UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Fisica e Astronomia "Galileo Galilei"

Master Degree in Physics

Final Dissertation

Transport properties of the Sachdev-Ye-Kitaev model

Thesis supervisor

Prof. Luca Dell'Anna

Candidate

Daniele Morotti

Academic Year 2024/2025

Thesis submission: 24 March 2025, Padua

Pan erwacht. Der Sommer marschiert ein
Was mir die Blumen auf der Wiese erzählen
Was mir die Tiere im Walde erzählen
Was mir der Mensch erzählt
Was mir die Engel erzählen
Was mir die Liebe erzählt

Pan si risveglia. Irrompe l'estate
Quel che mi raccontano i fiori del prato
Quel che mi raccontano gli animali della foresta
Quel che mi racconta l'uomo
Quel che mi raccontano gli angeli
Quel che mi racconta l'amore

Oh Mensch! Gib Acht!
Was spricht die tiefe Mitternacht?
Ich schlief, ich schlief,
Aus tiefem Traum bin ich erwacht:
Die Welt ist tief,
Und tiefer als der Tag gedacht.

Oh uomo! Sii attento!
Che dice la profonda mezzanotte?
Io dormivo, io dormivo,
Da un profondo sogno mi sono destato:
Il mondo è profondo,
E più profondo di quanto il giorno pensasse.

Gustav Mahler, *Symphony No.3 in D minor.* Titles of the six movements.

Friedrich Nietzsche, from *Thus Spoke Zarathustra*, used by Gustav Mahler as the text of the 4th Movement in Symphony No.3.

Personal translation.

Abstract

The Sachdev–Ye–Kitaev (SYK) model has attracted a lot of interest in recent years. It is characterized by an incredible combination of properties, in fact it is an exactly solvable quantum many-body system, based on strongly coupled fermions, that shows chaotic dynamics and an emergent conformal symmetry. In addition, it displays very different features with respect to common Fermi liquids, for instance, the resistivity is linear with respect to the temperature and the entropy is an extensive property in the $T \to 0$ limit. Recent theoretical studies have been devoted to study many peculiar properties of this model, either at equilibrium and out of equilibrium, investigating also the mesoscopic physics. The aim of the thesis will be to carry out a systematic study of the electric and thermoelectric transport properties of the spinful complex version of the SYK model, when contacted with metallic reservoirs.

Contents

Introduction **1**

1 Keldysh formalism **5**
1.1 Keldysh formalism for a MQS 8
1.1.1 Partition function 9
1.1.2 Continuum field theory 11
1.1.3 Correlators 12
1.1.4 Larkin-Ovchinnikov transformation 15
1.1.5 Distribution function and Keldysh Action 17
1.2 Free fermions 19
1.3 External sources 20
1.4 Interactions 22
1.5 Dressed Green's function and Dyson equation 23

2 Transport properties of quantum dots **28**
2.1 Global non-equilibrium steady state 29
2.2 Quantum transport in nanoscale systems 33
2.2.1 Keldysh Field Theory formalism 37
2.2.2 Single Level quantum dot 47
2.2.3 Two Levels quantum dot 51

3 Sachdev–Ye–Kitaev models **57**
3.1 SYK - Majorana fermion version 59
3.1.1 Free theory 61
3.1.2 Quenched disorder and Replica Trick 64
3.1.3 Effective action 66
3.1.4 Dyson–Schwinger equation 70
3.1.5 IR and strong coupling limits 72
3.2 Spinless complex SYK 74
3.2.1 Low energy solutions 77
3.2.2 Particle-Hole Symmetry 82

3.3 Spinful complex SYK 85
3.3.1 Keldysh formalism for the spinful complex SYK 92

4 Transport properties of the spinful complex SYK model 99
4.1 Stationary current in NESS 103
4.2 Weak-Tunneling regime and the Two Limits 110
4.2.1 Limit $eV \gg k_B T$ 110
4.2.1.1 Analytical derivation 114
4.2.2 Limit $eV \ll k_B T$ 116
4.2.2.1 Analytical derivation 118
4.2.3 Dependence of current I on parameter N 120
4.3 Linear response regime 121

Conclusions 127

Bibliography 133

Introduction

Over the last decade, a great deal of scientific research has focused on the study of a model with atypical properties, namely the SYK model. This model was devised by Alexei Kitaev [1], generalising a model previously conceived by Subir Sachdev and Jinwu Ye [2] in 1993. The strong interest shown towards this model stems from a rare combination of properties. In fact, it is an exactly solvable quantum many-body system (in the large N and infrared limit), based on N strongly coupled Majorana fermions (where the couplings are Gaussian random), that shows chaotic dynamics and an emergent approximate conformal symmetry in the IR. This model was born initially as a toy model for *holography*, in the context of quantum gravity, in fact it is connected holographically to black holes with AdS_2 horizons and it is linked to 2-dimensional quantum gravity [3, 4, 5, 6]. The later study of this model and its generalizations (to complex fermions) showed that in reality it also has properties of relevant interest associated with further fields of research. The main ones are information scrambling [7, 8], chaos [9], quantum entanglement [10, 11, 12] and, last but not least, strange metals [13, 14, 15]. In this thesis, the last of these will be the most involved field of interest.

In fact, the aim of this work is to study the transport properties of the SYK model in its spinful complex fermion version. Specifically, the system studied corresponds to the case in which this SYK model is implemented as a quantum dot interacting with two metallic reservoirs. Differences between the values of the chemical potentials and/or of the temperatures associated with the two reservoirs will generate phenomena belonging to the field of non-equilibrium thermodynamics. Moreover, since the zero dimensional SYK model is considered in the large N limit, we are therefore in the world of mesoscopic physics. In particular, this thesis will focus on the study of global non-equilibrium steady state configurations. Thus, the studied current passing through the quantum dot will be stationary and so independent of time. One of the final results of this thesis will be a numerical and analytical study of the electric current passing through the SYK model in the weak-tunneling regime (quantum dot - electrode couplings are the smallest energy scales). The other results have been found by exploiting the peculiar properties of the linear response regime, in particular we studied the behaviours of the conductance and of the thermoelectric

coefficient associated with this version of the SYK model, finally obtaining information on its thermopower. It is important to emphasize that the results obtained in this thesis are found to be perfectly consistent with other sources of scientific literature on the subject [16, 17, 18].

Since transport phenomena are closely related to the field of non-equilibrium thermodynamics, these results were obtained by using a theoretical framework capable of describing the temporal evolution of systems out of equilibrium and calculating the expectation values of the related physical observables. The theoretical formalism which has been chosen to carry out all the treatment in this thesis work corresponds to the Keldysh's theory, specifically using the non-equilibrium Green's functions technique in its variant based on functional integral QFT formalism (Keldysh field theory [19, 20]). The thesis is structured in such a way that each chapter (preceding the final one) provides essential tools for obtaining the final results, calculated and presented in Chapter 4. The order of the chapters follows a process of gradual focusing which starts from the description of the Keldysh theory (valid for every type of physical system out of equilibrium), goes through the study of systems composed by a generic quantum dot interacting with metal reservoirs, focuses on the specific quantum dot cases described by the SYK models, up to the study of the transport properties of a specific version of the SYK model. Specifically, the thesis is organized as follows:

- In Chapter 1 we will present the Keldysh field theory formalism, focusing on the main concepts and especially on the tools that will be used in the following chapters, such as the partition function and the closed time contour C. We will study the properties of the three types of Green's functions that arise from this theory and how to calculate the expectation values of physical observables through the introduction of asymmetrical external sources. Finally, we will review the *Dyson* equation, which allows for the treatment of interaction terms and also time dependent perturbations.

- In Chapter 2 we will apply the Keldysh formalism in the study of a generic LCR system (one central generic quantum dot connected to two reservoirs) characterized by a NESS (non-equilibrium steady state) configuration. In fact, after initially delving into the topic of NESS, we will study the Keldysh partition function of the entire system and we will derive the general expression for the stationary total current, corresponding to the sum of electrical and thermoelectric contributions. In particular, we will be able to write it as a function of the exact Green's functions of the quantum dot (as well as of the couplings to the electrodes and of the leads' distribution function). Then, in order to practice using this formalism, we will analyse the two simplest quantum dots,

namely the single level and the two levels non-interacting quantum dots.

- In Chapter 3, we temporarily pause the non-equilibrium topic and we will review three different variants of the SYK model, all studied at thermodynamic equilibrium, namely the Majorana fermion version, the spinless and the spinful complex fermion versions. The latter model will be exactly what will be considered as a quantum dot in the discussion of the following chapter. The objective of Chapter 3 is to study all the main properties of SYK models, with the final aim of obtaining the analytical solutions of their three Keldysh Green's functions (at equilibrium) in the case of large N (number of SYK's sites), IR and strong coupling limits.

- In Chapter 4 we will focus on the final goal of this thesis, that is the study of the transport properties of a quantum dot described by the spinful complex fermion version of the SYK model. The first part of the discussion will be carried out considering the case where the two reservoirs are at equal temperature, but with different chemical potentials. We will find the expression of the stationary current in the NESS configuration as a function of the real and imaginary components of the retarded Green's function associated to the isolated SYK model at thermodynamic equilibrium (as well as, of course, the others system parameters) and it will be analysed both numerically and analytically in the weak-tunneling regime. Finally, in the context of the linear response regime, we will concentrate on the study of the conductance and the thermoelectric coefficient associated to the SYK model examined. From the study of these properties we will also obtain information on the *Seebeck* coefficient in a situation of particle-hole symmetry.

Chapter 1

Keldysh formalism

Keldysh's theory [19, 20, 21, 22, 23] was born from the desire to study and make predictions about the expectation values of physical observables, associated with systems that are not in thermodynamic equilibrium. Systems at equilibrium are perfectly described by the *Matsubara formalism*, but the aim of the Keldysh theory is to create a formalism suitable to represent both systems in thermodynamic equilibrium and those in non-equilibrium. The strengths of this theory are its extreme adaptability, which allows it to be applied to a wide range of physical systems, and enables it to work with a light and simple (in shape) integral functional formalism. The development of Keldysh theory stems from the search for a theory with the following characteristics:

- The expectation values of observables are obtained by trace operation of the corresponding operator and the dynamical time evolution of the density operator (Schrödinger picture).
- The dynamical temporal evolution of the density operator is regulated by the time evolution operator $\hat{U}(t)$, according to the relation $\hat{\rho} \to U(t)\hat{\rho}\hat{U}(t)^{\dagger}$.
- The initial state of the physical system, in a distant past $t = -\infty$, is known and it is assumed that at this time the system consisted of non-interacting particles. In addition, it is assumed that the interactions are activated adiabatically and that this activation process is completed before the observation operations of the system. An important detail is that the dynamics of the system can be influenced by the presence in the Hamiltonian of a term describing a time-dependent perturbation; this will be responsible for an evolution of the system out of equilibrium.
- The presence of generating functions that allow the immediate expression of expectation values of observables (at any time) in functions of physical parameters of the system under consideration.

To make it more concrete and visible, consider for example a quantum many-body system whose dynamic is regulated by the Hamiltonian $\hat{H}(t)$. Assume that at time $t = -\infty$ the system is described by an Hamiltonian of non-interacting particles and that the state $\hat{\rho}(-\infty)$ is represented by an equilibrium density matrix of $\hat{H}(-\infty)$. In the following times we have the adiabatic switching on of the interactions. The state of the system will then evolve until it corresponds to an equilibrium density matrix associated with the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_{int}$ (where $\hat{H}_0 \equiv \hat{H}(-\infty)$), once the operation of activation of the interactions is completed. At such a point in the Hamiltonian there may be a time-dependent perturbation that causes the evolution of the density matrix out of equilibrium. At each time, the temporal evolution of the density matrix is regulated by the Liouville-Von Neumann equation:

$$i\hbar\frac{\partial\hat{\rho}(t)}{\partial t} = [\hat{H}(t), \hat{\rho}(t)] \tag{1.1}$$

whose solution has the form $\hat{\rho}(t) = \hat{U}_{t,-\infty}\hat{\rho}(-\infty)[\hat{U}_{t,-\infty}]^\dagger$, where $\hat{U}_{t,t'}$ is the unitary evolution operator. The latter must respect the following equations (where it is imposed that $\hbar = 1$):

$$\frac{\partial\hat{U}_{t,t'}}{\partial t} = -i\hat{H}(t)\hat{U}_{t,t'} \qquad\qquad \frac{\partial\hat{U}_{t,t'}}{\partial t'} = i\hat{U}_{t,t'}\hat{H}(t') \tag{1.2}$$

and in order to satisfy those relations, the time evolution operator shall be:

$$\hat{U}_{t,t'} = \mathbb{T}exp\left(-i\int_{t'}^{t}\hat{H}(t)dt\right) \tag{1.3}$$

where $\mathbb{T}$ represents the time ordering operation, which is essential since $\hat{H}(t)$ being time dependent, if taken at different times, in general does not commute. All the elements necessary to express the expectation value of a generic observable $\hat{A}$, at time t, are now available:

$$\langle\hat{A}\rangle(t) \equiv \frac{Tr[\hat{A}\hat{\rho}(t)]}{Tr[\hat{\rho}(t)]} = \frac{Tr[\hat{U}_{-\infty,t}\hat{A}\hat{U}_{t,-\infty}\hat{\rho}(-\infty)]}{Tr[\hat{\rho}(t)]} \tag{1.4}$$

Note the peculiarity of such an expression, there is a temporal evolution of the state from the initial instant until the time t in which the measurement of the observable is performed. After this, there is a time evolution backwards until the time does not coincide with the initial instant. The nature of this double path, one forward and the other backward, is not observable when the system is in thermodynamic equilibrium. In fact, in the latter case it can be shown [20] that the expectation value of an observable can be rewritten as a function of only $\hat{U}_{t,t'}$ operators that evolve the state forward over time, so $t > t'$ (and therefore there is no term acting

on the return path).

Returning to the general case, if the identity $\mathbb{1} = \hat{U}_{t,+\infty}\hat{U}_{+\infty,t}$ is inserted into the equation (1.4), it is possible to rewrite the content of the trace with an evolution that after the time t does not immediately take the path back, but continues on the way up to $t = +\infty$ and only after reaching it takes the way back until reaching the initial instant:

$$\langle \hat{A} \rangle(t) = \frac{Tr[\hat{A}\hat{\rho}(t)]}{Tr[\hat{\rho}(t)]} = \frac{Tr[\hat{U}_{-\infty,+\infty}\hat{U}_{+\infty,t}\hat{A}\hat{U}_{t,-\infty}\hat{\rho}(-\infty)]}{Tr[\hat{\rho}(-\infty)]} \tag{1.5}$$

In the equation (1.5), the evolution along the path $(-\infty, +\infty) \cup (+\infty, -\infty)$ is clearly visible, the latter is often called as *closed time contour* C or *Keldysh contour* (Fig(1.1)).

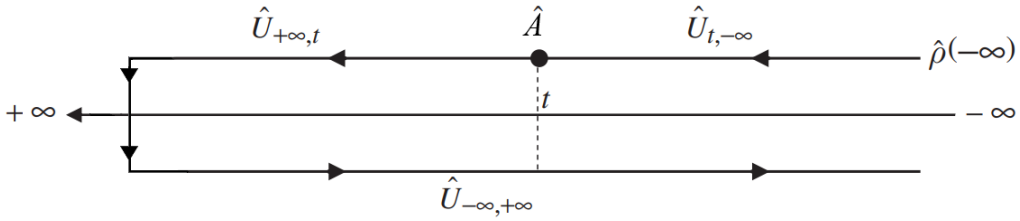


Figure 1.1: Graphical representation of the contour C and the temporal evolution associated with the equation (1.5).

Note that, unlike the case of equilibrium systems where their study is carried out by means of imaginary times, the Keldysh formalism is completely defined on real times. Taking equation (1.5) again and analysing its structure, consider the trivial case in which the operator $\hat{A}$ is the identity operator: in such a situation the initial state of the system is subjected to a temporal evolution along all the contour C, realized through the unitary operator $\hat{U}_C = \hat{U}_{-\infty,+\infty}\hat{U}_{+\infty,-\infty}$. It can be easily noticed that such operator $\hat{U}_C$ is nothing more than a representation of the identity operator, in fact for every infinitesimal operation of temporal evolution on the forward branch, there is an inverse in the backward branch which cancels its effect. All this leads to the understanding that in such a situation, the right hand side of equation (1.5) is always the unit. This is extremely useful for defining an object of absolute importance in Keldysh's theory, namely the partition function. It is defined as:

$$Z \equiv \frac{Tr[\hat{U}_C\hat{\rho}(-\infty)]}{Tr[\hat{\rho}(-\infty)]} \tag{1.6}$$

Note that this partition function, defined in this way, meets all four requirements written at the beginning of the chapter: it contains a dynamical time evolution operation of the initial state of the system, it is defined by means of unitary oper-

ators containing time-ordering operations, it carries information on the initial state of the system, it is by definition normalized to one and this is a property relevant for defining, starting from it, generating functions. It is essential to understand that the normalization of the partition function is a consequence of the equality of the Hamiltonian in the two branches of the contour C; if an asymmetry, between the two Hamiltonians associated with the two paths, is artificially introduced, the normalization property is lost. This phenomenon can be used to define the generating functions. For example, it is possible to insert an asymmetry by modifying the Hamiltonian as follows: $\hat{H}_V^{\pm}(t) \equiv \hat{H}(t) \pm \hat{A}V(t)$, where $\pm$ is associated with the two components of the contour, forward and backward. This asymmetry allows one to define the following non-trivial object, named as *generating function*:

$$Z[V] \equiv \frac{Tr[\hat{U}_C[V]\hat{\rho}(-\infty)]}{Tr[\hat{\rho}(-\infty)]} \tag{1.7}$$

It is now possible to express $\langle \hat{A} \rangle(t)$ by means of a simple and elegant expression, namely

$$\langle \hat{A} \rangle(t) = \left. \frac{i\delta Z[V]}{2\delta V(t)} \right|_{V=0} \tag{1.8}$$

It is important to note that the external time dependent potential V(t), in this context, has no physical role and does not affect the dynamics of the system in any way (at the end of the calculation it is set to zero), is simply a tool we use to describe the expectation value of observables as functional derivatives of generating functions.

Before moving on to the next section it is important to make a clarification. The Keldysh's theory can be formulated using different formalisms, the one presented and used in this thesis corresponds to the non-equilibrium Green's functions technique, specifically in its variant based on functional integral QFT formalism (Keldysh field theory [19, 20]).

1.1 Keldysh formalism for a MQS

To present all the real protagonists of Keldysh's theory and express their main relations, a miniature quantum system (MQS) is studied, specifically a single quantum state with energy ϵ_0 populated by spinless fermions. Considering such a simple system, it is possible to develop the relative functional integrals, without unnecessarily complicating the notation, in such a way that the attention can be focused on the most important features of Keldysh's theory, that are independent of the system under consideration. Consider a system described by the Hamiltonian $\hat{H} = \epsilon_0 \hat{c}^\dagger \hat{c}$, where $\hat{c}^\dagger$ and $\hat{c}$ are the usual creation and annihilation operators of fermions, in this

case on the state ϵ_0. Fermions are obliged to respect the Pauli principle, therefore the whole space, in which the states of the system live, is spanned by the two states $|0\rangle$ and $|1\rangle$. For the purposes of developing Keldysh's theory, it is essential to use the concept of coherent states $|\psi\rangle$. They are defined as the right eigenstates of the $\hat{c}$ operator and each of them is parameterized by a Grassmann number ψ, this number corresponds to its own eigenvalue. For the system under review, they correspond to:

$$|\psi\rangle = |0\rangle - \psi\,|1\rangle = e^{-\psi\hat{c}^\dagger}\,|0\rangle \tag{1.9}$$

It is important to note that such coherent states do not belong to the many-body space associated with the system, in fact these states are linear combinations of physical states, but the coefficients are not all complex numbers, given the presence of Grassmann numbers; they belong to the so-called generalized space. Similarly for $|\psi\rangle$, it is possible to define the left coherent states $\langle\psi|$ as the left eigenstates of the $\hat{c}^\dagger$ operator with eigenvalue $\bar{\psi}$. In this case they correspond to:

$$\langle\psi| = \langle 0| - \langle 1|\,\bar{\psi} = \langle 0|\,e^{-\hat{c}\bar{\psi}} \tag{1.10}$$

It is essential to remember, that unlike the bosonic case (where the eigenvalues of the coherent states are complex numbers), in the fermionic case the ψ and $\bar{\psi}$ eigenvalues (being Grassmann numbers) are completely unrelated. The extreme importance and usefulness of coherent states is given by the fact that through them one can write a closure relation, that is an identity operator acting on the physical space of the system:

$$\hat{\mathbb{1}} = \iint d\bar{\psi}d\psi e^{-\bar{\psi}\psi}\,|\psi\rangle\,\langle\psi| \tag{1.11}$$

Thanks to this closure relation, it is possible to express the trace of a generic $\hat{A}$ operator as a double integral on the space of the Grassmann numbers:

$$Tr[\hat{A}] = \sum_{n=0,1}\langle n|\,\hat{A}\,|n\rangle = \sum_{n=0,1}\iint d\bar{\psi}d\psi e^{-\bar{\psi}\psi}\,\langle n|\psi\rangle\,\langle\psi|\,\hat{A}\,|n\rangle = \iint d\bar{\psi}d\psi e^{-\bar{\psi}\psi}\,\langle\psi|\,\hat{A}\,|-\psi\rangle \tag{1.12}$$

1.1.1 Partition function

This equation is extremely useful for the calculation of the partition function, defined in (1.6), relative to the considered system. First, to simplify the treatment as much as possible, consider that at the initial instant (in a remote past) the state of the system is described by an equilibrium density matrix, so $\hat{\rho}(-\infty) = \hat{\rho}_0 = exp(-\beta(\hat{H} - \mu\hat{N})) = exp(-\beta(\epsilon_0 - \mu)\hat{c}^\dagger\hat{c}))$; in this case $Tr(\hat{\rho}_0) = 1 + \rho(\epsilon_0)$, where $\rho(\epsilon_0) = exp(-\beta(\epsilon_0 - \mu))$. The trick used to calculate the partition function consists

in dividing the contour C into $(2N-2)$ time intervals (each of width δt and with the impositions $t_1 = t_{2N} = -\infty$ and $t_N = t_{N+1} = +\infty$, Fig(1.2)) and consequently writing the operator $\hat{U}_C$ as a composition of time evolution operators (one for each interval), then insert $2N$ fermionic coherent state resolutions of unity (1.11) in the $2N$ points of the contour present after its subdivision.

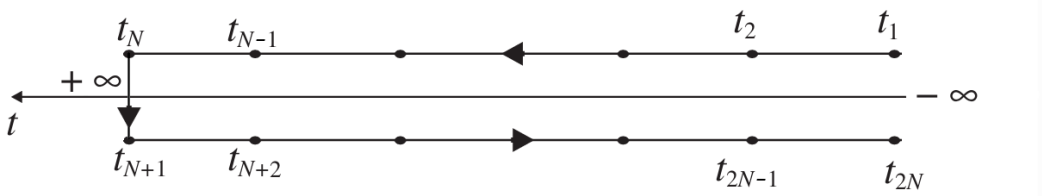


Figure 1.2: Subdivision of the contour C into $(2N-2)$ time intervals (each of width δt), with the impositions $t_1 = t_{2N} = -\infty$ and $t_N = t_{N+1} = +\infty$.

Leaving aside for the moment the exponential terms resulting from the introduction of identity operators, at this point the numerator $Tr[\hat{U}_C\hat{\rho}_0]$, after the trace operation, is equal to:

$$\langle\psi_{2N}|\,\hat{U}_{-\delta t}\,|\psi_{2N-1}\rangle \ldots \langle\psi_{N+2}|\,\hat{U}_{-\delta t}\,|\psi_{N+1}\rangle\,\langle\psi_{N+1}|\,\mathbb{1}\,|\psi_N\rangle\,\langle\psi_N|\,\hat{U}_{+\delta t}\,|\psi_{N-1}\rangle \ldots \langle\psi_1|\,\hat{\rho}_0\,|-\psi_{2N}\rangle \tag{1.13}$$

Taking a generic matrix element of the above expression (except that with $\hat{\rho}_0$ or $\mathbb{1}$) and considering that it has small time intervals ($\delta t \sim 1/N$), it corresponds to:

$$\begin{aligned}\langle\psi_i|\,\hat{U}_{\pm\delta t}\,|\psi_{i-1}\rangle &= \langle\psi_i|\,e^{\mp i\hat{H}(\hat{c}^\dagger,\hat{c})\delta t}\,|\psi_{i-1}\rangle \approx \langle\psi_i|\,(1\mp i\hat{H}(\hat{c}^\dagger,\hat{c})\delta t)\,|\psi_{i-1}\rangle \\ &= \langle\psi_i|\psi_{i-1}\rangle\,(1\mp iH(\bar{\psi}_i,\psi_{i-1})\delta t) \approx e^{\bar{\psi}_i\psi_{i-1}}e^{\mp iH(\bar{\psi}_i,\psi_{i-1})\delta t}\end{aligned} \tag{1.14}$$

This conclusion is possible because the Hamiltonian of the system under consideration is normally-ordered, so it is possible to write the above expression as a function of $H(\bar{\psi}_i,\psi_{i-1}) = \epsilon_0\bar{\psi}_i\psi_{i-1}$. The last matrix element to be studied is $\langle\psi_1|\,\hat{\rho}_0\,|-\psi_{2N}\rangle$ which can easily be shown to be equal to $exp(-\bar{\psi}_1\psi_{2N}\rho(\epsilon_0))$. Now that all the terms belonging to the partition function expression have been calculated, it can now be expressed as integral over a set of $2N\times 2$ Grassmann variables:

$$Z = \frac{1}{Tr[\hat{\rho}_0]}\iint\prod_{i=1}^{2N}[d\bar{\psi}_i d\psi_i]exp\left(i\sum_{i,i'=1}^{2N}\bar{\psi}_i G^{-1}_{ii'}\psi_{i'}\right) \tag{1.15}$$

where the term $iG^{-1}_{ii'}$ represents a $2N\times 2N$ matrix which, for the system under

consideration, corresponds to

$$iG^{-1}_{ii'} \equiv \left(\begin{array}{ccccc:ccccc} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\rho(\epsilon_0) \\ r_- & -1 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & \ddots \\ 0 & r_- & -1 & 0 & \ddots & 0 & 0 & 0 & 0 & \ddots \\ 0 & 0 & r_- & -1 & \ddots & 0 & 0 & 0 & 0 & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 & \ddots & \ddots & \ddots & \ddots \\ \hdashline 0 & 0 & 0 & 0 & +1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & r_+ & -1 & 0 & 0 & \ddots \\ 0 & 0 & 0 & 0 & \ddots & 0 & r_+ & -1 & 0 & \ddots \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 & r_+ & -1 & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 & 0 & 0 & \ddots & \ddots \end{array}\right) \tag{1.16}$$

with $r_\mp \equiv 1 \mp i\epsilon_0\delta t$ and where each sub-matrix delimited by the dashed lines is a $N \times N$ matrix.

1.1.2 Continuum field theory

So far, finite time intervals δt have been used and therefore the notation has been discrete; taking the limit $N \to \infty$ it is possible to bring the notation into a continuous formalism that allows for a much lighter and more compact notation. In this case, there is the substitution $\psi_i \to \psi(t)$ and the partition function is:

$$Z = \int D[\bar{\psi}\psi] exp\left(i \int_C dt[\bar{\psi}(t)\hat{G}^{-1}\psi(t)]\right) = \int D[\bar{\psi}\psi] exp(iS[\bar{\psi}, \psi]) \tag{1.17}$$

where the initial denominator $Tr[\hat{\rho}_0]$ has been absorbed by the measure $D[\bar{\psi}\psi]$ and the operator $\hat{G}^{-1} = i\partial_t - \epsilon_0$. It is immediately noticeable how with such continuous notation the information of the initial state of the system has been completely lost (and that of every other time, since the system is in thermodynamic equilibrium with $\hat{H}$ time independent). It is therefore only an abbreviation, to write in a more compact way; to make the calculations it is always necessary to return to the discrete case, conclude them and then go to the continuous limit. In the following sections it will be shown how to switch to a continuous notation that carries with it the information of the system's distribution function (boundary conditions), solving this annoying problem. To avoid working on the C contour, it is often convenient to rewrite it as an integral on the set of real numbers; this is possible by splitting the Grassmann field $\psi(t)$ into two parts: $\psi^+(t)$ and $\psi^-(t)$. The first is related to the times of the forward path, while the second to those of backward path. Thanks

to this substitution, the action becomes:

$$S[\bar{\psi},\psi] = \int_{-\infty}^{+\infty} dt[\bar{\psi}^{+}(t)(i\partial_t - \epsilon_0)\psi^{+}(t) - \bar{\psi}^{-}(t)(i\partial_t - \epsilon_0)\psi^{-}(t)] \tag{1.18}$$

Note that even with this notation, the information of the initial state is not visible, indeed such notation gives the impression that $\psi^{+}(t)$ and $\psi^{-}(t)$ have no relation, while in reality they are related by the element $(1 \times 2N)$ of the matrix described by (1.16).

1.1.3 Correlators

Given the problems of continuous notation, using discrete notation, elementary contractions are now calculated (using the properties of Gaussian integrals):

$$\langle \psi_i \bar{\psi}_{i'} \rangle \equiv \int D[\bar{\psi}\psi]\psi_i \bar{\psi}_{i'} exp\left(i \sum_{t,t'=1}^{2N} \bar{\psi}_t G^{-1}_{tt'} \psi_{t'} \right) = iG_{ii'} \tag{1.19}$$

Now, let's describe the correlator of the two Grassmann fermionic fields, using the notation $\pm$ for the fields (presented in the eq (1.18)); in this way the field's subscript i can only take values between 1 and N (included). This notation will emphasize how, given a pair of times (purely the numerical values, without the information of their position along the contour C), there are four different types of correlator, which arise from the four possible configurations of such times above the contour C. The correlators and their expressions for the system under review are presented below:

$$\begin{aligned}
\langle \psi_i^{+} \bar{\psi}_{i'}^{-} \rangle &\equiv iG^{<}_{ii'} = i\frac{(-\rho(\epsilon_0))r_{+}^{N-i'} r_{-}^{i-1}}{1+\rho(\epsilon_0)(r_{+}r_{-})^{N-1}} \\
\langle \psi_i^{-} \bar{\psi}_{i'}^{+} \rangle &\equiv iG^{>}_{ii'} = i\frac{r_{-}^{N-i'} r_{+}^{i-1}}{1+\rho(\epsilon_0)(r_{-}r_{+})^{N-1}} \\
\langle \psi_i^{+} \bar{\psi}_{i'}^{+} \rangle &\equiv iG^{\mathbb{T}}_{ii'} = ir_{-}^{i-i'}\left(\Theta(i-i') - \frac{\rho(\epsilon_0)(r_{-}r_{+})^{N-1}}{1+\rho(\epsilon_0)(r_{-}r_{+})^{N-1}} \right) \\
\langle \psi_i^{-} \bar{\psi}_{i'}^{-} \rangle &\equiv iG^{\tilde{\mathbb{T}}}_{ii'} = ir_{+}^{i-i'}\left(\Theta(i-i') - \frac{\rho(\epsilon_0)(r_{+}r_{-})^{N-1}}{1+\rho(\epsilon_0)(r_{+}r_{-})^{N-1}} \right)
\end{aligned} \tag{1.20}$$

where $\Theta(n)$ is the Heaviside step function. Let's ignore the indexes in the notation presented above for the moment, it will be clear in a little while. First of all, it is noted that these four objects are strongly linked to each other, in fact the following properties apply:

$$[G^{\mathbb{T}}]^{\dagger} = -G^{\tilde{\mathbb{T}}} \qquad\qquad [G^{<(>)}]^{\dagger} = -G^{<(>)} \tag{1.21}$$

Attention should be paid to the operation of Hermitian conjugation, which is applied to the objects described in eq (1.23) understood as $N \times N$ matrices. Now consider the continuum limit ($N \to \infty$, with the condition that $N\delta t$ is constant), in this way the propagators (or correlation functions) become:

$$\begin{aligned}
&\langle \psi^+(t)\bar{\psi}^-(t')\rangle \equiv iG^<(t,t') = -n_f exp(-i\epsilon_0(t-t')) \\
&\langle \psi^-(t)\bar{\psi}^+(t')\rangle \equiv iG^>(t,t') = (1-n_f) exp(-i\epsilon_0(t-t')) \\
&\langle \psi^+(t)\bar{\psi}^+(t')\rangle \equiv iG^{\mathbb{T}}(t,t') = \Theta(t-t')iG^>(t,t') + \Theta(t'-t)iG^<(t,t') \\
&\langle \psi^-(t)\bar{\psi}^-(t')\rangle \equiv iG^{\widetilde{\mathbb{T}}}(t,t') = \Theta(t'-t)iG^>(t,t') + \Theta(t-t')iG^<(t,t')
\end{aligned} \tag{1.22}$$

where n_f is the Fermi occupation number defined as $n_f \equiv \rho(\epsilon_0)(1+\rho(\epsilon_0))^{-1}$, it corresponds to the expectation value of the operator $\hat{c}^\dagger\hat{c}$ at each instant. This is a consequence of the fact that the system under consideration is in thermodynamic equilibrium and describes free fermions. To understand the notation used to describe the four correlators, these propagators are expressed through operatorial formalism below:

$$\begin{aligned}
iG^<(t,t') &= -\langle \hat{c}^\dagger(t')\hat{c}(t)\rangle \\
iG^>(t,t') &= +\langle \hat{c}(t)\hat{c}^\dagger(t')\rangle \\
iG^{\mathbb{T}}(t,t') &= +\langle \mathbb{T}[\hat{c}(t)\hat{c}^\dagger(t')]\rangle \\
iG^{\widetilde{\mathbb{T}}}(t,t') &= +\langle \widetilde{\mathbb{T}}[\hat{c}(t)\hat{c}^\dagger(t')]\rangle
\end{aligned} \tag{1.23}$$

where $\langle ... \rangle$ corresponds to the average operation, using the Heisenberg picture and imposing as reference time the initial one, therefore $-\infty$ in the case under consideration. For example, the expectation value of a physical observable $\hat{A}$ at time t corresponds to $\langle \hat{A}(t)\rangle = Tr[\hat{\rho}_0\hat{A}(t)]$ where $\hat{A}$ is the observable in the Heisenberg picture. Beware, however, that this $\hat{\rho}_0$ is not exactly the same as defined at the beginning of the chapter, indeed with the operatorial formalism the density matrix has to be normalized, so the initial $\hat{\rho}_0$ must be divided by the trace of itself. Now it is possible to understand the notation used to differentiate the four types of correlators: the $<$ ($>$) expresses the order of t and t' along the contour C, while $\mathbb{T}$ and $\widetilde{\mathbb{T}}$ represent the time ordering and the time anti-ordering operations. To have a more compact and simple notation in literature it can been found also the following formalism: $G^{++} \equiv G^{\mathbb{T}}$, $G^{--} \equiv G^{\widetilde{\mathbb{T}}}$, $G^{+-} \equiv G^<$, $G^{-+} \equiv G^>$. Now it is necessary to remember that these four correlators represent, in the discrete notation, the four possible configurations of the original $iG_{ii'}$ matrix (eq(1.34)). This matrix in the

continuum notation can be represented as follows:

$$\hat{G}(t,t') = \begin{pmatrix} G^{\mathbb{T}}(t,t') & G^{<}(t,t') \\ G^{>}(t,t') & G^{\widetilde{\mathbb{T}}}(t,t') \end{pmatrix} \tag{1.24}$$

This block Green's function matrix corresponds to the object that contains the four possible configurations (fixed times t and t') of a quantity called *contour-ordered Green's function* defined in this way:

$$iG_C(t,t') = +\langle \mathbb{T}_C[\hat{c}(t)\hat{c}^\dagger(t')]\rangle \tag{1.25}$$

When working with this tool, it must be taken into account that times are not simply real numbers, but also carry with them the information of the branch (in the contour C) to which they belong. Indeed $\mathbb{T}_C$, the contour-ordering operator, is nothing more than the operation of ordering the operators in relation to the position on the contour of their time arguments, in the sense that

$$\mathbb{T}_C[\hat{c}(t)\hat{c}^\dagger(t')] \equiv \begin{cases} +\hat{c}(t)\hat{c}^\dagger(t') & t >_c t' \\ -\hat{c}^\dagger(t')\hat{c}(t) & t' >_c t \end{cases} \tag{1.26}$$

where $t >_c t'$ signifies that t is further along the contour C than t'. If both times belong to the forward branch then operator $\mathbb{T}_C$ is equal to the operator $\mathbb{T}$ and so it turns out that $iG_C(t,t') = iG^{\mathbb{T}}(t,t')$, while if they both belong to the backward branch of the contour, it is equal to $iG^{\widetilde{\mathbb{T}}}(t,t')$. In the end, if the two times lie on two different branches of the contour, the two trivial cases are $iG^{<}(t,t')$ and $iG^{>}(t,t')$. The main and most important feature of the contour-ordered Green's function $iG_C(t,t')$, and therefore of the matrix $i\hat{G}(t,t')$, is the presence of the operator $\mathbb{T}_C$. Indeed, the presence of this operation allows a perturbative expansion of $iG_C(t,t')$ using Wick's theorem. According to the Keldysh theory, the study of non-equilibrium systems is carried out by means of a theoretical system structurally equal to that of equilibrium systems, but with two important substitutions:

- the imaginary and finite time interval (for fixed temperatures) is replaced by the infinite real contour C;
- the time-ordering operator $\mathbb{T}$ is substituted with the contour-ordering operator $\mathbb{T}_C$.

1.1.4 Larkin-Ovchinnikov transformation

Returning now to the equations (1.23), these correlators can be rewritten via a functional integral in this way:

$$\langle \psi^{\pm}(t)\bar{\psi}^{\pm}(t')\rangle = \int D[\bar{\psi}\psi]\psi^{\pm}(t)\bar{\psi}^{\pm}(t')exp(iS[\bar{\psi},\psi]) \tag{1.27}$$

where the action $S[\bar{\psi},\psi]$ corresponds to (1.18). As written earlier, here again it seems that the fields associated with two different branches have no correlation of any kind. Instead it has been shown that, by carrying out the calculations with discrete formalism, their correlators are different from zero. The desire/goal is to achieve a continuous formalism that brings with itself the information of correlations and so of the boundary conditions. This objective is achievable thanks to the fact that the four correlators defined in (1.23) are not independent, so the matrix $\hat{G}(t,t')$ has redundant information inside, indeed the following equation holds:

$$G^{\mathbb{T}}(t,t') + G^{\widetilde{\mathbb{T}}}(t,t') - G^{<}(t,t') - G^{>}(t,t') = 0 \tag{1.28}$$

This suggests that by means of a transformation one could obtain a matrix with only three non-zero elements, specifically this transformation was formulated by *Larkin* and *Ovchinnikov*, but it is commonly called *Keldysh rotation* (even if in the fermionic case (unlike the bosonic case) it is not a real rotation). This transformation is achieved by means of two matrices:

$$Q \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \sigma_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.29}$$

To make it clear how they are used, the following objects are defined:

$$\psi \equiv \begin{pmatrix} \psi^{+} \\ \psi^{-} \end{pmatrix} \qquad \psi' \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad \bar{\psi} \equiv \begin{pmatrix} \bar{\psi}^{+} & \bar{\psi}^{-} \end{pmatrix} \qquad \bar{\psi}' \equiv \begin{pmatrix} \bar{\psi}_1 & \bar{\psi}_2 \end{pmatrix} \tag{1.30}$$

The new Grassmann fields ψ_1, ψ_2, $\bar{\psi}_1$ and $\bar{\psi}_2$ are related to the initial fields by the following non-unitary transformations:

$$\psi' \equiv Q\psi \qquad \bar{\psi}' \equiv \bar{\psi}\sigma_3 Q^T \tag{1.31}$$

Note that the bar fields transform differently from non-bar fields, this is a consequence of the fact that the $\bar{\psi}$ have no relation to ψ , they are completely independent and therefore can transform in a different way. By performing the equations above,

the following relations are found:

$$\begin{aligned} \psi_1(t) &= \frac{1}{\sqrt{2}}(\psi^+(t) + \psi^-(t)) & \psi_2(t) &= \frac{1}{\sqrt{2}}(\psi^+(t) - \psi^-(t)) \\ \bar{\psi}_1(t) &= \frac{1}{\sqrt{2}}(\bar{\psi}^+(t) - \bar{\psi}^-(t)) & \bar{\psi}_2(t) &= \frac{1}{\sqrt{2}}(\bar{\psi}^+(t) + \bar{\psi}^-(t)) \end{aligned} \tag{1.32}$$

These field transformations are due to the choice of a specific transformation of the block Green's function $\hat{G}(t,t')$, that is $\hat{G} \to \hat{G}' \equiv Q\hat{G}\sigma_3 Q^T$. Thus, after this transformation, the block Green's function has the form:

$$\hat{G}'(t,t') = \begin{pmatrix} G^R(t,t') & G^K(t,t') \\ 0 & G^A(t,t') \end{pmatrix} \tag{1.33}$$

In this way, a block Green's function has been obtained with zero redundancy. The correlators associated with the four new types of fields are:

$$\langle \psi_a(t)\bar{\psi}_b(t')\rangle \equiv \int D[\bar{\psi}'\psi']\psi_a(t)\bar{\psi}_b(t')exp\left(i\iint dtdt'\bar{\psi}'(t){\hat{G}'}^{-1}(t,t')\psi'(t')\right) = iG'_{ab}(t,t') \tag{1.34}$$

where the subscripts a and b can be 1 or 2. From now on it will be assumed we are working with such formalism (subscripts 1 and 2), to lighten the notation the field's superscripts " $'$ " will not be used anymore (in order to understand the field's notation is essential to check the integration domain, that is if it is C or $\mathbb{R}$). The relations between the new $\hat{G}$ components and the initial ones are shown below:

$$\begin{aligned} G^R(t,t') &= G_{11}(t,t') = \Theta(t-t')(G^> - G^<) = -i\Theta(t-t')\langle\{\hat{c}(t), \hat{c}^\dagger(t')\}\rangle \\ G^A(t,t') &= G_{22}(t,t') = -\Theta(t'-t)(G^> - G^<) = +i\Theta(t'-t)\langle\{\hat{c}(t), \hat{c}^\dagger(t')\}\rangle \\ G^K(t,t') &= G_{12}(t,t') = G^> + G^< = -i\langle[\hat{c}(t), \hat{c}^\dagger(t')]\rangle \end{aligned} \tag{1.35}$$

where $\{,\}$ denotes the anti-commutator operation and $[,]$ the commutator operation. Looking at the eq(1.66) results, it is now also clear why the R and A superscripts were chosen. Indeed they indicate the retarded and advanced Green's functions. The $G^K(t,t')$ component is known as Keldysh Green's function. These components can be easily demonstrated to satisfy the following equations :

$$G^A = [G^R]^\dagger \qquad\qquad G^K = -[G^K]^\dagger \tag{1.36}$$

in which these Green's functions must be thought of as time-indices matrices. Performing the calculations, one finds for our miniature system example:

$$G^R(t,t') = -i\Theta(t-t')e^{-i\epsilon_0(t-t')} \quad \rightarrow \quad G^R(w) = (w-\epsilon_0+i0)^{-1}$$

$$G^A(t,t') = +i\Theta(t'-t)e^{-i\epsilon_0(t-t')} \quad \rightarrow \quad G^A(w) = (w-\epsilon_0-i0)^{-1}$$

$$G^K(t,t') = -i(1-2n_f(\epsilon_0))e^{-i\epsilon_0(t-t')} \quad \rightarrow \quad G^K(w) = -2\pi i(1-2n_f(w))\delta(w-\epsilon_0) \tag{1.37}$$

In the right part above are show their Fourier transformation with respect to the variable $(t-t')$, this is possible because the system under examination is stationary, indeed at each moment it is always described by the same density matrix.

1.1.5 Distribution function and Keldysh Action

Note that the Keldysh component $G^K(t,t')$ contains information about the initial state of the system, while the retarded and advanced components contain only information about the Hamiltonian spectrum of the system. Focus now on $G^K(w)$, it can be reformulated as follows:

$$G^K(w) = tanh\left(\frac{w-\mu}{2T}\right)[G^R(w) - G^A(w)] \tag{1.38}$$

where $T = \beta^{-1}$. This equation is of extreme relevance because it is valid for all systems in thermal equilibrium and not only for the toy model taken as an example. Indeed the (1.38) relation is a representation of the *fluctuation-dissipation theorem*, valid for every type of system in thermodynamic equilibrium. However, this is a very specific case, so it is more appropriate to define a more general parametrization of $G^K(t,t')$, valid in any type of context. For this purpose, a Hermitian matrix F (temporal indices, $F(t,t')$) is introduced, for which the following relation is valid:

$$G^K = G^R \circ F - F \circ G^A \tag{1.39}$$

where the circular sign denotes the convolution operation, that is the product between matrices in the time domain. F is an extremely important object in Keldysh theory, indeed it carries with it the information about the initial state of the system, or rather its *distribution*. Generally, $F(t,t')$ is not very widely used , but it is

preferable to work with its *Wigner transform*, often called *distribution function*:

$$F(w,t) \equiv \int dt' e^{iwt'} F(t+t'/2, t-t'/2) \tag{1.40}$$

This function describes the instantaneous quasi-particle distribution function at time t. If, as in the analysed toy model, there is a time-independent system, then in that case one gets $F(w,t) = F(w)$ which corresponds to the Fourier transform of $F(t,t') = F(t-t')$. In our case, characterized by thermodynamic equilibrium, it has $F(w) = 1 - 2n_f(w) = tanh((w-\mu)/2T)$. This last result, as written above, is valid for any system in thermodynamic equilibrium; therefore, in this situation, it is immediately clear that $G^K(w)$ is always totally defined by $G^R(w)$ and $G^A(w)$. Now, after presenting these last objects, consider the equation (1.34), the matrix $\hat{G}^{-1}$ has the following form:

$$\hat{G}^{-1} = \begin{pmatrix} [G^{-1}]^R & [G^{-1}]^K \\ 0 & [G^{-1}]^A \end{pmatrix} = \begin{pmatrix} [G^R]^{-1} & [G^{-1}]^K \\ 0 & [G^A]^{-1} \end{pmatrix} = \begin{pmatrix} [G^R]^{-1} & ([G^R]^{-1} \circ F - F \circ [G^A]^{-1}) \\ 0 & [G^A]^{-1} \end{pmatrix} \tag{1.41}$$

where the diagonal elements are calculated from the relation $[G^{-1}]^{R(A)} \circ G^{R(A)} = \mathbb{1}$ and for our system they are:

$$[G^{-1}]^{R(A)}(t,t') = \delta(t-t')(i\partial_{t'} - \epsilon_0 \pm i0^+) \quad \rightarrow \quad [G^{-1}]^{R(A)}(w) = w - \epsilon_0 \pm i0^+ \tag{1.42}$$

In a different way, $[G^{-1}]^K$ is found using the equation (1.64) and from the condition $G^K \circ [G^A]^{-1} + G^R \circ [G^{-1}]^K = 0$. At this point, all the objects necessary to represent the action S of our system have already been calculated, using the energy representation it corresponds to:

$$S[\psi, \bar{\psi}] = \int \frac{dw}{2\pi} \begin{pmatrix} \bar{\psi}_1(w) & \bar{\psi}_2(w) \end{pmatrix} \begin{pmatrix} w - \epsilon_0 + i0^+ & 2i0^+ tanh(\frac{w-\mu}{2T}) \\ 0 & w - \epsilon_0 - i0^+ \end{pmatrix} \begin{pmatrix} \psi_1(w) \\ \psi_2(w) \end{pmatrix} \tag{1.43}$$

in which the equality $[G^{-1}]^K = 2i0^+ F$ is used (this identity holds only if F commutes with $[G^{R(A)}]^{-1}$). Note that the action S contains information about the initial state of the system (in this specific case it is also the state at any other time, since the system is in thermodynamic equilibrium). In addition, the fact that $[G^{-1}]^K$ is different from zero, is extremely important, in fact it allows the functional integrals of eq(1.34) to converge. It is important to stress that for our system the $[G^{-1}]^K$ is infinitesimal, but in interacting systems such object becomes finite because they acquires a self-energy contribution.

