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Study of a Schrödinger-like Equation for the Collisionless Dynamics of Large-Scale Cosmological Structure in Phase Space

Studio di un'equazione Schrödinger-like per la dinamica collisionless della struttura cosmologica su grande scala nello spazio delle fasi

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Sommario in lingua italiana: Questa tesi ha come oggetto lo studio dell'instabilità gravitazionale nella struttura cosmologica su larga scala tramite l'uso delle equazioni della meccanica quantistica. Lo sviluppo delle perturbazioni di densità energetica in un universo dominato da materia fredda può essere modellato con le equazioni della fluidodinamica classica, che possono poi essere usate per produrre dei metodi di approssimazione analitica, qui presentati. In particolare, l'approssimazione di Zel'dovich permette di considerare come inerziali le particelle del fluido tramite una riparametrizzazione del tempo e permette di seguire la materia fino al collasso gravitazionale con un buon riscontro rispetto alle simulazioni numeriche. Tramite la trasformazione di Madelung, il campo di densità e quello di velocità del fluido possono essere rappresentati da un'equazione d'onda la cui evoluzione è data da una versione modificata dell'equazione di Schrödinger. Nel secondo capitolo è introdotto un formalismo per rappresentare la teoria di base della meccanica quantistica nello spazio delle fasi. Le funzioni di distribuzione di probabilità classiche non potranno essere usate, ma per descrivere gli stati si useranno invece delle quasi-distribuzioni di probabilità ("quasi" perchè private di alcune proprietà tipiche delle distribuzioni di probabilità, come ad esempio la non-negatività) con ruolo simile e con proprietà diverse legate a una scelta di quantizzazione. L'ordinamento anti-normale porta a una quasi-distribuzione naturalmente adatta allo studio di sistemi non-lineari che prende il nome di funzione di Husimi, e questa può essere calcolata in maniera analiticamente semplice facendo uso degli stati coerenti canonici. Lo sviluppo temporale della funzione di Husimi può essere studiato tramite un integrale di percorso definito nella rappresentazione degli stati coerenti, e il suo utilizzo nel caso di un'Hamiltoniana di particella libera porta a risultati consistenti con l'approssimazione di Zel'dovich.

Abstract: This thesis investigates a formalism based on a Schrödinger-like equation to describe the collisionless dynamics of large-scale cosmological structure. The approach provides an alternative way to study the evolution of the system in phase space and to analyse the formation and evolution of cosmic structures. Particular attention is devoted to the connection between this formalism and the classical descriptions of gravitational dynamics in cosmology.

Introduction

The formation of large-scale structure in our universe is linked to the presence of a Cold Dark Matter (CDM) component alongside observable matter. Baryonic matter may have started clustering only after matter-radiation decoupling at time¹ $z \sim 1100$, as photon coupling creates pressure strong enough to deny growth of the perturbations. If this was the case, density fluctuations in the Cosmic Microwave Background (which were observed to be of relative order $\sim 10^{-5}$ with high precision in the Planck mission [1]) would not be sufficient to justify the presence of structure. CDM, not interacting with photons and having negligible pressure, may instead have started notable growth of energy density perturbations as soon as the universe stopped being radiation-dominated ($z \sim 3400$); the gravitational wells thus formed would then be catalysts for the congregation of baryonic matter [2]. Significance in the study of gravitational instability in collisionless, cold matter-dominated universes is then clear: starting from the primordial density perturbations seen in the CMB, this model should replicate the formation of the chassis upon which large-scale structure is formed. In this thesis, we will explore some of the analytic approximations schemes proposed for the evolution of non-relativistic collisionless (i.e. pressureless) matter by using Newtonian fluid mechanics, starting from the linear-order Eulerian approximation which is used to show these conditions actually imply perturbation growth. We will then go over a more advanced scheme proposed by Zel'dovich, which yields better results in non-linear regimes and gives a characterization of density singularities; in this approximation, the particles will be assumed to follow inertial trajectories after a time transformation and, as such, will not stop upon clustering. An artificial adhesion term may be employed to stabilize produced structure, which can then be compared with numerical simulations (Figure 1). Another approximation will be introduced by applying the Madelung transformation and representing the density and velocity field of the fluid as a wavefunction which evolves according to a modified version of the Schrödinger equation; in analogy to the Zel'dovich approximation, the wavefunction is then evolved as a free-particle system. The focus of this thesis is to extend this formalism to phase space. This is not straightforward because quantum mechanics does not allow for probability distribution functions in configuration space, as their presence requires simultaneous measurement of both position and momentum, which are not compatible physical observables. To proceed, quantum mechanics is given a new formulation in which observable operators are transformed to phase-space functions uniquely by choice of an

¹Cosmological time is often expressed by the redshift parameter z , expressing the ratio of light wavelength at current time in reference to the original wavelength at production. Intuitively, it shows how much the universe expanded since the time of the referred event.

ordering rule; the state will then be represented by a quasi-distribution function allowing for the computation of expectation values in analogy to classical observables; “quasi” because this function will lack fundamental properties of properly defined probability densities. The highly-symmetrical Weyl quantization shows many desirable properties in its quasi-distribution function (commonly referred to as the Wigner function [3]), but will not be used; we will instead base our approach on the anti-normal quantization and the Husimi quasi-distribution function [4], which is shown to have useful qualities for the description of non-linear systems. Its evolution will be studied in free-particle conditions through the use of a path-integral propagator in the canonical coherent-state representation. The objective of this work will be initial characterization of the Husimi function formalism; we will show consistency between the trajectories obtained using this new-found version of the free-particle approximation in the classical limit and the ones from the classical Zel’dovich approximation.

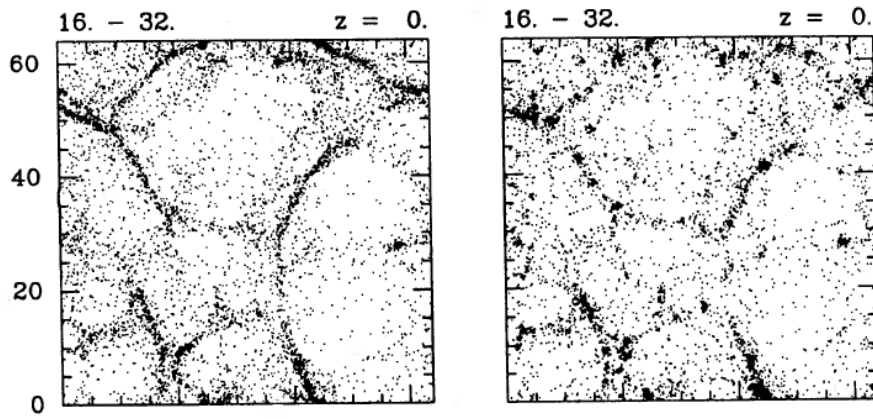


Figure 1: Comparison of density distribution obtained through the adhesion approximation (left) and a numerical simulation (right) from the same starting energy density conditions. [5]

1

Gravitational instability approximations

In this section we present a variety of known approximations schemes for the modelling of gravitational collapse for cold collisionless matter in an expanding universe. In doing so we will be employing Newtonian fluid dynamics; the considered fluid particles may be given different definitions based on the problem at hand, and the exposition will be general for a range of scales in which Newtonian mechanics is a valid approximation of general relativity. In this chapter we study the first-order solutions of the energy density fluid equations, showing how a small perturbation in the density field may get amplified to non-linear regime. We then explore a more sophisticated scheme in the form of the Zel'dovich approximation, also going over the effects of an added artificial adhesion term. We present a Schrödinger-like formalism for the problem, and go over the free-particle approximation. Along those mentioned during the chapter, in this exposition we made use of the following references: [6] [7] [8] [5] [9] [10] [11] [12].

1.1 Preliminaries

We formulate the problem in a flat, homogeneous and isotropic expanding universe endowed with a Friedmann-Robertson-Walker metric [13]:

$$ds^2 = -c^2 dt^2 + a(t) (dx^2 + dy^2 + dz^2)$$

Where $a(t) > 0$ is a time-dependent scale factor, and c is the speed of light. We assume homogeneity and isotropy to hold at scales larger than the considered structures, which may locally show neither property. The coordinates $\mathbf{x} = (x, y, z)$ follow the expansion dictated by the scale factor and are known as *comoving coordinates*. The physical coordinates $\mathbf{r}$ and the physical velocity $\mathbf{v}_r$ are then represented as:

$$\mathbf{r} = a(t)\mathbf{x} \quad \mathbf{v}_r = \dot{\mathbf{r}} = \dot{a}\mathbf{x} + a\dot{\mathbf{x}} = H(t)\mathbf{x} + \mathbf{v}$$

With $H(t) = \dot{a}/a$, the *Hubble constant*, representing the rate of expansion of the universe, and $\mathbf{v} = a\dot{\mathbf{x}}$ the *peculiar velocity*, which is the measured physical velocity of an object when the distance from an observer is negligible $H|\mathbf{x}| \approx 0$. We also get, for the partial derivatives:

$$\nabla_{\mathbf{r}} = \frac{1}{a(t)} \nabla_{\mathbf{x}} \quad \left. \frac{\partial}{\partial t} \right|_{\mathbf{r}} = \left. \frac{\partial}{\partial t} \right|_{\mathbf{x}} - H(\mathbf{x} \cdot \nabla_{\mathbf{x}}) \quad (1.1)$$

Where the time derivative can be obtained by considering the total derivative of a smooth function $f(\mathbf{x}, t)$ along a smooth curve $\gamma(t)$:

$$\frac{df(\gamma(t), t)}{dt} = \frac{\partial f}{\partial t} \Big|_{\mathbf{x}} + \dot{\mathbf{x}}(\gamma(t)) \cdot \nabla_{\mathbf{x}} f = \frac{\partial f}{\partial t} \Big|_{\mathbf{r}} + \left(a\dot{\mathbf{x}}(\gamma(t)) + \dot{a}\mathbf{x}(\gamma(t)) \right) \cdot \frac{1}{a} \nabla_{\mathbf{x}} f$$

The application of the Einstein field equations yields the Friedmann equation and the following form of the energy conservation law [13]:

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho_0 \quad (1.2)$$

$$\dot{\rho}_0 = -3 \frac{\dot{a}}{a} (p + \rho_0) \quad (1.3)$$

Along with a state equation $p = p(\rho)$, these equations form a closed system which expresses the dynamics for $a(t)$, homogeneous energy density $\rho_0(t)$ and pressure $p(t)$. It can be shown [6] that Newtonian physics is an appropriate approximation as long as the scales considered are less than the Hubble horizon $\sim c/H$ and more than the largest compact objects:

$$10^{-5} \text{pc} \ll \Delta L \ll 10^9 \text{pc} \quad (\text{for the present epoch})$$

Therefore we model energy density as a Newtonian fluid under the effect of self gravitation, and use the Euler equation [14] along with the continuity and Poisson equation for gravitational potential¹ for the evolution of the velocity field $\mathbf{v}_{\mathbf{r}}$, the perturbed energy density $\rho(\mathbf{r}, t) = \rho_0(t) + \delta\rho(\mathbf{r}, t)$ and the specific gravitational potential $\phi(\mathbf{r}, t) = \phi_0(\mathbf{r}, t) + \delta\phi(\mathbf{r}, t)$:

$$\frac{\partial \mathbf{v}_{\mathbf{r}}}{\partial t} \Big|_{\mathbf{r}} + (\mathbf{v}_{\mathbf{r}} \cdot \nabla_{\mathbf{r}}) \mathbf{v}_{\mathbf{r}} = -\frac{1}{\rho} \nabla_{\mathbf{r}} p - \nabla_{\mathbf{r}} \phi \quad (1.4)$$

$$\frac{\partial \rho}{\partial t} \Big|_{\mathbf{r}} + \nabla_{\mathbf{r}} \cdot (\rho \mathbf{v}_{\mathbf{r}}) = 0 \quad (1.5)$$

$$\nabla_{\mathbf{r}}^2 \phi = 4\pi G \rho \quad (1.6)$$

We expect the unperturbed system to yield:

$$\mathbf{v} = 0 \quad \rho(\mathbf{r}, t) = \rho_0(t)$$

along with a constant pressure and $\phi_0(\mathbf{r}, t)$ satisfying (1.6) for ρ_0 . We now switch to comoving coordinates in order to highlight the behaviour of the perturbations; equation (1.1) is used on the Euler equation:

$$\begin{aligned} \frac{\partial(\dot{a}\mathbf{x} + \mathbf{v})}{\partial t} \Big|_{\mathbf{x}} - H(\mathbf{x} \cdot \nabla_{\mathbf{x}})(\dot{a}\mathbf{x} + \mathbf{v}) + \frac{1}{a} ((\dot{a}\mathbf{x} + \mathbf{v}) \cdot \nabla_{\mathbf{x}})(\dot{a}\mathbf{x} + \mathbf{v}) &= -\frac{1}{a\rho} \nabla_{\mathbf{x}} p - \frac{1}{a} \nabla_{\mathbf{x}} \phi \\ \frac{\partial(\dot{a}\mathbf{x} + \mathbf{v})}{\partial t} \Big|_{\mathbf{x}} + H\mathbf{v} + \frac{1}{a} (\mathbf{v} \cdot \nabla_{\mathbf{x}}) \mathbf{v} &= -\frac{1}{a\rho} \nabla_{\mathbf{x}} p - \frac{1}{a} \nabla_{\mathbf{x}} \phi \end{aligned} \quad (1.7)$$

¹Homogeneous (and thus spatially infinite) mass densities should lead to a divergent gravitational potential. This flaw, commonly referred to as *Jeans Swindle* by the name of its initial perpetrator, is often incorporated in this kind of derivation still allowing for meaningful results despite the inconsistency. More recently, it was shown that the diverging terms are nullified in an expanding universe, meaning the Swindle is justified. [15]

The unperturbed solution satisfies:

$$\left. \frac{\partial(\dot{\mathbf{x}})}{\partial t} \right|_{\mathbf{x}} = -\frac{1}{a} \nabla_{\mathbf{x}} \phi_0$$

Subtracting this from (1.7) we get:

$$\left. \frac{\partial \mathbf{v}}{\partial t} \right|_{\mathbf{x}} + H \mathbf{v} + \frac{1}{a} (\mathbf{v} \cdot \nabla_{\mathbf{x}}) \mathbf{v} = -\frac{1}{a \rho} \nabla_{\mathbf{x}} p - \frac{1}{a} \nabla_{\mathbf{x}} \delta \phi \quad (1.8)$$

For the continuity equation:

$$\begin{aligned} \left. \frac{\partial \rho}{\partial t} \right|_{\mathbf{x}} - H(\mathbf{x} \cdot \nabla_{\mathbf{x}}) \rho + \frac{1}{a} \nabla_{\mathbf{x}} \cdot (\rho(\dot{\mathbf{x}} + \mathbf{v})) &= 0 \\ \left. \frac{\partial \rho}{\partial t} \right|_{\mathbf{x}} + 3H\rho + \frac{1}{a} \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v}) &= 0 \end{aligned}$$

And, by subtracting the equation for the unperturbed solution and using $\delta = \delta \rho / \rho_0$:

$$\begin{aligned} (1 + \delta) \left(\left. \frac{\partial \rho_0}{\partial t} \right|_{\mathbf{x}} + 3H\rho_0 \right) + \rho_0 \left. \frac{\partial \delta}{\partial t} \right|_{\mathbf{x}} + \frac{\rho_0}{a} \nabla_{\mathbf{x}} \cdot ((1 + \delta) \mathbf{v}) &= 0 \\ \left. \frac{\partial \delta}{\partial t} \right|_{\mathbf{x}} + \frac{1}{a} \nabla_{\mathbf{x}} \cdot ((1 + \delta) \mathbf{v}) &= 0 \end{aligned} \quad (1.9)$$

Finally, for the Poisson equation:

$$\nabla_{\mathbf{x}}^2 \phi = 4\pi G a^2 \rho \quad \Rightarrow \quad \nabla_{\mathbf{x}}^2 \delta \phi = 4\pi G a^2 \delta \rho \quad (1.10)$$

Forward on, partial derivatives will always be in respect to comoving coordinates, unless explicitly stated otherwise.

