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A Non-Renormalization Theorem at Finite Temperature

Supervisor

Dr. Elli Pomoni (DESY)

Internal Supervisor

Dr. Davide Cassani (INFN)

Candidate

Enrico Marchetto

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Abstract

In this thesis we prove a non-renormalization theorem for the 3-points functions of chiral, scalar superconformal primaries in the four-dimensional $\mathcal{N} = 4$ SYM at finite temperature. The theorem relies on a known, analogous proof done at zero temperature. In this work we adapt the proof to a finite temperature scenario; in particular, we discuss how to recover the conformal invariance and how to get rid of a soft, thermal breaking term appearing in the superconformal Ward identity, the principal tool exploited in the theorem.

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List of Symbols

$\eta_{\mu\nu}$	Metric of the flat space
$\Re[\dots]$	Real part
$\Im[\dots]$	Imaginary part
$\mathcal{O}_i(x_i)$	Generic local operator computed in the point x_i
$\lvert\text{vac}\rangle$	Vacuum state
$\hat{a}, \hat{a}^\dagger$	Destruction and creation operators
S	Classical action
$\mathcal{Z}$	Partition function
β	Inverse of the temperature T
g	Yang-Mills coupling
Θ	Coupling of the Yang-Mills operator $F\tilde{F}$
$\sigma^\mu_{\alpha\dot{\alpha}}, \bar{\sigma}^{\mu,\dot{\alpha}\alpha}$	Sigma matrices
$\varepsilon^{\mu\nu\rho\sigma}$	Levi-Civita symbol
tr	Trace (the trace target is always specified in the text)
$\theta_\alpha, \bar{\theta}^{\dot{\alpha}}$	Fermionic coordinates in the superspace
∂_μ	4-derivative in the spacetime
$D_\alpha, \bar{D}^{\dot{\alpha}}$	Covariant derivative in the superspace
D_μ	Yang-Mills covariant derivative
∇_τ	Connection over the conformal manifold (with respect to the coupling τ)
$\mathcal{N}$	Number of supersymmetries in a theory
$\mathcal{Q}^I_\alpha, \bar{\mathcal{Q}}^{\dot{\alpha}}_J$	Supercharges
$\mathcal{S}^\alpha_J, \bar{\mathcal{S}}^I_{\dot{\alpha}}$	Hermitian conjugates of the supercharges (in the radial quantization)
$M_{\mu\nu}$	Lorentz transformations generator
P_μ	Translations generator
$\mathcal{D}$	Dilatations generator
Δ	Conformal dimension (of a given operator)
K_μ	Special conformal transformations generator
I	Inversion of the coordinates
$[j,\bar{j}]^{\mathcal{R},r}_\Delta$	A full superconformal representation: $j, \bar{j}$ are Lorentz labels; $\mathcal{R}, r$ are respectively the Dynkin labels and the R-charge

$\mathcal{C}(j,\bar{j})$	Eigenvalue of the Lorentz Casimir operator in the $[j,\bar{j}]$ Lorentz representation
$\lvert j,\bar{j}\rangle_{\Delta}^{\mathcal{R},r}$	State generated by an operator sitting in the $[j,\bar{j}]_{\Delta}^{\mathcal{R},r}$ representation
$\lvert\text{h.w.}\rangle$, $\lvert\text{l.w.}\rangle$	States generated by the operators sitting the highest and lowest weights of their representations
$(a\ b\ c)$	Dynkin labels of a $su(4)$ representation
$\phi^{(\mathcal{R},\vec{m})}(x)$	Superconformal scalar primary operator, sitting in the $\mathcal{R}$ representation of $su(4)$, with a weight $\vec{m}$
$\phi^{(k,\vec{m})}(x)$	Superconformal scalar primary operator, sitting in the $(0\,k\,0)$ representation of $su(4)$, with a weight $\vec{m}$
$\phi^{(2,+)}(x)$	Superconformal scalar primary operator, sitting in the highest weight of the $(0\,2\,0)$ representation of $su(4)$
t	Generator of a unitary group $u(n)$ or of a special unitary group $su(n)$ (in the text it is decorated with the proper indices)
U	Finite transformation belonging to a unitary group $u(n)$ or to a special unitary group $su(n)$ (in the text it is decorated with the proper indices)
$R^{\xi\mu}$	R-symmetry current
$G^{\mu}_{\alpha I}, \bar{G}^{\mu\dot{\alpha}J}$	Supercurrents
X	Conserved charge associated to a generic classically conserved current (in the text it is decorated with the proper indices)
S_{∞}	Sphere at the infinity
S_i	Sphere centered in the point x_i
$d\Sigma_{\mu}$	Infinitesimal element of a 3-surface
$d\vec{\Sigma}$	Infinitesimal element of a 2-surface
$[\mathcal{A},\mathcal{B}\}$	If $\mathcal{A},\mathcal{B}$ are both operators in a spinorial Lorentz representation, it is an anticommutator; otherwise, it is a commutator
$[\dots]_{\text{lin}}$	It is attached to a quantity that must be expanded up to the linear order in a infinitesimal parameter
$T_c^{\mu\nu}$	Canonical stress-energy tensor

Contents

Chapter 1

Introduction

The premises

During the last decades, the framework of the *Quantum Field Theories* (QFT) has revealed itself to be the most promising laboratory for the theoretical study of the fundamental interactions. The astoundingly high precision of its physical predictions consecrated it as a starting point for any attempt to the expansion of our knowledge about the extremely microscopical physical laws. In particular, Quantum Field Theories have proved to be very successful at imposing specific *symmetries* to our models. The best example is represented by the description of three of the four known fundamental forces of our Universe: electromagnetic, weak, strong force. The QFT formalism describes these forces with the *gauge theories*. A gauge theory is a gauge invariant QFT, a model which enjoys a local, internal symmetry under the action of a semisimple continuous group. To be more precise, the fundamental forces are described by the so called *Yang-Mills* (YM) theories, gauge theories symmetric under the action of a special unitary group $su(n)$. For instance, the electroweak theory is a YM theory invariant under the action of $su(2)$, while the strong theory is a YM theory invariant under the action of $su(3)$.

A QFT not only describes the dynamics of the fields in our models, but also the interactions among them. Two interacting fields are said to be *coupled*: the strength of the interaction is controlled by a parameter called *coupling*. A generic coupling can be very small (the interaction is said to be weak) or very big (in this case, the interaction is strong). Usually, at this level we are only able to perform perturbative computations with QFTs describing weakly coupled fields, exploiting the coupling as a perturbative expansion parameter. However, in some cases it is possible to extract *non-perturbative results* from our theories. An important example is represented by the Conformal Field Theories (CFT), a class of QFTs enjoying the *conformal invariance*, an enhancement of the usual Poincaré invariance which also accounts for dilatations of the spacetime. CFTs do not contain any mass coefficient or dimensional coupling: this makes their predictions true at any energy scale. The conformal invariance highly constrains the theory and completely fixes some of its aspects. For instance, we might consider a scalar 3-points correlation function

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle \, :$$

in a CFT, its structure is completely fixed up to an overall factor

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}},$$

where $x_{ij} \equiv x_i - x_j$. The expression above is an example of a non-perturbative result in QFT.

Another highly-constraining symmetry is *supersymmetry*. A QFT enjoying supersymmetry is called *Supersymmetric QFT* (SQFT); a YM theory which is also supersymmetric is called *Supersymmetric YM theory* (SYM). Roughly speaking, supersymmetry acts on the fields and transforms the bosonic into fermionic ones (and viceversa). At the formal level, it is introduced as an enhancement of the Poincaré symmetry, called *SuperPoincaré symmetry* (cfr. [1], [2], [3]). A SQFT is not only invariant under the action of the Poincaré group, but also under the action of a pair of fermionic operators $\mathcal{Q}$ and $\bar{\mathcal{Q}}$, called *supercharges*. If our theory does not describe the gravitational interaction, we can introduce up to four pairs of supercharges. $\mathcal{Q}$ and $\bar{\mathcal{Q}}$, when applied to a state of the SQFT, respectively raise and lower the helicity/spin of the state by a factor of 1/2. Despite being conceptually very simple, supersymmetry might be a possible breakthrough in the study of the fundamental interactions. From a phenomenological point of view, a supersymmetric Standard Model would elegantly solve the Hierarchy problem, while a supersymmetric Grand Unified Theory would witness a perfect unification of all its gauge coupling constants at the energy scale $\Lambda \sim 10^{15}$GeV (cfr. [3]). Unfortunately, supersymmetry has not been experimentally verified, yet. However, despite the lack of evidence of its existence, it is still very attractive from a formal point of view because it highly constrains the structure of the classical action. In particular, the introduction of four supersymmetries ($\mathcal{N} = 4$) completely fixes all the operators allowed in the action.

Having understood the peculiarities of conformal invariance and supersymmetry, it is natural to study a class of QFTs enjoying both of them: the *Superconformal QFTs* (SCQFT). These models are highly constrained by their symmetries and they represent the ideal laboratory for the derivation of non-perturbative results. For this reason, SCQFTs have played an important role in theoretical physics for the last two decades, taking the exploration of the QFTs framework to a higher level. Moreover, SCQFTs also have an important physical role, working as holographic duals in the AdS/CFT conjecture (cfr. [4]). A very special toolbox for proving non-perturbative QFT results is provided by the $\mathcal{N} = 4$ SYM theory: not only it is a YM theory,i.e. it is able to describe fundamental forces, but it also enjoys the conformal invariance and the maximum amount of supersymmetries allowed for a non-gravitational theory.

The goal of the thesis

In the article [5] the authors prove a non-perturbative result in the framework of the $\mathcal{N} = 4$ theory: a *non-renormalization theorem*. The interactions in the $\mathcal{N} = 4$ theory are controlled by one marginal, complex parameter τ. In principle, τ can assume a continuous set of values: we will call this set *conformal manifold*. A renormalization of the theory transforms τ into a running parameter: the running can be described as a trajectory over the conformal manifold. The authors consider a special 3-points function made of superconformal, chiral, scalar primary operators

$$\langle \phi_1\, \phi_2\, \phi_3 \rangle = C_{123} \frac{\mathcal{G}(\text{group theory labels})}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3} |x_{23}|^{\Delta_2+\Delta_3-\Delta_1} |x_{13}|^{\Delta_1+\Delta_3-\Delta_2}}.$$

The structure of the correlation functions is almost entirely fixed by group theory (the factor $\mathcal{G}$) and by conformal invariance (the denominator, which contains all the dependence on the coordinates). Then, the dependence on the coupling τ is hidden for sure inside the overall factor C_{123}. When τ runs on a trajectory over the conformal manifold, the behavior of the overall factor C_{123}, and so of the entire correlation function, is described by a connection ∇_τ

$$\nabla_\tau \langle \phi_1\, \phi_2\, \phi_3 \rangle = \cdots \times \nabla_\tau C_{123}.$$

In the article, the authors succeeded at proving that $\nabla_\tau C_{123} = 0$, hence $\nabla_\tau \langle \phi_1\, \phi_2\, \phi_3 \rangle = 0$. The conclusion is: if the coupling τ undergoes a renormalization process and travels on a trajectory over the conformal manifold, the explicit expression of the 3-points correlation function does not change. This is a stunning result: not only the structure of the correlator has been fixed, but such structure does not depend on the strength of the interactions. However, the proof is conducted at *zero temperature*, which is the standard regime in which QFTs are studied. The zero temperature scenario is perfect for studying the interactions between a small amount of particles. Although this condition is clearly important, being verified inside the particle colliders, our most advanced experimental laboratories of fundamental Physics, it is not realized outside of them. Systems which are not man-made do not account for a few particles, but for a very large amount of them, living in a *ensemble*. The notion of particles ensemble allow us to define a temperature: we enter in a *finite temperature* scenario. The goal of this thesis is to show that the result of the article [5] holds at finite temperature, too. Obviously, there are non trivial problems to solve. For instance, the introduction of a finite temperature moves our theory from the usual spacetime manifold $\mathbb{R}^4$ to the non-topologically trivial manifold $\mathbb{R} \times S^1$. Moreover, the massless fields gain *thermal masses*, dependent on the temperature T. The main consequence is that supersymmetry, conformal invariance and superconformal symmetry are broken. However, in this thesis we will show that even at finite temperature it is possible to recover all the necessary hypothesis of the theorem, making its proof possible also in the new scenario. The result is important especially from a formal point of view: not only the finite temperature is a very peculiar way to break supersymmetry, but we also provide examples on how to recover the broken Ward identities. Moreover, studying the behavior of finite temperature SCQFTs is fundamental in the context of AdS/CFT, as we anticipated: in particular, the introduction of a finite temperature in the CFT is represented in the gravity dual by placing a Schwarzschild black hole in the origin of the AdS spacetime (cfr. the article [6]).

Organization of the work

In the chapter 2 we introduce the supersymmetry and we construct the supersymmetric multiplets needed in the thesis. Moreover, we also introduce the R-symmetry. We discuss the construction of a general supersymmetric model with the superspace formalism and we write down the $\mathcal{N} = 1$ SYM theory. At the end of the chapter, we introduce the $\mathcal{N} = 4$ SYM theory and we provide the reader with a definition of soft supersymmetry breaking.
In the chapter 3 we introduce the conformal symmetry, along with the conformal algebra. We learn how to construct the conformal multiplets and we discuss how the conformal invariance completely fixes the kinematic structure of the scalar 2-points and 3-points functions. We discuss the state-operator correspondence, the Operator Product Expansion and the unitarity bounds.
In the chapter 4 we introduce the superconformal symmetry along with the superconformal algebra. The radial quantization introduces a new definition of hermitian conjugation, so we define the new superconformal charges $\mathcal{S}, \bar{\mathcal{S}}$. We discuss the construction of the superconformal multiplets and we introduce the idea of shortening condition, commenting on a particular superconformal multiplet which will be employed in the thesis.
In the chapter 5 we introduce the general structure of a Ward identity. We discuss the structure of a Ward identity associated to a classical symmetry and the structure of softly broken one, computing the scale invariance Ward identities for a free massless and massive scalar theory.
In the chapter 6 we review the non-renormalization theorem at zero temperature exposed in the article [5]. The framework is the $\mathcal{N} = 4$ theory: we show that the 3-points functions of three superconformal chiral scalar primaries are not renormalized.
In the chapter 7 we describe the procedure to follow in order to turn on a finite temperature. We argue that imposing periodic boundary conditions on the fermionic fields makes a fermionic mass term appear in the action. Although the fermionic term would be sufficient for going through the following discussion, we can be more general introducing not

only the fermionic, but also the scalar and the vector mass operator. We interpret the mass operators as relevant deformations of our original, conformal and supersymmetric theory. We compute the supersymmetry and the superconformal symmetry Ward identities at finite temperature and we highlight their soft breaking terms. We compute the R-symmetry Ward identity at finite temperature and we verify that it is not broken. We make use of the R-symmetry Ward identity to get rid of the soft breaking terms.
In the chapter 8 we derive the scale invariance Ward identity at finite temperature for the $\mathcal{N} = 4$ theory. In order to remove the breaking terms, we introduce a conformal compensator. We expose the reasoning with the help of a simple toy model, then we repeat it using the $\mathcal{N} = 4$ theory. We repeat the proof of the non-renormalization theorem, but at finite temperature.
In the appendix A we expose the basic conventions employed in the thesis.
In the appendix B we review the most important aspects of the Cartan subalgebras and we discuss the construction of the $su(4)$ R-symmetry representations.
In the appendix C we prove two useful lemmas used in the non-renormalization theorem.

Chapter 2

Supersymmetric Field Theories

In this chapter we introduce the supersymmetry and the supersymmetric field theories. The supersymmetry is a symmetry which transforms bosonic degrees of freedom in fermionic degrees of freedom, and viceversa. This fundamental feature discloses a plethora of properties : for instance, a good UV behavior and the prediction of many undiscovered particles, the superpartners. However, experimental measures neglected their existence at the currently explored energy levels, thus supersymmetry, if it exists, *must be broken*. From the point of view of the field theory, supersymmetry is a great enhancement of the symmetry amount present in the theory. The Lagrangians of the supersymmetric field theories are strongly constrained, thus it is easier to work with them. In this chapter we are particularly interested in the Yang-Mills supersymmetric field theories. We will discuss all the tools required in order to construct the minimally supersymmetric YM theory, then we will introduce the maximum amount of supersymmetry in absence of gravity, writing down the $\mathcal{N} = 4$ SYM *theory*. In this chapter we will follow the references [2], [1] and [3].

2.1 Supersymmetric multiplets

Supersymmetry is a global, continuous symmetry and it is associated to a conserved charge $\mathcal{Q}$ which turns bosons into fermions and viceversa. In order to do so, the operator $\mathcal{Q}$ must sit in a spinorial representation of the Lorentz group: in 4 dimensions we have two different spinorial representation $[1, 0]$ and $[0, 1]$: the former hosts the supercharge $\mathcal{Q}_\alpha$ and the latter its hermitian conjugate $\bar{\mathcal{Q}}^{\dot{\alpha}}$, equipped respectively with an *undotted* and a *dotted Weyl spinor index*.

Let's consider a generic field theory: due to the Coleman-Mandula theorem, we know that, if its hypothesis are satisfied, the maximal symmetry algebra of the theory must be given by the *Poincaré algebra* in direct product with a finite-dimensional symmetry group (a *gauge group* or a *flavor group*, for instance). We focus on the Poincaré algebra: in order to include the supercharges, we promote it to the *superPoincaré (super)algebra*. The superPoincaré (super)algebra is a set of commutation and anticommutation rules between the generators

$$P_\mu, M_{\mu\nu}, \mathcal{Q}_\alpha, \bar{\mathcal{Q}}^{\dot{\alpha}}, \tag{2.1}$$

which are

$$\begin{aligned}
&\left[P_\mu, P_\nu\right] = 0, \quad \left[M_{\mu\nu}, P_\rho\right] = i\eta_{\rho[\nu}P_{\mu]}, \qquad \left[M_{\mu\nu}, M_{\rho\sigma}\right] = i\eta_{\nu\rho}M_{\mu\sigma} + \text{(cyclic permutations)},\\
&\left[P_\mu, \mathcal{Q}_\alpha\right] = 0, \quad \left[M_{\mu\nu}, \mathcal{Q}_\alpha\right] = i{\sigma_{\mu\nu\alpha}}^{\beta}\mathcal{Q}_\beta, \qquad \left\{\mathcal{Q}_\alpha, \bar{\mathcal{Q}}^{\dot{\beta}}\right\} = 2\sigma^\mu_{\alpha\dot{\gamma}}\varepsilon^{\dot{\gamma}\dot{\beta}}P_\mu.
\end{aligned}$$

Notice that the last relation is an anticommutation relation, as required by the spin-statistic theorem. It is possible to include in the superPoincaré (super)algebra more than one pair of supercharges. Theories with more than one pair of supercharges are called *extended* supersymmetric theories. The number of supercharges $\mathcal{N}$ is encoded in an index $I = 1, \dots, \mathcal{N}$. The enhanced commutation and anticommutation relations for the extended supersymmetric theories are

$$\left[P_\mu, \mathcal{Q}^I_\alpha\right] = 0, \qquad \left[M_{\mu\nu}, \mathcal{Q}^I_\alpha\right] = i\sigma_{\mu\nu\alpha}{}^{\beta}\mathcal{Q}^I_\beta, \qquad \left\{\mathcal{Q}^I_\alpha, \bar{\mathcal{Q}}^{\dot{\beta}}_J\right\} = 2\sigma^\mu_{\alpha\dot{\gamma}}\varepsilon^{\dot{\gamma}\dot{\beta}}P_\mu\delta^I_J. \tag{2.2}$$

If we have more than one pair of supercharges, we need to take another anticommutation relation into account

$$\left\{\mathcal{Q}^I_\alpha, \mathcal{Q}^J_\beta\right\} = \varepsilon_{\alpha\beta}Z^{IJ}. \tag{2.3}$$

The operator Z^{IJ} commutes with all the generators of the full superPoincaré (super)algebra, thus it is called *central charge*.

From the algebraic point of view, there is no limit to the number of supercharges we can introduce in the theory. However, physical theories impose two constraints. Each application of a supercharge $\mathcal{Q}^I_\alpha$ on a generic field ϕ generates a new field with different statistic and a helicity raised by a factor of $\frac{1}{2}$ (or lowered, if we apply $\bar{\mathcal{Q}}^{\dot{\alpha}}_J$), thus

- in a theory without gravity, the highest modulus allowed for a particle helicity is 1, hence the maximum amount of supersymmetry is $\mathcal{N} = 4$;

- in a theory with gravity, the highest modulus allowed for a particle helicity is 2, hence the maximum amount of supersymmetry is $\mathcal{N} = 8$.

In this thesis we won't consider theories in which gravity is involved, thus the maximal supersymmetric theory will be the $\mathcal{N} = 4$ theory. The amount of symmetry in the $\mathcal{N} = 4$ theory is so high that the Lagrangian structure is totally constrained.

The introduction of the superPoincaré (super)algebra requires the fields to be represented not only under the action of the finite gauge symmetry group and of the Lorentz group, but also under supersymmetry. Superpartners, i.e. fields which are connected by supersymmetry transformations, must belong to the same *supersymmetric multiplet*. Supersymmetric multiplets, also called supermultiplets, enjoy two fundamental properties:

- all the fields belonging to the same supermultiplet are associated to particles with the same mass. Let's consider the Fock space of a supersymmetric theory: we build a generic state in the Fock space applying a field operator φ to the vacuum

  $$\varphi \left|\text{vac}\right\rangle = \left|\varphi\right\rangle. \tag{2.4}$$

  Applying a supersymmetry charge, we generate a state belonging to the same supersymmetry multiplet

  $$\mathcal{Q}_\alpha\varphi \left|\text{vac}\right\rangle = \left|\mathcal{Q}_\alpha\varphi\right\rangle. \tag{2.5}$$

  The mass (squared) of the field φ is defined as the eigenvalue of the operator $P^2 = P_\mu P^\mu$ of the eigenstate $\left|\varphi\right\rangle$

  $$P^2\varphi \left|\text{vac}\right\rangle = P^2\left|\varphi\right\rangle = m^2\left|\varphi\right\rangle. \tag{2.6}$$

  Now, exploiting the commutation rule between $\mathcal{Q}_\alpha$ and P^2 we obtain

  $$P^2\mathcal{Q}_\alpha\varphi \left|\text{vac}\right\rangle = \mathcal{Q}_\alpha P^2\varphi\left|\text{vac}\right\rangle = \mathcal{Q}_\alpha m^2\left|\varphi\right\rangle = m^2\left|\mathcal{Q}_\alpha\varphi\right\rangle. \tag{2.7}$$

  As a direct consequence, supersymmetry is necessarily broken if two fields in the same supersymmetric multiplet acquire different masses;

- every supermultiplet must contain an equal number of bosonic and fermionic degrees of freedom. We define the fermionic number operator N_f: its eigenvalue is +1 when it acts on a fermionic state, 0 when it acts on a bosonic state. We construct the operator $(-)^{N_f}$: its eigenvalue is -1 when it acts on a fermionic state and +1 when it acts on a bosonic state. The definition of the operator $(-)^{N_f}$ requires that it anticommutes with the supercharges. Let's consider a generic finite-dimensional supersymmetric multiplet: if we consider a trace tr over the states of a supermultiplet, the object

$$\mathrm{tr}(-)^{N_f} \tag{2.8}$$

counts the difference between the number of fermionic states and bosonic states in the supermultiplet. We have (cfr. [2])

$$\begin{aligned} 0 &= \mathrm{tr}\left[-(-)^{N_f}\mathcal{Q}_\alpha\bar{\mathcal{Q}}^{\dot{\beta}} + (-)^{N_f}\mathcal{Q}_\alpha\bar{\mathcal{Q}}^{\dot{\beta}}\right] = \mathrm{tr}\left[-\bar{\mathcal{Q}}^{\dot{\beta}}(-)^{N_f}\mathcal{Q}_\alpha + (-)^{N_f}\mathcal{Q}_\alpha\bar{\mathcal{Q}}^{\dot{\beta}}\right] \\ &= \mathrm{tr}\left[(-)^{N_f}\bar{\mathcal{Q}}^{\dot{\beta}}\mathcal{Q}_\alpha + (-)^{N_f}\mathcal{Q}_\alpha\bar{\mathcal{Q}}^{\dot{\beta}}\right] = \mathrm{tr}\left[(-)^{N_f}\left\{\bar{\mathcal{Q}}^{\dot{\beta}},\mathcal{Q}_\alpha\right\}\right] \\ &= 2\sigma^\mu_{\alpha\dot{\gamma}}\varepsilon^{\dot{\gamma}\dot{\beta}}\mathrm{tr}\left[(-)^{N_f}P_\mu\right]. \end{aligned}$$

The only possibility for the identity to be true is

$$\mathrm{tr}(-)^{N_f} = 0. \tag{2.9}$$

We want to explicitly write down the $\mathcal{N} = 1$ massless supermultiplets and the (unique) $\mathcal{N} = 4$ massless vector supermultiplet. In order to do so, we follow sistematic procedure. Central charges are equal to zero in a theory with massless supermultiplets, thus supercharges of the same chirality anticommute. In order to study the fields representations, we place ourselves in a reference frame where $P_\mu = E\,(1,0,0,1)$. In this reference frame

$$\left\{\mathcal{Q}^I_\alpha, \bar{\mathcal{Q}}^{\dot{\beta}}_J\right\} = 2\sigma^\mu_{\alpha\dot{\gamma}}\varepsilon^{\dot{\gamma}\dot{\beta}}P_\mu\delta^I_J = \begin{pmatrix}0 & 0\\ 0 & 4E\end{pmatrix}_\alpha^{\ \dot{\beta}}\delta^I_J. \tag{2.10}$$

In particular, we have that $\left\{\mathcal{Q}^I_1, \bar{\mathcal{Q}}^{\dot{1}J}\right\} = 0$. Let's consider a generic state $|\phi\rangle$ belonging to the Hilbert space: if we impose the positivity of the norm we obtain

$$0 = \left\langle\phi\,\middle|\,\left\{\mathcal{Q}^I_1, \bar{\mathcal{Q}}^{\dot{1}}_I\right\}\,\middle|\,\phi\right\rangle = \left|\mathcal{Q}^I_1\,|\phi\rangle\right|^2 + \left|\bar{\mathcal{Q}}^{\dot{1}}_I\,|\phi\rangle\right|^2 \tag{2.11}$$

so the only possibility is to impose $\mathcal{Q}^I_1 = \bar{\mathcal{Q}}^{\dot{1}}_I = 0$ at the operatorial level. We got rid of half of the supersymmetric generators and we are left with the supercharges $\mathcal{Q}^i_2$ and $\bar{\mathcal{Q}}^{\dot{2}}_i$. We set up the following normalization, which defines a set of creation and a set of destruction operators

$$a^I \equiv \frac{1}{\sqrt{4E}}\mathcal{Q}^I_2, \qquad a^\dagger_I \equiv \frac{1}{\sqrt{4E}}\bar{\mathcal{Q}}^{\dot{2}}_I. \tag{2.12}$$

Thanks to the normalization, the operators a^I and $a^\dagger_I$ behaves exactly like ladder operators

$$\left\{a^I, a^\dagger_J\right\} = \delta^I_J, \qquad \left\{a^I, a^J\right\} = 0, \qquad \left\{a^\dagger_I, a^\dagger_J\right\} = 0. \tag{2.13}$$

We start by choosing a state which is annihilated by all the destruction operators a^I, then we employ the creation operators $a^\dagger_I$ in order to generate all the states belonging to the same supermultiplet. Notice that the same creation operator cannot be applied twice: this is due to the fermionic nature of the supercharges. The initial state sits on a Poincaré representation, hence it carries not only a mass eigenvalue (which is zero, in our case), but also an helicity λ. The application of a creation operator raises the helicity by a factor $\frac{1}{2}$, while the application of a destruction operator annihilates the state

$$|\lambda\rangle \to a_I^\dagger |\lambda\rangle \sim \left|\lambda + \frac{1}{2}\right\rangle_I, \qquad a^I |\lambda\rangle = 0. \tag{2.14}$$

We can determine the dimension of the supermultiplet: for each helicity value $\lambda + \frac{k}{2}$, we can obtain a state with such helicity applying k different creation operators in whichever order we desire (different creation operators anticommute, hence changing their order simply generates the same state up to an overall sign). Thus, we have a total of $\binom{\mathcal{N}}{k}$ states for each helicity value $\lambda + \frac{k}{2}$. The total number of states is $\sum_{k=0}^{\mathcal{N}} \binom{\mathcal{N}}{k} = 2^{\mathcal{N}}$, equally splitted into $2^{\mathcal{N}-1}$ bosonic states and $2^{\mathcal{N}-1}$ fermionic states. Now we are ready to discuss the $\mathcal{N} = 1$ and the $\mathcal{N} = 4$ supermultiplets:

- $\mathcal{N} = 1$: the generic supermultiplet counts 2 states

$$\left(\lambda, \lambda + \frac{1}{2}\right). \tag{2.15}$$

  In order to keep into account the CPT symmetry, we need to add the CPT conjugates of the states to the supermultiplet

$$\left(\lambda, \lambda + \frac{1}{2}\right) \oplus \left(-\lambda - \frac{1}{2}, -\lambda\right). \tag{2.16}$$

  In a theory with no gravity, the fundamental $\mathcal{N} = 1$ supermultiplets are the *chiral multiplet* and the *vector multiplet*

$$\left(0, \frac{1}{2}\right) \oplus \left(-\frac{1}{2}, 0\right), \qquad \left(\frac{1}{2}, 1\right) \oplus \left(-1, -\frac{1}{2}\right). \tag{2.17}$$

  The chiral multiplet describes two scalar degrees of freedom and two spinorial degrees of freedom, thus it is associated to a complex scalar and a (on-shell) Weyl spinor; the vector multiplet describes two spinorial degrees of freedom and two vectorial degress of freedom, thus it is associated to a (on-shell) Weyl spinor and to a (on-shell) vector boson;

- $\mathcal{N} = 4$: the vector supermultiplet counts $2^4 = 16$ states. In a theory without gravity (hence, all the helicities must be less or equal than 1), there is only one supermultiplet

$$\left(-1, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 0, 0, 0, 0, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1\right), \tag{2.18}$$

  which is also self-conjugated under CPT. The $\mathcal{N} = 4$ vector supermultiplet describes six scalar degrees of freedom, eight spinorial degrees of freedom and two vectorial degrees of freedom. The associated field content is given by three complex scalar fields (or: six real scalar fields), four (on-shell) Weyl fermions and one (on-shell) vector boson.

2.1.1 The R-symmetry

Under the hypothesis that the central charges Z^{IJ}, introduced in the equation (2.3), are equal to zero, the set of commutators and anticommutators in which the supercharges play a role is

$$\left[P_\mu, \mathcal{Q}^I_\alpha\right] = 0, \quad \left[M_{\mu\nu}, \mathcal{Q}^I_\alpha\right] = i\sigma_{\mu\nu\alpha}{}^{\beta}\mathcal{Q}^I_\beta, \quad \left\{\mathcal{Q}^I_\alpha, \bar{\mathcal{Q}}^{\dot\beta}_J\right\} = 2\sigma^\mu_{\alpha\dot\gamma}\varepsilon^{\dot\gamma\dot\beta}P_\mu\delta^I_J, \quad \left\{\mathcal{Q}^I_\alpha, \mathcal{Q}^J_\beta\right\} = 0.$$

There is room for a new global symmetry, called *R-symmetry*. The R-symmetry acts on the indices I, J of the supercharges and it leaves the (super)algebra intact. The action of the R-symmetry group on the supercharges is

$$\mathcal{Q}'^I_\alpha = U^I{}_J \mathcal{Q}^J_\alpha, \qquad \bar{\mathcal{Q}}'^{\dot\alpha}_I = \bar{\mathcal{Q}}^{\dot\alpha}_J U^{\dagger J}{}_I. \tag{2.19}$$

Imposing the anticommutation rule

$$\left\{ \mathcal{Q}^I_\alpha, \bar{\mathcal{Q}}^{\dot\beta}_J \right\} = 2\sigma^\mu_{\alpha\dot\gamma}\varepsilon^{\dot\gamma\dot\beta} P_\mu \delta^I_J \tag{2.20}$$

to be invariant under the R-symmetry requires the matrices U to satisfy the constraint

$$U^I{}_K U^{\dagger L}{}_J \delta^K_L = \delta^I_J, \tag{2.21}$$

so the R-symmetry group has to be the unitary group $u(\mathcal{N})$, with dimension equal to the number of supersymmetries $\mathcal{N}$ present in the theory. In particular:

- if we have only one set of supercharges, $\mathcal{N} = 1$ and the R-symmetry group reduces to $u(1)$: the supercharges are defined up to a phase;
- in the $\mathcal{N} = 4$ theory we have four sets of supercharges, so the R-symmetry group should be $u(4)$. Actually, the group can be reduced to $su(4)$ (cfr. the section 4.1). The supercharges $\mathcal{Q}^I_\alpha$ sit in the fundamental representation of $su(4)$, while the supercharges $\bar{\mathcal{Q}}^{\dot\alpha}_I$ sit in the anti-fundamental representation of $su(4)$.

2.2 Construction of supersymmetric Lagrangian models

In the previous section we reviewed how to describe the field contents of supersymmetric and extended supersymmetric theories through the use of the supermultiplets. The next step is to learn how to write down the Lagrangian of a supersymmetric theory. The most direct way to implement supersymmetry in a field theory is through the *superspace formalism*. Geometrically, a supersymmetry transformation can be seen as a translation in the superspace, an extended version of the spacetime: in addition to the usual x^μ coordinates, in the superspace we have four *spinorial* coordinates θ_α and $\bar\theta^{\dot\alpha}$. The aim is to develop all the relevant computations in the manifestly supersymmetric superspace formalism and then to integrate out the fermionic coordinates, obtaining an action living in the spacetime.

In the superspace, the action of the supercharges can be explicitly realized as the action of differential operators describing a *superspace translation*. The supercharge $\mathcal{Q}_\alpha$ describes a translation in the spacetime and in the θ_α coordinate, while the supercharge $\bar{\mathcal{Q}}^{\dot\alpha}$ describes a translation in the spacetime and in the $\bar\theta^{\dot\alpha}$ coordinate. Their generic structures are

$$\mathcal{Q}_\alpha = i\frac{\partial}{\partial\theta^\alpha} + ic\sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu, \qquad \bar{\mathcal{Q}}^{\dot\alpha} = -i\frac{\partial}{\partial\bar\theta_{\dot\alpha}} + ic^*\bar\sigma^{\mu\dot\alpha\beta}\theta_\beta\partial_\mu. \tag{2.22}$$

The constant c can be found imposing the anticommutation relation

$$\left\{ \mathcal{Q}_\alpha, \bar{\mathcal{Q}}^{\dot\alpha} \right\} = 2i\Im(c)\sigma^\mu_{\alpha\dot\gamma}\epsilon^{\dot\gamma\dot\alpha}\partial_\mu = 2\Im(c)\sigma^\mu_{\alpha\dot\gamma}\epsilon^{\dot\gamma\dot\alpha}P_\mu. \tag{2.23}$$

If we impose $\Im(c) = 1$, we obtain the correct result: we can choose $c = i$. Thus, the explicit realization of the supercharges as differential operators in the superspace is

$$\mathcal{Q}_\alpha = i\frac{\partial}{\partial\theta^\alpha} - \sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu, \qquad \bar{\mathcal{Q}}^{\dot\alpha} = -i\frac{\partial}{\partial\bar\theta_{\dot\alpha}} + \bar\sigma^{\mu\dot\alpha\beta}\theta_\beta\partial_\mu. \tag{2.24}$$

In the superspace we can define *superfields*, i.e. fields which depends not only on the spacetime coordinates, but also con the fermionic coordinates of the superspace. Thanks to the properties of the *Grassmann variables*, the most general superfield can be at most quadratic in θ or $\bar{\theta}$

$$F\left(x,\theta,\bar{\theta}\right) = a(x) + \theta\xi(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta b(x) + \bar{\theta}\bar{\theta}c(x) + \theta\sigma^{\mu}\bar{\theta}v_{\mu}(x) + \\ + \bar{\theta}\bar{\theta}\theta\eta(x) + \theta\theta\bar{\theta}\bar{\zeta}(x) + \theta\theta\bar{\theta}\bar{\theta}d(x), \quad (2.25)$$

where a, b, c, d are scalar spacetime fields, v_{μ} is a vector spacetime field, ξ, η are left-chirality Weyl spinor spacetime fields and $\bar{\chi}, \bar{\zeta}$ are right-chirality Weyl spinor spacetime fields. We can completely integrate out the fermionic variables of the superfield $F\left(x,\theta,\bar{\theta}\right)$ in two ways:

- we can integrate out all the fermionic coordinates at once

$$\int d^2\theta d^2\bar{\theta}\; F\left(x,\theta,\bar{\theta}\right) = d(x); \quad (2.26)$$

- we can integrate out only two fermionic coordinates with the same chirality

$$\int d^2\theta\; F\left(x,\theta,\bar{\theta}\right) = b(x) + \bar{\theta}\bar{\zeta}(x) + \bar{\theta}\bar{\theta}d(x), \quad (2.27)$$

$$\int d^2\bar{\theta}\; F\left(x,\theta,\bar{\theta}\right) = c(x) + \theta\eta(x) + \theta\theta d(x). \quad (2.28)$$

  Although for a general superfield this method does not provide the desired result, if we choose superfields which depend only on θ (or only on $\bar{\theta}$), the fermionic coordinates are totally integrated out

$$\int d^2\theta F\left(x,\theta\right) = b(x), \quad (2.29)$$

$$\int d^2\bar{\theta} F\left(x,\bar{\theta}\right) = c(x). \quad (2.30)$$

The action of the supercharges (2.24) on the superfield (2.25) gives us the supersymmetry transformation of each spacetime component of $F(x,\theta,\bar{\theta})$. We introduce an infinitesimal spinorial parameter ϵ_{α} which parametrizes the translation in the superspace. The supersymmetry variation of the superfield is

$$\begin{aligned}
\delta F\left(x,\theta,\bar{\theta}\right) &= \left(i\epsilon Q - i\bar{\epsilon}\bar{Q}\right) F\left(x,\theta,\bar{\theta}\right) \\
&= \left(-\epsilon^{\alpha}\frac{\partial}{\partial\theta^{\alpha}} + i\epsilon^{\alpha}\sigma^{\mu}_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_{\mu} - \bar{\epsilon}_{\dot{\alpha}}\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} + i\bar{\epsilon}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\beta}\theta_{\beta}\partial_{\mu}\right) F\left(x,\theta,\bar{\theta}\right) \\
&= \left(-\epsilon^{\alpha}\frac{\partial}{\partial\theta^{\alpha}} - \bar{\epsilon}_{\dot{\alpha}}\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} + i\left[\bar{\epsilon}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\beta}\theta_{\beta} + \epsilon^{\alpha}\sigma^{\mu}_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\right]\partial_{\mu}\right) F\left(x,\theta,\bar{\theta}\right) \\
&= F\left(x^{\mu} + i\bar{\epsilon}\bar{\sigma}^{\mu}\theta + i\epsilon\sigma^{\mu}\bar{\theta}, \theta - \epsilon, \bar{\theta} - \bar{\epsilon}\right) - F\left(x,\theta,\bar{\theta}\right),
\end{aligned}$$

so the supersymmetry transformations are associated to the infinitesimal superspace translation

$$\begin{aligned}
x^{\mu} &\to x^{\mu} + i\bar{\epsilon}\bar{\sigma}^{\mu}\theta + i\epsilon\sigma^{\mu}\bar{\theta}, \\
\theta_{\alpha} &\to \theta_{\alpha} - \epsilon_{\alpha}, \\
\bar{\theta}^{\dot{\alpha}} &\to \bar{\theta}^{\dot{\alpha}} - \bar{\epsilon}^{\dot{\alpha}}.
\end{aligned}$$

We report the explicit transformations of the spacetime components of the superfield (2.25). Although supersymmetry is very simply realized in the superspace, it stops being manifest once it is transposed in the spacetime

$$\delta a = -\epsilon\xi - \bar{\epsilon}\bar{\chi} \tag{2.31}$$
$$\delta\xi = -2\epsilon b - \sigma^\mu\bar{\epsilon}\left(v_\mu + i\partial_\mu a\right), \tag{2.32}$$
$$\delta\bar{\chi} = -2\bar{\epsilon}c + \bar{\sigma}^\mu\epsilon\left(v_\mu - i\partial_\mu a\right), \tag{2.33}$$
$$\delta b = -\bar{\epsilon}\bar{\zeta} + \frac{i}{2}\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\xi, \tag{2.34}$$
$$\delta c = -\epsilon\eta + \frac{i}{2}\epsilon\sigma^\mu\partial_\mu\bar{\chi}, \tag{2.35}$$
$$\delta v^\mu = \epsilon\sigma^\mu\bar{\zeta} - \bar{\epsilon}\bar{\sigma}^\mu\eta - \frac{i}{2}\bar{\epsilon}\bar{\sigma}^\nu\sigma^\mu\partial_\nu\bar{\chi} + \frac{i}{2}\epsilon\sigma^\nu\bar{\sigma}^\mu\partial_\nu\xi, \tag{2.36}$$
$$\delta\eta = -2\epsilon d + \frac{i}{2}\epsilon\bar{\sigma}^\mu\sigma^\nu\partial_\mu v_\nu - i\sigma^\mu\bar{\epsilon}\partial_\mu c, \tag{2.37}$$
$$\delta\bar{\zeta} = -2\bar{\epsilon}d - \frac{i}{2}\bar{\epsilon}\sigma^\mu\bar{\sigma}^\nu\partial_\mu v_\nu - i\bar{\sigma}^\mu\epsilon\partial_\mu b, \tag{2.38}$$
$$\delta d = \frac{i}{2}\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\eta + \frac{i}{2}\epsilon\sigma^\mu\partial_\mu\bar{\zeta}. \tag{2.39}$$

In order to correctly treat superfields in the superspace formalism, we need to introduce a derivative operator D_α which anticommutes with the supersymmetry generators

$$\delta\left(D_\alpha F\right) = D_\alpha\left(\delta F\right). \tag{2.40}$$

We can set the ansatz

$$D_\alpha = m\frac{\partial}{\partial\theta^\alpha} + n\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu, \qquad \bar{D}^{\dot{\alpha}} = m^*\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} + n^*\bar{\sigma}^{\mu\dot{\alpha}\beta}\theta_\beta\partial_\mu \tag{2.41}$$

and impose

$$\left\{D_\alpha, Q_\beta\right\} = \left\{D_\alpha, \bar{Q}^{\dot{\beta}}\right\} = \left\{\bar{D}^{\dot{\alpha}}, Q_\beta\right\} = \left\{\bar{D}^{\dot{\alpha}}, \bar{Q}^{\dot{\beta}}\right\} = 0. \tag{2.42}$$

The constraints above are satisfied by

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu, \qquad \bar{D}^{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} - i\bar{\sigma}^{\mu\dot{\alpha}\beta}\theta_\beta\partial_\mu. \tag{2.43}$$

2.2.1 Chiral superfield

A chiral superfield Φ is defined imposing the following constraints

$$\bar{D}^{\dot{\alpha}}\Phi = 0, \qquad D_\alpha\bar{\Phi} = 0. \tag{2.44}$$

The solution of these constraints is difficult to find if we place ourselves in a superspace with coordinates $x^\mu, \theta_\alpha, \bar{\theta}^{\dot{\alpha}}$: in fact, $D_\alpha x^\mu \neq 0$. This is an hint to the fact that the most natural coordinate choice in the superspace is given by the coordinates $\theta_\alpha, \bar{\theta}^{\dot{\alpha}}$ and by the coordinate $y^\mu \equiv x^\mu + i\theta\sigma^\mu\bar{\theta}$. In fact

$$\bar{D}^{\dot{\alpha}}y^\mu = i\sigma^\mu\bar{\theta} - i\sigma^\mu\bar{\theta} = 0. \tag{2.45}$$

If we consider the superfield $\Phi = \Phi(y^\mu, \theta, \bar{\theta})$, the constraint (2.44) is easily solved by

$$\Phi(y^\mu, \theta, \bar{\theta}) = \phi(y) + \theta\psi(y) + \theta\theta f(y). \tag{2.46}$$

We can go back to the x^μ coordinates expanding around them: for a generic spacetime field g we have

$$\begin{aligned} g(y) &= g(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu g(x) - \frac{1}{2}\theta\sigma^\mu\bar{\theta}\theta\sigma^\nu\bar{\theta}\partial_\mu\partial_\nu g(x) \\ &= g(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu g(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^\mu\partial_\mu g(x), \end{aligned}$$

thus the correct expression for the chiral superfield is

$$\Phi(x,\theta,\bar{\theta}) = \phi(x) + \theta\psi(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) + \theta\theta f(x) + \frac{i}{2}\theta\theta\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^\mu\partial_\mu\phi(x). \tag{2.47}$$

The superfield Φ contains the following spacetime fields: a complex scalar ϕ, a left-chirality Weyl spinor ψ and a complex scalar f. It is the correct embedding in a superfield of the $\mathcal{N} = 1$ chiral supermultiplet; the scalar f is an auxiliary field that is necessary in order to match the off-shell degrees of freedom (4 spinorial off-shell d.o.f. and 4 scalar off-shell d.o.f.). If we compare the chiral superfield (2.47) with the most general superfield (2.25), we immediately derive the supersymmetry transformations of the spacetime components of Φ

$$\begin{aligned} \delta\phi &= -\epsilon\psi, \\ \delta\psi &= -2i\sigma^\mu\bar{\epsilon}\partial_\mu\phi - 2\epsilon f, \\ \delta f &= i\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\psi. \end{aligned}$$

2.2.2 Vector superfield

A vector superfield V is defined by the following constraint

$$V = V^*. \tag{2.48}$$

If we consider the superfield (2.25), the constraint imposes the following structure

$$V\left(x,\theta,\bar{\theta}\right) = a + \theta\xi + \bar{\theta}\bar{\xi} + \theta\theta b + \bar{\theta}\bar{\theta}\bar{b} + \theta\sigma^\mu\bar{\theta}v_\mu + \bar{\theta}\bar{\theta}\theta\eta + \theta\theta\bar{\theta}\bar{\eta} + \theta\theta\bar{\theta}\bar{\theta}d, \tag{2.49}$$

with a, d, v_μ real fields. We redefine three of the spacetime-dependent components in the following way

$$v_\mu \equiv A_\mu, \qquad \eta \equiv \lambda - \frac{i}{2}\sigma^\mu\partial_\mu\bar{\xi}, \qquad d \equiv \frac{1}{2}D + \frac{1}{4}\partial_\mu\partial^\mu a. \tag{2.50}$$

After a substitution, the vector superfield turns out to be

$$\begin{aligned} V\left(x,\theta,\bar{\theta}\right) = a + \theta\xi + \bar{\theta}\bar{\xi} + \theta\theta b + \bar{\theta}\bar{\theta}\bar{b} + \theta\sigma^\mu\bar{\theta}A_\mu + \bar{\theta}\bar{\theta}\theta\left(\lambda - \frac{i}{2}\sigma^\mu\partial_\mu\bar{\xi}\right) + \\ + \theta\theta\bar{\theta}\left(\bar{\lambda} - \frac{i}{2}\bar{\sigma}^\mu\partial_\mu\xi\right) + \theta\theta\bar{\theta}\bar{\theta}\left(\frac{1}{2}D + \frac{1}{4}\partial_\mu\partial^\mu a\right). \end{aligned} \tag{2.51}$$

