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Ricci-flat Kähler manifolds: Beauville's decomposition theorem and special holonomy manifolds

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Contents

Introduction **3**

1 Preliminaries **5**
- 1.1 Kähler Manifolds . . . 5
- 1.2 Connections . . . 11
- 1.3 Chern Classes . . . 21
  - 1.3.1 The Chern-Weil homomorphism . . . 21
  - 1.3.2 Integer Chern classes . . . 24
  - 1.3.3 First Chern class for Kähler manifolds . . . 33

2 Beauville's decomposition theorem **36**
- 2.1 Parallel transport and holonomy . . . 36
- 2.2 De Rham Decomposition . . . 41
- 2.3 Covering spaces and holonomy . . . 44
- 2.4 Vanishing results . . . 47
- 2.5 The main theorem . . . 49

3 Special holonomy manifolds **56**
- 3.1 K3 surfaces . . . 56
- 3.2 Calabi-Yau manifolds . . . 62
- 3.3 Irreducible symplectic manifold . . . 66
  - 3.3.1 Douady spaces . . . 69
  - 3.3.2 Generalized Kummer manifolds . . . 77
  - 3.3.3 A sketch of the construction of O'Grady's examples . . . 80

Bibliography **84**

Introduction

In this work we have analyzed some properties of compact Ricci-flat Kähler manifolds, which by Calabi-Yau theorem coincide with the class of compact complex manifolds which admit a Kähler metric and have $c_1^{\mathbb{R}} = 0$.
Since the publication of Calabi's article in 1954, it has been clear that compact Kähler manifolds are particularly well-behaved, being the compact complex manifols for which riemannian geometry and complex geometry interact in the best possible way. Kähler geometry has always been a very active field of research, which has attracted both algebraic and differential geoemeters and yielded a great interaction between the two fields.
The first chapter of thesis recalls some notions of riemannian geometry, with a special focus on Kähler case. Then Chern classes are defined, we tried to provide some geometric intution together with the needed formalism. Also in this case, a section is devoted to discuss Kähler case.
In the second chapter one finds the main result, Beauville's decomposition theorem. The main ideas of the proof come from riemannian geometry, in particular from the theory of holonomy representations, and at a first sight it is a riemannian geometry theorem. The Bochner principle together with Calabi-Yau theorem, though, allow to transfer this theorem in a metric-free world, much closer to "pure" algebraic geometry.
The combination of these two results shows that, for Kähler manifolds, having a certain holonomy representation (for a fixed metric) is equivalent to a certain structure of cohomology of holomorphic sheaves (which does not depend on the metric) and allows to state the decomposition theorem in a metric-free context, with isometries replaced by biholomoprhisms.
This led algebraic geometers to extend some definitions (K3 surfaces, Calabi-Yau and irreducible symplectic manifolds) born in riemannian setting to algebraic varieties over arbitrary fields. In the last chapter these three classes of manifolds, which arise from Beauville's decomposition, are studied. We point out that the nomenclature is highly non standard, throughout the work Calabi-Yau manifolds will be complex manifolds with trivial canonical bundle and $h^{p,0} = 0$ for $0 < p < \dim(X)$ whereas irreducible-symplectic manifolds will be simply connected complex manifolds, with a holomorphic symplectic form ϕ. In this chapter differential tools are widely used, but the results will be mainly algebraic and topological.
The first class has been studied in depth in the last decades. Recently the re-

search on moduli spaces of sheaves on $K3$ surfaces is very alive (see [Saw16] for a recent survey) and keeps providing new results and insights.
Calabi-Yau manifolds have been studied intensively since the '60s and found many applications in theoretical physics (see for example [Gre97]). In particular Calabi-Yau threefolds are at the basis of the main models of string theories, where the conditions of compactness, existence of a Kähler metric and Ricci-flatness all come naturally from phyisical motivations (in physics often Calabi-Yau=Ricci-flat, as this class behave well enough for string theory).
The last class is instead more mysterious: topological properties of irreducible symplectic manifolds are quite well-understood, but there is great uncertainty about any kind of classification. Only four deformation classes have been found, but it has not been established if these are indeed the only ones.

The nature of the subject brought the necessity to work with tools from both riemannian geometry and complex algebraic geometry. Many standard results have been assumed and different theorems are not proved. We have presented, though, some quite detailed examples, especially in the second part, in order to help the reader to have a concrete view of the objects we have dealt with.
We have tried to summarize the main nowadays literature on the subject, presenting also some very recent results (see 3.3.3) which can be obtained from Beauville's theorem.
We also would like to remark that the discussion on the existence of a Kähler metric for the Douady space $X^{[n]}$ at the end of the paragraph 3.3.1, although based on known results, has not been found by us in a complete and detailed form anywhere in the literature.

Chapter 1

Preliminaries

1.1 Kähler Manifolds

In this paragraph we will briefly outline some definitions of Kähler manifolds, focusing in particular on the riemannian properties.
On a differentiable manifold M there is a natural operation on the graded algebra of vector-valued differential forms, known as Frölicher–Nijenhuis bracket[1]

$$[-,-] : \Omega^k(M,TM) \times \Omega^l(M,TM) \to \Omega^{k+l}(M,TM)$$

which for $k = l = 1$ reads

$$\begin{aligned}[K,L](X,Y) = &[KX,LY] + [LX,KY] + (KL + LK)[X,Y] \\ &- K([LX,Y] + [X,LY]) - L([KX,Y] + [X,KY])\end{aligned}$$

where $K, L \in \Omega^1(M,TM)$ and X, Y are vector fields. Taking $K = L = I$ one has

$$2[I,I](X,Y) = [X,Y] - [IX,IY] + I\big([IX,Y] + [X,IY]\big)$$

Definition 1.1.1. An almost complex structure on a real n-dimensional manifold M is an automorphism I of TM such that $I^2 = -Id$. It is said integrable if $[I,I] = 0$. An almost complex manifold is a couple (M,I).

The word integrable refers to the fact that $[I,I]$ measures the obstruction to solve a PDE on M. Indeed the almost complex structure I provides an well-known $\mathbb{R}$-linear bundle decomposition $TM \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$ with fibers

$$\begin{aligned}T^{1,0}_pM &:= \{v \in T_{p,\mathbb{C}}M \,|\, I_p(v) = iv\} \\ T^{0,1}_pM &:= \{v \in T_{p,\mathbb{C}}M \,|\, I_p(v) = -iv\}\end{aligned}$$

[1]See [Mic] for details

where I is the $\mathbb{C}$-linear extension.
If a real base for T_pM is given by

$$\Big(\frac{\partial}{\partial x_1}|_p, \dots, \frac{\partial}{\partial x_n}|_p, \frac{\partial}{\partial y_1}|_p, \dots, \frac{\partial}{\partial y_n}|_p\Big)_{j=1,\dots,n}$$

where $\frac{\partial}{\partial y_j}|_p = I(\frac{\partial}{\partial x_j}|_p)$, the map

$$\begin{aligned} T_{p,\mathbb{C}}M &\xrightarrow{\cong} T_p^{1,0}M \oplus T_p^{0,1}M \\ v &\mapsto \big(v - iI(v), v + iI(v)\big) \end{aligned}$$

shows that in any point

$$\Big(\frac{\partial}{\partial x_j}|_p - i\frac{\partial}{\partial y_j}|_p\Big)_{j=1,\dots,n}$$

is a complex base of $T_p^{1,0}M$ whereas

$$\Big(\frac{\partial}{\partial x_j}|_p + i\frac{\partial}{\partial y_j}|_p\Big)_{j=1,\dots,n}$$

is a complex base of $T_p^{0,1}M$. Then I is integrable if and only if

$$\frac{\partial}{\partial z_j} := \frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}$$

is a local frame for $T^{1,0}M$ and

$$\frac{\partial}{\partial \bar{z}_j} := \frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}$$

is a local frame for $T^{0,1}M$. These results are summarized in

Theorem 1.1.2 (Newlander–Nirenberg)**.** *For an almost complex manifold* (M, I) *the following are equivalent:*

- $[I, I] = 0$
- $T^{1,0}M$ *is spanned by* $\frac{\partial}{\partial z_j}$
- $T^{1,0}M$ *is closed for Lie bracket*
- $T^{1,0}M$ *is closed by parallel transport*
- $\bar{\partial}^2 = 0$

If one of the property holds, then there exists a unique complex atlas on M *whose underlying almost complex structure is* I*.*

It is in general a difficult problem to establish whether a differentiable manifold admits an almost complex structure and when this is integrable.
This is why it is usually more practical to consider complex manifolds in the first place.
We shall now give the definition of Kähler manifold from a riemannian point of view

Definition 1.1.3. A riemannian almost complex manifold (M, g, I) is Kähler if I preserves g (i.e. $g(u, v) = g(Iu, Iv)$ for any vector fields u, v) and $\nabla I = 0$ for the Levi-Civita connection.

It is not necessary to require the integrability of I because if the connection is torsion free, flat tensor fields are always integrable in the generalized sense.[2]

Hermitian structures
Let $E \to X$ be a complex vector bundle, then a hermitian metric h is a C^∞ assignment $x \mapsto h_x$ where h_x is an hermitian form on E_x. Now let $E = T_X$, in this case a hermitian metric induces naturally a riemannian structure on the underlying differentiable manifold by

$$g := \mathcal{R}h$$

which in this way is automatically I-compatible.
The converse holds true as well via

$$h := g + ig(-, I(-))$$

The difference between these two structures is then encoded in a 2-form

$$\omega := g(-, I(-))$$

If one extends g by $\mathbb{C}$-sesquilinearity to $\tilde{g}$, then an easy computation shows that

$$T^{1,0}M \oplus^{\perp} T^{0,1}M$$

and the $\mathbb{C}$-bilinear extension of ω to a complexified 2-form turns out to be in $\Omega^{1,1}(X)$. Notice also that $\tilde{g}$ on the complexified tangent space coincides with h on the holomorphic tangent bundle, because

$$\tilde{g}(u - iI(u), v - iI(v)) = 2h(u, v)$$
$$\tilde{g}(u + iI(u), v + iI(v)) = 2\overline{h(u, v)}$$

The following result is well known

[2]For example if $D \subset TM$ is a parallel distribution and X, Y are sections of D, then

$$[X, Y] = \nabla_X Y - \nabla_Y X$$

and the term on right hand side belongs to D, since

$$\nabla_X Y|_q = \lim_{t \to 0} \frac{1}{t} \left(P_{\gamma(t)}(Y) - Y|_q\right)$$

where $\gamma(t)$ is any integral curve for X with $\gamma(0) = q$ and $P_{\gamma(t)}(Y) \in D$ by hypothesis (and analogously for $\nabla_Y X$).

Lemma 1.1.4. *On a complex manifold X there is a one to one correspondence between hermitian metrics, compatible riemannian metrics (regarding X as a differentiable manifold) and elements of $\Omega^{1,1}(X) \cap \Omega^2_{\mathbb{R}}(X)$.*

Proof. For the two first families consider:

$$h \mapsto g := \mathcal{R}h$$

then g is clearly real, symmetric and positive definite. The inverse is given by

$$g \mapsto h := g - ig(I(-), -)$$

and such an h satisfies $h(u,v) = \overline{h(v,u)}$, is $\mathbb{C}$-linear in the first argument and $\mathbb{C}$-antilinear in the second one and is positive definite.
For the first one and the last one consider:

$$h \mapsto \omega := -\mathcal{I}\ h$$

with inverse

$$\omega \mapsto h := \omega(-, I(-)) - i\omega$$

□

It is often convenient to perform computations in coordinates: let $(z^1, \ldots, z^n)$ be holomoprhic coordinates, then

$$h\Big(\alpha^i \frac{\partial}{\partial z^i}, \beta^j \frac{\partial}{\partial z^j}\Big) = \alpha^i \overline{\beta^j} h_{ij}$$

i.e.

$$h = h_{i\bar{j}} dz^i \otimes d\bar{z}^j$$

which implies

$$\omega = \frac{i}{2} h_{i\bar{j}} dz^i \wedge d\bar{z}^j$$

The second and most common definition of Kähler metric is

Definition 1.1.5. A hermitian metric (or the corresponding compatible riemannian metric) over the complex manifold X is said Kähler if $d\omega = 0$.

A useful characterization of Kähler metrics in terms of their Taylor expansion if the following

Lemma 1.1.6. *An hermitian metric h over X is Kähler if and only if for every $p \in X$ there are local holomorphic coordinates $(z^1, \ldots, z^n)$ such that h reads*

$$h(z) = Id_n + O(\|z\|^2)$$

Proof. If h has such a local expression, then clearly $d\omega = 0$. Conversly, first choose a local chart centered in p such that $h(p) = \mathrm{Id}$, then

$$h = \sum_i dz^i \otimes d\overline{z}^i + \sum_{ij} \epsilon_{ij} dz^i \otimes d\overline{z}^j + O(\|z\|^2)$$

where $\epsilon_{ij} = \frac{\partial h_{ij}}{\partial z^k}(0) z^k + \frac{\partial h_{ij}}{\partial \overline{z}^l}(0)\overline{z}^l$ and since h is hermitian

$$\frac{\partial h_{ij}}{\partial z^k} = \overline{\Big(\frac{\partial h_{ji}}{\partial \overline{z}^k}\Big)}$$

Thus ω can be written as:

$$\omega = \frac{i}{2}\big(\delta_{ij} + \frac{\partial h_{ij}}{\partial z^k}(0) z^k + \frac{\partial h_{ij}}{\partial \overline{z}^l}(0)\overline{z}^l\big) dz^i \wedge d\overline{z}^j + O(\|z\|^2)$$

The fact the $d\omega = 0$ implies that

$$\partial\Big(\frac{\partial h_{ij}}{\partial z^k}\Big|_{z=0} z^k dz^i \wedge d\overline{z}^j\Big) = \frac{\partial h_{ij}}{\partial z^l}\Big|_{z=0} dz^l \wedge dz^i \wedge d\overline{z}^j$$

hence

$$\frac{\partial h_{ij}}{\partial z^l}\Big|_{z=0} = \frac{\partial h_{lj}}{\partial z^i}\Big|_{z=0}$$

Thus there exist holomorphic functions $\phi_j : \mathbb{C}^n \to \mathbb{C}$ such that

$$\frac{\partial h_{ij}}{\partial z^l}\Big|_{z=0} z^l = \frac{\partial \phi_j}{\partial z^i}$$

One can clearly choose ϕ_j vanishing in 0 and in this way it is possible to define new coordinates in a neighbourhood of the origin $w^j = z^j + \phi^j$, since

$$\phi_j(z) = \frac{\partial \phi_j}{\partial z^i \partial z^k}\Big|_{z=0} z^i z^k + O(\|z\|^3) = \frac{\partial h_{ij}}{\partial z^k}\Big|_{z=0} z^k + O(\|z\|^3)$$

In these new coordinates, then:

$$dw_i = dz_i + \frac{\partial \phi_i}{\partial z^k} dz^k$$

Thus one obtains:

$$\begin{aligned}\sum_i dw_i \wedge d\overline{w}_i &= \sum_i dz_i \wedge d\overline{z}_i + \sum_{i,k}\big(\frac{\partial \phi_i}{\partial z^k} dz_k \wedge d\overline{z}_i + \overline{\frac{\partial \phi_i}{\partial z^k}} dz_i \wedge d\overline{z}_k\big) + O(\|z\|^2) \\ &= \sum_i dz_i \wedge d\overline{z}_i + \sum_{i,k} \epsilon_{ki} dz_k \wedge d\overline{z}_i + O(\|z\|^2)\end{aligned}$$

and this last expression equals $\frac{2}{i}\omega + O(\|z\|)^2$. Now since the expansion of the change of coordinates Φ has order at most two around 0, if $f(z) = O(\|z\|)^2$, then $f(\Phi(w)) = O(\|w\|)^2$. This shows that

$$\omega = \frac{i}{2}\sum_i dw_i \wedge d\overline{w}_i + O(\|w\|)^2$$

hence

$$h = \sum_i dw_i \otimes d\overline{w}_i + O(\|w\|)^2$$

□

This result immediately implies the following important characterization

Theorem 1.1.7. *For a hermitian manifold (M, h) (or equivalently for a riemannian almost complex manifold (M, g, I)) the following are equivalent:*

1. *h is Kähler*
2. *I is flat for ∇^{LC}, i.e. $\nabla^{LC}(IZ) = I(\nabla^{LC}Z)$ for any real vector field Z*
3. *After the canonical identification (in the cateogry of C^∞-vector bundles over X) of T_X and $T_{\mathbb{R}}X$, $\nabla^{Ch} = \nabla^{LC}$*

Proof. Since ∇^{Ch} is a complex connection it is $\mathbb{C}$-linear and under the identification $\mathrm{Re} : T_X \to T_{\mathbb{R}}X$ one has $i = I$, hence 3. implies 2..
To prove that 2. implies 1., for the Levi-Civita connection

$$d(g(W, Z)) = g(\nabla W, Z) + g(W, \nabla Z)$$

holds by definition, so if ∇^{LC} commutes with I one has

$$d(\omega(W, Z)) = \omega(\nabla W, Z) + \omega(W, \nabla Z)$$

then for three vector fields Y, W, Z

$$Y(\omega(W, Z)) = \omega(\nabla_Y W, Z) + \omega(W, \nabla_Y Z)$$

Now the well known formula for exterior derivative

$$\begin{aligned} d\omega(Y, W, Z) =& Y(\omega(W, Z)) - W(\omega(Y, Z)) + Z(\omega(Y, W)) \\ & - \omega([Y, W], Z) + \omega(Y, [W, Z]) + \omega([Y, Z], W) \end{aligned}$$

yields $d\omega = 0$, writing all the brackets in terms of covariant derivatives.
For the last implication, note that ∇^{Ch} and ∇^{LC} coincide in the trivial case $X = \mathbb{C}^n$ and $h = \sum dz^i \otimes d\overline{z}^i$. Moreover, both the connections at each point depend on the metrics only up to the first order.
Now by the previous lemma if h is Kähler then in local coordinates it is trivial to the first order, which means that the two corresponding two connections coincide. □

The previous theorem shows that Kähler condition is really a natural one to ask.

1.2 Connections

Let X be a complex manifold and $E \to X$ a complex vector bundle, the bundle of E-valued k-forms will be denoted by$\mathcal{A}^k_{X,E} = \mathcal{A}^k_X \otimes E$ or simply $\mathcal{A}^k_E$, where $\mathcal{A}^k_X$ is the k-th exterior power of the complexified cotangent bundle. The bundle of holomorphic forms will be denoted as Ω^k_X and the spaces of global section as $\mathcal{A}^k(X)$ or $\Omega^k_X(X)$. In particular $\mathcal{A}^0_{X,E}(U) = C^\infty(U, E)$ for U open in X.

Definition 1.2.1. A connection ∇ is a $\mathbb{C}$-linear sheaf homomorphism

$$\nabla : \mathcal{A}^0_E \to \mathcal{A}^1_E$$

satisfying Leibniz rule

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla(\sigma)$$

for any section σ and any $f \in C^\infty(X;\mathbb{C})$, where for any differential form α and any Y section of $T_{\mathbb{C}}X$ one sets:

$$(\alpha \otimes \sigma)(Y) = (\alpha(Y))\sigma$$

The definition of real connection is analogous, by replacing $\mathbb{C}$ with $\mathbb{R}$. For a holomorphic vector bundle $E \to X$ there is a notion of holomorphic connection, defined replacing $\mathcal{A}^i_E$ with $\Omega^i_X \otimes E$, but it will not be used in this work.
A connection is not not C^∞-linear but the difference of two connections always is, indeed the space of connections for $E \to X$ is an affine space over $\mathcal{A}^1(X) \otimes \mathrm{End}(E)$. In particular given a connection ∇ all the other connections will be of the form $\nabla' = \nabla + \phi$ with $\phi \in \mathcal{A}^1(X) \otimes \mathrm{End}(E)$.
It is useful to have a local expression: if $U \subset X$ is a trivializing open set for E, with basis given by $(s_1, \ldots, s_k)$, then

$$\nabla(f^i s_i) = df^i \otimes s_i + f^i \nabla s_i$$

Set $\nabla s_i = \omega_i^j s_j$ where $\omega_i^j \in \mathcal{A}^1(X)$. The 1-forms ω_i^j completely determine the connection.
Since for X one has the projection operators

$$\mathrm{pr}_{1,0} : \mathcal{A}^1(X) \to \mathcal{A}^{1,0}(X) \qquad \mathrm{pr}_{0,1} : \mathcal{A}^1(X) \to \mathcal{A}^{0,1}(X)$$

any connection can be decomposed as:

$$\nabla^{1,0} = \mathrm{pr}_{1,0} \circ \nabla : \mathcal{A}^0_E(X) \to \mathcal{A}^{1,0}_E(X)$$

and

$$\nabla^{0,1} = \mathrm{pr}_{0,1} \circ \nabla : \mathcal{A}^0_E(X) \to \mathcal{A}^{0,1}_E(X)$$

Even though an exterior derivative for E-valued forms

$$\mathcal{A}^k_E(X) \to \mathcal{A}^{k+1}_E(X)$$

is a priori not defined, if E is holomorphic, $s \in H^0(X, \mathcal{A}_E^{0,q})$ and $(e_1 \dots, e_k)$ is a local trivialization over $U \subset X$, one has $s = \alpha^i \otimes e_i$ with $\alpha^i \in \mathcal{A}^{0,q}(U)$. Then one defines

$$(\bar{\partial}_E)_U(\alpha) := (\bar{\partial}\alpha_1, \dots, \bar{\partial}\alpha_k)$$

and this operator is well defined, indeed:

Lemma 1.2.2. *If U, V are two overlapping open subsets of X, $(E_1, \dots, E_k)$ a trivialization over U and $(F_1, \dots, F_k)$ a trivialization over V, then for each $\alpha \in \mathcal{A}_E^{0,q}(U \cap V)$ one has*

$$(\bar{\partial}_E)_U(\alpha) = (\bar{\partial}_E)_V(\alpha)$$

Proof. Let $\alpha = \alpha^i \otimes E_i = \beta^j \otimes F_j$ and M be the transition matrix between the two local frame. Then $\beta = M\alpha$ and:

$$\begin{aligned}(\bar{\partial}_E)_V(a) &= (\bar{\partial}_E)_V(\beta_1, \dots, \beta_k)\\ &= (\bar{\partial}\beta_1, \dots, \bar{\partial}\beta_k)\\ &= M(\bar{\partial}\alpha_1, \dots, \bar{\partial}\alpha_k)\end{aligned}$$

since $\bar{\partial}\beta^i = \bar{\partial}(\sum_j M_{ij}\alpha^j) = \sum_j M_{ij}\bar{\partial}(a^j)$. □

The definition exploits that transition matrix has holomorphic coefficients, thus is annihilated by $\bar{\partial}$. It is not possible to define $\partial_E : \mathcal{A}_E^{p,0}(X) \to \mathcal{A}_E^{p+1,0}(X)$ in an analogous way. It is now natural to relate $\bar{\partial}_E$ and $\nabla^{0,1}$.

Definition 1.2.3. A connection ∇ for a holomorphic vector bundle $E \to X$ is said compatible with the holomorphic structure if $\nabla^{0,1} = \bar{\partial}_E$.

If a complex vector bundle $E \to X$ has a hermitian structure, it is instead natural to require that

Definition 1.2.4. ∇ is said hermitian (or compatible with h) if

$$d(h(\sigma, \tau)) = h(\nabla\sigma, \tau) + h(\sigma, \nabla\tau)$$

where the natural extension of h is considered:

$$\begin{aligned}h(\alpha \otimes \sigma, \tau) &= \alpha h(\sigma, \tau)\\ h(\sigma, \alpha \otimes \tau) &= \bar{\alpha} h(\sigma, \tau)\end{aligned}$$

for each $\sigma, \tau \in \Omega^0(E)$ and $\alpha \in \Omega^1(X)$.

If ∇ is hermitian and $\phi \in \mathcal{A}^1(X)$, then $\nabla' = \nabla + \phi$ is hermitian if and only if

$$h(\phi(\sigma), \tau) + h(\sigma, \phi(\tau)) = 0$$

Then one sets

Definition 1.2.5. Define $\mathrm{End}(E, h)$ the subsheaf of E of sections ϕ which satisfies

$$h(\phi(\sigma), \tau) + h(\sigma, \phi(\tau)) = 0$$

Clearly the space of connections compatible with the holomorphic structure is an affine space modelled over $\mathcal{A}^{1,0}(X) \otimes \mathrm{End}(E, h)$ whereas the space of hermitian connections is an affine space modelled over $\mathcal{A}^1(X) \otimes \mathrm{End}(E, h)$. This leads to

Lemma 1.2.6. *For a holomorphic hermitian vector bundle $E \to X$ there exists a unique connection ∇, called Chern connection, compatible with the complex structure and with the metric.*

Proof. Let $s_1, \dots, s_k$ be local holomorphic sections which are a local basis for E. Compatibility condition reads:

$$d(h(s_i, s_j)) = h(\nabla s_i, s_j) + h(s_i, \nabla s_j)$$

Since $\nabla = d + \omega$ and the s_i are holomorphic, ω is a matrix of elements of $\mathcal{A}^{1,0}(X)$. Then

$$\partial(h(s_i, s_j)) = h(\nabla^{1,0} s_i, s_j)$$

or in matrix notation

$$\partial h = \omega^t h$$

which implies

$$\omega = \bar{h}^{-1} \partial(\bar{h})$$

□

Example 1.2.7. Let $(L, h) \to X$ be a holomorphic hermitian line bundle. Clearly h corresponds to a C^∞ function $X \to \mathbb{R}^+$. Then the Chern connection is given by $\nabla = d + \partial \log(h)$.

In real riemannian geometry one can proceed analogously and look for connections compatible with the structures on the manifold. For arbitrary real riemannian vector bundles $(E, g) \to X$ (i.e. whose structure group can be reduced to $O(n)$) it makes sense to ask for compatibility with g, but this does not determine a unique connection. Indeed the real vector space $\mathcal{A}^1(X) \otimes \mathrm{End}(E, g)$ is not trivial (this is again related to the fact that there is a priori no natural way to differentiate E-valued form). If $E = TX$, though, it makes sense to ask for compatibility of the connection with the differentiable structure.

Definition 1.2.8. Let M be a differentiable manifold with a connection ∇ on TM. Its torsion is defined as

$$T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$$

for any X, Y section of TM.

Notice that if the torsion vanishes, then $\mathcal{L}_X Y = \nabla_X Y - \nabla_Y X$.

Definition 1.2.9. For a riemannian manifold (M, g) the unique connection on TM compatible with metric and without torsion is called Levi-Civita connection.

The existence and uniqueness can be proved easily.
Given a connection it is possible to extend it to a differential operator

$$\nabla : \mathcal{A}^k_E(X) \to \mathcal{A}^{k+1}_E(X)$$

by setting, for a k-form σ and τ a section of E

$$\nabla(\sigma \otimes \tau) = d\sigma \otimes \tau + (-1)^{\deg(\sigma)} \wedge \nabla(\tau)$$

Then one has

Definition 1.2.10. The curvature of ∇ is defined as

$$R_\nabla = \nabla \circ \nabla : \mathcal{A}^0_E(X) \to \mathcal{A}^2_E(X)$$

Unlike connections, it is easy to check that curvature is C^∞-linear and this allows to view it as a global section of $\mathcal{A}^2(X) \otimes \mathrm{End}(E)$ (the so-called curvature form).
If a local frame $(s_1, \dots, s_k)$ is fixed for E, then

$$(R_\nabla s_i)(X, Y) = \Omega^j_i(X, Y) s_j$$

for any X, Y sections of $T_{\mathbb{C}}X$. Again the collection of 2-forms $(\Omega^j_i)^{\dim(X)}_{i,j=1}$ completely determines the curvature. By definition

$$R_\nabla(s_i) = (d\omega^j_i + \omega^j_k \wedge \omega^k_i) s_j$$

which implies the following formula, known as Cartan's structure equation

$$\Omega^j_i = d\omega^j_i + \omega^j_k \wedge \omega^k_i$$

If moreover one fixes local coordinates $(z_1, \dots, z_n)$ for X, then

$$\Omega^j_i = R^j_{i\alpha\beta} dz^\alpha \wedge dz^\beta + R^j_{i\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta + R^j_{i\bar{\alpha}\bar{\beta}} d\bar{z}^\alpha \wedge d\bar{z}^\beta$$

With the coordinate expression is easy to see that if ∇ is a connection on the tangent bundle, its associated curvature reads

$$R_\nabla(X, Y)(\sigma) = \nabla_X \nabla_Y \sigma - \nabla_Y \nabla_X \sigma - \nabla_{[X,Y]} \sigma$$

for any vector fields X, Y, Z.

Example 1.2.11. Let $(L, h) \to X$ be a hermitian line bundle. Then $\nabla(s) = \omega s$ hence $R_\nabla = d\omega$. If ∇ is the Chern connection, in particular, the coordinate expression reads

$$R_\nabla = \bar{\partial}\partial log(h)$$

whereas for an arbitrary vector bundles $R_\nabla = \bar{\partial}(\bar{h}^{-1}\partial(\bar{h})))$.