1.1.a Einstein-De Sitter universe

In the later exposition we'll make use of the results obtained from (1.2) and (1.3) in the hypothesis of a flat universe dominated by pressureless non-relativistic matter (usually referred to as *dust*), known as an *Einstein-De Sitter universe*. The kinetic energy of the particles is negligible, so the energy density ρ_0 may be expected to depend only on the mass density ($\rho = \rho_m c^2$). When pressure is null we are able to solve (1.3) for a :

$$\dot{\rho}_0 = -3 \frac{\dot{a}}{a} \rho_0 \quad \Rightarrow \quad \rho_0(t) = c_1 a^{-3}(t)$$

Then with the Friedmann equation:

$$\dot{a} = \sqrt{\frac{8\pi G c_1}{3}} a^{-1/2} \quad \Rightarrow \quad a(t) = (18\pi G c_1)^{3/7} t^{2/3}$$

The Hubble constant can be used to fix the constant for ρ_0 :

$$H = \frac{\dot{a}}{a} = \frac{2}{3t} \quad H^2 = \frac{8\pi G}{3} \rho_0$$

Therefore the evolution of the Einstein-De Sitter universe is given by:

$$a(t) \propto t^{2/3} \quad H = \frac{2}{3t} \quad \rho_0(t) = \frac{1}{6\pi G t^2} \quad (1.11)$$

The Einstein-De Sitter solution is appropriate when radiation and cosmological constant terms in the Friedmann equation are negligible, i.e. the epoch of matter domination, and will be the setting for the upcoming approximation scheme review.

1.2 Linear approximation

As a first approach, we will solve the fluid equations (1.8), (1.9) and (1.10) considering only the first order in v , $\delta\rho$ and $\delta\phi$:

$$\frac{\partial \mathbf{v}}{\partial t} + H\mathbf{v} = -\frac{1}{a\rho_0}\nabla p - \frac{1}{a}\nabla_{\mathbf{x}}\delta\phi \quad (1.12)$$

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\nabla \cdot \mathbf{v} = 0 \quad (1.13)$$

$$\nabla^2 \delta\phi = 4\pi G a^2 \rho_0 \delta \quad (1.14)$$

The fluid is ideal and barotropic (p is a function of ρ , which we are not yet assuming to be null); by Kelvin circulation theorem [14], the following integral is unchanging if its support curve $\Gamma(t)$ moves along with fluid flow:

$$\begin{aligned} 0 &= \frac{d}{dt} \oint_{\Gamma(t)} \mathbf{v}_{\mathbf{r}} \cdot d\mathbf{l} = \frac{d}{dt} \int_{\Sigma(t)} (\nabla_{\mathbf{r}} \times \mathbf{v}_{\mathbf{r}}) \cdot d\mathbf{l} = \\ &= \int_{\Sigma(t)} \left[\frac{\partial}{\partial t} (\nabla_{\mathbf{r}} \times \mathbf{v}_{\mathbf{r}}) + \nabla_{\mathbf{r}} \times (\mathbf{v}_{\mathbf{r}} \times (\nabla_{\mathbf{r}} \times \mathbf{v}_{\mathbf{r}})) \right] \cdot d\mathbf{l} \end{aligned}$$

Where we also applied Reynolds transport theorem in the case of a surface; in the linear regime the second term inside the surface integral is negligible and thus it is immediate to see the velocity field in the linear solution is irrotational as long as we assume it to be so at starting time. To solve the linear fluid equations we work in Fourier space:

$$\tilde{f}(\mathbf{k}, t) \equiv \mathbb{F}[f(\mathbf{x}, t)](\mathbf{k}) := \left(\frac{1}{2\pi} \right)^3 \int d^3x e^{-i\mathbf{k} \cdot \mathbf{x}} f(\mathbf{x}, t) \quad \mathbb{F}[\nabla f(\mathbf{x}, t)](\mathbf{k}) = i\mathbf{k} \tilde{f}(\mathbf{k}, t)$$

The transform thus defined allows us to express the fluid equations as:

$$\frac{\partial \tilde{\mathbf{v}}}{\partial t} + H\tilde{\mathbf{v}} = -\frac{i\mathbf{k}}{a\rho_0} \frac{dp}{d\rho} \Big|_{\rho_0} \rho_0 \tilde{\delta} - \frac{i\mathbf{k}}{a} \tilde{\delta\phi} = -\frac{i\mathbf{k}}{a} \left(c_s^2 \tilde{\delta} + \tilde{\delta\phi} \right) \quad (1.15)$$

$$\frac{\partial \tilde{\delta}}{\partial t} + \frac{i\mathbf{k}}{a} \cdot \tilde{\mathbf{v}} = 0 \quad (1.16)$$

$$-k^2 \tilde{\delta\phi} = 4\pi G a^2 \rho_0 \tilde{\delta} \quad (1.17)$$

Where c_s is the speed of sound in the fluid. In equation (1.15) the LHS is parallel to $\mathbf{k}$. This implies that the component of $\tilde{\mathbf{v}}$ perpendicular to $\mathbf{k}$ develops according to:

$$\frac{\partial \tilde{\mathbf{v}}_{\perp}}{\partial t} + H \tilde{\mathbf{v}}_{\perp} = 0 \quad \Rightarrow \quad \tilde{\mathbf{v}}_{\perp}(t) = \frac{1}{a(t)} \tilde{\mathbf{v}}_{\perp}(0)$$

By time-differentiating Equation (1.16) and substituting the other Fourier-transformed equations we find the equation regulating the evolution of δ in the linear approximation:

$$\begin{aligned} \frac{\partial^2 \tilde{\delta}}{\partial t^2} - \frac{iH}{a} \mathbf{k} \cdot \tilde{\mathbf{v}} + \frac{i\mathbf{k}}{a} \cdot \frac{\partial \tilde{\mathbf{v}}}{\partial t} &= 0 \\ \frac{\partial^2 \tilde{\delta}}{\partial t^2} - \frac{iH}{a} \mathbf{k} \cdot \tilde{\mathbf{v}} + \frac{i\mathbf{k}}{a} \cdot \left(-H \tilde{\mathbf{v}} - \frac{i\mathbf{k}}{a} (c_s^2 \tilde{\delta} + \delta \tilde{\phi}) \right) &= 0 \\ \frac{\partial^2 \tilde{\delta}}{\partial t^2} - 2 \frac{iH}{a} \mathbf{k} \cdot \tilde{\mathbf{v}} + \frac{c_s^2 k^2}{a^2} \tilde{\delta} + \frac{k^2}{a^2} \delta \tilde{\phi} &= 0 \\ \frac{\partial^2 \tilde{\delta}}{\partial t^2} + 2H \frac{\partial \tilde{\delta}}{\partial t} + \left(\frac{c_s^2 k^2}{a^2} - 4\pi G \rho_0 \right) \tilde{\delta} &= 0 \end{aligned} \quad (1.18)$$

This equation shows resemblance to that of a damped harmonic or hyperbolic oscillator, depending on the value assumed by the time-constant factor of $\tilde{\delta}$. It is negative if:

$$k > k_J := \frac{4\pi G \rho_0 a^2}{c_s^2}$$

In this case the forcing term will smother the perturbation δ . Therefore we need $k < k_J$ to have a growing δ . The quantity k_J regulates the behaviour of the system and is known as the *Jeans wavenumber*; working with cold collisionless matter we can assume p (and c_s) to be negligible, and so we also take $k \ll k_J$:

$$\frac{\partial^2 \tilde{\delta}}{\partial t^2} + 2H \frac{\partial \tilde{\delta}}{\partial t} - 4\pi G \rho_0 \tilde{\delta} = 0$$

The evolution of the perturbation depends on the chosen cosmological solution; in a Einstein-De Sitter universe we have:

$$\frac{\partial^2 \tilde{\delta}}{\partial t^2} + \frac{4}{3t} \frac{\partial \tilde{\delta}}{\partial t} - \frac{2}{3t^2} \tilde{\delta} = 0 \quad \Rightarrow \quad \tilde{\delta}(t) = c_1 \frac{1}{t} + c_2 t^{2/3} \quad (1.19)$$

Where we have used the ansatz $\tilde{\delta} \propto t^m$. The $1/t$ term starts as a negligible quantity and remains so, while the second term shows polynomial growth. The other dynamical variables $\mathbf{k} \cdot \mathbf{v}$ and $\delta \tilde{\phi}$ are given by (1.16) and (1.17), in particular:

$$\mathbf{k} \cdot \mathbf{v} \propto a \frac{\partial \delta}{\partial t} \propto t^{1/3} \quad (1.20)$$

The linear approximation holds as long as $|\delta| \ll 1$; when system evolution leads to $|\delta| \approx 1$, the density may even become negative. The start of gravitational collapse from small density perturbations is predicted, but by no means this approximation scheme is adequate in showing the dynamics of the system in more advanced stages of structure formation.

1.3 Zel'dovich approximation

Next, we will give a review of the Zel'dovich approximation [16]. Introduced in 1969, it employs a Lagrangian approach and shows accurate results to N-body simulations near $|\delta| \approx 1$, until the formation of singularities in the energy density. We now assume the Einstein-De Sitter solution ($a = \mu t^{2/3}$ for a non-zero constant μ , $\rho_0 \propto a^{-3}$) and the matter to be pressureless; in these conditions, for an ideal fluid the velocity field can be assumed to remain irrotational (i.e. can be written as the gradient of a velocity potential); this can be shown by taking the curl of (1.4) and remains valid until collapse. First we rescale the full fluid equations (1.8), (1.9) and (1.10) by using $a(t)$ as a time variable; the gravitational potential perturbation $\delta\phi$ is also rescaled by a constant for convenience:

$$\frac{\partial}{\partial t} = \dot{a} \frac{\partial}{\partial a} \quad \mathbf{u} = \frac{\partial \mathbf{x}}{\partial a} = \frac{1}{a} \frac{\partial \mathbf{x}}{\partial t} = \frac{\mathbf{v}}{a\dot{a}} \quad \varphi = \frac{3}{2\mu^3} \delta\phi$$

We show passages required to transform the Euler equation (1.8):

$$\begin{aligned} \dot{a}^2 \mathbf{u} + a\ddot{a} \mathbf{u} + a\dot{a} \frac{\partial \mathbf{u}}{\partial a} + \dot{a}^2 \mathbf{u} + a\dot{a}^2 (\mathbf{u} \cdot \nabla) \mathbf{u} &= -\frac{2\mu^3}{3} \frac{\nabla \varphi}{a} \\ \dot{a} = \frac{2a}{3t} \quad \ddot{a} = -\frac{2a}{9t^2} \quad \frac{4}{9} \frac{a^2}{t^2} \left[a \left(\frac{\partial \mathbf{u}}{\partial a} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) + \frac{3}{2} \mathbf{u} \right] &= -\frac{2\mu^3}{3} \frac{\nabla \varphi}{a} \\ a \left(\frac{\partial \mathbf{u}}{\partial a} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) + \frac{3}{2} \mathbf{u} &= -\frac{3\mu^3}{2} \frac{t^2}{\mu^3 t^2} \frac{\nabla \varphi}{a} \end{aligned}$$

The new system for the fluid equations is:

$$\frac{\partial \mathbf{u}}{\partial a} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \frac{3}{2a} (-\mathbf{u} - \nabla \varphi) \quad (1.21)$$

$$\frac{\partial \delta}{\partial a} + \nabla \cdot ((1 + \delta) \mathbf{u}) = 0 \quad (1.22)$$

$$\nabla^2 \varphi = \frac{\delta}{a} \quad (1.23)$$

Where the rescaled velocity is still irrotational: $\mathbf{u} = -\nabla \mathcal{U}$. Notice that the linear solution produced in the previous section would have made $\mathbf{u}$ constant in the Eulerian sense, as $\mathbf{u} \propto \mathbf{v}/a\dot{a} \propto t^{1/3}t^{-1/3} = 1$. The Zel'dovich approximation consists in adopting a Lagrangian viewpoint, considering the particle starting at comoving space position $\mathbf{q}$ and assuming it moves at constant velocity $\mathbf{u}$ in a -time (normalized so that at initial time $a(t_i) = 1$):

$$\mathbf{x}(\mathbf{q}, a) = \mathbf{q} + (a - 1)\mathbf{u}(\mathbf{q}) \quad (1.24)$$

In contrast to the linear solution which assumes $\partial \mathbf{u} / \partial a = 0$, this is equivalent to assuming the material derivative of $\mathbf{u}$ (i.e. the LHS of (1.21), $d/da := \partial/\partial a + \mathbf{u} \cdot \nabla$) equals 0; the Lagrangian particles in the Zel'dovich solution are in a state of inertial motion in a -time; this makes (1.21) yield:

$$\mathbf{u} = -\nabla \varphi \quad \mathcal{U} = \varphi$$

Conservation of mass can be expressed by considering x :

$$\rho(\mathbf{x}, t) a^3 \det \left[\frac{d\mathbf{x}}{d\mathbf{q}}(\mathbf{q}, t) \right] d^3q = \rho(\mathbf{q}, t_i) d^3q \Rightarrow (1 + \delta(\mathbf{x}, t)) \det \left[\frac{d\mathbf{x}}{d\mathbf{q}}(\mathbf{q}, t) \right] = (1 + \delta(\mathbf{q}, t_i))$$

$$\delta(\mathbf{x}, t) = \frac{(1 + \delta(\mathbf{q}, t_i))}{\det \left[\frac{d\mathbf{x}}{d\mathbf{q}} \right]} - 1 \quad (1.25)$$

