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Non-Abelian orbifolds in string theory

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A Tommaso
luce dei miei occhi in questa nera possibilità
solo una
di altre infinite in cui ci stiamo guardando

A questo lavoro
che durante il suo tempo mi ha nascosto il dolore
facendomi immaginare un'altra realtà
tanto più bella rispetto a quella che si vede con gli occhi
da non volerla lasciare

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"Cinque anatre volano a sud:
Molto prima del tempo l'inverno è arrivato
Cinque anatre in volo vedrai
Contro il sole velato, contro il sole velato"
"... ... ..."
"Cinque anatre andavano a sud:
Forse una soltanto vedremo arrivare,
Ma quel suo volo certo vuole dire
Che bisognava volare, che bisognava volare,
Che bisognava volare, che bisognava volare.."
F.Guccini

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Introduction

String theory is currently one of the best candidate to give a unified model of Nature's forces, including gravity, in a quantum-mechanical framework. The idea is to imagine the matter at fundamental level not described by point-particles but as tiny strings that moving through the space-time span two-dimensional surfaces called *World-sheet*. A general feature of string theory is that in the low energy limit General Relativity and Yang-Mills theories naturally appear.
There are different string theories, the simplest being is surely the *Bosonic String* theory. However, it has some problems that lead to physical inconsistencies. The presence in the spectrum of particles, called *Tachyons*, with negative mass squared leads to instability of the vacuum, the absence of fermionic particles, necessary constituents of the matter, makes the theory difficult to apply.
A way to solve some failure of Bosonic string is to introduce a new symmetry on the theory, the *supersymmetry* (SUSY), characterized by fermionic generators that act on the Hilbert space exchanging bosons and fermions. The commutation rules of the new generators, carrying a fermionic index, with the generators of space-time symmetry imply that the pairs boson/fermion under SUSY must have equal mass. However, no known bosonic particle has the same mass as an electron. Therefore, if the fundamental interaction theory is supersymmetric then the supersymmetry must be spontaneously broken in the observable range of energy.
Five different models of *Superstring* theories have been constructed: *Type I*, *Type IIA*, *Type IIB*, *Heterotic* $SO(32)$ and *Heterotic* $E_8 \times E_8$. They can been seen as the same theory at different regimes under a certain duality operation.
Another of the main features of string theory is the presence of extra-dimensions respect to observed four-dimensional space-time in General Relativity. Indeed the Bosonic String and the Superstring theories are defined in spaces-time with $D = 26$ and $D = 10$ dimensions respectively. The idea is to split the full space-time into a d-dimensional Minkowski space (e.g. $d = 4$) and some $D - d$ dimensional compact Riemannian space. The physics at length scales much larger than the size of compact space is the same as a d-dimensional Minkowski space: the remaining $D - d$ dimensions have been *compactified*.
However, just some compact manifold provide consistent theories. One of the simplest examples of compactification is the Superstring compactification of the six extra-dimensions on a flat torus $\mathcal{M}^{1,3} \times T^6$, that still leads to phenomenologies that cannot be applied. These phenomenologies do not reproduce the Standard Model at low energy. Generally, compactification schemes that lead to realistic phenomenologies involve more complicate mathematical objects. Among these, one of the most important classes of compactification manifold are the *Calabi-Yau* ones. Calabi-Yau surfaces CY_n are $2n$-dimensional compact Ricci-flat manifolds with Holonomy group $SU(n)$. The reduced Holonomy group $SU(n) \subset SO(2n)$ preserves only a certain number of supersymmetries on extended dimensions. In particular, the class of CY_2 surfaces $K3$ plays an important role in various aspects of mathematics and string theory.
Type IIA string compactifications on $K3 \times T^d \times \mathbb{R}^{5-d,1}$ are one of the first examples of holografic