Since the curvature can be seen as a two form, it is natural to ask in which component of $\mathcal{A}^2(X) \otimes \mathrm{End}(E, h)$ it lives. For Chern connection the following holds

Lemma 1.2.12. *Let ∇ be the the Chern connection for a holomorphic vector bundle $(E,h) \to X$, then*

$$R_\nabla \in \mathcal{A}^{1,1}_{i\mathbb{R}}(X) \otimes End(E,h)$$

Proof. A simple computation yields

$$h(R_\nabla(s_1), s_2) + h(s_1, R_\nabla(s_2)) = 0$$

The fact that R_∇ has no $(0,2)$-part follows from the definition

$$R_\nabla = (\nabla^{1,0} + \bar{\partial})(\nabla^{1,0} + \bar{\partial})$$

and this also implies that $h(R_\nabla(s_1), s_2)$ has no $(0,2)$ part whereas $h(s_1, R_\nabla(s_2))$ has no $(2,0)$ part, then they necessarily both have only $(1,1)$ part.
To see that R_∇ is pure-imaginary, consider a local ortonormal frame $(s_1, \dots s_k)$ for E, so that in coordinates $h = \mathrm{Id}$. Then for any vector fields X, Y one has

$$(R_\nabla(X,Y))_{ij} = h\big(R_\nabla(X,Y)(s_i), s_j\big) = -h\big(s_i, R_\nabla(X,Y)(s_j)\big) = -\overline{(R_\nabla(X,Y))}_{ij}$$

□

Definition 1.2.13. A section s of E is said parallel if $\nabla s = 0$.

If ψ is a fixed vector field on X, then the covariant derivative along ψ is defined as

$$\begin{aligned} \nabla_\psi : \mathcal{A}^0_E(X) &\to \mathcal{A}^0_E(X) \\ \sigma &\mapsto \nabla_\psi(\sigma) = \nabla(\sigma)(\psi) \end{aligned}$$

and $\nabla(\sigma)(-)$ is not only $\mathbb{C}$-linear but C^∞-linear, i.e. it is a morphism of sheaves of C^∞-modules. In particular $\big(\nabla_\psi(\sigma)\big)_p$ depends only on the pointwise value of ψ. This reinforces the idea that connections are derivations of tensor fields along a chosen direction, as

- If X and X' are two vector fields with $X|_p = X'|_p$, then $\nabla_X T|_p = \nabla_{X'} T|_p$
- If T and T' are two tensor fields with $T|_{\gamma(t)} = T'|_{\gamma(t)}$ where γ is an integral curve for X, then $\nabla_X T|_{\gamma(t)} = \nabla_X T'|_{\gamma(t)}$

Another classical object of riemannian geometry is

Definition 1.2.14. Given a connection ∇ for $E \to M$ and a local trivialization $(e_1, \dots, e_k)$, the Christoffel Symbols associated to the connection and the trivialization are given by:

$$\nabla_{\partial_j}(e_k) = \Gamma^i_{jk} e_i$$

We now focus on the case in which $E = TX$. A connection for this bundle can be interpreted as a way to differentiate vector fields along a fixed vector field ψ.

Definition 1.2.15. The Ricci curvature tensor of a connection ∇ on TM with associated curvature R is defined as

$$\mathrm{Ric}(X,Y) = \mathrm{tr}(R(-,X)Y)$$

In a local orthonormal frame $(e_1,\dots,e_n)$ for TM, the above formula yields:

$$\begin{aligned}\mathrm{Ric}(X,Y) &= \mathrm{tr}(R(-,X)Y)\\ &= \sum_i g(R(E_i,X)Y,E_i)\end{aligned}$$

which implies in particular that $\mathrm{Ric}(X,Y) = \mathrm{Ric}(Y,X)$. The term $g(R(E_i,X)Y,E_i)$ is often written as $R(E_i,X,Y,E_i)$. and is knwon by riemannian geometry as sectional curvature (see [DF92] for details). In local coordinates one has

$$\mathrm{Ric}(X,Y) = R_{ij}dx^i \otimes dx^j(X,Y)$$

and a simple computation shows that $R_{ij} = R^k_{ikj}$.
The geometric intuition behind Ricci tensor is the following: fix $p \in M$ and let $(x_1,\dots,x_n)$ be normal geodesic coordinates around p (these are coordinates associated to a chart ϕ centered in p, such that $\phi^{-1}x_i$ are geodesic and $g(\partial x_i|_p, \partial x_j|_p) = \delta_{ij}$), then the volume form can be locally written as:

$$dV_g = \big(\mathrm{Id} - \frac{1}{6}\mathrm{Ric}(x_i,x_j) + O(\|x^3\|)\big)dx_1\dots dx_n$$

thus the Ricci tensor is the main contribution of the curvature in the distorsion of the volume.
Given a connection ∇ on $E \to X$, we will now define induced connections on tensor bundles, exterior power bundles and the dual bundle. They actually descend from a unique object, indeed if $E \to X$ is a rank k complex vector bundle and $Fr(E) \to X$ is the underlying $\mathrm{GL}(\mathbb{C}^k)$-principal bundle, all these bundles are obtained from $\mathrm{Fr}(E)$ via different representations.
If $E \backsimeq \mathrm{Fr}(E) \times_\rho \mathbb{C}^k$, then

$$\begin{aligned}E^{\otimes n} &\backsimeq Fr(E) \times_{\rho^{\otimes n}} (\mathbb{C}^k)^{\otimes n}\\ \bigwedge^n E &\backsimeq Fr(E) \times_{\rho^{\wedge n}} \bigwedge^n \mathbb{C}^k\\ E^* &\backsimeq Fr(E) \times_{\rho^*} (\mathbb{C}^k)^*\end{aligned}$$

and the formulas below descend from this.

Definition 1.2.16. If ∇ is a connection on $E \to X$, then the natural induced connections are defined on generators as

- On $E^{\otimes n}$, $\nabla^{\otimes n}_X(s_1 \otimes \cdots \otimes s_n) = \sum_i (s_1 \otimes \cdots \otimes \nabla_X s_i \otimes \cdots \otimes s_n)$
- On $\bigwedge^n E$, $\nabla^{\wedge n}_X(s_1 \wedge \cdots \wedge s_n) = \sum_i (s_1 \wedge \cdots \wedge \nabla_X s_i \wedge \cdots \wedge s_n)$

- On E^*, the induced connection satisfies $X(\xi(s)) = \xi(\nabla_X s) + (\nabla^*_X \xi)(s)$ for any ξ section of E
- On $\mathrm{End}(E) = E^* \otimes E$, $\nabla^{\mathrm{End}}_X(f)(s) = \nabla_X(f(s)) - f(\nabla_X s)$

Example 1.2.17. Let ∇ be a connection on $TM \to M$, then the natural extension of ∇ to all the tensor bundles $T^p_q M = TM^{\otimes p} \otimes T^*M^{\otimes q}$ is given by

$$\begin{aligned}(\nabla_X T)(\omega_1, \ldots, \omega_p, Y_1, \ldots, Y_q) =& X\big(T(\omega^1, \ldots, \omega_p, Y_1, \ldots, Y_q)\big) \\ &- \sum_i T(\omega_1, \ldots, \nabla_X \omega^i, \ldots, \omega_p, Y_1, \ldots, Y_q) \\ &- \sum_j T(\omega_1, \ldots, \omega_p, Y_1, \ldots, \nabla_X Y_j, \ldots, Y_q)\end{aligned}$$

where $(\nabla_X \omega^i)(Z) = X(\omega_i(Z)) - \omega_i(\nabla_X Z)$.

The following property yields some useful symmetries of the curvature tensor. Since $R_\nabla(X,Y)$ is a section of $\mathrm{End}(E)$ and this bundle has a natural induced connection $\widetilde{\nabla}$, it is possible to apply this connection to $R_\nabla(X,Y)$. The precise expression is:

$$(\widetilde{\nabla}_W R_\nabla(X,Y))(s) = \nabla_W\Big(R_\nabla(X,Y)(s)\Big) - R_\nabla(X,Y)\Big(\nabla_W(s)\Big)$$

for any vector field W. Notice in particular that $(\nabla(R_\nabla))(s) \neq \nabla(R_\nabla(s))$.

Lemma 1.2.18 (Bianchi Identity)**.** *Any connection ∇ on a vector bundle $E \to X$ satisfies $\tilde{\nabla}(R_\nabla) = 0$.*

For any torsion-free connection on the tangent bundle, Bianchi identity specifies to

Lemma 1.2.19 (First Bianchi identity in vector form)**.** *If ∇ is without torsion, then for the curvature tensor R one has:*

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

Proof.

$$\begin{aligned}&\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\ &+ \nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X - \nabla_{[Y,Z]} X \\ &+ \nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y - \nabla_{[Z,X]} Y \\ =& \nabla_X [Y,Z] + \nabla_Y [Z,X] + \nabla_Z [X,Y] \\ &- \nabla_{[X,Y]} Z - \nabla_{[Y,Z]} X - \nabla_{[Z,X]} Y \\ =& [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] \\ =& 0\end{aligned}$$

$\square$

If one has also a metric tensor g, consider the tensor

$$R(X,Y,Z,W) = g(R(X,Y)Z,W)$$

for any X,Y,Z,W sections of TM. If the connection is also compatible with the metric (i.e. it is the Levi-Civita connection) one has

Lemma 1.2.20 (Second Bianchi Identity)**.**

$$(\nabla_X R)(Y,Z,T,W) + (\nabla_Y R)(Z,X,T,W) + (\nabla_Z R)(X,Y,T,W) = 0$$

Proof. Since $\nabla_X R$ is C^∞-linear in all the four arguments, it suffices to prove the identity in a point p for X,Y,Z,T,W coordinate vector fields which are orthogonal in p. In this way $(\nabla_X Y)|_p = 0$. With this assumption

$$(\nabla_X R)(Y,Z,T,W) = X(R(Y,Z,T,W)) = g(\nabla_X(R(Y,Z)T),W)$$

and the last term equals

$$g(\nabla_X\nabla_Y\nabla_Z T - \nabla_X\nabla_Z\nabla_Y T, W)$$

The conclusion follows by permuting ciclically X,Y,Z and summing everything together. □

Now let X be a Kähler manifold. Since I is flat for the Levi-Civita connection

$$R(X,Y)(IZ) = \nabla_X\nabla_Y IZ - \nabla_Y\nabla_X IZ - \nabla_{[X,Y]}IZ = IR(X,Y)Z$$

and since g is compatible

$$R(X,Y,IZ,IT) = R(X,Y,Z,T) = R(IX,IY,Z,T)$$

which implies

$$\begin{aligned}
\mathrm{Ric}(IX,IY) &= \sum_i gR(e_i,IX)IY,e_i) \\
&= \sum_i g(R(Ie_i,X)Y,Ie_i) \\
&= \mathrm{Ric}(X,Y)
\end{aligned}$$

since if (e_i) is an orthonormal frame, so is (Ie_i). This allows to define an important object, called Ricci Form, by

$$\rho(X,Y) = \mathrm{Ric}(IX,Y)$$

From now on we will not distinguish between the real Ricci-form and its $\mathbb{C}$-bilinear extension. The first property of Ricci form

Lemma 1.2.21. *For a Kähler manifold X, the form ρ is in $\mathcal{A}^{1,1}_X$ and is closed.*

Proof. One has:

$$\begin{aligned}\mathrm{Ric}(I\frac{\partial}{\partial z_i},\frac{\partial}{\partial z_j}) =&\mathrm{Ric}(\frac{\partial}{\partial y_i}+i\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}-i\frac{\partial}{\partial y_j})\\ =&\mathrm{Ric}(\frac{\partial}{\partial y_i},\frac{\partial}{\partial x_j})+\mathrm{Ric}(\frac{\partial}{\partial x_i},\frac{\partial}{\partial y_j})\\ &+i\Big(\mathrm{Ric}(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j})-\mathrm{Ric}(\frac{\partial}{\partial y_i},\frac{\partial}{\partial y_j})\Big)\\ =&0\end{aligned}$$

and by conjugation one sees that ρ is a $(1,1)$-form.
To show that ρ is closed:

$$\begin{aligned}\mathrm{Ric}(X,Y) &= \sum_i R(e_i,X,Y,e_i)\\ &= \sum_i R(e_i,X,IY,Ie_i)\\ &= \sum_i -\big(R(X,IY,Ie_i,e_i)+R(IY,e_i,X,Ie_i)\big)\\ &= \sum_i R(X,IY,e_i,Ie_i)+R(Y,Ie_i,X,Ie_i)\\ &= \mathrm{tr}(R(X,IY)\circ I)-\mathrm{Ric}(X,Y)\end{aligned}$$

which yields, writing $2\rho(X,Y)=\mathrm{tr}(R(X,Y)\circ I)$:

$$\begin{aligned}2d\rho(X,Y,Z) &= 2\big((\nabla_X\rho)(Y,Z)+(\nabla_Y\rho)(Z,X)+(\nabla_Z\rho)(X,Y)\big)\\ &= 2\mathrm{tr}\big((\nabla_X R)(Y,Z)\circ I+(\nabla_Y R)(Z,X)\circ I+(\nabla_Z R)(X,Y)\circ I\big)\\ &= 0\end{aligned}$$

by the second Bianchi identity. $\square$

For Kähler manifolds the $\partial\bar{\partial}$-lemma ensures that there exists a function $f\in C^\infty(X,\mathbb{C})$ such that $\rho=\partial\bar{\partial}f$. We will now show that this primitive is strongly related to the metric structure.
Let $(x_1,\ldots,x_n,y_1,\ldots,y_n)$ with $y_i=I(x_i)$ be real coordinates with associated complex coordinates $(z_1,\ldots,z_n,\bar{z}_1,\ldots,\bar{z}_n)$, then

$$\begin{aligned}\rho_{i\bar{j}}-\rho_{\bar{j}i}=\rho(\frac{\partial}{\partial z_i},\frac{\partial}{\partial \bar{z}_j}) &= \mathrm{Ric}(I\frac{\partial}{\partial z_i},\frac{\partial}{\partial \bar{z}_j})\\ &= i\mathrm{Ric}(\frac{\partial}{\partial z_i},\frac{\partial}{\partial \bar{z}_j})=i(\mathrm{Ric}_{i\bar{j}}+\mathrm{Ric}_{\bar{j}i})\end{aligned}$$

since Ric is symmetric. Thus we have found that if $\mathrm{Ric}=R_{i\bar{j}}dz^i\otimes d\bar{z}^j$, then

$$\rho=\sqrt{-1}R_{i\bar{j}}dz^i\wedge d\bar{z}^j$$

We will show that curvature tensor for Kähler manifolds has many symmetries. Greek indices will range from 1 to n whereas the Latin ones from 1 to $2n$. In particular $Z_i = \frac{\partial}{\partial z_i}$ if $i \leq n$ and $Z_i = \frac{\partial}{\partial \bar{z}_i}$ if $i \geq n+1$. Complexified Christoffel symbols are given by

$$\nabla_{Z_i} Z_j = \Gamma_{ij}^k Z_k$$

Since the connection is torsion free, they are symmetric in the bottom indices and since $T^{1,0}$ is ∇-parallel

$$\Gamma_{i\bar{\beta}}^{\alpha} = 0$$

This all together shows that the only non-vanishing components are:

$$\Gamma_{\beta\gamma}^{\alpha} \qquad \Gamma_{\bar{\beta}\bar{\gamma}}^{\bar{\alpha}}$$

Now recall that the $\nabla_X^{LC} Y = \nabla_{X-iIX}^{Ch}(Y - iIY)$ for any X, Y sections of TX. Then

$$\frac{\partial h_{\beta\bar{\delta}}}{\partial z_\alpha} = \frac{\partial}{\partial z_\alpha} h(Z_\beta, Z_{\bar{\delta}}) = h(\nabla_{Z_\alpha} Z_\beta, Z_{\bar{\delta}}) = \Gamma_{\alpha\beta}^{\gamma} h_{\gamma\bar{\delta}}$$

by compatibility of h and the fact that $\nabla_{Z_\alpha} Z_{\bar{\delta}} = 0$. Thus one has the formula

$$\Gamma_{\alpha\beta}^{\epsilon} = \frac{\partial h_{\beta\bar{\delta}}}{\partial z_\alpha} h^{\bar{\delta}\epsilon}$$

In the same spirit, for the complexified curvature tensor

$$R(Z_a, Z_b)Z_c = R_{abc}^d Z_d$$

Since $T^{1,0}M$ is parallel, $R_{ab\bar{\delta}}^{\gamma} = 0$. For the only non-vanishing components, one has

$$R_{\alpha\bar{\beta}\gamma}^{\delta} Z_\delta = -\nabla_{Z_{\bar{\beta}}} \nabla_{Z_\alpha} Z_\gamma = -\nabla_{Z_{\bar{\beta}}} \left(\Gamma_{\alpha\gamma}^{\delta} Z_\delta\right) = -\frac{\partial \Gamma_{\alpha\gamma}^{\delta}}{\partial \bar{z}_\beta} Z_\delta$$

thus

$$R_{\alpha\bar{\beta}\gamma}^{\delta} = -\frac{\partial \Gamma_{\alpha\gamma}^{\delta}}{\partial \bar{z}_\beta}$$

To get the associated complexified Ricci tensor one has to sum over the upper index with the first lower one, obtaining

$$\mathrm{Ric}_{\bar{\beta}\gamma} = -\frac{\partial \Gamma_{\alpha\gamma}^{\alpha}}{\partial \bar{z}_\beta}$$

Together with the previuos formula this yields

$$\Gamma_{\alpha\gamma}^{\alpha} = h^{\bar{\delta}\alpha} \frac{\partial h_{\gamma\bar{\delta}}}{\partial z_\alpha} = \frac{1}{\det(h)} \frac{\partial \det(h)}{\partial z_\alpha}$$

where the last equality follows from Jacobi formula for the determinant. Then

$$\mathrm{Ric}_{\alpha\bar{\beta}} = -\frac{\partial^2 \log\big(\det(h)\big)}{\partial z_\alpha \partial \bar{z}_\beta}$$

and

$$\rho = i\bar{\partial}\partial \log\big(\det(h)\big)$$

Remark 1.2.22. If ∇ is a connection on a complex vector bundle $E \to X$, we have seen that the natural connection on $\det(E) \to X$ is given by

$$\nabla(s_1 \wedge \cdots \wedge s_n) = \sum_i (s_1 \wedge \cdots \wedge \nabla s_i \wedge \cdots \wedge s_n)$$

which implies that if the local expression for the connection is $\nabla s_i = \omega_i^j s_j$, for $\det(E)$ is

$$\nabla(s_1 \wedge \cdots \wedge s_n) = (\omega_1^1 + \cdots + \omega_n^n) s_1 \wedge \cdots \wedge s_n$$

Thus the induced curvature on $\det(E)$ is $d(\omega_1^1 + \cdots + \omega_n^n)$.
By taking the trace of the curvature of E, given by $\Omega_j^i = d\omega_j^i + \omega_j^k \wedge \omega_k^i$, one finds

$$\sum_i \Omega_i^i = \sum_i d\omega_i^i + \sum_{i,k} \omega_i^k \wedge \omega_k^i = \sum_i d\omega_i^i$$

where the second summand vanishes since

$$\sum_{i,k} \omega_i^k \wedge \omega_k^i = \sum_{k,i} \omega_k^i \wedge \omega_i^k = -\sum_{k,i} \omega_i^k \wedge \omega_k^i$$

We thus have two ways of getting a 2-form on X, by considering two different traces of the curvature: the Ricci-form ρ and the curvature form on the determinant $\sum_i \Omega_i^i$. These are different objects in general, but in the next paragraph we will see that for Kähler manifolds they coincide. This is a highly non trivial property, it is indeed equivalent to the fact that ρ coincides, up to a constant, with the first Chern class $c_1^{\mathbb{R}}(X)$.

1.3 Chern Classes

Chern Classes are a particular type of characteristic classes, whose general definition is the following

Definition 1.3.1. Let $\mathrm{Vect}_k : \mathbf{Man} \to \mathbf{Set}$ be the functor which associates to M the set of iso-classes of rank k vector bundles over M and let $H^* : \mathbf{Man} \to \mathbf{Set}$ be the cohomology functor. Then a characteristic class is a natural transformation $\mathrm{Vect}_k \implies H^*$.

This definition provides a first intuition, but is clearly too abstract to be useful in practice. We will first define Chern classes in terms of the curvature. This approach provides in particular a way to compute these objects. At the end of the paragraph we will give the intersection-theoretic definition of Chern classes, which helps to understand the geometric idea behind them.

1.3.1 The Chern-Weil homomorphism

Let $E \to X$ be a k-vector bundle and ∇ a connection on X. If Ω_i^j are the 2-forms of the curvature in a local frame, under a change of local frame for E,

$\tilde{e}_j = A^i_j e_i$, one has $\tilde{\Omega} = A^{-1}\Omega A$, indeed

$$A^l_j \Omega^k_l e_k = R_\nabla(A^l_j e_l) = R_\nabla(\tilde{e}_j) = \tilde{\Omega}^i_j \tilde{e}_i = \tilde{\Omega}^i_j A^k_i e_k$$

Consider now a polynomial $p \in k[x_1, \dots, x_{r^2}]$ for an arbitrary field, which can be identified as a polynomial function $M_{r\times r}(k) \to k$. Then p is said invariant if

$$p(A^{-1}XA) = p(X) \quad \text{for any } A \in \mathrm{GL}(r \times r, k)$$

Two easy examples are $\mathrm{tr}_k(X) = \mathrm{tr}(X^k)$ and by the coefficients of $\det(X + \lambda Id)$, seen as a polynomial in λ. If one takes $k = \mathbb{C}$, the latter is in some sense the most important invariant polynomial, since

Theorem 1.3.2. *The ring of invariant complex polynomials on $gl(r, \mathbb{C})$ is generated as an algebra over $\mathbb{C}$ by the coefficients $f_i(X)$ of $det(\lambda Id + X)$. Thus,*

$$Inv(gl(r, \mathbb{C})) = \mathbb{C}[f_1(X), \dots, f_r(X)]$$

For a proof see [Tu17, p. 313].

Consider now a homogeneous invariant polynomial of degree k on $gl(r, \mathbb{C})$, fix a point $p \in U$, with U trivializing E, and consider the $\mathbb{C}$-algebra

$$\mathcal{A} = \bigoplus_{i=0}^{dim X} \bigwedge^{2i} T^*_{\mathbb{C},p} X$$

The matrix $\Omega|_p$ clearly belongs to $\mathcal{A}$ and one has

$$p(\tilde{\Omega}|_p) = p(\Omega|_p) \qquad \text{for any } p \in U$$

We have defined in this way a local $2k$-differential form over U. Since p is invariant, if U_α andU_β are two overlapping trivializing open set for E, $p(\Omega)$ is still well defined in $U_\alpha \cap U_\beta$, because the induced action of the structure group $GL(r, \mathbb{C})$ on $\mathcal{A}^{2k}$ is given by adjoint action.

Therefore one has a well defined $2k$-form over X. The important result is that

Theorem 1.3.3. *The $2k$-form $p(\Omega)$ is closed and its cohomology class is independent of the connection.*

Proof. To prove that $d(p(\Omega)) = 0$ we use the fact that if V is a complex vector space of dimension n, the $\mathbb{C}$-algebra $\mathrm{Sym}^k(V^*)$ of symmetric multilinear forms of degree k is isomorphic to the algebra of d-homogeneous polynomials $\mathbb{C}[x_1, \dots, x_n]_d$. Now let $V = gl(r, \mathbb{C})$ and let $\mathrm{GL}(r, \mathbb{C})$ act on it by adjoint representation. This induces a left action on $\mathrm{Sym}^k(V^*)$ and on $\mathbb{C}[x_1, \dots, x_n]_d$ via

$$(L_A\sigma)(v_1, \dots, v_k) = \sigma(L_A v_1, \dots, L_A v_k)$$
$$(L_A p)(x_1, \dots, x_n) = p(L_A(x_1, \dots, x_n))$$

and the isomoprhism above is equivariant with respect to these two actions. This allows to identify the $\mathrm{GL}(r, \mathbb{C})$-invariant subalgebra of $\mathrm{Sym}^k(V^*)$ with the

$\mathrm{GL}(r,\mathbb{C})$-invariant subalgebra of $\mathbb{C}[x_1,\dots,x_n]_d$.
Now if f is an invariant element of $\mathrm{Sym}^k(V^*)$,

$$d(f(\Omega,\dots,\Omega)) = \tilde{\nabla}(f(\Omega,\dots,\Omega)) = kf(\hat{\nabla}\Omega,\dots,\Omega) = 0$$

where ∇ is the trivial connection on $X \times \mathbb{C}$ and $\hat{\nabla}$ is the induced connection on $\mathrm{End}(TM)$. The second equality follows from symmetry and the the third one by general Bianchi identity.
Thanks to the above isomoprhisms, we conclude that $p(\Omega)$ is closed.
If ∇_0 and ∇_1 are two different connection, since the space of connection is an affine $\mathbb{C}$-vector space over $\mathcal{A}^1(M,TM)$, consider the segment of connections

$$\nabla_t = t\nabla_0 + (1-t)\nabla_1 \qquad t \in [0,1]$$

Then $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$ and as $\Omega_t = d\omega_t + \omega_t \wedge \omega_t$

$$\begin{aligned}\Big(\frac{d}{dt}\Omega_t\Big)dt &= d(\omega_1-\omega_0) + (\omega_1-\omega_0)\wedge\omega_t + \omega_t \wedge (\omega_1-\omega_0)\\ &= \nabla_t(\omega_1-\omega_0)dt\\ &= 0\end{aligned}$$

Set $\phi = k \cdot \int_0^1 f((\omega_1-\omega_0),\Omega_t,\dots,\Omega_t)dt$, since $(\omega_1-\omega_0)$ transforms in the same way as the curvature, the integrand does not depend on the choice of local frame and $f((\omega_1-\omega_0),\Omega_t,\dots,\Omega_t)$ is a well-defined $2k-1$ form and so is ϕ.
As $\nabla_t\Omega_t = 0$, we find

$$\begin{aligned}k \cdot df((\omega_1-\omega_0),\Omega_t,\dots,\Omega_t) &= k \cdot f(\nabla_t(\omega_1-\omega_0),\Omega_t,\dots,\Omega_t)\\ &= k \cdot f\Big(\frac{d}{dt}(\Omega_t),\Omega_t,\dots,\Omega_t\Big)\\ &= \frac{d}{dt} f\Big(\Omega_t,\Omega_t,\dots,\Omega_t\Big)\end{aligned}$$

hence

$$d\phi = k \cdot \int_0^1 \frac{d}{dt} f(\Omega_t,\Omega_t,\dots,\Omega_t)dt$$

which shows that if f is symmetric $f(\Omega_1,\dots,\Omega_1)$ and $f(\Omega_0,\dots,\Omega_0)$ are in the same cohomology class. The result for arbitrary invariant polynomials follows by the above isomorphism. $\square$

This allows to define a $\mathbb{C}$-algebra homomorphism

$$\begin{aligned}c_E : \mathrm{Inv}(\mathrm{gl}(r,\mathbb{C})) &\to H^*(X)\\ p(X) &\mapsto [p(\Omega)]\end{aligned}$$

called Chern-Weil homomorphism. Complex characteristic classes are then the elements of the image of c_E.
We see that although characteristic classes can be defined in term of a chosen

connection, they depend only on some features that all the possible connections detect. Indeed, as we will see later, characteristic classes depend only on the topology of the vector bundle.
We can now give

Definition 1.3.4. Let $E \to X$ be a complex vector bundle which admits a connection ∇. Then the real Chern classes of E $c_i(E)$ are defined as

$$\det\left(\mathrm{Id} + \frac{i}{2\pi}\Omega\right) = 1 + c_1^{\mathbb{R}}(E) + \cdots + c_r^{\mathbb{R}}(E)$$

The term real refers to the fact that $c_i(E)$ as defined here are actually the image of a more fundamental object under the map

$$H^*(X, \mathbb{Z}) \to H^*(X, \mathbb{R}) = H^*(X, \mathbb{Z}) \otimes \mathbb{R}$$

1.3.2 Integer Chern classes

We recall that a complex vector bundle over a complex manifold X is said topological, differentiable or holomoprhic if the cocycles are respectively continuous, smooth or holomoprhic. Every total space over X is then in a natural way respectively a topological manifold, a real differentiable manifold or a complex manifold. A morphism over X is respectively a continuous, smooth or holomorphic map $E \to F$ making the obvious diagram commute.

