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Families of compact complex tori

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Contents

Bibliography 58

Introduction

A compact complex torus of dimension g is a complex Lie group isomorphic to V/Λ, where V is a $\mathbb{C}$-vector space of dimension g and Λ is a lattice in V.
Although all compact complex tori of dimension g are isomorphic in the category of real differentiable manifolds (to the product $(\mathbb{S}^1)^{2g}$), they can have non-isomorphic structures of complex manifolds.
As common in algebraic geometry, it is possible to simplify the study of compact complex tori using a categorical approach. One possibility is through a functor $F: \mathcal{T} \to \mathcal{H}$ yielding an equivalence of categories between $\mathcal{T}$, the category of compact complex tori of dimension g, and $\mathcal{H}$, the category of triples (Λ, V, γ), where Λ is a free $\mathbb{Z}$-module of rank $2g$, V is a $\mathbb{C}$-vector space of dimension g and $\gamma: \Lambda \to V$ is a lattice inclusion.
This is important for several reasons.
The first reason is that the treatment of compact complex tori under the categorical point of view allows us to focus our attention more on the relations and morphisms between them, rather then on the objects themselves.
The second reason is that an equivalence between $\mathcal{T}$ and $\mathcal{H}$ implies that the two categories satisfy the same properties. So, in order to understand compact complex tori, it is enough to understand $\mathcal{H}$, that is an easier category to handle.

The aim of this thesis is to extend F to a functor $F_B: \mathcal{T}_B \to \mathcal{H}_B$ yielding an equivalence of categories between $\mathcal{T}_B$, the category of families of compact complex tori of dimension g over a fixed complex manifold B, and $\mathcal{H}_B$, the category of triples (Λ, V, γ), where Λ is a locally constant B-Lie group with structural group $\mathbb{Z}^{2g}$, V is a holomorphic vector bundle of rank g over B and $\gamma: \Lambda \to V$ is a morphism of B-Lie groups, such that it yields a lattice inclusion fiberwise. If B is just a point, the categories $\mathcal{T}_B$ and $\mathcal{H}_B$ coincide with $\mathcal{T}$ and $\mathcal{H}$.

In Chapter 1 we study compact complex tori and define F, as particular case of the more general theory. We begin by defining and proving some properties of lattices and complex Lie groups, then we deal with de Rham Theorem and Hodge decomposition in degree 1 for compact complex tori.

In Chapter 2 and Chapter 3 the goal is to present prerequisities for Chapter 4. More precisely:

in Chapter 2 we begin by stating Ehresmann theorem, then we apply it to families of compact complex manifolds;

in Chapter 3, given a fixed complex manifolds B, we define B-Lie groups and discuss the link between holomophic vector bundles over B and locally free $\mathcal{O}_B$-modules of finite rank.

In Chapter 4 we generalize the results of Chapter 1 to families of compact complex tori.

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Chapter 1

Compact complex tori

We begin this chapter by defining lattices and stating a characterization of them.
Then, we define complex Lie groups and show how to define a complex manifold's structure on a quotient by a discrete group's action. In particular, we will study the quotient V/Λ, where V is a finite dimensional $\mathbb{C}$-vector space and $\Lambda \subset V$ a lattice, acting on V by translation.
We deal with cohomology, in order to prove de Rham Theorem and Hodge decomposition in degree 1 for compact complex tori.
Finally, we define the functor $F\colon \mathcal{T} \to \mathcal{H}$ and prove it yields an equivalence of categories.

1.1 Lattices

Definition 1.1.1. *A subgroup Λ of a finite dimensional real vector space V is a lattice if it is discrete and cocompact in V.*

Proposition 1.1.1. *A subgroup Λ of a finite dimensional real vector space V is a lattice if and only if Λ is a finitely generated abelian group such that every $\mathbb{Z}$-basis $\lambda_1, \dots, \lambda_n$ of Λ is an $\mathbb{R}$-basis of V.*
In particular, if Λ is a lattice in V, then V/Λ is diffeomorphic to the real n-torus $\mathbb{R}^n/\mathbb{Z}^n$.

Proof. Let Λ be a subgroup of a finite dimensional real vector space V, with $\dim_{\mathbb{R}} V = n$. Since V is a free abelian group, Λ is free and abelian too. Since there is a linear isomorphism $V \cong \mathbb{R}^n$, we can suppose $\Lambda \subseteq \mathbb{R}^n$.
Suppose that every $\mathbb{Z}$-basis $\lambda_1, \dots, \lambda_n$ of Λ is an $\mathbb{R}$-basis of $\mathbb{R}^n$. Then, we prove that Λ is a lattice.
In fact, let $\lambda_1, \dots, \lambda_n$ be a basis of Λ and $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be the linear isomorphism defined by $f(e_i) = \lambda_i$, for $i = 1, \dots, n$, where $e_1, \dots, e_n$ denotes the canonical basis of $\mathbb{R}^n$. Then, f induces a commutative diagram of abelian groups

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathbb{Z}^n & \longrightarrow & \mathbb{R}^n \\ & & \sim\big\downarrow f_{|\mathbb{Z}^n} & & \sim\big\downarrow f \\ 0 & \longrightarrow & \Lambda & \longrightarrow & \mathbb{R}^n \end{array}$$

with exact rows, where the arrows $\Lambda \to \mathbb{R}^n$ and $\mathbb{Z}^n \to \mathbb{R}^n$ are the inclusions, $f_{|\mathbb{Z}^n}$ is the restriction of f to $\mathbb{Z}^n$. By commutativity, the diagram induces an isomorphism $\overline{f}\colon \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\Lambda$ at the level of Cokernels. At the level of topological spaces, f is an homeomorphism, so that $f_{|\mathbb{Z}^n}$ and $\overline{f}$ are homeomorphisms too. Then, $\Lambda \subseteq \mathbb{R}^n$ is a lattice because it is discrete, since $\mathbb{Z}^n$ is discrete, and $\mathbb{R}^n/\Lambda$ is compact, since $\mathbb{R}^n/\mathbb{Z}^n$ is compact.
To conclude, we prove compactness of $\mathbb{R}^n/\mathbb{Z}^n$. Since $[0,1]^n \subset \mathbb{R}^n$ is compact and a fundamental set for $\mathbb{Z}^n$, $\mathbb{R}^n/\mathbb{Z}^n$ is compact too.
Conversely, suppose that Λ is a lattice. Then, we prove that every $\mathbb{Z}$-basis $\lambda_1, \dots, \lambda_n$ of Λ is an $\mathbb{R}$-basis of $\mathbb{R}^n$.
This is due to the following theorem:

Theorem 1.1.1. *Let $\Lambda \subseteq \mathbb{R}^n$ be a discrete subgroup of $\mathbb{R}^n$. Then, there exist $\lambda_1, \dots, \lambda_m \in \Lambda$ linearly independent over $\mathbb{R}$ (so that $m \le n$) such that*

$$\Lambda = \bigoplus_{i=1}^{m} \mathbb{Z}\lambda_i.$$

Moreover, Λ is cocompact if and only if $n = m$.

Proof. Let $\Lambda \subseteq \mathbb{R}^n$ be a discrete subgroup. If $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \subseteq \mathbb{R}^n$, then

$$\mathbb{R}^n/\Lambda \cong V/\Lambda \oplus \mathbb{R}^n/V. \tag{1.1}$$

Once the first statement of the theorem is proved, in order to prove the latter one we fix $\lambda_1, \dots, \lambda_m$ generators of Λ linearly independent over $\mathbb{R}$, so that $V = \oplus_{i=1}^m \mathbb{R}\lambda_i$. Then, $V/\Lambda \cong \mathbb{R}^m/\mathbb{Z}^m$ is compact and $\mathbb{R}^n/V$ is an $\mathbb{R}$-vector space of dimension $n-m$. By (1) we conclude that Λ is cocompact if and only if $n = m$.
By (1.1), we can assume that $V = \mathbb{R}^n$. Let $\Lambda^{'} \subseteq \Lambda$ be the discrete subgroup generated by a basis of $\mathbb{R}^n$ composed with elements of Λ. By definition $\Lambda^{'} \cong \mathbb{Z}^n$. Let us prove that Λ/Λ' is finite, so that $\Lambda \cong \mathbb{Z}^n$ and, as subgroup of $\mathbb{R}^n$, it is necessarely generated by a $\mathbb{R}$-basis of $\mathbb{R}^n$.
To conclude, let $F \subset \mathbb{R}^n$ be a compact subset and a fundamental set for $\Lambda^{'}$. Note that F exists since $[0,1]^n$ satisfies these properties for $\mathbb{Z}^n$. Then, $S = F \cap \Lambda$ is compact (it is finite) and surjects continuously in $\Lambda/\Lambda^{'}$. So, $\Lambda/\Lambda^{'}$ is finite because it is discrete (quotient of a discrete set) and compact (image under a continuous map of a compact set).

□

Finally we deduce that the previous homeomorphism $\overline{f}\colon \mathbb{R}^n/\Lambda \to \mathbb{R}^n/\mathbb{Z}^n$ is a diffeomorphism.
We fix $x \in \mathbb{R}^n/\Lambda$, a local chart (U, ϕ) of $\mathbb{R}^n/\Lambda$, with $x \in U$, and a local chart (V, ψ) of $\mathbb{R}^n/\mathbb{Z}^n$, with $\overline{f}(x) \in V$ and $\psi(\overline{f}(x)) = f(\phi(x))$, where f is the homeomorphism defined before. Computations show that the differential $D(\psi \circ \overline{f} \circ \phi^{-1})(\phi(x)) = A$, where A is the matrix associated to f. Then, $\overline{f}$ is a diffeomorphism because $A \in \mathrm{GL}_n(\mathbb{R})$ and all the transition maps of the atlas of $\mathbb{R}^n/\Lambda$ and $\mathbb{R}^n/\mathbb{Z}^n$ are C^∞-maps. □

1.2 Complex Lie groups

Definition 1.2.1. *Let G be a group and a complex manifold at the same time. Then G is called a complex Lie group if the map $G \times G \to G$, $(x, y) \mapsto x \cdot y^{-1}$ is holomorphic.*

Definition 1.2.2. *Let G and H be complex Lie groups. A morphism of complex Lie groups from G to H is a map $f : G \to H$ such that f is holomorphic and a group homomorphism.*

1.2.1 Quotients by a discrete group

Let X be a topological space and let G be a group that acts continuously on X, i.e. there exists an action $G \times X \to X$ such that, for any $g \in G$, the induced map $g : X \to X$ is continuous.
We will denote this action by $g \cdot x$, for all $g \in G$, $x \in X$.
The quotient space (or orbit space) X/G is endowed with a topology such that the projection map $X \to X/G$ is continuous, by saying that $V \subset X/G$ is open if and only if $\pi^{-1}(V) \subset X$ is open.

Definition 1.2.3. *Consider the following two properties:*

i. for all $x \in X$, there exists an open neighborhood Ω of x in X such that $g \cdot \Omega \cap \Omega = \emptyset$, for all $1_G \neq g \in G$;

ii. for all $(x, x^{'}) \in X \times X$, $x^{'} \notin G \cdot x$, there are open neighborhoods Ω and $\Omega^{'}$ of x and $x^{'}$, respectively, such that $g \cdot \Omega \cap \Omega^{'} = \emptyset$, for all $g \in G$.

The action of G on X is free and discontinuous (resp. free and properly discontinuous) if it satisfies i. (resp. i. and ii.).

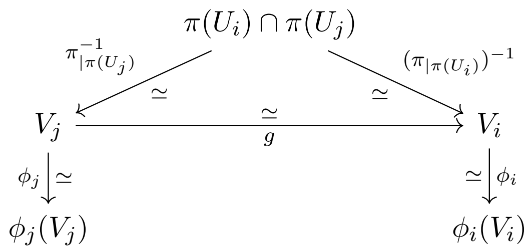
Definition 1.2.4. *Let X be a complex manifold and G be a complex Lie group. The action of G is holomorphic if the map $G \times X \to X$ is holomorphic.*

In particular, if the action of a complex Lie group G on a complex manifold X is holomorphic, then, for all $g \in G$, the induced map $g : X \to X$ is biholomorphic.
In order to prove this, fix $g \in G$. Then, the induced map $g : X \to X$ is holomorphic.
This is a biholomorphism since the map induced by $g^{-1} : X \to X$ is the inverse of the map induced by g.

Proposition 1.2.1. *Let X be a complex manifold and G be a discrete group, whose action on X is holomorphic, free and properly discontinuous. Then, the quotient X/G is a complex manifold in a natural way and the quotient map $\pi : X \to X/G$ is holomorphic.*

Proof. By covering space theory [1, 164-166], X/G is Hausdorff and $\pi : X \to X/G$ is a covering map. Then, there exists an open covering $X = \bigcup U_i$ by charts (U_i, ϕ_i) such that $g \cdot U_i \cap U_i = \emptyset$, for all $1_G \neq g \in G$. Hence, the restriction $\pi_{|U_i} : U_i \to \pi(U_i)$, of π to U_i, is bijective and $\pi(U_i)$ is open in X/G, since $\pi^{-1}(\pi(U_i)) = \bigsqcup_{g \in G} g \cdot U_i$ (where $\bigsqcup$ denotes that the union is disjoint). Thus, holomorphic charts for the quotients are given by $(\pi(U_i), \psi_i := \phi_i \circ (\pi_{|\pi(U_i)})^{-1})$. Indeed, for i, j the transitions functions $\psi_{ij} := \psi_i \circ \psi_j^{-1} : \psi_j(U_i \cap U_j) \to \psi_i(U_i \cap U_j)$ are holomorphic. In fact, if i and j are such that

$\pi(U_i) \cap \pi(U_j) \neq \emptyset$, there exist $V_i \subseteq U_i$ and $V_j \subseteq U_j$ such that $\pi(V_i) = \pi(V_j)$, so that there exists $g \in G$ such that $g \cdot V_j = V_i$. Then, we have a diagram



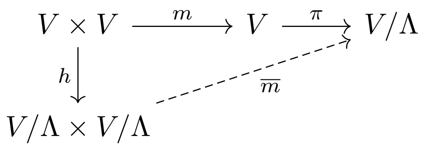
in which the triangle commutes. Hence, ψ_{ij} is holomorphic since it is equal to $\phi_i \circ g \circ \phi_j^{-1}$.
Finally, we show that $\pi\colon X \to X/G$ is holomorphic.
We fix $x \in X$, a local chart (U_i, ϕ_i) of X, with $x \in U_i$, and $(\pi(U_i), \psi_i)$ as local chart of $\pi(x) \in X/G$. Then, $\psi_i \circ \pi \circ \phi_i^{-1} = \phi_i \circ (\pi_{|\pi(U_i)})^{-1} \circ \pi \circ \phi_i^{-1} = \mathrm{Id}_{\phi_i(U_i)}$. Since $\psi_i \circ \pi \circ \phi_i^{-1}$ is holomorphic and all the transition maps of the atlas of X and X/G are holomorphic, we conclude that $\pi\colon X \to X/G$ is holomorphic. □

1.2.2 Example

Let V be a complex vector space of finite dimension and $\Lambda \subset V$ be a lattice, acting on V by translation. Then, the quotient V/Λ is a complex Lie group. In fact:

i. *V/Λ is a complex manifold.*
In fact, Λ is discrete and its action on V is free and properly discontinuous. Then, by Proposition 1.2.1, V/Λ is a complex manifold.

ii. *V/Λ is a complex Lie group.*
Let $m\colon V \times V \to V$ be the map on V defined by $(x, y) \mapsto x - y$, for all x, $y \in V$. This is holomorphic because V is a Lie group, since if we fix coordinates of V the map m is defined on $\mathbb{C}^n$, for $n = \dim_{\mathbb{C}} V$, which is a complex Lie group. The composite $\pi \circ m$ is constant on each fiber of the projection $h\colon V \times V \to V/\Lambda \times V/\Lambda$, because π is a group homomorphism. Then, there exists a unique continuous map $\overline{m}$ making the following diagram



commutative.
Finally, we prove that $\overline{m}$ is holomorphic.
As in the proof of Proposition 1.2.1, fix an open covering $V = \bigcup U_i$ by charts (U_i, ϕ_i) such that $\{(\pi(U_i), \psi_i)\}_i$ is an atlas of V/Λ, with $\psi_i := \phi_i \circ (\pi_{|\pi(U_i)})^{-1}$ for all i. In particular, $\{(U_i \times U_j, \phi_i \times \phi_j)\}_{i,j}$ is an atlas of $V \times V$ and $\{(\pi(U_i) \times \pi(U_j), \psi_i \times \psi_j)\}_{i,j}$ is an atlas of $V/\Lambda \times V/\Lambda$.

Fix $\pi(x)$, $\pi(y) \in V/\Lambda$, local charts $(\pi(U_i), \psi_i)$ and $(\pi(U_j), \psi_j)$ of V/Λ, with $x \in U_i$, $y \in U_j$, and $(\pi(U_k), \psi_k)$ as local chart of $\pi(x - y) \in V/\Lambda$, with $x - y \in U_k$. Then, $\psi_k \circ \overline{m} \circ (\psi_i^{-1} \times \psi_j^{-1}) = \phi_k \circ m \circ (\phi_i^{-1} \times \phi_j^{-1})$ is holomorphic. Since all the transition maps of the atlas of $V/\Lambda \times V/\Lambda$ and V/Λ are holomorphic, we conclude that $\overline{m}\colon V/\Lambda \times V/\Lambda \to V/\Lambda$ is holomorphic.