duality in Ads/CFT correspondance, and they provided a suitable background for a microscopic description for the Bekenstein-Hawking law of Black Hole entropy. [35] Moreover they lead to exact results on the study of spectrum of states.
From a geometric point of view there are interesting connections between $K3$ symmetries and the 26 sporadic simple groups: [36] [37] Mukai's theorem provides a classification of geometrical symmetries of $K3$ surfaces in terms of subgroups of the Mathieu group $\mathbb{M}_{23}$. This connection has been implemented by the discovery of new moonshine phenomena, initiated by an observation of Eguchi, Ooguri and Tachikawa [19]: the Elliptic Genus of $K3$ encodes an infinite-dimensional graded representation of the largest Mathieu sporadic group $\mathbb{M}_{24}$ (of which $\mathbb{M}_{23}$ is a subgroup). These observations led to the study of geometrical and non-geometrical symmetries of NLSMs with $K3$ target space and it led to new conjectures, such as Conway moonshine relating the stringy $K3$ symmetries to the finite *Conway* group C_{O_0}.
A natural formalism to describe the worldsheet dynamics of open and closed strings is the *conformal field theory* (CFT) and its supersymmetric extension (SCFT) for the Superstrings. In particular the dynamics of Superstrings compactified in some manifold is described by a two-dimensional superconformal theory called *Non Linear Sigma Model* (NLSM).
Type IIA and IIB Supertrings compactified on $\mathcal{M}^{1,3} \times T^2 \times K3$ lead to interesting NLSMs, that are $\mathcal{N} = (4,4)$ superconformal theories on the $K3$ surface. Because of the difficulty to study string theories for a generic K3 surface, it is useful to study it for some special examples of NLSMs on K3. In particular toroidal orbifolds T^4/G, where G is a finite group, are special points of *moduli space* of K3 in which it is possible to obtain exact results (spectrum, boundary states, symmetries...) for string applications. The drawback of these special models is that they are rather special, so they might not be well-suited to study the typical proprieties of a generic model.
So far the study focused on conformal theories on cyclic orbifolds or non-abelian orbifolds target spaces constructed through groups G that act geometrically on the torus T^4. In order to closely approach to more generic model, in this thesis we will concentrate on the study of NLSMs on toroidal orbifolds constructed by groups G with non-geometric action, i.e. those that do not descend from symmetries of T^4 space.

In the first chapter we introduce the *Conformal Symmetry* in a generic D-dimensional space, and we analyse more specifically its main proprieties on $D = 2$ dimensions.

In the second chapter we describe the Bosonic String Theory as a classical field theory, its symmetries and its Quantization through the different canonical, Lightcone and path integral formalisms.

In the third chapter we solve the failure of Bosonic String Theory introducing the Supersymmetry into the string theory through the *Ramond-Neveu-Schwarz procedure*. In particular we study $\mathcal{N} = 1, 2, 4$ superconformal theories.

In the fourth chapter we explore the principal schemes of compactification in String Theory. We give the first definitions of *Calabi-Yau* manifolds and $K3$ surfaces. We also provide the construction of *Moduli Space* of complex structures, Einstein metrics and NLSMs on $K3$ surfaces.

In the fifth chapter we describe the *Orbifolds* technique to construct new conformal theories and its consequences on the Hilbert space, where we are forced to introduce new sectors, *Twisted*, for the theory to be consistent. In order to study the spectrum of these theories, we introduce new functions as the *Elliptic* and *Twining Genus*. We compute explicitly them for the model $T^4/2.A_5$.

In the last chapter we present the *Boundary States* formalism to describe the D-branes on orbifold target spaces. First, we perform the calculation on $T^4/2.A_4$, where $2.A_4$ is the subgroup of $2.A_5$ containing only elements that act geometrically on T^4. Afterwards we proceed with a symmetrization of the resulting branes with the action of non-geometric elements of $2.A_5$.

Chapter 1

The Conformal Group

In this first chapter we would like to provide a brief introduction to conformal symmetry in arbitrary dimension, then focus our attention on CFT in two dimensions because it turns out to be of great interest to study world-sheet of both open and closed string. In this latter case there exists an infinite variety of locally conformal coordinate transformations: holomorphic maps from the complex plane into itself. Among this maps, we can identify 6 *global* parameters through which it is possible to define a *conformal group*, made of one-to-one mappings of the complex plane into itself. [1] [2] [3] [4]

1.1 The Conformal Symmetry

Let us consider the metric tensor $g_{\mu\nu}$ in a d-dimensional space-time. Let's define *conformal transformation* of the coordinates as a invertible map $x^\mu \to x'^\mu$ such that:

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x).$$

The set of this transformations forms the *conformal group*, and setting $\Lambda(x) = 1$ we see that, for flat space-time, it has the Poincarè group as its subgroup.
Let's consider the infinitesimal transformation:

$$x^\mu \quad \to \quad x'^\mu = x^\mu + \epsilon^\mu(x), \tag{1.1}$$

through which the metric changes, at first order, as follows:

$$g'_{\mu\nu} \Longrightarrow g_{\mu\nu} - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu).$$

In order to (1.1) be conformal the variation of the metric must be linear in the metric itself:

$$(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = f(x) g_{\mu\nu}, \tag{1.2}$$

We assume that a conformal transformation is an infinitesimal deformation of the standard metric $\eta_{\mu\nu}$, Euclidean or Minkowskian; after some manipulations we find

$$(2-d)\partial_\mu \partial_\nu f(x) = \eta_{\mu\nu} \partial^2 f(x)$$

contracting with $\eta_{\mu\nu}$

$$(d-1)\partial^2 f(x) = 0.$$

Now we can derive the explicit form of transformations in d dimensions.
If $d = 1$, there are no constraints on the function $f(x)$, therefore any smooth map is a conformal map.
If $d \geq 3$ we have the condition $\partial_\mu \partial_\nu f(x) = 0$, which tells us that $f(x)$ is a multilinear function of the form $f(x) = A + B_\mu x^\mu$. This implies that $\partial_\mu \partial_\nu \epsilon_\rho = const$, which means that ϵ_μ is at most quadratic in the coordinates, in particular we can write the following ansatz:

$$\epsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho \qquad c_{\mu\nu\rho} = c_{\mu\rho\nu}.$$

Analyzing the conformal constraints for the various terms we obtain the following splitting of transformations:

i) The constant term $\epsilon_\mu = a_\mu$ corresponds to constant translation.

ii) The linear term $\epsilon_\mu = m_{\mu\nu} x^\nu$, where $m_{\mu\nu} = -m_{\nu\mu}$, corresponds to infinitesimal rigid rotation.

iii) The linear term $\epsilon_\mu = \lambda x_\mu$, coming from the part of pure trace on $b_{\mu\nu}$, corresponds to infinitesimal scale transformation.

iv) The quadratic term $\epsilon_\mu = c_{\mu\nu\rho} x^\nu x^\rho$ corresponds to *special conformal transformation* (SCT).

Through exponentiations we find the finite transformations summarized as follows:

Translation	$x'^\mu = x^\mu + a^\mu$
Dilatation	$x'^\mu = \alpha x^\mu$
Rigid rotation	$x'^\mu = M^\mu_\nu x^\nu$
SCT	$x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2\mathbf{b}\cdot\mathbf{x} + b^2 x^2}$

The corresponding infinitesimal generators on the coordinates space are:

Translation	$P_\mu = -i\partial_\mu$
Dilatation	$D = -ix^\mu \partial_\mu$
Rotation	$L_{\mu\nu} = i\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right)$
SCT	$K_\mu = -i\left(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu\right)$

satisfying the following commutation relations:

$$\begin{aligned}
&[D, P_\mu] = iP_\mu \\
&[D, K_\mu] = -iK_\mu \\
&[K_\mu, P_\nu] = 2i\left(\eta_{\mu\nu} D - L_{\mu\nu}\right) \\
&[K_\rho, L_{\mu\nu}] = i\left(\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu\right) \\
&[P_\rho, L_{\mu\nu}] = i\left(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu\right) \\
&[L_{\mu\nu}, L_{\rho\sigma}] = i\left(\eta_{\nu\rho} L_{\mu\sigma} + \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho}\right).
\end{aligned} \tag{1.3}$$

Furthermore, it is possible to simplify these rules, therefore we define the new generators:

$$J_{\mu\nu} = L_{\mu\nu} \qquad J_{-1\mu} = \frac{1}{2}\left(P_\mu - K_\mu\right)$$
$$J_{-10} = D \qquad J_{0\mu} = \frac{1}{2}\left(P_\mu + K_\mu\right)$$

