tau, z) = \sum_{i=1}^{4} Z_g^{(i)}(\tau, z) \tag{7.36}$$

We can notice that the decomposition we saw in (7.16) holds also in this case, namely after the insertion of the symmetry g in the Elliptic Genus. We thus can start computing the ground states contribution to $Z_g^{(1)}(\tau, z)$. Again, let us take the state $|-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\rangle \equiv |gs\rangle$ as the state from which we can generate $\mathcal{H}_{gs}$. Applying the operators in the trace to the $|gs\rangle$ state we obtain:

$$(-1)^F \; y^{J_0^3} \; \zeta_L^{A_0^3} \; \zeta_R^{\tilde{A}_0^3} \, |gs\rangle = (+1) \cdot (y^{-1}) \cdot (+1) \cdot (+1) \, |gs\rangle = y^{-1} \, |gs\rangle$$

Acting now for example with the creation operator $\chi_0^{(1)*}$ on $|gs\rangle$, calling it for simplicity $|\chi_0^{(1)*}\rangle$, we can find:

$$\begin{aligned} (-1)^F \; y^{J_0^3} \; \zeta_L^{A_0^3} \; \zeta_R^{\tilde{A}_0^3} \, |\chi_0^{(1)*}\rangle = (-1)^F \; y^{J_0^3} \; \zeta_L^{A_0^3} \; \zeta_R^{\tilde{A}_0^3} \left| +\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right\rangle = \\ = (-1) \cdot (+1) \cdot (\zeta_L) \cdot (+1) \, |\chi_0^{(1)*}\rangle = -\zeta_L \, |\chi_0^{(1)*}\rangle \end{aligned}$$

using now $|\chi_0^{(2)*}\rangle$ we find:

$$\begin{aligned} (-1)^F \; y^{J_0^3} \; \zeta_L^{A_0^3} \; \zeta_R^{\tilde{A}_0^3} \, |\chi_0^{(2)*}\rangle = (-1)^F \; y^{J_0^3} \; \zeta_L^{A_0^3} \; \zeta_R^{\tilde{A}_0^3} \left| -\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right\rangle = \\ = (-1) \cdot (+1) \cdot (\zeta_L^{-1}) \cdot (+1) \, |\chi_0^{(2)*}\rangle = -\zeta_L^{-1} \, |\chi_0^{(2)*}\rangle \end{aligned}$$

and analogously with $|\chi_0^{(2)*} \chi_0^{(1)*}\rangle$ we get:

$$(-1)^F \; y^{J_0^3} \; \zeta_L^{A_0^3} \; \zeta_R^{\tilde{A}_0^3} \, |\chi_0^{(2)*} \chi_0^{(1)*}\rangle = y \, |\chi_0^{(2)*} \chi_0^{(1)*}\rangle$$

Acting now also with all the possible combinations of right-moving operators, we can obtain the remaining contributions. By computing then the trace what we can obtain is:

$$Z_g^{(1)gs}(\tau, z) = y^{-1}(1-\zeta_L y)(1-\zeta_L^{-1} y)(1-\zeta_R)(1-\zeta_R^{-1}) \qquad (7.37)$$

Let us now compute $Z_g^{(1)osc}(\tau, z)$. In order to simplify the calculations, it is useful to compute firstly all the eigenvalues of the g symmetry operator when they act on the oscillator terms of the bosonic and fermionic fields. The eigenvalues can be explicitly computed using the fact that the four supercurrents we defined in (7.1) have to be invariant under the action of the g symmetry. For simplicity we provide the tables of the eigenvalues of both $y^{J_0^3}$ and g operators. For the fermionic fields we can find:

	$\chi^{(1)*}$	$\chi^{(2)*}$	$\chi^{(1)}$	$\chi^{(2)}$	$\tilde{\chi}^{(1)*}$	$\tilde{\chi}^{(2)*}$	$\tilde{\chi}^{(1)}$	$\tilde{\chi}^{(2)}$
$y^{J_0^3}$	y	y	y^{-1}	y^{-1}	1	1	1	1
g	ζ_L	ζ_L^{-1}	ζ_L^{-1}	ζ_L	ζ_R	ζ_R^{-1}	ζ_R^{-1}	ζ_R

while for the bosonic fields we get:

	$\partial Z^{(1)*}$	$\partial Z^{(2)*}$	$\partial Z^{(1)}$	$\partial Z^{(2)}$	$\tilde{\partial} Z^{(1)*}$	$\tilde{\partial} Z^{(2)*}$	$\tilde{\partial} Z^{(1)}$	$\tilde{\partial} Z^{(2)}$
$y^{J_0^3}$	1	1	1	1	1	1	1	1
g	ζ_L	ζ_L^{-1}	ζ_L^{-1}	ζ_L	ζ_R	ζ_R^{-1}	ζ_R^{-1}	ζ_R

We are now ready to proceed with the computation of $Z_g^{(1)osc}(\tau, z)$. Let us start with the factor of the Hilbert space generated by the $\partial Z_{-n}^{(1)}$ operators. Analogously to the Elliptic Genus case, we can start computing, using the tables above:

$$\begin{gathered}
(-1)^F\, q^{L_0}\, \bar{q}^{\bar{L}_0}\, y^{J_0^3}\, g\, |\Omega\rangle = (+1)\cdot|\Omega\rangle = |\Omega\rangle \\
(-1)^F\, q^{L_0}\, \bar{q}^{\bar{L}_0}\, y^{J_0^3}\, g\, \partial Z_{-n}^{(1)}\, |\Omega\rangle = \zeta_L^{-1} q^n\, \partial Z_{-n}^{(1)}\, |\Omega\rangle \\
(-1)^F\, q^{L_0}\, \bar{q}^{\bar{L}_0}\, y^{J_0^3}\, g\, (\partial Z_{-n}^{(1)})^2\, |\Omega\rangle = \zeta_L^{-2} q^{2n}\, (\partial Z_{-n}^{(1)})^2\, |\Omega\rangle \\
\ldots
\end{gathered}$$

The contribution to the trace then can be expressed in a compact way:

$$1 + \zeta_L^{-1} q^n + \zeta_L^{-2} q^{2n} + \ldots = \sum_{l=0}^{+\infty} (\zeta_L^{-1} q^n)^l = \frac{1}{1-\zeta_L^{-1} q^n}$$