Definition 1.3.5. Let $L \to X$ be a holomorphic line bundle over a smooth complex manifold X. Then c_1 is defined by the long exact sequence

$$\cdots \to H^1(X; \mathcal{O}_X) \to H^1(X; \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X; \mathbb{Z}) \to H^2(X; \mathcal{O}_X) \to \dots$$

coming from the exponential sequence of sheaves

$$0 \to \underline{\mathbb{Z}} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$$

The group $H^1(X; \mathcal{O}_X^*)$ is in bijection with the set of isomorphism classes of holomoprhic line bundles (see [Voi02]), endowing the latter with tensor product one has a group isomorphism $H^1(X; \mathcal{O}_X^*) \backsimeq \mathrm{Pic}(X)$. Thus c_1 is invariant by isomorphism, but it is not a complete invariant, as in general c_1 is not injective. Indeed, $\ker(c_1)$ is a interesting invariant of the manifold X. If X is algebriac, $\ker(c_1)$ has a natural algebraic structure and is known as Picard variety of X, $\mathrm{Pic}^0(X)$.
We know give a necessary and sufficient condition for $\mathrm{Pic}^0(X)$ not to be trivial.

Theorem 1.3.6. *The variety $Pic^0(X)$ is trivial if and only if $h^{0,1}(X) = 0$.*

Proof. If X is compact the map $H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X^*)$ is surjective, thus $H^0(X, \mathcal{O}_X^*) \to H^1(X, \mathbb{Z})$ is the zero map, which implies that $H^1(X, \mathbb{Z}) \to$

$H^1(X,\mathcal{O}_X)$ is injective. Now $\ker(c_1) = \mathrm{Im}(H^1(X,\mathcal{O}_X) \to H^1(X,\mathcal{O}_X^*))$ and by exactness

$$\ker(c_1) \backsimeq \frac{H^1(X,\mathcal{O}_X)}{\mathrm{Im}(H^1(X,\mathbb{Z}) \to H^1(X;\mathcal{O}_X))} \backsimeq \frac{H^1(X,\mathcal{O}_X)}{H^1(X,\mathbb{Z})}$$

It can be proved that $H^1(X,\mathbb{Z})$ is a lattice inside $H^1(X,\mathcal{O}_X)$, hence the latter group is a complex torus of dimension $h^{0,1}(X)$. $\square$

Algebraic varieties with this property have a particular name

Definition 1.3.7. An algebraic variety X (over an arbitrary field) is said without irregularities if $H^1(X,\mathcal{O}_X) = 0$.

Alternatively definition one might define c_1 via another exponential sequence, namely

$$0 \to \underline{\mathbb{Z}} \xrightarrow{2\pi i} C_X \xrightarrow{\exp} C_X^* \to 0$$

where C_X is the sheaf of continuous $\mathbb{C}$-valued functions. Then c_1 is the map

$$H^1(X;C_X^*) \xrightarrow{c_1} H^2(X;\mathbb{Z})$$

which is now an isomorphism, since C_X is a flasque sheaf, hence $H^i(X;C_X) = 0$ if $i > 0$. The advantage of this second definition is that it applies to any topological complex vector bundle.

As before, $H^1(X;C_X^*)$ is in bijection with the isomorphism classes of complex topological line bundles. The fact the c_1 is not injective in the first case but it is in the second reflects the existence of line bundles which are topologically trivial but not holomorphically trivial.

This definition also allows to interpret geometrically c_1: a topological line bundle $L \to X$ is uniquely determined by the continuous cocycles $t_{ij} : U_{ij} \to \mathbb{C}^*$ and these can be identified with an element of $H^1(X,C_X^*)$. Equivalently, a line bundle is defined by a choice of an open cover U_i (which can be assumed finite) and some continuous functions $f_i : U_i \to \mathbb{C}$ such that $f_i/f_j = t_{ij}$. These f_i are clearly not uniquely determined, but if one requires that they intersect the zero section in a real codimension 2 topological submanifold, with finitely many connected components, then the associated fundamental class $\mathcal{Z} = [Z(f_i)+\cdots+Z(f_k)]$ is uniquely determined. Moreover, two cocycles gives rise to the same $\mathcal{Z}$ if and only if they are isomorphic, thus coincides with $\mathcal{Z} = c_1(L)$.

In particular, in holomorphic setting we have the important identity

$$c_1(\mathcal{O}(D)) = [D]$$

We now give two examples: in the first one $c_1(L) = 0$ even though L is not trivial and in the second one $c_1^{\mathbb{R}}(L) = 0$ whereas $c_1(L) \neq 0$.

Example 1.3.8. Let X a genus one Riemann surface and take $x, y \in X$. Then $\mathcal{O}(x-y)$ is not holomorphically trivial, since zero is the only global section. Indeed $\mathbb{P}\big(H^0(X;\mathcal{O}(x-y)\big) \backsimeq |x-y|$ and the latter is trivial. To prove that $c_1(\mathcal{O}(x-y)) = 0$, we use the fact $c_1(\mathcal{O}([D]) = [D]$. In this case $[x-y] = [x]-[y]$ as taking fundamental class is linear, and $[x] \backsimeq [y]$ since X is path-connected.

Example 1.3.9 (Enriques Surfaces)**.** Let X be a complex surface with K_X non trivial but $K_X^{\otimes 2} \backsimeq \mathcal{O}_X$ and without irregularities, so that in particular c_1 is an isomorphism. We have that $H^0(X, K_X) = 0$ as any non zero global section s should satisfy $s \otimes s \in \mathbb{C}$, but this would imply that s is itself trivial (see at level at stalks) and non zero. In particular $c_1(K_X) \neq 0$. But by what we will prove later, $0 = c_1(K_X)^{\otimes 2} = 2c_1(K_X)$, which implies that $c_1^{\mathbb{R}}(K_X) = 0$.

The quickest way to define Chern Classes is by axioms.

Definition 1.3.10. Let X be a real topological manifold. Then the Chern class map is the unique map c which associates a complex vector bundle $E \to X$ an element (called Chern polynomial)

$$c(E) \in H^*(X; \mathbb{Z})[t]$$

with $\deg_{2i}(c(E)) = c_i(E)$ satisfying:

1. If $\operatorname{rank}(E) = 1$, $c(E) = 1 + tc_1(E)$
2. For any map $f : Y \to X$, one has $c(f^*E) = f^*(c(E))$ (naturality)
3. If E and F are two vector bundles over X, then $c(E \oplus F) = c(E)c(F)$ where the ring structure of $H^*(X; \mathbb{Z})[t]$ is used (Whitney sum)

Whitney sum implies that $c_k(E \oplus F) = \sum_i^k c_i(E)c_{k-i}(F)$.
To prove the existence, given a rank $k+1$ complex vector bundle $E \to X$ with $\dim_{\mathbb{C}} X = n$, the associated projective bundle $\mathbb{P}(E) \to X$ is defined as

$$\frac{\coprod_i (U_i \times \mathbb{P}^k)}{\backsim}$$

where $\{U_i\}$ is a trivializing cover for $E \to X$ and $(x, [\alpha]) \backsim (x, [t_{ij}(\alpha)])$ for any $x \in U_i \cap U_j$ and with t_{ij} transition functions of $E \to X$. In this way $\mathbb{P}(E)$ is a $(n+k)$-complex manifol with the obviuos charts and it naturally gives rise to a fiber bundle $\mathbb{P}(E) \to X$. Thus one can consider the pullback diagram

$$\begin{tikzcd}
f^*E \arrow[r, "\tilde{f}"] \arrow[d, "\pi"] & E \arrow[d, "p"] \\
\mathbb{P}(E) \arrow[r, "f"] & X
\end{tikzcd}$$

where

$$f^*E = \{\big((x,l), e\big) \in \mathbb{P}(E) \times E : x = p(e)\}$$

and $\pi\big((x,l), e\big) = (x,l)$, which shows that $f^*E \to \mathbb{P}(E)$ is a rank k vector bundle. Now define

$$\mathcal{O}_E(-1) := \{\big((x,l), e\big) \in f^*E : e \in l\}$$

which is a subbundle of f^*E with one dimensional fiber, known as the tautological line bundle associated with $E \to X$. Over each direction in E_x, which is a point in $\mathbb{P}(E)$, the fiber is exactly the line which had been identified; in some sense this recovers the information lost with the projectivization.
We will denote its dual by

$$\mathcal{O}_E(1) = \mathcal{O}_E(-1)^*$$

Let $\eta = c_1(\mathcal{O}_E(1)) \in H^*(\mathbb{P}(E))$, we have that $\eta = [1] \in H^2(\mathbb{P}(E))$ and $[1]$ can be identified with the Poincaré dual of the submanifold of $\mathbb{P}(E)$

$$Y = \{(x, l) \in \mathbb{P}(E) : l \neq l_0\}$$

where l_0 is a fixed line in $\mathbb{C}$ and we use the same trivialization $\phi_i^{-1} : U_i \times \mathbb{C} \to E|_{U_i}$ to determine the two lines in $\mathbb{P}(E)$.
The ring homomorphism

$$h^* : H^*(X, \mathbb{Z}) \to H^*(\mathbb{P}(E), \mathbb{Z})$$

turns $H^*(\mathbb{P}(E), \mathbb{Z})$ into a $H^*(X, \mathbb{Z})$-algebra. The following theorem states that this module is free and finetely generated and that a basis is given by monomials of η.

Theorem 1.3.11. *The cohomology classes $1, \eta, \ldots, \eta^{k-1}$ are a basis of $H^*(\mathbb{P}(E))$ over $H^*(X)$. In particular h^* is injective.*

Theorem 1.3.12. *The map c defined by axioms above always exists.*

Proof. The identity

$$\eta^k + f^*c_1(E)\eta^{k-1} + \cdots + f^*c_k(E) = 0$$

in $H^*(\mathbb{P}(E), \mathbb{Z})$ clearly determines $c_i(E) \in H^{2i}(X, \mathbb{Z})$, after having set $c_0(E) = 1$.
We claim that it satisfies the axioms and that $c(E)$ defined in this way is the unique polynomial satisfying them.
The first condition is satisfied if E has rank 1: $\mathbb{P}(E)$ is isomorphic to X via f and $S = f^*E$. Then by definition

$$\eta = -c_1(\mathcal{O}_E(-1)) = -c_1(E)$$

and this agrees with $\eta + f^*c_1(E) = 0$. If the rank is greater than one, one cocncludes with the splliting construction below.
For the second condition, let $g : Y \to X$. Then

$$\mu^k + h^*c_1(g^*E)\mu^{k-1} + \cdots + h^*c_k(g^*E) = 0$$

with $h : \mathbb{P}(g^*E) \to Y$ and $1, \mu, \ldots, \mu^r$ a basis of $H^*(\mathbb{P}(g^*E))$. Let u be the natural map $\mathbb{P}(g^*E) \to \mathbb{P}(E)$, then $\mu^i = u^*\eta^i$ for any i and $fu = gh$. By injectivity of f^* and u^* one sees that the same relation is satisfied by $g^*c(E)$.

For the last condition, let $E = F \oplus G$. Then $\mathbb{P}(E)$ contains to projective subbundles $\mathbb{P}(F)$ and $\mathbb{P}(G)$ which do not intersect. Let $Z_F = \mathbb{P}(E) \setminus \mathbb{P}(F)$ and $Z_G = \mathbb{P}(E) \setminus \mathbb{P}(G)$ and consider $\pi_F : Z_G \to \mathbb{P}(F)$, $\pi_G : Z_F \to \mathbb{P}(G)$.
The tautological bundle $\mathcal{O}_E(-1)$ restricted to Z_G is isomoprhic to $\pi_F^*(\mathcal{O}_F(-1))$ and the restriction to Z_F is isomoprhic to $\pi_G^*(\mathcal{O}_G(-1))$. Consider now the monic polynomial $c' = t^k c(\frac{1}{t})$ which by definition annihilates h, in particular $c'(F)(h_F) = 0$ in $H^{2\cdot \mathrm{rk}(F)}(\mathbb{P}(F))$ and $c'(G)(h_G) = 0$ in $H^{2\cdot \mathrm{rk}(G)}(\mathbb{P}(G))$. then

$$\begin{aligned} \pi_F^*\big(c'(F)(h_F)\big) &= c'(F)(h_E)|_{Z_G} = 0 \qquad \text{in } H^{2\cdot \mathrm{rk}(F)}(\mathbb{P}(G)) \\ \pi_G^*\big(c'(G)(h_G)\big) &= c'(G)(h_E)|_{Z_F} = 0 \qquad \text{in } H^{2\cdot \mathrm{rk}(G)}(\mathbb{P}(F)) \end{aligned}$$

As $\mathbb{P}(E) = Z_F \cup Z_G$, the cup product $c'(F)(h_E) \cup c'(G)(h_E)$ vanishes in $H^{2\cdot \mathrm{rk}(E)}(\mathbb{P}(E))$, since the two relations above shows that $c'(F)(h_E)$ and $c'(G)(h_E)$ can be represented by cochains supported respectively in an open neighborhood of $\mathbb{P}(G)$ and $\mathbb{P}(F)$. As the polynomial $c'(F)c'(G)$ annihilates h_E and is monic of degree k, it must be equal to $c'(E)$. Thus $c(E) = c(F)c(G)$.
We also have to prove that c defined in this wat is unique. For this we need the following two lemmas. $\square$

Theorem 1.3.13 (Splitting Construction)**.** *Let $E \to X$ be a complex vector bundle of rank k, with X a topological (differentiable) manifold. Then there exists a topological (differentiable) manifold $Fl(E)$, called flag manifold associated to $E \to x$ and a continuous (smooth) map*

$$f : Fl(E) \to X$$

such that:

- *The pullback $f^*E \to Fl(E)$ is isomorphic to a direct sum of k line bundles*
- *The map $f^* : H^*(X) \to H^*(Fl(E))$ is injective*

Proof. By induction on $\mathrm{rank}(E)$. If E is a line bundle there is nothing to prove. If $\mathrm{rank}(E) = k$, consider the tautological and the quotient bundle over $\mathbb{P}(E)$ defined by

$$0 \to \mathcal{O}_E(-1) \to f^*E \to Q_E \to 0$$

Since Q_E has $\mathrm{rank}(Q_E) = k - 1$, it can be pulled back to a direct sum of $k-1$ line bundles. The diagram depicts the situation

$$\begin{tikzcd} \mathcal{O}_E(-1) \oplus L_1 \oplus \cdots \oplus L_k \arrow[d] & \mathcal{O}_E(-1) \oplus Q_E \backsimeq g^*E \arrow[d] & E \arrow[d, "p"] \\ \mathbb{P}(Q_E) \arrow[r, "h"] & \mathbb{P}(E) \arrow[r, "g"] & X \end{tikzcd}$$

where we used that short exact sequences of complex vector bundles always split. Thus $f = g \circ h$ is the map which realizes the splitting.
To prove injectivity of f, by the very definition of Chern classes one has

$$H^*(\mathbb{P}(E)) = \frac{H^*(X)[x]}{(x^k + c_1(E)x^{k-1} + \cdots + c_k(E))}$$
$$H^*(\mathbb{P}(Q_E)) = \frac{H^*(\mathbb{P}(E))[y]}{(y^{k-1} + c_1(Q_E)x^{k-2} + \cdots + c_{k-1}(Q_E))}$$

and combining the two one finds

$$H^*(\mathbb{P}(Q_E)) = \frac{H^*(X)[x,y]}{(x^k + c_1(E)x^{k-1} + \cdots + c_k(E), y^{k-1} + c_1(Q_E)x^{k-2} + \cdots + c_{k-1}(Q_E))}$$

□

The theorem is particularly convenient because Chern classes of line bundles are simple objects. It naturally yields the following

Theorem 1.3.14 (Splitting Principle)**.** *If a polynomial identity between Chern classes holds in $H^*(Fl(E))$, then it also holds in $H^*(X)$.*

Indeed if one wants to verify a polynomial identity $p(c_1(E), \ldots, c_k(E)) = 0$ in $H^*(X)$, consider the pullback

$$f^*p(c_1(E), \ldots, c_k(E)) = p(f^*c_1(E), \ldots, f^*c_k(E)) = p(c_1(f^*E), \ldots, c_k(f^*E))$$

Thus if the identity holds for direct sums of line bundles

$$p(c_1(E), \ldots, c_k(E)) = 0$$

by injectivity of f.

Proof of the unicity. Let $f : Fl(E) \to X$ be as in the lemmas, then

$$f^*(c(E)) = c(f^*E) = c(\oplus_{i=1}^k L_i) = \prod_{i=1}^k (1 + tc_1(L_i))$$

and injectivity of f^* completely determines $c(E)$. □

We will now give some consequences of the splitting principle.

Lemma 1.3.15 (Vanishing)**.** *If $E \to X$ has rank k, then $c_i(E) = 0$ for any $i > k$.*

Proof. Since the term on the right is a polynomial of degree at most k, so must be $f^*c(E)$. □

Lemma 1.3.16. *If $E \to X$ of rank k admits a never vanishing section s, then $c_r(E) = 0$.*

Proof. In this case E can be written as $E = \mathbb{C} \oplus F$, where the trivial bundle is spanned by s. Then the result follows by Whitney sum property. $\square$

Alternatively, onw may observe that the vector bundle $f^*E \otimes \mathcal{O}_E(1)$ admits a never vanishing global section, given by gluing the local sections $s \otimes s^*$, where s is in the image of $\mathcal{O}_E(-1) \hookrightarrow f^*E$, thus $c_k(f^*E \otimes \mathcal{O}_E(1)) = 0$ and this is equivalent to the above relation.
It is not immediate to provide a geometric intuition to Chern classes. The above lemma shows that c_k measures the obstruction to the existence of a never vanishing section. One can indeed prove (see [EH16]) that for a complex topological vector bundle $E \to X$ of rank k

$$c_i(E) = [D(s_0, \dots, s_{k-i}] \in H^{2i}(X, \mathbb{Z})$$

where s_0 are continuous global sections which fail to be independent in a codimension i complex topological submanifold and D is exactly this locus.
The splitting principle also provides a way to compute Chern classes of tensor bundles. The general formula is rather complicated and can be found in [EH16].

Lemma 1.3.17. *If E and F are two complex vector bundles over X of rank e and f respecively, one has*

$$c_1(E \otimes F) = f \cdot c_1(E) + e \cdot c_1(F)$$

Proof. If $E = \oplus_i L_i$ and $F = \oplus_j M_j$ where L_i and M_j are line bundles, then

$$E \otimes F = \bigoplus_{i,j} L_i \otimes M_j$$

and consequently

$$c(E \otimes F) = \prod_{i,j} c(L_i \otimes M_j)$$

Extracting the degree two part yields

$$c_1(E \otimes F) = \prod_{i,j} c_1(L_i \otimes M_j) = f \cdot c_1(L_i) + e \cdot c_1(M_j)$$

where in the second equality we used the line bundle definition of c_1.
For arbitrary E and F, let $u^*E \backsimeq \oplus_i L_i$ and $v^*F \backsimeq \oplus_j M_j$ with u, v defined as in previous lemmas. Then

$$c(u^*E \otimes v^*F) = c(u^* \otimes v^*(E \otimes F)) = u^* \otimes v^*(c(E \otimes F))$$

and if u^*, v^* are injective, so is $u^* \otimes v^* : H^*(E \otimes F) \to H^*(X)$. $\square$

This lemma implies in particular that if L is a line bundle, $c_1(L^*) = -c_1(L)$. If E is not invertible, one again uses splitting principle: if $E \backsimeq \bigoplus_i L_i$, then

$E^* = \bigoplus_i L_i^*$ hence $c(E^*) = \prod_i (1 - c_1(L_i))$ which implies $c_i(E^*) = (-1)^i c_i(E)$; if $f^*E \backsimeq \bigoplus_i L_i$, then

$$f^*c(E^*) = \prod_i (1 - c_1(L_i)) = -f^*c(E)$$

hence the conlcusion.

Lemma 1.3.18. *Let $E \to X$ be a rank k complex vector bundle. Then*

$$c_1(\det(E)) = c_1(E)$$

Proof. If $E \backsimeq \bigoplus_i L_i$ then $\det(E) = \otimes_i L_i$. This is easy to see this by checking the cocycles. Then

$$c_1(\det(E)) = \sum_i c_1(L_i) = c_1(E)$$

If $f^*E \backsimeq \bigoplus_i L_i$, one has the conclusion as $f^*c_1(\det(E)) = f^*c_1(E)$. $\square$

This result is indeed equivalent to $c_1(E) = [D(s_1, \ldots, s_{\mathrm{rank}(E)})]$.
We conclude the paragraph with two examples. Let X be a complex manifold, Chern classes can be computed for some special bundles on it: the holomorphic tangent bundle T_X, its dual Ω_X, the canonical bundle K_X and the anticanonical bundle K_X^*. Chern classes of the holomoprhic tangent bundle are denoted as $c_i(X) = c_i(T_X)$.

Example 1.3.19. For $X = \mathbb{P}^n$, the so called Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1} \to T_{\mathbb{P}^n} \to 0$$

is built in the following way. On a vector space V, let

$$q : V \setminus \{0\} \to \mathbb{P}V$$

be the quotient map and

$$dq_v : V_v^\times \to T_{[v]}\mathbb{P}^n$$

its differential. One clearly has that $\ker(dq_v) = \mathcal{L}(v)$, since radial derivation is mapped to the trivial one. Thus one has an isomorphism

$$V/\mathcal{L}(v) \to T_{[v]}\mathbb{P}^n$$

which depends on v. It is possible to make it canonical by normalizing: for any linear form $\alpha : \mathcal{L}(v) \to k$, $\alpha(v)dq_v$ does not depend on v anymore, but only on its class. This yields a canonical identification

$$\mathcal{L}(v)^* \otimes V/\mathcal{L}(v) \to T_{[v]}\mathbb{P}^n$$

on fibers, which comes from the sheaf morhism

$$\mathcal{O}_{\mathbb{P}^n}(1) \otimes Q \to T_{\mathbb{P}^n}$$

which implies the Euler sequence.
One can also build the dual sequence as

$$0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0$$

Thus by Whitney formula one finds

$$c(T_{\mathbb{P}^n}) = (1+\zeta)^{n+1}$$

where $\zeta = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$.
This implies in particualr that every section of $T_{\mathbb{P}^n}$ vanishes in some point, since

$$c_n(T_{\mathbb{P}^n}) = (n+1)\zeta^n$$

Example 1.3.20. Let $Y \subset X$ be a complex submanifold, then there is an injective holomorphic map of bundles $T_Y \to T_X|_Y$. This yields an exact sequence of bundles over Y

$$0 \to T_Y \to T_X|_Y \to N_{Y/X} \to 0$$

where $N_{Y/X}$ is known as normal bundle.
Then the so called adjunction formula holds, where the isomorphism is in the holomorhic category

$$K_Y \backsimeq K_X|_Y \otimes \det(N_{Y/X})$$

If Y is a hypersurface, then $N_{Y/X} \backsimeq \mathcal{O}(Y)|_Y$ where $\mathcal{O}(Y)$ is the line bundle on X associated to the divisor Y. The normal sequence splits in the differentiable category, thus

$$c(i^*T_X) = c(T_Y) \cdot c(i^*\mathcal{O}(Y))$$

or more explicitly

$$c(T_Y) = i^*\Big(c(T_X)(1 - c_1(\mathcal{O}(Y))\Big)$$

and in particular

$$c_1(T_Y) = i^*\Big(c_1(T_X) - c_1(\mathcal{O}(Y))\Big)$$

Indeed the formula for the first chern clss could also have been obtained by the adjunction formula, as

$$c_1(T_Y) = -c_1(K_X) = -\Big(c_1(i^*K_X) + c_1(i^*\mathcal{O}(Y))\Big)$$

If $Y \subset \mathbb{P}^n$ is of he form $Y = V(F)$, with F a homogeneous polynomial of degree d, then Y is the vanishing locus of some $s \in H^0(\mathbb{P}^n; \mathcal{O}(d))$, since $H^0(\mathbb{P}^n; \mathcal{O}(d))$ is naturally identified with $\mathbb{C}[x_0, \dots, x_n]_d$.
Then by adjunction formula $K_Y = K_{\mathbb{P}^n}|_Y \otimes \det(\mathcal{O}(d)) = \mathcal{O}(d-n-1)|_Y$. Thus

$$c_1(Y) = -c_1(K_Y) = c_1(i^*\mathcal{O}(n+1-d))$$

In particular if $d = n+1$, $c_1(Y) = 0$. We will see later on that this class of algebraic hypersurfaces is particularly relevant.

Example 1.3.21. With the above nation, fix $n = 3, d = 4$, then $c_2(Y) = i^*c_2(\mathbb{P}^3)$ and

$$c_2(T_{\mathbb{P}^3}) = c_2(\mathcal{O}(1)^{\oplus 3+1}) = 6(c_1(\mathcal{O}(1)))^2$$

by the Euler sequence and the Whitney sum property.
Then one can compute

$$\begin{aligned}\int_Y c_2(Y) = \int_Y i^*c_2(\mathbb{P}^3) &= \int_{\mathbb{P}^3} c_2(\mathbb{P}^3)c_1(\mathbb{P}^3)\\ &= \int_{\mathbb{P}^3} c_2(\mathbb{P}^3)c_1(\mathbb{P}^3) = 24\int_{\mathbb{P}^3}(c_1(\mathcal{O}(1))^3\end{aligned}$$

where we used that $c_1(\mathbb{P}^3) = 4(c_1(\mathcal{O}(1))$ and that $c_1(\mathcal{O}(1))$ belongs to the hyperplane class of $\mathbb{P}^n$, which is by definition the cohomology class of $\mathrm{Pd}(\mathbb{P}^{n-1})$.
Thus one has

$$\int_{\mathbb{P}^3}(c_1(\mathcal{O}(1))^3 = \int_{\mathbb{P}^2} i^*(c_1(\mathcal{O}(1))^2 = \int_{\mathbb{P}^1} j^*(c_1(\mathcal{O}(1)) = 1$$

where we used that

$$i^*(c_1(\mathcal{O}_{\mathbb{P}^3}(1)) = \frac{1}{4}i^*c_1(\mathbb{P}^3) = \frac{1}{4}\Big(c_1(\mathbb{P}^2) + c_1(\mathcal{O}_{\mathbb{P}^2}(1))\Big) = c_1(\mathcal{O}_{\mathbb{P}^2}(1))$$

and the last two equalities holds since

$$i^*\mathcal{O}_{\mathbb{P}^3}(\mathbb{P}^2) \backsimeq \mathcal{O}_{\mathbb{P}^2}(1)$$

which can be checked comparing the cocycles.
The result $\int_Y c_2(Y) = 24$ has a really surprising geometric meaning: it is the Euler characteristic of Y, and by Poincaré-Hopf theorem it coincides with the sum of the indices of zeros of a generic differentiable vector field.

1.3.3 First Chern class for Kähler manifolds

For a complex vector bundle $E \to X$ over an arbitrary complex manifold X, Chern-Weil theory allows to compute $c_i^{\mathbb{R}}(E)$, after the choice of any connection. If $E = T_X$, then $c_i^{\mathbb{R}}(X) = c_i^{\mathbb{R}}(T_X)$ by definition. In particular

$$c_i^{\mathbb{R}}(X) = \mathrm{tr}\Big(\frac{i}{2\pi}\Omega^i_j\Big) = \frac{i}{2\pi}\sum_j \Omega^j_j$$

and we have seen that $\sum_j \Omega^j_j$ is the curvature form on $\det(T_X)$. This agress with the fact that $c_1(E) = c_1(\det(T_X))$.
If X in endowed with a Kähler metric h, there is a peculiar representative for $c_1^{\mathbb{R}}(X)$, given by a multiple of the Ricci form induced by the Levi-Civita (equivalently Chern) connection, i.e. $\frac{\rho}{2\pi}$. This is easy to see thanks to the computation of the previous paragraph, indeed

$$\rho = i\bar{\partial}\partial\log(\det(h)) \qquad \sum_j \Omega^j_j = \bar{\partial}\partial\log(\tilde{h}) = \bar{\partial}\partial\log(\det(h))$$

where $\tilde{h} = \det(h)$ is the induced metric on $-K_X$. This is a surprising fact since $c_1^{\mathbb{R}}(X)$ does not depend at all on the metric structure.
Thus if X is a (not necessarily compact) Kähler manifold then $\rho = 2\pi c_1^{\mathbb{R}}(X)$ in real cohomology. If one restricts consider compact Kähler manifolds, this famous result holds

Theorem 1.3.22 (Calabi, Yau, [Yau78])**.** *Let (X, ω) be a compact Kähler manifold. Then for any closed form $\alpha \in \mathcal{A}^{1,1}_{\mathbb{R}}(X)$ in the class of $2\pi c_1^{\mathbb{R}}(X)$, there exists a unique Kähler form ω_α in the class of ω, whose Ricci form $\rho_{\omega_\alpha} = \alpha$.*

In other words if ω is fixed, there is a bijection between the class of $2\pi c_1^{\mathbb{R}}(X)$ and Kähler forms in the class of $[\omega]$. If instead one has a complex manifold X which admits a Kähler metric, but this is not a priory fixed, for every representative α of $2\pi c_1^{\mathbb{R}}(X)$ there exists a Kähler metric h with $\rho_\alpha = 2\pi c_1^{\mathbb{R}}(X)$. This Kähler metric is not uniquely determined, unless one restricts to a particular cohomology class of the corresponding ω.