We want to employ the vector superfield to embed the field content of the $\mathcal{N} = 1$ vector supermultiplet in the superspace: a vector boson A_μ, a left-chirality Weyl spinor λ_α and an auxiliary field D which makes the degrees of freedom match off-shell (4 spinorial off-shell d.o.f. and 3+1 scalar off-shell d.o.f.: one vectorial d.o.f. is cancelled by the gauge invariance). We have a redundance of spacetime components in the vector superfield (2.51). We can remove it in the following way: let's consider the chiral superfield Ω, then we can build the superfield

$$\Lambda \equiv i\left(\Omega - \bar{\Omega}\right), \tag{2.52}$$

which satisfies the constraint (2.48), hence it is a vector superfield. Then we can redefine the superfield (2.51) adding the newly introduced superfield (2.52), obtaining another vector superfield. Recalling the explicit expression (2.47) of a generic chiral superfield, we get

$$\Lambda = -2\Im(\phi) + i\theta\psi - i\bar{\theta}\bar{\psi} - 2\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}\Re(\phi) + i\theta\theta f - i\bar{\theta}\bar{\theta}\bar{f} + \frac{1}{2}\theta\theta\bar{\theta}\bar{\sigma}^{\mu}\partial_{\mu}\psi - \frac{1}{2}\bar{\theta}\bar{\theta}\theta\sigma^{\mu}\partial_{\mu}\bar{\psi} - \\ - \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\partial_{\mu}\partial^{\mu}\Im(\phi)\,. \tag{2.53}$$

The modified vector multiplet is

$$V + \Lambda = (a - 2\Im(\phi)) + \theta(\xi + i\psi) + \bar{\theta}(\bar{\xi} - i\bar{\psi}) + \theta\theta(b + if) + \\ + \bar{\theta}\bar{\theta}(\bar{b} - i\bar{f}) + \theta\sigma^{\mu}\bar{\theta}(A_{\mu} - 2\partial_{\mu}\Re(\phi)) + \bar{\theta}\bar{\theta}\theta\left(\lambda - \frac{i}{2}\sigma^{\mu}\partial_{\mu}\bar{\xi} - \frac{1}{2}\sigma^{\mu}\partial_{\mu}\bar{\psi}\right) + \\ + \theta\theta\bar{\theta}\left(\bar{\lambda} - \frac{i}{2}\bar{\sigma}^{\mu}\partial_{\mu}\xi + \frac{1}{2}\bar{\sigma}^{\mu}\partial_{\mu}\psi\right) + \theta\theta\bar{\theta}\bar{\theta}\left(\frac{1}{2}D + \frac{1}{4}\partial_{\mu}\partial^{\mu}a - \frac{1}{2}\partial_{\mu}\partial^{\mu}\Im(\phi)\right). \tag{2.54}$$

It is now possible to set

$$a - 2\Im(\phi) = 0, \tag{2.55}$$
$$\xi + i\psi = 0, \tag{2.56}$$
$$b + if = 0, \tag{2.57}$$

which returns the following vector multiplet, said to be in the *Wess-Zumino gauge*

$$V' = \theta\sigma^{\mu}\bar{\theta}(A_{\mu} - 2\partial_{\mu}\Re(\phi)) + \bar{\theta}\bar{\theta}\theta\lambda + \theta\theta\bar{\theta}\bar{\lambda} + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D. \tag{2.58}$$

The vector superfield (2.58) correctly embeds the field content of the vector supermultiplet, matching the fermionic and bosonic degrees of freedom off-shell. Notice that we still have the freedom to perform an (abelian) gauge transformation on the vector boson. Once we have fixed the vector field with our favourite gauge, the final expression is

$$V'' = \theta\sigma^{\mu}\bar{\theta}A_{\mu} + \bar{\theta}\bar{\theta}\theta\lambda + \theta\theta\bar{\theta}\bar{\lambda} + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D. \tag{2.59}$$

In conclusion, if we compare the vector superfield (2.59) with the most general superfield (2.25), we immedately derive the supersymmetry transformations for the components of V''

$$\delta A_{\mu} = \epsilon\sigma^{\mu}\bar{\lambda} - \bar{\epsilon}\bar{\sigma}^{\mu}\lambda,$$
$$\delta\lambda = -\epsilon D + \frac{i}{2}\epsilon\bar{\sigma}^{\mu}\sigma^{\nu}\partial_{\mu}A_{\nu},$$
$$\delta D = i\bar{\epsilon}\bar{\sigma}^{\mu}\partial_{\mu}\lambda + i\epsilon\sigma^{\mu}\partial_{\mu}\bar{\lambda}.$$

2.2.3 Writing down a supersymmetric Lagrangian

Vector superfields, chiral superfields and their superspace derivatives can be composed in order to construct other vector and chiral superfields. A generic supersymmetric action can be written as follows

$$S = \int d^4x \int d^2\theta d^2\bar{\theta}\; S(x, \theta, \bar{\theta}). \tag{2.60}$$

This action is *manifestly* supersymmetric. In the superspace, the supercharges are realized as the differential operators (2.24), hence when we apply them to the integrand of the integral

(2.60) we obtain the same action up to superspace total derivatives. We ask the action S to be real, thus the superfield $S(x,\theta,\bar{\theta})$ must be a vector superfield. In the most general case, $S(x,\theta,\bar{\theta})$ is made up of a vector superfield and the sum of a chiral superfield and its conjugate

$$\begin{aligned} S &= \int d^4x \int d^2\theta d^2\bar{\theta}\; V(x,\theta,\bar{\theta}) + \int d^4x \int d^2\theta d^2\bar{\theta} \left[\delta^{(2)}\left(\bar{\theta}\right)\Phi(x,\theta) + \delta^{(2)}\left(\theta\right)\bar{\Phi}(x,\bar{\theta})\right] \\ &= \int d^4x \int d^2\theta d^2\bar{\theta}\; V(x,\theta,\bar{\theta}) + \int d^4x \int d^2\theta\; \Phi(x,\theta) + \int d^4x \int d^2\bar{\theta}\; \bar{\Phi}(x,\bar{\theta}). \end{aligned}$$

If we have

$$V(x,\theta,\bar{\theta}) = \cdots + \theta\theta\bar{\theta}\bar{\theta}\mathcal{L}_v(x), \quad \Phi(x,\theta) = \cdots + \theta\theta\mathcal{L}_c(x), \quad \bar{\Phi}(x,\bar{\theta}) = \cdots + \bar{\theta}\bar{\theta}\bar{\mathcal{L}}_c(x), \tag{2.61}$$

then, recalling the integrals (2.26), (2.29) and (2.30), we can derive the spacetime action of the theory, paying the price of losing the manifest supersymmetry

$$S = \int d^4x \left[\mathcal{L}_v(x) + \mathcal{L}_c(x) + \bar{\mathcal{L}}_c(x)\right] = \int d^4x \mathcal{L}(x). \tag{2.62}$$

2.3 The $\mathcal{N} = 1$ SYM theory

The $\mathcal{N} = 4$ theory is a SYM theory, so we need to learn how to write SYM supersymmetric theories. The non-supersymmetric SYM action is

$$S = \int d^4x \left[-\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu}\right]. \tag{2.63}$$

In this section we will add one supersymmetry to this action, writing down the $\mathcal{N} = 1$ SYM theory. The $\mathcal{N} = 4$ theory can be seen as a special case of the $\mathcal{N} = 1$ SYM theory.

We start considering a vector superfield V, defined as in (2.51), and a chiral superfield Ω, defined as in (2.47). We define the superfields e^V and $e^{i\Omega}$. We introduce a generic *gauge group* G of dimension N, generated by the generators $t^1,\ldots,t^N$. All the superfields will sit in the *adjoint representation* of the gauge group: for the generic superfield F, it will be implicit that $F = F^a t^a$. We already witnessed the fact that the addition of the vector field (2.52) to V can reproduce the abelian gauge transformation of the vector boson. In order to mimic the non-abelian gauge transformations, we consider the following transformation in superspace

$$e^{V'} = e^{-i\bar{\Omega}}e^V e^{i\Omega}. \tag{2.64}$$

Expanding the exponential we can convince ourselves that the transformation (2.64) is a generalization of the transformation (2.52)

$$1 + V' + \cdots = (1 - i\bar{\Omega} + \ldots)(1 + V + \ldots)(1 + i\bar{\Omega} + \ldots) = 1 + V + i(\Omega - \bar{\Omega}) + \ldots \tag{2.65}$$

The transformation (2.64) is called *supergauge transformation*: it represents the correct encoding of the non-abelian transformation of the gauge vector in the superspace formalism. From now on we set ourselves in the Wess-Zumino gauge, choosing the correct Ω. With the superfield V and the superspace derivatives (2.43) we define the chiral superfield

$$W_\alpha \equiv -\frac{1}{4}\bar{D}\bar{D}\left(e^V D_\alpha e^{-V}\right). \tag{2.66}$$

It can be shown that the transformation (2.64) acts on the chiral superfield (2.66) as follows

$$W'_\alpha = e^{-i\bar{\Omega}}W_\alpha e^{i\Omega}.$$

It follows that the chiral superfield

$$\text{tr}\,[W_\alpha W^\alpha] \tag{2.67}$$

is invariant under the supergauge transformation (2.64); the trace is taken over the indices of the gauge group G. We are now able to obtain the $\mathcal{N} = 1$ SYM theory, exploiting the procedure described in the section 2.2.3: the spacetime action can be obtained writing down the superspace action

$$S_{SYM} = \int d^4x \int d^2\theta\ \text{tr}\,[W_\alpha W^\alpha] + h.c. \tag{2.68}$$

which leads to the supersymmetric SYM action, with explicit color indices, once the θ coordinates are integrated out

$$S_{SYM} = \int d^4x \Big[-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + i\bar{\lambda}^a \bar{\sigma}^\mu D_\mu \lambda^a + \frac{1}{2} D^2 + \frac{i}{8} \varepsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma} \Big], \tag{2.69}$$

where the covariant derivative is given by

$$D_\mu Y^a = \partial_\mu Y^a + f^{abc} A^b_\mu Y^c \tag{2.70}$$

and the field strength is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu. \tag{2.71}$$

We immediately notice that we recovered the action (2.63): in addition, we have the kinetic sector of the supersymmetric partner of the gauge vector boson, the gaugino λ, and the auxiliary field term $\frac{1}{2}D^2$. Moreover, the action (2.69) was easily derived in the Wess-Zumino gauge, but the the superspace action (2.67) is supergauge invariant, hence the expression (2.69) is always valid. At this point, we are able to introduce a coupling in the theory

$$\tau \equiv \frac{1}{g^2} - i\frac{\Theta}{8\pi^2}. \tag{2.72}$$

The τ coupling encodes the strength of the gauge boson self-interactions and, thanks to supersymmetry, of the gaugino-gauge boson interactions; moreover, it also takes into account the possibility of topological contributions coming from the operator

$$\frac{\Theta}{64\pi^2} \varepsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma}. \tag{2.73}$$

The coupling is inserted in the theory as a multiplicative overall factor

$$S_{SYM} = \Re \left\{ \tau \int d^4x \int d^2\theta\ \text{tr}\,[W_\alpha W^\alpha] + h.c. \right\}. \tag{2.74}$$

Usually, the Θ angle is set to zero in order to neglect the topological effects and the coupling constant g is absorbed inside the fields content of the theory

$$A^a_\mu \to g A^a_\mu, \qquad \lambda^a \to g\lambda^a, \qquad D^a \to g D^a. \tag{2.75}$$

So that we recover the expression

$$S_{SYM} = \int d^4x \Big[-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + i\bar{\lambda}^a \bar{\sigma}^\mu D_\mu \lambda^a + \frac{1}{2} D^2 \Big]. \tag{2.76}$$

2.4 The $\mathcal{N} = 4$ theory

In this section we discuss two different ways to obtain the $\mathcal{N} = 4$ SYM theory, the model studied in this thesis. The starting point is the $\mathcal{N} = 1$ SYM action (2.76). However, recalling the specific form of the $\mathcal{N} = 4$ unique supermultiplet (2.18), we immediately realize that the number of degrees of freedom must be raised:

- the first approach is to couple the SYM theory to a *matter sector* in order to describe the full $\mathcal{N} = 4$ field content.The R-symmetry is imposed with the introduction of a superpotential;
- the second approach is to define the $\mathcal{N} = 1$ SYM action not in 4 dimensions, but in 10 dimensions, instead. The theory is then compactified over a 6-dimensional torus. After having performed the compactification, the additional degrees of freedom naturally appear.

2.4.1 $\mathcal{N} = 4$ theory: non manifest R-symmetry

Introducing a matter sector

In this section we couple a chiral superfield (or a set of chiral superfields) to the action (2.76). In the end, all the newly introduced field content will have to be contained in the $\mathcal{N} = 4$ supermultiplet, hence all the fields introduced must be represented in the adjoint representation of the gauge group. We introduce a chiral superfield Φ with the structure (2.47) sitting in the adjoint representation of the gauge group. The supergauge transformations of the chiral superfield are

$$\Phi' = e^{i\Omega}\Phi e^{i\bar{\Omega}}, \qquad \bar{\Phi}' = e^{-i\Omega}\bar{\Phi}e^{-i\bar{\Omega}} \tag{2.77}$$

We want to couple this superfield to the vector superfield constructing a superspace operator invariant under supergauge transformations. The candidate is

$$S_{\text{matter}} = \int d^4x \int d^2\theta d^2\bar{\theta} \; \text{tr}\left[e^V\bar{\Phi}e^{-V}\Phi\right]. \tag{2.78}$$

Expanding the integrand and integrating out the fermionic coordinates we obtain

$$S_{\text{matter}} = \int d^4x \left[-D_\mu\bar{\phi}^a D^\mu\phi^a + i\bar{\psi}^a\bar{\sigma}^\mu D_\mu\psi^a - \bar{f}f + (\text{interactions})\right], \tag{2.79}$$

The procedure can be easily exploited in order to introduce more than one chiral multiplet. In particular, we introduce three sets of chiral multiplets Φ^I, $\bar{\Phi}_I$, where $I = 1,2,3$. Inserting also the coupling τ in the model, we get

$$S_{\text{matter}} = \Re\left\{\tau \int d^4x \int d^2\theta d^2\bar{\theta} \; \text{tr}\left[e^V\bar{\Phi}_I e^{-V}\Phi^I\right]\right\}, \tag{2.80}$$

which returns

$$S_{\text{matter}} = \int d^4x \left[-D_\mu\bar{\phi}_I D^\mu\phi^I + i\bar{\psi}^a_I\bar{\sigma}^\mu D_\mu\psi^{aI} - \bar{f}_I f^I + (\text{interactions})\right], \tag{2.81}$$

where we absorbed the coupling constant with a field redefinition

$$\phi^{aI} \to g\phi^{aI}, \qquad \psi^a \to g\psi^{aI}, \qquad f^{aI} \to gf^{aI}. \tag{2.82}$$

Introducing a superpotential

In the previous section we introduced three chiral supermultiplets in the theory and we took into account their interactions with the fields embedded in the vector supermultiplets. We are still allowed to introduce interactions among the chiral multiplets: this can be done promoting a function of the chiral superfields to a *superpotential*. The superpotential must be an *holomorphic* function of the chiral superfields and it must not depend on the anti-chiral superfields

$$\mathcal{W} = \mathcal{W}\left(\Phi_I\right). \tag{2.83}$$

The new sector of the action is

$$S_{\text{superpotential}} = \int d^4x \int d^2\theta\, \mathcal{W}\left(\Phi_I\right) + h.c. \tag{2.84}$$

The $\mathcal{N} = 4$ theory

We impose the following constraints on the superpotential $\mathcal{W}\left(\Phi_I\right)$: we require the invariance under a global $su(3)$ which acts on the index I, rotating the three chiral superfields Φ_I but leaving the vector superfield V invariant, we require the renormalizability and the holomorphicity:

- renormalizability and holomorphicity constrain the structure of the superpotential

$$\mathcal{W}\left(\Phi_I\right) = \text{tr}\left[a_I\Phi^I + b_{IJ}\Phi^I\Phi^J + c_{IJK}\Phi^I\Phi^J\Phi^K + d_{IJKL}\Phi^I\Phi^J\Phi^K\Phi^L\right]. \tag{2.85}$$

  The kinetic factor of the chiral multiplets, which appears in the equation (2.81), fixes the mass dimension of the scalar fields ϕ_I to $[\phi_I] = 1$. Given that $\Phi = \phi + \dots$, a renormalizable superpotential can be a polynomial of order four or less in the superfields Φ_I;

- the $su(3)$ invariance totally constrains the superpotential (cfr. [8]): only one tensor in the equation (2.85) can be realized as a $su(3)$ invariant tensor

$$a_I \to \mathbf{3}, \tag{2.86}$$
$$b_{IJ} \to \bar{\mathbf{3}} + \mathbf{6}, \tag{2.87}$$
$$c_{IJK} \to \boxed{\mathbf{1}} + \mathbf{8}\times 2 + \mathbf{10}, \tag{2.88}$$
$$d_{IJKL} \to \mathbf{3}\times 3 + \bar{\mathbf{6}}\times 2 + \mathbf{15}\times 4. \tag{2.89}$$

  The only singlet is associated to $c_{IJK} = c\,\varepsilon_{IJK}$, where c is an overall numerical multiplicative factor.

In conclusion, we must pick the superpotential

$$\mathcal{W}\left(\Phi_I\right) = c\varepsilon_{IJK}\text{tr}\left[\Phi^I\Phi^J\Phi^K\right]. \tag{2.90}$$

Plugging the superpotential (2.90) in the action, we obtain the $\mathcal{N} = 4$ action.

This procedure is useful in order to show the origin of the $\mathcal{N} = 4$ SYM theory as a special case of the $\mathcal{N} = 1$ SYM theory; the uniqueness of the chosen superpotential is a symptom of the uniqueness of the $\mathcal{N} = 4$ theory. However, through this method we are not able to recover the full $su(4)$ R-symmetry: the action is manifestly invariant under $su(3)\times u(1)$, i.e. the global symmetry which rotates the chiral superfields and the $\mathcal{N} = 1$ R-symmetry.

2.4.2 $\mathcal{N}=4$ theory: manifest R-symmetry

In this section we give the outlines of the procedure we should follow in order to recover the manifestly $su(4)$ invariant $\mathcal{N}=4$ SYM theory (cfr. [26],[33])

The $\mathcal{N}=4$ theory in 4 dimensions can be seen as the result of a Kaluza-Klein compactification of the $\mathcal{N}=1$ SYM theory in 10 dimensions

$$S_{10d} = \int d^{10}x \ \mathrm{tr}\left[\frac{1}{2}F_{mn}F^{mn} - i\bar{\lambda}\Gamma^m D_m\lambda - \sum_{r=1}^{7} D_r^2\right], \tag{2.91}$$

where the fields are defined in 10 dimensions and Γ^m are the gamma matrices in 10 dimensions. The seven auxiliary fields $D_1, \ldots, D_7$ are necessary in order to be able to match the off-shell degrees of freedom of the fields inserted in the action: the vector boson has 9 d.o.f. (after the gauge fixing), which added to the 7 scalar d.o.f. are exactly equal to 16, the real dimension of the 10-dimensional Weyl spinor (cfr. [40]). The 10-dimensional theory is then compactified over a torus T^6: the fields sit in representations of $so(6) \sim su(4)$. After the compactification, we obtain a manifestly $su(4)$ invariant $\mathcal{N}=4$ action

$$\begin{aligned} S_{\mathcal{N}=4} = \int d^4x \Big[&-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + i\bar{\lambda}^a_i\bar{\sigma}^\mu D_\mu\lambda^{ai} - \frac{1}{4}D_\mu\bar{X}^a_{ij}D^\mu X^{aij} + \\ &+ \frac{\sqrt{2}}{2}gf^{abc}\lambda^{ai}\lambda^{bj}\bar{X}^c_{ij} + \frac{\sqrt{2}}{2}gf^{abc}\bar{\lambda}^a_i\bar{\lambda}^b_j X^{cij} - \frac{1}{16}g^2 f^{abc}f^{aeg}X^{bij}X^{ckl}\bar{X}^e_{ij}\bar{X}^g_{kl}\Big]. \end{aligned} \tag{2.92}$$

The action (2.92) is written in the conventions of the article [21], which we adopted.

2.5 Soft Supersymmetry breaking

Supersymmetry has never been observed in the experiments, so there are for sure regimes in which it behaves like a broken symmetry. Supersymmetry can be broken in different ways. For instance, it might be *spontaneously broken*, like the electroweak $su(2) \times u(1)$ gauge symmetry in the Standard Model: the classical action S is invariant under the action of supersymmetry, but the vacuum of theory is not. In this case, perturbative computations around the physical vacuum necessarily break the supersymmetry, even if at the formal level the symmetry has never been destroyed. In this thesis we will not consider a spontaneous breaking: the supersymmetry will be *explicitly* broken by finite temperature effects. The breaking will not be spontaneous, but manifest at the Lagrangian level through the introduction of a new sector, in particular *mass operators*. A mass operator, or, in general, a relevant operator, which explicitly breaks the supersymmetry is called *soft* operator and the breaking caused by it is addressed as *soft breaking* (cfr. [2], [1], [36],[29]).

Chapter 3

Conformal Field Theories

In this chapter we introduce the conformal invariance, a symmetry of the spacetime coordinates which also transforms the field content of the theory. The most peculiar property of the CFTs is the *scale invariance*: this theories depend only on adimensional couplings, so their physical predictions are valid at every scale of energy. From a phenomenological point of view, it is crucial to understand the behavior of the CFTs because they are good approximations of the models when they reach their critical point. From a field-theoretical point of view, the conformal invariance is extremely precious because it highly constrains some aspects of the theory: for instance, 2-points and 3-points functions are completely solved up to numerical multiplicative constants. Another interesting aspect is the implementation of the unitarity: in order to correctly define the Hilbert space of a quantum conformal field theory, the fields and the operators must satisfy some constraints on their mass dimensions, called *unitarity bounds*. In this chapter we will follow the references [7], [4], [8] and [9].

3.1 Conformal multiplets

The first step towards a basic comprehension of the conformal invariance is to understand how to construct the irreducible representations of the conformal symmetry group. We will enhance the Poincaré algebra, introducing new generators, and we will discuss the procedure to follow in order to build the conformal multiplets.

3.1.1 Conformal algebra

The conformal symmetry is a spacetime, continuous symmetry which can be seen as an enhancement of the usual Poincaré algebra. In particular, it requires the addition of the *dilatations* generator $\mathcal{D}$. In general, an infinitesimal transformation of the spacetime coordinates is described by an infinitesimal translation

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x). \tag{3.1}$$

The transformation (3.1) describes a conformal transformation if it satisfies

$$\eta_{\mu\nu}\partial_{\rho}x'^{\mu}\partial_{\sigma}x'^{\nu} = \Omega(x)^2\eta_{\rho\sigma}, \tag{3.2}$$

where $\Omega(x)$ is a function of the coordinates and is called *scale factor*. In 4 dimensions it can be shown that the entire class of the conformal transformations can be generated by the action of four kinds of generators:

- if $\Omega(x) = 1$, we immediately recover the definition of the Poincaré symmetry, i.e. of the symmetry encoding the isometries of the spacetime. The symmetry is described by the Lorentz generators $M^{\mu}{}_{\nu}$ and the translations generators P^{μ}: their actions on the coordinates are

$$x'^{\mu} = x^{\mu} + \omega^{\mu}{}_{\nu} x^{\nu}, \qquad x'^{\mu} = x^{\mu} + c^{\mu}, \tag{3.3}$$

where $\omega_{\mu\nu}$ is an antisimmetric tensor describing an infinitesimal rotation and c^{μ} an infinitesimal constant vector;

- if $\Omega(x) = \lambda \neq 1$, the conformal transformation is generated by the dilatation generator $\mathcal{D}$, whose action on the coordinates is

$$x'^{\mu} = x^{\mu} + a x^{\mu}, \tag{3.4}$$

where a is a numerical infinitesimal constant;

- there is another class of conformal transformations called *special conformal transformations*: their infinitesimal action on the coordinates is given by

$$x'^{\mu} = x^{\mu} + 2 b_{\rho} x^{\rho} x^{\mu} + b^{\mu} x^2, \tag{3.5}$$

where b^{μ} is an infinitesimal constant vector. The finite action of a special conformal transformation on the coordinates is

$$x'^{\mu} = \frac{x^{\mu} + b^{\mu} x^2}{1 + 2 b_{\rho} x^{\rho} + b^2 x^2}. \tag{3.6}$$

It is interesting to notice that the transformation (3.6) can also be obtained from the composition

$$I \circ \text{Translation} \circ I, \tag{3.7}$$

where I stands for the *inversion* of the coordinates

$$I(x^{\mu}) = \frac{x^{\mu}}{x^2}. \tag{3.8}$$

If we consider a translation parametrized by the constant vector b^{μ}, then

$$x^{\mu} \xrightarrow{I} \frac{x^{\mu}}{x^2} \xrightarrow{\text{Tr}} \left(\frac{x^{\mu}}{x^2} + b^{\mu} \right) \xrightarrow{I} \frac{\frac{x^{\mu}}{x^2} + b^{\mu}}{\left(\frac{x^{\rho}}{x^2} + b^{\rho} \right) \left(\frac{x_{\rho}}{x^2} + b_{\rho} \right)} = \frac{x^{\mu} + b^{\mu} x^2}{1 + 2 b_{\rho} x^{\rho} + b^2 x^2}. \tag{3.9}$$

In conclusion, we might look at the special conformal transformation as a way to introduce the inversion of the coordinates in the conformal group. The inversion, being a discrete generator, cannot be a generator of the continuous conformal symmetry group, while the special conformal transformations can be infinitesimal, being connected to the identity of the symmetry group.

We can completely describe the conformal algebra with the following generators

$$P_{\mu}, M_{\mu\nu}, \mathcal{D}, K_{\mu}, \tag{3.10}$$

where $\mathcal{D}$ is the dilatations generator and K_{μ} the special conformal transformations generator. Notice that dilatations and special conformal transformations are respectively parametrized by a scalar and a vector, so their generators have respectively no indices and one vector index. The complete set of the significative commutation relations is (cfr. [4])

$$\begin{aligned} &[M_{\mu\nu}, P_{\rho}] = \eta_{\rho[\nu} P_{\mu]}, \quad [M_{\mu\nu}, K_{\rho}] = \eta_{\rho[\nu} K_{\mu]}, \quad [M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho} M_{\mu\sigma} + (\text{cyclic permutations}), \\ &[\mathcal{D}, P_{\mu}] = P_{\mu}, \qquad [\mathcal{D}, K_{\mu}] = -K_{\mu}, \qquad [K_{\mu}, P_{\nu}] = 2 \left(\eta_{\mu\nu} \mathcal{D} + M_{\mu\nu} \right). \end{aligned}$$

Now we study the explicit action of the generators of the conformal algebra on a generic operator $\mathcal{O}^{l}(0)$, where l stands for a set of Lorentz indices. The operator is computed in the origin, so, following [9]:

- the Lorentz generators $M_{\mu\nu}$ act as follows

$$\left[M_{\mu\nu}, \mathcal{O}^l\right](0) = \left(S_{\mu\nu}\right)_m^{\;\;l} \mathcal{O}^m(0), \tag{3.11}$$

  where $\left(S_{\mu\nu}\right)_m^{\;\;l}$ is the $M_{\mu\nu}$ generator realized in the same representation as $\mathcal{O}^l(0)$;

- the dilatations generator $\mathcal{D}$ acts as follows

$$\left[\mathcal{D}, \mathcal{O}^l\right](0) = \Delta \mathcal{O}^l(0), \tag{3.12}$$

  where Δ is the mass dimension of the operator $\mathcal{O}^l(0)$;

- the translations generator acts as follows

$$\left[P^\mu, \mathcal{O}^l\right](0) = -\left(\partial^\mu \mathcal{O}^l\right)(0); \tag{3.13}$$

- the special conformal transformations generator has a fundamental property: it acts as a ladder operator for the conformal dimension Δ of the operator $\mathcal{O}^l(0)$. Making use of the Jacobi identity

$$\begin{aligned} \left[\mathcal{D}, \left[K_\mu, \mathcal{O}^l\right]\right](0) &= \left[K_\mu, \left[\mathcal{D}, \mathcal{O}^l\right]\right](0) + \left[\left[\mathcal{D}, K_\mu\right], \mathcal{O}^l\right](0) \\ &= \Delta \left[K_\mu, \mathcal{O}^l\right](0) - \left[K_\mu, \mathcal{O}^l\right](0) \\ &= (\Delta - 1)\left[K_\mu, \mathcal{O}^l\right](0), \end{aligned}$$

  so the operator $\left[K_\mu, \mathcal{O}^l\right]$ has a conformal dimension equal to $(\Delta - 1)$: the action of the generator K_μ lowers the conformal dimension by a factor of 1. We want to consider only physical operators, so their conformal dimensions must be positive, hence there must be a finite number n such that, applying n times the generator K_μ to the operator $\mathcal{O}^l(0)$

$$\left[K_{\mu_1}, \ldots, \left[K_{\mu_n}, \mathcal{O}^l\right] \ldots\right](0) = 0. \tag{3.14}$$

3.1.2 Constructing a conformal multiplet

We call *conformal primary operator* an operator $\mathcal{O}^l_{\mathrm{Pr}}(0)$ such that

$$\left[K_\mu, \mathcal{O}^l_{\mathrm{Pr}}\right](0) = 0. \tag{3.15}$$

The conformal primary operator is the lowest weight state of a *conformal multiplet*, an infinite tower of operators generated by the action of the momentum operator P^μ. The operator P^μ acts as a derivative, i.e. as a raising operator for the conformal dimension Δ

$$\mathcal{O}^l_{\mathrm{Pr}}(0) \xrightarrow{P^\mu} \partial^\mu \mathcal{O}^l_{\mathrm{Pr}}(0) \xrightarrow{P^\nu} \partial^\nu \partial^\mu \mathcal{O}^l_{\mathrm{Pr}}(0) \ldots \tag{3.16}$$

$$\Delta \quad \xrightarrow{P^\mu} \quad \Delta + 1 \quad \xrightarrow{P^\nu} \quad \Delta + 2 \ldots \tag{3.17}$$

The operators $\partial^\mu \mathcal{O}^l_{\mathrm{Pr}}(0)$ and $\partial^\nu \partial^\mu \mathcal{O}^l_{\mathrm{Pr}}(0)$, along with the other ones inside the conformal multiplet, are called *conformal descendants*. We can reintroduce the coordinate dependence in the operator $\mathcal{O}^l_{\mathrm{Pr}}(x)$ expressing it as a linear combination of the conformal primary and its descendants

$$\mathcal{O}^l_{\mathrm{Pr}}(x) = \mathcal{O}^l_{\mathrm{Pr}}(0) + x^\mu \partial_\mu \mathcal{O}^l_{\mathrm{Pr}}(0) + \frac{1}{2} x^\mu x^\nu \partial_\mu \partial_\nu \mathcal{O}^l_{\mathrm{Pr}}(0) + \ldots \tag{3.18}$$

In conclusion, we introduced the conformal algebra and a class of operators, the conformal primaries, annihilated by the generator K_μ: they are the lowest weights of the conformal multiplets generated by P_μ. A generic conformal multiplet can be infinite, but the chain of descendants stops if unitarity is violated.

3.2 Unitarity

In this section we discuss the unitarity constraints on the conformal multiplets. The constraints arise from the requirement that the Hilbert space of the theory has a positive norm. We will introduce a Hilbert space describing how to construct a state in a conformal field theory, then we will derive some of the unitarity conditions which can be imposed on the conformal multiplets.

3.2.1 Radial quantization and the state-operator correspondence

In order to define an Hilbert space for our CFT, we need to introduce the *radial quantization*. Usually, we consider a spacetime $\mathbb{R}^{1,3}$ and we foliate it in hypersurfaces at constant *time*: in this picture, each state lives on a hypersurface; the evolution of the states is controlled by a time evolution operator. The generator of the infinitesimal time evolution is the Hamiltonian H. However, the spacetime might be foliated in a different way. In the Euclidean signature, we choose a specific point to be the origin of the spacetime and we foliate it in concentric hyperspheres, each one with a fixed radius. The states now live on the hyperspheres and their evolution is not controlled anymore by the time-evolution generator, but by the dilatations generator $\mathcal{D}$. The choice of the origin is arbitrary and does not affect any physical property, but it changes the ordering of the operators inside the correlators . In the usual quantization, the operators involved in the correlation functions are time-ordered, while in the radial quantization the operators are radius-ordered: the distance between the origin and the point in which each operator is computed changes along with the chosen origin. Operators living on the same hypersphere commute among each other.

In general, (local) operators and states are very different objects: the operators are computed in a specific point of the spacetime, while the states live on hypersurfaces orthogonal to a chosen direction (the radial direction, for instance). Let's consider an hypersurface Σ, a point $x = (r, \vec{\theta}) \in \Sigma$, a state $|\psi\rangle$ living on Σ and a local operator $\mathcal{O}(r, \vec{\theta})$, computed in $x = (r, \vec{\theta})$: the operator can act on the state and the notation is

$$\mathcal{O}(x)\,|\psi\rangle\,. \tag{3.19}$$

We would like to define a *vacuum state* $|\text{vac}\rangle$ and act on it with the local operators. The vacuum is a state, so it lives on an hypersurface. In the usual quantization, we can define the vacuum as a state living on an hypersurface at constant time, such that no local operator appears at a smaller time. The same procedure can be applied to the radial quantization: we define the vacuum as a state living on an empty hypersphere; then, we can dilatate the sphere until it reaches a local operator, which will act on the vacuum state.

From operators to states

Let's introduce the operator $\mathcal{O}(r)$ in the radially quantized spacetime (we neglect the dependence on the angular coordinates $\vec{\theta}$). Following [9], we assume that the operator can be applied to the vacuum state

$$\mathcal{O}(r)\,|\text{vac}\rangle\,. \tag{3.20}$$

We would like to understand which state lives on the hypersphere Σ with radius $r + \epsilon$, where ϵ is an infinitesimal distance between Σ and the hypersphere where the operator is computed. Thanks to the dilatation invariance, the hypersphere with radius r can be shrinked to the origin. As a consequence, the operator now is applied to the origin

$$\mathcal{O}(r)\left|\mathrm{vac}\right\rangle \to \mathcal{O}(0)\left|\mathrm{vac}\right\rangle . \tag{3.21}$$

However, the operator computed in the origin can be seen as a state living on the origin, seen as a degenerate hypersurface

$$\mathcal{O}(0)\left|\mathrm{vac}\right\rangle = \left|\mathcal{O}(0)\right\rangle . \tag{3.22}$$

Making use of the dilatation invariance, we expand hypersphere from the origin to Σ

$$\left|\mathcal{O}(0)\right\rangle \to \left|\mathcal{O}(r+\epsilon)\right\rangle . \tag{3.23}$$

In conclusion, thanks to the dilatation invariance we have

$$\mathcal{O}(r)\left|\mathrm{vac}\right\rangle \to \left|\mathcal{O}(r+\epsilon)\right\rangle . \tag{3.24}$$

From states to operators

First of all, we need to understand how a state is realized over an hypersphere with radius r. We will follow the reference [10]. In one-dimensional quantum mechanics, a system described by the classical action S evolves according to the propagator

$$G(x_i, x_f) = \int_{x_i(t_i)}^{x_f(t_f)} [\mathcal{D}x]\, e^{iS}, \tag{3.25}$$

so the wavefunction at x_f is given by

$$\psi(x_f, t_f) = \int dx_i\; G(x_i, x_f)\psi(x_i, t_i), \tag{3.26}$$

where the initial position x_i is free. Analogously, in QFT a state can be written as follows

$$\left|\psi\right\rangle_f = \int [\mathcal{D}\phi_i] \int_{\phi_i(t_i)=\phi_i}^{\phi_f(t_f)=\phi_f} [\mathcal{D}\phi]\, e^{iS[\phi]} \left|\phi\right\rangle_i , \tag{3.27}$$

where the initial field configuration is free. In the radial quantization, the equation (3.27) becomes

$$\left|\psi\right\rangle_f = \int [\mathcal{D}\phi_i] \int_{\phi_i(r_i)=\phi_i}^{\phi_f(r_f)=\phi_f} [\mathcal{D}\phi]\, e^{-S[\phi]} \left|\phi\right\rangle_i . \tag{3.28}$$

Now we consider a state defined on a hypersphere Σ

$$\left|\psi\right\rangle_\Sigma = \int [\mathcal{D}\phi_i] \int_{\phi_i(r_i)=\phi_i}^{\phi_f(r_\Sigma)=\phi_f} [\mathcal{D}\phi]\, e^{-S[\phi]} \left|\phi\right\rangle_i . \tag{3.29}$$

Thanks to the dilatation invariance, we can bring $r_i \to 0$, so that the initial state is defined in the origin, interpreted as a degenerate hypersurface

$$\left|\psi\right\rangle_\Sigma = \int [\mathcal{D}\phi_i] \int_{\phi_i(0)=\phi_i}^{\phi_f(r_\Sigma)=\phi_f} [\mathcal{D}\phi]\, e^{-S[\phi]} \left|\phi\right\rangle_i . \tag{3.30}$$

Now the state $\left|\phi\right\rangle_i$ can be seen as a local operator $\mathcal{O}$ computed in the origin and applied to the vacuum state

$$\left|\psi\right\rangle_\Sigma = \int_{\phi_i(0)=\mathrm{vacuum}}^{\phi_f(r_\Sigma)=\phi_f} [\mathcal{D}\phi]\, e^{-S[\phi]}\mathcal{O}(0)\left|\mathrm{vac}\right\rangle . \tag{3.31}$$

Applying the dilatation symmetry, we can compute the operator on a generic hypersphere of radius $r < r_\Sigma$

$$|\psi\rangle_\Sigma = \int_{\phi_i(r)=\text{vacuum}}^{\phi_f(r_\Sigma)=\phi_f} [\mathcal{D}\phi]\, e^{-S[\phi]} \mathcal{O}(r)\, |\text{vac}\rangle\,. \tag{3.32}$$

The state $|\psi\rangle_\Sigma$ can be interpreted as an operator inside the hypersphere Σ

In conclusion, in a radially-quantized conformal field theory states and operators, which in principle are completely different objects, are in a *correspondence*: every operator generates a state on a hypersphere containing it, while each state living on a surface can be seen as generated by an operator living inside it. We would have not been able to achieve this result in the usual quantization: translating in time, the operators would have been brought not to a degenerate hypersurface (the origin), but to a proper hypersurface (the hyperplane at $t = 0$). In this case there is no correspondence between operators and states: the former are computed on specific points, while the latter are defined over the whole hyperplane at $t = 0$.

3.2.2 The hermitian conjugation

The states of a quantum theory live in a Hilbert space, equipped with a norm. In a QFT the unitarity condition requires all the physical states to have a positive norm. In order to be able to compute the norms of the states in the radial quantization, we have to define a new hermitian conjugation. It can be shown that in the radial quantization, given the operator $\mathcal{O}(x)$, its hermitian conjugate operator is given by

$$\mathcal{O}^\dagger(x) = I \circ \mathcal{O}(x) \circ I. \tag{3.33}$$

This result can be used to compute the hermitian conjugates of the conformal algebra generators

$$M^\dagger_{\mu\nu} = -M_{\mu\nu}, \qquad P^\dagger_\mu = K_\mu, \qquad K^\dagger_\mu = P_\mu, \qquad \mathcal{D}^\dagger = \mathcal{D}. \tag{3.34}$$

3.2.3 Unitarity bounds

The procedure to adopt in order to derive the unitarity bounds is explained in [4]. First of all, we introduce a primary operator $\mathcal{O}(0)$, sitting in a representation of the conformal group

$$[j, \bar{j}]_\Delta\,, \tag{3.35}$$

where $j, \bar{j}$ are the Lorentz quantum numbers and Δ is the conformal dimension. Making use of the states-operators correspondence, we define the state

$$\mathcal{O}(0)\,|\text{vac}\rangle \to |O\rangle = \left|j, \bar{j}\right\rangle_\Delta. \tag{3.36}$$

The action of a conformal generator G on the operator $\mathcal{O}$ can be translated in the states picture as follows

$$[G, \mathcal{O}]\,(0)\,|\text{vac}\rangle \to |[G, \mathcal{O}]\rangle = G\left|j, \bar{j}\right\rangle_\Delta. \tag{3.37}$$

We stated that the state $\left|j, \bar{j}\right\rangle_\Delta$ is the highest weight of a conformal multiplet, so the first descendant is given by

$$a^\mu P_\mu \left|j, \bar{j}\right\rangle_\Delta, \tag{3.38}$$

where a^μ is a constant vector, useful to contract the vector index of the momentum operator. We normalize the norm of the primary state to $\left\langle j, \bar{j}\,\middle|\, j, \bar{j}\right\rangle_\Delta = 1$ and we compute the norm of the first descendant

$$\begin{aligned}
\bar{a}^\mu a^\nu \left\langle j, \bar{j}\right| K_\mu P_\nu \left|j, \bar{j}\right\rangle_\Delta &= \bar{a}^\mu a^\nu \left\langle j, \bar{j}\right| P_\nu K_\mu \left|j, \bar{j}\right\rangle_\Delta + 2a^\mu a^\nu \left\langle j, \bar{j}\right| \delta_{\mu\nu}\mathcal{D} + M_{\mu\nu} \left|j, \bar{j}\right\rangle_\Delta \\
&= 2\bar{a}^\mu a^\nu \left(\delta_{\mu\nu}\Delta \left\langle j, \bar{j}\,\middle|\, j, \bar{j}\right\rangle_\Delta + \left\langle j, \bar{j}\right| M_{\mu\nu} \left|j, \bar{j}\right\rangle_\Delta\right) \\
&= 2\bar{a}^\mu a^\nu \left(\delta_{\mu\nu}\Delta + \left\langle j, \bar{j}\right| M_{\mu\nu} \left|j, \bar{j}\right\rangle_\Delta\right).
\end{aligned}$$

We need to compute the term

$$\langle j, \bar{j} \,|\, M_{\mu\nu} \,|\, j, \bar{j} \rangle_{\Delta} . \tag{3.39}$$

In order to do so, we rewrite the Lorentz generators as follows

$$M_{\mu\nu} = \frac{1}{2} \left(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} \right) M_{\rho\sigma} = \left(\mathcal{M}_{\mu\nu} \right)_{\rho\sigma} \otimes M_{\rho\sigma}, \tag{3.40}$$

where the operator $\left(\mathcal{M}_{\mu\nu} \right)_{\rho\sigma}$ is the Lorentz generator acting on operators sitting in the vectorial representation and $\otimes$ is the product between different representations. The product (3.40) can be rewritten as follows

$$\left(\mathcal{M}_{\mu\nu} \right)_{\rho\sigma} \otimes M_{\rho\sigma} = \frac{1}{2} \left(\left(\left(\mathcal{M}_{\mu\nu} \right)_{\rho\sigma} \otimes \mathbf{1} + \mathbf{1} \otimes M_{\rho\sigma} \right)^2 - \left(\left(\mathcal{M}_{\mu\nu} \right)_{\rho\sigma} \otimes \mathbf{1} \right)^2 - \left(\mathbf{1} \otimes M_{\rho\sigma} \right)^2 \right) \tag{3.41}$$

Plugging the result inside the term (3.39), we obtain

$$\bar{a}^{\mu} a^{\nu} \langle j, \bar{j} \,|\, M_{\mu\nu} \,|\, j, \bar{j} \rangle_{\Delta} = |a|^2 \frac{1}{2} \left(\mathcal{C}(j', \bar{j}') - \mathcal{C}(1,1) - \mathcal{C}(j, \bar{j}) \right), \tag{3.42}$$

where $\mathcal{C}(1,1)$ is the eigenvalue of the Lorentz Casimir operator in the vectorial representation, $\mathcal{C}(j,\bar{j})$ in the representation of the primary and $\mathcal{C}(j',\bar{j}')$ in a representation contained in the product $[1,1] \otimes [j,\bar{j}]$. Eventually, the norm of the first descendant of a generic conformal primary state is

$$\bar{a}^{\mu} a^{\nu} \langle j, \bar{j} \,|\, K_{\mu} P_{\nu} \,|\, j, \bar{j} \rangle_{\Delta} = 2|a|^2 \left(\Delta + \frac{1}{2} \left(\mathcal{C}(j', \bar{j}') - \mathcal{C}(1,1) - \mathcal{C}(j, \bar{j}) \right) \right). \tag{3.43}$$

The first descendant state belongs to the Hilbert space if and only if the following *unitarity bound* holds

$$\Delta \geq \frac{1}{2} \left(\mathcal{C}(1,1) + \mathcal{C}(j, \bar{j}) - \mathcal{C}(j', \bar{j}') \right). \tag{3.44}$$

Now we are interested in computing three unitarity bounds, making use of the formula (3.44):

- we consider a *scalar* primary operator: then $[j,\bar{j}] = [0,0]$. There is only one possible product representation: $[j',\bar{j}'] = [1,1]$. The scalar Casimir eigenvalue is $\mathcal{C}(0,0) = 0$, while the vector one is $\mathcal{C}(1,1) = 3$. Thus, the unitarity bound is

  $$\Delta \geq 0. \tag{3.45}$$

  We might want to go on and derive another unitarity bound, obtained computing the norm of the second descendant of a scalar conformal primary. It can be shown that the result is

  $$\Delta \geq 1. \tag{3.46}$$

  One can show that this is the strongest unitarity bound for scalar primaries;

- we consider a *vector* primary operator: then $[j,\bar{j}] = [1,1]$. In order to derive the unitarity bound, we need to choose the lowest product representation: $[j',\bar{j}'] = [0,0]$. The unitarity bound is

  $$\Delta \geq 3. \tag{3.47}$$

  it can be shown that this is the strongest unitarity bound for vector primaries. If we consider a current J^{μ} with conformal dimension equal to $[J^{\mu}] = 3$, then its first descendant has zero norm, hence it is the null vector

  $$\left[P_{\mu}, J^{\mu} \right] |\text{vac}\rangle = \partial_{\mu} J^{\mu} |\text{vac}\rangle = \left| \partial_{\mu} J^{\mu} \right\rangle = \text{Null vector}; \tag{3.48}$$

- we consider a *spin* $\frac{3}{2}$ primary operator: then $\left[j,\bar{j}\right]=[2,1]$ (or: $\left[j,\bar{j}\right]=[1,2]$). Among all the possibilities, we choose the lowest product representation: $\left[j',\bar{j}'\right]=[1,0]$. The spinorial Casimir eigenvalue is $\mathcal{C}(1,0)=\frac{3}{2}$, while the spin $\frac{3}{2}$ one is $\mathcal{C}(2,1)=\frac{11}{2}$. Thus, the unitarity bound is

$$\Delta \geq \frac{7}{2}. \tag{3.49}$$

It can be shown that this is the strongest unitarity bound for spin $\frac{3}{2}$ primaries. If we consider a supercurrent G^{μ}_{α} with conformal dimension equal to $[J^{\mu}]=\frac{7}{2}$, then its first descendant has zero norm, hence it is the null vector

$$\left[P_{\mu},G^{\mu}_{\alpha}\right]|\text{vac}\rangle=\partial_{\mu}G^{\mu}_{\alpha}|\text{vac}\rangle=\left|\partial_{\mu}G^{\mu}_{\alpha}\right\rangle=\text{Null vector}. \tag{3.50}$$

We conclude this section highlighting a fundamental consequence of the unitarity bounds. Primary operators which *saturate* the unitarity bounds generate conformal multiplets containing null vectors. Null vectors does not appear in the Hilbert space, so they are not considered in the quantum theory. Suppose that the primary operator acquires an anomalous dimension, so that it does not saturate the unitarity bound anymore. This is forbidden: the null state would become a physical state, but it would not be elements of the Hilbert space. In conclusions, the conformal dimensions of primary operators which saturate the unitarity bounds is protected, i.e. it is unaffected by any anomalous dimension effect.