1.2 Free fermions

Before moving to the interactive systems, the intermediate level of complexity is presented, that is always a system of free particles but with many degrees of freedom and not only one as in the toy model used until now. In this case each fermion will live in a Hilbert space spanned by a base of momentum and spin eigenstates, so each element of such a base will be identified by the $\mathbf{k}$ and $\sigma = \pm 1/2$ indices; its Hamiltonian, in second quantization, corresponds to

$$\hat{H} = \sum_{\mathbf{k}} \sum_{\sigma} \epsilon_{\mathbf{k}} \hat{c}^{\dagger}_{\mathbf{k}\sigma} \hat{c}_{\mathbf{k}\sigma} \tag{1.44}$$

Given all these premises, it is understood that during the computation of the partition function, in the process of insertion of the identities (eq(1.11)) the used coherent states must be eigenstates of all $\hat{c}_{\mathbf{k}\sigma}$ operators (where obviously the corresponding eigenvalues can be different). The Keldysh's action associated with such a system is as follows:

$$\begin{aligned} S[\bar{\psi}, \psi] &= \sum_{\mathbf{k}} \sum_{\sigma} \int_C dt\, \bar{\psi}_{\mathbf{k}\sigma}(t)(i\partial_t - \epsilon_{\mathbf{k}})\psi_{\mathbf{k}\sigma}(t) \\ &= \sum_{\mathbf{k}} \sum_{\sigma} \int_{-\infty}^{+\infty} dt\, [\bar{\psi}^{+}_{\mathbf{k}\sigma}(t)(i\partial_t - \epsilon_{\mathbf{k}})\psi^{+}_{\mathbf{k}\sigma}(t) - \bar{\psi}^{-}_{\mathbf{k}\sigma}(t)(i\partial_t - \epsilon_{\mathbf{k}})\psi^{-}_{\mathbf{k}\sigma}(t)] \\ &= \sum_{\mathbf{k}} \sum_{\sigma} \iint_{-\infty}^{+\infty} dt dt' \sum_{a,b=1}^{2} \bar{\psi}_{a,\mathbf{k}\sigma}(t)[\hat{G}^{-1}_{\mathbf{k}}(t,t')]_{ab}\, \psi_{b,\mathbf{k}\sigma}(t') \end{aligned} \tag{1.45}$$

where in the third row the Keldysh rotation took place and the matrix $\hat{G}^{-1}_{\mathbf{k}}$ is of the type eq(1.41). This matrix, in energy representation, is identical to the matrix present in the equation (1.43) after having performed the substitution $\epsilon_0 \to \epsilon_{\mathbf{k}}$ (considering that the system is at thermodynamic equilibrium at the initial time). The action S can also be written in an equivalent way using as coherent states the eigenstates of field operators that are the position-spin annihilation operators $\hat{c}_{\sigma}(\mathbf{r})$; to move to this representation, one only needs to transform the fields $\psi_{\mathbf{k}\sigma}(t)$ via Fourier transformation, so

$$\psi_{\mathbf{k}\sigma}(t) \to \psi_{\sigma}(\mathbf{r}, t) = \sum_{\mathbf{k}} \psi_{\mathbf{k}\sigma}(t) e^{i\mathbf{k}\mathbf{r}} \tag{1.46}$$

while the $\bar{\psi}_{\mathbf{k}\sigma}(t)$ is forced to transform through the opposite Fourier transformation ($i \to -i$). In this case we no longer have a sum over the discrete values of $\mathbf{k}$, but an integral resulting from the continuous spectrum of the position operator. It will

therefore take the form:

$$S[\bar{\psi},\psi] = \sum_{\sigma} \iint_{\Omega} d\mathbf{r}d\mathbf{r}' \iint_{-\infty}^{+\infty} dtdt' \sum_{a,b=1}^{2} \bar{\psi}_{a,\sigma}(\mathbf{r},t)[\hat{G}^{-1}(\mathbf{r},t;\mathbf{r}',t')]_{ab}\,\psi_{b,\sigma}(\mathbf{r}',t') \tag{1.47}$$

where Ω is the volume of the system.

1.3 External sources

So far this thesis has mainly focused on the partition function Z, which however, as explained at the beginning of the section, is always equal to 1 by definition; this means that the "bare" Z does not carry any physical information. In order to find physical information, in particular the expectation values of physical observables, it is necessary the introduction of *source field* in the theoretical framework. Consider, for example, the artificial introduction of a potential term $V(\mathbf{r},t)$ in a free particle system (described by eq(1.45), but considering the spinless case to simplify the notation). This operation consists in modifying the system's Hamiltonian by adding to the kinetic term a potential term corresponding to

$$\hat{H}_V(t) = \int_{\Omega} d\mathbf{r} V(\mathbf{r},t)\hat{c}^{\dagger}(\mathbf{r})\hat{c}(\mathbf{r}) \tag{1.48}$$

and consequently in the total action there will appear the term:

$$S_V = -\int_{\Omega} d\mathbf{r} \int_{C} V(\mathbf{r},t)\bar{\psi}(\mathbf{r},t)\psi(\mathbf{r},t) \tag{1.49}$$

(be careful as this field notation does not refer to eq(1.30), there is the integration on C, so it is in the formalism of the type eq(1.17)). If S_V is reformulated in relation to the $\pm$ components of the fields and then to $1,2$ notation components, the following equality relation holds:

$$\begin{aligned} S_V &= -\int_{\Omega} d\mathbf{r} \int_{-\infty}^{+\infty} dt[V^{+}\bar{\psi}^{+}\psi^{+} - V^{-}\bar{\psi}^{-}\psi^{-}] \\ &= -\int_{\Omega} d\mathbf{r} \int_{-\infty}^{+\infty} dt[V^{cl}(\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2) + V^{q}(\bar{\psi}_1\psi_2 + \bar{\psi}_2\psi_1)] \\ &= -2\int_{\Omega} d\mathbf{r} \int_{-\infty}^{+\infty} dt[V^{cl}\rho^{q} + V^{q}\rho^{cl}] \end{aligned} \tag{1.50}$$

where

$$V^{cl(q)} \equiv (V^{+} \pm V^{-})/2 \qquad \rho^{cl(q)} \equiv \frac{1}{2}(\bar{\psi}^{+}\psi^{+} \pm \bar{\psi}^{-}\psi^{-}) \tag{1.51}$$

are called the classical and quantum components of the source and the density. With the artificial addition of S_V in the total action, the original partition function becomes the so-called generating function $Z[V]$. However note that if the potential $V(\mathbf{r},t)$ is identical in both branches of the contour ($V^+ = V^-$), then $Z[V]$ is still identically equal to 1 and therefore the physical information cannot be extracted from it. A solution to circumvent this problem is to consider a potential $V(\mathbf{r},t)$ that is different on the two branches of the contour (and therefore for the moment does not represent a physical potential); therefore it is considered to be in the situation where $V^+ \neq V^-$ and only at the end of the calculation their equality is imposed. As can be seen from the last line of eq(1.50), the classical source component V^{cl}, equal for the two branches ($V^{\pm} \equiv V^{cl} \pm V^q$), is coupled to the quantum component of density ρ^q, while V^q is coupled to the classical component ρ^{cl} also called the physical density of fermions because its functional average corresponds to the expectation value of the fermionic density operator $\hat{c}^\dagger(\mathbf{r})\hat{c}(\mathbf{r})$ at the time considered (up to an additive constant, which depends on the type of Keldysh rotation carried out). Note that if at the end of the calculation process of the expectation values of observable there is the imposition $V^q = 0$, then V^{cl} corresponds to the external physical scalar potential. Now there are all the tools to define the generating function:

$$Z[V^{cl}, V^q] \equiv \langle e^{iS_V} \rangle \tag{1.52}$$

where the brakets indicate as always the functional average over the Grassmann fields weighted by the exponential of iS_0 where S_0 is the free action (eq(1.47)). It can be verified, by a process of expansion in series, that $Z[V^{cl}, 0] \equiv 1$ and this is what is expected for all previous reasoning. V^q has no particular physical significance, it is simply an artificially introduced tool which allows one to obtain functional derivatives useful for the purpose of calculating physical quantities; however, its existence is essential in order to violate causality and to obtain a generating function. The introduction of $\hat{H}_V(t)$ in the total Hamiltonian allows one to calculate the expectation value at time t and position $\mathbf{r}$ of the fermionic density operator:

$$\langle \rho^{cl}(\mathbf{r},t) \rangle = \frac{i}{2} \frac{\delta}{\delta V^q(\mathbf{r},t)} Z[V^{cl}, V^q] \Big|_{V^q=0} \tag{1.53}$$

Note that what has just been calculated, does not relate to the case of a free particle system, but to a system in which fermions are immersed in a time-dependent scalar potential $V^{cl}(\mathbf{r},t)$, indeed after equating V^q to zero, the potential V^{cl} becomes the physical potential present in the system.

1.4 Interactions

The last ingredient to be added to the theoretical framework is the presence of interactions between fermions. As explained at the beginning of the section, in Keldysh's theory it is always assumed that at the initial time $-\infty$ the system consists of free fermions. The interactions are then successively adiabatically switched on. Consider for example the case of interacting fermions by means of a two-body potential operator (spin independent) described by the potential energy function $U(\mathbf{r}-\mathbf{r}')$, with Fourier transform $U(\mathbf{q})$. The component of the total Hamiltonian that describes this interaction, in second quantization, is:

$$\hat{H}_{int} = \frac{1}{2}\sum_{\mathbf{k},\mathbf{k}',\mathbf{q}}\sum_{\sigma\sigma'} U(\mathbf{q})\hat{c}^{\dagger}_{\mathbf{k}\sigma}\hat{c}^{\dagger}_{\mathbf{k}'\sigma'}\hat{c}_{\mathbf{k}'+\mathbf{q}\sigma'}\hat{c}_{\mathbf{k}-\mathbf{q}\sigma} \tag{1.54}$$

Its existence introduces into the partition function a term equal to $e^{iS_{int}}$, where the action S_{int} corresponds to:

$$S_{int}[\bar{\psi},\psi] = -\frac{1}{2}\int_C dt \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}}\sum_{\sigma\sigma'} U(\mathbf{q})\bar{\psi}_{\mathbf{k}\sigma}(t)\bar{\psi}_{\mathbf{k}'\sigma'}(t)\psi_{\mathbf{k}'+\mathbf{q}\sigma'}(t)\psi_{\mathbf{k}-\mathbf{q}\sigma}(t) \tag{1.55}$$

The interaction term (at full capacity) should only affect matrix elements (see equation (1.14)) associated with times far from the initial one, so at times when the operation of adiabatic switching on of the interaction is already concluded; note, however, that in the equation above the interaction is present at all times in the contour C, while it is expected that, in the times immediately following the initial one, in the matrix elements there is a term which realizes the adiabatic switching. Equation (1.55) is not wrong because it can be shown that the stages of adiabatic switching of the interactions can be neglected (or better, in them the interaction term can be considered as already fully active), the important thing is to always calculate\work in finite time conditions and never approach the limit $-\infty$. If we consider a total Hamiltonian $\hat{H}(t)$ in which the term $\hat{H}_{int}$ is slowly activated, the whole theoretical framework would certainly be more complex, but such formal precision would have no consequence on the calculation of physical information associated with times far from the initial (and therefore to any finite time). It should also be remembered that the adiabatic switching on process of the interactions has a precise effect on the temporal evolution of the system, indeed if initially the system is described by an equilibrium density matrix (associated with the Hamiltonian describing free particles), once the process of activation of the interactions is completed the system will still be described by an equilibrium density matrix, but now associated with the total Hamiltonian (all this applies if there are no time-dependent

perturbations, in this case the system is driven away from equilibrium). In addition, in the next sections of the thesis it will be shown how the process of adiabatic switching is very useful to obtain the non-equilibrium steady states of particular systems. At the end, it can be shown that the introduction of the interaction term in the total action does not spoil the normalization of Z, which always remains equal to 1. One method of proving this is to expand in series the term $e^{iS_{int}}$ and show that $\langle S_{int}\rangle = \langle S_{int}^2\rangle = \langle S_{int}^3\rangle = ... = 0$ with the help of the Gaussian action S_0 and its integration properties [20].

1.5 Dressed Green's function and Dyson equation

So far only the correlators associated with a system of free fermions have been studied and presented; in this section instead there is the introduction of the Green's functions associated to systems of interacting fermions and the objective will be to show how they are related to those of the non interacting case. These new correlators are also called *dressed* or $full$ Green's functions and they are defined in the following way [20]:

$$G_{ab}(\mathbf{r},t;\mathbf{r}',t') \equiv -i\int D[\bar{\psi}\psi]\psi_a(\mathbf{r},t)\bar{\psi}_b(\mathbf{r}',t')e^{i(S_0+S_{int})} \tag{1.56}$$

where as usual $a,b=(1,2)$. The matrix $\hat{G}$ formed by these correlators corresponds, according to the usual construction, to the Keldysh rotation of the matrix which has for elements the dressed Green's functions in $\pm$ notation:

$$G^{\pm\pm}(\mathbf{r},t;\mathbf{r}',t') \equiv -i\langle\psi^{\pm}(\mathbf{r},t)\bar{\psi}^{\pm}(\mathbf{r}',t')\rangle = -i\int D[\bar{\psi}\psi]\psi^{\pm}(\mathbf{r},t)\bar{\psi}^{\pm}(\mathbf{r}',t')e^{i(S_0+S_{int})} \tag{1.57}$$

The equation above represents the four possible configurations of the corresponding contour-ordered Green's function:

$$G_C(\mathbf{r},t;\mathbf{r}',t') = -i\langle\mathbb{T}_C[\hat{c}(\mathbf{r},t)\hat{c}^{\dagger}(\mathbf{r}',t')]\rangle \rightarrow \substack{inter.\\picture} \rightarrow -i\langle\mathbb{T}_C[e^{-i\int_C dt_1\hat{H}_{int}(t_1)}\hat{c}(\mathbf{r},t)\hat{c}^{\dagger}(\mathbf{r}',t')]\rangle \tag{1.58}$$

where the average operation is always made on the initial non interacting state. Also note that the time dependence of $\hat{H}_{int}$ is not only due to the use of the *interaction picture*, but also because it contains the term of adiabatic switching. As mentioned above, the presence of the contour-ordering operator $\mathbb{T}_C$ is extremely important and useful in the Keldysh's theoretical framework; indeed, thanks to its presence, the series expansion of the exponential term present in eq(1.56) can be treated and simplified using the Wick theorem. Actually, to perform all the steps necessary to

exploit the Wick theorem, it is not possible to study a single component G_{ab} (taken alone). It is necessary to work with the contour-ordered Green's function G_C, write the series expansion of the exponential term present in eq(1.57), apply the Wick theorem, and then, by Keldysh rotation, go to the formalism a, b (obtaining $\hat{G}$). This procedure allows one to express each $G_{ab}(\mathbf{r}, t; \mathbf{r}', t')$ as a sum of infinite contributions, each of which can be represented by a specific Feynman diagram that has two external legs described by the correlators $G_{ac,0}(\mathbf{r}, t; \mathbf{r_1}, t_1)$ and $G_{db,0}(\mathbf{r_2}, t_2; \mathbf{r}', t')$. Where the $'0'$ subscript indicates that they are associated to the non interacting case. The important relation that is obtained after all the steps described, is the so-called *Dyson* equation:

$$\begin{aligned}\hat{G} &= \hat{G}_0 + \hat{G}_0 \circ \hat{\Sigma} \circ \hat{G}_0 + \hat{G}_0 \circ \hat{\Sigma} \circ \hat{G}_0 \circ \hat{\Sigma} \circ \hat{G}_0 + \hat{G}_0 \circ \hat{\Sigma} \circ \hat{G}_0 \circ \hat{\Sigma} \circ \hat{G}_0 \circ \hat{\Sigma} \circ \hat{G}_0 + ... \\ &= \hat{G}_0 + \hat{G}_0 \circ \hat{\Sigma} \circ \hat{G}\end{aligned} \tag{1.59}$$

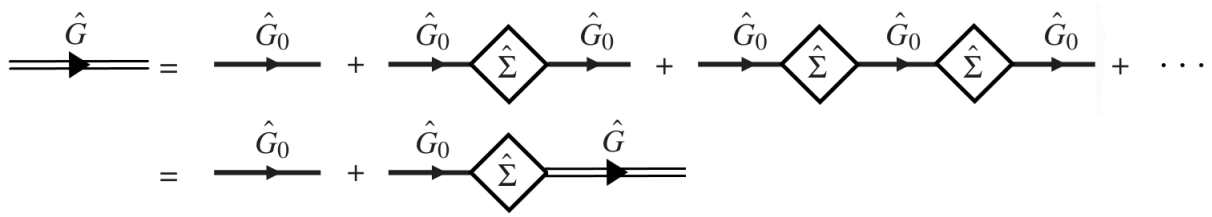


Figure 1.3: Graphical representation of the Dyson equation (1.59)

where $\hat{G}$ is the full Green's functions 2×2 matrix with elements described by eq(1.56), $\hat{G}_0$ is of the same nature but refers to the non-interacting system case (therefore corresponds to eq(1.56), where it is imposed $S_{int} = 0$), while $\hat{\Sigma}$, called *self-energy*, is a (2×2)-matrix in the Keldysh space that corresponds to the sum of terms associated to the inner part of all *irreducible* Feynman diagrams (with two external legs) of the interacting theory studied. Remember that the irreducible diagrams are all those diagrams from which it is not possible to generate two disconnected Feynman diagrams by just cutting an internal single line. To understand the meaning of the equation (1.59), one must pay attention to the convolution operation inside it: in this case it does not represent only the integrations on internal time variables, but also on the spatial ones and especially the matrix multiplication between 2×2 matrices in the Keldysh space. Note also that this equation has exactly the same structure as the Dyson equation for systems in thermodynamic equilibrium, the only difference is that now the interpreters are matrices. Often in this report the Dyson equation will also be presented as follows:

$$(\hat{G}_0^{-1} - \hat{\Sigma}) \circ \hat{G} = \hat{\mathbb{1}} \tag{1.60}$$

Although apparently $\hat{\Sigma}$ does not seem to respect particular properties, it can be shown that incredibly it has the same structure as $\hat{G}_0^{-1}$ and $\hat{G}_0$, that is

$$\hat{\Sigma} = \begin{pmatrix} \Sigma^R & \Sigma^K \\ 0 & \Sigma^A \end{pmatrix} \tag{1.61}$$

and consequently also the dressed Green's function $\hat{G}$ will have the same exact structure! The $\hat{\Sigma}$ components satisfy the following properties:

$$\Sigma^R(\mathbf{r_1}, t_1; \mathbf{r_2}, t_2) = [\Sigma^A(\mathbf{r_2}, t_2; \mathbf{r_1}, t_1)]^* \qquad \Sigma^K(\mathbf{r_1}, t_1; \mathbf{r_2}, t_2) = -[\Sigma^K(\mathbf{r_2}, t_2; \mathbf{r_1}, t_1)]^* \tag{1.62}$$

So the retarded and advanced components are mutually Hermitian conjugated (it can also be shown that they are lower and upper triangular matrices in time space), while Σ^K is an anti-Hermitian matrix (the asterisk represents the complex conjugate). Developing the eq(1.60), the K component of the difference $\hat{G}_0^{-1} - \hat{\Sigma}$ can be approximated with only the self-energy component Σ^K (if it is non zero), this is because, as has been shown previously, the term $[G_0^{-1}]^K$ is equal to $2i0^+F$, so it can be neglected.

Now consider the case of a system of interacting fermions governed by an Hamiltonian that is time independent and space translationally invariant. Thanks to these features, the dressed Green's function $\hat{G}(\mathbf{r}, t; \mathbf{r}', t')$ is equal to $\hat{G}(|\mathbf{r} - \mathbf{r}'|, t - t')$ and using the relation $[G_0^{-1}]^{R(A)} = [G_0^{R(A)}]^{-1}$, passing to the momentum-energy space via Fourier transform, and solving the Dyson equation finally one finds:

$$G^{R(A)}(\mathbf{k}, w) = \left(w - \frac{\mathbf{k}^2}{2m} - \Sigma^{R(A)}(\mathbf{k}, w) \right)^{-1} \tag{1.63}$$

Note that for the retarded and advanced components the structure of the Dyson equation (1.60) is still valid, whereas it is not valid for the K-component (easily provable). Returning to the equation above, it must be remembered that $\Sigma^{R(A)}$ are complex numbers, whose real parts are identical, while the imaginary parts are opposite: in both cases, the real part carries out an operation of renormalization of the particle's dispersion relation ($w_{\mathbf{k}} = \frac{\mathbf{k}^2}{2m}$), while the imaginary part corresponds to the decay rate of a particle in a given (renormalized) eigenstate $\mathbf{k}$ of the free fermions system. As regards the K component of the dressed Green's function, it is desirable to parametrize it in the same way that was presented in the case of the one level toy model, that is by introducing an $F(\mathbf{r}, t; \mathbf{r}', t')$ matrix (Hermitian in space and time domain):

$$G^K = G^R \circ F - F \circ G^A \tag{1.64}$$

Developing the Dyson equation, it is found that Σ^K must comply with the following

equation:

$$-\left[\left(i\partial_t + \frac{\nabla^2}{2m}\right), F\right] = \Sigma^K - (\Sigma^R \circ F - F \circ \Sigma^A) \tag{1.65}$$

where the $[,]$ corresponds to the commutation operation. This equation is called the quantum kinetic equation for the fermionic distribution matrix $F(\mathbf{r}, t; \mathbf{r}', t')$: the first member is the *kinetic* term, while the second is the *collision* term. For a system in thermodynamic equilibrium the kinetic term is zero and so the K component of the self-energy has the same structure of G^K described in the eq(1.64), then in equilibrium one has $\Sigma^K = \Sigma^R \circ F - F \circ \Sigma^A$. It is important to note that the Dyson equation is not valid only in a, b notation, but is equally valid for block Green's functions in $\pm$ notation, obviously in this last case the self-energy $\hat{\Sigma}$ will have a different form than that presented in eq.(1.61). The self-energy's components in the two different notations are linked by the following equations:

$$\begin{aligned}
\Sigma^R(\mathbf{r}, t; \mathbf{r}', t') &= \Theta(t - t')(\Sigma^>(\mathbf{r}, t; \mathbf{r}', t') - \Sigma^<(\mathbf{r}, t; \mathbf{r}', t')) \\
\Sigma^A(\mathbf{r}, t; \mathbf{r}', t') &= -\Theta(t' - t)(\Sigma^>(\mathbf{r}, t; \mathbf{r}', t') - \Sigma^<(\mathbf{r}, t; \mathbf{r}', t')) \\
\Sigma^K(\mathbf{r}, t; \mathbf{r}', t') &= \Sigma^>(\mathbf{r}, t; \mathbf{r}', t') + \Sigma^<(\mathbf{r}, t; \mathbf{r}', t')
\end{aligned} \tag{1.66}$$

So in conclusion, it is possible to say that the structure of the Dyson equation is identical both in the formalism of Keldysh and in the Matsubara formalism (able to describe only systems in thermodynamic equilibrium), the only difference is the transition to Keldysh space and thus to the matrix nature of the protagonists of the equation. Obviously, while for systems in equilibrium the choice of which theoretical framework to use is mostly a matter of personal taste, in the case of systems in non-equilibrium only the Keldysh formalism is suitable.

Chapter 2

Transport properties of quantum dots

The whole previous section has been dedicated to the presentation of the Keldysh theory which, as explained several times, arises from the need to have a theoretical framework describing systems in a condition of thermodynamical non-equilibrium. As written in the introduction of this paper, the objective of this thesis is to study the transport properties of the SYK model. The transport phenomena are closely related to the branch of non-equilibrium thermodynamics, or better the former can only exist in the presence of non-equilibrium configurations. A system in thermodynamic equilibrium is described by intensive parameters that assume the same value at all spatial points of the system and at all times (the entropy of the system remains constant over time), so there are no asymmetries/gradients that induce an ordered movement of matter or energy, namely a transport phenomenon. Therefore a non-equilibrium configuration is needed to study transport phenomena (in this case the entropy of the system will not be constant but constantly increasing). Keldysh formalism allows for the description of the expectation values of the physical observables, associated with such transport phenomena, in function of the Keldysh Green's functions.

Specifically, this thesis is concerned with the transport properties of the SYK model (fermionic with semi-integer spin) in a configuration where it is considered coupled to two metallic leads that act as semi-infinite thermal reservoirs. An important premise to be made is that this thesis will focus solely on the configurations of global non-equilibrium steady state (NESS) of the system just presented (the transition phase leading to NESS is not analysed). The study of this system using the SYK model will be preceded by the analysis of the case with a simplified quantum dot, in order to present more effectively the other elements of the entire physical system, but first of all this, it is necessary to deepen the theme of the NESS, essential for the understanding of the physical phenomenon under consideration.

2.1 Global non-equilibrium steady state

The system used to carry out the NESS study corresponds to a central quantum system (C) connected to two baths (the treatment can be generalised to the case with a greater number of bath [24, 25, 26]). The baths are considered to be semi-infinite fermionic leads, which are identified by the letters L and R (to indicate their position, left or right, Fig(2.1)). The leads act on the system as reservoirs that keep their corresponding parameters of temperature $T_{L,R}$ and chemical potential $\mu_{L,R}$ constant over time. As such they can exchange both energy and particles with the central system, they are therefore responsible for the evolution of its state until reaching the NESS. For such systems, regardless of the initial state of the central system, after a sufficiently long time the system reaches or better tends to a global non-equilibrium steady state (NESS); in this context the whole system is described by a time independent density matrix and consequently the expectation values of each physical observable are constant in time. Obviously, if the two leads have the same temperatures and chemical potentials, the asymptotic state achieved is that of thermodynamic equilibrium.

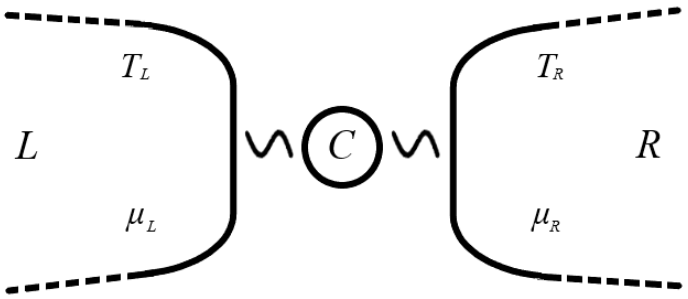


Figure 2.1: Diagrammatic illustration of a generic LCR system considered in this section.

The system described above has as its total Hamiltonian operator $\hat{H} = \hat{H}_0 + \hat{H}_{int}$, where $\hat{H}_0$ is the sum of the Hamiltonians of the two leads and the central system (all decoupled), so $\hat{H}_0 = \hat{H}_L + \hat{H}_R + \hat{H}_C$, and $\hat{H}_{int}$ describes the terms of interaction between them. Therefore $\hat{H}_{int} = \hat{V}_{LC} + \hat{V}_{RC}$ (it is assumed that all these terms conserve the total number of particles). As for the initial state of the whole system, consider that at initial time $t_0 = -\infty$ the three components are decoupled and that obviously the two leads are in thermodynamic equilibrium, so their statistical operators correspond to

$$\hat{\rho}_a = \frac{e^{-\beta_a(\hat{H}_a - \mu_a \hat{N}_a)}}{Z_a} \qquad with \qquad Z_a = Tr[e^{-\beta_a(\hat{H}_a - \mu_a \hat{N}_a)}] \tag{2.1}$$

where $a = L, R$ and $\hat{N}_a$ is the total particle number operator for the lead a. As regards the initial state of the central system, it is considered to be described by a generic statistical operator $\hat{\rho}_C$. So the total system at the initial time t_0 is described by the state

$$\hat{\rho}(t_0) = \hat{\rho}_L \otimes \hat{\rho}_C \otimes \hat{\rho}_R \tag{2.2}$$

Note that for the initial state of the central system no specific request has been made. This is due to the fact that we are interested in the NESS configuration that is completely independent from $\hat{\rho}_C$. Immediately after the initial time the interaction terms between the three components are active and consequently the density matrix (in the Schrödinger picture) that describes the whole system evolves over time according to the usual relation $\hat{\rho}_S(t) = \hat{U}(t, t_0)\hat{\rho}_S(t_0)\hat{U}^\dagger(t, t_0)$, where $\hat{U}(t, t_0) = e^{-i\hat{H}(t-t_0)}$ (of course coupling terms are considered constant over time, otherwise the system would not reach a NESS configuration). To deal with the time evolution of the state of such a system, it is advisable to switch to the interaction picture (with respect to $\hat{H}_0$), choosing as time of coincidence, between the two different pictures, the time $t = 0$. Using this theoretical framework the statistical operator at time t is linked to the S-picture counterpart by the equation $\hat{\rho}_I(t) = \hat{U}_0^\dagger(t, 0)\hat{\rho}_S(t)\hat{U}_0(t, 0)$, where $\hat{U}_0(t, 0) = e^{-i\hat{H}_0 t}$. In the interaction picture, the time evolution at time t of the statistical operator is related to that associated with the initial instant by the following relation:

$$\hat{\rho}_I(t) = \hat{U}_I(t, t_0)\hat{\rho}_I(t_0)\hat{U}_I^\dagger(t, t_0) \qquad with \qquad \hat{U}_I(t, t_0) = \hat{U}_0^\dagger(t, 0)\hat{U}(t, t_0)\hat{U}_0(t, 0) \tag{2.3}$$

In the transient phase, before the NESS is reached, the expectation value of each physical observable (not explicitly time-dependent) is time-dependent and corresponds to

$$\langle \hat{A}(t) \rangle = Tr[\hat{A}_I(t)\hat{\rho}_I(t)] \qquad where \qquad \hat{A}_I(t) = \hat{U}_0^\dagger(t, 0)\hat{A}\hat{U}_0(t, 0) \tag{2.4}$$

After a sufficiently long time, the NESS is reached and the expectation value of $\hat{A}$ is defined by means of a limit operation, i.e.

$$\langle \hat{A} \rangle_{ness} = \lim_{|t-t_0| \to \infty} \langle \hat{A}(t) \rangle \tag{2.5}$$

it is time independent, since in such a configuration the statistical operator is constant over time. Specifically in the theoretical framework used here, $t_0 \to -\infty$ was imposed and t was chosen to be equal to zero. With this choice, since that at $t = 0$

the two pictures coincide, one can describe the steady state as follows:

$$\hat{\rho}_{ness} \equiv \hat{U}_I(0,-\infty)\hat{\rho}_0\hat{U}_I^{-1}(0,-\infty) \tag{2.6}$$

To obtain the expression of $\hat{\rho}_{ness}$ as a function of the Hamiltonians of the three components of the system and the thermodynamic parameters describing the reservoirs, it is appropriate to exploit the power of the adiabatic switching process. It is therefore assumed that the interaction term is not present at the initial instant t_0 and that it is switched on very slowly until it is fully activated at $t = 0$. It is therefore assumed that the Hamiltonian describing the system has been modified and now corresponds to $\hat{H}_\epsilon(t) = \hat{H}_0 + e^{-\epsilon|t|}\hat{H}_{int}$, where ϵ is a random small positive constant. It is important to emphasize that the adiabatic limit of the process is obtained by applying the limit $\epsilon \to 0^+$ at the end of the calculations; this limit is essential because it returns the system to be described by the initial Hamiltonian. So, using this framework, the steady state can be described as

$$\hat{\rho}_{ness} \equiv \lim_{\epsilon\to 0^+} \hat{U}_{I,\epsilon}(0,-\infty)\hat{\rho}_0\hat{U}_{I,\epsilon}^{-1}(0,-\infty) \tag{2.7}$$

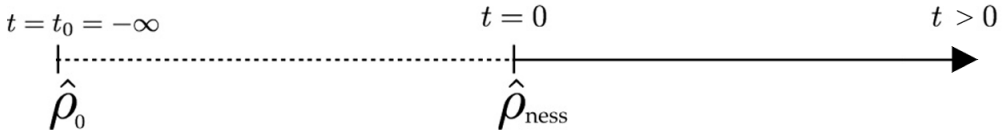


Figure 2.2: Starting from the initial time $t_0 = -\infty$, by an adiabatic switching on of the interactions, a NESS configuration is reached at the instant t=0. This state remains unchanged at any time $t > 0$.

Specifically, by carrying out the calculations [27, 26, 28, 29], it has been shown to be equal to:

$$\hat{\rho}_{ness} = \frac{e^{-\bar{\beta}(\hat{H}-\bar{\mu}\hat{N})+\hat{\Sigma}}}{Z_{ness}} \tag{2.8}$$

called the *McLennan-Zubarev form*. In order to understand the equation (2.8), it should be specified that $\bar{\beta} = (\beta_L + \beta_R)/2$, $\bar{\mu} = (\beta_L\mu_L + \beta_R\mu_R)/(\beta_L + \beta_R)$, $\hat{N}$ is the total particle number operator for the leads and $\hat{\Sigma}$ is the entropy production operator defined as $\hat{\Sigma} = \frac{1}{2}[(\beta_L\mu_L - \beta_R\mu_R)(\hat{N}_L - \hat{N}_R)_+ - (\beta_L - \beta_R)(\hat{H}_L - \hat{H}_R)_+]$, where the + subscript stands for the operation $\hat{O}_+ = \lim_{\epsilon\to 0^+} \epsilon \int_{-\infty}^{0} dt e^{\epsilon t}\hat{O}_H(t)$, with $\hat{O}_H(t) = e^{i\hat{H}t}\hat{O}e^{-i\hat{H}t}$. To avoid misunderstandings it is important to specify that the statistical operator $\hat{\rho}_{ness}$ describes the whole system and not only the central component. Finally, it is important to note that for the systems presented in this

section, namely the LCR configuration, the most significant physical observables are associated with operators describing the energy and particle currents ($\hat{J}_E$, $\hat{J}_Q$) passing through the central system that acts as the crossroads site between the two leads. In the NESS configuration, the expectation values of these operators are finite and constant over time. So for a generic $t > 0$ the following equation is valid:

$$\langle \hat{J}_{E,Q}(t) \rangle = \lim_{\epsilon \to 0^+} Tr[\hat{U}_\epsilon^\dagger(t,0)\hat{J}_{E,Q}\hat{U}_\epsilon(t,0)\hat{\rho}_\epsilon] \equiv \langle \hat{J}_{E,Q} \rangle \tag{2.9}$$

where $\hat{U}_\epsilon(t,0)$ is the evolution operator in the H picture relative to the case $\hat{H}_\epsilon(t)$ and $\hat{\rho}_\epsilon \equiv \hat{U}_\epsilon(0,-\infty)\hat{\rho}_0\hat{U}_\epsilon^\dagger(0,-\infty)$ which describes the state at time zero in the case of adiabatic switching but in the situation where the limit operation ($\epsilon \to 0^+$) has not yet been carried out. Therefore it is clear that with such formalism the steady state $\hat{\rho}_{ness}$ is equal to $lim_{\epsilon \to 0^+}\hat{\rho}_\epsilon$. An important note to be stressed is how, in the equation (2.9), the adiabatic limit should be performed as the last operation, so at the end of the calculations.

This whole discussion is fundamental to understand what will be presented in the next section, that is the study of a quantum dot interacting with two reservoirs. Such a system will be studied using the Keldysh formalism presented in the first chapter and, as was shown, in this theoretical framework it is always assumed that the terms of interaction/coupling in the Hamiltonian are activated adiabatically. So as shown in this section, it is assumed that at the initial time, in a remote past, the whole system is decoupled and subsequently the interactions are turned on adiabatically until the state of the system reaches the steady state (and it is in this state that the properties of the system are studied). There is, however, a point to be made, whereas in this section it has always been shown that the steady state is reached at a finite time of t=0, in the Keldysh formalism used in this thesis, it is always assumed that the process of adiabatic switching on of the interactions is always completed sometime prior to the observation time. This time of conclusion is never specified and it is also considered to belong to a remote past that is always prior to the finite times in which measurements are carried out. Also, as explained in the previous chapter, in the calculation of the partition function in the Keldysh formalism there never appear explicitly terms related to the adiabatic switching process (even if they exist), as in the calculation of the integral on the contour C, they can be replaced by "complete" interaction terms without any kind of consequence. Having made all these clarifications, it is understood that, in the next sections where LCR systems will be studied by means of Keldysh formalism, at any finite time t the whole system is considered to be in the corresponding NESS.

2.2 Quantum transport in nanoscale systems

This section examines the specific case in which the LCR system is a nanoscale system, namely a system with an essentially discrete energy spectrum in some parts [30]. In the case presented here, the central system is coupled to two infinitely large leads with continuous energy spectra, this means that the central system shall be described by a discrete spectrum Hamiltonian (from the point of view of theoretical treatment they are often regarded as point-like or zero dimensional). The two leads considered are ideal electrodes meaning that they are reservoirs of noninteracting electrons in thermodynamical equilibrium with fixed chemical potentials and temperatures over time. If there is a difference of non-zero temperature or chemical potential between the two reservoirs, then the central system will be in a non-equilibrium configuration. These systems are often called *nanojunctions*, while the central system is often referred to as a *quantum dot*. In literature, especially when it comes to the study of transport phenomena, it is quite common that the prefix *nano* is replaced with *meso*, since the line of demarcation between nanoscale systems and mesoscopic ones is often not extremely clear. In fact, the latter are characterized by dimensions that are between the microscopic (atomic) scale and the macroscopic (bulk) scale. More precisely, we can define mesoscopic systems as those systems for which the dephasing times over which quantum phase coherence is lost (as a result of interaction with the environment) is larger than the time scales related to transport phenomena of the system [21]. An example of uncertainty in terms of terminology is the model that will be studied in the last chapters of this thesis, namely the SYK model. Indeed this model can be seen as a nanoscale system if the number of sites N constituting it is finite and small; in our case, however, its study will be within the very large N limit, so in this case it will be appropriate to see it as a mesoscopic system. Now focusing on nanoscale systems, in order to study such types of systems there are different theoretical approaches, as written at the beginning of this thesis work, the one used here is provided by a non-equilibrium Green's functions technique, specifically in its variant based on functional integrals QFT formalism (Keldysh field theory) [19, 20, 23]. To demonstrate the power and effectiveness of such a theoretical framework, a nanoscale quantum dot LCR model is presented and studied below; it will be considered a completely generic quantum dot so that the theoretical treatment can be extended to any central system with a discrete energy spectrum.

Consider a LCR system consisting of two metal leads (described as non-interacting electron systems) and a generic quantum dot containing electron interaction terms

[31]. The coupling between the dot and the reservoirs is done by means of a tunneling term which describes the phenomenon of instantaneous passage of an electron from one end of the leads to a discrete energy level of the dot (each lead has a single-channel point contact with the dot, Fig(2.3)).

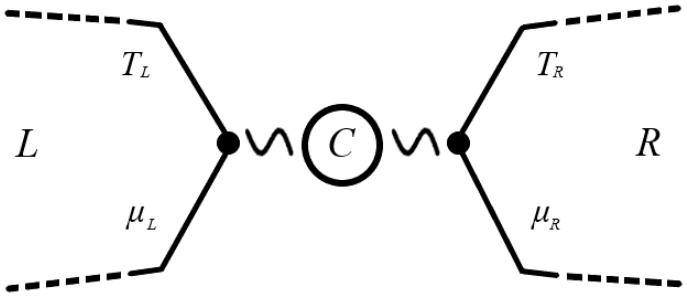


Figure 2.3: The tunneling phenomena involve only the ends of the two electrodes (regarding the leads).

The total Hamiltonian describing the whole system is therefore the sum of three contributions:

$$\hat{H} = \hat{H}_{leads} + \hat{H}_{dot} + \hat{H}_T = \sum_{a=L,R} \hat{H}_a + \hat{H}_{dot} + \hat{H}_T \tag{2.10}$$

Using the formalism of second quantization, the creation and annihilation operators associated with the base of the Hilbert space relative to an electron belonging to the lead-a (consisting of eigenstates of spin-momentum), are represented as $\hat{c}^\dagger_{\mathbf{k}\sigma a}$ and $\hat{c}_{\mathbf{k}\sigma a}$. Here $\mathbf{k}$ is the corresponding momentum eigenvalue and σ stands for the spin. Regarding the dot, those operators associated with the base constituted by the non-interacting energy eigenstates that are identified by the index $n = 0, 1, 2, 3, ...$ (relative to the non-interacting energy eigenvalue ϵ_n) and σ (spin of the eigenstate), are presented as $\hat{d}^\dagger_{n\sigma}$ and $\hat{d}_{n\sigma}$ and are used as ladder operators. For clarity, the following is the total Hamiltonian of the system in which the operator dependencies are explicitly expressed:

$$\hat{H} = \sum_{a=L,R} \hat{H}_a(\{\hat{c}^\dagger_{\mathbf{k}\sigma a}; \hat{c}_{\mathbf{k}\sigma a}\}) + \hat{H}_{dot}(\{\hat{d}^\dagger_{n\sigma}; \hat{d}_{n\sigma}\}) + \hat{H}_T(\{\hat{c}^\dagger_{\mathbf{k}\sigma L}; \hat{c}_{\mathbf{k}\sigma L}; \hat{c}^\dagger_{\mathbf{k}\sigma R}; \hat{c}_{\mathbf{k}\sigma R}; \hat{d}^\dagger_{n\sigma}; \hat{d}_{n\sigma}\}) \tag{2.11}$$

The above ladder operators satisfy among themselves the usual rules of fermionic anticommutation:

$$\{\hat{f}_i, \hat{f}^\dagger_j\} = \delta_{ij} \qquad \{\hat{f}_i, \hat{f}_j\} = \{\hat{f}^\dagger_i, \hat{f}^\dagger_j\} = 0 \tag{2.12}$$

where $f = c, d$ and $i, j = k\sigma a, n\sigma$. Note that these last assumptions means that

$\hat{H}_L, \hat{H}_L, \hat{H}_{dot}, \hat{H}_T$ are being considered as independent Hamiltonians with independent Hilbert spaces and this is only possible if we consider, as a coupling term between dot and leads, an instantaneous tunneling term (this is a strong assumption); indeed, if the electron, in the passage between lead and dot, could be found in an intermediate region between the two, such anticommutation rules would not be valid. Now that all the necessary preconditions have been made, the analytical forms of the three components of $\hat{H}$ are presented:

$$\begin{aligned}
\hat{H}_{leads} &= \sum_{\mathbf{k}} \sum_{\sigma} [w_{\mathbf{k}L} \hat{c}^{\dagger}_{\mathbf{k}\sigma L} \hat{c}_{\mathbf{k}\sigma L} + w_{\mathbf{k}R} \hat{c}^{\dagger}_{\mathbf{k}\sigma R} \hat{c}_{\mathbf{k}\sigma R}] \\
\hat{H}_{dot} &= \left(\sum_{n} \sum_{\sigma} \epsilon_n \hat{d}^{\dagger}_{n\sigma} \hat{d}_{n\sigma} \right) + \hat{V}(\hat{d}^{\dagger}_{n\sigma}, \hat{d}_{n\sigma}) \\
\hat{H}_T &= \sum_{a=L,R} \sum_{\mathbf{k}} \sum_{n} \sum_{\sigma} [W_{\mathbf{k}a,n} \hat{c}^{\dagger}_{\mathbf{k}\sigma a} \hat{d}_{n\sigma} + W^{*}_{\mathbf{k}a,n} \hat{d}^{\dagger}_{n\sigma} \hat{c}_{\mathbf{k}\sigma a}]
\end{aligned} \tag{2.13}$$

While the first two components do not need further details for now, it is good to stop for a moment to pay attention to the features of the tunneling term $\hat{H}_T$. From the above statement concerning the validity of the anticommutation properties, it is understood that the term of tunneling $\hat{H}_T$, presented in the equation (2.13), is not the "true" Hamiltonian, but an Hamiltonian whose expression allows one to easily describe currents between the leads and the dot. However, if the modulus of the tunneling matrix elements $W_{\mathbf{k}a,n}$ are sufficiently small, as in the so called weak coupling case, the correction to the commutation relations are negligible. The above expression of $\hat{H}_T$ is correct if the system can be well described by a tight-binding formulation. It is good to remember that the use of a tunneling term in the total Hamiltonian of the system must be thought of as a phenomenological microscopic approach that allows a multitude of transport phenomena to be described with high precision.

