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Mean field control with moderate interactions

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Abstract

We study the connection between an optimal control problem for a large system of particles interacting in a moderate way and a limit McKean-Vlasov control problem with a local dependence on the limit measure. Precisely, we prove the convergence of the value function associated with the N-particles problem towards the value function of the limit McKean-Vlasov control problem as N tends to infinity. We start by heuristically deriving the limit McKean-Vlasov control problem. Then, we prove the existence, uniqueness, and regularity of the optimal control for the limit problem, under a convexity assumption on the value function and by using the link with the associated MFG system. Finally, we prove the convergence by using regularization arguments on one side and the local regularity property of the empirical density associated to the particles system controlled with the limit optimal control.

Contents

1 Introduction

The topic of this work is the study of an optimal control problem with a large number of moderated interacting particles.

We consider the situation in which a central planner aims to minimize the average cost of a particles system in interaction by choosing all of the controls for all of the particles. The peculiarity of the model lies in the way this interaction occurs. Indeed, we assume that each particle interacts with the others in an increasingly localized way as the number of particles grows. In particular, denoting with N the number of particles, we consider the following optimal control problem:

$$\inf_{\bar{\alpha}^N} \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}\Big[\int_{t_0}^{T} \{\frac{|\alpha_t^{N,i}|^2}{2} + \int_{\mathbb{R}^d} F(x, V^N * m_t^N(x)) V^N * m_t^N(x) dx\} dt\Big] \tag{1}$$

where $\bar{\alpha}^N = (\alpha^{N,1}, ..., \alpha^{N,N})$ and the particles $(X^{N,i})_{i=1,...,N}$ follow the system of controlled SDEs :

$$dX_t^{N,i} = \alpha_t^{N,i} dt + \sqrt{2} dW_t^{N,i}, \quad X_{t_0}^{N,i} = Z^i \tag{2}$$

We leave for now the definition of the working assumptions.

As we can see from (1), each particle pays a price which depends quadratically on the choice of the control and a price that depends on its position and on the distribution of the other particles. In this last term, the assumption of moderate interaction is described by the convolution:

$$V^N * m_t^N(x) = \frac{1}{N} \sum_{i=1}^{N} V^N(x - X_t^{N,i}).$$

m_t^N is the empirical measure associated to the particles system (2), while V^N is defined by:

$$V^N(z) = N^{\beta} V(z N^{\frac{\beta}{d}})$$

where $V \in C^1(\mathbb{R}^d)$ is a symmetric density with compact support and $\beta \in (0, 1)$. With this particular choice of β, the support of V^N becomes smaller and smaller as N grows, and as a consequence, the interaction among the particles in the cost functional becomes more and more localized. This type of regime is called in the literature a regime of moderate interaction.

1.1 Our result

We study problem (1) when the number of particles is large. Specifically, for every initial time $t_0 \in [0, T]$ and vector of initial random i.i.d. positions $\bar{Z} = (Z^1, ...Z^N)$ we prove the convergence as $N \to +\infty$ of the value function $\mathcal{V}^N(t_0, \bar{Z})$ associated with the $N-$particles problem (1):

$$\mathcal{V}^N(t_0,\bar{Z}) = \inf_{\bar{\alpha}^N\in\mathcal{A}^N} \frac{1}{N}\sum_{i=1}^N \mathbb{E}\Big[\int_{t_0}^T \{\frac{|\alpha_t^{N,i}|^2}{2} + \int_{\mathbb{R}^d} F(x, V^N * m_t^N(x))V^N * m_t^N(x)dx\}dt\Big]$$

towards the value function associated to a McKean-Vlasov control problem with a local dependence on the limit measure:

$$\mathcal{U}(t_0,m_0) = \inf_{(\alpha,m)\in\mathcal{A}}\Big[\int_{t_0}^T\int_{\mathbb{R}^d}\frac{|\alpha(t,x)|^2}{2}m(t,x) + F(x,m(t,x))m(t,x)dxdt\Big] \tag{3}$$

where $(t_0,m_0)\in[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$ and m solves in the sense of distribution the Fokker-Plank equation:

$$\begin{cases}\partial_t m - \Delta m + div(m\alpha) = 0 & (t_0,T)\times\mathbb{R}^d\\ m(t_0) = m_0\end{cases} \tag{4}$$

We will postpone for a moment the definitions of the control sets $\mathcal{A}^N$, $\mathcal{A}$, as well as the working hypotheses.

1.2 Background and related literature

The convergence of a general class of $N-$particles optimal control problems towards the associated limit McKean-Vlasov optimal control problems was proven by Lacker in [11], under the key assumption of weak or nonlocal interactions, meaning that the dependence on the measure in the cost functional and in the coefficients of the dynamics is continuous with respect to the convergence in Wasserstein spaces. This kind of regime corresponds to the choice of $\beta = 0$ in the model presented in chapter 1.1 . Differently from our case, with this choice of parameter the radius of the interaction in the cost functional remains fixed as N-grows. As a consequence, the dependence on the measure in the limit cost functional is expected to be non-local, meaning that the second variable of F in the definition (3), is expected to be a measure instead of a density. Lacker's arguments are based on a reformulation of the state equation both for the N-particles problem and the McKean-Vlasov control problem as controlled martingale problems with a relaxed notion of controls. These formulations enjoy compactness properties. Together with the continuity properties of the cost functional and the coefficients of the dynamics with respect convergence in Wasserstein spaces, they ensure existence of optimal control for both the N-particles problem and the McKean-Vlasov problem and the desired convergence result. Recently Djete, Possamaï and Tan [8] extended the result by Lacker [11] to problems with a common noise.

Many other authors have contributed to the study of the mean field limit of optimal control problems under the assumption of weak interactions. In particular, Fornasier, Lisini, Orrieri and Savaré studied in [10] the problem for deterministic dynamics by using $\Gamma-$ convergence argument.
Recently, for the regime of weak interactions, a rate of convergence of the value function

$\mathcal{V}^N$ associated to the N-particles problem towards the value function $\mathcal{U}$ associated with the limit McKean-Vlasov control problem has been proven by Cardaliaguet, Daudin, Jackson, Souganidis [2] using PDE techniques in the presence of both idiosyncratic and common noises. Again, these arguments require at least the continuity of $\mathcal{U}$ in Wasserstein spaces in order to write a dynamic programming principle and thus to establish a link between $\mathcal{U}$ and an Hamilton-Jacobi equation in the space of measures.

Our work gives a first convergence result of $\mathcal{V}^N$ towards $\mathcal{U}$ for the regime of moderate interactions ($\beta \in (0,1)$). As said before, with this choice of parameter we do not have continuity properties in Wassertein spaces for the cost functional associated to $\mathcal{V}^N$ and $\mathcal{U}$ and this makes it impossible to use the same techniques previously presented for the weak or non-local regime.

A first propagation of chaos result for a system of uncontrolled, moderately interacting particles was proved by Oelschläger in [13] under a Lipschitz condition on the drift of the dynamics.

Recently, a Mean Field Games model with the assumption of moderate interactions among the players has been studied by Flandoli, Ghio and Livieri in [9]. In particular, they proved that any optimal control in feedback form for the MFG problem, induces a sequence of ϵ-Nash equilibrium for the $N-$player games. Their argument is based on an extension of the propagation of chaos result by [13]. In particular, the characterization of the limit density is given by regularity estimates on $V^N * m_t^N(\cdot)$ uniform in N, under the restriction $\beta \in (0, \frac{1}{2})$ and an integrability requirement on the initial distribution m_0.

Our work can be considered as the cooperative version of [9]. Indeed, in [9] the authors study the existence of approximate Nash equilibrium for $N-$player non-cooperative games when N is large, while in our work we are interested in the optimal control, i.e., in the minimizer of a global cost functional representing the average behaviour of the system of particles. For more details about the differences between Mean Field Control (MFC) and Mean Field Games (MFG) problems, we refer to [4] chapter 6.
Despite the different nature of the two problems, the regularity estimates on $V^N * m_t^N(\cdot)$ given in [9] have been useful in our convergence result.

1.3 Our strategy

As said in the previous paragraph, the local nature of our problem does not allow us to have continuity properties in Wasserstein spaces and for this reason, we used different techniques compared to those previously presented for the case of non-local interactions.

Our work is organized as follows:

- In chapter 2 we introduce rigorously the N-particles problem and we derive heuristically the limit problem. For this derivation we follows [13], where the regime of moderate interactions is studied for uncontrolled dynamics.
- In chapter 3 we prove existence and uniqueness of regular optimal control for the limit problem. Under the assumption of convexity of the limit cost functional, we

prove that the minimizer can be constructed by the solution of a (MFG) system. In particular, we prove the existence and uniqueness of regular solutions for this (MFG) system following some ideas from [5].

- Chapter 4 is devoted to the main results of this work: the convergence of the N-particles problem towards the limit problem. The strategy is based on the convergence of upper and lower sequences of the value function associated with the N-particle problem towards the value function of the limit problem. The lower sequence is given by a regularization argument that permits to use the result from [11]. As we will see, for this step the convexity assumption on the cost functional will play a crucial role. The convergence of the upper sequence is determined by the particular choice of the optimal control for the limit problem and by the regularity of the empirical density proved in [9]

- Finally, in the last section we analyze possible extensions of our result to more general problems and discuss important questions that remain open.

2 Model setup and main result

In this chapter we introduce the N-particles problem, the limit problem and the working hypotheses.

N-particles problem:

We consider a finite horizon $T > 0$. For every $t_0 \in [0, T]$ and for every N-vector of i.i.d random variable $\bar{Z} = (Z^1, ..., Z^N)$, we consider the N-particles value function:

$$\mathcal{V}^N(t_0, \bar{Z}) = \inf_{\bar{\alpha}^N \in \mathcal{A}^N} J^N(\bar{\alpha}) \tag{P}$$

where:

$$J^N(\bar{\alpha}) = \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}\Big[\int_{t_0}^{T} \{\frac{|\alpha_t^{N,i}|^2}{2} + \int_{\mathbb{R}^d} F(x, V^N * m_t^N(x))V^N * m_t^N(x)dx\}dt\Big]. \tag{5}$$

$\mathcal{A}^N$ is the set of controls $\bar{\alpha}^N = (\alpha^{N,1}, ..., \alpha^{N,N}) \in L^2((0,T) \times \overline{\Omega}, (\mathbb{R}^d)^N)$ progressively measurable, and for each $i = 1, ..., N$, the particle $(X_t^{N,i})_{t \in [t_0,T]}$ follows the controlled SDE:

$$dX_t^{N,i} = \alpha_t^{N,i}dt + \sqrt{2}dW_t^{N,i}, \quad X_{t_0}^{N,i} = Z^i \tag{6}$$

where $(W^{N,i})_{i=1,...,N}$ are independent $d-$dimensional brownian motion defined on a fixed filtered probability space $(\overline{\Omega}, \overline{\mathcal{F}}, (\overline{\mathcal{F}}_t)_t, \overline{\mathbb{P}})$.

In the definition of (P), $m_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{N,i}}$ is the empirical measure associated to the system (6). The peculiarity of problem (P) lies in the presence of the convolution $V^N * m_t^N(\cdot)$ between the empirical measure and a regularized symmetric density:

$$V^N(z) = N^{\beta}V(zN^{\frac{\beta}{d}})$$

where V is a C^1 symmetric density with compact support on $\mathbb{R}^d$ and $\beta \in (0, \frac{1}{2})$.

We work in the following setting of **HYPOTHESES (H)**:

A) $F : \mathbb{R}^d \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ is bounded and there exist $L > 0$ such that:

$$|F(x,p) - F(y,q)| \leq L(|x-y| + |p-q|)$$

B) The function $m \longrightarrow \mathcal{F}(x,m) = F(x,m)m$ is convex in m and its derivative $f(x,m) = \partial_m \mathcal{F}(x,m)$ is given by

$$f(x,m) = g(m) + h(x)$$

where g and h satisfy:

(a) $g \in C^2((0,+\infty))$ and there exists a positive constant c such that for every $m \in \mathbb{R}_+$ and uniformly in x:

$$0 \le g(m) \le cm^{\alpha} \text{ , with } \begin{cases} \frac{2}{d} \le \alpha \le \frac{2}{d-2} & if \quad d \ge 3 \\ \alpha < +\infty & otherwise \end{cases}$$

(b) $h \in C_b^2(\mathbb{R}^d)$, $h \ge 0$ on $\mathbb{R}^d$

C) The law of the initial position has density m_0 such that : $m_0 \in C_b^4(\mathbb{R}^d)$, $m_0 > 0$, $\int_{\mathbb{R}^d} m_0(x)dx = 1$ and $\int_{\mathbb{R}^d} e^{\lambda|x|} m_0(x)dx < +\infty$ for every $\lambda > 0$

D) $V^N(z) = N^{\beta} V(zN^{\frac{\beta}{d}})$ where V is a C^1 symmetric density with compact support on $\mathbb{R}^d$ and $\beta \in (0, \frac{1}{2})$.

Remark 1. The decomposition in hypothesis B) is satisfied by $\mathcal{F}(x,m) = F(x,m)m$ with F of the form:

$$F(x,m) = \int_0^m g(r)dr + h(x)$$

Remark 2. The convexity of $\mathcal{F}$ in the second variable, implies that g is increasing in m.

Remark 3. The restriction of β to $(0, \frac{1}{2})$ and the integrability assumption on the moment of m_0 at point C), will be necessary to establish the regularity of the empirical density in chapter 4.

Limit problem:

As explained in the introduction, with the choice of $\beta \in (0, \frac{1}{2})$ the interaction in the cost functional becomes more and more local as N-grows. In this regime, but in the case of uncontrolled dynamics, it has been proved in [13] the convergence in law of the empirical measure towards a deterministic measure with (Lebesgue) density.
This suggest to consider a limit McKean-Vlasov problem with a local dependence on the limit measure.

We introduce the value function for this problem in its analytic formulation. For $(t_0, m_0) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)$, we define the value function for the limit problem as:

$$\mathcal{U}(t_0, m_0) = \inf_{(\alpha,m)\in\mathcal{A}} \left[\int_{t_0}^T \int_{\mathbb{R}^d} \frac{|\alpha(t,x)|^2}{2} m(t,x) + F(x, m(t,x))m(t,x)dxdt \right] \tag{LP}$$

where $\mathcal{A}$ is the set of of couples (m, α) where $m(t,x) : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}$, $\alpha : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ such that $\alpha \in L_m^2([0,T] \times \mathbb{R}^d)$, $m \in L^1([0,T] \times \mathbb{R}^d)$, $m(t,x) \ge 0$, $\int_{\mathbb{R}^d} m(t,x)dx = 1$ for a.e $t \in (0,T)$ and m satisfies in the sense of distributions the Fokker-Plank equation:

$$
\begin{cases}
\partial_t m - \Delta m + div(m\alpha) = 0 & (t_0, T) \times \mathbb{R}^d \\
m(t_0) = m_0
\end{cases}
\tag{7}
$$

Our convergence result:

Now that the problems are well defined, we are ready to state our main result:

Theorem 2.1. *Under hypotheses (H), for every initial time $t_0 \in [0,T]$ and vector of i.i.d random initial position $\bar{Z} = (Z^{N,1}, ..., Z^{N,N})$ with law $m_0(\cdot)dx$, we have:*

$$
\lim_{N \to +\infty} \mathcal{V}^N(t_0, \bar{Z}) = \mathcal{U}(t_0, m_0). \tag{8}
$$

The rest of the work is devoted to the proof of this result, following the outline presented at the end of section 1.3.

3 Study of the limit problem

In this chapter we prove existence and uniqueness of smooth minimizer for the limit problem (LP).

3.1 Optimality conditions

Following [3], the idea is to reformulate problem (LP) as a convex minimization problem and than find sufficient optimality conditions for it.