Lemma 1.2.1. *Let V and $V^{'}$ be two $\mathbb{C}$-vector spaces of finite dimension, $\Lambda \subset V$ and $\Lambda^{'} \subset V^{'}$ be two lattices, $f\colon V/\Lambda \to V^{'}/\Lambda^{'}$ be a holomorphic map such that $f(0) = 0$. Then f is a morphism of complex Lie groups. Moreover, there exists a unique $\mathbb{C}$-linear map $F\colon V \to V^{'}$ with $F(\Lambda) \subseteq \Lambda^{'}$ inducing f.*

Proof. Note that $(V, 0) \to (V/\Lambda, 0)$ is a pointed universal covering of V/Λ. Covering space theory [1, 158-161] implies that there exists a unique continuous map $F\colon (V, 0) \to (V^{'}, 0)$ making the following diagram

$$\begin{array}{ccc} (V,0) & \xrightarrow{F} & (V^{'},0) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi^{'}} \\ (V/\Lambda,0) & \xrightarrow[f]{} & (V^{'}/\Lambda^{'},0) \end{array}$$

commutative, where π and $\pi^{'}$ are the canonical projections. It is holomorphic. In fact, locally $\pi^{'}$ is a biholomorphism (since it is a covering map), thus, locally, F is given by the composition $(\pi^{'}_{|})^{-1} \circ f_{|} \circ \pi_{|}$, which is holomorphic since composition of holomorphic functions (here $\pi^{'}_{|}$, $f_{|}$, $\pi_{|}$ denote the restrictions of $\pi^{'}$, f, π to good open subsets).
Commutativity of the diagram implies that

$$f(v + \Lambda) = F(v) + \Lambda^{'}, \quad \forall \quad v \in V.$$

The result will follow if we show that F is $\mathbb{C}$-linear. For all $\lambda \in \Lambda$, define

$$F_\lambda\colon V \to V^{'}$$

by

$$v \mapsto F_\lambda := F(v + \lambda) - F(v), \quad \forall \quad v \in V.$$

We note that F_λ is continuous, since F is continuous, and that its image is contained in $\Lambda^{'}$, by commutativity of the previous diagram. Since V is connected, F_λ has to be constant. We compute the constant in $v = 0$:

$$F_\lambda(0) = F(\lambda) + F(0) = F(\lambda).$$

Thus, for all $v \in V$, $\lambda \in \Lambda$, $F(v + \lambda) = F(v) + F(\lambda)$. This implies that the derivatives of F are periodic, hence constant by Liouville's Theorem [2, 4], so that F is $\mathbb{C}$-linear. ☐

Let $X = V/\Lambda$, $X^{'} = V^{'}/\Lambda^{'}$.
Denote by $\mathrm{Hom}(X, X^{'})$ the set of morphisms of complex Lie group from X to $X^{'}$. We endow $\mathrm{Hom}(X, X^{'})$ with a structure of abelian group $(\mathrm{Hom}(X, X^{'}), +)$ by defining

$$(f + g)(x) = f(x) + g(x), \quad \text{for all} \quad x \in X$$

for all $f,\, g \in \mathrm{Hom}(X, X^{'})$.
Let $\mathrm{Hom}_{\mathbb{C}}(V, V^{'})$ be the $\mathbb{C}$-vector space of $\mathbb{C}$-linear morphisms from V to $V^{'}$ and $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, \Lambda^{'})$ be the abelian group of $\mathbb{Z}$-linear morphisms from Λ to $\Lambda^{'}$.
We define ρ_a *the analytic representation of $Hom(X, X^{'})$*

$$\rho_a\colon \mathrm{Hom}(X, X^{'}) \to \mathrm{Hom}_{\mathbb{C}}(V, V^{'})$$

by sending

$$f \mapsto F.$$

This is well defined by Lemma 1.2.1.
The uniqueness part of the Lemma implies that it is injective and a group homomorphism.
We define ρ_r *the rational representation of $Hom(X, X^{'})$*

$$\rho_r\colon \mathrm{Hom}(X, X^{'}) \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \Lambda^{'})$$

by sending

$$f \mapsto F_{|\Lambda}.$$

By Lemma 1.2.1, $F(\Lambda) \subseteq \Lambda^{'}$ and F is $\mathbb{C}$-linear, thus the restriction is well defined and is $\mathbb{Z}$-linear.
This is a group homomorphism since ρ_a is a group homomorphism and we have

$$\rho_r(f + g) = \rho_a(f + g)_{|\Lambda} = (\rho_a(f) + \rho_a(g))_{|\Lambda} = \rho_a(f)_{|\Lambda} + \rho_a(g)_{|\Lambda} = \rho_r(f) + \rho_r(g)$$

for f and $g \in \mathrm{Hom}(X, X^{'})$.
It is injective because if f and $g \in \mathrm{Hom}(X, X^{'})$ are such that $\rho_r(f) = \rho_r(g)$, then

$$\rho_a(f)_{|\Lambda} = \rho_a(g)_{\Lambda} \Rightarrow \rho_a(f) = \rho_a(g) \Rightarrow f = g.$$

The first arrow follows from the fact that $\rho_a(f)$ is uniquely determined by the image of Λ, because Λ generates V as $\mathbb{R}$-vector space and $\rho_a(f)$ is $\mathbb{R}$-linear, since it is $\mathbb{C}$-linear. The second arrow follows from injectivity of ρ_a.

1.2.3 Compact complex tori

Definition 1.2.5. *A compact complex torus of dimension g is a complex Lie group isomorphic to V/Λ, where V is a $\mathbb{C}$-vector space of dimension g and Λ is a lattice in V.*

By Lemma 1.2.1 it follows that if $\phi\colon X \to X^{'}$ is a holomorphic map between complex tori such that $\phi(0_X) = 0_{X'}$, then it is a morphism of complex Lie groups.

Proposition 1.2.2. *If X is a compact complex torus of dimension g, then*

$$\mathbb{Z}^{2g} \cong \pi_1(X, *) \xrightarrow{\simeq} H_1(X; \mathbb{Z}).$$

Proof. By Proposition 1.1.1 it follows that, as differentiable manifolds,

$$X \cong \mathbb{R}^n/\mathbb{Z}^n \cong (\mathbb{R}/\mathbb{Z})^n \cong (\mathbb{S}^1)^{2g}$$

so that

$$\pi_1(X,*) \cong \pi_1((\mathbb{S}^1)^{2g},*) \cong \pi_1(\mathbb{S}^1,*)^{2g} \cong \mathbb{Z}^{2g}.$$

By Hurewicz theorem [4, 80-84]

$$\pi_1(X,*)^{\text{ab}} \xrightarrow{\simeq} H_1(X;\mathbb{Z}) \ .$$

Since $\pi_1(X,*)$ is abelian, $\pi_1(X,*)^{\text{ab}} = \pi_1(X,*)$ and, then, we obtain the conclusion. □

Let X be a compact complex torus.
We want to compute generators of $H_1(X;\mathbb{Z})$ and then, by tensoring with $-\otimes_{\mathbb{Z}}\mathbb{R}$, also of $H_1(X;\mathbb{R}) = H_1(X;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{R}$.
We fix a diffeomorphism $\phi\colon X \xrightarrow{\simeq} (\mathbb{S}^1)^{2g}$, so that it induces a group isomomorphism $f_*\colon H_1(X;\mathbb{Z}) \xrightarrow{\simeq} H_1((\mathbb{S}^1)^{2g};\mathbb{Z})$. Thus, it is enough to compute generators of $H_1((\mathbb{S}^1)^{2g};\mathbb{Z})$.
For $1\le i\le 2g$, let $p_i\colon (\mathbb{S}^1)^{2g}\to\mathbb{S}^1$ be the projection to the i-th component of the direct product. By homotopy theory [3, 76-77], we have the isomorphism

$$\pi_1((\mathbb{S}^1)^{2g},*) \xrightarrow{\simeq} \pi_1(\mathbb{S}^1,*)^{2g}$$

induced by the projections $\{p_i\}_{1\le i\le 2g}$. Combining this with Hurewicz's map, we obtain

$$H_1((\mathbb{S}^1)^{2g};\mathbb{Z}) \xrightarrow[\simeq]{(p_{i_*})_i} H_1(\mathbb{S}^1;\mathbb{Z})^{2g}$$

where, for all $1\le i\le 2g$, p_{i_*} is the map induced by p_i at the level of homologies.
Then, fixed $\alpha\colon [0,1]\to\mathbb{S}^1$ a path such that $[\alpha]$ generates $H_1(\mathbb{S}^1;\mathbb{Z})$, the paths

$$\alpha_i\colon [0,1]\to(\mathbb{S}^1)^{2g}, \quad i=1,\dots,2g$$

defined by

$$(p_j\circ\alpha_i)(t) = \delta_{i,j}\alpha(t), \quad \text{for all} \quad 1\le i,j\le 2g$$

are such that their equivalence classes $[\alpha_1],\dots,[\alpha_{2g}]$ generate $H_1((\mathbb{S}^1)^{2g};\mathbb{Z})$.

Definition 1.2.6. *A framed compact complex torus is (X,ϕ), where X is a compact complex torus and ϕ is an isomorphism $\phi\colon H_1(X;\mathbb{Z}) \xrightarrow{\simeq} \mathbb{Z}^{2g}$.*

1.3 De Rham Theorem

1.3.1 Smooth singular homology

Let X be a differentiable manifold and A be a commutative ring. Let S_* denote the singular chain complex of X with coefficients A

$$\dots \longrightarrow S_{p+1} \xrightarrow{\delta_{p+1}} S_p \xrightarrow{\delta_p} S_{p-1} \longrightarrow \dots .$$

For $p \in \mathbb{N}$, let S_p^{smooth} be the free group over A generated by the simplexes $\sigma\colon \Delta_p \to X$, such that there exists an open neighborhood $U \subseteq \mathbb{R}^p$ of Δ_p such that σ is the the restriction of a smooth function $\tilde{\sigma}\colon U \to X$. Denote by S_*^{smooth} the chain complex

$$\dots \longrightarrow S_{p+1}^{smooth} \xrightarrow{\delta_{p+1}^s} S_p^{smooth} \xrightarrow{\delta_p^s} S_{p-1}^{smooth} \longrightarrow \dots .$$

where the maps δ_p^s are given by the restriction of δ_p, for all $p \in \mathbb{N}$. The natural inclusion $S_p^{smooth}(X) \hookrightarrow S_p(X)$, for all $p \in \mathbb{N}$, induces a diagram

$$\begin{array}{ccccccccc} \dots & \longrightarrow & S_{p+1}^{smooth} & \xrightarrow{\delta_{p+1}^s} & S_p^{smooth} & \xrightarrow{\delta_p^s} & S_{p-1}^{smooth} & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & S_{p+1} & \xrightarrow{\delta_{p+1}} & S_p & \xrightarrow{\delta_p} & S_{p-1} & \longrightarrow & \dots \end{array}$$

in which every square commutes. Then, by [5, 291] the chain map

$$S_*^{smooth}(X) \hookrightarrow S_*(X)$$

given by the inclusion, induces an isomorphism of A-modules

$$H_*^{smooth}(X;A) \xrightarrow{\simeq} H_*(X;A).$$

1.3.2 De Rham pairing

Let X be a differentiable manifold.
We define a map

$$H_1(X;\mathbb{R})^{smooth} \times H^1_{\text{dR}}(X;\mathbb{R}) \to \mathbb{R}$$

by

$$([\sigma],[\omega]) \mapsto \int_\sigma \omega$$

for all $[\omega] \in H^1_{\text{dR}}(X;\mathbb{R})$ and $[\sigma] \in H_1(X;\mathbb{R})^{smooth}$.
This map is well-defined. In fact, for $[\sigma] \in H_1(X;\mathbb{R})^{smooth}$ and $[\omega] \in H^1_{\text{dR}}(X;\mathbb{R})$, the integral does not depend on the representatives of $[\sigma]$ and $[\omega]$:
if we choose $\sigma + \delta\sigma_1$ as representative of $[\sigma]$, then

$$\int_{\sigma+\delta\sigma_1} \omega = \int_\sigma \omega + \int_{\delta\sigma_1} \omega = \int_\sigma \omega + \int_{\sigma_1} d\omega = \int_\sigma \omega$$

because $\int_{\delta\sigma_1} \omega = \int_{\sigma_1} d\omega$, by Stokes' Theorem [6, 67-269], and $\int_{\sigma_1} d\omega = 0$, since ω is closed.
If we choose $\omega + d\omega_1$ as representative of $[\omega]$, then

$$\int_\sigma \omega + d\omega_1 = \int_\sigma \omega + \int_\sigma \omega_1 = \int_\sigma \omega + \int_{\delta\sigma} \omega_1 = \int_\sigma \omega$$

because $\int_\sigma d\omega_1 = \int_{\delta\sigma} \omega_1$, by Stokes, and $\int_{\delta\sigma} \omega_1 = 0$, since σ is closed.
Since this map is bilinear, this yields an $\mathbb{R}$-linear map

$$H_1(X;\mathbb{R})^{smooth} \otimes_{\mathbb{R}} H^1_{\text{dR}}(X;\mathbb{R}) \to \mathbb{R}.$$

Remark 1.3.1. *It is important to notice that the isomorphism*

$$\iota\colon H_1(X;\mathbb{R})^{smooth} \xrightarrow{\simeq} H_1(X;\mathbb{R})$$

allows us to define integrals of elements $[\omega] \in H^1_{dR}(X;\mathbb{R})$ *along* $[\gamma] \in H_1(X;\mathbb{R})$ *by*

$$\int_\gamma \omega := \int_\sigma \omega$$

where $\iota([\gamma]) = [\sigma]$*. We will always use this convention.*

Then we define the bilinear map

$$\Psi\colon H_1(X;\mathbb{R}) \otimes_{\mathbb{R}} H^1_{\mathrm{dR}}(X;\mathbb{R}) \to \mathbb{R}$$

by

$$([\sigma],[\omega]) \mapsto \int_\sigma \omega$$

for all $[\omega] \in H^1_{\mathrm{dR}}(X;\mathbb{R})$ and $[\sigma] \in H_1(X;\mathbb{R})$.

In particular, Ψ induces the $\mathbb{R}$-linear maps

$$\Psi_1\colon H_1(X;\mathbb{R}) \to H^1_{\mathrm{dR}}(X;\mathbb{R})^\vee := \mathrm{Hom}_{\mathbb{R}}(H^1_{\mathrm{dR}}(X;\mathbb{R}),\mathbb{R})$$

defined by

$$[\sigma] \mapsto \int_\sigma \quad \text{for all} \quad [\sigma] \in H_1(X;\mathbb{R})$$

and

$$\Psi_2\colon H^1_{\mathrm{dR}}(X;\mathbb{R}) \to H^1(X;\mathbb{R}) = \mathrm{Hom}_{\mathbb{R}}(H_1(X;\mathbb{R}),\mathbb{R})$$

defined by

$$[\omega] \mapsto \int \omega \quad \text{for all} \quad [\omega] \in H^1_{\mathrm{dR}}(X;\mathbb{R}).$$

1.3.3 Invariant forms

Let X be a differentiable manifolds.

Definition 1.3.1. *The translation by an element* $x_0 \in X$ *is defined to be the holomorphic map* $t_{x_0}\colon X \to X$, $x \mapsto x + x_0$.

Let $\mathcal{A}^1(X)$ be the sheaf of 1-forms on X (see Def. [7, 282])

Definition 1.3.2. *We define the* $\mathbb{R}$*-subvector space over of the invariant forms of* $\mathcal{A}^1(X)$ *as*

$$IF(X) := \{\omega \in \mathcal{A}^1(X) : t_x^*\omega = \omega \quad \forall \quad x \in V\}$$

where $t_x^*\omega$ *is the pull back of* ω *under* t_x*, the translation by* x*.*

Proposition 1.3.1. *If* X *is a differentiable manifold, then the* $\mathbb{R}$*-vector spaces* $IF(X)$ *and* $Hom_{\mathbb{R}}(T_0X,\mathbb{R})$ *are isomorphic, where* T_0X *is the tangent space at the point* $0 \in X$ *.*

Proof. Since every element of $IF(X)$ is uniquely determined by its value at the point $0 \in X$, the evaluation at 0

$$\mathrm{ev}_0 \colon IF(X) \to \mathrm{Hom}_{\mathbb{R}}(T_0X, \mathbb{R})$$

defined by

$$\omega \mapsto \omega(0)$$

for all $\omega \in IF(V)$, defines an isomorphism of $\mathbb{R}$-vector spaces

$$IF(X) \xrightarrow{\simeq} \mathrm{Hom}_{\mathbb{R}}(T_0X, \mathbb{R}).$$

□

Proposition 1.3.2. *Let V be a $\mathbb{C}$-vector space of dimension g, $\Lambda \subset V$ be a lattice in V and $\pi \colon V \to V/\Lambda$ be the canonical projection. Then, the pull back map*

$$\pi^* \colon IF(V/\Lambda) \to IF(V)$$

induced by π, is an isomorphism of $\mathbb{R}$-vector spaces.

Proof. Since V is an $\mathbb{R}$-vector space of dimension $2g$, we have a canonical isomorphism $T_0V \cong \mathbb{R}^{2g}$, so that

$$\mathrm{Hom}_{\mathbb{R}}(T_0V, \mathbb{R}) \cong \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^{2g}, \mathbb{R}) \cong \mathbb{R}^{2g}.$$

Thus, $IF(V) \cong \mathbb{R}^{2g}$. If we fix coordinates $x_1, \dots, x_{2g}$ of V, $dx_1, \dots, dx_{2g}$ are $2g$ linearly independent one-forms over $\mathbb{R}$ and invariant by translations, so that they generate $IF(V)$ over $\mathbb{R}$.
To conclude, consider $\pi^* \colon IF(V/\Lambda) \to IF(V)$. It is $\mathbb{R}$-linear and injective, because if ω_1 and $\omega_2 \in IF(V/\Lambda)$ are such that $\pi^*(\omega_1) = \pi^*(\omega_2)$, then

$$\omega_1(\pi(0)) = ev_0(\pi^*(\omega_1)) = ev_0(\pi^*(\omega_2)) = \omega_2(\pi(0))$$

so that $\omega_1 = \omega_2$.
It is also surjective because the dx_i's define one-forms on V/Λ (which we will still denote by dx_i) satisfying $\pi^*(dx_i) = dx_i$, for all $i = 1, \dots, 2g$.
Thus π^* is an isomorphism. In particular, $dx_1, \dots, dx_{2g}$ form a basis of $IF(V/\Lambda)$ over $\mathbb{R}$. □

Proposition 1.3.3. *Let X be a compact complex torus of dimension g. Then, there exists an isomorphism of $\mathbb{R}$-vector spaces*

$$IF(X) \xrightarrow{\simeq} H^1(X; \mathbb{R}).$$

Fixed real coordinates of X, it is defined by sending dx_i to $\Psi_2([dx_i])$, for $i = 1, \dots, 2g$.