3.3 Scalar correlation functions

One of the most interesting aspects of working with CFTs is the fact that 2-points and 3-points functions have a completely fixed kinematic (i.e. dependent on the coordinates) structure, up to multiplicative constants. In this section we briefly show how to derive the explicit structure of the 2-points and 3-points scalar correlation functions, following [43].

3.3.1 The embedding formalism

The most efficient way to compute the correlation functions in a CFT is to exploit the embedding formalism: the Euclidean spacetime $\mathbb{R}^4$ is embedded in a bigger spacetime $\mathbb{R}^{1,5}$, where the computations are done; in order to go back to the original spacetime, the results are projected on a section of the embedding space, called *Poincaré section*. This procedure is motivated by the fact that the conformal algebra in $\mathbb{R}^{1,3}$ can be rewritten as the Lorentz algebra in $\mathbb{R}^{1,5}$ in terms of the antisymmetric generators J^{mn}

$$[J^{mn},J^{rs}]=i\eta^{mr}J^{ns}+\text{Cyclic permutations}, \tag{3.51}$$

where m,n,r,s run over $0,1,\ldots,5$, $\eta^{mr}=\text{diag}(+,+,+,+,+,-)$ and

$$J^{\mu\nu}=M^{\mu\nu}, \qquad J^{56}=\mathcal{D}, \qquad J^{5\mu}=\frac{1}{2}\left(P^{\mu}-K^{\mu}\right), \qquad J^{6\mu}=\frac{1}{2}\left(P^{\mu}+K^{\mu}\right). \tag{3.52}$$

In the embedding space $\mathbb{R}^{1,5}$ we work with light-cone coordinates

$$\mathcal{X}^m=\left(\mathcal{X}^+,\mathcal{X}^-,\mathcal{X}^{\mu}\right), \tag{3.53}$$

where $\mathcal{X}^{\pm}=\mathcal{X}^5\pm\mathcal{X}^6$. The scalar product is

$$\mathcal{V}^m\mathcal{W}_m=\mathcal{V}^{\mu}\mathcal{W}_{\mu}-\frac{1}{2}\left(\mathcal{V}^-\mathcal{W}^++\mathcal{V}^+\mathcal{W}^-\right), \tag{3.54}$$

so $\mathcal{X}^2 = \mathcal{X}^\mu \mathcal{X}_\mu - \mathcal{X}^- \mathcal{X}^+$. The embedded spacetime is identified by the coordinates $\mathcal{X}_P^m = (1, x^2, x^\mu)$, which lead to $\mathcal{X}_P^m \mathcal{X}_{P,m} = 0$. The set of the $\mathcal{X}_P^m$ points is called Poincaré section. An important relationship on the Poincaré section is

$$\begin{aligned}(\mathcal{X}_P^m - \mathcal{Y}_P^m)^2 &= \mathcal{X}_P^2 + \mathcal{Y}_P^2 - 2\mathcal{X}_P^m \mathcal{Y}_{P,m} = -2\mathcal{X}_P^m \mathcal{Y}_{P,m} \\ &= \left[(0, x^2 - y^2, x^\mu - y^\mu)\right]^2 = (x^\mu - y^\mu)^2 ,\end{aligned}$$

so we obtain

$$\mathcal{X}_P^m \mathcal{Y}_{P,m} = -\frac{1}{2}(x^\mu - y^\mu)^2 . \tag{3.55}$$

In the embedding space we will work with scalar operators $\Phi(\mathcal{X})$ such that $\Phi(\mathcal{X}_P) = \phi(x)$, where $\phi(x)$ is a scalar operator defined in the spacetime.

3.3.2 Scalar correlation functions

We start considering the following correlator, defined in the embedding space

$$\langle \Phi_1(\mathcal{X}_1)\Phi_2(\mathcal{X}_2)\rangle . \tag{3.56}$$

The conformal symmetry requires the imposition of the following constraints:

- the correlator must be invariant under the action of the conformal algebra. Given that the conformal algebra in the embedding space is the Lorentz algebra, the correlator must be a function of the Lorentz invariants $\mathcal{X}_1^2$, $\mathcal{X}_2^2$ and $\mathcal{X}_1^m \mathcal{X}_{2,m}$. Once the result is projected on the Poincaré section, $\mathcal{X}_1^2 = \mathcal{X}_2^2 = 0$, so their contributions can be neglected;
- the correlator must be invariant under the scale transformation $\Phi(\lambda\mathcal{X}) = \lambda^{-\Delta}\Phi(\lambda\mathcal{X})$, where Δ is the conformal dimension of the field Φ and of its projection on the Poincaré section ϕ.

There is only one possibility which satisfies both requirements: if g_{12} is a numerical coefficient and $x_1^\mu - x_2^\mu \equiv x_{12}^\mu$, we can make use of the formula (3.55), obtaining

$$\langle \Phi_1(\mathcal{X}_1)\Phi_2(\mathcal{X}_2)\rangle = \delta_{\Delta_1,\Delta_2}\frac{g_{12}}{\mathcal{X}_1^m \mathcal{X}_{2,m}} \xrightarrow{\text{Poincaré section}} \langle \phi_1(x_1)\phi_2(x_2)\rangle = \delta_{\Delta_1,\Delta_2}\frac{g_{12}}{x_{12}^2}. \tag{3.57}$$

Now we consider the correlator

$$\langle \Phi_1(\mathcal{X}_1)\Phi_2(\mathcal{X}_2)\Phi_3(\mathcal{X}_3)\rangle . \tag{3.58}$$

the procedure is the same as the one adopted for the 2-points function. In the embedding space the correlator has the structure

$$\langle \Phi_1(\mathcal{X}_1)\Phi_2(\mathcal{X}_2)\Phi_3(\mathcal{X}_3)\rangle = \frac{C_{123}}{\left(\mathcal{X}_1^m \mathcal{X}_{2,m}\right)^{\frac{1}{2}(\Delta_1+\Delta_2-\Delta_3)}\left(\mathcal{X}_1^m \mathcal{X}_{3,m}\right)^{\frac{1}{2}(\Delta_1+\Delta_3-\Delta_2)}\left(\mathcal{X}_2^m \mathcal{X}_{3,m}\right)^{\frac{1}{2}(\Delta_2+\Delta_3-\Delta_1)}} \tag{3.59}$$

where C_{123} is a numerical coefficient; in the spacetime

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}x_{13}^{\Delta_1+\Delta_3-\Delta_2}x_{23}^{\Delta_2+\Delta_3-\Delta_1}}. \tag{3.60}$$

In the chapter 6 we will study a specific 3-points scalar correlation function: the result (3.60) completely fixes the dependence on the coordinates, so we know for sure that the numerical factor C_{123} is not kinematic.

3.4 The Operator Product Expansion

We conclude this chapter with a discussion on the Operator Product Expansion (OPE), following [9]. We consider a radially quantized spacetime and an hypersphere Σ. The hypershpere contains two local operators $\mathcal{O}_1(x_1)$ and $\mathcal{O}_2(x_2)$. The state $|\psi\rangle$ living on Σ is

$$|\psi\rangle = \int_{\phi_i(0)=\text{vacuum}}^{\phi_f(r_\Sigma)=\phi_f} [\mathcal{D}\phi]\, e^{-S[\phi]} \mathcal{O}_2(x_2)\mathcal{O}_1(x_1)\,|\text{vac}\rangle\,, \quad 0 < x_1 < x_2 \tag{3.61}$$

We know that the state $|\psi\rangle$ can be written as a linear combination of some conformal primary states $|\mathcal{O}_k\rangle$ and their descendants. The coefficients $c_k(x_1, x_2, P_\mu)$ of the sum inherits the dependence on the coordinates x_1, x_2 and also depend on the momentum operator P_μ, which generates the descendants

$$|\psi\rangle = \sum_k c_k(x_1, x_2, P_\mu)\,|\mathcal{O}_k\rangle\,. \tag{3.62}$$

Applying the states-operators correspondence to the primary states, we end up with

$$|\psi\rangle = \sum_k \int_{\phi_i(0)=\text{vacuum}}^{\phi_f(r_\Sigma)=\phi_f} [\mathcal{D}\phi]\, e^{-S[\phi]} c_k(x_1, x_2, P_\mu)\mathcal{O}_k(0)\,|\text{vac}\rangle\,. \tag{3.63}$$

Comparing the equations (3.61) and (3.63) we obtain the OPE

$$\mathcal{O}_2(x_2)\mathcal{O}_1(x_1) = \sum_k c_k(x_1, x_2, P_\mu)\mathcal{O}_k(0). \tag{3.64}$$

If two operators are near enough to be contained in the same hypersphere, leaving the other operators outside, the OPE exchanges them for a linear combination of terms, each one containing only one operator (primary or descendant). A specific situation in which this technique can be useful is in the simplification of long correlators like

$$\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle\,. \tag{3.65}$$

If two operators in the list are close to each other, then in the radial quantization they can be contained inside the same hypersphere, while all the other operators are left outside. In this conditions, we can apply the OPE. The correlator is simplified: although the OPE introduces a sum, it reduces the length by a unity. In general, it is difficult to determine the terms contained in the infinite sum (3.64) generated by the OPE: in the following chapters we will derive a few of them exploiting the conformal dimensions and the indices of the parents operators $\mathcal{O}_2(x_2)$ and $\mathcal{O}_1(x_1)$.

Chapter 4

Superconformal Field Theories

In the chapter 2 we discussed the supersymmetric QFTs, in particular the strong constraints that supersymmetry imposes to their Lagrangians. In the chapter 3 we introduced some of the most interesting properties of the CFTs: in particular, we realized that it is possible to have a very good handle on the kinematic structure of the correlators thanks to the scale invariance. A theory which enjoys both supersymmetry and conformal invariance is called *superconformal*. The combination of the two symmetries allows to completely solve some aspects of the models: for this reason, the SCFTs are the perfect workbench for developing or proving important non-perturbative properties of the QFTs. In a SCFT the conformal algebra is enhanced by the addition of the supercharges and the generators of the R-symmetry, which becomes a fundamental tool for the algebraic structure of the representations.

4.1 Superconformal (super)algebra

In this section we will follow [4] and we will include the supercharges and the R-symmetry generators in the conformal algebra. The result will be the superconformal (super)algebra, valid for any number of supercharges $\mathcal{N}$.

Conformal algebra

First of all, we modify the generators of the conformal algebra, exchanging their vectorial Lorentz indices for spinorial Lorentz indices. This is possible thanks to the Sigma matrices $\sigma^{\mu}_{\alpha\dot{\beta}}$ and $\bar{\sigma}^{\mu\dot{\beta}\alpha}$, equipped with both types of indices:

$$
\begin{aligned}
P_{\alpha\dot{\beta}} &\equiv \sigma^{\mu}_{\alpha\dot{\beta}} P_{\mu}, & K^{\dot{\beta}\alpha} &\equiv \bar{\sigma}^{\mu\dot{\beta}\alpha} K_{\mu}, \\
M_{\alpha}{}^{\beta} &\equiv -\frac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\beta}\sigma^{\nu}_{\alpha\dot{\alpha}} M_{\mu\nu}, & \bar{M}^{\dot{\alpha}}{}_{\dot{\beta}} &\equiv -\frac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\sigma^{\nu}_{\alpha\dot{\beta}} M_{\mu\nu}.
\end{aligned}
$$

Switching to the spinorial representation, the generators $M_{\mu\nu}$ split up in two different generators $M_{\alpha}{}^{\beta}$ and $M^{\dot{\alpha}}{}_{\dot{\beta}}$: the former acts on the left-chirality Weyl spinors, the latter on the right-chirality Weyl spinors. We write again the conformal algebra in the new notation

$$\begin{aligned}
\left[M_\alpha{}^\beta, M_\gamma{}^\delta\right] &= \delta^\beta_\gamma M_\alpha{}^\delta - \delta^\delta_\alpha M_\gamma{}^\beta, & \left[\bar{M}^{\dot\alpha}{}_{\dot\beta}, \bar{M}^{\dot\gamma}{}_{\dot\delta}\right] &= -\delta^{\dot\alpha}_{\dot\gamma}\bar{M}^{\dot\gamma}{}_{\dot\beta} + \delta^{\dot\gamma}_{\dot\beta}\bar{M}^{\dot\alpha}{}_{\dot\delta}, \\
\left[M_\alpha{}^\beta, P_{\gamma\dot\gamma}\right] &= \delta^\beta_\gamma P_{\alpha\dot\gamma} - \frac{1}{2}\delta^\beta_\alpha P_{\gamma\dot\gamma}, & \left[\bar{M}^{\dot\alpha}{}_{\dot\beta}, P_{\gamma\dot\gamma}\right] &= \delta^{\dot\alpha}_{\dot\gamma} P_{\gamma\dot\beta} - \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta} P_{\gamma\dot\gamma}, \\
\left[M_\alpha{}^\beta, K^{\dot\gamma\gamma}\right] &= -\delta^\gamma_\alpha K^{\dot\gamma\beta} + \frac{1}{2}\delta^\beta_\alpha K^{\dot\gamma\gamma}, & \left[\bar{M}^{\dot\alpha}{}_{\dot\beta}, K^{\dot\gamma\gamma}\right] &= -\delta^{\dot\gamma}_{\dot\beta} K^{\dot\alpha\gamma} + \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta} K^{\dot\gamma\gamma}, \\
[\mathcal{D}, P_{\alpha\dot\alpha}] &= P_{\alpha\dot\alpha}, & [\mathcal{D}, K^{\dot\alpha\alpha}] &= -K^{\dot\alpha\alpha}, \\
\left[K^{\dot\alpha\alpha}, P_{\beta\dot\beta}\right] &= 4\delta^\alpha_\beta\delta^{\dot\alpha}_{\dot\beta}\mathcal{D} + 4\delta^\alpha_\beta \bar{M}^{\dot\alpha}{}_{\dot\beta} + 4\delta^{\dot\alpha}_{\dot\beta} M_\beta{}^\alpha. & &
\end{aligned}$$

Supercharges

Now we introduce the commutation relations between the supercharges and the conformal generators. Part of the them have already been covered in the chapter 2, where we introduced the Poincaré (super)algebra. This section adds new relations, not only because we have the generators $\mathcal{D}$ and $K^{\dot\beta\alpha}$ at our disposal, but also because in the Euclidean spacetime the supercharges $\mathcal{Q}^I_\alpha$ and $\bar{\mathcal{Q}}^{\dot\beta}_J$ are independent, so we need to introduce their new hermitian conjugates. In the radial quantization we can follow the definition (3.33), already used to define $K^{\dot\alpha\beta}$

$$\bar{\mathcal{S}}^J_{\dot\beta} \equiv I \circ \bar{\mathcal{Q}}^{\dot\beta}_J \circ I, \qquad \mathcal{S}^\alpha_I \equiv I \circ \mathcal{Q}^I_\alpha \circ I. \tag{4.1}$$

The relations of the superconformal (super)algebra involving the supercharges and the conformal generators are

$$\begin{aligned}
\left[M_\alpha{}^\beta, \mathcal{Q}^I_\gamma\right] &= \delta^\beta_\gamma \mathcal{Q}^I_\alpha - \frac{1}{2}\delta^\beta_\alpha \mathcal{Q}^I_\gamma, & \left[\bar{M}^{\dot\alpha}{}_{\dot\beta}, \bar{\mathcal{Q}}_{\dot\gamma J}\right] &= \delta^{\dot\alpha}_{\dot\gamma}\bar{\mathcal{Q}}_{\dot\beta J} - \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\bar{\mathcal{Q}}_{\dot\gamma I}, \\
\left[M_\alpha{}^\beta, \mathcal{S}^\gamma_J\right] &= -\delta^\gamma_\alpha \mathcal{S}^\beta_J + \frac{1}{2}\delta^\beta_\alpha \mathcal{S}^\gamma_J, & \left[\bar{M}^{\dot\alpha}{}_{\dot\beta}, \bar{\mathcal{S}}^{\dot\gamma I}\right] &= -\delta^{\dot\gamma}_{\dot\beta}\bar{\mathcal{S}}^{\dot\alpha I} + \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\bar{\mathcal{S}}^{\dot\gamma I}, \\
\left[\mathcal{D}, \mathcal{Q}^I_\alpha\right] &= \frac{1}{2}\mathcal{Q}^I_\alpha, & [\mathcal{D}, \bar{\mathcal{Q}}^{\dot\alpha}_J] &= \frac{1}{2}\bar{\mathcal{Q}}^{\dot\alpha}_J, \\
\left[\mathcal{D}, \mathcal{S}^\beta_J\right] &= -\frac{1}{2}\mathcal{S}^\beta_J, & \left[\mathcal{D}, \bar{\mathcal{S}}^I_{\dot\alpha}\right] &= -\frac{1}{2}\bar{\mathcal{S}}^I_{\dot\alpha}, \\
\left[P_{\alpha\dot\alpha}, \mathcal{S}^\beta_J\right] &= -2\delta^\beta_\alpha \bar{\mathcal{Q}}_{\dot\alpha J}, & \left[P_{\alpha\dot\alpha}, \bar{\mathcal{S}}^{\dot\beta I}\right] &= -2\delta^{\dot\beta}_{\dot\alpha}\mathcal{Q}^I_\alpha, \\
\left[K^{\dot\alpha\alpha}, \mathcal{Q}^I_\beta\right] &= 2\delta^\alpha_\beta \bar{\mathcal{S}}^{I\dot\alpha}, & \left[K^{\dot\alpha\alpha}, \bar{\mathcal{Q}}_{\dot\beta J}\right] &= 2\delta^{\dot\alpha}_{\dot\beta}\mathcal{S}^\alpha_J, \\
\left\{\mathcal{Q}^I_\alpha, \bar{\mathcal{Q}}_{\dot\beta J}\right\} &= \frac{1}{2}\delta^I_J P_{\alpha\dot\beta}, & \left\{\bar{\mathcal{S}}^{\beta I}, \mathcal{S}^\alpha_J\right\} &= \frac{1}{2}\delta^I_J K^{\dot\beta\alpha}.
\end{aligned}$$

R-symmetry

In a generic supersymmetric theory the R-symmetry is not a mandatory feature; however, in a SCFT the generators of the R-symmetry group appear in the anticommutators between the $\mathcal{Q}$ and the $\mathcal{S}$ supercharges, so the superconformal (super)algebra requires them in order to close. The R-symmetry group for a theory with $\mathcal{N}$ supersymmetries is $u(\mathcal{N})$. The abstract generators of the R-symmetry group are the objects $t^I{}_J$, labelled by the indices I, J, both running from 1 to $\mathcal{N}$. The R-symmetry algebra is defined by the commutator

$$\left[t^I{}_J, t^K{}_L\right] = \delta^K_J t^I{}_L - \delta^I_L t^K{}_J. \tag{4.2}$$

The R-symmetry generators are involved in the following relations

$$\begin{aligned}
\left[t^I{}_J, \mathcal{Q}^K_\alpha\right] &= \delta^K_J \mathcal{Q}^I_\alpha - \frac{1}{4}\delta^I_J \mathcal{Q}^K_\alpha, &\qquad \left[t^I{}_J, \bar{\mathcal{Q}}^{\dot\alpha}_K\right] &= -\delta^I_K \bar{\mathcal{Q}}^{\dot\alpha}_J + \frac{1}{4}\delta^I_J \bar{\mathcal{Q}}^{\dot\alpha}_K, \\
\left[t^I{}_J, \mathcal{S}^\alpha_K\right] &= -\delta^I_K \mathcal{S}^\alpha_J + \frac{1}{4}\delta^I_J \mathcal{S}^\alpha_K, &\qquad \left[t^I{}_J, \bar{\mathcal{S}}^K_{\dot\alpha}\right] &= \delta^K_J \bar{\mathcal{S}}^I_{\dot\alpha} - \frac{1}{4}\delta^I_J \bar{\mathcal{S}}^K_{\dot\alpha}, \\
\left\{\mathcal{Q}^I_\alpha, \mathcal{S}^\beta_J\right\} &= \delta^I_J M_\alpha{}^\beta + \frac{1}{2}\delta^I_J \delta^\beta_\alpha \mathcal{D} - \delta^\beta_\alpha t^I{}_J, &\qquad \left\{\bar{\mathcal{S}}^{\dot\alpha I}, \bar{\mathcal{Q}}_{\dot\beta J}\right\} &= \delta^I_J \bar{M}^{\dot\alpha}{}_{\dot\beta} + \frac{1}{2}\delta^I_J \delta^{\dot\alpha}_{\dot\beta} \mathcal{D} + \delta^{\dot\alpha}_{\dot\beta} t^I{}_J.
\end{aligned}$$

Finally, we highlight the fact that in the $\mathcal{N} = 4$ theory the generator $t^I{}_I$ is central, so it can be removed from the superconformal (super)algebra. This implies that the $\mathcal{N} = 4$ theory R-symmetry group is simply $su(4)$ and that the operators defined in the model, although sitting in a $su(4)$ representation, do not carry any R-charge (differently from the $\mathcal{N} = 1$, $\mathcal{N} = 2$ and $\mathcal{N} = 3$ theories).

4.2 Constructing a superconformal multiplet

Looking at the commutation and anticommutation relations introduced in the previous section, we immediately notice that the generator $K^{\dot\alpha\beta}$ has a conformal dimension equal to -1 and the supercharges $\mathcal{S}^\beta_J$ and $\bar{\mathcal{S}}^I_\alpha$ have conformal dimensions equal to $-\frac{1}{2}$. As we did in the section 3.1.2, we require our physical operators to have a positive conformal dimension. The action of the generators $K^{\dot\alpha\beta}, \mathcal{S}^\beta_J$ and $\bar{\mathcal{S}}^I_\alpha$ lowers it at every step, so we necessarily reach a state $\left|j,\bar{j}\right\rangle^{\mathcal{R}}_\Delta$ such that

$$K^{\dot\alpha\beta}\left|j,\bar{j}\right\rangle^{\mathcal{R}}_\Delta = 0, \qquad \mathcal{S}^\beta_J\left|j,\bar{j}\right\rangle^{\mathcal{R}}_\Delta = 0, \qquad \bar{\mathcal{S}}^{I\dot\alpha}\left|j,\bar{j}\right\rangle^{\mathcal{R}}_\Delta = 0, \tag{4.3}$$

where $\mathcal{R}$ stands for all the R-symmetry quantum numbers (Dynkin labels, R-charge). The state is called *superconformal primary state* and it is the lowest weight of a superconformal multiplet. If we act on it with the momentum operator P_μ, we construct a tower of conformal descendants; if we act with the supercharges $\mathcal{Q}$ or $\bar{\mathcal{Q}}$, we construct a set of *superconformal descendants*. Unlike the conformal descendant, the superconformal descendants cannot be infinite in number: each supercharge can be applied only once, due to its fermionic nature. The maximum number of superconformal descendants in a superconformal multiplet is $2^{4\mathcal{N}}$ states; however, it is common for this number to be reduced by the requirement of unitarity. In the language of superconformal multiplets, unitarity is provided by the so called *shortening conditions*, which we will discuss in the next section. Whenever we act with another supercharge $\mathcal{Q}$ or $\bar{\mathcal{Q}}$ on a superconformal descendant, we can act in a symmetrized way, which creates the operator P_μ, or in an antisymmetrized way, which keeps us in the plane of the superconformal descendants. For instance, if we apply a supercharge $\mathcal{Q}$ to a first superconformal descendant

$$\begin{aligned}
\mathcal{Q}^I_\beta \bar{\mathcal{Q}}_{\dot\alpha J}\left|\text{h.w.}\right\rangle &= \frac{1}{2}\left\{\mathcal{Q}^I_\beta, \bar{\mathcal{Q}}_{\dot\alpha J}\right\}\left|\text{h.w.}\right\rangle + \frac{1}{2}\left[\mathcal{Q}^I_\beta, \bar{\mathcal{Q}}_{\dot\alpha J}\right]\left|\text{h.w.}\right\rangle = \\
&= \frac{1}{4}\delta^I_J P_{\beta\dot\alpha}\left|\text{h.w.}\right\rangle + \frac{1}{2}\left[\mathcal{Q}^I_\beta, \bar{\mathcal{Q}}_{\dot\alpha J}\right]\left|\text{h.w.}\right\rangle,
\end{aligned} \tag{4.4}$$

so the application of a second "lowering" $\mathcal{Q}$ operator returns the linear combination of a conformal descendant and a superconformal descendant.

Eventually, the superconformal multiplet is completed attaching to each state the R-symmetry multiplet to which it belongs. In the appendix B the construction of the R-symmetry multiples is explained in detail and a few examples of R-symmetry multiplets are provided.

4.3 Shortening conditions

If a certain combination of $\mathcal{Q}$ and $\bar{\mathcal{Q}}$ supercharges annihilates the superconformal primary state, the superconformal multiplet is said to be *short*; if this does not happens, the superconformal multiplet contains all the possible $2^{4\mathcal{N}}$ superconformal descendants and it is said to be *long*. In the chapter 6 we will make use of some operators belonging to a $\mathcal{N} = 4$ superconformal multiplet named $\mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{(0\,2\,0)}$. However, it is possible to show that the superconformal representations of the $\mathcal{N} = 4$ theory can be decomposed into $\mathcal{N} = 2$ superconformal representations (cfr. [8], [11]). For this reason, we will discuss the $\mathcal{N} = 2$ shortening conditions in order to better understand the structure of the superconformal multiplet in which we are interested.

$\mathcal{N} = 2$ shortening conditions

In the following, the $\mathcal{N} = 2$ superconformal multiplets will be denoted as $\mathcal{X}_{R,r,(j,\bar{j})}$, where $R, r, j, \bar{j}$ stand respectively for the $su(2)$ Dynkin label, the R-charge and the Lorentz quantum numbers. There are three basic types of shortening (cfr. [8], [11]):

- $\mathcal{A}$ type: this is the class of the long superconformal multiplets, denoted with $\mathcal{A}_{R,r,(j,\bar{j})}$;
- $\mathcal{B}^I$ type: the highest weight state is annihilated by $\mathcal{Q}^I_1$ and $\mathcal{Q}^I_2$

$$\mathcal{Q}^I_1 \,|\text{h.w.}\rangle = \mathcal{Q}^I_2 \,|\text{h.w.}\rangle = 0. \tag{4.5}$$

  this shortening condition is only possible when $j = 0$. An analogous shortening condition is the $\bar{\mathcal{B}}_I$ type, where the highest weight state is annihilated by $\bar{\mathcal{Q}}^{\dot{1}}_I$ and $\bar{\mathcal{Q}}^{\dot{2}}_I$ and it is possible only if $\bar{j} = 0$. Both these shortening conditions can be split into the weaker shortening conditions $\mathcal{B}^1$, $\mathcal{B}^2$ and $\bar{\mathcal{B}}_1$, $\bar{\mathcal{B}}_2$;
- $\mathcal{C}^I$ type: the highest weight state is annihilated by

$$\varepsilon^{\alpha\beta}\mathcal{Q}^I_\beta \,|\text{h.w.}\rangle = 0. \tag{4.6}$$

  If $j = 0$, the condition is replaced by

$$\left(\mathcal{Q}^I_\beta\right)^2 |\text{h.w.}\rangle = 0. \tag{4.7}$$

  This condition can be split into two weaker conditions: $\mathcal{C}^1$ and $\mathcal{C}^2$. Similarly to the $\mathcal{B}$ type conditions, it is possible to construct two conjugated shortening conditions $\bar{\mathcal{C}}_1$ and $\bar{\mathcal{C}}_2$ which make use of the supercharges $\bar{\mathcal{Q}}$.

A superconformal multiplet can be subjected to more than one shortening condition.

The $\mathcal{N} = 2$ decomposition of the $\mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{(0\,2\,0)}$ multiplet

The $\mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{(0\,2\,0)}$ superconformal multiplet can be decomposed in a sum of $\mathcal{N} = 2$ superconformal multiplets as follows (cfr. [8], [11])

$$\mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{(0\,2\,0)} \simeq 3\hat{\mathcal{B}}_2 \oplus \mathcal{E}_{2(0,0)} \oplus \bar{\mathcal{E}}_{-2(0,0)} \oplus \hat{\mathcal{C}}_{0(0,0)} \oplus 2\left(\mathcal{D}_{1(0,0)} \oplus \bar{\mathcal{D}}_{1(0,0)}\right), \tag{4.8}$$

where:

- $\hat{\mathcal{B}}_2$: the superconformal primary state is a Lorentz scalar ($j = \bar{j} = 0$) sitting in the triplet representation of $su(2)$, with no R-charge and conformal dimension equal to 2. It is subject to the shortening condition $\mathcal{B}_1 \cap \bar{\mathcal{B}}_2$;

- $\mathcal{E}_{2(0,0)}$: the superconformal primary state is a Lorentz scalar ($j = \bar{j} = 0$) sitting in the singlet representation of $su(2)$, with R-charge and conformal dimension equal to 2. It is subject to the shortening condition $\mathcal{B}_1 \cap \mathcal{B}_2$; the superconformal multiplet $\bar{\mathcal{E}}_{-2(0,0)}$ is the multiplet generated by the hermitian conjugate of the superconformal primary of $\mathcal{E}_{2(0,0)}$;

- $\hat{\mathcal{C}}_{0(0,0)}$: the superconformal primary state is a Lorentz scalar ($j = \bar{j} = 0$) sitting in the singlet representation of $su(2)$, with no R-charge and conformal dimension equal to 2. It is subject to the shortening condition $\mathcal{C}_1 \cap \mathcal{C}_2 \cap \bar{\mathcal{C}}_1 \cap \bar{\mathcal{C}}_2$;

- $\mathcal{D}_{1(0,0)}$: the superconformal primary state is a Lorentz scalar ($j = \bar{j} = 0$) sitting in the fundamental representation of $su(2)$, with R-charge equal to 1 and conformal dimension equal to 2. It is subject to the shortening condition $\mathcal{B}_1 \cap \bar{\mathcal{C}}_2$;

- $\bar{\mathcal{D}}_{1(0,0)}$: the superconformal primary state is a Lorentz scalar ($j = \bar{j} = 0$) sitting in the fundamental representation of $su(2)$, with R-charge equal to -1 and conformal dimension equal to 2. It is subject to the shortening condition $\bar{\mathcal{B}}_2 \cap \mathcal{C}_1$.

The importance of this superconformal multiplet is given by the fact that it contains the operators employed in the proof of the non-renormalization theorem: for instance, it contains the stress-energy tensor, the supercurrents and the Lagrangian of the theory itself.

Chapter 5

Ward Identities

QFTs are usually equipped with symmetries and the physics they describe must be invariant under the action of the associated symmetry groups. At the classical level, if we consider a *continuous* symmetry group, the symmetry manifests itself as a *continuity equation* involving a conserved current J^μ, as prescribed by the Noether theorem. However, the theorem crucially relies on the on-shellness conditions, thus it cannot be employed at the quantum level, where the classical equations of motion are not valid. At the quantum level, the symmetries of the theory are properly described not by the continuity equations, but by the *Ward identities*. In this chapter we will study how to derive a generic Ward identity with the Path Integral formalism. If a symmetry is broken at the classical or at the quantum level, the Ward identity develops some *breaking terms* which change its structure. The analysis of the breaking terms will be crucial in the following.

5.1 General structure of a Ward identity

In this section we derive the general structure of a Ward identity, following the procedure exposed in [7]. In the Path Integral formalism, we consider a continuous symmetry group. The generic infinitesimal action of the group on the coordinates and on a classical field $\varphi(x)$ is given by

$$x'^\mu = x^\mu + \omega_a \delta_a x^\mu, \qquad \varphi'(x') = \varphi(x) + \omega_a \delta_a \varphi(x), \tag{5.1}$$

where ω_a represents a set of infinitesimal parameters. The index a runs from 1 to N, where N is the dimension of the symmetry group. Making use of the transformations (5.1), we can write down

$$\begin{aligned}
\varphi'(x') - \varphi(x) &= \omega_a \delta_a \varphi(x) \\
&= \varphi'(x') - \varphi(x') + \partial_\mu \varphi(x') \omega_a \delta_a x^\mu + O(\omega_a^2) \\
&= \delta\varphi(x') + \partial_\mu \varphi(x') \omega_a \delta_a x^\mu + O(\omega_a^2),
\end{aligned}$$

and we obtain

$$\delta\varphi(x') = \omega_a \delta_a \varphi(x) - \partial_\mu \varphi(x') \omega_a \delta_a x^\mu + O(\omega_a^2), \tag{5.2}$$

which, up to the second order in ω_a, becomes

$$\delta\varphi(x) = \omega_a \left(\delta_a \varphi(x) - \partial_\mu \varphi(x) \delta_a x^\mu \right). \tag{5.3}$$

We can now consider a generic correlation function

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle = \frac{1}{\mathcal{Z}} \int [\mathcal{D}\phi]\, \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) e^{-S[\phi]}, \tag{5.4}$$

where $\mathcal{Z}$ represents the partition function of the theory, $S[\varphi]$ the classical action and $\mathcal{O}_i(x_i)$ the i-th operator which appears inside the correlator. In the following, we won't keep track of the specific representations in which the operators $\mathcal{O}$ sit: their indices will be considered implicit. Inside the Path Integral we make a change of the functional integration variables

$$\varphi(x) \to \varphi'(x), \tag{5.5}$$

so that

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle = \frac{1}{\mathcal{Z}} \int \left[\mathcal{D}\varphi'\right] \mathcal{O}_1'(x_1) \dots \mathcal{O}_n'(x_n) e^{-S'[\varphi']}, \tag{5.6}$$

where the operators $\mathcal{O}_1, \dots, \mathcal{O}_n$ and the action $S[\varphi]$ are signed, too. Now, we assume that the functional measure of the Path Integral is invariant under the action of the symmetry group: without this hypothesis, the Ward identity develops an anomalous term (cfr. the articles [12] and the lectures [13])

$$\left[\mathcal{D}\varphi'\right] = \left[\mathcal{D}\varphi\right]. \tag{5.7}$$

The relation between the operators written in the old variables and the operators written in the new ones is given by the transformation (5.3)

$$\mathcal{O}'_i(x_i) - \mathcal{O}_i(x_i) = \delta\mathcal{O}_i(x_i) = \omega_a \left(\delta_a \mathcal{O}_i(x_i) - \partial_\mu \mathcal{O}_i(x) \delta_a x_i^\mu\right). \tag{5.8}$$

The action can also be interpreted as an object transformed by the continuous symmetry group

$$S'[\varphi'] = S[\varphi] + \delta S[\varphi] + O(\omega_a^2), \tag{5.9}$$

where the term δS is linear in the infinitesimal parameter ω_a. In the thesis, special care will be put in the computation of the correct $\delta S[\varphi]$. In order to keep a compact notation, we temporarily define

$$\mathbb{O} = \mathbb{O}(x_1, \dots, x_n) \equiv \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n). \tag{5.10}$$

The change of variables act on $\mathbb{O}$ as follows

$$\mathbb{O}' = \mathbb{O} + \delta\mathbb{O} + O(\omega_a^2), \tag{5.11}$$

where the compact notation $\delta\mathbb{O}$ stands for

$$\delta\mathbb{O} \equiv \sum_{i=1}^{n} \mathcal{O}_1(x_1) \dots \delta\mathcal{O}_i(x_i) \dots \mathcal{O}_n(x_n), \tag{5.12}$$

where the variation $\delta\mathcal{O}_i(x_i)$ has been specified in the equation (5.8). Expanding the integrand of the r.h.s. in the equation (5.6) in powers of ω_a we obtain

$$\begin{aligned} \langle \mathbb{O} \rangle &= \frac{1}{\mathcal{Z}} \int \left[\mathcal{D}\varphi\right] \left(\mathbb{O} + \delta\mathbb{O} + O(\omega_a^2)\right) \left(1 - \delta S[\varphi] + O(\omega_a^2)\right) e^{-S[\varphi]} \\ &= \langle \mathbb{O} \rangle + \langle \delta\mathbb{O} \rangle - \langle \mathbb{O}\delta S[\varphi] \rangle + O(\omega_a^2), \end{aligned}$$

so, at first order in ω_a, we get

$$\langle \delta S[\varphi] \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle = \sum_{i=1}^{n} \langle \mathcal{O}_1(x_1) \dots \delta\mathcal{O}_i(x_i) \dots \mathcal{O}_n(x_n) \rangle ., \tag{5.13}$$

which is a completely general result.

5.2 Ward identity of a classical symmetry

In this section we specialize the general result (5.13) to the case of a classical symmetry of the theory. When we apply the transformation rules (5.3) to the classical action $S[\varphi]$, we obtain (cfr. [7])

$$\delta S = -\int d^4x \; J^\mu_a \partial_\mu \omega_a = \int d^4x \; \partial_\mu J^\mu_a \omega_a, \tag{5.14}$$

where in the last equality we integrated by parts, assuming trivial boundary conditions. J^μ_a is a classically conserved current. In fact, at the classical level, we can impose the on-shellness condition $\delta S = 0$: if the field configurations obey the classical equations of motions, the action is stationary against any variation of the fields. This immediately leads to the continuity equation of the current J^μ

$$\delta S = 0 \rightarrow \partial_\mu J^\mu_a = 0. \tag{5.15}$$

At the quantum level, this is not allowed and the continuity equation holds at the *operatorial* level, encoded in a Ward identity. Plugging the infinitesimal variation of the action (5.14) in the general result (5.13), we obtain

$$\int d^4x \; \partial_\mu \left\langle J^\mu_a(x)\omega_a(x)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\right\rangle = \sum_{i=1}^n \left\langle \mathcal{O}_1(x_1)\dots\delta\mathcal{O}_i(x_i)\dots\mathcal{O}_n(x_n)\right\rangle . \tag{5.16}$$

We define $X_a \equiv \int d^3x \; J^0_a$, the classically conserved charge, and we rewrite

$$\delta\mathcal{O}_i(x_i) = i\omega_a(x_i)\left[X_a, \mathcal{O}_i\right\}(x_i), \tag{5.17}$$

where we adopted notation $[\mathcal{A}, \mathcal{B}\}$, which can stand for both $[\mathcal{A}, \mathcal{B}]$ and $\{\mathcal{A}, \mathcal{B}\}$, depending on the fermionic or bosonic nature of the operators $\mathcal{A}, \mathcal{B}$. The Ward identity becomes

$$\int d^4x \; \partial_\mu \left\langle J^\mu_a(x)\omega_a(x)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\right\rangle = i\sum_{i=1}^n \left\langle \mathcal{O}_1(x_1)\dots\omega_a(x_i)\left[X_a, \mathcal{O}_i\right\}(x_i)\dots\mathcal{O}_n(x_n)\right\rangle . \tag{5.18}$$

We introduce a set of δ distributions which localize the parameter $\omega_a(x)$ on the points $x_1, \dots, x_n$, so we can get rid of it

$$\int d^4x \; \partial_\mu \left\langle J^\mu_a(x)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\right\rangle = i\sum_{i=1}^n \int d^4x \; \delta(x-x_i)\left\langle \mathcal{O}_1(x_1)\dots\left[X_a, \mathcal{O}_i\right\}(x)\dots\mathcal{O}_n(x_n)\right\rangle . \tag{5.19}$$

Thanks to the property of the δ, we obtain the *unintegrated Ward identity*

$$\int d^4x \; \partial_\mu \left\langle J^\mu_a(x)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\right\rangle = i\sum_{i=1}^n \left\langle \mathcal{O}_1(x_1)\dots\left[X_a, \mathcal{O}_i\right\}(x_i)\dots\mathcal{O}_n(x_n)\right\rangle . \tag{5.20}$$

It is interesting to notice that the integral in the l.h.s. of the equation (5.20) might in principle diverge when $x \to x_i$, $i = 1, \dots, n$, but the divergence is cured by the *contact terms* in the right hand side. This can be explicitly verified considering the integral over a 4-dimensional volume $\mathcal{V}$ containing all the points $x_1, \dots, x_n$

$$\int_{\mathcal{V}} d^4x \; \partial_\mu \left\langle J^\mu_a(x)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\right\rangle . \tag{5.21}$$

The integrand diverges whenever $x \to x_i$, $i = 1, \dots, n$, thus we must regularize the integral cutting out from the integration domain n spheres $S_1, \dots, S_n$ with an infinitesimal radius of modulus equal to ϵ. Each sphere S_i is centered on the point x_i in which the operator $\mathcal{O}_i$ is computed. Now we turn the volume integral (5.21) into a surface integral

$$\int_{\partial\mathcal{V}} d\Sigma_\mu \partial_\mu \left\langle J_a^\mu(x)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\right\rangle = \\ = \int_{S_\infty} d\Sigma_\mu \left\langle J_a^\mu(x)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\right\rangle - \sum_{i=1}^{n}\int_{S_i} d\Sigma_\mu \left\langle J_a^\mu(x)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\right\rangle \quad (5.22)$$

We consider the integral over S_∞. If the current $J_a^\mu(x)$ has a conformal dimension Δ, then the integrand falls at least as fast as $\frac{1}{|x|^{2\Delta}}$ for $x \to \infty$. Then, if $\Delta > \frac{3}{2}$, on the 3-sphere S_∞ the integral goes to zero. We are left with

$$\int_{\partial\mathcal{V}} d\Sigma_\mu \left\langle J_a^\mu(x)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\right\rangle = -\sum_{i=1}^{n}\int_{S_i} d\Sigma_\mu \left\langle J_a^\mu(x)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\right\rangle. \quad (5.23)$$

Now we can consider the OPE between the current $J_a^\mu(x)$ and the operator $\mathcal{O}_i$

$$J_a^\mu(x)\mathcal{O}_i(x_i) = \dots - \frac{i}{2\pi^2}\frac{(x-x_i)^\mu}{|x-x_i|^4}\left[X_a,\mathcal{O}_i\right\}(x_i) + \dots, \quad (5.24)$$

where the kinematic factor fixes the conformal dimension and the indices of the right hand side. We plug this result in the r.h.s. of the equation (5.22)

$$-\sum_{i=1}^{n}\int_{S_i} d\Sigma_\mu \left\langle \mathcal{O}_1(x_1)\dots - \frac{i}{2\pi^2}\frac{(x-x_i)^\mu}{|x-x_i|^4}\left[X_a,\mathcal{O}_i\right\}(x_i)\dots\mathcal{O}_n(x_n)\right\rangle. \quad (5.25)$$

It is important to notice that, although the OPE gives birth to an infinite sum, the only term which contributes to the integral is

$$-\frac{i}{2\pi^2}\frac{(x-x_i)^\mu}{|x-x_i|^4}\left[X_a,\mathcal{O}_i\right\}(x_i). \quad (5.26)$$

In fact, a different term in the sum (5.24):

- might make the integral fall to zero in the limit $\epsilon \to 0$;
- might make the integral diverge: in this case, we adopt a *renormalization scheme* (cfr. [14], [15]) in which we simply subtract the divergences adding specific counterterms.

Applying the renormalization scheme, the right hand side of the equation (5.23) is properly defined. We define $r_i^\mu \equiv \left(x^\mu - x_i^\mu\right)$,

$$i\sum_{i=1}^{n}\lim_{r_i\to\epsilon}\left[r_i^2\frac{r_i^\mu r_{i,\mu}}{r_i^4}\left\langle \mathcal{O}_1(x_1)\dots\left[X_a,\mathcal{O}_i\right\}(x_i)\dots\mathcal{O}_n(x_n)\right\rangle\right]. \quad (5.27)$$

The final result does not depend on the regulator ϵ

$$i\sum_{i=1}^{n}\left\langle \mathcal{O}_1(x_1)\dots\left[X_a,\mathcal{O}_i\right\}(x_i)\dots\mathcal{O}_n(x_n)\right\rangle. \quad (5.28)$$

We obtained exactly the integral of the contact terms in the r.h.s. of the hard Ward identity (5.20).

5.2.1 Integrated Ward identities

If we are working with a CFT, the current J^μ_a is an operator which saturates the unitarity bound (3.47). Its first conformal descendant is an operator $\mathcal{D}_a$ defined as follows

$$\mathcal{D}_a = \left[P_\mu, J^\mu_a\right\} \sim \partial_\mu J^\mu_a. \tag{5.29}$$

If the unitarity bound is saturated, the operator $\mathcal{D}_a$ generates a null vector when it is applied to the vacuum state

$$\mathcal{D}_a \left|\text{vac}\right\rangle = \left|\mathcal{D}_a\right\rangle, \qquad \left\langle \mathcal{D}_a \,\middle|\, \mathcal{D}_a\right\rangle = 0. \tag{5.30}$$

The positivity condition of the Hilbert space imposes

$$\left|\mathcal{D}_a\right\rangle = \left|\partial_\mu J^\mu_a\right\rangle = \text{Null vector} \Rightarrow \left\langle \partial_\mu J^\mu_a\right| = \text{Null vector}. \tag{5.31}$$

If we work with a CFT, the l.h.s. of the unintegrated Ward identity (5.20) is equal to zero

$$\partial_\mu \left\langle J^\mu_a(x)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\right\rangle = \left\langle \partial_\mu J^\mu_a(x) \,\middle|\, \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\right\rangle = 0 \tag{5.32}$$

and we are left with the *integrated Ward identity*

$$\sum_{i=1}^{n} \left\langle \mathcal{O}_1(x_1)\dots\left[X_a, \mathcal{O}_i\right\}(x_i)\dots\mathcal{O}_n(x_n)\right\rangle = 0. \tag{5.33}$$

Integrated Ward identity for a global $u(1)$ symmetry group

We specialize the previous discussion for an abelian, continuous symmetry group $u(1)$. The action of the symmetry group on the fields is the following

$$\varphi'(x) = \varphi(x) - iq\omega\varphi(x), \tag{5.34}$$

where q represents the numerical charge of the field φ and ω is an infinitesimal parameter. The coordinates are not transformed because the symmetry is global. There is only one generator: the identity. q and ω have conformal dimensions equal to zero. The infinitesimal variation of the action is

$$\delta S = \int d^4x\, \partial_\mu J^\mu \omega. \tag{5.35}$$

We notice that the conformal dimension of the current is $[J^\mu] = 3$: if we work with a CFT, the unitarity bound is satisfied (cfr. the section 3.44). Then, the integrated Ward identity is, according to the equation (5.33)

$$\sum_{i=1}^{n} \left\langle \mathcal{O}_1(x_1)\dots q_i\mathcal{O}_i(x_i)\dots\mathcal{O}_n(x_n)\right\rangle = 0, \tag{5.36}$$

where q_i is the charge of the operator $\mathcal{O}_i$. The charge is a constant number, thus it can be extracted from the correlator

$$\sum_{i=1}^{n} q_i \left\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_i(x_i)\dots\mathcal{O}_n(x_n)\right\rangle = \left\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_i(x_i)\dots\mathcal{O}_n(x_n)\right\rangle \left(\sum_{i=1}^{n} q_i\right) = 0. \tag{5.37}$$

In conclusion, if the correlator is not equal to zero and it does not violate any other symmetry of the theory, we must have

$$\sum_{i=1}^{n} q_i = 0. \tag{5.38}$$

The constraint (5.38) encodes every global $u(1)$ classical symmetry transposed to the quantum theory: it forbids the correlators to be globally charged.