Repeating the same calculations for the $\partial Z^{(1)*}$ case, we get:

$$\begin{gathered}
(-1)^F\, q^{L_0}\, \bar{q}^{\bar{L}_0}\, y^{J_0^3}\, g\ |\Omega\rangle = (+1)\cdot|\Omega\rangle = |\Omega\rangle \\
(-1)^F\, q^{L_0}\, \bar{q}^{\bar{L}_0}\, g\ \partial Z_{-n}^{(1)*}\, |\Omega\rangle = \zeta_L\ q^n\ \partial Z_{-n}^{(1)*}\, |\Omega\rangle \\
(-1)^F\, q^{L_0}\, \bar{q}^{\bar{L}_0}\, y^{J_0^3}\, g\ (\partial Z_{-n}^{(1)*})^2\, |\Omega\rangle = \zeta_L^2\ q^{2n}\ (\partial Z_{-n}^{(1)*})^2\, |\Omega\rangle \\
\ldots
\end{gathered}$$

Again we can use the geometric series to express the previous results:

$$1 + \zeta_L \; q^n + \zeta_L^2 \; q^{2n} + ... = \sum_{l=0}^{+\infty} (\zeta_L q^n)^l = \frac{1}{1-\zeta_L q^n}$$

The same can be done for the $\partial Z^{(2)}$ and $\partial Z^{(2)*}$ fields.
Let us now consider the fermionic contributions. We can start by considering the $\chi^{(1)}$ contribution, therefore what we obtain is:

$$(-1)^F \; q^{L_0} \; \bar{q}^{\bar{L}_0} \; y^{J_0^3} \; \zeta_L^{A_0^3} \; \zeta_R^{\tilde{A}_0^3} \; |\Omega\rangle = |\Omega\rangle$$
$$(-1)^F \; q^{L_0} \; \bar{q}^{\bar{L}_0} \; y^{J_0^3} \; \zeta_L^{A_0^3} \; \zeta_R^{\tilde{A}_0^3} \; \chi_{-n}^{(1)} \, |\Omega\rangle = -y^{-1}\zeta_L^{-1} q^n \, |\Omega\rangle$$

The contribution to the trace therefore becomes: $1-y^{-1}\zeta_L^{-1}q^n$. The considerations about right-moving operators hold also in this case, therefore we will not consider the anti-chiral field contributions to the oscillator term. What we have finally is:

$$Z_g^{(1)osc}(\tau,z) = \prod_{n=1}^{\infty} \frac{(1-y^{-1}\zeta_L^{-1}q^n)(1-y^{-1}\zeta_L q^n)(1-y\zeta_L q^n)(1-y\zeta_L^{-1}q^n)}{(1-\zeta_L^{-1}q^n)^2(1-\zeta_L q^n)^2} \tag{7.38}$$

In order now to find the final expression of $Z_g^{(1)}(\tau,z)$, we still need to compute the contribution to the trace coming from the vertex operators.
The action of the symmetry g on the charge of the vertex operator, namely the momentum $\vec{\lambda} \in \Gamma^{4,4}$, is $g : \vec{\lambda} \to g(\vec{\lambda}) \equiv \vec{\mu}$. Considering the action of g on the entire set of Hilbert subspaces, we obtain $g : \mathcal{H}_{\vec{\lambda}} \to \mathcal{H}_{g(\vec{\lambda})}$. If we represent g as a matrix that maps a Hilbert subspace $\mathcal{H}_{\vec{\lambda}\in\Gamma^{4,4}}$ to another point $\mathcal{H}_{\vec{\mu}\in\Gamma^{4,4}}$, the only contributions to the trace will come from the elements on the diagonal of the matrix. The problem therefore reduces to find the fixed points of the g symmetry acting on $\vec{\lambda}$. A trivial solution is clearly $\vec{\lambda} = 0$, namely a obvious contribution to the trace is given by the calculations performed in the Hilbert space where there are no vertex operators acting, but that is what we have already computed.
We can now show that, if $\zeta_L = 1$, there exists a vector $\vec{\lambda}'$, belonging to the lattice $\Gamma^{4,4}$, that is fixed.
Let us consider a symmetry g of order two[8]. We are always allowed to take a generic vector $\vec{\lambda} \in \Gamma^{4,4}$ and build a new vector $\vec{\lambda}' \in \Gamma^{4,4}$ of the form $\vec{\lambda}' = \vec{\lambda} + g(\vec{\lambda})$, since g is an automorphism of the lattice. We notice that $\vec{\lambda}'$ is invariant under the action of g by construction. If now all the eigenvalues of g are -1, namely the eigenspace relative to the eigenvalue -1 is the whole space, then $\vec{\lambda}' = 0$, because it necessary holds that $g(\vec{\lambda}) = -\vec{\lambda}$. Instead, if g admits at least an eigenvalue $+1$, then, since $\vec{\lambda}$ will not even be an eigenvector of g, $\vec{\lambda}'$ will be a non-vanishing g-invariant vector of $\Gamma^{4,4}$. We can thus conclude that, if g admits at least an eigenvalue $+1$, then there always exists a vector of the lattice $\Gamma^{4,4}$ that it is fixed under the action of g.
The contribution to the final result, given by the existence of this fixed vector, is a phase multiplied by the conformal weight of the vector $|\lambda\rangle$, namely:

$$Z_g^{(1)\lambda}(\tau,z) = \sum_{\text{fixed } \vec{\lambda}=(\vec{\lambda}_L,\vec{\lambda}_R)\in\Gamma^{4,4}} q^{\lambda_L^2/2} \; \bar{q}^{\lambda_R^2/2} \; e^{i\phi}$$

[8] The order of the symmetries we consider is indicated in Table 2 of [16], pag.20, as $o(\pm g_0)$.