The Jacobian in the above equation is known as the *deformation tensor*; its determinant expresses the expansion of particles included in a starting volume d^3q after time $t - t_i$. Working in a (1+1)-dimensional Einstein-de Sitter, equation (1.24) can be derived from the fluid equations, meaning the approximation is exact for 1-dimensional space or planar symmetrical configurations of 3-dimensional space [8]; by using (1.25)² in the Poisson equation (1.23):

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{1}{a} \left(\frac{\partial q}{\partial x} - 1 \right) \Rightarrow \frac{\partial \varphi}{\partial x} = \frac{1}{a} (q - x)$$

There is no space-wise integration constant, as homogeneity requires the spatial average of all the other terms to be null. Now using (1.21) with the material derivative:

$$\frac{du}{da} = \frac{3}{2a} \left(-u - \frac{1}{a} (q - x) \right) \quad 2a^2 \frac{d^2 x}{da^2} + 3a \frac{dx}{da} - 3(x - q) = 0$$

$$\text{with } y = x - q \quad 2a^2 \frac{d^2 y}{da^2} + 3a \frac{dy}{da} - 3y = 0 \Rightarrow x - q = c_1 a + \frac{c_2}{a^{2/3}} \quad (1.26)$$

Having used the same ansatz as (1.19). Keeping the growing term, the only choice for c_1 compatible with the starting behaviour yields the Zel'dovich condition. We can explicitly state the Jacobian for the transformation between Eulerian and Lagrangian coordinates (be $\delta_{i,j}$ the Kronecker delta):

$$\frac{dx_i}{dq_j} = \delta_{i,j} + (a(t) - 1) \frac{du_i}{dq_j} = \delta_{i,j} + (a(t) - 1) \frac{d^2 \mathcal{U}}{dq_j dq_i}$$

The Jacobian is symmetrical if $\mathcal{U}$ is smooth enough; we can then change Lagrangian coordinates to match the eigenvectors of the deformation tensor at any given starting point $\mathbf{q}$; we label the eigenvalues $\beta_i(\mathbf{q})$, $i = 1, 2, 3$. This is relevant in (1.25):

$$\delta(\mathbf{x}(\mathbf{q}, t), t) = \frac{(1 + \delta(\mathbf{q}, t_i))}{\prod_i (1 - (a(t) - 1) \beta_i(\mathbf{q}))} - 1$$

The Zel'dovich solution gives insight into the dynamics of collapse. If we have a strict maximum positive eigenvalue, the volume element will collapse into a surface perpendicular to its eigenvector as δ diverges. Similarly, if 2 or 3 eigenvalue share the same maximum positive value, the collapse will come about a 1 or 0 dimensional support respectively. These structures are referred

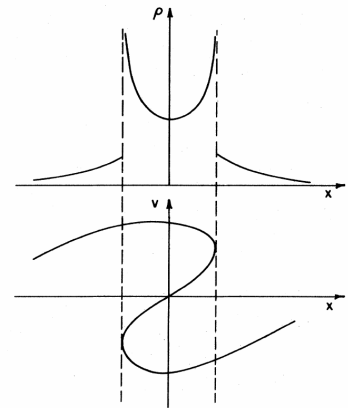


Figure 1.1: One-dimensional phase-space plot of caustic formation after shell-crossing. [11]

²We approximate $\delta(\mathbf{q}, t_i) \approx 0$. Including it would amount to a factor ≈ 1 to q in the final (1.26) result which does not evolve in time.

to as *caustics* in analogy with geometric optics; they are precursors to walls, filaments and nodes in the constitution of the cosmic web. The approximation is known to hold before *shell-crossing* occurs, (i.e. the collapse takes place and the velocity field becomes multivalued); at later times, the structures formed are dissolved as the particles continue moving inertially through caustics, and the particle density remains singular for further development (Figure 1.1).

1.4 Adhesion approximation

The adhesion approximation introduces a viscosity term in order to get matter to “stick” after shell-crossing; this is not introduced to model a physical principle, but rather because of its convenient analytical consequences; we build upon the Zel’dovich approximation, adding the new term in the RHS of the Euler equation:

$$\frac{\partial \mathbf{u}}{\partial a} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \nu \nabla^2 \mathbf{u} \quad (1.27)$$

For the velocity field before shell-crossing, the RHS will be negligible and the particles will follow Zel’dovich trajectories; near shell crossing, the viscous term becomes relevant and stops the particles allowing for permanence of structure. With potential flow $\mathbf{u} = -\nabla \mathcal{U}$ we integrate the equation and get:

$$\frac{\partial \mathcal{U}}{\partial a} - \frac{1}{2} (\nabla \mathcal{U})^2 = \nu \nabla^2 \mathcal{U}$$

This is the 3-dimensional Burgers equation, and it can be solved by applying the Hopf-Cole transform:

$$\begin{aligned} \mathcal{K} = e^{-\frac{\mathcal{U}}{2\nu}} \quad \frac{2\nu}{\mathcal{K}} \frac{\partial \mathcal{K}}{\partial a} - \frac{2\nu^2}{\mathcal{K}^2} (\nabla \mathcal{K})^2 &= 2\nu^2 \left(-\frac{1}{\mathcal{K}^2} (\nabla \mathcal{K})^2 + \frac{1}{\mathcal{K}} \nabla^2 \mathcal{K} \right) \\ \Rightarrow \quad \frac{\partial \mathcal{K}}{\partial a} &= \nu \nabla^2 \mathcal{K} \end{aligned} \quad (1.28)$$

The transformed potential follows the diffusion equation and development at any time t may be computed by the known propagator formula. The velocity potential $\mathcal{U}$ is defined at any time a . The assigned value of ν has effect on the dynamics, suppressing densi-

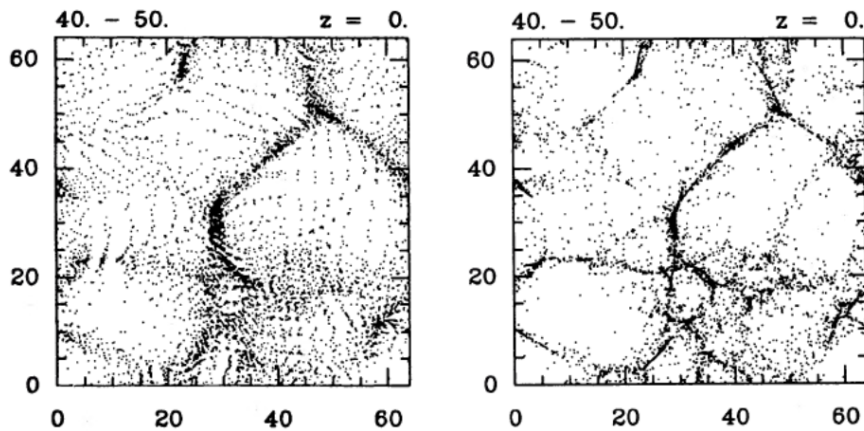


Figure 1.2: Comparison of adhesion energy density simulations with equal initial conditions; (left) uses a viscosity parameter 9 times higher than (right). [5]

ty perturbations on a spatial scale $\propto \nu^{1/2}$ (Figure 1.2). By working in the limit $\nu \rightarrow 0$, the approximation gets closer to Zel'dovich until shell-crossing, where the particles get stopped in position upon reaching the caustics, thus forming a proper "skeleton" for large scale structure in good correspondence with N-body simulations. The approximation breaks at times beyond shell-crossing; further approximations have been proposed, where a viscosity term of non-constant ν emerges naturally as a consequence of multiple flow (although requiring the idealized hypothesis of velocity dispersion remaining approximately isotropic through time evolution) [9].

1.5 Wave-mechanical formalism and free-particle approximation

In 1993 Widrow and Kaiser [10] proposed a description of the density and velocity fields through a wavefunction obeying a differential equation similar to the Schrödinger equation of quantum dynamics. As we will see, this approach is free from some of the problems encountered in the previously mentioned approximation schemes, such as the energy density unphysically becoming negative through time evolution, and is naturally better suited to describe multiple flow. We start again by assuming an ideal fluid with irrotational flow ($\mathbf{u} = -\nabla\mathcal{U}$) in the Einstein-De Sitter universe:

$$\begin{aligned}\frac{\partial\mathcal{U}}{\partial a} - \frac{1}{2}(\nabla\mathcal{U})^2 &= \frac{3}{2a}(-\mathcal{U} + \varphi) := V(\mathbf{x}, t) \\ \frac{\partial\delta}{\partial a} + \nabla \cdot ((1 + \delta)\nabla\mathcal{U}) &= 0 \\ \nabla^2\varphi &= \frac{\delta}{a}\end{aligned}$$

Where we have gathered the RHS terms of the Euler equation into an effective potential. We introduce the Madelung transformation:

$$\begin{cases} \psi(\mathbf{x}, t) := R(\mathbf{x}, t)e^{-i\frac{\mathcal{U}(\mathbf{x}, t)}{\nu}} \\ R^2(\mathbf{x}, t) := (1 + \delta(\mathbf{x}, t)) = |\psi(\mathbf{x}, t)|^2 \end{cases} \quad (1.29)$$

Where $\nu > 0$ is a free parameter in analogy to the Planck reduced constant $\hbar$, as we will soon ascertain. The density here is included in the real non-negative amplitude of the wavefunction R ; the inverse transformation is regular when $R > 0$ and it will always yield a positive density. We get the partial derivatives of ψ :

$$\begin{aligned}\nabla\psi &= \left(\nabla R - \frac{i}{\nu}R\nabla\mathcal{U} \right) e^{-i\frac{\mathcal{U}}{\nu}} \\ \nabla^2\psi &= \left(\nabla^2 R - \frac{i}{\nu} (2\nabla R \cdot \nabla\mathcal{U} + R\nabla^2\mathcal{U}) - \frac{1}{\nu^2}R(\nabla\mathcal{U})^2 \right) e^{-i\frac{\mathcal{U}}{\nu}} = \\ &= \left(\nabla^2 R - \frac{i}{\nu R} \nabla \cdot (R^2 \nabla\mathcal{U}) - \frac{1}{\nu^2}R(\nabla\mathcal{U})^2 \right) e^{-i\frac{\mathcal{U}}{\nu}} \\ \frac{\partial\psi}{\partial a} &= \left(\frac{\partial R}{\partial a} - \frac{i}{\nu}R\frac{\partial\mathcal{U}}{\partial a} \right) e^{-i\frac{\mathcal{U}}{\nu}}\end{aligned}$$

Then using the continuity equation:

$$\frac{\partial R^2}{\partial a} + \nabla \cdot (R^2 \nabla \mathcal{U}) = 0$$

We get the components for the Euler equation in wavefunction components:

$$\begin{aligned} (\nabla \mathcal{U})^2 &= -\frac{\nu^2}{R} e^{i\frac{\mathcal{U}}{\nu}} \nabla^2 \psi + \frac{\nu^2}{R} \nabla^2 R - \frac{2i\nu}{R} \frac{\partial R}{\partial a} \\ \frac{\partial \mathcal{U}}{\partial a} &= \frac{i\nu}{R} e^{i\frac{\mathcal{U}(\mathbf{x},t)}{\nu}} \frac{\partial \psi}{\partial a} - \frac{i\nu}{R} \frac{\partial R}{\partial a} \end{aligned}$$

And the Euler equation becomes:

$$\begin{aligned} \frac{i\nu}{R} e^{i\frac{\mathcal{U}(\mathbf{x},t)}{\nu}} \frac{\partial \psi}{\partial a} - \frac{i\nu}{R} \frac{\partial R}{\partial a} + \frac{\nu^2}{2R} e^{i\frac{\mathcal{U}}{\nu}} \nabla^2 \psi - \frac{\nu^2}{2R} \nabla^2 R + \frac{i\nu}{R} \frac{\partial R}{\partial a} &= V \\ (R \neq 0) \quad i\nu e^{i\frac{\mathcal{U}(\mathbf{x},t)}{\nu}} \frac{\partial \psi}{\partial a} + \frac{\nu^2}{2} e^{i\frac{\mathcal{U}}{\nu}} \nabla^2 \psi - \frac{\nu^2}{2} \nabla^2 R &= V R \\ i\nu \frac{\partial \psi}{\partial a} = -\frac{\nu^2}{2} \nabla^2 \psi + \left(\frac{\nu^2}{2R} \nabla^2 R + V \right) R e^{-i\frac{\mathcal{U}(\mathbf{x},t)}{\nu}} \end{aligned} \quad (1.30)$$

Thus leading to the Schrödinger-Poisson system:

$$i\nu \frac{\partial \psi}{\partial a} = -\frac{\nu^2}{2} \nabla^2 \psi + \left(\frac{\nu^2}{2R} \nabla^2 R + V \right) \psi \quad (1.31)$$

$$\nabla^2 \left(\frac{2}{3} V - \frac{\nu}{a} \arg \psi \right) = \frac{|\psi|^2 - 1}{a^2} \quad (1.32)$$

The wavefunction obtained from transforming a density and a velocity field following the Euler equation and the continuity equation thus follows a modified Schrödinger equation with an added term usually referred to as *quantum pressure*, as it penalizes the formation of sharp gradients in the energy density. Notice that for a real wavefunction (or equivalently a constant-phase one) this term should not be ignored [17] as it cancels out the kinetic energy operator leaving only the potential energy term. For a wavefunction obtained from (1.29), this is the case of a static solution. The requirement for single-flow is only required to define the wavefunction in (1.29), but the system we found may well be applied to more general forms of ψ . By subtracting the quantum pressure term from (1.31), the standard Schrödinger equation is restored and its equivalent fluid system follows a modified version of the Euler equation:

$$i\nu \frac{\partial \psi}{\partial a} = -\frac{\nu^2}{2} \nabla^2 \psi + V \psi \quad \Rightarrow \quad \frac{\partial \mathcal{U}}{\partial a} - \frac{1}{2} (\nabla \mathcal{U})^2 = V(\mathbf{x}, t) - \frac{\nu^2}{2\sqrt{1+\delta}} \nabla^2 \left(\sqrt{1+\delta} \right) \quad (1.33)$$

The new term has a regularizing effect on the singularities beyond shell-crossing as found in [18], and as $\nu \rightarrow 0$, classical behaviour is restored until the instant of collapse, similarly to what we have already mentioned with the viscous term in the adhesion approximation (although the qualitative effect is different, the density doesn't "stick" here). The analogue of the Zel'dovich approximation in this formalism is the free-particle approximation,

obtained by setting $V \equiv 0$ everywhere:

$$i\nu \frac{\partial \psi}{\partial a} = -\frac{\nu^2}{2} \nabla^2 \psi \quad (1.34)$$

$$-\nu \nabla^2 (\arg \psi) = \frac{|\psi|^2 - 1}{a} \quad (1.35)$$

Equation (1.34) is the known equation for the quantum free particle, and can be solved exactly. It is decoupled from (1.35); using that as a constraint on the starting wavefunction $\psi(t_i)$, we can check that the Poisson equation remains approximately satisfied for linear order single-flow solutions (1.19):

$$\frac{\partial}{\partial a} \left(\nabla^2 \mathcal{U} - \frac{\delta}{a} \right) \approx -\nabla \cdot \frac{\partial \mathbf{u}_{\text{lin}}}{\partial a} + \frac{\delta_{\text{lin}}}{a^2} - \frac{1}{a} \frac{\partial \delta_{\text{lin}}}{\partial a} + \mathcal{O}(\nu) = 0 + \frac{\delta_{\text{lin}} - \delta_{\text{lin}}}{a^2} + \mathcal{O}(\nu) = \mathcal{O}(\nu)$$

Figure 1.3 shows the development of the free-particle and the Zel'dovich approximation. Results are similar before shell-crossing, but the free-particle solution shows no singularities afterwards and deviates from the Zel'dovich solution.