5.3 Broken Ward identities

In the previous section we derived the structure (5.20) of a general unintegrated Ward identity which encodes a classical symmetry at the quantum level. It is possible that, at the classical level, the action of a group is not an actual symmetry of the theory: however, the action of the group can still be encoded in a Ward identity, even if it won't represent a symmetry of the model. In this thesis we will have to deal with a specific kind of Ward identity: the *softly broken* Ward identities. A softly broken Ward identity is very similar to a Ward identity associated to a classical symmetry of the theory, but it presents additional (breaking) terms controlled by mass parameters (cfr. the section 2.5). In this section we analyze an example of soft breaking studying the scale invariance Ward identity.

5.3.1 The scale invariance Ward identity

We want to derive the scale invariance Ward identity. We consider a conformal invariant field theory in 4 dimensions, described by the classical action

$$S = \int d^4x \; \mathcal{L}\left[\varphi, \partial\varphi\right] \tag{5.39}$$

where the whole field content of the theory is represented by $\varphi(x)$. The action of the dilatation symmetry on the fields and on the coordinates is

$$x'^{\mu} = \lambda x^{\mu}, \tag{5.40}$$

$$\varphi'(x') = \lambda^{-\Delta_\varphi}\varphi(x). \tag{5.41}$$

We can consider $\lambda = 1 + \alpha$, where α is an infinitesimal parameter. Then, expanding up to the linear order in α and following the convention exposed in the equation (5.3), we obtain

$$\delta x^{\mu} = \alpha x^{\mu}, \tag{5.42}$$

$$\delta\varphi(x) = -\alpha\Delta_\varphi\varphi(x) - \alpha x^{\mu}\partial_\mu\varphi(x). \tag{5.43}$$

It is useful to define also the infinitesimal transformations without the infinitesimal parameter

$$\delta_\alpha x^{\mu} = x^{\mu}, \tag{5.44}$$

$$\delta_\alpha\varphi(x) = -\Delta_\varphi\varphi(x) - x^{\mu}\partial_\mu\varphi(x), \tag{5.45}$$

and the infinitesimal variation

$$\delta'_\alpha\phi = \frac{\partial}{\partial\alpha}\left[\varphi'(x') - \varphi(x)\right]_{\text{lin}} = -\Delta_\varphi\varphi(x). \tag{5.46}$$

In order to write down the Ward identity, we need to compute the infinitesimal variation of the classical action δS under the action of the scale invariance. The formula which returns our desired result is

$$\delta S = -\int d^4x \; W^{\mu}\partial_\mu\alpha = \int d^4x \; \partial_\mu W^{\mu}\alpha, \tag{5.47}$$

where the dilatation current W^{μ} is given by (cfr. [7])

$$W^{\mu} = \left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\partial_\nu\varphi - \delta^{\mu}_{\nu}\mathcal{L}\right]\delta_\alpha x^{\nu} - \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\delta'_\alpha\varphi = \left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\partial^{\nu}\varphi - \eta^{\mu\nu}\mathcal{L}\right]x_\nu + \Delta_\varphi\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\varphi. \tag{5.48}$$

In the r.h.s. we can substitute the definition of the *canonical stress-energy tensor*

$$T_c^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\partial^{\nu}\varphi - \eta^{\mu\nu}\mathcal{L}, \tag{5.49}$$

so the generic dilatation current is given by

$$W^{\mu} = x_{\nu} T_{c}^{\mu\nu} + \Delta_{\varphi} \frac{\partial \mathcal{L}}{\partial \partial_{\mu} \varphi} \varphi. \tag{5.50}$$

Thanks to the general result (5.20), we can immediately write down the unintegrated Ward identity encoding the classical symmetry at the quantum level

$$\int d^4x \; \alpha \partial_{\mu} \langle W^{\mu} \mathcal{O}_1 \dots \mathcal{O}_n \rangle = i\alpha \sum_{l=1}^{n} \langle \mathcal{O}_1 \dots [\mathcal{D}, \mathcal{O}_l\} \dots \mathcal{O}_n \rangle \,, \tag{5.51}$$

where $\mathcal{D}$ is dilatations generator. Its action on the generic operator $\mathcal{O}_l$ is depicted by the equations (5.17) and (5.43). If we compute the derivative in the l.h.s. of the Ward identity and we define

$$T^{\mu}{}_{\mu} \equiv T^{\mu}{}_{\mu,c} + \Delta_{\varphi} \partial_{\mu} \left(\frac{\partial \mathcal{L}}{\partial \partial_{\mu} \varphi} \varphi \right), \tag{5.52}$$

we obtain

$$\int d^4x \, \langle T^{\mu}{}_{\mu} \mathcal{O}_1 \dots \mathcal{O}_n \rangle + \int d^4x \, \langle x_{\nu} \partial_{\mu} T_{c}^{\mu\nu} \mathcal{O}_1 \dots \mathcal{O}_n \rangle =$$

$$= - \sum_{l=1}^{n} \left(\Delta_{\mathcal{O}_l} + x_l^{\mu} \frac{\partial}{\partial x_l^{\mu}} \right) \langle \mathcal{O}_1 \dots \mathcal{O}_l(x_l) \dots \mathcal{O}_n \rangle \,. \tag{5.53}$$

The structure of the Ward identity can be simplified employing the unintegrated translations Ward identity. The action of a translation on the coordinates and on the fields is

$$x'^{\mu} = x^{\mu} + \kappa^{\mu}, \tag{5.54}$$

$$\varphi'(x') = \varphi(x). \tag{5.55}$$

where κ^{μ} is an infinitesimal, constant vector. The infinitesimal versions are

$$\delta x^{\mu} = \kappa^{\mu}, \tag{5.56}$$

$$\delta \varphi(x) = -\kappa^{\mu} \partial_{\mu} \varphi(x). \tag{5.57}$$

The current associated to the translations is the canonical stress-energy tensor $T_{c}^{\mu\nu}$ and the unintegrated Ward identity is

$$\int d^4x \, \langle \partial_{\mu} T^{\mu}{}_{\nu\, c} \mathcal{O}_1 \dots \mathcal{O}_n \rangle = - \sum_{l=1}^{n} \frac{\partial}{\partial x_l^{\nu}} \langle \mathcal{O}_1 \dots \mathcal{O}_l(x_l) \dots \mathcal{O}_n \rangle \,. \tag{5.58}$$

Now we consider the expression

$$\int d^4x \left\{ \langle x_{\nu} \partial_{\mu} T_{c}^{\mu\nu} \mathcal{O}_1 \dots \mathcal{O}_n \rangle + \sum_{l=1}^{n} \left(x_l^{\mu} \frac{\partial}{\partial x_l^{\mu}} \right) \langle \mathcal{O}_1 \dots \mathcal{O}_l(x_l) \dots \mathcal{O}_n \rangle \right\}, \tag{5.59}$$

which can be easily modified as follows

$$\int d^4x \left\{ \langle x^{\nu} \partial_{\mu} T^{\mu}{}_{\nu,c} \mathcal{O}_1 \dots \mathcal{O}_n \rangle + \sum_{l=1}^{n} \delta(x - x_l) \left(x^{\nu} \frac{\partial}{\partial x^{\nu}} \right) \langle \mathcal{O}_1 \dots \mathcal{O}_l(x) \dots \mathcal{O}_n \rangle \right\}; \tag{5.60}$$

collecting an overall factor x^{ν} we get

$$\int d^4x \; x^{\nu} \left\{ \langle \partial_{\mu} T^{\mu}{}_{\nu,c} \mathcal{O}_1 \dots \mathcal{O}_n \rangle + \sum_{l=1}^{n} \delta(x - x_l) \left(\frac{\partial}{\partial x^{\nu}} \right) \langle \mathcal{O}_1 \dots \mathcal{O}_l(x) \dots \mathcal{O}_n \rangle \right\} = 0, \tag{5.61}$$

where the equality in the last step is possible thanks to the translations Ward identity (5.58). Plugging the result (5.61) in the unintegrated Ward identity (5.53), the final expression becomes

$$\int d^4x \left\langle T^{\mu}{}_{\mu} \mathcal{O}_1 \ldots \mathcal{O}_n \right\rangle = -\sum_{l=1}^{n} \Delta_{\mathcal{O}_l} \left\langle \mathcal{O}_1 \ldots \mathcal{O}_l(x_l) \ldots \mathcal{O}_n \right\rangle . \tag{5.62}$$

In conclusion, not only we studied an explicit encoding of a symmetry into an unintegrated Ward identity, but we also witnessed how a Ward identity can modify the structure of another Ward identity, if the symmetries that they describe hold at the same time.

5.3.2 Broken scale invariance Ward identity at zero temperature

In this section we study a soft breaking of the unintegrated Ward identity (5.62). We will consider a free scalar theory, to which we will add a mass operator. We will obtain an unintegrated Ward identity with a structure similar to (5.62), but with an additional, soft breaking term. We start with the classical action

$$S = -\int d^4x \left[\partial_\mu \bar{\phi} \partial^\mu \phi \right] . \tag{5.63}$$

We derive the canonical symmetric stress-energy tensor $T_c^{\mu\nu}$

$$T_c^{\mu\nu} = \eta^{\mu\nu} \partial_\mu \bar{\phi} \partial^\mu \phi - \partial^\mu \bar{\phi} \partial^\nu \phi - \partial^\nu \bar{\phi} \partial^\mu \phi. \tag{5.64}$$

The trace is

$$T^{\mu}{}_{\mu,c} = \eta_{\mu\nu} T^{\mu\nu} = 2\partial_\mu \bar{\phi} \partial^\mu \phi, \tag{5.65}$$

so, recalling the definition (5.52), we obtain

$$T^{\mu}{}_{\mu} = 2\partial_\mu \bar{\phi} \partial^\mu \phi + \partial_\mu \left(\partial^\mu \bar{\phi} \phi + \bar{\phi} \partial^\mu \phi \right) . \tag{5.66}$$

Looking at the equation (5.62), the scale invariance Ward identity is

$$\int d^4x \left\langle T^{\mu}{}_{\mu} \mathcal{O}_1 \ldots \mathcal{O}_n \right\rangle = -\sum_{l=1}^{n} \Delta_{\mathcal{O}_l} \left\langle \mathcal{O}_1 \ldots \mathcal{O}_l(x_l) \ldots \mathcal{O}_n \right\rangle \tag{5.67}$$

The equation (5.67) encodes the scale invariance of the action (5.63) at the quantum level. Now we introduce a mass operator and the action becomes

$$S_{\text{full}} = S + S_{\text{mass}} = -\int d^4x \left[\partial_\mu \bar{\phi} \partial^\mu \phi + m^2 \bar{\phi} \phi \right] . \tag{5.68}$$

Following the general result (5.13), we get

$$\begin{aligned} &\langle \delta S(x) \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle + \langle \delta S_{\text{mass}}(x) \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle = \\ &\qquad = -\alpha \sum_{l=1}^{n} \left(\Delta_{\mathcal{O}_l} + x_l^\mu \frac{\partial}{\partial x_l^\mu} \right) \langle \mathcal{O}_1 \ldots \mathcal{O}_l(x_l) \ldots \mathcal{O}_n \rangle \end{aligned} \tag{5.69}$$

and we already know that the first term can be rewritten as follows

$$\langle \delta S(x) \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle = \alpha \int d^4x \left\langle T^{\mu}{}_{\mu} \mathcal{O}_1 \ldots \mathcal{O}_n \right\rangle + \alpha \int d^4x \left\langle x_\nu \partial_\mu T_c^{\mu\nu} \mathcal{O}_1 \ldots \mathcal{O}_n \right\rangle , \tag{5.70}$$

where $T_c^{\mu\nu}$ and $T^\mu{}_\mu$ are given by the expressions (5.64) and (5.66). We explicitly compute the term δS_{mass} acting on the fields with the transformation (5.45)

$$\begin{aligned}\delta S_{\text{mass}} = \left(S'_{\text{mass}}[\phi'] - S_{\text{mass}}[\phi]\right)|_{\text{lin}} &= -m^2 \int d^4x \left[\bar{\phi}'(x)\phi'(x) - \bar{\phi}(x)\phi(x)\right]|_{\text{lin}} \\ &= -m^2 \int d^4x \big[\left(\bar{\phi}(x) + \delta\bar{\phi}(x)\right)\left(\phi(x) + \delta\phi(x)\right) - \\ &\qquad\qquad - \bar{\phi}(x)\phi(x)\big]|_{\text{lin}} \\ &= -m^2 \int d^4x \left[\bar{\phi}(x)\delta\phi(x) + \delta\bar{\phi}(x)\phi(x)\right] \\ &= \alpha m^2 \int d^4x \left[2\bar{\phi}(x)\phi(x) + x^\mu\partial_\mu\left(\bar{\phi}(x)\phi(x)\right)\right],\end{aligned}$$

so the complete unintegrated Ward identity becomes

$$\begin{aligned}&\int d^4x \left\langle T^\mu{}_\mu \mathcal{O}_1 \ldots \mathcal{O}_n \right\rangle + \int d^4x \left\langle x_\nu \partial_\mu T^{\mu\nu} \mathcal{O}_1 \ldots \mathcal{O}_n \right\rangle + \\ &\quad + 2m^2 \int d^4x \left\langle \bar{\phi}(x)\phi(x)\mathcal{O}_1 \ldots \mathcal{O}_n \right\rangle + m^2 \int d^4x \left\langle x^\mu\partial_\mu\left(\bar{\phi}(x)\phi(x)\right)\mathcal{O}_1 \ldots \mathcal{O}_n \right\rangle = \\ &\quad = -\sum_{l=1}^{n}\left(\Delta_{\mathcal{O}_l} + x_l^\mu \frac{\partial}{\partial x_l^\mu}\right)\left\langle \mathcal{O}_1 \ldots \mathcal{O}_l(x_l) \ldots \mathcal{O}_n \right\rangle.\end{aligned} \tag{5.71}$$

The final expression can be significantly simplified making use of the unintegrated translations Ward identity. The canonical stress-energy tensor $T_{\text{full}}^{\mu\nu}$ associated to the action S_{full} is

$$T_{\text{full}}^{\mu\nu} = T_c^{\mu\nu} + m^2\eta^{\mu\nu}\bar{\phi}\phi, \tag{5.72}$$

and the translations Ward identity leads to the identity

$$\begin{aligned}&\int d^4x\, x^\nu \left\langle \partial_\mu T^\mu{}_{\nu,c} \mathcal{O}_1 \ldots \mathcal{O}_n \right\rangle + m^2 \int d^4x\, x^\nu \left\langle \partial_\nu\left(\bar{\phi}(x)\phi(x)\right)\mathcal{O}_1 \ldots \mathcal{O}_n \right\rangle = \\ &\quad = -\sum_{l=1}^{n} x_l^\nu \frac{\partial}{\partial x_l^\nu}\left\langle \mathcal{O}_1 \ldots \mathcal{O}_l(x_l) \ldots \mathcal{O}_n \right\rangle.\end{aligned} \tag{5.73}$$

Keeping into account the identity (5.73), the final result of this section is

$$\begin{aligned}&\int d^4x \left\langle T^\mu{}_\mu \mathcal{O}_1 \ldots \mathcal{O}_n \right\rangle + 2m^2 \int d^4x \left\langle \bar{\phi}(x)\phi(x)\mathcal{O}_1 \ldots \mathcal{O}_n \right\rangle = \\ &\quad = -\sum_{l=1}^{n} \Delta_{\mathcal{O}_l}\left\langle \mathcal{O}_1 \ldots \mathcal{O}_l(x_l) \ldots \mathcal{O}_n \right\rangle.\end{aligned} \tag{5.74}$$

The new Ward identity is similar to (5.67), but it is softly broken by the term

$$2m^2 \int d^4x \left\langle \bar{\phi}(x)\phi(x)\mathcal{O}_1 \ldots \mathcal{O}_n \right\rangle. \tag{5.75}$$

Chapter 6

A Non-Renormalization Theorem at $T = 0$

In this chapter we review the proof proposed in the article [5] of a non-renormalization theorem in 4 dimensions. The theorem is set in the framework of the $\mathcal{N} = 4$ maximally supersymmetric theory and it shows a property of a specific kind of *3-points functions*, constructed with the field content of the $\mathcal{N} = 4$ theory. We already know that, in a generic CFT, the 3-points functions are completely fixed up to a multiplicative factor (cfr. section 3.3.2). In principle, the multiplicative factor might depend on the couplings of the CFT, so it should be the subject of a change if the coupling is modified (for instance, as a consequence of a renormalization process): it is said to be *dynamical*. However, if we consider the correlation functions between three superconformal scalar primaries belonging to chiral superconformal multiplets of the $\mathcal{N} = 4$ theory, the non-renormalization theorem states that the multiplicative factor does not depend on the coupling of the theory.

6.1 Marginal operators in the $\mathcal{N} = 4$ theory

For every superconformal theory it is possible to define a *conformal manifold*, which is the continuous set of all the possible values that the couplings of the theory can assume. A trajectory over the conformal manifold is described by a deformation of the theory, induced by a *marginal* operator. If we are studying the $\mathcal{N} = 4$ theory in 4 dimensions, the marginal operators sit on a specific superconformal representation: any marginal operator must have, in fact, all the properties of a Lagrangian density. Hence, a marginal operator must sit in the scalar representation of the Lorentz group, in the singlet representation of the R-symmetry group, it must be chargeless under the action of any $u(1)$ global symmetry group (included the $u(1)$ generated by the R-symmetry charge operator) and its conformal dimension must be equal to 4. In 4 dimensions, the full set of the representations in which a generic operator in the $\mathcal{N} = 4$ theory sits is represented by the notation

$$[j, \bar{j}]_{\Delta}^{(a\ b\ c)} \tag{6.1}$$

where we recognize the structure already studied in the section 3.2.3. Given that the notation (6.1) denotes a superconformal representation, we added the Dynkin labels of the $su(4)$ R-symmetry representation (cfr. the appendix B for a basic introduction to the $su(N)$ representations and the meaning of the Dynkin labels). The representation of a marginal operator in the $\mathcal{N} = 4$ theory is condensed in the notation

$$[0, 0]_4^{(0\ 0\ 0)}\ . \tag{6.2}$$

$$
\begin{tikzcd}[column sep=tiny]
 & & & {[0,0]_2^{(0\,2\,0)}} \arrow[dl] \arrow[dr] & & & \\
 & & {[1,0]_{\frac{5}{2}}^{(011)}} \arrow[dl] \arrow[dr] & & {[0,1]_{\frac{5}{2}}^{(1\,1\,0)}} \arrow[dl] \arrow[dr] & & \\
 & \begin{array}{c}[2,0]_3^{(0\,1\,0)}\\ {[0,0]_3^{(0\,0\,2)}}\end{array} \arrow[dl] \arrow[dr] & & {[1,1]_3^{(1\,0\,1)}} \arrow[dl] \arrow[dr] & & \begin{array}{c}[0,2]_3^{(0\,1\,0)}\\ {[0,0]_3^{(2\,0\,0)}}\end{array} \arrow[dl] \arrow[dr] & \\
{[1,0]_{\frac{7}{2}}^{(0\,0\,1)}} \arrow[d] & & {[2,1]_{\frac{7}{2}}^{(1\,0\,0)}} \arrow[dr] & & {[1,2]_{\frac{7}{2}}^{(0\,0\,1)}} \arrow[dl] & & {[0,1]_{\frac{7}{2}}^{(1\,0\,0)}} \arrow[d] \\
{[0,0]_4^{(0\,0\,0)}} & & & \begin{array}{c}[2,2]_4^{(0\,0\,0)}\\ -[0,0]_4^{(1\,0\,1)}\end{array} \arrow[dl] \arrow[dr] & & & {[0,0]_4^{(0\,0\,0)}} \\
 & & -[1,0]_{\frac{9}{2}}^{(1\,0\,0)} \arrow[dr] & & -[0,1]_{\frac{9}{2}}^{(0\,0\,1)} \arrow[dl] & & \\
 & & & -[1,1]_5^{(0\,0\,0)} & & &
\end{tikzcd}
$$

Figure 6.1: The complete superconformal short multiplet $\mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{(0\,2\,0)}$ (cfr. the article [11]).

In a superconformal theory, we know that the marginal operators must be superconformal descendants of superconformal primaries. In particular, marginal operators belonging to the $\mathcal{N} = 4$ theory appear as superconformal descendants in the superconformal short multiplet $\mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{(0\,2\,0)}$, already examined in the section 4.3. The $\mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{(0\,2\,0)}$ superconformal multiplet is characterized by the following properties:

- let's consider the following basis for the set of the $\mathcal{N} = 4$ supercharges (cfr. the section B.3.3),
$$\mathcal{Q}^1_\alpha,\, \mathcal{Q}^2_\alpha,\, \mathcal{Q}^3_\alpha,\, \mathcal{Q}^4_\alpha;\, \bar{\mathcal{Q}}^{1,\dot{\alpha}},\, \bar{\mathcal{Q}}^{2,\dot{\alpha}},\, \bar{\mathcal{Q}}^{3,\dot{\alpha}},\, \bar{\mathcal{Q}}^{4,\dot{\alpha}} : \tag{6.3}$$
then, the shortening conditions for the $\mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{(0\,2\,0)}$ superconformal multiplet are
$$\mathcal{Q}^1_\alpha\,|\text{h.w.}\rangle = 0, \qquad \mathcal{Q}^2_\alpha\,|\text{h.w.}\rangle = 0, \qquad \bar{\mathcal{Q}}^{3,\dot{\alpha}}\,|\text{h.w.}\rangle = 0, \qquad \bar{\mathcal{Q}}^{4,\dot{\alpha}}\,|\text{h.w.}\rangle = 0, \tag{6.4}$$
where the notation $|\text{h.w.}\rangle$ stands for the highest weight state of the superconformal multiplet;

- the superconformal primary sits in the representation
$$[0,0]_2^{(0\,2\,0)}\,. \tag{6.5}$$
The $su(4)$ representation is codified via the three Dynkin labels $(0\;2\;0)$. The superconformal primary is associated to the highest weight state of the representation $|\text{h.w.}\rangle$ via the state-operator correspondence.

The complete superconformal multiplet $\mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{(0\,2\,0)}$ is pictured in the figure 6.1. The representation $[0,0]_2^{0\,2\,0}$ hosts the superconformal primary. It is important to notice that the two marginal operators, associated to the two $[0,0]_4^{(0\,0\,0)}$ representations,are generated by the same superconformal primary.

Now we construct the operator which sits the $[0,0]_2^{(0\,2\,0)}$ representation. The conformal dimension of the operator forbids us to use fermionic fields, which have conformal dimensions equal to $3/2$; the Lorentz representation forbids us to mix vector and scalar fields, so the non-trivial R-symmetry representation forces us to construct the operator with only the scalar fields. The superconformal primary is realized as the product of two $\mathcal{N}=4$ scalar fields X^{IJ} and X^{KL}. The symmetry properties of the indices possessed by the operator we are constructing can be visualized with the Young tableaux formalism

$$\begin{array}{|c|}\hline I\\\hline J\\\hline\end{array} \times \begin{array}{|c|}\hline K\\\hline L\\\hline\end{array} = \mathbf{1} + \begin{array}{|c|c|}\hline I & K\\\hline J & L\\\hline\end{array} + \begin{array}{|c|c|}\hline I & L\\\hline J \\\cline{1-1} K \\\cline{1-1}\end{array} + \begin{array}{|c|c|}\hline I & K\\\hline J \\\cline{1-1} L \\\cline{1-1}\end{array} . \tag{6.6}$$

The Dynkin labels $(0\ 2\ 0)$ are associated to the tableau $\begin{array}{|c|c|}\hline I & K\\\hline K & L\\\hline\end{array}$. Thus, the superconformal primary operator is

$$\mathrm{tr}\left[X^{IJ}X^{KL}\right], \tag{6.7}$$

where I,J and K,L are antisymmetric pairs of indices, I,K and J,L are symmetric pairs of indices and the trace is taken over the gauge indices, in order to make the operator gauge invariant.

6.2 Chiral primaries 3-points functions

As we mentioned at the beginning of this chapter, the central objects of the proof are the 3-points functions

$$\left\langle \phi_1^{(\mathcal{R}_1,\vec{m}_1)}(x_1)\,\phi_2^{(\mathcal{R}_2,\vec{m}_2)}(x_2)\,\phi_3^{(\mathcal{R}_3,\vec{m}_3)}(x_3)\right\rangle . \tag{6.8}$$

The operator $\phi^{(\mathcal{R},\vec{m})}(x)$ is the chiral primary operator of a short superconformal multiplet and it is annihilated by the supercharges $\mathcal{S}_I^{\alpha}$ and $\bar{\mathcal{S}}_{\dot{\alpha}}^{J}$. The shortening conditions are the ones listed in (6.4). The property of being the superconformal primary of a short multiplet is fundamental for the proof of the non-renormalization theorem: the conformal dimension of the operator is protected by a unitarity condition and does not evolve along any trajectory over the conformal manifold (cfr. the articles [5], [4]). The operators appearing in the 3-points function (6.8) sit in a representation with the structure

$$[0,0]_k^{(0\,k\,0)}, \tag{6.9}$$

where $(0\ k\ 0)$ are the Dynkin labels of the highest weight state. The three primaries inside the correlation function (6.8) are associated to three different (or even equal) integers k_1, k_2, k_3. Each primary is then labeled by a vector $\vec{m}$, which represents its weight in the $su(4)$ representation where it sits.

The conformal symmetry of the $\mathcal{N}=4$ theory fixes the explicit expression of the correlation function (6.8)

$$\langle \phi_1\,\phi_2\,\phi_3\rangle = C_{123}\frac{\mathcal{G}(k_1,k_2,k_3,\vec{m}_1,\vec{m}_2,\vec{m}_3)}{|x_{12}|^{k_1+k_2-k_3}|x_{23}|^{k_2+k_3-k_1}|x_{13}|^{k_1+k_3-k_2}}, \tag{6.10}$$

where the denominator encodes the coordinate dependence of the correlation function, $\mathcal{G}$ is a group-theoretical factor which depends only on the R-symmetry representations and C_{123} is the multiplicative factor of the 3-points function. Only C_{123} can carry a coupling dependence. The group-theoretical factor $\mathcal{G}$ can be fixed via a specific choice of the representations (fixing k_1,k_2,k_3) and of the weights (fixing $\vec{m}_1,\vec{m}_2,\vec{m}_3$). We will choose

$$\vec{m}_1=\vec{m}, \qquad \vec{m}_2 = \text{highest weight} \equiv +, \qquad \vec{m}_3 = \text{lowest weight} \equiv -. \tag{6.11}$$

In conclusion, we notice that in this notation the superconformal primary (6.7) is written

$$\phi^{(2,+)}. \tag{6.12}$$

6.3 Marginal Operators Deformations

In general, a QFT depends on some coupling constants. If these coupling constants are adimensional, they are associated to marginal operators. Changing the value of a given coupling constant does not modify the structure of the theory, but it can deeply modify its physics. A weak coupling allows us to compute the correlation functions via perturbative approaches, while a strong coupling often requires non-perturbative techniques. We already mentioned that each point on the conformal manifold is associated to a different coupling and that a marginal operator can be the generator of a flow over the conformal manifold. We provide the reader with a heuristical motivation of this fact: let's assume that the marginal operator $\mathcal{O}_g$ appears in the classical action

$$S = \int d^4x \left[\cdots + g\mathcal{O}_g\right]. \tag{6.13}$$

Then we consider a generic correlation function

$$\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle = \frac{1}{\mathcal{Z}}\int [\mathcal{D}\phi]\, \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)e^{-\int d^4x[\cdots+g\mathcal{O}_g]} \tag{6.14}$$

and we can apply a derivative to both sides of the equation

$$\begin{aligned}\frac{\partial}{\partial g}\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle &\sim \frac{1}{\mathcal{Z}}\int [\mathcal{D}\phi]\left(\int d^4x\, \mathcal{O}_g(x)\right)\mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)e^{-\int d^4x[\cdots+g\mathcal{O}_g]}\\ &\sim \int d^4x\, \langle \mathcal{O}_g(x)\mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle\,.\end{aligned}$$

Heuristically, the dependence of a correlation function on one marginal coupling can be studied computing a volume integral of another correlation function, equal to the original apart from the presence of the marginal operator $\mathcal{O}_g$ associated to the coupling g. However, in general the operators $\mathcal{O}_1,\ldots,\mathcal{O}_n$ are sections of bundles defined over the conformal manifold. Hence, the correct way to describe the dependence on the marginal coupling g is to define a connection ∇_g over the conformal manifold (cfr. the articles [16], [14]). The flow then is described by the equation

$$\nabla_g \langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle = \int d^4x\, \langle \mathcal{O}_g(x)\mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle\,. \tag{6.15}$$

We can consider the equation (6.15) as the *definition* of the differential operator ∇_g. In the $\mathcal{N} = 4$ theory we have only one coupling, introduced in the section 2.3

$$\tau = \frac{1}{g^2} - i\frac{\Theta}{8\pi^2}, \tag{6.16}$$

so there is only one flow equation which needs to be considered

$$\nabla_\tau \langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle = \int d^4x\, \langle \mathcal{O}_\tau(x)\mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle\,. \tag{6.17}$$

Notice that, looking at the equations (2.74) and (2.80), the marginal operators $\mathcal{O}_\tau$ can be identified as the Lagrangian density of the $\mathcal{N} = 4$ theory.

6.4 The Superconformal Ward Identity

The most important computational tool we will need in this chapter is the superconformal Ward identity. In particular, if we are studying a SCFT, the superconformal Ward identity encodes the superconformal symmetry at the quantum level. In this section we will derive the identity following the article [14]. The superconformal symmetry is associated to the conservation of the fermionic supercharges $\mathcal{Q}^I_\alpha$, $\bar{\mathcal{Q}}^{\dot\beta}_J$, $\mathcal{S}^\alpha_I$ and $\bar{\mathcal{S}}^J_{\dot\beta}$, which can be derived from the supercurrents $G^I_{\mu\alpha}$, $\bar{G}^{\dot\beta}_{\mu J}$. The supercurrents appear in the superconformal multiplet $\mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{(0\,2\,0)}$: their representations are $[2,1]^{(1\,0\,0)}_{\frac{7}{2}}$ and $[1,2]^{(0\,0\,1)}_{\frac{7}{2}}$ in the picture 6.1. The supercurrent (and its hermitian conjugate) enjoys two important properties:

- it fulfills the unitarity bound for the operators sitting in the representation $[2,1]$ of the Lorentz group, hence it is annihilated by the momentum operator

$$\left[P_\mu, G^{I\mu}_\alpha\right] \sim \partial_\mu G^{I,\mu}_\alpha = 0; \tag{6.18}$$

- in the superconformal multiplet $\mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{(0\,2\,0)}$ one place is occupied by the stress-energy tensor $T^{\mu\nu}$: in the picture 6.1 it is associated to the representation $[2,2]^{(0\,0\,0)}_4$. In fact, the stress-energy tensor sits in the singlet representation of $su(4)$ (it does not have any R-symmetry index) and in the $[2,2]$ representation of the Lorentz group. We immediately notice that it is a superconformal descendant of the supercurrent: up to some multiplicative factors we have

$$\left\{\bar{\mathcal{Q}}^{\dot\beta}_J, G^{\mu I}_\alpha\right\} \sim T^{\mu\nu}\bar{\sigma}^{\dot\beta\beta}_\nu \varepsilon_{\beta\alpha}\delta^I_J. \tag{6.19}$$

  We apply a σ_μ matrix to both sides of the equation

$$\left\{\bar{\mathcal{Q}}^{\dot\beta}_J, G^{\mu I}_\alpha\right\}\varepsilon^{\alpha\gamma}\sigma_{\mu\gamma\dot\gamma} \sim \delta^I_J T^{\mu\nu}\bar{\sigma}^{\dot\beta\beta}_\nu \varepsilon_{\beta\alpha}\varepsilon^{\alpha\gamma}\sigma_{\mu\gamma\dot\gamma} = \delta^I_J T^{\mu\nu}\bar{\sigma}^{\dot\beta\beta}_\nu \sigma_{\mu\beta\dot\gamma}. \tag{6.20}$$

  We take the trace over the free spinorial indices $\dot\beta$ and $\dot\gamma$

$$\left\{\bar{\mathcal{Q}}^{\dot\gamma}_J, G^{\mu\gamma I}\right\}\sigma_{\mu\gamma\dot\gamma} \sim \delta^I_J T^{\mu\nu}\bar{\sigma}^{\dot\gamma\beta}_\nu \sigma_{\mu\beta\dot\gamma} = \delta^I_J T^{\mu\nu} g_{\mu\nu} = \delta^I_J T^\mu{}_\mu = 0, \tag{6.21}$$

  where the last equality holds at the *operatorial* level in a CFT. We have

$$\bar{\mathcal{Q}}^{\dot\gamma}_J G^{\mu\gamma I}\sigma_{\mu\gamma\dot\gamma} + G^{\mu\gamma I}\sigma_{\mu\gamma\dot\gamma}\bar{\mathcal{Q}}^{\dot\gamma}_J = 0, \tag{6.22}$$

  which is satisfied if

$$G^{\mu\gamma I}\sigma_{\mu\gamma\dot\gamma} = 0. \tag{6.23}$$

We have introduced the supercurrents and their properties. Now we are interested in building a vector current out of them. The result can be easily obtained by contracting the spinorial index of the supercurrent with the index of a spinor ψ^α, which might depend on the coordinates

$$J^{\mu I}(x) \equiv \psi^\alpha(x) G^{\mu I}_\alpha(x). \tag{6.24}$$

We impose the *operatorial* identity

$$\partial_\mu J^{\mu I} = 0, \tag{6.25}$$

which is satisfied if

$$\left(\partial_\mu \psi^\alpha\right) G^{\mu I}_\alpha + \psi^\alpha \partial_\mu G^{\mu I}_\alpha = 0. \tag{6.26}$$

Applying the first property of the supercurrents, we realize that we just need to search for a class of spinors satisfying the constraint

$$\left(\partial_\mu \psi^\alpha\right) G^{\mu I}_\alpha = 0. \tag{6.27}$$

We have two possibilities:

- $\partial_\mu \psi^\alpha(x) = 0$: hence
$$\psi^\alpha(x) = \lambda^\alpha, \tag{6.28}$$
where λ^α is a constant spinor;

- $\partial_\mu \psi^\alpha(x) = \bar{\mu}_{\dot{\beta}} \bar{\sigma}_\mu^{\dot{\beta}\alpha}$: hence
$$\psi^\alpha(x) = \bar{\mu}_{\dot{\beta}} \bar{x}^{\dot{\beta}\alpha}, \tag{6.29}$$
where $\bar{\mu}_{\dot{\beta}}$ is a constant spinor and $\bar{x}^{\dot{\beta}\alpha} \equiv x^\mu \bar{\sigma}_\mu^{\dot{\beta}\alpha}$.

It is important to notice that the spinors λ^α and $\bar{\mu}_{\dot{\beta}}$ are not dynamical degrees of freedom of the theory, so they are not required to have a specific conformal dimension (which would be fixed by their kinetic sectors). The most general choice is given by the sum of the two contributions listed above
$$\psi^\alpha(x) = \lambda^\alpha + \bar{\mu}_{\dot{\beta}} \bar{x}^{\dot{\beta}\alpha}. \tag{6.30}$$
The spinor (6.30) is called *conformal Killing spinor*. The two distinct contributions to the conformal Killing spinor are associated to the conserved supercharges that the current $J^{\mu I}$ is able to generate
$$\mathcal{Q}^I_\alpha = \int d^3x \; G^{0I}_\alpha, \qquad \bar{\mathcal{S}}^{\dot{\beta} I} = \int d^3x \; \bar{x}^{\dot{\beta}\alpha} G^{0I}_\alpha. \tag{6.31}$$
We have all the tools required to explicitly compute the integrated Ward identity associated to the superconformal symmetry. Following the article [14], we ca start with the general structure of the unintegrated Ward identity

$$\int d^4x \left\{ \partial_\mu \left\langle J^{\mu I}(x) \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \right\rangle + [\text{Contact terms}] \right\} \rho_I = 0, \tag{6.32}$$

where ρ_I is an infinitesimal parameter carrying an $su(4)$ index, needed to contract the $su(4)$ index of the current $J^{\mu I}$. The integral of the 4-divergence diverges whenever x gets near to one of the points $x_1, \dots, x_n$: the [Contact terms] are responsible for the regularization of these divergences. We know that the identity (6.25) holds: plugging it in the equation (6.32) we obtain the integrated Ward identity

$$[\text{Contact terms}] = 0. \tag{6.33}$$

Comparing the equation (6.33) with the equation (5.37), we realize that

$$[\text{Contact terms}] = \sum_{i=1}^{n} \langle \mathcal{O}_1(x_1) \dots [X_a, \mathcal{O}_i\} \, (x_i) \dots \mathcal{O}_n(x_n) \rangle \,, \tag{6.34}$$

so this procedure for deriving the Ward identity is equivalent to the one adopted in the chapter 5. We want to explicitly compute the [Contact terms]. In order to do so, we need to understand the behavior of the integrand when it is computed near the points $x_1, \dots, x_n$. We already know how to decompose the volume integral in two surface integrals: the first one must be computed on the boundary surface of the integration domain; the second one is a sum of surface integrals, each one over a small sphere around each point $x_1, \dots, x_n$. The first contribution is equal to zero: in fact, the boundary of the integration domain is the 3-dimensional sphere at infinity. At infinity, the current $J^{\mu I}$ goes like $J^{\mu I} \sim r G^{\mu I} \sim \frac{r^\mu}{r^{7/2}} \sim \frac{1}{r^{5/2}}$, because the conformal dimension of the supercurrent is $\left[G^{\mu I}_\alpha \right] = \frac{7}{2}$. Thus, the integrand goes to zero at least as fast as $\frac{1}{r^5}$ and

$$\int_{S_\infty} d\Sigma_\mu \left\langle J^{\mu I}(x) \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \right\rangle \rho_I \sim \lim_{r \to \infty} 2\pi^2 r^3 \frac{1}{r^5} = 0. \tag{6.35}$$

This allows us to write

$$\int d^4x\, \partial_\mu \left\langle J^{\mu I}(x)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\right\rangle \rho_I = \\ = -\sum_{l=1}^{n}\int_{S_l} d\Sigma_\mu \left\langle J^{\mu I}(x)\mathcal{O}_1(x_1)\dots\mathcal{O}_l(x_l)\dots\mathcal{O}_n(x_n)\right\rangle \rho_I. \tag{6.36}$$

We can make use of the supercharges (6.31) to write the OPE between the supercurrent $G^{\mu I}_\alpha$ and a generic operator $\mathcal{O}_l$

$$G^{\mu I}_\alpha(x)\mathcal{O}_l(x_l) = \dots - \frac{i}{2\pi^2}\frac{(x-x_l)^\mu}{|x-x_l|^4}\left[\mathcal{Q}^I_\alpha, \mathcal{O}_l\right\}(x_l)+ \\ -\frac{i}{2\pi^2}\frac{(x-x_l)^\mu}{|x-x_l|^6}(x-x_l)_{\alpha\dot\beta}\left[\bar{\mathcal{S}}^{\dot\beta I}, \mathcal{O}_l\right\}(x_l)+\dots, \tag{6.37}$$

where the coefficients dependent on the coordinates adjust of the conformal dimensions and of the free indices of the objects involved. We use the expression above to find the OPE between a generic operator $\mathcal{O}_l$ and the current $J^{\mu I}$

$$J^{\mu I}(x)\mathcal{O}_l(x_l) = \psi^\alpha(x)G^{\mu I}_\alpha(x)\mathcal{O}_l(x_l). \tag{6.38}$$

Near the puncture x_l the conformal Killing spinor can be written as

$$\psi^\alpha(x) = \psi^\alpha(x_l) + \left(\partial_\mu\psi^\alpha\right)(x_l)\,(x-x_l)^\mu + O\left((x-x_l)^2\right) \tag{6.39}$$

If we redefine $(x-x_l)\equiv r_l$, the conformal Killing spinor can be written as

$$\psi^\alpha(x) = \psi^\alpha(x_l) + r_l^\mu\left(\partial_\mu\psi^\alpha\right)(x_l) + O\left(r_l^2\right). \tag{6.40}$$

We can put together the equations (6.37) and (6.40) to obtain

$$J^{\mu I}(x)\mathcal{O}_l(x_l) = \dots - \psi^\alpha(x_l)\frac{i}{2\pi^2}\frac{r_l^\mu}{r_l^4}\left[\mathcal{Q}^I_\alpha, \mathcal{O}_l\right\}(x_l)+ \\ -\left(\partial_\nu\psi^\alpha\right)(x_l)\frac{i}{2\pi^2}\frac{r_l^\nu r_l^\mu}{r_l^6}r_{l\alpha\dot\beta}\left[\bar{\mathcal{S}}^{\dot\beta I}, \mathcal{O}_l\right\}(x_l) + O\left(\frac{1}{r_l^2}\right)+\dots \tag{6.41}$$

In order to compute each term of the sum in (6.36), we consider

$$\int_{S_l} d\Sigma_\mu \left\langle \mathcal{O}_1(x_1)\dots J^{\mu I}(x)\mathcal{O}_l(x_l)\dots\mathcal{O}_n(x_n)\right\rangle \rho_I \tag{6.42}$$

and we plug the infinitesimal surface element and the OPE (6.41)

$$\lim_{r_l\to 0} 2\pi^2 r_l^2 r_{l\mu}\Big\langle \mathcal{O}_1(x_1)\dots - \psi^\alpha(x_l)\frac{i}{2\pi^2}\frac{r_l^\mu}{r_l^4}\left[\mathcal{Q}^I_\alpha, \mathcal{O}_l\right\}(x_l)+ \\ -\left(\partial_\nu\psi^\alpha\right)(x_l)\frac{i}{2\pi^2}\frac{r_l^\nu r_l^\mu}{r_l^6}r_{l\alpha\dot\beta}\left[\bar{\mathcal{S}}^{\dot\beta I}, \mathcal{O}_l\right\}(x_l) + O\left(\frac{1}{r_l^2}\right)\dots\mathcal{O}_n(x_n)\Big\rangle. \tag{6.43}$$

We apply the renormalization scheme mentioned in the section 5.1 and we get rid of all the divergences, so the final result is

$$-i\psi^\alpha(x_l)\left\langle \mathcal{O}_1(x_1)\dots\left[\mathcal{Q}^I_\alpha, \mathcal{O}_l\right\}(x_l)\dots\mathcal{O}_n(x_n)\right\rangle + \\ -i\left(\partial^\mu\psi^\alpha\right)(x_l)\left\langle \mathcal{O}_1(x_1)\dots\lim_{r_l\to 0}\left[\frac{r_{l\mu}r_{l\nu}}{r_l^2}\right]\sigma^\nu_{\alpha\dot\beta}\left[\bar{\mathcal{S}}^{\dot\beta I}, \mathcal{O}_l\right\}(x_l)\dots\mathcal{O}_n(x_n)\right\rangle. \tag{6.44}$$

We introduce a vector with unitary norm n_μ suche that $r^\mu = rn^\mu$ and we obtain an explicit expression for every term in the sum (6.36)

$$\begin{aligned} &- i\psi^\alpha(x_l)\left\langle \mathcal{O}_1(x_1)\dots\left[\mathcal{Q}^I_\alpha,\mathcal{O}_l\right\}(x_l)\dots\mathcal{O}_n(x_n)\right\rangle + \\ &\qquad - i\left(\partial^\mu\psi^\alpha\right)(x_l)n_\mu n_\nu\sigma^\nu_{\alpha\dot\beta}\left\langle \mathcal{O}_1(x_1)\dots\left[\bar{\mathcal{S}}^{\dot\beta I},\mathcal{O}_l\right\}(x_l)\dots\mathcal{O}_n(x_n)\right\rangle . \end{aligned} \tag{6.45}$$

The integrated superconformal Ward identity is

$$\begin{aligned} &\sum_{l=1}^n \psi^\alpha(x_l)\left\langle \mathcal{O}_1(x_1)\dots\left[\mathcal{Q}^I_\alpha,\mathcal{O}_l\right\}(x_l)\dots\mathcal{O}_n(x_n)\right\rangle + \\ &\qquad + \left(\partial^\mu\psi^\alpha\right)(x_l)n_\mu n_\nu\sigma^\nu_{\alpha\dot\beta}\left\langle \mathcal{O}_1(x_1)\dots\left[\bar{\mathcal{S}}^{\dot\beta I},\mathcal{O}_l\right\}(x_l)\dots\mathcal{O}_n(x_n)\right\rangle = 0. \end{aligned} \tag{6.46}$$

This fundamental tool depends on two sets of degrees of freedom: the spinor λ^α and the spinor $\bar\mu^\beta$:

- if we set $\bar\mu^{\dot\beta} = 0$, we simply end up with the integrated Ward identity associated to supersymmetry

$$\sum_{l=1}^n \left\langle \mathcal{O}_1(x_1)\dots\left[\mathcal{Q}^I_\alpha,\mathcal{O}_l\right\}(x_l)\dots\mathcal{O}_n(x_n)\right\rangle = 0; \tag{6.47}$$

- if the operators $\mathcal{O}_1,\dots,\mathcal{O}_n$ are superconformal primaries, then they are annihilated by the supercharges $\bar{\mathcal{S}}^{\dot\beta I}$ and we obtain a richer version of the identity (6.47)

$$\sum_{l=1}^n \psi^\alpha(x_l)\left\langle \mathcal{O}_1(x_1)\dots\left[\mathcal{Q}^I_\alpha,\mathcal{O}_l\right\}(x_l)\dots\mathcal{O}_n(x_n)\right\rangle = 0. \tag{6.48}$$

  The spinor $\psi^\alpha(x_l)$ can be chosen to be equal to zero for a given x_l. This makes the contribution to the Ward identity of one of the operators $\mathcal{O}_1,\dots,\mathcal{O}_n$ equal to zero.

6.5 The theorem

We have all the tools required to write the statement of the non-renormalization theorem. Applying the equation (6.17) to the equation (6.10), we obtain

$$\begin{bmatrix}\text{kinetic and group-}\\ \text{theoretical factors}\end{bmatrix} \times \nabla_\tau C_{123} = \int d^4x\left\langle \mathcal{O}_\tau(x)\phi_1^{(\mathcal{R}_1,\vec{m}_1)}(x_1)\,\phi_2^{(\mathcal{R}_2,\vec{m}_2)}(x_2)\,\phi_3^{(\mathcal{R}_3,\vec{m}_3)}(x_3)\right\rangle . \tag{6.49}$$

The statement of the theorem is

Theorem. *The coefficient C_{123} associated to the correlators (6.8) does not depend on the position over the conformal manifold, i.e. it does not depend on τ, the $\mathcal{N} = 4$ theory coupling constant.*

$$\nabla_\tau C_{123} = 0. \tag{6.50}$$

Looking at the flow equation (6.49), we immediately realize that the theorem can be immediately proven if the show the following lemma

Lemma. *The correlator*

$$\mathcal{I}(x,x_1,x_2,x_3) = \left\langle \mathcal{O}_\tau(x)\phi_1^{(\mathcal{R}_1,\vec{m}_1)}(x_1)\,\phi_2^{(\mathcal{R}_2,\vec{m}_2)}(x_2)\,\phi_3^{(\mathcal{R}_3,\vec{m}_3)}(x_3)\right\rangle \tag{6.51}$$

is identically equal to zero.

