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Quantitative Entanglement Witnesses for Qubit and Cavity

Entanglement Witnesses Quantitative per un Qubit e una Cavità

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Abstract

The existence of entangled systems is one of the most interesting aspects of quantum mechanics: studying non-classical correlations allows a better understanding of the non-local structure of quantum theory but also leads to important challenges in the development of new technologies in quantum information. Nowadays there are many ways to detect entanglement but they are not so efficient because of the number of measurements required and the numerical complexity to characterize general states. However using a specific class of observables, called Entanglement Witnesses, generalizations of the well-known observables used in Bell's Inequalities, we can obtain lower bounds for entanglement and so, using Quantitative Entanglement Witness and the related bounding function, actually a quantification. This work aims to apply this method to verify and quantify the presence of entanglement in a bipartite system of a qubit and a harmonic oscillator, following the already known case of two qubits.

L'esistenza di sistemi entangled rappresenta uno degli aspetti più interessanti della meccanica quantistica: lo studio di correlazioni non classiche difatti non solo permette di indagare la fondamentale struttura non locale di tale teoria ma è anche una sfida ancora aperta per poter utilizzare questa preziosa risorsa nello sviluppo di tecnologie di informazione quantistica. Nonostante attualmente siano conosciuti diversi metodi per l'individuazione e la stima dell'entanglement questi risultano in molti casi poco efficienti a causa della quantità di misure necessarie e della complessità numerica per la caratterizzazione di stati generici. Tuttavia tramite delle particolari osservabili, dette Entanglement Witnesses, generalizzazioni delle ben note osservabili usate nelle disuguaglianze di Bell, si possono ricavare dei limiti inferiori per l'entanglement e dunque, tramite Quantitative Entanglement Witness e relativa bounding function, un'effettiva quantificazione. In questo lavoro si prende in esame un sistema bipartito composto da un qubit ed un oscillatore armonico proponendosi, tramite tale metodologia, di verificare la presenza di correlazioni quantistiche e di quantificarle, sul modello di quanto già noto per un sistema a 2 qubits.

Indice

Introduction

Quantum entanglement is one of the most characterizing features of quantum physics: it is not only interesting to explore the nature of the deep connections between quantistic systems but it is also a very significant resource in many quantum information's aspects such as processing data, quantum errors correction [1], quantum simulators, machine learning [2], cryptography and teleportation. For this reason, there is a growing interest in theoretical and experimental methods of entanglement analysis, classification and detection.

Entanglement detection in particular is a challenging task in many experimental configurations because of the elevated number of measurements required to obtain a quantum tomography: these measurements are not always available, they can affect the measured states themselves and they could be very expensive if we consider multipartite systems, as the required measurements grow exponentially.

A first approach could be to measure the expectation value of a particular kind of observables called Entanglement Witness (EW), from which it is possible, in some cases, to realize if the state is entangled or not. This method anyway does not always work well and it does not provide any quantitative information: a more useful solution is to consider a different family of observables, the Quantitative Entanglement Witness (QEW), which is able to highlight lower bounds for entanglement value [3].

In this work, we will use QEW to study entanglement bounds for states of a qubit-cavity system, following the already known case of two qubits. The qubit-cavity system is significant as it is used in many quantum information devices, for example for the control of qubits: entangling a qubit with a resonator, described as a harmonic oscillator, permits in some systems to read out the qubits' information or to transmit to the qubits the information to proceed. Moreover, a cavity system can be entangled to a qubit to implement some operations that change during algorithms and so permits a more flexible calculation, useful to speed up the process but also necessary for quantum machine learning [2]. The qubit-cavity type of entanglement is in addition more stable and resistant to decoherence process.

In chapter 1 we give some basic notions of QEW and cavity's coherent states and we derive and justify, in analogy with the two-qubits example, the expression of the observables that composed the QEW of this system.

In chapter 2 we apply the QEW to entangled cat-states and we find the bounding functions for two measures of entanglement, entanglement negativity and entanglement of formation.
We also explore the possibility to use this witness for different cavity's states.

Capitolo 1

Theoretical background and derivation of witness

In this chapter we introduce some basic notions about Quantitative Entanglement Witness and we deduce its expression for a qubit-cavity system.

1.1 Quantitative Entanglement Witnesses

Entanglement Witnesses (EW) are a class of observables (hermitian operators) $\hat{\mathbf{W}}$ that are able, with only a few measurements, to give useful information about the entanglement of a certain class of states.

In fact, if we consider the expectation value of this observable on a generic mixed state ρ of a certain class of states, obtainable by evaluating the trace of this observable $\hat{\mathbf{W}}$ applied to ρ, we can obtain a sufficient condition for having entanglement [4]:

$$Tr[\hat{\mathbf{W}}\rho] < 0. \tag{1.1.1}$$

On the other hand, we have a necessary condition for separability. It is worth reminding [4] that for a bipartite system pure states are called separable if they can be written as a tensor product of two states from subsystem A and B, so their Schmidt rank is equal to one, whether for mixed states ρ_{AB} it means that they are writeable as:

$$\rho_{AB} = \sum_{k=0}^{D} p_k \rho_{A,k} \otimes \rho_{B,k} \tag{1.1.2}$$

in which D is the dimension of the subspaces A and B, p_k are the coefficients of the linear combination of k subspace and are probabilities, so normalized and with $p_k \in [0, 1]$.

If a state is separable so it means it is not entangled and it respects the necessary condition:

$$Tr[\hat{\mathbf{W}}\rho] \geq 0. \tag{1.1.3}$$

Condition in Eq.(1.1.1) is sufficient but not necessary so we can detect through this method only a part of all the entangled states: in fact, there are some entangled states for which $Tr[W\rho] \geq 0$.

To provide this action of entanglement's detection, EWs have to respect some properties, such as having at least a negative eigenvalue and a non-negative mean value of product states [5]. For each class of states exist an observable of this kind that can detect its entanglement, even if it is difficult to construct in some cases. There are many ways to find out a witness and often it concerns finding the optimal witness or decomposing it into local observables, for better experimental applications.

Moreover, to obtain more information about the entanglement of a state it is possible to upgrade the EW's method to the use of Quantitative Entanglement Witnesses (QEW) [3, 6].
This is a class of observables that has the same features as EWs but it is also able to provide, using so-called Bounding Functions (b(x)), lower bounds to the entanglement's value of a state: these functions can connect the information provided by QEW to a measure of bipartite entanglement.
Finding the most suitable bounding function concern an optimization problem and so it is not always reachable but for some classes of states, such as isotropic states or Werner states, it can be computed exactly [6, 7] as :

$$\mathcal{E}(\rho) > b(Tr[\hat{\mathbf{W}}\rho]) \tag{1.1.4}$$

providing a lower bound to the entanglement measure $\mathcal{E}$ of the state. Different kinds of entanglement measures can be chosen and the bounding function is different for different measures of entanglement.
In this work, we use two kinds of measures: Entanglement negativity ($\mathcal{E}_{\text{Neg}}$) and Entanglement of Formation ($\mathcal{E}_{\text{oF}}$).
This type of analysis can be applied to a bipartite system of two qubits and in particular to the class of isotropic states, which is defined as the family of pure and mixed states parameterized by a probability p as:

$$\rho_{iso}(p) = p\left|\Phi^+\right\rangle\left\langle\Phi^+\right| + \frac{(1-p)}{4}\mathbb{I} \tag{1.1.5}$$

with $p \in [0,1]$ and $|\Phi^+\rangle$ the Bell's state:

$$|\Phi^+\rangle = \frac{1}{2}(|00\rangle + |11\rangle). \tag{1.1.6}$$

For these states the Quantitative Entanglement Witness can be written as:

$$\hat{\mathbf{W}} = \hat{\mathbb{I}} - \hat{\sigma}_x \otimes \hat{\sigma}_x - \hat{\sigma}_z \otimes \hat{\sigma}_z + \hat{\sigma}_y \otimes \hat{\sigma}_y \tag{1.1.7}$$

in which $\hat{\sigma}_i$ are Pauli's matrices.
Applying Eq.(1.1.4) for this QEW and two different entanglement measures, $\mathcal{E}_{\text{Neg}}$ and $\mathcal{E}_{\text{oF}}$, it is shown [6] that isotropic states are entangled only if $p > \frac{1}{3}$ and the bounding functions result to be:

$$b_{Neg}(x) = \begin{cases} 0 & x > 1 \\ -\frac{x}{2} & x \leq 1 \end{cases} \tag{1.1.8}$$

$$b_{oF}(x) = \begin{cases} 0 & x > 1 \\ -\frac{1}{4}h_2(\sqrt{x} + \sqrt{2-x}) & x \leq 1 \end{cases} \tag{1.1.9}$$

where h_2 is defined as

$$h_2(t) = -t\log_2(t) - (1-t)\log_2(1-t) \tag{1.1.10}$$

and x as

$$x = Tr[\hat{\mathbf{W}}\rho_{iso}(p)]. \tag{1.1.11}$$

The following paragraphs aim to give some theoretical notions about quantum harmonic oscillator to define the analogous problem for states of a bipartite system of a qubit and a cavity.

1.2 Entangled system of a qubit and a cavity

A quantum cavity is a system, practically composed of a set of mirrors, that works as a resonator for some selected frequencies. Doing QED in the cavity the electromagnetic field can be quantized and this system for these frequencies can be described mathematically as a quantum harmonic oscillator.

To analyze the entanglement between a qubit and a harmonic oscillator, we need to choose a basis for the states of the harmonic oscillator.

1.2.1 Coherent states

The states of a quantum harmonic oscillator can be represented in a quantum phase-space where, differently from classic phase-space, they are not points but a larger portion of space, according to Heisenberg's indetermination principle.

The two observables used to describe the states, the position $\hat{\mathbf{X}}$ and the linear momentum $\hat{\mathbf{P}}$, which are respectively represented on the x and y axis, do not commute and so it is possible only to study a sort of probability distribution of the states, using the so-called Wigner's function [8].