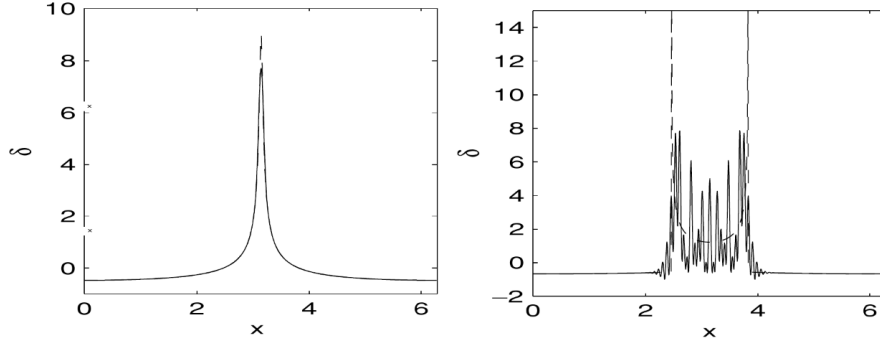


Figure 1.3: The development of the Zel'dovich approximation (dashed line) and the free-particle approximation (solid line) are shown for a small initial perturbation in one dimension before and after shell-crossing (respectively, left and right) [17]

We still have no clear way to reconstruct the values of multiple velocities at a given position once shell-crossing occurs. In the following chapters we will introduce the necessary formalism to extend the wavefunction representation for the collisionless cold matter model to phase space. We are expecting the function representing energy density in this setting to follow a law similar to Vlasov equation for the phase-space density distribution function $f(\mathbf{x}, \mathbf{p}, t)$ for this system:

$$\frac{df}{dt} := \mathbf{p} \cdot \nabla_{\mathbf{x}} f - \nabla_{\mathbf{x}} V \cdot \nabla_{\mathbf{p}} f + \frac{\partial f}{\partial t} = 0$$

2

Quantum mechanics in phase space

In this chapter, we will introduce the tools needed to bring the waveform formulation of Chapter 1 in phase space. Classical mechanics finds a natural formulation in a position-momentum space. Physical observables are defined as a real smooth functions on the phase space. The study of statistical mechanical systems is completed through the use of a density distribution function $f(\mathbf{x}, \mathbf{p})$, which has the physical meaning of the probability of finding one of the particles of the system in a infinitesimal region near the point $(\mathbf{x}, \mathbf{p})$, thus allowing predictions based on the likability of each allowed kinematical status of the particle ensemble. Newtonian mechanics is expressed and generalized in configuration space to Hamiltonian mechanics, in which an Hamiltonian function $H(\mathbf{x}, \mathbf{p})$ dictates the time evolution. On the other hand, in quantum mechanics observable quantities are defined as self-adjoint operators on a Hilbert space $\mathbf{H}$ representing the system under consideration. Time evolution of a state (a vector $|\psi\rangle \in \mathbf{H}$) is given by the Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle$$

Where $\hat{H}$ is a self-adjoint operator with the same role as its classical counterpart $H(\mathbf{x}, \mathbf{p})$. When $\mathbf{H} = L_2(\mathbb{R}^n)$ (i.e. a particle in n -dimensional space), the status of the system $|\psi\rangle \in \mathbf{H}$ can be represented as a wavefunction $\psi(\mathbf{x}) \equiv \langle \mathbf{x} | \psi \rangle$, and the probability density $f(\mathbf{x})$ of measuring the particle at position $\mathbf{x}$ is given by the Born postulate:

$$f(\mathbf{x}) = |\psi(\mathbf{x})|^2$$

Yet, there is no direct quantum-mechanical equivalent of the phase-space distribution function: $\hat{X}$ and $\hat{P}$ are not compatible and, by the uncertainty principle, it is not possible to measure with arbitrary precision both the position $\mathbf{x}$ and the momentum $\mathbf{p}$ simultaneously, so the phase space coordinates of a quantum particle are not defined at the same time. We are not able to define a conditional probability function $f(\mathbf{x}|\mathbf{p})$ (or $f(\mathbf{p}|\mathbf{x})$) because the position cannot be determined exactly with a fixed $\mathbf{p}$ (or $\mathbf{x}$); it is therefore impossible to define a quantum version of $f(\mathbf{x}, \mathbf{p})$ while also requiring it to be well-defined probability distribution function. In the following chapter we introduce a self-consistent formulation of quantum mechanics in phase space, establishing how a 1-on-1 correspondence between quantum-mechanical operators and configuration-space functions is set by choosing an ordering rule (i.e. a *quantization*) for the operators. These functions will be referred to as *operator symbols*. In particular, we will find the symbols of the density operator $\hat{\rho}$ to have properties which make them proper analogues of $f(\mathbf{x}, \mathbf{p})$. The chosen ordering rule will

also fix different properties on the symbols. We study the cases of the Weyl quantization as well as a generalized version of normal and anti-normal quantization, based on the ordering of creation and annihilation operators. References [19] [20] [21] [22] [23] [24] were employed in this chapter.

2.1 Symbol of an operator

In this section we give the definition of operator symbol in a given quantization. To start we outline the motivation leading to the main definition. We work in (1+1)-dimensional phase space for simplicity. We are looking to assign to each self-adjointed operator $\hat{A}$ a phase-space function (or temperate distribution) $A(x, p)$ invertibly. The symbol may have dependence on $\hbar$. There are some properties which are required to set up a correspondence principle:

- the assignment should be linear: $(\hat{A} + \hat{B}) \mapsto A(x, p) + B(x, p)$
- if $\hat{A}$ is a function of the position and momentum operators $\hat{X}$ and $\hat{P}$ (i.e. $\hat{A} = \overline{A}(\hat{X}, \hat{P})$) then in the classical limit the symbol $A(x, p)$ should take its classical value for any possible ordering rule, in the sense:

$$\lim_{\hbar \rightarrow 0} A(x, p) = \overline{A}(x, p)$$

- the symbols should be equipped with an associative product $\star : A(x, p) \times B(x, p) \mapsto (A \star B)(x, p)$, (known as *Moyal product*) mirroring the general non-commutativity of the operator product:

$$[A \ B]_{\star}(x, p) := (A \star B)(x, p) - (B \star A)(x, p) = -[B \ A]_{\star}(x, p)$$

- the Moyal commutator should amount to a Poisson bracket in the classical limit¹:

$$\lim_{\hbar \rightarrow 0} \frac{1}{i\hbar} [A \ B]_{\star}(x, p) = \{A \ B\}(x, p) := \frac{\partial A}{\partial x}(x, p) \frac{\partial B}{\partial p}(x, p) - \frac{\partial A}{\partial p}(x, p) \frac{\partial B}{\partial x}(x, p)$$

- parameter derivatives on the symbols should follow the Leibniz rule associated with the Moyal product (and not the scalar product, as is customary with general functions) [25].

$$\frac{\partial (A(t) \star B(t))}{\partial t}(x, p) = \left[\left(\frac{\partial A}{\partial t}(t) \star B(t) \right) + \left(A(t) \star \frac{\partial B}{\partial t}(t) \right) \right](x, p)$$

There is an intuitive way to assign a symbol satisfying the first 2 points to a generically-ordered monomial $A(\hat{X}, \hat{P})$ in $\hat{X}$ and $\hat{P}$, once an ordering rule is chosen; we just need to use the commutators of $\hat{X}$ and $\hat{P}$ to permute $A(\hat{X}, \hat{P})$:

$$[\hat{X} \ \hat{X}] = 0 \quad [\hat{P} \ \hat{P}] = 0 \quad [\hat{X} \ \hat{P}] = i\hbar$$

The monomial can rearranged as $A(\hat{X}, \hat{P}) = \tilde{A}(\hat{X}, \hat{P}) + B(\hat{X}, \hat{P})$, where both $\tilde{A}$ and B respect the ordering rule, and B is a polynomial such that $B(\hat{X}, \hat{P}) = \mathcal{O}(\hbar)$. For exam-

¹A theorem from Groenewold [19] proves it is not possible to define a symbol such that $[A \ B]_{\star} = \{A \ B\}$ in general.

ple, using the xp-ordering rule: $\hat{P}\hat{X}\hat{P}\hat{X} = \hat{X}^2\hat{P}^2 - 3i\hbar\hat{X}\hat{P} - \hbar^2$. For each ordering and each monomial $A(x, p)$ there is a unique $\tilde{A}(\hat{X}, \hat{P})$; so we set up the transformation and its inverse:

$$A(\hat{X}, \hat{P}) \mapsto \tilde{A}(x, p) + B(x, p) \quad A(x, p) = \tilde{A}(x, p) \xrightarrow{-1} \tilde{A}(\hat{X}, \hat{P})$$

Now we need to extend this from polynomials to all operators. Formally and already expecting this to be invertible, we may try using a Fourier transform [26] to get the inverse transformation:

$$\begin{aligned} \hat{f} &= \int_{\mathbb{R}^2} dx dp \delta(X - \hat{x}) \delta(p - \hat{P}) f(x, p) = \\ &= \left(\frac{1}{2\pi}\right)^2 \int_{\mathbb{R}^4} dx dp dv du \hat{T}(v, u) \exp(-i(vx + up)) f(x, p) \end{aligned} \quad (2.1)$$

It turns out that this is a proper way to set up the transformation, provided that the operator component of the term in parenthesis is the "clean" operator version of the phase factor of the Fourier transform satisfying:

$$\hat{T}(v, u) \mapsto e^{i(vx+up)} \quad (2.2)$$

This is the exponential operator whose series' terms all respect the ordering rule. We prove this only for the x-p quantization (extension to other ordering choices is straightforward) and for momentum and position operators polynomials (the extension to general physically relevant operators is beyond the scope of this thesis; we will simply assume the upcoming operators to be expandable as a polynomial series). The $\hat{T}_{xp}(v, u)$ is thus:

$$\hat{T}_{xp} = e^{iv\hat{X}} e^{iu\hat{P}} = \left[\sum_{n=0}^{\infty} \frac{1}{n!} (iv\hat{X})^n \right] \left[\sum_{n=0}^{\infty} \frac{1}{n!} (iu\hat{P})^n \right]$$

The transformation is linear, so it is sufficient to prove its validity for a monomial $x^m p^n$:

$$\begin{aligned} &\left(\frac{1}{2\pi}\right)^2 \int dx dp dv du e^{iv\hat{X}} e^{iu\hat{P}} e^{-i(vx+up)} x^m p^n = \\ &\left(\frac{1}{2\pi}\right)^2 \int dx dv e^{iv\hat{X}} e^{-i(vx)} x^m \int dp du e^{iu\hat{P}} e^{-i(up)} p^n \\ &\text{for the first integral: } \int dx dv e^{iv\hat{X}} e^{-i(vx)} x^m = 2\pi i^m \int dv e^{iv\hat{X}} \delta^{(m)}(v) = \\ &2\pi i^m \int dv \left(-\frac{\partial}{\partial v}\right)^m e^{iv\hat{X}} \delta(v) = 2\pi i^m (-i\hat{X})^m = 2\pi \hat{X}^m \end{aligned}$$

Where $\int y^n e^{-iyp} dy = 2\pi i^n \delta^{(n)}(p)$, and the delta is differentiated in the distribution sense $\int dx f(x) \delta^{(n)}(x) = \int dx (-\partial_x)^n f(x) \delta(x)$. Finally we show:

$$\left(\frac{1}{2\pi}\right)^2 \int dx dp dv du e^{iv\hat{X}} e^{iu\hat{P}} e^{-i(vx+up)} x^m p^n = \hat{X}^m \hat{P}^n$$

It is immediate that functions of isolated phase-space coordinates $f(x, p) \equiv f(x)$ or $f(x, p) \equiv f(p)$ are associated through (2.1) to $f(\hat{X})$ or $f(\hat{P})$ respectively independently of the ordering rule. In a more general sense (without requiring the properties mentioned at the start of this section), two symbol transformations on $\hat{A}$ may be defined in tandem by two

operators $\hat{\Delta}^\pm(x, p)$ in order to also get an expression for the trace of an operator product:

$$\begin{aligned} A^+(x, p) &= \text{Tr}(\hat{A}\hat{\Delta}^-(x, p)) & A^-(x, p) &= \text{Tr}(\hat{A}\hat{\Delta}^+(x, p)) \\ \hat{A} &= \int dx dp A^\pm(x, p)\hat{\Delta}^\pm(x, p) & \text{Tr}(\hat{A}\hat{B}) &= \int dx dp A^+(x, p)B^-(x, p) \end{aligned}$$

This allows for the computation of expectation values; the association created by $\hat{\Delta}^\pm$ is invertible and the counterpart operator $\hat{\Delta}^\mp(x, p)$ must satisfy [27]:

$$\text{Tr}(\hat{\Delta}^+(x, p)\hat{\Delta}^-(x', p')) = \delta(x - x')\delta(p - p') \quad \forall x, x', p, p' \in \mathbb{R} \quad (2.3)$$

In our case, (2.1) represents $\hat{\Delta}^\pm$, and the above condition is satisfied as long as $\hat{T}(v, u) \neq 0$ for any choice of (u, v) . This also means that the computation of expectation values may require the usage of two different ordering rules, "reciprocal" in the sense that their $\hat{T}$ operators will need to satisfy (2.3) (This will not be necessary for Weyl condition, we will see that $\hat{\Delta}_W^+ = \hat{\Delta}_W^-$). Generalization to higher dimension is easily accomplished by raising the degree of emerging normalization factors accordingly and changing standard multiplication to the scalar product for vectors when multiplied quantities become multi-dimensional.