For simplicity we consider $\zeta_L \neq 1$ and $\zeta_R \neq 1$, therefore what we obtain is $Z_g^{(1)\lambda}(\tau, z) = 1$.
We can now use the special functions defined in the Appendix (7.3), in particular the relation (7.50), to write in a useful way the final expression of $Z_g^{(1)}(\tau, z)$. What we can finally obtain thus is:

$$Z_g^{(1)}(\tau, z) = \frac{\theta_1(z + r_L|\tau)\ \theta_1(z - r_L|\tau)}{\theta_1(r_L|\tau)\ \theta_1(-r_L|\tau)} \cdot (2 - \zeta_L - \zeta_L^{-1})(2 - \zeta_R - \zeta_R^{-1}) \tag{7.39}$$

where we defined $\zeta_{L,R} \equiv e^{2\pi i r_{L,R}}$, with $r_{L,R} \in \mathbb{Q}/\mathbb{Z}$.
Since now the effect of the h operator can be equivalently thought as flipping the sign of the eigenvalues of the g operator, namely mapping $\zeta_{L,R} \to -\zeta_{L,R}$, it is very easy to find the expression also of $Z_g^{(2)}(\tau, z)$. What we can indeed obtain, through the relation (7.51), is:

$$Z_g^{(2)}(\tau, z) = \frac{\theta_2(z + r_L|\tau)\ \theta_2(z - r_L|\tau)}{\theta_2(r_L|\tau)\ \theta_2(-r_L|\tau)} \cdot (2 + \zeta_L + \zeta_L^{-1})(2 + \zeta_R + \zeta_R^{-1}) \tag{7.40}$$

Notice that if we impose the condition $g \equiv 1$, namely $\zeta_{L,R} = 1$ and $r_{L,R} = 0$, for the expressions (7.39) and (7.40), we respectively recover (7.18) an (7.21).

Let us now focus on the h-twisted sector terms, namely on $Z_g^{(3)}(\tau, z)$ and $Z_g^{(4)}(\tau, z)$. We can start computing the $Z_g^{(3)osc}(\tau, z)$ term first and, following the previous considerations, obtain also the $Z_g^{(4)osc}(\tau, z)$ term. Just like we did for the untwisted sector, we can start computing the $\partial Z^{(1)}$ contribution to the trace:

$$\begin{gathered}
(-1)^F\ q^{L_0}\ \bar{q}^{\bar{L}_0}\ y^{J_0^3}\ g\ |\Omega\rangle = |\Omega\rangle \\
(-1)^F\ q^{L_0}\ \bar{q}^{\bar{L}_0}\ y^{J_0^3}\ g\ \partial Z_r^{(1)}\ |\Omega\rangle = \zeta_L^{-1} q^r\ \partial Z_r^{(1)}\ |\Omega\rangle \\
(-1)^F\ q^{L_0}\ \bar{q}^{\bar{L}_0}\ y^{J_0^3}\ g\ (\partial Z_r^{(1)})^2\ |\Omega\rangle = \zeta_L^{-2} q^{2r}\ (\partial Z_r^{(1)})^2\ |\Omega\rangle \\
\ldots
\end{gathered}$$

The contribution to the trace can be written again as:

$$1 + \zeta_L^{-1} q^r + \zeta_L^{-2} q^{2r} + \ldots = \sum_{l=0}^{\infty} (\zeta_L^{-1} q^r)^l = \frac{1}{1 - \zeta_L^{-1} q^r}$$

Proceeding now analogously with the remaining bosonic and fermionic chiral fields, the oscillator term assume a very similar form to the term in (7.38), namely:

$$Z_g^{(3)osc}(\tau, z) = \prod_{r \in \mathbb{N}+1/2} \frac{(1 - y^{-1}\zeta_L^{-1} q^r)(1 - y^{-1}\zeta_L q^r)(1 - y\zeta_L q^r)(1 - y\zeta_L^{-1} q^r)}{(1 - \zeta_L^{-1} q^r)^2 (1 - \zeta_L q^r)^2}$$

Using the functions defined in the Appendix (7.3), in particular the relation (7.49), we can rewrite the previous term as:

$$Z_g^{(3)osc}(\tau, z) = \frac{\theta_4(z + r_L|\tau)\ \theta_4(z - r_L|\tau)}{\theta_4(r_L|\tau)\ \theta_4(-r_L|\tau)} \tag{7.41}$$

The $Z_g^{(4)osc}(\tau, z)$ term can now be immediately obtained by considering the fact that, as we did previously, the h operator flips the sign of the eigenvalues of g. What we can finally obtain thus, using the relation (7.48) in the Appendix, is:

$$Z_g^{(4)osc}(\tau, z) = \frac{\theta_3(z + r_L|\tau)\ \theta_3(z - r_L|\tau)}{\theta_3(r_L|\tau)\ \theta_3(-r_L|\tau)} \tag{7.42}$$

7.5.1 Action of g on h-twisted Ground States

Now, since we chose for simplicity to take $\zeta_{L,R} \neq 1$, the only terms that we need to compute are $Z_g^{(3)gs}(\tau, z)$ and $Z_g^{(4)gs}(\tau, z)$. We need therefore to present how the allowed discrete symmetries $g \in G$ act on those states. We know that any symmetry $g \in G$ of our NLSM commuting with the symmetry h, with respect to which we take the orbifold, induces a symmetry of the orbifold[9]. An easy proof is the following: consider a field $\phi(\sigma + 2\pi l, \tau) \in \mathcal{H}_{h-tw}$, what we can write therefore is

$$\phi(\sigma + 2\pi l, \tau) = h \cdot \phi(\sigma, \tau).$$

If we now consider:

$$g \cdot \phi(\sigma + 2\pi l, \tau) = g \cdot h\ \phi(\sigma, \tau) = ghg^{-1} \cdot g\phi(\sigma, \tau)$$

we obtain that the symmetry g maps a field from $\mathcal{H}_{h-tw}$ to $\mathcal{H}_{ghg^{-1}-tw}$, therefore, if $gh = hg$, we obtain that g is an endomorphism of $\mathcal{H}_{h-tw}$, from which follows what we initially stated.

As we will explicitly see however, it will not be unique but fixed up to an overall sign, so there will be actually two different symmetries.

Let us take a representation $\rho(g)$ of the g symmetry on the twisted ground states. From the definition in (7.24), we have that:

$$g(e_\lambda) = \rho(g) e_\lambda \rho(g)^{-1} = \kappa_g(\lambda) e_{g(\lambda)} \tag{7.43}$$

where $\kappa_g(\lambda)$ is a phase that is completely determined, up to a sign, after choosing a basis of the lattice $\Gamma^{4,4}$. Imposing this condition we can fix $\rho(g)$ up to a non-zero rescaling constant. We notice also that we can choose a basis of $\mathcal{H}^{tw}_{gs}$ in which the corresponding fields $\Phi_k(z, \bar{z})$, with $k = 1, ..., 16$, obey the following OPE:

$$\Phi_i(z, \bar{z}) \Phi_j(w, \bar{w}) = \frac{\delta_{ij}}{|z - w|} + ...$$

since the conformal weight of the twisted ground states is $(1/4, 1/4)$. Here $\rho(g) \in SO(16, \mathbb{R})$, so its eigenvalues will be roots of unity and $\mathrm{Tr}\rho(g) \in \mathbb{R}$.

