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Holographic effective field theories and the AdS_4/CFT_3 correspondence

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Abstract

The identification of the low-energy effective field theory associated with a given microscopic strongly interacting theory constitutes a fundamental problem in theoretical physics, which is particularly hard when the theory is not sufficiently constrained by symmetries. Recently, a new approach has been proposed, which addresses this problem for a large class of four-dimensional superconformal field theories, admitting a dual weakly coupled holographic description in string theory. This approach provides a precise prescription for the holographic derivation of the associated effective field theories. The aim of the thesis is to explore the generalization of this approach to the three-dimensional superconformal field theories admitting a dual M-theory description, by focusing on a specific model whose effective field theory has not been investigated so far.

Contents

Introduction

Quantum Field Theories (QFTs) are currently the best way to describe fundamental interactions. However, they are affected by some formal illnesses: for instance, Haag's theorem put a strain on the perturbative approach, since it states the non-existence of the interaction picture[1]. Even if we accept to work with perturbation theory there could be problems: indeed, QFTs have typically both weakly-coupled and strongly-coupled energy regimes. While we can use perturbative technologies for the former, the latter is quite challenging to deal with. A practical example is the theory of strong interactions: at low energies it is strongly coupled and hence one should invoke non-perturbative methods in order to get informations. By the way, it turns out that the low-energy behavior of this theory can be described by an Effective Field Theory (EFT): namely, we can build an effective Lagrangian in order to perform calculations in the strongly-coupled regime[2]. It is important to stress that this Lagrangian contains informations about the degrees of freedom relevant at low energies, for example pions, and has the most general expression compatible with the symmetries of the problem. At this point, one could ask if the UV completion of this theory is actually Quantum Chromodynamics (QCD), which is the current gauge theory of strong interactions, or alternatively if integrating out high-momentum degrees of freedom leads to the EFT we are talking about. The answer should be positive and this is supported by numerical results and experimental observations, together with basic theoretical considerations (i.e. symmetry consistence). But what about other theories? Is it always like the QCD case? Does the EFT Lagrangian exist? How can we build it? A general answer has not been found yet, however in certain cases there are prescriptions that lead exactly to the effective Lagrangian. Within this context, supersymmetry is a useful implementation to furtherly constrain a theory. Since supersymmetric particles have not been discovered yet, we should try to introduce a minimal amount of supersymmetry in order to deal with pseudo-realistic theories.

Recently, a novel approach for building EFT Lagrangians of minimally supersymmetric theories has been found in [1],[2]: it exploits the power of the so called gauge-gravity correspondence. This technology comes from a seemingly unrelated area of physics, i.e. String

[1]The original result from Rudolf Haag dates back to 1955 and can be found in "On quantum field theories", Matematisk-fysiske Meddelelser, 29, 12.

[2]See for example [4, 5].

Theory, and the basic idea is to study the gravity dual of the gauge theory: indeed, the former is typically weakly-coupled when the latter is strongly-coupled. In these cases we talk about Holographic Effective Field Theories (HEFT), "holography" being a key word when dealing with this particular kind of duality. Actually, this term perfectly describes the situation: the "hologram" is the gravity theory, also called the "bulk" side of the duality, and leaves in one dimension higher than the gauge theory, also known as the "boundary" side. More precisely, the "bulk" side has also extra dimensions which can be compactified in order to have an AdS_{d+1}/CFT_d duality, where the AdS stands for anti-de Sitter while the CFT is a QFT having an additional conformal symmetry. But why have we invoked String Theory? The reason is that the dynamics of branes, which are the multi-dimensional generalization of point-particles, is described by a gauge theory supported on their worldvolumes. In a large class of models, this gauge theory flows under the renormalization group to a non-trivial CFT and its strongly-coupled regime can be studied switching to the holographic dual. One can place a stack of branes on different background geometries: this will give rise to a family of field theories. The typical spacetime splitting is $\mathbb{R}^d \times X$, where $\mathbb{R}^d$ is "parallel" to the branes (and it is identified with the gauge theory spacetime) while X is a transversal manifold, usually a cone, whose dimension sums with d to ten or eleven, depending on whether we are working in a superstring or a M-theory context respectively. At this point, the stack generates a seemingly black hole configuration: if we study the near-horizon geometry we will find an $AdS_{d+1} \times Y$ splitting, where Y is the (compact) base of the cone X. The most notorious example is the Maldacena duality [3], which relates IIB superstring theory set on the $AdS_5 \times \mathbb{S}^5$ background with maximally supersymmetric Yang-Mills theory on $\mathbb{R}^{1,3}$. In this case we are dealing with AdS_5/CFT_4 duality, where the extra-dimensions of $\mathbb{S}^5$ are compactified. Since this model is maximally supersymmetric it is quite constrained: one among possible generalizations consists in replacing $\mathbb{S}^5$ with another five-dimensional compact manifold Y_5. This would typically lead to theories with less supersymmetries, the amount of supersymmetry being encoded in the geometrical structure of Y_5. The purpose of this thesis is to investigate a further generalization, namely the correspondence between three-dimensional superconformal field theories and their holographic dual. In this case the gravity side is M-theory, which may be interpreted as the strongly-coupled limit of IIA superstring theory. The spacetime splitting is $\mathbb{R}^{1,2} \times X_8$, whose near-horizon limit becomes $AdS_4 \times Y_7$. A natural question would be why are we interested in these kind of models, namely AdS_4/CFT_3? Firstly, toy-model three-dimensional theories have been important for the study of strong coupling features. For instance, in M-Theory context CFTs dual to gravity theories obtained by placing a stack of branes on a particular background lack an adjustable coupling constant and hence are necessarily strongly-coupled: so, holography could shed light on strong coupling phenomena. Secondly, there could exist untreatable condensed matter three-dimensional models which can be studied from the holographic point of view[3]. Last but not least, three-dimensional theories with $\mathcal{N}=2$ supersymmetries are obtained from dimensional reduction of four-dimensional theories with $\mathcal{N}=1$ (because they have the same number of supercharges), which are actually the most realistic models since they are minimally supersymmetric and four-dimensional.

In order to compute the HEFT, this thesis will adopt the strategies discussed in [1],[2]. The first step is the identification of "moduli", i.e. parameters characterizing the family of M-theory background geometries: these are related to the geometric features of the background and to branes positions on it. A fundamental concept is the so called "moduli space", the space of inequivalent vacua: one should check that the moduli space of the gravity side coincides with the field theory moduli space. Indeed, M-Theory moduli will correspond to scalar fields in the dual field theory description: since this is supersymmetric, moduli turn out to be components of chiral or vector supermultiplets. Then, the HEFT Lagrangian should describe the dynamics of these moduli fields. The situation is to some extent similar to massless QCD, i.e. the gauge theory describing two light quarks $m_u \sim 0 \sim m_d$ in four dimensions[4]. At high energy this theory has a global symmetry $SU(2)_L \times SU(2)_R \simeq SO(4)$ called "chiral symmetry". However, this symmetry is spontaneously broken down to $SU(2) \simeq SO(3)$ by VEVs of operators constructed using quark-antiquark pairs, like $\langle u\bar{u}\rangle$ or $\langle d\bar{d}\rangle$. Hence, from Goldstone's Theorem there should be exactly $dim[SO(4)/SO(3)] = dimSO(4) - dimSO(3) = 6-3 = 3$ Goldstone bosons, which are massless states. In the low-energy theory, where the original symmetry is spontaneously broken, only massless modes survive and one can build an EFT Lagrangian for pions π, which are actually the Goldstone bosons of the model at hand. At two-derivative order this takes the quite famous form $\mathcal{L}_{\text{EFT}} = f_\pi^2\, \mathrm{Tr}[\partial_\mu U^\dagger \partial^\mu U]$ and it is also known as the "Chiral Lagrangian"[5]. It is possible to show that the Chiral Lagrangian can be recast in a "geometrized" form called "nonlinear sigma model": this is characterized by non-trivial kinetic terms due to the presence of an overall curved metric, namely

$$\mathcal{L}_{\text{EFT}} = -\frac{1}{2} g_{ab}(\pi)\partial_\mu \pi^a \partial^\mu \pi^b, \qquad g_{ab}(\pi) = \delta_{ab} - \frac{\pi_a \pi_b}{f_\pi^2 - \vec{\pi}\cdot\vec{\pi}}.$$

So, the low-energy physics can be "geometrized": the dynamics of massless pions is described by a nonlinear sigma model and their interactions are encoded in the overall metric[6]. This metric is actually the one on the space of field theory vacua $\frac{SO(4)}{SO(3)} = \mathbb{S}^3$, which is parametrized by the three pions themselves. In a similar way, massless moduli parametrize a particular manifold, which is actually the moduli space, and the HEFT is described by a nonlinear sigma model too. The main difference is that while the Chiral Lagrangian can be obtained using purely

[3] As suggested for example in [40].

[4] See for example [4, 5].

[5] The quantity f_π is the pion decay constant and has the dimension of a mass, while the U function is adimensional and $U = U(\pi/f_\pi)$.

[6] Besides, notice that since the dimensionful f_π appears with negative powers in the interaction terms, the effective Lagrangian is non-renormalizable: this is the price to pay if we want to exploit the EFT at low-energies.

field-theoretical tools, the HEFT Lagrangian requires holography[7]: we will find an expression similar to $\mathcal{L}_{\text{EFT}}$, namely

$$\mathcal{L}^{bosonic}_{\text{HEFT}} = -K_{A\bar{B}}(\Phi,\bar{\Phi})\partial_\mu\Phi^A\partial^\mu\bar{\Phi}^{\bar{B}}, \qquad K_{A\bar{B}}(\Phi,\bar{\Phi}) = \frac{\partial^2 K}{\partial\Phi^A\partial\bar{\Phi}^{\bar{B}}},$$

where Φ are the massless moduli parametrizing the moduli space and $K_{A\bar{B}}$ is the metric on it. Its explicit expression depends on a function of moduli K: this is called "Kähler potential" of the moduli space and clearly plays a crucial role.

The original contribution of this work is the application of the aforementioned construction to a specific model, namely the Q^{111} theory. M2-branes are placed on a $\mathbb{R}^{1,2} \times X_8$ background geometry, X_8 being the cone over the seven-manifold Q^{111}, whose geometrical structure give rise to $\mathcal{N} = 2$ three-dimensional theories. After the classification of moduli in the gravity side, the HEFT Lagrangian is obtained by expanding the supergravity action: its truncation to two-derivatives order leads to the corresponding nonlinear sigma model. The dual theory is actually the IR fixed point of a "quiver", i.e. a gauge theory with matter in the adjoint and bifundamental representations. Its moduli space is shown to reproduce the cone over Q^{111}, as expected. Subtleties involving matter fields in the (anti)fundamental representation are highlighted since they correspond to "flavors" in the field theory and to geometrical D6-brane solutions in the IIA String Theory side (see for example [6]).

The thesis will be structured as follows. In the first chapter we want to introduce some basic concepts necessary to understand the main topic of this work: we will talk about supersymmetry (SUSY), conformal field theories and anti-de Sitter spacetime. Chapter two is dedicated to complex differential geometry because it is really the language required for this kind of study: we will try to be "not-so-rigorous" and our attention should be oriented towards the physical sense of using Calabi-Yau (CY) cones. Then, in the following chapter we present M-Theory, focusing on M2-branes solutions. We are particularly interested in backgrounds containing a Calabi-Yau cone CY_4: the near-horizon geometry is then investigated. Besides, the crucial points of gauge/gravity correspondence are illustrated, for example the natural presence of a gauge theory on the worldvolume of branes. Chapter four is a complete review of the Q^{111} quiver field theory: we will introduce the concept of moduli space and we will show how it can be obtained with different methods. In the fifth chapter we present the HEFT machinery, i.e. the identification and parametrization of moduli and the Lagrangian describing them, together with issues about the so called "$\mathcal{S}$-operation". Finally, chapter six contains the original contribution of this work, namely the HEFT for the Q^{111} model. The explicit metric of the moduli space is found using a suitable parametrization and this allows the construction of the HEFT Lagrangian as a nonlinear sigma model. Then we will carry out the matching with the field theory side, checking that the moduli space is actually the same.

[7]Indeed, $\mathcal{L}_{\text{EFT}}$ is almost completely fixed by symmetry arguments: the problem is that these arguments are not sufficient in supersymmetric cases. Moreover, field theories under exam are strongly coupled: this "suggests" that holography may be a possible solution for building an effective theory at two-derivatives, i.e. an HEFT.

Chapter 1

SUSY, CFT and AdS

The aim of this chapter is to collect the basic ingredients for this work. We start presenting the main features of supersymmetry (also known as SUSY), starting from a review on $d = 4$ $\mathcal{N} = 1$ which is propaedeutical to the $d = 3$ $\mathcal{N} = 2$ case. Indeed, the latter can be seen as a dimensional reduction of the former. We will follow [7, 4] for the first part and then [8, 9, 10]. After the SUSY introduction we will present the conformal group as an extension of the Poincaré one, together with some issues about scale/conformal invariance: we will follow [12, 13, 14, 15]. The next step is to consider a further extension of Poincaré algebra, taking into account both SUSY and conformal generators: the superconformal algebra. In the end, AdS-spacetimes are introduced. For these last topics we will consider [16, 17, 18].

1.1 Basics of SUSY

In the last few decades supersymmetry has played an important role not only in purely theoretical contexts but also in particle physics phenomenology.

This new symmetry made is first appearance in the seventies in String Theory context as a symmetry of the two-dimensional worldsheet. The first version of String Theory was purely bosonic and this led to two problems: there were tachyons, i.e. unphysical particles with negative mass, and there were not any fermions, which is unrealistic for phenomenological applications. Including SUSY in the description solve both of this problems. Indeed, SUSY is a symmetry which relates bosons and fermions such that every boson has a fermionic "partner". Moreover, it can be shown that the resulting (Super)String Theory lacks tachyons. It was then realized that SUSY could be a powerful tool for studying QFTs and hence it could be relevant for elementary particle physics. Since then, physicists proposed a lot of supersymmetric theories: minimal ($\mathcal{N} = 1$) SUSY, extended ($\mathcal{N} > 1$) SUSY, gauged SUSY (i.e. Supergravity). The most realistic SQFT should be a four-dimensional minimally supersymmetric theory representing the extension of the Standard Model, which is the current theory describing nature. Actually, there are several reasons to require SUSY in a phenomenological theory. First of all, the introduction of supersymmetric partners induces loop-cancellations. As a consequence, certain small or vanishing classical quantities will remain so once loop-corrections are taken into account. Furthermore, it seems that SUSY is necessary (although not sufficient for the last two of the following points) to solve some famous problems like:

- the running of Standard Model coupling constants, allowing the three couplings to meet at a specific "unifying" scale;
- the hierarchy problem, i.e. the big gap between Planck scale and Electroweak symmetry breaking scale;
- the smallness of the cosmological constant predicted by QFTs compared with experimental values;
- the renormalization procedure of quantum gravity.

However, LHC runs have not discovered supersymmetric particles yet. This means that SUSY must be broken at experimental energy scales since otherwise some of the predicted partners should be found. By the way, in this thesis we are not so interested in phenomenological results: SUSY should be regarded as a "simplifying assumption", constraining our models in such a way that they become "easy" to study.

1.2 Rigid SUSY in d=3+1

SUSY can be seen as an extension of the Poincaré algebra by generators commuting with translations[1]. These new elements of the algebra have "anticommuting grading", which means that infinitesimal parameters associated to supersymmetric variations are Grassmann variables. Supercharges transform either as dotted $\bar{Q}^I_{\dot{\alpha}}$ or undotted Q^I_α spinors under the Lorentz group and satisfy the following algebra:

$$\begin{aligned}
\left[P_\mu, Q^I_\alpha\right] &= 0, \\
\left[P_\mu, \bar{Q}^I_{\dot{\alpha}}\right] &= 0, \\
\left[M_{\mu\nu}, Q^I_\alpha\right] &= i(\sigma_{\mu\nu})_\alpha^{\beta} Q^I_\beta, \\
\left[M_{\mu\nu}, \bar{Q}^{I\dot{\alpha}}\right] &= i(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}_{\dot{\beta}} \bar{Q}^{I\dot{\beta}}, \\
\left\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\right\} &= 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^{IJ}, \\
\left\{Q^I_\alpha, Q^J_\beta\right\} &= \epsilon_{\alpha\beta} Z^{IJ}, \\
\left\{\bar{Q}^I_{\dot{\alpha}}, \bar{Q}^J_{\dot{\beta}}\right\} &= \epsilon_{\dot{\alpha}\dot{\beta}} (Z^{IJ})^*.
\end{aligned} \tag{1.2.1}$$

[1] We are not going to derive the superalgebra but we will make some dictated comments.

The index I runs over $\mathcal{N}$, which is the number of supersymmetries. It can be shown that $\mathcal{N}$ is related to the number of supercharges $\#$ by $\mathcal{N} = \frac{\#}{d_R}$, where d_R is the real dimension of the smallest irreducible spinorial representation of $SO(1, d-1)$[2]. Z^{IJ} are the so called central charges and they commute with every generator of the algebra by definition. Because of the grading, Z is antisymmetric and hence vanishes in the minimal SUSY case. From an algebraic point of view there is no reason to limit $\mathcal{N}$: however, it can be shown that consistent QFTs must have $\mathcal{N} \leq 8$ if gravity is taken into account and $\mathcal{N} \leq 4$ if we don't consider particles with spin larger than one.

Since the full SUSY algebra contains the Poincaré one, any representation of the superalgebra gives a representation of the Poincaré algebra, although in general a reducible one. It is well known that irreducible representations (irreps from now on) of Poincaré algebra correspond to what we commonly call particles: instead, an irrep of the superalgebra is associated to several particles organized in a supermultiplet. The corresponding states are related to each other by supercharges: since "$Q(fermions) = bosons$", states in the same supermultiplet may differ by one-half spin units.

From the superalgebra (1.2.1) one can obtain three fundamental features of supersymmetric theories:

1. Supermultiplets always contain an equal number of bosonic and fermionic degrees of freedom. Moreover, every field in a supermultiplet transform under the same representation.

2. All particles in a supermultiplet have the same mass. This is because P^2 is a Casimir also in the SUSY case, i.e. it commutes with every generator of the superalgebra[3]. However, they do not have the same spin.

3. The energy P_0 of any state in the Fock space is never negative.

The most important massless supermultiplets in the minimal global SUSY case, after integrating out auxiliary fields, are:

- the chiral multiplet $\Phi = (\varphi, \psi)$, containing a complex scalar and a Weyl spinor;

- the vector multiplet $V = (\chi, A)$, containing a gauge boson and a Weyl fermion (both in the adjoint representation of the gauge group).

Since we want to build SQFTs, we have to find representations of the superalgebra on fields: the most elegant way to achieve this is the so called superspace formalism. The basic idea is to interpret supercharges as generators of translations in some Grassmannian coordinate, in

[2]We address the reader to appendix B of [11] for a complete description of spinors in various dimensions.

[3]There are subtleties in the SCFT case because in conformal field theories P^2 is no more a good quantum number.

the same way as momentum generates spacetime translations. Then, the spacetime is enlarged using these "fermionic coordinates" and becomes a "superspace" parametrized by $(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\beta}})$.

Using this formalism, supercharges act on functions of the superspace variables[4] as derivative operators:

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu, \qquad \bar{Q}_{\dot{\beta}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\beta}}} + i\theta^\alpha\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu. \tag{1.2.2}$$

Since Grassmannian variables anticommutes, any product involving two or more of them vanishes. So, one can Taylor-expand a generic superspace (scalar) function, i.e. a superfield, as:

$$\begin{aligned} Y(x,\theta,\bar{\theta}) =& f(x) + \theta\psi(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta m(x) + \bar{\theta}\bar{\theta}n(x) + \\ &+ \theta\sigma^\mu\bar{\theta}v_\mu(x) + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\rho(x) + \theta\theta\bar{\theta}\bar{\theta}d(x). \end{aligned} \tag{1.2.3}$$

This expansion easily generalizes to tensors, with $Y_{...}$ carrying the same index structure of its components $(f_{...}, \psi_{...}, ...)$. However, a generic superfield contains too many degrees of freedom to represent an irrep of the superalgebra. Hence, we should impose some SUSY-invariant condition such that the number of degrees of freedom are lowered. In order to do this we first define some "covariant derivatives" D_α and $\bar{D}_{\dot{\beta}}$, anticommuting with SUSY generators:

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu, \qquad \bar{D}_{\dot{\beta}} = -\frac{\partial}{\partial\theta^{\dot{\beta}}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu. \tag{1.2.4}$$

At this point, $\bar{D}_{\dot{\beta}}\Phi = 0$ is a SUSY-invariant condition and it turns out that it effectively reduce the number of degrees of freedom in the generic superfield. Actually, a chiral superfield Φ is defined by

$$\bar{D}_{\dot{\beta}}\Phi = 0 \tag{1.2.5}$$

and admits the following expansion:

$$\begin{aligned} \Phi(x,\theta,\bar{\theta}) =& \phi(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\phi(x) + \\ &+ \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^\mu\bar{\theta} + \theta\theta F(x). \end{aligned} \tag{1.2.6}$$

Notice that it is expressed in terms of x-spacetime coordinate. Instead, the constraint (1.2.5) is easily solved if we define an y-spacetime coordinate as a shift of the x:

$$y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}, \qquad \bar{y}^\mu = x^\mu - i\theta\sigma^\mu\bar{\theta}. \tag{1.2.7}$$

Moreover, the chiral superfield expansion is now dependent only on θ and y, while the $\bar{\theta}$ dependence is hidden inside y:

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y). \tag{1.2.8}$$

[4]The convention on products of grassmannian variables we will follow is $\theta_\alpha\bar{\theta}_\beta = -\frac{1}{2}\epsilon_{\alpha\beta}\theta\bar{\theta}$, which is the one of [10].

This expansion (1.2.8) underlines the fact that the chiral supermultiplet contains a complex scalar ϕ, a Weyl spinor ψ and an auxiliary non-propagating scalar F.

Besides, we are interested in (abelian) vector superfields V. These satisfy the reality condition:

$$V = V^{\dagger}. \tag{1.2.9}$$

Its superfield expansion in the so called Wess-Zumino gauge, i.e. partially gauge-fixed in such a way that some undesired components are eliminated, is:

$$V_{WZ}(x,\theta,\bar{\theta}) = -\theta\sigma^{\mu}\bar{\theta}A_{\mu}(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x), \tag{1.2.10}$$

where A_{μ} is the gauge potential, λ is the Weyl spinor of the vector multiplet and D is a real auxiliary scalar. From expansion (1.2.10) it follows that $V_{WZ}^{n} = 0$ for $n \geq 3$, which turns out to be useful. Indeed, for non-abelian gauge theories the basic object is e^{V} rather than V itself and in the WZ-gauge one has:

$$e^{V} = 1 + V + \frac{V^2}{2}. \tag{1.2.11}$$

Superfield strengths are then defined as

$$W_{\alpha} = -\frac{1}{4}\bar{D}\bar{D}\left(e^{-V}D_{\alpha}e^{V}\right) \tag{1.2.12}$$

and one can easily check that they are chiral fields, i.e. satifying (1.2.5).

1.2.1 Supersymmetric Lagrangians in d=3+1

The reason why superspace is such an elegant formalism is that SUSY is manifest at Lagrangian level. For instance, any Lagrangian of the form

$$\int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} Y(x,\theta,\bar{\theta}) + \int \mathrm{d}^2\theta W(\Phi) + \int \mathrm{d}^2\bar{\theta}\,[W(\Phi)]^{\dagger} \tag{1.2.13}$$

is automatically SUSY-invariant since it transforms at most by a total spacetime derivative. Actually, the first term in (1.2.13) can be seen as a kinetic term while the other two are superpotentials, i.e. products of (anti)chiral superfields which are (anti)chiral superfields too. Notice that in the kinetic term there are both θ and $\bar{\theta}$ measures, while in the superpotiential there is only one of them. The reason is that for the former Y has $\theta\theta\bar{\theta}\bar{\theta}$ components while for the latter W has at most $\theta\theta$ components, being a chiral superfield. Then, Grassmann-integration rules pick only these particular components of the expansion for the total action functional.

To be more precise, let us write down some explicit Lagrangians. The most general renormalizable kinetic Lagrangian describing matter in a supersymmetric gauge theory is

$$\mathcal{L}_{kin} = \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \sum_i \Phi_i^{\dagger} e^{V} \Phi_i, \tag{1.2.14}$$

while for the kinetic term of gauge fields one has

$$\mathcal{L}_{gauge} = \frac{1}{g^2_{(i)}} \int \mathrm{d}^2\theta W^{(i)}_\alpha W^{\alpha(i)}, \tag{1.2.15}$$

where index i runs over all the matter multiplets. Together with the superpotential terms of (1.2.13), which we call $\mathcal{L}_W$, the total Lagrangian density is given by:

$$\mathcal{L} = \mathcal{L}_{kin} + \mathcal{L}_W + \mathcal{L}_{gauge}. \tag{1.2.16}$$

Occasionally, there could be "Fayet-Iliopoulos terms" in the Lagrangian. These are related to $U(1)$ factors in the gauge group and for each of them we can include

$$\mathcal{L}_{FI} = \sum_a \zeta^a \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} V^a, \tag{1.2.17}$$

where V is the abelian vector superfield associated to the $U(1)$.

Lagrangians above are the most general renormalizable ones in four dimensions[5]. However, we can drop the renormalizability principle and write the following $\mathcal{N} = 1$ superymmetric theory:

$$\mathcal{L} = \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} K\left((\Phi^\dagger e^{2gV})_i, \Phi_i\right) + \int \mathrm{d}^2\theta W(\Phi) + \int \mathrm{d}^2\theta f_{(ab)}(\Phi_i) W^{\alpha(a)} W^{(b)}_\alpha + c.c \tag{1.2.18}$$

where $K\left((\Phi^\dagger e^{2gV})_i, \Phi_i\right)$, the so called "Kähler potential", gives rise to kinetic terms while the $f_{(ab)}(\Phi_i)$ is a function of the chiral fields only and $W(\Phi)$ is generic. Typically (1.2.18) do not describe a fundamental, i.e. microscopic, theory because we dropped the renormalizability assumption. Nevertheless, it can describe an effective field theory valid at low energies only: renormalizability is no longer a criterion and one can build a Lagrangian containing no more than two spacetime-derivatives, while possible higher-order terms give subleading effects. In absence of vector fields (1.2.18) contains chiral multiplets only and it is globally supersymmetric. In this work we will come across an effective Lagrangian like[6] $\mathcal{L} = \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} K(\bar{\Phi}, \Phi)$: expanding the Kähler potential we are led to the SUSY version of the "nonlinear sigma model", i.e. a Lagrangian with nontrivial kinetic term describing interactions between low-energy degrees of freedom in a "geometric" way[7]. In the following chapters we will deepen this relation between physics and geometry in order to explicitly find the Kähler potential of the Q^{111} model.

[5] This is true if $W(\Phi)$ is at most cubic.

[6] More precisely, it will be a three-dimensional model while here we are discussing four-dimensional theories.

[7] As an anticipation, the aforementioned expansion will give something like

$$\mathcal{L} = \int \mathrm{d}^4\theta K(\bar{\Phi}, \Phi) = -K_{A\bar{B}}(\bar{\Phi}, \Phi) \partial_\mu \Phi^A \partial^\mu \bar{\Phi}^{\bar{B}} + ..., \qquad K_{A\bar{B}} = \frac{\partial^2 K}{\partial \Phi^A \partial \bar{\Phi}^{\bar{B}}},$$

where $K_{A\bar{B}}$ is the nontrivial metric of the nonlinear sigma model.

The scalar potential

We said that F and D are auxiliary component fields of chiral and vector supermultiplets respectively. Expanding the supermultiplets one can identify a "scalar potential" $\mathcal{V}$ that takes the form:

$$\begin{aligned}\mathcal{V}(\phi^\dagger, \phi) &= F^\dagger F + \frac{1}{2} D^2 = \\ &= \sum_i \left| \frac{\partial W}{\partial \phi^i} \right|^2 + \frac{1}{2} \sum_a \left| g^a \left(\phi^\dagger T^a \phi + \zeta^a \right) \right|^2 ,\end{aligned} \tag{1.2.19}$$

where the second equality comes from the equation of motion for the auxiliary fields and T^a are generators of the gauge group in some representation. In general, supersymmetric theories do not have isolated vacua: instead, they exhibit a continuous family of connected vacua. We call "moduli space" of inequivalent vacua the set of all zero-energy field configurations (modulo gauge transformations if any) for which (1.2.19) (or its generalizations as we will see) vanishes. Indeed, notice that since the scalar potential is a sum of squares, vacua are configurations such that $\mathcal{V} = 0$, i.e. $\langle F \rangle = 0 = \langle D \rangle$.

$\mathcal{R}$-Simmetry

Supersymmetric theories have additional global symmetries which can be seen as "supercharge-rotations": this is the reason why they are called $\mathcal{R}$-symmetries. It is important to stress that $\mathcal{R}$-symmetries are not supersymmetries, i.e. there are no related supercharges entering SUSY algebra. Defining the transformation of θ and $\bar{\theta}$ as

$$\theta \to e^{iq} \theta, \qquad \bar{\theta} \to e^{-iq} \bar{\theta}, \tag{1.2.20}$$

SUSY generators transform as

$$Q \to e^{-iq} Q, \qquad \bar{Q} \to e^{iq} \bar{Q}. \tag{1.2.21}$$

We anticipate that $\mathcal{R}$-symmetries are crucial because the scaling dimensions of chiral fields at nontrivial fixed points are fixed by their $\mathcal{R}$-charges.

1.3 Rigid SUSY in d=2+1

Although the three-dimensional case with $\mathcal{N} = 2$ supersymmetries can be obtained from dimensional-reduction of the minimally supersymmetric four-dimensional one[8], there exist some differences between them. First of all, the gauge coupling is dimensionful in three dimensions[9].

[8] They both posses four supercharges because their number is given by $\# = d_R \mathcal{N}$.

[9] We anticipate that this means that there cannot exist conformal field theories outside the infrared fixed point: we will be more clear and provide an intuitive explanation later on.

Secondly, the superalgebra changes a bit: there are no more dotted indexes because three-dimensional Poincaré group is $SL(2,\mathbb{R})$ instead of $SL(2,\mathbb{C})$ and hence the fundamental representation acts on real (Majorana) spinors. Moreover, the anticommuting rule for supercharges in (1.2.1) becomes

$$\left\{Q_\alpha, \bar{Q}_\beta\right\} = 2\gamma^\mu_{\alpha\beta} P_\mu + 2i\epsilon_{\alpha\beta} Z, \tag{1.3.1}$$

where $\gamma = (i\sigma_2, \sigma_3, \sigma_1)$ are real. The central charge Z can be interpreted as the dimensional-reduced momentum along the third space-dimension, namely the P_3 component of the four-momentum. SUSY generators Q and $\bar{Q}$ are complex now, so they include twice the minimal amount of supersymmetry in three dimensions. As in the four-dimensional case, there is a $U(1)_\mathcal{R}$ symmetry rotating supercharges.

Chiral superfield condition is actually the undotted version of the four-dimensional case (1.2.5) while vector superfield condition is the very same of (1.2.9): they both contain two real bosonic and two Majorana fermionic degrees of freedom on-shell. In addition, vector superfields V may be expressed in terms of linear superfields Σ satisfying

$$\epsilon^{\alpha\beta} D_\alpha D_\beta \Sigma = \epsilon^{\alpha\beta} \bar{D}_\alpha \bar{D}_\beta \Sigma = 0, \qquad \Sigma^\dagger = \Sigma, \tag{1.3.2}$$

whose lowest component is a scalar field instead of a spinor. More precisely, the vector supermultiplet contains a gauge field A_μ, a two-component complex spinor λ (the gaugino), a real scalar field σ (that can be interpreted as the dimensional-reduced component A_3 of the four-dimensional gauge field) and an auxiliary real scalar D. In Wess-Zumino gauge the expansions are:

$$V = -i\theta\bar{\theta}\sigma - \theta\gamma^\mu\bar{\theta}A_\mu + i\theta\theta\bar{\theta}\bar{\lambda} + i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D \tag{1.3.3}$$

and[10]

$$\begin{aligned}\Sigma &= -\frac{i}{2}\epsilon^{\alpha\beta}\bar{D}_\alpha D_\beta V = \\ &= \sigma + \theta\bar{\lambda} + \bar{\theta}\lambda + i\theta\bar{\theta}D + \frac{1}{2}\theta\gamma^\mu\bar{\theta}J_\mu - \frac{i}{2}\theta\theta\bar{\gamma}^\mu\partial_\mu\bar{\lambda} + \frac{i}{2}\bar{\theta}\bar{\theta}\theta\gamma^\mu\partial_\mu\lambda + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\sigma.\end{aligned} \tag{1.3.4}$$

The J field in (1.3.4) is the so called "dual field strength". Indeed, in three spacetime dimensions there exist a duality between the vector-photon A_μ and the scalar-photon τ such that

$$J_\mu = \partial_\mu\tau = \epsilon_{\mu\nu\rho}F^{\nu\rho}, \tag{1.3.5}$$

where $F^{\nu\rho}$ is the field strength of A_μ. Furthermore, it is possible to dualize the whole linear multiplet into a chiral multiplet Ψ having $\sigma + i\tau$ as its lowest component.

[10] We should mention that the following linear superfield expansion works for abelian gauge groups. In this thesis we will actually use Σ only in this case.

1.3.1 Supersymmetric Lagrangians in d=2+1

Kinetic Lagrangians for matter and gauge fields in three dimensions are the same of (1.2.14) and (1.2.15) respectively. An alternative for abelian vector superfields is to use the linear multiplet description with a kinetic Lagrangian like

$$\mathcal{L}_{lin} = \frac{1}{g^2}\int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\Sigma^2. \tag{1.3.6}$$

In three dimensions we can also include topological "Chern-Simons terms" since they are gauge-invariant. These take the form

$$\mathcal{L}_{CS} = \sum_i \frac{k_i}{4\pi}\,\mathrm{Tr}\left(\epsilon^{\mu\nu\rho}\left(A_\mu^{(i)}\partial_\nu A_\rho^{(i)} + \frac{2i}{3}A_\mu^{(i)}A_\nu^{(i)}A_\rho^{(i)}\right) + 2D^{(i)}\sigma^{(i)} - \bar{\lambda}^{(i)}\lambda^{(i)}\right), \tag{1.3.7}$$

where $k_i \in \mathbb{Z}$ are the so called "Chern-Simons levels" and the index i runs over the factors of the gauge group. In superspace notation we can rewrite (1.3.7) more compactly as

$$\mathcal{L}_{CS} = \frac{k}{4\pi}\int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\,\mathrm{Tr}\,\Sigma V. \tag{1.3.8}$$

Notice that for abelian factors (1.3.8) seems like a Fayet-Iliopoulos term: this can be seen if we consider the linear multiplet as an external field, i.e. non dynamical. Then, turning off every component field but the scalar σ, the Lagrangian (1.3.8) becomes exactly (1.2.17) provided that $\zeta = k\sigma$.

Another characteristic of three-dimensional SUSY theories is the distinction between real and complex masses. The latter are parameters entering in the Lagrangian via superpotential terms like $W_{\mathbb{C}} = m_{\mathbb{C}}\Phi^\dagger\Phi$, while the former are "induced" from external vector supermultiplets. Consider one such background vector $V_{bg} = -i\theta\bar{\theta}\sigma_{bg} + ...$ and imagine that the scalar component takes a real VEV $\langle\sigma_{bg}\rangle = m_{\mathbb{R}}$, whereas the others are all turned off. Then, a Lagrangian like

$$\mathcal{L}_{\mathbb{R}} = \int \mathrm{d}^4\theta \Phi^\dagger e^{V_{bg}}\Phi \tag{1.3.9}$$

clearly give rise to a mass term $m_{\mathbb{R}}^2|\phi|^2$ for the scalar in the chiral multiplet Φ and $m_{\mathbb{R}}\bar{\psi}\psi$ for the fermionic component. We want to point out that (1.3.9) can be interpreted as a modification to (1.2.14): indeed, expanding in component fields, the "effective" mass is given by $m = m_{\mathbb{R}} + \langle\sigma\rangle$, where σ is the scalar in the vector supermultiplet appearing in (1.2.14). We will see that the Q^{111} is in fact characterized by a real mass.

1.4 Conformal Field Theories

A conformal field theory is a quantum field theory invariant under the conformal group. We usually deal with the Poincaré group as the symmetry group of relativistic theories in flat spacetime. The explicit form of Poincaré transformations is

$$x^\mu \to \Lambda^\mu_\nu x^\nu + a^\mu,$$

that is a combination of Lorentz transformations and spacetime translations. This kind of transformations preserve distances. We can extend the spacetime symmetry group in such a way that angles between vectors are preserved: this is the conformal group, which clearly include the Poincaré one.

The most intuitive such transformation is the dilatation, a rescaling of spacetime coordinates such that

$$x^\mu \to \lambda x^\mu .$$

It is evident that this is not a Poincaré transformation since the metric does change:

$$\eta_{\mu\nu} \to \lambda^{-2}\eta_{\mu\nu}.$$

We can say that conformal transformations are generalizations of these scale transformations such that

$$x \to \tilde{x}(x), \qquad \eta_{\mu\nu} \to f(x)\eta_{\mu\nu}.$$

1.4.1 The conformal group

First of all in what follows we will deal with $d \geq 3$ spacetime dimensions, having finite-dimensional conformal group[11]. In order to obtain it we should start from conformal transformations. These consist of a Weyl transformation, i.e. local rescaling of the metric like

$$g'_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x), \tag{1.4.1}$$

combined with a coordinate diffeomorphism such that the metric is left invariant, namely:

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\hat{\Omega}^2(x) g_{\rho\sigma}(x) = g_{\mu\nu}(x), \tag{1.4.2}$$

where the last equality has to be read as "must be equal to" and $\Omega \neq \hat{\Omega}$ are arbitrary functions of the coordinates. Let us consider flat spacetime $g'_{\mu\nu} = g_{\mu\nu} = \eta_{\mu\nu}$. We can rewrite (1.4.2) as

$$\eta_{\rho\sigma}\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu} = \hat{\Omega}^2(x)\eta_{\mu\nu}. \tag{1.4.3}$$

When $\hat{\Omega}^2 = 1$ the Poincaré transformation condition is reproduced. Consider instead an infinitesimal coordinate transformation of the form

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu + O(\epsilon^2). \tag{1.4.4}$$

[11] It is important to mention that the $d = 2$ case is very interesting not only because it plays a crucial role in String Theory but also because the conformal group is infinite-dimensional. Besides, the $d = 1$ case seems to be a "conformal quantum mechanics": we will not enter in these topics.

Under (1.4.4) the LHS of (1.4.3) becomes at first order

$$\eta_{\rho\sigma}\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}=\eta_{\rho\sigma}\left(\delta^\rho_\mu+\frac{\partial\epsilon^\rho}{\partial x^\mu}\right)\left(\delta^\sigma_\nu+\frac{\partial\epsilon^\sigma}{\partial x^\nu}\right)= \\ =\eta_{\mu\nu}+\left(\frac{\partial\epsilon_\mu}{\partial x^\nu}+\frac{\partial\epsilon_\nu}{\partial x^\mu}\right).$$

Comparing with (1.4.3), it is clear that at first order in ϵ we must have

$$\partial_\mu\epsilon_\nu+\partial_\nu\epsilon_\mu=\omega(x)\eta_{\mu\nu}, \tag{1.4.5}$$

where ω is such that $\hat\Omega^2=1+\omega+...$ at infinitesimal level. At this stage we can further simplify (1.4.5): indeed, tracing both sides we get $\omega=\frac{2}{d}\partial^\mu\epsilon_\mu$. Substituting back into (1.4.5) we finally obtain

$$\partial_\mu\epsilon_\nu+\partial_\nu\epsilon_\mu-\frac{2}{d}(\partial^\rho\epsilon_\rho)\eta_{\mu\nu}=0, \tag{1.4.6}$$

which is the fundamental equation identifying conformal transformations (at infinitesimal level).