Idea of the proof. We follow the original Yau's paper [Yau78].
If ρ_{ω_α} represents $2\pi c_1^{\mathbb{R}}(X)$, which we know is represented by ρ, then by $\partial\bar{\partial}$-lemma

$$\rho_{\omega_\alpha} - \rho = \bar{\partial}\partial F$$

for some $F \in C^\infty(X, \mathbb{C})$.
If ρ_{ω_α} is the Ricci form of $\tilde{h}$, then

$$\rho_{\omega_\alpha} = i\bar{\partial}\partial \log(\det(\tilde{h}))$$

implies that

$$\bar{\partial}\partial\Big(i \log \frac{\det(\tilde{h})}{\det(h)} - F\Big) = \bar{\partial}\partial\phi = 0$$

Now if $\partial\bar{\partial}\phi = 0$, locally $\bar{\partial}\phi = \partial\psi$, with ψ another smooth function, but this says that ϕ is holomorphic, as $\mathcal{A}^{1,0}(U) \cap \mathcal{A}^{0,1}(U) = 0$ for any open set U. Since M is compact, ϕ is constant, i.e.

$$\det(\tilde{h}) = c \exp(-iF) \det(h)$$

for some $c \in \mathbb{C}$.
If ω_α is in the same class of ω, then

$$\tilde{h}_{i\bar{j}} = h_{i\bar{j}} + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}$$

and the above equation becomes

$$\det\left(h_{i\bar{j}} + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}\right) = C \exp(F) \det(h_{s\bar{t}})$$

Then solving the Calabi conjecture is equivalent to finding for some C a smooth ϕ such that the corresponding $\tilde{h}$ is Kähler. Moreover such C is actually uniquely determined by

$$\begin{aligned} C \int_X & \exp(F) \det(h)\, dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \\ &= \int_X \det(\tilde{h})\, dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \end{aligned}$$

and $\det(h)\, dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n$ is modulo constant the volume form, indeed if $\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i} = I \frac{\partial}{\partial x_i}\right)$ is a coordinate frame orthonormal in p,

$$\begin{aligned} & \det(h)|_p\, dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \\ &= \det(h)|_p\, dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n \end{aligned}$$

which equals 1 when evaluated against $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}\right)$. $\square$

Chapter 2

Beauville's decomposition theorem

2.1 Parallel transport and holonomy

Parallel transport is a foundamental object of riemannian geometry. If a riemannian manifold (M, g) is the configuration space of a physical system, with g being the kynetic energy tensor, parallel transport can be interpreted as an energy minimizing way to move vectors, covectors and more in general sections of bundles along a given path on M.

Definition 2.1.1. Let $E \to X$ be a differentiable vector bundle and $s \in E_p$. Given a curve $\gamma(t)$ in the base space, the parallel transport of s along $\gamma(t)$ is defined as

$$P_{\gamma(t)} = S|_{\gamma(t)}$$

where S is the unique parallel extension of s.

To show that such an extension always exists and that it is unique, if $s = s^i e_i$ for a local frame $(e_i)_{i=1}^k$, one has to consider the ODE

$$\frac{d}{dt}(s^i \circ \gamma)(t) + \sum_j \sum_h (\Gamma^i_{jh} \circ \gamma)(\gamma^j)' s^h = 0$$

with the initial conditions $(s^i(\gamma(0)) = s^i$ for $i = 1, \ldots, n$.
A special example of parallel transport, for a connection ∇ on the tangent bundle, is provided by transporting a vector along its integral curve. These are indeed special objects.

Example 2.1.2. A geodesic is a curve $\gamma(t)$ such that $\nabla_{\gamma'}\gamma' = 0$.

Geodesics in particular provide a diffeomorphism, for any point p, between an open subset $U \subset T_pM$ and a neighborhood of p, via the so called exponential map

$$\begin{aligned}\exp_p : U \subset T_pM &\to M\\ v &\mapsto \gamma_v(1)\end{aligned}$$

where $\gamma_v(t)$ is the unique geodesic with $\gamma_v(0) = p$, $\gamma_v'(0) = 0$.

Definition 2.1.3. A riemannian manifold is said geodesically complete if every solution to the geodesic ODE can be extended to the whole $\mathbb{R}$, or equivalently if at each point the exponential map is defined on the whole tangent space.

An important theorem relates geodesic completeness with metric completeness

Theorem 2.1.4 (Hopf-Rinow)**.** *A riemannian manifold M is geodesically complete if and only if M is complete as a metric space.*

If the manifold is compact, Hopf-Rinow theorem ensures that this is always the case.
Fix a point $p \in M$ and a loop γ based in p and consider $\alpha \in E_p$, then parallel transport of α along γ provides an automorphism ϕ_γ of E_p: it is indeed easy to see that parallel transport of α along γ^{-1} provides the inverse transformation of the fiber ϕ_γ^{-1}. Since the set of loops based in p has a natural group structure, it is natural to organize this into a group representation

$$\mathrm{Hol}_p \to \mathrm{GL}(n)$$

called holonomy representation in p.
If M is path connected, for any $x, y \in M$ one has $\mathrm{Hol}_x \backsimeq \mathrm{Hol}_y$ via conjugation, making possible to consider $\mathrm{Hol} \to \mathrm{GL}(n)$ independently from the point. We will denote Hol_p^0 the subgroup of Hol_p of contractible loops.
We shall recall the following definition from representation theory

Definition 2.1.5. A group representation $\rho : H \to \mathrm{GL}(n, k)$ is said reducible if it admits a proper subrepresentation, i.e. there is a proper subspace of the underlying k-vector space which is stable by the action of H. If ρ is not reducible, it is said irreducible.

Reducible representations are somehow not optimal. The holonomy representation is not irreducible in general, but we will see that whenever they are reducible they are also decomposable, i.e. if $V \subset T_pM$ is H-stable then $T_pM \setminus V$ is H-stable too.
We will study holonomy representations only for the Levi-Civita conenection. In this case parallel transport is an isometry on tangent space and accordingly on all the tensor bundles. Indeed one has for any $X, Y \in T_pM$

$$\begin{aligned}\frac{d}{dt} g(P_{\gamma(t)}X, P_{\gamma(t)}Y) &= \gamma'(t)\Big(g(P_{\gamma(t)}X, P_{\gamma(t)}Y)\Big)\\ &= g(\nabla_{\gamma'(t)}P_{\gamma(t)}X, P_{\gamma(t)}Y) + g(P_{\gamma(t)}X, \nabla_{\gamma'(t)}P_{\gamma(t)}Y)\\ &= 0\end{aligned}$$

which means that Hol $\to$ GL(n) actually factors through O(n).
This is a first example of a deep principle, which will be central in the following, that structures on a manifold and subgroups of GL(n) through which holonomy representation factors are strongly related.
We present now a useful lemma whose proof is just definition rearrangement.

Lemma 2.1.6 (Holonomy Principle)**.** *For a riemannian manifold (M, g) and a fixed point $p \in M$, a parallel tensor field of a fixed type on M is equivalent to a tensor of the same type in p which is invariant by the holonomy action.*

Proof. The non trivial part of the statment is that any tensor in p invariant by holonomy extends in a unique way to a parallel tensor field on M. This can be realized by setting $\tau_q = P_{\gamma(1)}\tau_p$ where γ is a path connecting p and q: since τ_p is invariant by holonomy, τ_q does not depend on the choice of γ. The resulting tensor field is smooth, since it locally solves a linear ODE with smooth coefficients. $\square$

This simple lemma allows to characterize structures on a manifold in terms of factorization of the holonomy representation. This is very similar (and actually related) to reduction of the structure group for the associated frame bundle.

Theorem 2.1.7. *Let (M, g) be a connected n-dimensional riemannian manifold with Levi-Civita connection. Then the following classes of manifolds are characterized by the following factorizations of the holonomy.*

1. *Orientable manifolds: $Hol \subseteq SO(n)$*
2. *Flat manifolds: $Hol_0 = Id$*
3. *Kähler manifolds: $Hol \subseteq U(m)$*
4. *Weakly Calabi-Yau manifolds: $Hol \subseteq SU(m)$*
5. *Hyperkähler manifolds: $Hol \subseteq Sp(r)$*

For the last class of manifolds, recall that the compact simplectic group of dimension n is defined as

$$\mathrm{Sp}(n) = \{M \in \mathrm{GL}(n;\mathbb{H}) \text{ such that } M^t H M = H\}$$

and H is the standard hermitian metric on $\mathbb{H}^n$, i.e.

$$x^t H y = x_1\overline{y}_1 + \cdots + x_n\overline{y}_n$$

and the conjugation operation is the obvious one. Equivalently $\mathrm{Sp}(n) = U(n, \mathbb{H})$.
Notice that $U(\frac{n}{2}, \mathbb{C}) = O(n) \cap Gl(\frac{n}{2}, \mathbb{C})$ and $U(\frac{n}{4}, \mathbb{H}) = O(n) \cap Gl(\frac{n}{4}, \mathbb{H})$.

Definition 2.1.8. A Riemannian manifold (M, g) is said hyperkähler[1] if it is endowed with three integrable almost complex structures I, J, K such that $I^2 = J^2 = K^2 = IJK = -1$ and the three corresponding Kähler forms are all closed.

[1] Notice that sometimes in the literature hyperkähler is used in place of irreducible symplectic.

The three almost complex structures give to each T_pM the structure of a free $\mathbb{H}$-module of dimension $\frac{1}{4}\dim_{\mathbb{R}} M$.

Proof. 1. M is orientable if and only if Ω^n is trivial. In this case a global section is given by the volume form ν_g, which is preserved by parallel transport as the connection is riemannian. By the holonomy principle, the induced representation $\tilde{\rho}$ on $\Lambda^n T_p^* M$ is trivial. If $\rho : \mathrm{Hol} \to \mathrm{GL}(n)$ is the holonomy representation for T_pM, $1 = \tilde{\rho}(\gamma) = \det(\rho(\gamma))$ for any loop γ.

2. Given a contractible loop one can always assume that it is contained in a single chart. [2] The manifold is flat if and only if in each chart

$$\Big(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\Big)$$

is an orthonormal frame, which is equivalent to have all the $\frac{\partial}{\partial x_i}$ parallel for the connection. By the holonomy principle, this happens if and only if $\frac{\partial}{\partial x_i}\big|_p$ is invariant by restricted holonomy. Then $\mathrm{Hol}^0(p) = \mathrm{Id}$.
Notice that this does not hold in general for $\mathrm{Hol}(p)$: the Klein bottle, obtained by isometrically twisting a square, has by previuos point $\mathrm{Hol} \nsubseteq \mathrm{SO}(n)$ (in this case $\mathrm{Hol} = \mathbb{Z}_2$) but the induced connection is flat.

3. Let $n = 2m$. Fix a point $p \in M$ and choose coordinates such that $g|_p = \mathrm{Id}$. Then, after the choice of an endomorphism $I_p \in O(T_pM)$, with $I_p^2 = -\mathrm{Id}$, $\mathrm{U}(m)$ can be identified with the matrices $U \in \mathrm{O}(n)$ with $UI_p = I_pU$, which is the definition of I_p invariant by holonomy.
Thus by the holonomy principle $\mathrm{Hol} \subseteq U(m)$ if and only if there exists a almost complex structure on M which is compatible with g and parallel. Such an almost complex structure is always integrable, thus M is Kähler.

4. Let (M, g, I) be Kähler. Since

$$SU(m) = U(m) \cap SL(n, \mathbb{R})$$

$SU(m)$ is the subgroup of matrices whose n-th Krönecker product is the identity, i.e. the ones which preserve an alternating complex n-form. By the holonomy principle, this is equivalent to the existence of $\alpha \in \Omega^n_X$, $\alpha \neq 0$, parallel by ∇. This implies that the canonical bundle of X is trivial and viceversa if $K_X \backsimeq \mathcal{O}_X$, the trivial connection on $\mathcal{O}_X$ provides such a form for K_X.
If instead $\mathrm{Hol}_0 \subseteq SU(m)$, then the induced contractible representation on Ω^n_X is the identity and by the previous point Ω^n_X is flat. Since the curvature of Ω^n_X is up to a scalar the Ricci curvature of X, this is the class of

[2] The idea is the following: let γ be a loop based in p, $p \in U_i$ and $h(x,t)$ a homotopy between γ and ϵ_p. One can assume (see ......) that h smooth and that $h(x,t) \subset U_i$ for $t \in (1-\epsilon, 1]$ with ϵ small enough and that for such t is an isotopy. Then the inverted isotopy $h(x, 1-t)$, $t \in [\epsilon, 1]$, allows to extend the local coordinates of U_i in a neighborhood of γ.

Ricci-flat Kähler metrics.
The two definitions clearly coincide in the case of simply connected manifolds but they do not coincide in general. See the example on Enriques surfaces.

5. Assume that $n = 4r$. By definition that $\mathrm{Sp}(r)$ is the group of $\mathbb{H}$-linear automorphisms of $\mathbb{H}^n$ preserving H. One can decompose

$$H = h + \phi J$$

where h is a complex hermitian form on $(\mathbb{H}^n, I)$ and ϕ is a $\mathbb{C}$-linear 2-form, using that any quaternion h can be written in a unique way as $h = z + Jw$, with $z, w \in \mathbb{C}$ via

$$h = \alpha + I\beta + J\gamma + K\delta = (\alpha + I\beta) + J(\gamma - J\delta)$$

thus $H(\alpha + J\beta, \gamma + J\delta) = H(\alpha, \gamma) + H(\beta, \delta) + J(H(\beta, \gamma) - H(\alpha, \delta))$ and setting

$$\begin{cases} h((\alpha + J\beta, \gamma + J\delta) = H(\alpha, \gamma) + H(\beta, \delta) \\ \phi(\alpha + J\beta, \gamma + J\delta) = H(\beta, \gamma) - H(\alpha, \delta) \end{cases}$$

it is easy to verify that h is hermitian and ϕ is $\mathbb{C}$-bilinear and alternating. Then $\mathrm{Sp}(n)$ is the subgroup of $\mathrm{U}(2n)$ whose elements preserve ϕ and by holonomy principle a Kähler manifold X has $\mathrm{Hol} \subset \mathrm{Sp}(n)$ if and only if it admits a holomorphic parallel 2-form α, with $\alpha \neq 0$.

$\square$

We remark the three classes of manifolds

$$\left\{\begin{matrix} \mathrm{Hol} = SU(m) \\ \text{Calabi-Yau} \\ \text{manifolds} \end{matrix}\right\} \subset \left\{\begin{matrix} \mathrm{Hol} \subset SU(m) \\ K_X \backsimeq \mathcal{O}_X, c_1(X) = 0 \\ \text{weakly Calabi-Yau} \\ \text{manifolds} \end{matrix}\right\} \subset \left\{\begin{matrix} \mathrm{Hol}_0 = SU(m) \\ c_1^{\mathbb{R}}(X) = 0 \\ \text{Ricci-flat} \\ \text{manifolds} \end{matrix}\right\}$$

The first inclusion is strict, too. We will see later some examples.
A fundamental result in the theory of riemannian holonomy is Berger's theorem, which shows that for a riemannian manifold only few subgroups of $\mathrm{O}(n)$ can occur as the restricted holonomy. For a proof we refer to the original paper [Ber55] or for a more geometric approach to the recent [Olm05].

Theorem 2.1.9 (Berger, 1955)**.** *Let (M, g) be a n-dimensional riemannian manifold. If M is not locally symmetric*[3]*, then Hol_0 is isomorphic to one of the following subgroups of $O(n)$:*

- $SO(n)$

[3] M is said locally symmetric in p if all the local diffeomoprhisms ϕ which fix p and reverse all the geodesics passing through p are isometries. M is said locally symmetric if this holds for any $p \in M$. It can be proven that M is locally symmetric if and only if $\nabla(R(x, y)z) = 0$ for any vector fields x, y, z.

- $U(m)$, $2m = n$
- $SU(m)$, $2m = n$
- $Sp(r)$, $4r = n$
- $Sp(1) \cdot Sp(r)$, $4r = n$
- $Spin(7)$, $n = 8$
- $Spin(9)$, $n = 16$
- G_2, $n = 7$

All the four last groups can not be embedded in $\mathrm{U}(m)$, thus they are a priori excluded in Kähler setting. It in interesting to notice that Berger did not establish which one of these groups may indeed occur: for G_2, for example, an example was found only in , a compact example in 1994. A major step was done in 2015, see [Cor+15].

2.2 De Rham Decomposition

In this paragraph we will present the powerful De Rham's decomposition theorem, which is one of the main tools in Beauville's proof. We will write only the proof of the main theorem, which is based on Pantilie's article [Pan92]. All the preparatory results can be founded with detailed proofs in [KN63] and [KN69]. We first recall some definitions.

Definition 2.2.1. Let τ be a subbundle of TM of rank k. A submanifold N is said integral for τ around x if there exists a neighborhood U of x such that $T_yN = \tau_y$ for any $y \in U$. If x is fixed, all such neighborhoods U are partially ordered by inclusion and for any ascending chain $(U_i)_{i=1}^k$ their union is still integral. From this one has the definition of maximal integral submanifold.

Definition 2.2.2. A submanifold N of (M, ∇) is said totally geodesic at a point x if for any $v \in T_xN$, the associated geodesic $\gamma(t)$ with $\gamma(0) = x$ and $\gamma'(0) = v$ is contained in N if t is small enough. It is said totally geodesic if it is totally geodesic at any point.

It can be shown that if ∇ is Levi-Civita and N is totally geodesic, then every geodesic in N with respect to the induced metric is a geodesic in M. This is sometimes taken as the definition of totally geodesic submanifold, but is actually stronger.
The statement of the decomposition theorem reads

Theorem 2.2.3 (De Rham, 1952)**.** *Let (M, g) be a simply connected complete Riemannian manifold. Then M is isometric to a product $M_0 \times M_1 \times \cdots \times M_n$ where M_0 is euclidean and the M_i are all irreducible (i.e. have irreducible holonomy representation). The holonomy of M splits for this decomposition, which is unique up to order.*

In order to prove it, some lemmas are needed. First one has to show that a reducible holonomy representations always splits, then that the decomposition of the tangent space in a point extends to a global decomposition of the tangent bundle.

Lemma 2.2.4. *Let (M, g) be a complete reducible riemannian manifold. Then for each $x \in M$ the decomposition of T_xM induces a local decomposition of the tangent bundle. Each factor of this decomposition is involutive and the corresponding maximal integral submanifold around x is a complete totally geodesic submanifold.*

Now let T''_xM be the orthogonal complement to T'_xM. It is easy to see that one has an orthogonal bundle decomposition $T'M \oplus T''M$. A refinement of Frobenius theorem allows to prove

Lemma 2.2.5. *If $T'M, T''M$ are two involutive distributions complementary at each point then there exist local coordinates such that $x^i = c^i$, $i = 1, \dots k$, is an integral submanifold for $T'M$ and $x^i = c^i$, $i = k+1, \dots n$, is an integral submanifold for $T''M$.*

This particual choice of basis is particularly convenient and is used in the proof of the following lemma.

Lemma 2.2.6. *Let $T'M, T''M$ and M'_x, M''_x defined as above. Then there exist three open neighborhoods V, V', V'' of $x \in M$ contained respectively in M, M'_x, M''_x such that $g_V = g_{V'} \times g_{V''}$.*

The local decomposition $V = V' \times V''$ allows to prove the existence of a pointwise decompositon of the holonomy representation. The idea is first to decompose an arbitrary loop γ in x in a finite product of special loops γ_i (here it is crucial that M is simply connected), such that studying parallel transport along γ_i reduces to studying parallel transport of a loop β contained in V.

Lemma 2.2.7. *Let $T'M$ and $T''M$ be as above. If M is simply connected, then the holonomy representation in x is the direct product of the two holonomy representations on T'_xM and T''_xM.*

In particular it follows that if γ is a loop based in x contained in M' (resp. M''), then it acts trivially on T''_xM (resp. T'_xM). Although we only need this version, the loop factorization above is necessary: wihout it the holonomy factorization holds only for loops contained in $V = V' \times V''$.

Proof of the existence. Fix $x \in M$. We will show that M is isometric to the product of the maximal integral submanifolds M'_x and M''_x. Let $u \in T'_xM$ and $v \in T''_xM$. Then the main part of the proof consists in showing the for any two curves $\gamma' \subset M'$ and $\gamma'' \subset M''$ connecting respectively x and $\exp u$, x and $\exp v$, one has

$$\exp P_{\gamma'(1)} v = \exp P_{\gamma''(1)} u$$

This is the main ingredient used by Pantilie, we refer to the original article [Pan92] for a proof.
Now fix $a, b \in M$ such that they are both contained M'' and a is contained also in M'. Let $v \in T''_a M$ such that $b = \exp v$. For arbitrary $x \in M'$, $\gamma' \subset M'$ connecting a and x, $u \in T'_a$ such that $x = \exp u$ and $\gamma'' \subset M''$ connecting a and b one has

$$\exp P_{\gamma'(1)} v = \exp P_{\gamma''(1)} u$$

Thus it is possible to identify isometrically different maximal integral submanifold by

$$\begin{aligned} \phi_a^b : M'_a &\to M'_b \\ x &\mapsto \exp P_{\gamma(1)} w \end{aligned}$$

where γ is any curve connecting a and b contained in M'' and $w \in T'_a M$ is such that $x = \exp(w)$. This map is well defined and depends only on a and b, moreover $\phi_b^c \circ \phi_a^b = \phi_a^c$. One can build analogously for $b, c, d \in M'$ an isometry

$$\psi_c^d : M''_c \to M''_d$$

such that $\psi_b^c \circ \psi_a^b = \psi_a^c$.
By what we formula above it follows that $\phi_a^b(c) = \psi_a^c(b)$ for any $a \in M$, $b \in M''_a$ and $c \in M'_a$. Moreover ϕ_a^b is an isometry since

$$d\phi_a^b|_x = P_{x,\phi_a^b(x)} : T'_x M \to T'_{\phi_a^b(x)} M$$

where $\delta \subset M''_x$ is any curve connecting x and $\phi_a^b(x)$.
Thus for a fixed $a \in M$ the map

$$\begin{aligned} H_a : M'_a \times M''_a &\to M \\ (x, y) &\mapsto \phi_a^y(x) = \psi_a^x(y) \end{aligned}$$

is well defined and is a local isometry. In particular $M'_a \times M''_a$ is connected and complete, thus it is a Riemannian covering map. Since M is simply connected, H_a is an isometry. □

In the last part of the proof we have used

Theorem 2.2.8 (Ambrose). *A local isometry $f : M \to N$ between riemannian manifolds with N connected and M complete is a covering map.*

To find a decomposition in irreducible factors as in the statement, one only has to iterate the costruction. The euclidean term M_0 appears if and only if there is a flat vector field and its dimension is the number of such invariant vector fields.
The unicity of the irreducible factors has to intended in this precise sense. For a proof see [KN63, p. 185].

Theorem 2.2.9. *If there is an isometry*

$$\phi : M_0 \times \prod_{i=1}^{k} M_i \to N_0 \times \prod_{i=1}^{j} N_i$$

between two irreducible factorizations, where the holonomy action on M_0 and N_0 is trivial, then $\dim(M_0) = \dim(N_0)$, $k = j$ and there is a permutation σ of $1, 2, \ldots, j$ such that

$$\phi(m_0, m_1, \ldots, m_n) = (u_0(m_0), u_{\sigma(1)}(m_{\sigma(1)}), \ldots, u_{\sigma(j)}(m_{\sigma(j)}))$$

for some isometries u_i.

If M admits a Kähler metric, the following result holds, whose proof is straightforward. It is based on the fact that a g compatible almost complex structure I which is flat for ∇ restricts to almost complex structures I_j on each factor, which are invariant by the restricted metrics g_j and flat for the restricted conections ∇_j. See [KN69, pp. 171–173] for details.

Theorem 2.2.10. *With the above notation, if (M, g) is Kähler, then all the (M_i, g_i) are Kähler and the decomposition is a holomorphic isometry.*

Thanks to this theorem it is possible to restrict the study of Hol_0 to irreducible representations.
The following is another importnat theorem which will be used in the proof. A geodesic line is a geodesic $\gamma(t)$ with the property that

$$\mathrm{dist}(\gamma(t_0), \gamma(t_1)) = |t_1 - t_0| \qquad \text{for any } t_0, t_1$$

Theorem 2.2.11 (Cheeger, Gromoll)**.** *Let M be a complete and connected riemannian manifold with $Ric \geq 0$.Then M isometric to a product $M \times \mathbb{R}^k$ where M contains no geodesic lines and $\mathbb{R}^k$ has its standard flat metric.*

2.3 Covering spaces and holonomy

In this paragraph we will prove the first version of the main theorem of this work. We now recall the main results of the theory of topologial covering spaces. All the details can be found for example in [Hat02].
A map $p : Y \to X$ between topological spaces is a covering map if for any $p \in X$ there exists an open neighborhood U such that $p^{-1}(U) \cong U \times F$ where F is a discrete non empty topological space called fiber of the covering. If X is path connected, F does not depend on U or the point x. The cover is said finite if $|F| < \infty$ and infinite otherwise.
Covering spaces over X are strongly related to the group $\pi_1(X)$. Indeed if X is path-connected, locally path-connected and semilocally simply-connected [4],

[4] Spaces with these extra properties are often called in the literature good spaces.

which are properties that all topological manifolds have, there is a correspondence between the subgroup lattice structure of $\pi_1(X, x_0)$ and isomorpshism classes of covering spaces.
The intuition is that each cover provides a way to unwind non-contractible loops in the base: fix $x_0 \in X$ and consider a loop γ based in x_0. It can be proved that for any y in the fiber $p^{-1}(x_0)$ there exists a unique lift $\tilde{\gamma}_y$ of γ such that $\tilde{\gamma}_y(0) = y$. Moreover if γ^{-1} is the inverse loop and $\tilde{\gamma}_y(1) = \xi$, then $\widetilde{\gamma^{-1}}_\xi(1) = y$. Thus one has a well defined map

$$\begin{aligned} \phi^y_{x_0} : \{\text{loops based in } x_0\} &\to \mathrm{Aut}(p^{-1}(x)) \\ \gamma &\mapsto [y \mapsto \tilde{\gamma}_y(1)] \end{aligned}$$

This map factors through $\pi_1(X, x_0)$, providing a group representation known as monodromy representation and one can prove that $\ker(\phi^y_{x_0}) = p_*\pi_1(Y, y)$.
It is natural to define the category $\mathrm{Cov}(X)$ of covers of X, whose morphisms for two objects $Y \to X$ and $Z \to X$ are continuous maps $Y \to Z$ preserving fibers. One can show that if X is a good space then $\mathrm{Cov}(X)$ has an initial object $U \to X$, known as universal cover. The theory shows that U is necessarily simply connected.
The universal cover for a X is obtained by unwinding all the non contractible loops in the base, while the other covers are obtained unwinding only some of them. Indeed the construction of universal cover first defines U as the set of all homotopy classes (with fixed endpoints) of paths starting at a point x_0, then puts a suitable topology on it and define the projection as $[\gamma] \mapsto \gamma(1)$.
To be more precise, let $p : Y \to X$ be an arbitrary cover, fix $x_0 \in X$ and consider the group homomorphism $p_* : \pi_1(Y, y) \to \pi_1(X, x)$ given for any $y \in p^{-1}(x)$ by $p_*([\alpha]) = [p \circ \alpha]$. One can prove p_* is injective and that $\bigcup_{y \in p^{-1}(x)} p_*(\pi_1(Y, y))$ is a conjugacy class in $\pi_1(X, x)$. The first bijection is given by

$$\{\text{subgroups of } \pi_1(X, x_0)\} \leftrightarrow \left\{ \begin{array}{c} \text{iso-classes of path-connected pointed} \\ \text{covering spaces over } X \\ (Y, y_0) \to (Z, z_0) \end{array} \right\}$$

where to $H \leq \pi_1(X, x_0)$ one associates a covering space $p : Y_H \to X$ with $p_*(\pi_1(Y_H, y_0)) = \pi_1(X, x_0)$ for some y_0 in the fiber of x_0. Such a space is defined starting with U and making H act on it: $[\gamma] \backsim [\gamma']$ if and only if $\gamma(1) = \gamma'(1)$ and $[\gamma * \gamma'^{-1}] \in H$. Then $p_*(\pi_1(Y_H, y_0)) = \pi_1(X, x_0)$ if one chooses as y_0 the class of constant path ϵ_{x_0}.
The association is injective since one can prove a map $f : Y \to Z \in \mathrm{Cov}(X)$ which takes a point y_0 to a point z_0 (both in the fiber of x_0) is an isomoprhism if and only if $p_{Y*}(\pi_1(Y, y_0)) = p_{Z*}(\pi_1(Z, z_0))$.
If one removes the marking in the fibers the following bijection holds

$$\{\text{conjugacy classes of } \pi_1(X, x_0)\} \leftrightarrow \left\{ \begin{array}{c} \text{iso-classes of path-connected} \\ \text{covering spaces over } X \\ Y \to Z \end{array} \right\}$$

where the partition of $\mathrm{Cov}(X)$ in iso-classes is now coarser.
Among all the covering spaces, some have the properties that their automorphism group acts transitively on each fiber. They are called normal (or Galois) covers and the name comes from the fact that each pointed iso-class of normal cover corresponds to a normal subgroup of $\pi_1(X, x_0)$. In particular the universal cover is always normal. Notice also that in this case the covering can be seen as an $\mathrm{Aut}(F)$-principal bundle.
The group of automoprhism of $p : Y \to X$ (i.e. fiber preserving homeomorphisms of Y) is denoted as $\mathrm{Deck}(p)$ and acts always properly on Y. For Galois covers, one has that

$$\mathrm{Deck}(p) \backsimeq \frac{\pi_1(X, x)}{p_*\pi_1(Y, y)}$$

and one can prove that p is Galois if and only if $X \backsimeq Y/\mathrm{Deck}(p)$ and the following diagram commutes, or equivalently if and only if the action of $\mathrm{Deck}(p)$ is transitive on each fiber.

$$\begin{tikzcd}
Y \arrow[r, equal] \arrow[d] & Y \arrow[d, "p"] \\
Y/\mathrm{Deck}(p) \arrow[r, "\backsimeq"] & X
\end{tikzcd}$$

In the following we will restrict our attention to riemannian covers, which are covers between riemannian manifolds which are also local isometry. Once that $p : Y \to (X, g)$ is a fixed differentable cover, $h = p^*g$ is the unique metric on Y making the cover riemannian, indeed

$$h(X, Y) = g(p_*X, p_*Y) = (p^*g)(X, Y)$$

Moreover in this case all the smooth cover automorphisms $\phi : Y \to Y$ respect the metric, as

$$h(\phi_*X, \phi_*Y) = (p^*g)(\phi_*X, \phi_*Y) = (p\phi)^*g(X, Y) = g(p_*X, p_*Y) = h(X, Y)$$

If X is a differentiable manifold and $q : U \to X$ is its universal topological cover, one can pullback the differentiable structure of X and thus obtain the smooth universal cover for X. The remarkable fact is that all the smooth covers over X are obtained by the action of $\pi_1(X)$ on U, which is a smooth action with respect to the induced differentiable structure.
Analogously, one defines the universal riemannian cover and the action of $\pi_1(X)$ on U is a local isometry with respect to the induced metric.
The following lemma will be important later on.