Which obey to the $so(d+1,1)$ (respectively $so(d-1,2)$) commutation relations:

$$[J_{ab}, J_{cd}] = i\left(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}\right),$$

where $a, b, c, d = -1, 0, 1, ..., d$ and $\eta_{ab} = diag\left(-1, 1, 1, ..., 1\right)$ (otherwise an additional component is negative), if space-time is Euclidean (respectively Minkowskian). This latter relation shows the isomorphism between $so(d+1,1)$, with $\frac{1}{2}\left(d+2\right)\left(d+1\right)$ parameters, and the conformal algebra on d dimensions.
For the case of dimensions $d = p + q \geq 3$, the conformal group of $\mathbb{R}^{p,q}$ is $SO(p+1, q+1)$.
If $d = 2$ the conformal invariance plays a special role: there exists an infinite dimensional group of coordinates transformations that, although not everywhere well-defined, are locally conformal. Let us consider the infinitesimal conformal condition (1.1), for $d = 2$ and $\eta_{\mu\nu} = \delta_{\mu\nu}$ we find the Cauchy-Riemann conditions:

$$\begin{aligned} &(i) \quad \partial_1\epsilon_1 = \partial_2\epsilon_2 \\ &(ii) \quad \partial_1\epsilon_2 = -\partial_2\epsilon_1 \end{aligned}$$

Writing $\epsilon(z) = \epsilon^1 + i\epsilon^2$ and $\overline{\epsilon}(\overline{z}) = \epsilon^1 - i\epsilon^2$, and using the complex coordinates $z = x_1 + ix_2$ and $\overline{z} = x_1 - ix_2$, the Cauchy-Riemann conditions in two dimensions define holomorphic and anti-holomorphic analytic transformations on the complex plane $\mathbb{C}^2$:

$$z \quad \rightarrow \quad f(z) \qquad\qquad \overline{z} \quad \rightarrow \quad f(\overline{z}).$$

This implies that the metric tensor transforms under these maps as:

$$ds^2 = dzd\overline{z} \quad \rightarrow \quad \frac{df}{dz}\frac{df}{d\overline{z}}dzd\overline{z} = \Lambda(z)dzd\overline{z}.$$

In order to calculate the algebra of the group, we introduce a complete basis for these analytic infinitesimal functions:

$$z \longrightarrow z' = z + \epsilon_n(z) \qquad \overline{z} \longrightarrow \overline{z}' = \overline{z} + \overline{\epsilon}_n(\overline{z}) \tag{1.4}$$

where:

$$\epsilon_n(z) = -\epsilon_n z^{n+1} \qquad \overline{\epsilon}_n(\overline{z}) = -\overline{\epsilon}_n\overline{z}^{n+1}.$$

Therefore the corresponding infinitesimal generators are:

$$l_n = -z^{n+1}\partial_z \qquad \bar{l}_n = -\overline{z}^{n+1}\partial_{\overline{z}} \tag{1.5}$$

which satisfy the following commutation relations:

$$\begin{aligned} &[l_m, l_n] = (m-n)l_{m+n} \\ &\left[\bar{l}_m, \bar{l}_n\right] = (m-n)\bar{l}_{m+n} \\ &\left[l_n, \bar{l}_m\right] = 0. \end{aligned} \tag{1.6}$$

Each of the first two relations define one copy of the so-called *Witt algebra*: the full algebra is the direct sum of two copies of the Witt algebra commuting with each other. Formally we could regard z and $\overline{z}$ as independent variables and the transformations (1.4) as maps on $\mathbb{C}^2$.
Since $n \in \mathbb{Z}$ in (1.5), the number of independent infinitesimal conformal transformations is infinite: this property is special in two dimensions and leads to important consequences.
It is important to note that not all the algebra generators are globally well-defined: so far we have required only local constraints. Holomorphic conformal transformations are generated by vector fields:

$$v(z) = -\sum_n a_n l_n = \sum_n a_n z^{n+1}\partial_z,$$

requiring $v(z)$ be non-singular in the limit $z \to 0$ allows us $a_n \neq 0$ only for $n \geq -1$. In order to analyze the behavior of $v(z)$ at $z \to \infty$, let us take the change of variables $z \to -1/w$:

$$v(z) = \sum_n a_n \left(-\frac{1}{w}\right)^{n+1} \left(\frac{\partial z}{\partial w}\right)^{-1} \partial_w = \sum_n a_n \left(-\frac{1}{w}\right)^{n+1} \partial_w$$