For $d=2$ there are infinite solutions for (1.4.6), while $d=1$ is a singular case: however, recall our interest in $d\geq 3$. It can be shown that the solution $\epsilon_\mu(x)$ is at most quadratic in x^ν and thus will take the form

$$\epsilon_\mu(x)=a_\mu+b_{\mu\nu}x^\nu+c_{\mu\nu\rho}x^\nu x^\rho. \tag{1.4.7}$$

Notice that for $b=0=c$ we recover infinitesimal translations, having momentum operator $P_\mu=i\partial_\mu$ as generator. Inserting the linear term of (1.4.7) into (1.4.6) gives

$$b_{\mu\nu}+b_{\nu\mu}=\frac{2}{d}(\eta^{\rho\sigma}b_{\rho\sigma})\eta_{\mu\nu}$$

so that we can split the b-coefficient in symmetric and antisymmetric parts like

$$b_{\mu\nu}=\alpha\eta_{\mu\nu}+m_{\mu\nu}.$$

The $m_{\mu\nu}$ tensor corresponds to infinitesimal Lorentz transformations, whose generator is $M_{\mu\nu}=i(x_\mu\partial_\nu-x_\nu\partial_\mu)$. The symmetric part correspond instead to infinitesimal dilatations with generator $D=ix^\mu\partial_\mu$.

The last class of solutions are the quadratic ones: these correspond to the so called "special conformal transformation"[12] and one can show that they are generated by $K_\mu=i(2x_\mu x^\nu\partial_\nu-x^2\partial_\mu)$.

[12] There are issues with finite special conformal transformations since they are not globally defined: one should consider conformal compactifications of spacetime, including points at infinity, but again this is a subtlety we do not investigate in this work.

Having generators we can introduce the conformal algebra:

$$\begin{aligned}
[P_\mu, P_\nu] &= 0 \\
[M_{\mu\nu}, M_{\rho\sigma}] &= -i(\eta_{\mu\rho} M_{\nu\sigma} + permutations) \\
[M_{\mu\nu}, P_\rho] &= i(\eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu) \\
[P_\mu, D] &= i P_\mu \\
[M_{\mu\nu}, D] &= 0 \\
[K_\mu, D] &= -i K_\mu \\
[P_\mu, K_\nu] &= 2i(\eta_{\mu\nu} D + M_{\mu\nu}) \\
[K_\mu, K_\nu] &= 0 \\
[M_{\rho\sigma}, K_\mu] &= i(\eta_{\mu\rho} K_\sigma - \eta_{\mu\sigma} K_\rho).
\end{aligned} \tag{1.4.8}$$

Notice that scale invariance, i.e. dilatation, is necessary for conformal invariance because D closes the algebra: so, conformal invariance implies scale invariance. The converse is not (totally) true: scale invariance does not imply conformal invariance. However, in many field theories the full conformal group seems to emerge from scale invariance only: we will try to give a partial explanation soon after, but we stress that it is still an open problem. Before doing this, we should point out that (1.4.8) algebra is isomorphic to $so(d, 2)$, which is the Lorentz algebra in mixed signature $(d, 2)$. Indeed, conformal generators can be identified with Lorentz ones as follows

$$J_{\mu\nu} = M_{\mu\nu}, \quad J_{\mu+} = P_\mu, \quad J_{\mu-} = K_\mu, \quad J_{+-} = D, \tag{1.4.9}$$

so that the algebra is exactly the Lorentz one:

$$[J_{MN}, J_{RS}] = -i(\eta_{MR} J_{NS} + permutations). \tag{1.4.10}$$

This allows us to anticipate a crucial point of AdS/CFT duality right here: the conformal group $SO(2, d)$ of d-dimensional flat spacetime is exactly the isometry group of AdS-spacetime in one dimension higher.

1.4.2 Local Field Representations

We all know that irreps of the Poincaré group are interpreted as particles in a quantum field theory. However, for a conformal invariant theory the "mass" P^2 is no more a Casimir and one should replace it with a better quantum number: this leads to the concept of "unparticles".

Recall that we can realize the conformal algebra on spacetime functions as differential operators:

$$\begin{aligned} P_\mu &= i\partial_\mu \\ M_{\mu\nu} &= i(x_\mu\partial_\nu - x_\nu\partial_\mu) \\ D &= ix^\mu\partial_\mu \\ K_\mu &= i(2x_\mu x^\rho\partial_\rho - x^2\partial_\mu). \end{aligned} \tag{1.4.11}$$

In QFTs we should realize these symmetries as operators acting on Hilbert spaces (Schrödinger picture) or on local operators (Heisenberg picture). Focusing on the latter, the action of generators (1.4.11) on local fields[13] reads

$$\begin{aligned} [P_\mu, O(x)] &= -i\partial_\mu O(x) \\ [M_{\mu\nu}, O(x)] &= -i(\Sigma_{\mu\nu} + x_\mu\partial_\nu - x_\nu\partial_\mu)O(x) \\ [D, O(x)] &= -i(\Delta + x^\mu\partial_\mu)O(x) \\ [K_\mu, O(x)] &= -i(2x_\mu\Delta + 2x^\lambda\Sigma_{\lambda\mu} + 2x_\mu x^\rho\partial_\rho - x^2\partial_\mu)O(x), \end{aligned} \tag{1.4.12}$$

where Δ is the scaling dimension of the operator $O(x)$ and $\Sigma_{\mu\nu}$ is the finite dimensional spin matrix of the Lorentz group. Actually, we have not formally defined the scaling dimension of an operator yet. So, consider a field operator O and a scale transformation with λ parameter: the scaling dimension Δ of O is defined according to

$$x \to \lambda x, \qquad O(x) \to O(\lambda x) = \lambda^{-\Delta} O(x) \tag{1.4.13}$$

and Δ turns out to be a good quantum number for the purpose of labeling irreps of the conformal group, together with Lorentz spin j. More precisely, we have the following eigenvalue equations:

$$D\,|\Delta, j\rangle = i\Delta\,|\Delta, j\rangle\,, \qquad M_{\mu\nu}\,|\Delta, j\rangle = \Sigma_{\mu\nu}\,|\Delta, j\rangle\,. \tag{1.4.14}$$

Let us consider now a local operator $O_\Delta(x)$ having scaling dimension Δ. When $x = 0$, this creates a state $|\Delta\rangle = O_\Delta(0)\,|0\rangle$ with scaling dimension Δ. Instead, if we consider the operator at $x \neq 0$ we will have:

$$|\chi\rangle \equiv O_\Delta(x)\,|0\rangle = e^{iPx} O_\Delta(0) e^{-iPx}\,|0\rangle = e^{iPx}\,|\Delta\rangle\,, \tag{1.4.15}$$

where in the last equality we have used vacuum invariance under translations. At this stage it is clear why we have problems interpreting particles as vacuum excitations: if we expand the exponential in (1.4.15) we end up with a superposition of states having different scaling dimensions, i.e. different Δ eigenvalues. To be more clear, notice that from (1.4.8) generators P_μ and K_μ act as ladder operators for dilatations, rising and lowering the scaling dimension respectively. Hence, when the momentum operator in e^{iPx} acts on $|\Delta\rangle$ it give rise to a superposition of states and $|\chi\rangle$ will not have definite scaling dimension. Anyway, an operator

[13] More precisely, they should be operators which rescale in a homogeneous way.

annihilated by the lowering operator K_μ is usually called "primary" while the ones obtained by applying the rising operator P_μ are called "descendant". One should act with K_μ until the lowest value of Δ, thus finding a primary operator with scaling dimension Δ annihilated by K_μ, and from there start to classify operators with (Δ, j).

1.4.3 The stress-energy tensor

Symmetries in quantum field theories constitute an algebra of conserved charges acting on Hilbert states, as we already stated. We usually say that these symmetries are realized by local conserved currents $\partial^\mu j_\mu = 0$ and that one can build conserved charges integrating j_0 over space. The existence of currents rather than charges is not necessary for symmetries: however, the so called "Noether assumption" is quite useful when studying conformal theories. For instance, using Noether assumption, the translational invariance is encoded in a conserved stress-energy tensor, i.e. $\partial^\mu T_{\mu\nu} = 0$. Lorentz invariance further require this stress-energy tensor to be symmetric so that the "Lorentz current" $J^{\mu\nu}_{(L)\rho} = x^{[\mu} T^{\nu]}_\rho$ is conserved. We want to focus on conformal symmetries.

Recall that the variation of an action under infinitesimal transformations $x^\mu \to x^\mu + \epsilon^\mu(x)$ in Noether theorem is given by

$$\delta S = -\int \mathrm{d}^d x j^\mu_a \partial_\mu \epsilon_a. \tag{1.4.16}$$

When we deal with diffeomorphism, (1.4.16) takes the form

$$\delta S = -\frac{1}{2}\int \mathrm{d}^d x T^{\mu\nu}(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) \tag{1.4.17}$$

and using (1.4.6) we arrive to

$$\delta S = -\frac{1}{d}\int \mathrm{d}^d x T^\mu_\mu \partial^\nu \epsilon_\nu. \tag{1.4.18}$$

So, it seems that in order to have conformal symmetry the stress-energy tensor must be traceless $T^\mu_\mu = 0$. Now, to some extent tracelessness corresponds to scale invariance. More precisely, the current associated to scale invariance is shown to be $J_{(D)\mu} = x^\rho T_{\mu\rho} - J_{(V)\mu}$, where $J_{(V)\mu}$ is known as the "virial current". Notice that in order to have $\partial^\mu J_{(D)\mu} = 0$ it must be $T^\mu_\mu = \partial^\mu J_{(V)\mu}$. Then, if the stress-energy tensor can be redefined such that its trace is $T' \equiv T - \partial^\mu J_{(V)\mu}$, the conservation of the dilatation currents, i.e. scale invariance, would correspond to the tracelessness of the improved stress-energy tensor $T'_{\mu\nu}$. Following this rather naive-classical argument, (1.4.18) is telling us that scale invariance implies conformal invariance when the virial current can be reabsorbed into an improved stress-energy tensor satisfying $T' = 0$. However, the problem about the enhancement of scale invariance to conformal invariance is a lot more subtle than this and it is still an open one. Besides, there is another argument we can follow, which require some fundamentals of Renormalization-Group (RG) flow.

The Renormalization-Group flow

The RG-flow is the study of how a QFT evolves from the UV to the IR regimes. A QFT has usually an ultraviolet cutoff Λ, which is the energy scale beyond which new degrees of freedom are necessary in the description: RG-flow let us quantify this "ignorance". One starts with some field content ϕ and some coupling g: we want to relate the coupling of the theory with Λ cutoff to the coupling of the theory with $b\Lambda$ cutoff, $b < 1$. In a path integral approach, redefining $\phi \to \phi + \phi'$, where only ϕ has non-zero Fourier modes in $|k| < b\Lambda$, and integrating out ϕ' gives us an "effective" theory expressed in terms of ϕ. The "integrating out procedure" corresponds to a motion through the space of possible Lagrangians: this is the idea of RG-flow.

A fundamental object in the study of RG-flow is the "beta-function", defined as:

$$\beta(g) \equiv \Lambda \frac{\partial g}{\partial \Lambda}. \tag{1.4.19}$$

A positive sign for $\beta(g)$ means that the coupling increases with energy, while if it is negative the coupling becomes smaller as the energy increases. When $\beta(g) = 0$ we talk about fixed points: the coupling remains fixed with energy and since there is no "typical" scale Λ the resulting theory is at least scale-invariant. More precisely, there exist "true" invariant theories and theories for which $\beta(g^*) = 0$ only for particular values of the coupling $g = g^*$. The latter case is the one of theories flown to fixed points, like the three-dimensional one of this thesis which has an infrared fixed point. There, the spectrum is continuous and there will be no well-defined particles, as we already seen for conformal theories. When the theory flows, operators acquire anomalous dimension $\gamma(g)$ which "freezes" at fixed points:

$$\Delta = \Delta_0 + \gamma(g^*), \tag{1.4.20}$$

where Δ_0 is the classical canonical dimension of the operator. Using perturbation theory it is possible to find a relation between the stress-energy tensor and the beta-function, namely

$$T^\mu_\mu \propto \beta(g). \tag{1.4.21}$$

It is then clear that the theory is at least scale-invariant, and hopefully conformal-invariant, at fixed points because of the vanishing of the β-function.

1.4.4 Superconformal algebra

Now we want to include supersymmetry in a conformal theory: it can be shown that the SUSY extension of the conformal algebra is only possible for $d \leq 6$ spacetime dimensions. The bosonic sector of the superconformal algebra has the form $\mathcal{G}_C \oplus \mathcal{G}_\mathcal{R}$, where $\mathcal{G}_C$ is the conformal algebra and $\mathcal{G}_\mathcal{R}$ is the $\mathcal{R}$-symmetry algebra acting on the superspace Grassmann-variables. In the three-dimensional case we have $o(2,3) \oplus o(\mathcal{N}) \subset so(2,3|\mathcal{N})$[14]. In order to have the complete

[14]There exists an isomorphism $so(2,3) \simeq sp(4,\mathbb{R})$ such that the superconformal algebra is actually $Osp(\mathcal{N}|4)$: we will see the rising of this group when dealing with brane solutions.

superconformal algebra one should include fermionic generators, namely the supercharges Q^a_α together with a new class of generators S^a_α called "superconformal charges": they are necessary to close the superalgebra. The relevant commutation relations are:

$$\begin{aligned}
\{S^a_\alpha, \bar{S}^b_\beta\} &= 2\delta^{ab}\gamma^\mu_{\alpha\beta}K_\mu \\
\{Q^a_\alpha, \bar{S}^b_\beta\} &= -i\delta^{ab}(2\delta_{\alpha\beta}D + (\gamma^{[\mu}\gamma^{\nu]})_{\alpha\beta}M_{\mu\nu}) + 2i\delta_{\alpha\beta}R^{ab} \\
[M_{\mu\nu}, S^a_\alpha] &= \frac{i}{2}(\gamma_{[\mu}\gamma_{\nu]})_{\alpha\beta}S^{a\beta} \\
[K_\mu, S^a_\alpha] &= 0 \\
[P_\mu, S^a_\alpha] &= -\gamma^{\alpha\beta}_\mu Q^a_\beta \\
[K_\mu, Q^a_\alpha] &= -\gamma^{\alpha\beta}_\mu S^a_\beta \\
[D, Q^a_\alpha] &= \frac{i}{2}Q^a_\alpha \\
[D, S^a_\alpha] &= -\frac{i}{2}S^a_\alpha \\
[D, R^{ab}] &= 0 \\
[R_{ab}, R_{cd}] &= i(\delta_{ac}R_{bd} + permutations) \\
[R_{ab}, Q^c_\gamma] &= i(\delta^c_a\delta_{bd} - \delta^c_b\delta_{ad})Q^d_\gamma \\
[R_{ab}, S^c_\gamma] &= i(\delta^c_a\delta_{bd} - \delta^c_b\delta_{ad})S^d_\gamma \\
[P_\mu, R^{ab}] &= [K_\mu, R^{ab}] = [M_{\mu\nu}, R^{ab}] = 0,
\end{aligned} \tag{1.4.22}$$

where R_{ab} are generators of $o(\mathcal{N})$. Notice that superconformal charges Q and S are also ladder operators for dilatations, acting as rising and lowering operators respectively. So, superconformal representations have primary operators annihilated by both the lowering operators K_μ, S.

Scaling dimensions

In four dimensions one can find that $[\theta] = [\bar{\theta}] = -\frac{1}{2}$ is the mass dimension of Grassmannian coordinates[15], while $[\Phi]_4 = 1$ from consistency. In the three-dimensional case, Grassmannian coordinates have the same mass dimension but the canonical dimension of component fields is lowered by one-half: this is because we have $\int d^3x$ instead of $\int d^4x$ for the kinetic actions and hence $[\Phi]_3 = \frac{1}{2}$. Having defined Δ as the scaling dimension of a field operator, it can be shown that for any $\mathcal{N} = 2$ three dimensional theory at the fixed point of RG-flow all operators satisfy

$$\Delta \geq |\mathcal{R}|\,, \tag{1.4.23}$$

where $\mathcal{R}$ is the charge under $U(1)_\mathcal{R}$ symmetry. Inequality (1.4.23) is saturated for chiral primary fields, which means that $\mathcal{R}$-symmetry fixes scaling dimensions at fixed points: this is a useful feature if we want to check that our theories are conformal, or at least scale-invariant. Actually,

[15] Which means that $[d^2\theta] = [d^2\bar{\theta}] = 1$.

theories we are considering in this thesis are $\mathcal{N}=2$ three-dimensional nonlinear sigma models: in [41] it was shown that if these theories are scale-invariant then they are also superconformal. So, even if there is no generalized proof, scale invariance is enhanced to superconformal invariance in some cases and hence it is sufficient to prove the former to obtain the latter. In order to better understand this statement, recall that nonlinear sigma models in this thesis are characterized by some function $K(\bar{\Phi},\Phi)$ called Kähler potential. Our effective action takes the schematic form $\int \mathrm{d}^3x\mathrm{d}^4\theta K$ and in order to prove scale-invariance we should find that the scaling dimension of the effective action is zero. Since $\Delta_{\mathrm{d}^3x}=-3$ and $\Delta_{\mathrm{d}^4\theta}=2$, it must be $\Delta_K=1$ in order for our theory to be scale-invariant. In [41] this condition is exactly the one required for a $\mathcal{N}=2$ $d=3$ nonlinear sigma model to be superconformal.

1.5 Anti de Sitter spacetime

Anti de Sitter spacetimes are maximally symmetric solutions to Einstein equations $R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\Lambda g_{\mu\nu}=T_{\mu\nu}$, where $T_{\mu\nu}=0$, and the cosmological constant Λ is negative. These spaces AdS_n admit the maximal number of Killing vectors $\frac{n(n+1)}{2}$ and are the minkowskian counterpart of euclidean hyperbolic spaces since they have negative curvature. n-dimensional anti de Sitter spacetime comes with a length scale L and is defined as the set of all points $(X^0,...,X^n)$ in a $(n+1)$-dimensional Minkowski spacetime $\mathbb{R}^{n-1,2}$ satisfying

$$-X_0^2+\sum_{i=1}^{n-1}X_i^2-X_n^2=-L^2. \tag{1.5.1}$$

Notice that the action of $SO(n-1,2)$ preserves (1.5.1) and that this group acts transitively on AdS_n, i.e. it is its isometry group. Besides, a point on AdS_n is left invariant by the action of $SO(n-1,1)$, i.e. it is the isotropy group. So, we can identify anti de Sitter spacetimes as coset manifolds

$$AdS_n=\frac{SO(n-1,2)}{SO(n-1,1)}, \tag{1.5.2}$$

making evident that $SO(n-1,2)$ is the isometry group. We can rewrite (1.5.1) more compactly as $\eta^{\mu\nu}X_\mu X_\nu-W^2=-L^2$, where we have defined $W=X_n$. By differentiation we obtain the following metric:

$$ds^2=\eta^{\mu\nu}dX_\mu dX_\nu-dW^2=\left(\eta^{\mu\nu}-\frac{\eta^{\mu\lambda}\eta^{\nu\rho}X_\lambda X_\rho}{\eta^{\alpha\beta}X_\alpha X_\beta+L^2}\right)dX_\mu dX_\nu. \tag{1.5.3}$$

At this stage we can calculate "curvatures", which take the form:

$$\begin{aligned}R_{\mu\nu\rho\sigma}&=-\frac{1}{L^2}(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho})\\ R_{\mu\nu}&=-\frac{n-1}{L^2}g_{\mu\nu}\\ R&=-\frac{n(n-1)}{L^2}\end{aligned} \tag{1.5.4}$$

Let us focus on AdS_4 for a moment. The line element reads

$$ds^2 = -(dT^2 + dW^2) + (dX^2 + dY^2 + dZ^2) \tag{1.5.5}$$

and there seems to be two time-coordinates T and W, while we would like to have only one. Replacing (T, W) with (ρ, t) such that $(T = \rho \sin t, W = \rho \cos t)$ and using canonical spherical parametrization for the remaining three spacial coordinates gives

$$ds^2 = -(d\rho^2 + \rho^2 dt^2) + (d\hat{r}^2 + \hat{r}^2 d\Omega^2) \tag{1.5.6}$$

Setting $L = 1$, (1.5.1) corresponds to the constraint $\rho^2 - \hat{r}^2 = 1$ and by differentiation and following insertion in (1.5.6) we und up with

$$ds^2 = -(1 + \hat{r}^2)dt^2 + \frac{d\hat{r}^2}{1 + \hat{r}^2} + \hat{r}^2 d\Omega^2_{n-2}, \tag{1.5.7}$$

which correctly have only one time-coordinate.

It is now possible to express the metric in conformal coordinates. Set $\hat{r} = \tan\psi$ so that (1.5.7) becomes

$$ds^2 = \frac{1}{\cos^2\psi}(-dt^2 + d\Omega^2_{n-1}). \tag{1.5.8}$$

Notice that AdS_n is conformally equivalent to $\mathbb{R}^{n-1,1}$. However, the ψ coordinate ranges from 0 to $\pi/2$ and not π, which means that space covers only one hemisphere: it is then improper to define $d\Omega^2_{n-1} = d\psi^2 + \sin^2\psi d\Omega^2_{n-2}$. Thus, we say that spatial sections of AdS_n are bounded by $\mathbb{S}^{n-2}$, which may be considered as euclidean spaces with a point at infinity. Together with the time coordinate t the Minkowski $\mathbb{R}^{1,n-2}$ is restored and appears as a boundary.

Consider $n = d + 1$, where d is the dimension of a spacetime with conformal symmetry. Then (1.5.2) reads

$$AdS_{d+1} = \frac{SO(d,2)}{SO(d,1)}, \tag{1.5.9}$$

from which it is clear that the isometry group of AdS_{d+1} coincide with the conformal group of its boundary $\mathbb{R}^{1,d-1}$: this is only one of the interesting aspects regarding AdS/CFT duality.

It is worth mentioning some coordinate systems which become very useful when dealing with AdS/CFT duality: the Poincaré charts. Taking the following definitions

$$\begin{aligned} &T = t/w, \qquad X = x/w, \qquad Y = y/w, \\ &W^+ \equiv W + Z = \frac{1}{w}(x^2 + y^2 - t^2) + w, \\ &W^- \equiv W - Z = \frac{1}{w}, \end{aligned} \tag{1.5.10}$$

for AdS_4, which is actually the case of interest in this thesis, the metric (1.5.5) becomes:

$$ds^2 = \frac{L^2}{w^2}(-dt^2 + dx^2 + dy^2 + dw^2) = \frac{L^2}{w^2}(dw^2 + dx^\mu dx_\mu), \tag{1.5.11}$$

which is the analogue of the metric for Poincaré half-plane and the boundary is at $w = 0$. With a further change of coordinates $w = e^{-\tilde{r}}$, the boundary appears at infinity and the metric reads

$$ds^2 = L^2(d\tilde{r}^2 + e^{2\tilde{r}}dx^\mu dx_\mu), \tag{1.5.12}$$

which is a nonsingular form of the anti-de Sitter metric in Poincaré coordinates. Notice that the metric (1.5.11) has a very important feature: it is invariant under dilatation

$$(x_\mu, w) \to (\lambda x_\mu, \lambda w). \tag{1.5.13}$$

This is crucial in AdS/CFT correspondence because radial coordinates in the gravity side are typically associated to some energy scale in the dual field theory. For instance, if we introduce $u = \frac{1}{w}$ then (1.5.11) becomes

$$ds^2 = L^2\left(\frac{du^2}{u^2} + u^2 dx^\mu dx_\mu\right) \tag{1.5.14}$$

and u can be identified as an energy scale. The boundary region of AdS is $w \ll 1$ and corresponds to $u \gg 1$, which is the UV regime of the dual CFT. On contrary, the horizon region $w \gg 1$ is equivalent to $u \ll 1$, so it correspond to low energies, i.e. the IR regime of the CFT. Taking again $L = 1$ for clarity, a form of the metric we will come across later on in this thesis is

$$ds^2 = \frac{dr^2}{r^2} + r^4 ds^2(\mathbb{R}^{1,2}), \tag{1.5.15}$$

obtained from $w = \frac{1}{u} = \frac{1}{r^2}$. So, if u has to be identified with some energy scale, the "scaling dimension" of the new radial coordinate r should be $\frac{1}{2}$: this number is very important in the conformal check at the end of this work because it allows us to assign the correct scaling dimensions Δ of the fields in the HEFT description. Moreover, the coordinate r is actually the radial coordinate of the conical structure wich we are going to introduce in the next chapter.

Chapter 2

Complex geometry handbook

This chapter is a mathematical parenthesis on the geometric objects we will come across throughout this thesis. Even if it seems a "technical vocabulary", especially in the first part, we will try to be "not-so-rigorous": the purpose is to present the most relevant facts concerning Calabi-Yau (CY) manifolds, one of the main characters in this work. However, we will see complex geometry in action, together with its physical importance, only in the next chapters, when we will introduce branes, M-Theory and finally holography. We will closely follow [19] here, with something from [20], but lots of information and applications can be found in papers cited in the next chapter.

2.1 Basics of differential geometry

The first thing we want to point out is that the concept of manifold do not coincide with the concept of metric. A manifold endowed with a metric is called Riemannian manifold, but we can in principle have different metrics for the same manifold: for instance, we anticipate that one of the most important calculations in this work is to find a particular metric on a given manifold. With this statement in mind, we can start to collect the basic geometrical object we will encounter.

Definition: A complex manifold is a topological space together with a holomorphic atlas.
Example: complex projective space $\mathbb{CP}_n$.
The n-dimensional projective space is the space of complex lines through the origin in $\mathbb{C}^{n+1}/\{0\}$, that is the set $(z_1, ..., z_{n+1})$ where $z_i \neq 0$, together with the identification $(z_1, ..., z_{n+1}) \approx \lambda(z_1, ..., z_{n+1})$ for any non-zero complex λ. We can take sets $U_j = \{z^j \neq 0\}$ as coordinate neighborhoods and choose coordinates $\zeta_j^l = \frac{z^l}{z^j}$ within each U_j. On the overlap $U_j \cap U_k$ we have

$$\zeta_j^l = \frac{z^l}{z^j} = \frac{\frac{z^l}{z^k}}{\frac{z^j}{z^k}} = \frac{\zeta_k^l}{\zeta_k^j}. \tag{2.1.1}$$

In this thesis we are mainly interested in $\mathbb{CP}_1$ spaces because they will emerge when dealing with the Q^{111} model. The unidimensional projective space is covered by two coordinate patches U_1 and U_2 with coordinate $\zeta_1 \equiv \zeta_1^2 = \frac{z_2}{z_1}$ and $\zeta_2 \equiv \zeta_2^1 = \frac{z_1}{z_2}$, respectively. In the overlap region $U_1 \cap U_2$ we have $\zeta_1 = \frac{1}{\zeta_2}$ and we see that the unidimensional complex projective space is actually the Riemann sphere $\mathbb{S}^2$.

Now we introduce the language of differential forms because it will be widely used in this work.

Definition: A p-form is a totally antisymmetric covariant tensor of rank p defined as

$$\alpha_p = \frac{1}{p!}\alpha_{m_1,\dots,m_p} dx^{m_1} \wedge \cdots \wedge dx^{m_p}, \tag{2.1.2}$$

where the symbol "$\wedge$" is called "wedge-product". This is the natural product between a p-form and a q-form and it gives a $(p+q)$-form.

Definition: The exterior derivative d is a map from the space of p-forms to the space of $(p+1)$-forms defined as

$$d\alpha_p = \frac{1}{p!}\partial_m \alpha_{m_1 \dots m_p} dx^m \wedge dx^{m_1} \wedge \cdots \wedge dx^{m_p}. \tag{2.1.3}$$

Definition: The Hodge-star $\star$ is a map from p-forms to $(n-p)$-forms defined as

$$\star\, \alpha_p = \frac{\sqrt{|detg|}}{p!(n-p)!}\epsilon^{m_1 \dots m_p}_{l_1 \dots l_{n-p}} \alpha_{m_1 \dots m_p} dx^{l_1} \wedge \cdots \wedge dx^{l_{n-p}}, \tag{2.1.4}$$

where g is some metric. This map let us define an inner product on the space of real forms as

$$(\alpha_p, \beta_q) = \int \alpha_p \wedge \star\beta_q. \tag{2.1.5}$$

Given an inner product we can define the adjoint of the exterior derivative $d^\dagger$ such that $(\alpha_p, d\beta_{p-1}) = (d^\dagger \alpha_p, \beta_{p-1})$. One can show that

$$\begin{aligned} d^\dagger &= \star d \star \qquad \text{if n even} \\ d^\dagger &= (-1)^p \star d \star \qquad \text{if n odd} \end{aligned} \tag{2.1.6}$$

and moreover $dd = 0 = d^\dagger d^\dagger$. The action of $d^\dagger$ on a p-form is given by

$$d^\dagger \alpha_p = -\frac{1}{(p-1)!}\nabla^k \alpha_{k m_2 \dots m_p} dx^m \wedge dx^{m_2} \wedge \cdots \wedge dx^{m_p}. \tag{2.1.7}$$

Definition: The Hodge-deRham operator is the second order differential operator defined as $\Delta \equiv dd^\dagger + d^\dagger d$ and it is the covariant generalization of the Laplacian.

Definition: A p-form ω is called harmonic if $\Delta\omega = 0$. Using the inner product one can show that a form on a compact manifold is harmonic if and only if it is both closed and co-closed, i.e. it satisfies $d\omega = 0$ and $d^\dagger\omega = 0$ respectively[1]. The existence of harmonic forms is related to global properties of the manifold on which they are defined. Indeed, Hodge has shown that a differential form on a compact manifold can always be written in terms of harmonic, closed and co-closed components in a unique way $\omega = \alpha + d\beta + d^\dagger\gamma$. Similarly, a closed form can always be written as $\omega = \alpha + d\beta$ with α harmonic and $d\beta$ exact. Surely an exact form is automatically closed since $d^2 = 0$, but the converse is not generally true: a closed form is exact only if the harmonic part is zero. Actually, given a closed form ω it is always possible to find a form β such that $\omega = d\beta$ within any coordinate patch of the manifold. However, there is no guarantee that β transform properly on the overlap region between two different patches and hence it cannot be globally defined in general. So, a closed form is exact only locally.

The study of harmonic forms is matter of Homology and Cohomology: we will only introduce the basic concepts for them relatively to a generic n-dimensional manifold M.
Definition: A p-chain a_p is a sum $a_p = \sum_i c_i N_i$, where N_i are p-dimensional oriented submanifolds of M. An integral over the chain can be expressed as $\int_{\sum_i c_i N_i} = \sum_i c_i \int_{N_i}$.
Definition: The boundary operator ∂ associates a manifold M with its boundary ∂M. The boundary operator acting on p-chains gives $(p-1)$-chains $\partial a_p = \sum_i c_i \partial N_i$.
Definition: A p-cycle C_p is a p-chain with no boundary, i.e. it satisfies $\partial C_p = 0$.
Definition: Let Z_p be the set of p-cycles and let B_p be the set of p-chains which are boundaries of $(p+1)$-chains, namely $a_p = \partial a_{p+1}$. The (simplicial) homology of M is the quotient set $H_p = Z_p/B_p$. In other words, H_p is the set of p-cycles with two cycles considered equivalent if they differ by a boundary, i.e. $a_p \sim a_p + \partial a_{p+1}$.
Example: the torus $\mathbb{T}^2$.
The bidimensional torus is shown to admit two non-trivial harmonic one-forms. So to speak, this is because there are "two basic curves which are not boundaries". Since it is a two-dimensional manifold we can only consider H_p with $p = 0, 1, 2$. Zero-chains are points, which have no boundary: they are then zero-cycles too. Notice that any two points form the boundary of a curve. Hence, H_0 consists of multiples of some representative point, i.e. $H_0 \simeq \mathbb{R}$. H_1 consists instead of the two independent cycles so that $H_1 \simeq \mathbb{R} \oplus \mathbb{R}$, while $H_2 \simeq \mathbb{R}$ because the only two-chain without boundary is $\mathbb{T}^2$ itself.

Definition: Let Z^p be the set of closed p-forms and let B^p be the set of exact p-forms. The "de Rham cohomology" is the quotient $H^p = Z^p/B^p$, i.e. H^p is the set of closed p-forms where

[1] It is important to stress that in the case of non-compact manifolds, such as cones we are going to deal with, the "if and only if" is not appropriate. Instead, a form being both closed and co-closed is surely harmonic by definition. Indeed, since $\Delta = dd^\dagger + d^\dagger d$, if $d\omega = 0 = d^\dagger\omega$ then $\Delta\omega = 0$.

two elements are considered equivalent if they differ by an exact form $\omega_p \sim \omega_p + d\beta_{p-1}$.
Theorem: de Rham showed that the two vector spaces H_p and H^p are dual to each other and hence isomorphic.

Definition: Betti numbers $b_p = dim H^p$ are topological quantities that give the amount of linearly independent harmonic p-forms.

Theorem: Given a p-cycle C_p there exists an $(n-p)$-form α_{n-p}, called the "Poincaré dual" of C_p, such that

$$\int_{C_p} \omega_p = \int_M \alpha_{n-p} \wedge \omega_p \tag{2.1.8}$$

for any closed p-form ω_p.

2.2 Riemannian manifolds

Given a manifold M and a metric tensor g on it, the couple (M, g) is called "Riemannian manifold" (but we will refer to it using M only). On M there exists a unique linear connection ∇ which is also torsion free, i.e. $[X, Y] = \nabla_X Y - \nabla_Y X$ for any vector fields X, Y on M. Moreover, it preserves the metric $\nabla g = 0$. This is called "Levi-Civita connection" and relatively to a local chart x^a it is defined with "Christoffel symbols" Γ^c_{ab} by $\nabla_{\partial_a}\partial_b = \Gamma^c_{ab}\partial_c$. Christoffel symbols can also be expressed in terms of the metric in the unique way:

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd}(\partial_a g_{db} + \partial_b g_{ad} - \partial_d g_{ab}). \tag{2.2.1}$$

Using ∇ we can introduce the notion of parallel transport. Given a curve $t \to \gamma(t)$ on M with velocity $\dot{\gamma}$, we say that a vector field X is parallel along γ if $\nabla_{\dot{\gamma}} X = 0$. Relatively to a local chart x^a we can write this equation as:

$$\dot{\gamma}^a \nabla_a X^b = \dot{\gamma}^a(\partial_a X^b + \Gamma^b_{ac} X^c) = \dot{X}^b + \Gamma^b_{ac}\dot{\gamma}^a X^c = 0. \tag{2.2.2}$$

A curve is called "geodesic" if its velocity is self-parallel, i.e. $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$. In the same fashion of (2.2.2) we get the geodesic equation:

$$\ddot{\gamma}^c + \Gamma^c_{ab}\dot{\gamma}^a\dot{\gamma}^b = 0 \tag{2.2.3}$$

Integration of (2.2.2) bring to the concept of parallel transport. More precisely, consider the curve $\gamma : [0, 1] \to M$ and the linear map $P_\gamma : T_{\gamma(0)}M \to T_{\gamma(1)}M$ taking vectors tangent to M at the point $\gamma(0) \in M$ to vectors tangent to M at $\gamma(1)$. If $X \in T_pM$ is a tangent vector to M at $p = \gamma(0)$ we define the "parallel transport" $P_\gamma(X)$ relative to γ by first extending X to a vector field along γ in such a way that solves (2.2.2) and then evaluating the vector field at $\gamma(1)$. Now we fix a point $p \in M$ and let γ be a differentiable curve which starts and ends at p. Then P_γ is a linear map from the tangent space T_pM to itself. Notice that we can both decompose and invert those maps: therefore P_γ forms a group. Restricting to contractible loops, the group of linear transformations

$$Hol(p) = \{P_\gamma \mid \gamma \text{ contractible loop based at } p\}$$

is called "(restricted) holonomy group at p" of the connection ∇. The holonomy group is a very important concept[2] and in the case of a riemannian manifold M it is shown to be isomorphic to $SO(n)$, where $n = dim\, M$. Besides, it is interesting that the Lie algebra of $Hol(p)$ is generated by the Riemann curvature tensor and hence the holonomy group somehow "measures" how much a space is curved. Indeed, we can fix two vectors X, Y on M and define a linear map as follows:

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}. \tag{2.2.4}$$

Relatively to a coordinate basis, the linear map (2.2.4) may be written as a tensor R^d_{abc} defined by

$$R(\partial_a, \partial_b)\partial_c = \partial_d \tag{2.2.5}$$

and hence having components

$$R^d_{abc} = \partial_a \Gamma^d_{bc} - \partial_b \Gamma^d_{ac} + \Gamma^e_{bc}\Gamma^d_{ae} - \Gamma^e_{ac}\Gamma^d_{be}. \tag{2.2.6}$$

Then, the Lie algebra is spanned by "curvature operators" $R_{ab} : \partial_c \to R^d_{abc}\partial_d$ and for a riemannian manifold this is actually $so(n)$ since R_{ab} is antisymmetric.

2.3 Kähler geometry

Definition: Let M be an n-dimensional complex manifold and let z^μ be local coordinates. We define the tensor I^m_n by

$$I = i\, dz^\mu \frac{\partial}{\partial z^\mu} - i\, dz^{\bar\mu} \frac{\partial}{\partial z^{\bar\mu}}. \tag{2.3.1}$$

I is called "complex structure" and it is a linear map from the tangent space to itself obeying $I^2 = -1$, i.e. in component $I^n_m I^p_n = -\delta^p_m$. This gives to each tangent space the structure of a complex vector space and hence the n must be even. Complexifying the tangent space we can diagonalize I immediately finding its eigenvalues $\pm i$. At this point one can identify two kinds of complex vector fields Z: type $(1,0)$ (or holomorphic) satisfy $IZ = iZ$, whereas type $(0,1)$ (or antiholomorphic) satisfy $IZ = -iZ$. Moreover, we can define two operators $P = \frac{1}{2}(\mathbb{1} - iI)$ and $Q = \frac{1}{2}(\mathbb{1} + iI)$ projecting out respectively the holomorphic and antiholomorphic components

[2]For instance, we will see that it is physically related to supersymmetry.

of a tensor. A complex k-form, with $k = p + q$, can be then decomposed in p-holomorphic and q-antiholomorphic parts in the following way:

$$\omega = \sum_{p+q=k} \omega^{(p,q)}. \tag{2.3.2}$$

Moreover, when the exterior derivative d acts on a (p, q)-form it gives a linear combination of forms having different type. This is because $d = \partial + \bar{\partial}$, where "Dolbeault operators" are defined as $\partial \equiv Pd$ and $\bar{\partial} \equiv Qd$. We can think of these operators as type $(1, 0)$ and type $(0, 1)$ parts of the exterior derivative d. Indeed:

$$\begin{aligned} \partial\omega^{(p,q)} &= (d\omega)^{(p+1,q)} \\ \bar{\partial}\omega^{(p,q)} &= (d\omega)^{(p,q+1)}. \end{aligned} \tag{2.3.3}$$

Each Dolbeaut operator defines its cohomology group, whose complex dimension is called "Hodge number".

Definition: A complex manifold is called hermitian if it is endowed with a metric of the form $ds^2 = g_{\mu\bar{\nu}} dz^\mu dz^{\bar{\nu}}$. A hermitian metric satisfies $g_{mn} = I^k_m I^l_n g_{kl}$: we say that the complex structure is compatible with the metric. Using the properties of I and hermiticity we can find that $g_{mk} I^k_n = -g_{nk} I^k_m$, which means that hermitian manifolds have always a natural two-form[3] $J_{mn} = g_{mk} I^k_n = -J_{nm}$.

We have just seen that complex manifolds admit globally defined tensors I which square to minus the identity. What if a real manifold admits such a tensor?

Definition: If a real manifold M admits a globally defined tensor I, which in this case is called "almost complex structure", such that $I^n_m I^p_n = -\delta^p_m$, then M is called almost complex. If in addition the metric is hermitian then M is called almost hermitian.