In order to prove the lemma, we explicitly construct the marginal operator $\mathcal{O}_\tau$, along with its hermitian conjugate, as a superconformal descendant. Looking at the superconformal multiplet in figure 6.1, it is easy to see that

$$\mathcal{O}_\tau = \left\{\mathcal{Q}^4_1, \left[\mathcal{Q}^4_2, \left\{\mathcal{Q}^3_1, \left[\mathcal{Q}^3_2, \phi^{(2,+)}\right]\right\}\right]\right\}, \quad \bar{\mathcal{O}}_\tau = \left\{\bar{\mathcal{Q}}^{1,\dot{1}}, \left[\bar{\mathcal{Q}}^{1,\dot{2}}, \left\{\bar{\mathcal{Q}}^{2,\dot{1}}, \left[\bar{\mathcal{Q}}^{2,\dot{2}}, \phi^{(2,+)}\right]\right\}\right]\right\}. \tag{6.52}$$

we recall that in a CFT the central charges Z^{IJ} are equal to zero, so the supercharges with the same chirality anticommute with each other: the order of the supercharges in the marginal operators (6.52) is not influent. In the following, we consider only the operator $\mathcal{O}_\tau$, but the lemma could be shown with the operato $\bar{\mathcal{O}}_\tau$, too. The crucial property of the marginal operator is that it can be written as

$$\mathcal{O}_\tau = \{\mathcal{Q}^*, \Gamma\}, \tag{6.53}$$

where $\mathcal{Q}^*$ is one of the left chirality supercharges appearing in the definition (6.52). The integrand (6.51) can be rewritten as follows

$$\mathcal{I}(x, x_1, x_2, x_3) = \left\langle \{\mathcal{Q}^*, \Gamma\}(x)\, \phi_1^{(k_1,\vec{m})}(x_1)\, \phi_2^{(k_2,+)}(x_2)\, \phi_3^{(k_3,-)}(x_3) \right\rangle. \tag{6.54}$$

We want to exploit the superconformal Ward identity to move the supercharge $\mathcal{Q}^*$ from the operator Γ to the primaries inside the braket. We make some preliminary considerations:

- ϕ_1, ϕ_2 and ϕ_3 are superconformal primary operators, so they are annihilated by the supercharges $\mathcal{S}^\alpha_I$ and $\bar{\mathcal{S}}^I_{\dot{\beta}}$;
- if $\mathcal{Q}^*$ is a left-chirality supercharge, then in the superconformal Ward identity the right-chirality $\bar{\mathcal{S}}^*$ supercharge appears. We consider

  $$\{\bar{\mathcal{S}}^*, \Gamma\} = \left\{\bar{\mathcal{S}}^*, \left[\mathcal{Q}', \left\{\mathcal{Q}'', \left[\mathcal{Q}''', \phi^{(2,+)}\right]\right\}\right]\right\}, \tag{6.55}$$

  where we renamed as $\mathcal{Q}', \mathcal{Q}'', \mathcal{Q}'''$ the three left-chirality supercharges left, different from $\mathcal{Q}^*$. Thanks to the state-operator correspondence, we can write

  $$\left\{\bar{\mathcal{S}}^*, \left[\mathcal{Q}', \left\{\mathcal{Q}'', \left[\mathcal{Q}''', \phi^{(2,+)}\right]\right\}\right]\right\} \to \bar{\mathcal{S}}^*\, \mathcal{Q}'\, \mathcal{Q}''\, \mathcal{Q}'''\, |\mathrm{h.w.}\rangle. \tag{6.56}$$

  The right-chirality supercharge anticommutes with all the left-chirality supercharges

  $$\left\{\bar{\mathcal{S}}^{\dot{\alpha}}_I, \mathcal{Q}^J_\alpha\right\} = 0, \tag{6.57}$$

  so we have

  $$\bar{\mathcal{S}}^*\, \mathcal{Q}'\, \mathcal{Q}''\, \mathcal{Q}'''\, |\mathrm{h.w.}\rangle = -\mathcal{Q}'\, \mathcal{Q}''\, \mathcal{Q}'''\, \bar{\mathcal{S}}^*\, |\mathrm{h.w.}\rangle. \tag{6.58}$$

  Applying the state-operator correspondence a second time, we obtain

  $$-\mathcal{Q}'\, \mathcal{Q}''\, \mathcal{Q}'''\, \bar{\mathcal{S}}^*\, |\mathrm{h.w.}\rangle \to \left\{\mathcal{Q}', \left[\mathcal{Q}'', \left\{, \mathcal{Q}''' \left[\bar{\mathcal{S}}^*, \phi^{(2,+)}\right]\right\}\right]\right\} = 0 \tag{6.59}$$

  because the superconformal primary state is annihilated by the $\bar{\mathcal{S}}^*$ supercharge;
- the operator $\phi_3^{(k_3,-)}$, sitting in the lowest weight of its R-symmetry representation, is annihilated by the supercharges $\mathcal{Q}^4_\alpha$ and $\mathcal{Q}^3_\alpha$

  $$\left[\mathcal{Q}^3_\alpha, \phi_3^{(k_3,-)}\right] = 0, \qquad \left[\mathcal{Q}^4_\alpha, \phi_3^{(k_3,-)}\right] = 0. \tag{6.60}$$

  The proof of this statement is reported in the appendix C.

Following the considerations listed above, the superconformal Ward identity applied to the correlator

$$\left\langle \Gamma(x)\, \phi_1^{(k_1,\vec{m})}(x_1)\, \phi_2^{(k_2,+)}(x_2)\, \phi_3^{(k_3,-)}(x_3) \right\rangle \tag{6.61}$$

returns the following identity

$$\psi(x)\, \langle \{\mathcal{Q}^*, \Gamma\}\,(x)\, \phi_1\, \phi_2\, \phi_3 \rangle + \psi(x_1)\, \langle \Gamma\, [\mathcal{Q}^*, \phi_1]\,(x_1)\, \phi_2\, \phi_3 \rangle + \\ + \psi(x_2)\, \langle \Gamma\, \phi_1\, [\mathcal{Q}^*, \phi_2]\,(x_2)\, \phi_3 \rangle = 0. \tag{6.62}$$

We impose $\psi(x)$ to be equal to zero when it is computed in x_2. We are left with

$$\psi(x)\, \langle \{\mathcal{Q}^*, \Gamma\}\,(x)\, \phi_1\, \phi_2\, \phi_3 \rangle = -\psi(x_1)\, \langle \Gamma\, [\mathcal{Q}^*, \phi_1]\,(x_1)\, \phi_2\, \phi_3 \rangle \tag{6.63}$$

and recalling the definition of the integrand (6.51) we can write

$$\psi(x)\, \mathcal{I}(x, x_1, x_2, x_3) = -\psi(x_1)\, \langle \Gamma\, [\mathcal{Q}^*, \phi_1]\,(x_1)\, \phi_2\, \phi_3 \rangle \tag{6.64}$$

We succedeed at moving the supercharge $\mathcal{Q}^*$ from the operator Γ to the operator ϕ_1. In order to conclude the proof, we make use of the following lemma

Lemma. *For a chiral primary with a generic weight $\vec{m}$, the following relation holds*

$$\left[\mathcal{Q}^*, \phi^{(k,\vec{m})}\right] = \sum_{\star \neq *} \left[\mathcal{Q}^\star, \phi^{(k,\vec{m}_\star)}\right]. \tag{6.65}$$

We call this relation null condition.

The lemma is proven in the appendix C. Plugging the null condition in the identity (6.64) we obtain

$$\psi(x)\, \mathcal{I}(x, x_1, x_2, x_3) = -\psi(x_1) \sum_{\star \neq *} \left\langle \Gamma\, \left[\mathcal{Q}^\star, \phi_1^{(k_1,\vec{m}_\star)}\right](x_1)\, \phi_2\, \phi_3 \right\rangle. \tag{6.66}$$

Now we can focus on the single term of the sum

$$\left\langle \Gamma(x)\, \left[\mathcal{Q}^\star, \phi_1^{(k_1,\vec{m}_\star)}\right](x_1)\, \phi_2^{(k_2,+)}(x_2)\, \phi_3^{(k_3,-)}(x_3) \right\rangle. \tag{6.67}$$

We apply again the superconformal Ward identity in order to move the supercharge $\mathcal{Q}^\star$ from the operator $\phi_1^{(k_1,\vec{m}_\star)}$ to the other operators. We already know that we can neglect the contribution given by the $\bar{\mathcal{S}}^\star$ supercharge. We have two possibilities:

- $\mathcal{Q}^\star$ annihilates the highest weight of a generic $(0\ k\ 0)$ representation of $su(4)$. Thus, the operator $\phi_2^{(k_2,+)}$ is annihilated by $\mathcal{Q}^\star$. We introduce a new conformal Killing spinor $\chi(x)$ and we impose $\chi(x_3) = 0$ in order to get rid of the contribution given by the operator $\phi_3^{(k_3,-)}$. We are left with the following identity

$$\chi(x_1) \left\langle \Gamma(x)\, \left[\mathcal{Q}^\star, \phi_1^{(k_1,\vec{m}_\star)}\right](x_1)\, \phi_2\, \phi_3 \right\rangle = -\chi(x) \left\langle \{\mathcal{Q}^\star, \Gamma\}\,(x)\, \phi_1^{(k_1,\vec{m}_\star)}(x_1)\, \phi_2\, \phi_3 \right\rangle. \tag{6.68}$$

  The operator $\{\mathcal{Q}^\star, \Gamma\}$ can be rewritten as

$$\left\{\mathcal{Q}^\star, \left[\mathcal{Q}', \left\{\mathcal{Q}'', \left[\mathcal{Q}''', \phi^{(2,+)}\right]\right\}\right]\right\}. \tag{6.69}$$

  We know by hypothesis that $\mathcal{Q}^\star \neq \mathcal{Q}', \mathcal{Q}'', \mathcal{Q}'''$ and, by construction, that $\mathcal{Q}^\star \neq \mathcal{Q}^*$. Thanks to the state-operator correspondence we can write

$$\left\{\mathcal{Q}^\star, \left[\mathcal{Q}', \left\{\mathcal{Q}'', \left[\mathcal{Q}''', \phi^{(2,+)}\right]\right\}\right]\right\} \to \mathcal{Q}^\star \mathcal{Q}' \mathcal{Q}'' \mathcal{Q}''' \, |\text{h.w.}\rangle\,. \tag{6.70}$$

We recall that in a CFT the supercharges with the same chirality anticommute

$$\mathcal{Q}^{\star}\,\mathcal{Q}'\,\mathcal{Q}''\,\mathcal{Q}'''\,|\text{h.w.}\rangle = -\mathcal{Q}'\,\mathcal{Q}''\,\mathcal{Q}'''\,\mathcal{Q}^{\star}\,|\text{h.w.}\rangle = 0, \tag{6.71}$$

where in the last step we used the fact that $\mathcal{Q}^{\star}$ annihilates the highest weight state by hypothesis;

- $\mathcal{Q}^{\star}$ annihilates the lowest weight of a generic $(0\;k\;0)$ representation of $su(4)$.Thus, the operator $\phi_3^{(k_3,-)}$ does not contribute. In this case, we can impose $\chi(x_2) = 0$ in order to get rid of the contribution given by the operator $\phi_2^{(k_2,+)}$. We are left with the identity

$$\chi(x_1)\left\langle\Gamma(x)\,\left[\mathcal{Q}^{\star},\phi_1^{(k_1,\vec{m}_{\star})}\right](x_1)\,\phi_2\,\phi_3\right\rangle = -\chi(x)\left\langle\{\mathcal{Q}^{\star},\Gamma\}\,(x)\,\phi_1^{(k_1,\vec{m}_{\star})}(x_1)\,\phi_2\,\phi_3\right\rangle. \tag{6.72}$$

In the r.h.s. of the previous equation the following operator appears

$$\left\{\mathcal{Q}^{\star},\left[\mathcal{Q}',\left\{\mathcal{Q}'',\left[\mathcal{Q}''',\phi^{(2,+)}\right]\right\}\right]\right\}. \tag{6.73}$$

By definition $\mathcal{Q}^{\star} \neq \mathcal{Q}^{*}$. However, $\mathcal{Q}^{\star}$ annihilates the lowest weight state. Recalling the equations (6.60), $\mathcal{Q}^{\star}$ is necessarily equal to one among $\mathcal{Q}'$, $\mathcal{Q}''$ and $\mathcal{Q}'''$. Switching from the operator to the state picture we have

$$\mathcal{Q}^{\star}\,\mathcal{Q}'\,\mathcal{Q}''\,\mathcal{Q}'''\,|\text{h.w.}\rangle = \left(\mathcal{Q}'\right)^2\,\mathcal{Q}''\,\mathcal{Q}'''\,|\text{h.w.}\rangle = 0, \tag{6.74}$$

where we assumed $\mathcal{Q}^{\star} = \mathcal{Q}'$ without any loss of generality.

We proved that the 4-points function (6.67) is always equal to zero, thus, recalling (6.66), every term in the sum is equal to zero. This means that

$$\psi(x)\,\mathcal{I}(x,x_1,x_2,x_3) = -\psi(x_1)\sum_{\star\neq *}\left\langle\Gamma\,\left[\mathcal{Q}^{\star},\phi_1^{(k_1,\vec{m}_{\star})}\right]\,\phi_J\,\phi_K\right\rangle = 0 \Rightarrow \mathcal{I}(x,x_1,x_2,x_3) = 0. \tag{6.75}$$

We succedeed at proving the lemma. The proof of the theorem then is trivial, looking at the equation (6.49).

Chapter 7

The $\mathcal{N} = 4$ theory at Finite Temperature

In this chapter we consider the $\mathcal{N} = 4$ SYM theory at finite temperature and we derive the superconformal Ward identity at finite temperature. We will start from the known results at zero temperature: the introduction of a finite temperature modifies the original theory in two different ways:

- it moves the theory from the spacetime manifold $\mathbb{R}^4$ to the non-topologically trivial manifold $\mathbb{R}^3 \times S^1$;
- it introduces new operators in the Lagrangian. In particular, we will show that it is possible to introduce a tree level fermionic mass in the Lagrangian. Recalling the section 2.5, this can lead to a soft breaking of the supersymmetry.

The superconformal Ward identity is a fundamental tool employed in the proof of the non-renormalization theorem at zero temperature. We want to make use of it at finite temperature, too. However, supersymmetry and superconformal symmetry are broken due to the introduction of a finite temperature, so we will carefully highlight the breaking terms of the superconformal Ward identity . We will show that the R-symmetry is preserved at the quantum level: we will make use of it to reshape the superconformal Ward identity according to our desire.

In this chapter we will understand how to introduce a finite temperature and we will discuss the consequences on our theory. Then, our goal will be the derivation of the superconformal Ward identity at finite temperature. We will proceed gradually: as a simpler but instructive example, in the first place we will study the supersymmetry Ward identity at finite temperature; then, in a second moment, we will move to the superconformal Ward identity. In both cases, we will make use of the R-symmetry Ward identity at finite temperature in order to get rid of the softly breaking terms.

7.1 Introduction of a finite temperature

In this section we discuss the effects of turning on a finite temperature $T \neq 0$ in the system, following [17], [18] and [19]. In particular, we want to study the changes in the global properties of the spacetime and in the behavior of the bosonic and the fermionic local operators. From a phenomenological point of view, the usual quantum field theory has to be interpreted as a theory living at zero temperature: for instance, it is useful for deriving predictions verified in the particle colliders, where two free particles interact in a singular point of the spacetime. The correct way to describe an ensemble of particles continuously interacting with each other is to use the framework of the *finite temperature quantum field theories*.

7.1.1 Quantum mechanics at finite temperature

In this section we briefly discuss the introduction of a finite temperature in a quantum mechanical system . The model is an ensemble of particles at the temperature $T \neq 0$, with $\beta \equiv 1/T$. The dynamics of the system is described by the Hamiltonian operator $\hat{H}$. It is possible to introduce a partition function

$$\mathcal{Z}(T) = \text{tr}\left[e^{-\beta\hat{H}}\right], \tag{7.1}$$

where the trace is taken over all the states belonging to the Hilbert space of the theory. In the x basis

$$\mathcal{Z}(T) = \int dx \left\langle x \middle| e^{-\beta\hat{H}} \middle| x \right\rangle = \int dx \left\langle x \middle| e^{-\epsilon\hat{H}} \dots e^{-\epsilon\hat{H}} \middle| x \right\rangle, \tag{7.2}$$

where the operator $e^{-\beta\hat{H}}$ has been splitted in $N \gg 1$ copies of the operator $e^{-\epsilon\hat{H}}$, where $\epsilon = \beta/N$. The splitting is possible thanks to the Trotter formula (cfr. [17]): if $\hat{A}$ and $\hat{B}$ are two bounded operators, then

$$e^{\hat{A}+\hat{B}} = \lim_{N\to\infty}\left(e^{\frac{\hat{A}}{N}}e^{\frac{\hat{B}}{N}}\right)^N; \tag{7.3}$$

considering that

$$\hat{H}(\hat{x},\hat{p}) = \frac{\hat{p}^2}{2m} + V(\hat{x}), \tag{7.4}$$

we can apply the Trotter formula and obtain the expression (7.2). Now we consider the completeness relations

$$\int dx_i \left|x_i\right\rangle \left\langle x_i\right| = \mathbb{1}, \qquad \int \frac{dp_i}{2\pi}\left|p_i\right\rangle\left\langle p_i\right| = \mathbb{1}, \tag{7.5}$$

where the index i runs from 1 to N. The former relation is inserted at the left of each operator $e^{-\epsilon\hat{H}}$, the latter at the right: the result is

$$\begin{aligned}\mathcal{Z}(T) = \int dx \left(\prod_{i=1}^{N}\int\frac{dx_i dp_i}{2\pi}\right) \langle x\,|\,p_N\rangle \langle p_N|\,e^{-\epsilon\hat{H}}\,|x_N\rangle \dots \langle x_{i+1}\,|\,p_i\rangle \langle p_i|\,e^{-\epsilon\hat{H}}\,|x_i\rangle \dots \\ \dots \langle x_2\,|\,p_1\rangle \langle p_1|\,e^{-\epsilon\hat{H}}\,|x_1\rangle \langle x_1\,|\,x\rangle .\end{aligned} \tag{7.6}$$

The partition function (7.6) can be simplified with the following relations

$$\langle x_m\,|\,x_n\rangle = \delta(x_m - x_n), \quad \langle x_{i+1}\,|\,p_i\rangle = e^{ip_i x_{i+1}}, \quad \langle p_i|\,e^{-\epsilon\hat{H}}\,|x_i\rangle = e^{-\epsilon H(x_i,p_i)}e^{-ip_i x_i}, \tag{7.7}$$

where the last one was obtained thanks to the fact that, as prescribed by the Trotter formula (7.3), the Hamiltonian is splitted in its $\hat{p}$ dependent part, with eigenstates $\langle p_i|$, and its $\hat{x}$ dependent part, with eigenstates $|x_i\rangle$. The new intermediate result is

$$\begin{aligned}\mathcal{Z}(T) = \int dx \left(\prod_{i=1}^{N}\int\frac{dx_i dp_i}{2\pi}\right)\Big\{e^{-[\epsilon H(x_N,p_N)+ip_N(x_N-x)]}\dots e^{-[\epsilon H(x_i,p_i)+ip_i(x_i-x_{i+1})]}\dots \\ \dots e^{-[\epsilon H(x_1,p_1)+ip_1(x_1-x_2)]}\Big\}\delta\left(x_1 - x\right).\end{aligned} \tag{7.8}$$

The integral in x can be immediately computed

$$\mathcal{Z}(T) = \left(\prod_{i=1}^{N} \int \frac{dx_i dp_i}{2\pi}\right) \left[e^{-\sum_{j=1}^{N}\left[\epsilon H(x_j,p_j)+ip_j(x_j-x_{j+1})\right]}\right]_{x_{N+1}=x_1,\epsilon=\beta/N}. \tag{7.9}$$

The Hamiltonian operator has been entirely replaced by the Hamiltonian function

$$H(p_j, x_j) = \frac{p_j^2}{2m} + V(x_j), \tag{7.10}$$

which is quadratic in the momenta. The integrals in the momenta turn out to be Gaussian integrals, so they can be explicitly computed

$$\mathcal{Z}(T) = \left(\prod_{i=1}^{N} \int dx_i\right) \left\{\left(\frac{m}{2\pi\epsilon}\right)^{\frac{N}{2}} e^{-\epsilon\sum_{j=1}^{N}\left[\frac{1}{2\epsilon}m(x_{j+1}-x_j)^2+V(x_j)\right]}\right\}_{x_{N+1}=x_1,\epsilon=\beta/N}. \tag{7.11}$$

Introducing a continuous coordinate τ such that $x = x(\tau)$ and $x(0) = x(\beta)$, then we can take the continuous limit of the partition function (7.11)

$$\mathcal{Z}(T) = C\int [\mathcal{D}x]\, e^{-\int_0^\beta d\tau\left[\frac{1}{2}m\left(\frac{dx}{d\tau}\right)^2+V(x(\tau))\right]}, \tag{7.12}$$

where the overall factor C is given by

$$C = \lim_{N\to\infty}\left(\frac{mN}{2\pi\beta}\right)^{\frac{N}{2}}. \tag{7.13}$$

The C factor is divergent, but it depends neither on the potential nor on the coordinate $x(\tau)$, so it can simply be absorbed in the functional measure. Eventually, the partition function of an ensemble of quantum particles in equilibrium at the finite temperature $T \neq 0$ is

$$\mathcal{Z}(T) = \int [\mathcal{D}x]\, e^{-\int_0^\beta d\tau\left[\frac{1}{2}m\left(\frac{dx}{d\tau}\right)^2+V(x(\tau))\right]}. \tag{7.14}$$

This partition function must be compared with the Path Integral in the usual Minkowski spacetime

$$\mathcal{Z} = \int [\mathcal{D}x]\, e^{i\int dt\left[\frac{1}{2}m\left(\frac{dx}{dt}\right)^2-V(x(t))\right]}. \tag{7.15}$$

We can easily identify the partition function of the ensemble with the Path Integral of the theory: it is sufficient to perform the Wick rotation $t = -i\tau$ of the time coordinate, switching from the Minkowskian signature to the Euclidean signature, and to integrate τ not from $+i\infty$ to $-i\infty$, but over the interval $[0, i\beta]$; given that the dynamical variable x is periodic in the τ variable, the interval is a circle of length β.

In general, the procedure to adopt to turn on a finite temperature $T \neq 0$ is the following:

1. switch from the Minkowski signature to the Euclidean signature and introduce the imaginary time variable τ;

2. compactify the time dimension, parametrized by τ, on a circle of length $\beta = 1/T$;

3. identify the periodicity conditions over the time circle of all the variables in the model
.

This procedure goes under the name of *imaginary time formalism*. The global structure of the spacetime is deeply modified: we move from the Euclidean spacetime $\mathbb{R}^4$ to the manifold $\mathbb{R}^3 \times S^1$, which is not topologically trivial. The principal consequence is the modification of the boundary conditions of the variables: in a finite temperature field theory, the fields do not fall to zero on the time dimension, the circle S^1. However, even if the global structure of the spacetime is modified, the local structure is not. In fact, the flat metric is not compromised by the compactification, so the curvature of the finite temperature spacetime is still trivial.

7.1.2 Scalar fields at finite temperature

In this section we show that the imaginary time formalism holds even for a scalar field theory. Even if we adopt a scalar field, the same conclusions hold for a generic bosonic field. If we are dealing with a bosonic quantum field theory, the partition function can be written as follows

$$\mathcal{Z}(T) = \int d\phi \left\langle \phi \left| e^{-\beta \int d^3x \mathcal{H}(\hat{\phi},\hat{\pi})} \right| \phi \right\rangle, \tag{7.16}$$

where $\hat{\pi}$ is the conjugate momenta of the field $\hat{\phi}$. The integrand can be splitted similarly to the equation (7.2), making use of the Trotter formula and introducing the parameter $\epsilon = \beta/N$. Then, thanks to the completeness relations

$$\int d\phi_i \left|\phi_i\right\rangle \left\langle\phi_i\right| = \mathbb{1}, \qquad \int \frac{d\pi_i}{2\pi} \left|\pi_i\right\rangle \left\langle\pi_i\right| = \mathbb{1}, \tag{7.17}$$

we obtain the intermediate result

$$\begin{aligned}\mathcal{Z}(T) = \int d\phi \left(\prod_{i=1}^{N} \int \frac{d\phi_i d\pi_i}{2\pi}\right) \left\langle \phi \mid \pi_N \right\rangle \left\langle \pi_N \right| e^{-\epsilon \int d^3x \mathcal{H}(\hat{\phi},\hat{\pi})} \left|\phi_N\right\rangle \dots \\ \dots \left\langle \phi_{i+1} \mid \pi_i \right\rangle \left\langle \pi_i \right| e^{-\epsilon \int d^3x \mathcal{H}(\hat{\phi},\hat{\pi})} \left|\phi_i\right\rangle \dots \left\langle \phi_2 \mid \pi_1 \right\rangle \left\langle \pi_1 \right| e^{-\epsilon \int d^3x \mathcal{H}(\hat{\phi},\hat{\pi})} \left|\phi_1\right\rangle \left\langle \phi_1 \mid \phi \right\rangle.\end{aligned} \tag{7.18}$$

The following identities hold

$$\left\langle \phi_m \mid \phi_n \right\rangle = \delta(\phi_m - \phi_n), \ \left\langle \phi_{i+1} \mid \pi_i \right\rangle = e^{i\pi_i\phi_{i+1}}, \ \left\langle \pi_i \right| e^{-\epsilon \int d^3x \mathcal{H}(\hat{\phi},\hat{\pi})} \left|\phi_i\right\rangle = e^{-\epsilon \int d^3x \mathcal{H}(\phi_i,\pi_i)} e^{-i\pi_i\phi_i}, \tag{7.19}$$

so we end up with

$$\mathcal{Z}(T) = \left(\prod_{i=1}^{N} \int \frac{d\phi_i d\pi_i}{2\pi}\right) \left[e^{-\sum_{j=1}^{N}\left[\epsilon \int d^3x \mathcal{H}(\phi_j,\pi_j) + i\pi_j(\phi_j - \phi_{j+1})\right]}\right]_{\phi_{N+1}=\phi_1, \epsilon=\beta/N}. \tag{7.20}$$

Similarly to the previous section, if the Hamiltonian density function is quadratic in the conjugated momenta, we can compute the integrals in $d\pi_i$. Eventually, taking the continuum limit, we obtain

$$\mathcal{Z}(T) = \int [\mathcal{D}\phi]\, e^{-\int_0^\beta d\tau \int d^3x \left[\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + V(\phi)\right]}, \tag{7.21}$$

which is exactly the result we would have obtained if we had applied the imaginary time formalism to the Path Integral of a generic scalar field theory. Finally, notice that all the steps were completely analogous to those encountered in the quantum mechanical case, so the periodicity conditions of the scalar field over the time circle is

$$\phi(\vec{x}, 0) = \phi(\vec{x}, \beta). \tag{7.22}$$

It could be shown that the same applies to every bosonic field.

7.1.3 Fermionic fields at finite temperature

In this section we verify the validity of the imaginary time formalism for a fermionic quantum field theory. The interesting difference with the previous cases lies in the fact that the fermionic fields are Grassmann variables, hence they anticommute. The starting point is still the partition function of the canonical ensemble

$$\mathcal{Z}(T) = \mathrm{tr}\left[e^{-\beta \hat{H}}\right]. \tag{7.23}$$

The fermionic fields involved are ψ and $\bar{\psi}$, which constitute two independent variables; the functional measure will be always written with the following convention

$$\int d\bar{\psi} d\psi. \tag{7.24}$$

In order to do the trace over all the states in the Hilbert space, it is useful to understand how the Hilbert space is composed. Let's suppose we have at our disposal the creation and destruction operators $\hat{a}, \hat{a}^\dagger$, tied together by the relations

$$\left\{\hat{a}, \hat{a}^\dagger\right\} = 1, \qquad \left\{\hat{a}^\dagger, \hat{a}^\dagger\right\} = 0, \qquad \{\hat{a}, \hat{a}\} = 0. \tag{7.25}$$

Then, we have a vacuum state $|0\rangle$: the only other state that we are able to construct is $|1\rangle \equiv a^\dagger |0\rangle$. All the other trials end up with a zero or with the $|0\rangle$ state, due to the anticommutation relations (7.25). Then, the completeness relation is simply

$$|0\rangle \langle 0| + |1\rangle \langle 1| = \mathbb{1}, \tag{7.26}$$

while the trace of a generic operator $\mathcal{O}$ is given by

$$\mathrm{tr}\,[\mathcal{O}] = \langle 0 \,|\, \mathcal{O} \,|\, 0\rangle + \langle 1 \,|\, \mathcal{O} \,|\, 1\rangle\,. \tag{7.27}$$

We want to rewrite the completeness relation and the trace over the Hilbert space in a more convenient way. In order to do so, we need to give the following definitions

$$|\psi\rangle \equiv e^{-\psi \hat{a}^\dagger} |0\rangle = \left(1 - \psi \hat{a}^\dagger\right)|0\rangle\,, \qquad \langle\psi| \equiv \langle 0|\, e^{-\hat{a}\bar{\psi}} = \langle 0| \left(1 - \hat{a}\bar{\psi}\right), \tag{7.28}$$

so that

$$\langle \psi_m \,|\, \psi_n\rangle = \left\langle 0 \left| e^{-\hat{a}\bar{\psi}_m} e^{-\psi_n \hat{a}^\dagger} \right| 0 \right\rangle = e^{\bar{\psi}_m \psi_n}. \tag{7.29}$$

The completeness relation can be rewritten as follows

$$\begin{aligned}
\mathbb{1} &= |0\rangle \langle 0| + |1\rangle \langle 1| = |0\rangle \langle 0| + \int d\bar{\psi} d\psi \; \psi \hat{a}^\dagger |0\rangle \langle 0| \,\hat{a}\bar{\psi} = \\
&= \int d\bar{\psi} d\psi \,(1 - \bar{\psi}\psi) \left(1 - \psi\hat{a}^\dagger\right)|0\rangle \langle 0| \left(1 - \hat{a}\bar{\psi}\right) = \int d\bar{\psi} d\psi \; e^{-\bar{\psi}\psi} |\psi\rangle \langle\psi|\,,
\end{aligned}$$

while the trace of a *bosonic* operator (the exponential of the Hamiltonian, for instance) becomes

$$\begin{aligned}
\mathrm{tr}\,[\mathcal{O}] &= \langle 0 \,|\, \mathcal{O} \,|\, 0\rangle + \langle 1 \,|\, \mathcal{O} \,|\, 1\rangle = \langle 0 \,|\, \mathcal{O} \,|\, 0\rangle - \left\langle 1 \left| \int d\bar{\psi} d\psi \; \bar{\psi}\psi\mathcal{O} \right| 1 \right\rangle \\
&= \langle 0 \,|\, \mathcal{O} \,|\, 0\rangle - \int d\bar{\psi} d\psi \left\langle 0 \left| \hat{a}\bar{\psi}\mathcal{O}\psi\hat{a}^\dagger \right| 0 \right\rangle \\
&= \int d\bar{\psi} d\psi \,(1 - \bar{\psi}\psi) \left\langle 0 \left| \left(1 + \hat{a}\bar{\psi}\right)\mathcal{O}\left(1 - \psi\hat{a}^\dagger\right) \right| 0 \right\rangle
\end{aligned}$$

if we define $\langle -\psi | \equiv \langle 0 | e^{\hat{a}\bar{\psi}}$, we get

$$\mathrm{tr}\,[\mathcal{O}] = \int d\bar{\psi} d\psi \; e^{-\bar{\psi}\psi} \, \langle -\psi \,|\, \mathcal{O} \,|\, \psi \rangle \,. \tag{7.30}$$

We are finally able to compute the partition function

$$\mathcal{Z}(T) = \mathrm{tr}\left[e^{-\beta \hat{H}}\right] = \int d\bar{\psi} d\psi e^{-\bar{\psi}\psi} \left\langle -\psi \,\middle|\, e^{-\beta \hat{H}} \,\middle|\, \psi \right\rangle . \tag{7.31}$$

Making use of the Trotter formula we split the operator into N terms and we insert $N-1$ completeness relations in the following way

$$\begin{aligned} \mathcal{Z}(T) = \int d\bar{\psi} d\psi \left(\prod_{i=2}^{N} \int d\bar{\psi}_i d\psi_i \right) e^{-\bar{\psi}\psi} \, \langle -\psi | \, e^{-\epsilon \int d^3x \mathcal{H}(\hat{\bar{\psi}}, \hat{\psi})} e^{-\bar{\psi}_N \psi_N} \, |\psi_N\rangle \ldots \\ \ldots e^{-\bar{\psi}_i \psi_i} \, |\psi_i\rangle \, \langle \psi_i | \, e^{-\epsilon \int d^3x \mathcal{H}(\hat{\bar{\psi}}, \hat{\psi})} e^{-\bar{\psi}_{i-1} \psi_{i-1}} \, |\psi_{i-1}\rangle \ldots \\ \ldots e^{-\bar{\psi}_2 \psi_2} \, |\psi_2\rangle \, \langle \psi_2 | \, e^{-\epsilon \int d^3x \mathcal{H}(\hat{\bar{\psi}}, \hat{\psi})} \, |\psi\rangle \,. \end{aligned} \tag{7.32}$$

Using the identity (7.29) and defining $\psi \equiv \psi_1$, we simplify the expression above

$$\begin{aligned} \mathcal{Z}(T) = \left(\prod_{i=1}^{N} \int d\bar{\psi}_i d\psi_i \right) e^{-\bar{\psi}_1 \psi_1} \, \langle -\psi_1 | \, e^{-\epsilon \int d^3x \mathcal{H}(\hat{\bar{\psi}}, \hat{\psi})} \, |\psi_N\rangle \ldots \\ \ldots e^{-\epsilon \int d^3x \mathcal{H}(\bar{\psi}_i, \psi_{i-1})} e^{-\bar{\psi}_{i+1}(\psi_{i+1} - \psi_i)} \ldots e^{-\epsilon \int d^3x \mathcal{H}(\bar{\psi}_2, \psi_1)} e^{-\bar{\psi}_2(\psi_2 - \psi_1)} \end{aligned} \tag{7.33}$$

We focus on the first term of the integrand

$$\int d\bar{\psi}_1 d\psi_1 \int d\bar{\psi}_N d\psi_N \; e^{-\bar{\psi}_1 \psi_1} \, \langle -\psi_1 | \, e^{-\epsilon \int d^3x \mathcal{H}(\hat{\bar{\psi}}, \hat{\psi})} \, |\psi_N\rangle \,, \tag{7.34}$$

which turns out to be

$$\int d\bar{\psi}_1 d\psi_1 \int d\bar{\psi}_N d\psi_N \; e^{-\epsilon \int d^3x \mathcal{H}(-\bar{\psi}_1, \psi_N)} e^{\bar{\psi}_1(-\psi_1 - \psi_N)} . \tag{7.35}$$

The final expression of the discretized partition function is

$$\mathcal{Z}(T) = \left(\prod_{i=1}^{N} \int d\bar{\psi}_i d\psi_i \right) \left[e^{-\epsilon \sum_{j=1}^{N} \int d^3x \mathcal{H}(\bar{\psi}_{j+1}, \psi_j) + \bar{\psi}_{j+1} \frac{\psi_{j+1} - \psi_j}{\epsilon}} \right]_{\psi_{N+1} = -\psi_1, \bar{\psi}_{N+1} = -\bar{\psi}_1, \epsilon = \beta/N} . \tag{7.36}$$

Taking the continuum limit, we obtain the Path Integral in the imaginary time formalism

$$\mathcal{Z}(T) = \int [\mathcal{D}\bar{\psi}] \, [\mathcal{D}\psi] \left[e^{-\int_0^{i\beta} d\tau \left(\int d^3x \mathcal{H}(\bar{\psi}(\tau), \psi(\tau)) \right) + \bar{\psi} \frac{d\psi(\tau)}{d\tau}} \right] = \int [\mathcal{D}\bar{\psi}] \, [\mathcal{D}\psi] \left[e^{-\int_0^{\beta} d\tau \int d^3x \mathcal{L}_{\mathrm{Eucl}}} \right] \tag{7.37}$$

where we had to impose the following antiperiodicity conditions on the fermionic fields

$$\psi(\vec{x}, 0) = -\psi(\vec{x}, \beta), \qquad \bar{\psi}(\vec{x}, 0) = -\bar{\psi}(\vec{x}, \beta). \tag{7.38}$$

In conclusion, whenever we want to describe a canonical ensemble of relativistic quantum particles at a finite temperature $T \neq 0$, we must adopt a quantum field theory at finite temperature. The imaginary time formalism provides us with a simple procedure for turning on the temperature: the partition function of our theory is the usual Path Integration computed in a Euclidean spacetime where the time dimension has been compactified over a circle of length $\beta = 1/T$; the bosonic fields are *periodic* over the newly introduced time circle, while the fermionic fields are *antiperiodic*.

7.2 Ward identities at zero temperature

In this section we will make use of the results of the chapter 5 in order to derive the unintegrated and integrated Ward identities associated to the supersymmetry and the R-symmetry in the $\mathcal{N} = 4$ theory. These results will constitute a starting point for the computations at finite temperature. We start by recalling the full $\mathcal{N} = 4$ action, discussed in the section 2.4.2

$$S = \int d^4x \Big[-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + i\lambda^{aI} \sigma^\mu D_\mu \bar{\lambda}^a_I - \frac{1}{4} D_\mu \bar{X}^a_{IJ} D^\mu X^{aIJ} + \\ + \frac{\sqrt{2}}{2} g f^{abc} \lambda^{aI} \lambda^{bJ} \bar{X}^C_{IJ} + \frac{\sqrt{2}}{2} g f^{abc} \bar{\lambda}^a_I \bar{\lambda}^b_J X^{cIJ} - \frac{1}{16} g^2 f^{abc} f^{aeg} X^{bIJ} X^{cKL} \bar{X}^e_{IJ} \bar{X}^g_{KL} \Big]. \tag{7.39}$$

7.2.1 R-symmetry Ward Identity at $T = 0$

In this section we consider the action of the global $su(4)$ R-symmetry on the classical action (7.39). In the following, the generators of $su(4)$ are called $t^{\xi I}{}_J$, where the indices I, J are the the indices of the fundamental representation of $su(4)$ and the index ξ labels the specific generator; the indices I, J run from 1 to 4, while the index ξ runs from 1 to 15. The field transformations are

$$\delta A^a_\mu = 0, \tag{7.40}$$

$$\delta \lambda^{aI} = \omega_\xi t^{\xi I}{}_J \lambda^{aJ}, \tag{7.41}$$

$$\delta X^{aIJ} = 2\omega_\xi t^{\xi [I}{}_K X^{aKJ]}, \tag{7.42}$$

where we introduced the set of infinitesimal, bosonic parameters ω_ξ. We want to obtain the infinitesimal variation of the classical action (7.39): under the action of a continuous symmetry, we know that the the final result has the structure

$$\delta S = -\int d^4x \left[R^{\xi\mu} \partial_\mu \omega_\xi \right], \tag{7.43}$$

where $R^{\xi\mu}$ is the R-symmetry current, conserved at the classical level. The current $R^{\xi\mu}$ can be explicitly expressed in terms of the Lagrangian and of the field transformations (7.40), (7.41) and (7.42). If φ represents a generic field which appears in the theory, then

$$R^{\xi\mu} = -\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \varphi \right)} \delta^\xi_\omega \varphi, \tag{7.44}$$

where

$$\delta^\xi_\omega \varphi = \frac{\partial}{\partial \omega_\xi} \delta \varphi. \tag{7.45}$$

We apply the formula (7.44) to the action (7.39) and we obtain the current $R^{\xi\mu}$

$$R^{\xi\mu} = \frac{1}{4} D^\mu \bar{X}^a_{IJ} \delta^\xi_\omega X^{aIJ} + \frac{1}{4} \delta^\xi_\omega \bar{X}^a_{IJ} D^\mu X^{aIJ} - i\lambda^{aI} \sigma^\mu \delta^\xi_\omega \bar{\lambda}^a_I; \tag{7.46}$$

after some simplifications, the final expression turns out to be

$$R^{\xi\mu} = -\frac{1}{2} D^\mu \bar{X}^a_{JI} t^{\xi I}{}_K X^{aKJ} + \frac{1}{2} \bar{X}^a_{JK} t^{\xi K}{}_I D^\mu X^{aIJ} - i\bar{\lambda}^a_I \bar{\sigma}^\mu t^{\xi I}{}_J \lambda^{aJ}. \tag{7.47}$$

We perform an integration by parts in the equation (7.43) and we obtain

$$\delta S = \int d^4x \left[\omega_\xi \partial_\mu R^{\xi\mu} \right]. \tag{7.48}$$

Recalling the general result (5.20), the unintegrated R-symmetry Ward identity is

$$\int d^4x \left\langle \partial_\mu R^{\xi\mu} \mathcal{O}_1 \dots \mathcal{O}_n \right\rangle \omega_\xi = \delta_{\text{R-symm}} \left\langle \mathcal{O}_1 \dots \mathcal{O}_n \right\rangle , \tag{7.49}$$

where $\delta_{\text{R-symm}} \langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle$ represents the action of the R-symmetry over the operators insider the correlator $\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle$

$$\delta_{\text{R-symm}} \langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle = \sum_{i=1}^{n} \langle \mathcal{O}_1 \dots \delta\mathcal{O}_i \dots \mathcal{O}_n \rangle , \tag{7.50}$$

where $\delta\mathcal{O}_i$ is the infinitesimal variation of the operator $\mathcal{O}_i$ under the action of the R-symmetry (cfr. the trasnformation (5.3)).

From the general Ward identity (7.49), we want to derive a more specific one. The $su(4)$ R-symmetry group possesses the abelian subgroup $u(1) \times u(1) \times u(1)$, generated by the Cartan generators of $su(4)$. We want to specialize the Ward identity (7.49) to one of the abelian subgroups of $su(4)$. We adopt the following convention for the Cartan generators (cfr. [20])

$$t^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \quad t^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \quad t^{15} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} . \tag{7.51}$$

We call q_1, q_2, q_3 the charges associated respectively to the symmetries generated by t^3, t^8, t^{15}. Each field appearing in the $\mathcal{N}=4$ theory comes with a set of charges q_1, q_2, q_3, listed in the table 7.1. The vector boson A_μ is not charged under the action of the R-symmetry, so it does not carry any charge associated to its abelian subgroup. The λ spinors sit in the fundamental representation, thus their charges lie on the diagonals of the generators t^3, t^8 and t^{15}. The charges of the X scalars can be derived from the equation (7.42), once it is adapted to a specific Cartan generator t^i

$$\delta X^{aIJ} = \omega_i \left({t^{iI}}_I + {t^{iJ}}_J \right) X^{aIJ} = \omega_i q(I,J) X^{aIJ}, \tag{7.52}$$

where $i = 1,2,3$ and the indices I, J are not contracted in this case, but they are referred to specific values among $I, J = 1,2,3,4$. Each scalar X^{IJ} has a different set of charges, depending on its indices I and J. Recalling the equation (7.50) and the general result (5.38), if the current $R^{\xi\mu}$ is conserved at the operatorial level, we have

$$\sum_{i=1}^{n} q_i = 0, \tag{7.53}$$

where q_i is a charge under the action of the Cartan t^i. This shows that, in a R-symmetry invariant theory, the physical correlators must be globally discharged under the action of a $u(1)$ symmetry group generated by one of the three Cartan generators (7.51).

7.2.2 Supersymmetry Ward Identity at $T = 0$

In this subsection we consider the action of the supersymmetry on the classical action (7.39). The field transformations are (cfr. [21])

$$\delta A_\mu^a = -i\bar{\epsilon}_I \bar{\sigma}_\mu \lambda^{aI} + i\bar{\lambda}_I^a \bar{\sigma}_\mu \epsilon^I, \tag{7.54}$$

$$\delta\lambda^{aI} = -\frac{1}{2}\sigma^{\mu\nu} F_{\mu\nu}^a \epsilon^I + i\sqrt{2}\bar{\epsilon}_J \bar{\sigma}^\mu D_\mu X^{aJI} + g f^{abc} X^{bIJ} \bar{X}_{JK}^c \epsilon^K, \tag{7.55}$$

$$\delta X^{aIJ} = \sqrt{2}\left(\epsilon^I \lambda^{aJ} - \epsilon^J \lambda^{aI} + \varepsilon^{IJKL} \bar{\epsilon}_K \bar{\lambda}_L^a\right) \tag{7.56}$$

	q_1	q_2	q_3
A_μ	0	0	0
λ^1	1	$1/\sqrt{3}$	$1/\sqrt{6}$
λ^2	-1	$1/\sqrt{3}$	$1/\sqrt{6}$
λ^3	0	$-2/\sqrt{3}$	$1/\sqrt{6}$
λ^4	0	0	$-3/\sqrt{6}$
X^{01}	0	$2/\sqrt{3}$	$2/\sqrt{6}$
X^{02}	1	$-1/\sqrt{3}$	$2/\sqrt{6}$
X^{03}	1	$1/\sqrt{3}$	$-2/\sqrt{6}$
X^{12}	-1	$-1/\sqrt{3}$	$2/\sqrt{6}$
X^{13}	-1	$1/\sqrt{3}$	$-2/\sqrt{6}$
X^{23}	0	$-2/\sqrt{3}$	$-2/\sqrt{6}$

Table 7.1: Flavor charges of the field content of the $\mathcal{N} = 4$ theory under the action of t^1, t^2 and t^3, the Cartan generators of $su(4)$.

where we introduced a set of infinitesimal spinors ϵ^I_α, provided with a $su(4)$ fundamental index I. The currents associated to supersymmetry are the supercurrent $G^{\mu I}_\alpha(x)$, which sits in the $[2,1]$ representation of the 4-dimensional Lorentz group, and $\bar{G}^{\mu\dot{\alpha}}_J(x)$ which sits in the $[1,2]$ representation. We want to obtain the infinitesimal variation of the classical action S: under the action of a continuous symmetry

$$\delta S = -\int d^4x \left[G^\mu_I \partial_\mu \epsilon^I + h.c. \right]. \tag{7.57}$$

The supercurrent is given by

$$G^\mu_I = -\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \varphi\right)} \delta_{\epsilon,I}\varphi, \tag{7.58}$$

where

$$\delta_{\epsilon,I}\varphi = \frac{\partial}{\partial \epsilon^I}\delta\varphi. \tag{7.59}$$

Applying the formula (7.58) to the action (7.39), we obtain the explicit expression of the supercurrent

$$G^\mu_I = \frac{1}{4} D^\mu \bar{X}^a_{HJ} \delta_{\epsilon,I} X^{aHJ} + \frac{1}{4}\delta_{\epsilon,I}\bar{X}^a_{HJ} D^\mu X^{aHJ} - i\lambda^{aH}\sigma^\mu \delta_{\epsilon,I}\bar{\lambda}^a_H + F^{a\mu\nu}\delta_{\epsilon,I}A^a_\nu. \tag{7.60}$$

After some computations, the final result is given by the expression

$$G^\mu_I = \sqrt{2} D^\mu \bar{X}^a_{IJ}\lambda^{aJ} + \sqrt{2} D_\nu \bar{X}^a_{IJ}\sigma^\mu\bar{\sigma}^\nu\lambda^{aJ} + i\bar{\lambda}^a_I \bar{\sigma}_\nu F^{a\mu\nu}, \tag{7.61}$$

where we employed the duality relation $\bar{X}^a_{IJ} = \frac{1}{2}\varepsilon_{IJKL}X^{aKL}$ (cfr. [8], [21]). We perform an integration by parts in the infinitesimal variation (7.57) and we obtain

$$\delta S = \int d^4x \left[\partial_\mu G^\mu_I \epsilon^I + h.c. \right]. \tag{7.62}$$

The unintegrated supersymmetry Ward identity is given by

$$\int d^4x \left\langle \partial_\mu G^\mu_I \mathcal{O}_1 \ldots \mathcal{O}_n \right\rangle \epsilon^I + h.c. = \delta_{\text{susy}} \left\langle \mathcal{O}_1 \ldots \mathcal{O}_n \right\rangle, \tag{7.63}$$

where δ_{susy} is a compact notation representing the action of the supersymmetry on the operators inside the correlator $\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle$

$$\delta_{\text{susy}} \langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle = \sum_{i=1}^{n} \langle \mathcal{O}_1 \ldots \delta\mathcal{O}_i \ldots \mathcal{O}_n \rangle \,, \tag{7.64}$$

where $\delta\mathcal{O}_i$ is the infinitesimal variation of the operator $\mathcal{O}_i$ under the action of the supersymmetry (cfr. the trasnformation (5.3)).