Non-singularity as $z \to \infty$ ($w \to 0$) allows us $a_n \neq 0$ only for $n \leq 1$. Finally, only conformal transformations generated by l_{-1}, l_0, l_1 are globally well defined. The same considerations can be applied to the anti-holomorphic part.
The *(Global) Conformal Group* in two dimensions is the group of conformal transformations well-defined and invertible on the Riemann sphere. This is generated by the globally defined infinitesimal generators $\{l_{-1}, l_0, l_1\} \bigcup \{\bar{l}_{-1}, \bar{l}_0, \bar{l}_1\}$. On this set we can identify this generators as:

$$\begin{aligned} l_{-1}, \bar{l}_{-1} \qquad & \textit{Translations} \\ l_0 + \bar{l}_0 \qquad & \textit{Dilatations} \\ i(l_0 - \bar{l}_0) \qquad & \textit{Rotations} \\ l_1, \bar{l}_1 \qquad & \textit{SCT} \end{aligned} \tag{1.7}$$

The finite form of this transformations is:

$$\begin{aligned} z \quad &\longrightarrow \quad f(z) = \frac{az+b}{cz+d} \\ \overline{z} \quad &\longrightarrow \quad f(\overline{z}) = \frac{\overline{az}+\overline{b}}{\overline{cz}+\overline{d}} \end{aligned} \tag{1.8}$$

where $a, b, c, d \in \mathbb{C}$ such that $ad - bc = 1$. In the end, the conformal group of the Riemann sphere is $SL(2, \mathbb{C})/\mathbb{Z}_2 \sim SO(3, 1)$.
The distinction encountered here between global and local conformal groups is unique to two dimensions, in higher dimensions there exists only a global conformal group.
The Witt algebra admits a central extension, called *Virasoro algebra* with *central charge c*, whose generators L_n obey:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \tag{1.9}$$

Of course, a similar analysis can be carried out for the generators $\overline{L}_n$.
For $m, n = -1, 0, 1$, L_{-1} generates translations, L_0 generates dilations and rotations, and L_1 generates Special Conformal Transformations: therefore also $\{L_{-1}, L_0, L_1\}$ are generators of $SL(2, \mathbb{C})/\mathbb{Z}_2$ transformations.

1.2 Conformal invariance in Field Theory

A classical field theory has conformal invariance if its action S is invariant under conformal transformation. Except for some pathological case, the theories possessing scalar invariance and Poincarè-invariance are conformally invariant. This last fact is important when we want to realize the conformal invariance at quantum level. In fact, even when we start from a quantum theory with a conformally invariant bare action, in the renormalized theory we are usually forced to introduce a scale that breaks the conformal symmetry, except for particular values of parameters, which constitute a renormalization group fixed point.
Les us first describe how classical fields transform under an element of the conformal group, after which we analyze its behavior at quantum level. Let $\Phi(x)$ be a generic field defined on the space-time with coordinates $\{x^\mu\}$. Let us denote by T_g the generators of the conformal algebra on the fields space and by ω_g the corresponding infinitesimal parameter. A multicomponent field $\Phi(x)$ transforms as:

$$\Phi'(x') = (1 - i\omega_g T_g)\,\Phi(x).$$

Besides the action of the T_g generators on the field, we must include the transformation of the argument. In order to find the form of this generators we start by studying the subgroup that leaves the origin $x = 0$ invariant. This subgroup is generated by rotation, dilatation and SCT. We denote by $S_{\mu\nu}, \tilde{\Delta}, \kappa_\mu$ the values of $L_{\mu\nu}, D, K_\mu$ at $x = 0$: using the commutation rules (1.3) and the Baker-Campbell-Hausdorff formula: [1]

$$\begin{aligned}
e^{ix^\rho P_\rho} L_{\mu\nu} e^{-ix^\rho P_\rho} &= S_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu \\
e^{ix^\rho P_\rho} D e^{-ix^\rho P_\rho} &= D + x^\nu P_\nu \\
e^{ix^\rho P_\rho} K_\mu e^{-ix^\rho P_\rho} &= K_\mu + 2x_\mu \tilde{\Delta} - 2x^\nu L_{\mu\nu} + 2x_\mu (x^\nu P_\nu) - x^2 P_\mu
\end{aligned}$$

from which we arrive finally at the following extra transformation rules:

$$\begin{aligned}
&P_\mu \Phi(x) = -i\partial_\mu \Phi(x) \\
&L_{\mu\nu}\Phi(x) = i\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right)\Phi(x) + S_{\mu\nu}\Phi(x) \\
&D\Phi(x) = \left(-ix^\nu \partial_\nu + \tilde{\Delta}\right)\Phi(x) \\
&K_\mu \Phi(x) = \left[\kappa_\mu + 2x_\mu \tilde{\Delta} - x^\nu S_{\mu\nu} - 2ix_\mu x^\nu \partial_\nu + ix^2 \partial_\mu\right]\Phi(x)
\end{aligned}$$

If we require that $\Phi(x)$ belongs to an irreducible representation of the Lorentz group, then every matrix commuting with $S_{\mu\nu}$ must be a multiple of the identity, in particular $\tilde{\Delta}$, this enforce the matrices K_μ to vanish. Therefore $\tilde{\Delta} = -i\Delta$, where Δ is the scaling of field, and its Eigenvalues are not real, then $\tilde{\Delta}$ is not hermitian and its representation is not unitary.
Now we can define the finite transformations rules of fields. The case of a spinless field ($S_{\mu\nu} = 0$) is particularly interesting, it transforms under conformal transformations according to:

$$\Phi(x) \quad \rightarrow \quad \Phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/d} \Phi(x) \tag{1.10}$$

where the Jacobian is related to scalar factor through $\left|\frac{\partial x'}{\partial x}\right| = \Lambda(x)^{-d/2}$, and the field is called *quasi-primary* .

[1] The Baker-Campbell-Hausdorff formula is $e^{-A}Be^{A} = B + [B;A] + \frac{1}{2}\,[[B;A]\,;] + ...$
This expression guarantees that the left-hand side is valued in the conformal algebra.

We know that under a arbitrary infinitesimal coordinates transformation $x^\mu \longrightarrow x^\mu + \epsilon^\mu$, the matter action changes as:

$$\delta S = \int d^d x T^{\mu\nu} \partial_\mu \epsilon_\nu = \frac{1}{2} \int d^d x T^{\mu\nu} \left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu\right),$$

where the symmetric tensor $T_{\mu\nu}$ is the *energy-momentum* tensor. If we consider the conformal infinitesimal transformation (1.2), then the previous relation becomes:

$$\delta S = \frac{1}{d} \int d^d x T^\mu_\mu \partial_\rho \epsilon^\rho.$$

From this last form we infer that a theory with traceless energy-momentum tensor must be a theory conformally invariant, but the converse is not necessary true.
Under rather general condition in $d = 2$ scalar invariant theories owns traceless energy-momentum tensors, and the whole conformal invariance follows from Poincarè and scalar invariances. [5]
Now we specialize our treatment to bidimensional case. Let's start with an Euclidean two dimensional space $\mathbb{R}^2$ and perform the complexification $\mathbb{R}^2 \to \mathbb{C}^2$ previously described, so z and $\overline{z}$ are considered independent complex variables and the fields ϕ of our theory can be written as:

$$\phi(x^0, x^1) \quad \to \quad \phi(z, \overline{z}).$$

Fields depending only on z, i.e. $\phi(z)$, are called *chiral fields* or *holomorphic*, and fields $\phi(\overline{z})$ depending only on $\overline{z}$ are called *anti-chiral fields* or *anti-holomorphic*.
According to (1.10), a field is called *primary* of *conformal dimension* $\left(h, \overline{h}\right)$ if it transforms as:

$$\phi(z, \overline{z}) \quad \to \quad \phi'(z, \overline{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \overline{f}}{\partial \overline{z}}\right)^{\overline{h}} \phi(f(z), \overline{f}(\overline{z})), \tag{1.11}$$

under a conformal transformation. If the (1.11) holds only for global transformations, i.e. $f(z) \in SL(2, \mathbb{C})/\mathbb{Z}_2$, then ϕ is called *quasi-primary* field.
Not all fields in a CFT are primary or quasi-primary: the other fields are called *secondary* fields.
Since the algebra of infinitesimal conformal transformations in two dimensions is infinite dimensional, and the energy-momentum tensor encodes the behaviour of the theory under these transformations, it is possible to study such a conformal theory without knowing the explicit form of its action, but only the energy-momentum tensor.
Using the fact that the energy-momentum tensor for a conformal field theory is traceless and its translational invariance, i.e. $\partial^\mu T_{\mu\nu} = 0$, we conclude that in two dimensional CFTs with Euclidean signature on complex coordinate its only non-vanishing components are a chiral and an anti-chiral field:

$$T_{zz}(z, \overline{z}) = T(z) \qquad T_{\overline{z}\overline{z}}(z, \overline{z}) = \overline{T}(\overline{z}).$$