2.2 Weyl quantization

We now consider the results obtained with the *Weyl quantization*, i.e. the maximally symmetric ordering rule assigning to each monomial $x^m p^n$ an average of all permutations of the appropriate operator polynomials; as an example:

$$x^2 p^2 \xrightarrow{-1} \frac{1}{6} \left(\hat{X}^2 \hat{P}^2 + \hat{X} \hat{P} \hat{X} \hat{P} + \hat{X} \hat{P}^2 \hat{X} + \hat{P} \hat{X}^2 \hat{P} + \hat{P} \hat{X} \hat{P} \hat{X} + \hat{P}^2 \hat{X}^2 \right)$$

In this case, the $\hat{T}$ operator of (2.1) is given by

$$\hat{T}_W(v, u) = e^{i(v\hat{X}+u\hat{P})} \quad \hat{f} = \left(\frac{1}{2\pi} \right)^2 \int dx dp dv du e^{i(v\hat{X}+u\hat{P})} e^{-i(vx+up)} f_W(x, p)$$

2.2.a Properties of Weyl symbols

We will need the Baker-Campbell-Hausdorff formula [19] for operators commuting with their commutator:

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2}[\hat{A}, \hat{B}]} \Rightarrow \quad \hat{T}_W(v, u) = e^{iv\hat{X}} e^{iu\hat{P}} e^{\frac{i\hbar}{2}vu} = e^{iu\hat{P}} e^{iv\hat{X}} e^{-\frac{i\hbar}{2}vu} \quad (2.4)$$

The position and momentum generalized basis of $\mathbf{H}$ is given by:

$$\langle x|x' \rangle = \delta(x - x') \quad \langle p|p' \rangle = \delta(p - p') \quad \langle x|p \rangle = \frac{e^{\frac{i}{\hbar}xp}}{\sqrt{\hbar}} \quad (2.5)$$

We can compute the integral kernel in the position representation using (2.1) with the Weyl ordering rule:

$$\begin{aligned}
 K_{\hat{f}}(x, y) &:= \langle x | \hat{f} | y \rangle = \left(\frac{1}{2\pi} \right)^2 \int dx' dp dv du \langle x | e^{iu\hat{P}} e^{iv\hat{X}} | y \rangle e^{-\frac{i\hbar}{2}vu} e^{-i(vx'+up)} f_W(x', p) \\
 &\langle x | e^{iu\hat{P}} e^{iv\hat{X}} | y \rangle = \langle x | e^{iu\hat{P}} | y \rangle e^{ivy} = \int dp' \langle x | p' \rangle \langle p' | y \rangle e^{iup'} e^{ivy} = \\
 &= \frac{1}{h} \int dp' \exp \left[ip' \left(\frac{x-y}{h} + u \right) \right] e^{ivy} = \frac{2\pi}{h} e^{ivy} \delta \left(u - \left(\frac{y-x}{h} \right) \right) \\
 \Rightarrow K_{\hat{f}}(x, y) &= \frac{1}{2\pi h} \int dx' dp dv \exp \left[iv \left(\frac{x+y}{2} - x' \right) - \frac{i}{h}(y-x)p \right] f_W(x', p) = \\
 &= \frac{1}{h} \int dx' dp \delta \left(x' - \frac{x+y}{2} \right) e^{-\frac{i}{h}(y-x)p} f_W(x', p) = \\
 &\quad \frac{1}{h} \int dp e^{-\frac{i}{h}(y-x)p} f_W \left(\frac{x+y}{2}, p \right) \tag{2.6}
 \end{aligned}$$

The rate of change of the phase grows linearly with distance, and the symbol has the midway point of x and y as the argument. We are able to express the explicit phase-space formulas for:

- the direct Weyl transformation, which resembles a symmetrized form of the Fourier transform on the operators:

$$\begin{aligned}
 f_W(x, p) &= \int dk \delta(k-p) f_W(x, k) = \frac{1}{h} \int dk dy e^{\frac{i}{h}y(k-p)} f_W(x, k) = \\
 &= \int dy e^{-\frac{i}{h}yp} \left(\frac{1}{h} \int dk e^{\frac{i}{h}y \cdot k} f_W(x, k) \right) = \int dy e^{-\frac{i}{h}yp} K_{\hat{f}} \left(x + \frac{y}{2}, x - \frac{y}{2} \right) = \\
 &= \int dy e^{-\frac{i}{h}yp} \langle x + y/2 | \hat{f} | x - y/2 \rangle \tag{2.7}
 \end{aligned}$$

- the action of operators on the states in the x-representation:

$$\langle x | \hat{f} | \psi \rangle = \int d^d y \langle x | \hat{f} | y \rangle \psi(y) = \frac{1}{h} \int d^d y \int dp e^{-\frac{i}{h}(y-x)p} f_W \left(\frac{x+y}{2}, p \right) \psi(y) \tag{2.8}$$

- the trace of an operator:

$$\text{Tr}[\hat{A}] = \int dx \langle x | \hat{A} | x \rangle = \int dx K_{\hat{A}}(x, x) = \frac{1}{h} \int dx dp A_W(x, p) \tag{2.9}$$

As a consequence of the Weyl quantization, (2.7) has an almost identical form in the p-representation:

$$\begin{aligned}
 (2.7) &= \frac{1}{h} \int dy dp' dp'' \exp \left[i \frac{y}{h} \left(-p + \frac{p' + p''}{2} \right) + i \frac{x}{h} (p' - p'') \right] \langle p' | \hat{A} | p'' \rangle = \\
 &= 2 \int dp' dp'' \delta(p' - (2p - p'')) e^{i \frac{x}{h} (p' - p'')} \langle p' | \hat{A} | p'' \rangle = 2 \int dp'' e^{i \frac{x}{h} (2p - 2p'')} \langle 2p - p'' | \hat{A} | p'' \rangle = \\
 &\quad k = 2p - 2p'' \quad = \int dk e^{i \frac{x}{h} k} \langle p + k/2 | \hat{A} | p - k/2 \rangle \tag{2.10}
 \end{aligned}$$

We are thus able to properly define the Moyal product for the Weyl quantization:

$$\begin{aligned}
 (A \star B)_W(x, p) &= \int dy e^{-\frac{i}{\hbar}yp} K_{\hat{A}\hat{B}}\left(x + \frac{y}{2}, x - \frac{y}{2}\right) = \\
 &= \int dy dz e^{-\frac{i}{\hbar}yp} K_{\hat{A}}\left(x + \frac{y}{2}, z\right) K_{\hat{B}}\left(z, x - \frac{y}{2}\right) = \\
 x' &= \frac{2(x+z)+y}{4} \quad x'' = \frac{2(x+z)-y}{4} \quad dx' dx'' = \left(\frac{h}{2}\right)^2 dy dz \\
 &\stackrel{(2.6)}{=} \left(\frac{2}{h}\right)^2 \int dx' dx'' dp' dp'' A_W(x', p') B_W(x'', p'') \exp\left(\frac{i}{\hbar}V_2(x, x', x'', p, p', p'')\right) \quad (2.11)
 \end{aligned}$$

Where V_2 is the polynomial:

$$V_2(x, x', x'', p, p', p'') = 2[(x'' - x')p + (x - x'')p' + (x' - x)p''] = 2 \det \begin{vmatrix} 1 & 1 & 1 \\ x & x' & x'' \\ p & p' & p'' \end{vmatrix}$$

2.2.b Expectation values and Wigner function

The expectation value of an operator $\hat{A}$ can be evaluated in the state $\hat{\rho}$ using the Weyl symbol formalism:

$$\begin{aligned}
 \langle \hat{A} \rangle_{\hat{\rho}} &= \text{Tr}[\hat{\rho}\hat{A}] = \frac{1}{h} \int dx dp (A_W \star \rho_W)(x, p) = \\
 &= \frac{1}{h} \left(\frac{2}{h}\right)^2 \int dx' dx'' dp' dp'' A_W(x', p') \rho_W(x'', p'') \int dx dp \exp\left(\frac{i}{\hbar}V_2(x, x', x'', p, p', p'')\right) \\
 &\quad \int dx dp \exp\left(\frac{i}{\hbar}V_2(x, x', x'', p, p', p'')\right) = \left(\frac{h}{2}\right)^2 \delta(x'' - x') \delta(p'' - p') \\
 &\Rightarrow \langle \hat{A} \rangle_{\hat{\rho}} = \frac{1}{h} \int dx dp A_W(x, p) \rho_W(x, p) \quad (2.12)
 \end{aligned}$$

This is the same equation used in classical mechanics to compute expectation values. The analogue to the classical distribution function $f(x, p)$ for the quantum system is then:

$$f_W(x, p) = \frac{\rho_W}{h}(x, p) \quad (2.13)$$

This is known as the *Wigner function* [3]. We showcase some of its properties:

- $f_W(x, p)$ is normalized and can express the purity condition on the trace:

$$\int dx dp f_W(x, p) = \text{Tr}[\hat{\rho}] = 1 \quad \int dx dp f_W^2(x, p) = \frac{1}{h} \text{Tr}[\hat{\rho}^2] \stackrel{\text{purity}}{=} \frac{1}{h}$$

- $f_W(x, p)$ is real:

$$\begin{aligned}
 f_W(x, p) &\stackrel{(2.7)}{=} \int dy e^{-\frac{i}{\hbar}yp} \langle x + y/2 | \frac{\hat{\rho}}{h} | x - y/2 \rangle \stackrel{y'=-y}{=} \int dy e^{\frac{i}{\hbar}yp} \langle x - y/2 | \frac{\hat{\rho}}{h} | x + y/2 \rangle = \\
 &\stackrel{\hat{\rho}=\hat{\rho}^\dagger}{=} \int dy \left(e^{-\frac{i}{\hbar}yp} \langle x + y/2 | \frac{\hat{\rho}}{h} | x - y/2 \rangle \right)^* = f_W^*(x, p)
 \end{aligned}$$

- the Wigner function can be reduced back to the standard quantum-mechanical x-probability or p-probability density when integrated over the other configuration-space variable:

$$\begin{aligned} \int dp f_W(x, p) &= \frac{1}{h} \int dp dy e^{-\frac{i}{h}yp} \langle x + y/2 | \hat{\rho} | x - y/2 \rangle \stackrel{(2.7)}{=} \\ &\int dy \delta(y) \langle x + y/2 | \hat{\rho} | x - y/2 \rangle = \langle x | \hat{\rho} | x \rangle \end{aligned} \quad (2.14)$$

$$\int dx f_W(x, p) \stackrel{(2.10)}{=} \frac{1}{h} \int dx dk e^{\frac{i}{h}xk} \langle p + k/2 | \hat{\rho} | p - k/2 \rangle = \langle p | \hat{\rho} | p \rangle \quad (2.15)$$

This also shows that the Weyl symbol of the projection operators $|x\rangle \langle x|$ and $|p\rangle \langle p|$ associated with measurement of the phase-space coordinates are respectively the Dirac delta $\delta(x)$ and $\delta(p)$.

The main difference between f_W and classical distribution functions is that the Wigner function can assume negative values². An example of this can be found in the Wigner functions from the eigenstates of the harmonic oscillator (Figure 2.1); in fact to obtain a proper distribution function (equations (2.14) and (2.15)), f_W needs to undergo integration on lines at fixed x or p . In general, f_W integrated with support on any line $ax + bp + c = 0$ yields non-negative result and can be used to generate a proper distribution function on the set of such lines. This is a peculiar characteristic of the Weyl quantization [27]. The Wigner function and its analogues in different quantizations are commonly referred to as *quasi-distribution function*, as a proper probability distribution function in quantum configuration space can never be achieved.

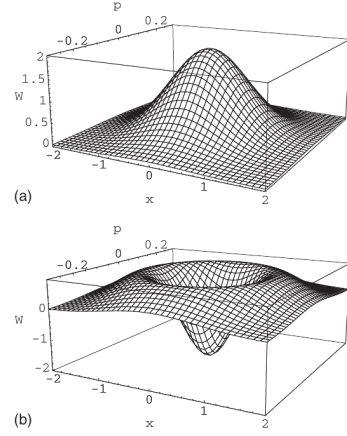


Figura 2.1: Wigner function of the ground state and the first excited state of the 1-d harmonic oscillator.[21]

2.2.c Time evolution

The Weyl-Moyal product can be expressed in the following differential form, as we show:

$$\begin{aligned} (A \star B)_W(x, p) &= A_W(x, p) e^{\frac{i\hbar}{2} (\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)} B_W(x, p) := \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{i\hbar}{2} \left(\frac{\partial}{\partial x_1} \frac{\partial}{\partial p_2} - \frac{\partial}{\partial p_1} \frac{\partial}{\partial x_2} \right) \right]^n A_W(x_1, p_1) B_W(x_2, p_2) \Big|_{\substack{x_1=x_2=x \\ p_1=p_2=p}} = \\ &= \int dv' du' du'' dv'' \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{i\hbar}{2} (v'u'' - u'v'') \right]^n e^{i(v'x+u'p)} e^{i(v''x+u''p)} \times \\ &\quad \times \tilde{A}_W(v', u') \tilde{B}_W(v'', u'') = \end{aligned} \quad (2.16)$$

²It can be proven that a pure-state wavefunction $\psi(x)$ leads to a fully non-negative Wigner function if and only if $\psi(x)$ is a Gaussian curve up to a constant factor. [28]

$$\begin{aligned}
 &= \left(\frac{2}{\hbar}\right)^2 \int du'' dv'' dx' dp' dx'' dp'' \delta\left(v'' - \frac{2}{\hbar}(p' - p)\right) \delta\left(u'' - \frac{2}{\hbar}(x - x')\right) \times \\
 &\quad \times \exp(i(v''x + u''p - v''x'' - u''p'')) A_W(x', p') B_W(x'', p'') = \\
 &= \left(\frac{2}{\hbar}\right)^2 \int dx' dx'' dp' dp'' A_W(x', p') B_W(x'', p'') \exp\left(\frac{i}{\hbar} V_2(x, x', x'', p, p', p'')\right)
 \end{aligned}$$

Thus re-obtaining the other expression for the Moyal product (2.11). To show an explicit equation for the evolution of f_W , we make use of the von Neumann equation for the evolution of density operators:

$$\frac{\partial}{\partial t} \hat{\rho}(t) = \frac{1}{i\hbar} [\hat{H}(t) \hat{\rho}(t)]$$

The symbol version of this is:

$$\begin{aligned}
 \frac{\partial}{\partial t} f_W(x, p, t) &= \frac{1}{i\hbar} \left((H_W \star f_W)(x, p, t) - (f_W \star H_W)(x, p, t) \right) = \\
 &= \frac{2}{\hbar} H_W(x, p, t) \frac{e^{\frac{i\hbar}{2}(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)} - e^{-\frac{i\hbar}{2}(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)}}{2i} f_W(x, p, t) = \\
 &= \frac{2}{\hbar} H_W(x, p, t) \sin\left(\frac{\hbar}{2}(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)\right) f_W(x, p, t) \quad (2.17)
 \end{aligned}$$

In the limit $\hbar \rightarrow 0$, if the operator components inside the sine behave well, this equation reduces to Liouville equation for the dynamics of phase-space distributions and the evolution of f_W is classical. This cannot be expected to happen in the general case. The terms of the series are proportional to $\hbar^{2n} \partial^{2n+1}$, and the derivatives may grow faster than the Planck constant suppression, as f_W cannot be assumed a priori to be regular and may have dependence on $\hbar$. When the Hamiltonian operator has natural mechanical form $\hat{H} = \hat{P}^2/2m + V(\hat{X})$:

$$\begin{aligned}
 \frac{\partial}{\partial t} f_W(x, p, t) &= \frac{p^2}{m\hbar} \sin\left(-\frac{\hbar}{2} \overleftarrow{\partial}_p \overrightarrow{\partial}_x\right) f_W(x, p, t) + \frac{2}{\hbar} V(x) \sin\left(\frac{\hbar}{2} \overleftarrow{\partial}_x \overrightarrow{\partial}_p\right) f_W(x, p, t) = \\
 &= -\frac{p}{m} \frac{\partial f_W}{\partial x}(x, p, t) + \frac{2}{\hbar} \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \left(\frac{\hbar}{2}\right)^{2n+1} \frac{\partial^{2n+1} V}{\partial x^{2n+1}}(x) \frac{\partial^{2n+1} f_W}{\partial p^{2n+1}}(x, p, t) \quad (2.18)
 \end{aligned}$$

For potentials of the type $\partial_x^n V \equiv 0 \ \forall n \geq 3$. like the harmonic oscillator or the free particle, the higher order terms of the series simply disappear and this reduces to the Liouville equation regardless of $\hbar$:

$$\frac{\partial}{\partial t} f_W(x, p, t) = -\frac{p}{m} \frac{\partial f_W}{\partial x}(x, p, t) + \frac{\partial V}{\partial x}(x) \frac{\partial f_W}{\partial p}(x, p, t)$$

This means that the Wigner function of these potentials evolves following the classical trajectories while remaining fully quantum; later we will use this property for the potential $V(x) = 0$:

$$\text{For a free-particle:} \quad f_W(x, p, t) = f_W\left(x_0 + \frac{p_0}{m}t, p_0, 0\right) \quad (2.19)$$

In the study of the gravitational instability with a general potential $V(x)$, the Wigner func-

tion is not an appropriate tool to study the evolution of the system, because both the potential and the energy distributions transformed to symbols through the Weyl quantization may be arbitrarily irregular, meaning that usage of a Vlasov-type equation (the truncated version of (2.18)) is not justified.