7.3 Ward identities at finite temperature

In this section we will derive the R-symmetry, the supersymmetry and the superconformal Ward identities at finite temperature, starting from the results at zero temperature. It is important to remark that at finite temperature the supersymmetry and the superconformal symmetry are not symmetries of the theory, hence the supercharges $\mathcal{Q}$ and $\mathcal{S}$ are not defined. However, we will show that the effect of the finite temperature is to generate a *tree-level fermionic mass operator* in the Lagrangian

$$\mathcal{L}^{\mathcal{N}=4} \xrightarrow{T \neq 0} \mathcal{L}^{\mathcal{N}=4} + \mathcal{O}_{\text{Fermionic mass}} \tag{7.65}$$

The supercharges $\mathcal{Q}$ and $\mathcal{S}$ are properly defined for the original Lagrangian $\mathcal{L}^{\mathcal{N}=4}$, along with all the structures and the objects we are able to build with them: supermultiplets, conformal multiplets, superconformal multiplets. The sector $\mathcal{L}^{\mathcal{N}=4}_{\text{Masses}}$ can be seen as a *relevant deformation* of the original theory around its conformal point.

7.3.1 A mass breaking $\mathcal{N} = 4$ softly

In this section we start from the $\mathcal{N} = 4$ theory and we learn how to deform it introducing a tree-level fermionic mass operator, motivated by the finite temperature. Although such operator would be sufficient for carrying out the discussion in the following, we will introduce mass operators for the scalars and the vector boson, too. Doing so, we will keep the discussion as general as possible. The newly-introduced mass operators will be multiplied by thermal masses, i.e. numerical coefficients dependent on T. The numerical values of the thermal masses depend on the loop order we are considering. We will provide the explicit value of the tree-level fermionic thermal mass and the 1-loop corrections, computed in [22].

Tree level thermal mass

First of all, we motivate the addition of a tree-level mass operator for the fermionic fields carrying out a brief computation. We start studying the $\mathcal{N} = 1$ free theory as a simpler toy model. The starting point is the sector S_{ferm} of the action, which describes the free fermionic field

$$S_{\text{ferm}} = \int d^4x \left[i\psi\sigma^{\mu}\partial_{\mu}\bar{\psi} \right] . \tag{7.66}$$

We want to compute the 2-points function in presence of a finite temperature T. We separate the temporal index from the spatial indices

$$S_{\text{ferm}} = \int dt \int d^3x \left[-i\psi\sigma_0\partial_0\bar{\psi} + i\psi\sigma_i\partial_i\bar{\psi} \right] . \tag{7.67}$$

We switch from the Minkowski action to the Euclidean action performing a Wick rotation of the time coordinate

$$t = -i\tau, \qquad k_0 = ik_{\tau}, \qquad \partial_0 = i\partial_{\tau}, \tag{7.68}$$

so the action becomes

$$S_{\text{ferm}} = \int_0^\beta d\tau \int d^3x \left[-i\psi\sigma_0\partial_\tau\bar{\psi} + \psi\sigma_i\partial_i\bar{\psi}\right]. \tag{7.69}$$

We introduce the Euclidean sigma matrix $\sigma_4 = -i\sigma_0$, so

$$S_{\text{ferm}} = \int_0^\beta d\tau \int d^3x \left[\psi\sigma_4\partial_\tau\bar{\psi} + \psi\sigma_i\partial_i\bar{\psi}\right]. \tag{7.70}$$

We substitute the spinors with their Fourier transforms

$$\psi(\tau, x_i) = T\sum_{k_\tau}\int \frac{d^3k}{(2\pi)^3}\tilde{\psi}(k_\tau, k_i)e^{ik_\tau\tau + ik_ix_i}, \qquad \bar{\psi}(\tau, x_i) = T\sum_{k_\tau}\int \frac{d^3k}{(2\pi)^3}\tilde{\bar{\psi}}(k_\tau, k_i)e^{-ik_\tau\tau - ik_ix_i}, \tag{7.71}$$

where the component k_τ of the Euclidean 4-momentum can acquire the discrete values (cfr. [18], [17], [19])

$$k_\tau = (2n+1)\pi i T, \qquad n \in \mathbb{Z}. \tag{7.72}$$

k_τ depends on odd integers because the fermionics fields are antiperiodic over the time circle; a periodicity condition would have lead to

$$k_\tau = 2n\pi i T, \qquad n \in \mathbb{Z}, \tag{7.73}$$

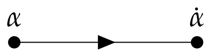
which is the case for the bosonic fields. The Euclidean action becomes

$$\begin{aligned} S_{\text{ferm}} &= T^2 \sum_{k_\tau, p_\tau} \int_0^\beta d\tau \int d^3x \int \frac{d^3k}{(2\pi)^3}\int \frac{d^3p}{(2\pi)^3} e^{i(k_\tau - p_\tau)\tau + i(k_i - p_i)x_i}\left[\tilde{\psi}\sigma_4\left(-ip_\tau\right)\tilde{\bar{\psi}} + \tilde{\psi}\sigma_i\left(-ip_i\right)\tilde{\bar{\psi}}\right] \\ &= -iT\sum_{k_\tau}\int d^3k \left[k_\tau\tilde{\psi}\sigma_4\tilde{\bar{\psi}} + k_i\tilde{\psi}\sigma_i\tilde{\bar{\psi}}\right] = T\sum_{k_\tau}\int d^3k\ \tilde{\psi}\left[-ik_\tau\sigma_4 - ik_i\sigma_i\right]\tilde{\bar{\psi}}. \end{aligned}$$

The Euclidean 2-points function at the tree level is

$$\langle\psi\bar{\psi}\rangle = i\frac{\delta^2}{\delta\tilde{\psi}\delta\tilde{\bar{\psi}}}S_{\text{ferm}} = \left[k_\tau\sigma_4 + k_i\sigma_i\right]. \tag{7.74}$$

The fermionic propagator is given by the inverse of the 2-points function



$$= \frac{1}{k_i\sigma_{i,\alpha\dot{\alpha}} + k_\tau\sigma_{4,\alpha\dot{\alpha}}}. \tag{7.75}$$

We notice that a term dependent on the temperature appeared

$$k_\tau\sigma_{4,\alpha\dot{\alpha}} = (2n+1)\pi T i\sigma_{4,\alpha\dot{\alpha}}. \tag{7.76}$$

The first important conclusion is that the Euclidean propagator of the free fermion, in 4 dimensions and at finite temperature, is equivalent to the Euclidean propagator of the free fermion in 3 dimensions, at zero temperature and with a T dependent correction. A possible way to deal with a 4-dimensional theory at finite temperature would be to compactify the fields over the time circle, exchanging each 4-dimensional field with an infinite tower of Matsubara modes (i.e. the coefficients of a discrete Fourier decomposition). As a consequence, the reduction of the theory on the 3-dimensional Euclidean space $\mathbb{R}^3$ would replace the Lagrangian with an infinite tower of Lagrangians, each one carrying the kinetic sector of a specific Matsubara mode and a (thermal) mass operator. However, in this thesis we will follow a different path.

We want to give an interpretation to the result (7.75). Let's assume that the fermionic fields have *periodic* boundary conditions over the time circle, instead of the (physically correct) antiperiodic ones. By doing this, we would obtain a slightly different result for the fermionic propagator

$$\overset{\alpha}{\bullet}\longrightarrow\overset{\dot\alpha}{\bullet} = \frac{1}{k_i\sigma_{i,\alpha\dot\alpha}+2n\pi i T\sigma_{4,\alpha\dot\alpha}}. \tag{7.77}$$

We recover the propagator of a fermionic field with antiperiodic boundary conditions and we rewrite its denominator as the denominator appearing in the propagator (7.77) plus a T dependent correction

$$\overset{\alpha}{\bullet}\longrightarrow\overset{\dot\alpha}{\bullet} = \frac{1}{k_i\sigma_{i,\alpha\dot\alpha}+(2n+1)\pi i T\sigma_{4,\alpha\dot\alpha}} = \frac{1}{(k_i\sigma_{i,\alpha\dot\alpha}+2n\pi i T\sigma_{4,\alpha\dot\alpha})+\pi i T\sigma_{4,\alpha\dot\alpha}}. \tag{7.78}$$

We notice that the propagator of a fermionic field with antiperiodic boundary conditions can be interpreted as the propagator (7.77), associated to periodic boundary conditions, provided with a thermal mass correction

$$\pi T\sigma_{4,\alpha\dot\alpha} = m\sigma_{4,\alpha\dot\alpha} = m_{\alpha\dot\alpha}. \tag{7.79}$$

Notice that whichever choice we make for the fermionic boundary conditions over the time circle, *supersymmetry is always explicitly broken*:

- if the fermions have *antiperiodic* boundary conditions, k_τ assumes the values $(2n+1)\pi i T$, with $n\in\mathbb{Z}$. The bosonic fields, instead, have periodic boundary conditions and k_τ assumes the values $2n\pi i T$. The different behavior of the fermionic and the bosonic fields over the time circle breaks the supersymmetry;
- if the fermions have *periodic* boundary conditions, their momentum component k_τ assumes the values $2n\pi i T$, with $n\in\mathbb{Z}$. The boundary conditions of the fermionic fields and of the bosonic fields over the time circle are the same. However, in this scenario, a fermionic mass operator naturally emerges in the tree-level Lagrangian and spoils the supersymmetry, breaking it softly.

We will adopt the second interpretation. Fermionic fields, from now on, will have *periodic boundary conditions* over the time circle and the original theory will be invariant under the action of the supersymmetry at finite temperature. However, the supersymmetry breaking will be inevitable because the theory will acquire an additional mass sector which will softly break the supersymmetry. The original free action becomes

$$S_{\text{ferm}} = \int_0^\beta d\tau \int d^3x\, [\psi\sigma_4\partial_\tau\bar\psi + \psi\sigma_i\partial_i\bar\psi + \psi m\bar\psi]\,, \tag{7.80}$$

where the mass operator is

$$\psi m\bar\psi = m\psi^\alpha\sigma_{4,\alpha\dot\alpha}\bar\psi^{\dot\alpha}. \tag{7.81}$$

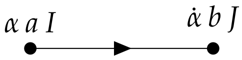
Now we can go back to the $\mathcal{N}=4$ theory and derive an analogous result. The computation is the same, but with the free fermionic sector of the $\mathcal{N}=4$ theory

$$S_{\text{ferm}} = \int d^4x\left[i\lambda^{aI}\sigma^\mu\partial_\mu\bar\lambda^a_I\right]. \tag{7.82}$$

The structure of the fermionic kinetic sector is identical to the one in the action (7.66), up to the R-simmetry indices. We can easily extract the fermionic propagator modifying the result (7.75)

$$i\frac{\delta}{\delta\tilde\lambda^{aI}\delta\tilde{\bar\lambda}^b_J}S_{\text{ferm}} = [ik_\tau\sigma_4 + ik_i\sigma_i]\,\delta_{ab}\delta^J_I. \tag{7.83}$$

The tree level fermionic propagator at finite temperature is



$$= \frac{\delta_{ab}\delta_I^J}{k_i\sigma_{i,\alpha\dot{\alpha}} + (2n+1)\pi T\sigma_{4,\alpha\dot{\alpha}}} = \frac{\delta_{ab}\delta_I^J}{(k_i\sigma_{i,\alpha\dot{\alpha}} + 2n\pi T\sigma_{4,\alpha\dot{\alpha}}) + \pi T\sigma_{4,\alpha\dot{\alpha}}}. \tag{7.84}$$

We adopt the periodic boundary conditions interpretation, so we add to the original action a fermionic mass operator

$$S_{\text{ferm}} = \int_0^\beta d\tau \int d^3x \left[\lambda^{aI}\sigma_4\partial_\tau\bar{\lambda}_I^a + \lambda^{aI}\sigma_i\partial_i\bar{\lambda}_I^a + i\lambda^{aI}m\bar{\lambda}_I^a\right] \tag{7.85}$$

where the mass operator is

$$\lambda^{aI}m\bar{\lambda}_I^a = m\lambda^{\alpha aI}\sigma_{4,\alpha\dot{\alpha}}\bar{\lambda}_I^{\dot{\alpha}a}. \tag{7.86}$$

One loop thermal masses

In the previous section we explicitly showed how a fermionic tree-level mass operator can emerge in the Lagrangian of the 4-dimensional $\mathcal{N} = 4$ theory. As we anticipated, this result would be sufficient in order to derive all the relevant conclusions in this thesis. However, we can do an exercise, introducing generic mass operators for the scalars and for the vector boson. The algebraic structures of these operators will be suggested by the one-loop results in the computations of the thermal masses, which we will take from [22].

- The *scalar* 2-points function acquires the following one loop correction

$$\left\langle \bar{X}_{IJ}^a X^{bKL}\right\rangle = \cdots + \frac{g^2}{12}T^2\delta_{[I}^{[K}\delta_{J]}^{L]}\delta^{ab}. \tag{7.87}$$

  From this one loop result we learn that the scalar thermal mass, up to one loop order, is equal to

$$M_X = \frac{g}{\sqrt{12}}T \tag{7.88}$$

  and that the correct algebraic structure of a generic scalar mass operator is

$$\bar{X}_{IJ}^a X^{aIJ}; \tag{7.89}$$

- The 2-points *fermionic* function acquires the following one loop correction

$$\left\langle \lambda^{aI}\bar{\lambda}_J^b\right\rangle = \cdots + g^2 T\frac{\log 2}{4\pi^2}\delta_J^I\delta^{ab}; \tag{7.90}$$

  We already knew from the tree level computation that the correct algebraic structure is

$$\lambda^{aI}\bar{\lambda}_I^a. \tag{7.91}$$

  The fermionic thermal mass at one loop, completed with the tree level mass, is equal to

$$m = \pi T + g^2 T\frac{\log 2}{4\pi^2}; \tag{7.92}$$

- The 2-points *vectorial* function acquires the following one loop correction

$$\left\langle A_\mu^a A_\nu^b\right\rangle = \cdots + \frac{g^2}{12}T^2\zeta_{\mu\nu}\delta^{ab}, \tag{7.93}$$

  where $\zeta_{\mu\nu}$ is a generic symmetric tensor; hence, the correct algebraic structure is

$$A_\mu^a A_\nu^a; \tag{7.94}$$

  and the one-loop thermal mass is equal to

$$M_A = \frac{g}{\sqrt{12}}T. \tag{7.95}$$

Structure of the mass operators

In this section we introduce three mass operators constructed with the field content of the $\mathcal{N} = 4$ theory. Later, we will make use of these operators to deform the $\mathcal{N} = 4$ theory, once the finite temperature is turned on. Once again, we stress the fact that the fermionic tree-level mass operator would be sufficient for our goals, while the general discussion is performed as a useful exercise.

It is important to notice that the explicit computations at one-loop level (7.89), (7.91) and (7.94) show that the correct algebraic structures, which are meant to represent the mass operators at any quantum order, are invariant under the action of the R-symmetry. In the following, we will compute the R-symmetry Ward identity at finite temperature and it won't be broken. This will be considered a proof *a posteriori* of the R-symmetry invariance at any loop order.

- In the $\mathcal{N} = 4$ theory the scalar degrees of freedom are framed in the $su(4)$ tensor X^{IJ}, which sits in the antisymmetric **6** representation of $su(4)$, in the singlet representation of the Lorentz group and in the adjoint representation of the gauge group. A candidate for the mass operator must be quadratic in the scalar fields and it is provided by the one loop result (7.89)

$$\mathrm{tr}\left[2M_X^2\bar{X}_{IJ}X^{IJ}\right] = -M_X^2\bar{X}^a_{IJ}X^{aIJ}; \tag{7.96}$$

- In the $\mathcal{N} = 4$ theory the fermionic degrees of freedom are framed in the $su(4)$ vectors λ^I_α and $\bar{\lambda}^{\dot{\alpha}}_I$, which sit in the fundamental representation of $su(4)$, in the $[1,0]$ and $[0,1]$ representations of the Lorentz group and in the adjoint representation of the gauge group. Gauge symmetry is trivially preserved taking the trace over the gauge indices. The mass operator must be quadratic, so we must combine two fermionic fields as it is suggested by the one loop result (7.91). We can write down a Majorana mass term

$$\mathrm{tr}\left[\bar{\Psi}M_\Psi\Psi\right] = -\frac{1}{2}\bar{\Psi}^a M_\Psi \Psi^a, \tag{7.97}$$

where the Weyl spinors are embedded as follows

$$\Psi^a = \begin{pmatrix} \lambda^{aI}_\alpha \\ \bar{\lambda}^{\dot{\alpha}a}_I \end{pmatrix} \tag{7.98}$$

and $\bar{\Psi} = \Psi^T\mathcal{C}$, where $\mathcal{C}$ is the charge conjugation matrix. The mass term can be expressed in terms of the Weyl spinors expanding the expression

$$-\frac{1}{2}\bar{\Psi}^a M_\Psi \Psi^a = -\frac{1}{2}\begin{pmatrix} \lambda^{\alpha aI} & \bar{\lambda}^a_{\dot{\alpha}I} \end{pmatrix}\begin{pmatrix} 0 & -m_{\alpha\dot{\alpha}} \\ \bar{m}^{\dot{\alpha}\alpha} & 0 \end{pmatrix}\begin{pmatrix} \lambda^{aI}_\alpha \\ \bar{\lambda}^{\dot{\alpha}a}_I, \end{pmatrix} \tag{7.99}$$

where the mass matrix is off-diagonal, in order to let left-chirality Weyl spinors couple to right-chirality ones, and viceversa. The mass term turns out to be

$$\frac{1}{2}\lambda^{\alpha aI}m_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}a}_I - \frac{1}{2}\bar{\lambda}^a_{\dot{\alpha}I}\bar{m}^{\dot{\alpha}\alpha}\lambda^{aI}_\alpha. \tag{7.100}$$

The mass coefficients inherits the spinor indices from a Sigma matrix (the identity σ_0, for example)

$$m_{\alpha\dot{\alpha}} \equiv m\sigma_{0,\alpha\dot{\alpha}}, \qquad \bar{m}^{\dot{\alpha}\alpha} \equiv m\bar{\sigma}_0^{\dot{\alpha}\alpha}. \tag{7.101}$$

Exploiting the properties of the Sigma matrices, we can write the fermionic mass operator as follows

$$\lambda^{\alpha aI}m_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}a}_I = \lambda^{aI}m\bar{\lambda}^a_I. \tag{7.102}$$

We succeeded in recovering the algebraic structure suggested by the tree-level and one loop results for the fermionic mass operator;

- In the $\mathcal{N}=4$ theory the vectorial degrees of freedom are grouped in the $su(4)$ singlet A^a_μ, which sits in the singlet representation of $su(4)$, in the $[1,1]$ representation of the Lorentz group and in the adjoint representation of the gauge group. The candidate for the mass operator is the following, as suggested by the one loop result (7.94)

$$\mathrm{tr}\left[M_A^{\mu\nu}A_\mu A_\nu\right] = -\frac{1}{2}M_A^{\mu\nu}A^a_\mu A^a_\nu. \tag{7.103}$$

The most general mass operator requires a mass tensor $M_A^{\mu\nu}$ with the following structure

$$M_A^{\mu\nu} = M^2_{A,T}\mathbb{P}_T^{\mu\nu} + M^2_{A,L}\mathbb{P}_L^{\mu\nu}. \tag{7.104}$$

The tensorial structure of the mass $M_A^{\mu\nu}$ keeps the longitudinal contribution to the mass separated from the transversal one: the tensor $\mathbb{P}_T^{\mu\nu}$ projects on the transversal direction while the tensor $\mathbb{P}_L^{\mu\nu}$ projects on the longitudinal direction.

The introduction of the vector mass operator requires a careful discussion of the BRST symmetry at finite temperature. The new operator, in fact, breaks the symmetry. In this thesis this issue will not be discussed.

7.3.2 Corrections to the Ward identities

The introduction of a finite temperature has important consequences on the theory. The finite temperature can modify the structure of a Ward identity, introducing breaking terms of different natures. In this section we will compute the Ward identities associated to the R-symmetry and to the supersymmetry at finite temperature, taking care of two different sources of breaking terms:

- the non trivial boundary conditions: switching on a finite temperature, we changed the spacetime manifold from $\mathbb{R}^4$ to $\mathbb{R}^3\times S^1$;
- a new (thermal) mass sector must be considered. As anticipated in the section 7.3.1, we will use the operators

$$S_{\text{mass}} = -i\int_0^\beta d\tau\int d^3x\left[-M_X^2\bar{X}^a_{IJ}X^{aIJ} - \lambda^{aI}m\bar{\lambda}^a_I - \frac{1}{2}M_A^{\mu\nu}A_\mu A_\nu\right]. \tag{7.105}$$

In the following, the full theory at finite temperature will be

$$S_{\text{full}} = S + S_{\text{mass}}. \tag{7.106}$$

R-symmetry Ward identity at finite temperature

In order to derive the corrections to the R-symmetry Ward identity, we want to consider all the possible corrections to the infinitesimal variation of the classical action S_{full}. First of all, we detach the original theory from the mass sector and we recover the zero temperature computation

$$\delta S = -\int d^4x\left[R^{\xi\mu}\partial_\mu\omega_\xi\right]. \tag{7.107}$$

The integration by parts must be handled carefully at finite temperature: we have to take into account the integral of the 4-divergence, too

$$\delta S = \int d^4x\left[\omega_\xi\partial_\mu R^{\xi\mu}\right] - \int d^4x\partial_\mu\left[R^{\xi\mu}\omega_\xi\right]. \tag{7.108}$$

The first term in the r.h.s. of the equation (7.108) is the one already present in the Ward identity at zero temperature. The second term cannot be trivially set to zero making use of the Gauss theorem: we turn the second term into a Euclidean integral and we evaluate it explicitly at finite temperature

$$\begin{aligned}\int_{\mathbb{R}^3\times S^1} d^4x\partial_\mu\left[R^{\xi\mu}\omega_\xi\right] &= \int_{\mathbb{R}^3\times S^1} -id\tau\, d^3x\left[i\partial_0\left(R_0^\xi\omega_\xi\right)+\vec{\nabla}\cdot\left(\vec{R}^\xi\omega_\xi\right)\right]\\ &= \int_{\mathbb{R}^3} d^3x\int_{S^1} d\tau\,\partial_0\left(R_0^\xi\omega_\xi\right) - i\int_{S^1} d\tau\int_{\mathbb{R}^3} d^3x\vec{\nabla}\cdot\left(\vec{R}^\xi\omega_\xi\right)\\ &= \int_{\mathbb{R}^3} d^3x\left[R_0^\xi\omega_\xi\right]_0^\beta - i\int_{S^1} d\tau\int_{\partial\mathbb{R}^3} d\vec{\Sigma}\cdot\left(\vec{R}^\xi\omega_\xi\right)\end{aligned}$$

The second integral in the r.h.s. can be computed: the boundary $\partial\mathbb{R}^3$ is the 2-sphere at infinity. We set the radius $\vec{r}$, with modulus r, representing the 3-dimensional distance from the origin to a generic point in $\mathbb{R}^3$. The conformal dimension of the current is $\left[\vec{R}^\xi\right]=3$, so the current goes as $\vec{R}^\xi\sim\frac{\vec{r}}{r^4}$ at the infinite

$$\int_{\partial\mathbb{R}^3} d\vec{\Sigma}\cdot\left(\vec{R}^\xi\omega_\xi\right)=\lim_{r\to\infty}4\pi r\vec{r}\cdot\frac{\vec{r}}{r^4}=0. \tag{7.109}$$

In the end, we just need to consider the integral

$$\int_{\mathbb{R}^3} d^3x\left[R_\tau^\xi\omega_\xi\right]_0^\beta \tag{7.110}$$

which is the only possible additional contribution to δS due to the new spacetime manifold $\mathbb{R}^3\times S^1$. However, the integrand is null because it is globally periodic over the time circle:

- the infinitesimal parameter ω_ξ is constant, hence it is trivially periodic over the time circle;
- the current (7.47) is periodic because each one of its terms contains a couple of fields with the same periodicity conditions over the time circle.

Hence, the non-trivial boundary conditions did not generate breaking terms in the R-symmetry Ward identity.

Now we consider the contribution given by the mass sector (7.105). In order to derive its contribution to δS_{full}, we apply the R-symmetry transformations to the fields

$$\begin{aligned}\delta S_{\text{mass}} &= -i\int_0^\beta d\tau\int d^3x\Big[-M_X^2\delta\bar{X}^a_{IJ}X^{aIJ}-M_X^2\bar{X}^a_{IJ}\delta X^{aIJ}-\delta\lambda^{aI}m\bar{\lambda}^a_I-\lambda^{aI}m\delta\bar{\lambda}^a_I-M_A^{\mu\nu}A_\mu\delta A_\nu\Big]\\ &= -i\int_0^\beta d\tau\int d^3x\Big[2\omega_\xi M_X^2\bar{X}^a_{IK}t^{\xi K}{}_J X^{aIJ}-2\omega_\xi M_X^2\bar{X}^a_{IJ}t^{\xi I}{}_K X^{aKJ}-\omega_\xi\lambda^{aJ}mt^{\xi I}{}_J\bar{\lambda}^a_I+\\ &\qquad +\omega_\xi\lambda^{aI}mt^{\xi J}{}_I\bar{\lambda}^a_J\Big]\\ &= 0.\end{aligned}$$

The infinitesimal variation of the mass sector under the action of the R-symmetry is equal to zero. Hence, the contribution of the mass sector to the infinitesimal variation of the classical action δS_{full} is equal to zero. The structure of the Ward identity is not modified and the R-symmetry holds at any loop level even at finite temperature

$$\int_0^\beta d\tau\int d^3x\left\langle\partial_\mu R^{\xi\mu}\mathcal{O}_1\ldots\mathcal{O}_n\right\rangle\omega_C=i\delta_{\text{R-symm}}\left\langle\mathcal{O}_1\ldots\mathcal{O}_n\right\rangle. \tag{7.111}$$

Supersymmetry Ward identity at finite temperature

In order to derive the corrections to the supersymmetry Ward identity, we consider again all the steps faced in the previous section for the R-symmetry Ward identity. First of all, we detach the original theory from the mass sector and we start from the zero temperature result

$$\delta S = -\int d^4x \left[G_I^\mu \partial_\mu \epsilon^I + h.c. \right] . \tag{7.112}$$

We perform the integration by parts and we carefully handle the total derivative

$$\delta S = \int d^4x \left[\partial_\mu G_I^\mu \, \epsilon^I + h.c. \right] - \int d^4x \partial_\mu \left[G_I^\mu \epsilon^I + h.c. \right] . \tag{7.113}$$

The second term can be explicitly evaluated

$$\begin{aligned}
\int_{\mathbb{R}^3\times S^1} d^4x \partial_\mu \left[G_I^\mu \epsilon^I + h.c. \right] &= \int_{\mathbb{R}^3\times S^1} -id\tau \, d^3x \left[i\partial_0 \left(G_{0I}\epsilon^I \right) + \vec{\nabla}\cdot\left(\vec{G}_I\epsilon^I\right) + h.c. \right] \\
&= \int_{\mathbb{R}^3} d^3x \int_{S^1} d\tau \, \partial_0 \left(G_{0I}\epsilon^I + h.c. \right) - i\int_{S^1} d\tau \int_{\mathbb{R}^3} d^3x \vec{\nabla}\cdot\left(\vec{G}_I\epsilon^I + h.c.\right) \\
&= \int_{\mathbb{R}^3} d^3x \left[G_{0I}\epsilon^I + h.c. \right]_\tau^\beta - i\int_{S^1} d\tau \int_{\partial\mathbb{R}^3} d\vec{\Sigma}\cdot\left(\vec{G}_I\epsilon^I + h.c.\right)
\end{aligned}$$

We show that the second term in the r.h.s. is equal to zero. The conformal dimension of the supercurrent is $\left[\vec{G}_i\right] = \frac{7}{2}$, thus the supercurrent goes as $\vec{G}_i \sim \frac{\vec{r}}{r^{9/2}}$ at the infinity

$$\int_{\partial\mathbb{R}^3} d\vec{\Sigma}\cdot\left(\vec{G}_I\epsilon^I + h.c.\right) = \lim_{r\to\infty} 4\pi r\vec{r}\cdot\frac{\vec{r}}{r^{9/2}} = 0. \tag{7.114}$$

We are left with the term

$$\int_{\mathbb{R}^3} d^3x \left[G_{0I}\epsilon^I + h.c. \right]_0^\beta . \tag{7.115}$$

The integrand is equal to zero:

- the infinitesimal parameter ϵ^I is constant, hence it must be periodic over the time circle;

- the behavior of the supercurrent component G_{0I} over the time circle has to be discussed. The explicit expression is (7.61): each term present in the expression is a composition of a bosonic field and a fermionic field. Hence, the boundary conditions of the supercurrent are the same of the fermionic fields. We recall the discussion at the end of the section 7.3.1: the fermionic fields are set to be periodic over the time circle in order to make the mass operator explicitly appear in the Lagrangian. Thus, the supercurrent is periodic over the time circle.

In the end, the non-trivial boundary conditions have no effect on the supersymmetry Ward identity.

Now we consider the contribution given by the mass sector (7.105). In order to obtain the contribution to the infinitesimal variation δS_{full}, we apply the supersymmetry transformations to the fields

$$\begin{aligned}\delta S_{\text{mass}} &= -i\int_0^\beta d\tau \int d^3x\Big[- M_X^2 \delta\bar{X}^a_{IJ} X^{aIJ} - M_X^2 \bar{X}^a_{IJ}\delta X^{aIJ} - \delta\lambda^{aI} m\bar{\lambda}^a_I - \lambda^{aI} m\delta\bar{\lambda}^a_I - M_A^{\mu\nu} A_\mu \delta A_\nu\Big] \\ &= -i\int_0^\beta d\tau \int d^3x\Big[-4\sqrt{2}M_X^2 \bar{X}^a_{IJ}\epsilon^I\lambda^{aJ} - \left(-\frac{1}{2}\sigma^{\mu\nu}F^a_{\mu\nu}\epsilon^I + gf^{abc}X^{bIJ}\bar{X}^c_{JK}\epsilon^K\right) m\bar{\lambda}^a_I - \\ &\qquad - i\sqrt{2}D_\mu \bar{X}^a_{IJ}\lambda^{aI} m\bar{\sigma}^\mu\epsilon^J - iM_A^{\mu\nu}A_\mu\bar{\lambda}^a_I\bar{\sigma}_\nu\epsilon^I\Big] + h.c. \\ &= -i\int_0^\beta d\tau \int d^3x\Big[-4\sqrt{2}M_X^2 \bar{X}^a_{IJ}\lambda^{aJ} - \frac{1}{2}\sigma^{\mu\nu}F^a_{\mu\nu}\bar{\lambda}^a_I\bar{m} + gf^{abc}X^{bIJ}\bar{X}^c_{JK}\bar{\lambda}^a_I\bar{m} + \\ &\qquad + i\sqrt{2}D_\mu \bar{X}^a_{IJ}\lambda^{aJ} m\bar{\sigma}^\mu - iM_A^{\mu\nu}A_\mu\bar{\lambda}^a_I\bar{\sigma}_\nu\Big]\epsilon^I + h.c. \\ &= -i\int_0^\beta d\tau \int d^3x\left[\mathcal{A}_I\epsilon^I + h.c.\right],\end{aligned}$$

where we defined the operator

$$\begin{aligned}\mathcal{A}_I = -4\sqrt{2}M_X^2\bar{X}^a_{IJ}\lambda^{aJ} - \frac{1}{2}\sigma^{\mu\nu}F^a_{\mu\nu}\bar{\lambda}^a_I\bar{m} + gf^{abc}X^{bIJ}\bar{X}^c_{JK}\bar{\lambda}^a_I\bar{m} + \\ + i\sqrt{2}D_\mu\bar{X}^a_{IJ}\lambda^{aJ}m\bar{\sigma}^\mu - iM_A^{\mu\nu}A_\mu\bar{\lambda}^a_I\bar{\sigma}_\nu.\end{aligned} \tag{7.116}$$

In the end, the complete variation of the full action is

$$\delta S_{\text{full}} = -i\int_0^\beta d\tau\int d^3x\left[\left(\partial_\mu G^\mu_I + \mathcal{A}_I\right)\epsilon^I + h.c.\right] \tag{7.117}$$

and the unintegrated supersymmetry Ward identity at finite temperature turns out to be

$$\int_0^\beta d\tau\int d^3x\,\langle\partial_\mu G^\mu_I\mathcal{O}_1\ldots\mathcal{O}_n\rangle\,\epsilon^I + \int_0^\beta d\tau\int d^3x\,\langle\mathcal{A}_I\mathcal{O}_1\ldots\mathcal{O}_n\rangle\,\epsilon^I + h.c. = i\delta_{\text{susy}}\,\langle\mathcal{O}_1\ldots\mathcal{O}_n\rangle\,. \tag{7.118}$$

7.3.3 The supersymmetry breaking term does not contribute

The supersymmetry Ward identity (7.118) at finite temperature includes the term

$$-\,i\int_0^\beta d\tau\int d^3x\,\langle\mathcal{A}_I\mathcal{O}_1\ldots\mathcal{O}_n\rangle\,\epsilon^I + h.c.. \tag{7.119}$$

In this section we want to show that the R-symmetry is sufficient to get rid of the integrand

$$\langle\mathcal{A}_I\mathcal{O}_1\ldots\mathcal{O}_n\rangle\,. \tag{7.120}$$

The idea is to show that the correlator is globally charged under the action of the R-symmetry: if the R-symmetry holds, this implies that the correlator is equal to zero. Actually, it is sufficient to show that the correlator (7.120) is charged under a particular subgroup of $su(4)$, i.e. the $u(1)$ subgroup generated by the Cartan generator t^{15}. In order to do so, we specialize the $su(4)$ unintegrated Ward identity to the $u(1)_{15}$ case. Recalling the section 5.2.1, we already know that the $u(1)_{15}$ Ward identity is equivalent to the following constraint, meant to hold at any loop order

$$\sum_{l=1}^n q_3^l = 0, \tag{7.121}$$

where q_3^l is the $u(1)_{15}$ charge of the operator $\mathcal{O}_l$ inside the correlator. As an hypothesis, we assume that the correlator

$$\langle\mathcal{O}_1\ldots\mathcal{O}_n\rangle \tag{7.122}$$

satisfies the constraint (7.121). This hypothesis is very specific and it is the easiest one to satisfy. However, it could be possible to choose a stronger hypothesis, too. For instance, the argument would still work if we imposed to the operators inside the correlator (7.122) to sit in a R-symmetry representation such that, once the operator $\mathcal{A}_I$ is introduced, the product cannot sit in a singlet representation.

The key step is to notice that the operator $\mathcal{A}_I$ must be charged under $u(1)_{15}$. Looking at the definition of the $\mathcal{A}_I$ operator, we can notice that it is composed of many terms: each term contains some bosonic operators plus a single fermionic operator. Recalling the table 7.1, we can see that (up to an overall factor $1/\sqrt{6}$) the bosonic fields have even q_3 charges, while the fermionic fields have odd q_3 charges. This proves that each term of the operator (7.116) is charged: hence, the entire operator $\mathcal{A}_I$ is charged. The correlator (7.120) does not respect the constraint (7.121), due to the addition of the charged operator $\mathcal{A}_I$ inside the braket: this proves that the integrand (7.120) is equal to zero.

In conclusion, if we impose the R-symmetry, the supersymmetry soft breaking term can be eliminated and the unintegrated supersymmetry Ward identity at finite temperature can be written as follows

$$\int_0^\beta d\tau \int d^3x \left\langle \partial_\mu G_I^\mu \mathcal{O}_1 \dots \mathcal{O}_n \right\rangle \epsilon^I + h.c. = i\delta \left\langle \mathcal{O}_1 \dots \mathcal{O}_n \right\rangle . \tag{7.123}$$

We conclude that the unintegrated supersymmetry Ward identity, if the theory is R-symmetry invariant, is *not* broken.

7.4 The Superconformal Ward identity at Finite Temperature

In the previous section we showed that it is possible to derive the supersymmetry Ward identity at finite temperature and that, even if the symmetry itself is broken, it is possible to write down the unintegrated Ward identity exactly with the same structure as at zero temperature. In order to set up all the tools for the proof of the non-renormalization theorem at finite temperature, we must be able to write down the correct Ward identity for the superconformal symmetry. The superconformal symmetry, like the supersymmetry, is broken at finite temperature, so the structure of its Ward identity includes some breaking terms, generated either by the newly introduced mass sector or by the non trivial boundary conditions. We want to show that the R-symmetry is able to cancel the breaking terms, showing us that the unintegrated superconformal Ward identity is not broken and that the integrated superconformal Ward identity has the same structure as (6.46).

First of all, we clarify that the superconformal symmetry is an *internal* symmetry with a spinorial parameter dependent on the coordinates: in fact, we recall that the superconformal charges $\mathcal{S}$ are defined as

$$\mathcal{S} \equiv I \circ \mathcal{Q} \circ I, \tag{7.124}$$

where I is the inversion operator acting on the coordinates. The operator $\mathcal{S}$ acts on the coordinates as follows

$$x^\mu \xrightarrow{I} \frac{x^\mu}{x^2} \xrightarrow{\mathcal{Q}} \frac{x^\mu}{x^2} \xrightarrow{I} x^\mu = x'^\mu. \tag{7.125}$$

The superconformal charges $\mathcal{S}$ act on the coordinates as the identity, hence the superconformal symmetry is an internal symmetry.

We can obtain the superconformal Ward identity considering the supersymmetry transformations and replacing the constant infinitesimal spinor ϵ^I with a conformal Killing spinor

$\psi^I(x)$, dependent on the coordinates. The superconformal transformations of the fields are

$$\delta A^a_\mu = -i\bar{\psi}_I(x)\bar{\sigma}_\mu\lambda^{aI} + i\bar{\lambda}^a_I\bar{\sigma}_\mu\psi^I(x), \tag{7.126}$$

$$\delta\lambda^{aI} = -\frac{1}{2}\sigma^{\mu\nu}F^a_{\mu\nu}\psi^I(x) + i\sqrt{2}\bar{\psi}_J(x)\bar{\sigma}^\mu D_\mu X^{aJI} + gf^{abc}X^{bIJ}\bar{X}^c_{JK}\psi^K(x), \tag{7.127}$$

$$\delta X^{aIJ} = \sqrt{2}\left(\psi^I(x)\lambda^{aJ} - \psi^J(x)\lambda^{aI} + \varepsilon^{IJKL}\bar{\psi}_K(x)\bar{\lambda}^a_L\right). \tag{7.128}$$

We apply the superconformal transformations to the action (7.39) in order to obtain the infinitesimal variation, which can be formally written as follows

$$\delta S = i\int_0^\beta d\tau\int d^3x\left[G^\mu_I\partial_\mu\psi^I(x) + h.c.\right]. \tag{7.129}$$

We proceed integrating by parts, obtaining

$$\delta S = i\int_0^\beta d\tau\int d^3x\left[\partial_\mu G^\mu_I\psi^I(x) + h.c.\right] - i\int_0^\beta d\tau\int d^3x\partial_\mu\left[G^\mu_I\psi^I(x) + h.c.\right]. \tag{7.130}$$

The second term must be evaluated with caution at finite temperature. Similarly to what we did for the supersymmetry, we have

$$\begin{aligned}
\int_{\mathbb{R}^3\times S^1} d^4x\partial_\mu\left[G^\mu_I\psi^I(x) + h.c.\right] &= \int_{\mathbb{R}^3\times S^1} -id\tau\, d^3x\left[i\partial_0\left(G_{0I}\psi^I(x)\right) + \vec{\nabla}\cdot\left(\vec{G}_I\psi^I(x)\right) + h.c.\right]\\
&= \int_{\mathbb{R}^3} d^3x\int_{S^1} d\tau\,\partial_0\left(G_{0I}\psi^I(x) + h.c.\right) -\\
&\quad - i\int_{S^1} d\tau\int_{\mathbb{R}^3} d^3x\vec{\nabla}\cdot\left(\vec{G}_I\psi^I(x) + h.c.\right)\\
&= \int_{\mathbb{R}^3} d^3x\left[G_{0I}\psi^I(x) + h.c.\right]_0^\beta -\\
&\quad - i\int_{S^1} d\tau\int_{\partial\mathbb{R}^3} d\vec{\Sigma}\cdot\left(\vec{G}_I\psi^I(x) + h.c.\right)
\end{aligned}$$

We evaluate explicitly the second term in the r.h.s.

$$\int_{\partial\mathbb{R}^3} d\vec{\Sigma}\cdot\left(\vec{G}_I\psi^I(x) + h.c.\right) \tag{7.131}$$

We already know that the supercurrent at the infinite goes as $\vec{G}_i \sim \frac{\vec{r}}{r^{9/2}}$; the explicit expression of the conformal Killing spinor is (6.30), thus $\psi_I \sim r$ at the infinite. The result is

$$\int_{\partial\mathbb{R}^3} d\vec{\Sigma}\cdot\left(\vec{G}_I\psi^I(x)\right) = \lim_{r\to\infty} 4\pi r^2\vec{r}\cdot\frac{\vec{r}}{r^{9/2}} = 0. \tag{7.132}$$

The contribution of the non trivial boundary conditions then is given by the integral

$$\int_{\mathbb{R}^3} d^3x\left[G_{0I}\psi^I(x) + h.c.\right]_0^\beta. \tag{7.133}$$

The integrand can be computed, substituting to $\psi^I(x)$ its explicit expression (6.30)

$$\left[G_{0I}\psi^I(x) + h.c.\right]_0^\beta = \left[G_{0\alpha I}\bar{\mu}^I_{\dot\beta}\bar{x}^{\dot\beta\alpha} + h.c.\right]_0^\beta = \left[iG_{0\alpha I}\bar{\mu}^I_{\dot\beta}\bar{\sigma}_4^{\dot\beta\alpha}\tau + h.c.\right]_0^\beta = 0, \tag{7.134}$$

where the last equality holds because

- the infinitesimal parameter $\bar{\mu}^I$ is constant, so it must be periodic over the time circle;

- the supercurrent component G_{0I} is periodic for the reasons discussed above for the supersymmetry Ward identity at finite temperature.

For these reasons, the contribution of the non trivial boundary conditions is equal to zero even for the superconformal Ward identity at finite temperature.