Lemma 2.3.1. *If $\pi : Y \to X$ is a riemannian cover, there is a canonical injection $j : Hol(Y) \hookrightarrow Hol(X)$ which sends $Hol_0(Y)$ onto $Hol^H(X)$, where*

$$Hol^H(X) = \{\gamma : [\gamma] \in H\}$$

and H is the conjugacy class corresponding to the cover.

Proof. The map j associates to a loop γ based in y the loop $\pi(\gamma)$ based in $x = \pi(y)$. If $\pi(\gamma)$ induces the trivial holonomy on T_xX, then γ induces the trivial holonomy on T_yY as well. Indeed it is a known result from the theory of covering spaces that in this case there exists a neighborhood V of $\pi(\gamma)$ such that $\pi^{-1}(V) = V \times F$ and γ is entirely contained in the sheet of y. Thus $\pi|_{\pi^{-1}(V)}$ is an isometry.
Now fix $\gamma \in \mathrm{Hol}^H(X)$. If $y \in Y$ is fixed, the loops in y correspond to the loops in x whose class are in one of the conjugated subgroups. Moving y along the fiber, one finds all the elements in the conjugacy class. $\square$

For example, if $U \to X$ is the universal cover, then $\mathrm{Hol}(X) = \mathrm{Hol}^0(Y) = \mathrm{Hol}^0(Y)$.

2.4 Vanishing results

The following results usually go under the name of Bochner's vanishing theorems. They prescribe, in general, vanishing of sections of bundles under some negativity (or positivity) assumption on the curvature. In its first version (see for details [Pet06]), the theory developed by Bochner, Yano, Lichnerowicz and others regarded riemannian manifolds and harmonic functions: for a riemannian manifold (X, g) and a smooth function $u : X \to \mathbb{R}$ Bochner discovered the following formula

$$\frac{1}{2}\Delta\|\mathrm{grad}(u)\|^2 = g(\mathrm{grad}(\Delta u), \mathrm{grad}u) + \|\mathrm{Hess}(u)\|^2 + \mathrm{Ric}(\mathrm{grad}(u), \mathrm{grad}(u))$$

where $\mathrm{grad}(u) = (du)^\sharp$, Δ is the Laplace-Beltrami operator ($\Delta(u) = \mathrm{div}(\mathrm{grad}(u)$ where the divergence is defined for any vector field X by $\mathrm{div}(X)\mathrm{vol}_g = \mathcal{L}_X \mathrm{vol}_g$) and the Hessian is defined by $\mathrm{Hess}(u) = \nabla\mathrm{grad}(u)$, with ∇ the Levi-Civita connection.
This formula can be manipulated to get all the vanishing result but the approach we will adopt now, mainly based on [Kob87], is more straightforward. We need two lemmas

Lemma 2.4.1. *For a holomorphic hermitian vector bundle $(E, h) \to X$, with X complex manifold X, the following formula holds*

$$\partial\bar{\partial}h(s, s) = h(\nabla^{1,0}s, \nabla^{1,0}s) - h(R_\nabla s, s)$$

for any $s \in H^0(X, E)$.

Proof. Consider first the case of a smooth $\mathbb{C}$-valued function f, regarded as a smooth section of the trivial line bundle over X. In this case

$$\bar{\partial}\partial s = \nabla^{0,1}\nabla^{1,0}s$$

Now let $f = h(s, s)$; since s is holomorphic $\nabla^{0,1}s = 0$, hence

$$\partial\bar{\partial}h(s, s) = -\nabla^{0,1}h(\nabla^{1,0}s, s) = h(\nabla^{1,0}s, \nabla^{1,0}s) - h(\nabla^{0,1}\nabla^{1,0}s, s)$$

where we used that

$$\nabla^{1,0}h(\alpha,\beta) = h(\nabla^{1,0}\alpha,\beta) + (-1)^{\deg(\alpha)}h(\alpha,\nabla^{0,1}\beta)$$
$$\nabla^{0,1}h(\alpha,\beta) = h(\nabla^{0,1}\alpha,\beta) + (-1)^{\deg(\alpha)}h(\alpha,\nabla^{1,0}\beta)$$

Since $R_\nabla(s) = \nabla\nabla s = \nabla^{0,1}\nabla^{1,0}s + \nabla^{1,0}\nabla^{0,1}s = \nabla^{0,1}\nabla^{1,0}s$, we have proved the formula. $\square$

We need to define a new riemannian object, as positivity of vector bundles will be always meant in terms of this tensor. Assume that the base X of the hermitian holomorphic vector bundle is endowed with an hermitian metric g. This provides a new way to contract the tensor: if $R^i_{j\alpha\bar{\beta}}$ is the local expression of the curvature, one defines

$$K^i_j = g^{\alpha\bar{\beta}}R^i_{j\alpha\bar{\beta}} \qquad K_{j\bar{k}} = h_{i\bar{k}}K^i_j$$

and the tensor $K = K_{i\bar{j}}dz^i \otimes d\bar{z}^j$ is known as mean curvature of $(E,h) \to (X,g)$. Now the formula of the previous lemma reads in coordinates

$$\frac{\partial^2 h(s,s)}{\partial z^\alpha \partial \bar{z}^\beta} = h_{i\bar{j}}\nabla_\alpha s^i \nabla_{\bar{\beta}}\bar{s}^j - h_{i\bar{k}}R^i_{j\alpha\bar{\beta}}s^j\bar{s}^k$$

Multiplying both sides by $g^{\alpha\bar{\beta}}$ and summing over repeated indices yields

$$g^{\alpha\bar{\beta}}\frac{\partial^2 h(s,s)}{\partial z^\alpha \partial \bar{z}^\beta} = \|\nabla^{1,0}s\|^2 - K_{j\bar{k}}s^j\bar{s}^k$$

One of the special features of Kähler manifolds is that if $E = TX$ and $g = \mathrm{Re}(h)$ then $K_{j\bar{k}} = \mathrm{Ric}_{j\bar{k}}$, since curvature tensor enjoys extra symmetries. The second lemma still holds for arbitrary of $(E,h) \to (X,g)$, we will next focus on the Kähler case.

Lemma 2.4.2. *Let (E,h) be an Hermitian vector bundle over a compact hermitian manifold (X,g). If $\widehat{K} \leq 0$, then every holomorphic section s is parallel and $\widehat{K}(s,s) = 0$.*

Proof. We will use the following maximum principle by Hopf

Lemma 2.4.3. *Let $U \subset \mathbb{R}^n$ be open and connected; let f and g^{ij}, $1 \leq i,j \leq n$, be smooth real function with (g^{ij}) symmetric and positive-definite. Set*

$$L(f) = g^{ij}\frac{\partial^2 f}{\partial x^i \partial x^j}$$

If $L(f) \geq 0$ and f attains a local maximum in the interior of U, then f is constant.

Now let $f = h(s,s)$. Since X is compact, f attains a maximum M. Its preimage is closed and we will show that it is also open. Let $x_0 \in f^{-1}(M)$ and consider a coordinate neighborhood U of x_0. Then we have

$$L(f) = g^{\alpha\bar{\beta}} \frac{\partial^2 h(s,s)}{\partial z^\alpha \partial \bar{z}^\beta} = \|\nabla^{1,0} s\|^2 - K_{j\bar{k}} s^j \bar{s}^k \geq 0$$

which means that $f = M$ on the whole U. Thus $f^{-1}(M) = X$, hence $L(f) = 0$ everywhere on X. Then $\nabla^{1,0} s = 0$ and since s is holomorphic $\nabla s = 0$. $\square$

We can finally present the main result

Theorem 2.4.4 (Bochner's principle for Kähler manifolds)**.** *Let X be a compact Kähler manifold with Ric ≥ 0. Then every $s \in H^0(X, \Omega_X^p)$ is parallel with respect to the Chern connection. If Ric ≤ 0, then every every $t \in H^0(X, T_X^{\otimes k})$ is parallel.*

Proof. We only have to apply the previous lemma: if Ric ≥ 0 (respectively Ric ≤ 0) then for $(TX, h) \to (X, h)$ one has that $\widehat{K} \geq 0$ (respectively $\widehat{K} \leq 0$) and for $(\Omega_X, h^*) \to (X, h)$ that $\widehat{K} \leq 0$ (respectively $\widehat{K} \geq 0$).
Since positivity and negativity of $\widehat{K}$ is preserved by taking tensor products and exterior powers, one has the thesis. $\square$

Clearly if Ric $= 0$, holomorphic sections of all the above bundles are parallel.

Corollary 2.4.5. *For a compact manifold X with a Ricci-flat Kähler metric h, the following map is a bijection*

$$\phi_x : H^0(X, \Omega_X^p) \to Inv(\Omega_{X,x}^p) \quad \textit{for any } x \in X$$

where $Inv(\Omega_{X,x}^p)$ is the subspace of forms of $\Omega_{X,x}^p$ invariant by holonomy.

2.5 The main theorem

Theorem 2.5.1 (Beauville, 1983)**.** *Let X be a Ricci-flat Kähler manifold. Then its Riemannian universal cover is holomorphically isometric to*

$$\mathbb{C}^k \times \prod_i V_i \times \prod_j W_j$$

where $\mathbb{C}^k$ has the standard flat hermitian metric, $Hol(V_i) = SU(n_i)$ and $Hol(W_j) = Sp(m_j)$. The decomposition is unique up to order.
From this one can obtain a finite riemannian cover

$$T \times \prod_i V_i \times \prod_j W_j$$

where T is a k-dimensional complex torus.

Proof. Let $\tilde{X}$ be the riemannian universal cover of X, which is itself a Kähler Ricci-flat manifold. De Rham decomposition provides a holomorphic isometry

$$\tilde{X} \to \mathbb{C}^k \times \prod_l M_l$$

where every M_l is irreducible. By theorem (...), $\prod_l M_l$ is compact. Since every M_l is Ricci-flat, its restricted holonomy is contained in $SU(n_l)$. Berger classification tells that $\mathrm{Hol}_0(M_l)$ can be either $SU(n_l)$ itself or $Sp(r_l)$, since if M_l were symmetric and compact it would have positive Ricci curvature. Being simply connected implies then that $\mathrm{Hol}(M_l)$ is either $SU(n_l)$ or $Sp(r_l)$.
For the construction of a finite cover, we will make act a suitable subgroup of $\pi_1(X)$ on $\tilde{X}$. We need an additional lemma.

Lemma 2.5.2. *Let X be a compact, simply connected and Ricci-flat Kähler manifold. Then the group of holomorphic automorphisms $Aut(X)$ is discrete and the subgroup of isometries $Isom(X)$ is finite.*

Proof. The group $\mathrm{Aut}(X)$ admits a complex Lie group structure and its Lie algebra can be identified with sections of the holomoprhic tangent bundle T_X (for a proof see [BM47]). Let σ be such a section. Bochner's principle ensures that σ is parallel, thus $\sigma(p)$ is invariant by holonomy for any $p \in X$. By the first part of Beauville's theorem, though, X is necessarily irreducible, which implies that σ is the zero section. Then $\mathrm{Aut}(X)$ is discrete and since $\mathrm{Isom}(X)$ is compact (see [KN63], p. 239) it is finite. □

Since the cover is riemannian, $\pi_1(X)$ acts isometrically on $\tilde{X}$ and by unicity of De Rham decomposition every u in the image of the representation split as

$$u(z,m) = (u_1(z), u_2(m)) \qquad \text{where } z \in \mathbb{C}^k,\ m \in \prod_l M_l$$

Consider the composition $\Phi : \pi_1(X) \to \mathrm{Isom}(\tilde{X}) \to \mathrm{Isom}(\prod_l M_l)$ and let $\Gamma = \ker \Phi$. The action of Γ is necessarily free on $\mathbb{C}^k$. The quotient $\mathbb{C}^k/\Gamma$ is compact, since $\pi_1(X)/\Gamma \leq \mathrm{Isom}(\prod_l M_l)$ and the latter is finite by previous lemma, and finite covering of compact spaces are compact. Notice also that if Γ were trivial, $\pi_1(X)$ would be finite, hence the covering $\tilde{X} \to X$.
By Bieberbach theorem, the subgroup of translations $\Gamma' \leq \Gamma$ is non trivial and of finite index, thus $\tilde{X}/\Gamma'$ is a finite cover and is isomorphic to $T \times \mathrm{Isom}(\prod_l M_l)$ □

Notice that the only "degeneracy" of uniqueness can happen for $\mathrm{SU}(2) = \mathrm{Sp}(1)$. We will later study in detail these manifolds, known as $K3$ surfaces.
Together with the results contained in the next chapter (which still follow from the decomposition theorem), the theorem implies the following results.

Corollary 2.5.3. *There is an exact sequence of groups $1 \to \mathbb{Z}^{2k} \to \pi_1(X) \to G \to 1$ where the group G is finite.*

Corollary 2.5.4. *If $\chi(\mathcal{O}_X) \neq 0$, then any finite covering space Y is without irregularities and $\pi_1(X)$ is finite.*

Corollary 2.5.5. *If dim*(X) *is odd, then* $\chi(\mathcal{O}_X) = 0$.

Proof. contenuto... $\square$

Corollary 2.5.6. *Let* $n = 2r = dim(X)$*. Then* $0 \leq \chi(\mathcal{O}_X \leq 2^r$*. The equality is obtained if and only if* $X = (K3)^r$.

Proof. Let $Y = \prod_i M_i \to X$ be the finite cover in the form of the theorem. One has that

$$0 \leq \chi(\mathcal{O}_{M_i})^{(m_i/2)+1} \leq 2^{(m_i/2)+1}$$

and one has an equality if and only if M_i is a $K3$. One concludes observing that

$$\chi(\mathcal{O}_X) \leq \chi(\mathcal{O}_Y) = \prod_i \chi(\mathcal{O}_{M_i})$$

$\square$

Corollary 2.5.7. *There is a bound* $h^{p,0}(X) \leq \binom{n}{p}$ *for all* p*. If one has an equality for some* $0 < p < n$*, then* X *is a torus.*

Proof. Let $T \times \prod_j M_j$ be the finite cover of X, where T is a torus. Then $h^{p,0}(M_i) \leq h^{p,0}(T_i)$ and the equality holds if and only if M_i is a torus. Thus $h^{p,0}(Y) \leq \binom{n}{p}$. Assume that the equality holds for p_0, then one sees immediately that Y has to be a complex torus. We shall show that in this case X is a torus as well.
Without loss of generality, we assume that the cover is Galois. Since $H^{p_0,0}(X) \simeq H^{p_0,0}(Y)^G$ and they have the same dimension, G acts trivially on the whole $H^{p_0,0}(Y)$. As Y is a complex torus and a basis for $H^{p_0,0}(Y)$ is provided by $dz_{i_1} \wedge \cdots \wedge dz_{i_{p_0}}$, the action of G on $H^{1,0}(Y)$ has to be the multiplication with a p_0-th root of unity. The Lefschetz fixed-point formula yields

$$1 - \binom{n}{1} L_g + \binom{n}{2} L_g^2 + \cdots + (-1)^n = (1 - L_g)^n = 0$$

which is verified if and only if $L_g = 1$ and the induced action on $H^{1,0}(Y)$ is trivial if and only if G acts by translation. Then $X = Y/G$ is a torus. $\square$

The original version of Beauville's theorem does not prescribe the unicity of the finite étale decomposition. Some improvements were presented shortly after by Beauville himself in [Bea83a]. First one has the following

Lemma 2.5.8. *Let* $X = T \times S$*, with* T *a complex torus and* S *a compact Kähler metric with* $b_1(S) = 0$*. Then any auotomorphism* u *of* X *is of the form* $u = (v, w)$ *with* $v \in Aut(T)$ *and* $w \in Aut(S)$*.*

We say that a Kähler manifolds X is split if $X = T \times S$ and that a finite covering $Y \to X$ is split if Y is split. A split covering $\pi : T \times S \to X$ is said minimal if it is a Galois covering and $\mathrm{Deck}(\pi)$ does not contain any element of the form (τ, Id_S) where τ is a translation of the torus. The term minimal is explained by the following theorem.

Theorem 2.5.9. *If a compact Kähler manifold X admits a finite split covering, then there exists a unique minimal split covering π. Any other split covering factors through π.*

Proof. Let $Y \to X$ be a finite split covering. Every finite cover $Z \to Y$ is split, indeed if $Y \backsimeq T \times S$ then Z is isomorphic to

$$\frac{U}{H} \times S \qquad \text{with } H \leq \pi_1(T)$$

where U is the universal cover of T. This is finite if and only if H has finite index, which implies that U/H is isomorphic to T, as every finite cover of a torus is isomorphic to a torus.
In particular there exists a finite Galois cover $p : T \times S \to X$. Let $K \leq \mathrm{Deck}(p)$ be the subgroup consisting of the automorphisms of the form $(\tau, 1)$. We have that K is normal in $\mathrm{Deck}(p)$, by the preovious lemma.
Let $\tilde{T}$ be the complex torus which stays over T, i.e. $\tilde{T}/K = T$. Then the cover $\tilde{T} \times S \to X$ is clearly finite and Galois with Galois group $\mathrm{Deck}(p)/K$, hence minimal.
Let $\pi' : T' \times S' \to X$ be another split cover of X (not necessarily normal), corresponding to $H \leq \pi_1(X)$. We shall prove that it factors through $\tilde{T} \times S \to X$. We will call $\widetilde{G}$ and G' the two subgroups of $\pi_1(X)$ corresponding to π and π'. We first build a Galois cover (not necessarily finite) $\pi'' : W \to X$ which corresponds to a normal subgroup $G'' \trianglelefteq \pi_1(X)$, with $G'' \subseteq \widetilde{G} \cap G'$. Then π'' factors through π and π', which imply that $S = S'$ and $W \backsimeq T'' \times S$.
The following diagram depicts the situation.

$$\begin{tikzcd}
 & T'' \times S \arrow[dl, "\rho"'] \arrow[dr, "\rho'"] \arrow[dd, "\pi''"] & \\
\tilde{T} \times S \arrow[dr, "\pi"'] & & T' \times S \arrow[dl, "\pi'"] \\
 & X &
\end{tikzcd}$$

The two Galois groups $\mathrm{Deck}(\rho)$ and $\mathrm{Deck}(\rho')$ are subgroups of $\mathrm{Deck}(\pi'')$, thus their elements are all of the form $(\tau, 1)$ (a finite cover $T \to T$ has always the translation as the only automorphisms, being unramified).
Since π is minimal, we have that $G' \subseteq \widetilde{G}$, hence π' factors through π. $\square$

In particular if X is a Ricci-flat manifold, there exists a unique finite Galois cover $T \times \prod_i V_i \times \prod_j W_j$, and the auotomorphisms of this cover never act simply by a translation on T. This result is clearly very convenient, as Galois covers are the easiest to deal with.
Some immediate consequences:

- If $\pi : T \times S \to X$ is a minimal Galois cover, $X \backsimeq (T \times S)/G$. Let ϕ be an automorphism of X, then

$$\begin{tikzcd}
T \times S \arrow[r, dashed, "\hat{\phi}"] \arrow[d, "\pi"] \arrow[dr] & T \times S \arrow[d, "\pi"'] \\
X \arrow[r, "\phi"] & X
\end{tikzcd}$$

where the dashed arrow exists by the factorization property. In particular any ϕ lifts to a unique automorphism of $T \times S$. Since $\phi G = G\hat{\phi}$, the group $\mathrm{Aut}(X)$ identifies with the normalizer of $G \leq \mathrm{Aut}(T) \times \mathrm{Aut}(S)$.

- If $S = \prod_i S_i^{n_i}$ where S_i are irreducible and pairwise non isomoprhic manifolds (not necessarily Kähler), then $\mathrm{Aut}(S) \backsimeq \prod_i \mathrm{Aut}(S_i^{n_i})$

The theorem shall be considered as a classification theorem of Ricci-flat Kähler manifolds, in low dimension we have:

- Riemann surfaces
  All Riemann surfaces are projective, thus in particular they admit a Kähler metric. The only admissible groups in the decomposition in dimension one are the trivial group and $\mathrm{SU}(1) = \mathbb{C}^*$. Even in the simple case of a Riemann surface X, it is not immediate to determine without the theorem whether X admits a Ricci-flat Kähler metric, except from some cases. If $g(X) = 0$, any such metric would induce the flat connection on the canonical bundle K_X; as X is simply connected, this would imply that K_X is trivial, but by Riemann-Roch theorem $H^0(X, K_X) = 0$.
  If $g(X) = 1$ the canonical bundle is trivial, hence a Ricci-flat Kähler metric exists by Calabi-Yau theorem.
  For higher genus, one could try to compute c_1, but the theorem immediately tells that $g(X) = 1$ is the only case, since for any finite cover only the torus term can appear and Riemann-Hurwitz formula.

- Kähler complex surfaces
  In dimension 2, the admissibile groups in the decomposition are the trivial one and $\mathrm{SU}(2) = \mathrm{Sp}(1)$ and clearly only one of them can appear.
  The situation is still simple: if there is a finite cover of the form $T^2 \to X$, which can be assumed Galois, then $X = T^2/\mathbb{Z}^4$. If the torus is projective, so is the quotient[5] and the resulting manifold is known as bi-elliptic surface. If not, one has a commutative diagram in the topological category

$$\begin{tikzcd}
T^2 \arrow[r, "\backsimeq"] \arrow[d] & T^2_{\mathrm{proj}} \arrow[d] \\
T^2/\mathbb{Z}^4 \arrow[r, "\backsimeq"'] & (T^2/\mathbb{Z}^4)_{\mathrm{proj}}
\end{tikzcd}$$

[5] In general if $f : S_1 \to S_2$ is a map between compact surfaces, S_1 is projective if and only if S_2 is projective (see 4,6.8 in [Bar+03]).

and as being Kähler is a topological property for complex surfaces, everything in the diagram is Kähler.
It can be proved that in both cases X has torsion canonical bundle (see,), thus they admit a Ricci-flat Kähler metric.
If the universal cover has the form $V \to X$, then either X is simply connected and $V = X$ (these are indeed the only compact, simply connected, Ricci-flat Kähler surfaces) or $V/\pi_1(X) \backsimeq X$ and $\pi_1(X)$ is necessarily finite. The theorem does not provide further constraints on $\pi_1(X)$. It is a difficult problem to determine all the groups which can act on a $K3$ addressed from example by (...) and a complete classification has not been obtained yet.
However if $\pi_1(X)$ acts on V, there is a natural action of $\pi_1(X)$ on all the $H^p(X, \mathbb{C})$. In particular X is without irregularities as well. By theorem 4.2.3, X Kähler and without irregularities is simply connected if the canonical bundle is trivial. This clearly happens if and only if $X = V$. If not, still $c_1^{\mathbb{R}}(X) = 0$ because X is Ricci-flat and some additional work shows that it can only be 2-torsion. Thus X is an Enriques surface.
In conclusion the Ricci-flat Kähler compact surfaces are: complex tori, bi-elliptic surfaces, $K3$ surfaces (the only simply connected) and Enrique surfaces.

Thanks to Calabi-Yau theorem it is possible to give a metric-free version of the decomposition theorem. The statement is based on the cohomological properties of the special manifolds appearing in the decomposition, which are studied in the next chapter.

Theorem 2.5.10. *Let X be a compact Kähler manifold with $c_1^{\mathbb{R}}(X) = 0$. Then its universal cover is biholomoprhic to*

$$\mathbb{C}^k \times \prod_i V_i \times \prod_j W_j$$

where each V_i is projective, has dimensione at least 3, trivial canonical bundle and $h^{p,0} = 0$ for any $0 < p < dim(V_i)$ and X_j is irreducible symplectic and admits a Kähler metric. The decomposition is unique up to reordering the factors. Moreover there exists a finite cover $Y \to X$ with

$$Y \backsimeq T \times \prod_i V_i \times \prod_j W_j$$

Proof. The existence is clear by Calabi-Yau theorem and (...).
To prove the unicity, notice that if Y and Z are compact, simply connected manifolds which admit a Kähler metric and with vanishing $c_1^{\mathbb{R}}$, then every biholomoprhism

$$u : \mathbb{C}^p \times Y \to \mathbb{C}^q \times Z$$

splits as $u = (u_1, u_2)$. In particular $p = q$. Indeed one can write $u(z, y) = (u_y(z), u_z(y))$, since w_z is a one-parameter automorphism of Y varying continuously with z and the automorphism group of Y is discrete, u_z does not depend

on z, thus $u(z,y) = (u_y(z), u_2(y))$. This implies that u_y also does not depend on y.
The result follows if we prove that given $Y_1, \dots, Y_k$ compact, simply connected, with vanishing $c_1^{\mathbb{R}}$ and $X = \prod_i Y_i$, any Ricci-flat Kähler metric g on X is of the form $\sum \mathrm{pr}_i^*(g_i)$, with g_i Ricci-flat Kähler metrics on the Y_i.
This is easy to prove: let g be as above, with corresponding Kähler form ω. Then $\omega = \sum \mathrm{pr}_i^*(\omega_i)$ by Kunneth formula (all the Y_i are simply connected), where each ω_i is a Kähler form for Y_i. Let g_i the corresponding Ricci-flat Kähler metric on each Y_i. Then $\sum_i \mathrm{pr}_i^*(g_i)$ coincides with g, by the unicity prescribed by Calabi-Yau theorem.
To conclude, if X is imply connected fix a Ricci-flat Kähler metric g, then any biholomoprhism $X \to \prod_i V_i \times \prod_j W_j$ is also an isometry, with respect to the induced Kähler metrics on each factor, and the unicity of the biholomoprhism thus follows by the unicity of the isometry of theorem (...).
If X is not simply connected, the splitting of u given above yields the conclusion. $\square$

Chapter 3

Special holonomy manifolds

3.1 K3 surfaces

We have defined $K3$ surfaces as the compact Kähler surfaces which are simply connected and have Hol = SU(2). The following definition is metric free and can also be adapted to arbitrary algebraic varieties.

Definition 3.1.1. A compact and connected complex surface[1] X is a $K3$ surface if $K_X \backsimeq \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.

To check for the equivalence, we need the following non trivial result (see [Bar+03, p. 144]).