1.3 Radial quantization

On the previous sections we have introduced the fundamental concepts of conformal invariance at the classical level. Now we would study the consequences of this invariance at the quantum level through the construction of a operatorial formalism.
In general the operatorial formalism distinguishes a time direction from a space direction: this is natural in a Minkowskian space-time but not for a Euclidean space-time. So let us consider our theory in an infinite space-time cylinder $S^1 \times \mathbb{R}$, where $\mathbb{R}$ parametrizes the Euclidean time

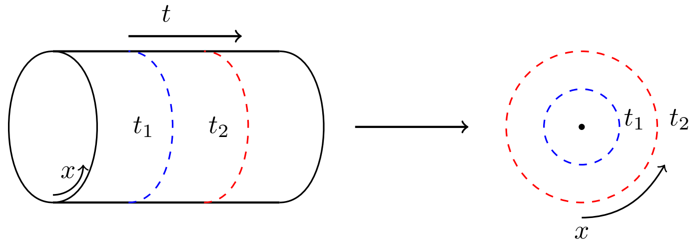

Figure 1.1: Mapping from the cylinder to the complex plane.

direction t, whereas the space direction x runs along a circle S^1, obtained by identifying the endpoints of a segment $[0; L]$. Through an analytic continuation of the space, we introduce a complex coordinate:

$$\xi = t + ix. \tag{1.12}$$

Now we map every point of cylinder to the complex plane:

$$z = e^{2\pi\xi/L}. \tag{1.13}$$

We note that the positions at $t \to -\infty$ are the origin $z = 0$ in the plane, while those at $t \to +\infty$ are the point at infinity on the Riemann sphere.

Following this map, the time translations $t \to t + a$ on the cylinder are mapped to complex dilation $z \to e^{2\pi a/L} z$ on the plane, and the space translations $x \to x + b$ are mapped to rotations $z \to e^{2\pi i b/L} z$. Since in QM the generator of time translations is the Hamiltonian which in the present case corresponds to the dilation operator, and the generator for space translations is the momentum operator corresponding to rotations, we find that:

$$H = L_0 + \overline{L}_0 \qquad P = i\left(L_0 - \overline{L}_0\right).$$

Let us consider the Hilbert space of asymptotic states at $t \to -\infty$, that corresponds to the point $z = 0$ on the plane. In order to construct a CFT on this space, it is necessary to assume the existence of a vacuum state on the Hilbert space $|0\rangle$ upon which be possible to build the whole space by the application of creation operators. Within radial quantization the different asymptotic "in" state are created acting on $|0\rangle$ with the fields in the $t \to -\infty$ limit:

$$|\phi_{in}\rangle = \lim_{z,\overline{z}\to 0} \phi(z,\overline{z})|0\rangle.$$

In order to define a bilinear product on this Hilbert space we must to introduce "out" states through the action of Hermitian conjugation on conformal fields. In the Minkowski space-time the Hermitian conjugation does not act on the space-time coordinates, but in the Euclidean space is different since the time $\tau = it$ is reversed upon conjugation. In radial quantization this correspond to the map $z \to 1/z^*$. Then we define:

$$[\phi(z,\overline{z})]^\dagger = \overline{z}^{-2h} z^{-2\overline{h}} \phi\left(\frac{1}{\overline{z}}, \frac{1}{z}\right), \tag{1.14}$$

where $\phi(z,\overline{z})$ is a quasi-primary field with conformal weights $(h, \overline{h})$. Then:

$$\begin{aligned}\langle\phi_{out}|\phi_{in}\rangle &= \lim_{z,\overline{z},w,\overline{w}\to 0} \overline{z}^{-2h} z^{-2\overline{h}} \langle 0|\phi(1/\overline{z}, 1/z)\phi(w,\overline{w})|0\rangle = \\ &= \lim_{\xi,\overline{\xi}\to\infty} \overline{\xi}^{2h} \xi^{2\overline{h}} \langle 0|\phi(\overline{\xi},\xi)\phi(0,0)|0\rangle\end{aligned}$$