Indeed, using the change of variable $w(t,x) = \alpha(t,x)m(t,x)$ problem (LP) can be rewritten as:

$$\inf_{(w,m)\in\mathcal{B}} \tilde{J}(w,m) = \inf_{(w,m)\in\mathcal{B}} \left[\int_{t_0}^{T}\int_{\mathbb{R}^d} \phi(m,w) + \mathcal{F}(x,m(t,x))dxdt\right] \tag{LP2}$$

where $\mathcal{B}$ is the set of function $m(t,x) : [0,T]\times\mathbb{R}^d \longrightarrow \mathbb{R}$, $w : [0,T]\times\mathbb{R}^d \longrightarrow \mathbb{R}^d$ such that $m, w \in L^1((0,T)\times\mathbb{R}^d)$, $m(t,x) \geq 0$, $\int_{\mathbb{R}^d} m(t,x)dx = 1$ for a.e $t \in (0,T)$ and satisfies in the sense of distributions the continuity equation:

$$\begin{cases} \partial_t m - \Delta m + div(w) = 0 & (t_0,T)\times\mathbb{R}^d \\ m(t_0) = m_0 \end{cases} \tag{9}$$

The function $\phi : \mathbb{R}_+ \times \mathbb{R}^d \longrightarrow \mathbb{R}$ is defined as follows:

$$\phi(m,w) = \begin{cases} \frac{1}{2}\frac{|w|^2}{m} & if \quad m > 0 \\ 0 & if \quad m = 0, \quad w = 0 \\ +\infty & otherwise \end{cases}$$

Lemma 3.1. *The function ϕ is convex.*

Proof. ϕ can be written as the supremum of linear functional (in m and w) as follow:

$$\phi(m,w) = \sup_{\beta\in\mathbb{R}^d}\{w\cdot\beta - \frac{m}{2}|\beta|^2\}$$

and thus it is convex. □

Now, since $\mathcal{F}$ is convex in m, ϕ is convex in both variables, the functional $\tilde{J}$ is convex. Moreover, the set $\mathcal{B}$ is convex so that (LP2) is a convex minimization problem.

The function ϕ is C^1 in the region $m > 0$. Thus, its sub-differential at a point $(\bar{m},\bar{w}) \in (0,+\infty)\times\mathbb{R}^d$ is:

$$\partial\phi(\bar{m},\bar{w}) = (\partial_m\phi, \partial_w\phi) = (\frac{\bar{w}}{\bar{m}}, -\frac{|\bar{w}|^2}{2\bar{m}^2})$$

Following [3], the system of optimality conditions for (LP2) is the following (MFG) system:

$$\begin{cases} -\partial_t u - \Delta u + \dfrac{|Du|^2}{2} = f(x, m(t,x)) & (0,T) \times \mathbb{R}^d \\ \partial_t m - \Delta m - div(mDu) = 0 & (0,T) \times \mathbb{R}^d \\ m(0,x) = m_0, \quad u(x,T) = 0 \end{cases} \tag{MFG}$$

where $f(x,m) = \partial_m \mathcal{F}(x,m)$.

As we will prove in the next chapter, under our hypothesis there exists an unique solution $(\bar{u}, \bar{m}) \in (C^{1,2}([0,T] \times \mathbb{R}^d))^2$.

It can be proved using Schauder estimates that this solution satisfies:

$$\partial_t \bar{u}, D\bar{u}, \partial_t D\bar{u}, \Delta\bar{u}, \bar{m}, \partial_t \bar{m}, \partial_t D\bar{m}, \Delta\bar{m} \in C_b([0,T] \times \mathbb{R}^d).$$

Moreover, $\bar{m} \in L^1((0,T) \times \mathbb{R}^d)$ and $|D\bar{u}|^2 \bar{m} \in L^1((0,T) \times \mathbb{R}^d)$ so that $(\bar{m}, -D\bar{u}) \in \mathcal{A}$ and in particular $(\bar{m}, -\bar{m}D\bar{u}) \in \mathcal{B}$.

Remark 4.

1. Thanks to the strong maximum principle, if $\bar{m}$ is the solution of the Fokker-Plank equation in (MFG), then $\bar{m} > 0$.
2. If we prove that $(-\bar{m}D\bar{u}, m)$ is a minimizer for (LP2), then $(-D\bar{u}, \bar{m})$ is a minimizer for (LP). Indeed, the minimization in (LP2) can be restricted to the couples $(m, w) \in \mathcal{B}$ such that $\frac{|w|^2}{m} \in L^1((0,T) \times \mathbb{R}^d)$ so that $\alpha = \frac{w}{m} \in L^2_m([0,T] \times \mathbb{R}^d)$.

In this chapter we will prove that $(-D\bar{u}, \bar{m})$ is the unique minimizer for (LP) in $\mathcal{A}$. We start by proving the following result for (LP2):

Proposition 3.2. *Let $(\bar{u}, \bar{m})$ be the solution of (MFG), then $(-\bar{m}D\bar{u}, \bar{m})$ minimizes (LP2).*

Proof. Let $(\bar{m}, \bar{w}) = (\bar{m}, -\bar{m}D\bar{u})$ and $(m, w) \in \mathcal{B}$ another competitor for the minimization.
Using the convexity of ϕ and $\mathcal{F}$ and the sub-differentiability of ϕ in $(\bar{m}, \bar{w})$ and $\mathcal{F}$ in $\bar{m}$ we have:

$$\begin{aligned} \tilde{J}(m,w) - \tilde{J}(\bar{m}, \bar{w}) &= \int_0^T \int_{\mathbb{R}^d} \phi(\bar{m}, \bar{w}) - \phi(m,w) + \mathcal{F}(x, \bar{m}) - \mathcal{F}(x, m) dxdt \\ &\geq \int_0^T \int_{\mathbb{R}^d} \frac{\bar{w}}{\bar{m}} \cdot (\bar{w} - w) - \frac{|\bar{w}|^2}{2\bar{m}^2}(\bar{m} - m) + f(x, \bar{m})(m - \bar{m}) dxdt. \end{aligned}$$

Now, using that $\bar{w} = -\bar{m}D\bar{u}$ and the fact that $\bar{u}$ solves the H-J equation in (MFG), we can rewrite this last integral as:

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} -D\bar{u} \cdot (\bar{w} - w) + \frac{|D\bar{u}|^2}{2}(\bar{m} - m) + (\partial_t \bar{u} + \Delta \bar{u} - \frac{|D\bar{u}|^2}{2})(\bar{m} - m) dxdt \\
&= \int_0^T \int_{\mathbb{R}^d} (\partial_t \bar{u} + \Delta \bar{u})(\bar{m} - m) - D\bar{u} \cdot (\bar{w} - w) dxdt \\
&= \int_0^T \int_{\mathbb{R}^d} (\partial_t \bar{u} + \Delta \bar{u} - D\bar{u} \cdot D\bar{u})\bar{m} dxdt - \int_0^T \int_{\mathbb{R}^d} (\partial_t \bar{u} + \Delta \bar{u} - D\bar{u} \cdot w) m dxdt
\end{aligned}
$$

Since $\bar{u}$ is C^2 we can use it as test function (up to an approximation with cut off functions) for the weak formulation of the Fokker-Plank equation in (MFG) and for the continuity equation (9), so that, since at time zero $\bar{m}(0,x) = m(0,x) = m_0(x)$ and at time T, $\bar{u}(T,x) = 0$ we have:

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} (\partial_t \bar{u} + \Delta \bar{u} - D\bar{u} \cdot D\bar{u})\bar{m} dxdt + \int_0^T \int_{\mathbb{R}^d} (\partial_t \bar{u} + \Delta \bar{u}) m - D\bar{u} \cdot w dxdt \\
&= \int_{\mathbb{R}^d} (\bar{u}(0,x) - \bar{u}(0,x)) m_0(x) dx = 0
\end{aligned}
$$

which proves that $\tilde{J}(m,w) \geq \tilde{J}(\bar{m},\bar{w})$ hence $(-\bar{m}D\bar{u}, \bar{m})$ minimize $\tilde{J}$ in $\mathcal{B}$. $\square$

Thanks to the convexity of $\mathcal{B}$ and to the strict convexity of ϕ, $(-\bar{m}D\bar{u}, \bar{m})$ is the unique minimizer for (LP) in $\mathcal{B}$. As a consequence, $(-D\bar{u}, \bar{m})$ is the unique minimizer for (LP).

3.2 Study of the MFG system

In this chapter we prove the existence of classical solution for the MFG system:

$$\begin{cases} -\partial_t u - \Delta u + \dfrac{|Du|^2}{2} = f(x, m(t,x)) & (0,T) \times \mathbb{R}^d \\ \partial_t m - \Delta m - div(mDu) = 0 & (0,T) \times \mathbb{R}^d \\ m(0,x) = m_0(x), \quad u(x,T) = 0 & x \in \mathbb{R}^d \end{cases} \qquad (MFG)$$

we recall that $f(x,m) = g(m) + h(x)$ where g and h satisfies hypothesis (B) and the m_0 satisfies hypothesis (C).

System (MFG) is a MFG system with local coupling defined on the whole space $\mathbb{R}^d$. The local nature of this problem gives problem in the proof of the existence of regular solution for the Hamilton-Jacobi equation since there is no smoothing effect as in the non-local case.
Moreover, working on $\mathbb{R}^d$ creates compactness issues for the fixed point map.

Our strategy is based on the work [5]. The idea is to obtain sufficient integrability on Du, thanks to the growth restriction on g, in order to obtain local Hölder regularity for m.

The key ingredient of the strategy is the following a priori estimate:

$$\int_0^T \int_{\mathbb{R}^d} \mu^{2\alpha+1} dxdt \leq C \qquad (10)$$

where μ is the solution for $\lambda \in (0,1)$ of the system:

$$\begin{cases} -\partial_t v - \Delta v + \dfrac{|Dv|^2}{2} = f(x, \mu(x,t)) & (0,T) \times \mathbb{R}^d \\ \partial_t \mu - \Delta \mu - div(\mu Dv) = 0 & (0,T) \times \mathbb{R}^d \\ \mu(0) = \lambda m_0, \quad v(x,T) = 0 & x \in \mathbb{R}^d \end{cases} \qquad \text{(S)}$$

which is obtain using first and second order estimates and the positive sign of $g'(m)$.

Then the proof of the existence of regular solutions follows by using Schaefer's fixed point theorem and a bootstrap procedure to gain regularity.

So, we start proving the a priori estimates (10).

3.2.1 A priori estimate

In this section we prove the following theorem:

Theorem 3.3. *Suppose* $(v, \mu) \in C([0,T], C_b^4(\mathbb{R}^d)) \times \Big(C_b^1([0,T] \times \mathbb{R}^d) \cap C([0,T], L^1(\mathbb{R}^d)) \Big)$ *solves system (S) for* $\lambda \in (0,1)$*. Then there exists a positive constant* $M = M(d, \alpha, ||m_0||_{L^{\alpha+1}(\mathbb{R}^d)})$

such that:

$$\int_0^T \int_{\mathbb{R}^d} \mu^{2\alpha+1}(t,x)dxdt \leq M$$

Proof. **Step 1.** (First order estimate) We multiply the Fokker-Plank equation in (S) by μ^α and we integrate by parts on the ball $\mathcal{B}_R$, $R>0$:

$$\int_0^t \int_{\mathcal{B}_R} \partial_t\mu\mu^\alpha dxds - \int_0^t \int_{\mathcal{B}_R} \Delta\mu\mu^\alpha dxds - \int_0^t \int_{\mathcal{B}_R} div(\mu Du)\mu^\alpha dxds = 0$$

$$\Longrightarrow \frac{1}{\alpha+1}\int_{\mathcal{B}_R} \mu^{\alpha+1}(t)dx + \alpha \int_0^t \int_{\mathcal{B}_R} \mu^{\alpha-1}|D\mu|^2 dxds =$$
$$- \alpha \int_0^t \int_{\mathcal{B}_R} Dv\cdot D\mu\mu^\alpha dxds + \frac{1}{\alpha+1}\int_{\mathcal{B}_R} \mu^{\alpha+1}(0)dx + G_R$$

where:

$$G_R = \int_0^t \int_{\partial\mathcal{B}_R} D\mu\mu^\alpha \cdot ndxdt + \int_0^t \int_{\partial\mathcal{B}_R} Dv\mu\mu^\alpha \cdot ndxds$$

Now, since μ, $D\mu$, Dv are bounded and $\mu \in C(L^1)$, the integrands of G_R belong to $L^1((0,T)\times\mathbb{R}^d)$. So, by lemma (A.1) in appendix A there exists a sequence $R_k \nearrow +\infty$ such that $G_{R_k} \longrightarrow 0$, so by dominated convergence we obtain that for every $t\in[0,T]$:

$$\frac{1}{\alpha+1}\int_{\mathbb{R}^d} \mu^{\alpha+1}(t)dx + \alpha \int_0^t \int_{\mathbb{R}^d} \mu^{\alpha-1}|D\mu|^2 dxds =$$
$$- \alpha \int_0^t \int_{\mathbb{R}^d} Dv\cdot D\mu\mu^\alpha dxds + \frac{1}{\alpha+1}\int_{\mathbb{R}^d} \mu^{\alpha+1}(0)dx$$

showing in particular that $\mu^{\alpha-1}|D\mu|^2 \in L^1((0,T)\times\mathbb{R}^d)$. Now $D\mu\mu^\alpha = \frac{1}{\alpha+1}D\mu^{\alpha+1}$. Moreover, μ, $D\mu$, Dv, D^2v are bounded and $\mu\in C(L^1)$. So, integrating by parts using lemma (A.2) in the appendix and thanks to the Hölder inequality we obtain:

$$- \alpha \int_0^t \int_{\mathbb{R}^d} Dv\cdot D\mu\mu^\alpha dxds$$
$$= \frac{\alpha}{\alpha+1}\int_0^t \int_{\mathbb{R}^d} \Delta v\mu^{\alpha+1}dxds \leq \frac{\alpha}{\alpha+1}\Big(\int_0^t \int_{\mathbb{R}^d} |D^2v|^2\mu dxds\Big)^{\frac{1}{2}}\Big(\int_0^t \int_{\mathbb{R}^d} \mu^{2\alpha+1}dxds\Big)^{\frac{1}{2}}$$

Now, $\mu^{\alpha-1}|D\mu|^2 = \frac{4}{(\alpha+1)^2}|D\mu^{\frac{\alpha+1}{2}}|^2$ and letting t vary in $[0,T]$:

$$\Big[\sup_{t\in[0,T]}\int_{\mathbb{R}^d} \mu^{\alpha+1}dx + \frac{4}{\alpha+1}\int_0^T \int_{\mathbb{R}^d} |D\mu^{\frac{\alpha+1}{2}}|^2 dxdt\Big]^2$$
$$\leq 2\Big(\int_{\mathbb{R}^d} \mu^{\alpha+1}(0)dx\Big)^2 + 2\alpha^2 (\int_0^T \int_{\mathbb{R}^d} |D^2v|^2\mu dxdt\Big)\Big(\int_0^T \int_{\mathbb{R}^d} \mu^{2\alpha+1}dxdt\Big)$$

Step 2.(Parabolic interpolation) We now use some estimates from [7]. In particular by proposition 3.1 in [7] with $v=\mu^{\frac{\alpha+1}{2}}, p=2, q=\frac{2(2\alpha+1)}{\alpha+1}$ and $\frac{d+m}{N}=\frac{2\alpha+1}{\alpha+1}$ there exists a constant $\tilde{C}=\tilde{C}(d,\alpha)$ such that:

$$(\int_0^T \int_{\mathbb{R}^d} \mu^{2\alpha+1}dxdt\Big)^{\frac{2}{q}} \leq \tilde{C}\Big[\int_0^T \int_{\mathbb{R}^d} |D\mu^{\frac{\alpha+1}{2}}|^2 dxdt + \Big(\sup_{t\in[0,T]}\int_{\mathbb{R}^d} \mu^\beta dx\Big)^{\frac{2}{m}}\Big]$$

where $\beta = m\frac{\alpha+1}{2} = \frac{\alpha N}{2}$ and

$$\frac{2}{q} = \frac{\alpha+1}{2\alpha+1} \in (\frac{1}{2}, 1).$$

Now if $d \leq 2$ and $\alpha \geq \frac{2}{d}$ or $d > 2$ and $\frac{2}{d} \leq \alpha \leq \frac{2}{d-2}$ then $1 \leq \beta \leq \alpha+1$. So by interpolation:

$$||\mu(t)||_{L^\beta(\mathbb{R}^d)} \leq ||\mu(t)||^{1-\theta}_{L^1(\mathbb{R}^d)} ||\mu(t)||^{\theta}_{L^{\alpha+1}(\mathbb{R}^d)}$$

for $\frac{1}{\beta} = 1 - \theta + \frac{\theta}{\alpha+1}$.