Those arguments fix the matrix $\rho(g)$ up to an overall sign and that is the best we can do.

The procedure of finding the matrix $\rho(g)$ is the following:

- Choose a basis for the matrix $\rho(g)$. For example, we could take the basis computed in (7.26).[10]

- Find the action of g on the basis w_i, m_i, $i = 1, ..., 4$ of the lattice $\Gamma^{4,4}$. Using the relation (7.43) and possibly fixing in a trivial way the arbitrary constants $\kappa_g(e_{m_i})$ and $\kappa_g(e_{w_i})$, one can find the corresponding relations $g(e_{m_i}) = \kappa_g(m_i) e_{g(m_i)}$ and $g(e_{w_i}) = \kappa_g(w_i) e_{g(w_i)}$.

- Find, up to a normalization constant, a simultaneous eigenvector $|\chi\rangle$ for the operators $g(e_{m_i})$.

[9] For more mathematical considerations see [23].

[10] For simplicity, we will always use it in our following examples.

- Act on $|\chi\rangle$ in all possible ways with the operators $g(e_{w_i})$ and find all the 16 simultaneous eigenvectors of $g(e_{m_i})$.

- Projecting now all those 16 states onto the basis we initially chose, we obtain the coefficient that, opportunely ordered, will form our final matrix.

Examples of $\rho(g)$ matrices: Let us make some explicit examples. We can take, as previously adviced, the basis written in (7.26), namely the one generated from the vector $|1,1,1,1\rangle$ by acting in all possible ways with the operators e_{w_i}. The tranformations of the windings and momenta w_i and m_i, basis of the lattice $\Gamma^{4,4}$, that we consider are:

$$\begin{aligned} g(w_1) &= -w_1 \\ g(w_i) &= w_i, \qquad i = 2,3,4 \\ g(m_j) &= m_j, \qquad j = 1,...,4 \end{aligned}$$

We have to notice now that the transformations above are not automorphisms of the lattice[11] and therefore they cannot lead to a symmetry of our model. We take this first example just as an exercise.
The corresponding transformations of the basis of $\mathcal{H}^{tw}_{gs}$, by fixing opportunely every arbitrary constant $\xi_g(\lambda)$, are, for $i = 2,3,4$ and $j = 1,...,4$:

$$\begin{aligned} &g(e_{w_1}) = -e_{-w_1} = -e_{w_1}^{-1} = -e_{w_1} && \kappa_g(w_1) = -1 \\ &g(e_{w_i}) = e_{w_i} && \kappa_g(w_i) = 1 \\ &g(e_{m_j}) = e_{m_j} && \kappa_g(m_j) = 1 \end{aligned}$$

We have therefore that:

$$\begin{aligned} \underbrace{g(e_{m_i})}_{\rho(g)e_{m_i}\rho(g)^{-1}} \quad \rho(g)\,|1,1,1,1\rangle &= \rho(g)e_{m_i}\,|1,1,1,1\rangle = \rho(g)\,|1,1,1,1\rangle \\ &= e_{m_i}\;\rho(g)\,|1,1,1,1\rangle \end{aligned}$$

from which we understand that the form of the eigenvector of $g(e_{m_i})$, $|\chi\rangle \equiv \rho(g)\,|1,1,1,1\rangle$ is $|\chi\rangle = |1,1,1,1\rangle$.
Following now the procedure we presented above, we should apply in all possible ways the operators $g(e_{w_i})$ and obtain 16 linearly independent states. Projecting every state onto the basis we initially chose, we obtain a list of coefficients that form the rows of the final matrix. In practice, the first eigenvector we have is $|\chi\rangle = |1,1,1,1\rangle$, therefore the first row of the matrix $\rho(g)$ is:

$$\rho(g)_{1,j} = (1, \underbrace{0,...,0}_{15 \text{ times}}) \qquad j = 1,...,16$$

In order thus to obtain the second row of our matrix we should apply the operator $g(e_{w_1})$ to the eigenvector $|\chi\rangle = |1,1,1,1\rangle$. What we obtain is:

$$g(e_{w_1})\,|\chi\rangle = g(e_{w_1})\,|1,1,1,1\rangle = -\;e_{w_1}\,|1,1,1,1\rangle = -\,|-1,1,1,1\rangle$$

[11]Indeed the relations in (7.25) are not preserved by the symmetry g.

The second row of the matrix $\rho(g)$ therefore is:

$$\rho(g)_{2,j} = (0, -1, \underbrace{0, ..., 0}_{14 \text{ times}}) \qquad j = 1, ..., 16$$

Proceeding in the same way for the remaining eigenvectors, the final matrix results:

$$\rho(g) = \pm \ \mathrm{diag}(1, -1, 1, 1, 1, -1, -1, -1, 1, 1, 1, -1, -1, -1, 1, -1)$$

It can be easily verified that $\mathrm{Tr}\rho(g) = 0$, $\det \rho(g) = 1$ and clearly $\rho(g)_{ij} \in \mathbb{R}$, $i, j = 1, ..., 16$.

2B Class

We can now start considering automorphisms of the lattice $\Gamma^{4,4}$ that lead to interesting symmetries of a suitable NLSM on $\mathbb{T}^4$. The first class of symmetries we want to consider is the 2B class in the notation of [16]. Let us therefore consider the following action on the basis of $\mathcal{H}^{tw}_{gs}$:

$$\begin{aligned} g(e_{m_1}) &= e_{m_2} & g(e_{w_1}) &= e_{w_2} \\ g(e_{m_2}) &= -e_{m_1}^{-1} = -e_{m_1} & g(e_{w_2}) &= -e_{w_1}^{-1} = -e_{w_1} \\ g(e_{m_3}) &= e_{m_4} & g(e_{w_3}) &= e_{w_4} \\ g(e_{m_4}) &= -e_{m_3}^{-1} = -e_{m_3} & g(e_{w_4}) &= -e_{w_3}^{-1} = -e_{w_3} \end{aligned}$$