Definition: The Nijenhuis tensor N_I of the almost complex structure I is defined as

$$N_I(X, Y) = I[IX, IY] + [X, IY] + [IX, Y] - I[X, Y]. \tag{2.3.4}$$

Theorem: An almost complex structure becomes a complex structure if and only if the associated Nijenhuis tensor vanishes. In that case, there exist a holomorphic atlas such that

$$I^\mu_\nu = i\delta^\mu_\nu, \quad I^{\bar{\mu}}_{\bar{\nu}} = -i\delta^{\bar{\mu}}_{\bar{\nu}}, \quad I^{\bar{\mu}}_\nu = 0 = I^\mu_{\bar{\nu}}. \tag{2.3.5}$$

Recall that the Christoffel connection is uniquely determined by two requirements: covariantly constant metric and symmetric connection. When we have a complex manifold it is quite natural to require the constant covariance of the complex structure, namely $\nabla I = 0$. A unique

[3]In order to avoid confusion we must distinguish the $(1, 1)$-tensor I^m_n and the $(2, 0)$-form J_{mn}. In a free-index notation, the former is defined from $I = I^m_n \partial_m \otimes dx^n$ while the latter corresponds to $J = \frac{1}{2} J_{mn} dx^m \wedge dx^n$.

connection is then singled out requiring the torsion tensor $\Gamma^r_{[mn]}$ to be pure in its lower indexes. It follows that all the mixed components of the connection vanish and hence that a hermitian connection is pure in its indexes. Using these facts, one can obtain:

$$\begin{aligned}\text{Pure connection}: &\quad \Gamma^\lambda_{\mu\nu} = g^{\lambda\bar{\rho}}\partial_\mu g_{\nu\bar{\rho}}\\ \text{Non-vanishing curvature}: &\quad R^{\bar{\sigma}}_{\mu\bar{\nu}\bar{\rho}} = -R^{\bar{\sigma}}_{\bar{\nu}\mu\bar{\rho}} = \partial_\mu\Gamma^{\bar{\sigma}}_{\bar{\nu}\bar{\rho}}\\ \text{Ricci form}: &\quad R = iR^{\bar{\rho}}_{\mu\bar{\nu}\bar{\rho}}dz^\mu\wedge dz^{\bar{\nu}} = i\partial\bar{\partial}\log\sqrt{detg}.\end{aligned} \tag{2.3.6}$$

Notice that the Ricci two-form is always closed, i.e. $dR = 0$, but it is not globally exact altough (2.3.6) holds globally. The Ricci form defines a particular cohomology class

$$c_1 = \left[\frac{1}{2\pi}R\right] \tag{2.3.7}$$

called "first Chern class". Actually, c_1 is a topological invariant and it does not change under smooth variation of the metric, which in contrast affect the Ricci form R.

Definition: A hermitian manifold is said to be Kähler if the natural two-form J is closed, i.e. $dJ = 0$. On a Kähler manifold J is called "Kähler form". From $dJ = 0$ it follows

$$\partial_\lambda g_{\mu\bar{\nu}} = \partial_\mu g_{\lambda\bar{\nu}}, \qquad \partial_{\bar{\rho}} g_{\mu\bar{\nu}} = \partial_{\bar{\nu}} g_{\mu\bar{\rho}}, \tag{2.3.8}$$

which translates into the fact that $g_{\mu\bar{\nu}} = \partial_\mu\partial_{\bar{\nu}}\varphi_j$ for some real scalar φ_j that can be defined on each patch U_j. These scalars are also known as "Kähler potentials" and we can write

$$J = i\partial\bar{\partial}\varphi_j \tag{2.3.9}$$

for each patch U_j, whereas in some intersection $U_j \cap U_k$ we have $\varphi_j = \varphi_k + f_{jk}(z)$ with a holomorphic transition function. We should stress that J is not exact. Indeed, for a n-dimensional manifold M the n-fold product $J\wedge\cdots\wedge J$ is proportional to the volume form d$vol(M)$: integration over the manifold M then gives its volume. If J is exact, for example $J = d\beta$, then this volume is always zero, which is clearly not true. Instead, since J is covariantly constant it is also co-closed: so, having both $dJ = 0 = d^\dagger J$, the Kähler form is harmonic.

Example: $\mathbb{CP}_1$ is a Kähler manifold.

This example will become useful later in this thesis. Recalling (2.1.1), set

$$\varphi_j = \log\left(\sum_{l=1}^{2}|\zeta^l_j|^2\right) = \log(|\zeta^1_j|^2 + |\zeta^2_j|^2) \tag{2.3.10}$$

as the Kähler potentials so that $\varphi_1 = \log(1 + |\zeta_1|^2)$ and $\varphi_2 = \log(1 + |\zeta_2|^2)$. On the overlap $U_1 \cap U_2$, since $\zeta_1 = \frac{1}{\zeta_2}$ we have $\varphi_1 = \varphi_2 - \log(|\zeta_2|^2)$ and hence $\partial\bar{\partial}\varphi_1 = \partial\bar{\partial}\varphi_2$. The metric generated by this potential is the "Fubini-Study" one

$$g_{\mu\bar{\nu}} = \partial_\mu\partial_{\bar{\nu}}\varphi_1 = \frac{1}{(1+|\zeta_1|^2)^2}\delta_{\mu\bar{\nu}}. \tag{2.3.11}$$

An alternative definition of a Kähler manifold is based on holonomy: a manifold (M, g, I) is said to be Kähler if its holonomy group lies in $U(n)$.

2.3.1 Ricci-flatness: the Calabi-Yau geometry

We can restrict the holonomy of a Kähler manifold by imposing constraints on the curvature, for example asking it to be Ricci-flat, i.e. the Ricci tensor vanishes. This request has an important physical meaning: transversal cones X in the background geometry $\mathbb{R}^{1,2} \times X_8$ that we will see in the next chapter must satisfy supergravity equation of motion in vacuum in order to be good stable backgrounds, i.e. X must be Ricci-flat.
Definition: A Calabi-Yau (CY) manifold is a Kähler manifold with vanishing Ricci form.
It can be shown that this is equivalent to demanding that the holonomy group is $SU(n)$ rather than $U(n) \simeq SU(n) \times U(1)$, the $U(1)$ factor being generated by the the Ricci tensor. So, if the manifold is Ricci-flat there is no $U(1)$ and the holonomy group is restricted. As we have previously seen, the Ricci form defines the first Chern class (2.3.7). Consider a generic metric g and a Ricci-flat metric g' on the Kähler manifold M. The associated Ricci forms are related by $R(g) = R(g') + \text{exact-form}$ and since $R(g') = 0$ we get $c_1 = 0$. This fact brought to the following crucial theorem, conjectured by Calabi and later proved by Yau.
Theorem: Given a complex manifold with vanishing first Chern class and any Kähler metric g with Kähler form J, there exists a unique Ricci-flat Kähler metric g' whose Kähler form J' is in the same cohomology class as J. The utility of this theorem is that one can construct CY manifolds by simply constructing $c_1 = 0$ manifolds.
Example: $\mathbb{CP}_n$ is not Ricci-flat.
The Ricci form for the projective space $\mathbb{CP}_n$ with the Fubini-Study metric (2.3.11) takes the form

$$R = -(n+1)J. \tag{2.3.12}$$

Since we know that the Kähler form is not exact, then it is clear from (2.3.7) that the first Chern class is nontrivial: this means that projective spaces cannot admit a Ricci-flat metric.

A fundamental property of CY manifolds is that they admit covariantly constant spinors $\nabla^{(X)}\eta = 0$, which have an important physical meaning: they are related to SUSY. Actually, we will see that not every brane-solution is supersymmetric, but demanding the preservation of some SUSY implies that the transversal cone X must admit some covariantly constant spinor, i.e. it must be a CY. So, the CY-condition on transverse space will ensure both a stable background geometry and the preservation of some supersymmetry.

Hodge numbers of a CY manifold

We stated that Betti numbers are topological numbers b_p giving the dimension of the p-th de Rham cohomology $H^p(M)$ of a manifold M. After the definition of a metric on M, b_p counts the number of linearly-independent harmonic p-forms and for a Kähler metric there exists a decomposition in terms of "Hodge numbers" $h^{p,q}$ such that $b_k = \sum_{p=0}^{k} h^{p,k-p}$, $h^{p,q}$ counting the number of harmonic (p,q)-forms on M. A CY n-fold has symmetries and dualities relating

Hodge numbers. For instance, one can prove that $h^{p,0} = h^{n-p,0}$ and $h^{p,q} = h^{q,p}$. Moreover, Poincaré duality gives $h^{p,q} = h^{n-p,n-q}$. We are particularly interested in the $n = 4$ case: these CY four-folds are characterized by three independent Hodge numbers $(h^{1,1}, h^{1,3}, h^{1,2})$. Then, one can compute Betti numbers, which in turn play an important role since they are associated to symmetries in the field theories we are going to deal with.

2.4 Calabi-Yau cones

As anticipated, in this work we will study branes on a background geometry $\mathbb{R}^{1,2} \times X_8$, where the branes are parallel to the $\mathbb{R}^{1,2}$ factor and can be considered as pointlike with respect to the transverse CY cone X_8: in this section we want to highlight some geometrical features about this cone. First of all, it has to be intended as a manifold $X_8 = \mathbb{R}_+ \times Y_7$ with metric $ds_8^2 = dr^2 + r^2 ds_7^2$, where Y_7 is the (compact) base of the cone. The point $r = 0$ is singular unless $Y_7 = \mathbb{S}^7$: we then talk about conical singularities. Let us be more precise.

Consider the riemannian manifold (Y, g_Y) and let $X = \mathbb{R}_+ \times Y$. We parametrise $\mathbb{R}_+$ by $r > 0$ and define the metric g_X on X such that

$$ds_X^2 = g_{mn}^X dx^m dx^n = dr^2 + r^2 ds_Y^2 = dr^2 + r^2 g_{ij}^Y dx^i dx^j. \tag{2.4.1}$$

The riemannian manifold (X, g_X) constructed in this way is called "metric cone" of (Y, g_Y). We will sometimes call $C(Y)$ the singular cone over Y.

One of the most important features of "conelike" metrics is the existence of a Killing vector generating a rescaling of the radial coordinate. This is usually called "Euler vector" and takes the form $\xi = r\partial_r$: it turns out to be essential for building some geometric structure on the base Y. We have mentioned earlier that CY manifolds are related to covariantly constant spinors, also called "parallel spinors", and that these are related to SUSY generators: we want to deepen the relation between the CY cone, its base and such spinors. Recall that a covariantly constant spinor η satisfies $\nabla^{(X)}\eta = 0$, i.e. it is invariant under parallel transport and hence its value at any point $p \in X$ is invariant under the holonomy group $Hol(p)$. A manifold admitting such spinor fields is necessarily Ricci-flat, otherwise there must be some rotation of the spinor after parallel transporting it around a closed loop. We are obviously interested in $Hol(p) = SU(n = 4)$, which is the CY four-fold case[4]. Now, it is possible to find a correspondence between parallel spinors on (X, g_X) and some Killing spinors on (Y, g_Y), namely

$$\nabla^{(X)}\eta = 0 \quad \longleftrightarrow \quad \nabla^{(Y)}\eta = \pm\frac{1}{2}\Gamma\eta, \tag{2.4.2}$$

where Γ are the (Dirac) gamma-matrixes on the base of the cone. This fact will let us see SUSY generators as related to Killing spinors on Y rather than parallel ones on X, so that we

[4]We should mention that there exists other three holonomy groups related to covariantly constant spinors, which in the four-fold case are: G_2, $Sp(2)$ and $Spin(7)$.

can use these two terms “interchangeably”. Letting $\mathcal{N}$ denote the dimension of the space of Killing spinors, one can find that the case of CY four-fold have $\mathcal{N} = 2$.[5] Besides, in terms of the seven-manifold at the base, the reduced holonomy of the CY cone implies the existence of some tensors that contracted with the Euler vector give rise to geometrical objects characterizing the base Y. For instance, one can build a “Sasakian structure” on Y and then, as usually found in literature, define a Sasaki space as the base of a Kähler cone. Since we are interested in Ricci-flat cones because they provide stable supergravity backgrounds, one can also find that the relative base is an Einstein space, i.e. it has a Ricci tensor proportional to the metric[6]. Summarizing, $C(Y)$ is Kähler if and only if Y is Sasaki, but since the cone is also Ricci-flat it follows that its base is also Einstein. So, a Calabi-Yau cone has a Sasaki-Einstein base and both of them are related to $\mathcal{N} = 2$.

Resolutions and moduli: a preview

Cones are singular manifolds with the singularity at the tip. String and M-Theory can be studied on such singular manifolds giving rise to new features with respect to the flat spacetime case. One of them is about “resolutions”: we can replace the singularity of the cone with a smooth manifold and this leads to the so called “resolved cone”. We will sometimes call $C(Y)$ the singular cone over the base Y, while X will be identified with the resolved cone. The resolution is more rigorously defined as a map $\pi : X \to C(Y)$ such that the singular point $\{r = 0\}$ of $C(Y)$ is effectively replaced by an higher-dimensional locus in X, called “exceptional set”. The metric on the resolved X is no more invariant under rescalings, but it should be a CY one approaching the CY metric of $C(Y)$ asymptotically[7]: there is a theorem that ensure this and it is to some extent a non-compact version of the aforementioned Calabi-Yau theorem. Indeed, if X is compact then the CY theorem implies that it admits a unique Ricci-flat Kähler metric: the non-compact version is implemented with suitable boundary conditions, namely that the metric should be asymptotic to the one on $C(Y)$.

Theorem: Given a singular cone $C(Y)$ with vanishing first Chern class and any Kähler metric g with Kähler form J, if $\pi : X \to C(Y)$ is a resolution of the singular cone then X admits a unique Ricci-flat Kähler metric g' which is asymptotic to g and whose Kähler form J' is in the same cohomology class of J.

Sometimes, as in the case of the Q^{111} model treated in this work, resolution manifolds, i.e. exceptional sets, are product of $\mathbb{CP}_1 \simeq \mathbb{S}^2$, whose volumes are regulated by some parameters.

[5]For completeness, we mention that $Spin(7)$ holonomy gives $\mathcal{N} = 1$, CY four-folds have $\mathcal{N} = 2$ and (“hyperkähler”) $Sp(2)$ holonomy gives $\mathcal{N} = 3$, whereas for $\mathcal{N} > 3$ the manifold is necessarily a quotient of $\mathbb{C}^4$.

[6]We can indeed consider a change of coordinates in (2.4.1) using $\phi = \ln r$ so that $ds_X^2 = e^{2\phi}(d\phi^2 + ds_Y^2)$, which is clearly conformally equivalent to the metric of a cylinder over Y, namely $d\phi^2 + g_{ij}^Y dx^i dx^j$. If g_{mn}^X in (2.4.1) is Ricci-flat, after the conformal transformation the Ricci tensor on the base turns out to be $R_{ij}^Y = 2g_{ij}^Y$, i.e. the space is Einstein.

[7]We will see the physical reason to require this when dealing with holography.

We anticipate that these parameters, together with the branes positions on X, give rise to a certain amount of "moduli" that will correspond to fields of a particular field theory.

Resolved cone: an example

Consider the three-dimensional complex cone $C(Y_5)$ treated in [39], where the base Y_5 has isometry group $SO(4) \times U(1)$. This is called Klebanov-Witten model and we anticipate now that it will come out later on in this thesis. As a complex manifold, it can be described by a quadric equation in $\mathbb{C}^4$, namely

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0. \tag{2.4.3}$$

Notice that $z_i \to \lambda z_i$ with $\lambda \in \mathbb{C}^*$ leaves (2.4.3) invariant, so that its real and positive part $s \in \mathbb{R}_+^*$ can be interpreted as the typical scaling parameter of a cone. We can find the base Y_5 quotienting by $\mathbb{R}_+^*$, which is equivalent to intersecting the cone with the unit sphere in $\mathbb{C}^4$:

$$|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 = 1. \tag{2.4.4}$$

Since $SO(4) \simeq SU(2) \times SU(2)$ acts transitively on (2.4.4) and any point in the base is invariant under a $U(1)$ action, the base is actually the coset manifold $Y_5 = \frac{SU(2) \times SU(2)}{U(1)}$, also known as T^{11}. Alternatively, we can rewrite (2.4.3) using an obvious change of coordinates as

$$uv - xy = 0. \tag{2.4.5}$$

The conifold equation (2.4.5) has an immediate solution taking

$$u = a_1 b_1, \qquad v = a_2 b_2, \qquad x = a_1 b_2, \qquad y = a_2 b_1 \tag{2.4.6}$$

and notice that the identification is unchanged if we perform a rescaling $a_i \to \lambda a_i, b_i \to \lambda^{-1} b_i$. Moreover, the $SO(4) \simeq SU(2) \times SU(2)$ isometry has a clear interpretation: one $SU(2)$ acts on a_i and the other one acts on b_i. Now, if we write $\lambda = s e^{i\alpha}$, with $s \in \mathbb{R}_+^*$ and α real, the parameter s can be chosen to set

$$|a_1|^2 + |a_2|^2 = |b_1|^2 + |b_1|^2 = 1, \tag{2.4.7}$$

which makes evident that the isometry group is $SU(2) \times SU(2) \simeq \mathbb{S}^3 \times \mathbb{S}^3$. Then, dividing by the remaining $U(1)$ acting as

$$a_i \to e^{i\alpha} a_i, \qquad b_i \to e^{-i\alpha} b_i \tag{2.4.8}$$

we find the same base manifold $Y_5 = \frac{SU(2) \times SU(2)}{U(1)} = \mathbb{S}^2 \times \mathbb{S}^3 = T^{11}$.

At this stage the conifold has a singularity at the tip and there are two different ways to "smoothen" it. Following [38], they consist in substituting the singular tip with either $\mathbb{S}^2$ or $\mathbb{S}^3$: in this work we are more interested in the former option. For the moment, we want to anticipate that (a_1, a_2, b_1, b_2) admit a field theoretical interpretation. Indeed, when studying field theory vacua via minimization of a scalar potential like (1.2.19), it may emerge an equation like

$$|a_1|^2 + |a_2|^2 - |b_1|^2 - |b_1|^2 = \zeta, \tag{2.4.9}$$

where ζ is a Fayet-Iliopoulos parameter. If $\zeta = 0$ this is exactly (2.4.7) so that one ends up with a conifold moduli space. If instead $\zeta \neq 0$ then the moduli space is a resolution of the cone. To some extent, external parameters like FI can be interpreted as resolution parameters "deforming" the conifold equation and hence "resolving" the singular cone. The generalization to the case of $C(Q^{111})$ treated in this thesis is not so straightforward and will be worked out in the last chapter.

Chapter 3

M-Theory and brane solutions

In this chapter we are going to introduce some basic aspects of M-Theory following [21], focusing on its effective field theory: the eleven-dimensional supergravity. Then, we will consider the generalization of point-particles in M-Theory, namely M-branes: we are interested in geometrical solution to Einstein equations preserving a fraction of the original supersymmetry. We will see that M-branes placed on some eleven-dimensional background geometry give rise to a "warped" geometry, whose near-horizon limit includes an AdS factor and an internal manifold. Such solutions are studied for example in [22, 23, 24]. We already mentioned that supersymmetry is related to the number of Killing spinors on internal manifolds but the presence of branes sometimes reduces the amount of SUSY: this is pointed out also in [25, 26]. Remember that our interest is oriented towards M-Theory on Calabi-Yau conical four-folds and hence we will explicitly face this problem only, following [27] and [30] for more general warped solutions. Besides, we shall get a glimpse on the field theories dual to brane-configurations, which are matter of the next chapter, starting with the most famous example: the Maldacena duality [3]. The gauge/gravity correspondence is then explained as in [28], together with possible generalizations.

3.1 Basics of M-Theory

M-Theory was firstly conjectured by Edward Witten as a theory unifying all the five consistent versions of superstring theory: type I, type IIA, type IIB, heterotic $E_8 \times E_8$ and heterotic $SO(32)$. These ten-dimensional theories are related by string-dualities, which means that there should be only one theory having different descriptions. On the other hand, M-Theory can be interpreted as a strong coupling limit of type IIA (or eventually $E_8 \times E_8$, but in this thesis we are more interested in the former scenario), which develops a new dimension and approaches an eleven-dimensional limit. It is important to stress that M-Theory is not a String-Theory: indeed, the extended objects generalizing the notion of point-particles are M-branes rather than strings, which are not present in the eleven-dimensional theory. In what follows we are going to work with the effective theory of M-Theory, namely the eleven-dimensional supergravity. The downside is that effective theories are non-fundamental by definition: however, we can study dualities and brane solutions even in the low-energy limit with important outcomes.

3.1.1 Field content and M-branes

The massless spectrum of eleven-dimensional supergravity is relatively simple.

- First of all, 11-dimensional supergravity contains gravity, so there is a graviton. This is represented by a symmetric traceless tensor of the little group $SO(D-2)$, with $D=11$. It has therefore $\frac{1}{2}(D-1)(D-2)-1=44$ physical degrees of freedom.

- A SUSY theory should contain some fermionic degrees of freedom: indeed, there is a gravitino ψ_M, the supersymmetric partner of the graviton, with spin $\frac{3}{2}$. One can show that it has 128 degrees of freedom organized in a 32-component Majorana spinor[1].

- In order for the theory to be supersymmetric, we must include other 128-44=84 bosonic degrees of freedom: an eleven-dimensional three-form A_3 is what we need. Indeed, massless p-forms in D-dimensional spacetimes have $\binom{D-2}{p}$ physical degrees of freedom.

In general, Mp-branes are extended objects having a $(1+p)$-dimensional worldvolume which hosts a gauge theory. They naturally couple to a gauge potential, more precisely to a $(1+p)$-form A_{1+p}. Since eleven-dimensional supergravity contains a three-form gauge potential A_3, there should exist some M2-branes that couple to it[2]. These fundamental constituents are also called electric branes, which are by themselves sources of gauge fields, and from electromagnetic duality we know that there should be also (magnetic) M5-branes in the theory. This is because the electromagnetic dual of A_{1+p}, which is a massless gauge potential, is $C_{D-(1+p)-2}$: with $p=2$ and $D=11$ we find C_6, which naturally couples to a five-dimensional extended object[3]. Moreover, they are stable solitonic solutions to supergravity equations, which means that they look like (extremal) black-holes and share some of their properties.

We mentioned that M-Theory can be interpreted as a strong coupling limit of type IIA and that M-Theory does not contain strings, even if type IIA is a String Theory. This sounds quite strange, but we can think that the fundamental string of IIA is actually a M2-brane with a spatial dimension wrapping a circular eleventh dimension. Indeed, one can obtain type IIA

[1] Actually, 32 is the real dimension of the smallest spinor representation of the eleven-dimensional Lorentz group. In order to get it we start writing $D=2k+2$ for even dimensions and $D=2k+3$ for odd dimensions. The Dirac spinor representation has complex dimension 2^{k+1}, so that the number of real parameters in the smallest representation must be doubled and then reduced by half for a Majorana condition and by half for a Weyl condition. Hence, the minimal Majorana spinor in eleven dimensions has $\frac{2^{4+1}\times 2}{2}=32$ components.

[2] Just like the four-dimensional photon A_1 is associated to point-particles: the $(1+p)$-dimensional worldvolume in that case is the worldline.

[3] By electromagnetic duality $F_7=dC_6=\star_{11}dA_3=\star_{11}F_4$.

from dimensional reduction of M-Theory over a circle. Analogously, D4-branes[4]correspond to M5-branes. Alternatively, an M2-brane not wrapping the eleventh dimension become a D2-brane in type IIA after dimensional reduction. This is shown to correspond to a D3-brane in type IIB, which is very important for Maldacena duality. Reduction to type IIA can also give rise to D6-branes, which to some extent are different from other D-branes because they correspond to "purely geometrical" M-Theory configurations, i.e. they do not correspond to any M-Theory localized extended object. The reason why we are interested in this reduction to type IIA is that there is more control of Superstrings rather than M-Theory: indeed, String Theory admits a perturbative microscopic description which is not available for M-Theory. Besides, D6-branes play an important role in dual field theory descriptions, as we shall see in the following chapters.

3.1.2 Supergravity action

Gauge invariance of A_3 together with general coordinate invariance, local Lorentz invariance and supersymmetry put strong constraints on the action. Its bosonic part takes the unique form

$$S_{11} = \int \mathrm{d}^{11}x\sqrt{-g}R_{scalar} - \frac{1}{2}\int F_4 \wedge \star F_4 - \frac{1}{6}\int A_3 \wedge A_3 \wedge F_4, \tag{3.1.1}$$

where R_{scalar} is the scalar curvature and F_4 is the field strenght of A_3. The first term of (3.1.1) is clearly the Einstein-Hilbert action, while the remaining parts are respectively the kinetic term of A_3 and a Chern-Simons term. The reason why we are considering the bosonic part is that we are mostly interested in classical solutions, i.e. with vanishing fermionic fields in the background. Hence we can focus on (3.1.1).

Mp-brane solutions

Equations of motion descending from (3.1.1) are satisfied by the following metric:

$$ds^2_{11} = h^{-\frac{\tilde{d}}{9}}(r)dx^I dx^J \eta_{IJ} + h^{\frac{d}{9}}(r)dy^a dy^b \delta_{ab}, \qquad h(r) = \left(1 + \frac{k}{r^{\tilde{d}}}\right), \tag{3.1.2}$$

together with a field strength

$$F_{p+2} = \mathrm{d}vol(\mathbb{R}^d) \wedge \mathrm{d}h^{-1}(r), \tag{3.1.3}$$

where $d = p+1$ and $\tilde{d} = 11 - d - 2$ are the worldvolume dimensions of the Mp-brane and its dual, while $r = \sqrt{y^a y^b \delta_{ab}}$ is the radial distance in the transverse space. Indeed, $I, J = 0, ..., d-1$ are spacetime indexes for the longitudinal part, i.e. parallel to the brane, while $a, b = d, ..., 10$

[4]For our purposes it is sufficient to know that Dp-branes are the extended objects of String Theory, rather than M-Theory, having $(1+p)$-dimensional worldvolume. Type IIA has "$p = even$ D-branes" while type IIB has "$p = odd$ D-branes": they both couple to suitable $(1+p)$-forms. We can also see D-branes as extended object on which open strings can end.

are space indexes for the transverse part. The constant k in (3.1.2) can be interpreted as the electric/magnetic charge of the Mp-brane, namely the flux-integral over a suitable cycle: we will specialize the solution soon after. The solution (3.1.2) together with (3.1.3) is written in a rather compact fashion but recall that the relevant eleven-dimensional solutions have $p = 2$ or $p = 5$. Take for instance the former one: we want to briefly explain why it can be interpreted as an M2-brane. First of all, (3.1.2) with $p = 2$ is invariant under translations and rotations along the directions (x^0, x^1, x^2). Moreover, it is also invariant under rotations along the "transverse" directions $(x^3, ..., x^{10})$. These two facts combined lead to the interpretation of the solution as an extended object, having three-dimensional worldvolume, localized at the origin of the transverse coordinates. Furthermore, the solution has (electric) charge k so that the interpretation of the solution as an extended object that couples to a gauge potential is quite appropriate.

Supersymmetric solutions

The complete eleven-dimensional supergravity action is invariant under the following local supersymmetry transformations:

$$\begin{aligned} \delta e^A_M &= \bar{\epsilon}\Gamma^A\psi_M, \\ \delta A_{MNR} &= -3\bar{\epsilon}\Gamma_{[MN}\psi_{R]}, \\ \delta\psi_M &= \nabla_M\epsilon - \frac{1}{288}\big(\Gamma_M^{PQRS} - 8\delta^P_M\Gamma^{QRS}\big)F_{PQRS}\epsilon, \end{aligned} \tag{3.1.4}$$

where e^A_M are the vielbeins[5], ψ_M is the gravitino, ϵ is an arbitrary point-dependent 11-dimensional Majorana spinor and ∇_M is the covariant derivative associated to the Christoffel connection. Γ-matrices satisfy the algebra $\{\Gamma_M, \Gamma_N\} = 2g_{MN}$.

Being interested in classical solutions, every fermionic field in (3.1.4) must be vanishing. Hence, every variation is zero and the only nontrivial equation among (3.1.4) is

$$\nabla_M\epsilon - \frac{1}{288}\big(\Gamma_M^{PQRS} - 8\delta^P_M\Gamma^{QRS}\big)F_{PQRS}\epsilon = 0. \tag{3.1.5}$$

This can be also rewritten as

$$\nabla_M\epsilon + \frac{1}{12}\big(\Gamma_M\boldsymbol{F}^{(4)} - 3\boldsymbol{F}^{(4)}_M\big)\epsilon = 0 \tag{3.1.6}$$

after the definitions

$$\boldsymbol{F}^{(4)} = \frac{1}{4!}F_{MNPQ}\Gamma^{MNPQ}, \qquad \boldsymbol{F}^{(4)}_M = \frac{1}{3!}F_{MNPQ}\Gamma^{NPQ}. \tag{3.1.7}$$

A nontrivial solution ϵ to (3.1.5) is a Killing spinor and the equation itself leads to constraints both on the metric and the field strength, as we will see. Remember that Killing spinors

[5] Indexes $M, N, ...$ are related to "curved space" while $A, B, ...$ are related to "flat space". The former transform under general coordinate transformations, whereas the latter transform under local Lorentz transformations.

are associated to supersymmetries and hence supersymmetric solutions should admit some of them. To some extent they are the SUSY-analogue of Killing vector fields since they can be interpreted as fermionic parameters for infinitesimal SUSY transformations under which fields are invariant. So, Killing spinors are associated to fermionic symmetries just like Killing vectors are associated to bosonic ones.

3.2 M-branes on conical backgrounds

Consider a pure eleven-dimensional Minkowski flat spacetime background: then the metric solution to supergravity equation takes the form (3.1.2). Interestingly, the near-horizon (NH) geometry ($r \to 0$) of this solution is the eleven-dimensional spacetime $AdS_{p+2} \times \mathbb{S}^{9-p}$, having $SO(2, p+1) \times SO(10-p)$ as isometry group. On the other hand, (3.1.2) is asymptotically Minkowski as $r \to \infty$. Notice that $\mathbb{S}^{9-p}$ is the base of the $\mathbb{R}^{10-p}$ (non-singular)cone: we should have considered some different transverse spaces as a generalization, giving brane-solutions with different isometry groups. Moreover, it turns out that such geometries may be no more asymptotically minkowskian, for example if we take them to be singular cones. In this thesis we will actually deal with a stack of N M2-branes placed on a conical background geometry like $\mathbb{R}^{1,2} \times X_8$, so we are going to analyze the corresponding brane configuration and its NH-limit, whose physical importance will be clarified in the next section.

3.2.1 M2-brane solutions and the near-horizon limit

When $p = 2$, the membrane solution (3.1.2) is:

$$ds^2_{11} = \left(1 + \frac{k}{r^6}\right)^{-\frac{2}{3}} dx^I dx^J \eta_{IJ} + \left(1 + \frac{k}{r^6}\right)^{\frac{1}{3}} dy^a dy^b \delta_{ab}, \tag{3.2.1}$$

together with a field strength

$$F_4 = \mathrm{d}vol(\mathbb{R}^{1,2}) \wedge \mathrm{d}h^{-1}(r). \tag{3.2.2}$$

The charge k can be actually identified with the sixth power of some radius R, whose meaning will be clear in a while, so we will write $k = R^6$. The metric (3.2.1) is referred to a Minkowski background: the generalization to a transverse manifold X_8 reads

$$ds^2_{11} = \left(1 + \frac{R^6}{r^6}\right)^{-\frac{2}{3}} ds^2(\mathbb{R}^{1,2}) + \left(1 + \frac{R^6}{r^6}\right)^{\frac{1}{3}} ds^2(X). \tag{3.2.3}$$

Recall that if X is a cone then $ds^2(X) = dr^2 + r^2 ds^2(Y)$, where Y is the base of the cone. The flux quantization condition of the four-form field strength in (3.2.2) then reads

$$\frac{1}{(2\pi l_P)^6} \int_Y \star_{11} F_4 = \frac{1}{(2\pi l_P)^6} \int_X \mathrm{d} \star_{11} F_4 = N \in \mathbb{Z}, \tag{3.2.4}$$

actually giving the number of M2-branes in the stack. Besides, (3.2.4) leads to the relation

$$R = 2\pi l_P \left(\frac{N}{vol(Y)}\right)^{\frac{1}{6}}, \tag{3.2.5}$$

between the eleven-dimensional Planck length l_P, the volume of the base, the radius R and N.

We are now ready to study the NH-limit of (3.2.3). This corresponds to placing N M2-branes at the tip of the cone and then looking at the metric near $r = 0$. More precisely, we study the $r \ll R$ limit and the result is

$$ds^2_{NH} = \left(\frac{r}{R}\right)^4 ds^2(\mathbb{R}^{1,2}) + \left(\frac{R}{r}\right)^2 dr^2 + R^2 ds^2(Y), \tag{3.2.6}$$

where the first two terms gives exactly the metric (1.5.15) if $R = L = 1$. Defining the "holographic coordinate" $z = \frac{R^2}{r^2}$, the metric (3.2.6) takes the form

$$\begin{aligned} ds^2_{NH} &= \frac{r^4}{R^4} ds^2(\mathbb{R}^{1,2}) + \frac{R^2}{r^2} dr^2 + R^2 ds^2(Y) = \\ &= \frac{R^2}{4}\left[\frac{1}{z^2}\left(dz^2 + ds^2(\mathbb{R}^{1,2})\right)\right] + R^2 ds^2(Y) = \\ &= R^2[ds^2(AdS_4) + ds^2(Y)], \end{aligned} \tag{3.2.7}$$

so that it is clear that the radius R is actually the AdS-radius.

So, we showed that the near-horizon geometry generated by a stack of N M2-branes placed at the tip of the cone X_8 in a $\mathbb{R}^{1,2} \times X_8$ background is $AdS_4 \times Y_7$ as expected. The general form of its isometry group is

$$SO(2, 3) \times G, \tag{3.2.8}$$

where G is the isometry group of the base Y_7. When $Y_7 = \mathbb{S}^7$ then $G = SO(8)$ so that the algebra of the isometry group coincides with the bosonic sector of the superconformal three-dimensional algebra $Osp(8|4)$. This suggests that the symmetry of the dual field theory gets enhanced to a superconformal symmetry only near the horizon. Besides, the 8 of the isometry group of the sphere coincide with the number of supersymmetries preserved: indeed, the case of M2-branes on $\mathbb{S}^7$ corresponds to $\mathcal{N} = 8$. In general, Y_7 is a coset manifold $Y = G/H$ admitting $\mathcal{N}$ Killing spinors, where G takes the form

$$G = G' \times SO(\mathcal{N}) \tag{3.2.9}$$

and G' corresponds to some global symmetry. The $\mathcal{R}$-symmetry factor $SO(\mathcal{N})$ then combines with the isometry group of AdS_4 producing $Osp(\mathcal{N}|4)$. Hence, the isometry group for the non-spherical case is

$$Osp(\mathcal{N}|4) \times G'. \tag{3.2.10}$$

We expect to see the first factor in (3.2.10) as the bosonic sector of the superconformal symmetry group of some dual field theory. In the next subsection we will deepen the relation between brane solutions and residual supersymmety.

3.2.2 Supersymmetric M2 solutions

We want to solve (3.1.5) in the case where the background splits as $\mathbb{R}^{1,2} \times X_8$. The first step is gamma-matrices decomposition. We can adopt the basis

$$\Gamma_M = (\Gamma_\mu, \Gamma_m) \sim (\gamma_\mu \otimes \gamma_9, \mathbb{1}_2 \otimes \gamma_m), \tag{3.2.11}$$

where γ_μ and $\mathbb{1}_2$ are 2×2 $SO(1,2) = \text{Poincaré}_3$ matrices, while γ_9 and γ_m are 16×16 matrices of the isometry group of X_8.[6] The most general eleven-dimensional spinor field consistent with the isometry group of the "warped geometry" (3.2.1) in this gamma-basis can be decomposed in the following way:

$$\epsilon_{11}(x,y) = \zeta_1(x) \otimes \eta_1(y) + \zeta_2(x) \otimes \eta_2(y), \tag{3.2.12}$$

where ζ_1, ζ_2 are three-dimensional 2-component anticommuting spinors, while $\eta = \eta_1 + i\eta_2$ is an eight-dimensional 16-component commuting spinor. More precisely, considering the warp factor $h(r)$, (3.2.11) reads

$$\Gamma_\mu = h^{-\frac{1}{3}}(r)(\gamma_\mu \otimes \gamma_9), \qquad \Gamma_m = h^{\frac{1}{6}}(r)(\mathbb{1}_2 \otimes \gamma_m). \tag{3.2.13}$$

Now, it can be useful to anticipate that the eleven-dimensional Killing spinor solution can be written as

$$\epsilon_{11} = \zeta_3^0 \otimes \hat{\eta} = h^{\frac{1}{6}} \zeta_3^0 \otimes \eta, \tag{3.2.14}$$

where ζ_3^0 is a constant three-dimensional spinor and $\hat{\eta} = h^{\frac{1}{6}}\eta$ turns out to be the Killing spinor on X_8.

When we consider the M2-brane solution on purely Minkowski background we should take into account the presence of projectors $P_\pm = \frac{1}{2}(\mathbb{1} \pm \gamma_9)$. Indeed, the action of one projector on the $SO(8)$ spinor η imposes a chirality condition, halving the number of components. For instance, consider that purely Minkowski background has the maximal amount of supersymmetry, encoded in the 32-component ϵ_{11} spinor, i.e. 32 supercharges. These can also be interpreted as $32 = 2 \times 8 + 2 \times 8$ using (3.2.12). When the M2-brane is introduced and warps the geometry, we are left with a three-dimensional constant spinor ζ_3^0 and an eight-dimensional spinor η whose components are halved, i.e. only $2 \times 8 = 16$ supercharges are conserved. For the moment, let us check if (3.2.14) is actually a Killing spinor in the case of vanishing fluxes, i.e. we focus on the first term $\nabla_M \epsilon_{11} = 0$ in (3.1.5). Since $\nabla_M = (\partial_\mu, \nabla_m)$, this is true because $\partial_\mu \zeta_3^0 = 0$ for a constant three-dimensional spinor and $\nabla_m \hat{\eta} = 0$ for a Killing spinor of the transverse manifold X_8. However, it is in general inconsistent to take vanishing fluxes for M-Theory solutions and hence we are going to describe the procedure that leads to a complete and consistent solution to (3.1.5).

Before doing this we want to point out that supermembrane solutions may lower the maximal amount of SUSY and the background can break some SUSY by itself, depending also on the

[6] $\gamma_9 = \gamma_3 \cdots \gamma_{10}$ and $\gamma_9^2 = \mathbb{1}_{16}$.

orientation of X_8, or more correctly its base Y_7 if we remind the arguments in the previous chapter. Calling η a Killing spinor on Y_7, we can expect $\mathcal{N}$ solutions to

$$\nabla^{(Y)}_{\tilde{m}}\eta - \frac{1}{2}\gamma_{\tilde{m}}\eta = 0, \tag{3.2.15}$$

where the index $\tilde{m}$ is referred to the seven-dimensional base. Then, since a generic eight-dimensional spinor has 16 components, we are left with a fraction $\mathcal{N}/16$ of the full 32 supercharges of maximally symmetric background without branes. For example, $Y_7 = \mathbb{S}^7$ admits $\mathcal{N} = 8+8$ Killing spinors, i.e. it is maximally supersymmetric. These spinors can be viewed as $\eta = \{\eta_+, \eta_-\}$, where the eight η_+ satisfy (3.2.15), while the eight η_- satisfy the same relation with a "plus" sign. The membrane introduction cuts the η_- (or the η_+ depending on conventions) because of the projectors, recovering the aforementioned halving of supersymmetries[7]. However, in this thesis we consider a Sasaki-Einstein base: this possesses $\mathcal{N} = 2$ Killing spinors and hence preserve $\frac{\mathcal{N}}{16} \times 32 = 4$ supercharges. This 4 is precisely the amount of supersymmetry of a $\mathcal{N} = 2$ three-dimensional theory. Looking at (3.2.12), the two three-dimensional (constant) spinors ζ_1, ζ_2 can be interpreted as the SUSY generators of such a field theory: indeed, they correspond to $2+2$ supercharges.