Theorem 3.1.2. *A compact complex surface X admits a Kähler metric if and only if $b_1(X)$ is even.*

The proof is analyitic and is based on the currents. We remark that analogous results are unknown for dimension greater than 2, it is indeed still an open problem if the existence of a Kähler metric is a topological property.
The two definitions of $K3$ surfaces coincide since if X is Kähler, simply connected and has holonomy SU(2), then $K_X \backsimeq \mathcal{O}_X$ and $h^{0,1} = 0$.
Viceversa if X has no irregularities, the long exact sequence

$$H^0(X, \mathcal{O}_X^*) \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*)$$

implies that $H^1(X, \mathbb{Z}) = 0$, thus X admits a Kähler metric h by the theorem. Since $K_X \backsimeq \mathcal{O}_X$, one has that Hol $\subseteq SU(2)$. This leaves as the only possibilities $X = K3$ or $X = T$, the latter is excluded as

$$H^1(T^2, \mathcal{O}_{T^2}) = H^0(T^2, \Omega_{T^2}) = \mathbb{C} \cdot dz + \mathbb{C} \cdot dw$$

[1] In the algebraic setting one replaces compact complex manifolds with complete non-singular varieties over a field k.

The first example of $K3$ surfaces (on an arbitrary field) is provided by a smooth quartic $X \subset \mathbb{P}^3$. Indeed, we have a short exact sequence on $\mathbb{P}^3$

$$0 \to \mathcal{O}(-4) \to \mathcal{O} \to \mathcal{O}_X \to 0$$

and

$$H^1(\mathbb{P}^3, \mathcal{O}) = H^2(\mathbb{P}^3, \mathcal{O}(-4)) = 0$$

where we use that $H^k(\mathbb{P}^n, \mathcal{O}(m)) \backsimeq 0$ if $0 < k < n$ (see for example [Har13, pp. 225–226]).
Then $H^1(X, \mathcal{O}_X) = 0$, whereas the triviality of the canonical bundle will be proved later in more generality.
We now collect some properties of a $K3$ surface.

Lemma 3.1.3. *Let X be a $K3$ surface. Then:*

1. *Pic(X) does not contain any non-trivial torsion line bundle.*
2. *The integral cohomology of X is given by* $\begin{cases} H^0(X,\mathbb{Z}) = \mathbb{Z} \\ H^1(X,\mathbb{Z}) = 0 \\ H^2(X,\mathbb{Z}) = 22\mathbb{Z} \\ H^3(X,\mathbb{Z}) = 0 \\ H^4(X,\mathbb{Z}) = \mathbb{Z} \end{cases}$
3. *The Hodge diamond numbers of X are $h^{0,0} = h^{2,2} = h^{2,0} = h^{0,2} = 1$, $h^{1,1} = 20$ and 0 otherwise.*

Proof. For the point 1., if $L = \mathcal{O}(D) \in \mathrm{Pic}(X)$ and has torsion, then Riemann-Roch formula for surfaces[2] yields $\chi(L) = 2$, hence either L or L^* corresponds to an effective divisor. Without loss of generality, we assume the first case. This implies that if $s \in H^0(X, L)$, then $Z(s) = Z(s^{\otimes k})$ for any $k > 0$, i.e. L is trivial.
For the point 2., the group H^1 has been computed before. For H^2, the long exact exponential sequence shows that

$$\mathrm{Pic}(X) = \ker\left(H^2(X,\mathbb{Z}) \to H^2(X, \mathcal{O}_X)\right)$$

and since both $\mathrm{Pic}(X)$ and $H^2(X, \mathcal{O}_X)$ are torsion-free (as $\mathbb{Z}$-modules), so is $H^2(X,\mathbb{Z})$. This implies immediately that $H^3(X, Z) = 0$, indeed Poincaré duality ensures that $H^3(X, Z) = 0$ up to torsion and $H^2(X,\mathbb{Z})_{\mathrm{tor}} = H^3(X,\mathbb{Z})_{\mathrm{tor}}$ by universal coefficient theorem.
It remains to compute the rank of H^2. We have already seen that $\int_X c_2(X) = 24$ and that this number equals the topological Euler characteristic, thus $b_2(X) = 22$.
For the point 3., the first chain of equalities is obvious by definition and Serre duality, whereas $h^{1,1} = 20$ and the vanishing of all the other groups follow by point 2.. ☐

[2] See for details

Example 3.1.4 (Kummer $K3$ surface)**.** We build an example of a non-projective $K3$ surface. It is the only simple example (see).
Let A be an abelian surface, which can be identified with a quotient $\mathbb{C}^2/\Gamma$, where Γ is a rank 4 lattice. On this manifold there is a natural involution map $\iota : x \mapsto -x$, which has 16 fixed points p_i, since $A \backsimeq (\mathbb{R}/\mathbb{Z})^4$ as a group.
Now we can proceed in two different ways. The first one consists of blowing up the 16 fixed points by $\epsilon : \widetilde{A} \to A$, then the involution ι lifts to an involution $\tilde{\iota}$ of $\widetilde{A}$: if x is not in the exceptional divisor, then $\tilde{\iota}(x) = \iota(\epsilon(x))$; if $x \in \epsilon^{-1}(p_i)$, fix (z, w) local coordinates on A around p_i, so that $\iota(z, w) = (-z, -w)$. Since the blow-up of $\mathbb{C}^2$ in the origin is

$$\mathrm{Bl}_0(\mathbb{C}^2) = \{((z, w), [t : s]) \in \mathbb{C}^2 \times \mathbb{P}^1 : zs = tw\}$$

a neighborhood of x is covered by two open subsets with coordinates (w, t) and (z, s) respectively. Then one sets

$$\tilde{\iota}(w, t) = (-w, t) \qquad \tilde{\iota}(z, s) = (-z, s)$$

which lifts ι locally and is well defined on the intersection.
Now let $\pi : \widetilde{A} \to X = \widetilde{A}/\iota$ be the projection. We shall show that X and π are smooth complex manifold.
For any $q \in \widetilde{A}$, $q \notin \epsilon^{-1}(p_i)$, there exists a neighborhood $U = U_{\pi(q)}$ with $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ a two sheet covering. If $q \in \epsilon^{-1}(p_i)$, $\tilde{\iota}(\alpha, \beta) = (-\alpha, \beta)$ shows that (β, α^2) are local coordinates for X around $\pi(q)$. The complex manifold X is called Kummer $K3$ surface.
An alternative way is to consider first the singular surface A/ι and then to blow-up the 16 singularities, one has $q : X \to A/\iota$. The following commutative diagram depicts the situation.

$$\begin{tikzcd} \widetilde{A} \arrow[r, "\pi"] \arrow[d, "\epsilon"] & X \arrow[d, "q"] \\ A \arrow[r] & A/\iota \end{tikzcd}$$

Let us compute K_X. On A there is a never vanishing form $\omega \in H^0(A, \Omega^2_A)$ and identifying $A = \mathbb{C}^2/\Gamma$, one can assume $\omega = dz \wedge dw$. Then $\iota^*(\omega) = \omega$ and $\tilde{\iota}^*\epsilon^*(\omega) = \epsilon^*(\omega)$. Indeed a holomorphic atlas for $\widetilde{A}$ around the exceptional divisor is given by

$$\{(U^0_{q_i}, \phi^0_i), (U^1_{q_i}, \phi^1_i)\}$$

with $q_i \in \epsilon^{-1}(p_i)$ for any i, $U^0_{q_i} \cap \epsilon^{-1}(p_j) = \emptyset$ and $U^1_{q_i} \cap \epsilon^{-1}(p_j) = \emptyset$ if $i \neq j$ and

$$\phi^0_i((z, w), [t : s]) = (z, s) \qquad \phi^1_i((z, w), [t : s]) = (t, w)$$

In this charts ϵ reads respectively $\epsilon(z, s) = (z, zs)$ and $\epsilon(t, w) = (tw, w)$ and consequently

$$\epsilon^*(dz \wedge dw) = dz \wedge zds \qquad \epsilon^*(dz \wedge dw) = wdt \wedge dw$$

which are both invariant under $\tilde{\iota}^*$. This implies that $\tilde{\iota}^*\epsilon^*(\omega)$ descends to the quotient A/ι and that $H^0(A, \Omega^2_{A/\iota})$ is trivial. Since blowing up does not affect H^2, K_X is trivial.
It can be proved without much trouble that X is also simply connected, see for example [Spa56]. Alternatively one observes that

$$\pi^* : H^1(X, \mathcal{O}_X) \to H^1(\widetilde{A}, \mathcal{O}_{\widetilde{A}})$$

has image contained in $H^1(\widetilde{A}, \mathcal{O}_{\widetilde{A}})^\iota$, hence $H^1(X, \mathcal{O}_X) = 0$.
We claim that X is projective if and only if A is.
If A is projective so is A/ι[3] and consequently also the blow-up X.
Viceversa, if X is projective then so is the double cover $\widetilde{A}$: an alternative definition for a line bundle $L \to X$ to be ample is that for any coherent sheaf $\mathcal{F} \in \mathrm{Coh}(X)$, there exists a positive integer n such that $\mathcal{F} \otimes L^{\otimes m}$ is generated by global sections as an $\mathcal{O}_X$-module, for any $m \geq n$ (see EGA II, 4.5.5). Now let $f : X \to Y$ a finite morphism (of schemes) and L ample on Y, thus for $\mathcal{F} \in \mathrm{Coh}(X)$ we have by hypothesis a surjection

$$\mathcal{O}_Y^{(I)} \twoheadrightarrow f_*\mathcal{F} \otimes_{\mathcal{O}_Y} L^{\otimes m}$$

for m big enough. Now by the projection formula we have

$$f_*\mathcal{F} \otimes_{\mathcal{O}_Y} L^{\otimes m} \backsimeq f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*L^{\otimes m})$$

and we apply f^* to the above surjection obtaining by right exactness

$$f^*\mathcal{O}_Y^{(I)} \twoheadrightarrow f^*f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*L^{\otimes m})$$

The term on the left is by definition $\mathcal{O}_X^{(I)}$ whereas the natural map

$$f^*f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*L^{\otimes m}) \to \mathcal{F} \otimes_{\mathcal{O}_X} f^*L^{\otimes m}$$

is surjective by the finiteness of f, which yields that $\widetilde{A}$ is projective.
The projectivity of A follows finally from the Castelnuovo's contraction criterion (see [Har13, p. 414]) which states that if Y is a rational curve curve on a algebraic surface X and $Y^2 = -1$ in the Chow ring, then there exists a blow-up $X \to X_0$, where X_0 is projective and smooth, whose exceptional divisor is Y. The variety X_0 is often called blow-down of X.
It can be proved that $\widetilde{A} \to A$ can be obtained as 16 blow-downs, thus A is projective.

A peculiar fact is that from a differential point of view, all $K3$ surfaces are actually the same. Indeed the following holds.

Theorem 3.1.5 (Kodaira, [Kod64]). *Any $K3$ surface can be realized by choosing some complex structure I on a fixed 4-dimensional differentiable manifold M.*

[3] In general given a finite map $f : S_1 \to S_2$ between complex surfaces, S_1 is projective if and only if S_2 is (see [Bar+03, p. 162]). One direction of the is proved below.

We have seen that a smooth quartic in $\mathbb{P}^3$ is a $K3$, thus we can take for example $M = V(x^4 + y^4 + z^4 + w^4) \subset \mathbb{P}^3$ (the Fermat quartic). The question whether any complex structure I on M gives rise to a $K3$ is answered by

Theorem 3.1.6 (Friedman, Morgan, 1994, [FM94])**.** *Every complex surface that is diffeomorphic to a $K3$ surface is a $K3$ surface.*

To complete the picture we also give

Theorem 3.1.7 (Kodaira and Michael Freedman)**.** *Any complex surface which has the homotopy type of a $K3$ is homeomorphic to a $K3$ surface. There exist complex surfaces which are homeomorphic but not diffeomorphic to a $K3$ surface. They are known as homotopy $K3$ surfaces.*

A detailed analysis of these manifolds can be found in [Kod70]. Here for example the following characterization is given: a simply connected compact complex surface is a homotopy $K3$ if and only if $p_g = 1$, $c_1^2 = 0$ and $c_1 \equiv 0 \mod 2$.
All the homotopy $K3$ surfaces can be obtained via logarithimic transform of an elliptic $K3$ and we will now present the construction.

Example 3.1.8 (Homotopy $K3$ surfaces)**.** We will now show how to obtain complex surfaces which are homeomorphic but not diffeomorphic to a $K3$ surface. We will actually build a whole pairwise non diffeomorphic family.
In general an elliptic complex surface is a complex surface X which admits a fibration whose general fiber is an elliptic curve, i.e. there exists a holomoprhic map $\pi : X \to C$, where C is a riemann surface and a finite subset $F \subset C$ such that $\pi|_{X \setminus \pi^{-1}(F)}$ is a submersion and each fiber over $F \subset C$ is biholomoprhic to a smooth cubic in $\mathbb{P}^2$.
Homotopy $K3$ surfaces are all obtained (see [Kod70]) by a surgery operation of an elliptic $K3$, known as logarithmic transform.
Let p_0, p_1 be two cubic homogeneous polynomials in (z_0, z_1, z_2), then define the hypersurface

$$E(2) = \{t_0^2 p_0(z_0, z_1, z_2) + t_1^2 p_1(z_0, z_1, z_2) = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^2$$

For generic p_0, p_1, E_2 is smooth and is a projective $K3$; the general fiber of the natural projection on $\mathbb{P}^1$ is a smooth cubic.
Since any smooth elliptic curve has genus 1, it is biholomoprhic with $\mathbb{C}/\{m + n\tau;$ with $m, n \in \mathbb{Z}$ and $\mathrm{Im}(\tau) > 0\}$. Thus one can form a holomoprhic family of smooth elliptic curves parametrized by X as

$$C_{\tau(x)} = \frac{\mathbb{C}}{\Lambda_{\tau(x)} = \{m + n\tau(x);\, m, n \in \mathbb{Z}\}}$$

where $\tau : X \to \mathbb{C} \cap \mathrm{Im} > 0$ is a holomoprhic map from an arbitrary complex manifold X.
Consider now an elliptic surface $\pi : X \to B$ and a regular value $b \in B$. It is possible to prove that locally the elliptic fibration π is of form C_τ, i.e. where

exists a holomoprhic chart (Δ, w) centered in b, a holomoprhic map $\tau : \Delta \to \mathbb{C} \cap \mathrm{Im} > 0$ and a biholomoprhism $F : \pi^{-1}(\Delta) \to C_\tau$ such that the following diagram commutes

$$\begin{tikzcd}
\pi^{-1}(\Delta) \arrow[r, "F"] \arrow[d, "\pi"] & C_\tau \arrow[d, "\pi_\tau"'] \\
\Delta \arrow[r, "1"] & \Delta
\end{tikzcd}$$

Define $\Sigma \subset C_\tau \times \Delta$ as the pullback of $C_\tau \to \Delta$ along the map $\Delta \to \Delta$ which maps $\zeta \mapsto \zeta^m$, explicitly

$$\Sigma = \{(z, \zeta) : \, z \in C_{\tau(\zeta^m)}, \, \zeta \in \Delta\}$$

The natural projection $\Sigma \to \Delta$ is an elliptic fibration and Σ has a natural automorphism

$$(z, \zeta) \mapsto \left(\left(z + \frac{\tau(\zeta^m)}{m} \right) \bmod \Lambda_{\tau(\zeta^m)}, \, e^{2\pi i/m} \zeta \right)$$

which gives rise to a group $\Gamma =< \phi >$ of order m acting freely on Σ. Define $\tilde{\Sigma} = \Sigma/\Gamma$. The map $\Sigma \to \Delta$ which maps (z, ζ) to ζ^m is invariant with respect to this action, thus it descends to a map $u : \tilde{\Sigma} \to \Delta$, which induces an elliptic structure on $\tilde{\Sigma}$.

In general the fiber over b of an elliptic fibration $M \to B$ is said to have multiplicity m if there is a holomoprhic chart (Δ, w) centered in b such that $\pi|_{\pi^{-1}(\Delta)} = g^m$ for some $g \in \mathrm{Hol}(\pi^{-1}(\Delta), \Delta)$ and g has finitely many critical points. We call the hypersurface $g^{-1}(0)$ the m-th root of the fiber $\pi^{-1}(b)$ [4] .

In the case of $u : \tilde{\Sigma} \to \Delta$, the fiber over 0 has multiplicity m and its m-th root is smooth and biholomoprhic to $C_{\tau(0)}$, whereas the fibers over the over points are smooth with multiplicity one. Consider the biholomoprhism

$$L_m : \tilde{\Sigma} \setminus u^{-1}(0) \to C_\tau \setminus C_{\tau(0)}$$

defined as the descent of the Γ-invariant map $\Sigma \setminus \zeta^{-1}(0) \to C_\tau \setminus C_{\tau(0)}$ given by

$$(z, \zeta) \mapsto \left(\left(z - \frac{\tau(\zeta^m)}{2\pi i} \log \zeta \right) \bmod \Lambda_{\tau(\zeta^m)}, \, \zeta^m \right)$$

We can finally perform the surgery in this way: fix $b \in B$ regular for π, so that $E(2)|_b$ is smooth. Choose a small holomoprhic chart (Δ, w) centered in b. Then

- Remove from $E(2)$ the set $\pi^{-1}(\Delta_{1/2^m})$.
- Glue the elliptic fibration $\tilde{\Sigma}$ through the map

$$L_m : \tilde{\Sigma}|_{\Delta \setminus \bar{\Delta}_{1/2}} \to \pi^{-1}(\Delta \setminus \bar{\Delta}_{1/2^m})$$

[4] In the literature this is usually known as the reduction of the fiber $\pi^{-1}(b)$.

We denote this operation as $\log(M, b, m)$ or $E(2, m)$ in the case $M = E(2)$. This construction, known as m-th logarithmic transform, yields a new fibration which differs only in a neighborhood of b and that is holomorphically equivalent over $B \setminus b$. The fiber over b contains m elliptic curves, is singular and can be resolved by taking the m-th square root.
It can be proved that two simply connected surfaces of the form $E(2; n_1, n_2)$ and $E(2; n_1', n_2')$ are homeomorphic if and only if

$$n_1 + n_2 \equiv n_1' + n_2' \qquad \text{mod } 2$$

and are diffeomeomorphic if and only if

$$(n_1, n_2) = (n_1', n_2')$$

In particular one can easily build with this method a countable family of complex surfaces homeomorphic but not diffeomorphic to a $K3$.

The two classes of manifold that we will analyze in the next two paragraphs shall be seen as two possible generalizations of $K3$ to higher dimensions.

3.2 Calabi-Yau manifolds

We begin with a generalization of theorem (...).

Theorem 3.2.1. *Let (X, g) be a compact Kähler manifold with $Hol = SU(m)$. Then $h^{p,0} = 0$ for any $0 < p < m$ and $\chi(\mathcal{O}_X) = 1 + (-1)^m$.*

Proof. The metric is Kähler by lemma (...). Let $x \in X$. The induced representation on $\Omega^p_{X,x}$ is isomorphic to $\Lambda^p \rho^*$, where $\rho : \mathrm{SU}(m) \to \mathrm{GL}(m, \mathbb{C})$ is the inclusion. It is irreducible for any p, thus it is non trivial if $p \neq 0, m$. By lemma (...), this implies that $H^0(X, \Omega^p_X) = 0$ if $p \neq 0, m$. The number $\chi(\mathcal{O}_X)$, also known as Holomorphic Euler characteristic of a topological manifold, is defined as

$$\chi(\mathcal{O}_X) = \sum_i (-1)^i \dim(H^i(X, \mathcal{O}_X)) = \sum_i (-1)^i h^{0,i}$$

and the result follows by Hodge duality and by the triviality of the canonical bundle. □

Corollary 3.2.2. *If moreover $\dim(X) \geq 3$, then X is algebraic.*

Proof. By Kodaira embedding theorem, X admits a holomoprhic embedding in some $\mathbb{P}^n$ as $h^{2,0} = 0$ and by Chow theorem X admits an algebraic structure. □

Thanks to Calabi-Yau theorem, we have the two following results.

Theorem 3.2.3. *Let X be a complex manifold of dimensione m which admits a Kähler metric h_0. Then X admits a Kähler metric h whose associated holonomy group is $SU(m)$ if and only if $K_X \backsimeq \mathcal{O}_X$ and $H^0(Y, \Omega^p_Y)$ for any $0 < p < m$, where $Y \to X$ is an arbitrary finite cover. Moreover $\pi_1(X)$ is finite in this case.*

Proof. If such a metric h exists, the canonical bundle is trivial by theorem (...) and $H^0(X;\Omega^p_X) = 0$ for any $0 < p < m$ by theorem (...). If $Y \to X$ is a finite cover, $\mathrm{Hol}Y = \mathrm{Hol}(X)$.
Conversely if $K_X \simeq \mathcal{O}_X$ then $c_1(X) = 0$ and Calabi-Yau theorem yields a Ricci-flat Kähler metric h. Let

$$Y = T^k \times \prod_i V_i \times \prod_j W_j \to X$$

be its Beauville decomposition. The condition $H^0(Y, \Omega^p_Y) = 0$ for any $0 < p < m$ and $\dim(Y) = \dim(X)$ implies that $k = j = 0$ and $i = 1$. Indeed all the manifolds appearing in the decomposition have trivial canonical bundle, hence for different values of i, k, j there would be a non trivial global section.
Thus $Y = V_1$, which implies $\mathrm{SU}(m) = \mathrm{Hol}_0(Y) = \mathrm{Hol}_0(X)$. Since $\mathrm{Hol}(X) \subseteq \mathrm{SU}(m)$, one has the conclusion.
The group $\pi_1(X)$ is finite and its cardinality clearly coincides with the degree of the covering. □

Corollary 3.2.4. *If m is even, X is simply connected.*

We conclude the paragraph with some examples in arbitrary dimension. The following theorem is an important tool which relates the topology of a projective variety and its hyperplane sections (for a proof, see [Voi03]).

Theorem 3.2.5 (Lefschetz' hyperplane section)**.** *Let $X \subset \mathbb{P}^N$ be a complex n-dimensional projective variety (not necessarily smooth) and Y a hyperplane section of X, with $U = X \setminus Y$ smooth and m-dimensional. Then*

$$H^k(X, \mathbb{Z}) \simeq H^k(Y, \mathbb{Z})$$
$$\pi_k(X) \simeq \pi_k(Y)$$

for any $k < n - 1$.

Example 3.2.6. Let $X = V(F) \subset \mathbb{P}^n$ be a non necessarily smooth hypersurface with $\deg(F) = d$.
If $d = 1$, then $X \simeq \mathbb{P}^{n-1}$. If $d \geq 1$ then $F \in H^0(\mathbb{P}^n, \mathcal{O}(d))$ and this line bundle is very ample. The Veronese embedding $\phi : \mathbb{P}^n \to \mathbb{P}^M$ maps $[z_0, \ldots, z_n]$ to all the possible monomials of degree d, thus $M = \mathrm{bin}(d+1, n+1)$ and $\phi^*(\mathcal{O}(1)) = \mathcal{O}(d)$ by computing the cocycles.
In particular $F \in H^0(\mathbb{P}^n, \phi^*(\mathcal{O}(1)))$, i.e. it is of the form $\phi^* s = \phi \circ s$ where s is a linear form in $\mathbb{P}^M$. As $Z(s)$ is a hyperplane in $\mathbb{P}^M$, one applies Lefschetz' theorem to $\phi(V(F)) = Z(s) \cap \mathbb{P}^n$.
Thus for example any projective hypersurface $X \subset \mathbb{P}^n$ with $n \geq 4$ is simply connected.

Lefschetz' theorem can be adapted to a hypersurface of a projective variety, $Y \subset X \subset \mathbb{P}^N$ with the condition that $\mathcal{O}_X(Y)$ is ample. Indeed such an Y is by definition $Y = Z(s)$ for some $s \in H^0(X, \mathcal{O}_X(Y))$ and $\mathcal{O}_X(Y)$ is ample. Let ϕ be the embedding $X \hookrightarrow \mathbb{P}^m$ such that $\phi^*(\mathcal{O}_{\mathbb{P}^m}(1)) = \mathcal{O}_X(Y)$, then $Z(s) = Z(\phi^*\alpha)$ which implies that $\phi(Z(s)) = Z(\alpha) \cap \phi(X)$.[5]

Example 3.2.8. Let $X = V(F_1, \ldots, F_r) \subset \mathbb{P}^n$ be a smooth complete intersection, with $F_i \in (\mathbb{C}[x_0, \ldots, x_n])_{d_i}$ and $\sum d_i = n+1$. The adjunction formula gives

$$\det(N^*_{X,\mathbb{P}^n}) \otimes K_X = K_{\mathbb{P}^n}|_X$$

We shall show that $N_{X,\mathbb{P}^n} \backsimeq= (\bigoplus_i \mathcal{O}(d_i))|_X$. We have already seen that $N_{V(F_i),\mathbb{P}^n} \backsimeq \mathcal{O}(d_i)$, thus if the following holds

$$N_{Y \cap V(F),\mathbb{P}^n} = N_{Y,\mathbb{P}^n}|_{Y\cap V(F)} \oplus N_{V(F),\mathbb{P}^n}|_{Y \cap V(F)}$$

with $Y \cap V(F)$ complete instersection, we can conclude by induction.
Foix a local chart $\phi_i : U_i \to \mathbb{C}^n$ such that

$$\begin{aligned} \phi_i(U_i \cap V(F)) &= \{(z_1, \ldots, z_n) \in \phi_i(U_i) : z_1 = 0\} \\ \phi_i(U_i \cap Y) &= \{(z_1, \ldots, z_n) \in \phi_i(U_i) : z_2 = \cdots = z_k = 0\} \end{aligned}$$

with $k-1 = \dim(Y)$. Over U_i the cokernel of $T_{V(F)}(U_i) \to T_{\mathbb{P}^n}(U_i)$ can be identified with sections over U_i of the form $z_1 = c$ with constant c, and analogously the cokernel of $T_Y(U_i) \to T_{\mathbb{P}^n}(U_i)$ with local sections of the form $z_j = c_j$ for any $j = 2, \ldots, k$.
Thus $N_{Y\cap V(F),\mathbb{P}^n}(U_i)$ is identified with local sections over U_i with constant first k coordinates.
If $\phi_{ij} : \phi_i(U_{ij}) \to \phi_j(U_{ij})$ is the change of coordinates, cocycles for $N_{V(F),\mathbb{P}^n}$ are given by $(\partial_{x_1}\phi^1_{ij}) \circ \phi_j$ whereas cocycles for $N_{Y,\mathbb{P}^n}$ by the $\dim(Y)$ square matrix $\left(\frac{\partial \phi^l_{ij}}{\partial x_m} \circ \phi_j\right)_{l,m}$. Since

$$\partial_{x_k}\phi^1_{ij} = 0 \quad \text{if } k \neq 1 \qquad\qquad \frac{\partial \phi^l_{ij}}{\partial x_m} = 0 \quad \text{if } l \text{ or } m \text{ are not in } 2, \ldots, k$$

one has the conclusion.
We have thus found that

$$\det(\bigoplus_i \mathcal{O}(-d_i))|_X \otimes K_X = \mathcal{O}(-n-1)|_X$$

---

[5] A weaker version of theorem is the following:

Theorem 3.2.7 (Nakano-Lefschetz)**.** *Let X be an n-dimensional projecive variety and $j : Y \to X$ a closed embedding of a smooth hypersurface. If $\mathcal{O}_X(Y)$ is ample, then $H^{p,q}(X) \backsimeq H^{p,q}(Y)$ whenever $p+q < n-1$. In particular*

$$j^* : H^k(X, \mathbb{C}) \to H^k(Y, \mathbb{C})$$

is an isomoprhism for any $k < n-1$.

It is strictly weaker as it requires the smoothness of Y and the result only holds for rational cohomology, but can be proved quickly via Nakano vanishing theorem. See [Voi03] for a proof.

which shows that $K_X \backsimeq \mathcal{O}_X$.
In order to show that X is simply connected, assume $n \geq 4$. We have already shown that for $V(F_1)$, thus it suffices to prove that $\mathcal{O}_{X_i}(V(F_{i+1}))$ is ample, where $X_i = V(F_1, \dots, F_i)$. Let $j_i : X_i \to X_{i-1}$, by definition

$$\mathcal{O}_{X_i}(V(F_{i+1})) = j_i^*(\mathcal{O}_{X_{i-1}}(V(F_{i+1}))$$

thus by induction

$$\mathcal{O}_{X_i}(V(F_{i+1})) = (j_i^* \dots j_1^*)(\mathcal{O}_{\mathbb{P}^N}(V(F_{i+1}))$$

The latter is ample and one concludes because ampleness is preserved by pull-backs along embeddings.