Now since $||\mu(t)||_{L^1(\mathbb{R}^d)} = ||\mu(0)||_{L^1(\mathbb{R}^d)} \leq ||m(0)||_{L^1(\mathbb{R}^d)} = 1$ we get:

$$(\int_0^T \int_{\mathbb{R}^d} \mu^{2\alpha+1} dxdt\Big)^{\frac{2}{q}} \leq \tilde{C}\Big[\int_0^T \int_{\mathbb{R}^d} |D\mu^{\frac{\alpha+1}{2}}|^2 dxdt + \Big(\sup_{t\in[0,T]} \int_{\mathbb{R}^d} \mu^{\alpha+1} dx\Big)^a\Big]$$

where $a = \frac{2\theta\beta}{m(\alpha+1)}$. Since $a < 1$ we have:

$$(\int_0^T \int_{\mathbb{R}^d} \mu^{2\alpha+1} dxdt\Big)^{\frac{4}{q}} \leq 2\tilde{C}^2\Big[\int_0^T \int_{\mathbb{R}^d} |D\mu^{\frac{\alpha+1}{2}}|^2 dxdt + \Big(\sup_{t\in[0,T]} \int_{\mathbb{R}^d} \mu^{\alpha+1} dx\Big)^a\Big]^2 + 2\tilde{C}^2$$

Therefore, plugging this last inequality in the result of Step 1. we obtain:

$$(\int_0^T \int_{\mathbb{R}^d} \mu^{2\alpha+1} dxdt\Big)^{\frac{4}{q}} \leq c\Big[\int_{\mathbb{R}^d} \mu^{\alpha+1}(0)dx + (\int_0^T \int_{\mathbb{R}^d} |D^2 v|^2 \mu dxdt\Big)\Big(\int_0^T \int_{\mathbb{R}^d} \mu^{2\alpha+1} dxdt\Big) + 1\Big] \tag{11}$$

where c depends only on N, α.

Step 3.(Second order estimate)

We want now to estimate $\int_0^T \int_{\mathbb{R}^d} |D^2 v|^2 \mu dxdt$. We compute the laplacian of the Hamilton-Jacobi equation:

$$\begin{aligned} -\partial_t \Delta v - \Delta\Delta v + |D^2 v|^2 + D(\Delta v)\cdot Dv &= div(\partial_m f(x,\mu) D\mu) + div(D_x f(x,\mu)) \\ &= div(g'(\mu) D\mu) + \Delta h(x) \end{aligned}$$

since by hypothesis $f(x,\mu) = g(\mu) + h(x)$.

Now we multiply the above equality by μ and we integrate by parts (again using lemma A.2) to obtain:

$$\begin{aligned} &-\int_0^T \int_{\mathbb{R}^d} \partial_t \Delta v \mu dxdt - \int_0^T \int_{\mathbb{R}^d} \Delta v(\Delta\mu + div(Dv\mu)) dxdt \\ &\quad + \int_0^T \int_{\mathbb{R}^d} |D^2 v|^2 \mu dxdt \\ &\quad = -\int_0^T \int_{\mathbb{R}^d} g'(\mu)|D\mu(t,x)|^2 dxdt + \int_0^T \int_{\mathbb{R}^d} \Delta h(x) \mu dxdt \end{aligned}$$

Using the Fokker-Plank equation, the left hand side can be rewritten as:

$$\begin{aligned}
&-\int_0^T\int_{\mathbb{R}^d}\partial_t\Delta v\mu dxdt-\int_0^T\int_{\mathbb{R}^d}\Delta v(\Delta\mu+div(Dv\mu)dxdt\\
&=\int_{\mathbb{R}^d}\Delta v(0)\mu(0)dx-\int_{\mathbb{R}^d}\Delta v(T)\mu(T)dx\\
&=\int_{\mathbb{R}^d}\Delta v(0)\mu(0)dx
\end{aligned}$$

$$\begin{aligned}
\implies \int_0^T\int_{\mathbb{R}^d}|D^2v|^2\mu dxdt\le\int_0^T\int_{\mathbb{R}^d}Dv(0)\cdot D\mu(0)dx-\int_0^T\int_{\mathbb{R}^d}g'(\mu)|D\mu(t,x)|^2dxdt\\
+\int_0^T\int_{\mathbb{R}^d}\Delta h(x)\mu dxdt
\end{aligned}$$

Now since by assumption g is increasing in the second variable

$$-\int_0^T\int_{\mathbb{R}^d}g'(\mu)|D\mu(t,x)|^2dxdt\le 0$$

and again by assumption $||\Delta h||_{L^\infty}\le c$ for some constant c independent on μ, we have :

$$\int_0^T\int_{\mathbb{R}^d}|D^2v|^2\mu dxdt\le\int_0^T\int_{\mathbb{R}^d}Dv(0)\cdot D\mu(0)dx+c$$

Now by lemma (A.5) in appendix A, if we denote by $\mathcal{F}(x,r)=\int_0^r f(x,m)dm$ the antiderivative of f, we have that for a.e. $t\in[0,T]$:

$$\int_{\mathbb{R}^d}Dv(t,x)D\mu(t,x)dx+\frac{1}{2}\int_{\mathbb{R}^d}|Du(t,x)|^2mdx-\int_{\mathbb{R}^d}\mathcal{F}(x,m(t,x))dx=constant$$

and so for $t=0$ we have:

$$\begin{aligned}
&\int_{\mathbb{R}^d}Dv(0)\cdot D\mu(0)dx+\frac{1}{2}\int_{\mathbb{R}^d}|Dv(0)|^2\mu(0))dx-\int_{\mathbb{R}^d}\mathcal{F}(x,\mu(0)dx\\
&=\int_{\mathbb{R}^d}Dv(T)\cdot D\mu(T)dx+\frac{1}{2}\int_{\mathbb{R}^d}|Dv(T)|^2\mu(T)dx-\int_{\mathbb{R}^d}\mathcal{F}(x,\mu(T))dx.
\end{aligned}$$

Since for every $x\in\mathbb{R}^d$ $v(T,x)=0$ and $\mathcal{F}\ge 0$ we obtain:

$$\int_{\mathbb{R}^d}Dv(0)\cdot D\mu(0)dx\le\int_{\mathbb{R}^d}\mathcal{F}(x,\lambda m(0))dx=c_2$$

where c_2 does not depend on μ. In the end we have $\int_0^T\int_{\mathbb{R}^d}|D^2v|^2\mu dxdt\le c$ with c dependent only on d,α. Plugging this last inequality in (11) we get:

$$(\int_0^T\int_{\mathbb{R}^d}\mu^{2\alpha+1}dxdt\Big)^{\frac{4}{q}}\le\tilde{C}\Big(1+\int_0^T\int_{\mathbb{R}^d}\mu^{2\alpha+1}dxdt\Big).$$

Since $\frac{4}{q}\ge 1$, by Young inequality we conclude that:

$$\int_0^T\int_{\mathbb{R}^d}\mu^{2\alpha+1}dxdt\le M=M(d,\alpha,||m_0||_{L^{\alpha+1}}).$$

□

Now we are ready for the proof of the existence.

3.2.2 Existence

Theorem 3.4. *Under the initial assumption, there exists $(\bar{u}, \bar{m})$ classical solution of the system (MFG).*

Proof. Using the Hopfe-Cole transform $w = e^{-\frac{u}{2}}$ we rewrite MFG system as:

$$\begin{cases} -\partial_t w - \Delta w + f(x,m)w = 0 \quad (0,T)\times\mathbb{R}^d \\ \partial_t m - \Delta m - div(\dfrac{Dw}{w}m) = 0 \qquad (0,T)\times\mathbb{R}^d \\ m(0) = m_0, \quad w(x,T) = 1 \quad x\in\mathbb{R}^d \end{cases} \tag{12}$$

For $\alpha \geq \frac{2}{d}$ if $d \leq 2$ or $\frac{2}{d} \leq \alpha \leq \frac{2}{d-2}$ if $d > 2$, we introduce $X = L^{2\alpha+1}((0,T)\times\mathbb{R}^d)$ endowed with the topology of the strong convergence and we consider the open set:

$$U_{2M} = \{m \in X | \int_0^T \int_{\mathbb{R}^d} m^{2\alpha+1} dxdt < 2M\}$$

where M is the a priori bound of the previous section. We introduce also $\overline{U}_{2M}$, the closure of U_{2M} with respect the strong convergence norm of X. For $m \in \overline{U}_{2M}$, we consider the triple (m, μ, w) solution of :

$$\begin{cases} -\partial_t w - \Delta w + f(x,m)w = 0 \quad (0,T)\times\mathbb{R}^d \\ \partial_t \mu - \Delta \mu - div(\dfrac{Dw}{w}\mu) = 0 \qquad (0,T)\times\mathbb{R}^d \\ \mu(0) = m_0, \quad w(x,T) = 1 \quad x\in\mathbb{R}^d \end{cases} \tag{S1}$$

and we define the map $\Phi : \bar{U}_{2M} \longrightarrow X$ as $\Phi(m) = \mu$.

Proposition 3.5. *The map Φ is well posed, continuous and compact.*

Proof. We start from the well posedness.

Given $m \in \bar{U}_{2M}$ we have $m^\alpha \in L^p((0,T)\times\mathbb{R}^d)$ for $p = \frac{2\alpha+1}{\alpha} > \frac{d+2}{2}$ and this implies that $f(m) \in L^p((0,T)\times\mathbb{R}^d)$ for $p = \frac{2\alpha+1}{2} > \frac{d+2}{2}$. Since $w(T) \in L^\infty(\mathbb{R}^d)$ thanks to Theorem 10.1 section 3 of [12], there exist a weak solution $w \in V_{2,loc}((0,T)\times\mathbb{R}^d) \cap L^\infty((0,T)\times\mathbb{R}^d)$. Moreover, by the comparison principle $w \geq \underline{w} = e^{-T\max_{\mathbb{R}^d} g(x)} \min_{\mathbb{R}^d} e^{-w(T,x)} > 0$. Thanks to lemma (A.3), we have also $Dw \in L^{2p}((0,T)\times\mathbb{R}^d)$.

We pass now to the Fokker-Plank equation. Since $\mu(0) \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ then $\mu(0) \in L^2(\mathbb{R}^d)$. Moreover $\frac{Dw}{w} \in L^{2p}((0,T)\times\mathbb{R}^d)$ for $2p > d+2$ so that we have existence of a weak solution $\mu \in V_2((0,T)\times\mathbb{R}^d) \cap L^\infty((0,T)\times\mathbb{R}^d)$ that is also locally Hölder continuous. Now since $\mu \in L^\infty((0,T)\times\mathbb{R}^d)$ and has finite first order moment,by interpolation we have that $\mu \in X$ i.e. $\Phi(m) \in X$ and Φ is well defined.

We study now the compactness of Φ.
By the previous arguments we have that there exist constants K, θ, with $\theta \in (0,1)$, which depend only on $M, d, \alpha, ||u(T)||_{C^2}, ||m_0||_{L^\infty}, ||m_0||_{L^1(\mathbb{R}^d)}, ||xm_0||_{L^1(\mathbb{R}^d)}$ such that for any $m \in \bar{U}_{2M}$:

$$||w||_{L^\infty}, ||\frac{Dw}{w}||_{L^{2p}(\mathbb{R}^d\times(0,T))}, ||\mu||_{L^\infty((0,T),L^q(\mathbb{R}^d))} \leq K$$

for all $q \in [1, +\infty]$ and

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^d} |x|\mu(x,t)dx, \quad \sup_{z\in\mathbb{R}^d} ||\mu||_{C^{\theta,\frac{\theta}{2}}_{\mathcal{B}_1(z)\times[0,T]}} \leq K$$

We use these last inequalities to get the compactness of Φ. Indeed let $(m_n)_n$ be sequence in $\bar{U}_{2M}$ and for every n consider $\mu_n = \Phi(m_n)$. We have to prove that $(\mu_n)_n$ is relatively compact on X endowed with the norm of the strong convergence. For $R > 0$ we have:

$$\sup_n \sup_{t\in[0,T]} \int_{|x|\geq R} \mu_n(t,x)dx \leq \frac{K}{R}.$$

Furthermore, since $(\mu_n)_n$ is equibounded, by interpolation between L^∞ and L^1 we have that for every ϵ there exist R such that:

$$\sup_n ||\mu_n||_{L^\infty((0,T),L^{2\alpha+1}(\mathbb{R}^d\setminus B_R))} \leq \epsilon$$

where B_R is the ball of center 0 and radius R. On the other hand, on a compact set we can use the uniform bound on the Hölder norm. Indeed, since μ_n is equibounded in $C^{\theta,\frac{\theta}{2}}_{\mathcal{B}_R(z)\times[0,T]}$, by Ascoli-Arezelà we can estract a sub-sequence μ_{n_k} that converges uniformly on $\mathcal{B}_R(z)\times[0,T]$ to some μ. Since the same estimates hold for μ, we have that for n large enough:

$$||\mu_{n_k} - \mu||_{L^\infty((0,T),L^{2\alpha+1}(\mathbb{R}^d))} \leq \epsilon$$

By a diagonalization argument, we can find a subsequence μ_{n_k} which converges in $L^\infty((0,T),L^q(\mathbb{R}^d))$ and therefore in $L^{2\alpha+1}(\mathbb{R}^d\times(0,T))$ which proves the compactness of Φ.

Finally, we prove the continuity of Φ with respect to the strong convergence of $L^{2\alpha+1}(\mathbb{R}^d\times(0,T))$.
Let $(m_n)_n \subset \bar{U}_{2M}$ converging to $m \in X$. Than $m^\alpha \longrightarrow m^\alpha$ in $L^p((0,T)\times\mathbb{R}^d)$ and thanks to the hypothesis on f, $f(m_n) \longrightarrow f(m)$ in $L^p((0,T)\times\mathbb{R}^d)$. We have to prove that $\mu_n = \Phi(m_n)$ converges to $\mu = \Phi(m)$. We need a stability argument. Let (m_n, w_n, μ_n) the triple solving (S), then $\tilde{w} = w - w_n$ solves:

$$-\partial_t\tilde{w} - \Delta\tilde{w} = w(f(x,m) - f(x,m_n)) + \tilde{w}f(x,m_n), \quad \tilde{w}(T) = 0.$$

By lemma (A.3), we have:

$$\begin{aligned} ||w - w_n||_{L^\infty(\mathbb{R}^d\times(0,T))} &\leq C||w(f(m_n) - f(x,m))||_{L^p(\mathbb{R}^d\times(0,T))} \\ &\leq CK||f(m_n) - f(m)||_{L^p(\mathbb{R}^d\times(0,T))} \end{aligned}$$

where C depends on K, d, T, p.
Again, by the same lemma we have :

$$\begin{aligned} ||Dw_n - Dw||_{L^{2p}(\mathbb{R}^d\times(0,T))} &\leq C||w(f(m) - f(m_n)) + f(m_n)\tilde{w}||_{L^p(\mathbb{R}^d\times(0,T))} \\ &\leq C(K||f(m) - f(m_n)||_{L^p(\mathbb{R}^d\times(0,T))} + M^{\frac{1}{p}}||w - w_n||_{L^\infty(\mathbb{R}^d\times(0,T))}) \end{aligned}$$

So that $w_n \longrightarrow w$ in $L^\infty(\mathbb{R}^d \times (0,T))$ and $Dw_n \longrightarrow Dw$ in $L^{2p}(\mathbb{R}^d \times (0,T))$.
Now thanks to the maximum principle $w_n, w \geq \underline{w} > 0$ we have:

$$\frac{Dw_n}{w_n} - \frac{Dw}{w} = \frac{1}{w}(Dw_n - Dw) + Dw_n(\frac{w - w_n}{w_n w})$$

which implies the convergence of $\frac{Dw_n}{w_n} \longrightarrow \frac{Dw}{w}$ in $L^{2p}(\mathbb{R}^d \times (0,T))$.