The system described above is now supplemented by the presence of an external voltage V applied to the electrodes, this voltage will regulate the difference in chemical potential between the two leads. Defining μ_a as the chemical potential of lead-a, their difference $\mu_L - \mu_R$ is equal to eV (the chemical potential μ_a is the sum of the internal contribution, equal for both the leads, and the contribution made by the external voltage). This external voltage is responsible for generating an electric current between the leads, passing through the dot, but it is not the only possible cause of the creation of a NESS. Indeed, an absence of external voltage with a temperature difference between the leads generates a thermoelectric current. To simplify the notation, μ_R is chosen as the energy origin, so $\mu_R = 0$ and therefore every energy that will be presented must be considered as corresponding to the "true" energy less the factor μ_R. Usually when physical quantities are calculated there is no need to

return to the "real" energies because the first ones are obtained by integrating over the space of all real numbers (frequency domain) and therefore the re-scaling is a simple translation operation which does not bring any concrete change. Due to the presence of this external voltage, $\hat{H}$ must be modified by adding the relative terms of coupling with the fermionic ladder operators. The presence of these new terms in $\hat{H}$ complicates from an analytical point of view the study of the properties of the system, it is therefore appropriate to carry out a specific gauge transformation on the ladder operators

$$\hat{c}_{\mathbf{k}\sigma a} \rightarrow \hat{c}_{\mathbf{k}\sigma a} e^{i\mu_a t} \qquad\qquad \hat{c}^{\dagger}_{\mathbf{k}\sigma a} \rightarrow \hat{c}^{\dagger}_{\mathbf{k}\sigma a} e^{-i\mu_a t} \tag{2.14}$$

that allows to transfer the information of the external voltage to the tunneling matrix elements, the cost to be paid is their transformation into time-dependent parameters according to the following relation:

$$W_{\mathbf{k}a,n} \rightarrow \textit{gauge transformation} \rightarrow W_{\mathbf{k}a,n}(t) = W_{\mathbf{k}a,n} e^{-i\mu_a t} \tag{2.15}$$

Now the total Hamiltonian of the system under consideration is complete and therefore there are all the tools to define central operators in the understanding of LCR systems, that are the current operators $\hat{J}_a$. The plural is used because there are two operators, one for each lead, as they describe the current from the lead-a to the quantum dot. Using the Heisenberg picture, they are defined as ($\hbar = 1$):

$$\begin{aligned}
\hat{J}_a(t) &= -e\frac{d}{dt}\hat{N}_a(t) = -ie[\hat{H}(t), \hat{N}_a(t)] = -ie\left[\hat{H}(t), \sum_{\mathbf{k}}\sum_{\sigma} \hat{c}^{\dagger}_{\mathbf{k}\sigma a}(t)\hat{c}_{\mathbf{k}\sigma a}(t)\right] \\
&= -ie\left[\hat{H}_T(t), \sum_{\mathbf{k}}\sum_{\sigma} \hat{c}^{\dagger}_{\mathbf{k}\sigma a}(t)\hat{c}_{\mathbf{k}\sigma a}(t)\right] \\
&= -ie\sum_{\mathbf{k},\sigma}\left[\sum_{\mathbf{k}',\sigma'}\sum_{n}[W_{\mathbf{k}'a,n}(t)\hat{c}^{\dagger}_{\mathbf{k}'\sigma' a}(t)\hat{d}_{n\sigma'}(t) + W^{*}_{\mathbf{k}'a,n}(t)\hat{d}^{\dagger}_{n\sigma'}(t)\hat{c}_{\mathbf{k}'\sigma' a}(t)], \hat{c}^{\dagger}_{\mathbf{k}\sigma a}(t)\hat{c}_{\mathbf{k}\sigma a}(t)\right] \\
&= ie\sum_{\mathbf{k}}\sum_{\sigma}\sum_{\mathbf{k}'}\sum_{n}\sum_{\sigma'}\delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'}[W_{\mathbf{k}'a,n}(t)\hat{c}^{\dagger}_{\mathbf{k}\sigma a}(t)\hat{d}_{n\sigma'}(t) - W^{*}_{\mathbf{k}'a,n}(t)\hat{d}^{\dagger}_{n\sigma'}(t)\hat{c}_{\mathbf{k}\sigma a}(t)] \\
&= ie\sum_{\mathbf{k}}\sum_{n}\sum_{\sigma}[W_{\mathbf{k}a,n}(t)\hat{c}^{\dagger}_{\mathbf{k}\sigma a}(t)\hat{d}_{n\sigma}(t) - W^{*}_{\mathbf{k}a,n}(t)\hat{d}^{\dagger}_{n\sigma}(t)\hat{c}_{\mathbf{k}\sigma a}(t)]
\end{aligned} \tag{2.16}$$

Note that the current operator $\hat{J}_a$ differs from the a-component of $\hat{H}_T$ only by a sign change between the terms in the square brackets (except for the constant ie).

2.2.1 Keldysh Field Theory formalism

Now that all the major operatorial terms associated with the system under consideration have been presented, it is possible to start studying its properties using the Keldysh Field Theory presented in the first chapter; indeed this theoretical framework is suitable for the study of non-equilibrium systems and therefore also quantum transport properties. The central element of the theory, from which information about the system can be extracted, is its partition function (eq(1.6)). Up to a normalization factor, it corresponds to the trace of the time evolution of the initial time density matrix along the entire contour of Keldysh C. At the initial time $t = -\infty$, the three components of the system are considered to be decoupled and without interaction terms (so the tunneling term $\hat{H}_T$ and $\hat{V}$ are not present) and the two leads are in thermal equilibrium; the central dot can be described by any state, this will have no impact on the analysis, because it will be limited to the study of the NESS configuration. The statistical operator describing the system at the initial instant will then be expressed by the equation (2.2). The tunneling term and the interactions in the quantum dot will then be turned on adiabatically and, once this process is completed (at an instant in the remote past), the entire system will find itself in a global NESS configuration. The partition function will be described as a functional integral of an exponential term with exponent equal to iS, where S is called the total action and corresponds to the sum of the actions associated with the three components of the Hamiltonian $\hat{H}$ (eq(2.10)), so $S = S_{leads} + S_{dot} + S_T$. For the study of S_{leads}, as seen before the action described by eq(1.45) is adequate to describe a single reservoir, with the difference that now the Grassmann variables $\bar{\psi}_{\mathbf{k}\sigma}, \psi_{\mathbf{k}\sigma}$ are replaced by $\bar{\chi}_{\mathbf{k}\sigma a}, \chi_{\mathbf{k}\sigma a}$, so it corresponds to:

$$\begin{aligned}
S_{leads} &= \sum_{a=L,R} \sum_{\mathbf{k}} \sum_{\sigma} \int_C dt\, \bar{\chi}_{\mathbf{k}\sigma a}(t)(i\partial_t - \omega_{\mathbf{k}a})\chi_{\mathbf{k}\sigma a}(t) \\
&= \sum_{a=L,R} \sum_{\mathbf{k}} \sum_{\sigma} \int_{-\infty}^{+\infty} dt\, [\bar{\chi}^{+}_{\mathbf{k}\sigma\mathbf{a}}(t)(i\partial_t - \omega_{\mathbf{k}a})\chi^{+}_{\mathbf{k}\sigma\mathbf{a}}(t) - \bar{\chi}^{-}_{\mathbf{k}\sigma\mathbf{a}}(t)(i\partial_t - \omega_{\mathbf{k}a})\chi^{-}_{\mathbf{k}\sigma\mathbf{a}}(t)] \\
&= \sum_{a=L,R} \sum_{\mathbf{k}} \sum_{\sigma} \iint_{-\infty}^{+\infty} dt dt' \sum_{i,j=1}^{2} \bar{\chi}^{i}_{\mathbf{k}\sigma a}(t)[\hat{g}^{-1}_{\mathbf{k}a}(t-t')]_{ij}\, \chi^{j}_{\mathbf{k}\sigma a}(t') \\
&= \sum_{a=L,R} \sum_{\mathbf{k}} \sum_{\sigma} \iint_{-\infty}^{+\infty} dt dt'\, \bar{X}^{T}_{\mathbf{k}\sigma a}(t)\hat{g}^{-1}_{\mathbf{k}a}(t-t')X_{\mathbf{k}\sigma a}(t')
\end{aligned} \tag{2.17}$$

where as usual the $\pm$ notation refers to the two branches of the contour C, while the notation i, j refers to the fields obtained after the Keldysh rotation, therefore $\chi^{1(2)} = (\chi^{+} \pm \chi^{-})/\sqrt{2}$ and $\bar{\chi}^{1(2)} = (\bar{\chi}^{+} \mp \bar{\chi}^{-})/\sqrt{2}$. At the last line of the equation

above, to make the notation more compact we have defined $X = \begin{pmatrix}\chi^1 & \chi^2\end{pmatrix}^T$ and $\bar{X}^T = \begin{pmatrix}\bar{\chi}^1 & \bar{\chi}^2\end{pmatrix}$. The 2×2 matrix $\hat{g}^{-1}_{\mathbf{k}a}$ is the inverse of the Green's function matrix $\hat{g}_{\mathbf{k}a}$ that has the usual following expression:

$$\hat{g}_{\mathbf{k}a} = \begin{pmatrix} g^R_{\mathbf{k}a} & g^K_{\mathbf{k}a} \\ 0 & g^A_{\mathbf{k}a}\end{pmatrix} \tag{2.18}$$

Exploiting the time invariance of the system, it is appropriate to write its components as a function of the frequency by performing its Fourier transformation, obtaining the results already shown in Section (1.2):

$$g^{R(A)}_{\mathbf{k}a}(w) = (w - w_{\mathbf{k}a} \pm i0)^{-1} \qquad g^K_{\mathbf{k}a}(w) = F_a(w)[g^R_{\mathbf{k}a}(w) - g^A_{\mathbf{k}a}(w)] \tag{2.19}$$

So for its inverse

$$\hat{g}^{-1}_{\mathbf{k}a}(w) = \begin{pmatrix} w - w_{\mathbf{k}a} + i0^+ & 2i0^+F_a(w) \\ 0 & w - w_{\mathbf{k}a} - i0^+\end{pmatrix} \tag{2.20}$$

where $F_a(w)$ is the distribution function of the lead-a. Knowing that leads are always in thermodynamic equilibrium at the respective temperature T_a, the distribution function corresponds to $F_a(w) \equiv 1-2f_a(w)$ where $f_a(w) \equiv (e^{w/T_a}+1)^{-1}$ is the Fermi function (note that the chemical potential terms are not present, this is because their information is contained in the tunneling term). Therefore, since the leads are in thermodynamic equilibrium conditions, the second equation of (2.19) represents the statement of the fluctuation–dissipation theorem (FDT). Let's now pass to the study of the action component associated with the quantum dot. This action will be written as a function of the new Grassmann fields $\psi_{n\sigma}$ and $\bar{\psi}_{n\sigma}$ associated with the ladder operators $\hat{d}_{n\sigma}$ and $\hat{d}^\dagger_{n\sigma}$. The structure of S_{dot} is found with a analogous procedure to that for S_{leads}, the main difference is the presence of the interaction term V:

$$\begin{aligned} S_{dot} &= \int_C dt \left\{ \left[\sum_n \sum_\sigma \bar{\psi}_{n\sigma}(t)(i\partial_t - \epsilon_n)\psi_{n\sigma}(t)\right] - V(\bar{\psi}_{n\sigma}, \psi_{n\sigma})\right\} \\ &= \int_{-\infty}^{+\infty} dt \left\{ \left[\sum_n \sum_\sigma \left(\bar{\psi}^+_{n\sigma}(t)(i\partial_t - \epsilon_n)\psi^+_{n\sigma}(t) - \bar{\psi}^-_{n\sigma}(t)(i\partial_t - \epsilon_n)\psi^-_{n\sigma}\right)\right] + \right. \\ &\quad \left. - \left(V(\bar{\psi}^+_{n\sigma}, \psi^+_{n\sigma}) - V(\bar{\psi}^-_{n\sigma}, \psi^-_{n\sigma})\right)\right\} \\ &= \iint_{-\infty}^{+\infty} dtdt' \left[\sum_n \sum_\sigma \bar{\Psi}^T_{n\sigma}(t)\hat{g}^{-1}_{n,0}(t-t')\Psi_{n\sigma}(t')\right] - \int_{-\infty}^{+\infty} dt\, V(\bar{\psi}^1_{n\sigma}, \psi^1_{n\sigma}, \bar{\psi}^2_{n\sigma}, \psi^2_{n\sigma}) \end{aligned} \tag{2.21}$$

where as before $\Psi = \begin{pmatrix}\psi^1 & \psi^2\end{pmatrix}^T$ and $\bar{\Psi}^T = \begin{pmatrix}\bar{\psi}^1 & \bar{\psi}^2\end{pmatrix}$ with $\psi^{1(2)} = (\psi^+ \pm \psi^-)/\sqrt{2}$ and $\bar{\psi}^{1(2)} = (\bar{\psi}^+ \mp \bar{\psi}^-)/\sqrt{2}$. The matrices $\hat{g}_{n,0}^{-1}$ represent the inverse of the Green's function related to the decoupled and non-interacting quantum dot, in the frequency domain they have the same structure seen in the case of leads, that is:

$$\hat{g}_{n,0}^{-1}(w) = \begin{pmatrix} w - \epsilon_n + i0^+ & 2i0^+ F_0(w) \\ 0 & w - \epsilon_n - i0^+ \end{pmatrix} \tag{2.22}$$

Note that F_0 corresponds to the distribution function describing the quantum dot at the initial instant (in thermodynamic equilibrium), when it is decoupled from leads and without interaction terms; as has been written previously, it is not necessary to know the initial state of the dot. As will be shown, after the interaction and coupling terms are activated (and the steady state is reached), the relative contributions of self-energy make F_0 completely negligible (since it is coupled to an infinitesimal 0^+). Note that the generic nature of F_0 (since the initial temperature of the system and its chemical potential are not fixed) is not a problem for the inverse of the retarded and advanced components, in fact they depend only on the spectrum of the Hamiltonian and not on the state of the system (remind that they are associated with a non-interacting fermions system). The last component of the total action that is missing to be calculated is the one associated with the tunneling term; in this case, no new Grassmann variable needs to be introduced, since $\hat{H}_T$ is written as a function of the ladder operators $\hat{c}^\dagger_{\mathbf{k}\sigma a}, \hat{c}_{\mathbf{k}\sigma a}, \hat{d}^\dagger_{n\sigma}, \hat{d}_{n\sigma}$, so repeating the usual procedure:

$$\begin{aligned} S_T &= -\sum_{a=L,R}\sum_{\mathbf{k}}\sum_{n}\sum_{\sigma}\int_C dt\left[W_{\mathbf{k}a,n}(t)\bar{\chi}_{\mathbf{k}\sigma a}(t)\psi_{n\sigma}(t) + W^*_{\mathbf{k}a,n}(t)\bar{\psi}_{n\sigma}(t)\chi_{\mathbf{k}\sigma a}(t)\right] \\ &= -\sum_{a=L,R}\sum_{\mathbf{k}}\sum_{n}\sum_{\sigma}\int_{-\infty}^{+\infty} dt\left[W_{\mathbf{k}a,n}(t)\bar{X}^T_{\mathbf{k}\sigma a}(t)\Psi_{n\sigma}(t) + W^*_{\mathbf{k}a,n}(t)\bar{\Psi}^T_{n\sigma}(t)X_{\mathbf{k}\sigma a}(t)\right] \end{aligned} \tag{2.23}$$

At this point all the elements necessary to write the partition function Z of the entire system under consideration are present:

$$Z = \int D[\bar{X}, X]D[\bar{\Psi}, \Psi]e^{i(S_{leads}[\bar{X},X]+S_{dot}[\bar{\Psi},\Psi]+S_T[\bar{X},X,\bar{\Psi},\Psi])} = 1 \tag{2.24}$$

where the integration measure is:

$$D[\bar{X}, X]D[\bar{\Psi}, \Psi] \equiv \frac{1}{Tr[\hat{\rho}(t_0)]}\prod_a\prod_{\mathbf{k}}\prod_n\prod_\sigma D[\bar{X}_{\mathbf{k}\sigma a}(t), X_{\mathbf{k}\sigma a}(t)]D[\bar{\Psi}_{n\sigma}(t), \Psi_{n\sigma}(t)] \tag{2.25}$$

As explained in the first chapter, the partition function taken individually is not particularly useful since it is normalized to 1, in order to create a generating function

it is needed the introduction of external source fields in the theoretical framework, only in this way the expectation values of physical observables can be calculated. To this end it is appropriate to consider the introduction of two source fields $A_a(t)$. Specifically the total Hamiltonian $\hat{H}$ is modified by the addition of the contribution:

$$\hat{H}_A(t) = \sum_{a=L,R} A_a(t)\hat{J}_a \tag{2.26}$$

so adding two terms of interaction between such time dependent fields and the current operators $\hat{J}_a$. To this new contribution of the total Hamiltonian corresponds an action S_A that will be added to the previous ones:

$$\begin{aligned} S_A &= -\sum_{a=L,R}\int_C dt\; A_a(t)J_a(t) \\ &= -ie\sum_{a=L,R}\sum_{\mathbf{k}}\sum_{n}\sum_{\sigma}\int_C dt\; A_a(t)\Big[W_{\mathbf{k}a,n}(t)\bar{\chi}_{\mathbf{k}\sigma a}(t)\psi_{n\sigma}(t) - W^*_{\mathbf{k}a,n}(t)\bar{\psi}_{n\sigma}(t)\chi_{\mathbf{k}\sigma a}(t)\Big] \end{aligned} \tag{2.27}$$

As explained in Section (1.3), it is advisable to switch to the cl, q formalism, this will make the calculation of the expectation value of the currents $\hat{J}_a$ easier. Starting as usual from the formalism $\pm$:

$$\begin{aligned} S_A &= -\sum_{a=L,R}\int_{-\infty}^{+\infty} dt\; [A^+_a(t)J^+_a(t) - A^-_a(t)J^-_a(t)] \\ &= -2\sum_{a=L,R}\int_{-\infty}^{+\infty} dt\; [A^{cl}_a(t)J^q_a(t) + A^q_a(t)J^{cl}_a(t)] \end{aligned} \tag{2.28}$$

where the classical and quantum components are defined in the following ways $J^{cl,q} = (J^+ \pm J^-)/2$ and $A^{cl,q} = (A^+ \pm A^-)/2$. As can seen easily, in the last line of the equation above the classical and quantum components of the source fields and current are mixed; it must be remembered that after the quantum components are set to zero, the relative classical components represent respectively the physical currents and the external physical potentials (equal in the two branches). Since the external voltage is already included in $\hat{H}$, precisely in the tunneling matrix elements of $\hat{H}_T$, the physical system does not need an additional external field, therefore it imposes $A^{cl}_a = 0$. The quantum component A^q_a, even if completely fictitious, is a mathematical artifact that for the moment is considered non-zero, indeed it is necessary in order to perform a functional derivative of the generating function with respect to it; after this operation it will be set to zero and therefore will not affect the dynamics of the system. Now it is of interest to express S_A as a function of A^q_a and the fields used to describe the other components of the total action,

$\bar{X}_{\mathbf{k}\sigma a}, X_{\mathbf{k}\sigma a}, \bar{\Psi}_{n\sigma}, \Psi_{n\sigma}$. This will make it easier to develop the calculations needed to find the expectation value of the currents. Starting from the general expression, after the imposition $A_a^{cl} = 0$:

$$
\begin{aligned}
S_A &= -2 \sum_{a=L,R} \int_{-\infty}^{+\infty} dt\ A_a^q(t) J_a^{cl}(t) \\
&= -\sum_{a=L,R} \int_{-\infty}^{+\infty} dt\ A_a^q(t)(J_a^+(t) + J_a^-(t)) \\
&= -ie \sum_{a=L,R} \sum_{\mathbf{k}} \sum_{n} \sum_{\sigma} \int_C dt\ A_a^q(t) \Big[W_{\mathbf{k}a,n}(t) \bar{\chi}^+_{\mathbf{k}\sigma a}(t) \psi^+_{n\sigma}(t) - W^*_{\mathbf{k}a,n}(t) \bar{\psi}^+_{n\sigma}(t) \chi^+_{\mathbf{k}\sigma a}(t) + \\
&\quad + W_{\mathbf{k}a,n}(t) \bar{\chi}^-_{\mathbf{k}\sigma a}(t) \psi^-_{n\sigma}(t) - W^*_{\mathbf{k}a,n}(t) \bar{\psi}^-_{n\sigma}(t) \chi^-_{\mathbf{k}\sigma a}(t) \Big] \\
&= -ie \sum_{a=L,R} \sum_{\mathbf{k}} \sum_{n} \sum_{\sigma} \int_{-\infty}^{+\infty} dt\ A_a^q(t) \Big\{ \frac{1}{2} W_{\mathbf{k}a,n}(t) \Big[(\bar{\chi}^+_{\mathbf{k}\sigma a}(t) - \bar{\chi}^-_{\mathbf{k}\sigma a}(t))(\psi^+_{n\sigma}(t) - \psi^-_{n\sigma}(t)) + \\
&\quad + (\bar{\chi}^+_{\mathbf{k}\sigma a}(t) + \bar{\chi}^-_{\mathbf{k}\sigma a}(t))(\psi^+_{n\sigma}(t) + \psi^-_{n\sigma}(t)) \Big] - \frac{1}{2} W^*_{\mathbf{k}a,n}(t) \Big[(\bar{\psi}^+_{n\sigma}(t) - \bar{\psi}^-_{n\sigma}(t))(\chi^+_{\mathbf{k}\sigma a}(t) - \chi^-_{\mathbf{k}\sigma a}(t)) + \\
&\quad + (\bar{\psi}^+_{n\sigma}(t) + \bar{\psi}^-_{n\sigma}(t))(\chi^+_{\mathbf{k}\sigma a}(t) + \chi^-_{\mathbf{k}\sigma a}(t)) \Big] \Big\} \\
&= -ie \sum_{a=L,R} \sum_{\mathbf{k}} \sum_{n} \sum_{\sigma} \int_{-\infty}^{+\infty} dt\ A_a^q(t) \Big\{ W_{\mathbf{k}a,n}(t) \Big[\bar{\chi}^1_{\mathbf{k}\sigma a}(t) \psi^2_{n\sigma}(t) + \bar{\chi}^2_{\mathbf{k}\sigma a}(t) \psi^1_{n\sigma}(t) \Big] + \\
&\quad - W^*_{\mathbf{k}a,n}(t) \Big[\bar{\psi}^1_{n\sigma}(t) \chi^2_{\mathbf{k}\sigma a}(t) + \bar{\psi}^2_{n\sigma}(t) \chi^1_{\mathbf{k}\sigma a}(t) \Big] \Big\} \\
&= -ie \sum_{a=L,R} \sum_{\mathbf{k}} \sum_{n} \sum_{\sigma} \int_{-\infty}^{+\infty} dt\ A_a^q(t) \Big[W_{\mathbf{k}a,n}(t) \bar{X}^T_{\mathbf{k}\sigma a}(t) \hat{\tau}_x \Psi_{n\sigma}(t) - W^*_{\mathbf{k}a,n}(t) \bar{\Psi}^T_{n\sigma}(t) \hat{\tau}_x X_{\mathbf{k}\sigma a}(t) \Big]
\end{aligned}
\tag{2.29}
$$

where $\hat{\tau}_x$ is the first Pauli matrix that is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Adding the action S_A to the total action $S = S_{leads} + S_{dot} + S_T$, the following generating function is defined:

$$
Z[A_a^q] = \int D[\bar{X}, X] D[\bar{\Psi}, \Psi] e^{i(S[\bar{X}, X, \bar{\Psi}, \Psi] + S_A[A_a^q, \bar{X}, X, \bar{\Psi}, \Psi])} \tag{2.30}
$$

which corresponds to the partition function if A_a^q is set to zero. The introduction of S_A, with its related generating function $Z[A_a^q]$, was done solely for the purpose of finding a simple expression for the expectation value of the two physical currents [20], namely:

$$
I_a = \langle \hat{J}_a \rangle_{ness} = \langle J_a^{cl}(t) \rangle = \left. \frac{i \delta Z[A_a^q]}{2 \delta A_a^q(t)} \right|_{A_a^q = 0, J_a^q = 0} \tag{2.31}
$$

where J_a^q is set to zero only for formal clarity, in practice this operation has already been carried out a priori with the initial equality $A_a^{cl} = 0$. Before performing this

functional derivative it is advisable to integrate out the degrees of freedom of the leads in the generating function $Z[A_a^q]$ and thus to obtain the effective action S_{eff}, indeed we are interested in the study of transport properties of the dot and therefore it is a priority to calculate its effective Green's functions. The logical process is as follows:

$$\begin{aligned} Z[A_a^q] &= \int D[\bar{X}, X] D[\bar{\Psi}, \Psi] e^{i(S_{leads}[\bar{X},X]+S_{dot}[\bar{\Psi},\Psi]+S_T[\bar{X},X,\bar{\Psi},\Psi]+S_A[A_a^q,\bar{X},X,\bar{\Psi},\Psi])} \\ &= \int D[\bar{\Psi}, \Psi] e^{iS_{dot}[\bar{\Psi},\Psi]} \int D[\bar{X}, X] e^{i(S_{leads}[\bar{X},X]+S_T[\bar{X},X,\bar{\Psi},\Psi]+S_A[A_a^q,\bar{X},X,\bar{\Psi},\Psi])} \\ &= \int D[\bar{\Psi}, \Psi] e^{i(S_{dot}[\bar{\Psi},\Psi]+S_x[A_a^q,\bar{\Psi},\Psi])} = \int D[\bar{\Psi}, \Psi] e^{iS_{eff}[A_a^q,\bar{\Psi},\Psi]} \end{aligned} \tag{2.32}$$

To perform the integration over $D[\bar{X}, X]$, the best method is to exploit the structure of the exponent $(S_{leads} + S_T + S_A)$ and then carry out a series of Gaussian integrals, one for each triplet $(a, \mathbf{k}, \sigma)$; indeed, the exponent is:

$$\begin{aligned} &S_{leads} + S_T + S_A = \\ &= \sum_{a=L,R} \sum_{\mathbf{k}} \sum_{\sigma} \Bigg\{ \iint_{-\infty}^{+\infty} dt dt' \; \bar{X}^T_{\mathbf{k}\sigma a}(t) \hat{g}^{-1}_{\mathbf{k}a}(t-t') X_{\mathbf{k}\sigma a}(t') - \int_{-\infty}^{+\infty} dt \Bigg[\bar{X}^T_{\mathbf{k}\sigma a}(t) \times \\ &\left(\sum_n W_{\mathbf{k}a,n}(t)[1 + ieA_a^q(t)\hat{\tau}_x]\Psi_{n\sigma}(t) \right) + \left(\sum_n W^*_{\mathbf{k}a,n}(t)\bar{\Psi}^T_{n\sigma}(t)[1 - ieA_a^q(t)\hat{\tau}_x] \right) X_{\mathbf{k}\sigma a}(t) \Bigg] \Bigg\} \\ &\equiv \sum_{a=L,R} \sum_{\mathbf{k}} \sum_{\sigma} S_{LTA(a,\mathbf{k},\sigma)} \end{aligned} \tag{2.33}$$

Now focus on the integration over $D[\bar{X}, X]$:

$$\begin{aligned} &\int D[\bar{X}, X] e^{i(S_{leads}+S_T+S_A)} = \\ &= \frac{1}{Tr[\hat{\rho}_{leads}(t_0)]} \int \prod_a \prod_{\mathbf{k}} \prod_{\sigma} D[\bar{X}_{\mathbf{k}\sigma a}(t), X_{\mathbf{k}\sigma a}(t)] exp\left(i \sum_a \sum_{\mathbf{k}} \sum_{\sigma} S_{LTA(a,\mathbf{k},\sigma)} \right) \\ &= \frac{1}{Tr[\hat{\rho}_{leads}(t_0)]} \prod_a \prod_{\mathbf{k}} \prod_{\sigma} \int D[\bar{X}_{\mathbf{k}\sigma a}(t), X_{\mathbf{k}\sigma a}(t)] \; exp[iS_{LTA(a,\mathbf{k},\sigma)}] \\ &= \frac{1}{Tr[\hat{\rho}_{leads}(t_0)]} \prod_a \prod_{\mathbf{k}} \prod_{\sigma} det[i\hat{g}^{-1}_{\mathbf{k}a}] exp[iS_{x(a,\mathbf{k},\sigma)}] = \prod_a \prod_{\mathbf{k}} \prod_{\sigma} exp[iS_{x(a,\mathbf{k},\sigma)}] = exp(iS_x) \end{aligned} \tag{2.34}$$

where

$$S_{x(a,\mathbf{k},\sigma)} = -\iint_{-\infty}^{+\infty} dtdt' \left[\sum_n W^*_{\mathbf{k}a,n}(t)\bar{\Psi}^T_{n\sigma}(t)[1 - ieA^q_a(t)\hat{\tau}_x]\right]\hat{g}_{\mathbf{k}a}(t-t')\times \left[\sum_m W_{\mathbf{k}a,m}(t')[1 + ieA^q_a(t')\hat{\tau}_x]\Psi_{m\sigma}(t')\right] \tag{2.35}$$

It should be noted that in the steps of the formula (2.34), the transformation from the exponential of a series to a product of an exponential has often been carried out; in general this property is not valid if the exponent contains Grassmann variables, but in the situation shown it is valid since in $S_{LTA(a,\mathbf{k},\sigma)}$ there are only products between two Grassmann fields (and these products commute each other). By summing up those contributions, for each possible triplet configuration $(a, \mathbf{k}, \sigma)$, gives:

$$\begin{aligned} S_x &= \sum_{a=L,R}\sum_{\mathbf{k}}\sum_{\sigma} S_{x(a,\mathbf{k},\sigma)} \\ &= -\sum_{a,\mathbf{k},\sigma}\sum_{n,m}\iint_{-\infty}^{+\infty} dtdt'\ \bar{\Psi}^T_{n\sigma}(t)[1 - ieA^q_a(t)\hat{\tau}_x]W^*_{\mathbf{k}a,n}(t)\hat{g}_{\mathbf{k}a}(t-t')W_{\mathbf{k}a,m}(t')\times \\ &\quad [1 + ieA^q_a(t')\hat{\tau}_x]\Psi_{m\sigma}(t') \\ &= -\sum_{a,\sigma}\sum_{n,m}\iint_{-\infty}^{+\infty} dtdt'\ \bar{\Psi}^T_{n\sigma}(t)[1 - ieA^q_a(t)\hat{\tau}_x]\hat{\Sigma}_{a,nm}(t,t')[1 + ieA^q_a(t')\hat{\tau}_x]\Psi_{m\sigma}(t') \end{aligned} \tag{2.36}$$

where

$$\hat{\Sigma}_{a,nm}(t,t') \equiv \sum_{\mathbf{k}} W^*_{\mathbf{k}a,n}(t)\hat{g}_{\mathbf{k}a}(t-t')W_{\mathbf{k}a,m}(t') \tag{2.37}$$

correspond to the self-energy contribution associated with the coupling of the quantum dot with the lead-a, given by the presence of the tunneling term $\hat{H}_T$. Indeed, note that considering the case of dot without interactions and setting $A^q_a = 0$, the effective action is:

$$S_{eff} = \iint_{-\infty}^{+\infty} dtdt'\ \sum_{n,m}\sum_{\sigma}\bar{\Psi}^T_{n\sigma}(t)\left[\delta_{nm}\hat{g}^{-1}_{n,0}(t-t') - \sum_a \hat{\Sigma}_{a,nm}(t,t')\right]\Psi_{m\sigma}(t') \tag{2.38}$$

finding exactly the typical structure of the Dyson equation. Now remember that the time dependence of the tunneling matrix elements is described by the equation $W_{\mathbf{k}a,n}(t) = W_{\mathbf{k}a,n}e^{-i\mu_a t}$, consequently the contribution of self-energy described above gains from these elements the time dependent factor $e^{i\mu_a(t-t')}$. This term is easily tractable if we move to the formalism in the frequency/energy domain, indeed in such a case $e^{i\mu_a(t-t')}$ performs an energy shift operation in $\hat{g}_{\mathbf{k}a}(\omega)$ of exactly μ_a, thus

obtaining

$$\hat{\Sigma}_{a,nm}(\omega) = \sum_{\mathbf{k}} W^*_{\mathbf{k}a,n} \hat{g}_{\mathbf{k}a}(\omega - \mu_a) W_{\mathbf{k}a,m} \tag{2.39}$$

$\hat{\Sigma}_{a,nm}(\omega)$ is the self-energy contribution resulting from the coupling between the quantum dot and the lead-a, with the latter being a reservoir such couple is in thermodynamic equilibrium at steady state (in the hypothetical system where there is not the second lead) and therefore for the fluctuation–dissipation theorem it is true that:

$$\Sigma^K_{a,nm}(w) = F_a(\omega - \mu_a)[\Sigma^R_{a,nm}(w) - \Sigma^A_{a,nm}(w)] \ . \tag{2.40}$$

The stationary current I_a described by the equation (2.31) is now calculated. Expressed as a function of the Green's functions and self-energies presented in the previous pages it is given by:

$$\begin{aligned}
I_a &= \left.\frac{i\delta Z[A^q_a]}{2\delta A^q_a(t)}\right|_{A^q_a=0} = \left.\frac{i\delta}{2\delta A^q_a(t)} \int D[\bar{\Psi},\Psi] e^{iS_{eff}[A^q_a,\bar{\Psi},\Psi]}\right|_{A^q_a=0} \\
&= \frac{i}{2}\int D[\bar{\Psi},\Psi](-i)\sum_{n,m}\sum_{\sigma}\Bigg\{\iint_{-\infty}^{+\infty} dt_1 dt_2 \Bigg[\bar{\Psi}^T_{n\sigma}(t_1)[-ie\delta(t_1-t)\hat{\tau}_x]\hat{\Sigma}_{a,nm}(t_1,t_2)\Psi_{m\sigma}(t_2) + \\
&\quad + \bar{\Psi}^T_{n\sigma}(t_1)\hat{\Sigma}_{a,nm}(t_1,t_2)[+ie\delta(t_2-t)\hat{\tau}_x]\Psi_{m\sigma}(t_2)\Bigg]\Bigg\} e^{iS_{eff}[A^q_a,\bar{\Psi},\Psi]\big|_{A^q_a=0}} \\
&= \frac{ie}{2}\int D[\bar{\Psi},\Psi]\sum_{n,m}\sum_{\sigma}\Bigg\{\int_{-\infty}^{+\infty} dt_1 \bar{\Psi}^T_{n\sigma}(t_1)\hat{\Sigma}_{a,nm}(t_1,t)\hat{\tau}_x\Psi_{m\sigma}(t) + \\
&\quad - \int_{-\infty}^{+\infty} dt_2 \bar{\Psi}^T_{n\sigma}(t)\hat{\tau}_x\hat{\Sigma}_{a,nm}(t,t_2)\Psi_{m\sigma}(t_2)\Bigg\} e^{iS_{eff}[A^q_a,\bar{\Psi},\Psi]\big|_{A^q_a=0}} \\
&= \frac{ie}{2}\int D[\bar{\Psi},\Psi]\sum_{n,m}\sum_{\sigma}\Bigg\{\int_{-\infty}^{+\infty} dt_1 Tr[\hat{\Sigma}_{a,nm}(t_1,t)\hat{\tau}_x\Psi_{m\sigma}(t)\bar{\Psi}^T_{n\sigma}(t_1)] + \\
&\quad - \int_{-\infty}^{+\infty} dt_2 Tr[\hat{\tau}_x\hat{\Sigma}_{a,nm}(t,t_2)\Psi_{m\sigma}(t_2)\bar{\Psi}^T_{n\sigma}(t)]\Bigg\} e^{iS_{eff}[A^q_a,\bar{\Psi},\Psi]\big|_{A^q_a=0}} \\
&= \frac{ie}{2}\sum_{n,m}\sum_{\sigma}\Bigg\{\int_{-\infty}^{+\infty} dt_1 Tr\Bigg[\hat{\Sigma}_{a,nm}(t_1,t)\hat{\tau}_x\int D[\bar{\Psi},\Psi]\Psi_{m\sigma}(t)\bar{\Psi}^T_{n\sigma}(t_1) e^{iS_{eff}[A^q_a,\bar{\Psi},\Psi]\big|_{A^q_a=0}}\Bigg] + \\
&\quad - \int_{-\infty}^{+\infty} dt_2 Tr\Bigg[\hat{\tau}_x\hat{\Sigma}_{a,nm}(t,t_2)\int D[\bar{\Psi},\Psi]\Psi_{m\sigma}(t_2)\bar{\Psi}^T_{n\sigma}(t) e^{iS_{eff}[A^q_a,\bar{\Psi},\Psi]\big|_{A^q_a=0}}\Bigg]\Bigg\} \\
&= \frac{e}{2}\sum_{n,m}\sum_{\sigma}\Bigg\{\int_{-\infty}^{+\infty} dt_1 Tr\Big[\hat{\Sigma}_{a,nm}(t_1,t)\hat{\tau}_x\hat{G}_{mn}(t,t_1)\Big] - \int_{-\infty}^{+\infty} dt_2 Tr\Big[\hat{\tau}_x\hat{\Sigma}_{a,nm}(t,t_2)\hat{G}_{mn}(t_2,t)\Big]\Bigg\} \\
&= \frac{e}{2}\sum_{n,m}\sum_{\sigma}\Bigg\{\int_{-\infty}^{+\infty} \frac{dw}{2\pi} Tr\Big[\hat{\Sigma}_{a,nm}(w)\hat{\tau}_x\hat{G}_{mn}(w)\Big] - \int_{-\infty}^{+\infty} \frac{dw}{2\pi} Tr\Big[\hat{\tau}_x\hat{\Sigma}_{a,nm}(w)\hat{G}_{mn}(w)\Big]\Bigg\} \\
&= e\sum_{n,m}\int_{-\infty}^{+\infty} \frac{dw}{2\pi} Tr\Big\{\Big[\hat{\Sigma}_{a,nm}(\omega),\hat{\tau}_x\Big]\hat{G}_{mn}(\omega)\Big\}
\end{aligned} \tag{2.41}$$

where $\hat{G}_{mn}$ are the full or dressed Green's functions of the quantum dots that store inside them the information of the interactions in the dot and of the coupling with the reservoirs. Precisely, they are defined as $\hat{G}_{nm} = -i\langle \Psi_{n\sigma} \bar{\Psi}^T_{m\sigma} \rangle_{S_{eff[A^q_a=0]}}$. To move from the third last line to the penultimate line, the properties of a NESS configuration have been exploited. Indeed both the self-energies and the exact Green's functions depend solely on the temporal differences and consequently, by Fourier transformation, the temporal dependence has been removed from the t parameter. Now it proceeds with the explicit calculation of the matrix product and the resulting trace, in this way we can express the currents as a function of the Green's functions and the distribution functions:

$$Tr\left\{\left[\hat{\Sigma}_{a,nm}(\omega), \hat{\tau}_x\right]\hat{G}_{mn}(\omega)\right\} = Tr\left\{\left[\begin{pmatrix}\Sigma^R_{a,nm}(\omega) & \Sigma^K_{a,nm}(\omega)\\ 0 & \Sigma^A_{a,nm}(\omega)\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right]\hat{G}_{mn}(\omega)\right\} =$$

$$= Tr\left\{\begin{pmatrix}\Sigma^K_{a,nm}(\omega) & \Sigma^R_{a,nm}(\omega) - \Sigma^A_{a,nm}(\omega)\\ \Sigma^A_{a,nm}(\omega) - \Sigma^R_{a,nm}(\omega) & -\Sigma^K_{a,nm}(\omega)\end{pmatrix}\begin{pmatrix}G^R_{mn}(\omega) & G^K_{mn}(\omega)\\ 0 & G^A_{mn}(\omega)\end{pmatrix}\right\}$$

$$\begin{aligned}
&= \Sigma^K_{a,nm}(\omega)(G^R_{mn}(\omega) - G^A_{mn}(\omega)) + (\Sigma^A_{a,nm}(\omega) - \Sigma^R_{a,nm}(\omega))G^K_{mn}(\omega)\\
&= F_a(\omega-\mu_a)[\Sigma^R_{a,nm}(w) - \Sigma^A_{a,nm}(w)](G^R_{mn}(\omega) - G^A_{mn}(\omega)) - (\Sigma^R_{a,nm}(\omega) - \Sigma^A_{a,nm}(\omega))G^K_{mn}(\omega)\\
&= [\Sigma^A_{a,nm}(w) - \Sigma^R_{a,nm}(w)]\{G^K_{mn}(\omega) - F_a(\omega-\mu_a)(G^R_{mn}(\omega) - G^A_{mn}(\omega))\}
\end{aligned} \tag{2.42}$$

Focusing now on the term in the first square bracket:

$$\begin{aligned}
\Sigma^A_{a,nm}(w) - \Sigma^R_{a,nm}(w) &= \left(\sum_{\mathbf{k}} W^*_{\mathbf{k}a,n} g^A_{\mathbf{k}a}(\omega-\mu_a) W_{\mathbf{k}a,m}\right) - \left(\sum_{\mathbf{k}} W^*_{\mathbf{k}a,n} g^R_{\mathbf{k}a}(\omega-\mu_a) W_{\mathbf{k}a,m}\right)\\
&= \sum_{\mathbf{k}} W^*_{\mathbf{k}a,n}\left(g^A_{\mathbf{k}a}(\omega-\mu_a) - g^R_{\mathbf{k}a}(\omega-\mu_a)\right)W_{\mathbf{k}a,m}\\
&= 2\pi i \sum_{\mathbf{k}} W^*_{\mathbf{k}a,n}\delta(w - w_{\mathbf{k}a} - \mu_a) W_{\mathbf{k}a,m} = i\Gamma^a_{nm}(w-\mu_a)
\end{aligned} \tag{2.43}$$

where the couplings $\Gamma^a_{nm}(w) \equiv 2\pi \sum_{\mathbf{k}} W^*_{\mathbf{k}a,n}\delta(w - w_{\mathbf{k}a}) W_{\mathbf{k}a,m}$ have been introduced. These are useful indicators to understand the energy scales related to the coupling between the quantum dot and the leads. Now all the necessary tools, to express current terms, are available (the presence of $\hbar$ is reintroduced):

$$I_a = \frac{ie}{h}\sum_{n,m}\int_{-\infty}^{+\infty} dw\, \Gamma^a_{nm}(w-\mu_a)\left\{G^K_{mn}(\omega) - F_a(\omega-\mu_a)\left[G^R_{mn}(\omega) - G^A_{mn}(\omega)\right]\right\} \tag{2.44}$$

As expected, it is easy to verify how, under thermodynamic equilibrium conditions (leads at the same temperature and chemical potential), the two currents are equal and null $I_L = I_R = 0$, since the term in the above curly braces is zero. The configuration of interest of this thesis is on the other hand that of the NESS, and in such a case one has $I_L = -I_R$, since if it were not so, there would be no stationarity, this means that the current passing from lead-L to the quantum dot is equal to the current passing from the latter to lead-R. All this leads quite naturally to the definition of the continuous current passing through the quantum dot in a NESS configuration of the LCR system is given by:

$$I = \frac{I_L - I_R}{2} = \frac{ie}{2h} \sum_{n,m} \int_{-\infty}^{+\infty} dw \Bigg[\Big(\Gamma^L_{nm}(w - eV) - \Gamma^R_{nm}(w) \Big) G^K_{mn}(\omega) + \\ - \Big(F_L(\omega - eV)\Gamma^L_{nm}(w - eV) - F_R(\omega)\Gamma^R_{nm}(w) \Big) [G^R_{mn}(\omega) - G^A_{mn}(\omega)] \Bigg] \tag{2.45}$$

In literature this equation is known as the *Meir-Wingreen-Jauho formula* for the current [32, 33]. It expresses how the electron current between two leads, passing through a quantum dot, is a function of the exact Green's functions of the dot, and also of the coupling matrices $\Gamma^a_{nm}(w)$ and of the leads' distribution function. It is important to note that this formula is valid only in the approximation for which the electrons in the leads are not interacting; also it is thanks to this approximation that the formula is valid for any type of quantum dot. As will be shown in the next pages, in the case where the dot has no interaction terms, the *Meir-Wingreen-Jauho* formula translates into the known *Landauer-Büttiker* formula [34]. Returning to the equation (2.45), note that the current is not zero even when there is no variation of chemical potential between the leads, the important thing is the existence of a temperature variation between the two. So this formula describes not only electric currents, but also thermoelectric currents (*Seebeck effect*).

One topic that has not been explored so far is the interaction term V in the quantum dot and the impact of its presence on the exact Green's functions. The Green's functions of the interacting and decoupled quantum dot are identified with the symbols $\hat{g}_{nm}$, thanks to the Dyson equation they are related to the Green's functions of the non-interacting and decoupled dot by the following equation:

$$\hat{g}^{-1}_{nm}(t, t') = \delta_{nm}\hat{g}^{-1}_{n,0}(t, t') - \Sigma_{dot,nm}(t, t') \tag{2.46}$$

where $\Sigma_{dot,nm}$ are the self-energy contributions related to all the interactions inside the uncoupled dot. Attention should be paid to the fact that, even in the space of frequencies, $\hat{g}^{-1}_{nm}(w)$ is not the inverse of the Green's function $\hat{g}_{nm}(w)$, indeed $\hat{g}^{-1}_{nm}(w)$

is the (n, m)-element of the inverse of the matrix with elements $\hat{g}_{nm}(w)$. Using this theoretical framework one can obtain a simple expression for the terms of total self-energy containing information both from the interactions inside the dot and from the couplings with leads:

$$\hat{\Sigma}_{nm} = \hat{\Sigma}_{dot,nm} + \hat{\Sigma}_{L,nm} + \hat{\Sigma}_{R,nm} \tag{2.47}$$

Now we have all the elements necessary to present the *Keldysh-Dyson equation*, thanks to which the Green's functions of the interacting and coupled quantum dot can be calculated:

$$\hat{G}_{nm}(w) = \hat{g}_{nm}(w) + \sum_{a=L,R} \sum_{i,j} \hat{g}_{ni}(w)\hat{\Sigma}_{a,ij}(w)\hat{G}_{jm}(w) \tag{2.48}$$

whose corresponding *Dyson* equation is:

$$\hat{G}^{-1}_{nm}(w) = \delta_{nm}\hat{g}^{-1}_{n,0}(w) - \Sigma_{dot,nm}(w) - \hat{\Sigma}_{L,nm}(w) - \hat{\Sigma}_{R,nm}(w) \tag{2.49}$$

Also in this case it is necessary to pay attention to the fact that the (2×2) Keldysh matrix $\hat{G}^{-1}_{nm}(w)$ is not the inverse of the (2×2) Keldysh matrix $\hat{G}_{nm}(w)$.