2.3 Generalized normal, anti-normal quantization and Husimi quasi-distribution function

In [10], it is proposed that the Schrodinger-Poisson system introduced in Chapter 1 may be described through the use of a quasi-distribution function well adapted to non-linear systems. For a given quantization Q and its operator symbols, the appropriate quasi-distribution function may be defined as the function $f_Q(x, p)$ such that:

$$\langle \hat{A} \rangle_{\hat{\rho}} = \text{Tr}(\hat{\rho} \hat{A}) = \int dx dp f_Q(x, p) A_Q(x, p) \quad (2.20)$$

We define the *generalized creation and annihilation operators* as:

$$\hat{a} = \frac{1}{\sqrt{2\hbar\kappa}}(\kappa\hat{X} + i\hat{P}) \quad \hat{a}^\dagger = \frac{1}{\sqrt{2\hbar\kappa}}(\kappa\hat{X} - i\hat{P}) \quad [\hat{a}, \hat{a}^\dagger] = 1 \quad (2.21)$$

Where $\kappa > 0$ is a fixed parameter whose significance will be shown later and is referred to as *coarse-graining parameter*; its dimension is $[M]/[T]$ and the usual harmonic oscillator operators may be restored by choosing $\kappa = m\omega$. The *normal* ordering rule is that for which $\hat{X}$ and $\hat{P}$ are expressed as a linear combination of $\hat{a}$ and $\hat{a}^\dagger$, and each monomial of the annihilation and creation operators takes the form $\hat{a}^{\dagger m} \hat{a}^n$. Instead the *anti-normal* ordering rule requires $\hat{a}^n \hat{a}^{\dagger m}$. For the anti-normal symbols, the relation (2.1) is expressed by the T-operator $\hat{T}_{AN}(v, u) = e^{-z^* \hat{a}} e^{z \hat{a}^\dagger}$, which we can relate back to the Weyl operator $\hat{T}_W$ through usage of the Baker-Campbell-Hausdorff equality (2.4):

$$\hat{T}_{AN}(v, u) = \hat{T}_W(v, u) e^{-\frac{\hbar}{4} \left(\frac{v^2}{\kappa} + \kappa u^2 \right)} := g_{AN}(v, u) \hat{T}_W(v, u) \quad z = \sqrt{\frac{\hbar}{2}} \left(-\sqrt{\kappa} u + i \frac{1}{\sqrt{\kappa}} v \right) \quad (2.22)$$

The quasi-distribution function for a ordering rule in the shape of $\hat{T}(u, v) = g(u, v) \hat{T}_W(u, v)$ (for $g \neq 0, \forall(u, v)$) can be expressed as an integral of the Wigner function [23]:

$$f(x, p) = \left(\frac{1}{2\pi} \right)^2 \int dx' dp' dv du e^{iv(x'-x) + iu(p'-p)} g(v, u) f_W(x', p') \quad (2.23)$$

For g_{AN} , the (v, u) -integral in the above formula is just the Fourier transform of a Gaussian function. The quasi-distribution function is given by:

$$f_{AN}(x, p) = \frac{2}{h} \int dx' dp' \exp \left[-\frac{1}{\hbar} \left[\kappa(x' - x)^2 + \frac{(p' - p)^2}{\kappa} \right] \right] f_W(x', p') \quad (2.24)$$

The resulting function f_{AN} is the convolution of the Wigner function with a Gaussian curve in phase space. It is usually referred to as *Husimi quasi-distribution function* [4]. The width of the Gaussian in phase space satisfies $\sigma_x \sigma_p = \sqrt{\hbar/2\kappa} \sqrt{\hbar\kappa/2} = \hbar/2$; its shape is

controlled by the κ parameter. The quasi-distribution f_{AN} is a smoothed version of the Wigner function; for $\kappa \gg 1$ the function is mostly smeared on momentum space and left approximately the same in position space, viceversa for $\kappa \approx 0$ (Figure 2.2). We will show the Husimi function is real and non-negative, but it is still not a distribution function; in fact the symbols for the projectors $|x\rangle\langle x|$ (or $|p\rangle\langle p|$) are not delta functions, and relations (2.14) and (2.15) of the Weyl quantization for $\langle x|\hat{\rho}|x\rangle$ ($\langle p|\hat{\rho}|p\rangle$) are lost [27]; smoothing of the quasi-distribution function is compensated by the presence of more intricate symbol relation for the operators, meaning the symbols are often more irregular than their Weyl counterpart. This way, no information about the physical state is lost.

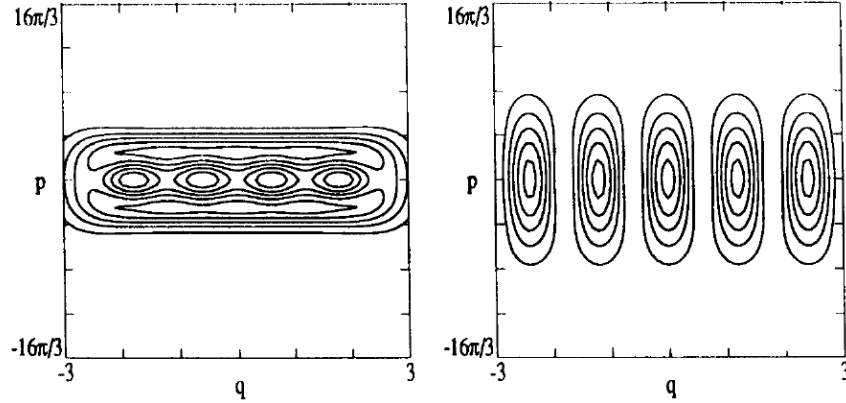


Figure 2.2: The contour plot of the fifth excited eigenstate of an infinite potential well is represented by a Husimi function with $\kappa = 3$ (left) and $\kappa = 30$ (right). [23]

The time evolution of f_{AN} can be obtained from a modified version of (2.17), by considering the expression for g_{AN} [23]. The deviation from classical time for a natural mechanical Hamiltonian can be estimated [29]:

$$\frac{\partial}{\partial t} f_{AN}(x, p, t) = \frac{2}{\hbar} H_{AN}(x, p, t) e^{\left(\frac{\hbar}{2} \left(\frac{1}{\kappa} \overleftarrow{\partial}_x \overrightarrow{\partial}_x + \kappa \overleftarrow{\partial}_p \overrightarrow{\partial}_p \right)\right)} \sin \left(\frac{\hbar}{2} (\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x) \right) f_{AN}(x, p, t) = \quad (2.25)$$

$$\frac{\partial}{\partial t} f_{AN}(x, p, t) = -\frac{p}{m} \frac{\partial f_{AN}}{\partial x}(x, p, t) + \frac{\partial V}{\partial x}(x) \frac{\partial f_{AN}}{\partial p}(x, p, t) + \mathcal{O} \left(\frac{\hbar}{\kappa X^2} \right) + \mathcal{O} \left(\frac{\hbar \kappa}{P^2} \right) \quad (2.26)$$

Where X and P are the characteristic lengths of f_{AN} in position and momentum space. The coarse-graining parameter grants control over higher order terms of the series, and an appropriate choice of κ may be used to make the time evolution of f_{AN} similar to the one predicted by classical mechanics. It can be proven [23] that the f_{AN} is expressed by the normal symbol of the density operator:

$$f_{AN}(x, p) = \frac{\rho_N}{\hbar}(x, p) \quad (2.27)$$

So we may think of showing its time development through the unitary time-evolution operator $\hat{U}(t, t')$ such that $|\psi(t)\rangle = \hat{U}(t, 0) |\psi(0)\rangle$; by using the formula for the time evolution of the density operator in the Schrödinger picture:

$$\hat{\rho}(t) = \hat{U}(t, 0) \hat{\rho}(0) \hat{U}^\dagger(0, t) \Rightarrow f_{AN}(x, p, t) = \frac{\rho_N}{\hbar}(x, p, t) = \frac{1}{\hbar} (U(t, 0) \star \rho_0 \star U^\dagger(0, t))_N(x, p)$$

We have a problem: unlike its reciprocal $g_{AN}(v, u)$, which is heavily suppressed at infinity, $g_N(v, u)$ has exponential growth in (v, u) and the integral kernel in the position representation computed like (2.6) would diverge. The position and momentum generalized basis are not suited to describe normally-ordered symbols ($\hat{X}$ and $\hat{P}$ operators are non-commuting). It is more convenient to work in the *canonical coherent state* basis, composed by eigenstates of the $\hat{a}$ operator:

$$\begin{aligned} \langle x | \alpha_{q,p} \rangle &= \left(\frac{2\kappa}{h} \right)^{1/4} \exp \left[-\frac{\kappa}{2h} (x - q)^2 + i \frac{p}{h} \left(x - \frac{q}{2} \right) \right] = \\ &= \left(\frac{2\kappa}{h} \right)^{1/4} \exp \left[-\frac{1}{2} \left(\sqrt{\frac{\kappa}{h}} x - \sqrt{2} \alpha_{q,p} \right)^2 + \frac{\alpha_{q,p}}{2} (\alpha_{q,p} - \alpha_{q,p}^*) \right] \end{aligned} \quad (2.28)$$

$$\hat{a} | \alpha_{q,p} \rangle = \alpha_{q,p} | \alpha_{q,p} \rangle \quad \alpha_{q,p} = \frac{1}{\sqrt{2h\kappa}} (\kappa q + ip) \quad (2.29)$$

Expression (2.28) is derived from the above form by completing the square in the exponent. This basis set is indexed by $\alpha \in \mathbb{C}$ and is over-complete (the states are not orthogonal).

$$\langle \alpha | \beta \rangle = e^{-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \alpha^* \beta} \quad \hat{\mathbb{1}} = \frac{1}{\pi} \int d^2 \alpha | \alpha \rangle \langle \alpha | \quad d^2 \alpha = d(\Re \alpha) d(\Im \alpha) = \frac{\pi}{h} dq dp \quad (2.30)$$

We can now find a straightforward expression for the normal symbol by noticing normally-ordered creation and annihilation monomials $\hat{a}^{\dagger m} \hat{a}^n$ output it for a given α when computing the expectation value in a α -coherent state:

$$\langle \alpha | \hat{a}^{\dagger m} \hat{a}^n | \alpha \rangle = \langle \alpha | \alpha \rangle \alpha^{*m} \alpha^n = \alpha^{*m} \alpha^n$$

In general, the normal symbol A_N of an operator is given by [23]:

$$A_N(q, p) = \langle \alpha_{q,p} | \hat{A} | \alpha_{q,p} \rangle =: A_N(\alpha_{q,p}^*, \alpha_{q,p}) \quad A_N(\alpha^*, \beta) = \frac{\langle \alpha | \hat{A} | \beta \rangle}{\langle \alpha | \beta \rangle} \quad (2.31)$$

The symbol $A_N(q, p)$ is always real for self-adjoint operators, as we can see it is obtained as the expectation value on a normalized, physical state. We can also see f_{AN} is always non-negative. A normalization factor is appropriate when changing the indexing of the symbol to complex space, but we will not factorize it here for later convenience. Normal symbols are generalized to functions A_N of 2 complex variables α and β . The new denominator in (2.31) reduces to 1 for $\alpha = \beta$ and its presence makes A_N an holomorphic function of α^* and β [24], as may be readily verified by using the position representation $\langle x | \alpha \rangle$ of (2.30). In the coherent state representation the Moyal product is easily computed:

$$\begin{aligned} (A \star B)_N(\alpha^*, \alpha) &= \langle \alpha | \hat{A} \hat{B} | \alpha \rangle = \\ &= \frac{1}{\pi} \int d^2 \beta \langle \alpha | \hat{A} | \beta \rangle \langle \beta | \hat{B} | \alpha \rangle = \frac{1}{\pi} \int d^2 \beta A_N(\alpha^*, \beta) B_N(\beta^*, \alpha) e^{-|\alpha - \beta|^2} \end{aligned} \quad (2.32)$$

3

Phase-space formulation of gravitational collapse

In this chapter, we use the quantum mechanics of symbols to convert the wave-mechanical formalism for the fluid dynamics of cold collisionless matter into configuration space, allowing for a representation based on both position and velocity in the classical limit $\nu \rightarrow 0$. The energy density will be represented by a quasi-distribution function. The Wigner function is viable for general densities only in the free-particle approximation, as the series for its time development may not be properly approximated when truncating higher order $\hbar$ terms, and so it is of meagre utility in the case of general Hamiltonians; we will instead focus on the Husimi function, which can be smooth (in a phase-space volume scale of $(\hbar/2)^n$) even when its corresponding Wigner function is not. Instead of equation (2.25), we will introduce a phase-space time propagator for the Husimi function using a path integral formalism based on coherent states. We will calculate the propagator in the case of the free-particle Hamiltonian, and we will show explicitly how the Zel'dovich condition is reproduced in this formalism. The following exposition is based on: [20] [24] [30].