M-Theory solutions preserving $\mathcal{N} = 2$

Let us focus on M-Theory "flux compactification[8]" to three-dimensional flat spacetime preserving $\mathcal{N} = 2$ supersymmetries. Our starting point is the warped metric

$$ds^2 = h^{-\frac{2}{3}}(y)\eta_{\mu\nu}dx^\mu dx^\nu + h^{\frac{1}{3}}(y)g_{mn}(y)dy^m dy^n, \tag{3.2.16}$$

where g_{mn} is the metric on the internal manifold X_8. We have seen that the emergence of $\mathcal{N} = 2$ in three dimensions corresponds to X_8 being a Calabi-Yau four-fold. Notice that the warp factor $h(r)$ has an important consequence: even if the background is a direct product, the introduction of M2-branes gives a spacetime which is no more a direct product but instead it is a warped version of it. This is sometimes indicated with $\mathbb{R}^{1,2} \times_w X_8$ and notice that if $N = 0$, i.e. there are no branes, the warp factor $h = 1 + \frac{R^6}{r^6}$ with R given by (3.2.5) is $h = 1$ and hence $\times_w \to \times$ in this case.

In order to work out the dimensional reduction of (3.1.5) we adopt the gamma-matrix decomposition (3.2.13) and the spinor decomposition $\epsilon(x, y) = \zeta(x) \otimes \eta(y)$. Besides, for the case at hand it can be shown that the only non-vanishing components of F_4 are

$$F_{mnpq}(y), \qquad F_{\mu\nu\rho m} = \epsilon_{\mu\nu\rho} f_m(y), \tag{3.2.17}$$

[7]There can be "extreme" situations: for example, if one considers the "squashed sphere", i.e. a round sphere with reversed orientation, then all supersymmetries are broken.

[8]We will not give a systematic presentation of this topic. We can say that "flux compactifications" are techniques employed to study the relation between a D-dimensional theory with fluxes, a field strength for example, and a d-dimensional one obtained from compactification of $D-d$ directions. In the case at hand, $D = 11$ and $d = 3$, but the $11-3 = 8$ "compact directions" are not compact: they make a cone. Nevertheless, these techniques are still called flux compactifications.

where $f_m(y)$ is an arbitrary function that we will determine soon. Now it is maybe useful to switch to the notation in (3.1.6) and (3.1.7) because using the gamma-matrix decomposition we find

$$\begin{aligned} \boldsymbol{F}^{(4)} &= h^{-2/3}(\mathbb{1}_2 \otimes \boldsymbol{F}) + h^{5/6}(\mathbb{1}_2 \otimes \gamma_9 \boldsymbol{f}), \\ \boldsymbol{F}^{(4)}_\mu &= h^{1/2}(\gamma_\mu \otimes \boldsymbol{f}), \\ \boldsymbol{F}^{(4)}_m &= -h f_m(\mathbb{1}_2 \otimes \gamma_9) + h^{-1/2}(\mathbb{1}_2 \otimes \boldsymbol{F}_m), \end{aligned} \tag{3.2.18}$$

where

$$\boldsymbol{F} = \frac{1}{24} F_{mnpq}\gamma^{mnpq}, \qquad \boldsymbol{F}_m = \frac{1}{6} F_{mnpq}\gamma^{npq}, \qquad \boldsymbol{f} = f_m \gamma^m. \tag{3.2.19}$$

At this stage we can analyze the internal and external components of $\delta\psi_M = 0$ separately.

For the external components $M = \mu$ we get

$$\delta\psi_\mu = \nabla_\mu \epsilon - \frac{1}{4} h^{-7/6}(\gamma_\mu \otimes \gamma_9 \gamma^m)\partial_m h^{2/3}\epsilon + \frac{1}{12}\big(\Gamma_\mu \boldsymbol{F}^{(4)} - 3\boldsymbol{F}^{(4)}_\mu\big)\epsilon = 0. \tag{3.2.20}$$

Since our three-dimensional external spacetime is minkowskian there always exists a covariantly constant spinor satisfying $\nabla_\mu \zeta(x) = 0$.[9] This let us simplify (3.2.20), which becomes

$$\gamma^m \partial_m h^{-1}\eta + \boldsymbol{f}\eta + \frac{1}{2} h^{-3/2}\boldsymbol{F}\eta = 0 \tag{3.2.21}$$

and leads to the constraints

$$\boldsymbol{F}\eta = 0, \qquad f_m(y) = -\partial_m h^{-1}(y). \tag{3.2.22}$$

Notice that the second equation of (3.2.22) provides a relation between some external component of the flux and the warp factor, hence it is evident that in the case of warping we cannot in general freely set fluxes to zero: the result would be inconsistent.

For the internal components $M = m$, using the same decompositions together with (3.2.18) we can turn $\delta\psi_m = 0$ into the expression

$$\nabla_m \eta + \frac{1}{4} h^{-2/3}\partial_m h^{2/3}\eta - \frac{1}{4} h^{-1/2}\boldsymbol{F}_m \eta = 0. \tag{3.2.23}$$

This equation is satisfied provided that

$$\boldsymbol{F}_m \hat{\eta} = 0, \qquad \nabla_m \hat{\eta} = 0, \tag{3.2.24}$$

where $\hat{\eta}$ is a nonvanishing covariantly constant complex spinor on the internal manifold X_8 and takes the form $\hat{\eta} = h^{1/6}\eta$. Notice that this calculation leads exactly to the anticipated (3.2.14). Moreover, in the case we are interested in, namely Q^{111}, the internal components of the flux can be freely set to zero, i.e. $F_{mnpq}(y) = 0$. Thus, the field strength (3.2.2) is completely characterized by the function $f_m(y)$ in (3.2.22).

[9] Indeed, one can take constant spinors ζ_3^0 in three-dimensional Minkowski spacetime so that $\nabla_\mu \zeta = \partial_\mu \zeta_3^0 = 0$.

The Q^{111} base manifold

There exist just three Sasaki-Einstein bases realized as coset manifolds G/H that give rise to $\mathcal{N}=2$ supersymmetries:

$$\begin{aligned} M^{ppr} &= \frac{SU(3)\times SU(2)\times U(1)\times U(1)}{SU(2)\times U(1)\times U(1)}, \\ Q^{ppp} &= \frac{SU(2)\times SU(2)\times SU(2)\times U(1)}{U(1)\times U(1)\times U(1)}, \\ V_{5,2} &= \frac{SO(5)}{SO(3)}. \end{aligned} \tag{3.2.25}$$

In this thesis we will deal with Q^{111} and so we give some previews on it[10]. First of all the isometry group has exactly the form (3.2.9), with the $\mathcal{R}$-symmetry group $SO(\mathcal{N}=2)\simeq U(1)_{\mathcal{R}}$ and $G'=SU(2)^3$. The metric on the cone over Q^{111} can be seen as a $\mathbb{C}$ bundle over $\mathbb{CP}^1\times\mathbb{CP}^1\times\mathbb{CP}^1$ or as a $\mathbb{C}^2$ bundle over $\mathbb{CP}^1\times\mathbb{CP}^1$. We will see that the latter structure is more appropriate for complex coordinates while using real coordinates the metric takes the form:

$$ds^2(Y=Q^{111}) = \frac{1}{16}\left(d\psi + \sum_{i=1}^{3}\cos\theta_i d\phi_i\right)^2 + \frac{1}{8}\sum_{i=1}^{3}\left(d\theta_i^2 + \sin^2\theta_i d\phi_i^2\right), \tag{3.2.26}$$

where (θ_i,ϕ_i) are standard coordinates on three copies of $\mathbb{CP}^1\simeq\mathbb{S}^2$, while ψ has 4π period

3.2.3 Warped CY_4 backgrounds and deformations

In the previous section we explored M2 solutions where N branes where organized in one stack set on a suitable M-Theory background, for example a conical one. However, there exist more general solutions in M-Theory where supergravity backgrounds take the form:

$$\begin{aligned} ds_{11}^2 &= h^{-2/3}ds^2(\mathbb{R}^{1,2}) + h^{1/3}ds^2(X), \\ F_4 &= \mathrm{d}vol(\mathbb{R}^{1,2})\wedge \mathrm{d}h^{-1}, \end{aligned} \tag{3.2.27}$$

where now we have a generic warp factor $h(r)$, i.e. it is not the one of (3.1.2). Actually, we can take the whole discussion of the previous section and repeat it for this generic warp factor $h(r)$: indeed, solutions preserving $\mathcal{N}=2$ supersymmetries in three dimensions only depend on the choice of the transverse manifold, which has to be a Calabi-Yau four-fold in our case. So, suppose that we are working with such a generic warped background $\mathbb{R}^{1,3}\times_w CY_4$. If $CY_4=\mathbb{R}^8$ or $CY_4=C(Y_7)$ then we already investigated what happens, in particular with N coincident M2-branes placed at $r=0$. However, we can think of "deforming" these kind of backgrounds in two quite natural ways: either allowing for M2-branes motion around CY_4 and resolving the singularity of $C(Y_7)$ using the $\pi : X\to C(Y)$ of the previous chapter. In the

[10]The p indicates a further quotient with some discrete group. For instance, Q^{222} is a Z_2 quotient of Q^{111}.

latter case it is crucial to require X to be a Calabi-Yau fourfold whose metric is asymptotic to the metric of a singular cone $C(Y)$ over a Sasaki-Einstein seven-dimensional base Y, i.e. $ds^2_X \to ds^2_{C(Y)} = dr^2 + r^2 ds^2_Y$. The physical reason is that such resolved cone X loses its scale-invariance, which in turn is restored only far away from the resolution manifold: this is very important in holography and we will use this property later in this thesis. Then, placing N coincident M2-branes at a point $y \in X$ leads to an equation for the warp factor

$$\Delta_X h = \frac{(2\pi l_P)^6 N}{\sqrt{detg_X}} \delta^8(x-y), \tag{3.2.28}$$

where Δ_X is the Laplacian on (X, g_X). This kind of equation typically arises from the supergravity equations of motion and requires the warp factor to be an harmonic function. Considering the motion of M2-branes on X, we can imagine that branes are point-like with respect to X and that they are sitting on it splitted in m stacks such that $N = \sum_{i=1}^{m} N_i$, with N_i M2-branes in the i-th stack. Defining $y_i \in X$ the position of every stack, (3.2.28) can be easily generalized to

$$\Delta_X h = \frac{(2\pi l_P)^6 N}{\sqrt{detg_X}} \sum_{i=1}^{m} \frac{N_i}{N} \delta^8(x-y_i). \tag{3.2.29}$$

A solution to (3.2.29) requires a particular CY metric g_X together with some boundary conditions. Since (X, g_X) is asymptotic to the singular cone we can require the large-r behavior of the warp factor to be $h \sim R^6/r^6$ so that it vanishes at infinity. These kind of backgrounds with asymptotically vanishing warp factor are called "asymptotically $AdS \times Y$". For instance, if we take $h = R^6/r^6$ and put it in (3.2.27) we will find the $AdS_4 \times Y_7$ only for large r, where $ds^2(X) \to dr^2 + r^2 ds^2(Y)$. In any case with $h \sim R^6/r^6$, (3.2.27) will be asymptotic to $AdS_4 \times Y_7$ with N units of $F_7 \sim \star_{11} F_4$ trough Y_7. Otherwise stated, given the asymptotically conical metric and the asymptotically vanishing warp factor, the M-Theory background (3.2.27) is asymptotic to the large r region of (3.2.7). To some extent, we can think of these asymptotically $AdS_4 \times Y_7$ backgrounds as supergravity solutions realizing the near-horizon physics of a M2-branes stack, i.e. $AdS_4 \times Y_7$, at large r. On the other side, at small r we have a metric $ds^2(X)$ which can be completely different from $dr^2 + r^2 ds^2(Y)$ and hence (3.2.27) is not $AdS_4 \times Y_7$ "everywhere". Recalling that the isometry group of AdS_4 coincides with the conformal group of a CFT_3, it is crucial to require the presence of some AdS_4 factor in the supergravity background if we want the AdS_4/CFT_3 duality to hold. For instance, in the case of a stack of M2-branes placed on the tip of $C(Y_7)$ the AdS_4 factor is found in the near-horizon limit, while for the "deformations" discussed here, giving asymptotically $AdS_4 \times Y_7$ solutions, the AdS_4 factor is found at infinity. Actually, it turns out that these kind of M-Theory vacua admit an interpretation in terms of vacua of a dual three-dimensional SCFT with $\mathcal{N} = 2$: this claim will be widely supported throughout the thesis, but we should start from the basis of gauge/gravity correspondence in the next section.

3.3 The gauge/gravity correspondence

The original Maldacena conjecture [3] states that the large N limit of certain (Super)CFTs is dual, i.e. equivalent, to a supergravity theory on a background metric containing an anti de Sitter spacetime and a transverse compact manifold. In particular, he argued that $\mathcal{N} = 4$ super Yang-Mills in four spacetime dimensions with gauge group $SU(N)$ is dual to type IIB supergravity on $AdS_5 \times \mathbb{S}^5$ background. The conjecture is motivated by considering N coincident branes in the String Theory and then taking a low-energy limit such that the gauge theory on the branes decouples from the physics in the bulk side. At the same time, branes produce a warped geometry whose near-horizon decouples from the bulk side in the same limit. The bulk then "factors-out", leaving the duality between the gauge theory and the near-horizon physics.

Gauge theories living on D-branes

We want to give an intuitive explanation of why gauge theories are hosted on the worldvolume of branes. First of all, we know that p-branes are coupled to generalized gauge potentials, A_{p+1}. These branes are in some sense stable, i.e. they do not "decay". Imagine two such branes living in a D-dimensional spacetime: there are open strings ending on the same brane and open strings connecting them. The former give rise to a massless vector while the latter give rise to a massive one. The reason is that strings stretching between branes have nonzero length, or equivalently nonzero tension $T \sim \frac{1}{\alpha'}$. The mass of these massive modes is given by:

$$\Delta M^2 \sim T^2 \sum_{i=p+1}^{D-1} |\phi_1^i - \phi_2^i|^2, \tag{3.3.1}$$

where ϕ are fields parametrizing the positions of branes in the transverse space. Notice that (3.3.1) is zero when the branes coincide, hence giving massless vectors. When a "stack" of N branes is considered, the generalization is straightforward. Indeed, there will be a $N \times N$ matrix of gauge vectors A_{ab} generating $U(N)$ gauge transformations, where the index a labels the starting branes and the index b labels the ending brane to which an open string is attached to. The mass term for these gauge vectors is shown to be

$$|A_{ab}|^2 \sum_{i=p+1}^{D-1} \frac{|\phi_a^i - \phi_b^i|^2}{\alpha'^2}, \tag{3.3.2}$$

so that diagonal components, i.e. with $a = b$, are massless while off-diagonal components acquire mass. Then, we can interpret diagonal gauge fields as open strings ending on the same D-brane and off-diagonal ones as open strings connecting different D-branes in the stack. Notice that since the mass is proportional to the distance between the D-branes, when branes coincide every massive vector become massless, which is a sort of "inverse-Higgs" mechanism. On the other hand, one can think of separating a collection of N_1 branes from the remaining

$N_2 = N - N_1$: this corresponds to an "Higgsing". For instance, if $U(N) \to U(N_1) \times U(N_2)$ then there will be $N^2 - (N_1^2 + N_2^2) = 2N_1N_2$ broken generators, giving an equal amount of massive vector bosons. By the way, we want to stress that while the gauge theory on a single brane is abelian, its generalization to a stack of N branes is non-abelian.

3.3.1 Maldacena duality

Consider N parallel D3-branes in ten-dimensional spacetime. If we consider the system at low energies then only the massless states are accounted for in the physics and we can write an effective supergravity Lagrangian for type IIB. Indeed, closed string massless states constitute a gravity supermultiplet in ten dimensions. On the other hand, open string massless states give a $\mathcal{N} = 4$ vector supermultiplet in four dimensions and their low-energy effective Lagrangian is the one of $\mathcal{N} = 4$ $U(N)$ super Yang-Mills (SYM).

The full effective action for masselss modes takes the form:

$$S = S_{bulk} + S_{brane} + S_{int}, \tag{3.3.3}$$

where S_{bulk} is the action of ten-dimensional supergravity, S_{brane} is the four-dimensional world-volume action containing the Yang-Mills theory and S_{int} describes the interaction between brane and bulk modes. The interaction Lagrangian is proportional to κ, which is the ten-dimensional gravitational coupling constant. The low-energy limit corresponds to $\kappa \to 0$, which translates into the decoupling of brane modes from bulk modes. More precisely, $\kappa \sim g_s\alpha'^2$ and the limit is actually $\alpha' \to 0$ with fixed string coupling $g_s \sim g_{YM}^2$. So, the theory in the low-energy limit describes two decoupled pieces: free IIB supergravity on flat ten-dimensional spacetime and $\mathcal{N} = 4$ SYM theory in four dimensions.

This decoupling argument can be repeated from a different point of view: warped geometry. The stack of D3-branes generates a supergravity solution that takes the form:

$$ds^2 = \left(1 + \frac{R^4}{r^4}\right)^{-\frac{1}{2}} ds^2(\mathbb{R}^{1,3}) + \left(1 + \frac{R^4}{r^4}\right)^{\frac{1}{2}} ds^2(\mathbb{R}^6), \tag{3.3.4}$$

where $R^4 \sim g_sN\alpha'^2$. So, in the low-energy limit $\alpha' \to 0$ it seems that (3.3.4) gives the flat spacetime metric. This situation is equivalent to consider $r \gg R$. Besides, there is another low-energy region. Since g_{tt} depends on r, the energy of an object measured by an observer at a constant position is affected by a redshift factor with respect to the energy measured at infinity, namely:

$$E_\infty = \left(1 + \frac{R^4}{r^4}\right)^{-\frac{1}{4}} E(r). \tag{3.3.5}$$

It is clear from (3.3.5) that an object brought close to $r = 0$ finds its energy reduced if that energy is measured by an observer far away from it. Hence, the other low-energy region is $r \ll R$: this actually corresponds to taking the near-horizon geometry of (3.3.4), which is

$AdS_5 \times \mathbb{S}^5$. In conclusion, the low-energy theory consists of two systems separed by a “barrier” that grows as $\alpha' \to 0$, making them decouple from each other. This two systems are IIB supergravity on flat ten-dimensional spacetime and IIB on the near-horizon $AdS_5 \times \mathbb{S}^5$.

At this stage, we have two decoupled systems both from the open strings field theory point of view and from supergravity solution point of view. Moreover, one of the decoupled systems is IIB supergravity on flat ten-dimensional spacetime from both sides. This suggests that the residual systems may be identified, giving an equivalence between $\mathcal{N} = 4$ SYM in four dimensions and type IIB on $AdS_5 \times \mathbb{S}^5$. This argument is further strengthened by symmetry considerations. Indeed, the isometry group of AdS_5 coincide with the conformal group of CFT_4, which is $SO(4, 2)$. Moreover, the $SO(6)$ isometry of $\mathbb{S}^5$ can be identified with the $\mathcal{R}$-symmetry group $SU(\mathcal{N} = 4)$ of the field theory. This means that some relevant properties of the low-energy description of the field theory living on branes actually corresponds to the near-horizon physics description.

Coupling constants and large N limit

We stressed that the gauge/gravity correspondence can only be trusted in the large N limit: we want to clarify this statement.

The dimensionless effective coupling of a SYM theory in a $(p + 1)$-dimensional spacetime is scale dependent [21], namely

$$g^2_{eff}(E) \sim g^2_{YM} N E^{p-3}. \tag{3.3.6}$$

This coupling is small for large energy in the $p < 3$ case and for small energy in the $p > 3$ case. The $p = 3$ case of D3-branes theory is exactly the $\mathcal{N} = 4$ SYM in four dimensions: this is a conformal field theory and indeed (3.3.6) becomes independent on energy scale and corresponds to the so called ’t Hooft coupling constant:

$$\lambda = g^2_{YM} N. \tag{3.3.7}$$

In the large N expansion we are going to introduce, (3.3.7) is held constant while $g^2_{YM} \sim g_s$ becomes small. Moreover, combining with $R^4 \sim g_s N \alpha'^2$ we obtain:

$$g_s \sim \frac{\lambda}{N}, \qquad \frac{R^4}{\alpha'^2} \sim \lambda. \tag{3.3.8}$$

So, the String Theory is weakly-coupled, i.e. $g_s \ll 1$, when N is large. Besides, a large coupling $\lambda \gg 1$, i.e. strongly-coupled field theory, corresponds to a big AdS-radius R in string length units $l_s \sim \sqrt{\alpha'}$ or alternatively to a fixed radius with $\alpha' \to 0$: this reminds the low-energy limit previously discussed. In conclusion, the combination (3.3.8) with $g_s \ll 1$ and $\lambda \gg 1$ corresponds to the duality between a strongly-coupled field theory and a weakly-coupled supergravity (effective) field theory. The strong-weak coupling duality feature is the whole point: it is this property that makes the correspondence useful.

From the field theory point of view, it is possible to show that Feynman diagrams are actually associated to two-dimensional Riemann surfaces with a determined Euler characteristic χ. If the diagram has V vertexes, E edges and F faces then $\chi = F - V + E$. Since each face is a loop, F corresponds to a factor of N, while each vertex and each edge corresponds to g_{YM}^2 and g_{YM}^{-2} respectively. Recalling that $g_{YM}^2 = \frac{\lambda}{N}$, the total contribution to an amplitude goes like $\lambda^{V-E} N^{F-V+E} = \lambda^{E-V} N^{\chi}$. For instance, this means that an observable $\mathcal{O}$ admits a power series expansion

$$\mathcal{O} = \sum_{g=0}^{\infty} f(\lambda) N^{2-2g}, \tag{3.3.9}$$

where we have expressed the Euler characteristic in terms of the genus g of the surface diagram, i.e. $\chi = 2 - 2g$. It is then clear from (3.3.9) that "higher-genus contributions" are strongly suppressed in the large N limit with respect to planar diagrams.

3.3.2 Generalizations of the conjecture

The Maldacena conjecture has found a lot of success not only because of its physical meaning but also because it is widely generalizable, hence providing a huge laboratory where the conjectured duality can be tested.

One among possible generalizations consists in studying a stack of D3-branes placed on the singularity of a $\mathbb{R}^{1,3} \times CY_3$ background, where CY_3 is the six-dimensional cone over a Sasaki-Einstein five-dimensional base Y_5: an example is the Klebanov-Witten theory described in [39]. Another kind of generalization consists in considering the M-Theory scenario with M2-branes probing a conical background $\mathbb{R}^{1,2} \times CY_4$. To some extent, the discussion for D3-branes in ten-dimensional spacetime can be carefully repeated for M2-branes in eleven-dimensional spacetime with suitable corrections: for instance, we have (3.2.3) instead of (3.3.4). Moreover, $p = 2$ and hence we see from (3.3.6) that the effective coupling constant increases as the energy decreases. In other words, the resulting field theory is not a conformal theory like $\mathcal{N} = 4$ SYM in four dimensions because the dimensionful coupling in three dimensions introduced a scale. Hence the field theory, which for the case at hand is a $d = 3$ $\mathcal{N} = 2$ quiver, whose structure will be explored in the next chapter, acquires the conformal symmetry only at the fixed point of an RG-flow: this is an infrared fixed point since it is found at low energies.

In general, the SCFT dual to the near-horizon physics on $AdS_4 \times Y_7$ can be thought of as the IR conformal fixed point of particular three-dimensional gauge theories with $\mathcal{N} = 2$ in the "far UV region", typically quiver gauge theories that are matter of the next chapter. One such microscopic theory has a non-trivial moduli space, whose points represent different field theory vacua. The crucial argument is that a particular region of this space is related to suitable background geometries, which are supergravity vacua. We can then say that there is a family of such geometries but only one point of the moduli space corresponds to the superconformal vacuum, for example $AdS_4 \times Y_7$. In that particular vacuum every operator we can build has zero

VEV since there cannot be any dimensionful scale in a conformally invariant theory. Instead, at a generic point of the moduli space the vacua spontaneously break the conformal symmetry because operators may acquire some VEV. We can imagine that the "motion" trough the moduli space corresponds to an RG-flow triggered by VEVs to some extent, but the RG-flow has to be intended in a two-step fashion and it is a very delicate issue which has not been rigorously described yet. We try to give the basic idea. From the far UV, where the theory is describable by a UV-quiver and it is not a SCFT, the flow is towards a "deep IR region" which is the true dual to the near-horizon physics, i.e. the IR fixed point. At this point we have $\langle O \rangle = 0$ for any operator O, but since there is no dimensionful scale then we can freely study the theory at the IR fixed point for any energy regime, even the UV: this however must not be confused with the "far UV" previously mentioned. Indeed, in the far UV the dual field theory is not a SCFT: it is required an RG-flow towards the IR as a first step and from there we can reach another UV region. This new UV region is the high-energy region of the SCFT which does not correspond to the high-energy region of the UV-quiver. Then, even if some operator acquires a VEV $\langle O \rangle \neq 0$ the superconformal symmetry should be present, at least via non-linear realization. In other words, the VEV $\langle O \rangle$ is interpreted as a spontaneous symmetry breaking scale and hence the conformal symmetry is spontaneously broken at a generic point of the moduli space. Now, consider one such field theory vacua with non-vanishing VEV. At energies well above this scale, let us say in the UV region, the conformal symmetry is "recovered": these vacua are then interpreted as supergravity vacua which are asymptotically $AdS_4 \times Y_7$ in the sense previously discussed. On the other hand, as energies become comparable with that scale the conformal symmetry starts to spontaneously break down and there will be massive states with a mass of order $\langle O \rangle$. So, at energies well below $\langle O \rangle$ we can integrate out these massive states and build an effective theory for massless modes only. In the branes picture, when the M2-branes are coincident and placed on the tip of the cone CY_4, the NH limit is dual to a SCFT, whose conformal symmetry is however spontaneously broken at a generic point of the moduli space. This point may represent the following (combination of) situations:

- the cone is resolved, hence schematically $\langle O \rangle = \langle resolutions \rangle \neq 0$;
- some or all the M2-branes are no more on the tip and instead are moving around the cone, hence schematically $\langle O \rangle = \langle positions \rangle \neq 0$.

Then, if $\tilde{N}$ is the number of mobile M2-branes on CY_4, one expects that a portion of the moduli space of vacua is the symmetrized product of CY_4 itself[11]. In the next chapter we will study the field theory side, focusing on the Q^{111} model of this thesis: we will actually find that a portion of the moduli space is in fact $Sym^{\tilde{N}} CY_4$, which can be parametrized by the positions of M2-branes on CY_4.

[11]This is because the moduli space of a point-like object on a manifold, like an M2-brane on CY_4, should contain the manifold itself. Thus, considering $\tilde{N}$ identical point-like objects, i.e. "branes indistinguishability", the moduli space should contain $Sym^{\tilde{N}} CY_4$.

Chapter 4

Quiver Field Theories

This chapter is dedicated to the field theories we are interested in, namely three-dimensional quiver gauge theories with Chern-Simons and matter content having $\mathcal{N} = 2$ supersymmetries. These describe the dynamics of M2-branes placed on a $\mathbb{R}^{1,2} \times CY_4$ background and hence they are supposed to RG-flow to a $\mathcal{N} = 2$ three-dimensional SCFT dual to M-Theory on $AdS_4 \times Y_7$, where Y_7 is the base of CY_4. We will work on a particular example of quiver gauge theory: the $Y_7 = Q^{111}$ model. Our attention should be oriented towards its moduli space $\mathcal{M}_{quiver}$, namely the space of inequivalent vacua of the field theory. Actually, we are interested in a particular branch $\mathcal{M} \subset \mathcal{M}_{quiver}$ of the full moduli space, the one that somehow reproduces the background cone. More precisely, if the $\mathcal{N} = 2$ Chern-Simons three-dimensional theory for Q^{111} is conjectured to describe the dynamics of $\tilde{N}$ mobile M2-branes on a CY_4, we expect that the moduli space of the field theory has a branch containing $\tilde{N}$ symmetrized copies of it, i.e. $\mathcal{M} = Sym^{\tilde{N}} CY_4$. It should be clear that if the branes are moving on a resolved version of CY_4 then we expect to find the resolved version inside the moduli space of the field theory:

$$\mathbb{R}^{1,2} \times C(Y_7) \leftrightarrow \mathcal{M} = Sym^{\tilde{N}} C(Y_7), \qquad \mathbb{R}^{1,2} \times X_8 \leftrightarrow \mathcal{M} = Sym^{\tilde{N}} X_8.$$

The moduli space is a Kähler manifold: while its complex structure is preserved under quantum corrections, its Kähler structure generally receives strong quantum corrections. In order to study the branch $\mathcal{M}$ from the field theory point of view we can adopt two strategies: a semiclassical calculation or a computation based on monopole operators. The former includes one-loop corrections and probes the Kähler structure while the latter is only aware of the complex structure, i.e. it does not "see" resolutions, but it is one-loop exact. We are very interested in the Kähler structure because it is the one giving rise to the Lagrangian for the effective field theory, as anticipated in (1.2.18). The Kähler metric for the nonlinear sigma model Lagrangian can in principle be calculated from the "far UV" theory: the problem is that the effective field theory is a low-energy theory where the coupling is strong and hence there are no direct ways to face the problem of finding the correct Kähler metric in this regime. Indeed, both the semiclassical and the monopole methods are subject to non-perturbative corrections and we can properly use them in the "far UV" region only. So, the problem is to compute the Kähler metric in the strongly coupled region where the quiver becomes a SCFT. We should mention that the moduli space of the SCFT obtained RG-flowing the UV-quiver is something different from $\mathcal{M}_{quiver}$: we can call it $\mathcal{M}_{SCFT}$. Nevertheless, we want to stress that, at least in the branch $\mathcal{M}$ we will work on, the complex structure is the very same and the information[1] about the Kähler structure we can obtain from the semiclassical computation in the UV are trustable even in the SCFT. The information we crucially lack is about the metric on $\mathcal{M}_{SCFT}$: it is exactly for this reason that we are going to use holography, whose techniques will be introduced in the next chapter, so that we can compute the metric on $\mathcal{M}_{SUGRA}$. Schematically, we can say that $\mathcal{M}_{quiver} \sim \mathcal{M}_{SCFT}$ from the complex point of view but in order to compute the metric we must switch to the holographic description, i.e. $\mathcal{M}_{SCFT} \to \mathcal{M}_{SUGRA}$. Quiver field theories and their duals are studied for example in [1, 2, 29, 30, 31, 32, 33, 34, 6, 35], together with their moduli spaces

4.1 The quiver structure

Let us start specifying that quivers are not field theories by themselves: they are graphs encoding informations about field theories.

More precisely, a quiver is a directed graph consisting of a set of nodes $V = \{v_1, v_2, ..., v_n\}$, a set of arrows A and two maps $s, t : A \to V$. For each $a \in A$ there is a node $s(a)$ called "source" and a node $t(a)$ called "target".

This structure turns out to be useful because the field content of certain field theories may be described in the following fashion:

- Each node $v_i \in V$ corresponds to a vector superfield in the adjoint representation of a Lie group G_i. The full gauge group of the theory is the product of these groups, namely $G_1 \times \cdots \times G_n$. In cases we are interested in we associate a gauge group factor $U(N_i)$, or $SU(N_i)$, to every node.

- Each arrow $a_{ij} \in A$, such that $s(a_{ij}) = v_i$ and $t(a_{ij}) = v_j$, corresponds to a chiral superfield Φ_{ij} transforming in the fundamental representation of the source (first index) and in the antifundamental of the target (second index). These are also called bifundamental fields: sometimes we will use a notation with only one index running over the arrows, namely Φ_a. The charge convention we will adopt is to assign it a $+1$ for the source group $U(N_i)_{s(a)}$ and a -1 for the target group $U(N_j)_{t(a)}$ so that a field that enters in a node brings a negative charge under that node. There can also exist arrows such that the source node

[1] We mean that even if the Kähler metric is strongly corrected using field theoretical techniques, i.e. it is not possible to directly compute it, the semiclassical method is useful to build a dictionary between resolution parameters in the field theory side and in the holographic counterpart.

coincides with the target one: these are associated to chiral fields transforming in the adjoint representation of that node.

Actually, it is possible that a subset of the nodes does not constitute a gauge group and it is instead identified with a flavor group. Consider $G+\tilde{G}$ nodes, where G is the number of gauge nodes and $\tilde{G} = \sum_{a\in\text{flavored}} h_a$ is the number of flavor nodes. The flavoring of a quiver consists in introducing h_a pairs of chiral fields (q_a, p_a), where the index a here runs over a subgroup of the arrow set A, and a flavor group $\prod_{a\in\text{flavored}} U(h_a)$. Then, an arrow having a flavor source with index $\hat{k}$ and a gauge target with index i corresponds to a chiral field $p_a = p_{\hat{k}i}$ transforming in the fundamental of the flavor source and antifundamental of the gauge target, while an arrow having a gauge source j and a flavor target $\hat{k}$ corresponds to a chiral field $q_a = q_{j\hat{k}}$ transforming in the fundamental of the gauge source and antifundamental of the flavor target.

We must point out that the Lagrangian of a quiver field theory is not completely fixed. Indeed, quivers lack information about superpotential and external parameters, like Fayet-Iliopoulos terms and real masses: these should be included by hand if any.

4.2 Chern-Simons coupled to matter

As we mentioned earlier, we are going to deal with quiver Chern-Simons (CS) theories having $\mathcal{N}=2$ supersymmetries in three spacetime dimensions. Typically, their gauge groups are product of G simple factors, for example $\prod_{i=1}^{G} U(N_i)$. There are standard kinetic terms for gauge fields but there are also CS terms like (1.3.8). Moreover there are always matter chiral superfields in the adjoint and bifundamental representations, but there could be also flavors in the sense previously discussed. The relation between these classes of field theories and the conical singularity is far from trivial and it is not completely understood yet, but there are some facts supporting the conjectured duality which we will point out step by step.

The full Lagrangian for the complete UV-quiver theory in superspace formulation is

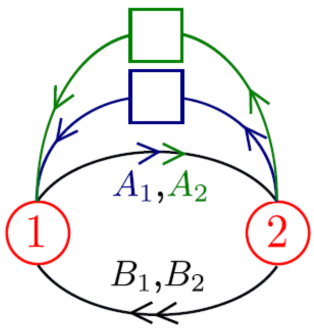
$$\begin{aligned}\mathcal{L} = &\int \mathrm{d}^4\theta \sum_{\Phi_{ij}} \mathrm{Tr}\, \Phi_{ij}^{\dagger} e^{-V_i} \Phi_{ij} e^{+V_j} + \sum_{i=1}^{G} \zeta_i \int \mathrm{d}^4\theta\, \mathrm{Tr}\, V_i + \\ &+ \sum_{i=1}^{G} k_i \int \mathrm{d}^4\theta \int_0^1 \mathrm{d}t\, \mathrm{Tr}\left(V_i \bar{D}^{\alpha} (e^{tV_i} D_{\alpha} e^{-tV_i})\right) - \\ &- \sum_{i=1}^{G} \left(\frac{1}{g_i^2} \int \mathrm{d}^2\theta\, \mathrm{Tr}\, W_i W^i + \int \mathrm{d}^2\theta W(\Phi, p, q) + c.c. \right) + \\ &+ \int \mathrm{d}^4\theta \sum_{p_{\hat{k}i}, q_{j\hat{k}}} \left(\mathrm{Tr}\, q_{j\hat{k}}^{\dagger} e^{-V_j} q_{j\hat{k}} e^{+V_{\hat{k}}} + \mathrm{Tr}\, p_{\hat{k}i}^{\dagger} e^{-V_{\hat{k}}} p_{\hat{k}i} e^{+V_i} \right),\end{aligned} \tag{4.2.1}$$

where the notation is borrowed from the first chapter. The first line involves the kinetic term for matter chiral fields in the bifundamental and Fayet-Iliopoulos terms for the $U(1)$ factors in the gauge group. The second line is the Chern-Simons term that give rise to (1.3.7) in components. In the third line, the first term is the kinetic term for gauge fields while the second term is the superpotential[2] one. The last line encodes the kinetic term and the real mass[3] term for the chiral flavor fields p and q.

We remind that k_i are the CS-levels, which are integers labeling every gauge group factor, i.e. $\prod_{i=1}^{G} U(N_i)_{k_i}$. The role played by CS-levels is far from trivial and it seems that for a correct duality between the gauge theory and the M-Theory they must sum to zero, i.e. $\sum_i k_i = 0$. Notice that if we rescale vector multiplets V_i with the respective dimensionful Yang-Mills coupling constants g_i as $V_i \to g_i V_i$, then "topological masses" $m_i = g_i^2 k_i$ arise for the fields component in the vector supermultiplets: at large coupling constants with finite CS-levels these masses are big. Hence, as stated in [40], at low energies compared with m_i these fields components can be integrated out leaving only the CS-terms in the action. So we expect to find "pure" CS theories, i.e. with kinetic term for gauge fields switched-off, in the low-energy region. Besides, CS-levels can be interpreted as discrete coupling constants since the dimensionless effective coupling of these theories turns out to be $g_{eff}^2 \sim \frac{1}{k}$: this has fundamental implication in the large-N argument. Indeed, the 't Hooft coupling is shown to be $\lambda = \frac{N}{k}$ and hence the holographic analysis is allowed when $k \ll N$, with $N \gg 1$: this is actually our situation.

4.3 The Q^{111} quiver theory

In order to give a concrete idea of the rather abstract structure of quivers we focus on the theory treated in this thesis, namely the Q^{111} model. This is the "far UV" theory supposed to RG-flow towards an IR fixed point where it becomes a SCFT, which is conjectured to be dual to M-Theory settled on the near-horizon geometry $AdS_4 \times Q^{111}$. The quiver diagram is the one borrowed from [6], namely



[2] The $W(\Phi, p, q)$ in (4.2.1) consists of an "unflavored" term depending on Φ and a coupling between the chiral bifundamental fields Φ and the flavor fields (q, p).

[3] Recall (1.3.9): here the external background fields are $V_{\hat{k}}$.

and it is characterized by:

- two gauge nodes (the $G = 2$ circles in the figure) so that gauge group is $U_1(N_1) \times U_2(N_2)$. Each of them is labelled by a CS-level, namely $\vec{k} = (k_1, k_2)$. Recall that $\sum_i k_i = 0$ is a necessary condition for the conjectured duality and hence it must be $k_1 = -k_2$: in the case at hand we take them to be $\vec{k} = (0, 0)$. We choose equal ranks $N_1 = N_2 = \tilde{N}$ for the gauge group factors so that for $G = 2$ we have

$$\prod_{i=1}^{G} U_i(N_i)_{k_i} = U_1(\tilde{N})_0 \times U_2(\tilde{N})_0. \tag{4.3.1}$$

  For reasons that will be clarified in a moment, we will focus on the abelian case where (4.3.1) is broken to $U(1)^{2\tilde{N}}$ so that the theory actually consists of $\tilde{N}$ copies of the same abelian field theory with gauge group

$$U_1(1)_0 \times U_2(1)_0. \tag{4.3.2}$$

- Two bi-arrows connecting the gauge nodes, corresponding to four bifundamental fields (A_1, A_2, B_1, B_2). The A-fields go from node 1 to node 2 whereas B-fields have opposite orientation. In terms of representations of the gauge group, we can say that $A_a \in (\tilde{N}, \underline{\tilde{N}})$ while $B_a \in (\underline{\tilde{N}}, \tilde{N})$.

- Two flavor nodes (the $\tilde{G} = h_1 + h_2 = 1 + 1 = 2$ squares in figure), where the flavoring at hand consists of two $U(1)$ flavor groups coupled to A-fields. The q-fields are arrows connecting a source gauge node and a target flavor node while the p-fields connect a source flavor node and a target gauge node. The former live in the (anti)fundamental of the (flavor)gauge group, while the latter live in the (anti)fundamental of the (gauge)flavor group. In terms of representations of the gauge group, $q_a \in (1, \tilde{N})$ while $p_a \in (\underline{\tilde{N}}, 1)$.