Proof. Since the statement involves only the differential point of view, we can suppose that $X = (\mathbb{S}^1)^{2g}$. In particular, using the same notations of previous sections, we know that

$$H_1(X; \mathbb{R}) = < [\alpha_1], \dots, [\alpha_{2g}] >_{\mathbb{R}} .$$

Fix real coordinates $x_1, \dots, x_{2g}$ of V. Since the dx_i's are closed one-forms, there exists an $\mathbb{R}$-linear map

$$\phi \colon IF(X) \to H^1_{\mathrm{dR}}(X; \mathbb{R})$$

defined on the basis by sending dx_i to its equivalence class $[dx_i]$ in $H^1_{\mathrm{dR}}(X; \mathbb{R})$, for all $i = 1, \dots, 2g$.
Then, we compose ϕ with the $\mathbb{R}$-linear map induced by Ψ

$$\Psi_2 \colon H^1_{\mathrm{dR}}(X; \mathbb{R}) \to H^1(X; \mathbb{R}).$$

Since $\int_{\alpha_i} dx_j = \delta_{ij}$, for $1 \leq i, j \leq 2g$, we conclude that the composition

$$IF(X) \xrightarrow{\phi} H^1_{\mathrm{dR}}(X; \mathbb{R}) \longrightarrow H^1(X; \mathbb{R})$$

is an isomorphism. □

1.3.4 De Rham Theorem

Note that in the proof of Proposition 1.4.1, we proved that

$$\Psi_2 \colon H^1_{\mathrm{dR}}(X; \mathbb{R}) \to H^1(X; \mathbb{R})$$

defined by sending $[\omega] \in H^1_{\mathrm{dR}}(X; \mathbb{R})$ to $\int \omega \in H^1(X; \mathbb{R})$, is surjective.

Theorem 1.3.1 (de Rham Theorem in degree 1 for compact complex tori)**.** *If X is a compact complex torus, the $\mathbb{R}$-linear map*

$$\Psi_2 \colon H^1_{dR}(X; \mathbb{R}) \to H^1(X; \mathbb{R})$$

is an isomorphism.

Proof. We need only to check the injectivity.
Let $[\omega] \in \mathrm{Ker}\Psi_2$. Then $\int_\gamma \omega = 0$ for all $[\gamma] \in H_1(X; \mathbb{R})$.
Fix $x_0 \in X$ and define a map

$$F \colon X \to \mathbb{R}$$

by

$$F(x) = \int_{x_0}^{x} \omega, \quad \text{for all} \quad x \in X$$

where the integral is over any path from x_0 to x.
The function F is well defined. In fact, let γ_1 and γ_2 be two paths from x_0 to x_1. Let $\overline{\gamma_2}$ be the inverse path of γ_2. Then

$$\int_{\gamma_1} \omega = \int_{\gamma_1} \omega + \int_{\overline{\gamma_2}\gamma_2} \omega = \int_{\gamma_1\overline{\gamma_2}} \omega + \int_{\gamma_2} \omega = \int_{\gamma_2} \omega$$

since the concatenation $\gamma_1\overline{\gamma_2}$ is a closed path.
By elementary calculus, F is smooth and its derivative is ω, so that $[\omega] = [dF] = 0 \in H^1_{\mathrm{dR}}(X; \mathbb{R})$. □

If we repeat the same construction over $\mathbb{C}$, we can define a $\mathbb{C}$-linear map

$$\Psi_{\mathbb{C}} \colon H_1(X;\mathbb{C}) \otimes_{\mathbb{C}} H^1_{\mathrm{dR}}(X;\mathbb{C}) \to \mathbb{C}$$

defined by

$$([\sigma],[\omega]) \mapsto \int_\sigma \omega$$

for all $[\omega] \in H^1_{\mathrm{dR}}(X;\mathbb{C})$ and $[\sigma] \in H_1(X;\mathbb{C})$, obtaining the following Theorem:

Theorem 1.3.2 (de Rham Theorem: complex case)**.** *If X is a compact comples torus, $\Psi_{\mathbb{C}}$ is an isomorphism.*

Proof. Same as the real case. □

Thus, if X is a compact complex torus, by de Rham Theorem the $\mathbb{C}$-linear maps induced by Ψ

$$\Psi_{1,\mathbb{C}} \colon H_1(X;\mathbb{C}) \to H^1_{\mathrm{dR}}(X;\mathbb{C})^\vee := \mathrm{Hom}_{\mathbb{C}}(H^1_{\mathrm{dR}}(X;\mathbb{C}),\mathbb{C})$$

defined by

$$[\sigma] \mapsto \int_\sigma \quad \text{for all} \quad [\sigma] \in H_1(X;\mathbb{C})$$

and

$$\Psi_{2,\mathbb{C}} \colon H^1_{\mathrm{dR}}(X;\mathbb{C}) \to \mathrm{Hom}_{\mathbb{C}}(H_1(X;\mathbb{C}),\mathbb{C}) = H^1(X;\mathbb{C})$$

defined by

$$[\omega] \mapsto \int \omega \quad \text{for all} \quad [\omega] \in H^1_{\mathrm{dR}}(X;\mathbb{C})$$

are isomorphisms.

1.3.5 Hodge decomposition

Let V be a $\mathbb{C}$-vector space of of dimension g and $\Lambda \subset V$ be a lattice in V. Fix $x_1,\ldots,x_{2g}$ real coordinates of V.
Consider the isomorphism

$$\mathrm{ev}_0 \colon IF(V) \xrightarrow{\simeq} \mathrm{Hom}_{\mathbb{R}}(T_0V,\mathbb{R})$$

given by the evaluation at $0 \in V$ and the composite

$$IF(V/\Lambda) \xrightarrow{\phi} H^1_{\mathrm{dR}}(V/\Lambda;\mathbb{R}) \xrightarrow[\simeq]{\Psi_2} H^1(V/\Lambda;\mathbb{R})$$

where ϕ is the $\mathbb{R}$-linear map

$$\phi \colon IF(V/\Lambda) \to H^1_{\mathrm{dR}}(V/\Lambda;\mathbb{R})$$

defined on the basis by sending dx_i to its equivalence class $[dx_i]$ in $H^1_{\mathrm{dR}}(V/\Lambda;\mathbb{R})$, for all $i = 1,\dots,2g$, and

$$H^1_{\mathrm{dR}}(V/\Lambda;\mathbb{R}) \xrightarrow[\simeq]{\Psi_2} H^1(V/\Lambda;\mathbb{R})$$

is the isomorphism of de Rham Theorem. Define

$$\Phi := \phi \circ \pi_*^{-1} \circ ev_0^{-1} \colon \ \mathrm{Hom}_{\mathbb{R}}(T_0V,\mathbb{R}) \longrightarrow H^1_{\mathrm{dR}}(V/\Lambda;\mathbb{R}) \ .$$

Proposition 1.3.4. *The morphism $\Phi\colon Hom_{\mathbb{R}}(T_0V,\mathbb{R}) \longrightarrow H^1_{dR}(V/\Lambda;\mathbb{R})$ is an isomorphism.*

Proof. We proved that ev_0 and π_* are isomorphisms. In order to conclude we have to prove that also ϕ is an isomorphism. This follows from the fact that $\Psi_2 \circ \phi$ and Ψ_2 are isomorphisms. □

Fix $z_1,\dots,z_g$ complex coordinates of V.

Proposition 1.3.5. *Let $\Omega^1_{V/\Lambda}$ be the sheaf of holomorphic one-forms on V/Λ. Then*

$$H^0(V/\Lambda,\Omega^1_{V/\Lambda}) =< dz_1,\dots,dz_g >_{\mathbb{C}}$$

Proof. The one-forms dz_j's are holomorphic, then $< dz_1,\dots,dz_g >_{\mathbb{C}} \subseteq H^0(V/\Lambda,\Omega^1_{V/\Lambda})$. Conversely, if $\omega \in H^0(V/\Lambda,\Omega^1_{V/\Lambda})$, then

$$\omega = f_1 dz_1 + \cdots + f_g dz_g$$

with $f_1,\dots,f_g \in \mathcal{O}_{V/\Lambda}(V/\Lambda)$. Since V/Λ is compact, $\mathcal{O}_{V/\Lambda}(V/\Lambda) = \mathbb{C}$, so that

$$H^0(V/\Lambda,\Omega^1_{V/\Lambda}) \subseteq< dz_1,\dots,dz_g >_{\mathbb{C}} \ .$$

□

If we tensor with $\mathbb{C}$ the isomorphism Φ, we obtain an isomorphism

$$\Phi_{\mathbb{C}}\colon \ \mathrm{Hom}_{\mathbb{R}}(T_0V,\mathbb{C}) \xrightarrow{\ \simeq\ } H^1_{\mathrm{dR}}(X;\mathbb{C}).$$

Since $dz_1,\dots,dz_g$ are $\mathbb{C}$-linearly independent elements of $\mathrm{Hom}_{\mathbb{R}}(T_0V,\mathbb{C})$, their images $\Phi_{\mathbb{C}}(dz_1) = [dz_1],\dots,\Phi_{\mathbb{C}}(dz_g) = [dz_g]$ are linearly independent elements of $H^1_{\mathrm{dR}}(X;\mathbb{C})$.

We define a $\mathbb{C}$-linear map

$$H^0(V/\Lambda,\Omega^1_{V/\Lambda}) \to H^1_{\mathrm{dR}}(V/\Lambda;\mathbb{C})$$

on generators, by sending

$$dz_i \mapsto [dz_i] \quad \text{for all} \quad 1 \le i \le g.$$

Since $[dz_1],\dots,[dz_g]$ are linearly-independent over $\mathbb{C}$, this map is injective. We will denote its image by $H^0(V/\Lambda,\Omega^1_{V/\Lambda})$.

Theorem 1.3.3 (Hodge decomposition in degree 1 for compact complex tori)**.** *Let X be a compact complex torus of dimension g. Then*

$$H^1_{dR}(X;\mathbb{C}) = H^0(X,\Omega^1_X) \oplus \overline{H^0(X,\Omega^1_X)}$$

where $\overline{H^0(X,\Omega^1_X)}$ denotes the complex conjugate of $H^0(X,\Omega^1_X)$.

Proof. Since $H^1_{\mathrm{dR}}(X;\mathbb{C})$ depends, modulo isomorphism, only on the differentiable structure of X, and since $H^0(X,\Omega^1_X)$ and $\overline{H^0(X,\Omega^1_X)}$ depend, modulo isomorphism, only on the complex one, we can suppose that $X = V/\Lambda$, with V a $\mathbb{C}$-vector space of dimension g and Λ a lattice in V.
Fix $z_1,\dots,z_g$ complex coordinates of V. The restriction $\Phi_|$ of Φ to the subset

$$\mathrm{Hom}_{\mathbb{C}}(T_0V,\mathbb{C}) =< dz_1,\dots,dz_g >_{\mathbb{C}} \subset \mathrm{Hom}_{\mathbb{R}}(T_0V,\mathbb{C})$$

yields an isomorphism

$$\Phi_|\colon\ \mathrm{Hom}_{\mathbb{C}}(T_0V,\mathbb{C}) \xrightarrow{\simeq} H^0(X,\Omega^1_X).$$

The decomposition $\mathrm{Hom}_{\mathbb{R}}(T_0V,\mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(T_0V,\mathbb{C}) \oplus \overline{\mathrm{Hom}_{\mathbb{C}}(T_0V,\mathbb{C})}$, where $\overline{\mathrm{Hom}_{\mathbb{C}}(T_0V,\mathbb{C})}$ denotes the complex conjugate of $\mathrm{Hom}_{\mathbb{C}}(T_0V,\mathbb{C})$ yields the Hodge decomposition in degree 1

$$H^1_{\mathrm{dR}}(X;\mathbb{C}) = H^0(X,\Omega^1_X) \oplus \overline{H^0(X,\Omega^1_X)}.$$

□

By taking the duals in the Hodge decomposition we obtain:

Corollary 1.3.1 (Dual Hodge decomposition in degree 1 for compact complex tori)**.** *Let X be a compact complex torus of dimension g. Then*

$$H^1_{dR}(X;\mathbb{C})^\vee = H^0(X,\Omega^1_X)^\vee \oplus \overline{H^0(X,\Omega^1_X)}^\vee.$$

1.4 Equivalence of categories for compact complex tori

Lemma 1.4.1. *Let A and B be two commutative rings, $\phi\colon A \to B$ be a ring homomorphism. Then, for all $M \in A - Mod$ and $N \in B - Mod$, we have a natural group isomorphism*

$$Hom_B(M \otimes_A B, N) \cong Hom_A(M, N_{|A})$$

where $N_{|A}$ is N with the structure of A-module induced by ϕ.

Let $\mathscr{T}$ be the category whose objects are compact complex tori of dimension g and the morphisms $\phi\colon X \to X'$ are morphisms of complex Lie groups.
Let $\mathscr{H}$ be the category whose objects are the triple (Λ, V, γ), where Λ is a free $\mathbb{Z}$-module of rank $2g$, V is a $\mathbb{C}$-vector space of dimension g and $\gamma\colon \Lambda \to V$ is a morphism of $\mathbb{Z}$-modules with the property that

$$\gamma_{\mathbb{R}}\colon \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \to V_{\mathbb{R}}$$

which is the morphism obtained by γ using Lemma 1.4.1, is an isomorphism of $\mathbb{R}$-vector spaces ($V_{\mathbb{R}}$ is just V viewed as an $\mathbb{R}$-vector space), and a morphism

$$\phi\colon (\Lambda, V, \gamma) \to (\Lambda^{'}, V^{'}, \gamma^{'})$$

is given by a couple (ϕ_1, ϕ_2), where $\phi_1 \colon \Lambda \to \Lambda^{'}$ is a morphism of $\mathbb{Z}$-modules and $\phi_2 \colon V \to V^{'}$ is a morphism of $\mathbb{C}$-vector spaces, such that they make the following diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{\gamma} & V \\ {\scriptstyle \phi_1}\downarrow & & \downarrow{\scriptstyle \phi_2} \\ \Lambda^{'} & \xrightarrow[\gamma^{'}]{} & V^{'} \end{array}$$

commutative.

Proposition 1.4.1. *Let Λ be a free $\mathbb{Z}$-module of rank $2g$, V be a $\mathbb{C}$-vector space of dimension g and $\gamma\colon \Lambda \to V$ be a morphism of $\mathbb{Z}$-modules.*
Then, $\gamma_{\mathbb{R}} \colon \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \to V_{\mathbb{R}}$ is an isomorphism of $\mathbb{R}$-vector spaces if and only if denoted by

$$\gamma_{\mathbb{C}} \colon \Lambda \otimes_{\mathbb{Z}} \mathbb{C} \to V,$$

the natural morphism obtained by γ using Lemma 3.1, the following decomposition holds:

$$\Lambda \otimes_{\mathbb{Z}} \mathbb{C} = Ker\gamma_{\mathbb{C}} \oplus \overline{Ker\gamma_{\mathbb{C}}}$$

where $\overline{Ker\gamma_{\mathbb{C}}}$ denotes the complex conjugate of $Ker\gamma_{\mathbb{C}}$.
Equivalently, upon choosing bases $\Lambda \cong \mathbb{Z}^{2g}$ and $V \cong \mathbb{C}^g$ to identify j with a $g \times 2g$ matrix (AB) for $A, B \in Mat_{g\times g}(\mathbb{C})$, the necessary and sufficient condition is that the matrix

$$\begin{pmatrix} A & B \\ \overline{A} & \overline{B} \end{pmatrix} \in Mat_{2g\times 2g}(\mathbb{C})$$

is invertible.

Proof. We note that we can factorize $\gamma_{\mathbb{C}}$ as

$$\begin{array}{ccc} \Lambda \otimes_{\mathbb{Z}} \mathbb{C} & \xrightarrow{\tilde{\gamma}} & V \oplus \overline{V} \\ & {\scriptstyle \gamma_{\mathbb{C}}}\searrow & \downarrow{\scriptstyle \pi} \\ & & V \end{array}$$

where $\overline{V} := \mathbb{C} \otimes_{\sigma,\mathbb{C}} V$ is the complex conjugate space of V (scalar extension by complex conjugation σ) and $\tilde{\gamma}\colon \Lambda \otimes_{\mathbb{Z}} \mathbb{C} \to V \oplus \overline{V}$ is defined by $\lambda \otimes c \mapsto (c\gamma(\lambda), \overline{c\gamma(\lambda)})$, for all $\lambda \otimes c \in \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$.
We divide the proof into two steps.

Step 1: We prove that $\gamma_{\mathbb{R}}$ is an isomorphism of $\mathbb{R}$-vector spaces if and only if $\tilde{\gamma}$ is an isomorphism of $\mathbb{C}$-vector spaces.
Since $\mathbb{C}$ is a faithfully flat $\mathbb{R}$-module, the condition that $\gamma_{\mathbb{R}}$ is an isomorphism is

equivalent to the isomorphism condition after applying scalar extension $\mathbb{R} \to \mathbb{C}$. But

$$\mathbb{C} \otimes_{\mathbb{R}} V = (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} V$$

with $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \times \mathbb{C}$ as $\mathbb{C}$-algebras via $a \otimes b \mapsto (ab, a\bar{b})$, for all $a \otimes b \in \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$. Hence, $\mathbb{C} \otimes_{\mathbb{R}} V$ is identified as a $\mathbb{C}$-vector space with $V \oplus \overline{V}$, and in this way the $\mathbb{C}$-linear scalar extension of $\gamma_{\mathbb{R}}$ is identified with $\tilde{\gamma}$.

Step 2: We prove that $\tilde{\gamma}$ is an isomorphism of $\mathbb{C}$-vector spaces if and only if we can decompose $\Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ as $\Lambda \otimes_{\mathbb{Z}} \mathbb{C} = \mathrm{Ker}\gamma_{\mathbb{C}} \oplus \overline{\mathrm{Ker}\gamma_{\mathbb{C}}}$.
If $\tilde{\gamma}$ is an isomorphism of $\mathbb{C}$-vector spaces, then

$$\mathrm{Ker}\gamma_{\mathbb{C}} = \mathrm{Ker}(\pi \circ \tilde{\gamma}) = \tilde{\gamma}^{-1}(\mathrm{Ker}\pi) = \tilde{\gamma}^{-1}(\overline{V})$$

and

$$\overline{\mathrm{Ker}\gamma_{\mathbb{C}}} = \overline{\tilde{\gamma}^{-1}(\overline{V})} = \tilde{\gamma}^{-1}(V).$$

Thus

$$\Lambda \otimes_{\mathbb{Z}} \mathbb{C} = \tilde{\gamma}^{-1}(V) \oplus \tilde{\gamma}^{-1}(\overline{V}) = \mathrm{Ker}\gamma_{\mathbb{C}} \oplus \overline{\mathrm{Ker}\gamma_{\mathbb{C}}}.$$

Conversely, if we can decompose $\Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ as $\mathrm{Ker}\gamma_{\mathbb{C}} \oplus \overline{\mathrm{Ker}\gamma_{\mathbb{C}}}$, we obtain that

$$\dim_{\mathbb{C}}\mathrm{Ker}\gamma_{\mathbb{C}} = \dim_{\mathbb{C}}\overline{\mathrm{Ker}\gamma_{\mathbb{C}}} = g$$

because

$$\Lambda \cong \mathbb{Z}^{2g} \Rightarrow \Lambda \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}^{2g} \Rightarrow \dim_{\mathbb{C}}\Lambda \otimes_{\mathbb{Z}} \mathbb{C} = 2g.$$

Thus, $\gamma_{\mathbb{C}}$ restricts to an isomorphism $\overline{\mathrm{Ker}\gamma_{\mathbb{C}}} \xrightarrow{\simeq} V$ (injective morphism of $\mathbb{C}$-vector spaces of the same dimension). Moreover, $\overline{\gamma(\overline{V})} = \gamma(V) = 0$, so that

$$\tilde{\gamma}(\overline{\mathrm{Ker}\gamma_{\mathbb{C}}}) = V$$

and

$$\tilde{\gamma}(\mathrm{Ker}\gamma_{C}) = \overline{\tilde{\gamma}(\overline{\mathrm{Ker}\gamma_{\mathbb{C}}})} = \overline{V}.$$

This proves that $\tilde{\gamma}$ is surjective. Since $\Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ and $V \oplus \overline{V}$ are $\mathbb{C}$-vector spaces of the same dimension, we conclude that $\tilde{\gamma}$ is an isomorphism.