3.1 Quasi-distribution representation of gravitational instability

We show the shape taken in configuration space by wavefunctions representing the fluid equations for the energy density of collisionless matter through the use of the Wigner function. The wavefunction for a system in conditions of irrotational flow is given by the Madelung representation (1.29):

$$\psi(\mathbf{x}, t) = \sqrt{1 + \delta(\mathbf{x}, t)} e^{-i \frac{\mathcal{U}(\mathbf{x}, t)}{\nu}}$$

Recalling ν has the same role as $\hbar$ and that we have set $m = 1$, we take the Wigner function of this waveform through (2.7):

$$\begin{aligned} f_w(\mathbf{x}, \mathbf{p}) &= \frac{1}{(2\pi\nu)^3} \int d^3y e^{-\frac{i}{\nu} \mathbf{y} \cdot \mathbf{p}} \psi(\mathbf{x} + \mathbf{y}/2) \psi^*(\mathbf{x} - \mathbf{y}/2) = \\ &= \frac{1}{(2\pi\nu)^3} \int d^3y e^{[-\frac{i}{\nu} (\mathbf{y} \cdot \mathbf{p} + \mathcal{U}(\mathbf{x} + \frac{\mathbf{y}}{2}) - \mathcal{U}(\mathbf{x} - \frac{\mathbf{y}}{2}))]} \sqrt{1 + \delta\left(\mathbf{x} + \frac{\mathbf{y}}{2}\right)} \sqrt{1 + \delta\left(\mathbf{x} - \frac{\mathbf{y}}{2}\right)} \end{aligned}$$

For each $\mathbf{x}$, the phase of the integrand Φ is stationary when:

$$\nabla_{\mathbf{p}} \Phi = \mathbf{y} = 0 \quad \nabla_{\mathbf{y}} \Phi = \mathbf{p} + \frac{1}{2} \left(\nabla \mathcal{U}(\mathbf{x} + \mathbf{y}/2) + \nabla \mathcal{U}(\mathbf{x} - \mathbf{y}/2) \right) = 0$$

$$\Rightarrow \mathbf{p} = -\nabla \mathcal{U}(\mathbf{x}) = \mathbf{u}(\mathbf{x})$$

As we expected, when $\nu \rightarrow 0$ for each $\mathbf{x}$ the Wigner function is concentrated near the curve of classical phase-space points describing its a -velocity field. In the free-particle approximation, the Wigner function has an explicit equation for time evolution (2.19), and each point evolves classically as:

$$f_W(\mathbf{x}, \mathbf{p}, t) = f_W(\mathbf{x}_0 + \mathbf{p}_0 t, \mathbf{p}_0, 0)$$

The formation of caustics in one dimension is represented by vertical inflexions of the classical curves in (x, p) -plane, and analogous structures in higher dimensions; singularities are not created in the Wigner function for this occurrence. Energy density flow in multiple directions can be easily represented with a mixed state composed by a sum of Madelung-form functions (1.29), which will have non-negligible f_W at different values of $\mathbf{p}$ in the same position $\mathbf{x}$. The Husimi function f_{AN} is a smoothed-out version of f_W , so it may hold similarity or not based on the value of the coarse-graining parameter.

3.2 Normal symbol path integral

In the last chapter we proposed an expression for the time development of f_{AN} :

$$f_{AN}(x, p, t) = \frac{1}{h} (U(t, 0) \star \rho_0 \star U(0, t))_N(x, p) \quad (3.1)$$

This is dependent on the expression for $U(x, p; t, 0)$, which is still not analytically viable in the general case; a path-integral approach sets the ground for approximation work. We will bring about an expression for the propagator:

$$\begin{aligned} K(z^*, w; t, 0) \quad \text{s.t.} \quad \langle z | \psi(t) \rangle &= \frac{1}{\pi} \int d^2 w K(z^*, w; t, 0) \langle w | \psi(0) \rangle = \\ &= \frac{1}{\pi} \int d^2 w \langle z | \hat{U}(t, 0) | w \rangle \langle w | \psi(0) \rangle \end{aligned}$$

through a discrete-time approximation in coherent-state representation. We work with a reasonable Hamiltonian $\hat{H}(t)$ (i.e. we assume it to be expandible in $\hat{a}$ and $\hat{a}^\dagger$, and to be well behaved in the time domain); $\hat{U}$ may be split as $\hat{U}(t + s, 0) = \hat{U}(t, s) \hat{U}(s, 0)$ an arbitrary number of times, and can be written through the use of the time-ordering operator $\hat{\mathcal{T}}$ as:

$$\begin{aligned} \hat{U}(t, 0) &= \hat{\mathcal{T}} \exp \left(-\frac{i}{\hbar} \int_0^t \hat{H}(t') dt' \right) = \prod_{n=0}^{N-1} \hat{\mathcal{T}} \exp \left(-\frac{i}{\hbar} \int_{n\tau}^{(n+1)\tau} \hat{H}(t') dt' \right) = \\ &= \prod_{n=0}^{N-1} \hat{\mathcal{T}} \exp \left(-\frac{i}{\hbar} \tau \hat{H}(t_n) \right) = \prod_{n=0}^{N-1} \exp \left(-\frac{i}{\hbar} \tau \hat{H}(t_n) \right) \end{aligned} \quad (3.2)$$

With $N \in \mathbb{N}$, $\tau = t/N$, and $t_n \in [n\tau, (n+1)\tau]$ such that the above equation holds; t_n is assumed to exist for each time interval for $N \gg 1$ through Lagrange-like arguments employing continuity. We find the propagator by using the coherent-state identity of (2.30) the same way the ordinary time-discrete path integral is derived in classical quantum

mechanics using the position-base identity:

$$\begin{aligned} K(z_N^*, z_0; t, 0) &= \langle z_N | \hat{U}(t, 0) | z_0 \rangle = \langle z_N | \prod_{n=0}^{N-1} \exp \left(-\frac{i}{\hbar} \tau \hat{H}(t_n) \right) | z_0 \rangle = \\ &= \int \left\{ \prod_{n=1}^{N-1} \frac{d^2 z_n}{\pi} \right\} \prod_{n=0}^{N-1} \langle z_{n+1} | \exp \left(-\frac{i}{\hbar} \tau \hat{H}(t_n) \right) | z_n \rangle := \int \left\{ \prod_{n=1}^{N-1} \frac{d^2 z_n}{\pi} \right\} e^f \end{aligned} \quad (3.3)$$

Where we have used the normal symbol definition (2.31). The expression for f can be approximated:

$$\begin{aligned} f &= \sum_{n=0}^{N-1} \ln \left\{ \langle z_{n+1} | \exp \left(-\frac{i}{\hbar} \tau \hat{H}(t_n) \right) | z_n \rangle \right\} \approx \sum_{n=0}^{N-1} \ln \left\{ \langle z_{n+1} | \left(1 - \frac{i}{\hbar} \tau \hat{H}(t_n) \right) | z_n \rangle \right\} = \\ &= \sum_{n=0}^{N-1} \ln \left\{ \langle z_{n+1} | z_n \rangle \left(1 - \frac{i}{\hbar} \tau H_N(z_{n+1}^*, z_n, t_n) \right) \right\} \approx \sum_{n=0}^{N-1} \left\{ \ln \langle z_{n+1} | z_n \rangle - \frac{i}{\hbar} \tau H_N(z_{n+1}^*, z_n, t_n) \right\} \end{aligned} \quad (3.4)$$

We use the equation for the scalar product of coherent states (2.30) and we are able to find the expression for the integrand (3.4):

$$\begin{aligned} K(z_N^*, z_0; t, 0) &\approx \int \left\{ \prod_{n=1}^{N-1} \frac{d^2 z_n}{\pi} \right\} \times \\ &\times \exp \left\{ \sum_{n=0}^{N-1} \left[-\frac{|z_{n+1}|^2 + |z_n|^2}{2} + z_{n+1}^* z_n - \frac{i}{\hbar} \tau H_N(z_{n+1}^*, z_n, t_n) \right] \right\} = \end{aligned} \quad (3.5)$$

$$= \int \left\{ \prod_{n=1}^{N-1} \frac{d^2 z_n}{\pi} \right\} \exp \left\{ \sum_{n=0}^{N-1} \left[-z_{n+1}^* \frac{z_{n+1} - z_n}{2} + z_n \frac{z_{n+1}^* - z_n^*}{2} - \frac{i}{\hbar} \tau H_N(z_{n+1}^*, z_n, t_n) \right] \right\} \quad (3.6)$$

The exponent is stationary in the complex variables of integration when:

$$\begin{aligned} \frac{\partial f}{\partial z_n} &= z_{n+1}^* - z_n^* - \frac{i}{\hbar} \tau \frac{\partial H_N}{\partial z_n}(z_{n+1}^*, z_n, t_n) = 0 \quad \forall n \in \{1, 2, \dots, N-1\} \\ \frac{\partial f}{\partial z_{n+1}^*} &= -z_{n+1} + z_n - \frac{i}{\hbar} \tau \frac{\partial H_N}{\partial z_{n+1}^*}(z_{n+1}^*, z_n, t_n) = 0 \quad \forall n \in \{0, 1, \dots, N-2\} \end{aligned} \quad (3.7)$$

where we have also taken advantage of H_N being holomorphic in its variables. Once the classical path $\{z_n\}_{n=0}^N$ satisfying these conditions is known, higher order expansion around the stationary point of the variables in the path may be considered, and second derivatives can be used to produce the complex analogue of a saddle-point approximation [30]. Care must be taken when sending $N \rightarrow \infty$. The problem is over-constricted: a trajectory starting at (q_0, p_0) will classically follow its Hamiltonian flux and it will end on the point (q_t, p_t) . Unless $z_N = z_t$, a saddle-point approximation may not be executed, as there will be no classical path to follow. We can solve this by noticing that the z_0^* and z_N never explicitly appear in (3.5). We can set new boundaries by considering z_n and z_n^* as

unrelated complex variables u_n and v_n , constrained by:

$$u_0 = z_0 \quad v_N = z_N^* \quad \text{but not necessarily} \quad u_N = z_N \quad v_0 = z_0^*$$

i.e. continuous $u(t)$ ignores the final point condition, while $v(t)$ ignores the first point condition. The two different paths start and end at different points in the general case. This occurrence is a consequence of relationship between annihilation and creation operators. They are not self-adjoint, and they do not commute despite one being the adjoint of the other, meaning annihilation eigenstates are not creation eigenstates and vice versa. As a reflection of this, normal symbols may be defined to depend on the variables $u(t)$ and $v(t)$ without requiring $v^* = u$. With these considerations in mind we can take the continuous limit of (3.6); we notice that the unbounded extremes appear in the terms $-u_0 v_0/2$ and $-u_N v_N/2$, and this is *wrong* in reference to (3.5), where the value of these terms is set only by z_N^* and z_0 . To restore the original meaning we have to substitute these terms with $-|u_0|^2/2$ and $-|v_N|^2/2$:

$$f = \int_0^t d\tau \left[\frac{1}{2} u(\tau) \dot{v}(\tau) - \frac{1}{2} v(\tau) \dot{u}(\tau) - \frac{i}{\hbar} H_N(v(\tau), u(\tau), \tau) \right] + \frac{u_0 v_0 + u_N v_N}{2} - \frac{|u_0|^2 + |v_N|^2}{2} \quad (3.8)$$

Conditions (3.7) become differential equations for the trajectories similar to Hamilton equations for the system:

$$\dot{v}(\tau) = \frac{i}{\hbar} \frac{\partial H_N}{\partial u}(v, u, \tau) \quad \dot{u}(\tau) = -\frac{i}{\hbar} \frac{\partial H_N}{\partial v}(v, u, \tau) \quad (3.9)$$

We omit the derivation, but the second-order saddle-point approximation can also be expressed in the continuous limit [30] leading to an approximated formula for the propagator once the exponent for the classical trajectory f_{cl} derived from (3.8) and (3.9) is known:

$$K(z_N^*, z_0; t, 0) \approx \exp \left\{ f_{cl} + \int_0^t d\tau \left[-\frac{i}{\hbar} \frac{\partial^2 H_N}{\partial u^2}(v(\tau), u(\tau), \tau) X(\tau) \right] \right\} \quad (3.10)$$

where $X(\tau)$ satisfies: $X(0) = 0 \quad \dot{X}(\tau) = -\frac{i}{\hbar} \left[\frac{1}{2} \frac{\partial^2 H_N}{\partial u^2} + 2 \frac{\partial^2 H_N}{\partial u \partial v} X(\tau) + 2 \frac{\partial^2 H_N}{\partial v^2} X^2(\tau) \right]$ (3.11)

If derivatives of the Hamiltonian at order ≥ 3 are null (as in the free-particle case), the discrete second-order expansion becomes exact and so does this expression of the propagator.

3.3 Free-particle propagator

To study the time development of the Husimi function in the free-particle approximation we compute the corresponding propagator $K(z^*, w; t, 0)$. We start by expressing the normal symbol of the Hamiltonian for a free particle by using the definition of creation and

annihilation operators (2.21):

$$\begin{aligned}\hat{H} &= \frac{\hat{P}^2}{2} = \frac{2\hbar\kappa}{2} \left(\frac{\hat{a} - \hat{a}^\dagger}{2i} \right)^2 = -\frac{\hbar\kappa}{4} (\hat{a}^2 - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} + \hat{a}^{\dagger 2}) = -\frac{\hbar\kappa}{4} (\hat{a}^2 - 2\hat{a}^\dagger\hat{a} + \hat{a}^{\dagger 2} - 1) = \\ &\Rightarrow H_N(v, u) = -\frac{\hbar\kappa}{4} (u^2 - 2uv + v^2 - 1) = -\frac{\hbar\kappa}{4} [(u - v)^2 - 1]\end{aligned}\quad (3.12)$$

We can then write the stationary-point equations (3.9) for the classical paths $u(\tau)$ and $v(\tau)$ with their respective boundaries:

$$\begin{aligned}\dot{v}(\tau) &= \frac{i\kappa}{2}(v - u) & \dot{u}(\tau) &= \frac{i\kappa}{2}(v - u) & \Rightarrow & \dot{v} - \dot{u} \equiv 0 & \dot{v} + \dot{u} &= i\kappa(v - u) \\ v - u &\equiv c_1 & v + u &= c_2 + i\kappa c_1 \tau & \Rightarrow & v(\tau) = \frac{c_1 + c_2}{2} + \frac{i\kappa c_1}{2} \tau & u(\tau) &= \frac{-c_1 + c_2}{2} + \frac{i\kappa c_1}{2} \tau\end{aligned}$$

We then apply the starting conditions $u(0) = w$ and $v(t) = z^*$ to fix the constants:

$$u(\tau) = w + i\kappa \frac{z^* - w}{2 + i\kappa t} \tau \quad v(\tau) = w + 2 \frac{z^* - w}{2 + i\kappa t} + i\kappa \frac{z^* - w}{2 + i\kappa t} \tau \quad (3.13)$$