Schematically, the charge content for the Q^{111} quiver is

	A_i	B_i	p_i	q_i
$U_1(1)_0$	1	-1	-1	0
$U_2(1)_0$	-1	1	0	1
$U(1)_{flavors}$	0	0	1	-1

(4.3.3)

We previously mentioned that the quiver diagram does not encode every information about the field theory. Indeed, the superpotential is added by hand and the same is true for parameters like Fayet-Iliopoulos and real masses. About the latter, we will see that the Q^{111} model is characterized by one FI and one real mass. Speaking of the former, the superpotential in (4.2.1) in this case requires some attention because of flavors. Typically, $W(\Phi)$ is a trace of product of chiral UV-quiver fields Φ_a: the "flavoring procedure" leads to a change in the typical superpotential. It consists in choosing a subset of the Φ_a bifundamental UV-quiver fields and introducing h_a pairs of chiral multiplets (q_a, p_a) coupled to them by the superpotential

$$W(\Phi, p, q) = W_0(\Phi) + \sum_{a \in \text{flavored}} p_a \Phi_a q_a = W_0 + \text{Tr}\, p_{\hat{k}i} \Phi_{ij} q_{j\hat{k}}, \tag{4.3.4}$$

W_0 being the unflavored term. For the Q^{111} quiver one starts from

$$W_0 = \text{Tr}\left(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1\right) \tag{4.3.5}$$

and then flavors the A-fields as described above, leading to the flavored superpotential

$$W = \text{Tr}\left(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1 + p_1 A_1 q_1 + p_2 A_2 q_2\right). \tag{4.3.6}$$

As we will see, inclusion of flavors is necessary for the correspondence with M-Theory to hold. In the Q^{111} model, even if we are interested in a particular branch of the moduli space such that $\langle p_{1,2} \rangle = \langle q_{1,2} \rangle = 0$, the presence of flavors is crucial both for the brane interpretation and for the characterization of the moduli space complex structure. Having identified the field content of the Q^{111} model we are ready to study its moduli space.

4.4 The moduli space of Q^{111}

The moduli space of a supersymmetric theory is the space of inequivalent vacua, i.e. vacuum configurations that cannot be mapped into each other using gauge transformations. This can be found minimizing a function that we call "scalar potential" $\mathcal{V}$ and then quotienting by the gauge group action in order to identify gauge-equivalent configurations. In a classical vacuum configuration, fermions are vanishing while bosonic scalar fields may acquire constant VEVs. We can identify the scalar potential of (4.2.1) from a "theta-expansion" of superfields: it consists in two pieces $\mathcal{V} = \mathcal{V}_D + \mathcal{V}_F$ that we call D-term and F-term contribution respectively. After integrating out auxiliary fields, and forgetting flavors for a moment, these take the component form

$$\mathcal{V}_F = \sum_{\Phi_{ij}} \left| \frac{\partial W}{\partial \Phi_{ij}} \right|^2 \tag{4.4.1}$$

and

$$\mathcal{V}_D = \sum_i g_i^2\, \text{Tr}\left(\zeta_i + k_i \sigma_i - \mu_i(\Phi)\right)^2 + \sum_{\Phi_{ij}} \text{Tr}(\sigma_i \Phi_{ij} - \Phi_{ij}\sigma_j)^\dagger (\sigma_i \Phi_{ij} - \Phi_{ij}\sigma_j), \tag{4.4.2}$$

where σ_i are scalar components[4] of the vector superfields while $\mu_i(\Phi)$ can be expressed as

$$\mu_i(\Phi) = \sum_j \left(\sum_{\Phi_{ij}} \Phi_{ij}\Phi_{ij}^\dagger - \sum_{\Phi_{ij}} \Phi_{ji}^\dagger \Phi_{ji} \right). \tag{4.4.3}$$

[4]Maybe we should point out that they are $\tilde{N} \times \tilde{N}$ matrices.

In the case at hand (4.4.3) reads

$$\begin{aligned} A_1 A_1^\dagger + A_2 A_2^\dagger - B_1^\dagger B_1 - B_2^\dagger B_2 = \mu_1, \\ B_1 B_1^\dagger + B_2 B_2^\dagger - A_1^\dagger A_1 - A_2^\dagger A_2 = \mu_2. \end{aligned} \tag{4.4.4}$$

It is easy to check that $\sum_i \mu_i(\Phi) = 0$ for a quiver theory: this is because each quiver field Φ appears exactly twice in the sum, once with a plus (when it exits from the source node) and once with a minus (when it enters in the target node). In other words, chiral fields are not charged under the "diagonal" $U_{diag}(1)$ and this translates into the fact that the vector associated to this gauge group, which we call "diagonal photon" $\mathcal{A}_{diag} = \sum_{i=1}^{G} \mathcal{A}_i$, is decoupled from matter[5].

Notice that the scalar potential is a sum of squares, so vacua can be found looking for the vanishing of both (4.4.1) and (4.4.2). More precisely, a proper supersymmetric vacuum must satisfy the following conditions, also called vacuum equations:

$$\begin{aligned} \partial_{\Phi_{ij}} W = 0, &\qquad \text{F-term}, \\ \mu_i(\Phi) = \zeta_i \mathbb{1}_{\tilde{N}} + k_i \sigma_i, &\qquad \text{D-term}, \\ \sigma_i \Phi_{ij} - \Phi_{ij} \sigma_j = 0, &\qquad \text{“Extra D-term”}, \end{aligned} \tag{4.4.5}$$

where in the second condition $k_i \sigma_i$ are not summed over the common index. The reason why we called the third condition in (4.4.5) "extra D-term" is that it arises in three-dimensional theories like Q^{111}, as opposed to the four-dimensional case whose moduli space is characterized by F-term and D-term conditions[6].

The solution to the F-term itself is an important object called "master space", while the full solution to (4.4.5) constitutes the total moduli space of the field theory $\mathcal{M}_{quiver}$. The latter is usually built quotienting the former by some subgroup of the gauge group: this is because we want to identify inequivalent vacua and hence we must mod by transformations mapping vacua into vacua. Let us stress that we are interested in a branch $\mathcal{M} \subset \mathcal{M}_{quiver}$ such that $\mathcal{M} = Sym^{\tilde{N}} CY_4$: this is because the moduli space $\mathcal{M}$ should be matched with the moduli space of supergravity in order for holography to hold. This branch is characterized by chiral flavors having vanishing VEV while the hermitian scalars in the vector supermultiplets are diagonalized using gauge transformations, namely

$$\langle q_a \rangle = 0 = \langle p_a \rangle, \qquad \sigma_i = diag(\sigma_n^{(i)}), \qquad n = 1, ..., \tilde{N}. \tag{4.4.6}$$

Furthermore, one can choose $\sigma_i = \sigma$ so that the "Extra D-term" in (4.4.5) is immediately satisfied provided that the chiral quiver bifundamental fields Φ_{ij} take diagonal VEVs too. The

[5] Recall that the vector supermultiplet, for example (1.3.3), has vector component $V = ... - \theta\gamma^\mu\bar{\theta}\mathcal{A}_\mu + ...$ We can in principle dualize $\mathcal{A}$ into a scalar τ, but the former must be decoupled from matter. So, only the diagonal combination admits a dualization into a scalar τ: this turns out to be crucial for the identification of the correct moduli space.

[6] In four-dimensional theories there are no CS-levels but the structure of F-term and D-term is the very same.

primary effect of such diagonal VEVs for $\sigma_i = \sigma$ is that the gauge group of the Q^{111} theory gets broken to an abelian subgroup:

$$U_1(\tilde{N}) \times U_2(\tilde{N}) \to \Big(U_1(1) \times U_2(1)\Big)^{\tilde{N}}. \tag{4.4.7}$$

The consequence is a factorization of the problem: the non-abelian theory on the branch defined by (4.4.6) becomes $\tilde{N}$ copies of the abelian $U(1)^2$ quiver theory. We expect that the moduli space of the $U(1)^2$ quiver theory reproduces CY_4 so that the branch (4.4.6) is actually $\mathcal{M} = Sym^{\tilde{N}} CY_4$. Otherwise stated, while the moduli space for the $U(1)^2$ quiver should reproduce the moduli space of 1 M2-brane probing CY_4, the latter being CY_4 itself, the moduli space for $U(1)^{2\tilde{N}}$ should reproduce the moduli space of $\tilde{N}$ M2-branes probing CY_4, the latter being $Sym^{\tilde{N}} CY_4$. In what follows we will focus on this branch for the abelian Q^{111} theory.

4.4.1 The abelian branch for Q^{111}

The F-term condition in (4.4.5) defines the master space as an affine variety

$$\mathcal{F} = \{\Phi_a \mid \partial_{\Phi_a} W = 0\} \subset \mathbb{C}^A, \tag{4.4.8}$$

where A, with a little abuse of notation, is the number of arrows in the quiver.

When the theory is abelian, the F-term is trivial because the superpotential is identically zero. For instance, considering vanishing VEVs for chiral flavors, (4.3.6) is a trace of the difference of two terms: since the theory is abelian, quiver fields are actually complex numbers and hence they commute giving trivially $W = 0$. So, in the abelian Q^{111} model we have $\mathcal{F} = \mathbb{C}^4$ parametrized by (A_1, A_2, B_1, B_2). This is exactly the same master space of the unflavored case since we are on a branch with vanishing VEVs for chiral flavor fields q, p.

On the other hand, the D-term is more complicated. In this abelian branch, (4.4.4) takes the form

$$\begin{aligned} |A_1|^2 + |A_2|^2 - |B_1|^2 - |B_2|^2 &= \mu_1, \\ -|A_1|^2 - |A_2|^2 + |B_1|^2 + |B_2|^2 &= \mu_2, \end{aligned} \tag{4.4.9}$$

where now

$$\mu_1 = \zeta_1 + k_1\sigma, \qquad \mu_2 = \zeta_2 + k_2\sigma. \tag{4.4.10}$$

At this stage it seems that since $k_1 = 0 = k_2$ the quiver condition $\sum_i \mu_i = 0$ is equivalent to imposing $\zeta_1 = -\zeta_2$ so that we have only one independent equation:

$$|A_1|^2 + |A_2|^2 - |B_1|^2 - |B_2|^2 = \zeta, \qquad \zeta = \zeta_1 = -\zeta_2. \tag{4.4.11}$$

However we must take into account a slight modification of D-terms due to loop-corrections of Chern-Simons levels k_i: this should be interpreted as a quantum correction of the "classical" moduli space.

Let us begin with saying that CS-levels k_i get shifted because of fermionic masses. More precisely, as reviewed in [36], integrating out massive fermions give rise to CS-terms at loop level and hence the "effective" levels are shifted with respect to the "bare" ones[7]. This immediately translates into a shift of the FI parameters and hence the first modification to (4.4.9) and (4.4.10) is

$$\begin{aligned} |A_1|^2 + |A_2|^2 - |B_1|^2 - |B_2|^2 &= \mu_1, \\ -|A_1|^2 - |A_2|^2 + |B_1|^2 + |B_2|^2 &= \mu_2, \end{aligned} \tag{4.4.12}$$

where

$$\begin{aligned} \mu_1 = \zeta_1^{\text{eff}}(\sigma) &= \zeta_1^{\text{bare}} + \Delta\zeta_1(\sigma), \\ \mu_2 = \zeta_2^{\text{eff}}(\sigma) &= \zeta_2^{\text{bare}} + \Delta\zeta_2(\sigma). \end{aligned} \tag{4.4.13}$$

The exact expression of $\Delta\zeta_i(\sigma)$ can be found for example in [2, 35]: we will not derive the whole procedure but we should highlight the crucial steps. First of all, in the case at hand the massive fermions that give rise to CS-shifts are most of the fermionic components of chiral superfields: A_a, B_a, q_a, p_a, $a = 1, 2$.

For bifundamental fields $\Phi_a = A_a, B_a$ in the quiver, the acquired mass is due to the scalar components of gauge vector superfields and it is given by

$$\delta M[(\Phi_{ij})_l^n] = \sigma_n^{(i)} - \sigma_l^{(j)}, \qquad n, l = 1, ..., \tilde{N}, \tag{4.4.14}$$

where the notation is the one of (4.4.6) and remember that we have chosen $\sigma^{(i)} = \sigma^{(j)} = \sigma$. So, only the off-diagonal components give rise to CS-shifts when integrated out: notice that shifts always depend on VEVs of the scalar components in gauge vector multiplets.

For chiral flavor fields the situation is a bit different because we have to take into account real masses. Indeed, from the last line of (4.2.1) we can obtain a total real mass

$$\delta M[(q_{j\hat{k}})_n] = \sigma_n - m_{\hat{k}}, \qquad \delta M[(p_{\hat{k}i})_n] = -\sigma_n + m_{\hat{k}}. \tag{4.4.15}$$

To be more explicit, the situation for Q^{111} is the following

$$\delta M[A, B] = \sigma_n - \sigma_l, \qquad \delta M[q_1] = \sigma_n - m_1 = -\delta M[p_1], \qquad \delta M[q_2] = \sigma_n - m_2 = -\delta M[p_2]. \tag{4.4.16}$$

At this stage, since σ_n are arbitrary VEVs we can freely redefine all of them to be $\sigma_n + m_1$ so that (4.4.16) becomes

$$\delta M[A, B] = \sigma_n - \sigma_l, \quad \delta M[q_1] = \sigma_n = -\delta M[p_1], \quad \delta M[q_2] = \sigma_n + m = -\delta M[p_2], \quad m = m_1 - m_2. \tag{4.4.17}$$

Furthermore, in the branch we are studying we repeat that our theory consists of $\tilde{N}$ copies of an abelian $U(1)^2$ quiver with diagonalized bifundamental fields. So, in (4.4.17) we chose one particular n and work with

$$\delta M[A, B]_{\text{diagonal}} = 0, \qquad \delta M[(q_1)] = \sigma = -\delta M[p_1], \qquad \delta M[q_2] = \sigma + m = -\delta M[p_2], \tag{4.4.18}$$

[7] In particular, we point out that the effective CS-levels are no more vanishing.

where now σ is a real parameter.

Another crucial aspect of FI quantum shifts $\Delta\zeta_i(\sigma)$ is that they inherit the property $\sum_i \Delta\zeta_i(\sigma) = 0$ from the "bare relation" $\sum_i k_i = 0$. So, summing the two equations in (4.4.13) we get again $\zeta_1^{\text{bare}} = -\zeta_2^{\text{bare}} = \zeta^{\text{bare}} = \zeta$ so that there is one independent FI parameter. Moreover it is clear that $\zeta_1^{\text{eff}}(\sigma) = -\zeta_2^{\text{eff}}(\sigma) = \zeta^{\text{eff}}(\sigma) = \zeta(\sigma)$. Now, considering the effective shifts, whose general formula can be found in [2, 35], the "bare" D-term condition (4.4.11) becomes

$$|A_1|^2 + |A_2|^2 - |B_1|^2 - |B_2|^2 = \zeta(\sigma), \qquad \zeta(\sigma) = \zeta + \frac{1}{2}|\sigma| + \frac{1}{2}|\sigma + m|. \tag{4.4.19}$$

The moduli space interpretation

The relation (4.4.19) encodes the structure of the moduli space. If we consider the master space $\mathcal{F} = \mathbb{C}^4$ parametrized by (A_1, A_2, B_1, B_2) and $\zeta(\sigma) = 0$ in (4.4.19) then we get exactly (2.4.7), which is the conifold of the Klebanov-Witten four-dimensional theory in [39]. Indeed, the gauge-invariant combinations[8] that we can build are

$$U = A_1 B_1, \qquad V = A_2 B_2, \qquad X = A_1 B_2, \qquad Y = A_2 B_1, \tag{4.4.20}$$

and actually satisfy the conifold equation $UV - XY = 0$. This conifold is the singular cone $C(T^{11})$ and it is a Calabi-Yau three-fold CY_3. There, D3-branes are placed on a background geometry $\mathbb{R}^{1,3} \times C(T^{11})$ and the corresponding field theory is studied. With only one brane probing that geometry, the moduli space of the field theory is shown to correspond to CY_3. Having $\tilde{N}$ such branes naturally gives $Sym^{\tilde{N}} CY_3$ in the case of non-coincident D3-branes.

Now consider $\zeta(\sigma) \neq 0$ with fixed σ: equation (4.4.19) becomes exactly[9] (2.4.9), so that the considered (sub)branch for the moduli space is a resolved version of CY_3. Finally, since σ parametrizes $\mathbb{R}$, F-term and D-term actually describe a resolved CY_3 fibered over $\mathbb{R}$. This is a seven-manifold, so we are lacking one direction to reproduce a CY_4. Remember at this point that we always have one scalar photon τ in our abelian $U(1)^2$ theory. Indeed, since $\sum_i \mu_i = 0$ there exits a $U(1)_{diag}$ under which matter is uncharged, i.e. there is no coupling between the diagonal photon $\mathcal{A}_{diag} = \sum_i \mathcal{A}_i$ and matter fields. So we can freely dualize the diagonal photon into a scalar photon τ. More precisely, the diagonal photon is only coupled to another gauge vector $\mathcal{B}$ via a Chern-Simons interaction

$$\frac{\tilde{k}}{G} \int \mathcal{B} \wedge \mathcal{F}_{diag}, \tag{4.4.21}$$

where $\mathcal{F}_{diag} = d\mathcal{A}_{diag}$, $\mathcal{B} = \tilde{k}^{-1} \sum_i k_i \mathcal{A}_i$ and $\tilde{k} = gcd\{k_i\}$. The equation of motion for $\mathcal{B}$ is

$$\mathcal{B} = G\tilde{k}^{-1} \star_3 d\mathcal{A}_{diag}. \tag{4.4.22}$$

[8] With respect to the charges in (4.3.3).

[9] Actually, the ζ here and there are not identified but we want to stress that the conifold equation gets deformed.

Now, as a standard procedure in electromagnetic duality, we can introduce a Lagrange multiplier term $-\int d\tau \wedge \mathcal{F}_{diag}$, τ being a scalar field, and consider $\mathcal{F}_{diag}$ as an unconstrained field, i.e. it is no more the field strength of $\mathcal{A}_{diag}$. Integrating out $\mathcal{F}_{diag}$ then leads to the identification

$$\mathcal{B} = G\tilde{k}^{-1}d\tau. \tag{4.4.23}$$

Comparing (4.4.22) and (4.4.23) we get the relation

$$d\tau = \star_3 d\mathcal{A}_{diag} = \star_3 \mathcal{F}_{diag} \qquad \longleftrightarrow \qquad \partial_\mu \tau = \epsilon_{\mu\nu\rho}\mathcal{F}^{\nu\rho}_{diag}, \tag{4.4.24}$$

which is exactly the conserved[10] current in (1.3.5). As a consequence of dualization, the scalar photon τ inherits a periodic behavior from the flux quantization condition of $\mathcal{F}_{diag} = d\mathcal{A}_{diag}$, namely

$$\int d\mathcal{A}_{diag} = 2\pi G n, \qquad n \in \mathbb{Z}. \tag{4.4.25}$$

So, if we have a $U(1)^G$ quiver we are always sure to have a diagonal photon that can be dualized into a scalar photon: this τ parametrizes a $U(1)$ due to its periodic behavior. Hence, the description of CY_4 can be completed: the branch we have considered is a $U(1)$ fibration, parametrized by τ, of a seven-manifold, the latter being a CY_3 fibered over a real line parametrized by σ. Since we have $\tilde{N}$ copies of the same abelian quiver, one finds $\tilde{N}$ copies of (CY_3, σ, τ) so that the moduli space is given by $\mathcal{M} = Sym^{\tilde{N}} CY_4 \subset \mathcal{M}_{quiver}$. Depending on the presence or not of effective Fayet-Iliopoulos parameters, the CY_4 is a resolved or singular version of $C(Q^{111})$ respectively.

4.4.2 The monopole method

The method previously discussed is the semiclassical computation of the moduli space, involving loop-corrected quantities. As we mentioned in the introduction to this chapter, we can obtain the same moduli space with a different strategy relying on the so called "monopole operators". They are very delicate objects and a complete introduction on them is beyond the aim of this thesis: we shall address the interested reader to [6, 35] and references therein. By the way, in what follows we want to give an operative definition of such monopoles and the related method for at least three reasons:

- quantum corrections to moduli space are taken into account. Moreover, as stated in [35] it gives a one-loop exact formulation of the moduli space;
- monopole operators seem to play a crucial role when dealing with flavored quivers;

[10] If it is not clear, the conservation of $J = dx^\mu J_\mu$ is due to the equation of motion for $\mathcal{F}_{diag}$. Indeed, let us call $J_\mu = \partial_\mu \tau$. Then $\partial^\mu J_\mu = \epsilon_{\mu\nu\rho}\partial^\mu \mathcal{F}^{\nu\rho}_{diag} = 0$ or in forms $dJ = d \star_3 \mathcal{F}_{diag} = 0$.

- we will use this method in order to match the complex structure of (one among the $\tilde{N}$ copies of) the abelian quiver moduli space $\mathcal{M}$ with the singular $CY_4 = C(Q^{111})$, which is actually the moduli space of one M2-brane moving on the singular cone.

The monopole method goes as follows. We define a chiral $\mathcal{N} = 2$ multiplet Ψ such that its lowest component is $\sigma + i\tau$, where σ and τ are exactly the same objects of the previous subsection. The scalar photon τ can be interpreted as a phase, i.e. it parametrizes a circle, due to its periodicity. At this stage monopole operators can be defined as $T^{(n)} = \exp(n\Psi)$, where n is the one of (4.4.25): notice that they are actually chiral superfields just like Ψ. We will always consider $n = \pm 1$ in this thesis and hence we define $T = T^{(1)}, \tilde{T} = T^{(-1)}$. The introduction of two new chiral fields $(T, \tilde{T})$ produces at least two effects:

- the master space (4.4.8) is clearly augmented because we have two new chiral fields, namely if we call $\mathcal{F}_T$ the master space in the monopole method then surely $\mathcal{F}_T \subset \mathbb{C}^{A+2}$;
- the D-term vacuum equations slightly change and hence one should include some additional condition on monopole operators in order to find the very same moduli space. Moreover, since the master space is enlarged we must mod by the full gauge group $U(1)^G$.

To clarify this point, in the semiclassical computation we had a master space $\mathcal{F} = \mathbb{C}^4$: this is the same for the three-dimensional Q^{111} theory and the four-dimensional Klebanov-Witten theory. There, we imposed the D-term condition and modded by $U(1)$: this two steps in sequence are usually indicated with $\mathcal{F}//U(1)$. This particular quotient "//" is called "Kähler quotient": it consists in imposing D-term condition and modding by a gauge group[11] . In our case we found $\mathcal{F}//U(1) = CY_3$ and the CY_4 was built using (σ, τ) as fibers. Here, with monopoles the master space is augmented to $\mathcal{F}_T = \mathbb{C}^6$ but we cannot use (σ, τ) with the same interpretation as before because they are "inside" monopole operators. So, in order to get the correct CY_4, at least with the right dimension, we must mod $\mathcal{F}_T$ by the full gauge group, namely $\mathbb{C}^6//U(1)^2$: this should reproduce the CY_4. A simpler but operative way to say this is: gauge-invariant operators built using UV-quiver bifundamental fields and monopoles should give a suitable parametrization of CY_4.

In the case of a flavored abelian quiver theory, if we introduce h_a pairs of flavors (q_a, p_a) coupled to some chiral fields Φ_a then the conjectured constraint on monopole operators reads

$$T\tilde{T} = \prod_{a \in \text{flavored}} \Phi_a^{h_a}. \tag{4.4.26}$$

This should be consistent with quiver charges and it can be shown that, due to flavoring, monopoles pick up a charge

$$Q[T^{(n)}] = \frac{|n|}{2} \sum_{a \in \text{flavored}} h_a Q[\Phi_a]. \tag{4.4.27}$$

[11] The Kähler quotient can also be seen as a quotient by the complexified gauge group: we do not enter in details, but it is sufficient to know that $dim_{\mathbb{C}}(A//B) = dim_{\mathbb{C}}A - dim_{\mathbb{C}}B$.

Consequently, the master space in the monopole method is obtained by adding $(T,\tilde{T})$ to the set of chiral fields, together with the "quantum F-term relation" (4.4.26), namely

$$\mathcal{F}_T = \{\Phi_a, T, \tilde{T} \mid \partial_{\Phi_a} W = 0, T\tilde{T} = \prod_{a\in\text{flavored}} \Phi_a^{h_a}\} \subset \mathbb{C}^{A+2}. \tag{4.4.28}$$

At this point, the moduli space[12] is obtained by a Kähler quotient of (4.4.28) with respect to the full gauge group $U(1)^G$, namely

$$\mathcal{M} = \mathcal{F}_T//U(1)^G. \tag{4.4.29}$$

Notice that from (4.4.28) we see that (4.4.29) has complex dimension $A+2-G$ and hence in order to reproduce the correct dimension of CY_4 it must be $A = G+2$. As we mentioned, even if this construction seems quite abstract it effectively reproduce the complex structure of CY_4 in a rather easy way[13]: let us specialize this machinery to the abelian Q^{111} quiver. The flavoring that gives (4.3.6) consists of $h_1 = h_2 = 1$ flavor pairs $(q_1,p_1),(q_2,p_2)$ coupled to A_1 and A_2 chiral fields respectively. Hence, the master space (4.4.28) for the Q^{111} quiver using the monopole method is

$$\mathcal{F}_T^{Q^{111}} = \{\Phi_a, T, \tilde{T} \mid \partial_{\Phi_a} W = 0, T\tilde{T} = A_1 A_2\} \subset \mathbb{C}^6, \tag{4.4.30}$$

which has to be modded by $U(1)^2$ in order to get the moduli space $\mathcal{M} = C(Q^{111})$. We recall that this method does not "see" resolutions, so we expect to find at most the singular cone in the moduli space. Nevertheless, we also expect that a "sub-branch" of this CY_4 reproduces the singular cone CY_3, namely $C(T^{11})$: this is indeed what happens. First of all, the charge matrix for UV-quiver fields and monopoles reads

$$\begin{array}{c|cccc|cc} & A_i & B_i & p_i & q_i & T & \tilde{T} \\ \hline U_1(1)_0 & 1 & -1 & -1 & 0 & 1 & 1 \\ U_2(1)_0 & -1 & 1 & 0 & 1 & -1 & -1 \end{array} \tag{4.4.31}$$

where the monopole charges are easily computed using (4.4.27). Then we have to build gauge-invariant combinations: there are eight of them, namely

$$\begin{aligned} w_1 &= \tilde{T}B_2, \quad w_2 = TB_1, \quad w_3 = A_1B_1, \quad w_4 = A_2B_2, \\ w_5 &= A_1B_2, \quad w_6 = TB_2, \quad w_7 = \tilde{T}B_1, \quad w_8 = A_2B_1. \end{aligned} \tag{4.4.32}$$

Now, the cone over the Sasaki-Einstein base Q^{111} can be parametrized using a set of eight affine coordinates $\{w_1, ..., w_8\}$ satisfying[14]

$$\begin{aligned} &w_1w_2 - w_3w_4 = w_1w_2 - w_5w_8 = w_1w_2 - w_6w_7 = 0, \\ &w_1w_3 - w_5w_7 = w_1w_6 - w_4w_5 = w_1w_8 - w_4w_7 = 0, \\ &w_2w_4 - w_6w_8 = w_2w_5 - w_3w_6 = w_2w_7 - w_3w_8 = 0. \end{aligned} \tag{4.4.33}$$

[12]We are talking about one of the $\tilde{N}$ copies. Alternatively, we can think that this moduli space is the one for 1 M2-brane probing CY_4.

[13]By identification of gauge-invariant operators and complex coordinates.

[14]See for example [34].

It is easy to check that the gauge-invariant operators in (4.4.32) satisfy the constraints (4.4.33), provided that we also use the "quantum F-term relation" $T\tilde{T} = A_1 A_2$. So, we found that the moduli space of the $U(1)^2$ quiver is $\mathcal{M} = C(Q^{111})$: in the case of $U(1)^{2\tilde{N}}$ we clearly obtain the symmetrized version $Sym^{\tilde{N}}C(Q^{111})$. On the M-Theory side, since the moduli space of $\tilde{N}$ mobile M2-branes on the transverse conical background $C(Q^{111})$ is $Sym^{\tilde{N}}C(Q^{111})$, the matching between the two sides is completed. Moreover, notice that if we consider the relations not involving monopoles among (4.4.32) we are left with $\{w_3, w_4, w_5, w_8\}$. Using (4.4.33) we see that they satisfy $w_3 w_4 - w_5 w_8 = 0$: recalling (2.4.5), this is exactly the equation for the three-dimensional conifold $C(T^{11})$. At this stage we have only sketched the procedure: a more consistent matching will be done in the last chapter, were we will introduce toric geometry and the dimensional reduction from M-Theory to type IIA. This will let us see how external parameters, like FI and real masses, are mathced with resolution parameters of the M-Theory background and moreover will provide an explanation of flavors in terms of D6-branes.

Chapter 5

Holographic Effective Field Theory

The purpose of this chapter is to illustrate the technologies introduced in [1] and developed in [2] to construct Holographic Effective Field Theories (HEFT), namely effective theories for strongly-coupled (S)CFTs admitting holographic dual descriptions. Let us recap the holography plot. The strongly coupled regime of a (supersymmetric) field theory is in general very difficult to treat. However, we have seen that this regime typically admits a dual description in String Theory or M-Theory: strictly speaking it is a supergravity (SUGRA), namely a low-energy version of String or M-Theory. Therefore, the low-energy dynamics of the field theory should be codified in an effective Lagrangian whose construction is necessarily based on the holographic dual theory. More precisely, the effective Lagrangian should describe the dynamics of "moduli fields", i.e. massless modes parametrizing the moduli space of vacua. Moreover, it is expected to be a nonlinear sigma model, i.e. a Lagrangian with nontrivial kinetic terms, that realizes the dynamics of moduli fields in a "geometrical way". Indeed, the nontriviality arises because of an overall metric: the one over the moduli space, which itself depends on moduli. As we mentioned in the introduction of the previous chapter, this moduli space is actually $\mathcal{M}_{SCFT}$. Even though we know its complex structure, we cannot compute the metric on it and hence we cannot obtain an effective field theory using pure field-theoretical tools. However, if we can check that $\mathcal{M}_{SCFT} \sim \mathcal{M}_{SUGRA}$ then we can switch to the holographic description and compute the metric on $\mathcal{M}_{SUGRA}$. Indeed, in the M-Theory side we have a crucial condition that we lack in the field theory side: the Ricci-flatness, which is required for every stable background geometry. Now, there must be something in the holographic description that correspond to moduli fields such that we can build a dual effective theory. As we shall see, the background geometry itself provides some dual moduli: for instance, branes positions on the cone and resolution parameters give rise to them. At this stage, the idea is that geometric moduli correspond to scalar fields, which are the lowest components of chiral or vector superfields in the field theory side. Since these moduli parametrize the moduli space of M-Theory vacua $\mathcal{M}_{SUGRA}$, the effective theory is an Holographic Effective Field Theory (HEFT) describing the physics of moduli as a nonlinear sigma model, whose non-trivial curved metric is the one on $\mathcal{M}_{SUGRA}$.

Summarizing, in order to find the HEFT there are in general three steps to follow:

- Identification of moduli both from the field theory side and the gravity side.
- Consistency check on the moduli space, namely $\mathcal{M}_{SUGRA} \sim \mathcal{M}_{SCFT}$.
- Holographic Lagrangian construction. Its explicit form can be found expanding a supergravity action, which is implicitly defined by a function called "Kähler potential": this will be presented in this chapter. Then, truncating it to second-derivative order leads to the so called nonlinear sigma model. This model is characterized by non-trivial kinetic terms, whose nontriviality is due to a curved overall metric. This metric is actually the one on the space of M-theory vacua, which in turn is equivalent, at least from the complex point of view, to the space of field theory vacua thanks to the consistency check. The main difference is that the metric cannot be calculated using the latter because of strong coupling issues: it is exactly for this reason that we must switch to the holographic weakly coupled description.

In the end, the HEFT is completely fixed by geometry.

5.1 Topology, Kähler moduli and harmonic forms

In this section we are going to use some concepts introduced in the chapter dedicated to complex geometry. Recall that our background cone $C(Y)$ is a Calabi-Yau eight-dimensional manifold, i.e. it is Ricci-flat and Kähler, with a Sasaki-Einstein seven-dimensional base. There is obviously a singularity at the tip, but we can consider resolutions: the singular point is effectively replaced by an higher-dimensional locus called "exceptional set" and the result is a resolved cone X. Even if we call it a resolved cone, it lacks a crucial characteristic of cones: its metric is no more invariant under dilatations. This fact has dramatic consequences in the dual field theory: the conformal simmetry, at least dilatations, seems lost. It is exactly for this reason that we ask for supergravity solutions with asymptotically $AdS \times Y$ behavior, so that the dual field theory "returns" conformal at high energy. Hence, one should check that the CY metric on the resolved cone approaches the one over the singular cone asymptotically: indeed, recall that these kind of vacua with $\langle O \rangle = \langle resolution \rangle \neq 0$ are dual to (S)CFTs where the conformal symmetry is "restored" in the UV, i.e. at energies well above $\langle O \rangle$, otherwise the conformal symmetry is spontaneously broken by $\langle O \rangle$. There is a theorem, similar to the Calabi-Yau one, which states that one such asymptotically conical metric always exists and moreover it is unique[1].

As we mentioned in the second chapter, some crucial topological quantities associated to manifolds are Betti numbers, which count the number of linearly independent harmonic forms

[1] See [34], but the proof should be found in mathematical literature.

on them. In the case at hand, it can be proved that

$$b_2(X) = b_2(Y) + b_6(X). \tag{5.1.1}$$

Relation (5.1.1) is equivalent to say that the number of harmonic 2-forms on the cone, or more precisely $(1,1)$-forms, split into two sets:

$$\omega_a = (\hat{\omega}_\alpha, \tilde{\omega}_\sigma), \quad a, b, ... = 1, ..., b_2(X), \quad \alpha, \beta, ... = 1, ..., b_6(X), \quad \sigma, \tau, ... = 1, ..., b_2(Y). \tag{5.1.2}$$

The main difference between the two sectors is the "normalizabilty". Indeed, $\hat{\omega}_\alpha$ are Poincaré dual to the $b_6(X)$ compact 6-cycles of the resolved cone and are L_2-normalizable, namely

$$\int_X \hat{\omega}_\alpha \wedge \star_X \hat{\omega}_\beta < \infty, \tag{5.1.3}$$

while $\tilde{\omega}_\sigma$ are Poincaré dual to the $b_2(Y)$ non-compact 6-cycles of the resolved cone and are L_2^w-normalizable, namely

$$\int_X e^{-6D} \tilde{\omega}_\sigma \wedge \star_X \tilde{\omega}_\tau < \infty. \tag{5.1.4}$$

The w in L_2^w stands for "warped" and indeed the $\tilde{\omega}_\sigma$ forms are normalizable only if we use a warped measure. More precisely, the warp factor in (5.1.4), which actually works as a damping factor, is related to the previously introduced warp-factor $h(r)$ in (3.1.2), specialized to the M2-brane case, by

$$e^{-6D(r)} \sim h(r) - 1 = \frac{R^6}{r^6}, \qquad r \to \infty. \tag{5.1.5}$$

Indeed, recall that we are working with asymptotically $AdS_4 \times Y_7$ backgrounds and hence the warp factor must behave like $\frac{R^6}{r^6}$ at large r in order for it to vanish at infinity[2].

Harmonic forms admit an interpretation as variation of the Kähler form J, namely

$$\omega_a = \frac{\partial J}{\partial v^a}. \tag{5.1.6}$$

Using harmonic forms as a basis, we can also expand the Kähler form in the following way:

$$J = J_0 + v^a \omega_a, \tag{5.1.7}$$

where v^a are the so called "Kähler moduli" of X_8 and J_0 is the exact component of the Kähler form. This in turn can be globally expressed as

$$J_0 = i\partial\bar{\partial}k_0 \tag{5.1.8}$$

[2] If we take for instance the warp factor (3.1.2) of the M2-brane solution, it is clear that we will have problems with (5.1.4) because of the constant 1. In general, we can imagine a different warp factor, like $h(r) = e^{-6D} = a + \frac{R^6}{r^6}\left(1 + o(r^{-1})\right)$: while the second term acts as a damping factor for (5.1.4), the constant a spoils normalizability and the result will be infinite. On the other hand, warp factors like $h(r) = e^{-6D} = \frac{R^6}{r^6}\left(1 + o(r^{-1})\right)$, which are exactly the ones consistent with asymptotically $AdS_4 \times Y_7$ backgrounds, ensure (5.1.4) to hold because of the choice $a = 0$.

for some globally defined function $k_0(z,\bar{z};v)$. Besides, in any local chart the harmonic forms are generated by "potentials" $\kappa_a(z,\bar{z};v)$ such that

$$\omega_a = i\partial\bar{\partial}\kappa_a. \tag{5.1.9}$$

Since Kähler moduli are actually parameters regulating the volume of some resolution 2-cycle C^a, i.e. $v^a \sim \int_{C^a} J$, there exists a sort of quantization condition $\int_{C^a} \omega_b \sim \delta^a_b \in \mathbb{Z}$. If we differentiate (5.1.7) with respect to v^b then it does emerge a consistency condition between (5.1.6) and (5.1.7), namely

$$\frac{\partial J_0}{\partial v^b} = -v^a \frac{\partial \omega_a}{\partial v^b}, \tag{5.1.10}$$

which translates into

$$\frac{\partial k_0}{\partial v^b} = -v^a \frac{\partial \kappa_a}{\partial v^b}. \tag{5.1.11}$$

More precisely, we can write the Kähler form as

$$J = i\partial\bar{\partial}k, \tag{5.1.12}$$

where $k(z,\bar{z};v)$ is the (total) Kähler potential defined as

$$k = k_0 + v^a\kappa_a, \quad \text{together with} \quad \kappa_a = \frac{\partial k}{\partial v^a}, \quad \frac{\partial \kappa_a}{\partial v^b} \to 0 \quad \text{for} \quad r \to \infty. \tag{5.1.13}$$

These conditions are supposed to be crucial for removing an ambiguity in the definition of potentials, namely k_0 and κ_a are defined up to coordinate-independent functions depending on Kähler moduli.

5.2 Chiral parametrization of moduli

On general grounds, the moduli characterizing M-Theory vacua include M2-branes positions on the cone and Kähler moduli v^a together with the so called "axionic moduli" of the M-Theory C_6 six-form[3]. The former admit a parametrization in terms of $4\tilde{N}$ complex coordinates z^i_I, where $i = 1, ..., 4$ and $I = 1, ..., \tilde{N}$. The latter admit a complex parametrization too, say with ρ_a where $a = 1, ..., b_2(X)$: the real part of ρ_a correspond to Kähler moduli and the imaginary part correspond to axionic moduli[4]. Then, both kind of coordinates are interpreted as chiral moduli fields. However, while positions moduli have a direct meaning as scalar component of a chiral superfield, resolution moduli are associated to the respective chiral superfield by the transformation

$$\operatorname{Re}\rho_a = \frac{1}{2}\sum_I \kappa_a(z_I, \bar{z}_I; v), \tag{5.2.1}$$

[3]Recall that this is the electromagnetic dual of the fundamental three-form A_3, i.e. $dC_6 = \star_{11} dA_3$.

[4]In what follows the imaginary part is not necessary and hence we address the reader to [1, 2] for an explanation of axionic moduli and their role.

whereas the imaginary part is not necessary for our purpose. An useful formula is

$$\frac{\partial \operatorname{Re}\rho_a}{\partial v^b} = -G_{ab} = \int_X e^{-6D}\omega_a \wedge \star_X \omega_b, \tag{5.2.2}$$

which let us invert (at least in principle) the relation (5.2.1) between resolution moduli v^a and their chiral coordinate counterpart $\operatorname{Re}\rho_a$.