Now, setting $\tilde{\mu} = \mu - \mu_n$ it satisfing:

$$-\partial_t \tilde{\mu} - \Delta\mu = -2div(\frac{Dw}{w}\tilde{\mu}) - 2div((\frac{Dw}{w} - \frac{Dw_n}{w_n})\mu_n) = 0, \;\; \tilde{\mu}(0) = 0.$$

As before, using lemma (A.3), we have that :

$$\begin{aligned} ||\mu - \mu_n||_{L^\infty(\mathbb{R}^d \times (0,T))} &\leq C||(\frac{Dw}{w} - \frac{Dw_n}{w_n})\mu_n||_{L^{2p}(\mathbb{R}^d \times (0,T))} \\ &\leq CK||\frac{Dw}{w} - \frac{Dw_n}{w_n}||_{L^{2p}(\mathbb{R}^d \times (0,T))} \end{aligned}$$

where C depends again only on K, d, T, p. This and the equiboundedness of μ_n, μ on $L^\infty((0,T), L^1(\mathbb{R}^d))$ gives the convergence of $(\mu_n)_n$ to μ in $L^{2\alpha+1}(\mathbb{R}^d \times (0,T))$ i.e. the continuty of Φ. □

We are now ready to prove the existence of smooth solution for (MFG). We use the following fixed point theorem (Schaefer's fixed point theorem).

Theorem 3.6. *Let X be a Banach space, $C \subset X$ closed and convex subset. Let U be an open subset of C and $\overline{U}$ its closure. Consider a function $\Phi : \overline{U} \longrightarrow C$ continuous and compact. Suppose that the following property holds:*

$$u = \lambda\Phi(u) \text{ for some } \lambda \in (0,1) \implies u \notin \partial U.$$

Then Φ has a fixed point.

We use the previous theorem with $C = X = L^{2\alpha+1}((0,T) \times \mathbb{R}^d)$, $U = U_{2M}$. The condition $u = \lambda\Phi(u)$ is equivalent to ask that (u, μ) is a solution of (S). In order to use the a priori estimate, we use a bootstrap procedure to prove that (u, μ) is a classical solution. Indeed, since μ is locally $C^{\theta, \frac{\theta}{2}}$, then w belongs to $C^{2+\theta, 1+\frac{\theta}{2}}$. This implies that μ solves a linear parabolic equation with $C^{\theta,\frac{\theta}{2}}$ coefficients so that $\mu \in C^{2+\theta, 1+\frac{\theta}{2}}$.

Then, using the a priori estimates, we have that $\int_0^T \int_{\mathbb{R}^d} \mu^{2\alpha+1} dxdt \leq M = M(d, \alpha, ||m_0||_{L^{\alpha+1}})$ i.e. $\mu \notin \partial U_{2M}$. So thanks to Scheafer's fixed point theorem system (S) has a classical solution (μ, w) and, in the end, $(\mu, u) = (\mu, -2log(w))$ is a classical solution of (MFG).

□

3.2.3 Uniqueness

The proof of the uniqueness is a consequence of the monotonicity of f in m.

Theorem 3.7. *There exist at most one classical solution to the MFG system (MFG).*

Proof. Suppose (u_1, m_1), (u_2, m_2) two classical solution of (MFG). Set $\bar{u} = u_1 - u_2$ and $\bar{m} = m_1 - m_2$. Then $\bar{u}$ solves:

$$-\partial_t \bar{u} - \Delta \bar{u} + \frac{1}{2}(|Du_1|^2 - |Du_2|^2) - [f(x, m_1(t,x)) - f(x, m_2(t,x))] = 0 \qquad (13)$$

while $\bar{m}$ solves:

$$\partial_t \bar{m} - \Delta \bar{m} - div(m_1 Du_1 - m_2 Du_2) = 0.$$

Now $\bar{u}$ is C^2 so, up to a truncation argument, we can use it as test function for the Fokker-Plank equation to obtain:

$$\int_0^T \int_{\mathbb{R}^d} (\partial \bar{u} + \Delta \bar{u}) \bar{m} dxdt - \int_0^T \int_{\mathbb{R}^d} \langle D\bar{u}, m_1 Du_1 - m_2 Du_2 \rangle = 0$$

since we have the same initial conditions.

We multiply (13) by $\bar{m}$, we integrate and we sum it to the previous equality to obtain:

$$\int_0^T \int_{\mathbb{R}^d} \frac{\bar{m}}{2}(|Du_1|^2 - |Du_1|^2) - \bar{m}[f(x, m_1(t,x)) - f(x, m_2(t,x))] - \langle D\bar{u}, m_1 Du_1 - m_2 Du_2 \rangle dxdt = 0$$

Now since f is non decreasing in the second variable

$$\int_{\mathbb{R}^d} [f(x, m_1(t,x)) - f(x, m_2(t,x))](m_1(t,x) - m_2(t,x)) dx \geq 0$$

so that :

$$\int_0^T \int_{\mathbb{R}^d} \frac{\bar{m}}{2}(|Du_1|^2 - |Du_1|^2) - \langle D\bar{u}, m_1 Du_1 - m_2 Du_2 \rangle dxdt \leq 0$$

Now,

$$\frac{\bar{m}}{2}(|Du_1|^2 - |Du_1|^2) - \langle D\bar{u}, m_1 Du_1 - m_2 Du_2 \rangle = -\frac{m_1 + m_2}{2}|Du_1 - Du_2|^2$$

so that :

$$\int_0^T \int_{\mathbb{R}^d} \frac{m_1 + m_2}{2} |Du_1 - Du_2|^2 dxdt \leq 0.$$

This implies that $Du_1 = Du_2$ m_1 and m_2 almost sure. This means that m_1 and m_2 solves the same Fokker-Plank equation and as a consequence u_1 and u_2 the same Hamilton-Jacobi equation. So, in the end $m_1 = m_2$ and $u_1 = u_2$ and the system (MFG) as unique solution.

□

4 Convergence of the N-particles problem

In this chapter we prove the convergence of the value function $\mathcal{V}^N$ associated to the N-particles problem towards the limit value function $\mathcal{U}$.
We recall that we denote by J^N:

$$J^N(\bar{\alpha}) = \mathbb{E}\Big[\frac{1}{N}\sum_{i=1}^{N}\int_{t_0}^{T}(\frac{|\alpha^{N,i}|^2}{2} + V^N * F(\cdot, V^N * m_t^N(\cdot))(X_t^{N,i})dt\Big]$$

the cost functional for the N-particles problem where $\bar{\alpha} = (\alpha^{N,1}, ..., \alpha^{N,N})$.

We start by considering a regularized version of the value function. In particular, for a given smooth symmetric kernel $\xi^\epsilon = \epsilon^{-d}\xi(\frac{x}{\epsilon})$, we consider the following regularized version of the initial local problem:

$$\mathcal{V}^{N,\epsilon}(t_0, \bar{Z}) = \inf_{\bar{\alpha}^N \in \mathcal{A}^N} \mathbb{E}\Big[\frac{1}{N}\sum_{i=1}^{N}\int_{t_0}^{T}(\frac{|\alpha^{N,i}|^2}{2} + V^N * \xi^\epsilon * F(\cdot, \xi^\epsilon * V^N * m_t^N(\cdot))(X_t^{N,i})dt\Big] \tag{Pϵ}$$

where the controls and the dynamics are the same as in (P).

Similarly, we consider the regularized version of (LP):

$$\mathcal{U}^\epsilon(t_0, m_0) = \inf_{(\alpha,m)\in\mathcal{A}}\Big[\int_{t_0}^{T}\int_{\mathbb{R}^d}\frac{|\alpha(t,x)|^2}{2}m(t,x) + F(x, \xi^\epsilon * m_t(x))\xi^\epsilon * m_t(x)dxdt\Big] \tag{LPϵ}$$

Note that for a smooth density $m(t,\cdot)$:

$$F(x, \xi^\epsilon * m_t(x))\xi^\epsilon * m_t(x) \to F(x, m(t,x))m(t,x)$$

as $\epsilon \to 0$.

As usual we denote by $\mathcal{F} : \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}$ the function:

$$\mathcal{F}(x,m) = F(x,m)m$$

and we introduce: $\mathcal{F}_\epsilon : \mathcal{P}_1(\mathbb{R}^d) \longrightarrow \mathbb{R}$ by:

$$\mathcal{F}_\epsilon(m) = \int_{\mathbb{R}^d}\mathcal{F}(x, \xi^\epsilon * m(x))dx = \int_{\mathbb{R}^d}F(x, \xi^\epsilon * m(x))\xi^\epsilon * m(x)dx$$

Thanks to the assumption on $\mathcal{F}$, $\mathcal{F}_\epsilon$ is convex on $\mathcal{P}_1(\mathbb{R}^d)$.

The proof of the convergence of $\mathcal{V}^N$ towards $\mathcal{U}$ is based on the following strategy:

1. prove the inequality:

$$\mathcal{V}^{N,\epsilon} \leq \mathcal{V}^N + c\epsilon \leq J^N(-D\bar{u}) + c\epsilon \tag{14}$$

   where $-D\bar{u}$ is the optimal control for the limit problem and ϵ does not depends on N

2. prove the convergence of $J^N(-D\bar{u}) \to \mathcal{U}$ as $N \to +\infty$.

3. for fixed ϵ use the convergence of $\mathcal{V}^{N,\epsilon}$ to $\mathcal{U}^\epsilon$ from [11]

4. prove the convergence of $\mathcal{U}^\epsilon$ to $\mathcal{U}$

Once the previous points have been shown it is easy to conclude that:

$$\limsup_{N\to+\infty} \mathcal{V}^N \leq \mathcal{U} \leq \liminf_{N\to+\infty} \mathcal{V}^N. \tag{15}$$

Indeed for every $\epsilon > 0$, by taking the lim sup in N on the right hand side of inequality (14), we obtain:

$$\limsup_N \mathcal{V}^N + c\epsilon \leq \mathcal{U} + c\epsilon$$

thanks to the convergence from point 2. Since this last inequality holds for every ϵ, letting ϵ goes to zero we obtain:

$$\limsup_{N\to+\infty} \mathcal{V}^N \leq \mathcal{U}.$$

Similarly, for fixed ϵ by taking the lim inf in N on the left hand side of (14) and by using the convergence from point 3, we obtain that for every $\epsilon > 0$:

$$\mathcal{U} \leq \liminf_N \mathcal{V}^N + c\epsilon.$$

Then, letting ϵ goes to zero, we obtain:

$$\mathcal{U} \leq \liminf_{N\to+\infty} \mathcal{V}^N$$

thanks to the convergence from point 4.

In the end, inequality (15) implies that:

$$\lim_{N\to+\infty} \mathcal{V}^N = \mathcal{U}$$

which is exactly the convergence result of Theorem 2.1.

Remark 5.
- The inequality $\mathcal{V}^N + c\epsilon \leq J^N(-D\bar{u}) + c\epsilon$ is trivial since $(\alpha_t^{N,i}) = -D\bar{u}(t,x)$ for $i = 1, ..., N$ are admissible controls for the N-particles problem.
- As we will prove, the inequality on the left hand side is given by the convexity assumption on $\mathcal{F}$ and by Jensen's inequality.

The rest of the chapter is devoted to the proofs of the points in the strategy.

4.1 Convergence of J^N towards $\mathcal{U}$

From the previous chapter we know that $-D\bar{u}$ is the optimal control for the limit McKean-Vlasov problem (LP).

In this chapter we will prove that the limit of the N-particles functional, when the particles are controlled by $\bar{\alpha}^N = (-D\bar{u}(t,x))_{i=1,\dots,N}$, is the limit value functional for (LP).

With this choice of controls, the N-particles functional reads:

$$J^N(-D\bar{u}) = \mathbb{E}\Bigg[\int_0^T \Bigg(\frac{1}{N}\sum_{i=1}^N \frac{|-D\bar{u}(t,X_t^{N,i})|^2}{2} + \int_{\mathbb{R}^d} F(x, V^N * m_t^N(x))V^N * m_t^N(x)dx\Bigg)dt\Bigg] \tag{16}$$

where each particle follows the SDE:

$$\begin{cases} dX_t^{N,i} = -D\bar{u}(t,X_t^{N,i})dt + \sqrt{2}dW_t^{N,i} \\ X_{t_0}^{N,i} = Z^i \end{cases} \tag{17}$$

Since $(Z^i)_i$ are i.i.d. random variables with law m_0 and the brownian motion $(W^{N,i})_i$ are independent, the particles $(X_t^{N,i})_{i=1,\dots,N}$ are i.i.d with $Law(X_t^{N,i}) = Law(X_t^{N,1}) = \bar{m}(t,x)dx$ where $\bar{m}$ is the solution of the Fokker-Plank equation associated to the (MFG) system.

So, the N-particles functional can be rewritten as:

$$J^N(-D\bar{u}) = \int_{t_0}^T \int_{\mathbb{R}^d} \{\frac{|-D\bar{u}(t,x)|^2}{2}\bar{m}(t,x)dxdt + \mathbb{E}\Bigg[\int_{\mathbb{R}^d} F(x, V^N * m_t^N(x))V^N * m_t^N(x)dx\Bigg]\}dt \tag{18}$$

where $m_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{N,i}}$ is the empirical measure associated to the particles system (17).

We want to prove the convergence of $J^N(-D\bar{u})$ to $\mathcal{U}(t_0,m_0)$ defined as:

$$\begin{aligned} \mathcal{U}(t_0,m_0) &= \inf_{(\alpha,m)\in\mathcal{A}} \Bigg[\int_{t_0}^T \int_{\mathbb{R}^d} \frac{|\alpha(t,x)|^2}{2} m(t,x) + F(x,m(t,x))m(t,x)dxdt\Bigg] \\ &= \int_{t_0}^T \int_{\mathbb{R}^d} \frac{|-D\bar{u}(t,x)|^2}{2}\bar{m}(t,x) + F(x,\bar{m}(t,x))\bar{m}(t,x)dxdt \end{aligned} \tag{19}$$

where again $(\bar{u},\bar{m})$ is the solution of the (MFG) system.

So, the only thing that we have to prove is the following convergence:

$$\mathbb{E}\Big[\int_{t_0}^T\int_{\mathbb{R}^d} F(x,V^N*m_t^N(x))V^N*m_t^N(x)dxdt\Big]\longrightarrow\int_{t_0}^T\int_{\mathbb{R}^d} F(x,\bar{m}(t,x))\bar{m}(t,x)dxdt \tag{20}$$

as $N\to+\infty$.

Remark 6. Since $(X_t^{N,i})_{i=1,\dots,N}$ are i.i.d with law $\bar{m}(t,\cdot)dx$, by Glivenko-Cantelli law of large numbers $m_t^N \rightharpoonup \bar{m}(t,x)dx$ a.s. and for every t, where by $\rightharpoonup$ we denote the weak convergence of measures. This implies that a.s. and for every t, $V^N*m_t^N\to\bar{m}(t,x)dx$. However, this is not enough to pass to the limit since F is not assumed to be linear in the second variable.