2.2.2 Single Level quantum dot

In this section the quantities, shown in the previous section, are calculated in the specific case where the dot is characterized by a single energy level ϵ_0 and there is no interaction term inside it ($\hat{\Sigma}_{dot,nm} = 0$), that is the simplest configuration that can exist in a quantum dot [31]. It is also assumed that the tunneling matrix elements are $\mathbf{k}$-independent, so $W_{\mathbf{k}a,0} = W_{a,0} \equiv W_a$. This configuration allows for the study of transport properties in a "protected environment", in this way, it is easier to present some key concepts which will also be present in the more complicated configurations which will be presented in the following sections of the thesis. In such a situation there is only one exact Green's function which is indicated by $\hat{G}_{00}(w)$; its retarded and advanced components $\hat{G}^{R(A)}_{00}(w)$ are easily calculable because for them the *Keldysh-Dyson equation* translates into an equation of identical structure:

$$\begin{aligned} G^{R(A)}_{00}(w) &= g^{R(A)}_{00}(w) + \sum_{a=L,R} g^{R(A)}_{00}(w)\Sigma^{R(A)}_{a,00}(w)G^{R(A)}_{00}(w) = \frac{1}{[g^{R(A)}_{00}(w)]^{-1} - \sum_{a=L,R}\Sigma^{R(A)}_{a,00}(w)} \\ &= \frac{1}{w - \epsilon_0 - \sum_{a=L,R}\sum_{\mathbf{k}} W^*_a g^{R(A)}_{\mathbf{k}a}(\omega - \mu_a)W_a} \\ &= \frac{1}{w - \epsilon_0 - |W_L|^2 g^{R(A)}_L(\omega - eV) - |W_R|^2 g^{R(A)}_R(\omega)} \end{aligned} \tag{2.50}$$

where $g_a^{R(A)}(w) \equiv \sum_{\mathbf{k}} g_{\mathbf{k}a}^{R(A)}(\omega)$. Instead, in order to calculate the Keldysh component $G_{00}^K(w)$, the following equation should be exploited, which can be easily obtained from the *Dyson* equation:

$$\begin{aligned}
G_{00}^K(w) &= G_{00}^R(w)\Big(\sum_{a=L,R}\Sigma_{a,00}^K(w)\Big)G_{00}^A(w) = \Big(\sum_{a=L,R}\Sigma_{a,00}^K(w)\Big)|G_{00}^R(w)|^2 \\
&= \Big(F_L(\omega-eV)[\Sigma_{L,00}^R(w)-\Sigma_{L,00}^A(w)] + F_R(\omega)[\Sigma_{R,00}^R(w)-\Sigma_{R,00}^A(w)]\Big)|G_{00}^R(w)|^2 \\
&= \Big(F_L(\omega-eV)|W_L|^2\Big[\sum_{\mathbf{k}}(-2\pi i)\delta(w-w_{\mathbf{k}a}-eV)\Big] + F_R(\omega)|W_R|^2\times \\
&\quad \Big[\sum_{\mathbf{k}}(-2\pi i)\delta(w-w_{\mathbf{k}a})\Big]\Big)|G_{00}^R(w)|^2 \\
&= -2\pi i\Big(F_L(\omega-eV)|W_L|^2\nu_L(w-eV) + F_R(\omega)|W_R|^2\nu_R(w)\Big)|G_{00}^R(w)|^2
\end{aligned} \tag{2.51}$$

where at the end is defined the *density of states* of the lead-a, given by $\nu_a(w) = \sum_{\mathbf{k}}\delta(w-w_{\mathbf{k}a})$. Having now the three components of the exact Green's function, it is possible to calculate the stationary average current passing through the quantum dot in the configuration considered, taking the *Meir-Wingreen-Jauho formula*, that is equation (2.45), one has:

$$\begin{aligned}
I &= \frac{ie}{2\hbar}\int_{-\infty}^{+\infty} dw\Big[\Big(\Gamma_{00}^L(w-eV)-\Gamma_{00}^R(w)\Big)G_{00}^K(\omega)+ \\
&\quad -\Big(F_L(\omega-eV)\Gamma_{00}^L(w-eV)-F_R(\omega)\Gamma_{00}^R(w)\Big)[G_{00}^R(\omega)-G_{00}^A(\omega)]\Big] \\
&= \frac{ie}{2\hbar}\int_{-\infty}^{+\infty} dw\Big\{\Big[2\pi|W_L|^2\nu_L(w-eV)-2\pi|W_R|^2\nu_R(w)\Big]\Big[-2\pi i\Big(F_L(\omega-eV)|W_L|^2\times \\
&\quad \nu_L(w-eV)+F_R(\omega)|W_R|^2\nu_R(w)\Big)|G_{00}^R(w)|^2\Big] - \Big[F_L(\omega-eV)2\pi|W_L|^2\nu_L(w-eV)+ \\
&\quad -F_R(\omega)2\pi|W_R|^2\nu_R(w)\Big][G_{00}^R(\omega)-G_{00}^A(\omega)]\Big\}
\end{aligned} \tag{2.52}$$

Focus now on the contribution of the last square bracket $[G_{00}^R(\omega)-G_{00}^A(\omega)]$. Being that one is the complex conjugate of the other, if we define d as the denominator of G_{00}^R expressed in the equation (2.50), one finds that $[G_{00}^R(\omega)-G_{00}^A(\omega)] = (d^*(w)-d(w))|G_{00}^R(w)|^2$, where $(d^*(w)-d(w))$ is equal to:

$$\begin{aligned}
&d^*(w)-d(w) = \\
&= [w-\epsilon_0-|W_L|^2g_L^R(\omega-eV)-|W_R|^2g_R^R(\omega)]^* - [w-\epsilon_0-|W_L|^2g_L^R(\omega-eV)-|W_R|^2g_R^R(\omega)]
\end{aligned}$$

$$
\begin{aligned}
&= [w - \epsilon_0 - |W_L|^2 g_L^A(\omega - eV) - |W_R|^2 g_R^A(\omega)] - [w - \epsilon_0 - |W_L|^2 g_L^R(\omega - eV) - |W_R|^2 g_R^R(\omega)] \\
&= |W_L|^2 [g_L^R(\omega - eV) - g_L^A(\omega - eV)] + |W_R|^2 [g_R^R(\omega) - g_R^A(\omega)] \\
&= |W_L|^2 \sum_{\mathbf{k}} [g_{\mathbf{k}L}^R(\omega - eV) - g_{\mathbf{k}L}^A(\omega - eV)] + |W_R|^2 \sum_{\mathbf{k}} [g_{\mathbf{k}R}^R(\omega) - g_{\mathbf{k}R}^A(\omega)] \\
&= -2\pi i \Bigg(|W_L|^2 \sum_{\mathbf{k}} \delta(w - w_{\mathbf{k}L} - eV) + |W_R|^2 \sum_{\mathbf{k}} \delta(w - w_{\mathbf{k}R}) \Bigg) \\
&= -2\pi i \Big(|W_L|^2 \nu_L(w - eV) + |W_R|^2 \nu_L(w) \Big)
\end{aligned}
\tag{2.53}
$$

Returning to the stationary current:

$$
\begin{aligned}
I = \frac{ie}{2h} \int_{-\infty}^{+\infty} dw \Bigg\{ (-4\pi^2 i) \Bigg[& |W_L|^4 F_L(w - eV) \nu_L^2(w - eV) + |W_L|^2 |W_R|^2 F_R(w) \times \\
& \nu_L(w - eV) \nu_R(w) - |W_L|^2 |W_R|^2 F_L(w - eV) \nu_L(w - eV) \nu_R(w) - |W_R|^4 F_R(w) \times \\
& \nu_R^2(w) \Bigg] |G_{00}^R(w)|^2 - (-4\pi^2 i) \Bigg[|W_L|^4 F_L(w - eV) \nu_L^2(w - eV) + |W_L|^2 |W_R|^2 \times \\
& F_L(w - eV) \nu_L(w - eV) \nu_R(w) - |W_L|^2 |W_R|^2 F_R(w) \nu_L(w - eV) \nu_R(w) - |W_R|^4 \times \\
& F_R(w) \nu_R^2(w) \Bigg] |G_{00}^R(w)|^2 \Bigg\} \\
= \frac{ie(-4\pi^2 i)}{2h} \int_{-\infty}^{+\infty} dw \Bigg(& 2|W_L|^2 |W_R|^2 F_R(w) \nu_L(w - eV) \nu_R(w) - 2|W_L|^2 |W_R|^2 F_L(w - eV) \times \\
& \nu_L(w - eV) \nu_R(w) \Bigg) |G_{00}^R(w)|^2 \\
= \frac{8\pi^2 e}{h} \int_{-\infty}^{+\infty} dw \, & [f_L(w - eV) - f_R(w)] |W_L|^2 |W_R|^2 \nu_L(w - eV) \nu_R(w) |G_{00}^R(w)|^2 \\
= \frac{2e}{h} \int_{-\infty}^{+\infty} dw \, & [f_L(w - eV) - f_R(w)] \mathcal{T}(w, V)
\end{aligned}
\tag{2.54}
$$

where $\mathcal{T}(w, V)$ is the so called *transmission coefficient* defined as follows:

$$
\mathcal{T}(w, V) = \frac{4\pi^2 |W_L|^2 |W_R|^2 \nu_L(w - eV) \nu_R(w)}{|w - \epsilon_0 - |W_L|^2 g_L^R(\omega - eV) - |W_R|^2 g_R^R(\omega)|^2} \tag{2.55}
$$

The equation (2.54) found above, which expresses the stationary current between two electrodes passing through a single level quantum dot, takes the known form of the *Landauer formula* [34] where the *transmission coefficient* $\mathcal{T}(w, V)$ describes the probability of transmission as a function of energy and voltage. One way to study the properties of the system effectively is to consider it in the context of the *linear response regime*. Indeed, considering V and $\Delta T = T_L - T_R$ as small parameters, by means of series expansions it is possible to calculate very good ap-

proximations of physical quantities. For example, the Fermi function associated to the lead-L can be expanded and at the first order it is equal to:

$$f_L(w - eV) \simeq f_R(w) - \frac{df_L}{dw}(w)\left[eV + w\frac{\Delta T}{T_L}\right] \tag{2.56}$$

Using this expression for the $f_L(w - eV)$ present in (2.54), we have that in the linear response regime the stationary current can be expressed as $I = GV + L\Delta T$, where G is defined as the *conductance* and L the *thermoelectric coefficient*. In order to simplify the calculations, in this regime it is appropriate to use the *wide-band approximation* [33] when we are near the Fermi energy; this approximation corresponds to the neglect of the real part of $\hat{g}_{\mathbf{k}a}(w)$ components and to consider a constant density of states ν_a evaluated at the Fermi energy. Consequently, applying these approximations, one finds:

$$\begin{aligned} g_a^R(w) &= \sum_{\mathbf{k}} g_{\mathbf{k}a}^R(\omega) = i\sum_{\mathbf{k}} Im[g_{\mathbf{k}a}^R(\omega)] = \frac{i}{2}\sum_{\mathbf{k}} Im[g_{\mathbf{k}a}^R(\omega) - g_{\mathbf{k}a}^A(\omega)] \\ &= \frac{i}{2}\sum_{\mathbf{k}} Im[-2\pi i\delta(w - w_{\mathbf{k}a})] = -i\pi\nu_a(w) \equiv -i\pi\nu_a \end{aligned} \tag{2.57}$$

Therefore the conductance G is equal to:

$$\begin{aligned} G &= -\frac{2e^2}{h}\int_{-\infty}^{+\infty} dw \left[\frac{df_L}{dw}(w)\right]\mathcal{T}(w, V) \\ &= -\frac{2e^2}{\hbar}\int_{-\infty}^{+\infty} dw \left[\frac{df_L}{dw}(w)\right]\frac{4\pi^2|W_L|^2|W_R|^2\nu_L\nu_R}{|w - \epsilon_0 + i\pi|W_L|^2\nu_L + i\pi|W_R|^2\nu_R|^2} \\ &= -\frac{2e^2}{h}\int_{-\infty}^{+\infty} dw \left[\frac{df_L}{dw}(w)\right]\frac{\Gamma^L\Gamma^R}{(w - \epsilon_0)^2 + (\frac{\Gamma^L}{2} + \frac{\Gamma^R}{2})^2} \end{aligned} \tag{2.58}$$

where the constant couplings Γ^a have been defined as $\Gamma^a \equiv 2\pi|W_a|^2\nu_a$. The presence of the f_L derivative in the above formula allows an easy calculation of G within the limit of $T \to 0$, indeed in that case the derivative corresponds to a negative Dirac delta, so:

$$G = \frac{2e^2}{h}\frac{\Gamma^L\Gamma^R}{\epsilon_0^2 + (\frac{\Gamma^L}{2} + \frac{\Gamma^R}{2})^2} \tag{2.59}$$

Both in the above solution and in that related to a generic T, a resonance phenomenon is visible known as *resonant tunneling*, where at $\epsilon_0 = 0$, so at the Fermi energy, the conductance is maximized (zero corresponds to both Fermi energies of the leads because we are assuming very small V and because all energy quantities are scaled by a factor μ_R). If we also consider the particular case in which there is a symmetry of coupling, so $\Gamma^L = \Gamma^R$, at $T = 0$ the maximum value of G corresponds to the *conductance quantum*, namely $G_0 = 2e^2/h$, which as can be seen is

completely independent of the coupling terms of the dot with leads. Now passing to the thermoelectric coefficient, it corresponds to:

$$\begin{aligned} L &= -\frac{2ek_B}{h}\int_{-\infty}^{+\infty} dw\left[\frac{df_L}{dw}(w)\right]\beta w\mathcal{T}(w,V) \\ &= -\frac{2ek_B}{h}\int_{-\infty}^{+\infty} dw\left[\frac{df_L}{dw}(w)\right]\frac{\beta w\Gamma^L\Gamma^R}{(w-\epsilon_0)^2+(\frac{\Gamma^L}{2}+\frac{\Gamma^R}{2})^2} \end{aligned} \tag{2.60}$$

Noting that w is an odd function, the derivative of the Fermi function is an even function, for $\epsilon_0 = 0$ the denominator is an even function and therefore the thermoelectric coefficient is zero at any temperature (this is therefore true if the energy associated with the single level corresponds to the Fermi energy and thus to the right chemical potential).

2.2.3 Two Levels quantum dot

Now consider the previous case of a non-interacting dot, but instead of a single energy level there are two discrete levels, that are ϵ_1 and ϵ_2. As before, it is assumed that the tunneling matrix elements are $\mathbf{k}$-independent, so $W_{\mathbf{k}a,i} \equiv W_{ai}$. Thus, in our case, the elements are W_{L1}, W_{L2}, W_{R1} and W_{R2}. The relative exact Green's functions are regulated by the *Keldysh-Dyson equation* (2.48), where $i,j = 1,2$. As always, the retarded and advanced components of the exact Green's functions satisfy an equation with the same structure of (2.48), so for the retarded components one has:

$$G^R_{nm}(w) = \delta_{nm}g^R_{n,0}(w) + \sum_{a=L,R}\sum_{i,j=1}^{2}\delta_{ni}g^R_{n,0}(w)\Sigma^R_{a,ij}(w)G^R_{jm}(w) \tag{2.61}$$

where using the usual notation $\Sigma^R_{a,ij}(w) = W^*_{ai}W_{aj}g^R_a(w-\mu_a)$. From the equation presented above, the four retarded Green's functions must satisfy the following equations:

$$\begin{aligned} G^R_{11}(w) &= g^R_{1,0}(w)\left[1+\sum_{a=L,R}\left(W^*_{a1}W_{a1}g^R_a(w-\mu_a)G^R_{11}(w)+W^*_{a1}W_{a2}g^R_a(w-\mu_a)G^R_{21}(w)\right)\right] \\ G^R_{12}(w) &= g^R_{1,0}(w)\sum_{a=L,R}\left(W^*_{a1}W_{a1}g^R_a(w-\mu_a)G^R_{12}(w)+W^*_{a1}W_{a2}g^R_a(w-\mu_a)G^R_{22}(w)\right) \\ G^R_{21}(w) &= g^R_{2,0}(w)\sum_{a=L,R}\left(W^*_{a2}W_{a1}g^R_a(w-\mu_a)G^R_{11}(w)+W^*_{a2}W_{a2}g^R_a(w-\mu_a)G^R_{21}(w)\right) \\ G^R_{22}(w) &= g^R_{2,0}(w)\left[1+\sum_{a=L,R}\left(W^*_{a2}W_{a1}g^R_a(w-\mu_a)G^R_{12}(w)+W^*_{a2}W_{a2}g^R_a(w-\mu_a)G^R_{22}(w)\right)\right] \end{aligned} \tag{2.62}$$

In order to find the $G^R_{nm}(w)$ explicitly, it is necessary to solve the system composed of these four equations. As an example of this, the steps for calculating $G^R_{22}(w)$ are reported below. The logical steps are elementary, first from the second equation $G^R_{12}(w)$ is found as a function of $G^R_{22}(w)$, then this is inserted into the fourth equation. So, by reformulating the second and the fourth equations:

$$\begin{aligned} G^R_{12}(w) &= \frac{G^R_{22}(w)\sum_a W^*_{a1}W_{a2}g^R_a(w-\mu_a)}{[g^R_{1,0}(w)]^{-1}-\sum_a W^*_{a1}W_{a1}g^R_a(w-\mu_a)} \\ G^R_{22}(w) &= \frac{1+G^R_{12}(w)\sum_a W^*_{a2}W_{a1}g^R_a(w-\mu_a)}{[g^R_{2,0}(w)]^{-1}-\sum_a W^*_{a2}W_{a2}g^R_a(w-\mu_a)} \end{aligned} \tag{2.63}$$

By making the substitution:

$$\begin{aligned} G^R_{22}(w) &= \frac{1+[\sum_a W^*_{a2}W_{a1}g^R_a(w-\mu_a)]\frac{G^R_{22}(w)\sum_a W^*_{a1}W_{a2}g^R_a(w-\mu_a)}{[g^R_{1,0}(w)]^{-1}-\sum_a W^*_{a1}W_{a1}g^R_a(w-\mu_a)}}{[g^R_{2,0}(w)]^{-1}-\sum_a W^*_{a2}W_{a2}g^R_a(w-\mu_a)} \\ &= \frac{[g^R_{1,0}(w)]^{-1}-\sum_a W^*_{a1}W_{a1}g^R_a(w-\mu_a)}{D^R(w,V)} \end{aligned} \tag{2.64}$$

where the denominator $D^R(w,V)$ corresponds to:

$$\begin{aligned} D^R(w,V) &= \left[[g^R_{1,0}(w)]^{-1}-\sum_a W^*_{a1}W_{a1}g^R_a(w-\mu_a)\right]\left[[g^R_{2,0}(w)]^{-1}-\sum_a W^*_{a2}W_{a2}g^R_a(w-\mu_a)\right]+ \\ &\quad -\left[\sum_a W^*_{a2}W_{a1}g^R_a(w-\mu_a)\right]\left[\sum_a W^*_{a1}W_{a2}g^R_a(w-\mu_a)\right] \\ &= \Big[w-\epsilon_1-|W_{L1}|^2g^R_L(w-eV)-|W_{R1}|^2g^R_R(w)\Big]\Big[w-\epsilon_2-|W_{L2}|^2g^R_L(w-eV)+ \\ &\quad -|W_{R2}|^2g^R_R(w)\Big]-|W_{L1}|^2|W_{L2}|^2|g^R_L(w-eV)|^2-|W_{R1}|^2|W_{R2}|^2|g^R_R(w)|^2+ \\ &\quad -2Re\Big[W^*_{L1}W_{L2}W_{R1}W^*_{R2}\Big]g^R_L(w-eV)g^R_R(w) \end{aligned} \tag{2.65}$$

Obviously, later on $G^R_{12}(w)$ is found by replacing the solution (2.64) in the second equation of eq(2.62). Repeating this procedure one obtains all the retarded compo-

nents, namely:

$$\begin{aligned} G^R_{11}(w) &= \frac{w-\epsilon_2-|W_{L2}|^2 g^R_L(w-eV)-|W_{R2}|^2 g^R_R(w)}{D^R(w,V)} \\ G^R_{12}(w) &= \frac{W^*_{L1}W_{L2}g^R_L(w-eV)+W^*_{R1}W_{R2}g^R_R(w)}{D^R(w,V)} \\ G^R_{21}(w) &= \frac{W^*_{L2}W_{L1}g^R_L(w-eV)+W^*_{R2}W_{R1}g^R_R(w)}{D^R(w,V)} \\ G^R_{22}(w) &= \frac{w-\epsilon_1-|W_{L1}|^2 g^R_L(w-eV)-|W_{R1}|^2 g^R_R(w)}{D^R(w,V)} \end{aligned} \tag{2.66}$$

The usual relation applies to the K components, for which it is:

$$G^K_{nm}(w) = \sum_{i,j=1}^{2} G^R_{ni}(w)\Big(\sum_{a=L,R}\Sigma^K_{a,ij}(w)\Big)G^A_{jm}(w) \tag{2.67}$$

where

$$\begin{aligned} \sum_{a=L,R}\Sigma^K_{a,ij}(w) &= F_L(\omega-eV)[\Sigma^R_{L,nm}(w)-\Sigma^A_{L,nm}(w)]+F_R(\omega)[\Sigma^R_{R,nm}(w)-\Sigma^A_{R,nm}(w)] \\ &= -2\pi i\Big[F_L(\omega-eV)W^*_{Li}W_{Lj}\nu_L(w-eV)+F_R(\omega)W^*_{Ri}W_{Rj}\nu_R(w)\Big] \end{aligned} \tag{2.68}$$

For the calculation of the stationary current it is considered to be in the regime of validity of the *wide-band approximation* (so at energy scales near the two Fermi energies, which must be approximately of the same magnitude), consequently g^R_a is a constant and equal to $-i\pi\nu_a$ and therefore also the Γ^a_{nm} parameters are constant and equal to $2\pi W^*_{an}W_{am}\nu_a = (\Gamma^a_{mn})^*$. It is also assumed that $\Gamma^a_{11}\Gamma^a_{22} = |\Gamma^a_{12}|^2$ and that we are in the presence of a symmetric junction, so $\Gamma^L_{mn} = \Gamma^R_{mn} \equiv \Gamma_{mn}$. The MWJ formula (2.45) is therefore taken up again and it can be easily seen that the contribution linked to G^K_{mn} is zero for the symmetry conditions explained above, so the stationary current corresponds to:

$$I = \frac{ie}{\hbar}\sum_{n,m=1}^{2}\int_{-\infty}^{+\infty} dw\,[f_L(\omega-eV)-f_R(\omega)]\Gamma_{nm}[G^R_{mn}(\omega)-G^A_{mn}(\omega)] \tag{2.69}$$

The focus is therefore on the element $\sum_{n,m=1}^{2}\Gamma_{nm}[G^R_{mn}(\omega)-G^A_{mn}(\omega)]$, in which the components R and A are complex conjugates of each other. The first step is to calculate the denominator $D^R(w)$ of the retarded components using all the

assumptions written above, so:

$$\begin{aligned} D^R(w) &= \Big[w-\epsilon_1+|W_{L1}|^2 i\pi\nu_L+|W_{R1}|^2 i\pi\nu_R\Big]\Big[w-\epsilon_2+|W_{L2}|^2 i\pi\nu_R+|W_{R2}|^2 i\pi\nu_R\Big]+ \\ &\quad +|W_{L1}|^2|W_{L2}|^2\pi^2\nu_L^2+|W_{R1}|^2|W_{R2}|^2\pi^2\nu_R^2+2Re\Big[W_{L1}^* W_{L2} W_{R1} W_{R2}^*\Big]\pi^2\nu_L\nu_R \\ &= [w-\epsilon_1+i\Gamma_{11}][w-\epsilon_2+i\Gamma_{22}]+\frac{\Gamma_{11}\Gamma_{22}}{4}+\frac{\Gamma_{11}\Gamma_{22}}{4}+\frac{1}{4}\Big(\Gamma_{12}(\Gamma_{12})^*+(\Gamma_{12})^*\Gamma_{12}\Big) \\ &= [w-\epsilon_1+i\Gamma_{11}][w-\epsilon_2+i\Gamma_{22}]+\Gamma_{11}\Gamma_{22} \\ &= (w-\epsilon_1)(w-\epsilon_2)+i[\Gamma_{11}(w-\epsilon_2)+\Gamma_{22}(w-\epsilon_1)] \end{aligned} \tag{2.70}$$

Now, any subtraction of the type $(G^R_{mn}-G^A_{mn})$ has the structure $(\frac{N}{D}-\frac{N^*}{D^*})=\frac{2iIm(ND^*)}{|D|^2}$, therefore for the case $m=1$ and $n=1$:

$$G^R_{11}(w)-G^A_{11}(w)=\frac{-2i\Gamma_{11}(w-\epsilon_2)^2}{[(w-\epsilon_1)(w-\epsilon_2)]^2+[\Gamma_{11}(w-\epsilon_2)+\Gamma_{22}(w-\epsilon_1)]^2} \tag{2.71}$$

and for symmetry reasons the numerator of $G^R_{22}(w)-G^A_{22}(w)$ is equal to $-2i\Gamma_{22}(w-\epsilon_1)^2$ (obviously the denominator is equal to that of the previous case). Instead for the mixed case:

$$G^R_{12}(w)-G^A_{12}(w)=\frac{-2i\Gamma_{12}(w-\epsilon_1)(w-\epsilon_2)}{[(w-\epsilon_1)(w-\epsilon_2)]^2+[\Gamma_{11}(w-\epsilon_2)+\Gamma_{22}(w-\epsilon_1)]^2} \tag{2.72}$$

and, as before, for symmetry reasons the numerator of $G^R_{21}(w)-G^A_{21}(w)$ is equal to $-2i\Gamma_{21}(w-\epsilon_1)(w-\epsilon_2)$. Finally, each of these subtractions must be multiplied by the corresponding factor Γ_{nm} and therefore performing these operations (and using the properties $\Gamma_{11}\Gamma_{22}=|\Gamma_{12}|^2$ and $\Gamma_{nm}=(\Gamma_{mn})^*$) it is found that the stationary current is described by the *Landauer* formula (the last line of the equation (2.54)) where the transmission coefficient corresponds to:

$$\begin{aligned} \mathcal{T}(w) &= \frac{[\Gamma_{11}(w-\epsilon_2)+\Gamma_{22}(w-\epsilon_1)]^2}{[(w-\epsilon_1)(w-\epsilon_2)]^2+[\Gamma_{11}(w-\epsilon_2)+\Gamma_{22}(w-\epsilon_1)]^2} \\ &= \frac{[\Gamma(w-\epsilon)-\frac{1}{2}\Delta\epsilon\Delta\Gamma]^2}{[(w-\epsilon)^2-\frac{1}{4}(\Delta\epsilon)^2]^2+[\Gamma(w-\epsilon)-\frac{1}{2}\Delta\epsilon\Delta\Gamma]^2} \end{aligned} \tag{2.73}$$

In the second line of the equation above $\mathcal{T}(w)$ is written according to new coupling and energy parameters, that is $\Gamma=\Gamma_{11}+\Gamma_{22}$, $\Delta\Gamma=\Gamma_{11}-\Gamma_{22}$ and $\epsilon=\frac{1}{2}(\epsilon_1+\epsilon_2)$, $\Delta\epsilon=\epsilon_2-\epsilon_1$. Note that in the case of $\Delta\epsilon=0$ and therefore when there is a degenerate single energy level with degeneration 2, the corresponding transmission

coefficient is equal to that obtained in the case of the single level with an effective coupling $\Gamma = \Gamma_{11} + \Gamma_{22}$. As in the previous section, in the linear response regime it is possible to calculate two important quantities: the conductance G and the thermoelectric coefficient L. For the first, if $T_R = 0$, by exploiting the presence of Dirac delta it is possible to obtain a simple and compact expression:

$$G = \frac{2e^2}{h} \frac{[\Gamma_{11}\epsilon_2 + \Gamma_{22}\epsilon_1]^2}{\epsilon_1^2\epsilon_2^2 + [\Gamma_{11}\epsilon_2 + \Gamma_{22}\epsilon_1]^2} = \frac{2e^2}{h} \frac{[\Gamma\epsilon + \frac{1}{2}\Delta\epsilon\Delta\Gamma]^2}{[\epsilon^2 - \frac{1}{4}(\Delta\epsilon)^2]^2 + [\Gamma\epsilon + \frac{1}{2}\Delta\epsilon\Delta\Gamma]^2} \tag{2.74}$$

It is immediately noticeable that G is maximized in the case where one of the two energy levels corresponds to the Fermi energy, in this case $G = G_0 = 2e^2/h$. As regards the thermoelectric coefficient, we know that it corresponds to

$$L = -\frac{2ek_B}{h} \int_{-\infty}^{+\infty} dw \left[\frac{df_L}{dw}(w)\right] \beta w \mathcal{T}(w, V) \tag{2.75}$$

It can be easily verified that it is zero in the specific conditions for which $\Delta\Gamma = \epsilon = 0$. In a configuration where only $\Delta\Gamma$ is set to zero, the thermoelectric coefficient L behaves similarly to what was observed in the single-level quantum dot. In fact, it is an odd function respect the ϵ parameter and this property allows for controlling the direction of the thermoelectric current by adjusting the mean energy ϵ.

Chapter 3

Sachdev–Ye–Kitaev models

As mentioned in the previous sections, the aim of this thesis is to study transport properties of a specific physical system, namely the spinful complex fermion version of the Sachdev-Ye-Kitaev model. Historically, its first version, devised by Kitaev [1] as a generalization of Sachdev–Ye model [2, 35], is based on Majorana fermions, while it was later extended to the case of complex fermions. The SYK models [1, 2, 36, 37] are quantum many-body systems based on strongly coupled fermions (which may occupy N different sites) characterized by random all-to-all interactions (quenched disorder), that show strongly chaotic dynamics [9, 38]. In the limit of large N and low temperature T, the model is solvable and is nearly conformally invariant. In these limits, the ratio of entropy to N is different from zero and the entropy is described by the so called Bekenstein-Hawking entropy [13, 35, 39], this is related to the fact that the SYK models are connected holographically to black holes with AdS_2 horizons [17]. In addition, being solvable models of holographic duality, they are linked to 2-dimensional quantum gravity [5, 6, 40, 41]. The fact that, under certain conditions, they are fully solvable systems, makes such models excellent building blocks for creating theories which effectively describe metals; in particular, SYK models show peculiar properties of non-Fermi liquids [42], for example, they show a resistivity that is linear with respect to the temperature (at low temperature). In the considered limits, the models do not have quasiparticles, therefore they fall into the category of the so-called *strange metals*, and this property makes these models an excellent instrument for the study of transport phenomena [17, 42, 43] and non-equilibrium dynamics and entanglement [10, 11, 12]. Finally, it is essential to say that in recent years several experimental realizations of the SYK models have been proposed [44, 45, 46, 47] and quantum simulations of their behaviour have been carried out [48, 49, 50].

In this chapter the study of three SYK models under thermodynamic equilibrium conditions will be presented: the initial version based on Majorana fermions, secondly the spinless version with complex fermions and third for the case of spinful

complex fermions (which be deepened in the next chapter under non-equilibrium conditions). Being in conditions of thermodynamic equilibrium it is convenient to use the Matsubara formalism and coherent state path integral formalism. In this theoretical framework the study of their properties resulting from their belonging to the category of quenched disordered systems will be carried out. Then the effect generated by the disorder average operation, that is the obtainment of an effective action will be considered. By studying the saddle point solutions of the effective action we will find the Dyson–Schwinger equations of the model which are solvable in the low frequency and strong coupling limits (with the presence of a conformal symmetry and, in the complex cases, also of a $U(1)$ gauge symmetry). Thus, the analytical expressions of the exact Green's functions associated with thermodynamic equilibrium conditions will be found (always within the limits considered). Finally, since the objective of this thesis is the study of the spinful complex SYK in non-equilibrium conditions, the operations of transition to the Keldysh formalism will be carried out.

3.1 SYK - Majorana fermion version

In this first treatment, the SYK model is studied in its version based on Majorana fermions interacting by means of quartic interaction terms [1, 6, 51, 52, 53]. The model has a quantum-mechanical nature and describes a system of N Majorana fermions ($N \gg 1$) with all-to-all random couplings involving 4 of these fermions at a time. The Hamiltonian describing such a system is:

$$\hat{H} = \frac{1}{4!} \sum_{i,j,k,l=1}^{N} J_{ijkl}\, \hat{\chi}_i \hat{\chi}_j \hat{\chi}_k \hat{\chi}_l \tag{3.1}$$

where $\hat{\chi}_i$ ($i = 1, ..., N$) are the Majorana fermion operators who have the property of being Hermitian and they satisfy the fermionic anticommutation rule, so:

$$\hat{\chi}_i = \hat{\chi}_i^{\dagger} \qquad\qquad \{\hat{\chi}_i, \hat{\chi}_j\} = \delta_{ij} \tag{3.2}$$

The nature of Majorana fermions is considerably different from that of complex fermions, indeed being hermitian they describe quasiparticles that are their own antiparticle; moreover as a consequence of this property it no longer makes sense to think of them in terms of creation and annihilation operators (being equal) and also falls the concept of occupation number operator. The model described is based on a $(0+1)$-dimensional space, that is a space with zero spatial dimension and one temporal dimension, therefore all j sites belong to the same point; alternatively, the N sites can be conceived as immersed in a larger dimensional space and interacting by means of infinite-range interactions. The coupling constants J_{ijkl} are real independent variables and they are distributed randomly, namely drawn from a random Gaussian distribution. Specifically, they are described by the probability density function [54]:

$$P(J_{ijkl}) \propto exp\left(-\frac{N^3 J_{ijkl}^2}{12 J^2}\right) \tag{3.3}$$

for each J_{ijkl}. This Gaussian distribution ensures that each coupling term has mean and variance equal to:

$$\overline{J_{ijkl}} = 0 \qquad\qquad \overline{J_{ijkl}^2} = \frac{3! J^2}{N^3} \tag{3.4}$$

Please note that no summation over i, j, k, l was implicitly assumed in the previous expressions. The variance of the coupling is characterized by the parameter J, equal for all couplings, which is an energy dimension-one constant, while the numerical factor and N^3 were simply introduced to simplify the calculations in the treatment of the large-N limit. So J takes on an important physical meaning as it regulates

the energy scale of all the interactions. It is important to note that to make an average, like those presented in (3.103), a large number of replicas of the same system must be created, where each of them can have different coupling configurations, but where the total system, consisting of all the replicas, follows the chosen probability distributions. Coming back to J_{ijkl}, note that the couplings are completely antisymmetric under the exchange of the indices i, j, k, l, this is a direct consequence of the anticommutation relations that the Majorana operators satisfy; this property reduces the number of independent non-zero couplings.

This section deals with the study of the SYK model in thermal equilibrium at a temperature T, consequently we will use the Matsubara formalism for such treatment since under conditions of thermal equilibrium it is much easier and lighter than the Keldysh formalism. However, there will be a focus on the link between the two formalisms, as in the following sections we will deal with the study of the SYK model in non-equilibrium conditions. In this theoretical framework, a role of great importance is played by the partition function Z (be careful not to mistake it with the Z present in Keldysh theory, they are two different objects) which, in the coherent states path integral formalism, corresponds to $Z = \int D\chi \, exp(-S[\chi])$, where the action $S[\chi]$ is equal to:

$$S[\chi] = \int d\tau \left[\frac{1}{2} \sum_{i=1}^{N} \chi_i(\tau)\dot{\chi}_i(\tau) + \frac{1}{4!} \sum_{i,j,k,l=1}^{N} J_{ijkl} \, \chi_i(\tau)\chi_j(\tau)\chi_k(\tau)\chi_l(\tau) \right] \tag{3.5}$$

where $\dot{\chi}_i = \frac{d\chi_i}{d\tau}$ and $D\chi = \prod_{i=1}^{N} D\chi_i(\tau)$ (with antiperiodic boundary conditions). In this theoretical framework τ indicates the Euclidean time (imaginary time formalism), it is linked to the real time formalism through the *Wick rotation* transformation $\tau = it$, where t is the Lorentzian time. Note that, by performing a dimensional analysis of the action's components written above, the Majorana fermion fields are dimensionless (in the 1-dim case). Another important clarification should be made on the integration interval of τ, in fact there are two different types, although they are closely related. The first case has an integration interval equal to $(-\infty, +\infty)$ and it is related to the zero temperature system, while the second is $[-\frac{\beta}{2}, +\frac{\beta}{2})$ and is related to the thermal state of the system at the finite temperature T, where $\beta = \frac{1}{T}$ (setting $k_B = 1$). The choice of such intervals allows one to have a simple map to pass from one set to another that preserves the order between times: $\tau_{inf} = tan(\frac{\pi \tau_{fin}}{\beta})$.

3.1.1 Free theory

We know that much of the physical information of an interacting system is contained in the corresponding Green's functions. It is therefore appropriate to start with the study of thermal two-point correlation functions in imaginary time related to the corresponding free theory. In the free theory, since there are no interaction terms, the Hamiltonian of the system is $\hat{H}_0 = 0$, consequently the Majorana operators are time independent even in Heisenberg picture:

$$\hat{\chi}_i(\tau) = e^{\tau \hat{H}_0} \hat{\chi} e^{-\tau \hat{H}_0} = \hat{\chi}(0) = \hat{\chi} \tag{3.6}$$

It is convenient to define the respective two-point correlators in imaginary time as:

$$G^{vac}_{0,ij}(\tau) \equiv + \langle 0 | \, \mathbb{T}[\hat{\chi}_i(\tau)\hat{\chi}_j(0)] \, |0\rangle \qquad\qquad G^{\beta}_{0,ij}(\tau) \equiv + \langle \mathbb{T}[\hat{\chi}_i(\tau)\hat{\chi}_j(0)] \rangle_\beta \tag{3.7}$$

where the first is concerned with the zero temperature case, while the second with the finite ones. Also $|0\rangle$ corresponds to the vacuum state in the free theory and the symbol $\langle ... \rangle_\beta$ describes the usual thermal average operation, so $\langle ... \rangle_\beta = \frac{Tr[exp(-\beta \hat{H}_0)...]}{Tr[exp(-\beta \hat{H}_0)]}$. Both correlators described in (3.7) are defined for τ values belonging to the $[-\beta, +\beta]$ interval, but as will be shown shortly, thanks to the antisymmetry properties of the correlators it will be possible to reduce the "effective" range to $[-\frac{\beta}{2}, +\frac{\beta}{2})$. To explicitly calculate these correlators it is preferable to consider an even number $N = 2K$ and define the following complex fermion ladder operators:

$$\hat{c}_i = \frac{\hat{\chi}_{2i} - i\hat{\chi}_{2i+1}}{\sqrt{2}} \qquad\qquad \hat{c}^\dagger_i = \frac{\hat{\chi}_{2i} + i\hat{\chi}_{2i+1}}{\sqrt{2}} \tag{3.8}$$

where $i = 1, .., K$. Indeed for these operators, it can be easily demonstrated that they satisfy the usual anticommutation relations, so $\{\hat{c}_i, \hat{c}^\dagger_j\} = \delta_{ij}$ and $\{\hat{c}_i, \hat{c}_j\} = \{\hat{c}^\dagger_i, \hat{c}^\dagger_j\} = 0$. Using these creation and annihilation complex fermion operators, it is possible to define the free vacuum $|0\rangle$ as the state that is annihilated by all $\hat{c}_i$ operators, therefore such that $\hat{c}_i \, |0\rangle = 0$ for each $i = 1, .., K$. The Fock space associated with such a system therefore has a basis composed by a set of 2^K elements, which correspond to all possible configurations in which complex fermions can be distributed in the K states; these elements can be identified with the vector $\{n_i\}$ where $n_i = 0, 1$ is the occupation number of the state related to $\hat{c}^\dagger_i$ and $i = 1, .., K$. The finite temperature two-point correlators in the free theory can now be calculated

using the relations with the complex fermion operators defined above :

$$\begin{aligned} G^{\beta}_{0,ij}(\tau) &= +\langle \mathbb{T}[\hat{\chi}_i(\tau)\hat{\chi}_j(0)]\rangle_\beta = \frac{Tr[e^{-\beta\hat{H}_0}\mathbb{T}(\hat{\chi}_i(\tau)\hat{\chi}_j(0))]}{Tr[e^{-\beta\hat{H}_0}]} = \frac{Tr[\mathbb{T}(\hat{\chi}_i(\tau)\hat{\chi}_j(0))]}{Tr[\mathbb{1}]} \\ &= \Theta(\tau)\frac{Tr[\hat{\chi}_i\hat{\chi}_j]}{Tr[\mathbb{1}]} - \Theta(-\tau)\frac{Tr[\hat{\chi}_j\hat{\chi}_i]}{Tr[\mathbb{1}]} \end{aligned} \tag{3.9}$$

It is easy to demonstrate that the correlators with $|i-j| > 1$ are equal to zero, indeed in these cases the related ladder operators anticommute with each other (since they have different subscripts). So focus now on the traces for which $|i-j| \leq 1$, there are four different configurations, but they can be grouped into the two following cases:

$$\begin{aligned} &Tr[\hat{\chi}_{2i}\hat{\chi}_{2i}] = Tr[\hat{\chi}_{2i+1}\hat{\chi}_{2i+1}] = \frac{1}{2}Tr[\hat{c}_i\hat{c}_i^\dagger + \hat{c}_i^\dagger\hat{c}_i] = \frac{1}{2}Tr[\mathbb{1}] = 2^{K-1} \\ &Tr[\hat{\chi}_{2i}\hat{\chi}_{2i+1}] = -Tr[\hat{\chi}_{2i+1}\hat{\chi}_{2i}] = \frac{i}{2}Tr[\hat{c}_i\hat{c}_i^\dagger - \hat{c}_i^\dagger\hat{c}_i] \\ &\qquad = \frac{i2^{K-1}}{2}\Big[\langle 0|\,\hat{c}_i\hat{c}_i^\dagger - \hat{c}_i^\dagger\hat{c}_i\,|0\rangle + \langle 1_i|\,\hat{c}_i\hat{c}_i^\dagger - \hat{c}_i^\dagger\hat{c}_i\,|1_i\rangle\Big] = 0 \end{aligned} \tag{3.10}$$

where $|1_i\rangle \equiv \hat{c}_i^\dagger\,|0\rangle$. Consequently the only non-zero correlators are those for which $i = j$ and knowing that $Tr[\mathbb{1}] = 2^K$, one gets:

$$G^{\beta}_{0,ij}(\tau) = \langle \mathbb{T}[\hat{\chi}_i(\tau)\hat{\chi}_j(0)]\rangle_\beta = \frac{1}{2}(\Theta(\tau) - \Theta(-\tau))\delta_{ij} = \frac{1}{2}sgn(\tau)\delta_{ij} \tag{3.11}$$

This result obtained in the case of finite temperatures is also equal to the case of zero temperature, so $G^{vac}_{0,ij}(\tau) = \frac{1}{2}sgn(\tau)\delta_{ij}$. It is important to remember that $G^{\beta}_{0,ij}(\tau)$ is an antiperiodic function under the change $\tau \to \tau + \beta$, so if we wanted to expand the $G^{\beta}_{0,ij}(\tau)$ to all the possible τ, it would be equal to:

$$G^{\beta}_{0,ij}(\tau) = \frac{1}{2}sgn\left(sin\frac{\pi\tau}{\beta}\right)\delta_{ij} \tag{3.12}$$

The important thing to remember is that, whenever an integration must be made over the $[-\frac{\beta}{2}, +\frac{\beta}{2})$ interval, the two Green's functions are equal and one has $G^{\beta}_{0,ij}(\tau) = G^{vac}_{0,ij}(\tau) = \frac{1}{2}sgn(\tau)\delta_{ij}$ which is immediately identifiable as an antisymmetric function. Now that all this has been calculated, it is useful to define the averaged two-point correlator:

$$G_0(\tau) \equiv \frac{1}{N}\sum_{i=1}^{N}\langle \mathbb{T}[\hat{\chi}_i(\tau)\hat{\chi}_i(0)]\rangle = \frac{1}{2}sgn(\tau) \tag{3.13}$$

that, in the $[-\frac{\beta}{2}, +\frac{\beta}{2})$ interval, is equal for both the $T = 0$ and finite T cases (for $T = 0$ it is exact at each τ). In the frequency domain, calculating the Fourier transformation, one obtains:

$$\begin{aligned} G_0(w) &= \int_{-\infty}^{+\infty} d\tau\, e^{iw\tau} \frac{1}{2} sgn(\tau) = -\frac{1}{iw} \qquad\qquad for\ T{=}0 \\ G_0(w_n) &= \int_{-\frac{\beta}{2}}^{+\frac{\beta}{2}} d\tau\, e^{iw_n\tau} \frac{1}{2} sgn(\tau) = -\frac{1}{iw_n} \qquad\qquad for\ T \neq 0 \end{aligned} \tag{3.14}$$

where $w_n = \frac{(2n+1)\pi}{\beta}$, with $n \in Z$, are the fermionic *Matsubara frequencies*. So the two correlators in the frequency domain are exactly equal, obviously the only distinction is in the domains that in the case of $T = 0$ is continuous, while for finite T it is discrete.

Before moving on to the next section, it is important to make a comment about the partition function Z associated with the action $S[\chi]$ (eq(3.5)). Indeed it should be noted that Z is defined in terms of a functional integration which is not well defined as a consequence of the fact that $\hat{\chi}_i^2 = \frac{1}{2}$. The anticommutation rules describing Majorana fermions are such that they cannot be described by either complex numbers or Grassmann numbers in the coherent state path integral formalism. This problem can be solved. The solution is given by the eq(3.8) which means that it is necessary to define the Majorana fermions in function of the complex ones, in order to obtain an integral over Grassmann variables. By analogy to what is written at the beginning of section, the Majorana fields can be expressed as follows

$$\chi_{2i} = \frac{\psi_i + \bar{\psi}_i}{\sqrt{2}} \qquad\qquad \chi_{2i+1} = i\frac{\psi_i - \bar{\psi}_i}{\sqrt{2}} \tag{3.15}$$

with $i = 1, .., K$ and the $\psi_i, \bar{\psi}_i$ being the Grassmann fields related to complex fermions. In this way we can define the measure of integration as

$$D\chi_{2i} D\chi_{2i+1} = \left| \frac{\delta(\chi_{2i}, \chi_{2i+1})}{\delta(\psi_i, \bar{\psi}_i)} \right| D\psi_i D\bar{\psi}_i = D\psi_i D\bar{\psi}_i \tag{3.16}$$

Consequently the partition function can be rewritten as a functional integral over Grassmann variables:

$$Z = \int \prod_{i=1}^{2K} D\chi_i\, e^{-S[\chi]} = \int \prod_{i=1}^{K} D\psi_i D\bar{\psi}_i\, e^{-S[\psi, \bar{\psi}]} \tag{3.17}$$

3.1.2 Quenched disorder and Replica Trick

The SYK model is a quantum mechanical model with random coupling constants (which follow a Gaussian distribution), consequently it is by definition a disordered system, specifically it belongs to the category of *quenched disordered* systems [21, 55], where the disorder is fully described by the probability distribution function of each coupling (the same for all couplings). The SYK model disorder is of the *quenched* type as it assumes that the couplings J_{ijkl} are constant over time, or rather, are constant on the timescale over which the other *dof* (χ_i) fluctuate. This situation could apparently generate much confusion, indeed every time someone calculates a physical quantity of the SYK model (with finite N), it would depend on the specific configuration of the couplings J_{ijkl} (which for each model is fixed in time); consequently the physical observables would be a function of the specific configuration of the couplings and for each SYK model they would have different expectation values. If so, it would pose a serious problem, since our goal is to have a theory that describes a generic SYK model and not one of its specific configuration. As often happens, these problems are solved if we assume that the system considered is large enough, that for the SYK model in zero spatial dimensions one may employ the limit $N \to \infty$; in this case, many of the physical observables no longer depend on the specific configuration of the couplings of the model considered, they are the so-called *self-averaging* quantities. An extremely important physical quantity, that satisfies this property, is the free energy per site:

$$\lim_{N\to\infty} F_N(\beta, \mathbf{J}) = F_\infty(\beta) \tag{3.18}$$

where $\mathbf{J}$ denote the set of couplings J_{ijkl} for a single SYK model. This means that, in the limit $N \to \infty$, the average over the disorder of a self-averaging quantity corresponds to its $\mathbf{J}$-independent case, thus for the free energy:

$$F = -\lim_{N\to\infty} \frac{1}{\beta N} \overline{logZ[\mathbf{J}]} = -\lim_{N\to\infty} \frac{1}{\beta N} \int D\mathbf{J}\; log \int D\chi\, e^{-S[\chi,\mathbf{J}]} = F_\infty(\beta) \tag{3.19}$$

where the average over the quenched disorder of a quantity $A(\mathbf{J})$ is defined as follows:

$$\overline{A} \equiv \int D\mathbf{J}\; A(\mathbf{J}) \quad with \quad D\mathbf{J} \equiv \prod_{i<j<k<l} \sqrt{\frac{N^3}{3!J^2}} \frac{dJ_{ijkl}}{\sqrt{2\pi}}\, exp\left[-\frac{N^3}{12J^2} \sum_{i<j<k<l} J_{ijkl}^2\right] \tag{3.20}$$

This means that the distributions of *self-averaging* quantities, in the limit $N \to \infty$, are peaked around their average values with zero variance. Now, studying the expression found for F just above, we notice immediately that as a first operation the partition function must be calculated, then its logarithm must be calculated

and only at the end this logarithm can be integrated (**J** dependent) on all the possible configurations **J** (weighing them appropriately); this last operation presents considerable technical difficulties. Note that by first performing the integration over the fields χ_i and only then over **J**, the configurations **J** are fixed during the χ_i integration, this means that **J** and the fields χ_i do not fluctuate together and this is a specific property of the *quenched disordered* . It would be much simpler if the logarithm were placed near the front of the equation, before any integration sign, but this is not our case since if it was then the system in question would not be in *quenched disordered* , but in *annealed disorder*. To deal with the technical difficulties arising from integrals that have a structure like that shown in equation (3.19), there are different possible approaches, here one which is probably the simplest (at least at the calculation level) will be presented, that is the so-called *replica trick* [56]. Consider a generic power of the partition function Z, namely Z^R, if R is an integer then Z^R can be interpreted as the partition function describing a system of R identical copies of the original SYK model (all with the same configuration **J**), hence the name *replica trick*.