A comment on moduli spaces and resolution parameters

With the chiral parametrization of moduli above introduced we have $4\tilde{N} + b_2(X)$ coordinates parametrizing $\mathcal{M}_{SUGRA}$, which means that $dim_{\mathbb{C}}\mathcal{M}_{SUGRA} = 4\tilde{N} + b_2(X)$. In particular, in (5.2.1) we are considering Kähler moduli as dynamical quantities. However, in the previous chapter the "resolution parameters" where non-dynamical constant: for instance, external background vectors gave rise to one Fayet-Iliopoulos and one real mass in the Q^{111} model. Then, the whole calculation for the moduli space there should be referred to as the "non-dynamical parameters case": the monopole method let us see $\mathcal{M} = Sym^{\tilde{N}}C(Y_7)$ while the semiclassical computation gave us the resolved version $\mathcal{M} = Sym^{\tilde{N}}X_8$. Both of them have clearly complex dimension $dim_{\mathbb{C}}\mathcal{M} = 4\tilde{N}$ and the dual interpretation is of $\tilde{N}$ M2-branes moving respectively on the cone or its resolved version. So, in the "non-dynamical parameters case" we have $\mathcal{M} = \mathcal{M}_{SUGRA} = Sym^{\tilde{N}}CY_4$ with CY_4 either singular or resolved. It is clear that there is a mismatch with the above setting: we should explain how the $b_2(X)$ new directions in the moduli space arise. As we will see later on in this chapter, it is possible to turn non-dynamical parameters into dynamical fields using the so called "$\mathcal{S}$-operation": while the former clearly do not affect the dimension of moduli space, the latter surely modify it. Indeed, when the $b_2(X)$ parameters become dynamical, either in the quiver theory or in the holographic counterpart, the moduli space develops $b_2(X)$ new directions: the result is that the new moduli space is a fibration of the old $Sym^{\tilde{N}}CY_4$ over these new $b_2(X)$ directions and hence it has the correct complex dimension. Even if we do not verify it, we think that the new fibered moduli spaces on both side of the duality should match. Besides, the physical implications will become clear in the next section writing down the HEFT Lagrangian.

5.3 The Holographic Effective Lagrangian

For the discussion of the low-energy effective theory we assume that M2-branes are not mutually coincident and that the two-derivative approximation can be trusted. The fundamental object is the Kähler potential on $\mathcal{M}_{SUGRA}$

$$K = 2\pi \sum_I k_0(z_I, \bar{z}_I; v), \tag{5.3.1}$$

which has to be considered as a function of chiral coordinates $\Phi^A = (\rho_a, z_I^i)$ obtained inverting (5.2.1) and hence expressing v^a as functions of ρ_a. By promoting the chiral coordinates to three-dimensional chiral superfields, the HEFT is then described by the effective action

$$S_{\text{HEFT}} = \int \mathrm{d}^3x \mathrm{d}^4\theta K(\Phi, \bar{\Phi}). \tag{5.3.2}$$

Expanding the Kähler potential to two-derivatives we get the bosonic effective Lagrangian, which is actually a nonlinear sigma model

$$\mathcal{L}^{bos}_{\text{HEFT}} = -K_{A\bar{B}}(\Phi, \bar{\Phi}) d\Phi^A \wedge \star_3 d\bar{\Phi}^{\bar{B}}, \qquad K_{A\bar{B}} = \frac{\partial^2 K}{\partial \Phi^A \partial \bar{\Phi}^{\bar{B}}}. \tag{5.3.3}$$

We would like a more explicit form for (5.3.3). First of all, we should derive K and hence k_0 in (5.3.1) with respect to moduli. In particular, we want to compute $\frac{\partial v^a}{\partial \operatorname{Re} \rho_b}$ and $\frac{\partial v^a}{\partial z_I^i}$. Let us start from the obvious identities $\delta_a^b = \frac{\partial \operatorname{Re} \rho_a}{\partial \operatorname{Re} \rho_b}$ and $0 = \frac{\partial \operatorname{Re} \rho_a}{\partial z_I^i}$. Using chain-derivatives and (5.2.2) we can write

$$\delta_a^b = \frac{\partial \operatorname{Re} \rho_a}{\partial \operatorname{Re} \rho_b} = \frac{\partial v^c}{\partial \operatorname{Re} \rho_b} \frac{\partial \operatorname{Re} \rho_a}{\partial v^c} = -G_{ac} \frac{\partial v^c}{\partial \operatorname{Re} \rho_b}, \tag{5.3.4}$$

so that

$$\frac{\partial v^a}{\partial \operatorname{Re} \rho_b} = -G^{ab}. \tag{5.3.5}$$

Then, if we define

$$\mathcal{A}^I_{ai} = \frac{\partial \kappa_a}{\partial z_I^i}, \tag{5.3.6}$$

we also get

$$0 = \frac{\partial \operatorname{Re} \rho_a}{\partial z_I^i} = \frac{\partial v^b}{\partial z_I^i} \frac{\partial \operatorname{Re} \rho_a}{\partial v^b} + \frac{\partial \operatorname{Re} \rho_a}{\partial z_I^i} = -G_{ab} \frac{\partial v^b}{\partial z_I^i} + \frac{1}{2} \mathcal{A}^I_{ai}, \tag{5.3.7}$$

so that

$$\frac{\partial v^a}{\partial z_I^i} = \frac{1}{2} G^{ab} \mathcal{A}^I_{bi}. \tag{5.3.8}$$

Using (5.3.5) and (5.3.8) we can find a clearer form of (5.3.3), namely

$$\mathcal{L}^{bos}_{\text{HEFT}} = -\pi G^{ab} \nabla \rho_a \wedge \star_3 \nabla \bar{\rho}_b - 2\pi \sum_I g_{i\bar{j}}(z_I, \bar{z}_I, v) dz_I^i \wedge \star_3 d\bar{z}_I^{\bar{j}}. \tag{5.3.9}$$

The covariant derivative in (5.3.9) is defined as

$$\nabla \rho_a = d\rho_a - \mathcal{A}^I_{ai} dz_I^i \tag{5.3.10}$$

and its presence is due to the fact that chiral coordinates ρ_a are actually function of z_I^i themselves. In other words, the parametrization of the moduli space $\mathcal{M}_{SUGRA}$ leads to nontrivial interactions between Kähler modes, corresponding to resolutions, and the modes associated to branes positions. Besides, there are different metrics in (5.3.9). The $g_{i\bar{j}}(z_I, \bar{z}_I, v)$ one is exactly the Calabi-Yau metric of the resolved cone X_8, which can be computed imposing Ricci-flatness on the Kähler form J. Notice that since it depends both on positions and Kähler modes, in the case of resolved cone it also give rise to interactions between Kähler moduli and branes positions. On the other hand, the G^{ab} one is the inverse of (5.2.2): notice that in order for it to be finite[5] it is crucial to require that the warp factor has asymptotic behavior (5.1.5). This is another reason to have an eleven-dimensional metric asymptotic to $AdS_4 \times Y_7$ and we should repeat that it is absolutely non-trivial to show that a Ricci-flat metric on resolved X_8 can always be found, and moreover it is unique, such that it asympotically approaches the one on the singular cone.

As a concluding comment on the HEFT Lagrangian, notice that in the case of non-dynamical Kähler moduli (5.3.9) is "quite trivial". Indeed, the first term does not appear and the second term has an immediate interpretation. Recall that in the "non-dynamical parameters case" the moduli space is given by $\mathcal{M}_{SUGRA} = Sym^{\tilde{N}} CY_4$. Then the HEFT Lagrangian describes $\tilde{N}$ copies of the same theory of a single M2-brane moving on a resolved cone. Every $g_{i\bar{\jmath}}$ have the same expression but depend on the I-th set of coordinates parametrizing the positions of the M2. Since we are interested in the non-trivial case with dynamical moduli, we must find something that let us go from one description to the other. Indeed, the moduli space check of the previous chapter actually works for the "trivial case". We have reasons to think that the check "is preserved" going from one description to the other, but we will see it in a moment.

The applicability regime of the HEFT

The HEFT Lagrangian provides a tool for studying the dynamics of low-energy degrees of freedom of a strongly coupled superconformal field theory. These degrees of freedom are massless modes: from a top-down perspective, one should reach this HEFT by integrating out massive modes with a mass of order $\langle O \rangle$. Then, we can properly use the HEFT only for energy regimes well below the scale set by $\langle O \rangle$. Recall that this is exactly the region of spontaneous symmetry breaking of the conformal symmetry: indeed, it is only at high energy, well above $\langle O \rangle$, that the conformal symmetry is restored. So, we stress that we can exclusively exploit the Lagrangian (5.3.9) in the phase where the conformal symmetry is spontaneously broken. The obvious consequence is that the superconformal symmetry is non-linearly realized on (5.3.9) and hence it is in general challenging to prove that it is actually related to a SCFT. As we will see in the last chapter, our strategy is to focus on dilatations only and check if (5.3.2) is scale invariant using asymptotic calculations.

5.3.1 A dual description with linear multiplets

Remember that in three spacetime dimensions there exists a scalar-vector duality which translates into a supersymmetric version, namely a duality between chiral and linear supermultiplets.

[5] See and compare with (5.1.4).

Imagine that we want to dualize some Kähler moduli ρ_a into Σ_a defined in (1.3.4) and satisfying (1.3.2). The chiral coordinates are now $\Phi^A = (\Sigma_a, z_I^i)$ and the dual formulation of (5.3.2) is

$$S_{\rm HEFT} = \int \mathrm{d}^3x \mathrm{d}^4\theta \mathcal{F}(\Phi, \bar{\Phi}), \tag{5.3.11}$$

where $\mathcal{F}$ is the Legendre transform of (5.3.1), namely

$$\mathcal{F}(z, \bar{z}, \Sigma) = K(z, \bar{z}, \operatorname{Re}\rho) + 4\pi\Sigma^a \operatorname{Re}\rho_a. \tag{5.3.12}$$

Then, the fundamental relations connecting dual Kähler potentials, chiral and linear multiplets are:

$$\Sigma^a = -\frac{1}{4\pi}\frac{\partial K}{\partial \operatorname{Re}\rho_a}, \qquad \operatorname{Re}\rho_a = \frac{1}{4\pi}\frac{\partial \mathcal{F}}{\partial \Sigma^a}. \tag{5.3.13}$$

Comparing (5.3.12) with (5.1.13), (5.2.1) and (5.3.1) we can write the dual Kähler potential as

$$\mathcal{F}(z, \bar{z}, \Sigma) = 2\pi \sum_I k(z_I, \bar{z}_I; \Sigma), \tag{5.3.14}$$

where the linear multiplets here have dynamical Kähler moduli v^a as lowest scalar component, i.e. $\Sigma^a = v^a + ...$

5.3.2 The $\mathcal{S}$-operation

In this subsection we want to give an idea of the role played by the so called "$\mathcal{S}$-operation" in this work. Roughly speaking, the $\mathcal{S}$-operation has two effects: it turns some non-dynamical parameters into dynamical parameters (or viceversa) and it turns some $U(1)$ gauge group into a global symmetry (or viceversa). It is beyond the aim of this thesis to explore the $\mathcal{S}$-operation pattern, but it is worth mentioning it because of the previous considerations on moduli spaces and resolution parameters, both in the quiver side and in the HEFT. In particular, it provides a "bridge" between the "non-dynamical parameters case" and the one with dynamical quantities.

We start saying that $b_2(Y)$ is a very important topological quantity: we know that it counts the harmonic two-forms on Y_7, but we should point out that this number is also related to some $U(1)^{b_2(Y)}$ "baryonic" symmetry group[6] in the dual CFT_3. Indeed, denoting ω_a these harmonic two-forms, with $a = 1, ..., b_2(Y)$ here, the M-Theory fundamental three-form can be written as $A_3 = A^a \wedge \omega_a$, where the $b_2(Y)$ massless $U(1)$ one-forms A^a can be obtained by integration over three-cycles C^a, namely

$$A^a = \int_{C^a} A_3. \tag{5.3.15}$$

[6]The term baryonic comes from the type IIB String Theory language and its AdS_5/CFT_4 version of the duality, see for instance [1, 34]. Here, we are dealing with M-Theory and AdS_4/CFT_3 correspondence. We mention that the main difference between the two cases is that in the CFT_3 at the infrared fixed point we can have the possibility of either gauged/ungauged $U(1)$ symmetries. On contrary, in the CFT_4 at the infrared fixed point the $U(1)$ symmetries are always global.

In the three-dimensional SUSY language, the gauge fields A^a are actually the vector components contained in vector supermultiplets V^a of the HEFT. Then, the field strengths for V^a are the linear multiplets $\Sigma^a = -\frac{i}{2}\epsilon^{\alpha\beta}\bar{D}_\alpha D_\beta V^a$, which can also be interpreted as topological current multiplets in the sense of (1.3.5): indeed, these currents are the ones associated to the $U(1)^{b_2(Y)}$ "baryonic" symmetries. Now, we claim that these symmetries can be either gauged or ungauged and the "bridge" between the two pictures is the $\mathcal{S}$-operation. We will follow [2], working at HEFT level and making some mandatory comments about the dual quiver interpretation.

Consider the HEFT action (5.3.11) with some external vector supermultiplet $\mathcal{A}_a$ gauge-invariantly coupled to dynamical linear multiplets such that

$$S_{\text{HEFT}}[\mathcal{A}_a] = \int \mathrm{d}^3x\mathrm{d}^4\theta\mathcal{F}(z,\bar{z},\Sigma) + \int \mathrm{d}^3x\mathrm{d}^4\theta\Sigma^a\mathcal{A}_a. \tag{5.3.16}$$

The $\mathcal{S}$-operation consists in promoting the $\mathcal{A}_a$ to dynamical gauge vector supermultiplets with a topological interaction. More precisely, a new set of external vector supermultiplets $\mathcal{B}_a$ is added and (5.3.16) becomes

$$S_{\text{HEFT}}[\mathcal{B}_a] = S_{\text{HEFT}}[\mathcal{A}_a] - \int \mathrm{d}^3x\mathrm{d}^4\theta\Theta^a\mathcal{B}_a, \tag{5.3.17}$$

where $\Theta_a = -\frac{i}{2}\epsilon^{\alpha\beta}\bar{D}_\alpha D_\beta\mathcal{A}_a$ are the field-strengths of the $b_2(Y)$ vector supermultiplets $\mathcal{A}_a$ and can be interpreted as topological conserved current multiplets too. The last term in (5.3.17) can be rewritten with an integration by parts as

$$-\int \mathrm{d}^3x\mathrm{d}^4\theta\Xi^a\mathcal{A}_a, \tag{5.3.18}$$

where Ξ^a is the "non-dynamical field-strength" of $\mathcal{B}_a$, namely

$$\Xi^a = -\frac{i}{2}\epsilon^{\alpha\beta}\bar{D}_\alpha D_\beta\mathcal{B}^a = \zeta^a + ... + \frac{1}{2}\theta\gamma^\mu\bar{\theta}J^a_\mu. \tag{5.3.19}$$

Then, the scalar components ζ^a of the linear multiplets Ξ^a can interpreted as Fayet-Iliopoulos parameters. However, as opposed to (1.2.17), here they should be considered as "point-dependent" FI parameters rather than constant. At this stage, since the $\mathcal{A}_a$ do not appear in the original action functional (5.3.11), which is the starting point of the procedure, we can integrate them out, leaving the relation between linear multiplets

$$\Sigma^a = \Xi^a \tag{5.3.20}$$

obtained from (5.3.16), (5.3.17) and (5.3.18). This means that the $\mathcal{S}$-operation in the low-energy region "freezes" the dynamical linear multiplets Σ^a, having v^a as scalar components, into background non-dynamical linear multiplets Ξ^a, having ζ^a as scalar components. The latter are effectively external current multiplets coupled to $b_2(Y)$ dynamical gauge vector supermultiplets $\mathcal{A}_a$: hence, the resulting theory has $U(1)^{b_2(Y)}$ additional gauge group. Geometrically, when the FI parameters are imposed to be constant, (5.3.20) corresponds to Kähler moduli being non-dynamical, i.e. $v^a = \zeta^a$.

The situation depicted till now at HEFT level has a dual interpretation. Consider a quiver field theory with generic gauge group $U(N)^G$ and $b_2(Y)$ independent non-dynamical FI parameters. The action (5.3.16) with dynamical linear multiplets Σ^a and external vector supermultiplets $\mathcal{A}_a$ does not correspond to that quiver. Instead, it is related to a quiver with gauge group $U(1)^{G-b_2(Y)} \times SU(N)^G$ and $U(1)^{b_2(Y)}$ global symmetry, together with dynamical FI parameters. Acting with the $\mathcal{S}$-operation on these global symmetries we promote them to gauge symmetries but at the same time we turn the dynamical FI into non-dynamical constants. The result is the $U(N)^G$ quiver gauge theory with $b_2(Y)$ independent FI constant parameters.

It is maybe useful to call "Theory A,B,C" the HEFT coupled to external vector supermultiplets $\mathcal{A}, \mathcal{B}, \mathcal{C}$. The Theory B, having non-dynamical Kähler parameters because of the aforementioned "freezing", corresponds to the quiver with gauge group $U(N)^G$ and non-dynamical FI parameters. In this case, (5.3.9) is the "trivial" one with the second term only: however, we want to study the non-trivial version of it having dynamical Kähler moduli and fibered moduli space. This should correspond to the other quiver with gauge group $U(1)^{G-b_2(Y)} \times SU(N)^G$ and $U(1)^{b_2(Y)}$ global symmetry, so let us see how one can obtain this quiver from the one with non-dynamical FI and full $U(N)^G$ gauge group. Restart from Theory B in (5.3.17) and apply $\mathcal{S}$-operation a second time. We promote $\mathcal{B}_a$ to dynamical vector supermultiplets and we also add a topological interaction between their field strengths Ξ^a and external vector supermultiplets $\mathcal{C}_a$ so that we arrive to Theory C:

$$S_{\text{HEFT}}[\mathcal{C}_a] = S_{\text{HEFT}}[\mathcal{B}_a] - \int \mathrm{d}^3x \mathrm{d}^4\theta \Xi^a \mathcal{C}_a. \tag{5.3.21}$$

Hence, it does emerge a term

$$\int \mathrm{d}^3x \mathrm{d}^4\theta(-\Theta^a \mathcal{B}_a - \Xi^a \mathcal{C}_a) = \int \mathrm{d}^3x \mathrm{d}^4\theta(-\Theta^a \mathcal{B}_a - \mathcal{B}_a \Omega^a), \tag{5.3.22}$$

where $\Omega_a = -\frac{i}{2}\epsilon^{\alpha\beta}\bar{D}_\alpha D_\beta \mathcal{C}_a$. Finally, integrating out $\mathcal{B}_a$ we are led to the identification $\mathcal{A}_a = -\mathcal{C}_a$. The interpretation is that we are back to Theory A with a slight change of sign for the external vector supermultiplets: this means that our quiver has a reduced gauge group with dynamical FI parameters, as we wanted.

$\mathcal{S}$-operation on flavors and real masses

The previous discussion seems to hold when dealing with unflavored quivers, i.e. the only external parameters are FI and the only $U(1)$ symmetries are "baryonic" ones[7]. If we have

[7] In what follows we want to evidence the difference between flavored and unflavored quivers: while the latter have only "baryonic symmetries", the former may have both "baryonic symmetries" and "flavor symmetries". For our purpose, we should intend the term "baryonic" as opposed to "flavor".

flavor symmetries and real masses as external parameters, like in the Q^{111} model, the story is a bit different. Indeed, in the above argument an ungauging of a $U(1)$ baryonic factor in the quiver, equivalent to go from Theory B to Theory C at HEFT level, corresponds to a new dynamical parameter. On contrary, in the case of a flavor symmetry $U(1)_F$ in the quiver it is a gauging of it that leads to a new dynamical parameter: we want to sketch this difference.

So, consider for instance the quiver gauge theory $U(\tilde{N})^2 = SU(\tilde{N})^2 \times U(1)_{diag} \times U(1)_{bar}^{gauge}$ with global $U(1)_F$ having one FI and one real mass as non-dynamical parameters: this is the Q^{111} model discussed in the previous chapter and we claim that it is properly dual to a "Theory B". At quiver level, we can act "on block" with $\mathcal{S}$-operation both on the $U(1)_{bar}^{gauge}$, which is a baryonic gauge symmetry, and on the global $U(1)_F$. The former becomes a global topological symmetry $U(1)_{bar}$ and the reasoning is the same of going from Theory B to Theory C at HEFT level: the FI becomes dynamical and the moduli space get larger. The latter becomes a $U(1)_F^{gauge}$ and the real mass becomes dynamical too: this is because we are promoting the background vector supermultiplet $V_{bg} = -im\theta\bar{\theta} + ...$ in the quiver[8], where m is the real mass, to a gauge vector multiplet. The reason why the moduli space get larger is that this mass, which is actually a new dynamical field, is a mass for chiral flavor fields: since we always consider vanishing VEVs for them, i.e. $\langle q \rangle = \langle p \rangle = 0$, that dynamical mass is allowed for being a new direction of the moduli space because it does preserve SUSY. In the end, considering the $\mathcal{S}$-operation as acting "on block" we have always a $U(\tilde{N})^2$ quiver gauge group but:

- "Theory B" is dual to $U(1)_F$ global and $SU(\tilde{N})^2 \times U(1)_{diag} \times U(1)_{bar}^{gauge}$ gauge, with non-dynamical parameters. The moduli space is the one giving a "trivial" (5.3.9), i.e. with second term only, and the consistency check surely holds because it is given by $Sym^{\tilde{N}} CY_4$ on both sides of the duality.

- "Theory C" is dual to $SU(\tilde{N})^2 \times U(1)_{diag} \times U(1)_F^{gauge}$ gauge and $U(1)_{bar}$ global, with dynamical parameters and hence a fibered moduli space, i.e. a non-trivial (5.3.9). Since this "Theory C" is obtained from "Theory B" applying $\mathcal{S}$-operation, we are led to think that the moduli space check is "preserved". We mean that even if the moduli space is no more $Sym^{\tilde{N}} CY_4$, the new fibered moduli spaces on the two sides of the duality should match[9].

The careful reader could be upset at this stage: the mismatch in the complex dimension of the moduli space is due to $b_2(X)$ new directions, but here with $\mathcal{S}$-operation we can only promote $b_2(Y)$ non-dynamical parameters to dynamical ones. This can be quite curious, but we anticipate that in the case of the Q^{111} model we have $b_2(X) = b_2(Y = Q^{111}) = 2$ and hence this problem does not affect us directly.

[8] Recall how real masses are introduced with (1.3.9).

[9] We do not explicitely verify it but in the next chapter we will carry out a matching between the parameters in the quiver field theory side and the resolution parameters in the holographic counterpart, namely $(\zeta, m) \leftrightarrow (v_1, v_2)$.

Chapter 6

The Q^{111} HEFT

We are finally ready to apply the techniques introduced in the previous chapters to the case of M2-branes probing a background geometry $\mathbb{R}^{1,2} \times CY_4$, where the transverse directions to M2-branes are either a cone $C(Q^{111})$ or its resolved version X. This is the original contribution of this work because the low-energy dynamics of the dual SCFT has not been investigated so far. Before rushing into the HEFT we should begin with a parenthesis on toric geometry, particularly focusing on a related tool called Gauged Linear Sigma Model (GLSM), because of two main reasons: the $C(Q^{111})$ is a toric manifold and throughout this chapter we will widely abuse of the GLSM. Indeed, this is an auxiliary field theory that turns out to be very useful in the matching between moduli spaces, especially if we consider the dimensional reduction from M-Theory to type IIA String Theory. Indeed, carrying out this reduction let us identify both the origin of flavors in the Q^{111} quiver from the brane point of view and construct a dictionary between the parameters in the quiver field theory, i.e. the FI and the real mass, and the Kähler parameters related to resolutions.

Then, we will proceed with the identification of general properties regarding $C(Q^{111})$, also specializing its GLSM. After this geometric preliminary, we will compute the metric of the resolved cone X: recall from the previous chapter that it must be Ricci-flat and it should approach the one on the singular cone asymptotically. A real-coordinates parametrization is useful for this purpose, see for instance [34], whereas a complex one let us see the Kähler structure in a clearer way. Hence, we express the metric in a suitable complex parametrization inspired by [1, 2, 37, 38]. Having the metric, or equivalently the Kähler form, we can obtain harmonic two-forms as in (5.1.6). Moreover, we can compute "potentials" k, k_0, κ_a together with the "non-trivial metric" G^{ab} and the "connection" $\mathcal{A}^I_{ai}$, respectively using (5.3.5) and (5.3.6). So, we can collect all the ingredients to build the HEFT Lagrangian (5.3.9) for the Q^{111} model. Besides, we want to perform new consistency checks between the quiver theory and the holographic counterpart. One of them is about the dimensional reduction from M-Theory to type IIA String Theory: in order to do this we will exploit the power of the aforementioned GLSM. After that, a check on the superconformal symmetry of the HEFT is done. Indeed, recall that the HEFT is trustable only when the conformal symmetry is spontaneously broken, i.e. the conformal symmetry should be non-linearly realized. An explicit check on non-linearly realized superconformal transformations can be in general very difficult: however, there are cases, like the one treated in this thesis, in which it is sufficient to check the scale-invariance of (5.3.2).

6.1 Toric geometry and the GLSM

As a premise, this section has not to be intended as an introduction on toric geometry. Instead, we want to collect at the beginning of this chapter the reasons why it is useful when dealing with holography and specifically in this work. Indeed:

- the case $C(Q^{111})$ object of the thesis is toric;
- "toricity" is relevant in quiver gauge theories because it furnishes an useful tool to operatively build the moduli space. The basic idea is to study a gauged linear sigma model (GLSM), which is an auxiliary theory that automatically reproduces the same moduli space of the quiver;
- it provides some kind of diagrams which are very useful to understand, at least pictorially, the dimensional reduction from M-Theory to type IIA that we will work out later on and in particular the origin of flavor symmetries in the Q^{111} quiver.

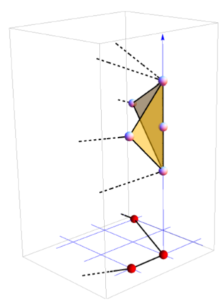


Figure 6.1: An example of (a portion of) toric diagram borrowed from [6].

The basic feature of a CY_4 toric manifold is that it can be "mapped" into a three-dimensional polyhedron called "toric diagram", like the one in figure. The physics behind this structure is that strictly external points[1] represent some particular submanifolds of CY_4, called "toric divisors", that can be wrapped by branes: this give rise to new features in the dual field description, for example flavors. It can happen that two such external points in the toric diagram are vertically aligned: then, as stated in [6], the vertical projection of the 3d diagram into a 2d diagram turns out to be equivalent to a dimensional reduction from M-Theory to type IIA along an eleventh compact direction[2]. The 2d diagram is actually the toric diagram associated to a different Calabi-Yau toric manifold, this time with one dimension less: this CY_3 is a suitable candidate for a conical background on which D-branes can be placed[3]. The reason to introduce

[1] Here, "strictly" means that these external points are actually vertexes of the polyhedron, i.e. a strictly external point never lie along a line connecting two external points nor inside a face of the toric diagram.

[2] This dimensional reduction is characterized by a "wisely chosen" M-Theory circle $U(1)_M$ as we will see in the explicit calculation.

[3] Moreover, this three-dimensional cone can be obtained from a Kähler quotient of the four-dimensional cone as $CY_4//U(1)_M = CY_3$, as we will explicitly see.

this rather abstract structure is the following. Imagine $1 + h$ vertically aligned points in a three-dimensional toric diagram related to a M-Theory background CY_4. Reduction to type IIA is interpreted as a projection of all $1 + h$ points down to only one strictly external point in the 2d toric diagram associated to CY_3. Following [6], this give rise to h coincident D6-branes wrapping the same toric divisor of CY_3. At this stage, in type IIA there are also D2-branes corresponding to dimensionally reduced M2-branes. Having both D2-branes and D6-branes we can imagine open strings connecting them[4]: this picture corresponds in fact to some $U(h)$ flavor symmetry in the field theory and hence to some couples of (q, p) chiral flavor fields discussed in the previous chapters.

Having sketched the utility of toric diagrams, some comments on the aforementioned gauged linear sigma model (GLSM) are really mandatory. We said that toric varieties, like $C(Q^{111})$, can be realized as the moduli space of an auxiliary model called GLSM. Indeed, there is a precise algorithm to write down the quiver gauge theory from toric data, see for example [31, 32, 33]. We will not deepen its construction but we will focus on its output, namely:

- A set of fields $\mathcal{P}_\rho$ called "perfect matchings", in terms of which the bifundamental chiral fields of the complete UV-quiver theory (together with monopole operators if considered) can be expressed:
$$\Phi_a = \prod_{\rho \in R(a)} \mathcal{P}_\rho, \tag{6.1.1}$$
where $R(a)$ is a subset of the perfect matchings.

- The 3d toric diagram of CY_4. Each perfect matching is mapped to a point of the toric diagram[5] and can be used as a field of a GLSM.

On the other hand, the GLSM can be exploited to characterize the M-Theory background too. More precisely, we can work out a matching between perfect matchings and complex coordinates parametrizing the complex cone as well as its resolutions.

Since throughout this chapter we will carry out a lot of different matchings between different sets of coordinates and/or fields exploiting the GLSM, we make a list of operations that the reader should take in mind before starting:

- identification of the correct set of perfect matchings and study of their GLSM;

- matching between perfect matchings and the set of complex chiral coordinates $\{z\}$ characterizing the position of one M2-brane on CY_4: this operation will give the M-Theory background CY_4 as a GLSM;

[4]Recall that D-branes are extended objects on which open strings can end.

[5]The correspondence is not one-to-one in general. Indeed, different perfect matchings may correspond to the same toric point.

- matching between perfect matchings and chiral fields in the UV-quiver using (6.1.1), taking into account monopole operators $(T, \tilde{T})$ in the homonymous method. This operation will give the moduli space of the quiver[6] as a GLSM;

- matching between complex chiral coordinates $\{z\}$ and chiral fields in the UV-quiver together with monopoles. Since these complex coordinates parametrize the position of one M2-brane on CY_4 and are in fact the low-energy degrees of freedom in the HEFT, this operation amounts to find how the degrees of freedom of the quiver, i.e. the far UV theory, are organized in the effective field theory[7];

- construction of gauge-invariant operators using chiral fields in the UV-quiver[8];

- construction of gauge-invariant operators using perfect matchings;

- check that these gauge-invariant combinations provide a suitable parametrization of CY_4 as an affine toric variety with coordinates $\{w\}$, namely gauge-invariant operators should satisfy some constraint equations defining the cone[9] just like in (4.4.33).

6.2 The internal M-Theory geometry

First of all, recall that Q^{111} is realized as the coset manifold

$$Q^{111} = \frac{SU(2) \times SU(2) \times SU(2) \times U(1)}{U(1) \times U(1) \times U(1)}. \tag{6.2.1}$$

This is the seven-dimensional Sasaki-Einstein base Y_7 of the Calabi-Yau cone $C(Y_7 = Q^{111})$ and hence it gives rise to $\mathcal{N} = 2$ supersymmetries in the dual field theory. The structure (6.2.1) suggests that the metric should be a $U(1)$ bundle over three spheres, as in (3.2.26), reflecting the isometry group $SU(2)^3 \times U(1)$.

Topologically, the resolved cone X is characterized by the following Betti numbers:

$$b_2(X) = 2, \qquad b_2(Y) = 2, \qquad b_6(X) = 0, \tag{6.2.2}$$

which let us specialize the relation (5.1.1). Recall that Betti numbers count the number of linearly independent harmonic forms. According to (6.2.2) and (5.1.2), the Q^{111} model is characterized by two non-normalizable, or better warp-normalizable in the sense of (5.1.4),

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[6] More precisely, one of the $\tilde{N}$ copies.

[7] This is to some extent equivalent to find out that pions in the low-energy regime of QCD are bound states of quarks, the latter being the high-energy degrees of freedom.

[8] Actually, we have already done this step in (4.4.32).

[9] A more rigorous statement is that the quantum chiral ring of the quiver must coincide with the ring of affine coordinates: this is a necessary condition in order for the gauge/gravity correspondence to hold.

harmonic forms $\tilde{\omega}_a$: since there are no $\hat{\omega}_a$ harmonic forms we will identify $\tilde{\omega}_a \equiv \omega_a$ in what follows. Consequently, the Kähler two-form should admit the expansion

$$J = J_0 + v^a \omega_a = J_0 + v^1 \omega_1 + v^2 \omega_2 = J_0 + b\omega_b + c\omega_c, \tag{6.2.3}$$

where $a = 1, 2$, $v^a = v^1, v^2 = b, c$ are the Kähler moduli and J_0 is an exact component. The fact that $b_2(X) = 2$ is telling us that there are two resolution parameters and indeed we will find out that the resolved cone admits a parametrization as a $\mathbb{C}^2$ vector bundle over the $\mathbb{CP}_1 \times \mathbb{CP}_1$ "base"[10]. More precisely, the resolved cone X is given by the total space of the vector bundle

$$\mathcal{O}_{\mathbb{CP}_1 \times \mathbb{CP}_1}(-1, -1) \oplus \mathcal{O}_{\mathbb{CP}_1 \times \mathbb{CP}_1}(-1, -1), \tag{6.2.4}$$

which is a notation we are going to explain soon after. Indeed, this is better understood using the GLSM of the next section. For now, we can imagine that the resolved cone is a product of two projective unidimensional spaces[11], where every point is actually a $\mathbb{C}^2$ space. Since the base of the bundle (6.2.4) consists of a pair of two-cycles $C^a = \{\mathbb{CP}_b, \mathbb{CP}_c\}$[12], the quantization condition of the harmonic forms reads

$$\int_{C^a} \omega_e = \delta^a_e, \qquad a, e = 1, ..., b_2(X) = 1, 2. \tag{6.2.5}$$

Thinking about (5.1.6), this implies that Kähler moduli $v^a = \{v_1, v_2\} = \{b, c\}$ can be identified as volumes of the resolution spheres, namely

$$v^a = \int_{C^a} J = vol(C^a). \tag{6.2.6}$$

6.2.1 The GLSM of $C(Q^{111})$: M-Theory analysis

The complex cone $C(Q^{111})$, as well as its resolutions, can be described by a GLSM with six fields $(a_1, a_2, b_1, b_2, c_1, c_2)$, which in turn are the perfect matchings of the model, and a $U(1)^2$ gauge group. What follows is a preliminary discussion on the GLSM which is useful to understand the bundle structure (6.2.4) and let us introduce the complex parametrization $\{z\}$ of the cone that we will use to compute our metric. We can think about this GLSM as an abelian gauge theory with the same gauge group of the quiver[13]. Their charge matrix takes the form

$$\begin{array}{c|cccccc} & a_1 & a_2 & b_1 & b_2 & c_1 & c_2 \\ \hline U(1)_I & -1 & -1 & 1 & 1 & 0 & 0 \\ U(1)_{II} & -1 & -1 & 0 & 0 & 1 & 1 \end{array} \tag{6.2.7}$$

[10]Here we hope that the term "base" does not generate confusion. We should distinguish the base of the cone, which is the Q^{111} manifold, from the base of the vector bundle, which is instead the resolution manifold $\mathbb{CP}_1 \times \mathbb{CP}_1$. After computing the metric this should be clear.

[11]Since we will always deal with unidimensional projective spaces, from now on we will write $\mathbb{CP}_1 \equiv \mathbb{CP}$ omitting its dimension.

[12]The a index runs over $b_2(X) = 2$. We will label the two projective spaces with b and c, as well as local coordinates parametrizing them, but these are not indexes like a, they are only names for the label.

[13]Clearly, we are thinking about the "single brane case" where we have only one M2 on the cone.

Since the theory is abelian, the F-term relations are trivial. Then, the two D-term equations of the GLSM reads

$$\begin{aligned} |b_1|^2 + |b_2|^2 - |a_1|^2 - |a_2|^2 = v_1, \\ |c_1|^2 + |c_2|^2 - |a_1|^2 - |a_2|^2 = v_2, \end{aligned} \tag{6.2.8}$$

where the resolutions parameters of the M-Theory background are interpreted as Fayet-Iliopoulos parameters for the GLSM. The singular cone is reproduced choosing vanishing FI parameters $v_1 = v_2 = 0$, while allowing for nonzero values we get resolutions. Let us consider $v_1, v_2 > 0$. Notice that for $a_1 = a_2 = 0$ in (6.2.8) we obtain the pair of spheres at the base of the bundle (6.2.4): indeed, we can imagine that $b_{1,2}$ and $c_{1,2}$ fields parametrize respectively $\mathbb{CP}_b$ and $\mathbb{CP}_c$, whereas the $a_{1,2}$ fields are the fiber coordinates of the $\mathbb{C}^2$. Let us clarify this structure.

The six perfect matchings parametrize the master space $\mathcal{F} = \mathbb{C}^6$, which has to be (Kähler) quotiented by the (complexified) gauge group $U(1)_I \times U(1)_{II}$ in order to find the moduli space of the GLSM[14]. Indeed, remember that we should identify gauge-equivalent combinations to characterize the space of inequivalent vacua. More precisely, in the case $v_1, v_2 > 0$ one has to subtract from $\mathbb{C}^6$ the set $Z = \{b_1 = b_2 = 0\} \cup \{c_1 = c_2 = 0\}$. Otherwise, if we consider a situation where $b_1 = b_2 = 0$ the first line in (6.2.8) clearly gives an absurd. We expect to find the moduli space (6.2.4) from the GLSM, which should match with the resolved cone $X \simeq \frac{\mathbb{C}^6 - Z}{(U(1)_I \times U(1)_{II})_{\mathbb{C}}}$. The action of the complexified gauge group on the master space reads

$$(a_1, a_2, b_1, b_2, c_1, c_2) \to (\xi_1^{-1}\xi_2^{-1}a_1, \xi_1^{-1}\xi_2^{-1}a_2, \xi_1 b_1, \xi_1 b_2, \xi_2 c_1, \xi_2 c_2), \qquad \xi_1, \xi_2 \in \mathbb{C}^*. \tag{6.2.9}$$

At this stage we can choose $\xi_1 = \frac{1}{b_2}$ and $\xi_2 = \frac{1}{c_2}$ so that

$$(a_1, a_2, b_1, b_2, c_1, c_2) \to (b_2 c_2 a_1, b_2 c_2 a_2, \frac{b_1}{b_2}, 1, \frac{c_1}{c_2}, 1), \tag{6.2.10}$$

provided that $b_2, c_2 \neq 0$. Then, defining

$$U = b_2 c_2 a_1, \quad Y = b_2 c_2 a_2, \quad \lambda_b = \frac{b_1}{b_2}, \quad \lambda_c = \frac{c_1}{c_2}, \tag{6.2.11}$$

which gives the identification between the perfect matchings and the complex coordinates $\{z\}$, (6.2.10) reads

$$(a_1, a_2, b_1, b_2, c_1, c_2) \to (U, Y, \lambda_b, 1, \lambda_c, 1). \tag{6.2.12}$$

The complex coordinates $\{z\} = (U, Y, \lambda_b, \lambda_c)$ provide a suitable parametrization of (a patch of) the resolved cone seen as a bundle (6.2.4). Indeed, (λ_b, λ_c) are local coordinates on the base $\mathbb{CP}_b \times \mathbb{CP}_c$ whereas (U, Y) are the fibral coordinates for $\mathbb{C}^2$. We must stress that it is not possible to find a globally well-defined metric on the resolved cone: however, we can focus

[14] Recall that the Kähler quotient "//" corresponds to imposing D-term conditions and then quotienting by the complexified gauge group.

on a particular patch and perform calculations. For instance, the (6.2.12) is related to the NORTH-NORTH patch of (6.2.4), while the SOUTH-SOUTH can be seen as

$$(a_1, a_2, b_1, b_2, c_1, c_2) \to (X, V, 1, \tilde{\lambda}_b, 1, \tilde{\lambda}_c), \tag{6.2.13}$$

where the set of coordinates $\{z\}$ is now given by[15]

$$X = b_1 c_1 a_1, \quad V = b_1 c_1 a_2, \quad \tilde{\lambda}_b = \lambda_b^{-1}, \quad \tilde{\lambda}_c = \lambda_c^{-1}. \tag{6.2.14}$$

Finally, we can give the idea of the notation in (6.2.4). The local coordinates (λ_b, λ_c) parametrize the base $\mathbb{CP}_1 \times \mathbb{CP}_1$, which is now identified with $\mathbb{CP}_b \times \mathbb{CP}_c$. The $(-1, -1)$ correspond to negative powers of these local coordinates in the following identifications

$$U = \frac{1}{\lambda_b}\frac{1}{\lambda_c} X, \qquad Y = \frac{1}{\lambda_b}\frac{1}{\lambda_c} V. \tag{6.2.15}$$

The fiber is actually a $\mathbb{C}^2$, parametrized by (U, Y) coordinates in the NN patch, with different combinations in other patches.