However, we can choose a class of states that minimize the uncertainty so that it can be easier, on one hand, to study the time evolution of the system and on the other hand to practically have more control over them experimentally.

These states are called coherent states $|\beta\rangle$ and they have the property of saturating Heisenberg's uncertainty principle:

$$\Delta\,\hat{\mathbf{Q}}_\beta \Delta\,\hat{\mathbf{P}}_\beta = \frac{\hbar}{2} \tag{1.2.1}$$

in which holds also:

$$\Delta\,\hat{\mathbf{Q}}_\beta = \Delta\,\hat{\mathbf{P}}_\beta. \tag{1.2.2}$$

Coherent states are defined as the eigenstates of the annihilation operator $\hat{\mathbf{a}}$ and so they respect the relations:

$$\hat{\mathbf{a}}\,|\beta\rangle = \beta\,|\beta\rangle \qquad\qquad \langle\beta|\,\hat{\mathbf{a}}^\dagger = \beta^*\,\langle\beta| \tag{1.2.3}$$

where $\beta,\ \beta^* \in \mathbb{C}$.

These states can be seen as a displaced vacuum state, which is the eigenstate of the harmonic oscillator's hamiltonian with the lower eigenvalue of energy, represented by $|0\rangle$ in the so-called Fock's basis [9, 10].

The displacement operator $\hat{\mathbf{D}}(\beta)$ is defined as:

$$\hat{\mathbf{D}}(\beta) = e^{\beta\,\hat{\mathbf{a}}^\dagger - \beta^*\,\hat{\mathbf{a}}} \tag{1.2.4}$$

and acts on the vacuum state as:

$$\hat{\mathbf{D}}(\beta)\,|0\rangle = |\beta\rangle \tag{1.2.5}$$

so β is defined through the mean value of position and momentum's displacement as:

$$\beta = \langle\beta|\,\hat{\mathbf{X}}\,|\beta\rangle + i\,\langle\beta|\,\hat{\mathbf{P}}\,|\beta\rangle = x_\beta + ip_\beta. \tag{1.2.6}$$

Time evolution of harmonic oscillator states can be seen as the rotation of the probability distribution in the quantum phase-space and it is regulated by the unitary operator $\hat{\mathbf{U}}$ defined as:

$$\hat{\mathbf{U}} = e^{-\frac{i\hat{\mathbf{H}}t}{\hbar}} \tag{1.2.7}$$

in which $\hat{\mathbf{H}}$ is the hamiltonian of the harmonic oscillator with $\hat{\mathbf{N}}$ the number operator and D the dimension of the harmonic oscillator, that we choose two-dimensional:

$$\hat{\mathbf{H}} = \hbar\omega(\hat{\mathbf{N}} + \frac{D}{2}). \tag{1.2.8}$$

For coherent states, time evolution is easier to describe because the wave's packet is not dispersive and so the mean values of $\hat{\mathbf{X}}$ and $\hat{\mathbf{P}}$ evolve classically: during the evolution the state takes only a phase, preserving the previous probability distribution's function.

Coherent states can be also expressed in Fock's basis. This basis is composed of the eigenstates $\{|n\rangle\}$ of number operators $\hat{\mathbf{N}}$, so they represent states at different levels of energy for the harmonic oscillator:

$$\hat{\mathbf{N}}\,|n\rangle = n\,|n\rangle \tag{1.2.9}$$

in which n represent the sum of the eigenvalues of each dimension number's operators.

A coherent state is a superposition of such states combined with different coefficients c_n:

$$|\beta\rangle = \sum_{n=0}^{\infty} c_n\,|n\rangle = \sum_{n=0}^{\infty} \langle n|\beta\rangle\,|n\rangle\,. \tag{1.2.10}$$

These coefficients can be calculated using normalization and so it is possible to express the state as:

$$|\beta\rangle = e^{-\frac{|\beta|^2}{2}} \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}}\,|n\rangle \tag{1.2.11}$$

From this expression, we can derive that coherent states are not orthogonal and so they do not form a basis for the harmonic oscillator's Hilbert space. Indeed, we can calculate scalar product between two different coherent states such as $|\beta\rangle$ and $|\alpha\rangle$ and the result is:

$$\langle\beta|\alpha\rangle = e^{-\frac{|\beta|^2}{2}} e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\beta^*)^n}{\sqrt{n!}} \frac{(\alpha)^m}{\sqrt{m!}}\,\langle n|m\rangle \tag{1.2.12}$$

which we can also rewrite as:

$$\langle\beta|\alpha\rangle = e^{-\frac{|\beta|^2+|\alpha|^2}{2}} e^{\beta^*\alpha} = e^{\frac{\beta^*\alpha-\beta\alpha^*}{2}} e^{\frac{|\alpha-\beta|^2}{2}} \tag{1.2.13}$$

in which the first term is just a phase so it is negligible.

1.2.2 Entangled cat-states

Using coherent states we can define the equivalent of the isotropic class of states for the bipartite system of a qubit and a cavity.

Concerning the harmonic oscillator, we have to choose two orthogonal states analogous to qubit's $|0\rangle$ and $|1\rangle$: a possibility is to choose two coherent states, that are often called cat-states [11], characterized by a real value of β, one placed on the positive x semi-axis and the other on the negative one. They can be named $|\beta\rangle$ and $|-\beta\rangle$.

As shown in Sec.1.2.1 two generic coherent states are not orthogonal and, in this case, Eq.(1.2.13) becomes:

$$\langle \beta | -\beta \rangle = e^{-|\beta|^2}. \tag{1.2.14}$$

If we calculate the probability transition between $|\beta\rangle$ and $|-\beta\rangle$ it results:

$$|\langle \beta | -\beta \rangle|^2 = e^{-2|\beta|^2} \tag{1.2.15}$$

so it is easy to understand that the probability of preparing a state $|\beta\rangle$ but finding it changed into $|-\beta\rangle$ decreases exponentially with the value of β.

To have two orthogonal states we should choose $|\beta| \to \infty$, which experimentally means we have to prepare states with $|\beta|$ as large as possible. As the trend is exponential, however, we can be confident that the non-orthogonality would affect only the states very near the origin and that can be neglected also for a small value of β.

To verify it we plot the value of the transition probability varying the value of β. As it is shown in Fig.1.1 this comes to zero rapidly, in fact the squared overlap is gaussian in β with a standard deviation of $\frac{1}{\sqrt{2}}$.

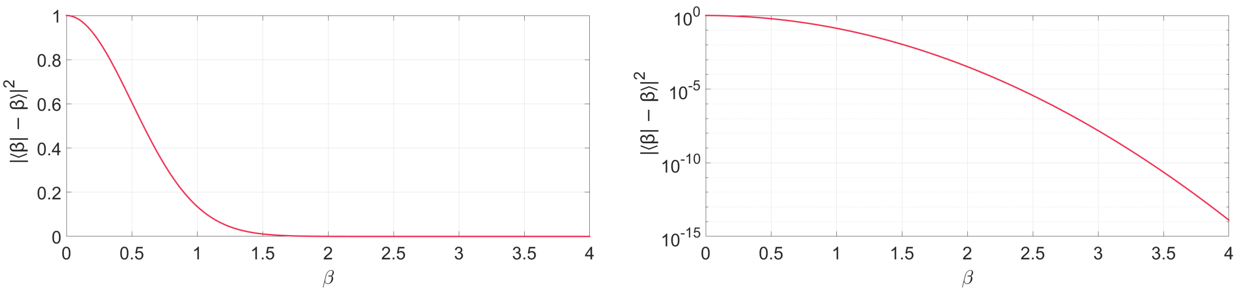

Figura 1.1: Linear and logarithmic squared overlap $|\langle \beta | -\beta \rangle|^2$ over β.

Assuming so that we can prepare two quasi-orthogonal coherent states $|\beta\rangle$ and $|-\beta\rangle$, it is possible to define an analogous of Bell's state for this bipartite system as:

$$|\Phi^+{}_{cat}\rangle = \frac{1}{2}(|0\rangle |\beta\rangle + |1\rangle |-\beta\rangle) \tag{1.2.16}$$

and the related class of states:

$$\rho_{cat}(p) = p\, |\Phi^+{}_{cat}\rangle \langle \Phi^+{}_{cat}| + \frac{(1-p)}{4} \hat{\mathbb{1}} \tag{1.2.17}$$

with $p \in [0, 1]$.

1.3 Quantitative Entanglement Witness for qubit and cavity

According to the features of coherent states, it is possible to assume that the QEW for the system of a qubit and a cavity has to be similar to one of two-qubits system.
We can rewrite, for this case, $\hat{\mathbf{W}}$ as:

$$\hat{\mathbf{W}} = \hat{\mathbb{I}} - \hat{\sigma}_x \otimes \hat{\mathbf{X}}_c - \hat{\sigma}_z \otimes \hat{\mathbf{Z}}_c + \hat{\sigma}_y \otimes \hat{\mathbf{Y}}_c \tag{1.3.1}$$

in which the observables $\hat{\mathbf{X}}_c$, $\hat{\mathbf{Y}}_c$, $\hat{\mathbf{Z}}_c$ are the equivalents of Pauli's matrices for the harmonic oscillator.
Starting from this hypothesis it is possible to deduce the structure of these new observables and verify their correctness by applying them to coherent states expressed on different bases.