Now we consider the contribution coming from the mass sector (7.105). In order to derive the infinitesimal variation δS_{mass}, we apply the infinitesimal superconformal transformations (7.126), (7.127) and (7.128) to the mass sector. Recalling the definition of the $\mathcal{A}_I$ operator (7.116), we obtain

$$\delta S_{\text{mass}} = -i \int_0^\beta d\tau \int d^3x \left[\mathcal{A}_I \psi^I(x) + h.c. \right], \tag{7.135}$$

which is the contribution of the mass sector. The full infinitesimal variation of the classical action S_{full} is

$$\delta S_{\text{full}} = -i \int_0^\beta d\tau \int d^3x \left[\partial_\mu G^\mu_I \psi^I(x) + \mathcal{A}_I \psi^I(x) + h.c. \right] \tag{7.136}$$

From this result, we can immediately derive the unintegrated superconformal Ward identity

$$\begin{aligned} &\int_0^\beta d\tau \int d^3x \left\langle \partial_\mu G^\mu_I \psi^I(x) \mathcal{O}_1 \dots \mathcal{O}_n \right\rangle + \\ &\qquad + \int_0^\beta d\tau \int d^3x \left\langle \mathcal{A}_I \psi^I(x) \mathcal{O}_1 \dots \mathcal{O}_n \right\rangle + h.c. = i\delta_{\text{sc}} \langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle, \end{aligned} \tag{7.137}$$

where the r.h.s. stands for

$$\delta_{\text{sc}} \langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle = \sum_{i=1}^n \langle \mathcal{O}_1 \dots \delta\mathcal{O}_i \dots \mathcal{O}_n \rangle, \tag{7.138}$$

where $\delta\mathcal{O}_i$ is the variation of the operator $\mathcal{O}_i$ under the action of the superconformal transformations. We focus on the first term of the identity

$$\int_0^\beta d\tau \int d^3x \left\langle \partial_\mu G^\mu_I \psi^I(x) \mathcal{O}_1 \dots \mathcal{O}_n \right\rangle + h.c. \tag{7.139}$$

First of all, we extract the partial derivative from of the correlator: this can be done if and only if

$$\int_0^\beta d\tau \int d^3x \left\langle G^\mu_I \partial_\mu \psi^I(x) \mathcal{O}_1 \dots \mathcal{O}_n \right\rangle + h.c. = 0. \tag{7.140}$$

We show that this is the case: if we insert the explicit expression of the conformal Killing spinor (6.30), we get

$$\int_0^\beta d\tau \int d^3x \left\langle \bar{\mu}^I_{\dot{\alpha}} \bar{\sigma}^{\dot{\alpha}\beta}_\mu G^\mu_{\beta I} \mathcal{O}_1 \dots \mathcal{O}_n \right\rangle + h.c. = -\int_0^\beta d\tau \int d^3x \left\langle G^{\mu\beta}_I \sigma_{\mu,\beta\dot{\alpha}} \bar{\mu}^{\dot{\alpha} I} \mathcal{O}_1 \dots \mathcal{O}_n \right\rangle + h.c. \tag{7.141}$$

The integrand in the r.h.s. is equal to zero: it is sufficient to apply the constraint (6.23)

$$G^{\mu\beta}_I \sigma_{\mu\beta\dot{\alpha}} = 0. \tag{7.142}$$

In the end, the first term in the l.h.s. of the equation (7.137) can be rewritten as

$$\int_0^\beta d\tau \int d^3x \, \partial_\mu \left\langle \psi^I(x) G^\mu_I \mathcal{O}_1 \dots \mathcal{O}_n \right\rangle + h.c. \tag{7.143}$$

We can recognize the conserved current (6.24) inside the correlator

$$\int_0^\beta d\tau \int d^3x \, \partial_\mu \langle J^\mu(x)\mathcal{O}_1 \ldots \mathcal{O}_n \rangle + h.c. \tag{7.144}$$

We already know how to deal with this integral: we integrate over a volume $\mathcal{V}$ which contains all the points $x_1, \ldots, x_n$, where the operators $\mathcal{O}_1, \ldots, \mathcal{O}_n$ are evaluated. The volume integral can be transformed into an integral over the surface $\partial\mathcal{V}$, which is composed of the 2-sphere at infinity times the two extrema of the time circle

$$\partial\mathcal{V} = S^2_\infty \times \{0, \beta\}\,. \tag{7.145}$$

Then we turn the volume integral (7.144) into the sum of two surface integrals

$$\int_{\partial\mathcal{V}} d\Sigma_\mu \, \langle J^\mu(x)\mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle - \sum_{l=1}^{n} \int_{S_l} d\Sigma_\mu \, \langle J^\mu(x)\mathcal{O}_1(x_1) \ldots \mathcal{O}_l(x_l) \ldots \mathcal{O}_n(x_n) \rangle + h.c. \tag{7.146}$$

The second term is identical to the integral (6.36). Each integral in the sum can be explicitly evaluated applying an OPE between the current J^μ and each operator $\mathcal{O}_l$: the OPE is localized around each puncture x_l, thus it is not influenced by the finite temperature effect, which determines a modification of the global structure of the spacetime, not of the local structure. Thus, we already know the replacement for the second term

$$\begin{aligned}
&-\sum_{l=1}^{n} \int_{S_l} d\Sigma_\mu \, \langle J^\mu(x)\mathcal{O}_1(x_1) \ldots \mathcal{O}_l(x_l) \ldots \mathcal{O}_n(x_n) \rangle + h.c. = \\
&\qquad = -\sum_{l=1}^{n} \psi_I^\alpha(x_l) \left\langle \mathcal{O}_1(x_1) \ldots \left[\mathcal{Q}^I_\alpha, \mathcal{O}_l \right\} (x_l) \ldots \mathcal{O}_n(x_n) \right\rangle + \\
&\qquad\quad + \left(\partial^\mu \psi_I^\alpha\right)(x_l) n_\mu n_\nu \sigma^\nu_{\alpha\dot\beta} \left\langle \mathcal{O}_1(x_1) \ldots \left[\bar{\mathcal{S}}^{\dot\beta I}, \mathcal{O}_l \right\} (x_l) \ldots \mathcal{O}_n(x_n) \right\rangle + h.c.
\end{aligned} \tag{7.147}$$

The only contribution left is given by the first integral in the expression (7.146). It is a surface integral evaluated over the boundary (7.145). Even at finite temperature, it can be proven that this contribution is equal to zero. We introduce the 4-vector n_μ, with a unitary module, such that $d\Sigma_\mu = d\Sigma n_\mu$

$$\int_{\partial\mathcal{V}} d\Sigma_\mu \, \langle J^\mu(x)\mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle = \int_{\partial\mathcal{V}} d\Sigma n_\mu \, \langle J^\mu(x)\mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle \tag{7.148}$$

The infinitesimal surface element can be splitted in two orthogonal contributions

$$\int_{\partial\mathcal{V}} d\Sigma n_\mu \, \langle J^\mu(x)\mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle = \int_{\partial\mathbb{R}^3 \times \{0,\beta\}} d\Sigma n_\mu \, \langle J^\mu(x)\mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle\,. \tag{7.149}$$

We can always choose to integrate first over the surface $\partial\mathbb{R}^3$. The τ component of the 4-vector n_μ, when it is evaluated on the 2-sphere at infinity, is null, so we will consider only its spatial components n_i. The current J^μ at infinity goes like $J \sim \frac{1}{r^{5/2}}$, so the integrand goes to zero at least as fast as $\frac{1}{r^5}$. Then, we have

$$\int_{\partial\mathbb{R}^3} d\Sigma n_i \, \langle J_i(x)\mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle \simeq \lim_{r\to\infty} r^2 \frac{1}{r^5} = 0 \tag{7.150}$$

In conclusion, the first term in the equation (7.137) can be rewritten as a sum of local contributions, where each term of the sum is evaluated in a specific point of the volume $\mathcal{V}$

$$\begin{aligned}
&\int_0^\beta d\tau \int d^3x \, \partial_\mu \langle J^\mu(x)\mathcal{O}_1 \ldots \mathcal{O}_n \rangle + h.c. = \\
&\qquad = -\sum_{l=1}^{n} \psi_I^\alpha(x_l) \left\langle \mathcal{O}_1(x_1) \ldots \left[\mathcal{Q}^I_\alpha, \mathcal{O}_l \right\} (x_l) \ldots \mathcal{O}_n(x_n) \right\rangle + \\
&\qquad\quad + \left(\partial^\mu \psi_I^\alpha\right)(x_l) n_\mu n_\nu \sigma^\nu_{\alpha\dot\beta} \left\langle \mathcal{O}_1(x_1) \ldots \left[\bar{\mathcal{S}}^{\dot\beta I}, \mathcal{O}_l \right\} (x_l) \ldots \mathcal{O}_n(x_n) \right\rangle + h.c.
\end{aligned} \tag{7.151}$$

The r.h.s. of the equation (7.151) are the [Contact terms] of the superconformal Ward identity. Recalling the equation (6.34), we can identify the r.h.s. of the equation (7.151) with the r.h.s. of the equation (7.137)

$$\delta_{\text{sc}} \langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle \,. \tag{7.152}$$

We are finally ready to write down the full superconformal Ward identity at finite temperature, provided with the contribution coming from the thermal mass sector

$$\begin{aligned} &\int_0^\beta d\tau \int d^3x \left\langle \partial_\mu G_I^\mu \psi^I(x) \mathcal{O}_1 \dots \mathcal{O}_n \right\rangle + \\ &\quad + \int_0^\beta d\tau \int d^3x \left\langle \mathcal{A}_I \psi^I(x) \mathcal{O}_1 \dots \mathcal{O}_n \right\rangle + h.c. = \\ &\qquad = - \sum_{l=1}^n \psi_I^\alpha(x_l) \left\langle \mathcal{O}_1(x_1) \dots \left[\mathcal{Q}_\alpha^I, \mathcal{O}_l \right\} (x_l) \dots \mathcal{O}_n(x_n) \right\rangle + \\ &\qquad\quad + \left(\partial_\mu \psi_I^\alpha \right)(x_l) n_\mu n_\nu \sigma^\nu_{\alpha\dot\beta} \left\langle \mathcal{O}_1(x_1) \dots \left[\bar{\mathcal{S}}^{\dot\beta I}, \mathcal{O}_l \right\} (x_l) \dots \mathcal{O}_n(x_n) \right\rangle + h.c. \end{aligned} \tag{7.153}$$

We consider the correlator

$$\left\langle \partial_\mu G_I^\mu \psi^I(x) \mathcal{O}_1 \dots \mathcal{O}_n \right\rangle . \tag{7.154}$$

This correlator contains the operator $\partial_\mu G_I^\mu$. Recalling the constraint (6.18), we know that in a theory with conformal symmetry the supercurrent G_I^μ fulfills a unitarity bound such that the state

$$\partial_\mu G_I^\mu \left|\text{vac}\right\rangle = \left|\partial_\mu G_I^\mu\right\rangle \tag{7.155}$$

is the null vector. Recalling the considerations made at the beginning of the section 7.3, this property is preserved at finite temperature: the state $\left|\partial_\mu G_I^\mu\right\rangle$ belongs to a Hilbert space constructed using the action S, not the full action S_{full}, comprehensive of the mass sector. Finally, we consider the correlator

$$\left\langle \mathcal{A}_I \psi^I(x) \mathcal{O}_1 \dots \mathcal{O}_n \right\rangle . \tag{7.156}$$

We substitute to $\psi^I(x)$ its explicit expression (6.30), obtaining

$$\left\langle \mathcal{A}_I \psi^I(x) \mathcal{O}_1 \dots \mathcal{O}_n \right\rangle = -\lambda^I_\alpha \left\langle \mathcal{A}_I^\alpha \mathcal{O}_1 \dots \mathcal{O}_n \right\rangle - \bar\mu_{\dot\beta} \bar{x}^{\dot\beta\alpha} \left\langle \mathcal{A}_{\alpha I} \mathcal{O}_1 \dots \mathcal{O}_n \right\rangle . \tag{7.157}$$

The first term in the r.h.s. has already been set to zero in the section 7.3.3 and the second term can be set to zero for the same reasons: the operator $\mathcal{A}_I$ is charged under the action of the group $u(1)_{15}$, so, under the correct hypothesis on the operators $\mathcal{O}_1, \dots \mathcal{O}_n$, the R-symmetry invariance sets the first and the second term to zero.

The final expression of the superconformal Ward identity at finite temperature, cleared from the finite temperature soft breaking term, is

$$\begin{aligned} &\sum_{l=1}^n \psi_I^\alpha(x_l) \left\langle \mathcal{O}_1(x_1) \dots \left[\mathcal{Q}_\alpha^I, \mathcal{O}_l \right\} (x_l) \dots \mathcal{O}_n(x_n) \right\rangle + \\ &\qquad + \left(\partial_\mu \psi_I^\alpha \right)(x_l) n_\mu n_\nu \sigma^\nu_{\alpha\dot\beta} \left\langle \mathcal{O}_1(x_1) \dots \left[\bar{\mathcal{S}}^{\dot\beta I}, \mathcal{O}_l \right\} (x_l) \dots \mathcal{O}_n(x_n) \right\rangle = 0. \end{aligned} \tag{7.158}$$

Chapter 8

A Non-Renormalization Theorem at $T \neq 0$

In this chapter we prove the non-renormalization theorem studied in the chapter 6 in a finite temperature scenario. We will not propose a different proof, but we will follow the same procedure employed at zero temperature. The adaptation is possible because, exploiting the results in the chapter 7, we will show that all the tools required to show the final result can be obtained at finite temperature, too. In particular, in this chapter we will focus on the conformal invariance. The issue will be tackled with the introduction of a conformal compensator, an auxiliary field coupled to the original theory. This new feature of the theory will be studied following the article [23].

8.1 Recovering the Conformal symmetry

In this section we will show the procedure to recover the conformal invariance, present in the original $\mathcal{N} = 4$ theory. The invariance is spoiled after the introduction of a finite temperature. In particular, the conformal invariance gets encoded in a Ward identity which is softly broken by the mass operators, turnt on by the finite temperature. The breaking terms are not charged under the R-symmetry, so we cannot use it to get rid of them, like we did in the chapter 7. In this case, it will be necessary to couple the $\mathcal{N} = 4$ theory to a *conformal compensator*. The coupling between the original theory and the new sector can be set arbitrarily weak: the only purpose of the newly introduced degree of freedom will be to explicitly recover the conformal invariance, removing the breaking terms from the scale invariance unintegrated Ward identity.

8.1.1 The scale invariance Ward identity at finite temperature

In the section 5.3.1 we derived the explicit expression of the unintegrated scale invariance Ward identity at zero temperature. We can take those results as a starting point before turning on the temperature: we need to study the effects of the new non trivial boundary conditions and of the (thermal) mass operators. In particular, we are interested in the possible breaking terms generated by those effects. We consider the $\mathcal{N} = 4$ theory and we explicitly compute the canonical stress-energy tensor with the formula

$$T_c^{\mu\nu} = \frac{\partial \mathcal{L}_{\mathcal{N}=4}}{\partial \partial_\mu X^{aIJ}} \partial^\nu X^{aIJ} + \frac{\partial \mathcal{L}_{\mathcal{N}=4}}{\partial \partial_\mu \bar{X}^a_{IJ}} \partial^\nu \bar{X}^a_{IJ} + \frac{\partial \mathcal{L}_{\mathcal{N}=4}}{\partial \partial_\mu \lambda^{aI}} \partial^\nu \lambda^{aI} + \frac{\partial \mathcal{L}_{\mathcal{N}=4}}{\partial \partial_\mu \bar{\lambda}^a_I} \partial^\nu \bar{\lambda}^a_I + \\ + \frac{\partial \mathcal{L}_{\mathcal{N}=4}}{\partial \partial_\mu A^a_\rho} \partial^\nu A^a_\rho - \eta^{\mu\nu} \mathcal{L}_{\mathcal{N}=4}. \quad (8.1)$$

The explicit expression of the canonical stress-energy tensor is

$$
\begin{aligned}
T_c^{\mu\nu} = -\eta^{\mu\nu}\Bigg[& -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + i\lambda^{ai}\sigma^\mu D_\mu \bar{\lambda}^a_i - \frac{1}{4}D_\mu \bar{X}^a_{IJ} D^\mu X^{aIJ} + \frac{\sqrt{2}}{2} g f^{abc}\lambda^{aI}\lambda^{bJ}\bar{X}^C_{IJ} + \\
& + \frac{\sqrt{2}}{2} g f^{abc}\bar{\lambda}^a_I \bar{\lambda}^b_J X^{cIJ} - \frac{1}{16}g^2 f^{abc}f^{aeg}X^{bIJ}X^{cKL}\bar{X}^e_{IJ}\bar{X}^g_{KL}\Bigg] - \frac{1}{4}D^\mu \bar{X}^a_{IJ}\partial^\nu X^{aIJ} - \frac{1}{4}\partial^\nu \bar{X}^a_{IJ} D^\mu X^{aIJ} + \\
& + i\lambda^{aI}\sigma^\mu \partial^\nu \bar{\lambda}^a_I - F^{a\mu\rho}\partial^\nu A^a_\rho, \qquad (8.2)
\end{aligned}
$$

while the trace of the canonical stress-energy tensor is given by

$$
T^\mu{}_{\mu,c} = -4\mathcal{L}_{\mathcal{N}=4} - \frac{1}{4}D^\mu \bar{X}^a_{IJ}\partial_\mu X^{aIJ} - \frac{1}{4}\partial^\mu \bar{X}^a_{IJ} D_\nu X^{aIJ} + + i\lambda^{aI}\sigma^\mu \partial_\mu \bar{\lambda}^a_I - F^{a\mu\rho}\partial_\mu A^a_\rho. \qquad (8.3)
$$

Recalling the definition (5.52), we can write

$$
\begin{aligned}
T^\mu{}_\mu = & -4\mathcal{L}_{\mathcal{N}=4} - \frac{1}{4}D^\mu \bar{X}^a_{IJ}\partial_\mu X^{aIJ} - \frac{1}{4}\partial^\mu \bar{X}^a_{IJ} D_\nu X^{aIJ} + i\lambda^{aI}\sigma^\mu \partial_\mu \bar{\lambda}^a_I - F^{a\mu\rho}\partial_\mu A^a_\rho - \\
& - \frac{1}{4}\partial_\mu\left(D^\mu \bar{X}^a_{IJ} X^{aIJ}\right) - \frac{1}{4}\partial_\mu\left(\bar{X}^a_{IJ} D_\nu X^{aIJ}\right) + \frac{3}{2}\partial_\mu\left(i\lambda^{aI}\sigma^\mu \bar{\lambda}^a_I\right) - \partial_\mu\left(F^{a\mu\rho}A^a_\rho\right). \qquad (8.4)
\end{aligned}
$$

In the following, we will call the action of the $\mathcal{N} = 4$ theory at zero temperature S_{CFT} (it will also be addressed as the *original* theory), the stress energy tensor (8.2) will be called $T^{\mu\nu}_{\text{CFT}}$ and the trace (8.4) will be called $T^\mu{}_{\mu,\text{CFT}}$. Turning on the finite temperature, the full theory becomes

$$
S_{\text{full}} = S_{\text{CFT}} + S_{\text{relevant}}, \qquad (8.5)
$$

where the sector S_{relevant} is

$$
S_{\text{relevant}} = -i\int_0^\beta d\tau \int d^3x \left[-M_X^2 \bar{X}^a_{IJ}X^{aIJ} - \lambda^{aI} m \bar{\lambda}^a_I - \frac{1}{2}M_A^{\mu\nu}A_\mu A_\nu\right]. \qquad (8.6)
$$

Recalling the general result (5.13), we need to compute the infinitesimal variation δS_{full} in order to derive the scale invariance Ward identity at finite temperature. We have that

$$
\delta S_{\text{full}} = \delta S_{\text{CFT}} + \delta S_{\text{relevant}}. \qquad (8.7)
$$

The variation δS_{CFT} has already been computed in the section 5.3.1

$$
\delta S_{\text{CFT}} = -\int d^4x\, W^\mu \partial_\mu \alpha = -\int d^4x\, \partial_\mu\left[W^\mu \alpha\right] + \int d^4x\, \partial_\mu W^\mu \alpha. \qquad (8.8)
$$

We carefully compute the integral of the total divergence

$$
\begin{aligned}
\int_{\mathbb{R}^3\times S^1} d^4x\, \partial_\mu\left[W^\mu\alpha\right] &= \int_{\mathbb{R}^3\times S^1} -id\tau\, d^3x \left[i\partial_0\left(W_0\alpha\right) + \vec{\nabla}\cdot\left(\vec{W}\alpha\right)\right] \\
&= \int_{\mathbb{R}^3} d^3x \int_{S^1} d\tau\, \partial_0\left(W_0\alpha\right) - i\int_{S^1} d\tau \int_{\mathbb{R}^3} d^3x\, \vec{\nabla}\cdot\left(\vec{W}\alpha\right) \\
&= \int_{\mathbb{R}^3} d^3x \left[W_0\alpha\right]_0^\beta - i\int_{S^1} d\tau \int_{\partial\mathbb{R}^3} d\vec{\Sigma}\cdot\left(\vec{W}\alpha\right).
\end{aligned}
$$

The second term in the r.h.s. can be set equal to zero: the conformal dimension of the current is $\left[\vec{W}\right] = 3$, so at infinity the current goes like $\vec{W} \sim \frac{\vec{r}}{r^4}$

$$
\int_{\partial\mathbb{R}^3} d\vec{\Sigma}\cdot\left(\vec{W}\alpha\right) = \lim_{r\to\infty} 4\pi r\vec{r}\cdot\frac{\vec{r}}{r^4} = 0; \qquad (8.9)
$$

the first term is equal to zero, too. In fact, knowing that the entire field content of the S_{CFT} theory is periodic over the time cirle and that the explicit expression for the scale invariance current is (5.50), we have

$$[W_0\alpha]_0^\beta = \left[x^\nu T^0{}_{\nu,\text{CFT}}\,\alpha\right]_0^\beta + \Delta_\varphi \left[\frac{\partial \mathcal{L}_{\text{CFT}}}{\partial \partial_0 \varphi}\varphi\alpha\right]_0^\beta = \left[\left(\tau T_{\text{CFT}}{}^0{}_0 + x_i T_{\text{CFT}}{}^0{}_i\right)\alpha\right]_0^\beta = 0. \tag{8.10}$$

where the variable φ stands for all the fields in the S_{CFT} theory. In conclusion, the non trivial boundary conditions has no effect on the infinitesimal variation of the classical action δS_{CFT}, so

$$\delta S_{\text{CFT}} = -i\alpha \int_0^\beta d\tau \int d^3x\, \partial_\mu W^\mu = -i\alpha \int_0^\beta d\tau \int d^3x\, (x_\nu \partial_\mu T^{\mu\nu}_{\text{CFT}}) - i\alpha \int_0^\beta d\tau \int d^3x\, T^\mu{}_{\mu,\text{CFT}}\,. \tag{8.11}$$

The variation $\delta S_{\text{relevant}}$ can be explicitly computed acting on the fields with the transformation (5.43)

$$\begin{aligned}\delta S_{\text{relevant}} = -i\alpha \int_0^\beta d\tau \int d^3x \Big\{ & 2M_X^2 \bar{X}^a_{IJ} X^{aIJ} + 3\lambda^{aI} m \bar{\lambda}^a_I + M_A^{\rho\sigma} A_\rho A_\sigma + \\ & + x_\nu \partial_\mu \left[-\eta^{\mu\nu}\left(-M_X^2 \bar{X}^a_{IJ} X^{aIJ} - \lambda^{aI} m \bar{\lambda}^a_I - \frac{1}{2} M_A^{\rho\sigma} A_\rho A_\sigma\right)\right]\Big\},\end{aligned} \tag{8.12}$$

so the complete infinitesimal variation of the full action is

$$\begin{aligned}\delta S_{\text{full}} = & -i\alpha \int_0^\beta d\tau \int d^3x \left[x_\nu \partial_\mu T^{\mu\nu}_{\text{full}}\right] - i\alpha \int_0^\beta d\tau \int d^3x\, T^\mu{}_{\mu,\text{CFT}} - \\ & - i\alpha \int_0^\beta d\tau \int d^3x \left[2M_X^2 \bar{X}^a_{IJ} X^{aIJ} + 3\lambda^{aI} m \bar{\lambda}^a_I + M_A^{\rho\sigma} A_\rho A_\sigma\right].\end{aligned} \tag{8.13}$$

It is useful to split the breaking term in four different breaking terms, each one multiplied by a mass coefficient

$$\begin{aligned}& - i\alpha \int_0^\beta d\tau \int d^3x \left[2M_X^2 \bar{X}^a_{IJ} X^{aIJ} + 3\lambda^{aI} m \bar{\lambda}^a_I + M_A^{\rho\sigma} A_\rho A_\sigma\right] = \\ & = -2i\alpha M_X^2 \int_0^\beta d\tau \int d^3x\, \mathcal{M}_X - 3i\alpha m \int_0^\beta d\tau \int d^3x\, \mathcal{M}_\lambda - \\ & - i\alpha M_{A,T}^2 \int_0^\beta d\tau \int d^3x\, \mathcal{M}_{A,T} - i\alpha M_{A,L}^2 \int_0^\beta d\tau \int d^3x\, \mathcal{M}_{A,L},\end{aligned} \tag{8.14}$$

where we defined

$$\mathcal{M}_X = \bar{X}^a_{IJ} X^{aIJ}, \quad \mathcal{M}_\lambda = \lambda^{aI}\sigma_0 \bar{\lambda}^a_I, \quad \mathcal{M}_{A,T} = -\frac{1}{2}\mathbb{P}_T^{\mu\nu} A^a_\mu A^a_\nu, \quad \mathcal{M}_{A,L} = -\frac{1}{2}\mathbb{P}_L^{\mu\nu} A^a_\mu A^a_\nu. \tag{8.15}$$

The final expression for the softly broken unintegrated scale invariance Ward identity can be recovered making use of the unintegrated translations Ward identity, as we did in the section 5.3.1. The result is

$$\begin{aligned}& \int_0^\beta d\tau \int d^3x \left\langle T^\mu{}_{\mu,\text{CFT}} \mathcal{O}_1 \ldots \mathcal{O}_n\right\rangle + 2M_X^2 \int_0^\beta d\tau \int d^3x \left\langle \mathcal{M}_X \mathcal{O}_1 \ldots \mathcal{O}_n\right\rangle + \\ & + 3m \int_0^\beta d\tau \int d^3x \left\langle \mathcal{M}_\lambda \mathcal{O}_1 \ldots \mathcal{O}_n\right\rangle + M_{A,T}^2 \int_0^\beta d\tau \int d^3x \left\langle \mathcal{M}_{A,T} \mathcal{O}_1 \ldots \mathcal{O}_n\right\rangle + \\ & + M_{A,L}^2 \int_0^\beta d\tau \int d^3x \left\langle \mathcal{M}_{A,L} \mathcal{O}_1 \ldots \mathcal{O}_n\right\rangle = -\sum_{l=1}^n \Delta_{\mathcal{O}_l} \left\langle \mathcal{O}_1 \ldots \mathcal{O}_l(x_l) \ldots \mathcal{O}_n\right\rangle.\end{aligned} \tag{8.16}$$

8.1.2 Introducing a conformal compensator

In this section we couple the theory at finite temperature S_{full} with a conformal compensator, following the procedure described in the article [23]. Our choice for the compensator will be a real, scalar field ω. The field is adimensional and transforms neither under the action of the supersymmetry charges $\mathcal{Q}$ and $\mathcal{S}$, nor under the action of the R-symmetry generators. After having coupled the compensator, we will have to deal with a theory described by the classical action

$$S_{\text{full+c.c.}} = S_{\text{CFT}} + S_{\omega} + S'_{\text{relevant}}, \tag{8.17}$$

where the action S_{ω} is simply the kinetic sector

$$S_{\omega} = -f^2 \int d^4x \left[\partial_\mu \omega \partial^\mu \omega\right], \tag{8.18}$$

and the sector S'_{relevant} is exactly the mass sector where every mass coefficient has been coupled to a copy of the compensator ω

$$S'_{\text{relevant}} = -i \int_0^\beta d\tau \int d^3x \left[-M_X^2 \omega^2 \bar{X}^a_{IJ} X^{aIJ} - \lambda^{aI} m \omega \bar{\lambda}^a_I - \frac{1}{2} M_A^{\mu\nu} \omega^2 A_\mu A_\nu\right]. \tag{8.19}$$

Notice that we have the freedom to tune arbitrarily the coefficient f: this feature will be crucial in the discussion. We can interpret the whole theory $S_{\text{full+c.c}}$ as an effective field theory of the conformal compensator, valid at an energy scale much smaller than a given cutoff Λ.

Toy model: scalar field theory

As an example, in this section we study the case of a simple scalar field theory. The starting point is the free scalar action

$$S_{\text{CFT}} = -\int d^4x \left[\partial_\mu \bar{\phi} \partial^\mu \phi\right]. \tag{8.20}$$

This theory will be addressed as the conformal fixed point. The scale invariance unintegrated Ward identity is (cfr. the section 5.3.2)

$$\int d^4x \left\langle T^\mu{}_{\mu,\text{CFT}} \mathcal{O}_1 \dots \mathcal{O}_n \right\rangle = -\sum_{l=1}^{n} \Delta_{\mathcal{O}_l} \left\langle \mathcal{O}_1 \dots \mathcal{O}_l(x_l) \dots \mathcal{O}_n \right\rangle, \tag{8.21}$$

where

$$T^\mu{}_{\mu,\text{CFT}} = 2\partial_\mu \bar{\phi} \partial^\mu \phi + \partial_\mu \left(\partial^\mu \bar{\phi} \phi + \bar{\phi} \partial^\mu \phi\right). \tag{8.22}$$

The theory S_{CFT} then is deformed by the addition of a mass operator, which triggers a relevant deformation of the theory

$$S_{\text{full}} = S_{\text{CFT}} + S_{\text{relevant}} = -\int d^4x \left[\partial_\mu \bar{\phi} \partial^\mu \phi + m^2 \bar{\phi} \phi\right]. \tag{8.23}$$

The unintegrated softly broken Ward identity then is given by (cfr. the section 5.3.2)

$$\int d^4x \left\langle T^\mu{}_{\mu,\text{CFT}} \mathcal{O}_1 \dots \mathcal{O}_n \right\rangle + 2m^2 \int d^4x \left\langle \bar{\phi}(x) \phi(x) \mathcal{O}_1 \dots \mathcal{O}_n \right\rangle = \\ = -\sum_{l=1}^{n} \Delta_{\mathcal{O}_l} \left\langle \mathcal{O}_1 \dots \mathcal{O}_l(x_l) \dots \mathcal{O}_n \right\rangle. \tag{8.24}$$

We want to recover the structure of the unintegrated Ward identity (5.62), so we introduce a conformal compensator

$$S_{\text{full+c.c.}} = -\int d^4x \left[\partial_\mu\bar{\phi}\partial^\mu\phi + m^2\omega^2\bar{\phi}\phi + f^2\partial_\mu\omega\partial^\mu\omega\right]. \tag{8.25}$$

The action (8.25) is meant to describe an *effective field theory* of the conformal compensator ω, thus it requires the existence of a cutoff Λ. If p represents the momenta circulating in the system, the theory is correct in the regime

$$p^2 \ll \Lambda^2. \tag{8.26}$$

The softly broken scale invariance Ward identity of the $S_{\text{full+c.c.}}$ theory is

$$\begin{aligned}\int d^4x \, \langle T^\mu{}_{\mu,\text{CFT}}\,\mathcal{O}_1\ldots\mathcal{O}_n\rangle + f^2\int d^4x\,\langle T^\mu{}_{\mu,\omega}\,\mathcal{O}_1\ldots\mathcal{O}_n\rangle +\\ +2m^2\int d^4x\,\langle\bar{\phi}(x)\phi(x)\omega^2(x)\mathcal{O}_1\ldots\mathcal{O}_n\rangle = -\sum_{l=1}^{n}\Delta_{\mathcal{O}_l}\,\langle\mathcal{O}_1\ldots\mathcal{O}_l(x_l)\ldots\mathcal{O}_n\rangle\,,\end{aligned} \tag{8.27}$$

where

$$T^\mu{}_{\mu,\omega} = 2\partial_\mu\omega\partial^\mu\omega + 2\partial_\mu\left(\omega\partial^\mu\omega\right). \tag{8.28}$$

Next, we collect the f^2 parameter, which has the dimension of a mass squared

$$\begin{aligned}\frac{1}{f^2}\int d^4x\,\langle T^\mu{}_{\mu,\text{CFT}}\,\mathcal{O}_1\ldots\mathcal{O}_n\rangle + \int d^4x\,\langle T^\mu{}_{\mu,\omega}\,\mathcal{O}_1\ldots\mathcal{O}_n\rangle +\\ +2\frac{m^2}{f^2}\int d^4x\,\langle\bar{\phi}(x)\phi(x)\omega^2(x)\mathcal{O}_1\ldots\mathcal{O}_n\rangle = -\frac{1}{f^2}\sum_{l=1}^{n}\Delta_{\mathcal{O}_l}\,\langle\mathcal{O}_1\ldots\mathcal{O}_l(x_l)\ldots\mathcal{O}_n\rangle\,.\end{aligned} \tag{8.29}$$

Recalling that the conformal dimension of the stress-energy tensor is equal to 4, then each term in the equation above has a conformal dimension equal to $(\sum_{l=1}^{n}\Delta_{\mathcal{O}_l}) - 2$. We are now free to make the following considerations (cfr. the article [23]):

- the theory (8.20) is our fixed conformal point. The mass operator is relevant, so it acts as the generator of a relevant flow. The introduction of the m^2 coefficient sets a fixed energy scale in the theory: we can define a UV cutoff Λ_{UV} such that

  $$m^2 \ll \Lambda_{UV}. \tag{8.30}$$

  In our scenario, at the beginning the theory (8.20) lives near the scale Λ_{UV}, then the relevant flow lowers the energy scale towards m^2. In this thesis we do not follow the flow, but we stay in proximity of the conformal point, so we are still in the UV regime, codified by

  $$m^2 \ll p^2 \lesssim \Lambda_{UV}^2; \tag{8.31}$$

- we are always allowed to make the coupling between the ϕ and the ω fields *arbitrarily* weak by properly tuning the f coefficient

  $$m^2 \ll f^2; \tag{8.32}$$

- as we have already stated, the full theory $S_{\text{full+c.c.}}$ is an effective theory of the conformal compensator ω and it makes sense only in the regime

  $$p^2 \ll \Lambda^2. \tag{8.33}$$

In conclusion, in the energy regime

$$m^2 \ll p^2 \lesssim \Lambda_{UV}^2 \ll \Lambda^2, f^2 \tag{8.34}$$

we can neglect the term multiplied by the factor $\frac{m^2}{f^2}$. Eventually, we obtain the Ward identity

$$\int d^4x \left\langle T^\mu{}_{\mu,\text{CFT}} \mathcal{O}_1 \ldots \mathcal{O}_n \right\rangle + f^2 \int d^4x \left\langle T^\mu{}_{\mu,\omega} \mathcal{O}_1 \ldots \mathcal{O}_n \right\rangle = -\sum_{l=1}^{n} \Delta_{\mathcal{O}_l} \left\langle \mathcal{O}_1 \ldots \mathcal{O}_l(x_l) \ldots \mathcal{O}_n \right\rangle , \tag{8.35}$$

and we recognize the Ward identity of a scale invariant theory containing the scalar field ϕ and the compensator ω

$$\int d^4x \left\langle T^\mu{}_{\mu,\text{CFT+c.c.}} \mathcal{O}_1 \ldots \mathcal{O}_n \right\rangle = -\sum_{l=1}^{n} \Delta_{\mathcal{O}_l} \left\langle \mathcal{O}_1 \ldots \mathcal{O}_l(x_l) \ldots \mathcal{O}_n \right\rangle , \tag{8.36}$$

Resuming, the complete procedure starts from an original theory S_{CFT} enjoying the scale invariance, codified by an unintegrated Ward identity with the structure (5.62). Then, the theory is deformed by adding a mass sector to the Lagrangian: the former Ward identity is substituted by a new softly broken one. The new Ward identity contains a breaking term controlled by a mass parameter m^2: we introduce a conformal compensator and we study the deformed theory in a specific energy regime (cfr.(8.34)), in which we are allowed to get rid of the soft breaking term. By doing this, we end up with an unintegrated Ward identity with the structure (5.62). In conclusion, the mass-deformed theory, coupled to the compensator, can be studied as a scale invariant theory in the specified energy regime.

Introducing the conformal compensator in the $\mathcal{N} = 4$ theory

We are ready to apply the same reasoning adopted in the previous section to the $\mathcal{N} = 4$ theory. The original theory S_{CFT} is deformed by the addition of the thermal mass operators, grouped in the sector S_{relevant}. We recover the definitions (8.15) and we introduce the conformal compensator

$$\begin{aligned} S_{\text{full+c.c}} = S_{\text{CFT}} + iM_X^2 \int_0^\beta d\tau \int d^3x \left[\mathcal{M}_X \omega^2\right] + im \int_0^\beta d\tau \int d^3x \left[\mathcal{M}_\lambda \omega\right] + \\ + iM_{A,T}^2 \int_0^\beta d\tau \int d^3x \left[\mathcal{M}_{A,T} \omega^2\right] + iM_{A,L}^2 \int_0^\beta d\tau \int d^3x \left[\mathcal{M}_{A,L} \omega^2\right] - \\ - if^2 \int_0^\beta d\tau \int d^3x \left[\partial_\mu \omega \partial^\mu \omega\right] . \end{aligned} \tag{8.37}$$

We can derive the new Ward identity (to be compared with the Ward identity (8.16))

$$\begin{aligned} \int_0^\beta d\tau \int d^3x \left\langle T^\mu{}_{\mu,\text{CFT}} \mathcal{O}_1 \ldots \mathcal{O}_n \right\rangle + f^2 \int_0^\beta d\tau \int d^3x \left\langle T^\mu{}_{\mu,\omega} \mathcal{O}_1 \ldots \mathcal{O}_n \right\rangle + \\ + M_{A,T}^2 \int_0^\beta d\tau \int d^3x \left\langle \mathcal{M}_{A,T} \omega^2 \mathcal{O}_1 \ldots \mathcal{O}_n \right\rangle + M_{A,L}^2 \int_0^\beta d\tau \int d^3x \left\langle \mathcal{M}_{A,L} \omega^2 \mathcal{O}_1 \ldots \mathcal{O}_n \right\rangle + \\ + 2M_X^2 \int_0^\beta d\tau \int d^3x \left\langle \mathcal{M}_X \omega^2 \mathcal{O}_1 \ldots \mathcal{O}_n \right\rangle + 3m \int_0^\beta d\tau \int d^3x \left\langle \mathcal{M}_\lambda \omega \mathcal{O}_1 \ldots \mathcal{O}_n \right\rangle = \\ = -\sum_{l=1}^{n} \Delta_{\mathcal{O}_l} \left\langle \mathcal{O}_1 \ldots \mathcal{O}_l(x_l) \ldots \mathcal{O}_n \right\rangle . \end{aligned} \tag{8.38}$$

After having collected a factor f^2, we study the Ward identity in the regime

$$m^2, M_X^2, M_A^2 \ll p^2 \lesssim \Lambda_{UV}^2 \ll \Lambda^2, f^2. \tag{8.39}$$

We are allowed to neglect the soft breaking terms, multiplied by the factors $\frac{M_X^2}{f^2}$, $\frac{m}{f}$, $\frac{M_{A,T}^2}{f^2}$ and $\frac{M_{A,L}^2}{f^2}$. The unintegrated scale invariance Ward identity turns out to be

$$\int d^4x \left\langle T^{\mu}{}_{\mu,\text{CFT+c.c.}} \mathcal{O}_1 \dots \mathcal{O}_n \right\rangle = -\sum_{l=1}^{n} \Delta_{\mathcal{O}_l} \left\langle \mathcal{O}_1 \dots \mathcal{O}_l(x_l) \dots \mathcal{O}_n \right\rangle . \tag{8.40}$$

In conclusion, in the energy regime (8.39) the $\mathcal{N} = 4$ theory, coupled to the conformal compensator, recovers the conformal invariance. Moreover, although the $\mathcal{N} = 4$ theory and the conformal compensator sector are coupled, we are able to set the strength of their coupling *arbitrarily* weak, tuning the f coefficient according to the energy regime (8.39).

8.1.3 Towards the non-renormalization theorem at finite temperature

In this section we set up all the tools required by the non-renormalization theorem.

Conformal invariance We recovered the conformal invariance introducing a conformal compensator coupled to the theory at finite temperature. The recovered conformal invariance allows us to completely fix the dependence on the coordinates of the scalar 3-points functions

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) \rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{13}^{\Delta_1+\Delta_3-\Delta_2} x_{23}^{\Delta_2+\Delta_3-\Delta_1}} . \tag{8.41}$$

We are sure that the overall coefficient C_{123} does not depend on the coordinates.

Connection over the conformal manifold Introducing a finite temperature, the global structure of the spacetime has been modified, so the definition of the connection over the conformal manifold obviously changes accordingly

$$\nabla_\tau \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle = \int_0^\beta d\tau \int d^3x \, \langle \mathcal{O}_\tau \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle . \tag{8.42}$$

Superconformal Ward identity Thanks to the R-symmetry, we recovered the superconformal Ward identity exactly with the needed structure. Although the full theory at finite temperature does not enjoy the superconformal symmetry, we showed that the following identity holds

$$\begin{aligned} &\sum_{l=1}^{n} \psi_I^\alpha(x_l) \left\langle \mathcal{O}_1(x_1) \dots \left[\mathcal{Q}_\alpha^I, \mathcal{O}_l \right\} (x_l) \dots \mathcal{O}_n(x_n) \right\rangle + \\ &\qquad + \left(\partial_\mu \psi_I^\alpha \right) (x_l) n_\mu n_\nu \sigma^\nu_{\alpha\dot\beta} \left\langle \mathcal{O}_1(x_1) \dots \left[\bar{\mathcal{S}}^{\dot\beta I}, \mathcal{O}_l \right\} (x_l) \dots \mathcal{O}_n(x_n) \right\rangle = 0. \end{aligned} \tag{8.43}$$

In the proof the operators $\mathcal{O}_l(x_l)$ will be superconformal primaries, thus the identity can be simplified

$$\sum_{l=1}^{n} \psi_I^\alpha(x_l) \left\langle \mathcal{O}_1(x_1) \dots \left[\mathcal{Q}_\alpha^I, \mathcal{O}_l \right\} (x_l) \dots \mathcal{O}_n(x_n) \right\rangle = 0. \tag{8.44}$$

Superconformal representations Although the full theory S_{full} at finite temperature does not enjoy the superconformal symmetry, one of its components, S_{CFT}, is superconformal invariant at zero temperature, i.e. in absence of the relevant deformations. Hence, we can construct all the superconformal representations with the superconformal theory S_{CFT}; moreover, we can also define the supercharges $\mathcal{Q}$ and $\mathcal{S}$. Thus, in the first place we can attribute all the superconformal properties to the operators appearing in the theorem and, only in a second moment, deform the theory, switching on the finite temperature.

8.2 The non-renormalization theorem at finite temperature

We are finally able to adapt the proof reviewed in the chapter 6 to a finite temperature scenario. The procedure is absolutely analogous to the one followed at zero temperature.

The object of our study is the dependence of the 3-points function

$$\left\langle \phi_1^{(\mathcal{R}_1,\vec{m}_1)}(x_1)\, \phi_2^{(\mathcal{R}_2,\vec{m}_2)}(x_2)\, \phi_3^{(\mathcal{R}_3,\vec{m}_3)}(x_3) \right\rangle \tag{8.45}$$

on the marginal coupling τ of the original $\mathcal{N} = 4$ theory. Thanks to the conformal invariance, the expression of the correlator can be exactly computed up to an overall coefficient C_{123}, as shown in the equation (6.10). Having defined the connection over the conformal manifold at finite temperature (cfr. the equation (8.42)), we can write

$$\begin{bmatrix}\text{kinetic and group-}\\ \text{theoretical factors}\end{bmatrix} \times \nabla_\tau C_{123} = \int_0^\beta d\tau \int d^3x \left\langle \mathcal{O}_\tau(x) \phi_1^{(\mathcal{R}_1,\vec{m}_1)}(x_1)\, \phi_2^{(\mathcal{R}_2,\vec{m}_2)}(x_2)\, \phi_3^{(\mathcal{R}_3,\vec{m}_3)}(x_3) \right\rangle . \tag{8.46}$$

The statement of the theorem is identical to the one adopted in the chapter 6:

Theorem. *The OPE coefficient associated to the correlators* (8.45) *does not depend on the position over the conformal manifold, i.e. it does not depend on τ, the $\mathcal{N} = 4$ theory coupling constant.*

$$\nabla_\tau C_{123} = 0. \tag{8.47}$$

As we have already highlighted, the proof of the theorem is trivial if we are able to show that

$$\mathcal{I}(x, x_1, x_2, x_3) = \left\langle \mathcal{O}_\tau(x) \phi_1^{(\mathcal{R}_1,\vec{m}_1)}(x_1)\, \phi_2^{(\mathcal{R}_2,\vec{m}_2)}(x_2)\, \phi_3^{(\mathcal{R}_3,\vec{m}_3)}(x_3) \right\rangle \stackrel{?}{=} 0, \tag{8.48}$$

where $\stackrel{?}{=}$ is the equality to prove. The marginal operator $\mathcal{O}_\tau$ can still be constructed as a superconformal descendant of the superconformal primary operator $\phi^{(2,+)}$ (as anticipated in the section 8.1.3)

$$\mathcal{O}_\tau = \left\{ \mathcal{Q}_1^4, \left[\mathcal{Q}_2^4, \left\{ \mathcal{Q}_1^3, \left[\mathcal{Q}_2^3, \phi^{(2,+)} \right] \right\} \right] \right\} \tag{8.49}$$

and can be written as

$$\mathcal{O}_\tau = \{ \mathcal{Q}^*, \Gamma \} , \tag{8.50}$$

where $\mathcal{Q}^*$ is one of the left chirality supercharges appearing in the definition (8.49). Now we can rewrite the integrand (8.48) as follows

$$\mathcal{I}(x, x_1, x_2, x_3) = \left\langle \{ \mathcal{Q}^*, \Gamma \} (x)\, \phi_1^{(k_1,\vec{m})}(x_1)\, \phi_2^{(k_2,+)}(x_2)\, \phi_3^{(k_3,-)}(x_3) \right\rangle \tag{8.51}$$

and we can make use of the superconformal Ward identity (8.44) in order to move the supercharge $\mathcal{Q}^*$ from the operator Γ to the superconformal primaries inside the braket. All the considerations made at zero temperature hold in this case, too. The result is

$$\begin{aligned}\psi(x) \langle \{ \mathcal{Q}^*, \Gamma \} (x)\, \phi_1\, \phi_2\, \phi_3 \rangle + \psi(x_1) \langle \Gamma\, [\mathcal{Q}^*, \phi_1]\, (x_1)\, \phi_2\, \phi_3 \rangle +& \\ + \psi(x_2) \langle \Gamma\, \phi_1\, [\mathcal{Q}^*, \phi_2]\, (x_2)\, \phi_3 \rangle = 0.&\end{aligned} \tag{8.52}$$

We have the freedom to impose $\psi(x)$ to be equal to zero when it is computed in the point x_2, so

$$\psi(x) \langle \{ \mathcal{Q}^*, \Gamma \} (x)\, \phi_1\, \phi_2\, \phi_3 \rangle = -\psi(x_1) \langle \Gamma\, [\mathcal{Q}^*, \phi_1]\, (x_1)\, \phi_2\, \phi_3 \rangle . \tag{8.53}$$

The integrand (8.51) can be written as follows

$$\psi(x)\, \mathcal{I}(x, x_1, x_2, x_3) = -\psi(x_1) \langle \Gamma\, [\mathcal{Q}^*, \phi_I]\, (x_1)\, \phi_2\, \phi_3 \rangle . \tag{8.54}$$

In order to conclude the proof, we make use of the null condition (6.65), proved in the appendix C

$$\psi(x)\,\mathcal{I}(x,x_1,x_2,x_3) = -\psi(x_1)\sum_{\star\neq *}\left\langle \Gamma\,\left[\mathcal{Q}^\star,\phi_1^{(k_1,\vec{m}_\star)}\right](x_1)\,\phi_2\,\phi_3\right\rangle . \tag{8.55}$$

For each term in the sum we use again the superconformal Ward identity in order to move the supercharges from the operator ϕ_1 to the others. As for the zero temperature case, we have two possibilities:

- $\mathcal{Q}^\star$ annihilates the highest weight of a generic $(0\;k\;0)$ representation of $su(4)$;
- $\mathcal{Q}^\star$ annihilates the lowest weight of a generic $(0\;k\;0)$ representation of $su(4)$.

Following the considerations already made in the section 6.5, we can prove that every term in the sum (8.55) is equal to zero. This means that

$$\psi(x)\,\mathcal{I}(x,x_1,x_2,x_3) = -\psi(x_1)\sum_{\star\neq *}\left\langle \Gamma\,\left[\mathcal{Q}^\star,\phi_1^{(k_1,\vec{m}_\star)}\right]\phi_2\,\phi_3\right\rangle = 0 \Rightarrow \mathcal{I}(x,x_1,x_2,x_3) = 0. \tag{8.56}$$

Having proved the lemma, the theorem is trivially proved, too. In conclusion, the non-renormalization theorem (8.47) not only holds at zero temperature, but also at finite temperature.

Chapter 9

Conclusions and Outlooks

In this thesis we started from the review of a non-perturbative result (derived in the article [5]), obtained at zero temperature, then we proved that the same result holds at finite temperature, too. Our conclusion was not trivial: when the temperature became finite, we witnessed the loss of important symmetries, like the conformal symmetry and the supersymmetry. The reason of such loss was the appearance of three thermal mass operators, which were interpreted as relevant deformations of the original theory enjoying all the symmetries listed above. The structure of the proof exposed in the original article [5] was easily adaptable to our scenario, but we needed to recover two fundamental tools employed in the theorem: the conformal invariance and the superconformal Ward identity. First of all, we learnt how to handle the broken and the unbroken symmetries with their Ward identities: in the thesis the Ward identities were *softly* broken, i.e. the broken nature of the symmetry manifested itself in the Ward identities through the appearance of some breaking terms multiplied by mass coefficients. We encountered two main (sets of) breaking terms to remove: the breaking term in the superconformal Ward identity and the breaking term in the scale invariance Ward identity. In order to get rid of them, we employed two different strategies. The superconformal Ward identity was recovered making use of the R-symmetry Ward identity, which was specialized to a very easy constraint: globally charged correlators, under the action of a specific Cartan generator of $su(4)$, has to be equal to zero. The breaking term of the superconformal Ward identity was globally charged, hence it was possible to remove it. In order to recover the scale invariance Ward identity, it was not possible to exploit the R-symmetry: the breaking terms were not globally charged under the R-symmetry group. We reached our goal introducing an auxiliary degree of freedom, the conformal compensator. Setting the theory in a specific energy regime, we were able to neglect the breaking terms and to recover the scale invariance Ward identity. Once the main tools had been restored, the proof of the theorem proceeded effortless.