For simplicity we define $p^N(t,x)=V^N*m_t^N(x)$.

We denote by $||f||_\gamma$ the Hölder norm

$$||f||_\gamma=||f||_\infty+[f]_\gamma$$

where $||f||_\infty=\sup_{x\in\mathbb{R}^d}|f(x)|$ and $[f]_\gamma=\sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|^\gamma}$

We follow [9]. The idea is to prove that $p^N(t,\cdot)$ enjoys local regularity properties uniformly in N. In particular we will use the following theorem.

Theorem 4.1. *Let $\beta\in(0,\frac{1}{2})$, then there exist $\gamma\in(0,1)$, $q\geq2$ and a constant $C_\gamma>0$ independent on N such that if $\sup_N||p^N(0)||_\gamma<+\infty$, then:*

$$\mathbb{E}[||p^N(t)||_\gamma^q]\leq C_\gamma.$$

Theorem 4.2. *Let $(\mu_N)_{N\in\mathbb{N}}$ be a sequence of random measure on $\mathcal{P}(\mathbb{R}^d)$ converging in law in the weak topology of $\mathcal{P}(\mathbb{R}^d)$ to a deterministic measure $\mu\in\mathcal{P}(\mathbb{R}^d)$ with continuous density p. Consider $p^N(x)=V^N*\mu_N(x)$. Suppose there exist $\gamma\in(0,1)$, $q\geq2$, $C>0$ such that:*

$$\mathbb{E}[||p^N(t)||_\gamma^q]\leq C$$

then for every $R>0$:

$$\lim_{N\to\infty}\mathbb{E}[||p^N-p||^{q'}_{C(\bar{B}_R(0))}]=0$$

for every $q'<q$.

We leave the proofs of the previous theorems to the end of the chapter and we prove now the convergence (20).

Theorem 4.3. *Assume that **HYPOTHESES (H)** hold and let $\beta\in(0,\frac{1}{2})$, γ and q as in Theorem 4.1. Then:*

$$\lim_{N\to+\infty}\mathbb{E}\Big[\int_{t_0}^T\int_{\mathbb{R}^d} F(x,V^N*m_t^N(x))V^N*m_t^N(x)dxdt\Big]=\int_{t_0}^T\int_{\mathbb{R}^d} F(x,\bar{m}(t,x))\bar{m}(t,x)dxdt$$

Proof. We study:

$$\begin{aligned}
&\left|\mathbb{E}\left[\int_{t_0}^T\int_{\mathbb{R}^d} F(x,V^N*m_t^N(x))V^N*m_t^N(x)dxdt-\int_{t_0}^T\int_{\mathbb{R}^d}F(x,\bar m(t,x))\bar m(t,x)dxdt\right]\right|\\
\leq&\mathbb{E}\left[\left|\int_{t_0}^T\int_{\mathbb{R}^d}F(x,p^N(t,x))p^N(t,x)dxdt-\int_{t_0}^T\int_{\mathbb{R}^d}F(x,\bar m(t,x))p^N(t,x)dxdt\right|\right]+\\
+&\mathbb{E}\left[\left|\int_{t_0}^T\int_{\mathbb{R}^d}F(x,\bar m(t,x))p^N(t,x)dxdt-\int_{t_0}^T\int_{\mathbb{R}^d}F(x,\bar m(t,x))\bar m(t,x)dxdt\right|\right]\\
&=(i)+(ii)
\end{aligned}$$

We start from (i). For a fixed $R>0$, we split the first integral as follows:

$$\begin{aligned}
&(i)=\mathbb{E}\left[\left|\int_{t_0}^T\int_{B_R}[F(x,p^N(t,x))-F(x,\bar m(t,x))]p^N(t,x)dxdt\right|\right]\\
&+\mathbb{E}\left[\left|\int_{t_0}^T\int_{\mathbb{R}^d\setminus B_R}[F(x,p^N(t,x))-F(x,\bar m(t,x))]p^N(t,x)dxdt\right|\right]\\
&\leq L\int_{t_0}^T\int_{B_R}\mathbb{E}[|p^N(t,x)-\bar m(t,x)|p^N(t,x)]dxdt+2||F||_{L^\infty}\mathbb{E}\left[\sup_{t\in[0,T]}\int_{\mathbb{R}^d\setminus B_R}p^N(t,x)dx\right]\\
&\leq L\int_{t_0}^T\int_{B_R}\mathbb{E}[||p^N(t,\cdot)-\bar m(t,\cdot)||^p_{C(\bar{\mathcal{B}}_R(0))}]^{\frac1p}\mathbb{E}[||p^N(t)||_\gamma^q]^{\frac1q}dxdt\\
&\quad+2||F||_{L^\infty}\mathbb{E}\left[\sup_{t\in[0,T]}\int_{\mathbb{R}^d\setminus B_R}p^N(t,x)dx\right]\\
&\leq CR^d\int_{t_0}^T\mathbb{E}[||p^N(t,\cdot)-\bar m(t,\cdot)||^p_{C(\bar{\mathcal{B}}_R(0))}]^{\frac1p}dt+2||F||_{L^\infty}\mathbb{E}\left[\sup_{t\in[0,T]}\int_{\mathbb{R}^d\setminus B_R}p^N(t,x)dx\right]
\end{aligned}$$

where q comes from Theorem 4.2 and $\frac1p+\frac1q=1$.

Thanks to Theorem 4.2, for every $\epsilon>0$ and for every R we can find N such that :

$$CR^d\int_{t_0}^T\mathbb{E}[||p^N(t,\cdot)-m(t,\cdot)||^p_{C(\bar{\mathcal{B}}_R(0))}]^{\frac1q}dt\leq\frac{\epsilon}{4}$$

For the integral on $\mathbb{R}^d\setminus B_R$, we prove that the second order moment of $p^N(t,x)dx$ is bounded uniformly in N. Indeed, since the particles $(\bar X_t^{N,i})_{i=1,..,N}$ are i.i.d, $\int_{\mathbb{R}^d}|x|^2m_0(x)<+\infty$ we have:

$$\mathbb{E}\left[\sup_{t\in[0,T]}\int_{\mathbb{R}^d}|x|^2m_t^N(dx)\right]\leq C$$

with C independent on N.

Then choosing $\tilde R>0$ such that $\overline{supp}V\subset\bar B_{\tilde R}(0)$ we have:

$$\int_{\mathbb{R}^d}|x|^2V^N(x-y)dx=\int_{\mathbb{R}^d}|x+y|^2V^N(z)dz\leq\sup_{|w|\leq\tilde R}|w+y|^2\leq 2(|y|^2+\tilde R^2)$$

This implies that:

$$\begin{aligned}\int_{\mathbb{R}^d} |x|^2 V^N * m_t^N(x)dx &\le \int_{\mathbb{R}^d} |x|^2 \Big(\int_{\mathbb{R}^d} V^N(x-y)m_t^N(y)dy\Big)dx \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |x|^2 V^N(x-y)dx m_t^N(dy) \\ &\le 2\tilde{R}^2 + 2\int_{\mathbb{R}^d} |y|^2 m_t^N(dy)\end{aligned}$$

$$\implies \mathbb{E}\Big[\sup_{t\in[0,T]} \int_{\mathbb{R}^d} |x|^2 V^N * m_t^N(x)dx\Big] \le 2\tilde{R}^2 + 2\mathbb{E}\Big[\sup_{t\in[0,T]} \int_{\mathbb{R}^d} |y|^2 m_t^N(y)dy\Big] \le 2\tilde{R}^2 + C \le K$$

for K independent on N. Thanks to this, we have:

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \int_{\mathbb{R}^d\setminus B_R} p^N(t,x)dx\Big] \le \frac{K}{R^2}.$$

So, choosing R big enough, for every $\epsilon > 0$ we can find $\bar{N}$ such that for every $N \ge \bar{N}$:

$$(i) \le \frac{\epsilon}{2}.$$

The convergence of (ii) is similar. Indeed:

$$(ii) \le ||F||_{L^\infty}\mathbb{E}\Big[\int_{t_0}^T \int_{\mathbb{R}^d} |p^N(t,x) - \bar{m}(t,x)|dxdt\Big]$$

Using the local uniform convergence from Theorem 4.2 and the bound on the second order moment of $p^N(t,x)dx$, and $\bar{m}(t,x)dx$, we can prove using the same argument as before that, for every $\epsilon > 0$ there exist R and $\tilde{N}$ such that for every $N \ge \tilde{N}$:

$$(ii) \le \frac{\epsilon}{2}.$$

In the end, putting the estimates together we obtain that for every $\epsilon > 0$ there exists $M = max\{\bar{N}, \tilde{N}\}$ such that for $N \ge M$:

$$\begin{aligned}&\Big|\mathbb{E}\Big[\int_{t_0}^T \int_{\mathbb{R}^d} F(x, V^N * m_t^N(x))V^N * m_t^N(x)dxdt - \int_{t_0}^T \int_{\mathbb{R}^d} F(x,\bar{m}(t,x))\bar{m}(t,x)dxdt\Big]\Big| \\ &\le (i) + (ii) \\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2} \le \epsilon\end{aligned}$$

which conclude the the proof of the convergence.

□

The rest of this section is dedicated to the proofs of Theorems 4.1 and 4.2.

4.1.1 Proofs

In this section we prove Theorem 4.1. We follow [9]. Let m_t^N the empirical measure associated to the particles system (17) and $\phi \in C_b^{1,2}$. By Ito's formula we have:

$$
\begin{aligned}
d\langle m_t^N, \phi(t,\cdot)\rangle &= \frac{1}{N}\sum_{i=1}^N d\phi(t, X_t^{N,i}) = \frac{1}{N}\sum_{i=1}^N (\partial_t \phi(t, X_t^{N,i}) + D\phi(t, X_t^{N,i}) \cdot D\bar{u}(t, X_t^{N,i}) + \Delta\phi(t, X_t^{N,i}))dt \\
&+ \frac{1}{N}\sum_{i=1}^N D\phi(t, X_t^{N,i}) \cdot dW_t^{N,i} \\
&= \langle \partial_t\phi(t) + D\phi(t)\cdot D\bar{u}(t) + \Delta\phi(t), m_t^N\rangle dt \\
&+ \frac{1}{N}\sum_{i=1}^N D\phi(t, X_t^{N,i}) \cdot dW_t^{N,i}.
\end{aligned}
$$

In the integral form the previous equality can be rewritten as:

$$
\begin{aligned}
\langle m_t^N, \phi(t,\cdot)\rangle = \langle m_0^N, \phi(0,\cdot)\rangle &+ \int_0^t \langle m_t^N, \partial_s\phi(s,\cdot) + D\phi(t,\cdot)\cdot D\bar{u}(t,\cdot) + \Delta\phi(s,\cdot)\rangle dt \\
&+ M_t^{\phi,N}
\end{aligned}
$$

where for every $\phi \in C_b^{1,2}$, $M_t^{\phi,N} = \int_0^T \frac{1}{N}\sum_{i=1}^N D\phi(t, X_t^{N,i}) \cdot dW_t^{N,i}$ is a martingale.

Now we prove that the empirical density $p^N(t,x) = V^N * m_t^N(x)$ solves an identity in mild form. We recall that we indicate by $G(t,x) = \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{2t}}$ the heat kernel. We consider also the semigroup $\mathcal{P}_t$ associated with the heat kernel, as the operator such that for every $h \in C_b^2(\mathbb{R}^d)$:

$$
(\mathcal{P}_t h)(x) = G_t * h(x) = \int_{\mathbb{R}^d} G(t, x-y)h(y)dy.
$$

Now, for a given $t \in [0,T]$ we rewrite the identity satisfies by the empirical measure, using as test function $\phi^{(t)}(s,x) = (\mathcal{P}_{t-s}(V^{N,-} * h))(x)$ for $s \in [0,t]$, $h \in C_b^2(\mathbb{R}^d)$ and $V^{N,-}(z) = V^N(-z)$ to obtain:

Lemma 4.4. *Let $p^N(t,x) = V^N * m_t^N(x)$. Then for every $t \in [0,T]$, $p^N(t)$ solves the following equation in mild form:*

$$
\begin{aligned}
p^N(t,x) = &\mathcal{P}_t p^N(0) + \int_0^t D\mathcal{P}_{t-s}(V^N * (D\bar{u}(s,\cdot)m_s^N))ds \\
&+ M_t^N(\cdot)
\end{aligned}
$$

where $M_t^N(\cdot) = \int_0^t \frac{1}{N}\sum_{i=1}^N \mathcal{P}_{t-s} DV^N(\cdot - X_s^{N,i}) \cdot dW_s^{N,i}$.

We study the regularity property of the martingale $M_t^N(\cdot)$. We recall the definition of Holder norm:

$$
||f||_\gamma = ||f||_\infty + [f]_\gamma
$$

where $||f||_\infty = \sup_{x\in\mathbb{R}^d} |f(x)|$ and $[f]_\gamma = \sup_{x\neq y \in \mathbb{R}^d} \frac{|f(x)-f(y)|}{|x-y|^\gamma}$. We now, prove the main theorem of this chapter:

Lemma 4.5. *Let $\beta \in (0, \frac{1}{2})$ in the definition of V^N. Then there exists $\gamma \in (0,1)$ such that for every $p \geq 2$, there is a constant $C_p > 0$ such that:*

$$\mathbb{E}[||M_t^N||_\gamma^p] \leq C_p$$

for every $N \in \mathbb{N}$ and $t \in [0,T]$.