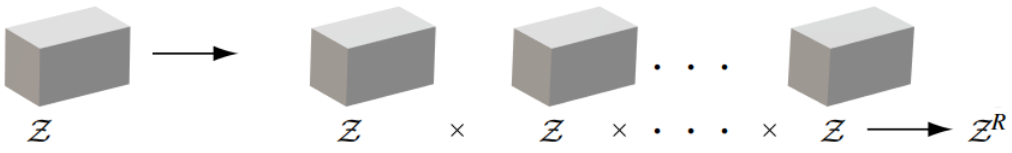


Figure 3.1: Graphical representation of the *replica trick*. A total system consisting of R identical copies of the original model is described by the partition function Z^R.

To understand one of the reasons why we are interested in the powers of Z, let's analyze the expression of the expectation value at thermal equilibrium of an observable $\hat{O}$ derived by means of an external source field V to which it is coupled:

$$\langle \hat{O} \rangle = -\frac{\delta}{\delta V} lnZ[V]\Big|_{V=0} = -\frac{\delta}{\delta V} \lim_{R\to 0} \frac{1}{R}(e^{RlnZ[V]} - 1)\Big|_{V=0} = -\frac{\delta}{\delta V} \lim_{R\to 0} \frac{1}{R} Z^R[V]\Big|_{V=0} \tag{3.21}$$

The last expression of the equation presented above therefore indicates that the expectation value of a physics observable can be expressed as a function of a power of Z and not its logarithm. This property is extremely useful at the computational level, because unlike logarithm, the power of Z is much easier to handle if expressed in the coherent state representation. So far, however, it has not yet been made clear what is the connection between the previous reasoning and the disorder average, this connection is given by the following identity:

$$\overline{logZ} = \lim_{R\to 0} \frac{1}{R} log\overline{Z^R} \tag{3.22}$$

where, if R is an integer number, one has

$$\overline{Z^R} = \int \prod_{r=1}^{R} D\chi^r \, \overline{e^{-\sum_{r=1}^{R} S[\chi^r, \mathbf{J}]}} \tag{3.23}$$

where r denotes the replica index and $D\chi^r \equiv \prod_{i=1}^{N} D\chi_i^r(\tau)$. It is important to note that, before the operation of averaging, in the total action the fields associated with different replicas do not interact with each other, so the action is *replica diagonal*. The latter expression is much easier to calculate than the corresponding logarithm. The only problem occurs in the limit $R \to 0$, in fact at the end of the calculation it is necessary to carry out the analytic continuation, but it is not guaranteed that the continuation will be analytical down to $R = 0$. For this reason, this method is called a "trick" and not a "theory", however it will be shown that in our use cases there will never be any problem with the calculation of the limit described above, therefore the *replica trick* will be perfectly valid and consistent. An equivalent expression to describe such a method can be written for a generic observable $\hat{O}$ and it is as follows:

$$\begin{aligned} \overline{\langle \hat{O} \rangle} &= \overline{\frac{1}{Z} \int D\chi O(\chi) e^{-S[\chi, \mathbf{J}]}} = \lim_{R \to 0} \overline{Z^{R-1} \int D\chi O(\chi) e^{-S[\chi, \mathbf{J}]}} \\ &= \lim_{R \to 0} \int \prod_{r=1}^{R} D\chi^r O(\chi^1) \overline{e^{-\sum_{r=1}^{R} S[\chi^r, \mathbf{J}]}} \end{aligned} \tag{3.24}$$

Note that the replica index "1" in $O(\chi^1)$ is completely arbitrary and at the end of the calculations the result must be independent from it. In the following sections it will be shown how this trick will allow us to calculate the effective action of the interacting SYK model (under disorder average) and consequently derive the *Dyson–Schwinger equation*.

3.1.3 Effective action

To calculate the disorder average of the SYK model there are two different possibilities: directly averaging the partition function, or averaging the free energy (i.e. calculating $\overline{logZ}$). With this second approach it will be necessary to exploit the replica trick, that is:

$$\overline{logZ} = \lim_{R \to 0} \frac{1}{R} log \overline{Z^R} = \lim_{R \to 0} \frac{\overline{Z^R} - 1}{R} = \lim_{R \to 0} \partial_R \overline{Z^R} \tag{3.25}$$

In the specific case of the SYK model, the two methods are equivalent and lead to the same effective action, this stems from the fact that in the replica action the

replica-nondiagonal contributions are eliminated by higher powers of $1/N$ [5, 57]. All this leads to the important relation:

$$\overline{Z^R} = (\overline{Z})^R + \mathcal{O}(1/N) \tag{3.26}$$

So consider a system consisting of R identical replicas of the same SYK model [52, 53, 58, 59, 60], all with the same configuration $\mathbf{J}$, and now we calculate the disorder average of the total partition function Z^R in terms of the χ_i^r fields, where $i = 1, .., N$ and $r = 0, .., R$.

$$\begin{aligned}
\overline{Z^R} &= \int D\mathbf{J}\; Z^R[\mathbf{J}] = \int D\mathbf{J}\; \prod_{r=1}^{R} D\chi^r\; e^{-\sum\limits_{r=1}^{R} S[\chi^r,\mathbf{J}]} \\
&= \int D\mathbf{J} \prod_{r=1}^{R} D\chi^r\; exp\Bigg[-\sum_{r=1}^{R}\int d\tau\Bigg(\frac{1}{2}\sum_{i=1}^{N}\chi_i^r(\tau)\partial_\tau\chi_i^r(\tau) + \frac{1}{4!}\sum_{i,j,k,l=1}^{N} J_{ijkl}\chi_i^r(\tau)\chi_j^r(\tau)\chi_k^r(\tau)\chi_l^r(\tau)\Bigg)\Bigg] \\
&= \int \prod_{i<j<k<l}\sqrt{\frac{N^3}{3!J^2}}\frac{dJ_{ijkl}}{\sqrt{2\pi}} \prod_{r=1}^{R} D\chi^r\; exp\Bigg[-\Bigg(\frac{N^3}{12J^2}\sum_{i<j<k<l} J_{ijkl}^2\Bigg) - \sum_{r=1}^{R}\int d\tau\Bigg(\frac{1}{2}\sum_{i=1}^{N}\chi_i^r(\tau)\partial_\tau\chi_i^r(\tau) + \\
&\quad + \frac{1}{4!}\sum_{i,j,k,l=1}^{N} J_{ijkl}\,\chi_i^r(\tau)\chi_j^r(\tau)\chi_k^r(\tau)\chi_l^r(\tau)\Bigg)\Bigg] \\
&= \int \prod_{r=1}^{R} D\chi^r exp\Bigg[-\frac{1}{2}\sum_{r=1}^{R}\sum_{i=1}^{N}\int d\tau\chi_i^r(\tau)\partial_\tau\chi_i^r(\tau) + \frac{J^2}{8N^3}\sum_{i,j,k,l=1}^{N}\Bigg(\sum_{r=1}^{R}\int d\tau\chi_i^r(\tau)\chi_j^r(\tau)\times \\
&\quad \chi_k^r(\tau)\chi_l^r(\tau)\Bigg)^2\Bigg] \\
&= \int \prod_{r=1}^{R} D\chi^r exp\Bigg[-\frac{1}{2}\sum_{r=1}^{R}\sum_{i=1}^{N}\int d\tau\chi_i^r(\tau)\partial_\tau\chi_i^r(\tau) + \frac{NJ^2}{8}\sum_{a,b=1}^{R}\int d\tau d\tau'\Bigg(\frac{1}{N}\sum_{i=1}^{N}\chi_i^a(\tau)\chi_i^b(\tau')\Bigg)^4\Bigg]
\end{aligned} \tag{3.27}$$

Now the bi-local replica collective fields $\mathcal{G}^{ab}(\tau,\tau')$ are defined as follows

$$\mathcal{G}^{ab}(\tau,\tau') \equiv \frac{1}{N}\sum_{i=1}^{N}\chi_i^a(\tau)\chi_i^b(\tau') \tag{3.28}$$

By analyzing the expression found for $\overline{Z^R}$ in equation (3.27), one can imagine that, to move towards its resolution, it is necessary to decouple the non linear terms $\mathcal{G}^4$. In order to do this, a possible strategy is to use the following identity, valid for a generic function f:

$$f(\mathcal{G}) = \int_{-\infty}^{+\infty} dx\; f(x)\delta(x-\mathcal{G}) = \frac{N}{2\pi}\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy\; f(x)e^{iNy(x-\mathcal{G})} \tag{3.29}$$

where, in our case, $f(\mathcal{G}) \equiv exp(\frac{NJ^2}{8}\mathcal{G}^4)$ and the x, y variables are defined as two-timepoint functional variables

$$x = G^{ab}(\tau, \tau') \qquad\qquad y = i\Sigma^{ab}(\tau, \tau') \tag{3.30}$$

So one has

$$exp\Big(\frac{NJ^2}{8}\sum_{a,b=1}^{R}\int d\tau d\tau' \mathcal{G}^{ab}(\tau,\tau')^4\Big) = \int \prod_{a,b=1}^{R}[DG^{ab}D\Sigma^{ab}]exp\Big\{N\sum_{a,b=1}^{R}\int_{\tau>\tau'} d\tau d\tau' \times \Big[\frac{J^2}{4}G^{ab}(\tau,\tau')^4 - \Sigma^{ab}(\tau,\tau')\Big(G^{ab}(\tau,\tau') - \mathcal{G}^{ab}(\tau,\tau')\Big)\Big]\Big\} \tag{3.31}$$

where the restriction to the domain $\tau > \tau'$ is allowed by the anti-symmetry property $\mathcal{G}^{ab}(\tau,\tau') = -\mathcal{G}^{ba}(\tau',\tau)$ and each integration measure $[DG^{ab}D\Sigma^{ab}]$ contains the factor $\frac{iN}{2\pi}$, indeed the following normalization condition is valid:

$$\int \prod_{a,b=1}^{R}[DG^{ab}D\Sigma^{ab}]exp\Big(-\frac{N}{2}\sum_{a,b=1}^{R}\int d\tau d\tau' \Sigma^{ab}(\tau,\tau')G^{ab}(\tau,\tau')\Big) = 1 \tag{3.32}$$

Now inserting the expression found in equation (3.31) into the equation (3.27), one gets:

$$\overline{Z^R} = \int \prod_{a,b=1}^{R}[DG^{ab}D\Sigma^{ab}]\prod_{r=1}^{R}D\chi^r\, exp\Big\{-\frac{1}{2}\sum_{a=1}^{R}\sum_{i=1}^{N}\int d\tau \chi_i^a(\tau)\partial_\tau\chi_i^a(\tau) + \frac{1}{2}\sum_{a,b=1}^{R}\sum_{i=1}^{N}\int d\tau d\tau' \times \Sigma^{ab}(\tau,\tau')\chi_i^a(\tau)\chi_i^b(\tau') + \frac{N}{2}\sum_{a,b=1}^{R}\int d\tau d\tau'\Big[\frac{J^2}{4}G^{ab}(\tau,\tau')^4 - \Sigma^{ab}(\tau,\tau')G^{ab}(\tau,\tau')\Big]\Big\} \tag{3.33}$$

Performing the integral over all the $\chi_i^r(\tau)$ fields one obtains the the N-th power of the Pfaffian of $[-\partial_\tau + \hat{\Sigma}]$, which can be visualized as a $R \times R$ matrix with elements $(-\delta_{ab}\partial_\tau - \Sigma^{ab})$ of indices (a, b) where each element corresponds to a matrix of indices (τ, τ'). In this way the disorder average of the total partition function corresponds to

$$\overline{Z^R} = \int \prod_{a,b=1}^{R}[DG^{ab}D\Sigma^{ab}](Pf[-\partial_\tau + \hat{\Sigma}])^N exp\Big\{\frac{N}{2}\sum_{a,b=1}^{R}\int d\tau d\tau'\Big[\frac{J^2}{4}G^{ab}(\tau,\tau')^4 + - \Sigma^{ab}(\tau,\tau')G^{ab}(\tau,\tau')\Big]\Big\} \equiv \int \prod_{a,b=1}^{R}[DG^{ab}D\Sigma^{ab}]\, e^{-S_R[G,\Sigma]} \tag{3.34}$$

where $S_R[G,\Sigma]$ is the effective action of system composed by R replicas after the disorder average, which is equal to

$$\begin{aligned} S_R[G,\Sigma] &= N\Bigg\{-ln\, Pf[-\partial_\tau+\hat{\Sigma}]+\frac{1}{2}\sum_{a,b=1}^{R}\int d\tau d\tau'\Bigg[\Sigma^{ab}(\tau,\tau')G^{ab}(\tau,\tau')-\frac{J^2}{4}G^{ab}(\tau,\tau')^4\Bigg]\Bigg\} \\ &= \frac{N}{2}\Bigg\{-Tr\, ln\Bigg[-\delta(\tau-\tau')\delta^{ab}\partial_\tau+\hat{\Sigma}^{ab}(\tau,\tau')\Bigg]+\sum_{a,b=1}^{R}\int d\tau d\tau'\Bigg[\Sigma^{ab}(\tau,\tau')G^{ab}(\tau,\tau')+ \\ &\quad -\frac{J^2}{4}G^{ab}(\tau,\tau')^4\Bigg]\Bigg\} \end{aligned} \tag{3.35}$$

where in the last row it has been chosen to show explicitly the elements of the matrix $[-\partial_\tau+\hat{\Sigma}]$ with respect to the indices a,b,τ,τ'. Now it is important to remember that the disorder average has physical correspondence only if the system is in the condition of large N (precisely in the limit $N\to\infty$) and in this limit, due to the presence of N in S_R, the partition function $\overline{Z^R}$ can be calculated by finding the saddle point solutions for the bilocal fields G^{ab},Σ^{ab}.

$$\overline{Z^R}\approx exp\Bigg(-\min_{\Sigma}\max_{G}S_R[G,\Sigma]\Bigg) \tag{3.36}$$

It can be shown [6, 37, 52, 54, 61] that within the limit of large N, the only relevant saddle point solutions are those belonging to the family of *replica-diagonal* saddle points, so:

$$G^{ab}(\tau,\tau')\equiv\delta^{ab}G(\tau,\tau') \qquad\qquad \Sigma^{ab}(\tau,\tau')\equiv\delta^{ab}\Sigma(\tau,\tau') \tag{3.37}$$

This is true only if the system under consideration does not show any spin-glass behaviour. In our case, the SYK model presents a transition to a glassy phase, but it takes place at $T_{glass}\sim Je^{-\sqrt{N}}$ [37, 52] and considering that we are working in the large N limit, this transition temperature is very low. It follows that the replica-diagonal solutions are an excellent approximation whenever we are considering a case in which N is large and $T>T_{glass}$; if we want to study the system at lower temperatures, also necessary to consider quantum fluctuations. It is important to note that these constraints still allow one to study the system at extremely low temperatures, since T_{glass} tends to zero in the limit $N\to\infty$. Now if we consider that the "relevant" (or *effective*) bilocal fields G^{ab} and Σ^{ab} have such expressions in equation (3.34), one has

$$\overline{Z^R}\approx\int\prod_{r=1}^{R}[DG^r D\Sigma^r]\, e^{-\sum_{r=1}^{R}S_{eff}[G^r,\Sigma^r]}=\Bigg[\int DG\, D\Sigma\, e^{-S_{eff}[G,\Sigma]}\Bigg]^R=\overline{Z}^R \tag{3.38}$$

with

$$S_{eff}[G,\Sigma] \equiv N\left\{-ln\,Pf[-\partial_\tau+\Sigma]+\frac{1}{2}\int d\tau d\tau'\left[\Sigma(\tau,\tau')G(\tau,\tau')-\frac{J^2}{4}G(\tau,\tau')^4\right]\right\} \tag{3.39}$$

where now ∂_τ and Σ have only time indices and the symbol $\approx$ means that the quantity is defined up to additive factors of the order $1/N^p$ with $p \geq 1$. Note that in the large N situation, the presence of the factor N in the effective action plays the role of $\hbar$ in the classical limit. Now, it is important to understand that equation (3.69) is valid only in the large N limit, indeed the exact relation is $\overline{Z^R} = (\overline{Z})^R + \mathcal{O}(1/N)$ and the replica-nondiagonal contributions are suppressed only for $N \to \infty$. In an equivalent way, the physical content of equation (3.69) can be rewritten in the following expression, obtained by considering the limit $R \to 0$:

$$\overline{lnZ} \approx ln\overline{Z} \approx -\min_{\Sigma}\max_{G} S_{eff}[G,\Sigma] \tag{3.40}$$

where the two logarithms differ by $\mathcal{O}(1/N^2)$.

3.1.4 Dyson–Schwinger equation

In this section the saddle point solutions of the disorder averaged partition function $\overline{Z}$ will now be focused on. In order to find them, we have to calculate the conditions under which the effective action $S_{eff}[G,\Sigma]$ is maximized with respect to G and minimized with respect to Σ. The quickest way to do this is to calculate the two respective differentials, so starting with the differential with respect to Σ, one gets

$$\begin{aligned}\delta_\Sigma S_{eff} &= N\left\{-\frac{1}{2}\int d\tau d\tau'\left[-G_0^{-1}+\Sigma\right]^{-1}(\tau,\tau')\delta\Sigma(\tau',\tau)+\frac{1}{2}\int d\tau d\tau' G(\tau,\tau')\delta\Sigma(\tau,\tau')\right\} \\ &= N\left\{+\frac{1}{2}\int d\tau d\tau'\left(\left[-G_0^{-1}+\Sigma\right]^{-1}(\tau,\tau')+G(\tau,\tau')\right)\delta\Sigma(\tau,\tau')\right\}\end{aligned} \tag{3.41}$$

where in the second line the property $\Sigma(\tau',\tau) = -\Sigma(\tau,\tau')$ is used and the inverse tree-level propagator is defined as:

$$G_0^{-1}(\tau,\tau') \equiv \delta(\tau-\tau')\partial_\tau \tag{3.42}$$

Consequently the minimum is obtained when Σ satisfies the following equation

$$\delta_\Sigma S_{eff} = 0 \iff \left[-G_0^{-1}+\Sigma\right]^{-1}(\tau,\tau') = -G(\tau,\tau') \tag{3.43}$$

which can be reformulated as

$$G^{-1}(\tau,\tau') = G_0^{-1}(\tau,\tau') - \Sigma(\tau,\tau') \tag{3.44}$$

In regards to the differential with respect to G, one get

$$\delta_G S_{eff} = +\frac{N}{2}\int d\tau d\tau' \Big[\Sigma(\tau,\tau') - J^2 G(\tau,\tau')^3\Big]\delta G(\tau,\tau') \tag{3.45}$$

So the the action is maximum for the G for which the following equation is true:

$$\delta_G S_{eff} = 0 \iff \Sigma(\tau,\tau') = J^2 G(\tau,\tau')^3 \tag{3.46}$$

The results just found are extremely important from a physical point of view, indeed equations (3.44) and (3.46) correspond to the *Dyson-Schwinger* equations that describe the exact Green's function of the SYK model in the case of large N, so using the mean field approximation. The "classic" derivation of the DS-equations is the so-called diagrammatic approach which is presented in [6, 53]. This means that the saddle point solutions of the bilocal fields (auxiliary variables) $G(\tau,\tau')$ and $\Sigma(\tau,\tau')$ have the physical meaning of the Euclidean Majorana fermion two-point function and the fermion self-energy respectively (it is important to stress that this is a particular case which stems from the specific model under consideration). So if we interpret the saddle point equations as relations concerning correlators, one may observe that G_0 (defined in the eq(3.42)) corresponds to the free propagator defined in eq(3.13), while the bilocal field G can be interpreted as the exact two-point correlator, that is:

$$G(\tau,\tau') \equiv \frac{1}{N}\sum_{i=1}^{N}\langle \mathbb{T}[\hat{\chi}_i(\tau)\hat{\chi}_i(\tau')]\rangle \tag{3.47}$$

where $\hat{\chi}_i(\tau) = e^{\hat{H}\tau}\hat{\chi}_i e^{-\hat{H}\tau}$ with $\hat{H}$ described in eq(3.1). Exploiting the translation symmetry of the system considered (since by assumption it is in thermal equilibrium), one finds that $G(\tau,\tau') = G(\tau-\tau') \equiv G(\tau)$ and by applying the same reasoning for the self-energy $\Sigma(\tau,\tau')$, the expression for the self-energy becomes

$$\Sigma(\tau) = J^2 G(\tau)^3 \tag{3.48}$$

while going to Fourier space the DS-equation becomes:

$$G^{-1}(w) = G_0^{-1}(w) - \Sigma(w) \qquad \rightarrow \qquad G^{-1}(w) = -iw - \Sigma(w) \tag{3.49}$$

where, in the last expression, the result described by eq(3.14) was used. From now on the symbols G, G_0, Σ will always be used to identify the correlators and the self-

energy (i.e. no longer used to identify the auxiliary variables used during the replica trick process).

3.1.5 IR and strong coupling limits

In the previous section it was determined that, within the limit of large N, the exact Green's functions of the SYK model are regulated by the following DS equations:

$$\begin{cases} G(\tau,\tau') = G_0(\tau,\tau') + \int d\tau_1 d\tau_2 G_0(\tau,\tau_1)\Sigma(\tau_1,\tau_2)G(\tau_2,\tau') \\ \Sigma(\tau,\tau') = J^2 G(\tau,\tau')^3 \end{cases} \tag{3.50}$$

In general conditions this system is only solvable numerically, however, under specific conditions, it is possible to obtain analytical solutions. Specifically, this condition corresponds to the case in which we are considering the low frequency limit ($|w| \ll J$, i.e. $|\tau J| \gg 1$, where τ is the time difference) and the strong coupling limit ($\beta J \gg 1$). Let us start by considering the zero temperature thermal equilibrium case. In this limit eq(3.49) becomes $G^{-1}(w) = -\Sigma(w)$ because the inverse of tree-level propagator can be neglected, in the Euclidean time domain this expression is equivalent to:

$$\int d\tau_1 \Sigma(\tau,\tau_1)G(\tau_1,\tau') = -\delta(\tau-\tau') \tag{3.51}$$

In the limit considered, this previous equation and the second equation of eq(3.50) are invariant under reparametrizations of time, namely $\tau \to f(\tau)$ with $f'(\tau) > 0$, provided that the bilocal fields transform as follows:

$$\begin{aligned} G(\tau,\tau') &\to [f'(\tau)f'(\tau')]^{\Delta} G(f(\tau),f(\tau')) \\ \Sigma(\tau,\tau') &\to [f'(\tau)f'(\tau')]^{3\Delta} \Sigma(f(\tau),f(\tau')) \end{aligned} \tag{3.52}$$

where $\Delta = \frac{1}{4}$. This means that when such reparametrizations are implemented, the following equation can be shown to be valid:

$$\int df(\tau_1)\Sigma[f(\tau),f(\tau_1)]G[f(\tau_1),f(\tau')] = -\delta(f(\tau)-f(\tau')) \tag{3.53}$$

The fact that in this limit the Majorana fermions acquire an anomalous dimension Δ, indicates that we have to search for solutions (of the DS eq) with the following form [53, 59]:

$$G_c(\tau,\tau') = b\frac{sgn(\tau-\tau')}{|J(\tau-\tau')|^{2\Delta}} \tag{3.54}$$

where b is a constant and the subscript "c" stands for "conformal", since in one dimension the group of reparametrizations is equal to the conformal group (for this

reason we will often call the considered regime as *conformal limit*). So the *DS* equations's solutions have this form and the factor b can be determined by inserting G_c in the eq(3.51) (using the explicit form of Σ); in doing so, we find $b = (4\pi)^{-\frac{1}{4}}$. So, in the IR and strong coupling limit, at $T = 0$ the exact Green's function has the following expression:

$$G_c(\tau) = \frac{1}{(4\pi)^{\frac{1}{4}}} \frac{sgn(\tau)}{|J\tau|^{2\Delta}} \tag{3.55}$$

The invariance under reparametrization described by eq(3.73) is extremely useful for finding the expression of the exact correlators in the case of finite temperature. In fact, it is important to remember that the case at zero temperature is related to the case at finite temperature by the mapping $\tau_{inf} = tan(\frac{\pi \tau_{fin}}{\beta})$; so we have to exploit the invariance using $f(\tau) = tan(\frac{\pi\tau}{\beta})$ (note that $f'(\tau) > 0$). In doing so, i.e. by applying the mapping to G_c, we obtain the exact two-point propagators at finite temperature:

$$G_c^\beta(\tau) = \frac{\pi^{\frac{1}{4}}}{\sqrt{2\beta J}} \frac{sgn(\tau)}{|sin\frac{\pi\tau}{\beta}|^{2\Delta}} \tag{3.56}$$

where, as always at finite temperature, $\tau \in [-\frac{\beta}{2}, +\frac{\beta}{2})$. It's important to remember that the correlators calculated in this limit, $G_c(\tau)$ and $G_c^\beta(\tau)$, tends to be the same as the exact ones, $G(\tau)$ and $G^\beta(\tau)$, only when $|\tau| \gg J^{-1}$. In the opposite case, $|\tau| \ll J^{-1}$, they tends to the bare ones, namely $G_0(\tau)$ and $G_0^\beta(\tau)$.

3.2 Spinless complex SYK

This section will present the main properties of the complex SYK model in the spinless case [13, 17, 42, 62], so as to lay the foundations for the study of the model in which we are interested, namely the spinful case. The system under consideration is very similar to that studied in the previous section, with the difference that now the orbitals can be occupied by spinless complex fermions and therefore the ladder operators, in the Hamiltonian, no longer refer to Majorana fermions. So this zero spatial dimensional system consists of N orbitals (denoted by the indices $i, j, ..$) that are all degenerate with non-interacting energy equal to zero. Each of these orbitals is associated with a pair of creation and annihilation operators, $\hat{c}_i^\dagger$ and $\hat{c}_i$ (in this case related to the i-th orbital), which change the number of spinless complex fermions in such an orbital. As in the Majorana fermion model, also here the orbitals interact with each other by means of a four-fermion random interaction. The Hamiltonian describing the so-called complex SYK model is as follows

$$\hat{H}_c = \sum_{i,j,k,l=1}^{N} J_{ijkl}\, \hat{c}_i^\dagger \hat{c}_j^\dagger \hat{c}_k \hat{c}_l \tag{3.57}$$

with

$$\{\hat{c}_i, \hat{c}_j\} = \{\hat{c}_i^\dagger, \hat{c}_j^\dagger\} = 0 \qquad\qquad \{\hat{c}_i, \hat{c}_j^\dagger\} = \delta_{ij} \tag{3.58}$$

The independent *complex* coupling constants J_{ijkl} follow a Gaussian distribution which ensures that

$$\overline{J_{ijkl}} = 0 \qquad\qquad \overline{|J_{ijkl}|^2} = \frac{2J^2}{N^3} \tag{3.59}$$

with the following constraints

$$J_{ijkl} = J_{klij}^* \qquad\qquad J_{ijkl} = -J_{jikl} \qquad\qquad J_{ijkl} = -J_{ijlk} \tag{3.60}$$

Studying such a model in the thermodynamic equilibrium, using the imaginary time formalism, we know that an important tool is the partition function $Z = Tr[e^{-\beta(\hat{H}-\mu\hat{N})}]$, where μ is the chemical potential of the grand canonical ensemble and $\hat{N} = \sum_i^N \hat{c}_i^\dagger \hat{c}_i$ is the total number fermionic operator. It is therefore convenient to work directly with the so-called grand canonical Hamiltonian defined as

$$\hat{H} = \sum_{i,j,k,l=1}^{N} J_{ijkl}\, \hat{c}_i^\dagger \hat{c}_j^\dagger \hat{c}_k \hat{c}_l - \mu \sum_{i}^{N} \hat{c}_i^\dagger \hat{c}_i \tag{3.61}$$

It is easy to see how the theory associated with this Hamiltonian has an $U(1)$ global symmetry where the related conserved $U(1)$ density Q corresponds to

$$Q = \frac{1}{N}\sum_{i}^{N}\langle \hat{c}_i^{\dagger}\hat{c}_i\rangle \tag{3.62}$$

where $\langle ...\rangle$ describes the thermal average in the grand canonical ensemble. So Q is equal to the average fermion number and, with the variation of the chemical potential μ, it can take values in the range $(0, 1)$. The imaginary time Green's functions that we use to describe such a system are defined in the following way

$$G(\tau,\tau') \equiv -\frac{1}{N}\sum_{i=1}^{N}\langle \mathbb{T}[\hat{c}_i(\tau)\hat{c}_i^{\dagger}(\tau')]\rangle \tag{3.63}$$

where, unlike the section concerning the Majorana fermion model, here we define them using the minus sign (only for the sake of consistency with the majority of the related scientific literature). In the case of the free theory the corresponding correlators (in the frequency domain) will have the following standard structure

$$G_0(w_n) = \frac{1}{iw_n + \mu} \tag{3.64}$$

In order to obtain the *Dyson–Schwinger equations* that describes the relation between the exact Green's function and the bare one, we have to repeat the same procedure as in the case of the Majorana fermion model. The theoretical framework describing the logical steps is exactly identical to that seen in the previous section and therefore the calculations will not be presented again. The first step would be to calculate the disorder average of the total partition function Z^R describing a system of R identical copies, so

$$\begin{aligned}\overline{Z^R} &= \int D\mathbf{J}\; Z^R[\mathbf{J}] = \int D\mathbf{J}\prod_{r=1}^{R} D[\bar{c}^r, c^r]\; e^{-\sum_{r=1}^{R} S[\bar{c}^r, c^r, \mathbf{J}]} \\ &= \int \prod_{r=1}^{R} D[\bar{c}^r, c^r] e^{-S_{rep}[\bar{c}^r, c^r]}\end{aligned} \tag{3.65}$$

with $D[\bar{c}^r, c^r] \equiv \prod_{i=1}^{N} D\bar{c}_i^r(\tau) Dc_i^r(\tau)$ Grassmann fields and where

$$S_{rep} = \sum_{r=1}^{R}\sum_{i=1}^{N}\int d\tau \bar{c}_i^r(\tau)\Big(\partial_\tau - \mu\Big)c_i^r(\tau) - \frac{J^2}{4N^3}\sum_{a,b=1}^{R}\int d\tau d\tau' \left|\sum_{i=1}^{N}\bar{c}_i^a(\tau)c_i^b(\tau')\right|^4 \tag{3.66}$$

Then we add to the theoretical framework the bilocal replica collective fields $\mathcal{G}^{ab}(\tau,\tau')$ and $\Sigma^{ab}(\tau,\tau')$ and integrating out the degrees of freedom of the fermions, one gets

$$\overline{Z^R} = \int \prod_{a,b=1}^{R} [DG^{ab} D\Sigma^{ab}]\, e^{-S_R[G,\Sigma]} \tag{3.67}$$

with

$$S_R[G,\Sigma] = N\Bigg\{-ln\, det\Bigg[\delta(\tau-\tau')\delta^{ab}[\partial_\tau - \mu] + \hat{\Sigma}^{ab}(\tau,\tau')\Bigg] - \sum_{a,b=1}^{R}\int d\tau d\tau' \Bigg[\Sigma^{ab}(\tau,\tau')\times G^{ab}(\tau',\tau) + \frac{J^2}{4} G^{ab}(\tau,\tau')^2 G^{ab}(\tau',\tau)^2\Bigg]\Bigg\} \tag{3.68}$$

The disorder average has physical correspondence only if $N \to \infty$ and in this limit, due to the presence of N in S_R, the partition function $\overline{Z^R}$ can be approximated by the saddle point solutions for the bilocal fields G^{ab}, Σ^{ab}. As in the case of Majorana fermions, in the limit of large N, the saddle point solutions are replica-diagonal and consequently

$$\overline{Z^R} \approx \Bigg[\int DG\, D\Sigma\, e^{-\,S_{eff}[G,\Sigma]}\Bigg]^R = \overline{Z}^R \tag{3.69}$$

with

$$S_{eff}[G,\Sigma] = N\Bigg\{-ln\, det\Bigg[\delta(\tau-\tau')[\partial_\tau - \mu] + \Sigma(\tau,\tau')\Bigg] - \int d\tau d\tau' \Bigg[\Sigma^{ab}(\tau,\tau')G^{ab}(\tau',\tau) + + \frac{J^2}{4} G^{ab}(\tau,\tau')^2 G^{ab}(\tau',\tau)^2\Bigg]\Bigg\} \tag{3.70}$$

Now we have all the tools needed to find the DS equation of the system under consideration, indeed as in the case of the model with Majorana fermions, they correspond to the saddle point solutions of $\bar{Z}$ (remember that this is only valid in the limit of large N). Therefore, calculating the S_{eff} differentials, one gets

$$\begin{aligned} \delta_\Sigma S_{eff} = 0 \iff G(iw_n) &= \frac{1}{iw_n + \mu - \Sigma(w_n)} \\ \delta_G S_{eff} = 0 \iff \Sigma(\tau) &= -J^2 G(\tau)^2 G(-\tau) \end{aligned} \tag{3.71}$$

where $G(\tau,\tau') = G(\tau-\tau')$, $\Sigma(\tau,\tau') = \Sigma(\tau-\tau')$ and w_n are the usual Matsubara frequencies. These two equations describe the relations between exact Green's functions and bare Green's functions, described in the eqs(3.141,3.64), by means of the self-energy term Σ. Note also that in the left limit $\tau = 0^-$, one gets $G(\tau = 0^-) = Q$.

3.2.1 Low energy solutions

If now we consider the low frequency limit ($|w| \ll J$, i.e. $|\tau|J \gg 1$) and the strong coupling limit ($\beta J \gg 1$), we expect to find the same conformal symmetry present in the Majorana fermion case. However, in the case of the complex SYK model the situation is more complicated, or rather, a modification on the theoretical framework is necessary. Indeed we are considering a system of complex fermions and we have to take into account the existence of the related $U(1)$ symmetry. In the IR and strong coupling limit, in addition to the existence of a conformal symmetry, we can see that the global $U(1)$ symmetry breaks down to a $U(1)$ gauge invariance; in fact neglecting the kinetic term of the Lagrangian one loses the term containing the time derivative, consequently now the invariance is preserved even in the case where the phases (associated to $U(1)$) are time dependent. This will lead to a change in the IR symmetry transformations (compared to the Majorana case). To understand all this, let's start by writing the DS equation in Euclidean time using the limit considered

$$\begin{aligned} \int d\tau_1 G(\tau,\tau_1)\Sigma(\tau_1,\tau') &= -\delta(\tau-\tau') \\ \Sigma(\tau,\tau') &= -J^2 G(\tau,\tau')^2 G(\tau',\tau) \end{aligned} \tag{3.72}$$

These equations are invariant under reparametrizations of times [13, 17], namely $\tau \to f(\tau)$, provided that the bilocal fields transform as follows:

$$\begin{aligned} G(\tau,\tau') &\to [f'(\tau)f'(\tau')]^{\Delta}\frac{g(\tau')}{g(\tau)}G(f(\tau),f(\tau')) \\ \Sigma(\tau,\tau') &\to [f'(\tau)f'(\tau')]^{3\Delta}\frac{g(\tau')}{g(\tau)}\Sigma(f(\tau),f(\tau')) \end{aligned} \tag{3.73}$$

where $\Delta \equiv \frac{1}{4}$. The $f(\tau)$ and $g(\tau)$ functions are completely arbitrary and closely related to the system symmetries that emerge in the limit considered, i.e. the conformal symmetry and the $U(1)$ gauge invariance respectively. In order to obtain the analytic expressions of $G(\tau,\tau')$ and $\Sigma(\tau,\tau')$ within the limit considered, we start by considering G in the frequency domain, namely (at zero temperature)

$$G(iw) = \int d\tau\, e^{iw\tau}G(\tau) \tag{3.74}$$

and we define its analytic continuation to all complex frequencies z as $G(z)$. Now, following the approach of Refs.[2, 37], we consider this specific ansatz for the exact

Green's function in complex frequency (in the considered limit):

$$G(z) = C\frac{e^{-i(\pi\Delta+\theta)}}{z^{1-2\Delta}} \qquad with \quad z \in \mathbb{C}, Im(z) > 0,\ |z| \ll J \tag{3.75}$$

where C and θ are real parameter with $C > 0$ (Δ is fixed to $1/4$). It can be shown that this general expression is the correct one for the correlator of a charged operator (with $\mu \neq 0$) in the presence of a conformal invariance and a $U(1)$ gauge symmetry [17]. Now it is useful to write the spectral representation of $G(z)$, so

$$G(z) = \int_{-\infty}^{+\infty} d\Omega \frac{\rho(\Omega)}{z-\Omega} \tag{3.76}$$

where the density of states is equal to $\rho(\Omega) = -\frac{1}{\pi} Im\, G(\Omega + i0^+) > 0$ for all real Ω. As a direct consequence of the constraint $\rho(\Omega) > 0$ (arising from unitarity), this translates to the condition

$$-\pi\Delta < \theta < \pi\Delta \tag{3.77}$$

In order to find the imaginary time dependent Green's function $G(\tau)$, we have to calculate the Fourier transformation of $G(z)$, so by identifying $i\omega$ with z, one gets

$$\begin{aligned} G(\tau) &= \int \frac{dz}{2\pi i} \int d\Omega \frac{\rho(\Omega)}{z-\Omega} e^{-z\tau} = \int d\Omega\, \rho(\Omega) \int \frac{dz}{2\pi i} \frac{e^{-z\tau}}{z-\Omega} \\ &= \int_0^{+\infty} d\Omega\, \rho(-\Omega) F(-\Omega, \tau) + \int_0^{+\infty} d\Omega\, \rho(\Omega) F(\Omega, \tau) \end{aligned} \tag{3.78}$$

where $F(\Omega, \tau) \equiv \int \frac{dz}{2\pi i} \frac{e^{-z\tau}}{z-\Omega}$, with $z = i\omega$. The fastest approach to calculating such an integral is by means of simple complex contour integration using appropriate paths, in this way we obtain:

$$F(-\Omega, \tau) = \begin{cases} e^{\Omega\tau} & \tau < 0 \\ 0 & \tau > 0 \end{cases} \qquad F(\Omega, \tau) = \begin{cases} 0 & \tau < 0 \\ -e^{-\Omega\tau} & \tau > 0 \end{cases} \tag{3.79}$$

So, substituting

$$G(\tau) = \begin{cases} +\int_0^{+\infty} d\Omega\, \rho(-\Omega) e^{\Omega\tau} & \tau < 0 \\ \\ -\int_0^{+\infty} d\Omega\, \rho(\Omega) e^{-\Omega\tau} & \tau > 0 \end{cases} \tag{3.80}$$

Now, using the expression (3.75), we can write the spectral density $\rho(\Omega)$ in the low frequency and strong coupling limit as follow

$$\rho(\Omega) = -\frac{1}{\pi} Im\, G(\Omega + i0^+) = \begin{cases} C\dfrac{sin(\pi\Delta - \theta)}{\pi|\Omega|^{1-2\Delta}} & -J \ll \Omega < 0 \\ \\ C\dfrac{sin(\pi\Delta + \theta)}{\pi|\Omega|^{1-2\Delta}} & 0 < \Omega \ll J \end{cases} \tag{3.81}$$

where, for negative Ω, the identity $cos(\pi\Delta + \theta) = sin(\pi\Delta - \theta)$ is used. The fact that θ has opposite sign according to the sign of Ω, indicates how it is related to the concept of particle-hole asymmetry, indeed the positive and negative frequencies are related to the propagation forward (particle) and backward (hole) in time. Now, replacing $\rho(\Omega)$ in eq(3.80) with the expressions written above, the low energy Green's functions corresponds to

$$G(\tau) = \begin{cases} +\dfrac{C\Gamma(2\Delta)sin(\pi\Delta - \theta)}{\pi|\tau|^{2\Delta}} & \tau \ll -J^{-1} \\ \\ -\dfrac{C\Gamma(2\Delta)sin(\pi\Delta + \theta)}{\pi|\tau|^{2\Delta}} & \tau \gg J^{-1} \end{cases} \tag{3.82}$$

The above solutions are also valid for a generic $\Delta = \frac{1}{q}$, where q represents the number of interacting fields in the random all-to-all q-body interactions. In our case $\Delta = 1/4$, so one gets

$$G(\tau) = \begin{cases} +\dfrac{Csin(\frac{\pi}{4} - \theta)}{\sqrt{|\pi\tau|}} & \tau \ll -J^{-1} \quad T = 0 \\ \\ -\dfrac{Csin(\frac{\pi}{4} + \theta)}{\sqrt{|\pi\tau|}} & \tau \gg J^{-1} \quad T = 0 \end{cases} \tag{3.83}$$

In order to calculate the constant C, it is advisable to first find the expression for the self energy $\Sigma(\tau)$ in the considered limits (low energy solutions and $T = 0$). To achieve this, the quickest way is to insert the above solution into the second D-S equation (3.72), in this way we obtain

$$\Sigma(\tau) = \begin{cases} +\dfrac{C^3J^2cos(2\theta)cos(\frac{\pi}{4} + \theta)}{2|\pi\tau|^{\frac{3}{2}}} & \tau \ll -J^{-1} \quad T = 0 \\ \\ -\dfrac{C^3J^2cos(2\theta)sin(\frac{\pi}{4} + \theta)}{2|\pi\tau|^{\frac{3}{2}}} & \tau \gg J^{-1} \quad T = 0 \end{cases} \tag{3.84}$$

Now to find the constant C it is convenient to switch to its analytic continuation to all complex frequencies, namely

$$\Sigma(z) = \int_{-\infty}^{+\infty} d\Omega \frac{\sigma(\Omega)}{z - \Omega} \tag{3.85}$$

where $\sigma(\Omega) = -\frac{1}{\pi} Im\, \Sigma(\Omega)$. In this way, by calculating the Fourier transformation of $\Sigma(\tau)$ and performing its analytic continuation, one gets

$$\Sigma(z) = -\frac{C^3 J^2 cos(2\theta)}{\pi} e^{i(\frac{\pi}{4}+\theta)} \sqrt{z} \qquad with \quad z \in \mathbb{C}, Im(z) > 0,\ |z| \ll J \tag{3.86}$$

By inserting this expression and that of eq(3.75) in the first equation of eq(3.71), we can see that at $T = 0$ the relation $\mu - \Sigma(0) = 0$ must be valid, since $G(z)$ diverges in the limit $|z| \to 0$. The second piece of information we can extract is the expression for the constant C, in fact the equation is consistent only when the coefficients of the divergent terms are equal (for $|z| \to 0$), i.e. when

$$\frac{J^2 C^4 cos(2\theta)}{\pi} = 1 \qquad \to \qquad C = \left(\frac{\pi}{J^2 cos(2\theta)} \right)^{\frac{1}{4}} \tag{3.87}$$

Until now the particle-hole asymmetry parameter θ has remained unknown, but luckily this problem can be resolved by a *generalized Luttinger's theorem* [17, 37, 42, 62]. In fact this theorem allows the fermion number density Q to be written as a function of only the θ parameter, more specifically it states that:

$$Q = \frac{1}{2} - \frac{\theta}{\pi} - \frac{sin(2\theta)}{4} \tag{3.88}$$

Note that the θ domain $(-\frac{\pi}{4}, \frac{\pi}{4})$ is in perfect agreement with the physical domain of Q, indeed such a domain implies that $0 < Q < 1$, in particular Q decreases monotonically in that range. In literature it is possible to find all the treatment of the complex SYK as a function of another particle-hole asymmetry parameter, i.e. $\mathcal{E}$, usually called *spectral asymmetry*, which is related to the asymmetry angle θ by the following definition

$$e^{2\pi\mathcal{E}} \equiv \frac{sin(\pi\Delta + \theta)}{sin(\pi\Delta - \theta)} \tag{3.89}$$

consequently it is a real number with domain $(-\infty, +\infty)$. Using this parameter the previous equation (3.88) becomes

$$Q = \frac{1}{4}\left(3 - tanh(2\pi\mathcal{E})\right) - \frac{1}{\pi} tan^{-1}\left(e^{2\pi\mathcal{E}}\right) \tag{3.90}$$

and in this case Q is still a monotonically decreasing function in $(-\infty, +\infty)$. It is important to stress that all the analytic results found in this section are valid in the low energy limit. However, it can be shown by numerical analysis [63] that, for a range of values around the particle-hole symmetric point $Q = 1/2$, these solutions have high energy continuations that satisfy the DS equations. (Note: in this thesis it was decided not to present the derivation of the (3.88) and (3.90) equations since all the analysis that will be carried out in the following sections will be limited to the particle-hole symmetric case). In order to obtain the non-zero temperature solution, as was done in the Majorana model, the case at zero temperature is related to the one at finite temperature by a specific conformal map f, in this case $\tau_{inf} = \frac{1}{\pi T} tan(\pi T \tau_{fin})$. Performing the transformation described by eq(3.73) to the solution (3.83) one gets

$$G^{\beta}(\tau) = \begin{cases} +Cg(\tau)cos(\frac{\pi}{4} + \theta)\left(\dfrac{T}{sin(|\pi T\tau|)}\right)^{\frac{1}{2}} & -\frac{1}{T} < \tau < 0 \\ \\ -Cg(\tau)sin(\frac{\pi}{4} + \theta)\left(\dfrac{T}{sin(|\pi T\tau|)}\right)^{\frac{1}{2}} & 0 < \tau < \frac{1}{T} \end{cases} \tag{3.91}$$

The function $g(\tau)$, related to the $U(1)$ gauge invariance, is unknown but it can be determined by inserting two elements into the theoretical framework: by choosing the normalization, i.e. $g(0) = 1$, and imposing the anti-periodic condition (KMS condition) to the exact Green's function, so $G^{\beta}(\tau) = -G^{\beta}(\tau + \beta)$, where $\beta = 1/T$ and $k_B = 1$. These two conditions imply that

$$g(\tau) = g(\tau + 1/T)tan(\pi/4 + \theta) \tag{3.92}$$

and the solution of the equation is simply

$$g(\tau) = e^{-2\pi\mathcal{E}T\tau} \tag{3.93}$$

So the low energy solution of the exact Green's function at finite temperature are equal to

$$G^{\beta}(\tau) = \begin{cases} +C\dfrac{e^{-2\pi\mathcal{E}T\tau}}{\sqrt{1 + e^{+4\pi\mathcal{E}}}}\left(\dfrac{T}{sin(|\pi T\tau|)}\right)^{\frac{1}{2}} & -\frac{1}{T} < \tau < 0 \\ \\ -C\dfrac{e^{-2\pi\mathcal{E}T\tau}}{\sqrt{1 + e^{-4\pi\mathcal{E}}}}\left(\dfrac{T}{sin(|\pi T\tau|)}\right)^{\frac{1}{2}} & 0 < \tau < \frac{1}{T} \end{cases} \tag{3.94}$$

As always, these solutions can be defined for $\tau \in \mathbb{R}$, we just need to exploit its anti-periodic condition.