1.3.1 $\hat{\mathbf{X}}_c$ observable

The observable $\hat{\mathbf{X}}_c$ has to reproduce the action of $\hat{\sigma}_x$ on the two orthogonal states chosen as the basis of the two-dimensional Hilbert space. The action of $\hat{\sigma}_x$ on the $\hat{\sigma}_z$ basis for the qubit, $|0\rangle$ and $|1\rangle$, is to flip the qubits:

$$\hat{\sigma}_x |0\rangle = |1\rangle \qquad\qquad \hat{\sigma}_x |1\rangle = |0\rangle \tag{1.3.2}$$

so the aim is to find a Hermitian operator that flips $|\beta\rangle$ in $|-\beta\rangle$ and vice versa.
Observing the quantum phase-space we can see that the state $|\beta\rangle$ changes into $|-\beta\rangle$ after a rotation of π and, considering the analogy with classical phase-space, this operation can be represented as the time evolution of $\frac{T}{2}$ of the coherent state expressed by Eq.(1.2.7):

$$\hat{\mathbf{U}}\left(\frac{T}{2}\right)|\beta\rangle = e^{-i\frac{\hbar\omega(\hat{\mathbf{N}}+1)\frac{T}{2}}{\hbar}} = e^{-i\pi(\hat{\mathbf{N}}+1)}|\beta\rangle . \tag{1.3.3}$$

It is clear that the factor $e^{-i\pi}$ impacts the result only by adding a phase so it is possible to ignore it to hypothesize a simpler form for the operator $\hat{\mathbf{X}}_c$, that so can be written as:

$$\hat{\mathbf{X}}_c = e^{-i\pi\hat{\mathbf{N}}}. \tag{1.3.4}$$

It is a unitary operator as:

$$\hat{\mathbf{X}}_c^\dagger \hat{\mathbf{X}}_c = e^{i\pi\hat{\mathbf{N}}} e^{-i\pi\hat{\mathbf{N}}} = \hat{\mathbb{I}}. \tag{1.3.5}$$

$\hat{\mathbf{X}}_c$ represents an observable as it is hermitian because it is an analytic function of $\hat{\mathbf{N}}$, that is an observable itself, but it is also easy to show it:

$$\hat{\mathbf{X}}_c^\dagger = e^{i\pi\hat{\mathbf{N}}} \tag{1.3.6}$$

$$\left(\hat{\mathbf{X}}_c^\dagger\right)^\dagger = \left(e^{i\pi\hat{\mathbf{N}}}\right)^\dagger = e^{-i\pi\hat{\mathbf{N}}} = \hat{\mathbf{X}}_c. \tag{1.3.7}$$

Moreover, because $\hat{\mathbf{X}}_c$ is hermitian and unitary, it is obvious that $\hat{\mathbf{X}}_c^2$ is the identity matrix.
The heuristic derivation of $\hat{\mathbf{X}}_c$ has to be justified by the application on coherent states to see if it acts in the same way as $\hat{\sigma}_x$, as we suppose.

1.3.2 Motivation of $\hat{\mathbf{X}}_c$ operator

To apply $\hat{\mathbf{X}}_c$ to coherent states it is useful to express them in Fock's basis, as already explained in Sec.1.2.1:

$$\hat{\mathbf{X}}_c \left|\beta\right\rangle = e^{-i\pi\hat{\mathbf{N}}} e^{-\frac{\beta^2}{2}} \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}} \left|n\right\rangle . \tag{1.3.8}$$

As $\{\left|n\right\rangle\}$ are eigenstates of number operator is possible to apply $\hat{\mathbf{X}}_c$ to each member of the sum:

$$\hat{\mathbf{X}}_c \left|\beta\right\rangle = e^{-\frac{\beta^2}{2}} \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}} e^{-i\pi n} \left|n\right\rangle \tag{1.3.9}$$

that can also be rewritten as:

$$\hat{\mathbf{X}}_c \left|\beta\right\rangle = e^{-\frac{\beta^2}{2}} \sum_{n=0}^{\infty} \frac{(-\beta)^n}{\sqrt{n!}} \left|n\right\rangle = \left|-\beta\right\rangle . \tag{1.3.10}$$

The same calculations can be carried on for the state $\left|-\beta\right\rangle$:

$$\hat{\mathbf{X}}_c \left|-\beta\right\rangle = e^{-i\pi\hat{\mathbf{N}}} e^{-\frac{\beta^2}{2}} \sum_{n=0}^{\infty} \frac{(-\beta)^n}{\sqrt{n!}} \left|n\right\rangle = e^{-\frac{\beta^2}{2}} \sum_{n=0}^{\infty} \frac{(-\beta)^n}{\sqrt{n!}} e^{-i\pi n} \left|n\right\rangle \tag{1.3.11}$$

$$\hat{\mathbf{X}}_c \left|-\beta\right\rangle = e^{-\frac{\beta^2}{2}} \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}} \left|n\right\rangle = \left|\beta\right\rangle . \tag{1.3.12}$$

So we have demonstrated that $\hat{\mathbf{X}}_c$ actually acts on coherent states as $\hat{\sigma}_x$.

Another way to analyze the action of this observable is to study how it acts on the real basis states.

We can consider the real basis as a discrete one because we assume to have an upper limit to energy, which is a realistic assumption in any experimental setting.

So this basis is constituted by a finite number of states $\{\left|q\right\rangle\}$ which are eigenstates of the position operator $\hat{\mathbf{Q}}$:

$$\hat{\mathbf{Q}} \left|q\right\rangle = q \left|q\right\rangle \tag{1.3.13}$$

with $q \in \mathbb{N}$.

So if we apply $\hat{\mathbf{X}}_c$ to this class of states the relation can be written as:

$$\hat{\mathbf{X}}_c \left|q\right\rangle = \frac{1}{q} \hat{\mathbf{X}}_c \hat{\mathbf{Q}} \left|q\right\rangle \tag{1.3.14}$$

that, thanks to the unitary of $\hat{\mathbf{X}}_c$, can be rewritten as:

$$\hat{\mathbf{X}}_c \left|q\right\rangle = \frac{1}{q} \hat{\mathbf{X}}_c \hat{\mathbf{Q}} \hat{\mathbf{X}}_c^{\dagger} \hat{\mathbf{X}}_c \left|q\right\rangle . \tag{1.3.15}$$

We can demonstrate that

$$\hat{\mathbf{X}}_c \hat{\mathbf{Q}} \hat{\mathbf{X}}_c^{\dagger} = -\hat{\mathbf{Q}} \tag{1.3.16}$$

using Hadamard's equation:

$$e^{-i\pi\hat{\mathbf{N}}} \hat{\mathbf{Q}} e^{i\pi\hat{\mathbf{N}}} = \sum_{n=0}^{\infty} [...[[\hat{\mathbf{Q}}, i\pi\hat{\mathbf{N}}], i\pi\hat{\mathbf{N}}]..., i\pi\hat{\mathbf{N}}]. \tag{1.3.17}$$

In fact, expressing $\hat{\mathbf{Q}}$ as $k(\hat{\mathbf{a}}^\dagger + \hat{\mathbf{a}})$, in which k is a constant, for $n = 0$ and $n = 1$ the Eq.(1.3.17) becomes:

$$[k(\hat{\mathbf{a}}^\dagger + \hat{\mathbf{a}}), i\pi\,\hat{\mathbf{N}}] = ki\pi([\hat{\mathbf{a}}^\dagger, \hat{\mathbf{N}}] + [\hat{\mathbf{a}}, \hat{\mathbf{N}}]) = ki\pi(-\,\hat{\mathbf{a}}^\dagger + \hat{\mathbf{a}}) \tag{1.3.18}$$

$$k(i\pi)^2[(-\,\hat{\mathbf{a}}^\dagger + \hat{\mathbf{a}}), \hat{\mathbf{N}}] = k(i\pi)^2([-\,\hat{\mathbf{a}}^\dagger, \hat{\mathbf{N}}] + [\hat{\mathbf{a}}, \hat{\mathbf{N}}]) = k(i\pi)^2(\hat{\mathbf{a}}^\dagger + \hat{\mathbf{a}}) \tag{1.3.19}$$

and so we can deduce that the general expression for Eq.(1.3.17) can be written as:

$$\hat{\mathbf{X}}_c\,\hat{\mathbf{Q}}\,\hat{\mathbf{X}}_c^\dagger = k\left(\sum_{n=0}^{\infty}\frac{(-i\pi)^n}{n!}\,\hat{\mathbf{a}}^\dagger + \sum_{n=0}^{\infty}\frac{(i\pi)^n}{n!}\,\hat{\mathbf{a}}\right) = k(e^{-i\pi}\,\hat{\mathbf{a}}^\dagger + e^{i\pi}\,\hat{\mathbf{a}}) = -k(\hat{\mathbf{a}}^\dagger + \hat{\mathbf{a}}) = -\,\hat{\mathbf{Q}}. \tag{1.3.20}$$

So Eq.(1.3.15) becomes:

$$\hat{\mathbf{X}}_c\,|q\rangle = -\frac{1}{q}\,\hat{\mathbf{Q}}\,\hat{\mathbf{X}}_c\,|q\rangle \tag{1.3.21}$$

and it is demonstrated that $\hat{\mathbf{X}}_c\,|q\rangle$ is equal to $|-q\rangle$ because Eq.(1.3.21) can be rewritten as:

$$\hat{\mathbf{Q}}\,\hat{\mathbf{X}}_c\,|q\rangle = -q\,\hat{\mathbf{X}}_c\,|q\rangle\,. \tag{1.3.22}$$

1.3.3 $\hat{\mathbf{Z}}_c$ observable

Working on the $\hat{\sigma}_z$ model we aim to find an observable $\hat{\mathbf{Z}}_c$ that associates the eigenvalues ± 1 respectively to its two eigenstates $|\beta\rangle$ and $|-\beta\rangle$.
In fact, $\hat{\sigma}_z$ acts in the 2-qubits space as:

$$\hat{\sigma}_z\,|0\rangle = |0\rangle \qquad\qquad \hat{\sigma}_z\,|1\rangle = -\,|1\rangle \tag{1.3.23}$$

so we look for an observable able to change sign only to eigenstates with negative values of β.
The idea so is to adapt the well-known position operator $\hat{\mathbf{Q}}$, which can be expressed in terms of creation and annihilation operators as:

$$\hat{\mathbf{Q}} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{\mathbf{a}}^\dagger + \hat{\mathbf{a}}). \tag{1.3.24}$$

This observable acts on coherent states as:

$$\hat{\mathbf{Q}}\,|\beta\rangle = \sqrt{\frac{\hbar}{2m\omega}}\beta\,|\beta\rangle \tag{1.3.25}$$

$$\hat{\mathbf{Q}}\,|-\beta\rangle = -\sqrt{\frac{\hbar}{2m\omega}}\beta\,|-\beta\rangle\,. \tag{1.3.26}$$

We are interested only in the sign of these eigenvalues so is proper to define the searched observable $\hat{\mathbf{Z}}_c$ as:

$$\hat{\mathbf{Z}}_c = \mathrm{sgn}(\hat{\mathbf{a}}^\dagger + \hat{\mathbf{a}}) \tag{1.3.27}$$

that acts on coherent states, real and with large values of β, as wanted:

$$\hat{\mathbf{Z}}_c\,|\beta\rangle = |\beta\rangle \qquad\qquad \hat{\mathbf{Z}}_c\,|-\beta\rangle = -\,|-\beta\rangle\,. \tag{1.3.28}$$

As seen for the previous observable, also $\hat{\mathbf{Z}}_c^2$ is the identity, simply because it is defined as the squared sign function that does not add any sign to the state.