In the end, we have found that the moduli space of the GLSM can be seen as the total space of the vector bundle (6.2.4), the latter being a complex description of the M-Theory background X. Recall that this GLSM should reproduce the same moduli space of the $U(1)^2$ quiver associated to the Q^{111} model. Since for this quiver the moduli space is actually the resolved cone, the matching between moduli spaces is complete. However, this situation corresponds to a single M2-brane probing the resolved background: its position is indeed parametrized by $\{z\} = (U, Y, \lambda_b, \lambda_c)$. In the case of non-coincident $\tilde{N}$ M2-branes on X we have $\tilde{N}$ copies of the GLSM and hence $\tilde{N}$ copies of the coordinates $\{z\}$ parametrizing the M2-branes positions on X. The M-Theory moduli space is now $Sym^{\tilde{N}} X$, as well as the moduli space of the $U(1)^{2\tilde{N}}$ quiver. In the above matching argument, the subtlety about dynamical resolution parameters v^a discussed in the previous chapter is not investigated: we know that the moduli spaces should match again but they are fibered versions of the $Sym^{\tilde{N}} X$ ones, where the dynamical parameters are new fibers.

6.2.2 The Ricci-flat Kähler metric

As we already mentioned, the cone over Q^{111} can be seen as an affine variety with affine coordinates $w_i \in \mathbb{C}$. This means that $C(Q^{111})$ can be identified using the following set of constraints:

$$\begin{aligned} &w_1 w_2 - w_3 w_4 = w_1 w_2 - w_5 w_8 = w_1 w_2 - w_6 w_7 = 0, \\ &w_1 w_3 - w_5 w_7 = w_1 w_6 - w_4 w_5 = w_1 w_8 - w_4 w_7 = 0, \\ &w_2 w_4 - w_6 w_8 = w_2 w_5 - w_3 w_6 = w_2 w_7 - w_3 w_8 = 0, \end{aligned} \tag{6.2.16}$$

[15]We hope that the $X \in \{z\}$ of the SS patch will not be confused with the resolved cone X having the same name.

where a suitable parametrization satisfying (6.2.16) is given by

$$\begin{aligned}
w_1 &= \sqrt{t}e^{\frac{i}{2}(\psi+\phi_1+\phi_2+\phi_3)}\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\cos\frac{\theta_3}{2}, & w_2 &= \sqrt{t}e^{\frac{i}{2}(\psi-\phi_1-\phi_2-\phi_3)}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\sin\frac{\theta_3}{2},\\
w_3 &= \sqrt{t}e^{\frac{i}{2}(\psi+\phi_1-\phi_2-\phi_3)}\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\sin\frac{\theta_3}{2}, & w_4 &= \sqrt{t}e^{\frac{i}{2}(\psi-\phi_1+\phi_2+\phi_3)}\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\cos\frac{\theta_3}{2},\\
w_5 &= \sqrt{t}e^{\frac{i}{2}(\psi+\phi_1+\phi_2-\phi_3)}\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\sin\frac{\theta_3}{2}, & w_6 &= \sqrt{t}e^{\frac{i}{2}(\psi-\phi_1+\phi_2-\phi_3)}\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\sin\frac{\theta_3}{2},\\
w_7 &= \sqrt{t}e^{\frac{i}{2}(\psi+\phi_1-\phi_2+\phi_3)}\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\frac{\theta_3}{2}, & w_8 &= \sqrt{t}e^{\frac{i}{2}(\psi-\phi_1-\phi_2+\phi_3)}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\frac{\theta_3}{2}.
\end{aligned} \tag{6.2.17}$$

The coordinate t introduced in (6.2.17) is the so called "radial coordinate" and it should satisfy

$$t = \sum_{i=1}^{8} |w_i|^2. \tag{6.2.18}$$

In what follows we are going to find a metric on the NN patch of the cone, so we must identify the coordinates $(U, Y, \lambda_b, \lambda_c)$ with combinations of the w_i. In other words, we perform a matching between $\{z\}$ and $\{w\}$.[16]

First of all, the coordinates parametrizing $\mathbb{CP}_b$ and $\mathbb{CP}_c$ are respectively:

$$\begin{aligned}
\lambda_b &= e^{-i\phi_2}\tan\frac{\theta_2}{2} = \frac{w_2}{w_6} = \frac{w_8}{w_4},\\
\lambda_c &= e^{-i\phi_3}\tan\frac{\theta_3}{2} = \frac{w_5}{w_1} = \frac{w_3}{w_7}.
\end{aligned} \tag{6.2.19}$$

Looking at (6.2.15) and using (6.2.17) we can thus see that a good identification is

$$U = w_1, \qquad Y = w_4, \qquad X = w_3 \qquad V = w_2. \tag{6.2.20}$$

Then, using (6.2.15), (6.2.19) and (6.2.20), we can obtain (in the NN patch)

$$\sum_{i=1}^{8} |w_i|^2 = (|U|^2 + |Y|^2)(1 + |\lambda_b|^2)(1 + |\lambda_c|^2), \tag{6.2.21}$$

so that a good radial coordinate t is given by (6.2.21).

We know that a Calabi-Yau metric, i.e. Ricci-flat and Kähler, can be found using $g_{m\bar{n}} = \partial_m\bar{\partial}_{\bar{n}}k$, where k is the Kähler potential of the resolved cone. A good ansatz for it should consider the presence of a radial coordinate t that measure the distance of a point along the $\mathbb{C}^2$ fiber from the base $\mathbb{CP}_b \times \mathbb{CP}_c$, as well as the resolutions. Hence our starting point is[17]

$$k(t; b, c) = F(t; b, c) + b\log(1 + |\lambda_b|^2) + c\log(1 + |\lambda_c|^2). \tag{6.2.22}$$

[16] The latter can be directly matched with gauge-invariant combinations of perfect matchings in the GLSM but here we want to focus on the computation of the metric, so we postpone the matching between $\{w\}$ and gauge-invariant combinations of $(a_1, ..., c_2)$.

[17] We will sometimes omit the arguments of the functions during our calculations.

Notice that we can see (6.2.22) as $k = F(t) + v^a k_a = F(t) + bk_b + ck_c$, where $v^a = \{b, c\}$ are the Kähler moduli regulating the volume of the resolutions two-cycles $C^a = \{\mathbb{CP}_b, \mathbb{CP}_c\}$ while k_a are the Kähler potentials of these cycles. We know from (2.3.10) that these take the form written in (6.2.22), where the ζ there are the λ here and the metric on $\mathbb{CP}$ is the Fubini-Study one. The relative Kähler two-forms are

$$\begin{aligned} j_b &= i\partial\bar{\partial}k_b = i\partial\bar{\partial}\log(1+|\lambda_b|^2) = ie^{-2k_b}d\lambda_b \wedge d\bar{\lambda}_b = i\frac{d\lambda_b \wedge d\bar{\lambda}_b}{(1+|\lambda_b|^2)^2}, \\ j_c &= i\partial\bar{\partial}k_c = i\partial\bar{\partial}\log(1+|\lambda_c|^2) = ie^{-2k_c}d\lambda_c \wedge d\bar{\lambda}_c = i\frac{d\lambda_c \wedge d\bar{\lambda}_c}{(1+|\lambda_c|^2)^2}. \end{aligned} \tag{6.2.23}$$

It is useful to identify the base of the bundle (6.2.4) with $B = \mathbb{CP}_b \times \mathbb{CP}_c$ so that its Kähler potential and form are respectively

$$k_B = \sum_a k_a = k_b + k_c, \qquad j_B = \sum_a j_a = j_b + j_c. \tag{6.2.24}$$

Moreover, the radial coordinates takes the more compact expression

$$t = (|U|^2 + |Y|^2)e^{k_B}. \tag{6.2.25}$$

Having defined (6.2.24), it is easy to check that

$$j_b \wedge j_c = -e^{-2k_B} d\lambda_b \wedge d\bar{\lambda}_b \wedge d\lambda_c \wedge d\bar{\lambda}_c. \tag{6.2.26}$$

WARNING: In what follows we will occasionally omit the "wedge" product since this is the natural product between forms. It should be clear from context whether we are using"$\wedge$", for example when working with J, or "$\otimes$", for example if we switch to the metric ds^2 notation. Indeed, recall that finding the Kähler form is equivalent to finding the metric for a Kähler manifold[18].

Using (5.1.12) on (6.2.22) we immediately obtain

$$J = i\partial\bar{\partial}F(t) + bj_b + cj_c \tag{6.2.27}$$

and the core of the calculation is the research of a good $F(t)$, i.e. such that the metric is also Ricci-flat.

Defining $' \equiv \frac{\mathrm{d}}{\mathrm{d}t}$, the Kähler form generated by the potential (6.2.22) is

$$\begin{aligned} J =&(b+F't)j_b + (c+F't)j_c - i(F''t)e^{k_B}\left[\frac{(UdY - YdU)(\bar{U}d\bar{Y} - \bar{Y}d\bar{U})}{|U|^2+|Y|^2}\right] + \\ &+ ie^{k_B}(F''t+F')(dU + U\partial k_B)(c.c.) + ie^{k_B}(F''t+F')(dY + Y\partial k_B)(c.c.). \end{aligned} \tag{6.2.28}$$

[18] As explained in the second chapter, the natural two-form of an hermitian complex manifold $J = J_{i\bar{j}}dz^i \wedge d\bar{z}^{\bar{j}}$ is related to the metric $ds^2 = g_{i\bar{j}}dz^i \otimes d\bar{z}^{\bar{j}}$ by $J_{i\bar{j}} = ig_{i\bar{j}}$.

If we now define $\gamma = F't$ and call $\eta^U = dU + U\partial k_B$, $\eta^Y = dY + Y\partial k_B$, together with their complex conjugate, then a convenient expression for (6.2.28) is

$$\begin{aligned} J =&(b+\gamma)j_b + (c+\gamma)j_c + ie^{k_B}\frac{\gamma}{t}\left[\frac{(U\eta^Y - Y\eta^U)(c.c.)}{|U|^2+|Y|^2}\right]- \\ &- i\gamma' e^{k_B}\left[\eta^U\eta^{\bar{U}} + \eta^Y\eta^{\bar{Y}} - \frac{(U\eta^Y - Y\eta^U)(c.c.)}{|U|^2+|Y|^2}\right]. \end{aligned} \tag{6.2.29}$$

The reason why (6.2.29) is useful is that it is straightforward to compute the volume form on the resolved cone X. Indeed, at this stage we should check that the Kähler form J is compatible with the Ricci flatness condition, i.e. the Ricci form in (2.3.6) must vanish. In other words, we must find a γ such that the determinant of the metric is constant. The easiest way to compute this determinant is to combine the formulas relating the volume form with the metric and the 4-fold product of the Kähler form, namely:

$$\frac{1}{4!}J\wedge J\wedge J\wedge J = \mathrm{d}vol(X_8) = \left(\frac{i}{2}\right)^4 (detg)d^8z, \tag{6.2.30}$$

where $d^8z = dUdYd\bar{U}d\bar{Y}d\lambda_b d\bar{\lambda}_b d\lambda_c d\bar{\lambda}_c$. Using (6.2.29) and (6.2.26) we can easily get

$$J\wedge J\wedge J\wedge J \sim (b+\gamma)(c+\gamma)\gamma'\frac{\gamma}{t}d^8z, \tag{6.2.31}$$

leading to the Calabi-Yau equation condition

$$(b+\gamma)(c+\gamma)\gamma'\gamma = \frac{3}{2}t. \tag{6.2.32}$$

An explicit solution to (6.2.32) is quite difficult to obtain. However, we can still work with an implicit expression in order to check some regularity behaviors: we will do this in a moment. Before that, we want to write down an explicit form for the CY metric on the resolved $C(Q^{111})$ in real coordinates in order to verify if it truly approach the one of the singular cone asymptotically. Using (6.2.17), (6.2.19), (6.2.20), (6.2.25) together with lengthy calculations we finally arrive to

$$ds_X^2 = \frac{1}{4}(b+\gamma)d\Omega_b^2 + \frac{1}{4}(c+\gamma)d\Omega_c^2 + \frac{1}{4}\gamma d\Omega^2 + \gamma' t\left[\frac{dt^2}{4t} + \frac{1}{4}\left(d\psi + \sum_{i=1}^{3}\cos\theta_i d\phi_i\right)^2\right], \tag{6.2.33}$$

where

$$d\Omega^2 = d\theta_1^2 + \sin^2\theta_1 d\phi_1^2, \qquad d\Omega_b^2 = d\theta_2^2 + \sin^2\theta_2 d\phi_2^2, \qquad d\Omega_c^2 = d\theta_3^2 + \sin^2\theta_3 d\phi_3^2. \tag{6.2.34}$$

With a change of variable $t = \rho^2$ we further obtain

$$\begin{aligned} ds_X^2 &= \frac{1}{4}(b+\gamma)d\Omega_b^2 + \frac{1}{4}(c+\gamma)d\Omega_c^2 + \frac{1}{4}\gamma d\Omega^2 + \gamma'\rho^2\left[\frac{d\rho^2}{\rho^2} + \frac{1}{4}\left(d\psi + \sum_{i=1}^{3}\cos\theta_i d\phi_i\right)^2\right] = \\ &= \frac{1}{4}(b+\gamma)d\Omega_b^2 + \frac{1}{4}(c+\gamma)d\Omega_c^2 + \frac{1}{4}\gamma d\Omega^2 + \gamma' d\rho^2 + \frac{\gamma'\rho^2}{4}\left(d\psi + \sum_{i=1}^{3}\cos\theta_i d\phi_i\right)^2. \end{aligned} \tag{6.2.35}$$

Finally, it is convenient to introduce a new radial coordinate and a function, namely

$$2\gamma = r^2, \qquad \Phi = \frac{2\rho^2\gamma'}{\gamma}, \tag{6.2.36}$$

so that (6.2.35) becomes

$$ds_X^2 = \frac{1}{8}(2b+r^2)d\Omega_b^2 + \frac{1}{8}(2c+r^2)d\Omega_c^2 + \frac{1}{8}r^2 d\Omega^2 + \Phi^{-1}dr^2 + \Phi\frac{r^2}{16}\left(d\psi + \sum_{i=1}^{3}\cos\theta_i d\phi_i\right)^2. \tag{6.2.37}$$

It can be shown, as in [34], that $\Phi \to 1$ at large r and hence (6.2.37) approaches asymptotically the metric of the singular cone, with base metric (3.2.26), namely

$$\begin{aligned} ds_X^2 \to & \frac{r^2}{8}\sum_{i=1}^{3}(d\theta_i^2 + \sin^2\theta_i d\phi_i^2) + dr^2 + \frac{r^2}{16}\left(d\psi + \sum_{i=1}^{3}\cos\theta_i d\phi_i\right)^2 = \\ & = dr^2 + r^2 ds^2(Y = Q^{111}) = ds^2_{C(Q^{111})}. \end{aligned} \tag{6.2.38}$$

On the other hand, if Kähler moduli are set to zero, i.e. $b = c = 0$ then $\Phi = 1$ and (6.2.37) is again the metric on the singular cone, whereas taking only a combination of resolution parameters different from zero one obtains different partial resolutions of $C(Q^{111})$. Besides, if $r = 0$ then we are sitting on the resolution manifold, as expected from the meaning of the radial coordinate itself.

6.3 The HEFT ingredients

In this section we are going to collect all the ingredients for the HEFT Lagrangian of the Q^{111}. We compute the harmonic forms ω_a as well as the potentials k, k_0, κ_a necessary for the HEFT. About the former, we will check that they are in fact harmonic, i.e. $\Delta\omega = 0$. For the latter we will need some asymptotic calculations which turn out to be crucial for our final check on the conformal symmetry. In both cases we are lacking an explicit solution $\gamma(t; b, c)$ to (6.2.32), so the typical objects appearing in the HEFT Lagrangian will be expressed in integral form.

6.3.1 Harmonic forms

In order to compute the harmonic two-forms we will use (5.1.6), with J given by

$$\begin{aligned} J =& (b+\gamma)j_b + (c+\gamma)j_c + ie^{k_B}\gamma'[(dU + U\partial k_B)(c.c.) + (dY + Y\partial k_B)(c.c.)] + \\ & + ie^{k_B}\left(\frac{\gamma}{t} - \gamma'\right)\left[\frac{(UdY - YdU)(\bar{U}d\bar{Y} - \bar{Y}d\bar{U})}{|U|^2 + |Y|^2}\right] \end{aligned} \tag{6.3.1}$$

and $v^a = (b, c)$. We easily find:

$$
\begin{aligned}
\omega_1 = \omega_b = \frac{\partial J}{\partial b} =& j_b + \frac{\partial \gamma}{\partial b} j_B + ie^{k_B} \frac{\partial \gamma'}{\partial b} [(dU + U\partial k_B)(c.c.) + (dY + Y\partial k_B)(c.c.)] + \\
&+ ie^{k_B} \left(\frac{1}{t}\frac{\partial \gamma}{\partial b} - \frac{\partial \gamma'}{\partial b} \right) \left[\frac{(UdY - YdU)(\bar{U}d\bar{Y} - \bar{Y}d\bar{U})}{|U|^2 + |Y|^2} \right], \\
\omega_2 = \omega_c = \frac{\partial J}{\partial c} =& j_c + \frac{\partial \gamma}{\partial c} j_B + ie^{k_B} \frac{\partial \gamma'}{\partial c} [(dU + U\partial k_B)(c.c.) + (dY + Y\partial k_B)(c.c.)] + \\
&+ ie^{k_B} \left(\frac{1}{t}\frac{\partial \gamma}{\partial c} - \frac{\partial \gamma'}{\partial c} \right) \left[\frac{(UdY - YdU)(\bar{U}d\bar{Y} - \bar{Y}d\bar{U})}{|U|^2 + |Y|^2} \right].
\end{aligned}
\tag{6.3.2}
$$

Then, we should check that (6.3.2) are truly harmonic, i.e. they are both closed $d\omega = 0$ and co-closed $d^\dagger \omega = 0$. The easiest way to prove it is to use the concept of "primitivity". As stated in [21], for a CY_4 the primitive (p, q)-forms $\omega^{p,q}$ satisfy

$$\underbrace{J \wedge \cdots \wedge J}_{5-p-q \text{ times}} \wedge \omega^{p,q} = 0. \tag{6.3.3}$$

The clue is that if a primitive form is closed, then it is also co-closed[19]: so we are going to check that (6.3.2) are both closed and primitive $(1, 1)$-forms.

Closure is quite obvious since $d\omega = d\big(\frac{\partial J}{\partial v}\big) = \frac{\partial}{\partial v}(dJ)$ and J being the Kähler form is closed, i.e. $dJ = 0$. For primitivity we must check that $J \wedge J \wedge J \wedge \omega = 0$. From (6.2.31) and (6.2.32) we know that $J \wedge J \wedge J \wedge J$ does not depend on resolution moduli, i.e. $\frac{\partial}{\partial v}(J \wedge J \wedge J \wedge J) = 0$. By using (anti)commutation rules for differential forms calculus this is equivalent to $0 = J \wedge J \wedge J \wedge \frac{\partial J}{\partial v} = J \wedge J \wedge J \wedge \omega$.

6.3.2 Asymptotic behaviors

First of all we go back to (6.2.32) and write it as

$$(b + \gamma)(c + \gamma)\gamma'\gamma = l_1 t, \qquad l_1 = const. \tag{6.3.4}$$

This can be easily integrated obtaining

$$\frac{1}{4}\gamma^4 + \frac{1}{3}(b + c)\gamma^3 + \frac{1}{2}bc\gamma^2 = l_2 + l_1\frac{1}{2}t^2, \qquad l_1, l_2 = const. \tag{6.3.5}$$

Since $F'(t)$ should be regular at $t = 0$, then it must be $\gamma(t = 0) = 0$ from $\gamma = F't$ and so the constant l_2 in (6.3.5) must vanish for consistency. Besides, differentiation of (6.3.5) with respect to t (two times) leads to $\gamma'(t = 0) = \sqrt{\frac{l_1}{bc}}$ and hence when $b, c > 0$ we must take $l_1 > 0$ in order to have $\gamma'(t) > 0$ everywhere. Then, the constant l_1 can always be reabsorbed into the

[19] In the case of CY_4, so $n = 8$ real dimensions, we have $d^\dagger \chi^{(1,1)} = \star_8 d \star_8 \chi^{(1,1)}$ for a $(1, 1)$-form χ. If χ is also primitive then $\star_8 \chi = \frac{1}{2} J \wedge J \wedge \chi$. If χ is also closed then $d\chi = 0$. So $d^\dagger \chi = \star_8 d(J \wedge J \wedge \chi)$: applying the Leibniz rule for the exterior derivative we find $d^\dagger \chi = 0$ because the Kähler form is closed too, i.e. $dJ = 0$.

radial variable t and so we choose it to be $l_1 = 1$. The fact that $\gamma(0) = 0$ and $\gamma'(t) > 0$ let us interpret γ as a radial variable itself[20], defined by

$$\frac{1}{2}t^2 = \frac{1}{4}\gamma^4 + \frac{1}{3}(b+c)\gamma^3 + \frac{1}{2}bc\gamma^2. \tag{6.3.6}$$

Notice that when $\gamma \gg b, c$ then from (6.3.6) we get $t^2 \sim \gamma^4$ and so $\gamma \sim t^{\frac{1}{2}}$ as $t \to +\infty$. Since we also introduced $2\gamma = r^2$ in (6.2.36), then we also get the behavior $t \sim r^4$ at large r. From $\gamma = F't$ by integration we obtain

$$F(t) = \int_0^t \frac{\mathrm{d}\tilde{t}}{\tilde{t}}\gamma(\tilde{t}) \sim r^2. \tag{6.3.7}$$

This asymptotic behavior for the Kähler potential $k \sim F(t) \sim r^2$, see (6.2.22), turns out to be the correct one for the scale invariance. Actually, $\gamma \gg b, c$ corresponds to a region where the "energy scale" γ is well above the scale set by some VEVs b, c. We can interpret this γ as a VEV itself, this time for the "radial position" of one of the $\tilde{N}$ mobile M2-branes on the resolved cone X. Then, for $\gamma \gg b, c$ we can imagine that the geometry that the M2-brane "sees" is the one of the singular cone because it is far away from the resolved singularity. We anticipate that this is useful because from there we can relate the scaling dimension of the radial coordinate r, which is known only in the large-r region, with the scaling dimensions of the chiral coordinates $\{z\}$. Since the latter are "pure coordinates" in the sense that their scaling dimensions do not depend on any asymptotic behavior, we can both make use of them "everywhere", even in the $\gamma \ll b, c$ region, and compare the result with the field theory predictions for the scaling dimensions of $\{z\}$.[21] On the other side, one can also explore the $\gamma \ll b, c$ region too. Here, from (6.3.6) we get $t^2 \sim bc\gamma^2$ and hence $F(t) \sim \frac{t}{\sqrt{bc}}$: this is good for two reasons. Firstly, $F(0) \to 0$ makes sense because, looking at (6.2.22), when $t \to 0$ we are "near" the resolution spheres and hence we expect that the Kähler potential reduces to $bk_b + ck_c$ just like the metric (6.2.37) reduces to $\frac{1}{4}(bd\Omega_b^2 + cd\Omega_c^2)$ as $r \to 0$. Secondly, it seems that even if the VEV γ for the "radial position" of our M2-brane is lower than the VEVs for the resolutions, i.e. the M2-brane "sees" the resolved geometry, we find a good scaling behavior for $F(t)$. This is curious because while scaling dimensions can be surely obtained in the asymptotic region far away from the resolutions, as the M2-brane approaches resolutions we have no right to surely state that such asymptotic scaling dimensions hold "everywhere". However, the fact that one can find the correct scaling dimensions for the chiral coordinates $\{z\}$, which are "asymptotic-independent", suggests that the asymptotic scaling dimensions in fact hold "everywhere". We will return on this topic when we will perform the final SCFT check.

It is useful to observe that one can write

$$\gamma(t) = t^{\frac{1}{2}}\tilde{\gamma}\left(\frac{b}{t^{\frac{1}{2}}}, \frac{c}{t^{\frac{1}{2}}}\right) = t^{\frac{1}{2}}\tilde{\gamma}(\alpha_b, \alpha_c), \tag{6.3.8}$$

[20] It varies monotonically from 0 to $+\infty$.

[21] Recall that we should find a matching between complex coordinates $\{z\}$ and chiral fields in the UV-quiver, together with monopoles: we will see an example in the next section.

and correspondingly (6.3.6) translates into

$$\frac{1}{2} = \frac{1}{4}\tilde{\gamma}^4 + \frac{1}{3}(\alpha_b + \alpha_c)\tilde{\gamma}^3 + \frac{1}{2}\alpha_b\alpha_c\tilde{\gamma}^2. \tag{6.3.9}$$

Then, we define $\tau = \tilde{t}^{\frac{1}{2}}$ so that (6.3.7) reads

$$F(t) = \int_0^{t^{\frac{1}{2}}} \mathrm{d}\tau\tilde{\gamma}(\frac{b}{\tau}, \frac{c}{\tau}). \tag{6.3.10}$$

Notice that (6.3.10) satisfies the homogeneity condition

$$F(\lambda^2 t; \lambda v) = \lambda F(t; v). \tag{6.3.11}$$

Before going on, we should mention that in our calculations we have omitted some normalization constants in order to be more clear. However, we want to be somehow more precise in the asymptotic behaviors, especially because they are crucial for the computation of potentials in the next section. Hence, we show how this constants can be fixed. First of all, one should require the quantization condition on harmonic forms, namely $\int_{C^a} \omega_e = \delta^a_e$. Let us consider the resolution parameter $v_1 = b$, but the same goes for $v_2 = c$. We can compute

$$\int_{\mathbb{CP}_b} J = \int_{\mathbb{CP}_b} (b+\gamma)j_b = \int_{\mathbb{CP}_b} (b+\gamma(t=0))j_b = b\int_{\mathbb{CP}_b} j_b, \tag{6.3.12}$$

with $j_b = -\frac{1}{2}\sin\theta_2 d\phi_2 \wedge d\theta_2$. It is then easy to check that $\int_{\mathbb{CP}_b} j_b = 2\pi$ and hence $b = \frac{1}{2\pi}\int_{\mathbb{CP}_b} J$. Thus, from (5.1.6) we get $\int_{\mathbb{CP}_b} \omega_b = 2\pi$: it is now clear that we have to normalize the Kähler form by 2π. Indeed, $J \to \frac{J}{2\pi}$ and hence $\omega_b \to \frac{\omega_b}{2\pi}$ so that $\int_{\mathbb{CP}_b} \omega_b = 1$ as we wanted. An alternative check is to make use of the first Chern class (2.3.7) and of the relation (2.3.12) for the case of n-dimensional projective space with $n = 1$. Then we have

$$\int_{\mathbb{CP}_b} j_b = -\frac{1}{2}\int_{\mathbb{CP}_b} R_{\mathbb{CP}_b} = -\pi\int_{\mathbb{CP}_b} c_1 = 2\pi \tag{6.3.13}$$

since $\int_{\mathbb{CP}_b} c_1 = -2$ for the unidimensional projective space. Hence we are led to the very same normalization. At the same time, since J is related to the metric we should consider a modification of radial variables too, namely:

$$\gamma = F't \to \gamma = 2\pi F't, \qquad 2\gamma = r^2 \to 2\gamma = \pi r^2. \tag{6.3.14}$$

Now we can complete the discussion on (6.3.10) and its homogeneity relation (6.3.11). First of all, (6.3.7) and (6.3.10) become respectively

$$F(t) = \frac{1}{2\pi}\int_0^t \frac{\mathrm{d}\tilde{t}}{\tilde{t}}\gamma(\tilde{t}), \qquad F(t) = \frac{1}{\pi}\int_0^{t^{\frac{1}{2}}} \mathrm{d}\tau\tilde{\gamma}(\frac{b}{\tau}, \frac{c}{\tau}). \tag{6.3.15}$$

Using (6.3.14), since at large t we have $\gamma \simeq 2^{\frac{1}{4}} t^{\frac{1}{2}}$, the relation between t and r takes the form $t \simeq \frac{\pi^2}{4\sqrt{2}} r^4$. Then, asymptotically integrating (6.3.15) we get:

$$k \simeq F(t) \simeq \frac{2^{\frac{1}{4}}}{\pi} t^{\frac{1}{2}} \simeq \frac{2^{\frac{1}{4}}}{\pi} \left(\frac{\pi^2}{4\sqrt{2}} r^4 \right)^{\frac{1}{2}} = \frac{1}{2} r^2. \tag{6.3.16}$$

Now we want to study the next order of the asymptotic expansion. For large t, or better for $t^{\frac{1}{2}} \gg v^a$, at first order in v^a/τ we can actually find

$$\gamma \simeq 2^{\frac{1}{4}} t^{\frac{1}{2}} - \frac{1}{3}(b+c) \qquad \rightarrow \qquad \tilde{\gamma} \simeq 2^{\frac{1}{4}} - \frac{1}{3}(\alpha_b + \alpha_c). \tag{6.3.17}$$

Then, using the correct normalization we get

$$F(t; v) \simeq \frac{1}{\pi} \left[2^{\frac{1}{4}} t^{\frac{1}{2}} - \frac{1}{6}(b+c) \log t \right] + \tilde{F}(v) + \sum_{n \geq 1} t^{-\frac{n}{2}} \tilde{f}_{(n)}(v), \tag{6.3.18}$$

where (6.3.11) requires

$$\tilde{F}(\lambda v) = \lambda \tilde{F}(v) + \frac{1}{3\pi}(b+c)\lambda \log \lambda, \qquad \tilde{f}_{(n)}(\lambda v) = \lambda^{n+1} \tilde{f}_{(n)}(v). \tag{6.3.19}$$

Moreover, since $F(t; v)$ is independent from λ, we can differentiate (6.3.11) in order to obtain

$$v^a \frac{\partial F}{\partial v^a} + 2t \frac{\partial F}{\partial t} = F \tag{6.3.20}$$

and from there, using $2t \frac{\partial F}{\partial t} = \frac{1}{\pi} \gamma(t; v)$, we get

$$v^a \frac{\partial F}{\partial v^a} - F = -\frac{1}{\pi} \gamma. \tag{6.3.21}$$

Now, deriving (6.3.21) with respect to moduli gives

$$v^b \frac{\partial^2 F}{\partial v^a \partial v^b} = -\frac{1}{\pi} \frac{\partial \gamma}{\partial v^a}. \tag{6.3.22}$$

Since $\frac{\partial \gamma}{\partial v^a} \to -\frac{1}{3}$ for large γ,[22] it is clear from (6.3.18) and (6.3.19) that

$$v^b \frac{\partial^2 \tilde{F}}{\partial v^a \partial v^b} = \frac{1}{3\pi} \tag{6.3.23}$$

and

$$\tilde{F}(v) = \frac{1}{3\pi}(b \log b + c \log c) + \tilde{\tilde{F}}(v) + v^a \tilde{l}_a + const., \tag{6.3.24}$$

where $\tilde{l}_a$ are constants and $\tilde{\tilde{F}}(\lambda v) = \lambda \tilde{\tilde{F}}(v)$ so that both (6.3.19) and (6.3.23) are satisfied[23].

[22]Take for instance b. Since $\frac{\partial}{\partial b}\left(\frac{t^2}{2}\right) = 0$ then we get $\frac{\partial \gamma}{\partial b} = \frac{-\frac{1}{3}\gamma^3 - \frac{1}{2}c\gamma^2}{\gamma^3 + \frac{1}{3}(b+c)\gamma^2 + bc\gamma}$ from (6.3.6).

[23]This is because the homogeneity of $\tilde{\tilde{F}}$ implies $v^a \tilde{\tilde{F}}_a = \tilde{\tilde{F}}$ and from there $v^b \frac{\partial^2 \tilde{\tilde{F}}}{\partial v^a \partial v^b} = 0$.

6.3.3 Potentials

Now we want to compute the explicit form of potentials in (5.1.13), namely k_0 and κ_a. First of all, we have to slightly modify our notation because the following calculation is very delicate. The Kähler potential in (6.2.22) is not the complete one because we could have added a function independent on complex coordinates such that the Kähler form $J = i\partial\bar{\partial}k$ would have been the same. So, in what follows (6.2.22) is renamed as $\hat{k}$ and using the correct normalization reads

$$\hat{k} = F(t;v) + \frac{1}{2\pi}v^a k_a = \frac{1}{2\pi}\int_0^{t^{\frac{1}{2}}} \frac{\mathrm{d}\tilde{t}}{\tilde{t}}\gamma(\tilde{t}) + \frac{1}{2\pi}v^a k_a. \tag{6.3.25}$$

We stress that $\hat{k} \neq k$ and the difference is due to the fact that potentials are defined modulo ambiguities depending on Kähler moduli, namely

$$k = \hat{k} + \Omega(v) = F(t;v) + \frac{1}{2\pi}v^a k_a + \Omega(v). \tag{6.3.26}$$

Recalling (5.1.13) we then obtain

$$\kappa_a = \frac{\partial k}{\partial v^a} = \frac{\partial F}{\partial v^a} + \frac{1}{2\pi}k_a + \Omega_a(v), \qquad \Omega_a = \frac{\partial \Omega}{\partial v^a}. \tag{6.3.27}$$

Since for large values of t we have $\frac{\partial^2 F}{\partial v^a \partial v^b} \to \frac{\partial^2 \tilde{F}}{\partial v^a \partial v^b}$, see for instance (6.3.18), the asymptotic condition in (5.1.13) is easily satisfied provided that

$$\Omega(v) = -\tilde{F}(v) = -\frac{1}{3\pi}(b\log b + c\log c) - \tilde{\tilde{F}}(v) - v^a\tilde{l}_a - const., \tag{6.3.28}$$

where we used (6.3.24). Thus, (6.3.26) is telling us that

$$k = \hat{k} + \Omega(v) = F(t;v) + \frac{1}{2\pi}v^a k_a - \frac{1}{3\pi}(b\log b + c\log c) - \tilde{\tilde{F}}(v) - v^a\tilde{l}_a - const., \tag{6.3.29}$$

whereas from (6.3.27) we get

$$\kappa_a = \frac{\partial k}{\partial v^a} = \frac{\partial F}{\partial v^a} + \frac{1}{2\pi}k_a - \frac{1}{3\pi}\log v^a - \frac{1}{3\pi} - \tilde{\tilde{F}}_a(v) - \tilde{l}_a, \qquad \tilde{\tilde{F}}_a = \frac{\partial \tilde{\tilde{F}}}{\partial v^a}. \tag{6.3.30}$$

Notice that the homogeneous function $\tilde{\tilde{F}}(v)$ remains undetermined: nevertheless, it disappears from the asymptotic expansion of the potentials[24], i.e. when $t^{\frac{1}{2}} \gg v^a$. Moreover, we can compute k_0 from (5.1.13) using (6.3.29), (6.3.30) and (6.3.20). The result reads

$$k_0 = k - v^a\kappa_a = \frac{1}{\pi}\gamma + \frac{1}{3\pi}(b+c), \tag{6.3.31}$$

where we also used the homogeneity relation $v^a\tilde{\tilde{F}}_a = \tilde{\tilde{F}}$. Recall that k_0 is very important in the HEFT because of (5.3.1) and (5.3.2).

[24]Indeed, for large t the F in (6.3.29) and (6.3.30) is the asymptotic expanded version (6.3.18). Using (6.3.24) it is clear that $\tilde{\tilde{F}}(v)$ goes away.

6.3.4 The G_{ab} metric and the $\mathcal{A}^I_{ai}$ connection

The only HEFT ingredients left are the G_{ab} matrix and the $\mathcal{A}^I_{ai}$ connection introduced in the previous chapter. From (5.2.1) and (5.2.2), together with (6.3.30), we obtain

$$\begin{aligned} G_{ab} &= -\frac{\partial \operatorname{Re}\rho_a}{\partial v^b} = -\frac{1}{2}\sum_I \frac{\partial \kappa_a}{\partial v^b} = \frac{1}{2}\sum_I \frac{\partial^2 k}{\partial v^a \partial v^b} = \\ &= -\frac{1}{2}\sum_I \left[\frac{1}{\pi}\int_0^{t_I^{\frac{1}{2}}} \mathrm{d}\tau \frac{\partial^2 \tilde{\gamma}}{\partial v^a \partial v^b} - \frac{1}{3\pi}\delta_{ab}\frac{1}{v^b} - \frac{\partial^2 \tilde{\tilde{F}}}{\partial v^a \partial v^b}\right], \end{aligned} \tag{6.3.32}$$

where the sum runs over the index I related to the number of branes in the theory[25].

On the other hand, connections are given by (5.3.6) and (6.3.30), namely

$$\mathcal{A}^I_{ai} = \frac{\partial \kappa_a}{\partial z^i_I} = \frac{1}{\pi}\left[\frac{\partial}{\partial z^i_I}\int_0^{t_I^{\frac{1}{2}}} \mathrm{d}\tau \frac{\partial \tilde{\gamma}}{\partial v^a} + \frac{1}{2}\frac{\partial}{\partial z^i_I}\log(1+|\lambda^I_a|^2)\right], \tag{6.3.33}$$

where $z^i_I = (U, Y, \lambda_b, \lambda_c)_I$ in the NN patch: schematically, we should distinguish the case $z = U, Y$ from $z = \lambda$.

First of all we rewrite[26] the first term in (6.3.33) as

$$\mathcal{A}^I_{ai} = \frac{1}{\pi}\frac{\partial t}{\partial z^i_I}\frac{\partial}{\partial t}\int_0^{t^{\frac{1}{2}}} \mathrm{d}\tau \frac{\partial \tilde{\gamma}}{\partial v^a} \tag{6.3.34}$$

and using (6.2.25) we can compute

$$\frac{\partial t}{\partial z} = \begin{cases} \bar{U}e^{k_B} & \text{if } z = U \\ \bar{Y}e^{k_B} & \text{if } z = Y \\ t\frac{\partial k_B}{\partial z} = t\frac{\bar{\lambda}}{1+|\lambda|^2} & \text{if } z = \lambda. \end{cases} \tag{6.3.35}$$

Now, the connections (6.3.33) reads

$$\begin{aligned} \mathcal{A}^I_{a(\lambda)} &= \frac{1}{\pi}\frac{\bar{\lambda}^I_a}{1+|\lambda^I_a|^2}\left[\frac{1}{2} + t\frac{\partial}{\partial t}\int_0^{t^{\frac{1}{2}}} \mathrm{d}\tau \frac{\partial \tilde{\gamma}}{\partial v^a}\right], && z = \lambda \\ \mathcal{A}^I_{a(U)} &= \frac{1}{\pi}\frac{\bar{U}^I}{|U^I|^2 + |Y^I|^2}\left[t\frac{\partial}{\partial t}\int_0^{t^{\frac{1}{2}}} \mathrm{d}\tau \frac{\partial \tilde{\gamma}}{\partial v^a}\right], && z = U \\ \mathcal{A}^I_{a(Y)} &= \frac{1}{\pi}\frac{\bar{Y}^I}{|U^I|^2 + |Y^I|^2}\left[t\frac{\partial}{\partial t}\int_0^{t^{\frac{1}{2}}} \mathrm{d}\tau \frac{\partial \tilde{\gamma}}{\partial v^a}\right], && z = Y. \end{aligned} \tag{6.3.36}$$

At large values of the radial variable where we can take (6.3.17), connections in (6.3.36) go like:

$$\mathcal{A}_{(\lambda)} \sim \frac{\bar{\lambda}}{1+|\lambda|^2}, \qquad \mathcal{A}_{(U)} \sim \frac{\bar{U}}{|U|^2+|Y|^2}, \qquad \mathcal{A}_{(Y)} \sim \frac{\bar{Y}}{|U|^2+|Y|^2}. \tag{6.3.37}$$

Even though these results are implicit, the HEFT Lagrangian at two-derivatives order is the nonlinear sigma model (5.3.9) with kinetic terms characterized by (6.3.32), (6.3.33) and (6.2.28).