The matrix interpretation is immediate by Step 1, upon identifying the $2g \times 2g$ matrix as computing the $\mathbb{C}$-linear map $\tilde{\gamma}$ relative to the $\mathbb{C}$-basis of $\mathbb{C} \otimes_{\mathbb{Z}} \Lambda$ coming from the chosen $\mathbb{Z}$-basis of Λ and the $\mathbb{C}$-basis of $V \oplus \overline{V}$ coming from the chosen basis of V and the corresponding conjugate basis of $\overline{V}$. $\square$

We define $F\colon \mathscr{T} \to \mathscr{H}$ in the following way: on the objects by $F(X) = (H_1(X;\mathbb{Z}), H^0(X, \Omega^1_X)^\vee, \gamma)$, where $\gamma\colon H_1(X;\mathbb{Z}) \to H^0(X, \Omega^1_x)^\vee$ is given by the composite:

$$\begin{array}{ccccc} H_1(X;\mathbb{Z}) & \overset{\iota}{\hookrightarrow} & H_1(X;\mathbb{C}) \xrightarrow[\simeq]{\Psi_{1_{\mathbb{C}}}} & H^1_{\mathrm{dR}}(X;\mathbb{C})^\vee = & H^0(X,\Omega^1_X)^\vee \oplus \overline{H^0(X,\Omega^1_X)}^\vee \\ & \searrow^{\gamma} & & & \downarrow \pi_1 \\ & & & & H^0(X,\Omega^1_X)^\vee \end{array}$$

where $\iota\colon H_1(X;\mathbb{Z}) \to H_1(X;\mathbb{C})$ is the inclusion, $\Psi_{1,\mathbb{C}}\colon H_1(X;\mathbb{C}) \to H^1_{\text{dR}}(X;\mathbb{C})^\vee$ is the de Rham isomorphism and $\pi_1\colon H^0(X,\Omega^1_X)^\vee \oplus \overline{H^0(X,\Omega^1_X)}^\vee \to H^0(X,\Omega^1_X)^\vee$ is the projection onto the first factor of the direct sum of Dual Hodge decomposition.

On morphisms F is defined in the following way: let $X \xrightarrow{f} X'$ be a morphism in $\mathscr{T}$, then we associate to it the couple $(H_1(f), H^0(f)^\vee)$, where

$$H_1(f) := H_1(f;\mathbb{Z})\colon H_1(X;\mathbb{Z}) \to H_1(X';\mathbb{Z})$$

and

$$H^0(f)^\vee := H^0(f,\Omega^1_X)^\vee\colon H^0(X,\Omega^1_X)^\vee \to H^0(X,\Omega^1_{X'})^\vee$$

are the natural maps induced by f.

Proposition 1.4.2. *$F\colon \mathscr{T} \to \mathscr{H}$ is a functor.*

Proof. First we prove that F is well-defined.

If $X \in \mathscr{T}$, then $F(X) = (H_1(X;\mathbb{Z}), H^0(X,\Omega^1_X)^\vee, \gamma) \in \mathscr{H}$. In fact, $H_1(X;\mathbb{Z})$ is a free $\mathbb{Z}$-module of rank $2g$, by Proposition 1.2.2, and $H^0(X,\Omega^1_X)$ is a $\mathbb{C}$-vector space of dimension g. We only have to prove that

$$\gamma_{\mathbb{R}}\colon H_1(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \to H^0(X,\Omega^1_X)^\vee_{\mathbb{R}}$$

is an isomorphism of $\mathbb{R}$-vector spaces. To do this, we use the equivalent condition of Proposition 1.4.1. We note that

$$\gamma \otimes \text{Id}_{\mathbb{C}} = (\pi_1 \circ \Psi_{1_{\mathbb{C}}} \circ i) \otimes \text{Id}_{\mathbb{C}} = (\pi_1 \otimes \text{Id}_{\mathbb{C}}) \circ (\Psi_{1,\mathbb{C}} \otimes \text{Id}_{\mathbb{C}}) \circ (\iota \otimes \text{Id}_{\mathbb{C}}).$$

Since $H_1(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = H_1(X;\mathbb{C})$, we have that

$$\iota \otimes \text{Id}_{\mathbb{C}}\colon H_1(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \to H_1(X;\mathbb{C})$$

is the identity. Moreover, $\pi_1 \otimes \text{Id}_{\mathbb{C}} = \pi_1$ and $\Psi_{1,\mathbb{C}} \otimes \text{Id}_{\mathbb{C}} = \Psi_{1,\mathbb{C}}$, since π_1 and $\Psi_{1,\mathbb{C}}$ are $\mathbb{C}$-linear morphisms. Then, $\gamma \otimes \text{Id}_{\mathbb{C}} = \pi_1 \circ \Psi_{1,\mathbb{C}}$.

By de Rham Theorem with coefficients in $\mathbb{C}$, $\Psi_{1,\mathbb{C}}$ is an isomorphism, so that

$$H_1(X;\mathbb{C}) = \Psi^{-1}_{1,\mathbb{C}}(H^0(X,\Omega^1_X)^\vee) \oplus \Psi^{-1}_{1,\mathbb{C}}(\overline{H^0(X,\Omega^1_X)}^\vee) \tag{1.2}$$

and $\text{Ker}(\pi_1 \circ \Psi_{1,\mathbb{C}}) = \Psi^{-1}_{1,\mathbb{C}}(\text{Ker}\pi_1) = \Psi^{-1}_{1,\mathbb{C}}(\overline{H^0(X,\Omega^1_X)}^\vee)$.

Since $\overline{\text{Ker}(\pi_1 \circ \Psi_{1,\mathbb{C}})} = \overline{\Psi^{-1}_{1,\mathbb{C}}(\overline{H^0(X,\Omega^1_X)}^\vee)} = \Psi^{-1}_{1,\mathbb{C}}(H^0(X,\Omega^1_X)^\vee)$, from (1.2) it follows that

$$H_1(X;\mathbb{C}) = \text{Ker}(\gamma_{\mathbb{C}}) \oplus \overline{\text{Ker}(\gamma_{\mathbb{C}})}$$

as we wanted.

If $X \xrightarrow{f} X'$ is a morphism in $\mathscr{T}$, then

$$H_1(f)\colon H_1(X;\mathbb{Z}) \to H_1(X';\mathbb{Z})$$

is a morphism of $\mathbb{Z}$-modules,

$$H^0(f)^\vee\colon H^0(X,\Omega^1_X)^\vee \to H^0(X,\Omega^1_{X'})^\vee$$

is a morphism of $\mathbb{C}$-modules and they make the following diagram

$$\begin{array}{ccc} H_1(X;\mathbb{Z}) & \xrightarrow{\gamma} & H^0(X,\Omega^1_X)^\vee \\ {\scriptstyle H_1(f)}\downarrow & & \downarrow{\scriptstyle H^0(f)^\vee} \\ H_1(X';\mathbb{Z}) & \xrightarrow[\gamma']{} & H^0(X',\Omega^1_{X'})^\vee \end{array}$$

commutative, by functoriality of $H_1(X;\mathbb{Z})$ and $H^0(X,\Omega^1_X)^\vee$.
Finally, in order to be a functor, F has to preservers composition of morphisms and identities. This is the case, by functoriality of $H_1(X;\mathbb{Z})$ and $H^0(X,\Omega^1_X)^\vee$. $\square$

1.4.1 The Albanese map

Let X be compact complex manifold satisfying Hodge decomposition in degree 1.
For a fixed base point $x_0 \in X$ one defines the *Albanese map*

$$\mathrm{alb}_{x_0}\colon X \to H^0(X,\Omega^1_X)^\vee/H_1(X;\mathbb{Z})$$

by

$$x \mapsto \left(\omega \mapsto \int_{x_0}^{x} \omega\right)$$

where the integral is over any path from x_0 to x and we identify $H_1(X;\mathbb{Z})$ with its image under the morphism $\gamma\colon H_1(X;\mathbb{Z}) \to H^0(X,\Omega^1_X)^\vee$ of Proposition 1.4.2.

The integral $\int_{x_0}^{x}\omega$ depend on the chosen path connecting x_0 and x, but for two different choices the difference is an integral over a closed path (same computations of Proposition 1.3.1). Hence, $\mathrm{alb}(x)$ is well-defined as an element of $H^0(X,\Omega^1_X)^\vee/H_1(X;\mathbb{Z})$.

Assume now that $X = V/\Lambda$, with V a $\mathbb{C}$-vector space of dimension g and Λ a lattice in V. Fix complex coordinates $z_1,\dots,z_g$ of V. Then

$$H^0(X,\Omega^1_X)^\vee = < dz_1,\dots,dz_g >^\vee_{\mathbb{C}}$$

and

$$H_1(X;\mathbb{Z}) = \left\{\left(\int_\sigma dz_1,\dots,\int_\sigma dz_g\right) : [\sigma] \in H_1(X;\mathbb{Z})\right\}$$

because the coordinates of the image of $[\sigma]$ in $H^0(X,\Omega^1_X)^\vee$ are given by its values on generators $dz_1,\dots,dz_g$.
The map

$$\mathrm{alb}_{x_0}\colon X \to H^0(X,\Omega^1_X)^\vee/H_1(X;\mathbb{Z})$$

is defined by

$$x \mapsto \left(\int_{x_0}^{x} dz_1,\dots,\int_{x_0}^{x} dz_g\right) \mod \left\{\left(\int_\sigma dz_1,\dots,\int_\sigma dz_g\right) : [\sigma] \in H_1(X;\mathbb{Z})\right\}.$$

Proposition 1.4.3. *Let X be a compact complex torus and $x_0 = 0 \in X$. The Albanese map $alb_{x_0} \colon X \to H^0(X, \Omega^1_X)^\vee / H_1(X; \mathbb{Z})$ is an isomorphism of complex Lie groups.*

Proof. Since $H_1(X; \mathbb{Z})$ depends, modulo isomorphism, only on the differentiable structure of X and $H^0(X, \Omega^1_X)^\vee$ depends, modulo isomorphism, only on the complex one, we can suppose that $X = V/\Lambda$.

i. alb_{x_0} is holomorphic.
For $1 \leq j \leq g$, define

$$\mathrm{alb}^j_{x_0}(x) := \int_{x_0}^{x} dz_j \mod \left\{ \int_\sigma dz_j : [\sigma] \in H_1(X; \mathbb{Z}) \right\}.$$

Then, alb_{x_0} is holomorphic if and only if $\mathrm{alb}^j_{x_0}$ is holomorphic, for all $1 \leq j \leq g$. By elementary calculus, $\mathrm{alb}^j_{x_0}$ is smooth with derivative dz_j. Since this is holomorphic, this implies that $\mathrm{alb}^j_{x_0}$ is holomorphic, for all $1 \leq j \leq g$.

ii. alb_{x_0} is a biholomorphism.

- alb_{x_0} is a covering map.
Since $\mathrm{alb}_{x_0}(0) = 0$ and it is holomorphic, it is a morphism of complex Lie groups. Let $F := \rho_a(\mathrm{alb}_{x_0})$. The Jacobian of F at $0 \in V$ is the identity, thus F is a local biholomorphism at 0. Since F is $\mathbb{C}$-linear, it is a local biholomorphism at each point $x \in V$. Thus, alb_{x_0} is a local biholomorphism at each point $x \in V/\Lambda$. In fact, let $\pi \colon V \to V/\Lambda$ and $\pi' \colon H^0(X, \Omega^1_X)^\vee \to H^0(X, \Omega^1_X)^\vee / H_1(X; \mathbb{Z})$ be the canonical projections. Since they are covering map, they are local biholomorphism. Thus, locally, alb_{x_0} is given by the composition of biholomorphisms $\pi'_| \circ F \circ (\pi_|)^{-1}$ (where $\pi'_|$, $F_|$, $\pi_|$ denote the restrictions of π', F, π open subsets over which they are biholomorphisms). By [8, 151], it implies that alb_{x_0} is a covering map.
- alb_{x_0} is invertible.
To do this, by covering space theory, it is enough to prove that the induced map
$$\pi_1(\mathrm{alb}_{x_0}) \colon \pi_1(V/\Lambda, 0) \to \pi_1(H^0(X, \Omega^1_X)^\vee / H_1(X; \mathbb{Z}), 0)$$
which is injective, is surjective.
Consider the commutative diagram [9, 81]
$$\begin{array}{ccc} \pi_1(V/\Lambda, 0) & \xrightarrow{\pi_1(\mathrm{alb}_{x_0})} & \pi_1(H^0(X, \Omega^1_X)^\vee / H_1(X; \mathbb{Z}), 0) \\ \simeq \downarrow & & \downarrow \simeq \\ H_1(V/\Lambda; \mathbb{Z}) & \xrightarrow[H_1(\mathrm{alb}_{x_0})]{} & H_1(H^0(X, \Omega^1_X)^\vee / H_1(X; \mathbb{Z}); \mathbb{Z}) \end{array}$$
where the vertical arrows are given by Hurewicz and $H_1(\mathrm{alb}_{x_0})$ is the map induced by alb_{x_0} at the level of homologies.

Commutativity implies that the surjectivity of $\pi_1(\mathrm{alb}_{x_0})$ is equivalent to the surjectivity of $H_1(\mathrm{alb}_{x_0})$. But this follows as there is a natural isomorphism

$$H_1(X;\mathbb{Z}) \cong \pi_1(H^0(X,\Omega^1_X)^\vee/H_1(X;\mathbb{Z})) \xrightarrow{\simeq} H_1(H^0(X,\Omega^1_X)^\vee/H_1(X;\mathbb{Z});\mathbb{Z})$$

and as, under this identification, $H_1(\mathrm{alb}_{x_0})([\gamma])$ is identified with $\left(\int_\gamma dz_1, \ldots, \int_\gamma dz_g\right)$.

□

1.4.2 Equivalence of categories

We define a functor $G\colon \mathscr{H} \to \mathscr{T}$ on the objects by $G(\Lambda, V, \gamma) = V/\gamma(\Lambda)$, on the morphisms in the natural way: let

$$(\phi_1, \phi_2)\colon (\Lambda, V, \gamma) \to (\Lambda', V', \gamma')$$

be a morphism in $\mathscr{H}$, to it we associate the unique morphism

$$\phi\colon V/\gamma(\Lambda) \to V'/\gamma'(\Lambda')$$

making the following diagram

$$\begin{array}{ccc} V & \xrightarrow{\phi_2} & V' \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ V/\gamma(\Lambda) & \xrightarrow[\phi]{} & V'/\gamma'(\Lambda') \end{array}$$

commutative, where $\pi_1\colon V \to V/\gamma(\Lambda)$ and $\pi_2\colon V' \to V'/\gamma'(\Lambda')$ are the natural projections. In this way, we define a functor.

In fact, if $(\Lambda, V, \gamma) \in \mathscr{H}$, then, by the condition on γ and Proposition 1.1.1, it follows that $\gamma(\Lambda)$ is a lattice in V, so that $V/\gamma(\Lambda)$ is a complex torus of dimension g.

If

$$(\phi_1, \phi_2)\colon (\Lambda, V, \gamma) \to (\Lambda', V', \gamma')$$

is a morphism in $\mathscr{H}$, then the morphism $\phi\colon V/\gamma(\Lambda) \to V'/\Lambda'$ exists and it is unique, since $\pi_2 \circ \phi_2$ preserves the fibers of π_1, and it is holomorphic since its composition with the local biholomorphism π_1 (it is holomorphic and locally invertible) is $\pi_2 \circ \phi_2$, which is holomorphic since ϕ_2 is $\mathbb{C}$-linear, thus holomorphic, and π_2 is holomorphic by Proposition 1.2.1). Moreover, by the condition that it makes commutative the diagram

$$\begin{array}{ccc} V & \xrightarrow{\phi_2} & V' \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ V/\gamma(\Lambda) & \xrightarrow[\phi]{} & V'/\gamma'(\Lambda') \end{array}$$

and that π_1, π_2 and ϕ_2 respect the origins, it follows that ϕ preserves the origins too.