The operator $(\hat{\mathbf{a}}^\dagger + \hat{\mathbf{a}})$ is Hermitian, actually as the position operator:

$$\left(\hat{\mathbf{a}}^\dagger + \hat{\mathbf{a}}\right)^\dagger = \left((\hat{\mathbf{a}}^\dagger)^\dagger + (\hat{\mathbf{a}})^\dagger\right) = \hat{\mathbf{a}} + \hat{\mathbf{a}}^\dagger \tag{1.3.29}$$

but it is not truly correct to state that also $\text{sgn}(\hat{\mathbf{a}}^\dagger + \hat{\mathbf{a}})$ is hermitian because sign is not an analytic function.

The sign function is not well defined near the origin and, for this reason, it does not admit Taylor's series expansion. This problem can be solved on one hand considering only states with $|\beta| \to \infty$, which is also a condition for orthogonality, as seen in Sec.1.2.1, or, on the other hand, using another function that approximates the sign, as we see afterwards.

1.3.4 Motivation of $\hat{\mathbf{Z}}_c$ operator

To better investigate how this operator acts on coherent states we can express it in two different bases: the real basis and Fock's basis.

To use the first one we have to rewrite $\hat{\mathbf{Z}}_c$ in function of $\hat{\mathbf{Q}}$ and is enough to remember how we can express annihilation and creation operators in function of $\hat{\mathbf{Q}}$ and $\hat{\mathbf{P}}$:

$$\hat{\mathbf{a}} = \frac{\hat{\mathbf{Q}} - i\,\hat{\mathbf{P}}}{\sqrt{2}} \qquad \hat{\mathbf{a}}^\dagger = \frac{\hat{\mathbf{Q}} + i\,\hat{\mathbf{P}}}{\sqrt{2}}. \tag{1.3.30}$$

So we obtain:

$$\hat{\mathbf{Z}}_c = \text{sgn}(\frac{\hat{\mathbf{Q}} + i\,\hat{\mathbf{P}} + \hat{\mathbf{Q}} - i\,\hat{\mathbf{P}}}{\sqrt{2}}) = \text{sgn}(\sqrt{2}\,\hat{\mathbf{Q}}). \tag{1.3.31}$$

The states $|\beta\rangle$ and $|-\beta\rangle$, as they are on the real axis, can be considered themselves eigenstates of $\hat{\mathbf{Q}}$ with respectively eigenvalues $\pm\beta$. Recalling the definition of β, Eq.(1.2.6), the real part of the value is represented by q_β and we can assume, for the reason that coherent states minimize the uncertainty relation, that the mean value of $\hat{\mathbf{Q}}$ calculated for the state represent effectively the position of the state.

So it is possible to apply the latest form of $\hat{\mathbf{Z}}_c$ to a coherent state and, as expected, the result is:

$$\hat{\mathbf{Z}}_c\,|\beta\rangle = \text{sgn}(\sqrt{2}\,\hat{\mathbf{Q}})\,|\beta\rangle = |\beta\rangle \tag{1.3.32}$$

$$\hat{\mathbf{Z}}_c\,|-\beta\rangle = \text{sgn}(\sqrt{2}\,\hat{\mathbf{Q}})\,|-\beta\rangle = -\,|-\beta\rangle\,. \tag{1.3.33}$$

It is also worth trying to express $\hat{\mathbf{Z}}_c$ in Fock's basis and apply it to coherent states.

To do this we consider the action of annihilation and creation operators on Fock's states:

$$\hat{\mathbf{a}}\,|n\rangle = \sqrt{n}\,|n-1\rangle \qquad \hat{\mathbf{a}}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,. \tag{1.3.34}$$

Even if $|n\rangle$ are not eigenstates of these operators it would be possible to calculate the result of the application of $(\hat{\mathbf{a}}^\dagger + \hat{\mathbf{a}})$ on them, which indeed is the position operator minus a multiplicative factor.

The problem regards, as said before, the sign function: it can not be expanded in Taylor's series and, for this reason, it is not possible to apply a term of this operator for each state $|n\rangle$ in the sum.

So if we want to apply $\hat{\mathbf{Z}}_c$ to a coherent state expressed in Fock's basis it is necessary to change the function sign with another function that returns as much as possible the same results of the previous one and that can be expanded in Taylor's series.

The idea is to use the hyperbolic function $\tanh(Gx)$, in which G is a constant, which approximates the sign function according to these relations:

$$\tanh(0) = 0 \qquad \lim_{x\to\pm\infty} \tanh(Gx) = \pm 1. \tag{1.3.35}$$

Its Taylor's series has two different expressions, for the positive and negative real parts of x:

$$\tanh(x) = \begin{cases} 1 - 2\sum_{k=0}^{\infty}(-1)^k e^{-2G(k+1)x} & Re(x) > 0 \\ -1 + 2\sum_{k=0}^{\infty}(-1)^k e^{2G(k+1)x} & Re(x) < 0 \end{cases} \tag{1.3.36}$$

If we substitute x with $(\hat{\mathbf{a}}^\dagger + \hat{\mathbf{a}})$ and apply these expressions to coherent states it results for $|\beta\rangle$:

$$\tanh(\hat{\mathbf{a}}^\dagger + \hat{\mathbf{a}})\,|\beta\rangle = \left(1 - 2\sum_{k=0}^{\infty}(-1)^k e^{-2G(k+1)(\hat{\mathbf{a}}^\dagger+\hat{\mathbf{a}})}\right) e^{-\frac{\beta^2}{2}} \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}}\,|n\rangle \tag{1.3.37}$$

$$\tanh(\hat{\mathbf{a}}^\dagger + \hat{\mathbf{a}})\,|\beta\rangle = |\beta\rangle - 2\sum_{k=0}^{\infty}(-1)^k e^{-2G(k+1)(\hat{\mathbf{a}}^\dagger+\hat{\mathbf{a}})}\,|\beta\rangle = |\beta\rangle - 2\sum_{k=0}^{\infty}(-1)^k e^{-2G(k+1)\beta}\,|\beta\rangle \tag{1.3.38}$$

and for $|-\beta\rangle$:

$$\tanh(\hat{\mathbf{a}}^\dagger + \hat{\mathbf{a}})\,|-\beta\rangle = \left(-1 + 2\sum_{k=0}^{\infty}(-1)^k e^{2G(k+1)(\hat{\mathbf{a}}^\dagger+\hat{\mathbf{a}})}\right) e^{-\frac{\beta^2}{2}} \sum_{n=0}^{\infty} \frac{-\beta^n}{\sqrt{n!}}\,|n\rangle \tag{1.3.39}$$

$$\tanh(\hat{\mathbf{a}}^\dagger + \hat{\mathbf{a}})\,|-\beta\rangle = -\,|-\beta\rangle + 2\sum_{k=0}^{\infty}(-1)^k e^{2G(k+1)(\hat{\mathbf{a}}^\dagger+\hat{\mathbf{a}})}\,|-\beta\rangle = -\,|-\beta\rangle + 2\sum_{k=0}^{\infty}(-1)^k e^{2G(k+1)\beta}\,|-\beta\rangle\,. \tag{1.3.40}$$

These relations show that, at the first order, we obtain the same result as using the function sign but it is worth studying whether the other orders of the expansion are negligible or not and in which situations.

As we see the terms of the sum depend exponentially on β so it is proper to suppose that this contribution has to be negligible for a large value of β, in which cases so $\tanh(Gx)$ represent a fair substitution of the sign function, but probably not for values of β near the origin.

Moreover, by increasing G we expect to obtain a function that better approximates $\mathrm{sgn}(x)$. To test this supposition it is possible varying β value to create a simulation of the trend of the state's correction β_c due to the application of $\tanh(Gx)$. To compute the sum, as it should be of infinite terms, we choose a cut-off at $N = 500$.

As we can see in Fig.1.2 the correction goes to zero fast and the β value at which the correction is negligible depends on the constant used so can be reduced as small as wanted. Moreover, remembering the condition for orthogonality tested in Fig.1.1, it seems that the $\tanh(Gx)$'s approximation does not work so well in the same region of non-orthogonality.

So to solve this problem it is sufficient to use, as said before, a large value of β and, in this case, the hyperbolic tangent provides a great approximation.

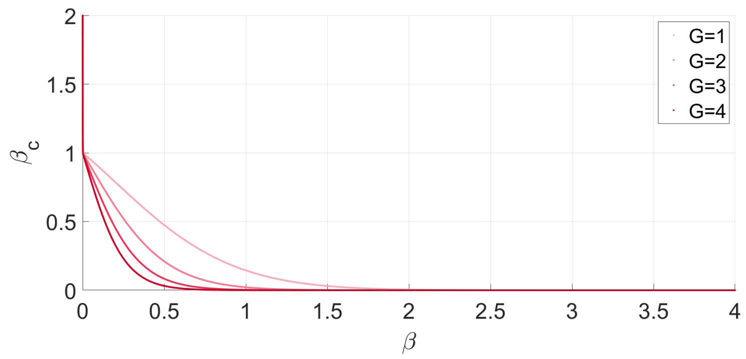

Figura 1.2: Correction to $\tanh(\hat{\mathbf{a}}^\dagger + \hat{\mathbf{a}}) |\beta\rangle$ over β.