As usual let us take as basis of our matrix, the one in (7.26). Let us now compute the simultaneous eigenvector of the operators $g(e_{m_i})$ with all the corresponding eigenvalues $+1$. Taking again $|\chi\rangle \equiv \rho(g)\,|1, 1, 1, 1\rangle$, we get:

$$\begin{aligned} g(e_{m_1})\,|\chi\rangle &= (+1) \cdot |\chi\rangle \\ &= e_{m_2}\,|\chi\rangle \end{aligned}$$

$$\begin{aligned} g(e_{m_2})\,|\chi\rangle &= (+1) \cdot |\chi\rangle \\ &= -e_{m_1}\,|\chi\rangle \end{aligned}$$

$$\begin{aligned} g(e_{m_3})\,|\chi\rangle &= (+1) \cdot |\chi\rangle \\ &= e_{m_4}\,|\chi\rangle \end{aligned}$$

$$\begin{aligned} g(e_{m_4})\,|\chi\rangle &= (+1) \cdot |\chi\rangle \\ &= -e_{m_3}\,|\chi\rangle \end{aligned}$$

All this conditions are simultaneously saisfied by the eigenvectors $|\chi\rangle = |-1, 1, -1, 1\rangle$. Projecting now this state onto the initial fixed basis, what we obtain is:

$$\rho_{1,j}(g) = \pm \ (0, 0, 0, 0, 0, 0, 1, 0, ..., 0) \qquad j = 1, ..., 16$$

The next eigenvector we have to compute is:

$$g(e_{w_1})\,|\chi\rangle = e_{w_2}\,|-1, 1, -1, 1\rangle = |-1, -1, -1, 1\rangle$$

therefore the second row of the matrix will be:

$$\rho_{2,j}(g) = \pm \ (0, ..., 0, 1, 0, 0, 0, 0) \qquad j = 1, ..., 16$$

Following the same procedure, what we can finally obtain is:

$$\begin{aligned}
g(e_{w_2})\,|\chi\rangle &= -e_{w_1}\,|-1,1,-1,1\rangle = -\,|1,1,-1,1\rangle\\
g(e_{w_3})\,|\chi\rangle &= e_{w_4}\,|-1,1,-1,1\rangle = |-1,1,-1,-1\rangle\\
g(e_{w_4})\,|\chi\rangle &= -e_{w_3}\,|-1,1,-1,1\rangle = -\,|-1,1,1,1\rangle\\
g(e_{w_1})g(e_{w_2})\,|\chi\rangle &= -e_{w_2}e_{w_1}\,|-1,1,-1,1\rangle = -\,|1,-1,-1,1\rangle\\
g(e_{w_1})g(e_{w_3})\,|\chi\rangle &= e_{w_2}e_{w_4}\,|-1,1,-1,1\rangle = |-1,-1,-1,-1\rangle\\
g(e_{w_1})g(e_{w_4})\,|\chi\rangle &= -e_{w_2}e_{w_3}\,|-1,1,-1,1\rangle = -\,|-1,-1,1,1\rangle\\
g(e_{w_2})g(e_{w_3})\,|\chi\rangle &= -e_{w_1}e_{w_4}\,|-1,1,-1,1\rangle = -\,|1,1,-1,-1\rangle\\
g(e_{w_2})g(e_{w_4})\,|\chi\rangle &= e_{w_1}e_{w_3}\,|-1,1,-1,1\rangle = |1,1,1,1\rangle\\
g(e_{w_3})g(e_{w_4})\,|\chi\rangle &= -e_{w_4}e_{w_3}\,|-1,1,-1,1\rangle = -\,|-1,1,1,-1\rangle\\
g(e_{w_1})g(e_{w_2})g(e_{w_3})\,|\chi\rangle &= -e_{w_2}e_{w_1}e_{w_4}\,|-1,1,-1,1\rangle = -\,|1,-1,-1,-1\rangle\\
g(e_{w_1})g(e_{w_2})g(e_{w_4})\,|\chi\rangle &= e_{w_2}e_{w_1}e_{w_3}\,|-1,1,-1,1\rangle = |1,-1,1,1\rangle\\
g(e_{w_1})g(e_{w_3})g(e_{w_4})\,|\chi\rangle &= -e_{w_2}e_{w_4}e_{w_3}\,|-1,1,-1,1\rangle = -\,|-1,-1,1,-1\rangle\\
g(e_{w_2})g(e_{w_3})g(e_{w_4})\,|\chi\rangle &= e_{w_1}e_{w_4}e_{w_3}\,|-1,1,-1,1\rangle = |1,1,1,-1\rangle\\
g(e_{w_1})g(e_{w_2})g(e_{w_3})g(e_{w_4})\,|\chi\rangle &= e_{w_2}e_{w_1}e_{w_4}e_{w_3}\,|-1,1,-1,1\rangle = |1,-1,1,-1\rangle
\end{aligned}$$

After the proper projections, the final matrix $\rho(g)$ becomes:

$$\rho(g_{2B}) = \pm\begin{pmatrix}
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0
\end{pmatrix}$$

where, again, can be easily checked that $\det\rho(g_{2B}) = 1$ and $\mathrm{Tr}\rho(g_{2B}) = 0$.

3A Class

Another interesting example of symmetry of our model is the one generated from the following transformations:

$$\begin{aligned}
g(e_{m_1}) &= -e_{m_1}^{-1}e_{m_2} & g(e_{w_1}) &= e_{w_2}\\
g(e_{m_2}) &= -e_{m_1}^{-1} & g(e_{w_2}) &= e_{w_1}^{-1}e_{w_2}^{-1}\\
g(e_{m_3}) &= -e_{m_3}^{-1}e_{m_4} & g(e_{w_3}) &= e_{w_4}\\
g(e_{m_4}) &= -e_{m_3}^{-1} & g(e_{w_4}) &= e_{w_3}^{-1}e_{w_4}^{-1}
\end{aligned}$$