Theorem 1.4.1. *(F, G) is an equivalence of categories between $\mathscr{T}$ and $\mathscr{H}$.*

Proof. We will prove that G is a quasi-inverse of F.
In the same notations of Proposition 1.4.2, for $X \in \mathscr{T}$, we have

$$X \cong (G \circ F)(X) = H^0(X, \Omega^1_X)/\gamma(H_1(X;\mathbb{Z})) \quad \text{in} \quad \mathscr{T}$$

via the Albanese map.
For $(\Lambda, V, \gamma) \in \mathscr{H}$, we have

$$(\Lambda, V, \gamma) \cong (F \circ G)(\Lambda, V, \gamma) = (H_1(V/\gamma(\Lambda);\mathbb{Z}), H^0(V/\gamma(\Lambda), \Omega^1_{V/\gamma(\Lambda)})^\vee, \phi_\gamma) \quad \text{in} \quad \mathscr{H}. \tag{1.3}$$

Let us prove the isomorphism in (1.3).
Let $x_0 = 0 \in V/\gamma(\Lambda)$ and

$$\mathrm{alb}_{x_0} \colon V/\gamma(\Lambda) \xrightarrow{\simeq} H^0(V/\gamma(\Lambda), \Omega^1_X)^\vee / H_1(V/\gamma(\Lambda);\mathbb{Z})$$

be the Albanese isomorphism.
The morphism $\rho_a(\mathrm{alb}_{x_0}) \colon V \to H^0(V/\gamma(\Lambda), \Omega^1_X)^\vee$ is $\mathbb{C}$-linear and it is an isomorphism. In fact,

$$\rho_a(\mathrm{alb}_{x_0}^{-1}) \circ \rho_a(\mathrm{alb}_{x_0}) \colon V \to V$$

and

$$\mathrm{Id}_V \colon V \to V$$

are two $\mathbb{C}$-linear lifts of $\mathrm{Id}_{V/\Lambda}$. Uniqueness of lift implies that

$$\rho_a(\mathrm{alb}_{x_0}^{-1}) \circ \rho_a(\mathrm{alb}_{x_0}) = \mathrm{Id}_V.$$

Similarly,

$$\rho_a(\mathrm{alb}_{x_0}) \circ \rho_a(\mathrm{alb}_{x_0}^{-1}) = \mathrm{Id}_{H^0(V/\Lambda;\Omega^1_{V/\Lambda})^\vee}$$

so that $\rho_a(\mathrm{alb}_{x_0})$ is an isomorphism.
The composition $\phi_\gamma^{-1} \circ \rho_r(\mathrm{alb}_{x_0}) \circ \gamma \colon \Lambda \to H_1(V/\Lambda;\mathbb{Z})$ is $\mathbb{Z}$-linear, since composition of $\mathbb{Z}$-linear morphisms, and it is an isomorphism, because $\gamma \colon V \to \gamma(\Lambda)$ and $\phi_\gamma \colon H_1(V/\Lambda;\mathbb{Z}) \to \phi_\gamma(H_1(V/\Lambda;\mathbb{Z}))$ are isomorphisms and $\rho_r(\mathrm{alb}_{x_0})$ is an isomorphism. In fact

$$\rho_a(\mathrm{alb}_{x_0})^{-1}_{|\gamma(\Lambda)} \circ \rho_a(\mathrm{alb}_{x_0})_{|\gamma(\Lambda)} = (\rho_a(\mathrm{alb}_{x_0})^{-1} \circ \rho_a(\mathrm{alb}_{x_0}))_{|\gamma(\Lambda)} = \mathrm{Id}_{\gamma(\Lambda)}$$

and

$$\rho_a(\mathrm{alb}_{x_0})_{|\phi_\gamma(H_1)} \circ \rho_a(\mathrm{alb}_{x_0})^{-1}_{|\phi_\gamma(H_1))} = (\rho_a(\mathrm{alb}_{x_0}) \circ \rho_a(\mathrm{alb}_{x_0})^{-1})_{|\phi_\gamma(H_1)} = \mathrm{Id}_{\phi_\gamma(H_1)}$$

where we denoted $\phi_\gamma(H_1(V/\Lambda;\mathbb{Z}))$ by $\phi_\gamma(H_1)$. Commutativity of the diagram

$$\begin{array}{ccc} V & \xrightarrow[\simeq]{\rho_a(\mathrm{alb}_{x_0})} & H^0(V/\gamma(\Lambda), \Omega^1_X)^\vee \\ \uparrow & & \uparrow \\ \gamma(\Lambda) & \xrightarrow[\rho_r(\mathrm{alb}_{x_0})]{\simeq} & \phi_\gamma(H_1(V/\Lambda);\mathbb{Z})) \end{array}$$

where the vertical arrows are given by inclusions, implies commutativity of

$$\begin{array}{ccc} V & \xrightarrow[\simeq]{\phi_2} & H^0(V/\gamma(\Lambda), \Omega^1_X)^\vee \\ {\scriptstyle\gamma}\uparrow & & \uparrow{\scriptstyle\phi_\gamma} \\ \Lambda & \xrightarrow[\phi_1]{\simeq} & H_1(V/\Lambda; \mathbb{Z}) \end{array}$$

where $\phi_1 = \phi_\gamma^{-1} \circ \rho_r(\mathrm{alb}_{x_0} \circ \gamma$ and $\phi_2 = \rho_a(\mathrm{alb}_{x_0})$. This yields an isomorphism in $\mathscr{H}$. By functoriality of $H_1(X;\mathbb{Z})$ and $H^0(X, \Omega^1_X)^\vee$ and the naturality of the isomorphisms in (1.3), it follows that the isomorphisms $X \cong (G \circ F)(X)$ and $(\Lambda, V, \gamma) \cong (F \circ G)(\Lambda, V, \gamma)$ are functorial in X and (Λ, V, γ), yielding the isomorphisms of functors

$$\mathrm{Id}_{\mathscr{T}} \cong G \circ F \quad \text{and} \quad \mathrm{Id}_{\mathscr{H}} \cong F \circ G.$$

□

Let $\mathscr{T}_f$ be the category defined as $\mathscr{T}$, but where we replace compact complex tori by framed compact complex tori, and let $\mathscr{H}_f$ be the category defined as $\mathscr{H}$, but where we replace Λ by $\mathbb{Z}^{2g}$. The same argument still yields an equivalence of categories between $\mathscr{T}_f$ and $\mathscr{H}_f$.

Chapter 2

Ehresmann Theorem and families of complex manifolds

In this chapter, we state Ehresmann Theorem and apply it to $X \to B$, a family of compact complex manifolds over a complex manifold B, by computing the cohomology groups $H^1(X_b; \mathbb{Z})$ for every $b \in B$, where X_b denotes the fiber over b, and by proving that the sheaf $R_1\phi_*\mathbb{Z}_X$ is a $\mathbb{Z}$-local system on B.

2.1 Ehresmann Theorem

Let X and B be differentiable manifolds and $f\colon X \to B$ be a C^1 morphism. Let $X_0 := f^{-1}(0)$ denote the fibre of f above the point $0 \in B$ and $X_U := f^{-1}(U)$ denote the of the subset $U \subseteq B$ by f.

Theorem 2.1.1 (Ehresmann)**.** *Let $f\colon X \to B$ be a proper submersion between two differentiable manifolds. Then, for any $0 \in B$ there exists an open $U \subseteq B$, with $0 \in U$, and a diffeomorphism*

$$T_U\colon X_U \xrightarrow{\simeq} X_0 \times U$$

over U, i.e. such that the following diagram

$$\begin{array}{ccc} X_U & \xrightarrow{\quad\simeq\quad} & X_0 \times U \\ & {\scriptstyle f_{|X_U}} \searrow \quad \swarrow {\scriptstyle pr_2} & \\ & U & \end{array}$$

commutes, where pr_2 is the projection onto the second factor and $f_{|X_U}$ is the restriction of f to X_U.

Proof. See [10, 220-221]. □

If $U \subseteq B$ realizes the isomorphism of Ehresmann's Theorem, X is said to be *topologically trivial* over U.

2.2 Families of compact complex manifolds

Let B be a fixed complex manifold.

Definition 2.2.1. *A complex manifold over B is (X, π), where X is a complex manifold and π is a holomorphic map $\pi\colon X \to B$.*

If we will not need to specify the morphism, we will denote (X, π) a complex manifold over B just by X.
If X and T are complex manifolds over B, we denote by $X(T)$ or $\mathrm{Hol}_B(T, X)$ the set of all holomorphic maps $f\colon T \to X$ making the following diagram

$$\begin{array}{ccc} T & \longrightarrow & X \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

commutative. We call B-morphisms from T to X the elements of $X(T)$.
Let (X, ϕ) be a complex manifold over B.

Definition 2.2.2. *We say that (X, ϕ) is a family of compact complex manifolds if ϕ is a proper holomorphic submersion.*

Ehresmann Theorem applies to families of complex manifolds.
Suppose that U is an open subset in B over which X is topologically trivial and $0 \in U$. Then there is an isomorphism

$$T_U\colon X_U \xrightarrow{\simeq} X_0 \times U$$

as differentiable manifolds.
Since complex manifolds are locally contractible, we can suppose that U is contractible. Then, for $b \in U$, the inclusion

$$X_0 \times \{b\} \xhookrightarrow{\mathrm{Id}_{X_0} \times \iota_b} X_0 \times U$$

is an homotopy equivalence.
The commutative diagram

$$\begin{array}{ccc} X_U & \xrightarrow[\simeq]{T_U} & X_0 \times U \\ {\scriptstyle j_b}\uparrow & & \uparrow{\scriptstyle \mathrm{Id}_{X_0} \times \iota_b} \\ X_b & \xrightarrow[T_{U_|}]{\simeq} & X_0 \times \{b\} \end{array}$$

where $T_{U_|}$ is the restriction of T_U to X_b, implies that the inclusion

$$X_b \xhookrightarrow{j_b} X_U$$

is an homotopy equivalence too. So, it induces an isomorphism

$$H^1(X_U; \mathbb{Z}) \xrightarrow[\simeq]{j_b^*} H^1(X_b; \mathbb{Z}).$$

at the level of cohomologies. Then, if s and $b \in U$, there is a natural isomorphism

$$j_s^* \circ (j_b^*)^{-1}\colon H^1(X_b; \mathbb{Z}) \xrightarrow{\simeq} H^1(X_s; \mathbb{Z}).$$

2.3 $\mathbb{Z}$-Local systems associated to a family of compact complex manifolds

Definition 2.3.1. *Let X be a locally connected topological space. An abelian sheaf $\mathcal{F}$ on X is a $\mathbb{Z}$-local system if it is locally isomorphic to a constant abelian sheaf of finite rank.*

Theorem 2.3.1 (Proper base-change)**.** *Let $f\colon X \to Y$ be a proper map between locally compact topological spaces. Let $\mathcal{F}$ be an abelian sheaf on X. For any $y \in Y$ and for all $n \geq 0$, there is a canonical isomorphism*

$$(R^n f_*(\mathcal{F}))_y \xrightarrow{\simeq} H^n(f^{-1}(y); \mathcal{F})$$

where $R^n f_*$ is the n-th derived functor of the functor f_* from the category of abelian sheaves on Y to the category of abelian sheaves on X.

Let $\phi\colon X \to B$ be a family of compact complex manifolds and fix $0 \in B$.
Let $R^1\phi_*\mathbb{Z}_X$ be the first derived functor of the functor ϕ_* and $\mathbb{Z}_X$ be the constant sheaf on X of stalk $\mathbb{Z}$.

Proposition 2.3.1. *The sheaf $R^1\phi_*\mathbb{Z}_X$ is a $\mathbb{Z}$-local system on B.*

Proof. Let U be an open subset of B, over which X is topologically trivial, with $0 \in U$. By proper base-change

$$(R^1\phi_*(\mathbb{Z}_X))_y \xrightarrow{\simeq} H^1(X_y; \mathbb{Z}_X)$$

for all $y \in U$. If we compose this isomorphism with

$$j_0^* \circ (j_y^*)^{-1}\colon H^1(X_y; \mathbb{Z}) \xrightarrow{\simeq} H^1(X_0; \mathbb{Z})$$

we obtain an isomorphism of sheaves on U

$$R^1\phi_*(\mathbb{Z}_X)_{|U} \cong (H^1(X_0; \mathbb{Z}_X)_B)|U$$

where $(H^1(X_0; \mathbb{Z}_X)_B)_{|U}$ is the restriction of the constant sheaf $H^1(X_0; \mathbb{Z}_X)_B$ on B of stalk $H^1(X_0; \mathbb{Z}_X)$. □

We define the dual

$$R_1\phi_*\mathbb{Z}_X = (R^1\phi_*\mathbb{Z}_X)^\vee := \mathcal{H}om_{\mathbb{Z}_X}(R^1\phi_*\mathbb{Z}_X, \mathbb{Z}_X)$$

where $\mathcal{H}om(R^1\phi_*\mathbb{Z}_X, \mathbb{Z}_X)$ is the sheaf of abelian groups over B defined by

$$U \mapsto \mathrm{Hom}_{\mathbb{Z}_U}((R^1\phi_*\mathbb{Z}_X)_{|U}, \mathbb{Z}_U)$$

for all $U \subseteq B$ open.
Since the dual sheaf of a constant sheaf is constant, we obtain the following result:

Proposition 2.3.2. *The sheaf $R_1\phi_*\mathbb{Z}_X$ is a $\mathbb{Z}$-local system on B.*

Chapter 3

Group schemes in the category of complex manifolds

In this chapter, given a fixed complex manifold B, we define B-Lie groups and show the link between holomorphic vector bundles over B and the category of locally free $\mathcal{O}_B$-modules of finite rank.

3.1 B-Lie groups

Let B be a complex manifold.

Definition 3.1.1. *i. A B-Lie group is a compact complex manifold X over B with the property that for any complex manifold T over B, the set $X(T)$ is equipped with a functorial group structure, i.e. there exists a group structure on $X(T)$ and, for any complex manifold $T^{'}$ over B and $f \in T(T^{'})$, the map induced by f*

$$f^{*}\colon X(T) \to X(T^{'})$$

is a group homomorphism;

ii. A B-Lie group (X, π, m, i, e) is a complex manifold (X, π) over B together with B-morphisms $m\colon X \times_B X \to X$ (group law, or multiplication), $i\colon X \to X$ (inverse) and $e\colon B \to X$ (identity section), such that the following identities of morphisms hold:

$$m \circ (m \times Id_X) = m \circ (Id_X \times m)\colon X \times_B X \times_B X \to X,$$

$$m \circ (e \times Id_X) = j_1\colon B \times_B X \to X,$$

$$m \circ (Id_X \times e) = j_2\colon X \times_B B \to X,$$

$$e \circ \pi = m \circ (Id_X \times i) \circ \Delta_{X/B} = m \circ (i \times Id_X \circ \Delta_{X/B})\colon X \to X,$$

where $j_1\colon B \times_B X \to X$ and $j_2\colon X \times_B B \to X$ are the canonical isomorphisms and $\Delta_{X/B}\colon X \to X \times_B X$ is the diagonal morphism.

Proposition 3.1.1. *The two definitions of B-group are equivalent.*

Proof. Iit follows from Yoneda Lemma (See [11, 31]). □

If $\pi\colon X \to B$ is a holomorphic vector bundle over B, then X is a B-group. We verify that it satisfies the first definition of B-group:

- if $\phi\colon T \to B$ is a complex manifold over B, the set $X(T)$ has a group structure defined as follows: for $f, g \in X(T)$, $f+g$ is the morphism defined by $(f+g)(t) = f(t)+g(t)$ for all $t \in T$, where the sum respects the $\mathbb{C}$-vector space structure of $X_{\phi(t)}$;
- if $T^{'}$ is another complex manifold over B and $f \in T(T^{'})$, the map induced by f

  $$f^{*}\colon X(T) \to X(T^{'})$$

  is a group homomorphism, because it maps $g_1+g_2 \in X(T)$ to the morphism defined, for all $t \in T$, by

  $$((g_1+g_2)\circ f)(t) = (g_1+g_2)(f(t)) = g_1(f(t)) + g_2(f(t)) = ((g_1\circ f) + (g_2 \circ f))(t)$$

  so that $f^{*}(g_1+g_2) = f^{*}(g_1) + f^{*}(g_2)$.

According to the two definitions of B-Lie groups, we define morphisms of B-groups.

Definition 3.1.2. *i. Let X and $X^{'}$ be two B-Lie groups. A homomorphism of B-Lie group from X to $X^{'}$ is a morphism $f \in X^{'}(X)$ such that the map induced by f*

$$f(T)\colon X(T) \to X^{'}(T)$$

is a group homomorphism and functorial in T, for any complex manifold T over B;

ii. Let (X, π, m, i, e) and $(X^{'}, \pi^{'}, m^{'}, i^{'}, e^{'})$ be two B-Lie groups. A homomorphism of B-Lie groups from X to $X^{'}$ is a morphism $f \in X^{'}(X)$ such that $f \circ m = m^{'} \circ (f \times f)\colon X \times_B X \to X^{'}$. (In particular, this condition implies that $f \circ e = e^{'}$ and $f \circ i = i^{'} \circ f$).

Let G be a complex Lie group.

Definition 3.1.3. *The B-Lie group with structural group G is the B-Lie group $(\bigsqcup_{g\in G} B, \pi)$, where $\pi\colon \bigsqcup_{g\in G} B \to B$ is the canonical projection.*

Definition 3.1.4. *A locally constant B-Lie group with structural group G is a complex manifold over B which is locally a constant B-Lie group with structural group G.*

Given $\pi\colon X \to B$ and $\pi^{'}\colon C \to B$ two complex manifolds over B, their fiber product yields a commutative diagram

$$\begin{array}{ccc} C \times_B X & \xrightarrow{pr_X} & X \\ {\scriptstyle pr_C}\downarrow & & \downarrow{\scriptstyle \pi} \\ C & \xrightarrow[\pi^{'}]{} & B \end{array}$$

where pr_X and pr_C denote the projections to X and C, respectively.
Since all the properties are stable under base-change, $(C \times_B X, \mathrm{pr}_C)$ is a complex manifold over C.

If $C = \{b\}$, with $b \in B$, and $\pi^{'}$ is the inclusion $\{b\} \hookrightarrow B$, $C \times_B X$ is canonically isomorphic to X_b.
Moreover, X_b is a complex Lie group. In fact, Definition *ii.* of $\{b\}$-group yields commutative diagrams of holomorphic maps

$$\begin{array}{ccc} X_b \times X_b & \xrightarrow{\quad m \quad} & X_b \\ & \searrow \quad \swarrow & \\ & \{b\} & \end{array}$$

and

$$\begin{array}{ccc} X_b & \xrightarrow{\quad i \quad} & X_b \\ & \searrow \quad \swarrow & \\ & \{b\} & \end{array}$$

which give holomorphic maps $m\colon X_b \times X_b \to X_b$ and $i\colon X_b \to X_b$. Since the map $X_b \times X_b \to X_b$ defined by $(x, y) \mapsto x \cdot y^{-1}$ is given by the composition of holomorphic maps

$$m \circ (\mathrm{Id}_{X_b} \times i)\colon X_b \times X_b \to X_b$$

it is holomophic too.

3.2 Holomorphic vector bundles and locally free sheaves

Let $\phi\colon X \to B$ be a holomorphic vector bundle.
We define a map by associating to each open subset $U \subseteq B$ the set

$$X(U) := \{s\colon U \to \phi^{-1}(U) : s \quad \text{is holomorphic}, \pi \circ s = \mathrm{Id}_U\}.$$

This association defines a presheaf of abelian groups over B, denoted by X.
The presheaf X is, in fact, a sheaf.
Indeed, let $U \subseteq B$ be an open subset. The functor X satisfies the uniqueness' condition of sheaves, because two sections s and $t \in X(U)$ coincide if and only if they coincide on an open covering of U. Moreover, X satisfies the gluing condition of sheaves, because given a family of sections defined on an open covering of U, whose restrictions coincide in the intersections, then it is possible to glue them in a section of $X(U)$.