1.3.5 $\hat{\mathbf{Y}}_c$ observable and relation with Pauli's Matrices

To find an expression also for the operator $\hat{\mathbf{Y}}_c$ we analyze the action of $\hat{\sigma}_y$ on $|0\rangle$ and $|1\rangle$:

$$\hat{\sigma}_y |0\rangle = i |1\rangle \qquad \hat{\sigma}_y |1\rangle = -i |0\rangle . \tag{1.3.41}$$

As it is shown the action consists in flipping the qubit but also adding a unitary phase to $|1\rangle$, so we can deduce that the action of $\hat{\mathbf{Y}}_c$ has to be a combination of the action of $\hat{\mathbf{X}}_c$ and $\hat{\mathbf{Y}}_c$.

More precisely we know that Pauli's matrices are related to each other through the relations:

$$[\hat{\sigma}_i, \hat{\sigma}_j] = 2i\epsilon_{ijk}\,\hat{\sigma}_k \tag{1.3.42}$$

$$\{\hat{\sigma}_i, \hat{\sigma}_j\} = 2\delta_{ijk}\,\hat{\mathbb{I}} \tag{1.3.43}$$

in which $[,]$ is the commutator and $\{,\}$ is the anticommutator.

We can suppose that these relations should be respected also by the new observables and so using Eq.(1.3.42) it is possible to obtain the expression of $\hat{\mathbf{Y}}_c$ in function of $\hat{\mathbf{X}}_c$ and $\hat{\mathbf{Z}}_c$:

$$\hat{\mathbf{Y}}_c = \frac{i[\hat{\mathbf{X}}_c, \hat{\mathbf{Z}}_c]}{2} = \frac{i[e^{i\pi\hat{\mathbf{N}}}, \mathrm{sgn}(\hat{\mathbf{a}}^\dagger + \hat{\mathbf{a}})]}{2}. \tag{1.3.44}$$

If we try to apply this operator to coherent states it shows the expected results:

for $|\beta\rangle$

$$\begin{aligned}\hat{\mathbf{Y}}_c |\beta\rangle &= \frac{i}{2}\left(e^{i\pi\hat{\mathbf{N}}}\,\mathrm{sgn}(\hat{\mathbf{a}}^\dagger + \hat{\mathbf{a}}) - \mathrm{sgn}(\hat{\mathbf{a}}^\dagger + \hat{\mathbf{a}})e^{i\pi\hat{\mathbf{N}}}\right)|\beta\rangle = \\ &= \frac{i}{2}\left(e^{i\pi\hat{\mathbf{N}}}|\beta\rangle - \mathrm{sgn}(\hat{\mathbf{a}}^\dagger + \hat{\mathbf{a}})|-\beta\rangle\right) = \frac{i}{2}\left(|-\beta\rangle + |-\beta\rangle\right) = i|-\beta\rangle\end{aligned} \tag{1.3.45}$$

and for $|-\beta\rangle$

$$\begin{aligned}\hat{\mathbf{Y}}_c |-\beta\rangle &= \frac{i}{2}\left(e^{i\pi\hat{\mathbf{N}}}\,\mathrm{sgn}(\hat{\mathbf{a}}^\dagger + \hat{\mathbf{a}}) - \mathrm{sgn}(\hat{\mathbf{a}}^\dagger + \hat{\mathbf{a}})e^{i\pi\hat{\mathbf{N}}}\right)|-\beta\rangle = \\ &= \frac{i}{2}\left(-e^{i\pi\hat{\mathbf{N}}}|-\beta\rangle - \mathrm{sgn}(\hat{\mathbf{a}}^\dagger + \hat{\mathbf{a}})|\beta\rangle\right) = \frac{i}{2}\left(-|\beta\rangle - |\beta\rangle\right) = -i|\beta\rangle .\end{aligned} \tag{1.3.46}$$

As the new operators act like Pauli's matrices it is interesting to verify if they respect also the other relations of SU(2) algebra.

First of all, they have to respect the relation:

$$\hat{\sigma}_i^2 = \hat{\mathbb{I}}. \tag{1.3.47}$$

As shown in sections 1.3.2 and 1.3.4, it is respected by $\hat{\mathbf{X}}_c$ and $\hat{\mathbf{Z}}_c$ and, because of SU(2) algebra, the relation is valid also for the operator $\hat{\mathbf{Y}}_c$.

Moreover, Eq.(1.3.42) and Eq.(1.3.43) can be demonstrated applying the definition to $|\beta\rangle$ and $|-\beta\rangle$, for example for $\hat{\mathbf{Z}}_c$:

$$\{\hat{\mathbf{Z}}_c, \hat{\mathbf{Z}}_c\}\,|\beta\rangle = (\hat{\mathbf{Z}}_c\,\hat{\mathbf{Z}}_c + \hat{\mathbf{Z}}_c\,\hat{\mathbf{Z}}_c)\,|\beta\rangle = 2\,|\beta\rangle = 2\,\hat{\mathbb{I}}\,|\beta\rangle \tag{1.3.48}$$

$$\{\hat{\mathbf{X}}_c, \hat{\mathbf{Z}}_c\}\,|\beta\rangle = (\hat{\mathbf{X}}_c\,\hat{\mathbf{Z}}_c + \hat{\mathbf{Z}}_c\,\hat{\mathbf{X}}_c)\,|\beta\rangle = (\hat{\mathbf{X}}_c\,|\beta\rangle + \hat{\mathbf{Z}}_c\,|-\beta\rangle) = (|-\beta\rangle - |-\beta\rangle) = 0. \tag{1.3.49}$$

It is possible to demonstrate in the same way that all the other combinations of operators respect these relations.

The real motivation of this analogy lies in the fact that new operators, as they act as Pauli's matrices on the basis $|\beta\rangle$ and $|-\beta\rangle$, have, in this basis, the same matrix form of $\hat{\sigma}_i$:

$$\hat{\mathbf{X}}_c = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{\mathbf{Z}}_c = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \hat{\mathbf{Y}}_c = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{1.3.50}$$

From this evidence, we could deduce that the Eq.(1.3.1) can be connected to Eq.(1.1.7) if the harmonic oscillator's observables are expressed in the basis of the two coherent states $|\beta\rangle$ and $|-\beta\rangle$. Anyway, as we have seen, this is not truly a basis because of the lack of perfect orthogonality so in the following section we will better justify in what sense this analogy is possible.

1.4 Witness decomposition

As we have shown in Sections 1.3.2, 1.3.4, the observables $\hat{\mathbf{X}}_c$ and $\hat{\mathbf{Z}}_c$ can both be expressed in the real basis and we have studied how they act on the eigenstates of position $\{|q\rangle\}$ as:

$$\hat{\mathbf{X}}_c\,|\pm q\rangle = |\mp q\rangle \qquad\qquad \hat{\mathbf{Z}}_c\,|\pm q\rangle = \pm\,|q\rangle\,. \tag{1.4.1}$$

From these relations we can deduce the one for $\hat{\mathbf{Y}}_c$:

$$\hat{\mathbf{Y}}_c\,|\pm q\rangle = \pm i\,|\mp q\rangle\,. \tag{1.4.2}$$

If we express these operators in their matrix form in the real basis, ordering the eigenvalues with increasing module (0,$\pm q$, $\pm 2q$, $\pm 3q$...), we obtain:

$$\hat{\mathbf{X}}_c = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & ... \end{pmatrix} \quad \hat{\mathbf{Z}}_c = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & ... \end{pmatrix} \quad \hat{\mathbf{Y}}_c = i\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & ... \end{pmatrix}. \tag{1.4.3}$$

As we can see they are block matrices and, if we exclude the matrix element $\langle 0|\,...\,|0\rangle$, we can notice that the blocks on the diagonal are Pauli's matrices.

So we can rewrite these operators using the tensor product as:

$$\hat{\mathbf{X}}_c = \hat{\mathbb{I}} \otimes \hat{\sigma}_x \qquad \hat{\mathbf{Z}}_c = \hat{\mathbb{I}} \otimes \hat{\sigma}_z \qquad \hat{\mathbf{Y}}_c = \hat{\mathbb{I}} \otimes \hat{\sigma}_y. \tag{1.4.4}$$

This decomposition allows us to consider the action of these observables on the states of the harmonic oscillator similar to the one of Pauli's matrices on the qubit: it underlines that these observables act on the cavity system as they could isolate a subsystem which responds to the entanglement detection like a qubit. So it seems that the entanglement information detected by the witness is only due to this qubit "inscribed" in the harmonic oscillator. This is a significant achievement as we can expect to have similar bounding functions also in the case of the system qubit-cavity. Otherwise, we should not forget that this decomposition keeps not track of the matrix element $\langle 0| \ldots |0\rangle$ and that it is not correct to use 2-qubits witness to test this type of system.

Capitolo 2

Application of the witness

In the previous chapter, we have found the expression for the witness of a bipartite system of a harmonic oscillator and a qubit. In this second chapter, we want to apply this result to the entangled cat states of the system and see whether it has the same behaviour as the two qubits' system.

2.1 Application of the witness on entangled cat states

Recalling the definition of entangled cat states, Eq.(1.2.17), we can apply the new witness, Eq. (1.3.1), to see for which values of p these states are entangled or not.

To do it we have to use the condition in Eq.(1.1.1) and so it is convenient to express $\rho_{cat}(p)$ and $\hat{\mathbf{W}}$ in their matrix forms.