The conditions needed to determine the simultaneous eigenvector are:

$$
\begin{aligned}
g(e_{m_1})\,|\chi\rangle &= -e_{m_1}e_{m_2}\,|\chi\rangle = |\chi\rangle \\
g(e_{m_2})\,|\chi\rangle &= -e_{m_1}\,|\chi\rangle = |\chi\rangle \\
g(e_{m_3})\,|\chi\rangle &= -e_{m_3}e_{m_4}\,|\chi\rangle = |\chi\rangle \\
g(e_{m_4})\,|\chi\rangle &= -e_{m_3}\,|\chi\rangle = |\chi\rangle
\end{aligned}
$$

from which we understand that $|\chi\rangle = |-1,1,-1,1\rangle$.
Now, as we have already seen previously, we have to build all the remaining 16 eigenstates. What we can obtain thus is:

$$
\begin{aligned}
g(e_{w_1})\,|h\rangle &= e_{w_2}\,|h\rangle = |-1,-1,-1,1\rangle \\
g(e_{w_2})\,|h\rangle &= e_{w_1}e_{w_2}\,|h\rangle = |1,-1,-1,1\rangle \\
g(e_{w_3})\,|h\rangle &= e_{w_4}\,|h\rangle = |-1,1,-1,1\rangle \\
g(e_{w_4})\,|h\rangle &= e_{w_3}e_{w_4}\,|h\rangle = |-1,1,1,-1\rangle \\
&\dots
\end{aligned}
$$

and so on. The resulting matrix $\rho(g)$ becomes:

$$
\rho(g_{3A}) = \pm \begin{pmatrix}
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}
$$

for which one can check that $\det\rho(g_{3A}) = 1$ e $\mathrm{Tr}\rho(g_{3A}) = 1$.

2E Class

The last class of interesting symmetries of our model we would like to present is the 2E Class. It arises from the transformations on the basis of $\mathcal{H}^{tw}_{gs}$:

$$
g(e_{m_i}) = e_{w_i} \qquad g(e_{w_i}) = e_{m_i}, \qquad i = 1, \dots, 4
$$

from which we can obtain, as usual, the form of the $|\chi\rangle$ eigenstate:

$$
|\chi\rangle = \frac{1}{\sqrt{16}}\Big(|1,1,1,1\rangle + e_{w_1}|1,1,1,1\rangle + e_{w_2}|1,1,1,1\rangle + \dots + e_{w_1}e_{w_2}e_{w_3}e_{w_4}|1,1,1,1\rangle \Big)
$$

With some tedious calculations one can apply the same arguments in order to build the 16 eigenstates and the project them. The final matrix one can find is:

$$\rho(g_{2E}) = \pm\left(\frac{1}{\sqrt{16}}\right)^{16} \cdot \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\
1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 \\
1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\
1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 \\
1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\
1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1
\end{pmatrix}$$

where again one can verify that $\det\rho(g_{2E}) = \frac{4294967296}{(\sqrt{16})^{16}} = \frac{16^8}{(\sqrt{16})^{16}} = 1$ and $\mathrm{Tr}\rho(g_{2E}) = 0$.

The Symmetry $\pi_{g_1} = 1^8\ 2^{-8}\ 4^8$

Let us now find the final result for some types of classified symmetries $g \in G$ but for which the corresponding matrix $\rho(g)$ has never been computed.
We want now to present the calculations about the symmetry corresponding to $\pi_{g_1} = 1^8\ 2^{-8}\ 4^8$, as can be seen from Table 3 in [19], pag.46. Now, the first thing we have to do is to correctly interpret the meaning of the values of π_g. The notation $\pi_g = n^a \cdot m^{-b}$ means that we have to consider a-times the n-roots of unity and remove from these b-times the m-roots of unity. The remaining set of values will be the eigenvalues of the symmetry g on the ground states of both untwisted and twisted sector.
Let us explicitly compute them:

$$\overbrace{\underbrace{+1,...,+1}_{8\text{ times}}}^{1^8} \cdot \overbrace{\underbrace{+1,...,+1}_{8\text{ times}},\underbrace{-1,...,-1}_{8\text{ times}}}^{2^{-8}} \cdot \overbrace{\underbrace{+1,...,+1}_{8\text{ times}},\underbrace{-1,...,-1}_{8\text{ times}},\underbrace{i,...,i}_{8\text{ times}},\underbrace{-i,...,-i}_{8\text{ times}}}^{4^8}$$

Removing from the set above the values corresponding to the 2^{-8} term, the eigenvalues we get[12] $1^8, i^8, -i^8$.
It can be shown that 1^8 are the eight eigenvalues corresponding to the matrix representation of g on untwisted ground states, while $i^8, -i^8$ are the sixteen eigenvalues corresponding to the matrix representation of g on twisted ground states, namely the already introduced $\rho(g)$ matrix[13].

---

[12]With this notation, this time, we indicate the value of the single eigenvalue with, as exponent, its multiplicity.

[13]The motivation is the fact that in the untwisted sector there are 16 ground states which, under the action of the h symmetry, 8 have eigenvalues $+1$ while 8 eigenvalues -1. Since we are looking for h-invariant states only, the 8 ones with negative eigenvalues are projected out. This does not happen with the 16 ground states in the twisted sector because they are taken, by definiton, all with $+1$ eigenvalues.

Following the already showed procedure, we can compute the $\rho(g)$ matrix for the 1A class using the following automorphisms of the $\Gamma^{4,4}$ lattice:

$$\begin{array}{ll} g(e_{m_1}) = -\ e_{m_1} & g(e_{m_i}) = \ e_{m_i} \\ g(e_{w_1}) = -\ e_{w_1} & g(e_{w_i}) = \ e_{w_i} \end{array} \quad i = 2,3,4$$

The matrix $\rho(g)$ thus results:

$$\rho(g_1) = \pm \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{pmatrix}$$

An easy and rapid check shows us that $\mathrm{Tr}\rho(g_1) = 0$, $\det\rho(g_1) = 1$ and the eigenvalues are $i^8, -i^8$. After the considerations we made previously, these eigenvalues are exactly the ones we expected, for the twisted sector ground states, of the symmetry $\pi_{g_1} = 1^8\ 2^{-8}\ 4^8$.