Example 3.2.9 (Hypersurfaces of Fano varieties)**.** In this example we modify our notation to distinguish between the canonical bundle ω_X and divisor K_X. Let X be a smooth projective variety with $\dim(X) \geq 3$ and ample anticanonical bundle $-\omega_X$.
Since X is projective, it is Kähler and one of the formulation of Kodaira theorem implies that $-\omega_X$ is positive, i.e. $c_1^{\mathbb{R}}(-\omega_X)$ can be represented by a real positive $(1,1)$-form α. By Calabi-Yau theorem, $\alpha = -\rho/2\pi$ with ρ the Ricci-form of some Kähler metric h and as $\rho = (i\mathrm{Ric})^{\mathrm{Alt}}$, for Fano varieties $\mathrm{Ric}(X) > 0$ for any Kähler metric. Then by Myers' theorem X is simply connected.
We will now show that any smooth divisor V in the linear system $|-K_X|$ is a special unitary manifold. First one has $\mathcal{O}_X(V) \backsimeq \mathcal{O}_X(-K_X) = -\omega_X$, thus Lefschetz' theorem implies that V is simply connected as well.
The conormal exact sequence

$$0 \to N^*_{V,X} \to \Omega_X|_V \to \Omega_V \to 0$$

where $N_{V,X} \backsimeq \mathcal{O}_X(V)|_V$ yields the adjunction formula

$$i^*\omega_X = \omega_V \otimes \mathcal{O}_X(-V)|_V$$

which implies $\omega_V \backsimeq \mathcal{O}_V$.
It remains to prove that $h^{p,0} = 0$ for any $0 < p < \dim(V)$. By Kodaira vanishing theorem

$$H^p(X, \mathcal{O}_X) = H^p(X, \omega_X \otimes -\omega_X) = 0 \quad \text{for any } p > 0$$

hence by Lefschetz' theorem $H^p(X, \mathbb{C}) = H^p(V, \mathbb{C})$ for any $0 < p < \dim(V)$. This yields the conclusion, as the isomorphism is an isomoprhism of Hodge structures.
In particular any smooth divisor in the anticanonical class of a Fano variety is a projective $K3$. Conersely, it is interesting to establish which projective $K3$ can be obtained in this way. A detailed discussion can be found in [Bea02].

3.3 Irreducible symplectic manifold

In this paragraph we will describe some features of Kähler manifolds with Hol $=$ Sp(r), usually known in the literature as irreducible symplectic manifold. [6] Their story is quite curious: at the beginning it was erronously suggested by Bogomolov in [Bog78] that none of them existed in dim > 2. Later on, Fujiki showed in [Fuj83] that the blow-up of $(K3 \times K3)/\mathrm{Sym}_2$ along its singular locus is irreducible symplectic, providing the first example. Soonly after, Beuaville in [Bea83b] generalized Fujiki's construction.

Theorem 3.3.1. *Let X be a compact Kähler metric with $dim_{\mathbb{C}}(X) = 2r$ and $Hol = Sp(r)$. Then X admits a non-degenerate form $\phi \in H^0(X, \Omega^2_X)$ and*

$$H^0(X, \Omega^p_X) = \begin{cases} 0 & \textit{if } p \textit{ is odd} \\ \mathbb{C} \cdot \phi^{\frac{q}{2}} & \textit{if } p \textit{ is even} \end{cases}$$

In particular $\chi(\mathcal{O}_X) = r + 1$ and X is simply connected.

Proof. The existence of ϕ has been proved in (...). A result from representation theory (See Bourbaki,....) states that if V is a complex vector space of dimension $2l$, ϕ is a non-degenerate alternating two-forms, then

$$\Lambda^k V = E_k \oplus X_-(E_{k-2}) \oplus X_-^2(E_{k-4}) \oplus \dots$$

where X_- is an endomorphism of ΛV increasing the degree by 2, $E_j \subseteq \Lambda^j V$ and are stable for the action of Sp(r), for any j.
One can also show that this decomposition induces a decomposition of the form

$$\Lambda^k V = P_k \oplus \phi(P_{k-2}) \oplus \phi^2(P_{k-4}) \oplus \dots$$

where still $P_j \subseteq \Lambda^j V$ and are stable for the action of Sp(r), for any j.
As multiplication by ϕ^k provides an equivariant isomorphism $\Lambda^{r-k} V \backsimeq \Lambda^{r+k} V$, the only invariant elements are ϕ and its powers.
By taking $\Lambda^r(V) = \Omega^r_X(x)$ and applying lemma (...) one has the conclusion.
The formula for the holomoprhic Euler characteristic is now obvious.
For the last point, if $\tilde{Y} \to X$ is the universal cover, then $\mathrm{Hol}(\tilde{Y}) = \mathrm{Hol}_0(X) \subseteq$ Sp(r), which leaves $\mathrm{Hol}(\tilde{Y}) =$ Sp(r) as the only possibility. As $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{\tilde{Y}}) = r + 1$, X is simply connected. □

The form ϕ defines a holomoprhic symplectic structure on X as it is automatically closed. It is not difficult to build Kähler manifolds with such a structure: analogously to the real case (see), the cotangent bundle Ω_X of any complex manifold admits a holomoprhic symplectic form, which in a local coordinate neighborhood reads $\sum_i d\alpha_i \wedge dz^i$ where z^i are the coordinates on X and α_i are the coordinates on the fiber of $\Omega_X \to X$ given by $\alpha_q(v_q) = (\alpha_i)_q v^i_q$ for

[6] Any Kähler manifold is trivially symplectic from the differentiable point of view, as the Kähler form is always non degenerate and closed.

any $v \in T_{X,q}$.
A manifold X admitting a holomorphic 2-form ω such that $\omega^{\dim(X)}$ is not the zero section provides another example. Indeed the restriction of ω to $Y \setminus \operatorname{div}(\omega^r)$ is non degenerate, as

$$\omega_p(u,-) = 0 \implies \omega_p^r(u,-,\dots,-) = 0$$

We will now show that admitting a holomoprhic symplectic structure, for manifolds admitting a Kähler metric, is equivalent to the existence of a Kähler metric whose holonomy contained in the symplectic group. If one requires that $\mathrm{Hol} = \mathrm{Sp}(r)$ or that X is simply connected, though, the number of examples drastically drop.

Theorem 3.3.2. *Let X be a $2r$-dimensional compact complex manifold, admitting a Kähler metric h_0. Then X admits a symplectic structure if and only if there exists a Kähler metric h such that $Hol \subseteq Sp(r)$, whereas there exists a Kähler metric h such that $Hol = Sp(r)$ if and only if $\pi_1(X) = 0$ and X admits a unique (modulo a constant) symplectic structure.*

Proof. If X admits a symplectic structure ω one has that $K_X \backsimeq \mathcal{O}_X$, hence $c_1^{\mathbb{R}}(X) = 0$ and by Calabi-Yau theorem there exists a Ricci-flat Kähler matric h. With respect to h, ω is parallel by Bochner principle and by theorem (..) $\mathrm{Hol} \subseteq \mathrm{Sp}(r)$. The other implication has been proved in (...).
If X admits a metric h with $\mathrm{Hol} = \mathrm{Sp}(r)$, then X is simply connected and admits a symplectic structure ω by the previous theorem, which is unique as if $\omega = f\tilde{\omega}$ then $0 = \bar{\partial}(f\tilde{\omega}) = \bar{\partial}(f) \wedge \omega$. Since ω is non degenerate, necessarily $\bar{\partial}(f) = 0$.
Conversely, if X admits a symplectic form ω then $\mathrm{Hol} \subseteq \mathrm{Sp}(r)$ for some Kähler metric h and X is isomorphic to its universal cover, thus

$$X \underset{\backsimeq}{\xrightarrow{\Phi}} \prod_i V_i \times \prod_j W_j$$

with $\mathrm{Hol}(V_i) = \mathrm{SU}(r_i)$ and $\mathrm{Hol}(W_j) = \mathrm{Sp}(r_j)$.
Then $\omega = \Phi^*\eta = \sum_i \Phi_i\eta_i + \sum_j \Phi_j\eta_j$, where η, η_i, η_j are symplectic form (Φ_i and Φ_j are local biholomoprhisms, thus they preserve non degeneracy). Since $\sum_i \lambda_i\Phi_i\eta_i + \sum_j \lambda_j\Phi_j\eta_j$ is symplectic for any $\lambda_1, \dots, \lambda_k \in \mathbb{C}^*$, $k = i + j$, the hypothesis implies that $k = 1$. Then necessarily the only term appearing in the decomposition is W_1, because $\mathrm{SU}(2r)$ is never a proper subgroup of $\mathrm{Sp}(r)$. $\square$

We now present a very recent alternative characterization of irreducible simplectic manifolds [Sch22]. We present the proof in details as it is an interesting example of application of Beauville's decomposition.

Theorem 3.3.3. *Let X be a compact Kähler manifold with $H^0(X; \Omega_X^2) \backsimeq \mathbb{C}\omega$. Then X is an irreducible symplectic manifold if and only if X has no irregularities.*

Proof. One direction is immediate: if X is simply connected then $H_1(X, \mathbb{C}) = 0$ by Hurewitz theorem and $H^1(X, \mathbb{C}) = 0$ by Poincaré duality and universal coefficient theorem. The result follows from $H^1 = H^{1,0} \oplus H^{0,1}$.
Conversely, we shall first prove that if X is holomorphic symplectic and without irregularities then it is simply connected or it is the quotient of a torus. Let $X' = T \times V \times W$ be its decomposition, with $V = \prod_i V_i$ and $W = \prod_j W_j$. By theorem (...), we can assume that the cover is Galois, thus $X'/G = X$ with $G \leq$ Aut(X').
We have the following isomorphisms

$$H^0(X, \Omega_X^2) \backsimeq H^0(X'/G, \Omega_{X'/G}^2) \backsimeq H^0(X', \Omega_{X'}^2)^G$$

thus $H^0(X'; \Omega_{X'}^2)^G = \mathbb{C}(\pi^*\omega)$. Now we have that

$$H^0(X', \Omega_{X'}^2) = H^0(T, \Omega_T^2) \oplus H^0(W, \Omega_W^2)$$

since $h^{1,0}(V) = h^{2,0}(V) = h^{1,0}(W) = 0$, thus $\pi^*\omega = \pi_T^*\eta + \pi_W^*\mu$ where π_T and π_W are the projections. Moreover $\pi^*\omega$ is non degenerate, as π is unramified, which implies that the Y factor do not appear and that η and μ are non degenerate as well.
If $u \in \mathrm{Aut}(X')$ then it splits as $u = (v, w)$, thus $\pi_T^*\eta$ and $\pi_W^*\mu$ are G-invariant. But $H^0(X, \Omega_X^2)$ has dimension one, then either T or W does not apper. In the latter case, we have concluded. Thus assume that $X = W/G$. We will now show that every action of G on W has a fixed point, which clearly implies that G acts trivially as X is smooth.
Let ω_i be the pullbacks of $\pi^*\omega$ to each factor W_i. By what we have seen in lemma (...), each $H^0(W, \Omega_W^{2p})$ is generated by the wedge products of ω_i and $H^{p,0}(W) = 0$ if p is odd. Every automorphism f of W preserves a symplectic form α and up to rescaling one can assume that $\alpha = \sum_i \omega_i$. By what we have observed in remark (...), f acts by permuting the factor and then on each single factor by f_i. In particular, there is a permutation σ such that $f^*\omega_i = \lambda_i \omega_{\sigma(i)}$ for any i, with $\lambda_i \in \mathbb{C}^*$. But the collection of ω_i has been chosen in a way that $f^*(\sum_i \omega_i) = \sum_i \omega_i$, then f^* is just a permutation and

$$\mathrm{tr}(f^*|_{H^{k,0}(W)}) \begin{cases} = 0 & \text{if } k \text{ is odd} \\ \geq 0 & \text{if } k \text{ is even} \\ = 1 & \text{if } k = 0, 2 \cdot \dim(X) \end{cases}$$

Then f has a fixed point by Lefschetz fixed point formula and $W = X$.
To conclude the proof, we admit the result of [Sch22] [7]

Lemma 3.3.4 (Schwald, 2022)**.** *Every smooth symplectic torus quotient with* $h^0(X, \Omega_X^2) = 1$ *is a two dimensional torus.*

This concludes the proof as for a two dimensional torus X one has $h^0(X, \mathcal{O}_X) = 2$. ∎

[7] The whole articol is devoted to analyze smooth quotient of torus, we refer to that for the details.

As we have anticipated, there known examples of irreducible symplectic manifold (up to deformation) are very few. They are:

- Douady spaces of a $K3$ surface.
- Generalized Kummer manifolds.
- Six dimensional OG-manifolds
- Ten dimensional OG-manifolds

We will now give the details of the construction of each of them.

3.3.1 Douady spaces

The first example of irreducible symplectic manifold if dimension higher than 2 was $K3^{[2]}$, found by Fujiki in 1982, in [Fuj82]. The following generalization is first found in Beauville's paper [Bea83b].
The construction of Douady spaces is analogous to the construction of Hilbert schemes, which we will now present.
Let X be a smooth projective scheme over $\mathbb{C}$. Let $X^{(r)}$ be the set which correponds to effective cycles of degree r. A more sophisticated way to parametrize the zero dimensional[8] subschemes $Z \subset X$ is by considering also the length of Z

$$l(Z) = \dim_{\mathbb{C}}(H^0(Z, \mathcal{O}_Z))$$

As a set, $\mathrm{Hilb}^n(X)$ is defined as the collection of all the zero-dimensional subschemes of length n.
If X is reduced and $x \in Z$ is closed, one defines the multiplicity of x in Z as $e_x(Z) = \dim_{\mathbb{C}}(\mathcal{O}_{Z,x})$. Then there is a map, called Hilbert-Chow map

$$\rho : \mathrm{Hilb}^n(X) \to X^{(n)}$$

which maps Z to

$$|Z| = \sum_{x \in X, x \text{ closed}} e_x(Z) \cdot x$$

Here is an easy example.

Example 3.3.5. Consider the affine scheme $V = \mathrm{Spec}(\mathbb{C}[x, y])$. There is a bijection between all the subschemes of V and the ideals of $\mathbb{C}[x, y]$ and $Z \subset V$ has dimension 0 if and only if the corresponding ideal I has height 2, i.e. each chain $p_0 \subset \cdots \subset p_n$, where p_i are prime ideals and $I \subseteq p_n$, has length two.
If $n = 1$ closed subschemes are simple points, as $\mathbb{C}[x, y]/I$ has length one if and only if it is isomorphic to $\mathbb{C}$.
If $\mathbb{C}[x, y]/I$ has length two, there is a relation $a + bx + cy \in I$ with $a, b, c \in \mathbb{C}$ and up to a coordinate change $y \in I$. Then $x \notin I$ and $x^2 + I$ has to be a $\mathbb{C}$-linear

[8] We recall that the dimension of a scheme V can be defined either as the topological dimension of the underlying space or as $\sup_{\xi \in V} \dim(\mathcal{O}_{V,\xi})$.

combination of 1 and x, thus $I = (y, (x-\alpha)(x-\beta))$. The quotient $\mathbb{C}[x,y]/I$ corresponds either to two points or to a 2-fat point, the latter in particular can be thought as a point with a tangent vector attached. Recalling how blowups are built, this suggests that

$$\mathrm{Hilb}^2(\mathbb{A}^2) \simeq \mathrm{Bl}_\Delta\Big((\mathbb{A}^2 \times \mathbb{A}^2)/(z,w) \backsim (w,z)\Big)$$

Notice that already in the case $n = 2$ the set of 0-dimensional subschemes of length n is richer than the set of effective 0-cycles of degree 2, because the information on the tangent vector is now included.
If $n = 3$ there are more possibilities. For a cycle $z+w+t$, the ideal has the form $I = (x, y(y-\alpha)(y-\beta)))$, for a cycle $2z + w$ then $I = (x, y^2) \cap (x-\alpha, y-\beta)$, i.e. a single point and a different point with a tangent vector attached, for a cycle $3z$ then $I = (x, y^3)$, which corresponds to a point with a first and second order vector. The only ideal we have not considered is $I = (x, y)^2$, which still corresponds to a cycle $3z$ and can be thought as

$$\lim_{\delta,\epsilon\to 0} z + (z + \epsilon e_1) + (z + \epsilon e_2)$$

i.e. a point with two different first order tangent vectors attached, which determines a cone.
Although it is less evident, one can prove that in general $\mathrm{Hilb}^k(\mathbb{A}^2)$ can be obtained as a sequence of blow-up along the diagonals

$$\mathrm{Hilb}^k(\mathbb{A}^2) \simeq \mathrm{Bl}_{\Delta_n}(\mathrm{Bl}_{\Delta_{n-1}}(...)) \to \cdots \to \mathrm{Bl}_{\Delta_2}((\mathbb{A}^2)^{(k)}) \to (\mathbb{A}^2)^{(k)}$$

where Δ_k contains the cycles of the form $k \cdot x_1 + x_2 + \ldots x_{n+k+1}$ and all the x_i are distinct.
A detailed construction for the algebraic case can be found in [Hai98], where a combinatorial approach is used. For the complex analytic case, some adjustments have to be made. We refer to [CM00] for the details.

We would like to show that the Hilbert scheme, in some situations, is an answer to a moduli problem, i.e. it is an object which parametrizes a family in a universal way.
We recall that a morphism $\phi : S \to T$ between schemes is said flat if the induced maps on stalks is a flat map of rings, whereas ϕ is said finite if there is an affine open cover $(V_i = \mathrm{Spec}(B_i))$ for T such that $(U_i = \phi^{-1}(V_i))$ is an affine open cover for S and $\phi|_{U_i}$ is a finite ring morphism. The degree of a flat morphism is defined as $\dim_{k(y)} \mathcal{O}(\phi^{-1}(y))$.
We also need the definition of Hilbert polynomial associated to $Z \subset X$, which is given by

$$P_Z(k) = \chi(\mathcal{O}_Z \otimes \mathcal{O}_X(kD)$$

where D is an ample divisor on X.
Consider now the following moduli problem. The functor $\mathrm{Hilb}^n_X : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ sends a scheme T to the set of isomorphism classes of flat morphisms $\pi : Z \to T$ whose fibers have Hilbert polynomial costantly n, where $Z \subseteq T \times X$ is a subscheme. The following important result holds.

Theorem 3.3.6 (Grothendieck). *The functor $Hilb_X^n$ is representable by the quasi-projective scheme $Hilb^n(X)$, which is then a fine moduli space for the above problem.*

If one restricts to X non-singular and connected algebraic surface, the following holds

Theorem 3.3.7 (Fogarty, [Fog68]). *The Hilbert scheme $Hilb^n(X)$ is indeed a smooth and connected algebraic variety of dimension $2n$.*

For higher dimensional varieties, not only in general $\mathrm{Hilb}^n(X)$ is no longer smooth, but in general the singular locus is more intricated than in $X^{(n)}$.

Consider now be a compact complex surface X and define the Barlet space as $S^{(r)} = S^r/\mathcal{S}_r$, where $\mathcal{S}_r$ is the symmetric group acting on $(x_1, \dots, x_r)$ by permutation. Again as a set $S^{(r)}$ corresponds to the set of effective 0-cycles of degree r, but it is in general a difficult problem to determine its global topology. For semplicity, we restrict now to the case $r = 2$ and show that $S^{(2)} \setminus \Delta$ is naturally a complex manifold and that Δ is singular. The irreducible symplectic manifold we are looking for is exactly the resolution of $S^{(r)}$.
Let $[(x, y)] \in S^{(2)} \setminus \Delta$, fix a neighborhood V not intersecting Δ and onsider U_1 and U_2 the corresponding neighborhood of respectively (x, y) and (y, x). Up to shrinking U_1 and U_2, they do not intersect. Then, in local coordinates

$$V \backsimeq \frac{\{(x_1, x_2, x_3, x_4) \in W \subset \mathbb{C}^4 : (x_1, x_2) \neq (x_3, x_4)\}}{(x_1, x_2, x_3, x_4) \backsim (x_3, x_4, x_1, x_2)}$$

which shows that V is biholomorphic with a open subset of $\mathbb{C}^4$.
Now fix $[(x, x)]$ and a neighborhood Z. With respect to the induced local coordinates of S, $[(x, x)]$ is a singular point, indeed if with respect to the canonical basis $(x_1, x_2, x_3, x_4) \mapsto (x_3, x_4, x_1, x_2)$, in the basis $(e_1+e_3, e_2+e_4, e_1-e_3, e_2-e_4)$ one has $(y_1, y_2, y_3, y_4) \mapsto (y_1, y_2, -y_3, -y_4)$. To conclude, regular functions on this space are now

$$\mathbb{C}[y_1, y_2] \otimes \mathbb{C}[y_3, y_4]^{\mathrm{Inv}}$$

and $\{p(y_3, y_4) = p(-y_3, -y_4)\} \backsimeq \mathbb{C}[y^2, z^2, yz] \backsimeq \mathbb{C}[u, v, z]/(z^2 - uv)$. Hence Z is biholomorphic to $\mathbb{C}^2 \times V(z^2 - uv)$ and in particular is not smooth in $[(x, x)]$.

Definition 3.3.8. Let X be a complex manifold. The r-Douady space $X^{[r]}$ contains as a set all the 0-dimensional complex submanifolds $Z \subset X$ with $\mathrm{length}(\mathcal{O}_Z) = r$.

One can still define a moduli problem to which $X^{[r]}$ is the solution, but the situation is more delicate as Hilbert polynomials are no more defined in this setting. A detailed discussion can be found in [KS58].
It can be proved that if $\dim(S) = 2$, then $X^{[r]}$ is naturally endowed with a smooth complex structure. As in the algebraic setting, one can define naturally the Hilbert-Chow map, which is now holomoprhic $\epsilon : S^{[r]} \to S^{(r)}$.
If $\dim(S) \leq 2$, ϵ is bimeromorphic. Thus $S^{[r]} \to S^{(r)}$ can be considered a

resolution of singularities and $\epsilon^{-1}(D)$ is an irreducible divisor. Consider in D the subset $D_* = \{2x_1 + x_2 + \cdots + x_{r-1} : x_i \neq x_j\}$ and define

$$S_*^{(r)} = S^{(r)} \setminus (D \setminus D_*)$$

Let $S_*^{[r]} = \epsilon^{-1} S_*^{(r)}$ and $S_*^r = \pi^{-1} S_*^{(r)}$, then $S^{[r]} \setminus S_*^{[r]}$ has codimension 2 in $S^{[r]}$, since $S^{[r]} \setminus S_*^{[r]} = \epsilon^{-1}(D \setminus D_*)$ and D is irreducible.
It is now easier to deal with $S_*^{[r]}$, indeed with the same kind of computation as above, we find that $S_*^{(r)}$ is locally isomoprhic around its singular points to $\mathbb{C}^{2r-2} \times Q$, where Q is a cone with vertex v and that under this identification D_* corresponds to $\mathbb{C}^{2r-2} \times \{v\}$. The restricion $\epsilon|_{S_*^{[r]}}$ is now the blow-up of D_*. In S_*^r, which is the subset of product containing r-uples with at most two repetitions, the diagonal is smooth with codimension 2.
If $\eta : \mathrm{Bl}_{\Delta_*}(S_*^r) \to S_*^r$ is the blow-up, the action of the symmetric group lifts to an action on $\mathrm{Bl}_{\Delta_*}(S_*^r)$ and in conclusion one has that $S_*^{[r]}$ and $\mathrm{Bl}_{\Delta_*}(S_*^r)/\mathcal{S}$ are biholomorphic. With this one can prove

Lemma 3.3.9. *If S has trivial canonical bundle, $S^{[r]}$ admits a holomorphic symplectic structure.*

Proof. The idea is to build such a structure on $S_*^{[r]}$ and then extend it, by Hartog's theorem, to the whole $S^{[r]}$. Let ω be a symplectic form on S and set

$$\psi = \mathrm{pr}_1^*\omega + \cdots + \mathrm{pr}_r^*\omega$$

which is symplectic on S^r. It restricts to a holomoprhic 2-form on S_*^r, which can be pulled back along $\eta : \mathrm{Bl}_{\Delta_*}(S_*^r) \to S_*^r$ to a form $\tilde{\omega}$, which is invariant by the action of the symmetric group as ψ is clearly invariant on S^r.
Thus $\tilde{\omega}$ descends to a holomoprhic two form ϕ on $S_*^{[r]}$. Indeed, $\tilde{\omega} = \rho^*\phi$, as there is a one to one correspondence between invariant forms on $\mathrm{Bl}_{\Delta_*}(S_*^r)$ and forms on $S_*^{[r]}$.
We shall prove that ϕ never vanishes. Recall that for an arbitrary ramified cover $f : X \to Y$ between complex varieties

$$\mathrm{div}(f^*K_Y) = f^*\mathrm{div}(K_Y) + R_f$$

Now the map $\rho : \mathrm{Bl}_{\Delta_*}(S_*^r) \to S_*^{[r]}$ has ramification divisor

$$\sum_{i<j} \eta^{-1}(V(z_i - z_j))$$

thus

$$\mathrm{div}(\rho^*\phi)^r = \rho^*\mathrm{div}(\phi^r) + \sum_{i<j} \eta^{-1}(V(z_i - z_j))$$

By definition $\rho^*\phi = \tilde{\omega} = \eta^*(\psi|_{S_*^r})$ and

$$\mathrm{div}(\eta^*(\psi|_{S_*^r})^r) = R_\eta$$

since $\mathrm{div}(\psi|_{S^r_*}) = 0$.
To conclude, the exceptional divisor of η is $R_\eta = \sum_{i<j} \eta^{-1}(V(z_i - z_j))$, hence

$$\rho^* \mathrm{div}(\phi^r) = 0$$

which concludes the proof. □

We shall now study the cohomology of the manifolds $S^{[r]}$.

Lemma 3.3.10. *If S is a compact complex surface, the manifolds $S^{[r]}$ and $S^{(r)}$ have the same fundamental group.*

Proof. The following picture suggests how to compute $\pi_1(S^{[r]})$ starting from $\pi_1(S^r)$.
Fix $\xi = (x_1, \dots, x_r)$ outside from the diagonal. Every loop γ based in ξ has to be identified in the quotient with $\sigma(\gamma)$, for any $\sigma \in \mathrm{Sym}_r$, thus the group can be presented as

$$\pi_1(S^{(r)}, \xi) =< \alpha_1 \dots \alpha_k |\, \alpha_i \in \pi_1(S^r, \xi),\, \sigma(\alpha_i)\alpha_i^{-1} = 1 >$$ [9]

The injection $S^r_* \hookrightarrow S^r$ induces an isomoprhism on fundamental groups, as the first space is obtained from the latter by removing a codimension 4 submanifold, and the quotient of S^r_* by the subgroup of transpositions equals the quotient of S^r by the action of Sym_r. Thus $S^{(r)}_* \hookrightarrow S^{(r)}$ induces an isomorphism of fundamental groups as well.
To conclude, we notice that both the blow-up $S^{[r]}_* \to S^{(r)}_*$ and $S^{[r]}_* \hookrightarrow S^{[r]}$ induces isomorphisms on the fundamental groups, again by codimension argument. □

In particular if S is simply connected, so is S^r, $S^{(r)}$ and $S^{[r]}$.
To conclude analysis of the manifolds $S^{[r]}$ it remains to investigate the unicity of the symplectic structure (see theorem) and whether $S^{[r]}$ admits a Kähler metric. For the first point, we will use some special property of the space $S^{[r]}$.
We recall that an integer pure Hodge structure on cohomology is the datum of a direct sum decomposition of complex vector spaces

$$H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X)$$

such that $H^{q,p}(X) = \overline{H^{p,q}(X)}$.

Theorem 3.3.11. *Let S be a complex compact surface, and $r \geq 2$. Then*

1. *Both $H^2(S^{(r)}, \mathbb{C})$ and $H^2(S^{[r]}, \mathbb{C})$ admit pure Hodge structures.*

---

[9] This surprisingly is isomorphic to the abelianization of $\pi_1(S)$, by

$$\pi_1(S^{(r)}, (x_0, \dots, x_0)) \to \mathrm{Ab}(\pi_1(S, x_0))$$
$$\alpha_i = (a_1, \dots, a_r) \mapsto a_1 \dots a_r$$

2. *The map $\pi^* : H^2(S^{(r)}, \mathbb{C}) \to H^2(S^r, \mathbb{C})$ is an isomoprhism onto the subspace of $H^2(S^r, \mathbb{C})$ formed by representatives invariant by Sym_r.*

3. *The sequence $0 \to H^2(S^{(r)}, \mathbb{C}) \xrightarrow{\epsilon^*} H^2(S^{[r]}, \mathbb{C}) \to \mathbb{C}[E] \to 0$ is a split exact sequence. and ϵ^* preserves them.*

4. *If $H^1(S, \mathbb{C})$ vanishes, pi* induces an isomoprhism of Hodge structures.*

Proof. The manifold $S^{(r)}$ can be covered by open holomorphic charts (attenzion, è singolare! spazio complesso, non manifold). (U_i, ϕ_i) such that $U_i \backsimeq W_i/H$ where W_i is an open subset of $\mathbb{C}^{2r}$ and $H \leq \mathrm{Gl}(2r, \mathbb{C})$. Complex manifolds satisfying this properties are known as V-manifolds. (....) They admit pure Hodge structures on each cohomology group and satisfy Poincaré duality, in particular

$$\begin{tikzcd}
H^2(S^{(r)}) \arrow[r, "\pi^*"] \arrow[d, "\backsimeq"] & H^2(S^r) \arrow[d, "\backsimeq"] \\
H_{2r-2}(S^{(r)}) & H_{2r-2}(S^r) \arrow[l, "\pi_*"]
\end{tikzcd}$$

is commutative.