Definition 3.2.1. *The sheaf X is called the sheaf of sections of ϕ.*

Let $\mathcal{O}_B$ be the sheaf of holomorphic functions on B, i.e. the sheaf on B defined by associating to each open subset $U \subseteq B$ the $\mathbb{C}$-algebrea of holomorphic functions over it.

Lemma 3.2.1. *The sheaf X is an $\mathcal{O}_B$-module.*

Proof. Let $U \subseteq B$ be an open subset. The abelian group $X(U)$ has a structure a structure of $\mathcal{O}_B(U)$-module: for $s \in B(U)$ and $f \in \mathcal{O}_B(U)$, $f \cdot s$ is the element defined by

$$(f \cdot s)(x) = f(x)s(x), \quad \text{for all} \quad x \in U.$$

The element $f \cdot s$ is such that $\pi \circ f \cdot s = \mathrm{Id}_U$, because $f(x) \in \mathbb{C}$, $s(x) \in X_x$ and X_x has the structure of a $\mathbb{C}$-vector space, so that the element $f(x)s(x)$ belongs to X_x for any $x \in B$. Moreover, $f \cdot s$ is holomorphic, since product of holomorphic functions. Thus, $f \cdot s \in X(U)$. The fact that $X(U)$ satisfies the axioms of a $\mathcal{O}_B(U)$-module follows from the fact that, for each $x \in B$, the fiber X_x satisfies the axioms of a $\mathbb{C}$-module.
If $V \subseteq U \subseteq B$ are open inclusions, the restriction morphism

$$X(U) \to X(V)$$

maps the element $f \cdot s$ to its restriction $(f \cdot s)_{|V}$ to V. Since $(f \cdot s)_{|V} = f_{|V} \cdot s_{|V}$, the restriction is $\mathcal{O}_B(U)$-linear. □

Proposition 3.2.1. *Let B be a complex manifold. Associating to a holomorphic vector bundle its sheaf of sections defines an equivalence of categories between the category of holomorphic vector bundles over B and the category of locally free $\mathcal{O}_B$-modules of finite rank.*

Proof. Idea of the proof (see [12, 72]): The sheaf of sections of a holomorphic vector bundle $\pi \colon X \to B$ of rank g is a locally constant $\mathcal{O}_B$-module, since locally on B, X is isomorphic to a product $U \times \mathbb{C}^g$, $U \subseteq B$ open.
Conversely, choosen trivialisations $\psi_i \colon \mathcal{F}_{|U_i} \xrightarrow{\simeq} \mathcal{O}_{U_i}^{\oplus g}$, denote by $(\psi_i)_{|ij}$ the restriction of ψ_i to $\mathcal{F}_{|U_i \cap U_j}$. Defined the transition maps

$$\psi_{ij} := (\psi_i)_{|ijj} \circ (\psi_j^{-1})_{|ij} \colon \mathcal{O}_{U_i \cap U_j}^{\oplus g} \xrightarrow{\simeq} \mathcal{O}_{U_i \cap U_j}^{\oplus g}$$

the maps $\psi_{ij}(U_i \cap U_j)$ are given with a matrix of holomorphic functions on $U_i \cap U_j$. Therefore, $\{(U_i, \psi_{ij}(U_i \cap U_j)\}$ can be used as a cocyle defining a holomorphic vector bundle over B. On the morphisms, this correspondence is defined by giving them locally and, then, by gluing them. Using the fact that holomorphic vector bundles and sheaves are uniquely determined, up to isomorphism, by their cocycles, one can check that this defines an equivalence of categories. □

Remark 3.2.1. *Similarly to Proposition 3.2.1, using cocycles, it is possible to associate to a $\mathbb{Z}$-local system $\mathcal{F}$ on B a locally constant B-Lie group with structural group an abelian group of rank $2g$ and to a morphism of $\mathbb{Z}$-local systems $\mathcal{F} \to \mathcal{G}$ on B a morphism on the associated B-Lie groups.*

Chapter 4

Families of compact complex tori

In this chapter we extend the results of chapter 1 to families of compact complex tori. Given a fixed complex manifold B, we define families of compact complex tori over B and study some properties of them.
We generalize the notion of quotient to $\gamma\colon \Lambda \to V$, where Λ is a locally constant B-Lie group with structural group $\mathbb{Z}^{2g}$, V is a holomorphic vector bundle of rank g and γ is a morphism of B-Lie groups, such that it yields a lattice inclusion fiberwise.
Finally, we extend F to a functor $F_B\colon \mathcal{T}_B \to \mathcal{H}_B$ and prove it yields an equivalence of categories.

4.1 Families of compact complex tori

Let B be a complex manifold.

Definition 4.1.1. *A family of compact complex tori of dimension g over B is a triple (X, π, σ), where:*

- *(X, π) is a family of compact complex manifolds over B;*
- *$\sigma\colon B \to X$ is a holomorphic section of π, also called the zero section, with the property that X_b is a compact complex torus of dimension g with zero $\sigma(b)$, for all $b \in B$.*

Definition 4.1.2. *A family of framed compact complex tori of dimension g over B is (X, π, σ, ϕ), where:*

- *(X, π, σ) is a family of compact complex tori of dimension g over B;*
- *ϕ is an isomorphism of abelian sheaves over B $\phi\colon (R^1\pi_*\mathbb{Z}_X)^\vee \xrightarrow{\simeq} \mathbb{Z}_B^{\oplus 2g}$.*

Definition 4.1.3. *A morphism of compact complex tori over B from (X', π', σ') to (X, π, σ) is a morphism $\psi\colon X' \to X$ of complex manifolds such that:*

- *the following diagram*

$$\begin{array}{ccc} X' & \xrightarrow{\psi} & X \\ & \searrow^{\pi'} \quad \swarrow_{\pi} & \\ & B & \end{array}$$

  commutes;

- *it preserves the zero sections, i.e.* $\psi \circ \sigma' = \sigma$.

Definition 4.1.4. *A morphism of framed compact complex tori* (X, π, σ, ϕ) *and* $(X', \pi', \sigma', \phi')$ *over* B *is a morphism* $\phi \colon X' \to X$ *such that:*

- *it is a morphism of compact complex tori from* (X', π', σ') *to* (X, π, σ)*;*
- *it induces the isomorphism between the framings.*

4.1.1 The sheaf $\phi_* \Omega^1_{X/B}$.

Let (X, ϕ, σ) be a family of compact complex tori over B.
Since ϕ is a submersion, by definition, the induced map at the level of complex tangent spaces

$$T_x X \to T_{\phi(x)} B$$

is surjective, for every $x \in X$. This implies that the map induced on the holomorphic tangent bundles (see Def [13, 71]) over X

$$TX \to \phi^* TB$$

is surjective, where $\phi^* TB$ denotes the pull back bundle of TB under ϕ^*. This is because, for every $x \in X$, the fiber at x of $(\phi^* TB)$ is canonically identified with the fiber at $\phi(x)$ of TB.
Moreover, the Kernel of $T_x X \to T_{\phi(x)} B$ is canonically identified with $T_x X_{\phi(x)}$. In fact, by holomorphic implicit function theorem [14, 11] it follows that, for every $x \in X$, there exists an open neighborhood $V \subseteq X$ of x and a biholomorphism $h := (\phi, h_F)$ from V to a product $\phi(V) \times F$ over $\phi(V)$. Thus

$$T_x X \cong T_x V \cong T_{h(x)}(\phi(V) \times F) \cong T_{\phi(x)} B \oplus T_{h_F(x)} F.$$

By considering the fibers over $\phi(x)$ on V, h yields the isomorphism

$$X_{\phi(x)} \cap V \xrightarrow[\simeq]{h} \{\phi(x)\} \times F \cong F$$

so that

$$T_x X_{\phi(x)} \cong T_x (X_{\phi(x)} \cap V) \cong T_{h_F(x)} F.$$

Define $T_{X/B} := \mathrm{Ker}(TX \to \phi^* TB)$. This is a holomorphic vector bundle over X, since the rank of the $\mathbb{C}$-linear map $T_x X \to T_{\phi(x)} B$ does not depend on x.
Define $\Omega^1_{X/B} := T^\vee_{X/B}$ to be the dual bundle. It is a holomorphic vector bundle over

X, with fibers canonically identified with $H^0(X_{\phi(x)}, \Omega^1_{X_{\phi(x)}})$, for all $x \in X$, through the isomorphisms

$$(\Omega^1_{X/B})_x \cong ((T_{X/B})_x)^\vee \cong (T_x X_{\phi(x)})^\vee \xrightarrow{\simeq} H^0(X_{\phi(x)}, \Omega^1_{X_{\phi(x)}}).$$

The last isomorphism is given by the composition $\Phi_| \circ Jt_x(0)^\vee$, where

$$\Phi_| \colon (T_x X_{\phi(x)})^\vee \xrightarrow{\simeq} H^0(X_{\phi(x)}, \Omega^1_{X_{\phi(x)}})$$

is the isomorphism yielding the Hodge decomposition and

$$Jt_x(0)^\vee \colon (T_x X_{\phi(x)})^\vee \to (T_0 X_{\phi(x)})^\vee$$

is the dual of the map induced by t_x, the translation by x, on the tangent spaces. Since t_x is a biholomorphism, the map induced on the tangent spaces is an isomorphism. So, $Jt_x(0)^\vee$ is an isomorphism too.
Denote still by $\Omega^1_{X/B}$ the associated sheaf, using Proposition 3.2.1. Let $\phi_*\Omega^1_{X/B}$ be the direct image of $\Omega^1_{X/B}$ through the functor ϕ_*. The following result holds:

Proposition 4.1.1. *The sheaf $\phi_*\Omega^1_{X/B}$ is a locally free $\mathcal{O}_B$-module of rank g.*

Proof. Let $\mathrm{Im}\sigma \subseteq X$, be the complex submanifold of X given by the image of the zero section σ and let $N^\vee$ be the dual of the normal bundle of $\mathrm{Im}\sigma$ in X. It is a holomorphic vector bundle over $\mathrm{Im}\sigma$ with fiber canonically isomorphic to $T_{\sigma(b)}X_b$ for avery $b \in B$. In particular, it has rank g.
In fact, by definition (see Def [13, 71]) there exists a short exact sequence of holomorphic vector bundles

$$0 \longrightarrow \mathcal{T}_{\mathrm{Im}\sigma} \longrightarrow \mathcal{T}_X|_{\mathrm{Im}\sigma} \longrightarrow N \longrightarrow 0$$

where $\mathcal{T}_{\mathrm{Im}\sigma}$ is the holomorphic tangent bundle of $\mathrm{Im}\sigma$ and $\mathcal{T}_X|_{\mathrm{Im}\sigma}$ is the holomorphic tangent bundle of X restricted to $\mathrm{Im}\sigma$.
Fix $b \in B$. Denoted by $\iota \colon \mathrm{Im}\sigma \hookrightarrow X$ the inclusion, note that $\mathcal{T}_X|_{\mathrm{Im}\sigma} = \iota^* TX$. Thus, $(\mathcal{T}_X|_{\mathrm{Im}\sigma})_{\sigma(b)}$ is canonically isomorphic to $T_{\sigma(b)}X$.
Exactness of the sequence implies that

$$0 \longrightarrow \mathcal{T}_{\sigma(b)}\mathrm{Im}\sigma \longrightarrow \mathcal{T}_{\sigma(b)}X \longrightarrow N_{\sigma(b)} \longrightarrow 0$$

is an exact sequence of $\mathbb{C}$-vector spaces, so that it splits. So,

$$\mathcal{T}_{\sigma(b)}X \cong \mathcal{T}_{\sigma(b)}\mathrm{Im}\sigma \oplus N_{\sigma(b)}.$$

Let V an open neighborood of $\sigma(b)$ such that there exists a biholomorphism $h = (\phi, h_F)$ to $\phi(V) \times F$ over $\phi(V)$ (this is possible by holomorphic implicit function theorem). Proceeding as for $T_{X/B}$, computations yield

$$T_{\sigma(b)}X \cong T_{\sigma(b)}V \xrightarrow{\simeq} T_b\phi(V) \oplus T_{\sigma(b)}X_b.$$

Since $\sigma\colon B \to \mathrm{Im}\sigma$ is a biholomorphism (holomorphic with inverse the restriction of π to $\mathrm{Im}\sigma$, which is holomorphic), it induces an isomorphism

$$T_b\phi(V) \xrightarrow{\simeq} T_{\sigma(b)}(V \cap \mathrm{Im}\sigma) \cong T_{\sigma(b)}\mathrm{Im}\sigma \ .$$

By considering the two decompositions of $T_{\sigma(b)}X$, we obtain $N_{\sigma(b)} \cong T_{\sigma(b)}X_b$.
Let $\sigma^*N^\vee$ be the pull back to B of $N^\vee$. By duality, it is a holomorphic vector bundle over B of rank g, with fiber $(\sigma^*N^\vee)_b \cong N^\vee_{\sigma(b)} \cong (T_{\sigma(b)}X_b)^\vee$, for every $b \in B$.
Fix $0 \in B$, U an open neighborhood of 0 trivializing $\sigma^*N^\vee$ and holomorphic sections $s_1, \dots, s_g \in H^0(U, \sigma^*N^\vee)$, such that $s_1(b), \dots, s_g(b)$ are linearly independent over $\mathbb{C}$, for every $b \in U$.
For every b, let

$$\Phi_{|b}\colon (T_{\sigma(b)}X_b)^\vee \xrightarrow{\simeq} H^0(X_b, \Omega^1_{X_b})$$

be the isomorphism yielding the Hodge decomposition and $\omega_{i,b} = \phi_{|b}(s_i(b))$, for $i = 1, \dots, g$.
We define holomorphic sections $\omega_1, \dots, \omega_g$ of $H^0(U, \phi_*\Omega^1_{X/B})$ by $\omega_i(x) := \omega_{i,\phi(x)}$, for every $x \in X_U$, and a morphism of sheaves on U

$$\sigma^*N^\vee_{|U} \to (\phi_*\Omega^1_{X/B})_{|U}$$

on U

$$\sigma^*N^\vee(U) \to \phi_*\Omega^1_{X/B}(U)$$

by sending

$$s_i \mapsto \omega_i, \quad \text{for all} \quad i = 1, \dots, g$$

and by extending it to a morphism of $\mathcal{O}_B(U)$-module. On the open subsets of U we define it by taking restrictions of functions.
This yields an isomorphism of $\mathcal{O}_B$-module, because at the level of stalks it coincides with the isomorphism Φ_b. In fact,

$$(\phi_*\Omega^1_{X/B})_b \cong (\sigma_*(\phi_*\Omega^1_{X/B}))_{\sigma(b)} \cong (\iota_*\Omega^1_{X/B})_{\sigma(b)} \cong (\Omega^1_{X/B})_{\sigma(b)}.$$

Since $\sigma^*N^\vee$ is a holomorphic vector bundle of rank g, its sheaf of section is a locally free $\mathcal{O}_B$-module of rank g. Thus, $\phi_*\Omega^1_{X/B}$ is a locally free $\mathcal{O}_B$-module of rank g too. □

Since the dual of a constant sheaf is constant and $\mathcal{O}_B^\vee = \mathcal{O}_B$, we obtain the following result:

Proposition 4.1.2. *The sheaf $(\phi_*\Omega^1_{X/B})^\vee$ is a locally free $\mathcal{O}_B$-module of rank g.*

4.2 Quotients

Let $\gamma\colon \Lambda \to V$ be a morphism of B-groups, where $\pi_V\colon V \to B$ is a holomorphic vector bundle of rank g and Λ is a locally constant B-Lie group with structural group $\mathbb{Z}^{2g}$. Suppose that, for every $b \in B$, the morphism

$$\gamma_b\colon \Lambda_b \to V_b$$

is such that the induced map

$$\gamma_{b_{\mathbb{R}}} : \Lambda_b \otimes_{\mathbb{Z}} \mathbb{R} \to V_{b_{\mathbb{R}}}$$

is an isomorphism of $\mathbb{R}$-vector spaces.
We define an equivalence relation $\sim$ on V by $v \sim v^{'}$ if and only if $\pi_V(v) = \pi_V(v^{'}) = b$ and $v - v^{'} \in \gamma_b(\Lambda_b)$.

Definition 4.2.1. *We define* $V/\Lambda := V/\sim$.

Let $\pi \colon V \to V/\Lambda$ be the canonical projection. Since π_V is constant on each fiber of π, there exists a unque continuous map ϕ making the following diagram

$$\begin{array}{ccc} V & \xrightarrow{\quad \pi_V \quad} & B \\ & \searrow^{\pi} \quad \quad \nearrow^{\phi} & \\ & V/\Lambda & \end{array}$$

commutative.