3.2.2 Particle-Hole Symmetry

In the previous section we saw how the angle θ is somehow related to particle-hole asymmetry, so now it is appropriate to demonstrate that it is in fact different from zero only in cases where the physical system under consideration is not symmetric under particle-hole transformation. The first thing to do is to define particle-hole transformation and, considering the Hamiltonian (3.61), we can define it as

$$\hat{c}_i \to \hat{c}_i^\dagger \qquad \hat{c}_i^\dagger \to \hat{c}_i \qquad J_{ijkl} \to J^*_{klij} \tag{3.95}$$

It is easy to note that the Hamiltonian (3.61) is not symmetric under such transformation, in fact all the terms related to chemical potential and those of interactions in which the operators $\hat{c}_i$ and $\hat{c}_i^\dagger$ (referring to the same site i) are coupled, generate new contributions in the Hamiltonian. Consequently, to make the Hamiltonian symmetric under particle-hole transformation, it would not be sufficient to impose the chemical potential at zero, but it is also necessary to anti-symmetrize the SYK interaction (by adding specific terms). The generic expression of the antisymmetrized Hamiltonian (also containing the chemical potential terms), has the following form [62, 64, 65]

$$\hat{H}_A = \sum_{i,j,k,l=1}^{N} J_{ijkl}\left(\hat{c}_i^\dagger \hat{c}_j^\dagger \hat{c}_k \hat{c}_l + \hat{A}_{ijkl}\right) - \mu \sum_{i}^{N}\left(\hat{c}_i^\dagger \hat{c}_i - \frac{1}{2}\right) \tag{3.96}$$

where

$$\hat{A}_{ijkl} = \frac{1}{2}\left[\delta_{ik}\hat{c}_j^\dagger \hat{c}_l - \delta_{il}\hat{c}_j^\dagger \hat{c}_k - \delta_{jk}\hat{c}_i^\dagger \hat{c}_l + \delta_{jl}\hat{c}_i^\dagger \hat{c}_k + \frac{1}{2}\delta_{il}\delta_{jk} - \frac{1}{2}\delta_{ij}\delta_{kl}\right] \tag{3.97}$$

This expression contains all the necessary terms to make the interaction term completely antisymmetric. Now it is important to note that $\hat{H}_A$ is manifestly particle-hole symmetric only in the case where $\mu = 0$. We now ask ourselves how this can be useful for the study of the SYK model, since the Hamiltonian $\hat{H}_A$ describes a different system than that corresponding to (3.61). The solution is to note that, in the case $\mu = 0$, the two Hamiltonians differ by terms which are of subleading order in N in the limit $N \to \infty$. This is absolutely not a problem for our treatment, since all the results found in the previous sections are based on the assumption of being

in the limit of large N. We can then state that, within that limit, the Hamiltonian

$$\hat{H} = \sum_{i,j,k,l=1}^{N} J_{ijkl}\, \hat{c}_i^\dagger \hat{c}_j^\dagger \hat{c}_k \hat{c}_l \tag{3.98}$$

is invariant under the particle-hole transformation. The fact that such an Hamiltonian is particle-hole symmetric ensures that the expectation values of the number operators $\hat{n}_i = \hat{c}_i^\dagger \hat{c}_i$ correspond to $\langle \hat{n}_i \rangle = \frac{1}{2}$ (in the case of a grand canonical ensemble with temperature T and chemical potential $\mu = 0$). So it follows that the average fermion number Q is also equal to $1/2$. Now the connection between the angle θ and particle-hole asymmetry is finally clear, in fact knowing the relation eq(3.88) it is immediately possible to verify that for $Q = 1/2$ one gets $\theta = 0$ and therefore also $\mathcal{E} = 0$. An angle θ different from zero is therefore indicative of an asymmetry between the density of particles and holes in the system. So in conclusion we see that at the particle-hole symmetric point $Q = 1/2$, $\theta = \mathcal{E} = 0$, considering $\mu = 0$ and the large N limit, the exact Green's function in the low frequency and the strong coupling limit at zero temperature are equal to

$$G(\tau) = \begin{cases} +\dfrac{B}{\sqrt{|\pi\tau|}} & \tau \ll -J^{-1} \qquad T = 0 \\ \\ -\dfrac{B}{\sqrt{|\pi\tau|}} & \tau \gg J^{-1} \qquad T = 0 \end{cases} \tag{3.99}$$

where $B \equiv \dfrac{\pi^{1/4}}{\sqrt{2J}}$. While the finite temperature solutions are

$$G^\beta(\tau) = \begin{cases} +B\left(\dfrac{T}{sin(|\pi T\tau|)}\right)^{\frac{1}{2}} & -\frac{1}{T} < \tau < 0 \\ \\ -B\left(\dfrac{T}{sin(|\pi T\tau|)}\right)^{\frac{1}{2}} & 0 < \tau < \frac{1}{T} \end{cases} \tag{3.100}$$

It is important to note that the solutions given above are exactly identical to those found in the Majorana fermion model (within the same limits), namely the expressions (3.55) and (3.56) (they are equal up to a multiplication factor (-1), this is simply due to the fact that they are defined with opposite sign). Thinking about it carefully, this is not too surprising, as the two different cases obey to the canonical anticommutation relations (although with some difference in the Majorana case); in addition, in the Majorana fermion model, by definition there are no contributions related to chemical potential because, in this context, it loses meaning, in fact it is

not possible to define number operators.

3.3 Spinful complex SYK

The model of interest is now introduced, namely the model whose transport properties, under NESS conditions, will be studied in the next chapter, where it will be used as a quantum dot between two leads. This model corresponds to the generalization to the spinful case of the model studied in the previous section, that is the spinless complex SYK model. So in this case the orbitals can be occupied by spin-1/2 complex fermions and the creation and annihilation operators, $\hat{c}_i^\dagger$ and $\hat{c}_i$ (which in this case reference the i-th orbital), are replaced by the operators $\hat{c}_{i\sigma}^\dagger$ and $\hat{c}_{i\sigma}$, where the spin index σ can assume the values $\pm 1/2$. Unlike the previous model, now we choose to consider the case in which four-fermion orbital interactions are characterized by *real* coupling constants J_{ijkl} that are also spin-independent. The Hamiltonian describing this model is as follows [66]:

$$\hat{H} = \frac{1}{2} \sum_{i,j,k,l=1}^{N} \sum_{\sigma\sigma'} J_{ijkl} \left[\hat{c}_{i\sigma}^\dagger \hat{c}_{j\sigma'}^\dagger \hat{c}_{k\sigma'} \hat{c}_{l\sigma} + \hat{c}_{l\sigma}^\dagger \hat{c}_{k\sigma'}^\dagger \hat{c}_{j\sigma'} \hat{c}_{i\sigma} \right] \tag{3.101}$$

with

$$\{\hat{c}_{i\sigma}, \hat{c}_{j\sigma'}\} = \{\hat{c}_{i\sigma}^\dagger, \hat{c}_{j\sigma'}^\dagger\} = 0 \qquad\qquad \{\hat{c}_{i\sigma}, \hat{c}_{j\sigma'}^\dagger\} = \delta_{ij}\delta_{\sigma\sigma'} \tag{3.102}$$

The independent *real* coupling constants J_{ijkl} follow a Gaussian distribution which ensures that

$$\overline{J_{ijkl}} = 0 \qquad\qquad \overline{J_{ijkl}^2} = \frac{J^2}{(4N)^3} \tag{3.103}$$

with the following constraints

$$J_{ijkl} = J_{lkji} \qquad\qquad J_{ijkl} = -J_{jikl} \qquad\qquad J_{ijkl} = -J_{ijlk} \tag{3.104}$$

In addition, we choose to consider the case in which the couplings J_{ijkl} are different from zero only if the four indexes are all different. Note that, as in the previous case, this Hamiltonian preserves the total number of fermions and is symmetrical under the time-reversal transformation. An important point to mention is that in this section the analysis of such a model will be limited to the study under conditions of thermodynamic equilibrium (grand canonical ensemble) and in the particle-hole symmetry case, therefore considering $\mu = 0$. So, as done previously, in order to obtain the *Dyson–Schwinger equations*, we have to calculate the disorder average of the total partition function Z^R describing a system of R identical copies, so

$$\overline{Z^R} = \int D\mathbf{J}\; Z^R[\mathbf{J}] = \int D\mathbf{J} \prod_{r=1}^{R} D[\bar{c}^r, c^r]\; e^{-\sum_{r=1}^{R} S[\bar{c}^r, c^r, \mathbf{J}]} \tag{3.105}$$

with $D[\bar{c}^r, c^r] \equiv \prod_{i=1}^{N} \prod_{\sigma=\pm 1/2} D\bar{c}^r_{i\sigma}(\tau) Dc^r_{i\sigma}(\tau)$ and where

$$S[\bar{c}^r, c^r, \mathbf{J}] = \int d\tau \left(\sum_{i=1}^{N} \sum_{\sigma} \bar{c}^r_{i\sigma}(\tau) \partial_\tau c^r_{i\sigma}(\tau) + H(\bar{c}^r(\tau), c^r(\tau)) \right) \tag{3.106}$$

with H given by eq(3.101). Performing the average over the Gaussian distribution coupling constants, one obtains

$$\int D\mathbf{J}\, e^{-\sum_{r=1}^{R} \int d\tau H(\bar{c}^r(\tau), c^r(\tau))} = exp\left\{ \frac{J^2}{4(4N)^3} \sum_{a,b=1}^{R} \int d\tau d\tau' \left[\sum_{i,j,k,l=1}^{N} \left(\mathcal{N}^{ab}_{ijkl}(\tau,\tau') + \mathcal{A}^{ab}_{ijkl}(\tau,\tau') \right) \right] \right\} \tag{3.107}$$

with

$$\mathcal{N}^{ab}_{ijkl}(\tau,\tau') = \sum_{\sigma\sigma'\rho\rho'} (\bar{c}^a_{i\sigma}(\tau) c^b_{i\rho}(\tau'))(\bar{c}^a_{j\sigma'}(\tau) c^b_{j\rho'}(\tau'))(\bar{c}^b_{k\rho'}(\tau') c^a_{k\sigma'}(\tau))(\bar{c}^b_{l\rho}(\tau') c^a_{l\sigma}(\tau))$$

$$\mathcal{A}^{ab}_{ijkl}(\tau,\tau') = \sum_{\sigma\sigma'\rho\rho'} (\bar{c}^a_{i\sigma}(\tau) \bar{c}^b_{i\rho}(\tau'))(\bar{c}^a_{j\sigma'}(\tau) \bar{c}^b_{j\rho'}(\tau'))(c^a_{k\sigma'}(\tau) c^b_{k\rho'}(\tau'))(c^a_{l\sigma}(\tau) c^b_{l\rho}(\tau')) \tag{3.108}$$

So, as in the previous cases, the disorder average generates 8-fermion terms that in this case we group into two contributions: the *normal* part $\mathcal{N}^{ab}_{ijkl}(\tau,\tau')$ and the *anomalous* part $\mathcal{A}^{ab}_{ijkl}(\tau,\tau')$. Now, instead of continuing with the calculations for the R-replicas system, let us use our knowledge of the SYK models' properties; in fact we know that $\overline{Z^R}$, in the limit $N \to \infty$, is governed by replica-diagonal saddle point solutions [6, 58, 67], since the replica off-diagonal terms have maximum order $\mathcal{O}(1/N)$, so $\overline{Z^R} = (\overline{Z})^R + \mathcal{O}(1/N)$. So we can perform the replica trick using only replica-diagonal terms and what we obtain is equivalent to the disorder average of the partition function related to a single SYK system (up to a R-exponential). Therefore, from now on, we choose to study $\overline{Z}$ and consequently replica indices are no longer present. In order to integrate the Grassmann fields $[\bar{c}, c]$ we have to insert bilocal fields (the Green's functions) and the corresponding self-energies in the theoretical framework. However unlike the cases previously studied, here the action also has the anomalous contribution and consequently, in addition to entering the normal Green's functions, it will also be necessary to introduce the anomalous ones. For introducing the Green's functions and their self-energies there is an alternative method to eq(3.29) (although obviously it is equivalent in terms of the result), in

fact it is possible to insert them in the partition function through the identity

$$\begin{aligned} 1 &= \int DG\, \delta\Bigg(NG(\tau,\tau') + \sum_{i=1}^{N} \bar{c}_{i\sigma}(\tau) c_{i\sigma}(\tau') \Bigg) \\ &= \int DG\, D\Sigma\, exp\Bigg(\int d\tau d\tau'\, \Sigma(\tau,\tau') \Bigg[NG(\tau,\tau') + \sum_{i=1}^{N} \bar{c}_{i\sigma}(\tau) c_{i\sigma}(\tau') \Bigg] \Bigg) \end{aligned} \tag{3.109}$$

This is just one of the four identities we need to use to integrate the Grassmann fields $[\bar{c}, c]$, specifically this identity is associated with the normal Green's function (in Grassmann variables) which is defined as follow

$$G(\tau,\tau') = -\frac{1}{N} \sum_{i=1}^{N} \bar{c}_{i\sigma}(\tau) c_{i\sigma}(\tau') \tag{3.110}$$

Note that the spin has not been explicitly stated, this is because for symmetry properties of the system we expect that the two exact normal Green's functions (intended as thermal averages) are independent of the spin projection, so we can treat $G_{\uparrow\uparrow}$ and $G_{\downarrow\downarrow}$ as if they were spin independent. The set of units to be inserted can be represented in a compact way by the following (2×2) identity matrix

$$\mathbf{1} = \int D\mathbf{G}\, D\mathbf{\Sigma}\, exp\Bigg(\sum_{i=1}^{N} \bar{\Psi}_i \mathbf{\Sigma} \Psi_i + N\, Tr[\mathbf{\Sigma G}] \Bigg) \tag{3.111}$$

where the following notation is used $\Psi_i = \begin{pmatrix} c_{i\uparrow} & \bar{c}_{i\downarrow} \end{pmatrix}^T$, $\bar{\Psi}_i = \begin{pmatrix} \bar{c}_{i\uparrow} & c_{i\downarrow} \end{pmatrix}$ and the matrices $\mathbf{G}(\tau,\tau')$, $\mathbf{\Sigma}(\tau,\tau')$ are respectively the Green's function matrix and the self-energy matrix, defined as follows

$$\mathbf{G}(\tau,\tau') = \begin{pmatrix} G(\tau,\tau') & \bar{F}(\tau,\tau') \\ F(\tau,\tau') & -G(\tau',\tau) \end{pmatrix} \qquad \mathbf{\Sigma} = \begin{pmatrix} \Sigma(\tau,\tau') & \bar{\Xi}(\tau,\tau') \\ \Xi(\tau,\tau') & -\Sigma(\tau',\tau) \end{pmatrix} \tag{3.112}$$

Note that in this theoretical framework, the $(1,1)$-element of the matrix $\mathbf{G}$ is associated with the spin configuration $\uparrow\uparrow$, while the $(2,2)$-element is associated with the $\downarrow\downarrow$ configuration, therefore the latter can be expressed as $-G(\tau',\tau)$ only if we assume that the respective saddle point solutions are spin independent (since we know a priori that, for SYK models, the saddle point solutions correspond to the DS-equations). From this follows the $-\Sigma(\tau',\tau)$ expression for the 22-element of the matrix $\mathbf{\Sigma}$. The fields $F, \bar{F}$ correspond to the anomalous Green's function and the identity (3.111) makes them equal to

$$F(\tau,\tau') = -\frac{1}{N} \sum_{i=1}^{N} c_{i\downarrow}(\tau) c_{i\uparrow}(\tau') \qquad \bar{F}(\tau,\tau') = -\frac{1}{N} \sum_{i=1}^{N} \bar{c}_{i\uparrow}(\tau) \bar{c}_{i\downarrow}(\tau') \tag{3.113}$$

and the relative anomalous self-energies are the dual fields $\Xi, \bar{\Xi}$, while Σ is the normal self-energy. Therefore, after inserting the identity and rewriting the terms $\mathcal{N}^{ab}_{ijkl}(\tau,\tau')$ and $\mathcal{A}^{ab}_{ijkl}(\tau,\tau')$ as s function of the normal and the anomalous Green's functions, one gets

$$\overline{Z} = \int D\mathbf{G}\, D\mathbf{\Sigma}\, D[\bar{c},c]\, e^{-S[\bar{c},c,\mathbf{G},\mathbf{\Sigma}]} \tag{3.114}$$

where

$$\begin{aligned} S[\bar{c},c,\mathbf{G},\mathbf{\Sigma}] = &\Bigg\{\sum_{i=1}^{N}\int d\tau d\tau' \left[\sum_{\sigma}\bar{c}_{i\sigma}(\tau')\Big(\delta(\tau-\tau')\partial_\tau + \Sigma(\tau',\tau)\Big)c_{i\sigma}(\tau) + \bar{\Xi}(\tau',\tau)\times\right.\\ &\left.\bar{c}_{i\uparrow}(\tau')\bar{c}_{i\downarrow}(\tau) + \Xi(\tau',\tau)c_{i\downarrow}(\tau')c_{i\uparrow}(\tau)\right]\Bigg\} - N\int d\tau d\tau' \left[2\Sigma(\tau',\tau)G(\tau,\tau') + \right.\\ &\left.\Xi(\tau',\tau)\bar{F}(\tau,\tau') + \bar{\Xi}(\tau',\tau)F(\tau,\tau') + \frac{J^2}{64}\Big(\bar{F}(\tau,\tau')^2F(\tau,\tau')^2 + G(\tau,\tau')^4\Big)\right]\\ = &\left(\sum_{i=1}^{N}\int d\tau d\tau'\, \bar{\Psi}_i\mathbf{M}\Psi_i\right) - N\int d\tau d\tau' \left[2\Sigma(\tau',\tau)G(\tau,\tau') + \Xi(\tau',\tau)\bar{F}(\tau,\tau') + \right.\\ &\left. + \bar{\Xi}(\tau',\tau)F(\tau,\tau') + \frac{J^2}{64}\Big(\bar{F}(\tau,\tau')^2F(\tau,\tau')^2 + G(\tau,\tau')^4\Big)\right] \end{aligned} \tag{3.115}$$

where

$$\mathbf{M} \equiv [\delta(\tau-\tau')\partial_\tau]\mathbf{1} + \mathbf{\Sigma}(\tau',\tau) \tag{3.116}$$

Note that the term $(\bar{F}^2F^2 + G^4)$ does not fully reproduce all contributions from $\sum_{i,j,k,l=1}^{N}(\mathcal{N}^{ab}_{ijkl} + \mathcal{A}^{ab}_{ijkl})$, this is not a problem because the related partition functions are identical, since the neglected contributions have no saddle point solutions associated, therefore giving zero contribution to $\overline{Z}$. The result is an action in which the Grassmann fields are now decoupled from each other and therefore it is possible to integrate them:

$$\overline{Z} = \int D\mathbf{G}\, D\mathbf{\Sigma}\, e^{-S_{eff}[\mathbf{G},\mathbf{\Sigma}]} \tag{3.117}$$

with

$$\begin{aligned} S_{eff}[\mathbf{G},\mathbf{\Sigma}] = N\Bigg\{&-ln\, det\,\mathbf{M} - \int d\tau d\tau' \left[2\Sigma(\tau',\tau)G(\tau,\tau') + \Xi(\tau',\tau)\bar{F}(\tau,\tau') + \right.\\ &\left. + \bar{\Xi}(\tau',\tau)F(\tau,\tau') + \frac{J^2}{64}\Big(\bar{F}(\tau,\tau')^2F(\tau,\tau')^2 + G(\tau,\tau')^4\Big)\right]\Bigg\} \end{aligned} \tag{3.118}$$

In order to find the saddle point equations, it is convenient to rewrite the term $ln\, det\,\mathbf{M}$ in the Matsubara frequency space by developing appropriate Fourier transformations, in fact we know that the Grassmann fields $[\bar{c},c]$ satisfy antiperiodic

boundary conditions and we also know that due to the time translation invariance one has $\mathbf{\Sigma}(\tau',\tau)=\mathbf{\Sigma}(\tau'-\tau)$, so using the usual relations (at finite temperature)

$$f(\tau)=\frac{1}{\beta}\sum_{w_n}e^{-iw_n\tau}f(w_n)\qquad\qquad f(w_n)=\int_0^\beta d\tau\, e^{iw_n\tau}f(\tau)\tag{3.119}$$

with $w_n=(2n+1)\pi/\beta$, the matrix $\mathbf{M}$ corresponds to

$$\mathbf{M}=\bigoplus_n\begin{pmatrix}-iw_n+\Sigma(w_n) & \bar{\Xi}(w_n)\\ \Xi(w_n) & -iw_n+\Sigma(w_n)\end{pmatrix}\tag{3.120}$$

(note that $\mathbf{M}$ is no longer a function in this representation [68, 69]). So the effective action can be written in the following form

$$S_{eff}[\mathbf{G},\mathbf{\Sigma}]=N\Bigg\{-\sum_n ln\Bigg[\Big(-iw_n+\Sigma(w_n)\Big)^2-\bar{\Xi}(w_n)\Xi(w_n)\Bigg]-\int d\tau d\tau'\Bigg[2\Sigma(\tau',\tau)G(\tau,\tau')+\\ +\Xi(\tau',\tau)\bar{F}(\tau,\tau')+\bar{\Xi}(\tau',\tau)F(\tau,\tau')+\frac{J^2}{64}\Big(\bar{F}(\tau,\tau')^2F(\tau,\tau')^2+G(\tau,\tau')^4\Big)\Bigg]\Bigg\}\tag{3.121}$$

By taking functional derivatives of the effective action, one obtains saddle-point equations, that we know correspond to the DS-equations. So starting with the differential with respect to $\bar{F}$, one gets

$$\delta_{\bar{F}}S_{eff}==N\Bigg\{-\int d\tau d\tau'\Big(\Xi(\tau',\tau)+\frac{J^2}{32}\bar{F}(\tau,\tau')F(\tau,\tau')^2\Big)\delta\bar{F}(\tau,\tau')\Bigg\}\tag{3.122}$$

$$\delta_{\bar{F}}S_{eff}=0\iff\Xi(\tau',\tau)=-\frac{J^2}{32}\bar{F}(\tau,\tau')F(\tau,\tau')^2\tag{3.123}$$

Furthermore, since we are considering the specific case in which $\mu=0$, equality $\Xi(\tau',\tau)=\Xi(\tau,\tau')$ is valid [66], so

$$\Xi(\tau,\tau')=-\frac{J^2}{32}\bar{F}(\tau,\tau')F(\tau,\tau')^2\tag{3.124}$$

For the differential with respect to G

$$\delta_G S_{eff}==N\Bigg\{-\int d\tau d\tau'\Big(2\Sigma(\tau',\tau)+\frac{J^2}{16}G(\tau,\tau')^3\Big)\delta G(\tau,\tau')\Bigg\}\tag{3.125}$$

$$\delta_G S_{eff}=0\iff\Sigma(\tau',\tau)=-\frac{J^2}{32}G(\tau,\tau')^3\tag{3.126}$$

Thanks to the fact that, in the case $\mu=0$, we have $\Sigma(\tau',\tau)=-\Sigma(\tau,\tau')$ (since in this case both the bare and the exact Green's function are odd functions), the

equation above corresponds to

$$\Sigma(\tau,\tau') = \frac{J^2}{32} G(\tau,\tau')^3 \tag{3.127}$$

With regards to the differential with respect to Σ, it is convenient to calculate the derivative in the space of the Matsubara frequencies, so

$$\begin{aligned}\frac{\delta S_{eff}}{\delta\Sigma(w_n)} &= N\left\{-\frac{2[-iw_n+\Sigma(w_n)]}{[-iw_n+\Sigma(w_n)]^2-\bar{\Xi}(w_n)\Xi(w_n)}+\frac{\delta}{\delta\Sigma(w_n)}\left(-2\sum_m \Sigma(w_m)G(w_m)\right)\right\}\\ &= N\left\{\frac{2[-iw_n+\Sigma(w_n)]}{[w_n+i\Sigma(w_n)]^2+\bar{\Xi}(w_n)\Xi(w_n)}-2G(w_n)\right\}\end{aligned} \tag{3.128}$$

So

$$\frac{\delta S_{eff}}{\delta\Sigma(w_n)} = 0 \iff G(w_n) = \frac{-iw_n+\Sigma(w_n)}{[w_n+i\Sigma(w_n)]^2+\bar{\Xi}(w_n)\Xi(w_n)} \tag{3.129}$$

Identical reasoning for the derivative with respect to $\bar{\Xi}$

$$\begin{aligned}\frac{\delta S_{eff}}{\delta\bar{\Xi}(w_n)} &= N\left\{-\frac{-\Xi(w_n)}{[-iw_n+\Sigma(w_n)]^2-\bar{\Xi}(w_n)\Xi(w_n)}+\frac{\delta}{\delta\bar{\Xi}(w_n)}\left(-\sum_m \bar{\Xi}(w_m)F(w_m)\right)\right\}\\ &= N\left\{\frac{-\Xi(w_n)}{[w_n+i\Sigma(w_n)]^2+\bar{\Xi}(w_n)\Xi(w_n)}-F(w_n)\right\}\end{aligned} \tag{3.130}$$

So

$$\frac{\delta S_{eff}}{\delta\bar{\Xi}(w_n)} = 0 \iff F(w_n) = \frac{-\Xi(w_n)}{[w_n+i\Sigma(w_n)]^2+\bar{\Xi}(w_n)\Xi(w_n)} \tag{3.131}$$

Studying the saddle-point equations (3.124) and (3.131), it can be seen that the anomalous Green's function F and self-energy Ξ admit only the zero solution, so the previous DS-equations are reduced to

$$\begin{cases} \Sigma(\tau,\tau') = \dfrac{J^2}{32} G(\tau,\tau')^3 \\ \\ G(w_n) = \dfrac{1}{iw_n - \Sigma(w_n)} \end{cases} \tag{3.132}$$

It is immediately possible to verify that they correspond exactly (up to a multiplicative constant) to the DS-equations (3.71) of the spinless complex SYK in the case $\mu = 0$; consequently, all the studies previously carried out within the limits of low frequency and strong coupling in the case of particle-hole symmetry, related to

the spinless complex SYK model, are also valid for the spinful model studied in this section. So the exact normal Green's function at $T = 0$, within the limits described above, corresponds to

$$G(\tau) = \begin{cases} +\dfrac{D}{\sqrt{|\pi\tau|}} & \tau \ll -J^{-1} \qquad T = 0 \\ \\ -\dfrac{D}{\sqrt{|\pi\tau|}} & \tau \gg J^{-1} \qquad T = 0 \end{cases} \tag{3.133}$$

where $D \equiv \dfrac{(32\pi)^{1/4}}{\sqrt{2J}}$. While the finite temperature solutions are

$$G^{\beta}(\tau) = \begin{cases} +D\left(\dfrac{T}{sin(|\pi T\tau|)}\right)^{\frac{1}{2}} & -\frac{1}{T} < \tau < 0 \\ \\ -D\left(\dfrac{T}{sin(|\pi T\tau|)}\right)^{\frac{1}{2}} & 0 < \tau < \frac{1}{T} \end{cases} \tag{3.134}$$

In the frequency domain, by performing the appropriate Fourier transformations, we have in the case $T = 0$

$$G(w) = -i\sqrt{2}D\frac{sgn(w)}{\sqrt{|w|}} \qquad \Sigma(w) = -\frac{i}{\sqrt{2}D}sgn(w)\sqrt{|w|} \tag{3.135}$$

while at finite temperature

$$\begin{aligned} G(\pm w_n) &= \mp\frac{iD}{\sqrt{\pi T}}\frac{\Gamma(n+3/4)}{\Gamma(n+5/4)} \\ \Sigma(\pm w_n) &= \mp\frac{iJ^2D^3}{8\sqrt{\pi}}\sqrt{T}\frac{\Gamma(n+5/4)}{\Gamma(n+3/4)} \end{aligned} \tag{3.136}$$

where $w_n = (2n+1)\pi T$ with $n \in \mathbb{N}$. These latter expressions can be rewritten as

$$\begin{aligned} G(w_n) &= -sgn(w_n)\frac{iD}{\sqrt{\pi T}}\frac{\Gamma(\frac{|w_n|}{2\pi T}+\frac{1}{4})}{\Gamma(\frac{|w_n|}{2\pi T}+\frac{3}{4})} \\ \Sigma(w_n) &= -sgn(w_n)\frac{iJ^2D^3}{8\sqrt{\pi}}\sqrt{T}\frac{\Gamma(\frac{|w_n|}{2\pi T}+\frac{3}{4})}{\Gamma(\frac{|w_n|}{2\pi T}+\frac{1}{4})} \end{aligned} \tag{3.137}$$

where $w_n = (2n+1)\pi T$ with $n \in \mathbb{Z}$. It is interesting to note that these expressions can be further simplified, in fact if we consider the analytic continuation of the $\Gamma(z)$

function (with $z \in \mathbb{C}, -z \notin \mathbb{N}$), the following relation is valid (with $n \in \mathbb{Z}$)

$$\frac{\Gamma(\frac{w_n}{2\pi T}+\frac{1}{4})}{\Gamma(\frac{w_n}{2\pi T}+\frac{3}{4})} = -\frac{\Gamma(\frac{-w_n}{2\pi T}+\frac{1}{4})}{\Gamma(\frac{-w_n}{2\pi T}+\frac{3}{4})} \tag{3.138}$$

So considering the analytic continuation of the $\Gamma(z)$, the ratio between the Γ functions behaves as if it already contained the information of the *sgn* function, therefore we have:

$$\begin{aligned} G(w_n) &= -\frac{iD}{\sqrt{\pi T}}\frac{\Gamma(\frac{w_n}{2\pi T}+\frac{1}{4})}{\Gamma(\frac{w_n}{2\pi T}+\frac{3}{4})} \\ \Sigma(w_n) &= -\frac{iJ^2D^3}{8\sqrt{\pi}}\sqrt{T}\frac{\Gamma(\frac{w_n}{2\pi T}+\frac{3}{4})}{\Gamma(\frac{w_n}{2\pi T}+\frac{1}{4})} \end{aligned} \tag{3.139}$$

where $w_n = (2n+1)\pi T$ with $n \in \mathbb{Z}$.

3.3.1 Keldysh formalism for the spinful complex SYK

Since we are interested in studying the spinful complex SYK model under non-equilibrium conditions, it is important to study the relations between the objects found so far, that are in the Matsubara formalism, with the tools of the Keldysh formalism. It is necessary to return to the use of an essential object, namely the Keldysh contour and it is important to remember that the two formalisms are based on two different types of time, one Euclidean and the other Lorentzian. So using the real time theoretical framework, we know that one of the most important tools, in the Keldysh theory, is the contour-ordered Green's function G_C; in the model under consideration, the four possible configurations of G_C are as follows:

$$\begin{aligned} G^{<}(t,t') &= G_C(t_+,t'_-) = +i\frac{1}{N}\sum_{i=1}^{N}\langle \hat{c}^{\dagger}_{i\sigma}(t')\hat{c}_{i\sigma}(t)\rangle \\ G^{>}(t,t') &= G_C(t_-,t'_+) = -i\frac{1}{N}\sum_{i=1}^{N}\langle \hat{c}_{i\sigma}(t)\hat{c}^{\dagger}_{i\sigma}(t')\rangle \\ G^{\mathbb{T}}(t,t') &= G_C(t_+,t'_+) = -i\frac{1}{N}\sum_{i=1}^{N}\langle \mathbb{T}[\hat{c}_{i\sigma}(t)\hat{c}^{\dagger}_{i\sigma}(t')]\rangle \\ G^{\widetilde{\mathbb{T}}}(t,t') &= G_C(t_-,t'_-) = -i\frac{1}{N}\sum_{i=1}^{N}\langle \widetilde{\mathbb{T}}[\hat{c}_{i\sigma}(t)\hat{c}^{\dagger}_{i\sigma}(t')]\rangle \end{aligned} \tag{3.140}$$

where the subscripts $\pm$ denote in which contour's branch the times are (+ forward branch, − backward branch). Regarding instead the formalism with Euclidean time, we have previously introduced the usual normal Green's function

$$G(\tau,\tau') \equiv -\frac{1}{N}\sum_{i=1}^{N}\langle \mathbb{T}[\hat{c}_{i\sigma}(\tau)\hat{c}^{\dagger}_{i\sigma}(\tau')]\rangle \tag{3.141}$$

In order to obtain the retarded, advanced and Keldysh Green's functions in real time, it is advisable to first calculate the four components of G_C listed above. The quickest way to get them is to carry out the process of analytic continuation of the imaginary time Green's function to real time, using the transformation called *Wick rotation*, which is $\tau \to it$. Specifically, by exploiting the time translation invariance [14], they correspond to

$$G^{<}(t) = \begin{cases} iG(\tau<0\to it) & t<0 \\ iG(\tau<0\to it) & t>0 \end{cases} \qquad G^{\mathbb{T}}(t) = \begin{cases} iG(\tau<0\to it) & t<0 \\ iG(\tau>0\to it) & t>0 \end{cases}$$

$$G^{>}(t) = \begin{cases} iG(\tau>0\to it) & t<0 \\ iG(\tau>0\to it) & t>0 \end{cases} \qquad G^{\widetilde{\mathbb{T}}}(t) = \begin{cases} iG(\tau>0\to it) & t<0 \\ iG(\tau<0\to it) & t>0 \end{cases} \tag{3.142}$$

Considering the case of thermodynamic equilibrium at finite temperature T, if we insert the solutions (3.134), we find the following expressions

$$G^{<}(t) = \begin{cases} iDe^{-i\frac{\pi}{4}}\left(\frac{T}{sinh(\pi T|t|)}\right)^{\frac{1}{2}} & t<0 \\ iDe^{i\frac{\pi}{4}}\left(\frac{T}{sinh(\pi Tt)}\right)^{\frac{1}{2}} & t>0 \end{cases}$$

$$G^{>}(t) = \begin{cases} -iDe^{i\frac{\pi}{4}}\left(\frac{T}{sinh(\pi T|t|)}\right)^{\frac{1}{2}} & t<0 \\ -iDe^{-i\frac{\pi}{4}}\left(\frac{T}{sinh(\pi Tt)}\right)^{\frac{1}{2}} & t>0 \end{cases}$$

$$G^{\mathbb{T}}(t) = \begin{cases} iDe^{-i\frac{\pi}{4}}\left(\frac{T}{sinh(\pi T|t|)}\right)^{\frac{1}{2}} & t<0 \\ -iDe^{-i\frac{\pi}{4}}\left(\frac{T}{sinh(\pi Tt)}\right)^{\frac{1}{2}} & t>0 \end{cases}$$

$$G^{\widetilde{\mathbb{T}}}(t) = \begin{cases} -iDe^{i\frac{\pi}{4}}\left(\frac{T}{sinh(\pi T|t|)}\right)^{\frac{1}{2}} & t<0 \\ iDe^{i\frac{\pi}{4}}\left(\frac{T}{sinh(\pi Tt)}\right)^{\frac{1}{2}} & t>0 \end{cases} \tag{3.143}$$

where the identity $sin(it) = i\,sinh(t)$ $(\forall t \in \mathbb{R})$ has been used and, in the calculation of $\sqrt{\pm i}$ factors, the principal value of the respective root has always been considered. We now have all the tools needed to calculate G^R,G^A and G^K, in fact one obtains for the retarded component

$$\begin{aligned} G^R(t) &= -i\Theta(t)\frac{1}{N}\sum_{i=1}^{N}\langle\{\hat{c}_{i\sigma}(t),\hat{c}^\dagger_{i\sigma}(0)\}\rangle = \Theta(t)(G^>(t)-G^<(t)) \\ &= \Theta(t)\left[-iD\left(\frac{T}{sinh(\pi Tt)}\right)^{\frac{1}{2}}\left(e^{-i\frac{\pi}{4}}+e^{i\frac{\pi}{4}}\right)\right] = -i\sqrt{2}D\,\Theta(t)\left(\frac{T}{sinh(\pi T|t|)}\right)^{\frac{1}{2}} \end{aligned} \tag{3.144}$$

and for the advanced

$$\begin{aligned} G^A(t) &= +i\Theta(-t)\frac{1}{N}\sum_{i=1}^{N}\langle\{\hat{c}_{i\sigma}(t),\hat{c}^\dagger_{i\sigma}(0)\}\rangle = -\Theta(-t)(G^>(t)-G^<(t)) \\ &= -\Theta(-t)\left[-iD\left(\frac{T}{sinh(\pi Tt)}\right)^{\frac{1}{2}}\left(e^{i\frac{\pi}{4}}+e^{-i\frac{\pi}{4}}\right)\right] = i\sqrt{2}D\,\Theta(-t)\left(\frac{T}{sinh(\pi T|t|)}\right)^{\frac{1}{2}} \end{aligned} \tag{3.145}$$

For the Keldysh Green's function, one obtains

$$\begin{aligned} G^K(t) &= -i\frac{1}{N}\sum_{i=1}^{N}\langle[\hat{c}_{i\sigma}(t),\hat{c}^\dagger_{i\sigma}(0)]\rangle = G^>(t)+G^<(t) \\ &= -\sqrt{2}D\,sgn(t)\left(\frac{T}{sinh(\pi T|t|)}\right)^{\frac{1}{2}} \end{aligned} \tag{3.146}$$

To rewrite these quantities in the frequency domain, we calculate the associated Fourier transformations or, in the case of $G^{R,A}$, we can perform the analytic continuation from Matsubara frequencies to real frequencies, through the transformation $iw_n \to w \pm i0^+$, this gives the following result

$$G^R(w) = -\frac{iD}{\sqrt{\pi T}}\frac{\Gamma(\frac{1}{4}-\frac{iw}{2\pi T})}{\Gamma(\frac{3}{4}-\frac{iw}{2\pi T})} \tag{3.147}$$

Of course, the usual relation $G^A(w) = [G^R(w)]^*$ is valid. Now that we have got the retarded component, the easiest way to get $G^K(w)$ is to use the Fluctuation-

Dissipation theorem, so

$$G^K(w) = tanh\left(\frac{w}{2T}\right)[G^R(w) - G^A(w)] = 2i\,tanh\left(\frac{w}{2T}\right)Im[G^R(w)] \tag{3.148}$$

that satisfies the property $G^K(w) = -[G^K(w)]^*$.

Now we have all the tools needed to write the DS equations, at thermal equilibrium, in Lorentzian time. In fact, performing the analytic continuation of the elements in the first eq(3.132), one gets

$$\Sigma^>(t,t') = \frac{J^2}{32}G^>(t,t')^3 \qquad\qquad \Sigma^<(t,t') = \frac{J^2}{32}G^<(t,t')^3 \tag{3.149}$$

where $\Sigma^>$ is the element with coordinate $(2,1)$ of the self-energy matrix $\hat{\Sigma}$ in the Keldish space (this is true if we are in $\pm$ field notation and so we have not used the Larkin-Ovchinnikov transformation), while $\Sigma^<$ is the $(1,2)$ element. Using the above written expressions and recalling that

$$\begin{aligned} G^R(t,t') &= \frac{1}{2}[G^{\mathbb{T}}(t,t') - G^{\widetilde{\mathbb{T}}}(t,t') + G^>(t,t') - G^<(t,t')] \\ G^A(t,t') &= \frac{1}{2}[G^{\mathbb{T}}(t,t') - G^{\widetilde{\mathbb{T}}}(t,t') - G^>(t,t') + G^<(t,t')] \\ G^K(t,t') &= \frac{1}{2}[G^{\mathbb{T}}(t,t') + G^{\widetilde{\mathbb{T}}}(t,t') + G^>(t,t') + G^<(t,t')] \end{aligned} \tag{3.150}$$

we can obtain the following useful equation (using temporal translation invariance)

$$\begin{aligned} \Sigma^R(t) &= \Theta(t)(\Sigma^>(t) - \Sigma^<(t)) \\ &= \frac{J^2}{32}\frac{1}{4}\left(G^K(t)^2[G^R(t)^* - 2G^R(t)] + G^R(t)^2G^R(t)^*\right) \end{aligned} \tag{3.151}$$

Regarding instead the analytic continuation of the second eq(3.132), being in the frequency space we obtain the retarded components, so

$$G^R(w)[w - \Sigma^R(w)] = 1 \tag{3.152}$$

To conclude the map that links the objects of the DS equations in real time, it is necessary to find the relation between $G^>$ and G^R. If the system under consideration is in thermal equilibrium then this relation will be governed, as usual, by the fluctuation-dissipation theorem and therefore one gets

$$G^>(w) = -\frac{i}{e^{-\beta w}+1}A(w) \qquad\qquad G^<(w) = +\frac{i}{e^{\beta w}+1}A(w) \tag{3.153}$$

where $A(w)$ is the spectral function and it is defined as $A(w) = -2ImG^R(w)$. To

demonstrate such relations, let us start by showing the following

$$\begin{aligned} G^{<}(-t) = G^{<}(-t,0) &= +i\frac{1}{N}\sum_{i=1}^{N}\langle \hat{c}^{\dagger}_{i\sigma}(0)\hat{c}_{i\sigma}(-t)\rangle = +i\frac{1}{N}\sum_{i=1}^{N}\langle \hat{c}^{\dagger}_{i\sigma}(t)\hat{c}_{i\sigma}(0)\rangle \\ &= +i\frac{1}{N}\sum_{i=1}^{N}\frac{1}{Tr[e^{-\beta\hat{H}}]}Tr[e^{-\beta\hat{H}}\hat{c}^{\dagger}_{i\sigma}(t)\hat{c}_{i\sigma}(0)] \\ &= +i\frac{1}{N}\sum_{i=1}^{N}\frac{1}{Tr[e^{-\beta\hat{H}}]}Tr[e^{i(t+i\beta)\hat{H}}\hat{c}^{\dagger}_{i\sigma}(0)e^{-i(t+i\beta)\hat{H}}e^{-\beta\hat{H}}\hat{c}_{i\sigma}(0)] \\ &= +i\frac{1}{N}\sum_{i=1}^{N}\frac{1}{Tr[e^{-\beta\hat{H}}]}Tr[e^{-\beta\hat{H}}\hat{c}_{i\sigma}(0)\hat{c}^{\dagger}_{i\sigma}(t+i\beta)] = -G^{>}(-t-i\beta) \end{aligned} \tag{3.154}$$

So if we then take the associated Fourier transformations, we find the equation

$$G^{<}(w) = -e^{-\beta w}G^{>}(w) \tag{3.155}$$

Studying the Fourier transformation of the retarded component

$$\begin{aligned} G^{R}(w) &= \int_{-\infty}^{+\infty} dt\, e^{iwt}\Theta(t)[G^{>}(t) - G^{<}(t)] \\ &= \int_{-\infty}^{+\infty}\frac{dw'}{2\pi}\, G^{>}(w')[1+e^{-\beta w'}]\int_{-\infty}^{+\infty} dt\, e^{i(w-w')t}\Theta(t) \\ &= \int_{-\infty}^{+\infty}\frac{dw'}{2\pi}\, G^{>}(w')[1+e^{-\beta w'}]\frac{i}{w-w'+i0^{+}} \end{aligned} \tag{3.156}$$

The term $G^{>}(w)$ is purely negative imaginary, so in order to calculate the imaginary component of $G^{R}(w)$, we have to consider only the real part of $i(w-w'+i0^{+})^{-1}$, namely $\pi\delta(w-w')$, in this way one gets

$$Im[G^{R}(w)] = -\frac{i}{2}G^{>}(w)[1+e^{-\beta w}] \tag{3.157}$$

Once this equation is obtained, it is straightforward to derive the expressions in eq(3.153). However, we are interested in the study of the system under non-equilibrium conditions, so the previous relation is not sufficient. In order to obtain a more general expression we can rewrite the second eq(3.132) in the imaginary time space and using the convolution operation we get

$$\partial_{\tau}G(\tau) = \delta(\tau) + \int d\tau'\Sigma(\tau-\tau')G(\tau') \tag{3.158}$$

By performing the analytical continuation operation on this equation, we obtain the

following two expressions [70, 71]:

$$\begin{aligned} i\partial_t G^{>}(t,t') &= \int_{-\infty}^{+\infty} dt_1 \Big[\Sigma^R(t,t_1)G^{>}(t_1,t') + \Sigma^{>}(t,t_1)G^A(t_1,t')\Big] \\ -i\partial_{t'} G^{>}(t,t') &= \int_{-\infty}^{+\infty} dt_1 \Big[G^R(t,t_1)\Sigma^{>}(t_1,t') + G^{>}(t,t_1)\Sigma^A(t_1,t')\Big] \end{aligned} \tag{3.159}$$

These are the so-called *Kadanoff–Baym* equations and they are extremely useful because they are valid in totally general conditions, therefore also for systems in non-equilibrium. However, as will be shown in the next chapter, our analysis will use other tools and not exploit these last two equations.

Chapter 4

Transport properties of the spinful complex SYK model

In this chapter we will address the final aim of this thesis, that is the study of the transport properties of a quantum dot described by the spinful complex fermion version of the SYK model. The global system under consideration corresponds to the setup outlined in Section (2.2), where the Hamiltonian of the dot is given by the eq(3.101), so

$$\hat{H}_{dot} = \frac{1}{2} \sum_{i,j,k,l=1}^{N} \sum_{\sigma\sigma'} J_{ijkl} \left[\hat{d}^{\dagger}_{i\sigma} \hat{d}^{\dagger}_{j\sigma'} \hat{d}_{k\sigma'} \hat{d}_{l\sigma} + \hat{d}^{\dagger}_{l\sigma} \hat{d}^{\dagger}_{k\sigma'} \hat{d}_{j\sigma'} \hat{d}_{i\sigma} \right] \tag{4.1}$$

This means that we are considering the case where the intrinsic chemical potential of the dot is zero and thus it shows a particle-hole symmetry. Furthermore, in order to use in such a study of the dot in non-equilibrium conditions, the analytic expressions of the real time SYK Green's functions (calculated in the previous chapter, Section (3.3.1)), it is necessary to limit ourselves to the study of the case where the two electrodes of the LCR system are at the same temperature T. The NESS configuration will be generated solely by the presence of an external voltage V applied to the electrodes. Obviously an additional necessary condition is to consider the case of large N and examine the system in the conformal limit, so under conditions for which $\beta J \gg 1$. The study is limited to the case of a single temperature, because in the case, where the temperatures of the leads are very different, it is therefore not possible to exploit the properties of the linear response regime. This situation presents considerable difficulties, due to the fact that the SYK model is characterized by strong quartic interactions, which make the introduction of reasonable approximations useful for the study of the model extremely complicated. An important note to consider during the reading of the following pages is that, in this chapter, we will use the same formalism and notation used in Section (2.2), so the

previously introduced objects will not be redefined and presented.