We can now work out the expression (3.8) for f_{cl} :

$$\begin{aligned}f_{\text{cl}} &= \int_0^t d\tau \left[\frac{1}{2} \left(i\kappa \frac{z^* - w}{2 + i\kappa t} \right) (u(\tau) - v(\tau)) - \frac{i}{\hbar} \left(-\frac{\hbar\kappa}{4} [(u(\tau) - v(\tau))^2 - 1] \right) \right] + \\ &\quad + \frac{1}{2} \left[w \left(w + 2 \frac{z^* - w}{2 + i\kappa t} \right) + z^* \left(z^* - 2 \frac{z^* - w}{2 + i\kappa t} \right) - |w|^2 - |z|^2 \right] = \\ &= \int_0^t d\tau \left[-\frac{i\kappa}{4} (u(\tau) - v(\tau))^2 + \frac{i\kappa}{4} ((u(\tau) - v(\tau))^2 - 1) \right] + \\ &\quad + \frac{1}{2} \left[w^2 + z^{*2} - \frac{2(z^* - w)^2}{2 + i\kappa t} - |w|^2 - |z|^2 \right] = \\ &= -\frac{i\kappa}{4} t + \frac{w^2}{2} + \frac{z^{*2}}{2} - \frac{(z^* - w)^2}{2 + i\kappa t} - \frac{|w|^2}{2} - \frac{|z|^2}{2}\end{aligned}\quad (3.14)$$

In the integral we used the expression for the difference $u(\tau) - v(\tau)$ extracted from (3.13). We also need the $X(t)$ function (3.11) satisfying:

$$X(0) = 0 \quad \dot{X}(\tau) = \frac{i\kappa}{2} \left[\frac{1}{2} - 2X(\tau) + 2X^2(\tau) \right]$$

This is easily solved by noticing $X(\tau) = 1/2$ is an equilibrium point:

$$\begin{aligned}Y(\tau) &:= X(\tau) - \frac{1}{2} & \dot{Y}(\tau) &= i\kappa Y^2(\tau) & \Rightarrow & -\frac{1}{Y(\tau)} + \frac{1}{Y(0)} = i\kappa \tau \\ Y(0) &\stackrel{Y(0)=1/2}{\Rightarrow} Y(\tau) = -\frac{1}{2 + i\kappa \tau} & \Rightarrow & X(\tau) = \frac{i\kappa \tau}{2(2 + i\kappa \tau)}\end{aligned}\quad (3.15)$$

The integral in (3.10) is then:

$$\begin{aligned}
 \int_0^t d\tau \left[-\frac{i}{\hbar} \frac{\partial^2 H_N}{\partial u^2}(v, u, \tau) X(\tau) \right] &= \int_0^t d\tau -\frac{i}{\hbar} \left(-\frac{\hbar \kappa}{2} \right) \left(\frac{i \kappa \tau}{2(2 + i \kappa \tau)} \right) = \\
 &= \frac{i \kappa}{4} \int_0^t d\tau \left[1 - \frac{2}{2 + i \kappa \tau} \right] = \frac{i \kappa}{4} t - \frac{i}{2} \int_0^{\kappa t/2} d\tau' \frac{1}{1 + i \tau'} = \frac{i \kappa}{4} t - \frac{i}{2} [-i \ln(1 + i \tau')]_0^{\kappa t/2} = \\
 \tau' &= \frac{\kappa}{2} \tau \quad d\tau' = \frac{\kappa}{2} d\tau \quad = \frac{i \kappa}{4} t - \frac{1}{2} \ln \left(1 + \frac{i \kappa t}{2} \right) \quad (3.16)
 \end{aligned}$$

The full expression for the propagator (3.10) is then:

$$K(z^*, w; t, 0) = \left[1 + \frac{i \kappa t}{2} \right]^{-1/2} \exp \left\{ \frac{w^2}{2} + \frac{z^{*2}}{2} - \frac{(z^* - w)^2}{2 + i \kappa t} - \frac{|w|^2}{2} - \frac{|z|^2}{2} \right\} \quad (3.17)$$

Where the complex square root in the first term is chosen by requiring it to approach 1 as $t \rightarrow 0$. This way, the instant $t = 0$ propagator is:

$$K(z^*, w; 0, 0) = \exp \left\{ \frac{1}{4\hbar} \left[-\kappa(q_z - q_w)^2 - \frac{1}{\kappa}(p_z - p_w)^2 + 2i\kappa[q_z p_w - q_w p_z] \right] \right\}$$

We do not have a delta function as is the case in the position-representation path integral; instead the module of this expression is a Gaussian peaked on the starting point (q_w, p_w) of width $\sigma_q \sigma_p = 2\hbar$.

3.4 Free-particle approximation for the Husimi function

Let us show the consistency of this approach for the time development of the Husimi function by applying the free-particle approximation and exploring the physical consequences of this approach on a wave-function. We work with a one-dimensional system as generalization to higher order is straightforward. Once again we use the Madelung form (1.29) for an irrotational flow as the initial state ψ_0 for the system, also assuming the energy density perturbation $\delta_0 \ll 1$:

$$\psi_0(x, t) = \sqrt{1 + \delta_0(x, t)} e^{-i \frac{\mathcal{U}_0(x, t)}{\nu}} \quad (3.18)$$

We start by applying the coherent-state identity (2.30) on the expression for f_{AN} at time t :

$$\begin{aligned}
 f_{AN}(z^*, z, t) &= \frac{1}{2\pi\nu} (U(t, 0) \star \rho_0 \star U(0, t))_N(z^*, z) = \langle z | \hat{U}(t, 0) \hat{\rho}_0 \hat{U}^\dagger(t, 0) | z \rangle = \\
 &= \frac{1}{2\pi^3\nu} \int d^2\alpha d^2\beta \langle z | \hat{U}(t, 0) | \alpha \rangle \langle \alpha | \hat{\rho}_0 | \beta \rangle \langle \beta | \hat{U}^\dagger(t, 0) | z \rangle = \\
 &= \frac{1}{2\pi^3\nu} \int d^2\alpha d^2\beta \langle z | \hat{U}(t, 0) | \alpha \rangle \langle \alpha | \hat{\rho}_0 | \beta \rangle \left[\langle z | \hat{U}(t, 0) | \beta \rangle \right]^* \quad (3.19)
 \end{aligned}$$

Each identity created a $1/\pi$ factor. We need to compute the product of the initial state and

a canonical coherent state (2.28):

$$\begin{aligned} \langle \alpha | \psi_0 \rangle &= \int dx \left(\frac{\kappa}{\pi\nu} \right)^{1/4} \exp \left[-\frac{1}{2} \left(\sqrt{\frac{\kappa}{\nu}} x - \sqrt{2} \alpha^* \right)^2 + \frac{\alpha^*}{2} (\alpha^* - \alpha) \right] \psi_0(x) = \\ x' = \sqrt{\frac{\kappa}{2\nu}} x &= \left(\frac{\kappa}{\pi\nu} \right)^{1/4} \sqrt{\frac{2\nu}{\kappa}} \exp \left(\frac{\alpha^{*2} - |\alpha|^2}{2} \right) \int dx' e^{-(x' - \alpha^*)^2} \psi_0 \left(\sqrt{\frac{2\nu}{\kappa}} x' \right) \end{aligned}$$

We now assume the characteristic length of ψ_0 to be much longer than the width of the Gaussian, requiring $\sqrt{2\nu/\kappa} \approx 0$; this is feasible by adjusting κ so that $\kappa \gg \nu$ (when working in the limit $\nu \rightarrow 0$, which will be used later, a balanced coarse-graining parameter $\kappa \approx 1$ is sufficient); considering also the module of ψ_0 is not singular anywhere, we have that ψ_0 is approximately constant:

$$\langle \alpha | \psi_0 \rangle \approx \left(\frac{\kappa}{\pi\nu} \right)^{1/4} \sqrt{\frac{2\nu}{\kappa}} \exp \left(\frac{\alpha^{*2} - |\alpha|^2}{2} \right) \psi_0 \left(\sqrt{\frac{\nu}{2\kappa}} (\alpha + \alpha^*) \right) \quad (3.20)$$

The ρ_0 term of the integral (3.19) is expressed:

$$\begin{aligned} \langle \alpha | \hat{\rho}_0 | \beta \rangle &= \langle \alpha | \psi_0 \rangle \langle \psi_0 | \beta \rangle = 2 \sqrt{\frac{\pi\nu}{\kappa}} \exp \left(\frac{\alpha^{*2} - |\alpha|^2}{2} + \frac{\beta^2 - |\beta|^2}{2} \right) \times \\ &\times \psi_0 \left(\sqrt{\frac{\nu}{2\kappa}} (\alpha + \alpha^*) \right) \psi_0^* \left(\sqrt{\frac{\nu}{2\kappa}} (\beta + \beta^*) \right) \end{aligned} \quad (3.21)$$

We now have explicit knowledge of all the terms in (3.19). Its integrand may be expressed by application of (3.17), (3.21) and (3.18) as the product of an exponential with some factors which are constant or varying weakly and slowly in both α and β (the amplitude terms of the wave-functions ψ_0 and ψ_0^* from (3.21)):

$$f_{AN}(z^*, z, t) \approx \text{const.} \times \int d^2\alpha d^2\beta e^{h(\alpha, \beta)}$$

The function $h(\alpha, \beta)$ is not holomorphic in either variable:

$$\begin{aligned} h(\alpha, \beta) &:= -|\alpha|^2 - |\beta|^2 + \frac{\alpha^2 + \alpha^{*2} + \beta^2 + \beta^{*2}}{2} - \frac{1}{2 + i\kappa t'} [(z^* - \alpha)^2 + (z - \beta^*)^2] + \\ &+ \frac{i}{\nu} \left[-\mathcal{U}_0 \left(\sqrt{\frac{\nu}{2\kappa}} (\alpha + \alpha^*) \right) + \mathcal{U}_0 \left(\sqrt{\frac{\nu}{2\kappa}} (\beta + \beta^*) \right) \right] \end{aligned} \quad (3.22)$$

Where $t' = a(t) - 1$. All of these terms have a dependency on the reciprocal of ν (when complex variables are converted back to (q, p) space), so in the limit $\nu \rightarrow 0$ the imaginary component of the exponent may cause the phase of the integrand to oscillate rapidly and cancel itself through integration. The main contribution will come from the region of integration for which $\Im h$ is stationary. The exponent has shape $h(\alpha, \beta) = h'(\alpha) + h'(\beta)$, so it is sufficient to look for stationarity in $\alpha = \alpha_1 + i\alpha_2$:

$$\Im h'(\alpha_1, \alpha_2) = \frac{4(-\alpha_1\alpha_2 + z_1\alpha_2 - z_2\alpha_1) + \kappa t'(\alpha_1^2 - \alpha_2^2 - 2z_1\alpha_1 - 2z_2\alpha_2)}{4 + \kappa^2 t'^2} - \frac{\mathcal{U}_0}{\nu} \left(\sqrt{\frac{2\nu}{\kappa}} (\alpha_1) \right)$$

(We have just rationalized the t' -dependent terms with $(2 + i\kappa t')^{-1} = (2 - i\kappa t')/(4 + \kappa^2 t'^2)$)

and developed complex multiplication before ignoring the real part). The stationary-phase conditions are then:

$$\frac{\partial \Im h'}{\partial \alpha_1} = \frac{-4\alpha_2 - 4z_2 + \kappa t'(2\alpha_1 - 2z_1)}{4 + \kappa^2 t'^2} - \sqrt{\frac{2}{\kappa \nu}} \frac{d\mathcal{U}_0}{dx} \left(\sqrt{\frac{2\nu}{\kappa}}(\alpha_1) \right) = 0 \quad (3.23)$$

$$\frac{\partial \Im h'}{\partial \alpha_2} = \frac{-4\alpha_1 + 4z_1 + \kappa t'(-2\alpha_2 - 2z_2)}{4 + \kappa^2 t'^2} = 0 \quad (3.24)$$

Equation (3.24) is equivalent to $2\alpha_1 - 2z_1 = -\kappa t'(\alpha_2 + z_2)$ which can be easily substituted in (3.23). Requiring satisfaction of both yields the system:

$$\frac{\alpha_2 + z_2}{2} + \frac{1}{\sqrt{2\kappa\nu}} \frac{d\mathcal{U}_0}{dx} \left(\sqrt{\frac{2\nu}{\kappa}}(\alpha_1) \right) = 0 \quad \alpha_1 - z_1 + \frac{\kappa t'}{2}(\alpha_2 + z_2) = 0$$

And finally we express these conditions in configuration space using expression (2.29):

$$\frac{p_\alpha + p_z}{2} = -\frac{d\mathcal{U}_0}{dx}(q_\alpha) = u_0(q_\alpha) \quad q_\alpha + p_\alpha \frac{t'}{2} = q_z - p_z \frac{t'}{2} \quad (3.25)$$

$$\Rightarrow \quad q_z = q_\alpha + u_0(q_\alpha)t' \quad (3.26)$$

We have found that for a free-particle system in the Husimi function representation, when conditions $0 \approx \nu \ll \kappa$ are met, a sufficient condition for the evolved f_{AN} at time t' to be non-negligible on a given z is the existence of a position-space point q_α such that $p_z = u_0(q_\alpha)$ and q_z respects (3.26), which is the same equation as the Zel'dovich condition (1.24). This shows the new version of the free-particle approximation is consistent with the classical Zel'dovich approximation. The analogy should hold beyond shell-crossing (i.e. the new approximation should not allow for permanence of created structure, just like in the Zel'dovich case) and in this occasion the phase-space density is not expected to show singularities. Just as in Section 3.1, a form of multi-flow may be represented by mixed states as a sum of Husimi functions in the Madelung form.

Conclusions

In this thesis we carried out an exploratory study of the use of Husimi functions as a representation for the dynamics of gravitational instability in cold collisionless matter which is naturally well-suited for describing multiple flow. To reach this formulation, we gave a review of the historical development of approximations for this problem up to the formulation of Schrödinger-Poisson formalism and the free-particle approximation. We went over the general reformulation of quantum mechanics in phase space in order to represent system states as quasi-distribution functions in correspondence with classical probability distribution functions. We showcased some properties of the Weyl quantization in the symbol formalism to better show both analogies and differences in respect to classical mechanics, and we introduced the generalized normal and anti-normal quantization linked to the Husimi function. Finally, we used coherent-state representation to define a path integral, which was then used to get saddle-point equations for the propagator of the Husimi function and study system evolution with dynamics regulated by a free-particle Hamiltonian. The main result we found is that the Husimi version of the free-particle approximation is consistent with behaviour observed in classical inertial-like approximation schemes. This validates the Husimi formalism in an elementary setting, and sets the ground for the general case $V \neq 0$, which may be built through perturbation theory on the free-particle case. Other utility-based approximations may be built on this result by inserting analytically tame additional terms in order to fabricate supplementary properties, much like how the adhesion approximation draws upon the Zel'dovich one makes structure stable. Additional examination of the computational viability of the Husimi formalism is still required, along with comparison to N-body simulations which may provide further validation.

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