To prove the second point, we replace $S^{(r)}$ with $S^{(r)}_*$ and $S^{[r]}$ with $S^{[r]}_*$ as the difference in codimension is higher than the one detected by H^2. We have the following commutative diagram

$$\begin{tikzcd}
H^2(\mathrm{Bl}_\Delta(S^r_*))^{\mathrm{Inv}} & H^2(S^r)^{\mathrm{Inv}} \arrow[l, "\eta^*"'] \\
H^2(S^{[r]}_*) \arrow[u, "\rho^*"] & H^2(S^{(r)}_*) \arrow[l, "\epsilon^*"] \arrow[u, "\pi^*"']
\end{tikzcd}$$

where the resctrictions to the invariant parts of $H^2(\mathrm{Bl}_\Delta(S^r_*))$ and $H^2(S^r)$ is due the fact that

We have seen that π^* is bijective and since $S^{[r]}_*$ is isomoprhic to $\mathrm{Bl}_\Delta(S^r_*))/\mathrm{Sym}_r$, the map ρ^* is bijective as well. The map

$$H_2(\mathrm{Bl}_\Delta(S^r_*)) \to H_2(S^r_*) = H_2(S^r)$$

is surjective and equivariant with respect to the action of Sym_r, thus after dualizing one has that η^* is injective and commutativity ϵ^* is injective, too. In particular

$$H^2(\mathrm{Bl}_\Delta(S^r_*))^{\mathrm{Inv}} = \mathrm{Im}(\eta^*) \oplus W$$

for some complex vector subspace[10] $W \subset H^2(\mathrm{Bl}_\Delta(S^r_*))^{\mathrm{Inv}}$ and

$$\frac{H^2(\mathrm{Bl}_\Delta(S^r_*))^{\mathrm{Inv}}}{\mathrm{Im}(\eta^*)} = \rho^* \frac{H^2(S^{[r]}_*)}{\mathrm{Im}(\epsilon^*)}$$

and the latter quotient is exactly the vector space generated by fundamental class $[E]$ of the exceptional divisor of $S^{[r]} \to S^{(r)}$.

For the last point, if $H^1(S, \mathbb{C})$ vasnishes, then

$$\phi : H_2(S^r, \mathbb{C}) \xrightarrow{\backsimeq} H_2(S)^{\oplus r}$$

by Kunneth formula and ϕ respects the Hodge structures.(...). We find the result by dualizing and from the obvious identity $H_2(S)^{\oplus r}/\mathrm{Sym}_r = H_2(S)$. $\square$

In particular if S is a $K3$ surface, $S^{[r]}$ is simply connected, there is an injective homomorphism $i : H^2(S, \mathbb{C}) \to H^2(S^{[r]}, \mathbb{C})$ which preserves the Hodge structure and

$$H^2(S^{[r]}, \mathbb{C}) \backsimeq i(H^2(S, \mathbb{C})) \oplus \mathbb{C}[E]$$

This implies that $\mathbb{C}[E] \subset H^{1,1}(S^{[r]})$ and that $H^{2,0}$ and $H^{0,2}$ coincide for $S^{[r]}$ and S.
where $i(H^2(S, \mathbb{C})) = \eta^*(H^2(S^r)^{\mathrm{Inv}})$. If $\alpha \in H^2(S, \mathbb{C})$ we have by commutativity

$$i\alpha = \eta^*\beta = \eta^*\pi^*\gamma = \rho^*\epsilon^*\gamma$$

Thus we have obtained that if S is a $K3$ surface, $h^{2,0}(S^{[r]}) = h^{2,0}(S) = 1$ and in particular the symplectic structure built in theorem (...) is unique up to a scalar.
At the time of the publication of Beauville's article it was not known if S Kähler would imply $S^{[r]}$, unless S is projective and consequently a quite sophisticated restriction was necessary (see [Bea83b, p. 768]).
The resolution of the problem was mainly due to Varouchas, who generalized in [Var84] and [Var89] the notion of Kähler metric to complex analytic spaces[11] and studied how certain morphisms preserves it. We first recall some definitions.

Definition 3.3.12. A continuous function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is said subharmonic (strictly subharmonic) if for any open ball $B_r(x)$ contained in U, whenever $\phi : \overline{B_r(x)} \to \mathbb{R}$ is harmonic and $f(y) \leq \phi(y)$ on the boundary, then $f \leq \phi$ $(f(y) < \phi(y))$ on the whole $\overline{B_r(x)}$.
A continuous function on a complex space $f : X \to \mathbb{R}$ is said:

- pluriharmonic if $f \circ \phi$ is harmonic

[10]Indeed all these vector spaces are finite dimensional

[11]A complex analytic space, which is the singular generalization of a complex manifold, is a locally ringed space $(X, \mathcal{O}_X)$ over $\mathbb{C}$ which is locally isomorphic as a ringed space to $(U, \mathcal{O}_U/I_Z)$ where $U \subset \mathbb{C}^n$ and I_Z is an ideal sheaf generated by $(f_1, \ldots, f_k)$ global holomorphic functions on $\mathbb{C}^n$. It is in particular inherited with all the $H^{p,q}$ groups.

- plurisubharmonic if $f \circ \phi$ is subharmonic
- strictly plurisubharmonic if $f \circ \phi$ is strictly subharmonic

for any holomorphic map $\phi : \overline{B_1(0)} \to X$.

It can be easily proved that a C^2-function is strictly plurisubharmonic if and only if $i\partial\bar{\partial} f$ is a positive definite $(1,1)$-form. Conversely, the new characterization of Kähler metrics is the following

Definition 3.3.13 ([Var84]). A Kähler cocycle (of class $0 \leq p \leq \infty$) is the datum of an open cover $\{U_i\}$ and some strictly plurisubharmonic functions $f_i : U_i \to \mathbb{R}$ of class C^p such that each $f_i - f_j$ is pluriharmonic on U_{ij}.

It can be shown (see [Var84] for details) that a continuous Kähler cocycle gives rise to a smooth Kähler cocycle, through a regularization argument, and that on a complex manifold there is a one to one correspondence between Kähler metrics and equivalence classes of smooth Kähler cocycles, where

$$\{(U_i, f_i)_{i\in I}\} \backsim \{(V_j, g_j)_{j\in J}\}$$

if and only if $(f_i - g_j)|_{U_i \cap V_j}$ is pluriharmonic for any $i \in I, j \in J$. In the singular case we can still define the Kähler form ω, it lives in $H^{1,1}$ but in general a singular Kähler form does not determine uniquely the metric.
These two theorems then follow, both from [Var89].

Theorem 3.3.14 (Varouchas). *If X is a Kähler space, then the symmetric product $X^{(n)}$ is Kähler.*

Theorem 3.3.15 (Varouchas). *If $f : X \to Y$ is a projective morphism and Y is a Kähler space, then so is X.*

The classical notion of a projective complex analytic morphism is a proper[12] holomoprhic map $f : X \to Y$ which has projective fibers and the embedding of each fiber has to be uniform, i.e. there is a line bundle L on X whose restriction to each fiber is ample.
Hence if one proves that the Douady-Barlet morphism $\epsilon : X^{[n]} \to X^{(n)}$ for a compact surface X is projective, the result follows.
In [CM00] one can find a very detailed analysis of the properties of ϵ. Its local projectivity follows from the projectivity of $(\mathbb{C}^2)^{[n]} \to (\mathbb{C}^2)^{(n)}$, which can be proved in coordinates or by observing that this diagram commutes

$$\begin{tikzcd}
(\mathbb{C}^2)^{[n]} \arrow[r, hook] \arrow[d] & (\mathbb{P}^2)^{[n]} \arrow[d] \\
(\mathbb{C}^2)^{(n)} \arrow[r, hook] & (\mathbb{P}^2)^{(n)}
\end{tikzcd}$$

[12]In the topological sense

Thus one has a finite cover U_i on $X^{[n]}$ with some line bundles L_i on U_i such that their restrictions to the fibers are ample. Since the exceptional divisor E is irreducible and the fibers outside the exceptional divisor are finite (thus any line bundle restricted here is ample), it is possible to modify and glue the line bundles L_i in a suitable way, to obtain a global line bundle, whose restrictions to the fibers are ample. Then we have

Corollary 3.3.16. *If X is a complex compact surface, then $X^{[n]}$ admits a Kähler metric.*

For a different approach we refer to [AS06], where an explicit Kähler metric is built on the Douady space, using some smoothing of currents.

3.3.2 Generalized Kummer manifolds

As we have seen in the previous paragraph, if S is a compact Kähler surface with a symplectic structure, then $S^{[r]}$ always admits a Kähler metric and a symplectic structure, whereas the unicity of this structure and the simply connectedness depend on the topology of S. The new class of manifold is built starting with $A^{[r]}$, with A a complex torus.
Since A is abelian, A^r is naturally isomorphic as an abelian group to $A^{\oplus r}$ and the geometric action of Sym_r is compatible with the group structure, i.e. $A^{(r)}$ is a quotient group. Then define the map

$$\begin{aligned} s : A^{(r+1)} &\to A \\ [p_0] + \cdots + [p_r] &\mapsto p_0 + \cdots + p_r \end{aligned}$$

and by pre-composition

$$S = s \circ \epsilon : A^{[r+1]} \to A$$

Notice that $A^{[r+1]}$ fails in general to be abelian. Define two actions of A: on A itself by $L_a(x) = x + (r+1)a$ and on $A^{[r+1]}$ by translation. The map S is isotrivial, i.e. the following diagram commutes

$$\begin{tikzcd} A \times K_r \arrow[r] \arrow[d, "\mathrm{pr}_1"'] & A^{[r+1]} \arrow[d, "S"] \\ A \arrow[r, "(r+1)"'] & A \end{tikzcd}$$

We will now show that $K_r = S^{-1}(0)$, called r-th generalized Kummer manifold, is irreducible symplectic. It is a hypersurface of $A^{[r+1]}$ and for $r = 1$ one finds exactly the construction of example (...), hence the name.
The canonical bundle of K_r is trivial, by the adjunction formula, thus the restriction of the symplectic form ψ of $A^{[r+1]}$ to ϕ is a symplectic form for K_r if and only if it is non degenerate.
Take $x \in K_r$ in the fiber of ϵ of a $r+1$-tuple of all distinct elements, then $T_x A^{[r+1]} \backsimeq (T_0 A)^{\oplus r+1}$ and the pushforward $S_{*,x} : T_x A^{[r+1]} \to T_{S(x)} A$ can be

identified with $(T_0A)^{\oplus r+1} \to T_0A$, mapping $(v_0, \dots, v_r)$ to $v_0 + \dots + v_r$.
Looking at the proof of (...), one sees that

$$\psi_x(v_i, v_j) = 0 \qquad \text{if } i \neq j$$

and $\psi_x|_{T_0A}$ is non degenerate. Then ψ_x restricted to $T_xK_r = \ker(S_{*,x})$ is non degenerate, indeed any non zero $u \in T_xK_r$ is the sum of at least two u_1, u_2 non zero vectors in different T_0A, hence

$$\psi_x(u_1 + u_2 + \dots, v) \neq 0$$

choosing for example v such that $\psi_x(u_1, v) \neq 0$.
It is also easy to see that K_r is simply connected, indeed from the fibration $S : A^{[r+1]} \to A$ one has the long exact sequence in homotopy

$$\dots \to \pi_2(A) \to \pi_1(K_r) \to \pi_1(A^{[r+1]}) \to \pi_1(A) \to \dots$$

and $\pi_1(A)$ is abelian, thus the map $\pi_1(A^{[r+1]}) \to \pi_1(A)$ is bijective, and $\pi_2(A) = 0$. In conclusion $\pi_1(K_r) = 0$.
We now describe in more details the topology of K_r.

Lemma 3.3.17. *For a generalized Kummer manifold K_r there is an injective homomoprhism $j : H^2(A, \mathbb{C}) \to H^2(K_r, C)$ compatible with Hodge structures and if $r \geq 2$*

$$0 \to H^2(A, \mathbb{C}) \to H^2(K_r, \mathbb{C}) \to \mathbb{C}[F] \to 0$$

is split exact, where F is the trace[13] *on K_r of the exceptional divisor of $A^{[r+1]} \to A^{(r+1)}$.*

Proof. Assume that the following sequence is exact

$$0 \to H^2(A, \mathbb{C}) \xrightarrow{S^*} H^2(A^{[r+1]}, \mathbb{C}) \xrightarrow{i^*} H^2(K_r, C) \to 0$$

where $i : K_r \hookrightarrow A^{[r+1]}$ is the canonical injection.
By theorem (...) we have an Hodge structure preserving isomorphism

$$\rho : H^2(A^{[r+1]}, \mathbb{C}) \to H^2(A^{r+1}, \mathbb{C})^{\mathrm{Inv}} \oplus \mathbb{C}[E]$$

The group $H^2(A^{r+1}, \mathbb{C})^{\mathrm{Inv}}$ can be easily computed passing to the dual (we omit $\mathbb{C}$ for brevity)

$$H_2(A^{r+1})/\mathrm{Sym}_{r+1} \backsimeq \left[\bigoplus_{i_0+\dots+i_r=2} H_{i_0}(A) \otimes \dots \otimes H_{i_r}(A) \right] / \mathrm{Sym}_{r+1}$$

[13]Given a closed embedding $i : Y \to X$ and V a linear system on X, which corresponds to a vector subspace $V \subseteq H^0(X, L)$ for some line bundle L, the trace of V on Y, denoted as $V|_Y$, is the linear system on Y corresponding to $W \subseteq H^0(Y, i^*L)$ such that W is the image of V under the natural map $H^0(X, L) \to H^0(Y, i^*L)$. The trace of a divisor D is the trace of the complete linear sistem $L(D)$. The trace of a complete linear system, in general, fails to be complete. If i^*L has no non-trivial global sections then $V|_Y = 0$ (which corresponds to negligible interesections (or empty?) between $Z(V)$ and Y.

and Sym_{r+1} acts in the natural way on the right hand side.
Thus $H_2(A^{r+1})/\mathrm{Sym}_{r+1}$ is generated by

$$\left\{ \sum_i l^i_* \omega : \omega \in H_2(A) \right\}$$

where l_i is the canonical i-th injection $A \hookrightarrow A^{r+1}$ and by the cycle represented by

$$\left\{ \sum_{i,j} l^i_* \alpha \cap l^j_* \beta : \alpha, \beta \in H_1(A) \right\}$$

By Poincaré duality one finds that $H^2(A^{r+1}, \mathbb{C})^{\mathrm{Inv}}$ is generated by

$$\left\{ \sum_i \mathrm{pr}_i^*[\omega] : [\omega] \in H^2(A) \right\} \qquad \left\{ \sum_{i,j} \mathrm{pr}_i^* u \wedge \mathrm{pr}_j^* v : u, v \in H^1(A) \right\}$$

where we used the fact that cup product (which corresponds to wedge product for De Rham cohomology) is dual to geometric intersection.
Since $\sum_{i,j} \mathrm{pr}_i^* u \wedge \mathrm{pr}_j^* v$ is the same as $(\rho \circ S^*)(u \wedge v)$, we have

$$H^2(A^{r+1}, \mathbb{C})^{\mathrm{Inv}} = \mu(H^2(A, \mathbb{C})) \oplus \mathrm{Im}(\rho \circ S^*)$$

where μ is an injective homomoprhism.
This yields the conclusion, as

$$\begin{aligned} H^2(K_r) \backsimeq \frac{H^2(A^{[r+1]})}{\mathrm{Im}(S^*)} &= \frac{\rho^{-1}(H^2(A^{r+1})^{\mathrm{Inv}} \oplus \mathbb{C}[E])}{\mathrm{Im}(S^*)} \\ &= \frac{(\rho^{-1} \circ \mu)(H^2(A, \mathbb{C}) \oplus \rho^{-1}\mathbb{C}[E]) \oplus \mathrm{Im}(S^*)}{\mathrm{Im}(S^*)} \end{aligned}$$

It remains to prove the exactness of the first sequence.
The injectivity of S^* is clear. For the surjectivity of i^*, we will consider a reduced map at the place of i. Let $N = \{(a_0, \ldots, a_r) : \sum a_i = 0\}$ and set

$$N_* = N \cap A_*^{r+1} \qquad K_* = K_r \cap A_*^{[r+1]} \qquad \delta_{ij} = N_* \cap \Delta_{ij}$$

so that $\Delta = \cup_{ij} \delta_{ij}$ is smooth and has codimension 2 in N_* and the following diagram commutes

$$\begin{tikzcd} \mathrm{Bl}_\Delta(N_*) \arrow[r, hook, "l"] \arrow[d] & \mathrm{Bl}_\Delta(A_*^{r+1}) \arrow[d] \\ K_* \arrow[r, hook, "i"'] & A_*^{[r+1]} \end{tikzcd}$$

In particular K_* is identified with $A_*^{[r+1]}/\mathrm{Sym}_{r+1}$. The action of A restricts to an action on $A_*^{[r+1]}$, thus the submanifold $K_r \setminus K_*$ has codimension 2 and the same holds for $N \setminus N_*$.
This allows to replace the 2-cohomology groups of all the spaces with the cohomology of the reduced ones, thus to prove the surjectivity of i^* it is enough to prove the surjectivity of $l^* : H^2(\mathrm{Bl}_\Delta(A_*^{r+1})) \to H^2(\mathrm{Bl}_\Delta(N_*))$.
A simple Mayer-Vietoris argument is not enough to compute $H^2(\mathrm{Bl}_\Delta(N_*))$. If N_* is Kähler, though, it is possible to exploit the integral Hodge structure to get (see [Voi02, p. 180] for details)

$$H^2(\mathrm{Bl}_\Delta(N_*)) \backsimeq H^2(N) \oplus H^0(\Delta)$$

where the isomorphism holds in integral cohomology as well and preserves Hodge structures.
This yields the surjectivity of l^*, as $\delta_{ij} = l^*\Delta_{ij}$ and for the summand $H^2(N)$ one can consider

$$\begin{tikzcd}
H^2(\mathrm{Bl}(A_*^{r+1})) \arrow[r, "l^*"] & H^2(\mathrm{Bl}(N_*)) \arrow[d, two heads, "\mathrm{pr}"] \\
H^2(A^{r+1}) \arrow[u] \arrow[r, dashed] & H^2(N)
\end{tikzcd}$$

where the dashed arrow is surjective because $N \hookrightarrow A^{r+1}$ admits a retraction and the diagram clearly commutes. $\square$

As we have seen for the Doaudy Spaces, the lemma shows the the symplectic structure is unique up to a scalar, indeed $\mathbb{C}[F]$ has to be conataned in $H^{1,1}$ and $h^{2,0}(K_r) = h^{2,0}(A) = 1$.

Remark 3.3.18. If $r \geq 2$, $S^{[r]}$ and K_r have different homotopy type. Indeed by lemmas (..) and (...) one has

$$b_2(S^{[r]}) = b_2(S) + 1 = 23 \qquad b_2(K_r) = b_2(A) + 1 = 7$$

3.3.3 A sketch of the construction of O'Grady's examples

These two classes of examples are built in a more sophisticated way. We mainly follow [OGr99; OGr03; OGr04]. We first show that both $S^{[r]}$ and K_r give actually rise to two uncountable families, both obtained by deformation of one of them.

Theorem 3.3.19 (Beauville, [Bea83b]). *Let $X \to B$ be a smooth family of Kähler manifolds and assume B connected. If X_{b_0} is irreducible symplectic, then all the fibers X_b are irreducible symplectic.*

It is instead not known whether an arbitrary deformation of an irreducible symplectic manifold is Kähler, hence irreducible symplectic by the theorem.

Theorem 3.3.20 (Beauville, [Bea83b]). *Let Def($S^{[r]}$) and Def(K_r)) be the universal deformation spaces of $S^{[r]}$ and K_r. They both have a complex manifold structure. In Def($S^{[r]}$) the elements of the form $S^{[r]}$ (K_r) form a smooth hypersurface, whereas in Def(K_r)) the elements of the form K_r form a countable union of smooth hypersurface.*

Indeed, it is possible to prove that

$$\dim(\mathrm{Def}(S^{[r]})) = h^{1,1}(S^{[r]}) = h^{1,1}(S) + 1 = \dim(\mathrm{Def}(S)) + 1 = 21$$
$$\dim(\mathrm{Def}(K_r)) = h^{1,1}(K_r) = h^{1,1}(A) + 1 = \dim(\mathrm{Def}(A)) + 1 = 5$$

where K_r is the generalized Kummer associated to A.
After the publication of Beauville's two families, Mukai (see [Muk84b; Muk84a; Muk84a]) showed that they could be seen as a manifestation of a more general behaviour. They proved that some moduli spaces of sheaves over a projective $K3$ surface or over an abelian surface A are irreducible symplectic manifolds.
Let S be a projective $K3$ and let $\mathcal{O}_S(D)$ be the ample line bundle defined by $\mathcal{O}_S(1) \otimes \mathcal{O}(D)$ for an ample divisor D. Then we consider the torsion-free sheaves[14] on S which are also $\mathcal{O}_S(D)$-semistable, i.e. they satisfy

$$\frac{1}{\mathrm{rk}(G)}\chi\left(G \otimes \mathcal{O}_S(mD)\right) \leq \frac{1}{\mathrm{rk}(F)}\chi\left(F \otimes \mathcal{O}_S(mD)\right)$$

whenever we have an injection of sheaves $G \subseteq F$ and G is not zero. A sheaf F is said stable if the inequality is strict whenever the injection has non trivial cokernel.
We put an equivalence relation on the family of all the $\mathcal{O}_S(D)$ semi-stable sheaves: it can be proved that any such sheaf F admits a filtration

$$0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_k = F$$

whose graded factors F_i/F_{i-1} are stable and

$$\frac{1}{\mathrm{rk}(F_i/F_{i-1})}\chi\left(F \otimes \mathcal{O}_S(mD)\right) = \frac{1}{\mathrm{rk}(F)}\chi\left(F \otimes \mathcal{O}_S(mD)\right)$$

for any i. The graded factors of the filtration are uniquely determined, up to the order, by F.
Then one sets $F_1 \backsim F_2$ if and only if they have the same graded factors. The relation coincides with isomorphism for stable sheaves, but is coarser for semi-stable ones.
An imporant theorem by Gieseker (see [Gie77]) shows that the set of equivalence classes of $\mathcal{O}_S(D)$ semi-stable sheaves, with fixed rank and Chern classes has a natural projective variety structure.
Consider the subset $M_{S,D}(r, c_1, s)$ of the classes of coherent, semi-stable and

[14] An $\mathcal{O}_X$-module F is torsion-free if none of its tensor power is isomoprphic to $\mathcal{O}_X$

pure[15] sheaves F, with

$$^{16}\mathrm{rk}(F) = r \qquad ^{17}c_1(F) = c_1 \qquad \chi(F) = r + s$$

Mukai proved in [Muk84b] that the open subset of stable sheaves $M^{\mathrm{st}}_{S,H}(r, c_1, s) \subseteq M_{S,D}(r, c_1, s)$ is smooth and if it is non empty, then

$$\dim M^{\mathrm{st}}_{S,H}(r, c_1, s) = 2 - 2rs + c_1^2$$

Now let ω be the symplectic form on S, then at a point $[F]$ of $M^{\mathrm{st}}_{S,H}(r, c_1, s)$, represented by a locally-free sheaf F one sets

$$\tilde{\omega}(\alpha \wedge \beta) = \int_S \omega \wedge \mathrm{Tr}(\alpha \wedge \beta)$$

for any $\alpha, \beta \in H^{0,1}(\mathrm{End}(F))$.[18] This induces a symplectic form on the whole space.
If r, c_1, s are chosen in such a way that $M^{\mathrm{st}}_{S,D}(r, c_1, s) = M_{S,D}(r, c_1, s)$ then the main result is

Theorem 3.3.21. *With the notation used above, $M_{S,D}(r, c_1, s)$ is a projective irreducible symplectic manifold. Moreover it is a deformation of $S^{[n]}$ with $n = 2 - 2rs + c_1^2$.*

If one starts with an abelian surface A in place of a $K3$ surface and proceed analogously (now set $\chi(F) = s$), it is necessary to perform some additional operations on $M_{A,D}(r, c_1, s)$. First consider the map

$$\begin{aligned} \Phi : M_{A,D}(r, c_1, s) &\to A \times \mathrm{Pic}^{c_1}(A) \\ [F] &\mapsto \left(\sum c_2^{\mathrm{rat}}(F), [\wedge^r F]\right) \end{aligned}$$

where $\mathrm{Pic}^{c_1}(A)$ is the component of $\mathrm{Pic}(A)$ of line bundles L with $c_1(L) = c_1$, $c_2^{\mathrm{rat}}(F)$ is the second Chern class in the Chow group $\mathrm{CH}^2(A)$ (see [EH16] for details) and the morphism $\mathrm{CH}^2(A)$ is induced by the group structure of A.
If $\dim M_{A,D}(r, c_1, s) \geq 4$, it can be proved that Φ is a submersion and the fibers are pairwise isomorphic. Thus we can consider

$$M_{A,D}(r, c_1, s)^0 = \Phi^{-1}(p, [\xi])$$

for some fixed $(p, [\xi]) \in A \times \mathrm{Pic}^{c_1}(A)$.
Then we have that $M_{A,D}(r, c_1, s)^0 \subset M_{A,D}(r, c_1, s)$ (still under the hypothesis

[15]The support of an abelian sheaf F on X is defined as the points of X where $F_x \neq 0$. If F is coherent, one say that F has pure dimension d if $\dim \mathrm{supp}(G) = d$ for any coherent subsheaf $G \subseteq F$.
[16]The rank of an $\mathcal{O}_X$-module in a point $x \in X$ is the dimension of the fiber.
[17]For a definition of Chern classes of coherent sheaves see [Gre80]
[18]It can be proved that the tangent space of $M_{S,H}(r, c_1, s)$ at $[F]$ is isomorphic to $H^{0,1}(\mathrm{End}(F))$ if $[F]$ is locally free and to $\mathrm{Ext}^1(F, F)$ otherwise.

$M^{\text{st}}_{S,D}(r, c_1, s) = M_{S,D}(r, c_1, s))$ is irreducible symplectic and deformation equivalent to K_n, where $n = 2 - 2rs + c_1^2$.
To get new deformation classes, one may hope to exploit the above construction: $M^{\text{st}}_{S,D}(r, c_1, s)$ has a symplectic form and is smooth. The theorem suggests to look at moduli spaces for which $M^{\text{st}}_{S,D}(r, c_1, s) \neq M_{S,D}(r, c_1, s)$. This is the case, for example, of $M_{S,D}(2, 0, -2)$.
If S is a $K3$ and D is chosen generically, then the strictly semistable sheaves in $M_{S,D}(2, 0, -2)$ are always represented by $I_Z \oplus I_W$ where $[Z], [W] \in S^{[2]}$. The singular locus of $M_{S,D}(2, 0, -2)$ is swept exactly by these sheaves and the symplectic form on $M^{\text{st}}_{S,D}(r, c_1, s)$ extends to the whole $M_{S,D}(r, c_1, s)$.
If S is a torus, the same holds but replacing I_Z and I_W with $(I_p \otimes L) \oplus (I_{p'} \otimes L')$ where $p, p' \in S$ and L, L' are in the Picard variety.
O'Grady in [OGr99; OGr03] built a resolution of singularities

$$\pi : \widetilde{M}_{S,D}(2, 0, -2) \to M_{S,D}(2, 0, -2)$$

for S either a $K3$ or a torus.
If S is a $K3$ π pullbacks the symplectic form to a symplectic form and one has that $\widetilde{M}_{S,D}(2, 0, -2)$ is projective, non-singular, irreducible symplectic of dimension 10. He moreover proved that [19]

$$b_2(\widetilde{M}_{S,D}(2, 0, -2)) \geq 24$$

which in particular implies that $\widetilde{M}_{S,D}(2, 0, -2)$ belongs to a new deformation class, as b_2 is clearly invariant by deformation.
If S is a torus, one has that

$$\widetilde{M}_{S,D}(2, 0, -2)^0 = \pi^{-1}(M_{S,D}(2, 0, -2)^0)$$

which is still projective, non-singular, irreducible symplectic and has dimension 6. In O'Grady's papers it was also proved that in this case

$$b_2(\widetilde{M}_{S,D}(2, 0, -2)^0) = 8$$

thus again it belongs to a new deformation class.
After the publication of the two O'Grady examples, it appeared natural to try to work in a similar way on other moduli spaces. It seemed promising, until

Theorem 3.3.22 (Kaledin, Lehn, Sorger, 2006 [KLS06])**.** *The only moduli spaces of semistable sheaves on a projective $K3$ or on an abelian surface admitting a symplectic resolution are the O'grady's ones.*

Since then, no other deformation classes have been found.

[19] It was later proved Rapagnetta in [Rap07] that equality holds.

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