Proposition 4.2.1. *The map* $\phi \colon V/\Lambda \to B$ *is proper.*

Proof. Since the property of being proper is local on B, we will work locally on B.
By considering an open subset of B trivializing Λ and V, we can assume that $\Lambda = B \times \mathbb{Z}^{2g}$ and $V = B \times \mathbb{C}^g$ with j corresponding to a $g \times 2g$ matrix (a_{hk}) of holomorphic functions $T = (a_{hk})$ such that the $2g \times 2g$ matrix

$$\begin{pmatrix} T(b) \\ \overline{T(b)} \end{pmatrix} \in \mathrm{Mat}2g \times 2g(\mathbb{C})$$

is invertible for each $b \in B$, by Proposition 1.4.1. In particular, the top $g \times 2g$ matrix $T(b)$ is surjective as a linear map $\mathbb{C}^{2g} \to \mathbb{C}^g$, so by equality of row rank and column rank it has an invertible $g \times g$ submatrix. Working locally around some $b_0 \in B$, we may rearrange the order of the trivialization of Λ, so that the left submatrix of $T(b_0)$ is invertible. By shrinking around b_0 we can then assume that the left $g \times g$ submatrix of $T(b)$ is invertible for all $b \in B$.
Writing $T = (A_1 \quad A_2)$ with $g \times g$ matrices A_1 and A_2 whose entries are holomorphic functions, we have arranged that A_1 is invertible, so by multiplying $V = B \times \mathbb{C}^g$ by $\mathrm{Id}_B \times A_1^{-1}$, we can arrange that $T = (1_g \quad Z)$ for some holomorphic map $Z \colon B \to \mathrm{Mat}_{g \times g}(\mathbb{C})$.
Consider the holomorphic map $\overline{Z} \colon B \to \mathrm{Mat}_{g \times g}(\mathbb{C})$. By Proposition 1.4.1, the holomorphic map

$$\begin{pmatrix} 1_g & Z \\ 1_g & \overline{Z} \end{pmatrix} : B \to \mathrm{Mat}_{2g \times 2g}(\mathbb{C})$$

is valued in $GL_{2g}(\mathbb{C})$. Equivalently, subtracting the top $g \times 2g$ block from the bottom one gives

$$\begin{pmatrix} 1_g & Z \\ 0 & \overline{Z} - Z \end{pmatrix}$$

so its invertibility is equivalent to that of the $g \times g$ matrix $\overline{Z} - Z = -2i\text{im}(Z)$. Thus, if we write $Z = A_1 + iA_2$ for holomorphic maps $A_1 \colon B \to \text{Mat}_g(\mathbb{R})$ and $A_2 \colon B \to \text{Mat}_g(\mathbb{R})$, A_2 is valued in $GL_g(\mathbb{R})$.
Consider the holomorphic map $\psi \colon B \times \mathbb{C}^g \to B \times \mathbb{C}^g$ over B defined by

$$(b, x + iy) \mapsto (b, x + (A_1(b) + iA_2(b))y) = (b, (x + A_1(b)y) + iA_2(b)y).$$

This is a biholomorphism, as it is invertible with inverse holomorphic

$$(b, u + iv) \mapsto (b, (u - A_1(b)A_2(b)^{-1}v) + iA_2(b)^{-1}v).$$

This isomorphism yields a commutative diagram

$$\begin{array}{ccc} B \times \mathbb{C}^g & \xrightarrow[\simeq]{\psi} & B \times \mathbb{C}^g \\ {\scriptstyle \iota_1}\big\uparrow & & \big\uparrow{\scriptstyle \iota_2} \\ B \times \mathbb{Z}^{2g} & \xrightarrow[\psi_|]{\simeq} & B \times (1_g \quad Z)\mathbb{Z}^{2g} \end{array}$$

where ι_1 and ι_2 are the inclusions and $\psi_|$ is the restriction of ψ. Thus, commutatitivity of the diagram implies that, denoted by $\overline{\psi}$ the map induced by ψ at the level of the quotients, it is an isomorphism of topological spaces

$$\overline{\psi} \colon B \times C^g/\mathbb{Z}^{2g} \cong (B \times \mathbb{C}^g)/(B \times \mathbb{Z}^2) \xrightarrow{\simeq} (B \times \mathbb{C}^g)/B \times (1_g \quad Z)\mathbb{Z}^{2g} \cong B \times V/\Lambda.$$

Moreover, since the previous diagram commutes with the projection onto B, $\overline{\psi}$ commutes with the projection onto B too.
Thus, topologically

$$\phi \colon V/\Lambda \to B$$

is precisely the projection onto the first factor $B \times \mathbb{C}/\mathbb{Z}^{2g} \to B$, which is proper. In fact, the inverse image of a compact subset $K \subseteq B$ is $K \times \mathbb{C}^g/\mathbb{Z}^{2g}$, which is compact in $B \times \mathbb{C}^g/\mathbb{Z}^{2g}$ because product of compact subsets. □

Proposition 4.2.2. *The quotient V/Λ has the structure of a complex manifold relative to which $\pi \colon V \to V/\Lambda$ is a covering map and then $\phi \colon V/\Lambda \to B$ yields a family of compact complex tori over B.*

Proof. First, we will work locally on B. By considering an open subset of B trivializing Λ and V, we can assume that $\Lambda = B \times \mathbb{Z}^{2g}$ and $V = B \times \mathbb{C}^g$ with j corresponding to a $g \times 2g$ matrix (a_{hk}) of holomorphic functions $T = (a_{hk}) = (1_g \quad Z)$, for some holomorphic map $Z \colon B \to \text{Mat}_{g \times g}(\mathbb{C})$ with $\text{im}Z$ invertible.
The group $\mathbb{Z}^{2g}$ acts on $B \times \mathbb{C}^g$ on the right by :

$$\lambda : (b, x) \mapsto (b, x + (1_g \quad Z(b))\lambda) \quad \text{for} \quad \lambda \in \mathbb{Z}^{2g}.$$

Since this is action is free and properly discontinuous, by Proposition 1.2.1, the quotient under this action is a complex manifold and the canonical projection from V to the quotient is a covering map. Since the quotient coincides with V/Λ and the canonical projection

with π, we obtain that π a covering map and V/Λ is a complex manifold.
Thus, we obtain a family of compact complex tori.
In fact, for each $b \in B$, the fiber $(V/\Lambda)_b$ coincides with $V_b/\gamma_b\Lambda_b \cong (\mathbb{C}^g/(1_g \quad Z(b))\mathbb{Z}^{2g})$, which is a compact complex torus of dimension g.
The map $\phi\colon V/\Lambda \to B$ is proper, by Proposition 4.2.1, and a submersion, since locally it is given by the composition $\pi_V \circ \pi_{|}^{-1}$, where $\pi_{|}$ is the restrion of π to an open subset of V over which is it a biholomorphism. The composite is a submersion, since composition of submersions.
Finally, the function

$$\tilde{\sigma}\colon B \to V$$

defined by

$$b \mapsto (b, 0) \in V$$

is holomorphic, since it is holomorphic componentwise, and a section of π_V. Then,

$$\sigma := \pi \circ \tilde{\sigma}$$

defines a holomorphic section of ϕ.
Let $B = \cup_{i\in I} U_i$ be an open covering of B with U_i as in the previous part. Then,

$$V/\Lambda = \phi^{-1}(B) = \phi^{-1}(\cup_{i\in I} U_i) = \cap_{i\in I}\phi^{-1}(U_i) = \cap_{i\in I} V_{U_i}/\Lambda_{U_i}$$

so that $\{V_{U_i}/\Lambda_{U_i}\}_{i\in I}$ is an open covering of V/Λ.
Since the restriction of π

$$\pi_{|}\colon \pi^{-1}(V_{U_i}/\Lambda_{U_i}) \to V_{U_i}/\Lambda_{U_i}$$

is a covering map, π is a covering map too.
Moreover, we can glue the V_{U_i}/Λ_{U_i} into a complex manifolds, since if i and $j \in I$ are such that $U_i \cap U_j \neq \emptyset$, then we define the function

$$\phi_{ij}\colon (V_{U_i}/\Lambda_{U_i})_{U_i\cap U_j} \to (V_{U_j}/\Lambda_{U_j})_{U_i\cap U_j}$$

as the unique biholomorphism making the following diagram commutative

$$\begin{array}{ccccc}
(\Lambda_{U_i})_{U_i\cap U_j} & \xhookrightarrow{\gamma} & (V_{U_i})_{U_i\cap U_j} & \xrightarrow{\pi} & (V_{U_i}/\Lambda_{U_i})_{U_i\cap U_j} \\
\simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow\phi_{ij} \\
(\Lambda_{U_j})_{U_i\cap U_j} & \xhookrightarrow[\gamma]{} & (V_{U_j})_{U_i\cap U_j} & \xrightarrow[\pi]{} & (V_{U_j}/\Lambda_{U_j})_{U_i\cap U_j}
\end{array}$$

where the vertical isomorphisms are given by the cocycles defining Λ and V.
For every $b \in B$ there exists U_i such that $b \in U_i$, so that the fiber over b is a compact complex torus of dimension g, by previous part.
Since the properties of properness and being a submersions are local, on B and on X, respectively, by the first part of the proof, ϕ is proper and a submersion.
Finally, we can glue the zero sections σ_i defined on U_i into a global section σ defined on B

because the functions ϕ_{ij} preserve the origins. In fact, if $b \in U_i \cap U_j$, the previous diagram yields a comutative diagram

$$\begin{array}{ccc} V_b & \xrightarrow{\pi_b} & V_b/\gamma_b(\Lambda_b) \\ \simeq\big\downarrow & & \simeq\big\downarrow \phi_{ij} \\ V_b & \xrightarrow[\pi_b]{} & V_b/\gamma_b(\Lambda_b). \end{array}$$

Since the vertical isomorphism $V_b \xrightarrow{\simeq} V_b$ is $\mathbb{C}$-linear and $\pi_b(0) = 0$, $\phi_{ij}(0) = 0$ by commutativity. □

4.3 Equivalence of categories for families of compact complex tori

Let $\mathscr{T}_B$ be the category whose objects are the compact complex tori of dimension g over B and morphisms are the morphisms of compact complex tori over B preserving origins. Let $\mathscr{H}_B$ be the category whose objects are the triple (Λ, V, γ), where Λ is a locally constant B-Lie group with structural group $\mathbb{Z}^{2g}$, V is a holomorphic vector bundle of rank g and $\gamma \colon \Lambda \to V$ is a morphism of B-Lie groups with the property that, denoted by $\gamma_b \colon \Lambda_b \to V_b$ the map induced by γ, for any $b \in B$, then

$$\gamma_{b_\mathbb{R}} \colon \Lambda_b \otimes_\mathbb{Z} \mathbb{R} \to V_{b_\mathbb{R}}$$

is an isomorphism of $\mathbb{R}$-vector spaces ($V_{b_\mathbb{R}}$ is just V_b viewed as an $\mathbb{R}$-vector space), and a morphism $\phi \colon (\Lambda, V, \gamma) \to (\Lambda', V, \gamma')$ is given by a couple (ϕ_1, ϕ_2), where $\phi_1 \colon \Lambda \to \Lambda'$ is a morphism of B-groups and $\phi_2 \colon V \to V'$ is a morphism of holomorphic vector bundles, such that they make the following diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{\gamma} & V \\ \phi_1\big\downarrow & & \big\downarrow \phi_2 \\ \Lambda' & \xrightarrow[\gamma']{} & V' \end{array}$$

commutative.

We define $F_B \colon \mathscr{T}_B \to \mathscr{H}_B$ in the following way: let (X, ϕ, σ) be a family of compact complex tori over B. To it, we associate the object $F_B(X) = (\Lambda, V, \gamma)$, where:

- Λ is the locally constant B-Lie group associated to the $\mathbb{Z}$-local system $R_1\phi_*\mathbb{Z}_X$ on B.
  Let $U \subseteq B$ be a contractible open subset over which X is topologically trivial and $0 \in U$. Then, the canonical map $\pi \colon \Lambda \to B$ yields the projection onto the first factor

  $$U \times H_1(X_0; \mathbb{Z}) \xrightarrow[\pi_|]{} U.$$

  Then, for $b \in U$, the natural isomorphism induced by the inclusion of b in U

  $$j_{b_*}^{-1} \circ j_{0_*} \colon H_1(X_0; \mathbb{Z}) \xrightarrow{\simeq} H_1(X_b; \mathbb{Z})$$

(the fact that j_{b_*} and j_{0_*} are isomorphisms can be proved by the same argument we used at the level of cohomologies) yields an isomorphism

$$\Lambda_b \xrightarrow{\simeq} H_1(X_b;\mathbb{Z}).$$

- V is the dual of the holomorphic vector bundle associated to the sheaf $\phi_*\Omega^1_{X/B}$;
- the map
$$\gamma\colon \Lambda \to V$$
is defined by sending
$$(b,[\sigma_b]) \mapsto (b, \gamma_{X_b}([\sigma_b]))$$
for all $b \in B$, $[\sigma_b] \in H_1(X_b;\mathbb{Z})$, where $\gamma_{X_b}\colon H_1(X_b;\mathbb{Z}) \to H^0(X_b,\Omega^1_{X_b})^\vee$ is the morphism of Proposition 1.4.2.

We will denote the map γ also by γ_X, to specify it is associated to the family $X \to B$.

If $\phi\colon X \to B$ and $\phi^{'}\colon X^{'} \to B$ are families of compact complex tori and $f\colon X \to X^{'}$ is a morphism in $\mathscr{T}_B$, let

$$\tilde{\phi}_1\colon R_1\phi_*\mathbb{Z}_X \to R_1\phi^{'}_*\mathbb{Z}_{X'}$$

and

$$\tilde{\phi}_2\colon (\phi_*\Omega^1_{X/B})^\vee \to (\phi^{'}_*\Omega^1_{X'/B})^\vee$$

be the morphisms induced by f at the level ol sheaves. These yield morphisms

$$\phi_1\colon \Lambda \to \Lambda^{'}$$

of B-Lie groups and

$$\phi_2\colon V \to V^{'}$$

of holomorphic vector bundles over B. We define $F(f) = (\phi_1,\phi_2)$.

Let $U \subseteq B$ be an open subset. If we fix coordinates of X_U and, for any $b \in B$, $\omega_{1,b},\dots,\omega_{g,b}$ generators of X_b, then γ_X has the form:

$$\gamma_X(b,[\sigma_b]) \mapsto \left(b, \left(\int_{\sigma_b}\omega_{1,b},\dots,\int_{\sigma_b}\omega_{g,b}\right)\right)$$

for all $b \in B$, $[\sigma_b] \in H_1(X_b;\mathbb{Z})$.

Theorem 4.3.1. *If $\phi\colon X \to B$ is a family of compact complex tori of dimension g, then the map γ_X is holomorphic.*

Proof. Fix $U \subseteq B$ an open subset such that it is contractible and X is topologically trivial over U. Let

$$T\colon X_U \xrightarrow{\simeq} X_0 \times U$$

be the diffeomorphism of Ehresmann Theorem. Fix Φ an isomorphism

$$\Phi\colon \mathbb{Z}^{2g} \xrightarrow{\simeq} H_1(X_U;\mathbb{Z})$$

and let $[a_i] := \Phi(e_i)$, for $i = 1, \dots, 2g$, where e_i is the element of $\mathbb{Z}^{2g}$ that has 1 in the i-th component and 0 in the other ones. For $b \in U$, let $[a_i(b)]$ be the image of $[a_i]$ through the isomorphism

$$H_1(X_U; \mathbb{Z}) \xrightarrow[\simeq]{j_{b*}^{-1}} H_1(X_b; \mathbb{Z})$$

induced by the inclusion

$$X_b \xhookrightarrow{j_b} X_U.$$

By shrinking U if necessary, it is possible to construct mappings $\alpha_1, \dots, \alpha_{2g} \colon \mathbb{S}^1 \times U \to X_U$ such that:

- $\alpha_1, \dots, \alpha_{2g}$ are continuous;
- they make the following diagram

$$\begin{array}{ccc} \mathbb{S}^1 \times U & \xrightarrow{\alpha_1 \dots \alpha_{2g}} & X_U \\ & {\scriptstyle pr_2} \searrow \quad \swarrow {\scriptstyle \phi} & \\ & U & \end{array}$$

  commutative;
- for each $b \in U$, the map
$$\alpha_{i_b} \colon \mathbb{S}^1 \to X_U$$
  defined by
$$\theta \mapsto \alpha_i(\theta, b)$$
  is a piecewise smooth representative of $[a_i(b)]$, for all $i = 1, \dots, 2g$;
- for each $\theta \in \mathbb{S}^1$, the map
$$\alpha_{i_\theta} \colon U \to X_U$$
  defined by
$$b \mapsto \alpha_i(\theta, b)$$
  is holomorphic, for all $i = 1, \dots, 2g$ (this is a consequence of implicit function theorem, so it is important that the morphism ϕ is a submersion)

This construction is a generalization of [15, 14] for $g \geq 1$.
Then, basic calculus implies that

$$\int_{\alpha_{i_b}} \omega_{j,b}$$

vary holomorphically with $b \in U$, for $i = 1, \dots, 2g$, $j = 1, \dots, g$, where $\omega_1, \dots, \omega_j$ are holomorophic sections of $H^0(U, \phi_* \Omega^1_{X/B})$ whose restrictions $\omega_{1,b}, \dots, \omega_{g,b}$ to $H^0(X_b, \Omega^1_{X_b})$ are linearly independent over $\mathbb{C}$.
Fix ω_j, for some $1 \leq j \leq 2g$. The integral

$$\int_{\sigma_b} \omega_{j,b}$$

vary holomorphically with $b \in U$. In fact, there exist $n_1, \dots, n_{2g} \in \mathbb{Z}$ such that

$$[\sigma_b] = n_1[\alpha_{1_b}] + \cdots + n_{2g}[\alpha_{2g_b}]$$

since the $[\alpha_{i_b}]$'s generate $H_1(X_b; \mathbb{Z})$, so that

$$\int_{\sigma_b} \omega_{j,b} = n_1 \int_{\alpha_{1_b}} \omega_{j,b} + \cdots + n_{2g} \int_{\alpha_{2g_b}} \omega_{j,b}$$

varies holomorphically with $b \in U$, since sum of terms varying holomorphically with $b \in U$. From this, it follows that

$$[\sigma_b] \mapsto \left(\int_{\sigma_b} \omega_{1,b}, \dots, \int_{\sigma_b} \omega_{g,b} \right)$$

varies holomorphically with $b \in B$. Thus $\gamma_X(b)$ vary holomorphically with $b \in B$.