Proof. It's enough to check condition a) and b) of lemma (B.2) in the appendix. We start from a). Let $\epsilon_N^{-1} = N^{\frac{\beta}{d}}$. Using the definition of $M_t^N(\cdot)$ we have:

$$\begin{aligned}
\mathbb{E}[|M_t^N(x)|^p] &= \frac{1}{N^p}\mathbb{E}\left[\left|\sum_{i=1}^N \int_0^t D\mathcal{P}_{t-s}V^N(x - X_s^{N,i})dW_s^{N,i}\right|^p\right] \\
&\leq \frac{C_p}{N^p}\mathbb{E}\left[\left|\sum_{i=1}^N \int_0^t |D\mathcal{P}_{t-s}V^N(x - X_s^{N,i})|^2 ds\right|^{\frac{p}{2}}\right] \\
&\leq \frac{C_p C_{T,R,V}^p \epsilon_N^{-pd-p\delta}}{N^p}\mathbb{E}\left[\left|\sum_{i=1}^N \int_0^t \frac{1}{(t-s)^{1-\delta}} e^{-\frac{|x-X_s^{N,i}|}{4T}} ds\right|^{\frac{p}{2}}\right] \\
&\leq \frac{\tilde{C}_{T,R,V}^p \epsilon_N^{-pd-p\delta}}{N^p} e^{-p\frac{|x|}{8T}}\mathbb{E}\left[e^{p\frac{||X^{N,i}||_{\infty,T}}{8T}}\left|\sum_{i=1}^N \int_0^t \frac{1}{(t-s)^{1-\delta}} ds\right|^{\frac{p}{2}}\right] \\
&\leq \frac{C'_{p,T,R,V,\delta}\epsilon_N^{-pd-p\delta}}{N^{\frac{p}{2}}} e^{-p\frac{|x|}{8T}}\mathbb{E}\left[e^{p\frac{||X^{N,i}||_{\infty,T}}{8T}}\right].
\end{aligned}$$

where we used lemma (B.3) in the second line. Now remember that by hypothesis $\int_{\mathbb{R}^d} e^{\lambda|x|}m_0 dx < \infty$ for every $\lambda > 0$, so that we have

$$\mathbb{E}\left[e^{p\frac{||X^{N,i}||_{\infty,T}}{8T}}\right] \leq C$$

where C does not depend on i and on N.Therefore:

$$\mathbb{E}[|M_t^N(x)|^p] \leq C''_{p,T,R,V,\delta}\frac{\epsilon_N^{-pd-p\delta}}{N^{\frac{p}{2}}} g^p(x)$$

where $g(x) = e^{-\frac{|x|}{8T}}$ is integrable at any power. Now since $\epsilon_N^{-1} = N^{\frac{\beta}{d}}$:

$$\frac{\epsilon_N^{-pd-p\delta}}{N^{\frac{p}{2}}} = \frac{N^{\frac{\beta}{2}(pd+p\delta)}}{N^{\frac{p}{2}}} = N^{(\frac{1}{2}-\beta)p - \frac{\beta p \delta}{d}}$$

which remains bounded for $\beta \in (0, \frac{1}{2})$ by choosing δ (depending on p) small enough. The proof of b) is similar to the previous one. We define:

$$\begin{aligned}
&\Delta_h M_t^N(x) = M_t^N(x) - M_t^N(x+h) \\
&\Delta_h \mathcal{P}_{t-s}V^N(x - X_s^{N,i}) = D\mathcal{P}_{t-s}V^N(x - X_s^{N,i}) - D\mathcal{P}_{t-s}V^N(x + h - X_s^{N,i})
\end{aligned}$$

Then we have:

$$\mathbb{E}[|\Delta_h M_t^N(x)|^p] = \frac{1}{N^p}\mathbb{E}\left[\left|\sum_{i=1}^N \int_0^t \Delta_h \mathcal{P}_{t-s}V^N(x-X_s^{N,i})dW_s^{N,i}\right|^p\right]$$
$$\leq \frac{C_p}{N^p}\mathbb{E}\left[\left|\sum_{i=1}^N \int_0^t |\Delta_h \mathcal{P}_{t-s}V^N(x-X_s^{N,i})|^2 ds\right|^{\frac{p}{2}}\right]$$
$$\leq \frac{C_p}{N^p}\mathbb{E}\left[\left|\sum_{i=1}^N \int_0^t \frac{C_{T,R,V}^2}{(t-s)^{1+\bar{\gamma}}}|h|^{2\gamma}\epsilon_N^{-2d-2\delta(1-\gamma)}e^{-2\lambda_{T,R,V}|x-X_s^{N,i}|}ds\right|^{\frac{p}{2}}\right]$$
$$\leq \frac{\tilde{C}_{p,T,R,V}\epsilon_N^{-pd-p\delta(1-\gamma)}}{N^{\frac{p}{2}}}|h|^{p\gamma}e^{-2\lambda_{T,R,V}|x|}\mathbb{E}\left[e^{p\lambda_{T,R,V}||X^{N,i}||_{\infty,T}}\right]$$

and we can conclude as for the previous term.

□

We are now ready to prove Theorem 4.1. Remember, we have to prove that if $\beta \in (0, \frac{1}{2})$ and $\sup_N ||p^N(0)||_\gamma^2 < +\infty$ then there exist $p \geq 2$, $\gamma \in (0,1)$ and a constant $C_\gamma > 0$ such that:

$$\mathbb{E}[||p^N(t)||_\gamma^p] \leq C_\gamma$$

Proof. [Theorem 4.1] By using the representation in mild form for $p^N(t)$ we have:

$$\mathbb{E}[||p^N(t)||_\gamma^p]^{\frac{1}{p}} \leq \mathbb{E}[||\mathcal{P}_t p^N(0)||_\gamma^p]^{\frac{1}{p}} + \int_0^t \mathbb{E}[||D\mathcal{P}_{t-s}(V^N * (D\bar{u}(s,\cdot)m_s^N))||_\gamma^p]^{\frac{1}{p}}ds$$
$$+ \mathbb{E}[||M_t^N||_\gamma^p]^{\frac{1}{p}}$$
$$\leq C + \int_0^t \frac{C}{(t-s)^{\frac{1+\gamma}{2}}}\mathbb{E}[||V^N * (D\bar{u}(s,\cdot)m_s^N||_\infty^p]^{\frac{1}{p}}ds + C$$

Now :

$$|V^N * (D\bar{u}(s,\cdot)m_s^N(x)| \leq \int_{\mathbb{R}^d} V^N(x-y)|D\bar{u}(s,y)|m_s^N(dy)$$
$$\leq ||D\bar{u}||_\infty \int_{\mathbb{R}^d} V^N(x-y)m_s^N(dy) = ||D\bar{u}(s,\cdot)||_\infty p^N(s,x)$$

so that

$$\mathbb{E}[||V^N * (D\bar{u}(s,\cdot)m_s^N||_\infty^p]^{\frac{1}{p}} \leq ||D\bar{u}||_\infty \mathbb{E}[||p^N(s)||_\gamma^p]^{\frac{1}{p}}.$$

Plugging this in the previous inequality we obtain:

$$\mathbb{E}[||p^N(t)||_\gamma^p]^{\frac{1}{p}} \leq C + C\int_0^t \frac{||D\bar{u}(s,\cdot)||_\infty}{(t-s)^{\frac{1+\gamma}{2}}}\mathbb{E}[||p^N(s)||_\gamma^p]^{\frac{1}{p}}ds.$$

Thanks to Gronwall's lemma we have:

$$\mathbb{E}[||p^N(t)||_\gamma^p]^{\frac{1}{p}} \leq C\exp\left(||D\bar{u}||_\infty \int_0^t \frac{1}{(t-s)^{\frac{1+\gamma}{2}}}ds\right)$$
$$\leq Cexp\left(-||D\bar{u}||_\infty t^{\frac{1-\gamma}{2}}\right)$$
$$\leq C_\gamma$$

since from lemma 4.5, $\gamma \in (0, 1)$.

□

Theorem 1.2 is a consequence of the previous result. For the proof we refer to [9] Lemma D.2 appendix D.

4.2 Proof of the inequality $\mathcal{V}^{N,\epsilon} \leq \mathcal{V}^N + o(\epsilon)$

We rewrite (LPϵ) as :

$$\mathcal{V}^{N,\epsilon}(t_0, \bar{Z}) = \inf_{\bar{\alpha}^N \in \mathcal{A}^N} \mathbb{E}\Big[\int_{t_0}^T \Big(\frac{1}{N}\sum_{i=1}^N \frac{|\alpha^{N,i}|^2}{2} + \int_{\mathbb{R}^d} F(x, \xi^\epsilon * V^N * m_t^N(x))\xi^\epsilon * V^N * m_t^N(x)dx\Big)dt\Big]$$

For every $\bar{\alpha}^N \in \mathcal{A}^N$ we compare $\mathcal{V}^{N,\epsilon}$ with:

$$\mathcal{V}^N(t_0, \bar{Z}) = \inf_{\bar{\alpha}^N} \mathbb{E}\Big[\int_{t_0}^T \Big(\frac{1}{N}\sum_{i=1}^N \frac{|\alpha^{N,i}|^2}{2} + \int_{\mathbb{R}^d} F(x, V^N * m_t^N(x))V^N * m_t^N(x)dx\Big)dt\Big]$$

In particular we want to prove that for every $\bar{\alpha}^N \in \mathcal{A}^N$ and a.s.:

$$\begin{aligned}
&\int_{\mathbb{R}^d} F(x, \xi^\epsilon * V^N * m_t^N(x))\xi^\epsilon * V^N * m_t^N(x)dx \\
&\leq \int_{\mathbb{R}^d} F(x, V^N * m_t^N(x))V^N * m_t^N(x)dx + c|\epsilon|
\end{aligned}$$

for some positive constant c independent of ϵ.

Using the convexity of $m \to F(x,m)m$ and thanks to Jensen's inequality, we can write:

$$\begin{aligned}
&F(x, \xi^\epsilon * V^N * m_t^N(x))\xi^\epsilon * V^N * m_t^N(x) \\
&= F(x, \int_{\mathbb{R}^d} V^N * m_t^N(x-y)\xi^\epsilon(y)dy)(\int_{\mathbb{R}^d} V^N * m_t^N(x-y)\xi^\epsilon(y)dy) \\
&\leq \int_{\mathbb{R}^d} \xi^\epsilon(y)F(x, V^N * m_t^N(x-y))V^N * m_t^N(x-y)dy
\end{aligned}$$

This implies that:

$$\begin{aligned}
&\int_{\mathbb{R}^d} F(x, \xi^\epsilon * V^N * m_t^N(x))\xi^\epsilon * V^N * m_t^N(x)dx \\
&\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \xi^\epsilon(y)F(x, V^N * m_t^N(x-y))V^N * m_t^N(x-y)dydx \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \xi^\epsilon(y)F(x+y, V^N * m_t^N(x))V^N * m_t^N(x)dydx
\end{aligned}$$

Now, using the lipschitzianity of F we can compare the last integral and the non-regularized term:

$$
\begin{aligned}
\Big| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} & \xi^\epsilon(y) F(x+y, V^N * m_t^N(x)) V^N * m_t^N(x) dx dy \\
& - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \xi^\epsilon(y) F(x, V^N * m_t^N(x)) V^N * m_t^N(x) dy dx \Big| \\
& \leq L \int_{\mathbb{R}^d} V^N * m_t^N(x) dx \int_{\mathbb{R}^d} \xi^\epsilon(y) |y| dy \\
& \leq L \int_{\mathbb{R}^d} \xi^\epsilon(y) |y| dy
\end{aligned}
$$

since $\int_{\mathbb{R}^d} V^N * m_t^N(x) dx = 1$. In the end, we obtain the desired inequality:

$$
\begin{aligned}
& \int_{\mathbb{R}^d} F(x, \xi^\epsilon * V^N * m_t^N(x)) \xi^\epsilon * V^N * m_t^N(x) dx \\
& \leq \int_{\mathbb{R}^d} F(x, V^N * m_t^N(x)) V^N * m_t^N(x) dx + L\epsilon
\end{aligned}
$$

and thus:

$$
\mathcal{V}^{N,\epsilon}(t_0, \bar{Z}) \leq \mathcal{V}^N(t_0, \bar{Z}) + L\epsilon.
$$

4.3 Convergence of $\mathcal{U}^\epsilon$ to $\mathcal{U}$

The proof of this convergence is based on stability properties of the MFG system. We start by looking for uniform in ϵ regularity properties of the minimizers for (LPϵ). Hypothesis B) implies that $\mathcal{F}_\epsilon$ is convex in $\mathcal{P}_1(\mathbb{R}^d)$. As a consequence, a minimizer for (LPϵ) is given by $(-Du_\epsilon, m_\epsilon)$ where (u_ϵ, m_ϵ) solves the MFG system:

$$\begin{cases} -\partial_t u - \Delta u + \dfrac{|Du|^2}{2} = \dfrac{\delta \mathcal{F}_\epsilon}{\delta m}(x, m(t)) \quad (0,T)\times\mathbb{R}^d \\ \partial_t m - \Delta m - div(mDu) = 0 \qquad (0,T)\times\mathbb{R}^d \\ m(0) = m_0, \quad u(x,T) = 0 \end{cases} \tag{$MFG1$}$$

where $\frac{\delta \mathcal{F}_\epsilon}{\delta m}(m,x) = \xi^\epsilon * F(\cdot, \xi^\epsilon * m(\cdot))(x) + \xi^\epsilon * (\partial_m F(\cdot, \xi^\epsilon * m(\cdot))\xi^\epsilon * m(\cdot))(x)$

Under the above hypothesis, for every ϵ, system $(MFG1)$ has a unique solution (u_ϵ, m_ϵ) in $C^{2+\theta, 1+\frac{\theta}{2}}(\mathbb{R}^d \times [0,T])$.

In order to prove the convergence of $\mathcal{U}^\epsilon$ to $\mathcal{U}$, we need local uniform convergence of m_ϵ and Du_ϵ to m and Du respectively, where (u,m) is the solution of the MFG system:

$$\begin{cases} -\partial_t u - \Delta u + \dfrac{|Du|^2}{2} = f(x, m(t,x)) \quad (0,T)\times\mathbb{R}^d \\ \partial_t m - \Delta m - div(mDu) = 0 \qquad (0,T)\times\mathbb{R}^d \\ m(0) = m_0, \quad u(x,T) = 0 \end{cases} \tag{$MFG2$}$$

for $f(x,m) = \partial_m \mathcal{F}(x,m) = F(x,m) + \partial_m F(x,m)m$.

Note that $\frac{\delta \mathcal{F}_\epsilon}{\delta m}(m,x) = \xi^\epsilon * f(\cdot, \xi^\epsilon * m(\cdot))(x)$.

By repeating exactly the same computations as in proof of the a priori estimate in Chapter 1, we prove that:

Proposition 4.6. *Under the Hypotheses (H), there exists a positive constant* $M = M(d, \alpha, ||m_0||_{L^{\alpha+1}(\mathbb{R}^d)})$ *independent on* ϵ*, such that if* m_ϵ *is the solution of the Fokker-Plank equation in* $(MFG1)$*, then:*

$$\int_0^T \int_{\mathbb{R}^d} (\xi^\epsilon * m_\epsilon(t,x))^{\alpha+1} dxdt \leq M \tag{21}$$

Proof. The proof is the same as in chapter 1). In particular, thanks to the structure of $\frac{\delta \mathcal{F}_\epsilon}{\delta m}(m,x)$ we can use again the positive sign of $\partial_m f(x,m)$ in the second order estimate. Indeed :

$$\int_0^T \int_{\mathbb{R}^d} div(\xi^\epsilon * (\partial_m f(\cdot, \xi^\epsilon * m_\epsilon(t,\cdot))\xi^\epsilon * Dm_\epsilon(t,\cdot))(x)) m_\epsilon(t,x) dxdt$$
$$= -\int_0^T \int_{\mathbb{R}^d} \partial_m f(x, \xi^\epsilon * m_\epsilon(t,x)) |\xi^\epsilon * Dm_\epsilon(t,x)|^2 dxdt \leq 0$$

and then the conclusion follows as in chapter 1. □

A first consequence of the previous bound, is the local uniform convergence of $(m_\epsilon)_\epsilon$. To see this, we study the stability properties of the system $(MFG1)$.

First, using the Hopf-Cole transform $w_\epsilon(t,x) = e^{-u_\epsilon(t,x)}$ we rewrite the system $(MFG1)$ as :

$$\begin{cases} -\partial_t w_\epsilon - \Delta w_\epsilon + f_\epsilon(x, m_\epsilon(t))w_\epsilon = 0 & (0,T)\times\mathbb{R}^d \\ \partial_t m_\epsilon - \Delta m_\epsilon - div(\frac{Dw_\epsilon}{w_\epsilon}m_\epsilon) = 0 & (0,T)\times\mathbb{R}^d \\ m(0) = m_0, \quad w_\epsilon(x,T) = 1 \end{cases} \qquad (MFG3)$$

where $f_\epsilon(x,m) = \frac{\delta\mathcal{F}_\epsilon}{\delta m}(x,m)$.

The regularity of m_ϵ is related to the integrability of $\frac{Dw_\epsilon}{w_\epsilon}$ and thus, since by the maximum principle $w_\epsilon \geq \delta > 0$, to the integrability of Dw_ϵ. So, we start by studying the Hamilton-Jacobi equation.

Thanks to Proposition 4.6 and to the hypotesis C, we have that $f_\epsilon(x,m(t)) \in L^p((0,T)\times\mathbb{R}^d)$ for $p = \frac{\alpha+1}{\alpha} > \frac{d+2}{2}$.