[25] We should point out that there is one radial coordinate t_I for every brane. Indeed, recall that there are also different sets of coordinates $\{z\}_I$.

[26] Dropping the index I from the radial coordinate t_I for clarity.

6.4 Final consistency checks

In this section we want to perform some consistency checks. One of them is about the moduli space and the dimensional reduction from M-Theory to type IIA String Theory, the other one is about the conformal symmetry of the HEFT. For the former we will exploit toric geometry and the GLSM while for the latter we will use the asymptotic behaviors worked out in the previous section.

6.4.1 From M-Theory to type IIA

As we mentioned, toric geometry and the GLSM provide an interpretation of flavors in the quiver theory in terms of branes. Here we will focus on the abelian $U(1)^2$ quiver theory, whose moduli space was shown to be $C(Q^{111})$ and was matched to the M-Theory moduli space, i.e. the moduli space of a single M2-brane probing $C(Q^{111})$ in this case, using (4.4.32) and (4.4.33). There, we built gauge-invariant combinations of the UV-quiver chiral fields and monopoles: these were matched to the set of coordinates $\{w\}$ parametrizing $C(Q^{111})$ as an affine variety. Recall that the monopole method was unable to "see" resolutions and that we argued that there should be a sector in the moduli space giving a three-dimensional CY cone $C(T^{11})$, the latter being the Klebanov-Witten model (KW) of [39]. An alternative version of the moduli space was obtained from the semiclassical method, see (4.4.19) and its interpretation, which in turn is "aware" of resolutions. There, the moduli space CY_4 was a resolved cone but it was not the one of (6.2.4). Indeed, the CY_4 was shown to be an $U(1)$ fibration, parametrized by the scalar photon τ, of a seven-manifold, the latter being a CY_3 fibered over the real line parametrized by σ. The CY_3 was a resolved version of $C(T^{11})$ with resolution parameter given by (4.4.19). However, the M-Theory background in (6.2.4), which is the moduli space[27] of one M2-brane probing the resolved cone X, is characterized by two resolution parameters $v_1, v_2 = b, c$. We would like to match these two pictures and in order to do this we should perform a dimensional reduction of M-Theory to type IIA along a circle $U(1)_M$, namely $CY_3 = CY_4//U(1)_M$. This is done by studying the GLSM and choosing a particular M-Theory circle so that $U(1)_M$ is interpreted as a new gauge group: we will follow [6, 35, 2].

Dimensional reduction: monopole method

We begin from the singular cone case. The GLSM is given by

$$\begin{array}{c|cccccc} & a_1 & a_2 & b_1 & b_2 & c_1 & c_2 \\ \hline U(1)_I & -1 & -1 & 1 & 1 & 0 & 0 \\ U(1)_{II} & -1 & -1 & 0 & 0 & 1 & 1 \\ \hline U(1)_M & 0 & 1 & 0 & 0 & 0 & -1 \end{array} \tag{6.4.1}$$

[27]We are again considering non-dynamical parameters.

where we also wrote the action of $U(1)_M$ on perfect matchings $(a_1, ..., c_2) \in \mathbb{C}^6$. There are eight gauge-invariant combinations of perfect matchings with respect to $U(1)_I \times U(1)_{II}$, namely

$$\begin{aligned} &w_1 = a_1 b_2 c_2, \quad w_2 = a_2 b_1 c_1, \quad w_3 = a_1 b_1 c_1, \quad w_4 = a_2 b_2 c_2, \\ &w_5 = a_1 b_2 c_1, \quad w_6 = a_2 b_2 c_1, \quad w_7 = a_1 b_1 c_2, \quad w_8 = a_2 b_1 c_2. \end{aligned} \tag{6.4.2}$$

Notice that these combinations satisfy the set of constraint (4.4.33) so that the GLSM (6.4.1) perfectly realizes the toric $C(Q^{111})$ as its moduli space: here, we matched the perfect matching with the set of coordinates $\{w\}$. Moreover, if we start from (6.2.19),(6.2.20) and try to build a dictionary between the coordinates $\{z\}$ parametrizing the position of the M2-brane on $C(Q^{111})$ and the perfect matchings variables we can identify the whole set of affine coordinates $\{w\}$ as a result.[28]

Now we can proceed with two calculations: identify the CY_3 in the dimensional reduction and try to match perfect matchings and chiral fields in the UV-quiver. We begin from the latter and report the gauge charges for clarity, namely

$$\begin{array}{c|cccc|cc} & A_i & B_i & p_i & q_i & T & \tilde{T} \\ \hline U_1(1)_0 & 1 & -1 & -1 & 0 & 1 & 1 \\ U_2(1)_0 & -1 & 1 & 0 & 1 & -1 & -1 \end{array} \tag{6.4.3}$$

Recall that in order to reproduce the correct moduli space monopole operators should satisfy the constraint (4.4.26), which in this case is

$$T\tilde{T} = A_1 A_2. \tag{6.4.4}$$

We can easily solve (6.4.4) via perfect matching variables in (6.4.1) as

$$A_1 = a_1 c_1, \quad A_2 = a_2 c_2, \quad B_1 = b_1, \quad B_2 = b_2, \quad T = a_2 c_1, \quad \tilde{T} = a_1 c_2, \tag{6.4.5}$$

which provides the identification of UV-quiver fields and monopoles with perfect matchings, in the sense of (6.1.1). Then, using the dictionary (6.4.5) we can translate the gauge-invariant combinations of perfect matching in (6.4.2) into gauge-invariant combinations of UV-quiver fields and monopoles: the result is exactly (4.4.32), namely

$$\begin{aligned} &w_1 = a_1 b_2 c_2 = \tilde{T} B_2, \quad w_2 = a_2 b_1 c_1 = T B_1, \quad w_3 = a_1 b_1 c_1 = A_1 B_1, \quad w_4 = a_2 b_2 c_2 = A_2 B_2, \\ &w_5 = a_1 b_2 c_1 = A_1 B_2, \quad w_6 = a_2 b_2 c_1 = T B_2, \quad w_7 = a_1 b_1 c_2 = \tilde{T} B_1, \quad w_8 = a_2 b_1 c_2 = A_2 B_1. \end{aligned} \tag{6.4.6}$$

We argued that the combinations without monopoles in (4.4.32) could give rise to the equation $w_3 w_4 - w_5 w_8 = 0$ defining the conifold $CY_3 = C(T^{11})$. Here we can be somehow more precise in

[28] For example, taking $U = a_1 b_2 c_2$ and $X = a_1 b_1 c_1$ we reproduce (6.2.15). Then we find out that $a_1 b_2 c_2$ and $a_1 b_1 c_1$ are actually gauge-invariant combinations of perfect matchings. So, we can identify them with w_1 and w_3 respectively.

saying this thanks to the dimensional reduction and the GLSM. Indeed, notice that according to the $U(1)_M$ charges in (6.4.1) we have

$$\begin{array}{c|cccccccc} & w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 \\ \hline U(1)_M & -1 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \end{array} \tag{6.4.7}$$

so that $\{w_3, w_4, w_5, w_8\}$ are in fact uncharged combinations under $U(1)_M$. Hence, the dimensional reduction let us clearly see the sector $w_3w_4 - w_5w_8 = 0$, i.e. the Klebanov-Witten conifold, "inside" the $C(Q^{111})$ model. As a side result, we can actually identify some of the $\{z\}$ with gauge invariant combinations of quiver fields. For instance, in the NN patch we have $Y = w_4 = A_2B_2$ and we know that Y is related to a chiral field in the HEFT Lagrangian (5.3.9) of the z-kind, i.e. not the ρ-kind one. This field is actually a low-energy degree of freedom in the HEFT and it turned out to be a combination of fields of the UV quiver theory, in this case A_2B_2. Here, we want to point a parallelism with QCD where pions, i.e. some low-energy degrees of freedom, are bound states of quarks, which are matter fields in the UV theory.

Toric diagram and brane interpretation

Now we want to give a pictorical idea of how flavors are related to D6-branes emerging in the dimensional reduction using toric diagrams introduced at the beginning of this chapter. A complete calculation is beyond the aim of the thesis, so we will refer the reader to [6] for a deeper analysis.

As noticed for example in [6], the quiver structure of Q^{111} is actually the ABJM one of [40] with the addition of flavors. The latter is another three-dimensional SUSY model but it has more supersymmetries than Q^{111}. Moreover, the ABJM quiver is the same of the KW one: even if they are theories in different spacetime dimensions, the quiver structure is the same. An interesting discussion involves toric geometry. The ABJM has a 3d toric diagram with four external points: they are

$$\{(1,0,0),(0,1,0),(1,1,0),(0,0,1)\} \qquad \text{ABJM toric diagram.} \tag{6.4.8}$$

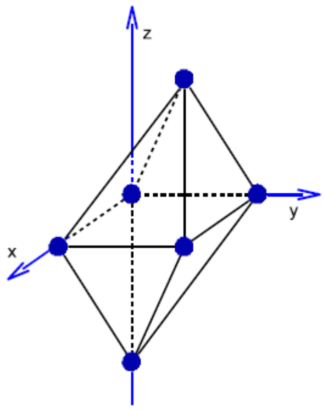
If we consider the flavoring of the Q^{111} quiver, which consists in $\tilde{G} = h_1 + h_2 = 1 + 1 = 2$ new flavor nodes with respect to ABJM quiver, it can be shown that two new points in the ABJM toric diagram are added: they are both "below" ABJM points in (6.4.8), giving

$$\{(1,0,0),(0,1,0),(1,1,0),(0,0,1),(0,0,0),(1,1,-1)\} \qquad C(Q^{111}) \text{ toric diagram.} \tag{6.4.9}$$

The six points in (6.4.9) are then related to six perfect matchings, which themselves enter in the GLSM description. With respect to (6.4.1) we shall rename these fields in order to make evident their positions in the diagram, namely

$$a_1 \to a_0, \quad a_2 \to c_0, \quad b_1 \to b_0, \quad b_2 \to d_0, \quad c_1 \to a_{-1}, \quad c_2 \to c_1, \tag{6.4.10}$$

so that the index is related to the z-quote. For clarity, we report the toric diagram of [6] for $C(Q^{111})$:



Then, we can express (6.4.6) using (6.4.10):

$$\begin{gathered} w_1 = a_0 d_0 c_1 = \tilde{T} B_2, \quad w_2 = c_0 b_0 a_{-1} = T B_1, \quad w_3 = a_0 b_0 a_{-1} = A_1 B_1, \quad w_4 = c_0 d_0 c_1 = A_2 B_2, \\ w_5 = a_0 d_0 a_{-1} = A_1 B_2, \quad w_6 = c_0 d_0 a_{-1} = T B_2, \quad w_7 = a_0 b_0 c_1 = \tilde{T} B_1, \quad w_8 = c_0 b_0 c_1 = A_2 B_1. \end{gathered} \tag{6.4.11}$$

As we mentioned earlier, a projection of the 3d toric diagram into a 2d diagram is equivalent to a suitable dimensional reduction of M-Theory to type IIA. Actually, projecting (6.4.9) into the $z = 0$ plane gives a 2d toric diagram with four points: they are

$$\{(1,0,0), (0,1,0), (1,1,0), (0,0,0)\} \qquad C(T^{11}) \text{ toric diagram} \tag{6.4.12}$$

and we should not be surprised that (6.4.12) is exactly the toric diagram related to the KW theory. Indeed, we have just found the $CY_3 = C(T^{11})$ "inside" the $CY_4 = C(Q^{111})$. Then, since the two points having $z \neq 0$, i.e. a_{-1} and c_1, are vertically aligned to other two points having $z = 0$, i.e. a_0 and c_0, the vertical projection, equivalent to the dimensional reduction, give rise to two "detached" D6-branes in type IIA. We mean that there will be D2-D6 systems with open strings connecting them that give rise to $U(h_1 = 1) \times U(h_2 = 1)$ flavor symmetry in the quiver field theory[29]. As an aside, notice that if $a_0 = a_{-1} = 0$ then the only surviving combinations in (6.4.11) are w_4 and w_8, whereas if $c_0 = c_1 = 0$ the surviving combinations are w_3 and w_5. It is not a case that the surviving combinations are exactly the uncharged ones under $U(1)_M$, see (6.4.7), but we will refer the reader to [6] for a complete treatment.

Dimensional reduction: semiclassical method

Here we will show that in order to have a consistent dimensional reduction, one such that the resolved CY_4 (6.2.4) takes the fibered form (CY_3, σ, τ) with CY_3 the resolved version of $C(T^{11})$,

[29] If instead we have $1+h$ vertically aligned point in the 3d toric diagram, there will be h coincident D6-branes in type IIA and hence a $U(h)$ flavor group.

we must identify resolution parameters on the M-Theory side, i.e. Kähler moduli v_1, v_2, with "resolution parameters" on the quiver side, i.e. the FI ζ and the real mass $m = m_1 - m_2$ in (4.4.19). We quickly remind that we are in the branch of the moduli space identified by (4.4.6) where moreover $\sigma_i = \sigma$, bifundamental chiral fields are diagonalized and the quiver is one of the $\tilde{N}$ copies of $U(1)^2$. Since the theory is abelian, the F-term condition in (4.4.5) is trivial. We report for clarity the D-term condition (4.4.19), namely

$$|A_1|^2 + |A_2|^2 - |B_1|^2 - |B_2|^2 = \zeta(\sigma) \tag{6.4.13}$$

where

$$\zeta(\sigma) = \zeta + \frac{1}{2}|\sigma| + \frac{1}{2}|\sigma + m| = \begin{cases} \zeta - \frac{1}{2}m - \sigma & \text{if}\, \sigma \leq -m \\ \zeta + \frac{1}{2}m & \text{if}\, -m \leq \sigma \leq 0 \\ \zeta + \frac{1}{2}m + \sigma & \text{if}\, \sigma \geq 0. \end{cases} \tag{6.4.14}$$

This picture characterize the moduli space of the quiver.

On the M-Theory side we exploit the GLSM (6.4.1) but we also add a "new resolution parameter" r_0, or better a new FI parameter for the GLSM theory, namely

$$\begin{array}{c|cccccc|c} & a_1 & a_2 & b_1 & b_2 & c_1 & c_2 & FI \\ \hline U(1)_I & -1 & -1 & 1 & 1 & 0 & 0 & v_1 \\ U(1)_{II} & -1 & -1 & 0 & 0 & 1 & 1 & v_2 \\ \hline U(1)_M & 0 & 1 & 0 & 0 & 0 & -1 & r_0 \end{array} \tag{6.4.15}$$

where $v_1, v_2 \geq 0$ and $r_0 \in \mathbb{R}$. The D-term equation for the GLSM (6.4.15) take the form

$$\begin{aligned} |b_1|^2 + |b_2|^2 - |a_1|^2 - |a_2|^2 &= v_1, \\ |c_1|^2 + |c_2|^2 - |a_1|^2 - |a_2|^2 &= v_2, \\ |a_2|^2 - |c_2|^2 &= r_0. \end{aligned} \tag{6.4.16}$$

At this point we try to rearrange the gauge groups of the GLSM, i.e. $U(1)_I, U(1)_{II}, U(1)_M$, and the FI of the GLSM, i.e. v_1, v_2, r_0, in such a way that they somehow reproduce (6.4.14): this procedure obviously depend on the range of v_1, v_2, r_0.

- If $r_0 \leq -v_2 \leq 0$ we can reorganize (6.4.15) as follows

$$\begin{array}{c|cccccc|c} & a_1 & a_2 & b_1 & b_2 & c_1 & c_2 & FI \\ \hline U(1)_I - U(1)_{II} - U(1)_M & 0 & -1 & 1 & 1 & -1 & 0 & v_1 - v_2 - r_0 \\ U(1)_{II} + U(1)_M & -1 & 0 & 0 & 0 & 1 & 0 & v_2 + r_0 \\ \hline U(1)_M & 0 & 1 & 0 & 0 & 0 & -1 & r_0 \end{array} \tag{6.4.17}$$

  Then (6.4.16) becomes

$$\begin{aligned} |b_1|^2 + |b_2|^2 - |a_2|^2 - |c_1|^2 &= v_1 - v_2 - r_0, \\ |c_1|^2 - |a_1|^2 &= v_2 + r_0, \\ |a_2|^2 - |c_2|^2 &= r_0. \end{aligned} \tag{6.4.18}$$

Notice that since $r_0 \leq 0$ and $v_2 + r_0 \leq 0$ we can eliminate the combinations

$$\begin{aligned} |a_1|^2 &= |c_1|^2 - (v_2 + r_0) \geq 0, \\ |c_2|^2 &= |a_2|^2 - r_0 \geq 0, \end{aligned} \tag{6.4.19}$$

which in turn are the uncharged fields under $U(1)_I - U(1)_{II} - U(1)_M$. Then, upon the identifications

$$A_1 \leftrightarrow b_1, \quad A_2 \leftrightarrow b_2, \quad B_1 \leftrightarrow c_1, \quad B_2 \leftrightarrow a_2, \tag{6.4.20}$$

the first equation in (6.4.18) is exactly (6.4.13) with

$$\zeta(\sigma) \leftrightarrow v_1 - v_2 - r_0. \tag{6.4.21}$$

- If $-v_2 \leq r_0 \leq 0$ we can reorganize (6.4.15) as follows

(6.4.22)

	a_1	a_2	b_1	b_2	c_1	c_2	FI
$U(1)_I$	-1	-1	1	1	0	0	v_1
$U(1)_{II} + U(1)_M$	-1	0	0	0	1	0	$v_2 + r_0$
$U(1)_M$	0	1	0	0	0	-1	r_0

Then (6.4.16) becomes

$$\begin{aligned} |b_1|^2 + |b_2|^2 - |a_2|^2 - |a_1|^2 &= v_1, \\ |c_1|^2 - |a_1|^2 &= v_2 + r_0, \\ |a_2|^2 - |c_2|^2 &= r_0. \end{aligned} \tag{6.4.23}$$

Notice that since $r_0 \leq 0$ and $v_2 + r_0 \geq 0$ we can eliminate the combinations

$$\begin{aligned} |c_1|^2 &= |a_1|^2 + (v_2 + r_0) \geq 0, \\ |c_2|^2 &= |a_2|^2 - r_0 \geq 0, \end{aligned} \tag{6.4.24}$$

which in turn are the uncharged fields under $U(1)_I$. Then, upon the identifications

$$A_1 \leftrightarrow b_1, \quad A_2 \leftrightarrow b_2, \quad B_1 \leftrightarrow a_1, \quad B_2 \leftrightarrow a_2, \tag{6.4.25}$$

the first equation in (6.4.23) is exactly (6.4.13) with

$$\zeta(\sigma) \leftrightarrow v_1. \tag{6.4.26}$$

- If $r_0 \geq 0$ we can reorganize (6.4.15) as follows

(6.4.27)

	a_1	a_2	b_1	b_2	c_1	c_2	FI
$U(1)_I + U(1)_M$	-1	0	1	1	0	-1	$v_1 + r_0$
$U(1)_{II} + U(1)_M$	-1	0	0	0	1	0	$v_2 + r_0$
$U(1)_M$	0	1	0	0	0	-1	r_0

Then (6.4.16) becomes

$$\begin{aligned} |b_1|^2 + |b_2|^2 - |c_2|^2 - |a_1|^2 &= v_1 + r_0, \\ |c_1|^2 - |a_1|^2 &= v_2 + r_0, \\ |a_2|^2 - |c_2|^2 &= r_0. \end{aligned} \tag{6.4.28}$$

Notice that since $r_0 \geq 0$ and $v_2 + r_0 \geq 0$ we can eliminate the combinations

$$\begin{aligned} &|c_1|^2 = |a_1|^2 + (v_2 + r_0) \geq 0, \\ &|a_2|^2 = |c_2|^2 + r_0 \geq 0, \end{aligned} \tag{6.4.29}$$

which in turn are the uncharged fields under $U(1)_I + U(1)_M$. Then, upon the identifications

$$A_1 \leftrightarrow b_1, \quad A_2 \leftrightarrow b_2, \quad B_1 \leftrightarrow a_1, \quad B_2 \leftrightarrow c_2, \tag{6.4.30}$$

the first equation in (6.4.28) is exactly (6.4.13) with

$$\zeta(\sigma) \leftrightarrow v_1 + r_0. \tag{6.4.31}$$

In the end we can compare (6.4.21), (6.4.26), (6.4.31) with (6.4.14)

$$\zeta(\sigma) = \begin{cases} \zeta - \frac{1}{2}m - \sigma & \text{if } \sigma \leq -m \\ \zeta + \frac{1}{2}m & \text{if } -m \leq \sigma \leq 0 \\ \zeta + \frac{1}{2}m + \sigma & \text{if } \sigma \geq 0 \end{cases} \quad \leftrightarrow \quad \zeta(r_0) = \begin{cases} v_1 - v_2 - r_0 & \text{if } r_0 \leq -v_2 \\ v_1 & \text{if } -v_2 \leq r_0 \leq 0 \\ v_1 + r_0 & \text{if } r_0 \geq 0 \end{cases} \tag{6.4.32}$$

so that the quiver picture and the M-Theory picture coincide provided that we identify

$$r_0 \leftrightarrow \sigma, \qquad v_2 \leftrightarrow m, \qquad v_1 \leftrightarrow \zeta + \frac{1}{2}m. \tag{6.4.33}$$

At this point, one can be upset because of the inclusion of a "new" FI parameter r_0 for the GLSM: it actually seems an unjustified artifact. However, if we eliminate r_0 from (6.4.18), (6.4.23), (6.4.28) we find the D-term equation (6.2.8) of the "original" GLSM for (6.2.4): so the procedure is consistent.

6.4.2 Superconformal invariance

We know that the AdS_4/CFT_3 correspondence translates into the fact that the field theory dual to M-Theory on the near-horizon background $AdS_4 \times Q^{111}$ acquires the superconformal symmetry. In other words, $AdS_4 \times Q^{111}$ corresponds to the superconformal vacuum of the $\mathcal{N} = 2$ field theory, which is only one point in the moduli space: this is the IR fixed point characterized by operators with exactly vanishing VEVs, i.e. $\langle O \rangle = 0$. Now, different points in the moduli space correspond to different vacua characterized by some non-vanishing VEV, i.e. $\langle O \rangle \neq 0$. So, our position on the moduli space is parametrized by the VEVs of some operators:

these are actually the chiral moduli fields. When all these VEVs are zero the interpretation is that every M2-brane is sitting on the tip of the singular cone in the M-Theory background: this is exactly the vacuum preserving the full superconformal symmetry. If (some of) these chiral operators acquire a VEV $\langle O \rangle \neq 0$ then the conformal symmetry is spontaneously broken by the new scale. Recall that from the M-Theory point of view, these field theory vacua with $\langle O \rangle \neq 0$ should be in one-to-one correspondence with asymptotically $AdS_4 \times Q^{111}$ backgrounds, the latter being related to either mobile M2-branes and resolutions. To be clearer, one should compare the energy of a typical process in the SCFT with the scale of spontaneous symmetry breaking set by these VEVs. At energies well above $\langle O \rangle$, i.e. the UV region, we expect a "pure" SCFT with the full superconformal symmetry: this is because in the holographic description the background is asymptotically $AdS_4 \times Q^{111}$, i.e. it has a "restored" conformal invariance. On contrary, at energies below $\langle O \rangle$ we expect a spontaneous symmetry breaking and, as a consequence, there will be massive states in the theory with a mass of order $\langle O \rangle$. It is only at energies well below $\langle O \rangle$ that we can consistently exploit the HEFT Lagrangian: indeed, this effective theory can be obtained by integrating out massive modes so that it describes massless fields only, i.e. moduli. In other words, the dynamics of moduli can be encoded in the HEFT Lagrangian only in the spontaneously broken phase, which is actually the low-energy region of a strongly coupled SCFT. Thus, we expect that the superconformal symmetry is non-linearly realized at this Lagrangian level: in general, it can be quite difficult to find out non-linear conformal transformations. However, as showed for example in [41], three-dimensional nonlinear sigma models with $\mathcal{N} = 2$ supersymmetries characterized by a Kähler potential K are automatically superconformal provided that $\Delta_K = 1$. We remind from the first chapter that this condition is in fact equivalent to the scale invariance of the theory and hence this is a case where the scale invariance is enhanced to the superconformal one. So, our strategy for the superconformal check on (5.3.2) is to compute the scaling dimensions Δ of the objects we are dealing with and see if the condition

$$\Delta_K = 1 \tag{6.4.34}$$

is satisfied. Before starting to do so, remember that our Kähler potential K is defined in (5.3.1) and depends on k_0, whose expression is (6.3.31). So, our goal is to find $\Delta_\gamma, \Delta_b, \Delta_c$: they should all be equal to one.

Recall that the fields populating the HEFT are the chiral coordinates $\{z\}$, parametrizing the positions of M2-branes, and the Kähler moduli ρ^a, the latter related to $v^a = b, c$. In the previous section we found that the chiral coordinates $\{z\}$ could be expressed in terms of chiral fields of the UV-quiver and monopole operators, for example we got $Y = A_2 B_2$. So, as a first step, we should compute the scaling dimensions of chiral fields in the UV-quiver, together with monopoles. We begin from the superpotential (4.3.5) of the unflavored three-dimensional quiver field theory. Since the Q^{111} quiver has an $U(1)_{\mathcal{R}}$ symmetry, every term in its action functional should be invariant under this $\mathcal{R}$-symmetry: we are particularly interested in the superpotential term $\int \mathrm{d}^3x\mathrm{d}^2\theta W$. Recalling (1.2.20) we get $\mathcal{R}_{\mathrm{d}^2\theta} = -2$, while obviously $\mathcal{R}_{\mathrm{d}^3x} = 0$. Thus, in order for the superpotential action to be invariant under $U(1)_{\mathcal{R}}$ it must be $\mathcal{R}_W = 2$. Since we are dealing with a quartic superpotential, i.e. $W \sim \Phi^4$, it is clear that chiral fields $\Phi = A_1, A_2, B_1, B_2$ must have $\mathcal{R}_\Phi = \frac{1}{2}$ by symmetry. At this stage, the $\mathcal{R}$-symmetry argument in (1.4.23) ensure us that the scaling dimensions of our fields at the IR fixed-point, i.e. where the quiver theory becomes a SCFT, are fixed by their charge under $U(1)_{\mathcal{R}}$: in particular $\Delta_\Phi = \mathcal{R}_\Phi = \frac{1}{2}$. For monopole operators the calculation is more subtle. Indeed, as stated in [6], with the flavoring procedure monopole operators get a $\mathcal{R}$-charge

$$\mathcal{R}[T^{(n)}] = \frac{|n|}{2} \sum_{a \in \text{flavored}} h_a \mathcal{R}[\Phi_a], \tag{6.4.35}$$

which is quite similar to (4.4.27). Since $n = \pm 1$ and $T^{(1)} = T, T^{(-1)} = \tilde{T}$, for the case of the flavored Q^{111} quiver theory we have

$$\mathcal{R}_T = \mathcal{R}_{\tilde{T}} = \frac{1}{2} h_1 \mathcal{R}_{A_1} + \frac{1}{2} h_2 \mathcal{R}_{A_2} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}, \tag{6.4.36}$$

which is just the same of a UV chiral field. Then, since monopoles are actually chiral fields too, they saturate (1.4.23) and hence $\Delta_T = \mathcal{R}_T = \frac{1}{2}$. We collect for clarity the scaling dimensions for chiral fields of the UV-quiver, namely:

$$\Delta_{A_1} = \Delta_{A_2} = \Delta_{B_1} = \Delta_{B_2} = \Delta_T = \Delta_{\tilde{T}} = \frac{1}{2}. \tag{6.4.37}$$

Now we can easily compute the scaling dimension of the chiral coordinates $\{z\}$. For instance, $\Delta_Y = \Delta_{A_2} + \Delta_{B_2} = 2\Delta_\Phi = 1$ and the same is true for the other fibral coordinates of (6.2.4), namely $\Delta_U = \Delta_V = \Delta_X = \Delta_Y = 1$. On the other hand, the local coordinates on the $\mathbb{CP}$ at the base of the bundle (6.2.4) have naturally $\Delta_\lambda = 0$.[30] In the end, having the complete set of scaling dimensions for $\{z\}$, namely

$$\Delta_U = \Delta_V = \Delta_X = \Delta_Y = 1, \qquad \Delta_\lambda = 0, \tag{6.4.38}$$

we can compute the scaling dimension for γ. Indeed, notice that according to (6.2.25), the radial coordinate t has $\Delta_t = 2\Delta_U = 2\Delta_Y = 2$. Moreover, in the limit $\gamma \gg b, c$ we found the asymptotic behavior $\gamma \sim t^{\frac{1}{2}}$ from (6.3.6) and hence $\Delta_\gamma = \frac{1}{2}\Delta_t = 1$ as expected. From the field theory point of view, we can also get quite easily the scaling dimensions for resolution parameters. Indeed, since they are the scalar components of linear multiplets, i.e. $\Sigma^a = v^a + \ldots$, which in turn can be interpreted as topological conserved current multiplets, their scaling

[30] Even if we have not carried out the whole matching between $\{z\}$ and $(\Phi_a, T, \tilde{T})$, it seems that fibral coordinates can be expressed as product of two chiral fields, like $Y = A_2 B_2$, while the λs are quotients. For instance, looking at (6.2.19) and (6.4.6) we can identify $\lambda_b = B_1 B_2^{-1}$ and $\lambda_c = A_1 \tilde{T}^{-1}$. So $\Delta_\lambda = \frac{1}{2} - \frac{1}{2} = 0$. Finding $\Delta_\lambda = 0$ is quite appropriate because local coordinates for the $\mathbb{CP}$ at the base of (6.2.4) are "fixed" at $r = 0$ and hence they do not scale at all.

dimension is always equal to one[31]. Thus $\Delta_b = \Delta_c = 1$. Actually, one can avoid using the asymptotic behavior $\gamma \sim t^{\frac{1}{2}}$: indeed, if we have $\Delta_t = 2$ from $\Delta_{\{z\}}$ and $\Delta_b = \Delta_c = 1$ from the consideration on conserved currents, we can obtain $\Delta_\gamma = 1$ from consistency with the complete (6.3.6), i.e. not the asymptotic version $\gamma \sim t^{\frac{1}{2}}$. In the end we get

$$\Delta_\gamma = \Delta_b = \Delta_c = 1. \tag{6.4.39}$$

A complementary check on this calculation consists in considering the "holographic coordinate" $z = \frac{R^2}{r^2}$ introduced to study the near-horizon geometry in (3.2.7) or alternatively to obtain the asymptotically $AdS_4 \times Q^{111}$ metric in the large r region from (3.2.27) with $h \sim R^6/r^6$. This coordinate is what we called w in (1.5.11) and (1.5.13) and has the fundamental property that a rescaling $(x_\mu, z) \to (\Lambda x_\mu, \Lambda z)$ leave the AdS_4 metric $ds^2_{AdS_4} = \frac{R^2}{z^2}(dz^2 + dx^\mu dx_\mu)$ invariant. So $\Delta_z = -1$ and hence the radial coordinate r has scaling dimension $\Delta_r = \frac{1}{2}$ as anticipated, at least asymptotically where we can deal with an AdS factor. Since $2\gamma = r^2$, see for instance (6.2.36), we also get $\Delta_\gamma = 2\Delta_r = 1$ and from the asymptotic behavior $\gamma \sim t^{\frac{1}{2}}$ we obtain $\Delta_t = 2$. We claim that this asymptotic behaviors for scaling dimensions hold everywhere, even for finite r, γ, t. Indeed, consistency with (6.3.6) fixes the scaling dimensions of Kähler parameters: since we found $\Delta_\gamma = 1$ and $\Delta_t = 2$ then it must be $\Delta_b = \Delta_c = 1$. The agreement with their interpretation as lowest components of linear multiplets $\Sigma^a = v^a + \ldots$, and hence with the scaling dimensions obtained from the field theory calculation, suggests that our claim is quite supported. Moreover, using (6.2.25) we can obtain $\Delta_U = \Delta_Y = 1$ and $\Delta_\lambda = 0$ from $\Delta_t = 2$: even if the latter is an asymptotic scaling dimension, the former hold everywhere since the scaling dimensions of the complex coordinates $\{z\}$ do not depend on any asymptotic behavior. Besides, the fact that they are the same scaling dimensions obtained from the field theory strengthen our claim. Actually, this claim seems to be supported by a further consideration. In the preceding discussion we started in the large-r region and worked with $\gamma \gg b, c$ in order to find scaling dimensions that hold everywhere, i.e. the one of the chiral coordinates $\{z\}$. We remind that in this limit we are comparing a "radial position" VEV γ for one of the $\tilde{N}$ M2-branes with the resolutions VEV b, c: the interpretation is that the geometry "seen" by this brane is the singular one and hence we are allowed to obtain the scaling dimensions of $\{z\}$ from the asymptotic scaling dimensions of r, γ, t. However, in the opposite region $\gamma \ll b, c$ it seems that this reasoning collapses: the M2-brane "sees" the resolved geometry and hence the identifications between coordinates may be questionable. Nevertheless, since we know that chiral coordinate $\{z\}$ have "asymptotic-independent" scaling dimensions, i.e. $\Delta_U = \Delta_Y = \Delta_X = \Delta_V = 1$ and $\Delta_\lambda = 0$, we can obtain $\Delta_t = 2$ "everywhere" from (6.2.25). Then, the scaling dimensions for γ and b, c are fixed "everywhere", both for $\gamma \gg b, c$ and $\gamma \ll b, c$, from consistency with the

[31] It is a known result, see for example [13], that conserved currents do not renormalize. This means that their anomalous dimension is zero and hence they have fixed scaling dimension equal to their canonical dimension, i.e. $\Delta_J = d - 1$ for $\partial_\mu J^\mu = 0$. Since $d = 3$ we easily get $\Delta_J = 2$. Then, looking at (1.3.4), it is clear that $\Delta_{\theta\bar{\theta}} = -1$ and hence $\Delta_\Sigma = 1$ as well as its scalar component field.

Ricci-flatness equation (6.3.6). Indeed, if $\Delta_t = 2$ it must be $\Delta_\gamma = 1$ and $\Delta_b = \Delta_c = 1$. So, if we start from $\Delta_{\{z\}}$ we find out that the asymptotic scaling dimensions hold everywhere[32].

In the end, we can check if (5.3.2) is in fact invariant under the superconformal group. Looking at (6.3.31) we find $\Delta_{k_0} = 1$ because $\Delta_\gamma = \Delta_b = \Delta_c = 1$. Then, from (5.3.1) we have

$$\Delta_K = \Delta_{k_0} = 1 \tag{6.4.40}$$

as we wanted: this concludes our series of consistency checks.

[32]As an aside, recall that the function $F(t)$ in (6.2.22) can be expressed in the integral form $F(t) = \int_0^t \frac{d\tilde{t}}{\tilde{t}}\gamma(\tilde{t})$. When $\gamma \gg b, c$ we found the asymptotic behavior $F(t) \sim r^2$ while for $\gamma \ll b, c$ we got $F(t) \sim \frac{t}{\sqrt{bc}}$. In the former case we have $\Delta_{F(t)} = 2\Delta_r = 1$, which is the same of the latter because $\Delta_{F(t)} = \Delta_t - \frac{1}{2}(\Delta_b + \Delta_c) = 1$. Since $\gamma = F't = \frac{dF}{dt}t$, then $\Delta_\gamma = \Delta_{F(t)} = 1$ both for $\gamma \gg b, c$ and for $\gamma \ll b, c$.

Chapter 7

Conclusions and closing remarks

In this thesis, following the holographic prescriptions proposed in [1, 2], we have successfully identified the effective field theory describing the low-energy dynamics of a strongly coupled three-dimensional SCFT with $\mathcal{N} = 2$ supersymmetries at a vacuum in the moduli space that spontaneously break the conformal symmetry. The SCFT under examination is the infrared fixed point of a microscopic theory, the Q^{111} quiver model. This in turn is engineered by placing a stack of coincident $\tilde{N}$ M2-branes on the tip of a Calabi-Yau cone $C(Q^{111})$ over the Sasaki-Einstein base Q^{111}. We underline the fact that the Q^{111} quiver is maybe the simplest model featuring flavor symmetries and real masses and it is nontrivial to check that it truly corresponds to our holographic description. However, our results for this case of AdS_4/CFT_3 correspondence are quite supported by the consistency checks performed at moduli space level. Indeed, both its complex structure and its Kähler structure, i.e. resolutions, are shown to match on the two sides of the duality. The monopole method seems to shine for the former check, while the semiclassical method is especially indicated for the latter. In particular, the dimensional reduction from M-Theory to type IIA results in a dictionary between external parameters in the quiver and Kähler parameters, i.e. $(\zeta, m) \leftrightarrow (v_1, v_2)$.

We stress that the fundamental correspondence is between M-Theory on $AdS_4 \times Q^{111}$ and the $\mathcal{N} = 2$ three-dimensional SCFT: indeed, we found a correspondence between M-Theory vacua admitting an AdS_4 factor and field theory vacua of the dual SCFT. If the former is exactly $AdS_4 \times Q^{111}$, i.e. the near-horizon limit of the stack of M2-branes placed on the tip of the singular cone $C(Q^{111})$, then the corresponding field theory vacuum is the only one preserving the full superconformal symmetry. On contrary, at a generic vacuum the conformal symmetry is spontaneously broken. Indeed, in the M-Theory side one can "lose" the AdS_4 structure, which in turn is recovered at infinity provided that our M-Theory backgrounds are chosen to be "asymptotically $AdS_4 \times Q^{111}$". Correspondingly, in the field theory side the conformal symmetry is spontaneously broken by VEVs that clearly have an holographic interpretation: resolutions and/or M2-branes motion. In these cases, the conformal group is restored at energies well above the scale set by these VEVs: this statement is to some extent "dual" to the one about the recovering of the AdS_4 structure at infinity.

We remind that our HEFT is trustable only at energies well below the scale set by these VEVs, which in turn are interpreted as spontaneous symmetry breaking scales. So, the HEFT describes the dynamics of a system where the conformal symmetry is spontaneously broken. Indeed, it is still non-linearly realized and we found that the HEFT action is in fact superconformal invariant as a further consistency check. We point out that the spontaneous breaking of conformal symmetry is a topic not completely understood in general and so these HEFT models could shed light on it. In particular, since conformal theories lack the usual concept of particles, a phenomenological realistic theory should require at least some kind of spontaneous breaking of the conformal symmetry.

Besides, recall that our HEFT is a two-derivative formulation: a possible direction of development is to study higher-derivative operators in the holographic Lagrangian and their dual interpretation. Moreover, we remark that our calculation required mutually non-coincident M2-branes: if we start with $\tilde{N}$ branes and allow for their motion around the transverse manifold, spontaneously breaking the dual conformal symmetry of the system, our HEFT corresponds to $\tilde{N}$ "stacks" consisting of only one brane. One could ask what happens when two or more M2-branes on $C(Q^{111})$ coincide: we think that our HEFT breaks down. For instance, if we place a stack of n M2-branes, with $1 \ll n < \tilde{N}$, on a non-singular point and "zoom in", then we should find an $AdS_4 \times \mathbb{S}^7$ structure because the n M2-branes are sitting on a "smooth point". We mean that the neighborhood of the stack is mildly curved and hence we expect a dual sector with some SCFT having $\mathcal{N} > 2$ supersymmetries in three spacetime dimensions. Besides, it could be interesting to investigate particular branches of the moduli space that we have not treated.

Another very important condition for our holographic calculation is the "large-$\tilde{N}$, large-λ" limit, where λ is the 't Hooft coupling constant. Roughly speaking, this means that we should take a large number of branes, which in turn correspond to a large number of "colors" in the dual field theory. We remind that in this limit the M-Theory is in fact a weakly-coupled eleven-dimensional supergravity and hence our HEFT is actually a perturbative result. It is worth mentioning that there exist different formulations of the AdS/CFT conjecture and that the strongest one would like to work with generic values of $\tilde{N}$ and λ: this means that we are out of the perturbative regime of supergravity and hence, as a possible development of this work, one may explore if non-perturbative effects can emerge. Indeed, hypothetical matchings regarding non-perturbative phenomena on the two sides of the duality are very important to provide evidences on the strongest form of the conjecture.

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