So, in order to study the transport properties of the dot, we are interested in calculating the expectation values of the related physical observables (under NESS conditions). As seen in previous chapters, they can be described as a function of the exact Green's functions of the interacting and coupled quantum dot; an example is the expression of the stationary current described by eq(2.45). Given the configuration of our system, we have previously derived that these exact Green's functions $\hat{G}_{nm}$ satisfy the following Keldysh-Dyson equation:

$$\hat{G}_{nm}(w) = \hat{g}_{nm}(w) + \sum_{a=L,R} \sum_{i,j=1}^{N} \hat{g}_{ni}(w)\hat{\Sigma}_{a,ij}(w)\hat{G}_{jm}(w) \tag{4.2}$$

whose corresponding *Dyson* equation is:

$$\hat{G}^{-1}_{nm}(w) = \hat{g}^{-1}_{nm}(w) - \hat{\Sigma}_{L,nm}(w) - \hat{\Sigma}_{R,nm}(w) \tag{4.3}$$

The symbol $\hat{g}_{nm}$ describes the (2×2) block Green's function matrix of the interacting and decoupled quantum dot related to the sites (n, m). Since, as seen in the previous chapter, the Green's functions of SYK models are different from zero only in the case where they are associated with two equal sites, one gets

$$\hat{g}_{nm}(w) = \delta_{nm}\hat{g}(w) \tag{4.4}$$

where $\hat{g}$ is the Keldysh (2×2) Green's functions matrix that describes each (n, n)-correlators related to the isolated spinful complex version of the SYK quantum dot at thermodynamic equilibrium. In fact, in the limit of large N, the symmetry of the SYK model makes all (n, n)-correlators equal. Displaying $\hat{g}$'s components, it corresponds to

$$\hat{g}(w) = \begin{pmatrix} g^R(w) & g^K(w) \\ 0 & g^A(w) \end{pmatrix} \tag{4.5}$$

where

$$g^R(w) = -\frac{iD}{\sqrt{\pi T}} \frac{\Gamma(\frac{1}{4} - \frac{iw}{2\pi T})}{\Gamma(\frac{3}{4} - \frac{iw}{2\pi T})} \qquad g^K(w) = 2i\, tanh\left(\frac{w}{2T}\right) Im[g^R(w)] \tag{4.6}$$

and $g^A(w) = [g^R(w)]^*$. Note that, although the above expressions are identical to those in Equations (3.147,3.148), the associated Green's functions are formally different. In fact, in the previous chapter we defined them using the formalism $\frac{1}{N}\sum_{i=1}^{N}$, whereas now the correlators are associated with a single site, without considering any average on different sites. However, the two objects have equal expression, because in the limit of large N each site of the SYK model is physically equivalent

to the others and consequently the associated correlators are also equal (obviously after performing the disorder average operation, which we take for granted in the configuration considered). Hence, considering that the Green's functions of the isolated dot are diagonal over the dot's sites, the Keldysh-Dyson equation corresponds to

$$\begin{aligned}\hat{G}_{nm}(w) &= \delta_{nm}\hat{g}(w) + \sum_{a=L,R}\sum_{i,j=1}^{N}\delta_{ni}\hat{g}(w)\hat{\Sigma}_{a,ij}(w)\hat{G}_{jm}(w) \\ &= \delta_{nm}\hat{g}(w) + \hat{g}(w)\sum_{a=L,R}\sum_{j=1}^{N}\hat{\Sigma}_{a,nj}(w)\hat{G}_{jm}(w)\end{aligned} \tag{4.7}$$

where

$$\hat{\Sigma}_{a,nm}(\omega) = \frac{1}{N}\sum_{\mathbf{k}} W^{*}_{\mathbf{k}a,n}\hat{g}_{\mathbf{k}a}(\omega-\mu_a)W_{\mathbf{k}a,m} \tag{4.8}$$

where the $1/N$ term comes from the rescaling of the tunneling matrix elements with a factor $1/\sqrt{N}$. We assumed that the tunneling matrix elements are $\mathbf{k}$-independent and, for symmetry conditions between the different dot's sites, we also assume that they are equal for all the possible sites n, so $W_{\mathbf{k}a,n} = W_{a,n} \equiv W_a$. In this way the (n,m)-self-energy contribution associated with the coupling of the quantum dot with the lead-a corresponds to

$$\hat{\Sigma}_{a,nm}(\omega) = \frac{|W_a|^2}{N}\sum_{\mathbf{k}}\hat{g}_{\mathbf{k}a}(\omega-\mu_a) = \frac{|W_a|^2}{N}\hat{g}_a(\omega-\mu_a) \tag{4.9}$$

where $\hat{g}_a(\omega-\mu_a) \equiv \sum_{\mathbf{k}}\hat{g}_{\mathbf{k}a}(\omega-\mu_a)$. Inserting this expression of the self-energy contributions into the equation (4.7), one gets

$$\begin{aligned}\hat{G}_{nm}(w) &= \delta_{nm}\hat{g}(w) + \hat{g}(w)\sum_{a=L,R}\frac{|W_a|^2}{N}\hat{g}_a(\omega-\mu_a)\sum_{j=1}^{N}\hat{G}_{jm}(w) \\ &= \delta_{nm}\hat{g}(w) + \hat{g}(w)\frac{\hat{A}(w)}{N}\sum_{j=1}^{N}\hat{G}_{jm}(w)\end{aligned} \tag{4.10}$$

where

$$\hat{A}(w) \equiv \sum_{a=L,R}|W_a|^2\hat{g}_a(\omega-\mu_a) \tag{4.11}$$

Using such formalism, the eq(4.3) can be rewritten as

$$\begin{aligned}\hat{G}^{-1}_{nm}(w) &= \hat{g}^{-1}_{nm}(w) - \hat{\Sigma}_{L,nm}(w) - \hat{\Sigma}_{R,nm}(w) = \delta_{nm}\hat{g}^{-1}(w) - \sum_{a=L,R}\hat{\Sigma}_{a,nm} \\ &= \delta_{nm}\hat{g}^{-1}(w) - \frac{\hat{A}(w)}{N}\end{aligned} \tag{4.12}$$

Note that $\hat{g}^{-1}_{nm}(w)$ is the (n,m)-element of the inverse of the matrix with elements $\hat{g}_{nm}(w)$, it is equal to $\delta_{nm}\hat{g}^{-1}(w)$ only because the matrix just mentioned is diagonal, in fact we know that $\hat{g}_{nm}(w) = \delta_{nm}\hat{g}(w)$. Let us now focus on the matrix $\hat{G}^{-1}$ with elements $\hat{G}^{-1}_{nm}(w)$. The above equation tells us that $\hat{G}^{-1}$ corresponds to the sum of a constant diagonal matrix and a constant matrix (with all elements equal), hence its inverse $\hat{G}$ can also be expressed as the sum of two matrices, specifically of a constant diagonal matrix and a constant matrix. So $\hat{G}$ will have all diagonal elements equal to each other and all off-diagonal elements equal to each other. In order to make the notation more compact we will use the following objects

$$\begin{aligned} &\hat{G}_{nn} = \hat{G}_d \quad \forall n \in [1,2,...,N] \\ &\hat{G}_{nm} = \hat{G}_{od} \quad \forall n,m \in [1,2,...,N] \;\; with \;\; n \neq m \end{aligned} \tag{4.13}$$

where $d = diagonal$ and od=*off-diagonal*. Adding this information to the eq(4.10), we derive the following two expressions

$$\begin{cases} \hat{G}_d(w) = \hat{g}(w) + \dfrac{1}{N}\hat{g}(w)\hat{A}(w)\Big[\hat{G}_d(w) + (N-1)\hat{G}_{od}\Big] \\ \\ \hat{G}_{od}(w) = \dfrac{1}{N}\hat{g}(w)\hat{A}(w)\Big[\hat{G}_d(w) + (N-1)\hat{G}_{od}\Big] \end{cases} \tag{4.14}$$

From such a system of equations, it is easy to realize that the diagonal component of the exact Green's function matrix is related to the off-diagonal one by the following simple relation

$$\hat{G}_d(w) = \hat{g}(w) + \hat{G}_{od}(w) \tag{4.15}$$

By doing this substitution in the first equation of the above system, one gets

$$\hat{G}_d(w) = \hat{g}(w) + \frac{1}{N}\hat{g}(w)\hat{A}(w)\Big[N\hat{G}_d(w) - (N-1)\hat{g}(w)\Big] \tag{4.16}$$

so

$$\begin{cases} \hat{G}_d(w) = \Big[1 - \hat{g}(w)\hat{A}(w)\Big]^{-1}\Big[1 + \Big(\frac{1}{N} - 1\Big)\hat{g}(w)\hat{A}(w)\Big]\hat{g}(w) \\ \\ \hat{G}_{od}(w) = \Big[1 - \hat{g}(w)\hat{A}(w)\Big]^{-1}\Big[1 + \Big(\frac{1}{N} - 1\Big)\hat{g}(w)\hat{A}(w)\Big]\hat{g}(w) - \hat{g}(w) \end{cases} \tag{4.17}$$

We know that, given the triangular structure of all the (2×2)-matrices in the above equations, these relations are also valid for the retarded and advanced components,

so

$$\begin{cases} G_d^{R(A)}(w) = \dfrac{1+\left(\frac{1}{N}-1\right)g^{R(A)}(w)A^{R(A)}(w)}{1-g^{R(A)}(w)A^{R(A)}(w)} \times g^{R(A)}(w) \\ \\ G_{od}^{R(A)}(w) = \dfrac{1+\left(\frac{1}{N}-1\right)g^{R(A)}(w)A^{R(A)}(w)}{1-g^{R(A)}(w)A^{R(A)}(w)} \times g^{R(A)}(w) - g^{R(A)}(w) \end{cases} \tag{4.18}$$

4.1 Stationary current in NESS

A quantity that is of central importance, in our analysis of the system, is the continuous stationary current of electrons passing through the quantum dot in a NESS configuration. In Section (2.2), we have managed to express its expectation value with the following sum of integrals in the frequency domain

$$I = \frac{I_L - I_R}{2} = \frac{ie}{2h}\sum_{n,m=1}^{N}\int_{-\infty}^{+\infty} dw \Bigg[\Big(\Gamma_{nm}^{L}(w-eV) - \Gamma_{nm}^{R}(w)\Big)G_{mn}^{K}(\omega) + \\ - \Big(F_L(\omega-eV)\Gamma_{nm}^{L}(w-eV) - F_R(\omega)\Gamma_{nm}^{R}(w)\Big)[G_{mn}^{R}(\omega) - G_{mn}^{A}(\omega)]\Bigg] \tag{4.19}$$

that corresponds to the so-called *Meir-Wingreen-Jauho formula*. In our case, the couplings Γ_{nm}^{a} can be expressed as

$$\Gamma_{nm}^{a}(w) = \frac{2\pi}{N}\sum_{\mathbf{k}} W_{\mathbf{k}a,n}^{*}\delta(w-w_{\mathbf{k}a})W_{\mathbf{k}a,m} = \frac{2\pi}{N}|W_a|^2\nu_a(w) \equiv \frac{C^a(w)}{N} \tag{4.20}$$

where $C^a(w) \equiv 2\pi|W_a|^2\nu_a(w)$. The expectation value of the current can be expressed as the sum of two contributions, one associated with diagonal terms (N factors) and one associated with off-diagonal terms (N^2-N factors), specifically it can be written as

$$I = \frac{ie}{2h}\Big[NI_d + (N^2-N)I_{od}\Big] \tag{4.21}$$

where

$$I_{d(od)} = \frac{1}{N}\int_{-\infty}^{+\infty} dw \Bigg[\Big(C^L(w-eV) - C^R(w)\Big)G_{d(od)}^{K}(\omega) + \\ - \Big(F_L(\omega-eV)C^L(w-eV) - F_R(\omega)C^R(w)\Big)[G_{d(od)}^{R}(\omega) - G_{d(od)}^{A}(\omega)]\Bigg] \tag{4.22}$$

The Keldysh components of the exact Green's functions can be formulated using the usual relation derived from the Dyson equation, which is

$$G^K_{mn}(w) = \sum_{i,j=1}^{N} G^R_{mi}(w)\left(\Sigma^K_{syk,ij}(w) + \sum_{a=L,R}\Sigma^K_{a,ij}(w)\right)G^A_{jn}(w) \tag{4.23}$$

where

$$\Sigma^K_{syk,ij}(w) \equiv \delta_{ij}\Sigma^K(w) \tag{4.24}$$

that being calculated at the thermodynamic equilibrium corresponds to

$$\Sigma^K(w) = tanh\left(\frac{w}{2T}\right)[\Sigma^R(w) - \Sigma^A(w)] = 2i\,tanh\left(\frac{w}{2T}\right)Im[\Sigma^R(w)] \tag{4.25}$$

with

$$\Sigma^R(w) = -\frac{iJ^2D^3}{8\sqrt{\pi}}\sqrt{T}\frac{\Gamma(\frac{3}{4} - \frac{iw}{2\pi T})}{\Gamma(\frac{1}{4} - \frac{iw}{2\pi T})} \tag{4.26}$$

The calculation of the stationary current can be simplified if we introduce the *wide-band approximation* for the two electrodes in our theoretical framework and if we consider that the LCR system is in the specific configuration of symmetric junctions. The first approximation is to neglect the real part of the $\hat{g}_{\mathbf{k}a}(w)$ components and to consider a constant density of states ν_a evaluated at the Fermi energy. Consequently one gets:

$$\begin{aligned} g^R_a(w) &= \sum_{\mathbf{k}} g^R_{\mathbf{k}a}(\omega) = i\sum_{\mathbf{k}} Im[g^R_{\mathbf{k}a}(\omega)] = \frac{i}{2}\sum_{\mathbf{k}} Im[-2\pi i\delta(w - w_{\mathbf{k}a})] \\ &= -i\pi\nu_a(w) \equiv -i\pi\nu_a \end{aligned} \tag{4.27}$$

Thus, it follows that

$$\Gamma^L_{nm}(w - eV) = \frac{2\pi}{N}|W_L|^2\nu_L \equiv \frac{C^L}{N} \qquad \Gamma^R_{nm}(w) = \frac{2\pi}{N}|W_R|^2\nu_R \equiv \frac{C^R}{N} \tag{4.28}$$

Next we impose the following assumption based on the symmetry of the junctions, so

$$\Gamma^L_{mn} = \Gamma^R_{mn} \equiv \Gamma_{mn} \equiv \frac{C}{N} \tag{4.29}$$

With these two assumptions, we find that the expression of $I_{d(od)}$ (described by eq(4.22)) is greatly simplified, in fact it becomes

$$I_{d(od)} = \frac{2C}{N}\int_{-\infty}^{+\infty} dw\Big(f_L(\omega - eV) - f_R(\omega)\Big)\Big[G^R_{d(od)}(\omega) - G^A_{d(od)}(\omega)\Big] \tag{4.30}$$

where the Fermi distribution functions of the leads correspond to

$$f_L(w-eV) = \left[exp\left(\frac{w-eV}{T}\right)+1\right]^{-1} \qquad f_R(w) = \left[exp\left(\frac{w}{T}\right)+1\right]^{-1} \tag{4.31}$$

Note that, as stated at the beginning of the chapter, we are considering the specific case in which both leads and dot are kept at the same temperature. The only physical element, that induces a dot's NESS configuration, is the presence of the external voltage $eV = \mu_L - \mu_R$ (where μ_R is set to zero). Recalling now the expressions of eq(4.18), we note that the following relation holds

$$G^R_{od}(\omega) - G^A_{od}(\omega) = \Big(G^R_d(\omega) - G^A_d(\omega)\Big) - \Big(g^R(w) - g^A(w)\Big) \tag{4.32}$$

It follows that the off-diagonal component I_{od} of the current can be expressed as

$$I_{od} = I_d - \frac{2C}{N}\int_{-\infty}^{+\infty} dw\Big(f_L(\omega - eV) - f_R(\omega)\Big)\Big[g^R(\omega) - g^A(\omega)\Big] \tag{4.33}$$

So the total stationary current in the NESS configuration is definable as

$$I = \frac{ie}{2h}\Big\{N^2 I_d - 2C(N-1)\int_{-\infty}^{+\infty} dw\Big(f_L(\omega - eV) - f_R(\omega)\Big)\Big[g^R(\omega) - g^A(\omega)\Big]\Big\} \tag{4.34}$$

The subtraction $(g^R - g^A)$ corresponds to $2iIm[g^R]$, so in order to calculate this object, let's start with the study of g^R:

$$g^R(w) = -\frac{iD}{\sqrt{\pi T}}\frac{\Gamma(\frac{1}{4} - \frac{iw}{2\pi T})}{\Gamma(\frac{3}{4} - \frac{iw}{2\pi T})} \tag{4.35}$$

Knowing that the gamma functions satisfy the identity $\Gamma(1-z)\Gamma(z) = \dfrac{\pi}{sin(\pi z)}$, by choosing $z = \frac{3}{4} - ix$, with $x = \frac{w}{2\pi T}$, one gets

$$\frac{1}{\Gamma(\frac{3}{4} - ix)} = \Gamma\left(\frac{1}{4} + ix\right)\frac{1}{\pi} sin\left(\frac{3\pi}{4} - i\pi x\right) \tag{4.36}$$

So the term in the expression for g^R which contains a ratio of gamma functions can be expressed a

$$\frac{\Gamma(\frac{1}{4} - ix)}{\Gamma(\frac{3}{4} - ix)} = \Gamma\left(\frac{1}{4} - ix\right)\Gamma\left(\frac{1}{4} + ix\right)\frac{1}{\pi} sin\left(\frac{3\pi}{4} - i\pi x\right) \tag{4.37}$$

Using the property $\Gamma(\bar{z}) = \overline{\Gamma(z)}$, the retarded g^R can be written as

$$g^R(w) = -\frac{iD}{\pi\sqrt{\pi T}}\left|\Gamma\left(\frac{1}{4}+ix\right)\right|^2 sin\left(\frac{3\pi}{4}-i\pi x\right) \tag{4.38}$$

Now using the following identity

$$sin(x+iy) = sin(x)cosh(y) + i\,cos(x)sinh(y) \tag{4.39}$$

one gets

$$\begin{aligned} Im[g^R(w)] &= -\frac{D}{\pi\sqrt{\pi T}}\left|\Gamma\left(\frac{1}{4}+ix\right)\right|^2 sin\left(\frac{3\pi}{4}\right)cosh(-\pi x) \\ &= -\frac{D}{\pi\sqrt{2\pi T}}\left|\Gamma\left(\frac{1}{4}+\frac{iw}{2\pi T}\right)\right|^2 cosh\left(\frac{w}{2T}\right) \end{aligned} \tag{4.40}$$

It is also useful to directly derive the real component of g^R, that is:

$$\begin{aligned} Re[g^R(w)] &= \frac{D}{\pi\sqrt{\pi T}}\left|\Gamma\left(\frac{1}{4}+ix\right)\right|^2 cos\left(\frac{3\pi}{4}\right)sinh(-\pi x) \\ &= \frac{D}{\pi\sqrt{2\pi T}}\left|\Gamma\left(\frac{1}{4}+\frac{iw}{2\pi T}\right)\right|^2 sinh\left(\frac{w}{2T}\right) \end{aligned} \tag{4.41}$$

Following the same logical steps and using the identity $cosh(2x) = cosh^2(x) + sinh^2(x)$, the imaginary component of Σ^R corresponds to:

$$Im[\Sigma^R(w)] = -\frac{J^2D^3}{4}\sqrt{\frac{\pi T}{2}}\frac{1}{|\Gamma(\frac{1}{4}+\frac{iw}{2\pi T})|^2}\frac{cosh(\frac{w}{2T})}{cosh(\frac{w}{T})} \tag{4.42}$$

So in order to obtain the stationary current under NESS conditions (in the chosen theoretical framework), the last component that needs to be calculated is I_d, which is

$$\begin{aligned} I_d = \frac{2C}{N}\int_{-\infty}^{+\infty} dw\Big(f_L(\omega-eV) - f_R(\omega)\Big)\Big\{\Big[1-g^R(w)A^R(w)\Big]^{-1}\Big[1+\Big(\frac{1}{N}-1\Big)g^R(w)\times \\ A^R(w)\Big]g^R(w) - \Big[1-g^A(w)A^A(w)\Big]^{-1}\Big[1+\Big(\frac{1}{N}-1\Big)g^A(w)A^A(w)\Big]g^A(w)\Big\} \end{aligned} \tag{4.43}$$

First, we focus on the retarded and advanced components of the matrix $\hat{A}$ which carry with them the information of the interaction's intensity between the quantum dot and the two electrodes. We start by calculating the retarded component, insert-

ing, in its initial expression, the information related to the considered assumptions:

$$\begin{aligned} A^R(w) &= \sum_{a=L,R} |W_a|^2 g_a^R(\omega-\mu_a) = \sum_{a=L,R} |W_a|^2 \sum_{\mathbf{k}} g_{\mathbf{k}a}^R(\omega-\mu_a) = \sum_{a=L,R} |W_a|^2(-i\pi\nu_a) \\ &= -i\pi(|W_L|^2\nu_L + |W_R|^2\nu_R) = -i\pi\left(\frac{C}{2\pi}+\frac{C}{2\pi}\right) = -iC \end{aligned} \tag{4.44}$$

Similarly, for the advanced component one gets

$$\begin{aligned} A^A(w) &= \sum_{a=L,R} |W_a|^2 g_a^A(\omega-\mu_a) = \sum_{a=L,R} |W_a|^2 \sum_{\mathbf{k}} g_{\mathbf{k}a}^A(\omega-\mu_a) \\ &= \sum_{a=L,R} |W_a|^2 \sum_{\mathbf{k}} [g_{\mathbf{k}a}^R(\omega-\mu_a)]^* = [A^R(w)]^* = iC \end{aligned} \tag{4.45}$$

Now we finally have all the tools needed to calculate I_d. The last operation to be carried out is that of formulating the content within the curly brackets of eq(4.43), as a function of the objects of which we have the analytical expression, i.e. the real and imaginary components of $g^R(w)$. We start by making the components of $\hat{A}$ explicit and writing the content as a single fraction:

$$g^R(w)\frac{1+\left(\frac{1}{N}-1\right)g^R(w)A^R(w)}{1-g^R(w)A^R(w)} - g^A(w)\frac{1+\left(\frac{1}{N}-1\right)g^A(w)A^A(w)}{1-g^A(w)A^A(w)} =$$

$$g^R(w)\frac{1-iC\left(\frac{1}{N}-1\right)g^R(w)}{1+iCg^R(w)} - g^A(w)\frac{1+iC\left(\frac{1}{N}-1\right)g^A(w)}{1-iCg^A(w)} =$$

$$\frac{g^R(w)\left(1-iCg^A\right)\left[1-iC(\frac{1}{N}-1)g^R(w)\right] - g^A(w)\left(1+iCg^R\right)\left[1+iC(\frac{1}{N}-1)g^A(w)\right]}{\left(1+iCg^R(w)\right)\left(1-iCg^A(w)\right)} \tag{4.46}$$

The denominator of this fraction corresponds to

$$den = 1+iC\left(g^R(w)-g^A(w)\right)+C^2|g^R(w)|^2 \tag{4.47}$$

while the numerator is equal to

$$\begin{aligned} num = g^R(w)\Bigg[1 - iC\bigg(\frac{1}{N} - 1\bigg)g^R(w) - iCg^A(w) - C^2\bigg(\frac{1}{N} - 1\bigg)g^R(w)g^A(w)\Bigg] + \\ - g^A(w)\Bigg[1 + iC\bigg(\frac{1}{N} - 1\bigg)g^A(w) + iCg^R(w) - C^2\bigg(\frac{1}{N} - 1\bigg)g^R(w)g^A(w)\Bigg] \end{aligned}$$

$$\begin{aligned} = g^R(w) - iC\bigg(\frac{1}{N} - 1\bigg)[g^R(w)]^2 - iCg^R(w)g^A(w) - C^2\bigg(\frac{1}{N} - 1\bigg)[g^R(w)]^2 g^A(w) + \\ - g^A(w) - iC\bigg(\frac{1}{N} - 1\bigg)[g^A(w)]^2 - iCg^R(w)g^A(w) + C^2\bigg(\frac{1}{N} - 1\bigg)g^R(w)[g^A(w)]^2 \end{aligned}$$

$$\begin{aligned} = \Big(g^R(w) - g^A(w)\Big) - iC\bigg(\frac{1}{N} - 1\bigg)\Big([g^R(w)]^2 + [g^A(w)]^2\Big) - 2iCg^R(w)g^A(w) + \\ - C^2\bigg(\frac{1}{N} - 1\bigg)g^R(w)g^A(w)\Big(g^R(w) - g^A(w)\Big) \end{aligned} \tag{4.48}$$

Exploiting the fact that $g^A(w) = [g^R(w)]^*$, we can rewrite this fraction as a function of the real and imaginary components of $g^R(w)$ (of which we have the analytical expressions). In this way the denominator is equal to

$$den = 1 - 2CIm[g^R(w)] + C^2\Big((Re[g^R(w)])^2 + (Im[g^R(w)])^2\Big) \tag{4.49}$$

while the numerator

$$\begin{aligned} num = 2iIm[g^R(w)] - iC\bigg(\frac{1}{N} - 1\bigg)2\Big[(Re[g^R(w)])^2 - (Im[g^R(w)])^2\Big] + \\ - 2iC|g^R(w)|^2 - C^2\bigg(\frac{1}{N} - 1\bigg)|g^R(w)|^2 2iIm[g^R(w)] \end{aligned} \tag{4.50}$$

Now, using the analytical expressions described by the equations (4.40,4.41), we will have that the squared modulus of g^R corresponds to

$$\begin{aligned} |g^R(w)|^2 &= (Re[g^R(w)])^2 + (Im[g^R(w)])^2 \\ &= \frac{D^2}{2\pi^3 T}\bigg|\Gamma\bigg(\frac{1}{4} + \frac{iw}{2\pi T}\bigg)\bigg|^4 cosh\bigg(\frac{w}{T}\bigg) = P(w)^2 cosh\bigg(\frac{w}{T}\bigg) \end{aligned} \tag{4.51}$$

where, in order to make the notation more compact, the following function has been defined:

$$P(w) = \frac{D}{\pi\sqrt{2\pi T}}\bigg|\Gamma\bigg(\frac{1}{4} + \frac{iw}{2\pi T}\bigg)\bigg|^2 \tag{4.52}$$

Now, by inserting these analytical solutions, we find that the denominator corresponds to

$$den = 1 + 2CP(w)cosh\left(\frac{w}{2T}\right) + C^2P(w)^2cosh\left(\frac{w}{T}\right) \tag{4.53}$$

while the numerator

$$\begin{aligned} num = &-2iP(w)cosh\left(\frac{w}{2T}\right) - 2iC\left(\frac{1}{N}-1\right)\left(P(w)^2sinh^2\left(\frac{w}{2T}\right) - P(w)^2cosh^2\left(\frac{w}{2T}\right)\right)+ \\ &- 2iCP(w)^2cosh\left(\frac{w}{T}\right) + 2iC^2\left(\frac{1}{N}-1\right)P(w)^3cosh\left(\frac{w}{T}\right)cosh\left(\frac{w}{2T}\right) \\ \\ = &\, 2iP(w)\Bigg[CP(w)\left(\frac{1}{N}-1\right) + C^2P(w)^2\left(\frac{1}{N}-1\right)cosh\left(\frac{w}{T}\right)cosh\left(\frac{w}{2T}\right)+ \\ &- cosh\left(\frac{w}{2T}\right) - CP(w)cosh\left(\frac{w}{T}\right)\Bigg] \end{aligned} \tag{4.54}$$

It follows that the diagonal component of the stationary current can be expressed as

$$I_d = \frac{2C}{N}\int_{-\infty}^{+\infty} dw\left(f_L(\omega - eV) - f_R(\omega)\right)\frac{num(w)}{den(w)} \tag{4.55}$$

where the components of the integrated fraction are described by the equations (4.53,4.54). We have therefore succeeded in finding an expression for the total stationary current (under NESS conditions), which is only a function of the coupling constants between the quantum dot and the leads, and of the real and imaginary components of the retarded Green's function associated to the isolated SYK model in thermodynamic equilibrium (as well as, of course, of the Fermi functions related to the leads). It indeed corresponds to:

$$I = \frac{ie}{2h}\left\{N^2I_d + 4iC(N-1)\int_{-\infty}^{+\infty} dw\left(f_L(\omega - eV) - f_R(\omega)\right)\frac{D}{\pi\sqrt{2\pi T}}\left|\Gamma\left(\frac{1}{4}+\frac{iw}{2\pi T}\right)\right|^2 cosh\left(\frac{w}{2T}\right)\right\} \tag{4.56}$$

where I_d is described by eq(4.55).

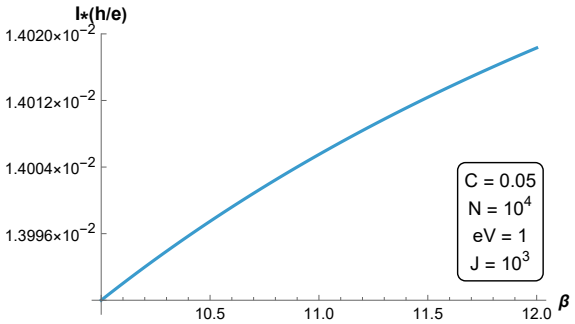
4.2 Weak-Tunneling regime and the Two Limits

In order to study the expression of the stationary current I as a function of the physical parameters N, T, C, eV, J, the operation NIntegrate from the software *Wolfram Mathematica* [72] was chosen to numerically calculate the integrals, in the frequency domain. Before proceeding with the analysis of the data obtained, it is important to point out one thing. Within the integrals of eq(4.55) and eq(4.56), the frequency w is always accompanied by the presence of $\hbar$, while the temperature T is always multiplied by k_B; in fact in the case of this thesis these constants have been set equal to 1. Consequently these physical quantities are energies, exactly like C, eV, J. It is easy to see how such quantities (except w) within the I expression are organized in such a way that they generate dimensionless fractions, hence the opportunity to treat β, C, eV, J as dimensionless parameters (where $\beta = (k_B T)^{-1}$). Another premise to be stressed is that the following data are all related to a very specific case, that is the *weak-tunneling regime*. This means that the LCR system is considered in the case where the quantum dot-electrodes coupling C is the smallest energy scale (compared with other energy parameters characterizing the system, namely $1/\beta, C, eV, J$). This regime is an excellent approximation to describe the LCR system in the context of scanning tunneling spectroscopy experiments and in the case where the electrodes are separated from the dot by an oxide barrier [16]. Obviously further conditions that must be fulfilled are that $\beta J \gg 1$, N large and that $J \gg eV$, where these last three restrictions stem from the fact that in our treatment we used the SYK's analytical solutions (which exist only within such limits). Specifically, the request that $J \gg eV$ is derived from the $J \gg w$ condition, since the subtraction between the two Fermi functions (in eq(4.55) and eq(4.56)) makes relevant only the w belonging to the $\sim [0, eV]$ range. In the *weak-tunneling regime*, we are interested in the study of the stationary current at the variation of two physical quantities that are easy to control experimentally, namely the temperature and the external voltage. In order to derive the dependence of the current from these parameters, it is advantageous to study the current at two different opposite limits: $eV \gg 1/\beta$ and $eV \ll 1/\beta$.

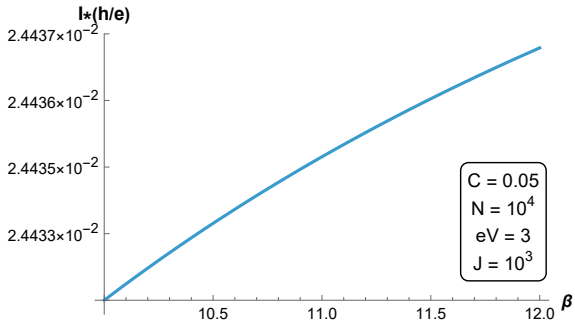
4.2.1 Limit $eV \gg k_B T$

Let's start by studying the first case. Below we present two graphs (4.1*a*, 4.1*b*) in which the dependence of the current is shown with respect to the β parameter in the range $[10, 12]$, for two different values of $eV = 1, 3$. Note that throughout this

chapter the decision was made to present the stationary current I such that it is always multiplied by the constant h/e; this object then represents an energy. In all the graphs presented in this chapter, the energy parameters (or their inverse, for instance in the case of β) are displayed as being dimensionless. The Cartesian axes associated with such quantities must therefore be thought of as properly rescaled with respect to the same energy unit.



(4.1a)



(4.1b)

Although at first sight there seems to be a strong dependence on temperature, simply looking at the scale on the y-axis actually reveals that the variation in current is extremely small as the temperature changes. In fact in this range, the percentage variation of the current is on the order of 10^{-5} and 10^{-6} respectively. In order to obtain other information on the possible dependence on temperature, one strategy is to study the ratio between the two currents shown above, so $I(eV = 3)/I(eV = 1)$, within the same temperature range.

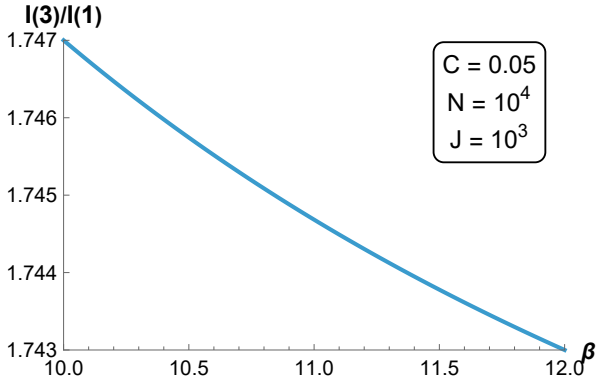


Figure 4.2

As can be easily seen from the Fig(4.2), the ratio of the currents varies very slowly with respect to a change in temperature; in fact, the relative variation of the ratio over the considered range is on the order of 10^{-3}. However, this does not rule out the possibility that a temperature dependence of the current exists (for example, a proportionality to a power of the temperature), which cancels out in the ratio. For now, note that the ratio of the currents takes values very close to $\sqrt{3}$, which

corresponds to the square root of the ratio between the two different considered eV. We are also interested in understanding how this ratio between currents varies as the coupling J changes, keeping in mind that as J increases (for a fixed β) the system progressively approaches the conformal regime, so where the analytical solutions of the SYK model are valid. In the Fig(4.3) below, four different cases are presented (including the case previously shown).

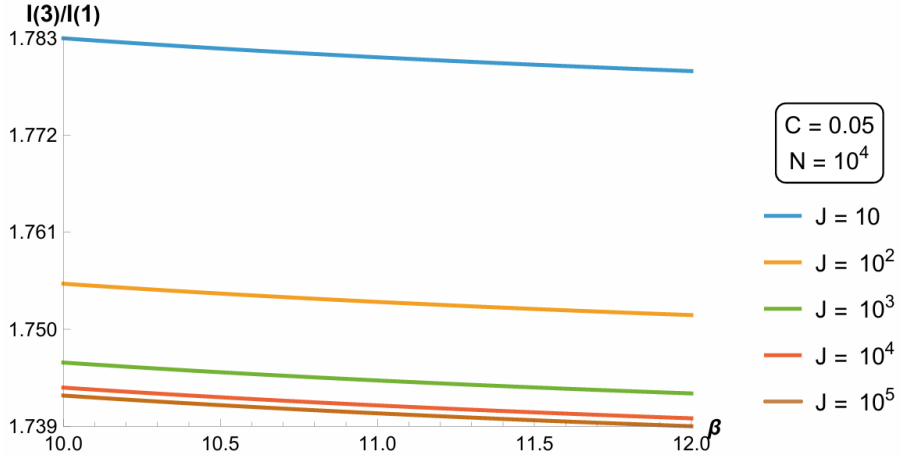


Figure 4.3

It is interesting to note that as J increases, so moving towards the regime where the solutions used for the SYK model are most accurate, the curves describing the ratios become closer to each other and, most importantly, approach the value $\sqrt{3}$ ever more closely. In addition, it can be observed that the two curves associated with the highest J values, approach $\sqrt{3}$ as the parameter β increases (which in this case also indicates a progression towards the conformal regime). For a better understanding of the characteristics of the possible temperature dependence, one possible approach is to compare the behavior of the current associated with different eV values within the same temperature range on the same graph (Fig(4.4)).

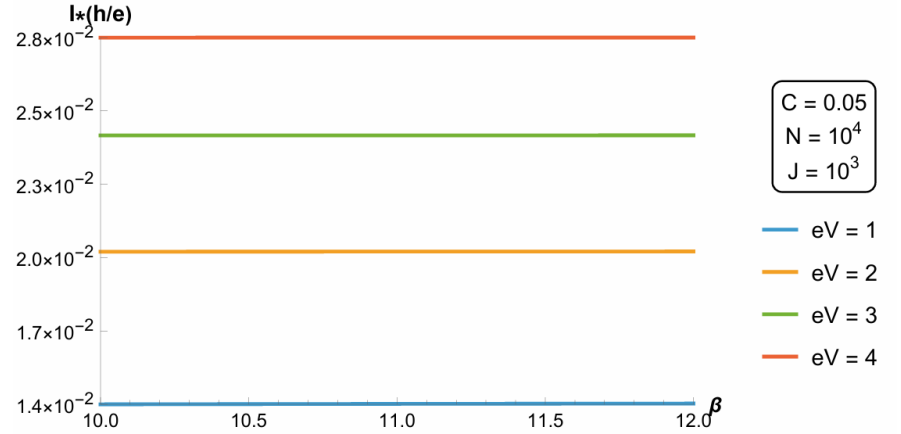


Figure 4.4

If we consider a range on the y-axis that includes the solutions associated with different voltages (which are of the same order of magnitude and thus not very different from each other), it is therefore observed that to a first approximation the currents appear to be independent of temperature. Moreover, continuing the previously thought experiment, it is easy to see that all possible ratios between the values of the four currents are always very similar to the ratios of the square roots of their respective eV values. It is immediately apparent, for example, to notice that the ratio between the "red" current and the "blue" one is approximately 2, which is also the square root of the ratio of the associated eV values, in fact $2 = \sqrt{4/1}$. Now, let's instead study the dependence of the current on the parameter eV (keeping the temperature fixed) considering a wide range of external voltage values. An example of this is the following Fig(4.5).

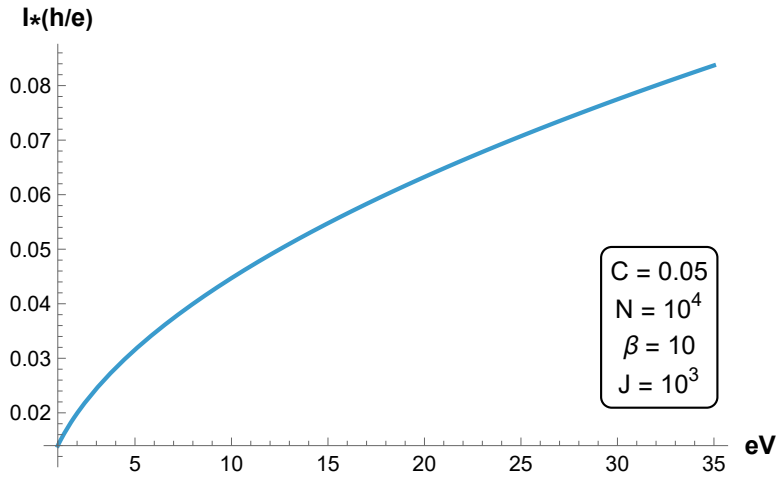


Figure 4.5

The trend shown above is highly consistent with a function corresponding to the square root of eV (appropriately rescaled), which is in perfect agreement with what was previously observed (the ratio properties between the currents). Moreover, if we tried to plot other currents with different temperatures in the graph above, they would completely overlap. In order to observe a deviation between the curves associated with currents at different temperatures, it is necessary to consider a small interval in eV, apply a relatively low external voltage and analyze the case where the two currents are related to significantly different temperatures. An example of this is shown in the graph on the following page.

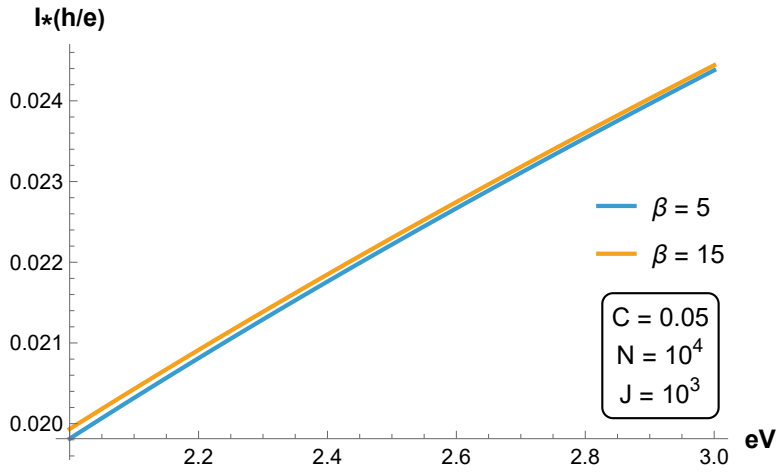


Figure 4.6

If we were to consider additional current curves associated with lower temperatures (and therefore larger β values), it can been verified that these curves would almost completely overlap with the one corresponding to $\beta = 15$. All the information we have gathered leads to the conclusion that if we consider the LCR system in the weak-tunneling regime and in the configuration where the SYK model is in the so-called conformal regime, in the limit of $eV \gg k_B T$ the stationary current I is, to a first approximation, independent of temperature and directly proportional to the square root of the external voltage. So

$$I \propto \sqrt{eV} \qquad when \qquad eV \gg k_B T \tag{4.57}$$

4.2.1.1 Analytical derivation

Now, we want to show how to analytically derive this proportionality relation for the current in the regimes and limits considered. First of all, let us remember that the expression for the stationary current (which we found earlier) corresponds to:

$$I = \frac{ie}{2h}\left\{N^2\left[\frac{2C}{N}\int_{-\infty}^{+\infty} dw\left(f_L(\omega - eV) - f_R(\omega)\right)\frac{num(w)}{den(w)}\right] + 4iC(N-1)\int_{-\infty}^{+\infty} dw \times \right.$$
$$\left.\left(f_L(\omega - eV) - f_R(\omega)\right)\frac{D}{\pi\sqrt{2\pi T}}\left|\Gamma\left(\frac{1}{4} + \frac{iw}{2\pi T}\right)\right|^2 cosh\left(\frac{w}{2T}\right)\right\} \tag{4.58}$$

In the $eV \gg k_B T$ limit, the Fermi functions behave like step functions; as a result, their difference selects a specific integration interval over the frequencies. This means that, to a first approximation, the expected value of the stationary current

corresponds to

$$I \simeq \frac{ie}{2h}\left\{N^2\left[\frac{2C}{N}\int_0^{eV} dw\ \frac{num(w)}{den(w)}\right] + 4iC(N-1)\int_0^{eV} dw \frac{D}{\sqrt{2\pi^3 T}}\left|\Gamma\left(\frac{1}{4}+\frac{iw}{2\pi T}\right)\right|^2 \times cosh\left(\frac{w}{2T}\right)\right\} \tag{4.59}$$

Let's start by examining the terms belonging to the second integral. Note that the frequency w is always accompanied by a β factor, which in our limit satisfies the relation $\beta \gg 1/eV$. Observing that the effective frequencies in the integration satisfy the relation $\beta w \gg 1$, we can approximate part of the integrand in the following way (which is valid only in the limit $\beta w \gg 1$):

$$\left|\Gamma\left(\frac{1}{4}+\frac{i\beta w}{2\pi}\right)\right|^2 cosh\left(\frac{\beta w}{2}\right) \simeq \frac{1}{\sqrt{\beta w}} \tag{4.60}$$

Using this approximation, the second integral in the expression of I would thus correspond to:

$$\int_0^{eV} dw \frac{D}{\sqrt{2\pi^3 T}}\frac{1}{\sqrt{\beta w}} \propto \sqrt{eV} \tag{4.61}$$

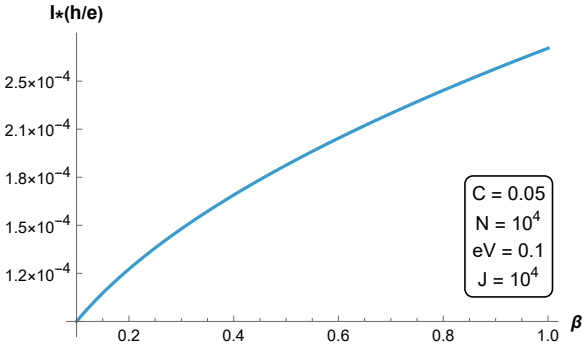
This involves the elimination of the temperature dependence of the current, and by performing the frequency integral, we recover the proportionality to the square root of eV. Now we move on to the study of the first integral in the expression of eq(4.59). Considering that we are in the weak-tunneling regime (C is the smallest energy scale), using the eq(4.61) approximation and observing that the term $C^2 P(w)^2 cosh(w/T)$ is also much smaller than 1 within the considered integration range, the dominant term in the denominator (4.53) corresponds to 1, thus $den(w) \simeq 1$. In a similar way, by analyzing the numerator described by the eq(4.54), it is easy to see that its dominant term (in the considered range) is given by the following factor

$$num(w) \simeq -2iP(w)cosh\left(\frac{w}{2T}\right) \tag{4.62}$$

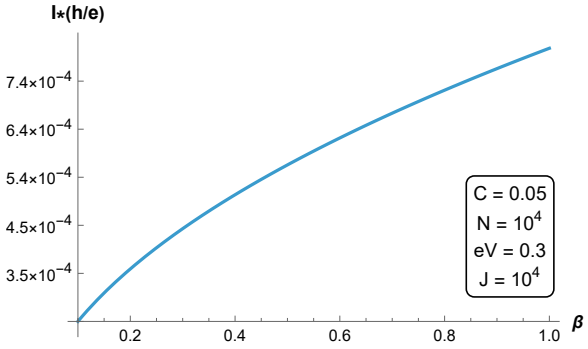
Remembering that $P(w)$ is described by the expression given in eq(4.52), the first integral in the expression (4.59) has the exact same structure (apart from multiplicative constants) as the second integral. Ergo, we know that in the considered regime, it is proportional to $\sqrt{eV}$. It has thus been shown that, in the considered regime, both integrals appearing in the expression for the stationary current I, are independent of temperature and directly proportional to $\sqrt{eV}$. Consequently, the stationary current will also exhibit these properties.

4.2.2 Limit $eV \ll k_B T$

Now, let us consider the system in the weak-tunneling and conformal regimes and analyze the behavior of the stationary current in the limit $eV \ll k_B T$. In order to make comparisons with the graphs analyzed in the previous section, we will try, whenever possible, to use the same parameters or at least similar ratios in this section. For example, the product βJ is always kept on the order of $10^3 - 10^4$, and the parameters C and N remain unchanged. We start by presenting two graphs below that show the behavior of the current as a function of the parameter β, in the interval $[0.1, 1]$, for two eV cases which are 0.1 and 0.3.



(4.7a)



(4.7b)

It is interesting to compare these graphs with the (4.1a) and (4.1b) ones. In fact, even though we are now considering a β interval that is smaller (precisely half, of the range considered previously), we can note that in this regime the current increases much faster as β varies. In fact, the growth rate of the current with respect to β is about $10 - 100$ times larger than the growth rate analyzed in the $eV \gg k_B T$ limit. As we did in the previous section, we can also derive information from studying the ratio between these curves ($I(0.3)/I(0.1)$), which is represented by the following Fig(4.8).