□

Proposition 4.3.1. $F_B \colon \mathscr{T}_B \to \mathscr{H}_B$ *is a functor.*

Proof. Let $\phi \colon X \to B$ be a family of compact complex tori and consider $F_B(X) = (\Lambda, V, \gamma)$. In order to prove that $F_B(X) \in \mathscr{H}_B$, we only have to show that γ satisifes the necessary properties.
It is a morphism of B-groups. In fact, since γ is defined fiber by fiber, it makes the following diagram

$$\begin{array}{ccc} \Lambda & \longrightarrow & V \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

commutative. It is holomorphic by the Theorem 4.3.1 and it can be checked that the induced map

$$\gamma(T) \colon \Lambda(T) \to V(T)$$

is a group homomorphism and functorial in T, for any complex manifold T over B.
Finally, the induced map $\gamma_b \colon \Lambda_b \to V_b$ yields an isomorphism

$$\gamma_{b_{\mathbb{R}}} \colon \Lambda_b \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\ \simeq\ } V_{b_{\mathbb{R}}}$$

for any $b \in B$, by Proposition 1.4.2.
Thus, $F_B(X)$ defines an object in $\mathscr{T}_B$.
If $\phi \colon X \to B$ and $\phi' \colon X' \to B$ are families of compact complex tori and $f \colon X \to X'$ is a morphism in $\mathscr{T}_B$, we have to show that, if $F_B(f) = (\phi_1, \phi_2)$, the diagram of B-Lie groups' morphisms

$$\begin{array}{ccc} \Lambda_X & \xrightarrow{\gamma_X} & V_X \\ {\scriptstyle \phi_1}\downarrow & & \downarrow{\scriptstyle \phi_2} \\ \Lambda_{X'} & \xrightarrow[\gamma_{X'}]{} & V' \end{array}$$

is commutative. This is true since, for every $b \in B$, it coincides with the commutative diagram

$$\begin{array}{ccc} H_1(X;\mathbb{Z}) & \xrightarrow{\gamma} & H^0(X,\Omega^1_X)^\vee \\ {\scriptstyle H_1(f)}\downarrow & & \downarrow{\scriptstyle H^0(f)^\vee} \\ H_1(X';\mathbb{Z}) & \xrightarrow[\gamma']{} & H^0(X',\Omega^1_{X'})^\vee. \end{array}$$

Finally, in order to be a functor, F has to preservers composition of morphisms and identities. This is the case, by functoriality of the construction. □

4.3.1 The Albanese map on families of compact complex manifolds

Let $\phi\colon X \to B$ be a family of compact complex manifolds such that, for each $b \in B$, X_b is a compact complex manifold satisfying Hodge decomposition in degree 1.
Fixed base points $x_b \in X_b$ for every $b \in B$, one defines the *Albanese map on the family* $\phi\colon X \to B$

$$\mathrm{alb}\colon X \to (\phi_*\Omega^1_{X/B})^\vee / R_1\phi_*\mathbb{Z}_X$$

fiber by fiber, by

$$x \mapsto \mathrm{alb}_{x_b}(x), \quad \text{if} \quad x \in X_b$$

where we identify the sheaves $\phi_*\Omega^1_{X/B}$, $R_1\phi_*\mathbb{Z}_X$ with the associated B-Lie groups and $R_1\phi_*\mathbb{Z}_X$ with its image under the morphsm $\gamma\colon R_1\phi_*\mathbb{Z}_X \to (\phi_*\Omega^1_{X/B})^\vee$.

Lemma 4.3.1. *Let $X \to B$ and $Y \to B$ be biholomorphic submersions between complex manifolds, and $f \in Y(X)$ such that the induced map $f_b\colon X_b \to Y_b$ between fibers is a biholomorphism, for every over $b \in B$. Then f is a biholomorphism.*

Proof. Since f is a biholomorphism on each fiber, it is bijective. So ti suffices to prove that it is a local isomorphism. By the holomorphic inverse function theorem, it is equivalent to prove that for each $b \in B$ and for each $x \in X_b$, the map induced on the complex tangent spaces $Jf(x)\colon T_xX \to T_{f(x)}X$ is an isomorphism.
Let $b \in B$ and $x \in X_b$. Since the maps $X \to B$ and $Y \to B$ are submersions, the induced maps on the complex tangent spaces

$$T_xX \to T_bB, \quad T_{f(x)}Y \to T_bB$$

are surjective and the respective kernels are identified with T_xX_b and $T_{f(x)}T_b$. By functoriality of derivaties (i.e. the Chain Rule), the map $Jf(x)$ commutes with the quotient maps onto T_bB and carries T_xX_b to $Y_{f(x)}Y_b$ via $Jf_b(x)$. That is, we have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_xX_b & \longrightarrow & T_xX & \longrightarrow & T_bB & \longrightarrow & 0 \\ & & {\scriptstyle Jf_b(x)}\downarrow & & {\scriptstyle Jf(x)}\downarrow & & \mathrm{Id}_{T_bB}\downarrow & & \\ 0 & \longrightarrow & T_{f(x)}Y_b & \longrightarrow & T_{f(x)}Y & \longrightarrow & T_bB & \longrightarrow & 0. \end{array}$$

But f_b is an isomorphism by hypothesis, so the left arrow is an isomorphism and hence so is the middle arrow, by Five-Lemma [16, 98]. □

Proposition 4.3.2. *Let (X, ϕ, σ) be a family of compact complex tori over B. The Albanese map on this family*

$$alb_\sigma \colon X \to (\phi_* \Omega^1_{X/B})^\vee / R_1\phi_*\mathbb{Z}_X$$

defined by

$$x \mapsto alb_{\sigma(\phi(x))}(x)$$

is an isomorphism of families of compact complex tori.

Proof. To prove holomorphicity, fix $0 \in B$ and $y \in X_0$. Let U be an open neighborhood of 0 and $\omega_{1,0}, \ldots, \omega_{g,0}$ be holomorphic one-forms on X_0, as in Theorem 4.3.1.
Then, for all $b \in U$ and $x \in X_b$,

$$\mathrm{alb}_\sigma(x) = \mathrm{alb}_{\sigma(b)}(x) = \left(\int_{\sigma(b)}^x \omega_{1,b}, \ldots, \int_{\sigma(b)}^x \omega_{g,b} \right) \mod \Lambda_b$$

where

$$\Lambda_b := \left\{ \left(\int_{\sigma_b} \omega_{1,b}, \ldots, \int_{\sigma_b} \omega_{g,b} \right) : [\sigma_b] \in H_1(X_b; \mathbb{Z}) \right\}.$$

There exists an open neighborood V of y, with $V \subseteq X_U$, such that, for all $x \in V$, there exists a path

$$\gamma_x \colon [0,1] \to X_{\phi(x)}$$

with

$$\gamma_x(0) = \sigma(\phi(x)), \quad \gamma_x(1) = x$$

piecewise smooth and varying holomorphically with $x \in V$.
Since $\omega_{1,b}, \ldots, \omega_{g,b}$ vary holomorphically with $b \in U$ and ϕ is holomorphic, they vary holomorphically with $x \in V$. Thus, by elementary calculus

$$x \mapsto \left(\int_{\gamma_x} \omega_{1,\phi(x)}, \ldots, \int_{\gamma_x} \omega_{g,\phi(x)} \right)$$

varies holomorphically with $x \in V$. By shrinking U if necessary, by Theorem 4.3.1,

$$\int_{\sigma_b} \omega_{1,b}, \ldots, \int_{\sigma_b} \omega_{g,b}, \quad \text{for} \quad [\sigma_b] \in H_1(X_b; \mathbb{Z})$$

vary holomorphically with $b \in U$, thus the lattices $\Lambda_{\phi(x)}$ vary holomorphically with $x \in V$.
Since $\mathrm{alb}_{\sigma(\phi(x))}$ does not depend on the path connecting $\sigma(\phi(x))$ and x, we obtain

$$\mathrm{alb}_\sigma(x) = \left(\int_{\gamma_x} \omega_{1,\phi(x)}, \ldots, \int_{\gamma_x} \omega_{g,\phi(x)} \right) \mod \Lambda_{\phi(x)}$$

which varies holomorphically with $x \in V$.
We now establish the existence of V and γ_x. Since $\phi \colon X \to B$ is a submersion, by holomorphic implicit function theorem, there exists an open neighborhood $V \subseteq X_U$ of y and a biholomorphism to a product

$$h \colon V \xrightarrow{\simeq} \phi(V) \times V'$$

over $\phi(V)$. Thus, $h = (\phi, h_1)$, with $h_1 \colon V \to V^{'}$ holomorphic. By passing to local charts, we can suppose that V, $\phi(V) \times V^{'}$ are subset of $\mathbb{C}^n$, for $n = \dim X$.
If $\sigma(\phi(x)) \in V$, we define

$$\gamma \colon [0,1] \times V \to \phi(V) \times V^{'}$$

by

$$(t, x) \mapsto (\phi(x), th_1(x) + (1-t)(h_1 \circ \sigma \circ \phi)(x)).$$

This is continuous, since continuous in each component, and the map

$$\gamma_t \colon X \to \phi(V) \times V^{'}$$

defined by

$$x \mapsto \gamma_t(x) = \gamma(t, x)$$

is holomorphic, for all $t \in [0,1]$. Thus the path

$$\gamma_x := h^{-1}(\gamma(-, x)) \colon [0,1] \to X$$

satisfies the required properties.
If $\sigma(\phi(x)) \notin V$, we can fix a path connecting $\sigma(\phi(x))$ and x. By compactness, we can cover it by finitely many open subsets V_i of X, for $i = 1, \dots, m$, to which we can apply holomorphic implic function theorem. We fix $y_i \in V_i \cap V_{i+1}$, for $i = 1, \dots, m-1$, $y_0 = \sigma(\phi(x))$ and $y_m = x$, and in each V_i we construct a path γ_i as before, connecting y_{i-1} and y_i, for $i = 1, \dots, m$. By gluing the γ_i's, we obtain a path γ piecewise smooth, connecting $\sigma(\phi(x))$ and x and varying homomorphically with $x \in V$.

This is a morphism over B, since it is defined fiber by fiber, and it preserves the zero sections, since, by $\mathbb{C}$-linearity, it preserves the origins fiber by fiber.
It is a biholomorphism by Lemma 4.3.1, because it is holomorphic and its restriction to X_b is the biholomorphism $\mathrm{alb}_{\sigma(b)}$, for all $b \in B$. □

4.3.2 Equivalence of categories

We define a functor $G_B \colon \mathscr{H}_B \to \mathscr{T}_B$ on the objects by $G(\Lambda, V, \gamma) = V/\Lambda$, on the morphisms in the natural way: let

$$(\phi_1, \phi_2) \colon (\Lambda, V, \gamma) \to (\Lambda^{'}, V^{'}, \gamma^{'})$$

be a morphism in $\mathscr{H}_B$, to it we associate the unique morphism of complex manifolds

$$\phi \colon V/\Lambda \to V^{'}/\Lambda^{'}$$

making the following diagram

$$\begin{array}{ccc} V & \xrightarrow{\phi_2} & V^{'} \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ V/\Lambda & \xrightarrow[\phi]{} & V^{'}/\Lambda^{'} \end{array}$$

commutative, where $\pi_1 \colon V \to V/\Lambda$ and $\pi_2 \colon V^{'} \to V^{'}/\Lambda^{'}$ are the natural projections.
In this way, we define a functor.

In fact, if $(\Lambda, V, \gamma) \in \mathscr{H}_B$, then, by Proposition 4.2.2, $V/\Lambda \to B$ is a family of complex tori of dimension g over B.
If

$$(\phi_1, \phi_2)\colon (\Lambda, V, \gamma) \to (\Lambda^{'}, V^{'}, \gamma^{'})$$

is a morphism in $\mathscr{H}_B$, then the morphism $\phi\colon V/\Lambda \to V^{'}/\Lambda^{'}$ exists and it is unique, since $\pi_2 \circ \phi_2$ preserves the fibers of π_1, and it is holomorphic, since its composition with the local biholomorphism π_1 (it is holomorphic and locally invertible, because it is a covering map) is $\pi_2 \circ \phi_2$, which is holomorphic since composition of holomorphic functions.
By the condition that ϕ_2, π_1 and π_2 are morphisms of B-groups, it follows that also ϕ is a morphism of B-groups. Thus, for every $b \in B$, there is a commutative diagram

$$\begin{array}{ccc} V_b & \xrightarrow{\phi_{2_b}} & V_b^{'} \\ {\scriptstyle \pi_{1_b}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{2_b}} \\ V_b/\gamma_b(\Lambda_b) & \xrightarrow[\phi_b]{} & V_b^{'}/\gamma_b^{'}(\Lambda_b^{'}) \end{array} .$$

Since ϕ_{2_b}, π_{1_b} and π_{2_b} respect the origins (ϕ_{2_b} because is $\mathbb{C}$-linear, since ϕ_2 is a morphism of holomorphic vector bundles, π_{1_b} and π_{2_b} by definition), it follows that ϕ_b preserves the origins for every $b \in B$. Thus, it preserves the zero sections.

Theorem 4.3.2. (F_B, G_B) *is an equivalence of categories between* $\mathscr{T}_B$ *and* $\mathscr{H}_B$*.*

Proof. We will prove that G_B is a quasi-inverse of F_B.
In the same notations of Proposition 4.3.1, for $X \in \mathscr{T}_B$, we have

$$X \cong (G_B \circ F_B)(X) \quad \text{in} \quad \mathscr{T}_B$$

via the Albanese map on families of compact complex tori.
For $(\Lambda, V, \gamma) \in \mathscr{H}_B$, we have

$$(\Lambda, V, \gamma) \cong (F \circ G)(\Lambda, V, \gamma) = ((\phi_* \Omega^1_{X/B})^{\vee}, R_1\phi_*\mathbb{Z}_X, \phi_\gamma) \quad \text{in} \quad \mathscr{H}_B \tag{4.1}$$

where $X := V/\Lambda$. Let us prove the isomorphism in (4.1).
Let σ and $\sigma^{'}$ be the zero sections of $\phi\colon V/\Lambda \to B$ and $\phi^{'}\colon \phi_*\Omega^1_{X/B}/R_1\phi_*\mathbb{Z}_X \to B$, respectively, and

$$\text{alb}_\sigma\colon V/\Lambda \xrightarrow{\ \simeq\ } (\phi_*\Omega^1_{X/B})^{\vee}/R_1\phi_*\mathbb{Z}_X$$

be the Albanese isomorphism.
Let $b \in B$, $0_V \in V$ and $0_{\phi_*} \in \phi_*\Omega^1_{X/B}$ be such that $\pi(0_V) = \sigma(b)$ and $\pi^{'}(0_{\phi^*}) = \sigma^{'}(b)$, where $\pi\colon V \to V/\Lambda$ and $\pi^{'}\colon (\phi_*\Omega^1_{X/B})^{\vee} \to (\phi_*\Omega^1_{X/B})^{\vee}/R_1\phi_*\mathbb{Z}_X$ are the canonical projections. Since

$$\text{alb}_\sigma(\pi(0_V)) = \text{alb}_{\sigma(b)}(\sigma(b)) = \sigma^{'}(b) = \pi^{'}(0_{\phi^*})$$

covering space theory implies that there exists a unique continuous map

$$\phi_2\colon (V, 0_V) \to ((\phi_*\Omega^1_{X/B})^{\vee}, 0_{\phi_*})$$

making the following diagram

$$\begin{array}{ccc} (V, 0_V) & \xrightarrow{\quad \phi_2 \quad} & ((\phi_*\Omega^1_{X/B})^\vee, 0_{\phi_*}) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ (V/\Lambda, \sigma(b)) & \xrightarrow[\mathrm{alb}_\sigma]{} & ((\phi_*\Omega^1_{X/B})^\vee / R_1\phi_*\mathbb{Z}_X, \sigma'(b)) \end{array}$$

commutative.
It is a isomorphism of holomorphic vector bundles over B:

- It is holomorphic: locally π' is a biholomorphism (since it is a covering map), thus, locally, ϕ_2 is given by the composition $(\pi'_|)^{-1} \circ \mathrm{alb}_{\sigma|} \circ \pi_|$, which is holomorphic since composition of holomorphic functions (here $\pi'_|$, $\mathrm{alb}_{\sigma|}$, $\pi_|$ denote the restrictions of π', alb_σ, π to open subsets over which they are biholomorphisms);
- Commutativity of the previous diagram and the fact that alb_σ is a morphism over B, imply that $\phi' \circ \phi_2 = \phi$;
- for all $b \in B$, the map induced by alb_σ on the fibers is the morphism of compact complex tori $\mathrm{alb}_{\sigma(b)}$. By uniqueness of the lift, the map induced by ϕ_2 has to be equal to the analytic representation $\rho_a(\mathrm{alb}_{\sigma(b)})$. Thus,

$$\phi_{2_b} \colon V_b \to (\phi_*\Omega^1_{X/B})^\vee_b$$

  is $\mathbb{C}$-linear, for all $b \in B$;
- since ϕ_2 is holomorphic and $\phi_{2_b} = \rho_a(\mathrm{alb}_{\sigma(b)})$ is an isomorphism, for all $b \in B$, ϕ_2 is a biholomorphism by Lemma 4.3.1.

Consider the restriction of ϕ_2 to $\gamma(\Lambda)$. It induces an isomorphism of B-Lie groups

$$\phi_{2|} \colon \gamma(\Lambda) \xrightarrow{\simeq} \phi_\gamma(R_1\phi_*\mathbb{Z}_X).$$

In fact, $\phi_{2|}$ is a morphism of B-Lie groups from $\gamma(\Lambda)$ to its image, because it is the restriction of a morphism of B-Lie groups. Since, for all $b \in B$, the restriction of ϕ_{2_b} to $(\gamma(\Lambda))_b = \gamma_b(\Lambda_b)$ is the isomorphism

$$\rho_r(\mathrm{alb}_{\sigma(b)}) \colon (\gamma(V/\Lambda))_b = \gamma_b(\Lambda_b) \xrightarrow{\simeq} \phi_{\gamma_b}(H_1(V_b/\Lambda_b); \mathbb{Z}) = (\phi_\gamma(R_1\phi_*\mathbb{Z}_X))_b$$

we deduce that the image $\phi_{2|}(\gamma(\Lambda))$ is $\phi_\gamma(R_1\phi_*\mathbb{Z}_X)$ and $\phi_{2|}$ is an isomorphism, since holomorphic and biholomorphic on each fiber.
The B-Lie groups morphisms $\gamma \colon \Lambda \to \gamma(\Lambda)$ and $\phi_\gamma \colon R_1\phi_*\mathbb{Z}_X \to \phi_\gamma(R_1\phi_*\mathbb{Z}_X)$ are isomorphisms, because holomorphic and isomorphisms fiber by fiber. Then, we define

$$\phi_1 := \gamma_X^{-1} \circ \phi_{2|} \circ \gamma \colon \Lambda \to R_1\phi_*\mathbb{Z}_X.$$

It is an isomorphism of B-Lie group, since composition of isomorphisms of B-Lie groups. Moreover, since the following diagram

$$\begin{CD} V @>{\phi_2}>{\simeq}> (\phi_*\Omega^1_{X/B})^\vee \\ @A{\gamma}AA @AA{\phi_\gamma}A \\ \Lambda @>{\simeq}>{\phi_1}> R_1\phi_*\mathbb{Z}_X \end{CD}$$

is commutative, (ϕ_1, ϕ_2) yields an isomorphism in $\mathscr{H}_B$.
By functoriality of $R_1\phi_*\mathbb{Z}_X$ and $\phi_*\Omega^1_{X/B}$ and the naturality of the isomorphisms in (4.1), it follows that the isomorphisms $X \cong (G_B \circ F_B)(X)$ and $(\Lambda, V, \gamma) \cong (F_B \circ G_B)(\Lambda, V, \gamma)$ are functorial in X and (Λ, V, γ), yielding the isomorphisms of functors

$$\mathrm{Id}_{\mathscr{T}_B} \cong G_B \circ F_B \quad \text{and} \quad \mathrm{Id}_{\mathscr{H}_B} \cong F \circ G.$$

$\square$

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