The Symmetry $\pi_{g_2} = 2^4\ 4^{-4}\ 8^4$

Always looking at the Table 3 in [19], pag.46, we would like to compute the matrix $\rho(g)$ corresponding to the symmetry $\pi_{g_2} = 2^4\ 4^{-4}\ 8^4$. The eigenvalues of the g_2 symmetry acting on twisted ground states, exactly like we did previously, can be easily computed and result: $e^{2\pi i/8}$, $e^{6\pi i/8}$, $e^{10\pi i/8}$, $e^{14\pi i/8}$, each taken with multiplicity four.
The matrix $\rho(g)$ satisfying our requests can be generated in different ways. We choose, in particular, to take the following transformations belonging to the 2E class[14]:

$$g(e_{m_1}) = -\ e_{w_1} \qquad \begin{array}{l} g(e_{m_i}) = \ e_{w_i} \\ g(e_{w_i}) = \ e_{m_i} \end{array} \quad i = 2,3,4 \tag{7.44}$$

A different matrix but with the same set of eigenvalues can be obtained, as we just said, by taking as transformations on the basis of $\mathcal{H}^{tw}_{gs}$:

$$\begin{array}{l} g(e_{w_1}) = -e_{m_1} \\ g(e_{w_3}) = -e_{m_3} \\ g(e_{w_4}) = -e_{m_4} \end{array} \qquad \begin{array}{l} g(e_{w_4}) = e_{m_4} \\ g(e_{m_i}) = e_{w_i} \end{array} \qquad i = 1, ..., 4$$

[14]The class symmetry notation will be always referred to the notation used in [16].

Let us now build the matrix $\rho(g)$ corresponding to the set of transformations in (7.44). It can be shown with the usual procedure that the eigenvector of the $g(e_{m_i})$ operators, with all its eigenvalues set to $+1$, $|\chi\rangle$ is:

$$|\chi\rangle = \frac{1}{\sqrt{16}}\Big(|1,1,1,1\rangle - e_{w_1}|1,1,1,1\rangle + e_{w_2}|1,1,1,1\rangle + ... - e_{w_1}e_{w_2}e_{w_3}e_{w_4}|1,1,1,1\rangle \Big)$$

In particular it appears a "$-$" sign every time an e_{w_1} operator is present in the previous formula. Again, the final matrix one can find is:

$$\rho(g_2) = \pm\left(\frac{1}{\sqrt{16}}\right)^{16} \cdot \begin{pmatrix}
1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 \\
1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 \\
1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\
1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\
1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 \\
1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1
\end{pmatrix}$$

This matrix has the property to have $\mathrm{Tr}\rho(g_2) = 0$, $\det\rho(g_2) = 1$ and its eigenvalues are exactly the ones corresponding to the symmetry $\pi_{g_2} = 2^4\ 4^{-4}\ 8^4$ for the twisted ground states.

The Symmetry $\pi_{g_3} = 1^4\ 2^2\ 4^4$

The eigenvalues corresponding to the symmetry $\pi_{g_3} = 1^4\ 2^2\ 4^4$, of the matrix $\rho(g_3)$ acting on the twisted ground states are: -1^2, 1^6, i^4, $-i^4$. Let us consider again an automorphism of the lattice $\Gamma^{4,4}$ corresponding to the 2E class. We can try with the following transformations:

$$\begin{aligned} g(e_{w_i}) &= e_{m_i} \\ g(e_{m_i}) &= -e_{w_i} \end{aligned} \qquad i = 1, ..., 4$$

The eigenstate $|\chi\rangle$ of can be easily computed:

$$|\chi\rangle = \frac{1}{\sqrt{16}}\Big(|1,1,1,1\rangle - e_{w_1}|1,1,1,1\rangle - e_{w_2}|1,1,1,1\rangle + ... - e_{w_2}e_{w_3}e_{w_4}|1,1,1,1\rangle + e_{w_1}e_{w_2}e_{w_3}e_{w_4}|1,1,1,1\rangle \Big)$$

where a "$-$" sign appears every time an operator e_{w_i} is present. With the usual procedure we can find the final matrix:

$$\rho(g_3) = \pm \left(\frac{1}{\sqrt{16}}\right)^{16} \cdot \begin{pmatrix}
1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\
1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 \\
1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\
1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 \\
1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 \\
1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}$$

with again the properties: $\mathrm{Tr}\rho(g_3) = \frac{16}{\sqrt{16}} = 4$, $\det \rho(g_3) = 1$. The set of eigenvalues is exactly -1^2, 1^6, i^4, $-i^4$, namely the ones of the symmetry $\pi_{g_3} = 1^4\ 2^2\ 4^4$ for the twisted ground states.

7.5.2 Final Results

Let us now compute the Twining Genus defined with the insertion of some of the symmetries we explicitly found above.
After all the considerations we made in this last chapter, the expression of the Twining Genus in (7.36), with the condition $\zeta_L \neq 1$, can be rewritten in a more useful form:

$$\begin{aligned}
Z_g^{orb}(\tau, z) = \frac{1}{2} \Bigg[& \frac{\theta_1(z + r_L|\tau)\ \theta_1(z - r_L|\tau)}{\theta_1(r_L|\tau)\ \theta_1(-r_L|\tau)} \cdot (2 - \zeta_L - \zeta_L^{-1})(2 - \zeta_R - \zeta_R^{-1}) \\
& + \frac{\theta_2(z + r_L|\tau)\ \theta_2(z - r_L|\tau)}{\theta_2(r_L|\tau)\ \theta_2(-r_L|\tau)} \cdot (2 + \zeta_L + \zeta_L^{-1})(2 + \zeta_R + \zeta_R^{-1}) \\
& + \left(\frac{\theta_4(z + r_L|\tau)\ \theta_4(z - r_L|\tau)}{\theta_4(r_L|\tau)\ \theta_4(-r_L|\tau)} + \frac{\theta_3(z + r_L|\tau)\ \theta_3(z - r_L|\tau)}{\theta_3(r_L|\tau)\ \theta_3(-r_L|\tau)} \right) \cdot \mathrm{Tr}_{\mathcal{H}_{gs}^{tw}}[\rho(g)] \Bigg]
\end{aligned} \tag{7.45}$$

We can now start computing, for example, the Twining Genus corresponding to the symmetry g_{2B}. By looking at Table 2 of [16], pag.20, we are allowed to choose the eigenvalues of g_{2B} when it acts on the fermions and bosons: $\zeta_L = \zeta_R = i$. By considering also the condition $\mathrm{Tr}\rho(g_{2B}) = 0$, from the expression (7.45), we get:

$$Z_{g_{2B}}^{orb}(\tau, z) = 2 \cdot \left[\frac{\theta_1(z + \frac{1}{4}|\tau)\ \theta_1(z - \frac{1}{4}|\tau)}{\theta_1(\frac{1}{4}|\tau)\ \theta_1(-\frac{1}{4}|\tau)} + \frac{\theta_2(z + \frac{1}{4}|\tau)\ \theta_2(z - \frac{1}{4}|\tau)}{\theta_2(\frac{1}{4}|\tau)\ \theta_2(-\frac{1}{4}|\tau)} \right]$$