For cat states we have:

$$\rho_{cat}(p) = \begin{pmatrix} \frac{1+p}{4} & 0 & 0 & \frac{p}{2} \\ 0 & \frac{1-p}{4} & 0 & 0 \\ 0 & 0 & \frac{1-p}{4} & 0 \\ \frac{p}{2} & 0 & 0 & \frac{1+p}{4} \end{pmatrix}. \tag{2.1.1}$$

For the different contributions to the witness we have:

$$\hat{\sigma}_x \otimes \hat{\mathbf{X}}_c = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \hat{\sigma}_z \otimes \hat{\mathbf{Z}}_c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \hat{\sigma}_y \otimes \hat{\mathbf{Y}}_c = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{2.1.2}$$

and so, according to Eq.(1.1.7), $\hat{\mathbf{W}}$ is:

$$\hat{\mathbf{W}} = \begin{pmatrix} 0 & 0 & 0 & -2 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix}. \tag{2.1.3}$$

If we apply the witness to the state and calculate the trace we obtain:

$$\hat{\mathbf{W}}\rho_{cat}(p) = -\begin{pmatrix} p & 0 & 0 & \frac{p+1}{2} \\ 0 & \frac{p-1}{2} & 0 & 0 \\ 0 & 0 & \frac{p-1}{2} & 0 \\ \frac{p+1}{2} & 0 & 0 & p \end{pmatrix} \tag{2.1.4}$$

$$Tr[\hat{\mathbf{W}}\rho_{cat}(p)] = 1 - 3p \tag{2.1.5}$$

so using Eq.(1.1.1) we can affirm that the state is entangled only for $p > \frac{1}{3}$.

We can use this result to obtain the bounding function of two different types of entanglement measure: the entanglement negativity $\mathcal{E}_{\text{Neg}}$ and the entanglement of formation $\mathcal{E}_{\text{oF}}$.

2.2 Entanglement negativity

The entanglement negativity is an entanglement measure defined as:

$$\mathcal{E}_{\text{Neg}}(\rho) = -\sum_{k=0}^{D} \min[eig(\rho^{Tp}), 0] \tag{2.2.1}$$

in which ρ^{Tp} refers to the partial transpose matrix of the state, D is the dimension of the state and *eig* stays for the eigenvalues of that matrix.

It is known, from Peres' criterion, that a necessary condition for separability is that the partial transpose matrix has to be a density operator so that it has no negative eigenvalues [5].

This is called also "PPT condition" and for a system of dimensions $2\otimes 2$ or $2\otimes 3$ it is a necessary and sufficient condition [12]: in fact also in Eq.(2.2.1) if all the eigenvalues are non-negative the state is not entangled.

So if we apply the definition in Eq.(2.2.1) to ρ_{cat} we obtain:

$$\rho_{cat}^{Tp}(p) = \begin{pmatrix} \frac{1+p}{4} & 0 & 0 & 0 \\ 0 & \frac{1-p}{4} & \frac{p}{2} & 0 \\ 0 & \frac{p}{2} & \frac{1-p}{4} & 0 \\ 0 & 0 & 0 & \frac{1+p}{4} \end{pmatrix} \tag{2.2.2}$$

and the eigenvalues result to be:

$$\lambda_{1,2,3} = \frac{1}{4}(1+p) \qquad\qquad \lambda_4 = \frac{1}{4}(1-3p). \tag{2.2.3}$$

As p is a probability, $\lambda_{1,2,3}$ can not be negative while λ_4 is negative if $p > \frac{1}{3}$, that is the same condition for entanglement of Eq.(2.1.5). So only λ_4 with $p > \frac{1}{3}$ contributes to the entanglement measure and Eq.(2.2.1) becomes:

$$\mathcal{E}_{\text{Neg}}(\rho_{cat}) = \frac{3p-1}{4}. \tag{2.2.4}$$

Because we expect for $p = 1$, the case of Bell's state, to have a maximum for the entanglement equal to the unit, we can add a multiplying factor 2 to the definition of entanglement negativity: it does not change the meaning of the measure, it is only a scale factor.

We so obtain:

$$\mathcal{E}_{\text{Neg}}(\rho_{cat}) = \frac{3p-1}{2}. \tag{2.2.5}$$

Now we want to link this measure with the witness' one and to do that we have to find a bounding function b_{Neg} that respects the equation:

$$\mathcal{E}_{\text{Neg}}(\rho_{cat}) = b_{\text{Neg}}(Tr[\hat{\mathbf{W}}\rho_{cat}(p)]). \tag{2.2.6}$$

As from Eq.(2.1.5) and Eq.(2.2.5) we know that it becomes:

$$\frac{3p-1}{2} = b_{\text{Neg}}(1-3p) \tag{2.2.7}$$

we can deduce the bounding function for the entanglement negativity:

$$b_{\text{Neg}}(x) = -\frac{x}{2} \tag{2.2.8}$$

in which $x = 1 - 3p$, for $p > \frac{1}{3}$.
If $p < \frac{1}{3}$ Eq.(2.1.5) and also Eq.(2.2.5) conditions impose null entanglement, so the complete buonding function has to be:

$$b_{\text{Neg}}(x) = \begin{cases} 0 & x > 1 \\ -\frac{x}{2} & x \leq 1 \end{cases} . \tag{2.2.9}$$

We have found so that the bounding function for the bipartite system of a qubit and a cavity is, for the entanglement negativity measure, equal to the one of the system of two qubits, in Eq.(1.1.8).

2.3 Entanglement of formation

The entanglement of formation is an entanglement measure that is based on the fact that any mixed states can be created by different combinations of projectors on pure states, that can be themselves entangled or not. A mixed state so can be decomposed into many different families of projectors weighted with some normalized coefficients and for each family we can write it as:

$$\rho_{AB} = \sum_{j=0}^{N} p_j \left|\phi_j\right\rangle \left\langle\phi_j\right| \tag{2.3.1}$$

in which $\{\left|\phi_j\right\rangle \left\langle\phi_j\right|\}$ is a family of projectors on pure states that are combined with different classical probabilities p_j.
For each possible family we can estimate the entanglement of the mixed state as the sum of the Von Neumann's entropy $S(\rho_{Aj})$ of each projector, weighted with p_j:

$$\mathcal{E}(\rho_{AB}) = \sum_{j=0}^{N} p_j S(\rho_{Aj}). \tag{2.3.2}$$

Von Neumann's entropy is used as an entanglement estimation for pure states and is defined as:

$$\mathcal{E}(\rho_{AB}) = S(\rho_{AB}) = -Tr[\rho_A \log_2(\rho_A)] \tag{2.3.3}$$

in which ρ_A is the partial trace of the state.
Considering the entanglement value of all the possible families we can calculate the entanglement of formation, which is defined as the minimum on all these configurations that describe our mixed state [1, 13]:

$$\mathcal{E}_{\text{oF}}(\rho) = \min \sum_{j=0}^{N} p_j S(\rho_{Aj}) \tag{2.3.4}$$

This calculation is a minimization problem and so it is hard to find a solution in general cases.
Fortunately, for bipartite systems of $2 \otimes 2$ it results to be easier: it can be computed through the so-called concurrence (C) [1].

This quantity C is defined as:

$$C(\rho_{AB}) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4) \tag{2.3.5}$$

in which λ_i, that are decreasingly ordered, are the square root of the eigenvalues α_i of a matrix R that is defined as:

$$R = \rho_{AB}\tilde{\rho}_{AB} \tag{2.3.6}$$

in which $\tilde{\rho}_{AB}$ is:

$$\tilde{\rho}_{AB} = \rho_{AB}(\hat{\sigma}_y \otimes \hat{\sigma}_y)\rho_{AB}(\hat{\sigma}_y \otimes \hat{\sigma}_y). \tag{2.3.7}$$

So using these elements the expression of the entanglement of formation can be written as:

$$\mathcal{E}_{\text{oF}}(t) = h_2(t) = -t\log_2(t) - (1-t)\log_2(1-t) \tag{2.3.8}$$

in which t is a variable that depends on the concurrence as:

$$t = \frac{1 + \sqrt{1 - C(\rho_{AB})^2}}{2}. \tag{2.3.9}$$

We aim to obtain this entanglement measure for a generic cat state so we start calculating R matrix for the states defined by Eq.(2.1.1).

We can see that $\tilde{\rho}_{cat}$ is equal to ρ_{cat}, so R becomes:

$$R(\rho_{cat}) = \rho_{cat}^2 = \frac{1}{16}\begin{pmatrix} 4p^2 + (p+1)^2 & 0 & 0 & 4p(p+1) \\ 0 & (1-p)^2 & 0 & 0 \\ 0 & 0 & (1-p)^2 & 0 \\ 4p(p+1) & 0 & 0 & 4p^2 + (p+1)^2 \end{pmatrix}. \tag{2.3.10}$$

The eigenvalues of this matrix are:

$$\alpha_{1,2,3} = \frac{(p-1)^2}{4} \qquad \alpha_4 = \frac{(3p+1)^2}{4}. \tag{2.3.11}$$

To calculate the concurrence we have to consider the square roots of these eigenvalues, which we call λ_i. We can notice that λ_4 is always positive, because of p definition, while if we calculate $\lambda_{1,2,3}$ we obtain:

$$\lambda_{1,2,3} = \sqrt{\frac{(p-1)^2}{4}} = \frac{|p-1|}{2} \tag{2.3.12}$$

so

$$\lambda_{1,2,3} = \begin{cases} \frac{p-1}{2} & p > 1 \\ \frac{1-p}{2} & p < 1 \end{cases}. \tag{2.3.13}$$

Because p can not be larger than the unit we have to choose the second definition and from that it is possible to calculate that is always true that $\lambda_4 > \lambda_{1,2,3}$ and apply the Eq.(2.3.5).