We need the following lemma from [5]:

Lemma 4.7. *Suppose z is a bounded weak solution of:*

$$\partial_t z - \Delta z = fz + g + div(hz)$$

where $f, g \in L^p((0,T)\times\mathbb{R}^d)$, $h \in L^{2p}((0,T)\times\mathbb{R}^d)$ for some $p > \frac{d+2}{2}$ then:

$$||z||_{L^\infty((0,T)\times\mathbb{R}^d))} \leq C(||f||_{L^p((0,T)\times\mathbb{R}^d)} + ||z(0)||_{L^\infty(\mathbb{R}^d)})$$

where the constant C remains bounded for bounded values of $||g||_{L^p}, ||h||_{L^{2p}}$.
Moreover, suppose $f, h \equiv 0$ then:

$$||Dz||_{L^{2p}((0,T)\times\mathbb{R}^d))} \leq C(||f||_{L^p((0,T)\times\mathbb{R}^d)} + ||Dz(0)||_{L^{2p}(\mathbb{R}^d)})$$

where C depends only on d, T, p.

We use the first inequality of the lemma with $g, h \equiv 0$ to obtain a bound on $||w_\epsilon||_{L^\infty((0,T)\times\mathbb{R}^d))}$ uniform in ϵ. As a consequence $||f_\epsilon w_\epsilon||_{L^p((0,T)\times\mathbb{R}^d}$ remains bounded uniformly in ϵ and using the second inequality with $f, h \equiv 0$ and $g = f_\epsilon w_\epsilon$ we obtain an uniform bound in $L^{2p}((0,T)\times\mathbb{R}^d)$ for Dw_ϵ and thus for $\frac{Dw_\epsilon}{w_\epsilon}$.

We can now study the uniform regularity of m_ϵ. Again, by lemma (4.7) with $f, g \equiv 0$ and $h = \frac{Dw_\epsilon}{w_\epsilon}$ we obtain a bound on $||m_\epsilon||_{L^\infty((0,T)\times\mathbb{R}^d))}$ uniform in ϵ.
Then thanks to Theorem 10.1 section 3 of [12] there exist $K, \theta \in (0,1)$ which do not depend on ϵ such that $||m_\epsilon||_{C^{\theta,\frac{\theta}{2}}_{loc}} \leq K$.

In particular by Ascoli-Arzelà, $(m_\epsilon)_\epsilon$ converges, up to sub sequences, locally uniformly on $[0,T]\times\mathbb{R}^d$ to some $\mu\in C([0,T]\times\mathbb{R}^d)$

Then $(x,t)\longrightarrow f_\epsilon(x,m_\epsilon(t))$ locally uniform converges to $(x,t)\longrightarrow f(x,\mu(t))$ and by stability argument for the Hamilton-Jacobi equation, one gets the local uniform convergence of u_ϵ to v solution of the (H-J) with coupling $f(x,\mu(t))$.

Now we study the uniform convergence of Du_ϵ. First, note that since $\frac{Dw_\epsilon}{w_\epsilon}=Du_\epsilon$ we have that $||Du_\epsilon||_{L^{2p}((0,T)\times\mathbb{R}^d))}$ is bounded uniformly in ϵ. This implies that u_ϵ solves the non-homogeneus heat equation:

$$\partial_t u_\epsilon-\Delta u_\epsilon=g_\epsilon,\quad u_\epsilon(T,x)=0$$

where $g_\epsilon(x,t)=f_\epsilon(x,m_\epsilon(t))-\frac{|Du_\epsilon|^2}{2}$ is bounded uniformly in $L^p((0,T)\times\mathbb{R}^d))$. By using Theorem 11.1 p. 211 of [12], we find a constant C which does not depend on ϵ such that :

$$||Du_\epsilon||_{C^{\theta,\frac{\theta}{2}}_{loc}}\le C(||u_\epsilon||_\infty+||g_\epsilon||_{L^p})$$

So, again by Ascoli-Arzelà, up to sub-sequence Du_ϵ locally uniform converges to some w. But since u_ϵ locally uniform converges to v then $w=Dv$.

Now passing to the limit on a convergent sub-sequence we obtain that (v,μ) solves:

$$\begin{cases}-\partial_t v-\Delta v+\dfrac{|Dv|^2}{2}=f(x,\mu(t,x)))\quad (0,T)\times\mathbb{R}^d\\ \partial_t\mu-\Delta\mu-div(\mu Du)=0\qquad (0,T)\times\mathbb{R}^d\\ \mu(0)=m_0,\quad v(x,T)=0\end{cases}\qquad (MFG)$$

hence, by uniqueness, $(v,\mu)=(u,m)$ and all the sequences $u_\epsilon,m_\epsilon,Du_\epsilon$ locally uniform converge to u,m,Du respectively.

The previous computations give also a bound on $\int_0^T\int_{\mathbb{R}^d}\frac{|Du_\epsilon|^2}{2}m_\epsilon(t,x)dxdt$. Indeed $||Du_\epsilon||_{L^{2p}((0,T)\times\mathbb{R}^d))}$ is uniformly bounded, $||m_\epsilon||_{L^\infty((0,T),L^q(\mathbb{R}^d))}$ is uniformly bounded for every $q\ge 1$ and so by interpolation we can find a positive which we still denoted by M, independent on ϵ such that:

$$\int_0^T\int_{\mathbb{R}^d}\frac{|Du_\epsilon|^2}{2}m_\epsilon(t,x)dxdt\le M\qquad (22)$$

As a consequence, we have an uniform in ϵ bound on the second order moment of $(m_\epsilon(t))_\epsilon$:

Corollary 4.8. *There exist a positive constant C independent on ϵ such that:*

$$\sup_{t\in[0,T]}\int_{\mathbb{R}^d}|x|^2 m_\epsilon(t,x)dxdt\le C$$

Proof. for every ϵ, $m_\epsilon(t) = Law(X_t^\epsilon)$ where X_t^ϵ solves:

$$\begin{cases} dX_t^\epsilon = -Du_\epsilon(t, X_t^\epsilon)dt + \sqrt{2}dB_t \\ X_{t_0}^\epsilon = Z \end{cases} \tag{23}$$

where $Law(Z) = m_0$. So for every $t \in [t_0, T]$ we have:

$$\int_{\mathbb{R}^d} |x|^2 m_\epsilon(t,x)dx = \mathbb{E}[|X_t^\epsilon|^2] \leq 2(\mathbb{E}\left[|Z|^2 + \int_{t_0}^T |Du_\epsilon(t, X_t^\epsilon)|^2 dt + 4T\right]$$
$$\leq 2\left(\int_{\mathbb{R}^d} |x|^2 m_0(x)dx + \int_{t_0}^T \int_{\mathbb{R}^d} |Du_\epsilon(t,x)|^2 m_\epsilon(t,x)dxdt + 4T\right) \leq C$$

thanks to the hypothesis on m_0 and to proposition 4.6. □

This result will be useful to prove the convergence of the value function $\mathcal{U}^\epsilon$ out of a compact set. Indeed, indicating with $\mathcal{B}_R$ the ball of center 0 and radius R:

$$\int_{t_0}^T \int_{\mathbb{R}^d \setminus \mathcal{B}_R} m_\epsilon(t,x)dxdt \leq \int_{t_0}^T \int_{\mathbb{R}^d \setminus \mathcal{B}_R} \frac{|x|^2}{R^2} m_\epsilon(t,x)dxdt \leq \frac{C}{R^2}.$$

We are now ready to prove the convergence of $\mathcal{U}^\epsilon$ to $\mathcal{U}$.
Fix $(t_0, m_0) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)$ and for every ϵ consider $(Du_\epsilon, m_\epsilon)$, minimizer of $\mathcal{U}^\epsilon$. By the previous computation we know that Du_ϵ,m_ϵ locally uniform converges to Du,m.
We fix a radius $R > 0$ and we split the space integral between $\mathbb{R}^d$ and $\mathbb{R}^d \setminus \mathcal{B}_R$:

$$\mathcal{U}^\epsilon(t_0, m_0) = \int_{t_0}^T \int_{\mathcal{B}_R} \frac{|Du_\epsilon(t,x)|^2}{2} m_\epsilon(t,x) + F(x, \xi^\epsilon * m_\epsilon(t,x))\xi^\epsilon * m_\epsilon(t,x)dxdt$$
$$+ \int_{t_0}^T \int_{\mathbb{R}^d \setminus \mathcal{B}_R} \frac{|Du_\epsilon(t,x)|^2}{2} m_\epsilon(t,x) + F(x, \xi^\epsilon * m_\epsilon(t,x))\xi^\epsilon * m_\epsilon(t,x)dxdt$$

The first integral converges to :

$$\int_{t_0}^T \int_{\mathcal{B}_R} \frac{|Du(t,x)|^2}{2} m(t,x) + F(x, m(t,x))m(t,x)dxdt$$

thanks to the local uniform convergence of Du_ϵ and m_ϵ and the continuity property of F.

For the convergence of the second integral we use the Corollary 4.8. Indeed since $Du_\epsilon, m_\epsilon, Du$ are uniformly bounded in $[0,T] \times \mathbb{R}^d$ by a constant, say K, independent on ϵ and on R, we can write:

$$
\begin{aligned}
\Bigg| \int_{t_0}^{T} \int_{\mathbb{R}^d \setminus \mathcal{B}_R} & \frac{|Du_\epsilon(t,x)|^2}{2} m_\epsilon(t,x) + F(x, \xi^\epsilon * m_\epsilon(t,x)) \xi^\epsilon * m_\epsilon(t,x) dxdt \\
& - \int_{t_0}^{T} \int_{\mathbb{R}^d \setminus \mathcal{B}_R} \frac{|Du(t,x)|^2}{2} m(t,x) + F(x, m(t,x)) m(t,x) dxdt \Bigg| \\
& \le 2\Big(\frac{K^2}{2} + \max_{[0,T]\times[0,K]} |F(x,m)|\Big) \Big[\int_{t_0}^{T} \int_{\mathbb{R}^d \setminus \mathcal{B}_R} m_\epsilon(t,x) dxdt + \int_{t_0}^{T} \int_{\mathbb{R}^d \setminus \mathcal{B}_R} m(t,x) dxdt \Big] \\
& \le \tilde{K} \frac{C}{R^2}
\end{aligned}
$$

Since $\tilde{K}$ and C do not depend on R, for every δ we can find $\bar{\epsilon}$ and R such that for very $\epsilon \geq \bar{\epsilon}$:

$$
\Big| \mathcal{U}_\epsilon(t_0, m_0) - \mathcal{U}(t_0, m_0) \Big| \le \delta
$$

that is :

$$
\lim_{\epsilon \to 0^+} \mathcal{U}_\epsilon(t_0, m_0) = \mathcal{U}(t_0, m_0).
$$

5 Final considerations and open questions

Our work provides a first convergence result for control problems on systems of particles with moderate interactions.
However, some technical improvements can be made, and more importantly, some questions remain open.

Let's start with the technical improvements. Firstly, the work can be extended without adding theoretical issues to models with more general Hamiltonian and coupling. This extensions make the existence proof of a regular solution for the MFG system more complicated since, for instance, we can no longer use the Hopf-Cole transformation to linearize it.
Secondly, the work can be extended to more general dynamics of the form:

$$\begin{cases} dX_t^{N,i} = b(t, X_t^{N,i}, V^N * m_t^N(X_t^{N,i}), \alpha_t^{N,i})dt + \sigma(t, X_t^{N,i}, V^N * m_t^N(X_t^{N,i}), \alpha_t^{N,i})dW_t^{N,i} \\ X_{t_0}^{N,i} = Z^i \end{cases} \tag{24}$$

i.e. to models where the particles also interact in the coefficients of the dynamics. This extensions makes the convergence in Chapter 4.1 more complicated since the particles are no longer i.i.d., and thus we can no longer use Glivenko-Cantelli law of large numbers. One solution, could be to use a more sophisticated propagation of chaos result such as those proven in [9].

In addition to these possible technical improvements, the question regarding the convergence rate remains open. In our work, we do not provide a rate of convergence of $\mathcal{V}^N$ towards $\mathcal{U}$.
Having a rate of convergence is of fundamental importance for numerical applications.

For models with nonlocal interactions, a rate of convergence is provided using PDEs techniques. The idea is to link $\mathcal{U}$ with an Hamilton-Jacobi equation in the space of measures and than by a comparison argument compare it with $\mathcal{V}^N$. This approach requires to have at least continuity of $\mathcal{U}$ in a Wassertsein spece in order to write a dynamic programming principle.

Since in our model $\mathcal{U}$ depends locally on the measure (the dependence is on the density) we don not have continuity property in general Wassertein spaces.

A possible solution could be to consider an Hamilton-Jacobi equation in a restricted space of measure to ensure smoothness properties for $\mathcal{U}$. Specifically, following [1] the idea could be to write an H-J equation in the space of measures with Lebesgue densities in suitable Sobolev spaces.

A Appendix. Some useful results and estimates

Lemma A.1. *Assume $h \in C([0,t] \times \mathbb{R}^d) \cap L^1([0,t] \times \mathbb{R}^d)$. Then there exist a sequence $R_k \nearrow +\infty$ such that :*

$$R_k \int_0^t \int_{\partial B_{R_k}} |h(t,x)| dxdt \to 0$$

as $k \to +\infty$.

Proof. see [5] lemma A.1. □

Lemma A.2. *Assume $Y, f \in C^1([0,T] \times \mathbb{R}^d)$ and:*

$$f divY, \quad Y \cdot Df, \frac{f|Y|}{1+|x|} \in L^1((0,T) \times \mathbb{R}^d).$$

The the following integration by parts formula holds:

$$\int_0^T \int_{\mathbb{R}^d} divY f dxdt = -\int_0^T \int_{\mathbb{R}^d} Y \cdot Df dxdt.$$

Proof. see [5] lemma A.2. □

Lemma A.3. *Suppose z is a bounded weak solution of:*

$$\partial_t z - \Delta z = fz + g + div(hz)$$

where $f, g \in L^p((0,T) \times \mathbb{R}^d)$, $h \in L^{2p}((0,T) \times \mathbb{R}^d)$ for some $p > \frac{d+2}{2}$ then:

$$||z||_{L^\infty((0,T)\times\mathbb{R}^d))} \leq C(||f||_{L^p((0,T)\times\mathbb{R}^d)} + ||z(0)||_{L^\infty(\mathbb{R}^d)})$$

where the constant C remains bounded for bounded values of $||g||_{L^p}, ||h||_{L^{2p}}$.

Moreover, suppose $f, h \equiv 0$ then:

$$||Dz||_{L^{2p}((0,T)\times\mathbb{R}^d))} \leq C(||f||_{L^p((0,T)\times\mathbb{R}^d)} + ||Dz(0)||_{L^{2p}(\mathbb{R}^d)})$$

where C depends only on d, T, p.

Proof. see [5] lemma A3. □

Lemma A.4. *Let m be a classical solution to the Fokker-Plank equation in (MFG), and suppose that $m, m_t, Dm, \Delta m, Du \in C_b([0,T] \times \mathbb{R}^d)$ and $m_0 \in L^1(\mathbb{R}^d)$, Then m is non-negative, $m \in C([0,T], L^1(\mathbb{R}^d))$ and $||m(t)||_{L^1(\mathbb{R}^d)} = ||m(0)||_{L^1(\mathbb{R}^d)}$ for all $t \in [0,T]$.*

Moreover, if $|x|m_0, |x|^2 m_0 \in L^1(\mathbb{R}^d)$ then:

$$\frac{|Dm|^2}{m}, |Du||Dm| \in L^1((0,T) \times \mathbb{R}^d)$$

Proof. see [5] lemma 2.3 □

Lemma A.5. *Let (u, m) be a classical solution of (MFG). Suppose that m, m_t, Dm, Δm, Du, $\partial_t u$, $\partial_t Du$, Δu, $\partial_t Dm \in C_b(\mathbb{R}^d \times [0,T])$. Then the following quantity is conserved:*

$$\int_{\mathbb{R}^d} Du(t) \cdot Dm(t) dx + \frac{1}{2} \int_{\mathbb{R}^d} |Du(t)|^2 m(t) dx + \int_{\mathbb{R}^d} F(x, m(t)) dx = E \in \mathbb{R}$$

for a.e. $t \in [0,T]$

Proof. We multiply the first equation in (MFG) by $\partial_t m$ and the second by $\partial_t u$.