Considering now the symmetry g_{3A}, we can analogously choose $\zeta_L = \zeta_R = e^{\frac{2\pi i}{3}}$ and impose the condition $\mathrm{Tr}\rho(g_{3A}) = 1$. From the expression (7.45) we get:

$$Z^{orb}_{g_{3A}}(\tau,z) = \frac{1}{2}\left[\frac{\theta_1(z+\frac{1}{3}|\tau)\ \theta_1(z-\frac{1}{3}|\tau)}{\theta_1(\frac{1}{3}|\tau)\ \theta_1(-\frac{1}{3}|\tau)}\cdot 9 + \frac{\theta_2(z+\frac{1}{3}|\tau)\ \theta_2(z-\frac{1}{3}|\tau)}{\theta_2(\frac{1}{3}|\tau)\ \theta_2(-\frac{1}{3}|\tau)} + \left(\frac{\theta_4(z+\frac{1}{3}|\tau)\ \theta_4(z-\frac{1}{3}|\tau)}{\theta_4(\frac{1}{3}|\tau)\ \theta_4(-\frac{1}{3}|\tau)} + \frac{\theta_3(z+\frac{1}{3}|\tau)\ \theta_3(z-\frac{1}{3}|\tau)}{\theta_3(\frac{1}{3}|\tau)\ \theta_3(-\frac{1}{3}|\tau)}\right)\right]$$

Let us now focus on the symmetry π_{g_1}. By looking again at Table 2 of [16], pag.20, we are allowed to choose the eigenvalues: $\zeta_{L,R} = -1$. By taking into account also the condition $\mathrm{Tr}\rho(g_1) = 0$, the Twining Genus for the π_{g_1} symmetry becomes simply:

$$Z^{orb}_{g_1}(\tau,z) = 8\cdot\frac{\theta_1(z+\frac{1}{2}|\tau)\ \theta_1(z-\frac{1}{2}|\tau)}{\theta_1(\frac{1}{2}|\tau)\ \theta_1(-\frac{1}{2}|\tau)}$$

Now, in the case of the π_{g_2} and π_{g_3} symmetries, since they belong to the 2E class, ζ_L or ζ_R may assume the value $+1$. We understand thus that, in this case, we cannot avoid the contribution from Vertex operators to the corresponding Twining Genus. We will not compute this kind of contribution explicitly therefore, for simplicity, we will only indicate it with a $\Theta^{\lambda}_{g_{2,3}}$, also called *Theta Series*.
Let us compute now the Twining Genus for the π_{g_2} symmetry. We are now allowed to choose $\zeta_L = +1$ and $\zeta_R = -1$. We notice that clearly the relation (7.45) does not hold anymore. By following the procedure we presented along this chapter, one can easily find also the expression of the Twining Genus[15] when $\zeta_L = +1$. By taking also into account the condition $\mathrm{Tr}\rho(g_2) = 0$, the Twining Genus for π_{g_2} becomes:

$$Z^{orb}_{g_2}(\tau,z) = -2\cdot\frac{\theta_1(z|\tau)^2}{\eta(\tau)^6}\cdot\ \Theta^{\lambda}_{g_2}$$

For the π_{g_3} symmetry we are again allowed to choose $\zeta_L = 1$ and $\zeta_R = -1$ with the condition that $\mathrm{Tr}\rho(g_3) = 4$. The Twining Genus for this symmetry thus becomes:

$$Z^{orb}_{g_3}(\tau,z) = -2\cdot\left[\frac{\theta_1(z|\tau)^2}{\eta(\tau)^6}\cdot\Theta^{\lambda}_{g_3} - \left(\frac{\theta_4(z|\tau)^2}{\theta_4(0|\tau)^2} + \frac{\theta_3(z|\tau)^2}{\theta_3(0|\tau)^2}\right)\right]$$

[15] One should only pay attention on substituting in the explicit calculaitons, the correct expression of the Theta and Dedekind functions.

7.6 Appendix: Dedekind and Theta Functions

In this appendix we collect some of the tools we used in the computation of the Twining Genus in Chapter 7. Let us start from the definition of the Theta functions:

$$\theta_1(z|\tau) = -iy^{1/2}\ q^{1/8}\prod_{n=1}^{\infty}(1-q^n)\prod_{n=0}^{\infty}(1-y\ q^{n+1})(1-y^{-1}q^n) \tag{7.46}$$

$$\theta_2(z|\tau) = y^{1/2}\ q^{1/8}\prod_{n=1}^{\infty}(1-q^n)\prod_{n=0}^{\infty}(1+y\ q^{n+1})(1+y^{-1}q^n) \tag{7.47}$$

$$\theta_3(z|\tau) = \prod_{n=1}^{\infty}(1-q^n)\prod_{r\in\mathbb{Z}+1/2}^{\infty}(1+y\ q^r)(1+y^{-1}q^r) \tag{7.48}$$

$$\theta_4(z|\tau) = \prod_{n=1}^{\infty}(1-q^n)\prod_{r\in\mathbb{Z}+1/2}^{\infty}(1-y\ q^r)(1-y^{-1}q^r) \tag{7.49}$$

where $y = e^{2\pi iz}$ and $q = e^{2\pi i\tau}$. Another very important function we used for other calculations, is the Dedekind function:

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$$

that is linked with the Theta functions by the following relation:

$$\eta(\tau)^3 = \frac{1}{2}\theta_2(\tau)\ \theta_3(\tau)\ \theta_4(\tau)$$

where $\theta_i(\tau) = \theta_i(\tau|0)$.
We notice that we can rewrite in a useful way the relations (7.46) and (7.47), namely:

$$\theta_1(z|\tau) = -i(y^{1/2}-y^{-1/2})\ q^{1/8}\prod_{n=1}^{\infty}(1-q^n)(1-y\ q^n)(1-y^{-1}q^n) \tag{7.50}$$

$$\theta_2(z|\tau) = (y^{1/2}+y^{-1/2})\ q^{1/8}\prod_{n=1}^{\infty}(1-q^n)(1+y\ q^n)(1+y^{-1}q^n) \tag{7.51}$$

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