So the concurrence results:

$$C(\rho_{cat}) = \begin{cases} \frac{3p-1}{2} & p > \frac{1}{3} \\ 0 & p < \frac{1}{3} \end{cases}. \tag{2.3.14}$$

We can so apply the first part of this result in Eq.(2.3.9) and Eq.(2.3.8), which after some calculations can be expressed as:

$$t(\rho_{cat}) = \frac{1+\sqrt{1-\left(\frac{3p-1}{2}\right)^2}}{2} = \frac{2+\sqrt{4-(3p-1)^2}}{4} \tag{2.3.15}$$

$$\mathcal{E}_{\text{oF}}(\rho_{cat}) = h_2\left(\frac{1+\sqrt{1-\left(\frac{3p-1}{2}\right)^2}}{2}\right). \tag{2.3.16}$$

Finally, we can deduce the bounding function for the entanglement of formation, using Eq.(2.1.5):

$$h_2\left(\frac{1+\sqrt{1-\left(\frac{3p-1}{2}\right)^2}}{2}\right) = b_{\text{oF}}(1-3p) \tag{2.3.17}$$

that becomes:

$$b_{\text{oF}}(x) = \begin{cases} h_2\left(\frac{2+\sqrt{4-x^2}}{4}\right) & x<0 \\ 0 & x\geq 0 \end{cases} \tag{2.3.18}$$

in which x=$1-3p$. It is worth noticing that this bounding function is analogous to the one reported for two qubits [6, 7].

2.4 Comparison between entanglement measures

The trend of the two entanglement measures when p increases is linear for $\mathcal{E}_{\text{Neg}}(\rho_{cat})$ while is logarithmic for $\mathcal{E}_{\text{oF}}(\rho_{cat})$: in fact they detect two different kinds of entanglement, that increase with probability in different ways. Anyway they both respect the condition of null entanglement for $p \leq \frac{1}{3}$ and saturate to the unit for $p=1$, as is shown in Fig.2.1.

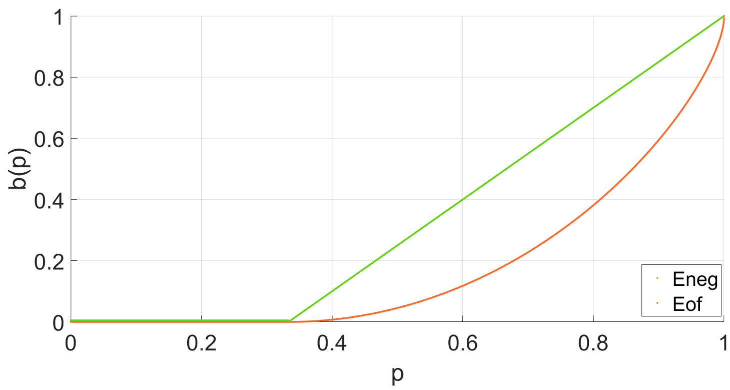

Figura 2.1: b_{Neg} and b_{oF} on p.

2.5 Application of the witness for small $|\beta|$ values

In the previous sections we have deduced the bounding functions for $\mathcal{E}_{\text{Neg}}$ and $\mathcal{E}_{\text{oF}}$ applying the witness to entangled cat-states: we have worked as $|\beta\rangle$ and $|-\beta\rangle$ constituted a basis but, as explained in Sec.1.2.1, it is an approximation valid only for large values of $|\beta|$.

In this section, we want therefore to study if the witness and the bounding functions found in this work are able to give us some information on the entanglement also outside this approximation. To do that we consider a pure entangled state, the analogous of Bell's state for a qubit-cavity system, and try to compute the entanglement negativity value by applying the definition. Then we compare the result to the expectation value of the witness inserted in the relative bounding function.

First of all the state that we consider is the one of Eq.(1.2.16) but as $|\beta\rangle$ and $|-\beta\rangle$ are not orthogonal it is better to express them in a different and really orthonormal basis, whose eigenvectors are:

$$|+\rangle = \frac{|\beta\rangle + |-\beta\rangle}{\sqrt{2(1+\chi)}} \qquad |-\rangle = \frac{|\beta\rangle - |-\beta\rangle}{\sqrt{2(1-\chi)}}. \tag{2.5.1}$$

The normalization factors are calculated by imposing $\langle +|+\rangle$ to be the unit and $\langle +|-\rangle$ to be zero, conditions to have an orthonormal basis, and the value called χ is the superposition $\langle \beta| - \beta\rangle$, already calculated in Eq.(1.2.14).

Using this basis $|\beta\rangle$ and $|-\beta\rangle$ become:

$$|\beta\rangle = \sqrt{\frac{(1+\chi)}{2}}\,|+\rangle + \sqrt{\frac{(1-\chi)}{2}}\,|-\rangle \tag{2.5.2}$$

$$|-\beta\rangle = \sqrt{\frac{(1+\chi)}{2}}\,|+\rangle - \sqrt{\frac{(1-\chi)}{2}}\,|-\rangle \tag{2.5.3}$$

and so the entangled state of Eq.(1.2.16) can be written as

$$|\Phi^+_{cat}\rangle = \frac{1}{2}\left(|0\rangle\left(\sqrt{(1+\chi)}\,|+\rangle + \sqrt{(1-\chi)}\,|-\rangle\right) + |1\rangle\left(\sqrt{(1+\chi)}\,|+\rangle - \sqrt{(1-\chi)}\,|-\rangle\right)\right). \tag{2.5.4}$$

We can therefore write the density matrix of the state, which is diagonal, as:

$$\rho_{cat}(\chi) = \frac{1}{4}\begin{pmatrix} (1+\chi) & \sqrt{(1+\chi)(1-\chi)} & (1+\chi) & -\sqrt{(1+\chi)(1-\chi)} \\ \sqrt{(1+\chi)(1-\chi)} & (1-\chi) & \sqrt{(1+\chi)(1-\chi)} & -(1-\chi) \\ (1+\chi) & \sqrt{(1+\chi)(1-\chi)} & (1+\chi) & -\sqrt{(1+\chi)(1-\chi)} \\ -\sqrt{(1+\chi)(1-\chi)} & -(1-\chi) & -\sqrt{(1+\chi)(1-\chi)} & -(1-\chi) \end{pmatrix} \tag{2.5.5}$$

and calculate the entanglement negativity varying χ value with the definition in Eq.(2.2.1).

On the other hand we want to calculate the expectation value of the witness, that corresponds to compute the mean values of observables $\{\hat{\mathbb{I}}, \hat{\sigma}_x \otimes \hat{\mathbf{X}}_c, \hat{\sigma}_y \otimes \hat{\mathbf{Y}}_c, \hat{\sigma}_z \otimes \hat{\mathbf{Z}}_c\}$.

To do that for small values of $|\beta|$ it is better to express states and operators in the real basis.

We know that coherent states can be expressed as displaced vacuum states and in the real basis $|0\rangle$ is represented by a normalized gaussian:

$$|0\rangle = \frac{e^{-x^2}}{\sqrt[4]{\pi}} \tag{2.5.6}$$

so

$$\Psi(x) = \langle x|\beta\rangle = \langle x|\,\hat{\mathbf{D}}(\beta)|0\rangle = \frac{e^{-(x-\beta)^2}}{\sqrt[4]{\pi}} \tag{2.5.7}$$

$$\Psi(-x) = \langle x| - \beta\rangle = \langle x|\,\hat{\mathbf{D}}(-\beta)|0\rangle = \frac{e^{-(x+\beta)^2}}{\sqrt[4]{\pi}}. \tag{2.5.8}$$

We know that the observables of the witness act on these states as:

$$\hat{\mathbf{X}}_c \Psi(x) = \Psi(-x) \tag{2.5.9}$$

$$\hat{\mathbf{Z}}_c \Psi(x) = \mathrm{sgn}(x)\Psi(x) \tag{2.5.10}$$

$$\hat{\mathbf{Y}}_c \Psi(x) = i\,\mathrm{sgn}(x)\Psi(-x) \tag{2.5.11}$$

so it is possible to calculate all the matrix elements of these observables, that we call $\hat{\mathbf{O}}$, computing the integrals:

$$\langle \pm\beta | \, \hat{\mathbf{O}} | \mp \beta \rangle = \int_{-\infty}^{+\infty} dx \Psi(\pm x)^* \, \hat{\mathbf{O}} \Psi(\mp x) \tag{2.5.12}$$

$$\langle \pm\beta | \, \hat{\mathbf{O}} | \pm \beta \rangle = \int_{-\infty}^{+\infty} dx \Psi(\pm x)^* \, \hat{\mathbf{O}} \Psi(\pm x). \tag{2.5.13}$$

Doing this calculation it is easy to show that for the observables $\hat{\mathbb{I}}$ and $\hat{\mathbf{X}}_c$ the matrix elements are:

$$\langle \pm\beta | \, \hat{\mathbb{I}} | \pm \beta \rangle = \langle \pm\beta | \, \hat{\mathbf{X}}_c | \mp \beta \rangle = 1 \tag{2.5.14}$$

$$\langle \pm\beta | \, \hat{\mathbb{I}} | \mp \beta \rangle = \langle \pm\beta | \, \hat{\mathbf{X}}_c | \pm \beta \rangle = \chi \tag{2.5.15}$$

while for $\hat{\mathbf{Y}}_c$ and $\hat{\mathbf{Z}}_c$ some of them are zero

$$\langle \pm\beta | \, \hat{\mathbf{Z}}_c | \mp \beta \rangle = \langle \pm\beta | \, \hat{\mathbf{Y}}_c | \pm \beta \rangle = 0 \tag{2.5.16}$$

but the others can not be analytically resolved because of the sign function.
It is anyway possible to express these contributions approximatively with a series expansion such as:

$$\langle \pm\beta | \, \hat{\mathbf{Z}}_c | \pm \beta \rangle = \langle \pm\beta | \, \hat{\mathbf{Y}}_c | \mp \beta \rangle = -\frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k \beta^{(2k+1)}}{(2k+1)k!}. \tag{2.5.17}$$

In posses of these values, we are so able to compute the expectation value of $\hat{\mathbf{W}}$ for the maximal entangled state of qubit-cavity system and compare it with the entanglement negativity calculated with the definition, making a simulation varying $|\beta|$ values.