The importance of our conclusion lies in the transposition of a powerful result at zero temperature into the same result, but in a finite temperature scenario. The original theorem showed that the explicit expression of a specific 3-points function in the context of the $\mathcal{N} = 4$ theory does not depend on the coupling of the theory: if the coupling change (we switch from a weak coupling to a strong coupling, or we renormalize the theory), the non-perturbative result does not change. However, this was true only at zero temperature: showing that the theorem can hold also at finite temperature made the statement valid not only for systems of few interacting particles, but for ensembles of particles, too.

The main outlook of this work is to lower the amount of supersymmetry enjoyed by the original theory. In particular, it would be interesting to consider a generic $\mathcal{N} = 2$ theory and try to obtain the same non-perturbative results derived in [14], turning on a finite temperature.

Appendix A

Conventions

In this appendix we go through all the principal conventions used in the thesis. We always make use of the *natural units*, imposing

$$c = \hbar = k_b = 1. \tag{A.1}$$

Th metric of the Minkowski spacetime is

$$\eta^{\mu\nu} = \mathrm{diag}\,(-,+,+,+)\,, \tag{A.2}$$

while the metric of the Euclidean spacetime is

$$\delta^{\mu\nu} = \mathrm{diag}\,(+,+,+,+)\,. \tag{A.3}$$

A.1 Indices

If it is not specified differently, we always adopt the Einstein convention over the repeated indices. We list all the different kinds of indices used in the thesis:

- Spacetime indices in 4 dimensions: $\mu, \nu, \rho, \sigma, \dots$;
- Spacetime indices in >4 dimensions: $m, n, r, s, \dots$;
- Undotted spinorial indices in 4 dimensions: $\alpha, \beta, \gamma, \delta, \dots$. Our convention is to always set these indices *low*: $\Gamma_\alpha, \Gamma_\beta, \Gamma_\gamma, \Gamma_\delta, \dots$;
- Dotted spinorial indices in 4 dimensions: $\dot\alpha, \dot\beta, \dot\gamma, \dot\delta, \dots$. Our convention is to always set these indices *high*: $\Gamma^{\dot\alpha}, \Gamma^{\dot\beta}, \Gamma^{\dot\gamma}, \Gamma^{\dot\delta}, \dots$;
- Gauge symmetry indices: $a, b, c, d, \dots$. Our convention is to always set these indices *high*: $\Gamma^a, \Gamma^b, \Gamma^c, \Gamma^d, \dots$;
- Spatial indices in 3 dimensions: $i, j, k, l, \dots$. Our convention is to always set these indices *low*: $\Gamma_i, \Gamma_j, \Gamma_k, \Gamma_l, \dots$;
- R-symmetry $su(4)$ indices: $I, J, K, L, \dots$. Our convention is to always set *high* the indices of an object sitting in the fundamental representation of $su(4)$ ($\Gamma^I, \Gamma^J, \dots$) and to always set *low* the indices of an object sitting in the anti-fundamental representation of $su(4)$ ($\Gamma_I, \Gamma_J, \dots$).

A.2 Yang-Mills theory

Whenever we deal with a YM field theory, in our conventions the algebra of the gauge symmetry group is defined by the relation

$$\left[t^a, t^b\right] = f^{abc} t^c \tag{A.4}$$

where f^{abc} is a real number and the generators t^a are anti-hermitian $(t^a)^\dagger = -t^a$. The trace over the gauge indices of two generators of the gauge group is given by

$$\text{tr}\left[t^a t^b\right] = -\frac{1}{2}\delta^{ab}. \tag{A.5}$$

Our definition of the covariant derivative is

$$D_\mu Y^a = \partial_\mu Y^a + f^{abc} A^b_\mu Y^c, \tag{A.6}$$

while our definition of the field strength is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu. \tag{A.7}$$

A.3 Weyl spinors and superspace coordinates

In the thesis the fermionic fields are represented by 2-components Weyl spinors. Weyl spinors in 4 dimensions can carry two different kinds of spinorial indices: an undotted index α and a dotted index $\dot{\alpha}$. This is due to the fact that the Lorentz group $so(1,3)$ can be split in two copies of $su(2)$: each copy provides its own spinorial representation. The left-chirality Weyl spinor and the right-chirality Weyl spinor are respectively

$$\psi_\alpha, \qquad \bar{\chi}^{\dot{\alpha}}. \tag{A.8}$$

Hermitian conjugation in Minkowski spacetime turns left-chirality spinors into right-chirality spinors

$$(\psi_\alpha)^\dagger = \bar{\psi}^{\dot{\alpha}}. \tag{A.9}$$

In the 4-dimensional Minkowski spacetime, we can define a Majorana spinor as follows

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}. \tag{A.10}$$

The convention on the positions of the spinorial indices is exposed in the section A.1. However, the position can be altered using the antisymmetric objects

$$\varepsilon^{\alpha\beta}, \qquad \varepsilon^{\dot{\alpha}\dot{\beta}}, \qquad \varepsilon_{\alpha\beta}, \qquad \varepsilon_{\dot{\alpha}\dot{\beta}}, \tag{A.11}$$

where the only non-zero components are

$$\varepsilon^{12} = \varepsilon^{\dot{1}\dot{2}} = \varepsilon_{21} = \varepsilon_{\dot{2}\dot{1}} = 1, \qquad \varepsilon^{21} = \varepsilon^{\dot{2}\dot{1}} = \varepsilon_{12} = \varepsilon_{\dot{1}\dot{2}} = -1. \tag{A.12}$$

The spinorial indices are lowered and raised as follows

$$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta, \qquad \psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta, \qquad \bar{\psi}^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\beta}}, \qquad \bar{\psi}_{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\beta}}. \tag{A.13}$$

Our convention on the Sigma matrices in the Minkowski spacetime is the following

$$\sigma^\mu = (\mathbb{1}, \vec{\sigma}), \qquad \bar{\sigma}^\mu = (-\mathbb{1}, \vec{\sigma}), \tag{A.14}$$

where $\vec{\sigma}$ represents the three Pauli matrices; in the Euclidean spacetime we have $\sigma_4 = -i\sigma_0$, so

$$\sigma_\mu = (\vec{\sigma}, i\mathbb{1})\,. \tag{A.15}$$

The Sigma matrices are naturally equipped with a couple of spinorial indices

$$\sigma^\mu_{\alpha\dot{\beta}}, \qquad \bar{\sigma}^{\mu\dot{\alpha}\beta}. \tag{A.16}$$

The two Sigma matrices are in the following relation

$$\bar{\sigma}^{\mu\dot{\alpha}\alpha} = \varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}}\sigma^\mu_{\beta\dot{\beta}}, \qquad \sigma^\mu_{\alpha\dot{\alpha}} = \varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\sigma}^{\mu\dot{\beta}\beta}. \tag{A.17}$$

Moreover, we can construct

$$\sigma^{\mu\nu}{}_\alpha{}^\beta = \frac{1}{4}\left(\sigma^\mu_{\alpha\dot{\gamma}}\bar{\sigma}^{\nu\dot{\gamma}\beta} - (\mu \leftrightarrow \nu)\right), \qquad \bar{\sigma}^{\mu\nu\dot{\alpha}}{}_{\dot{\beta}} = \frac{1}{4}\left(\bar{\sigma}^{\mu\dot{\alpha}\gamma}\sigma^\nu_{\gamma\dot{\beta}} - (\mu \leftrightarrow \nu)\right). \tag{A.18}$$

The principal spinorial identities used in the thesis are

$$\begin{aligned}
&\varepsilon^{\alpha\beta}\psi_\alpha\chi_\beta = \psi\chi = \chi\psi, && \chi\sigma^\mu\bar{\psi} = -\bar{\psi}\bar{\sigma}^\mu\chi, && \chi\sigma^\mu\bar{\sigma}^\nu\psi = \psi\sigma^\nu\bar{\sigma}^\mu\chi,\\
&\varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\alpha}}\bar{\chi}^{\dot{\beta}} = \bar{\psi}\bar{\chi} = \bar{\chi}\bar{\psi}, && (\chi\sigma^\mu\bar{\psi})^\dagger = \psi\sigma^\mu\bar{\chi}, && (\chi\sigma^\mu\bar{\sigma}^\nu\psi)^\dagger = \bar{\psi}\bar{\sigma}^\nu\sigma^\mu\bar{\chi}.
\end{aligned}$$

When we study a theory in the superspace, we have to deal with four fermionic coordinates $\theta_\alpha, \bar{\theta}^{\dot{\alpha}}$. They are Grassmann variables, hence a generic function can be at most quadratic in θ or $\bar{\theta}$. The spinorial relations are

$$\begin{aligned}
\theta^\alpha\theta^\beta &= -\frac{1}{2}\varepsilon^{\alpha\beta}\theta\theta, & \theta_\alpha\theta_\beta &= \frac{1}{2}\varepsilon_{\alpha\beta}\theta\theta,\\
\bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} &= \frac{1}{2}\varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta}, & \bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} &= -\frac{1}{2}\varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta},\\
\theta\sigma^\mu\bar{\theta}\theta\sigma^\nu\bar{\theta} &= -\frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\eta^{\mu\nu}, & \theta\psi\theta\chi &= -\frac{1}{2}\theta\theta\psi\chi.
\end{aligned}$$

Doing calculus in the superspace requires the knowledge of the following relations

$$\begin{aligned}
&\frac{\partial}{\partial\theta_\alpha}\theta_\beta = \delta^\alpha_\beta, && \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\bar{\theta}^{\dot{\beta}} = \delta^{\dot{\beta}}_{\dot{\alpha}}, && \left(\frac{\partial}{\partial\theta_\alpha}\right)^\dagger = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}},\\
&\int d^2\theta\ \theta\theta = 1, && \int d^2\bar{\theta}\ \bar{\theta}\bar{\theta} = 1, && \int d^2\theta d^2\bar{\theta}\ \theta\theta\bar{\theta}\bar{\theta} = 1.
\end{aligned}$$

Appendix B

$su(N)$ representations

In this appendix we provide the reader with all the algebraic tools required for the proof exposed in the chapters 6 and 8. First of all, we will introduce the definition of Cartan subalgebra and the classification of all the possible semi-simple Lie algebras. In particular, we will emphasize the fact that a generic semi-simple Lie algebra can be seen as a collection of $su(2)$ subalgebras non-trivially connected among each other. In this appendix we will follow the references [24] and [25]

B.1 The Cartan subalgebra

Let's consider a general N dimensional Lie algebra, identified by its abstract generators $t^1, \dots, t^N$ and the following relations

$$\left[t^a, t^b\right] = f^{ab}{}_c t^c, \qquad \left[\left[t^a, t^b\right], t^c\right] + (\text{cyclic permutations}) = 0. \tag{B.1}$$

In the following we will work with the *adjoint* representation of the algebra generators. We identify the *Cartan subalgebra*: it is composed of the generators $H^1, \dots, H^r$ satisfying the following relation

$$\left[H^i, H^j\right] = 0, \quad \forall i, j. \tag{B.2}$$

The relation (B.2) allows us to simultaneously diagonalize all the Cartan generators. In the new basis, the commutator between a Cartan generator and a generic generator of the full algebra t^a is

$$\left[H^i, t^a\right] = f^{ia}{}_b t^b = t^{ia}{}_b t^b = H^{ia}{}_b t^b = \beta^i(a)\delta^a_b t^b = \beta^i(a) t^a. \tag{B.3}$$

The vector $\vec{\beta}(a) = \left(\beta^1(a) \dots \beta^r(a)\right)$ is called *root*. The Cartan generators are generators with a root equal to zero. We rewrite the generator t^a as follows

$$t^a \equiv E_{\vec{\beta}(a)} = E_{\vec{\beta}}, \tag{B.4}$$

where we left the index a implicit in order to simplify the notation. If $E_{\vec{\beta}}$ is a generator of the algebra, it can be shown that $E_{-\vec{\beta}}$ is a generator of the algebra, too. In conclusion, the original set of abstract generators has been realized in the adjoint representation and divided in two distinct sets:

- the Cartan generators $H^1, \dots, H^r$: they have roots equal to zero. r is called *rank* of the Lie algebra;
- the *ladder* generators $E_{\vec{\beta}(1)}, E_{-\vec{\beta}(1)} \dots E_{\vec{\beta}(\frac{n-r}{2})}, E_{-\vec{\beta}(\frac{n-r}{2})}$: this set always has an even number of elements.

n	n'	$\theta_{\alpha\beta}$	length ratio
0	0	$\pi/2$	not fixed
1	1	$\pi/3$	1
1	2	$\pi/4$	$\sqrt{2}$
1	3	$\pi/6$	$\sqrt{3}$
-1	-1	$2\pi/3$	1
-1	-2	$3\pi/4$	$\sqrt{2}$
-1	-3	$5\pi/6$	$\sqrt{3}$

Table B.1: All the possible angles $\theta_{\alpha\beta}$ and length ratios between two generic roots.

Given two different ladder generators $E_{\vec{\alpha}}$ and $E_{\vec{\beta}}$, it can be shown that the following relations hold

$$2\frac{\vec{\alpha}\cdot\vec{\beta}}{\vec{\beta}\cdot\vec{\beta}} = p - q = n \in \mathbb{Z}, \qquad 2\frac{\vec{\alpha}\cdot\vec{\beta}}{\vec{\alpha}\cdot\vec{\alpha}} = p' - q' = n' \in \mathbb{Z}. \tag{B.5}$$

Multiplying the two relations we obtain

$$\frac{\left(\vec{\alpha}\cdot\vec{\beta}\right)^2}{\left(\vec{\beta}\cdot\vec{\beta}\right)\left(\vec{\alpha}\cdot\vec{\alpha}\right)} = \frac{nn'}{4} \Rightarrow \left(\cos\theta_{\alpha\beta}\right)^2 = \frac{nn'}{4} \leq 1. \tag{B.6}$$

The relations (B.5) and (B.6) strongly constrain the root system: only a finite number of angles $\theta_{\alpha\beta}$ and length ratios are allowed. In the table B.1 all the possible relations between two generic roots are listed.

B.1.1 A basis for the root system

The root system is composed of $N - r$ vectors; however, the roots are vectors with r components, thus only r roots can be linearly independent. We want to find a basis for the root system, so we need to identify r linearly independent roots. First of all, we can split the root system in two sets with equal cardinality. For every root $\vec{\beta} = \left(\beta^1 \dots \beta^r\right)$, starting from the component β^1, we consider the first non-zero component: if it is positive, the root is called *positive*. We focus on the positive roots and we define r (positive) *simple roots* with the following property: a (positive) root is simple if it cannot be expressed as a linear combination of other simple roots with positive coefficients . It can be shown that the scalar product between two simple roots is always negative or null. Thus, two simple roots can only be separated by a subset of the possible angles listed in the table B.1: $\left(\frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}\right)$. In the following, $\vec{\alpha}_1, \dots, \vec{\alpha}_r$ will denote the simple roots.

We impose a new normalization for the Cartan, the ladder and the simple root generators

$$h_i \equiv \frac{2}{\vec{\alpha}_i\cdot\vec{\alpha}_i}\vec{\alpha}_i\cdot\vec{H}, \qquad e_{\vec{\beta}} \equiv \sqrt{\frac{2}{\vec{\beta}\cdot\vec{\beta}}}E_{\vec{\beta}}, \qquad e_i \equiv \sqrt{\frac{2}{\vec{\alpha}_i\cdot\vec{\alpha}_i}}E_{\vec{\alpha}_i}, \tag{B.7}$$

In the following we will consider simple root systems with simple roots of different lengths. We normalize the length of the longer roots

$$\vec{\alpha}_i\cdot\vec{\alpha}_i = 1. \tag{B.8}$$

Then, the shorter roots norms are normalized according to the table B.1.

B.2 $su(2)$ subalgebras

The original Lie algebra has been decomposed into a set of Cartan generators plus a set of ladder generators. The ladder generators, in turn, host a subset of simple roots generators. Now it is possible to identify in the Lie algebra a collection of r copies of the $su(2)$ algebra, each one associated to one of the Cartan generators. Each $su(2)$ subalgebra is composed of a Cartan generator h_i, a simple root generator e_i and e_{-i}, where e_{-i} is associated to the opposite root with respect to $\vec{\alpha}_i$. The defining relations for each $su(2)$ algebra are

$$[h_i, e_i] = 2e_i, \qquad [h_i, e_{-i}] = -2e_{-i}, \qquad [e_i, e_{-i}] = h_i. \tag{B.9}$$

The information about the nature of a specific Lie algebra is encoded in the non trivial connections between the copies of $su(2)$. The following commutation relations codifies the connections

$$[h_i, h_j] = 0, \qquad [h_i, e_j] = A_{ij}e_j, \qquad [h_i, e_{-j}] = -A_{ij}e_{-j}, \qquad [e_i, e_j] = \pm e_{\vec{\alpha}_i+\vec{\alpha}_j}, \tag{B.10}$$

where A_{ij} is the *Cartan matrix* and can be written as a function of the simple roots $\vec{\alpha}_i$ and $\vec{\alpha}_j$

$$A_{ij} = 2\frac{\vec{\alpha}_i \cdot \vec{\alpha}_j}{\vec{\alpha}_i \cdot \vec{\alpha}_i}. \tag{B.11}$$

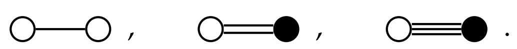
Notice that, by definition of simple root, the root $\vec{\alpha}_i + \vec{\alpha}_j$ is not a simple root.

The decomposition in copies of $su(2)$ and the connections between them can be graphically visualized with the help of *Dynkin diagrams*. They can be constructed following a few rules:

- each $su(2)$ copy is associated to a simple root $\vec{\alpha}_i$ and is represented by a dot

$$\bigcirc \ ;$$

- two dots are associated to two different simple roots $\vec{\alpha}_i$ and $\vec{\alpha}_j$. From the table 7.1 we can see that the square of the ratio of their lengths can be equal to 1,2 or 3. In a Dynkin diagram, the dots are connected by a number of lines determined by the squared ratio of the lengths of the two simple roots. The longer root is represented by a filled dot.



The following drawing rules can be derived from the properties of the simple roots:

- the Dynkin diagram of a Lie algebra of rank r can contain up to $r - 1$ single lines;
- a Dynkin diagram cannot contain a closed cycle;
- each dot cannot host more than three lines;
- removing a dot must generate a new, valid Dynkin diagram;
- replacing a linear chain of dots with a single dot must generate a new, valid Dynkin diagram.

There are only nine classes of connected Dynkin diagrams which satisfy all the rules listed above: this means that only nine classes of semi-simple Lie algebras exist. The complete classification is reported in the table B.2.

Ordinary algebras		
A_r	$su(r+1)$	
B_r	$so(2r+1)$	
C_r	$sp(2r)$	
D_r	$so(2r+4)$	
Exceptional algebras		
G_2		
F_4		
E_6		
E_7		
E_8		

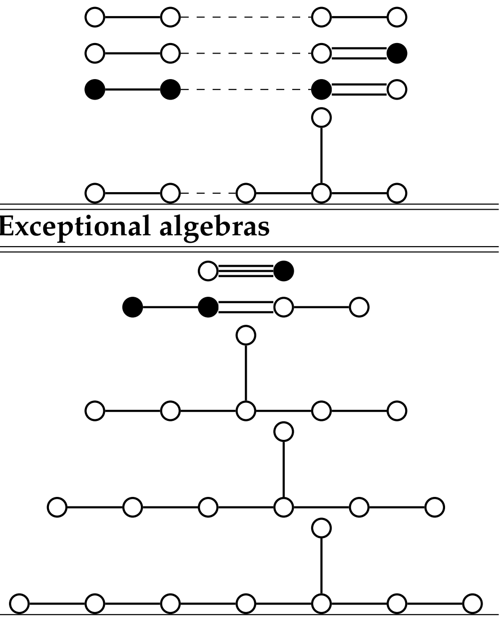
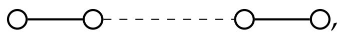


Table B.2: The complete classification of the Lie algebras, completed with their Dynkin diagrams. The r label, used in the section "Ordinary Algebras", stands for the number of Cartan generators, i.e. for the number of dots in the Dynkin diagrams.

B.3 $su(N)$

The general theory presented above can be applied to the specific case of the $su(N)$ algebra. It is crucial to understand its representations in order to study the superconformal field theories, where the $su(\mathcal{N})$ generators are included in the full SCFT superalgebra. This implies that the operators defined in a SCFT sit in specific representations of the $su(\mathcal{N})$ algebra. In the $su(r+1)$ case all the roots have the same length, as we can see from the table B.2. The Dynkin diagram of the $su(N) = su(r+1)$ algebra is



(B.12)

where the number of dots is equal to r. All the simple roots have the same length, hence we can employ the normalization $\vec{\alpha}_i \cdot \vec{\alpha}_i = 1$ for every $i = 1, \ldots, r$. The Cartan matrix (B.11) becomes symmetric and its explicit expression is $A_{ij} = 2\vec{\alpha}_i \cdot \vec{\alpha}_j$. Each dot in the diagram (B.12) is associated to one of the r copies of $su(2)$ in the $su(r+1)$ algebra

$$(e_1, e_{-1}, h_1), \ldots, (e_r, e_{-r}, h_r) . \tag{B.13}$$

A generic root can be written as a linear combination of the r simple roots

$$\vec{\lambda} = \lambda_1 \vec{\alpha}_1 + \cdots + \lambda_r \vec{\alpha}_r . \tag{B.14}$$

Recalling the constraint (B.5), we can assign to the root $\vec{\lambda}$ a list of integer numbers called *Dynkin labels*

$$2\frac{\vec{\lambda} \cdot \vec{\alpha}_i}{\vec{\alpha}_i \cdot \vec{\alpha}_i} = d_i . \tag{B.15}$$

We now apply a change of basis such that

$$\vec{\lambda} = \lambda_i \vec{\alpha}_i = d_j \vec{u}_j = 2\frac{\vec{\lambda} \cdot \vec{\alpha}_j}{\vec{\alpha}_j \cdot \vec{\alpha}_j} \vec{u}_j = 2\lambda_k \vec{\alpha}_k \cdot \vec{\alpha}_j \vec{u}_j = \lambda_k A_{kj} \vec{u}_j , \tag{B.16}$$

so we set

$$\vec{\alpha}_i = A_{ij}\vec{u}_j \tag{B.17}$$

and the components of the generic root $\vec{\lambda}$ simply become its Dynkin labels.

B.3.1 Building the $su(N)$ representations

A generic state in a $su(N)$ representation is identified by

- the representation $\mathcal{R}$ we are considering;
- a root $\vec{\lambda}$, called *weight*.

Different states belonging to the same representation can have different weights: for instance, in the $\mathcal{N} = 4$ theory, even though the whole spinor λ^I lies in the fundamental representation of $su(4)$, the different components $\lambda^1, \ldots, \lambda^4$ are associated to different weights. The notation for a generic state is

$$\left|\vec{\lambda}, \mathcal{R}\right\rangle = \left|e_{\vec{\lambda}}, \mathcal{R}\right\rangle . \tag{B.18}$$

Dropping the label $\mathcal{R}$, we study how the operators h_i, e_i and e_{-i} act on (B.18). Considering that all the operators are realized in the adjoint representation of the symmetry group

$$h_i \left|e_{\vec{\lambda}}\right\rangle = \left|[h_i, e_{\vec{\lambda}}]\right\rangle = d_i \left|e_{\vec{\lambda}}\right\rangle , \tag{B.19}$$

$$e_i \left|e_{\vec{\lambda}}\right\rangle = \left|[e_i, e_{\vec{\lambda}}]\right\rangle \simeq \left|e_{\vec{\lambda}+\vec{\alpha}_i}\right\rangle , \tag{B.20}$$

$$e_{-i} \left|e_{\vec{\lambda}}\right\rangle = \left|[e_{-i}, e_{\vec{\lambda}}]\right\rangle \simeq \left|e_{\vec{\lambda}-\vec{\alpha}_i}\right\rangle . \tag{B.21}$$

Now, let's consider a vector containing all the Dynkin labels of a state with root $\vec{\lambda}$

$$(d_1, \ldots, d_r) . \tag{B.22}$$

The action of the ladder operator e_{-i} on the Dynkin labels can be extracted from the following computation

$$\vec{\lambda} - \vec{\alpha}_k = d_i\vec{u}_i - A_{ki}\vec{u}_i = (d_i - A_{ki})\,\vec{u}_i. \tag{B.23}$$

The new set of Dynkin labels is obtained subtracting the k-th row of the Cartan matrix from the vector (B.22). From now on, the representation label $\mathcal{R}$ will be implicit and a generic state will be identified by its Dynkin labels. In order to build a $su(N)$ representation we can act on a vector of Dynkin labels with every lowering operator e_{-i}. The starting point of this process is the *highest weight state*, while the final point is the *lowest weight state*. The highest weight state is defined as the state annihilated by every raising operator; the lowest weight state is the state annihilated by every lowering operator. If we set a highest weight state associated to the root $\vec{\lambda}^*$, then

$$e_i \left|e_{\vec{\lambda}^*}\right\rangle = 0 \quad \forall i = 1, \ldots, r. \tag{B.24}$$

The highest weight state is associated to the Dynkin labels $(a_1, \ldots, a_r)$. However, if the constraint (B.24) holds, then it must be true also if we take the hermitian conjugate of both sides of the equation, which leads to

$$e_{-i} \left|e_{-\vec{\lambda}^*}\right\rangle = 0 \quad \forall i = 1, \ldots, r. \tag{B.25}$$

We are ready to construct representations of $su(N)$ using the following algorithm:

- we start from a highest weight state with positive or null Dynkin labels

$$(a_1, \dots, a_r)\,; \tag{B.26}$$

- we act with the lowering operators in all the possible ways, subtracting from the Dynkin labels vector the rows of the Cartan matrix. We discard all the states with a negative norm, or equal to zero. A good practical rule is to consider the Dynkin labels of the highest weight state and select highest one: a state with a Dynkin label higher in modulus has to be discarded;

- when we reach the lowest weight state, all the lowering operators annihilate it, so the procedure is concluded and we derived a complete representation of $su(N)$.

B.3.2 $su(2)$

In this section we construct some $su(2)$ representations. The $su(2)$ Lie algebra admits a trivial decomposition in copies of $su(2)$

$$(e_1, e_{-1}, h_1)\,, \tag{B.27}$$

so each state will be identified by only one Dynkin label; moreover, we have only one lowering operator, e_{-1}. The Cartan matrix is trivial

$$A_{su(2)} = 2, \tag{B.28}$$

so the action of e_{-1} on a generic state (a_1) is

$$(a_1) \xrightarrow{e_{-1}} (a_1 - 2)\,. \tag{B.29}$$

Singlet representation This representation is trivial

$$(0)\,. \tag{B.30}$$

The highest and the lowest weights coincide.

Doublet representation This is the fundamental representation

$$(1) \xrightarrow{e_{-1}} (\bar{1})\,, \tag{B.31}$$

where we adopted the notation $-k = \bar{k}$, which will be used in the following.

Triplet representation The structure becomes richer and the trend becomes clear

$$(2) \xrightarrow{e_{-1}} (0) \xrightarrow{e_{-1}} (\bar{2})\,. \tag{B.32}$$

B.3.3 $su(4)$

We are finally able to discuss the $su(4)$ representations. The decomposition in copies of $su(2)$ is

$$(e_1, e_{-1}, h_1)\,, (e_2, e_{-2}, h_2)\,, (e_3, e_{-3}, h_3)\,, \tag{B.33}$$

so each state will be identified by three Dynkin labels. We have three lowering operators e_{-1}, e_{-2} and e_{-3}: if the Cartan matrix is

$$A_{su(4)} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \tag{B.34}$$

then their actions on a generic Dynkin labels vector $(a_1\ a_2\ a_3)$ are

$$(a_1\ a_2\ a_3) \xrightarrow{e_{-1}} ((a_1-2)\ \ (a_2+1)\ \ a_3)\,, \tag{B.35}$$

$$(a_1\ a_2\ a_3) \xrightarrow{e_{-2}} ((a_1+1)\ \ (a_2-2)\ \ (a_3+1))\,, \tag{B.36}$$

$$(a_1\ a_2\ a_3) \xrightarrow{e_{-3}} (a_1\ \ (a_2+1)\ \ (a_3-2))\,. \tag{B.37}$$

Singlet representation This representation is trivial, but it is very important, since it hosts the gauge vector boson A^a_μ of the $\mathcal{N}=4$ SYM theory

$$(0\,0\,0)\,. \tag{B.38}$$

Fundamental representation This representation hosts all the operators defined in $\mathcal{N}=4$ theory with one, high $su(4)$ index (the supercharges $\mathcal{Q}^I_\alpha$, for instance)

$$\begin{array}{c} (1\,0\,0) \\ \downarrow \\ (\bar{1}\,1\,0) \\ \downarrow \\ (0\,\bar{1}\,1) \\ \downarrow \\ (0\,0\,\bar{1}) \end{array}$$

Anti-Fundamental representation Similarly to the previous one, this representation hosts all the operators with only one $su(4)$ index, but low (the conjugated spinor $\bar{\lambda}^{\dot{\alpha}a}_I$, for instance)

$$\begin{array}{c} (0\,0\,1) \\ \downarrow \\ (0\,1\,\bar{1}) \\ \downarrow \\ (1\,\bar{1}\,0) \\ \downarrow \\ (\bar{1}\,0\,0) \end{array}$$

Sextuplet representation This representation hosts the scalar fields X^{IJ} of the $\mathcal{N} = 4$ theory

$$
\begin{tikzcd}[column sep=tiny]
 & (0\,1\,0) \arrow[d] & \\
 & (1\,\bar{1}\,1) \arrow[dl] \arrow[dr] & \\
(\bar{1}\,0\,1) \arrow[dr] & & (1\,0\,\bar{1}) \arrow[dl] \\
 & (\bar{1}\,1\,\bar{1}) \arrow[d] & \\
 & (0\,\bar{1}\,0) &
\end{tikzcd}
$$

$(0\ 2\ 0)$ **representation** This representation hosts the superconformal primary $\phi^{(2,+)}$ (cfr. the section 6.2), which is crucially important in the proof of the non-renormalization theorem. In particular, the primary sits in the highest weight of this representation, as it is shown by the label $+$

$$
\begin{tikzcd}
 & (0\,2\,0) \equiv + \arrow[d] & \\
 & (1\,0\,1) \arrow[dl] \arrow[d] \arrow[dr] & \\
(\bar{1}\,1\,1) \arrow[d] \arrow[dr] & (2\,\bar{2}\,2) \arrow[dl] \arrow[dr] & (1\,1\,\bar{1}) \arrow[d] \arrow[dl] \\
(0\,\bar{1}\,2) \arrow[d] \arrow[dr] & (\bar{1}\,2\,\bar{1}) \arrow[d] & (2\,\bar{1}\,0) \arrow[d] \arrow[dl] \\
(\bar{2}\,0\,2) \arrow[d] & (0\,0\,0) \arrow[dl] \arrow[d] \arrow[dr] & (2\,0\,\bar{2}) \arrow[d] \\
(\bar{2}\,1\,0) \arrow[d] \arrow[dr] & (1\,\bar{2}\,1) \arrow[dl] \arrow[dr] & (0\,1\,\bar{2}) \arrow[d] \arrow[dl] \\
(\bar{1}\,\bar{1}\,1) \arrow[dr] & (\bar{2}\,2\,\bar{2}) \arrow[d] & (1\,\bar{1}\,\bar{1}) \arrow[dl] \\
 & (\bar{1}\,0\,\bar{1}) \arrow[d] & \\
 & (0\,\bar{2}\,0) \equiv - &
\end{tikzcd}
$$

The procedure can be repeated for every representation $(0\ k\ 0)$: the weight diagram then contains all the possible weights $\vec{m}$ of the operator $\phi^{(k,\vec{m})}$.

Appendix C

Two Lemmas for the Non-Renormalization Theorem

C.1 Proof of the relation (6.60)

Our goal is to prove the following identities, valid for a given basis of the supercharges

$$\left[\mathcal{Q}^1_\alpha, \phi^{(k,+)}\right] = \left[\mathcal{Q}^2_\alpha, \phi^{(k,+)}\right] = \left[\mathcal{Q}^3_\alpha, \phi^{(k,-)}\right] = \left[\mathcal{Q}^4_\alpha, \phi^{(k,-)}\right] = 0. \tag{C.1}$$

First of all, we consider a primary operator $\phi^{(k,\vec{m})}$ sitting in the highest weight state of the $(0\,k\,0)$ representation of $su(4)$, associated to the Young tableau

$$\underbrace{\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array} \cdots \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}}_{k} . \tag{C.2}$$

The left-chirality supercharges $\mathcal{Q}^I$ sit in the fundamental representation of $su(4)$, which is associated to the Young tableau

$$\square\,. \tag{C.3}$$

Now we introduce a basis for the left-chirality supercharges. We consider the fundamental representation of $su(4)$ and we assign a weight to each left-chirality supercharge. The lowering operators of the $su(4)$ algebra are $\{e_{-1}, e_{-2}, e_{-3}\}$, so we can realize the $su(4)$ supercharges quadruplet in the following way

$$\left|\mathcal{Q}^1\right\rangle = |\text{h.w.}\rangle\,, \qquad \left|\mathcal{Q}^2\right\rangle = e_{-1}\,|\text{h.w.}\rangle\,, \tag{C.4}$$

$$\left|\mathcal{Q}^3\right\rangle = e_{-2}e_{-1}\,|\text{h.w.}\rangle\,, \qquad \left|\mathcal{Q}^4\right\rangle = e_{-3}e_{-2}e_{-1}\,|\text{h.w.}\rangle = |\text{l.w.}\rangle\,, \tag{C.5}$$

The operators $\left[\mathcal{Q}^I, \phi^{(k,\vec{m})}\right]$ sits in a $su(4)$ representation which can be visualized via the following Young tableau product

$$\square \times \underbrace{\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array} \cdots \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}}_{k} = \cdots + \underbrace{\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array} \cdots \begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2} \end{array}}_{k+1} . \tag{C.6}$$

The representation which hosts the operator is $(1\ k\ 0)$ (cfr. the article [5]); the highest weight is $\left[\mathcal{Q}^1, \phi^{(k,+)}\right]$, while the lowest is $\left[\mathcal{Q}^4, \phi^{(k,-)}\right]$. They are both associated to superconformal states with zero norm:

- let's consider the state $\left|\left[\mathcal{Q}^1_\alpha, \phi^{(k,+)}\right]\right\rangle = \mathcal{Q}^1_\alpha\left|\phi^{(k,+)}\right\rangle = \mathcal{Q}^1_\alpha\,|\text{h.w.}\rangle$. Its norm is

$$\left|\mathcal{Q}^1\,|h.w.\rangle\right|^2 = \left\langle \text{h.w.}\,\middle|\,\mathcal{S}^\alpha_1\mathcal{Q}^1_\alpha\,\middle|\,\text{h.w.}\right\rangle = \left\langle \text{h.w.}\,\middle|\,\mathcal{S}^\alpha_1\mathcal{Q}^1_\alpha + \mathcal{Q}^1_\alpha\mathcal{S}^\alpha_1\,\middle|\,\text{h.w.}\right\rangle. \tag{C.7}$$

Recalling the structure of the anticommutator (cfr. the section 4.1)

$$\left\{\mathcal{Q}^I_\alpha, \mathcal{S}^\beta_J\right\} = \delta^I_J {M_\alpha}^\beta + \frac{1}{2}\delta^I_J\delta^\beta_\alpha\mathcal{D} - \delta^\beta_\alpha {t^I}_J, \tag{C.8}$$

we have

$$\left|\mathcal{Q}^1\,|\text{h.w.}\rangle\right|^2 = \left\langle \text{h.w.}\,\middle|\,{M_\alpha}^\alpha + \mathcal{D} - 2{t^1}_1\,\middle|\,\text{h.w.}\right\rangle \tag{C.9}$$

We consider each operator inside the braket: the primary operator does not transform under the action of the Lorentz group, so $\langle \text{h.w.}\,|\,{M_\alpha}^\alpha\,|\,\text{h.w.}\rangle = 0$; the conformal dimension of the primary is $\Delta = k$, thus $\langle \text{h.w.}\,|\,\mathcal{D}\,|\,\text{h.w.}\rangle = k$. Putting everything together we are left with

$$\left|\mathcal{Q}^1\,|\text{h.w.}\rangle\right|^2 = k - 2\left\langle \text{h.w.}\,\middle|\,{t^1}_1\,\middle|\,\text{h.w.}\right\rangle; \tag{C.10}$$

- let's consider the state $\left|\left[\mathcal{Q}^4_\alpha, \phi^{(k,-)}\right]\right\rangle = \mathcal{Q}^4_\alpha\left|\phi^{(k,-)}\right\rangle = \mathcal{Q}^4_\alpha\,|\text{l.w.}\rangle$. Similarly to the previous point, we obtain

$$\left|\mathcal{Q}^4\,|\text{l.w.}\rangle\right|^2 = k - 2\left\langle \text{l.w.}\,\middle|\,{t^4}_4\,\middle|\,\text{l.w.}\right\rangle. \tag{C.11}$$

We want $|\text{h.w.}\rangle$ and $|\text{l.w.}\rangle$ to be eigenstates of the operators ${t^1}_1$ and ${t^4}_4$. This is possible if and only if the operators ${t^1}_1$ and ${t^4}_4$ are linear combinations of the Cartan generators h_1, h_2 and h_3. Our ansatz is

$${t^1}_1 = A_1h_1 + A_2h_2 + A_3h_3, \qquad {t^4}_4 = B_1h_1 + B_2h_2 + B_3h_3. \tag{C.12}$$

In order to fix the coefficients of the linear combinations above, we recall the commutators written in the section 4.1

$$\left[{t^1}_1, \mathcal{Q}^K_\alpha\right] = \delta^K_1\mathcal{Q}^1_\alpha - \frac{1}{4}\mathcal{Q}^K_\alpha, \qquad \left[{t^4}_4, \mathcal{Q}^K_\alpha\right] = \delta^K_4\mathcal{Q}^4_\alpha - \frac{1}{4}\mathcal{Q}^K_\alpha. \tag{C.13}$$

Recalling the equation (B.19), the weight diagram of the fundamental representation of $su(4)$ drawn in the previous section and the fact that the supercharges sit in such representation, we can completely fix the two sets of coefficients. The explicit expressions of the two operators are

$${t^1}_1 = \frac{3}{4}h_1 + \frac{1}{2}h_2 + \frac{1}{4}h_3, \qquad {t^4}_4 = -\frac{1}{4}h_1 - \frac{1}{2}h_2 - \frac{3}{4}h_3. \tag{C.14}$$

Finally, recalling that the Dynkin labels of the highest weight state and of the lowest weight state are $(0\;k\;0)$ and $(0\;\bar{k}\;0)$, the two norms are

$$\left|\mathcal{Q}^1\,|\text{h.w.}\rangle\right|^2 = k - 2\left\langle \text{h.w.}\,\middle|\,\frac{1}{2}h_2\,\middle|\,\text{h.w.}\right\rangle = k - k = 0, \tag{C.15}$$

$$\left|\mathcal{Q}^4\,|\text{l.w.}\rangle\right|^2 = k + 2\left\langle \text{l.w.}\,\middle|\,\frac{1}{2}h_2\,\middle|\,\text{l.w.}\right\rangle = k + (-k) = 0. \tag{C.16}$$

The proof can be concluded making use of the Jacobi identity

$$\left[e_{-1}, \left[\mathcal{Q}^1, \phi^{(k,+)}\right]\right] - \left[\mathcal{Q}^1, \left[e_{-1}, \phi^{(k,+)}\right]\right] + \left[\phi^{(k,+)}, \left[e_{-1}, \mathcal{Q}^1\right]\right] = 0, \tag{C.17}$$

but $\left[\mathcal{Q}^1, \phi^{(k,+)}\right] = 0$ and $\left[e_{-1}, \mathcal{Q}^1\right] = \mathcal{Q}^2$; moreover

$$\left|\left[e_{-1}, \phi^{(k,+)}\right]\right\rangle = e_{-1}\left|\text{h.w.}\right\rangle, \tag{C.18}$$

so we can compute the norm

$$\langle \text{h.w.} \,|\, e_1 e_{-1} \,|\, \text{h.w.}\rangle = \langle \text{h.w.} \,|\, [e_1, e_{-1}] \,|\, \text{h.w.}\rangle = \langle \text{h.w.} \,|\, h_1 \,|\, \text{h.w.}\rangle = 0, \tag{C.19}$$

because the first Dynkin label of the highest weight is equal to 0. In conclusion, we have

$$\left[\mathcal{Q}^2, \phi^{(k,+)}\right] = 0. \tag{C.20}$$

In a completely analougous way we can prove that

$$\left[\mathcal{Q}^3, \phi^{(k,-)}\right] = 0. \tag{C.21}$$

C.2 Proof of the Null Condition lemma

In this section we prove the lemma (6.65). The lemma is fundamental in the proof of the non-renormalization theorem because it transforms the action of a given supercharge $\mathcal{Q}^*$ on a superconformal chiral primary in the sum of the actions of many supercharges, all different from $\mathcal{Q}^*$. The proof of this lemma can be obtained by induction:

- *First step*: We consider the operator $\phi^{(k,-)}$, associated to the lowest weight state of the $(0\ k\ 0)$ representation of $su(4)$. In the previous section of this appendix we showed that $\phi^{(k,-)}$ is annihilated by the left-chirality supercharges $\mathcal{Q}^3_\alpha$ and $\mathcal{Q}^4_\alpha$. Thus, given that $\mathcal{Q}^*$ is either one of the $\mathcal{Q}^3_\alpha$s supercharges or one of the $\mathcal{Q}^4_\alpha$s, we have

$$\left[\mathcal{Q}^*, \phi^{(k,-)}\right] = 0; \tag{C.22}$$

- *Second step*: We apply one of the three raising operators e_i contained in the $su(4)$ algebra to the lowest weight operator

$$\left[e_i, \phi^{(k,-)}\right], \tag{C.23}$$

  The following Jacobi identity holds

$$\left[\mathcal{Q}^*, \left[e_i, \phi^{(k,-)}\right]\right] - \left[e_i, \left[\mathcal{Q}^*, \phi^{(k,-)}\right]\right] - \left[\phi^{(k,-)}, \left[e_i, \mathcal{Q}^*\right]\right] = 0. \tag{C.24}$$

  Plugging the equation (C.22) in the identity, we obtain

$$\left[\mathcal{Q}^*, \left[e_i, \phi^{(k,-)}\right]\right] = -\left[\left[e_i, \mathcal{Q}^*\right], \phi^{(k,-)}\right]. \tag{C.25}$$

  The operator $[e_i, \mathcal{Q}^*]$, in the chosen basis for the supercharges, is equal either to $\mathcal{Q}^2$, if $\mathcal{Q}^* = \mathcal{Q}^3$, or to $\mathcal{Q}^3$, if $\mathcal{Q}^* = \mathcal{Q}^4$. Then, $[e_i, \mathcal{Q}^*] = \mathcal{Q}^\star \neq \mathcal{Q}^*$

$$\left[\mathcal{Q}^*, \left[e_i, \phi^{(k,-)}\right]\right] = -\left[\mathcal{Q}^\star, \phi^{(k,-)}\right]. \tag{C.26}$$

- *Third step*: First of all, an operator with the generic weight $\vec{m}$ can be constructed as follows

$$\phi^{(k,\vec{m})} = \left[e_{i_1}, \ldots \left[e_{i_n}, \phi^{(k,-)}\right] \ldots\right] \tag{C.27}$$

We consider the lemma (6.65) to be true for the operator (C.27)

$$\left[\mathcal{Q}^{*},\phi^{(k,\vec{m})}\right]=\left[\mathcal{Q}^{*},\left[e_{i_1},\dots\left[e_{i_n},\phi^{(k,-)}\right]\dots\right]\right]=\sum_{\star\neq *}\left[\mathcal{Q}^{\star},\phi^{(k,\vec{m}_{\star})}\right]. \tag{C.28}$$

Now we make the inductive step and we consider the operator

$$\phi^{(k,\vec{m}')}=\left[e_i,\left[e_{i_1},\dots\left[e_{i_n},\phi^{(k,-)}\right]\dots\right]\right]=\left[e_i,\phi^{(k,\vec{m})}\right]. \tag{C.29}$$

We apply the supercharge $\mathcal{Q}^{*}$ to the new operator (C.29), then we employ the Jacobi identity

$$\left[\mathcal{Q}^{*},\phi^{(k,\vec{m}')}\right]=\left[\mathcal{Q}^{*},\left[e_i,\phi^{(k,\vec{m})}\right]\right]=\left[e_i,\left[\mathcal{Q}^{*},\phi^{(k,\vec{m})}\right]\right]-\left[\left[e_i,\mathcal{Q}^{*}\right],\phi^{(k,\vec{m})}\right], \tag{C.30}$$

We define $\mathcal{Q}^{\diamond}\equiv[me_i,\mathcal{Q}^{*}]$ and we recall the equation (C.28), so we can rewrite the previous equation as follows

$$\begin{aligned}\left[\mathcal{Q}^{*},\phi^{(k,\vec{m}')}\right]&=\sum_{\star\neq *}\left[e_i,\left[\mathcal{Q}^{\star},\phi^{(k,\vec{m}_{\star})}\right]\right]-\left[\mathcal{Q}^{\diamond},\phi^{(k,\vec{m})}\right]\\&=\sum_{\star\neq *}\left\{\left[\mathcal{Q}^{\star},\left[e_i,\phi^{(k,\vec{m}_{\star})}\right]\right]+\left[\left[e_i,\mathcal{Q}^{\star}\right],\phi^{(k,\vec{m}_{\star})}\right]\right\}-\left[\mathcal{Q}^{\diamond},\phi^{(k,\vec{m})}\right].\end{aligned}$$

The operator $\phi^{(k,\vec{m}_{\times})}=\left[e_i,\phi^{(k,\vec{m}_{\star})}\right]$ has a higher weight and $\mathcal{Q}^{\times}\equiv[e_i,\mathcal{Q}^{\star}]$ is another element of the supercharges quadruplet different from both $\mathcal{Q}^{\star}$ and $\mathcal{Q}^{*}$, so

$$\left[\mathcal{Q}^{*},\phi^{(k,\vec{m}')}\right]=\sum_{\star\neq *}\left\{\left[\mathcal{Q}^{\star},\phi^{(k,\vec{m}_{\times})}\right]+\left[\mathcal{Q}^{\times},\phi^{(k,\vec{m}_{\star})}\right]\right\}-\left[\mathcal{Q}^{\diamond},\phi^{(k,\vec{m})}\right]. \tag{C.31}$$

The r.h.s. of the equation (C.31) can be condensed as follows

$$\left[\mathcal{Q}^{*},\phi^{(k,\vec{m}')}\right]=\sum_{\circ\neq *}\left[\mathcal{Q}^{\circ},\phi^{(k,\vec{m}_{\circ})}\right], \tag{C.32}$$

which is exactly the structure required by the lemma (6.65).

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