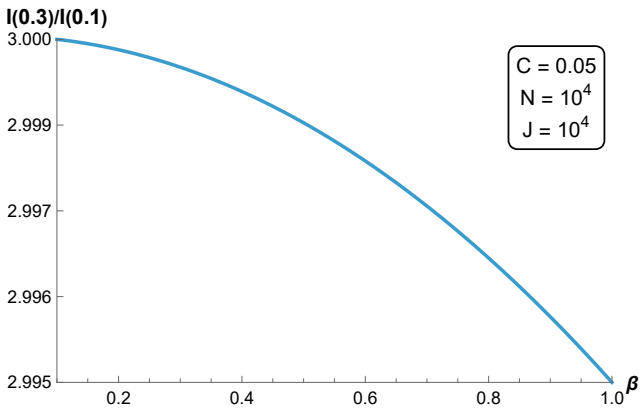


Figure 4.8

As can be verified, this ratio between currents has an extremely small dependence on temperature; this can be understood from the fact that its relative variation within the considered range is on the order of 10^{-3}. Obviously, this does not exclude a dependence of the current on temperature, on the contrary, the growth rate of the previous pair of graphs suggests the existence of such a dependence. Continuing with the analysis of the ratios between different currents (started in the previous section), we notice that all the ratios are extremely close to the value 3, which exactly corresponds to the quotient between the two different eV values considered. Moreover, the ratio gets closer and closer to 3 as β decreases (and consequently k_BT increases), which means that, in that direction, the $eV \ll k_BT$ condition becomes increasingly well satisfied. If we also attempted to plot the curves associated with different J values on the graph above (for example $10^2, 10^3, 10^4, 10^5$), what we would obtain would be a complete overlap of the associated curves, which becomes increasingly perfect in the limit of β tending towards zero (the corresponding graph is not shown, as it is not very significant). We are now interested in studying the dependence of the current on the parameter β. In order to do this, the behaviour of four currents, associated with different eV values, have been presented in the Fig(4.9) below as a function of β.

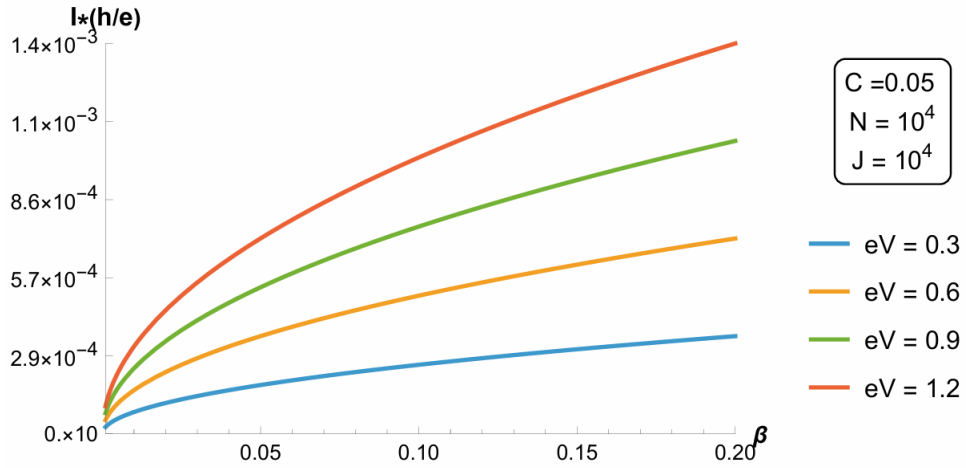


Figure 4.9

The four represented curves can be perfectly fitted by an appropriately rescaled $\sqrt{\beta}$ function; therefore we expect the stationary current in the NESS configuration to be proportional to $\sqrt{\beta}$. In addition, it is interesting to observe that in the graph presented above, for each fixed β, the four curves are equidistant from each other and this reflects the proportionality ratios of the respective eV values. This is a clear indication that the current is directly proportional to the parameter eV. So we now study the dependence of the current I on the parameter eV by plotting three different curves associated with different temperatures in the following Fig(4.10).

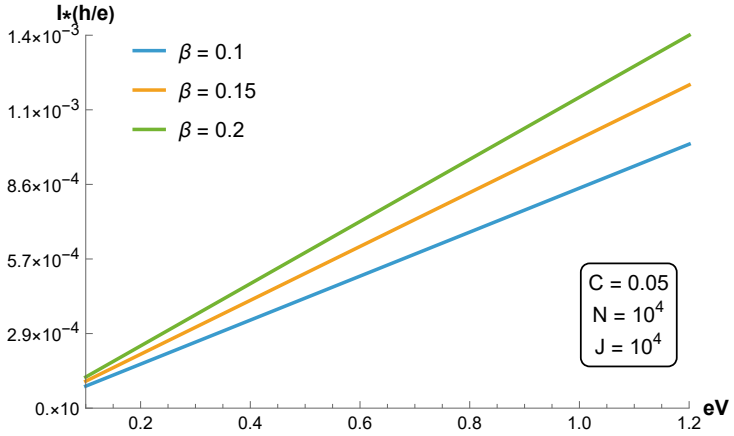


Figure 4.10

One can notice, both visually and through fitting operations, that these curves exhibit a strongly linear behaviour with respect to the parameter eV. In the graph above, the lower limit of the eV interval has been set to 0.1 to ensure that the weak-tunneling condition is satisfied. If we were to fit the three displayed curves, the solutions would correspond to a linear function passing through the origin of the axes. This confirms that the stationary current is directly proportional to the parameter eV, as the intercepts of the linear functions are zero. From all this, it follows that in the weak-tunneling and conformal regime and in the limit $eV \ll k_B T$, the stationary current in the NESS configuration is directly proportional to $\sqrt{\beta}$ and eV, thus:

$$I \propto \frac{eV}{\sqrt{T}} \qquad when \qquad eV \ll k_B T \tag{4.63}$$

4.2.2.1 Analytical derivation

Now, let's analytically derive this proportionality relation for the current in the regimes and limits considered. In the regime $eV \ll k_B T$, we can express the differences between the Fermi functions, appearing in the expression (4.58), using the following relation:

$$\lim_{eV \to 0} \frac{f_L(w - eV) - f_R(w)}{eV} = \frac{\beta}{4cosh^2(\beta w/2)} \tag{4.64}$$

Using this property in the considered regime, we obtain a stationary current found to be equal to

$$I = \frac{ie}{2h}\left\{ N^2 \left[\frac{2C}{N} \int_{-\infty}^{+\infty} dw \frac{eV\beta}{4\cosh^2(\beta w/2)} \frac{\text{num}(w)}{\text{den}(w)} \right] + 4iC(N-1) \int_{-\infty}^{+\infty} dw \times \right.$$
$$\left. \frac{eV\beta}{4\cosh^2(w\beta/2)} \frac{D}{\pi} \sqrt{\frac{\beta}{2\pi}} \left| \Gamma\left(\frac{1}{4} + \frac{iw\beta}{2\pi} \right) \right|^2 \cosh\left(\frac{w\beta}{2} \right) \right\} \tag{4.65}$$

Note that within the two integrals presented above, the variable w is always divided by the temperature T, in fact this occurs inside the Gamma functions, the *cosh* functions, and in both $num(w)$ and $den(w)$. Now, let us specifically consider the second integral in eq(4.65). If we carry out a variable change of the form $y = w\beta$, this integral becomes proportional to $eV/\sqrt{T}$ multiplied by a constant integral that is independent of the parameters eV and T. However, such reasoning is not sufficient to study the first integral in the above expression for the current effectively. In fact, within $num(w)$ and $den(w)$, there are factors associated with different powers of temperature, due to the fact that they are described through different powers of $P(w)$. The problem can be solved by identifying the dominant factors within them in the considered regime, namely under the weak-tunneling conditions, the conformal limit and for $eV \ll k_B T$. Examining the components of the numerator and denominator in the considered regimes, it follows that their dominant factors correspond to those previously found in Section(4.2.1.1). So one gets

$$\frac{num(w)}{den(w)} \simeq -2iP(w)cosh\left(\frac{w\beta}{2}\right) \tag{4.66}$$

Using this approximation, we find that the first integral of expression(4.65) takes the same structure (up to multiplicative constants) as the second integral. Hence, following the same reasoning used for the study of the second integral, we conclude that it is also proportional to $eV/\sqrt{T}$. Since the stationary current is given by the sum of these integrals, it will therefore satisfy the same proportionality properties in the considered regimes.

Before moving on to the next section, it is interesting to note one thing regarding the analysis of the stationary current in the two considered limits (namely all the treatment carried out in the Section (4.2)). In fact, the results obtained in the current section of this thesis are found to be perfectly consistent with other sources of scientific literature on the subject, especially with the paper [16]. Although the configuration of the LCR system used in that paper is partially different from the system used in this thesis, it is extremely interesting to note how the same behaviors of the stationary current are obtained. One of the main differences between the two models is that, unlike what was done in this thesis, in the paper [16] the tunneling matrix elements $W_{\mathbf{k}a,n}$ which are described by a Gaussian random distribution, with $\overline{|W_{\mathbf{k}a,n}|^2} = W^2$, were considered. Instead, in this thesis, we assumed that the tunneling matrix elements are **k**-independent and that they are equal for all possible sites n, consequently, only one coupling is associated with each electrode. It is as if a more "democratic", namely uniform, coupling distribution would give the same results as a Gaussian distribution.

4.2.3 Dependence of current I on parameter N

In this section, we want to explore the role of the parameter N within our theoretical framework and how the stationary current under NESS conditions is related to it. The first thing to focus on concerns some implications that arise from the analysis conducted in Sections (4.2.1.1) and (4.2.2.1). Specifically, we are interested in the consequences that follow from the fact that in both of these sections, it has been explained how (in the respective regimes considered) the term num/den is well approximated by the expression $-2iP(w)cosh(w\beta/2)$. In fact, if we insert this approximation into the expression for the stationary current described by the eq(4.58), we find:

$$I = \frac{ie}{2h}\Bigg\{N^2\Bigg[\frac{2C}{N}\int_{-\infty}^{+\infty} dw\Bigg(f_L(\omega - eV) - f_R(\omega)\Bigg)\Bigg(-2iP(w)cosh\Bigg(\frac{w\beta}{2}\Bigg)\Bigg)\Bigg] + \\ + 4iC(N-1)\int_{-\infty}^{+\infty} dw\Bigg(f_L(\omega - eV) - f_R(\omega)\Bigg)P(w)cosh\Bigg(\frac{w}{2T}\Bigg)\Bigg\} \tag{4.67}$$

It is easy to observe that, in the expression presented above, the contributions dependent on N cancel out, making the final expression for the current I independent of the number of sites N in the SYK model. It follows that I corresponds to

$$I = \frac{2eC}{h}\int_{-\infty}^{+\infty} dw\Bigg(f_L(\omega - eV) - f_R(\omega)\Bigg)P(w)cosh\Bigg(\frac{w}{2T}\Bigg) \tag{4.68}$$

and this has also been verified numerically. In fact, by studying the current I as the parameter N varies, one can confirm its complete independence. Clearly, it is essential to specify that this holds as long as the current is analyzed within the two previously examined regimes. If instead we examine the stationary current in completely different regimes, the approximation (4.66) is no longer valid, and as a result, a dependence of the current on the parameter N might emerge. From a numerical analysis of the current, it has been observed that a significant dependence on the parameter N can occur under critical conditions. Specifically, these critical conditions occur in situations where we move away from the weak-tunneling regime and the parameter C becomes the largest energy scale (compared to the other energy parameters of the system). The additional conditions found through the numerical analysis are $k_BT \ll 1$, $eV \gg 10$ and $J < 10^4$. An interesting situation occurs when we compare four currents associated with different N values (of different orders of magnitude) as a function of eV in the critical regime just described. In this situation, at the two extreme limits of eV (tending to zero and tending to infinity) it is possible to verify that the four curves overlap, while in an intermediate region, deviations

can be observed. An example of such a situation is shown in the Fig(4.11), where the focus is restricted to the deviation region.

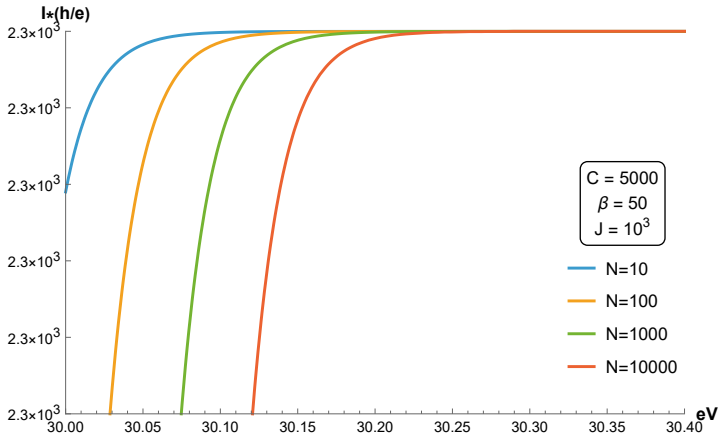


Figure 4.11

As a final note of this topic, it is important to make an appropriate clarification. In this section, we have studied the dependence of the stationary current on the number of sites N; however it is important to recall that our entire theoretical framework is based on the use of the exact solutions of the SYK model, which are valid only in the large N limit. For this reason, data corresponding to a relatively small N have no physical meaning within the theoretical framework used.

4.3 Linear response regime

Let us now study the LCR system under consideration in the context of the *linear response regime*. The study of our model in this regime is extremely useful; indeed using its properties, we can also derive information about the thermoelectric current, which manifests itself when the two leads differ by a small temperature variation. This would not be surprising if we had initially developed a theoretical framework that describes the case of electrodes at different temperatures. However, it is important to remember that this is not the case considered at the beginning of this chapter. In fact the entire theoretical framework in this chapter assumes a constant temperature, identical for both leads. Nevertheless, this is not a problem, as *linear response theory* effectively handles this issue by working with infinitesimal variations in temperature and voltage. So let us begin by considering the LCR system in the specific configuration where the two electrodes have temperatures differing by a very small positive value ΔT (equal to $T_L - T_R$), and their difference in chemical potentials, namely eV, is also infinitesimal. As previously seen in Section (2.2.2), in such a situation it is possible and convenient to perform the following

expansion of f_L:

$$f_L(w - eV) \simeq f_R(w) - \frac{df_L}{dw}(w)\left[eV + w\frac{\Delta T}{T_L}\right] \tag{4.69}$$

If we insert this expression into the eq(4.58), which describes the stationary current I under NESS conditions, we find that it can be formulated as $I = GV + L\Delta T$, where G is the *conductance* and L is the *thermoelectric coefficient*. To start, let us consider the study of the conductance G, which is described by the following expression:

$$G = -\frac{ie^2}{2h}\left\{N^2\left[\frac{2C}{N}\int_{-\infty}^{+\infty} dw\left[\frac{df_L}{dw}(w)\right]\frac{num(w)}{den(w)}\right] + 4iC(N-1)\int_{-\infty}^{+\infty} dw\left[\frac{df_L}{dw}(w)\right]\times \frac{D}{\pi\sqrt{2\pi T}}\left|\Gamma\left(\frac{1}{4} + \frac{iw}{2\pi T}\right)\right|^2 cosh\left(\frac{w}{2T}\right)\right\} \tag{4.70}$$

Here below, we present the results obtained through numerical analysis. In this analysis, we set the number of sites N to 10^4, the coupling J to 10^4 and weak coupling between the leads and the quantum dot was considered (so C is small). The first graph we present in Fig(4.12) shows the behavior of the conductance G (multiplied by the constant h/e^2) as a function of the parameter β for four different coupling values C (note that in this case the y-axis is associated with a dimensionless parameter).

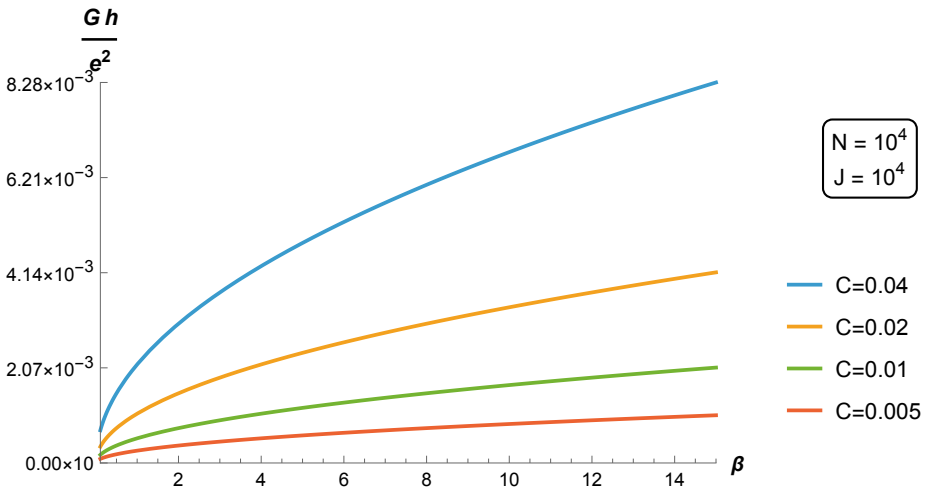


Figure 4.12

As one might expect intuitively, at a fixed temperature the conductance decreases as the lead-dot coupling C decreases until it reaches zero (at $C = 0$). Moreover, we observe an increase in conductance as the temperature decreases, considering

that the boundary condition for $T \to \infty$ corresponds to $G = 0$. Specifically, by performing fitting operations on the numerical data, we found that the observed trends are optimally compatible with appropriately rescaled square root functions of β. Thus, in the considered regime, we observe that $G \propto \sqrt{\beta}$. In addition, looking at Fig(4.12), we can observe that the ratios between the four represented curves follow the ratios of the corresponding coupling values C and this is a clear indication of a linear dependence of the current I on the coupling C. In order to deepen this dependence, we present the following graph (Fig(4.13)), which shows the behavior of the current I as a function of C for four different values of the parameter β.

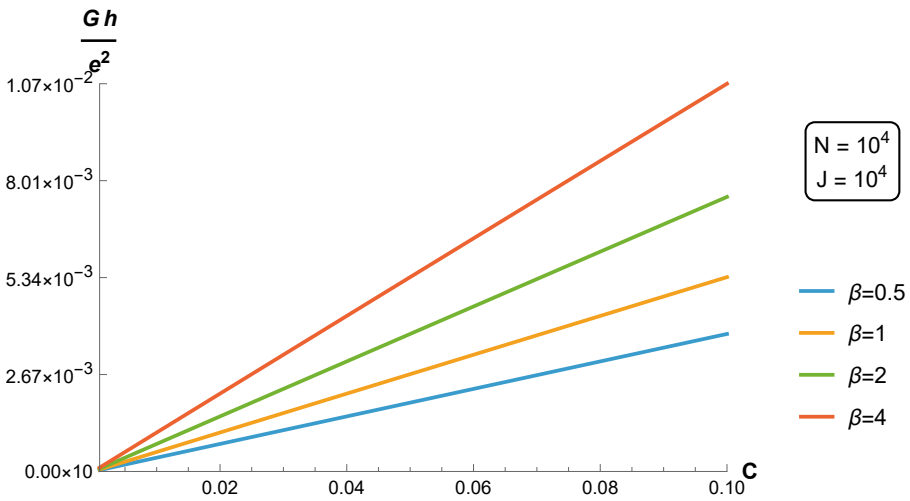


Figure 4.13

It is easy to observe that there is a strong linear relation between the current and the coupling parameter C, which is further confirmed by the excellent fits performed on the current data. Hence, combining this dependence with the previously found relation, we obtain that

$$G \propto \frac{C}{\sqrt{T}} \tag{4.71}$$

It is significant to note that the linearity of G with respect to C is in perfect agreement with the analysis conducted in Section (4.2.2.1). In fact, the conditions considered here comply with those in that section and therefore the approximation described by eq(4.66) is valid in the analysis of G. This indicates that all contributions dependent on C in the expressions of $num(w)$ and $den(w)$ (described by the equations (4.54,4.53)) are negligible (in the regime of interest), thus the only C parameters, appearing in the conductance expression, are those directly visible in eq(4.70). This explains why G is linear with respect to C. The proportionality of G with respect to $\sqrt{\beta}$ is also immediately apparent, considering that the stationary current is directly proportional to $eV\sqrt{\beta}$ in the regime $eV \ll k_B T$. It is also sig-

nificant to point out that these results, related to the conductance, are perfectly in agreement with those shown in the paper [16].

Let us now move on to the study of the thermoelectric coefficient L, which in our case is described by the following expression:

$$L = -\frac{iek_B}{2h}\left\{N^2\left[\frac{2C}{N}\int_{-\infty}^{+\infty} dw\left[\beta w\frac{df_L}{dw}(w)\right]\frac{num(w)}{den(w)}\right] + 4iC(N-1)\int_{-\infty}^{+\infty} dw\left[\beta w\frac{df_L}{dw}(w)\right]\times \right.$$
$$\left.\frac{D}{\pi}\sqrt{\frac{\beta}{2\pi}}\left|\Gamma\left(\frac{1}{4}+\frac{i\beta w}{2\pi}\right)\right|^2 cosh\left(\frac{\beta w}{2}\right)\right\} \tag{4.72}$$

As done for the conductance, we study the expression of L numerically and we consider the system under the same conditions as in the previous analysis, i.e. with the same values of the parameters N, J, C, β. The regime of interest is defined by the weak-tunneling conditions and by the conformal limit, so C and T are very small and J large (remember $eV = 0$). From the numerical analysis of the expression (4.72), it was noticed that the behaviour of the thermoelectric coefficient with respect to the β parameter is not well defined and L assumes extremely small values. A similar situation occurs if L is studied as a function of the parameter C. All this suggests that the code used to carry out the numerical analysis has shortcomings when used to calculate the associated integrals. It is therefore appropriate to proceed analytically; this conclusion is also supported by the fact that the values returned by the software are extremely small and therefore could, in reality, be zero. In fact, if we consider that we are in the regime described above, we know that, to a very good approximation, the stationary current I is described by the expression (4.68). Using this expression for the stationary current to derive the thermoelectric coefficient L, one finds

$$L = -\frac{2ek_BC}{h}\int_{-\infty}^{+\infty} dw\left[\beta w\frac{df_L}{dw}(w)\right]\frac{D}{\pi}\sqrt{\frac{\beta}{2\pi}}\left|\Gamma\left(\frac{1}{4}+\frac{iw\beta}{2\pi}\right)\right|^2 cosh\left(\frac{w\beta}{2}\right) \tag{4.73}$$

Let us now study the components of this integral. The derivative of the Fermi function with respect to w is an even function of its argument w and *cosh* is also an even function. Then, using the property $\Gamma(\bar{z}) = \overline{\Gamma(z)}$, we find that

$$\left|\Gamma\left(\frac{1}{4}-\frac{iw\beta}{2\pi}\right)\right|^2 = \left|\Gamma\left(\frac{1}{4}+\frac{iw\beta}{2\pi}\right)\right|^2 \tag{4.74}$$

and therefore the square module of Γ function is also an even function with respect to w. From all this, knowing that the integration range is symmetrical and that these even functions are multiplied by w (which is a odd function), we discover that the thermoelectric coefficient L is approximately equal to zero in the regime considered.

This is in perfect agreement with the literature on the thermoelectric properties of the SYK model [17, 18]. In fact from this literature, we know that the *Seebeck* coefficient (also called *thermopower*), that corresponds to the ratio between the thermoelectric coefficient L and the conductance G, satisfies the following equation at very low temperature:

$$\lim_{T\to 0} S = \frac{L}{G} = 2\pi\mathcal{E} \tag{4.75}$$

where $\mathcal{E}$ is the *spectral asymmetry* parameter defined in equation (3.89). This relation indicates that the thermopower is completely determined by the particle-hole asymmetry of the SYK model. In our case, it is assumed to be in the conditions where the intrinsic chemical potential of the SYK model is set equal to zero and consequently the model is in a situation of particle-hole symmetry. So, considering $\mu = 0$, the SYK quantum dot considered is at the particle-hole symmetric point, which is described by $Q = 1/2$ and $\theta = \mathcal{E} = 0$. From all this it follows that, under particle-hole symmetry conditions, the thermopower is identically zero at very low temperatures (which therefore satisfy the conformal limit).

Conclusions

The aim of this thesis is to study the transport properties of the SYK model in its spinful complex fermion version. In order to achieve this goal, a precise series of logical steps had to be followed, so that all the necessary tools, to deal with the transport phenomena related to the SYK model, were available and usable in the aforementioned treatment.

The first operation necessary for this purpose was the study of Keldysh's theory (presented in Chapter 1), which provides a theoretical framework that is able to describe the temporal evolution of systems in configurations of thermodynamic non-equilibrium. This theory can be formulated using several different approaches, the one used here is provided by a non-equilibrium Green's functions technique, specifically in its variant based on functional integral QFT formalism (Keldysh field theory). We have thus introduced some central tools in our discussion, such as the concept of partition function and the associated generating functions, both strongly linked to a peculiar object of this theory, namely the closed time contour C. We analyzed the properties of the three types of Green's functions that arise from this formalism and also how it is possible to calculate the expectation values of physical observables by introducing, into the theoretical framework, external sources that carry with them asymmetry contributions (between the two branches of the contour). Finally, the extraordinary power of the *Dyson* equation was shown, which allows for the treatment of interaction terms and also time dependent perturbations.

Furthermore, in this thesis, the study of transport properties of the SYK model was limited to the NESS conditions and restricted to the LCR configuration. For this reason, at the beginning of Chapter 2, we focused on the concept of the global non-equilibrium steady state, specifically by studying the physical system composed of a quantum dot interacting with two reservoirs at different temperatures and chemical potentials. The next step was then to create the whole theoretical framework, which allows us to study the transport properties of systems which consist of a quantum dot in contact (interacting) with two electrodes (corresponding to non-interacting fermions systems). Obviously the theoretical framework developed is based on the Keldysh Field Theory formalism. We have thus succeeded in expressing the stationary current between two leads, passing through a quantum dot, as a

function of the exact Green's functions of the dot (as well as of the couplings to the electrodes and of the leads' distribution function). The exact Green's functions of the quantum dots store inside them the information of the interactions in the dot and of the coupling with the reservoirs. Subsequently, in order to practice using this formalism before moving on to the SYK model, it was decided to study the two simplest quantum dots, namely the single level and the two levels non-interacting quantum dots. Specifically, the expressions of the associated stationary currents and, in the context of the linear response regime, also their conductances and thermoelectric coefficients were studied.

Now that all the theoretical tools, needed to study the transport properties of a quantum dot in LCR configuration, had been presented, the first version of the SYK model, based on Majorana fermions, was introduced at the beginning of Chapter 3. In fact, although our aim is the study of the SYK model with spinful complex fermions, it was considered extremely useful to arrive at the study of such a model by first passing through two additional steps, namely the Majorana fermion model and the spinless complex fermion model. Thus, the complexity of the models presented is constantly increasing and the comprehensibility of the text greatly benefits from this. Starting from the Majorana fermion model (at thermodynamic equilibrium and using the *Matsubara* formalism), we presented the operation of averaging over the quenched disorder related to the partition function and the associated replica trick. Using these tools, we succeeded in obtaining an effective action associated with the disorder averaged partition function. Studying its saddle point solutions, we find the *Dyson–Schwinger* equations that describe the exact Green's function of the SYK model in the case of large N (number of SYK's sites) and, in the IR and strong coupling limits, we obtain their analytical solutions. The whole procedure described here is then repeated for the case of the spinless complex fermion model. Unlike the previous case, with this model the study of the DS-equation solutions is slightly different. In fact, in the low frequency and strong coupling limit, besides the presence of a conformal symmetry, there is also the emergence of a $U(1)$ gauge invariance. This second symmetry introduces in the solutions of the exact Green's function a phase term θ that is strongly related to the particle-hole asymmetry of the system being considered. In this thesis we have chosen to focus on the specific case where the SYK model, used as a quantum dot, shows a particle-hole symmetry (thus $\theta = 0$) and this was achieved by setting the intrinsic chemical potential of the dot equal to zero. In this condition of symmetry, the study of the complex fermion model with half-integer spin was then carried out (always at thermodynamic equilibrium). The same procedure as in the two previous cases was used to obtain the analytical expressions of the correlators within the limits considered. The next step was to calculate the corresponding Green's functions in the Keldysh formalism (so in real

time), starting from those obtained with the *Matsubara* formalism. At this point, we finally obtained all the tools necessary to study transport properties of the spinful complex fermion version of the SYK model.

So in Chapter 4, a situation was considered where the quantum dot (described by the spinful complex SYK Hamiltonian) interacts with two reservoirs at equal temperature T and different chemical potentials, in the case where N is large and the SYK model is in the so-called conformal limit. Assuming that the LCR system is in the configuration of symmetric junctions, we find the expression of its stationary current in the NESS configuration as a function of the coupling constants between the quantum dot and the leads, and of the real and imaginary components of the retarded Green's function associated to the isolated SYK model at thermodynamic equilibrium (as well as, of course, of the Fermi functions related to the leads). This expression has been analysed both numerically and analytically in the weak-tunneling regime, with particular attention in the two limits $eV \gg k_B T$ and $eV \ll k_B T$ (where eV is the difference of the leads' chemical potentials). It was thus found that the stationary current obeys the following proportionality, namely $I \propto \sqrt{eV}$ and $I \propto eV/\sqrt{T}$, within the respective limits. The results of this analysis were in perfect agreement with those shown in the paper [16], although the configuration of the LCR system used in that paper is partially different from that used in this thesis. One of the main differences between the two models is that in the aforementioned paper it is assumed that the tunneling matrix elements are described by a Gaussian random distribution, while in this thesis, we assumed that they are constant and all equal for each electrode. Therefore, as noted previously, it is as if a more "democratic", namely uniform, coupling distribution would give the same results as a Gaussian distribution. It is thought that the correspondence between these two types of distributions should be further studied and analyzed, with the aim of understanding how the SYK model (used as a quantum dot) behaves as the type of distribution used to describe the tunneling matrix elements varies. Finally, the LCR system was studied both numerically and analytically in the context of the linear response regime. We have thus studied the dependencies of proportionality of the conductance, that is $G \propto C/\sqrt{T}$ (also in agreement with the results presented in the paper [16]), and it has been shown that within the limits considered the thermoelectric coefficient is approximately equal to zero. From this we derived that the *Seebeck* coefficient (or thermopower) is also zero and this is in perfect agreement with the literature on the thermoelectric properties of the SYK model [17, 18]. In fact, these papers show that, in the considered regime, the thermopower is directly proportional to the particle-hole asymmetry parameter $\mathcal{E}$ of the SYK model and in our case, it is assumed to be in a situation of particle-hole symmetry, so where $\mathcal{E} = 0$.

The combination of the non-equilibrium thermodynamics and the SYK models, generates an extremely wide field of research that offers opportunities for the continuation of the work undertaken in this thesis in various directions (many of which have already been undertaken in the scientific literature). For example, the first step that could be taken, to further develop this treatment, would be to consider the generic case in which the complex SYK model is associated with a particle-hole asymmetry parameter $\mathcal{E}$ different from zero (as done in [17, 18]). In this way one could study the thermoelectric coefficient and the thermopower. Specifically, the *Seebeck* coefficient analysis is of extreme interest since it is intimately linked to the entropy of the SYK model which is strongly connected to the *Bekenstein-Hawking* entropy of charged black holes. An additional topic that deserves to be explored is the behaviour of this model in the regime of strong coupling C, between the SYK quantum dot and the electrodes. In fact in this case, almost all the approximations made in the analytical study of the stationary current would no longer be valid and new contributions could dominate its expression. In this configuration (in a specific temperature range), it is also expected that the inelastic cotunneling phenomenon would be relevant, while the direct tunneling contributions are negligible [73]. Another field to be explored is the study of the LCR configuration in which the electrodes are not normal leads, but superconducting leads. In this case, in addition to the presence of electrical and thermoelectric current, there will also be a contribution of current generated by the Josephson effect [74]. The most ambitious direction to proceed is the study of the transport properties of the SYK model in the case where the temperatures of the leads are very different and it is therefore not possible to exploit the typical properties of the linear response regime. This situation entails notable challenges, due to the fact that the SYK model is defined by strong quartic interactions, which make the introduction of reasonable approximations useful for the study of the model particularly convoluted. Finally, another path that can be taken is the study of the SYK model quench-coupled with a cold bath and the analysis of its temporal evolution immediately after the coupling. In fact, this phenomenon is thought to describe some properties of the evaporation process of a black hole [70, 75].

Acknowledgements

Ringrazio il Professore Luca Dell'Anna per avermi accompagnato durante l'intero percorso di tesi ed, in special modo, per aver accresciuto il mio interesse verso questa branca incredibile della Fisica che è il non-equilibrio. Infine voglio ringraziare tutte le persone che direttamente o indirettamente hanno contribuito alla realizzazione di tale tesi. Felicità è, solo se condivisa.

Bibliography

[1] Alexei Kitaev. "A simple model of quantum holography". *Talks at KITP http://online.kitp.ucsb.edu/online/entangled15/kitaev/ and http://online.kitp.ucsb.edu/online/entangled15/kitaev2/*, April and May, 2015.

[2] Subir Sachdev and Jinwu Ye. "Gapless spin-fluid ground state in a random quantum Heisenberg magnet". *Physical review letters*, 70(21):3339, 1993.

[3] Juan Maldacena, Douglas Stanford, and Zhenbin Yang. "Conformal symmetry and its breaking in two-dimensional nearly anti-de Sitter space". *Progress of Theoretical and Experimental Physics*, 2016(12):12C104, 2016.

[4] Ahmed Almheiri and Joseph Polchinski. "Models of AdS_2 backreaction and holography". *Journal of High Energy Physics*, 2015(11):1–19, 2015.

[5] Antal Jevicki and Kenta Suzuki. "Bi-local holography in the SYK model: perturbations". *Journal of High Energy Physics*, 2016(11):1–28, 2016.

[6] Juan Maldacena and Douglas Stanford. "Remarks on the Sachdev-Ye-Kitaev model". *Physical Review D*, 94(10):106002, 2016.

[7] Yasuhiro Sekino and Leonard Susskind. "Fast scramblers". *Journal of High Energy Physics*, 2008(10):065, 2008.

[8] Leonard Susskind. "Addendum to fast scramblers". *arXiv preprint arXiv:1101.6048*, 2011.

[9] Juan Maldacena, Stephen Shenker, and Douglas Stanford. "A bound on chaos". *Journal of High Energy Physics*, 2016(8):1–17, 2016.

[10] Chunxiao Liu, Xiao Chen, and Leon Balents. "Quantum entanglement of the Sachdev-Ye-Kitaev models". *Physical Review B*, 97(24):245126, 2018.

[11] Yingfei Gu, Andrew Lucas, and Xiao-Liang Qi. "Spread of entanglement in a Sachdev-Ye-Kitaev chain". *Journal of High Energy Physics*, 2017(9):1–44, 2017.

[12] Yichen Huang and Yingfei Gu. "Eigenstate entanglement in the Sachdev-Ye-Kitaev model". *Physical Review D*, 100(4):041901, 2019.

[13] Subir Sachdev. "Bekenstein-Hawking entropy and strange metals". *Physical Review X*, 5(4):041025, 2015.

[14] Xue-Yang Song, Chao-Ming Jian, and Leon Balents. "Strongly correlated metal built from Sachdev-Ye-Kitaev models". *Physical Review Letters*, 119(21):216601, 2017.

[15] Sean Hartnoll, Andrew Lucas, and Subir Sachdev. *"Holographic quantum matter"*. MIT Press, 2018.

[16] Oguzhan Can, Emilian Nica, and Marcel Franz. "Charge transport in graphene-based mesoscopic realizations of Sachdev-Ye-Kitaev models". *Physical Review B*, 99(4):045419, 2019.

[17] Richard Davison, Wenbo Fu, Antoine Georges, Yingfei Gu, Kristan Jensen, and Subir Sachdev. "Thermoelectric transport in disordered metals without quasiparticles: The Sachdev-Ye-Kitaev models and holography". *Physical Review B*, 95(15):155131, 2017.

[18] Alexander Kruchkov, Aavishkar Patel, Philip Kim, and Subir Sachdev. "Thermoelectric power of Sachdev-Ye-Kitaev islands: Probing Bekenstein-Hawking entropy in quantum matter experiments". *Physical Review B*, 101(20):205148, 2020.

[19] Leonid Keldysh. "Diagram technique for nonequilibrium processes". In *Selected Papers of Leonid V Keldysh*. World Scientific, 2024.

[20] Alex Kamenev. *"Field theory of non-equilibrium systems"*. Cambridge University Press, 2023.

[21] Alexander Altland and Ben Simons. *"Condensed matter field theory"*. Cambridge University Press, 2010.

[22] Jørgen Rammer and Henrik Smith. "Quantum field-theoretical methods in transport theory of metals". *Reviews of Modern Physics*, 58(2):323, 1986.

[23] Lukas Sieberer, Michael Buchhold, and Sebastian Diehl. "Keldysh field theory for driven open quantum systems". *Reports on Progress in Physics*, 79(9):096001, 2016.

[24] Giacomo Guarnieri, Gabriel Landi, Stephen Clark, and John Goold. "Thermodynamics of precision in quantum nonequilibrium steady states". *Physical Review Research*, 1(3):033021, 2019.

[25] James McLennan Jr. "Statistical mechanics of the steady state". *Physical Review*, 115(6):1405, 1959.

[26] D.N. Zubarev, Vladimir Morozov, and Gerd Röpke. *"Statistical Mechanics of Nonequilibrium Processes, Volume 1: Basic Concepts, Kinetic Theory"*, volume 1. Wiley-VCH, 1996.

[27] D.N. Zubarev. "Nonequilibrium statistical operator as a generalization of Gibbs distribution for nonequilibrium case". *Cond. Matt. Phys*, 4(7), 1994.

[28] D.N. Zubarev. "The Method of the Non-Equilibrium Statistical Operator and its Applications. I". *Fortschritte der Physik*, 18(3):125–147, 1970.

[29] Abhishek Dhar, Keiji Saito, and Peter Hänggi. "Nonequilibrium density-matrix description of steady-state quantum transport". *Physical Review E — Statistical, Nonlinear, and Soft Matter Physics*, 85(1):011126, 2012.

[30] Dmitry Ryndyk. "Theory of quantum transport at nanoscale". *Springer Series in Solid-State Sciences*, 184:9, 2016.

[31] Marco Uguccioni and Luca Dell'Anna. "Keldysh field theory approach to electric and thermoelectric transport in quantum dots". *Physical Review B*, 110(20):205139, 2024.

[32] Yigal Meir and Ned Wingreen. "Landauer formula for the current through an interacting electron region". *Physical Review Letters*, 68(16):2512, 1992.

[33] Antti-Pekka Jauho, Ned Wingreen, and Yigal Meir. "Time-dependent transport in interacting and noninteracting resonant-tunneling systems". *Physical Review B*, 50(8):5528, 1994.

[34] Rolf Landauer. "Electrical resistance of disordered one-dimensional lattices". *Philosophical magazine*, 21(172):863–867, 1970.

[35] Subir Sachdev. "Holographic metals and the fractionalized Fermi liquid". *Physical Review Letters*, 105(15):151602, 2010.

[36] Olivier Parcollet and Antoine Georges. "Non-Fermi-liquid regime of a doped Mott insulator". *Physical Review B*, 59(8):5341, 1999.

[37] Antoine Georges, Olivier Parcollet, and Subir Sachdev. "Quantum fluctuations of a nearly critical Heisenberg spin glass". *Physical Review B*, 63(13):134406, 2001.

[38] Yiming Chen, Hui Zhai, and Pengfei Zhang. "Tunable quantum chaos in the Sachdev-Ye-Kitaev model coupled to a thermal bath". *Journal of High Energy Physics*, 2017(7):1–28, 2017.

[39] Subir Sachdev. "Strange metals and the AdS/CFT correspondence". *Journal of Statistical Mechanics: Theory and Experiment*, 2010(11):P11022, 2010.

[40] Hernán González, Daniel Grumiller, and Jakob Salzer. "Towards a bulk description of higher spin SYK". *Journal of High Energy Physics*, 2018(5):1–41, 2018.

[41] Marika Taylor. "Generalized conformal structure, dilaton gravity and SYK". *Journal of High Energy Physics*, 2018(1):1–41, 2018.

[42] Debanjan Chowdhury, Antoine Georges, Olivier Parcollet, and Subir Sachdev. "Sachdev-Ye-Kitaev models and beyond: Window into non-Fermi liquids". *Reviews of Modern Physics*, 94(3):035004, 2022.

[43] Yingfei Gu, Andrew Lucas, and Xiao-Liang Qi. "Energy diffusion and the butterfly effect in inhomogeneous Sachdev-Ye-Kitaev chains". *SciPost Physics*, 2(3):018, 2017.

[44] Ippei Danshita, Masanori Hanada, and Masaki Tezuka. "Creating and probing the Sachdev-Ye-Kitaev model with ultracold gases: Towards experimental studies of quantum gravity". *Progress of Theoretical and Experimental Physics*, 2017(8):083I01, 2017.

[45] D.I. Pikulin and Marcel Franz. "Black hole on a chip: Proposal for a physical realization of the Sachdev-Ye-Kitaev model in a solid-state system". *Physical Review X*, 7(3):031006, 2017.

[46] Anffany Chen, Roni Ilan, Fernando De Juan, D.I. Pikulin, and Marcel Franz. "Quantum holography in a graphene flake with an irregular boundary". *Physical Review Letters*, 121(3):036403, 2018.

[47] Aaron Chew, Andrew Essin, and Jason Alicea. "Approximating the Sachdev-Ye-Kitaev model with Majorana wires". *Physical Review B*, 96(12):121119, 2017.

[48] Laura García-Álvarez, Luis Egusquiza, Lucas Lamata, Adolfo Del Campo, Julian Sonner, and Enrique Solano. "Digital quantum simulation of minimal AdS/CFT". *Physical Review Letters*, 119(4):040501, 2017.

[49] Ryan Babbush, Dominic Berry, and Hartmut Neven. "Quantum simulation of the Sachdev-Ye-Kitaev model by asymmetric qubitization". *Physical Review A*, 99(4):040301, 2019.

[50] Zhihuang Luo, Yi-Zhuang You, Jun Li, Chao-Ming Jian, Dawei Lu, Cenke Xu, Bei Zeng, and Raymond Laflamme. "Quantum simulation of the non-Fermi-liquid state of Sachdev-Ye-Kitaev model". *npj Quantum Information*, 5(1):53, 2019.

[51] Alexei Kitaev. "Hidden correlations in the Hawking radiation and thermal noise, talk given at the Fundamental Physics Prize Symposium". In *SITP seminars https://online.kitp.ucsb.edu/online/joint98/kitaev/*, 2014.

[52] Alexei Kitaev and Josephine Suh. "The soft mode in the Sachdev-Ye-Kitaev model and its gravity dual". *Journal of High Energy Physics*, 2018(5):1–68, 2018.

[53] Dmitrii Trunin. "Pedagogical introduction to the Sachdev-Ye-Kitaev model and two-dimensional dilaton gravity". *Physics-Uspekhi*, 64(3):219, 2021.

[54] Joseph Polchinski and Vladimir Rosenhaus. "The spectrum in the Sachdev-Ye-Kitaev model". *Journal of High Energy Physics*, 2016(4):1–25, 2016.

[55] Tommaso Castellani and Andrea Cavagna. "Spin-glass theory for pedestrians". *Journal of Statistical Mechanics: Theory and Experiment*, 2005(05):P05012, 2005.

[56] S.F. Edwards and Phil Anderson. "Theory of spin glasses". *Journal of Physics F: Metal Physics*, 5(5):965, 1975.

[57] Gábor Sárosi. "AdS_2 holography and the SYK model". *arXiv preprint arXiv:1711.08482*, 2017.

[58] Dmitry Bagrets, Alexander Altland, and Alex Kamenev. "Sachdev-Ye-Kitaev model as Liouville quantum mechanics". *Nuclear Physics B*, 911:191–205, 2016.

[59] Irina Aref'eva, Mikhail Khramtsov, Maria Tikhanovskaya, and Igor Volovich. "Replica-nondiagonal solutions in the SYK model". *Journal of High Energy Physics*, 2019(7):1–59, 2019.

[60] Yingfei Gu, Xiao-Liang Qi, and Douglas Stanford. "Local criticality, diffusion and chaos in generalized Sachdev-Ye-Kitaev models". *Journal of High Energy Physics*, 2017(5):1–37, 2017.

[61] David Gross and Vladimir Rosenhaus. "All point correlation functions in SYK". *Journal of High Energy Physics*, 2017(12):1–58, 2017.

[62] Yingfei Gu, Alexei Kitaev, Subir Sachdev, and Grigory Tarnopolsky. "Notes on the complex Sachdev-Ye-Kitaev model". *Journal of High Energy Physics*, 2020(2):1–74, 2020.

[63] Tatsuo Azeyanagi, Frank Ferrari, and Fidel Schaposnik Massolo. "Phase diagram of planar matrix quantum mechanics, tensor, and Sachdev-Ye-Kitaev models". *Physical Review Letters*, 120(6):061602, 2018.

[64] Igor Klebanov, Alexey Milekhin, Grigory Tarnopolsky, and Wenli Zhao. "Spontaneous breaking of U(1) symmetry in coupled complex SYK models". *Journal of High Energy Physics*, 2020(11):1–29, 2020.

[65] Jonah Kudler-Flam, Ramanjit Sohal, and Laimei Nie. "Information scrambling with conservation laws". *SciPost Physics*, 12(4):117, 2022.

[66] Hanteng Wang, A.L. Chudnovskiy, Alexander Gorsky, and Alex Kamenev. "Sachdev-Ye-Kitaev superconductivity: Quantum Kuramoto and generalized Richardson models". *Physical Review Research*, 2(3):033025, 2020.

[67] Hanteng Wang, Dimitry Bagrets, A.L. Chudnovskiy, and Alex Kamenev. "On the replica structure of Sachdev-Ye-Kitaev model". *Journal of High Energy Physics*, 2019(9):1–31, 2019.

[68] Sharmistha Sahoo, Étienne Lantagne-Hurtubise, Stephan Plugge, and Marcel Franz. "Traversable wormhole and Hawking-Page transition in coupled complex SYK models". *Physical Review Research*, 2(4):043049, 2020.

[69] Étienne Lantagne-Hurtubise, Vedangi Pathak, Sharmistha Sahoo, and Marcel Franz. "Superconducting instabilities in a spinful Sachdev-Ye-Kitaev model". *Physical Review B*, 104(2):L020509, 2021.

[70] Ahmed Almheiri, Alexey Milekhin, and Brian Swingle. "Universal constraints on energy flow and SYK thermalization". *Journal of High Energy Physics*, 2024(8):1–43, 2024.

[71] Andreas Eberlein, Valentin Kasper, Subir Sachdev, and Julia Steinberg. "Quantum quench of the Sachdev-Ye-Kitaev model". *Physical Review B*, 96(20):205123, 2017.

[72] Wolfram Research, Inc. Mathematica, Version 14.2. Champaign, IL, 2024.

[73] Alexander Altland, Dmitry Bagrets, and Alex Kamenev. "Sachdev-Ye-Kitaev non-Fermi-liquid correlations in nanoscopic quantum transport". *Physical Review Letters*, 123(22):226801, 2019.

[74] Luca Dell'Anna. "Josephson current through the SYK model". *SciPost Physics*, 17(4):120, 2024.

[75] Pengfei Zhang. "Evaporation dynamics of the Sachdev-Ye-Kitaev model". *Physical Review B*, 100(24):245104, 2019.