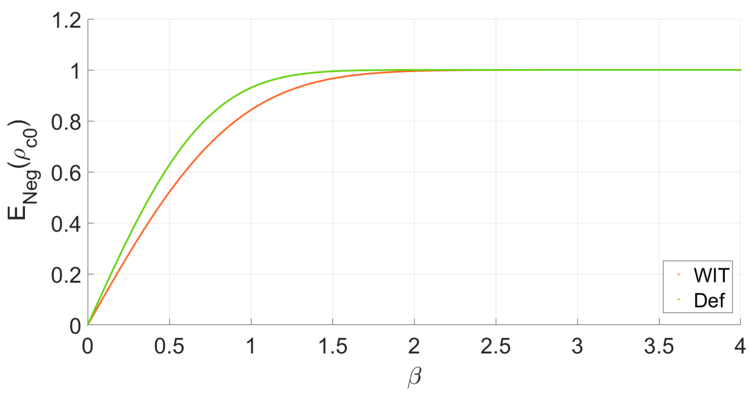

Figura 2.2: $\mathcal{E}_{\mathrm{Neg}}$ with definition and witness' method on β.

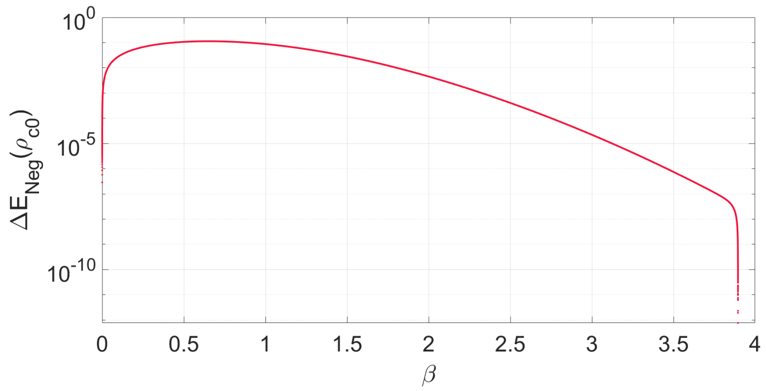

Figura 2.3: Logarithmic difference of $\mathcal{E}_{\text{Neg}}$ with definition and witness' method on β.

As expected entanglement is zero when $|\beta| \approx 0$ because in this case $|\beta\rangle$ and $|-\beta\rangle$ are approximatively the same state and so the state in Eq.(1.2.16) is separable.

On the other hand, increasing $|\beta|$ value, the state becomes maximally entangled as the coherent states are well separated and quasi-orthogonal. It is worth noticing that the witness follows the estimation of the definition with maximal discrepancies of 10^{-1}, this is an acceptable value if we consider that we use also some approximations for the $\text{sgn}(x)$ function.

This result allows us to assert that, also for a small value of β, it is valid to use the witness method for at least an approximate estimation of the entanglement of a qubit-cavity system.

Moreover, this analysis can guide us through the study of another kind of states, the so-called displaced number states, that, as the name says, are Fock's basis states displaced [14]:

$$|\beta, n\rangle = \hat{\mathbf{D}}(\beta)\, |n\rangle \,. \tag{2.5.18}$$

To test if the witness applies also to these states could be very useful because experimentally it is often the case that it is difficult to maintain the cavity states precisely on a certain level of energy: for this reason it is interesting to learn to work with states that are not only displacements of the vacuum and in general with the so-called thermal states, that are a superposition of states with different energies that follows Boltzmann's distribution. Following the example of what is explained in this section is possible to apply the analysis to states with different energy, easily changing the expression of $\Psi(x)$ and recalculating integrals.

In particular, if we apply this procedure to the case of n=1 the displaced state can be written as:

$$\Psi_1(x) = \langle x|\beta, 1\rangle = \langle x|\, \hat{\mathbf{D}}(\beta)|1\rangle = (x-\beta)\frac{e^{-(x-\beta)^2}}{\sqrt[4]{\frac{\pi}{4}}} \tag{2.5.19}$$

because in the real basis holds:

$$|1\rangle = x\frac{e^{-x^2}}{\sqrt[4]{\frac{\pi}{4}}}. \tag{2.5.20}$$

The value of χ in this case can be calculated by an integral of the displaced functions:

$$\chi' = \langle -\beta, 1|\beta, 1\rangle = \int_{-\infty}^{+\infty} 2(x-\beta)(x+\beta)\frac{e^{-(x-\beta)^2}}{\sqrt[4]{\pi}}\frac{e^{-(x+\beta)^2}}{\sqrt[4]{\pi}} = (1-\beta^2)e^{-\beta^2} \tag{2.5.21}$$

and substituted in Eq.(2.5.5) to calculate the entanglement negativity with the definition.

To apply the witness method it is sufficient to recalculate the integrals:

$$\langle \pm\beta | \hat{\mathbb{1}} | \pm \beta \rangle = - \langle \pm\beta | \hat{\mathbf{X}}_c | \mp \beta \rangle = 1 \tag{2.5.22}$$

$$\langle \pm\beta | \hat{\mathbb{1}} | \mp \beta \rangle = - \langle \pm\beta | \hat{\mathbf{X}}_c | \pm \beta \rangle = \chi' \tag{2.5.23}$$

$$\langle \pm\beta | \hat{\mathbf{Z}}_c | \pm \beta \rangle = - \langle \pm\beta | \hat{\mathbf{Y}}_c | \mp \beta \rangle = -\frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\beta \left((-1)^k \beta^{(2k)} - (1+2k)(-\beta^2)^{(k)} \right)}{\sqrt{\pi}(2k+1)k!}. \tag{2.5.24}$$

Using these results it is possible to create a simulation analogous to the case of n=0:

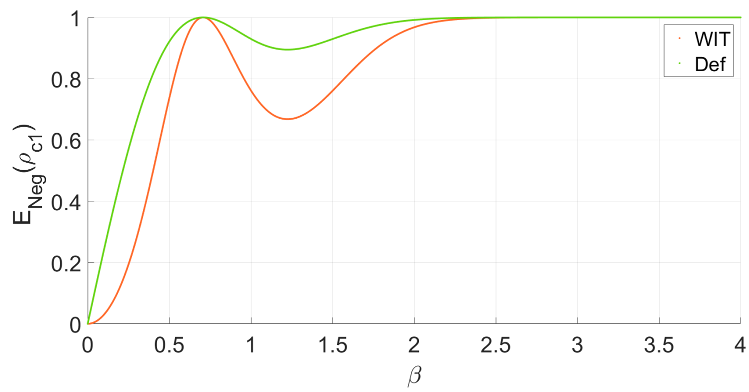

Figura 2.4: $\mathcal{E}_{\text{Neg}}$ for n=1 with definition and witness' method on β.

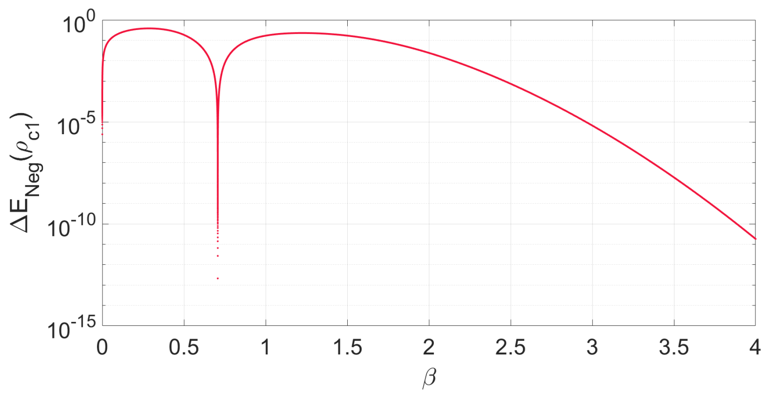

Figura 2.5: Logarithmic difference of $\mathcal{E}_{\text{Neg}}$ for n=1 with definition and witness' method on β.

Also in this case we can see that witness estimation is lower than the definition's one but the maximal discrepancies are of the order of 10^{-1}. According to these results so also in this case it is possible to use the witness method for an approximate estimation of the entanglement for small $|\beta|$ values.

Conclusion

In this work we have studied how it is possible to apply Quantitative Entanglement Witnesses to provide lower bounds to entanglement measures such as entanglement negativity and entanglement of formation.

The application of this procedure can be of fundamental importance for the detection of the entanglement in many aspects of quantum information, such as data processing and communication, because it is needed to know precisely if the systems we are working on are strongly entangled or not, doing as few measurements as possible.

In particular, in this study we have considered a prototypical example, a bipartite system composed of a qubit and a cavity, underlying the analogies with the two-qubits system. This kind of system results interesting because widely used in quantum devices, in which the estimation of entanglement allows for example to study the efficiency of the coupling between different device's components.

In chapter 1 we have analyzed some properties of the system and found out the expression of its witness. We have deduced three observables analogous to Pauli's matrices, justifying them using different bases of the cavity system, and we have also shown that they respect the same algebra of Pauli's matrices.

In chapter 2 we have applied the witness to entangled states of qubit-cavity and we have deduced the bounding functions for the entanglement negativity and the entanglement of formation measures: we have found that these functions are the same in the two-qubits case.

Moreover, we have shown that this procedure, firstly applied in the approximation of large $|\beta|$, can be used also in the limit of small $|\beta|$, studying more precisely the action of the witness' observables in the real basis.

This result can be very useful to extend witness analysis also to other kinds of experimental situations, for example with displaced number states or thermal states.

The previous work, in addition, could be expanded in many directions, considering for example other kinds of entanglement measures, such as relative entropy, or analyzing tripartite or multipartite systems: entanglement estimation also for these cases is not easily provided because a large number of measurements is required for a tomography.

For all these cases this study can be considered only a starting point from which to develop new theoretical tools for entanglement estimation that is nowadays more than ever necessary but not easily provided by the already known methodologies.

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