In order to perform several integration by parts, we multiply by $\phi_\epsilon = e^{-\epsilon \frac{|x|^2}{2}}$, $\epsilon > 0$. Integrating over $\mathbb{R}^d$ we obtain:

$$\int_{\mathbb{R}^d} [-\Delta u \partial_t m \phi_\epsilon - \Delta m \partial_t u \phi_\epsilon + \frac{1}{2} |Du|^2 \partial_t \phi_\epsilon - div(Dum) \partial_t u \phi_\epsilon - f(x,m) \partial_t m \phi_\epsilon] dx = 0 \tag{25}$$

Now, all the derivatives of ϕ_ϵ up to the second order are in $L^1((0,T) \times \mathbb{R}^d)$, so thanks to the boundedness of the derivatives of u and m, we can integrate by parts. We split the computation.

$$\begin{aligned}
&\int_{\mathbb{R}^d} -\Delta u \partial_t m \phi_\epsilon - \Delta m \partial_t u \phi_\epsilon dx \\
&= \int_{\mathbb{R}^d} Du \cdot D\partial_t m \phi_\epsilon + Du \cdot \partial_t m D\phi_\epsilon + Dm \cdot D\partial_t u \phi_\epsilon + Dm \cdot D\phi_\epsilon \partial_t u dx \\
&= \int_{\mathbb{R}^d} \partial_t (Du \cdot Dm \phi_\epsilon) + Du \cdot D\phi_\epsilon \partial_t m - mD\partial_t u \cdot D\phi_\epsilon - m \partial_t u \Delta \phi_\epsilon dx \\
&= \int_{\mathbb{R}^d} \partial_t (Du \cdot Dm \phi_\epsilon) + \partial_t (Du \cdot D\phi_\epsilon m) - 2mD\partial_t u D\phi_\epsilon - m \partial_t u \Delta \phi_\epsilon dx.
\end{aligned}$$

Then the second term :

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{1}{2} |Du|^2 \partial_t m \phi_\epsilon - div(Dum) \partial_t u \phi_\epsilon \\
&= \int_{\mathbb{R}^d} \frac{1}{2} |Du|^2 \partial_t m \phi_\epsilon + DumD\partial_t u \phi_\epsilon + Dum \partial_t u D\phi_\epsilon dx \\
&= \int_{\mathbb{R}^d} \partial_t (\frac{1}{2} |Du|^2 m \phi_\epsilon) + Dum \partial_t u D\phi_\epsilon dx.
\end{aligned}$$

And finally:

$$\int_{\mathbb{R}^d} f(x,m) \partial_t m \phi_\epsilon dx = \int_{\mathbb{R}^d} \partial_t F(x,m) \phi_\epsilon dx$$

So, plugging this estimates in (25) we obtain:

$$\begin{aligned}
&\int_{\mathbb{R}^d} \partial_t (Du \cdot Dm \phi_\epsilon + \frac{1}{2} |Du|^2 m \phi_\epsilon - F(x,m) \phi_\epsilon + DumD\phi_\epsilon) dx \\
&= \int_{\mathbb{R}^d} 2mD\partial_t u \cdot D\phi_\epsilon + m \partial_t u \Delta \phi_\epsilon - Dum \partial_t u D\phi_\epsilon dx.
\end{aligned}$$

Thanks to the integrability properties of ϕ_ϵ and its derivatives and thanks to the bounds on m, u an their derivatives we can write:

$$\begin{aligned}
&\partial_t \int_{\mathbb{R}^d} Du \cdot Dm\phi_\epsilon + \frac{1}{2}|Du|^2\partial_t\phi_\epsilon - F(x,m)\phi_\epsilon + DumD\phi_\epsilon dx \\
&= \int_{\mathbb{R}^d} m(2D\partial_t u \cdot D\phi_\epsilon + \partial_t u\Delta\phi_\epsilon - Du\partial_t uD\phi_\epsilon)dx.
\end{aligned}$$

so that integrating between t_1 and t_2 we obtain

$$\begin{aligned}
&\left[\int_{\mathbb{R}^d} Du \cdot Dm\phi_\epsilon + \frac{1}{2}|Du|^2 m\partial_t\phi_\epsilon - F(x,m)\phi_\epsilon + DumD\phi_\epsilon dx\right]_{t_1}^{t_2} \\
&= \int_{t_1}^{t_2}\int_{\mathbb{R}^d} m(2D\partial_t u \cdot D\phi_\epsilon + \partial_t u\Delta\phi_\epsilon - Du\partial_t uD\phi_\epsilon)dxdt.
\end{aligned}$$

Now since $\int_{\mathbb{R}^d} m(t)dx = 1$ for all t, and $m \in C_b((0,T)\times\mathbb{R}^d)$. Therefore, all the quantity $mD\partial_t u$, $m\partial_t u$, $mDu\partial_t$ lives in $L^1((0,T)\times\mathbb{R}^d)$, while $|Du(t)|^2 m(t)$, $F(x, m(t)) \in L^1(\mathbb{R}^d)$. Moreover, thanks to lemma (A.4) $DuDm(t) \in L^1(\mathbb{R}^d)$ for a.e. $t \in (0,T)$. So, by dominated convergence, letting ϵ goes to 0 we have that $\phi_\epsilon \to 1$, $D\phi_\epsilon, \Delta\phi_\epsilon \to 0$ uniformly, and:

$$\left[\int_{\mathbb{R}^d} Du(x) \cdot Dm(x)dx + \frac{1}{2}\int_{\mathbb{R}^d} |Du|^2 mdx - \int_{\mathbb{R}^d} F(x,m)dx\right]_{t_1}^{t_2} = 0$$

for a.e. t_1, t_2 in $(0,T)$ i.e. there exists $E \in \mathbb{R}$ such that:

$$\int_{\mathbb{R}^d} Du(t) \cdot Dm(t)dx + \frac{1}{2}\int_{\mathbb{R}^d} |Du(t)|^2 m(t)dx - \int_{\mathbb{R}^d} F(x,m(t))dx = E$$

for a.e. t. In particular, if $Dm_0 \in L^1(\mathbb{R}^d)$ than $Du(0)Dm_0 \in L^1(\mathbb{R}^d)$, so that the equality holds for $t = 0$. □

B Appendix. Hölder-type seminorm bounds

We present here the fundamental results regarding the Hölder type seminorm bound used in chapter 2 . We start with some definitions. For $s \in (0,1)$ and $p \in [1,+\infty)$, we define the space $W^{s,p}(\mathbb{R}^d)$ as:

$$W^{s,p}(\mathbb{R}^d) = \{f \in L^p(\mathbb{R}^d) : \frac{|f(x) - f(y)|}{|x-y|^{\frac{d}{p}+s}} \in L^p(\mathbb{R}^d \times \mathbb{R}^d)\}$$

endowed with the following norm:

$$||f||^p_{W^{s,p}} = \int_{\mathbb{R}^d} |f(x)|^p dx + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} dxdy = ||f||^p_{L^p(\mathbb{R}^d)} + [f]^p_{p,ps}.$$

For $p \in [1,+\infty)$ and $s \in (0,1)$ such that $sp > d$ it can be proven (see [6] theorem 8.2.) that there exists a constant $C = C(d,s,p)$ such that:

$$||f||_\infty + [f]_\gamma \leq C(||f||_{L^p} + [f]_{p,sp})$$

where $\gamma = \frac{sp-d}{p}$ and $sp > d$. The following estimates on $[f]_{p,ps}$ hold:

Lemma B.1. *Let $p \in [1,+\infty)$, $s \in (0,1)$ be such that $sp > d$, $d \in \mathbb{N}$. Then:*

$$[f]^p_{p,ps} \leq \int_{\mathbb{R}^d}\int_{|h|\leq 1} \frac{|f(y+h)-f(y)|^p}{|h|^{d+sp}} dhdy + 2C_{p,d,s}||f||^p_{L^p}.$$

Proof. see [9] Lemma C.1. □

We recall the definition of the martingale:

$$M^N_t(\cdot) = \int_0^t \frac{1}{N}\sum_{i=1}^N \mathcal{P}_{t-s} DV^N(\cdot - X^{N,i}_s) \cdot dW^{N,i}_s$$

where $\mathcal{P}_t$ is the semigroup associated to the heat kernel. Then the following properties hold.

Lemma B.2. *Assume that there exist a number $\epsilon > 0$ such that for $p \geq 2$ there exists a function $g_p > 0$ such that:*

a) $\mathbb{E}\left[|M^N_t(x)|^p\right] \leq g_p(x)$

b) $\mathbb{E}\left[|M^N_t(x) - M^N_t(x+h)|^p\right] \leq g_p(x)|h|^{\epsilon p}$

c) $\int_{\mathbb{R}^d} g_p(x)dx < +\infty$

for all $|h| \le 1$ and $x \in \mathbb{R}^d$. Then there is $\gamma > 0$ such that, for every $p \ge 2$, there exists a constant $C_p > 0$ such that:

$$\mathbb{E}\Bigg[||M_t^N||_\gamma^p\Bigg] \le C_p$$

Proof. The proof is a consequence of the previous lemma. First, note that thanks to the Hölder inequality its enough to prove the claim for a $\bar{p} \ge 2$ big enough. So fix $s \in (0, \epsilon)$ and take $\bar{p} \ge 2$ such that $s\bar{p} > d$.
By assumption $\mathbb{E}\Bigg[\int_{\mathbb{R}^d} |M_t^N(x)|^{\bar{p}} dx\Bigg] \le C$.
By the previous lemma:

$$\begin{aligned}
\mathbb{E}\Bigg[[M_t^N]_{\bar{p},s\bar{p}}^{\bar{p}}\Bigg] &\le \int_{\mathbb{R}^d}\int_{|h|\le 1} \frac{\mathbb{E}\Bigg[|M_t^N(y+h) - M_t^N(y)|^{\bar{p}}\Bigg]}{|h|^{d+s\bar{p}}} dh dy + 2C_{\bar{p},d,s}\mathbb{E}\Bigg[||M_t^N||_{L^{\bar{p}}}^{\bar{p}}\Bigg] \\
&\le \int_{\mathbb{R}^d}\int_{|h|\le 1} \frac{g_{\bar{p}}(y)|h|^{\epsilon\bar{p}}}{|h|^{d+s\bar{p}}} dh dy + C \\
&\le \Bigg(\int_{|h|\le 1} \frac{1}{|h|^{d-(\epsilon-s)\bar{p}}} dh\Bigg) \int_{\mathbb{R}^d} g_{\bar{p}}(y) dy + C \le C.
\end{aligned}$$

Using again that $\mathbb{E}[||M_t^N||_{L^{\bar{p}}}^{\bar{p}}] \le C$ and the previous lemma, we prove the claim for $\gamma = \frac{(s\bar{p}-d)}{\bar{p}}$. It can be proven that this constant does not depend on the $\bar{p}$ chosen. Indeed, chosen a $\bar{p}_0$ such that $s\bar{p}_0 > d$ and $\mathbb{E}[||M_t^N||_{\gamma_0}^{\bar{p}_0}] \le C_{\bar{p}_0}$ then for all $\bar{p} > \bar{p}_0$ we prove the inequality for $\bar{\gamma} = s - \frac{d}{\bar{p}}$ which is bigger than γ_0, therefore it holds for $\bar{\gamma}_0$ which can be taken as γ in the statement of the lemma. □

Now, since in our case $M_t^N(\cdot) = \int_0^t \frac{1}{N}\sum_{i=1}^N \mathcal{P}_{t-s} DV^N(\cdot - X_s^{N,i}) \cdot dB_s^{N,i}$, we need the following estimates on $(D\mathcal{P}_{t-s}V^N)(x)$.

Lemma B.3. *Let $N, d \in \mathbb{N}$, let $\mathcal{P}_t$ the semigroup associated to the density $G(t,x) = x + B_t$ with B_t standard brownian motionm $x \in \mathbb{R}^d$ and $t \in (0,T]$. Moreover, let $V \in C_c^1(\mathbb{R}^d) \cap \mathcal{P}(\mathbb{R}^d)$. Then:*

$$||\mathcal{P}_t h||_\gamma \le C||h||_\gamma.$$

Moreover, fix $R > 0$ such that the support of V is contained in $\mathcal{B}_R(0)$ and define $V^N = \epsilon_N^{-d} V(\epsilon_N^{-1} x)$. Then there exists two constant $C_{T,R,V} > 0$ and $\lambda_{T,R,V} > 0$ such that for every $\delta, \gamma \in (0,1)$, $x \in \mathbb{R}^d$, $|h| \le 1$ and $t \in [0,T]$:

$$|(D\mathcal{P}_t V^N)(x)| \le \frac{C_{T,R,V}}{t^{\frac{1-\delta}{2}}} \epsilon_N^{-d-\delta} e^{-\frac{|x|}{8T}}$$

$$|(D\mathcal{P}_t V^N)(x) - (D\mathcal{P}_t V^N)(x+h)| \le \frac{C_{R,T,V}}{t^{\frac{1}{2}(1+\gamma) - \frac{\delta}{2}(1-\gamma)}} |h|^\gamma \epsilon_N^{-d-\delta(1-\gamma)} e^{-\lambda_{T,R,V}|x|}.$$

Proof. see [9] Lemma C.3. □

References

[1] David M. Ambrose and Alpár R. Mészáros. Well-posedness of mean field games master equations involving non-separable local hamiltonians, 2022.

[2] Pierre Cardaliaguet, Samuel Daudin, Joe Jackson, and Panagiotis E Souganidis. An algebraic convergence rate for the optimal control of mckean–vlasov dynamics. *SIAM Journal on Control and Optimization*, 61(6):3341–3369, 2023.

[3] Pierre Cardaliaguet, P Jameson Graber, Alessio Porretta, and Daniela Tonon. Second order mean field games with degenerate diffusion and local coupling. *Nonlinear Differential Equations and Applications NoDEA*, 22:1287–1317, 2015.

[4] René Carmona and François Delarue. *Probabilistic Theory of Mean Field Games with Applications I*. Springer, 2018.

[5] Marco Cirant and Daria Ghilli. Existence and non-existence for time-dependent mean field games with strong aggregation. *Mathematische Annalen*, 383(3):1285–1318, 2022.

[6] Eleonora Di Nezza, Giampiero Palatucci, and Enrico Valdinoci. Hitchhikers guide to the fractional sobolev spaces. *Bulletin des sciences mathématiques*, 136(5):521–573, 2012.

[7] Emmanuele DiBenedetto. *Degenerate parabolic equations*. Springer Science & Business Media, 2012.

[8] Mao Fabrice Djete, Dylan Possamaï, and Xiaolu Tan. Mckean–vlasov optimal control: limit theory and equivalence between different formulations. *Mathematics of Operations Research*, 47(4):2891–2930, 2022.

[9] Franco Flandoli, Maddalena Ghio, and Giulia Livieri. N-player games and mean field games of moderate interactions. *Applied Mathematics & Optimization*, 85(3):38, 2022.

[10] Massimo Fornasier, Stefano Lisini, Carlo Orrieri, and Giuseppe Savaré. Mean-field optimal control as gamma-limit of finite agent controls. *European Journal of Applied Mathematics*, 30(6):1153–1186, 2019.

[11] Daniel Lacker. Limit theory for controlled mckean–vlasov dynamics. *SIAM Journal on Control and Optimization*, 55(3):1641–1672, 2017.

[12] OA Ladyzhenskaya, VA Solonnikov, and NN Uralceva. Linear and quasilinear equations of parabolic type, ams. *Trans. Math. Monograph*, 23, 1968.

[13] Karl Oelschläger. A law of large numbers for moderately interacting diffusion processes. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 69(2):279–322, 1985.