The conformal field $\phi(z,\overline{z})$ having conformal weights $(h,\overline{h})$ may be written through its Laurent expansion around $z_0=\overline{z}_0=0$ as follows:

$$\phi(z,\overline{z})=\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} z^{-m-h}\overline{z}^{-n-\overline{h}}\phi_{m,n} \tag{1.15}$$

where

$$\phi_{m,n}=\frac{1}{2\pi i}\int dz z^{m+h-1}\frac{1}{2\pi i}\int d\overline{z}\overline{z}^{n+\overline{h}-1}\phi(z,\overline{z}). \tag{1.16}$$

The quantisation of this field is achieved by promoting the Laurent modes $\phi_{m,n}$ to operators. Equating the hermitian conjugation on the real surface:

$$\phi(z,\overline{z})^\dagger=\sum_{n\in\mathbb{Z}}\overline{z}^{-m-h}z^{-n-\overline{h}}\phi^\dagger_{m,n}$$

with the definition (1.14):

$$\phi(z,\overline{z})^\dagger=\sum_{n\in\mathbb{Z}}\overline{z}^{-m-h}z^{-n-\overline{h}}\phi_{-m,-n}$$

we find that $\phi^\dagger_{m,n}=\phi_{-m,-n}$. Then the vacuum state must respect the following condition:

$$\phi_{m,n}|0\rangle=0 \qquad if \quad \left(m>-h,n>-\overline{h}\right),$$

and the definitions of asymptotic *in* and *out*-state can be simplified in the following way:

$$|\phi\rangle=\lim_{z,\overline{z}\to 0}\phi(z,\overline{z})|0\rangle=\phi_{-h,-\overline{h}}|0\rangle$$

$$\langle\phi|=\lim_{\frac{1}{z},\frac{1}{\overline{z}}\to 0}\left(\frac{1}{z}\right)^{2h}\left(\frac{1}{\overline{z}}\right)^{2\overline{h}}\langle 0|\phi(\frac{1}{z},\frac{1}{\overline{z}})=\langle 0|\phi_{+h,+\overline{h}}.$$

The convenience of CFTs in two dimension is the decoupling between the holomorphic and anti-holomorphic parts that allows us to treat them separately.
From the Noether's theorem applied to conformal symmetry, we can define a conserved current $J_\mu=T_{\mu\nu}\epsilon^\nu$ and a conserved charge Q at fixed time that within the radial quantization becomes defined at constant radius, i.e. $|z|=cost$:

$$Q=\frac{1}{2\pi i}\oint_{\mathcal{C}}\left(dzT(z)\epsilon(z)+d\overline{z}\overline{T}(\overline{z})\overline{\epsilon}(\overline{z})\right), \tag{1.17}$$

where the contour integral is performed over a circle of fixed radius and the sign conventions are taken positive in clokwise sense.
This conserved charge is the generator of symmetry transformations for a generic operator, in particular for the field $\phi(w,\overline{w})$ the infinitesimal transformations take the form:

$$\delta_{\epsilon\overline{\epsilon}}\phi(w,\overline{w})=\frac{1}{2\pi i}\oint_{\mathcal{C}}\left(dz\left[T(z)\epsilon(z);\phi(w,\overline{w})\right]+d\overline{z}\left[\overline{T}(\overline{z})\overline{\epsilon}(\overline{z});\phi(w,\overline{w})\right]\right). \tag{1.18}$$

The ambiguity in this latter expression is to decide whether w and $\overline{w}$ are inside or outside the contour $\mathcal{C}$. From QFT we know that correlation functions are only defined as a time ordered product, therefore, after the change of coordinates (1.13), in a CFT the time ordering becomes *Radial Ordering* induced through the operator R:

$$R\phi_1(z)\phi_2(w)=\begin{cases}\phi_1(z)\phi_2(w) & |w|<|z|\\ \pm\phi_2(w)\phi_1(z) & |w|>|z|\end{cases} \tag{1.19}$$

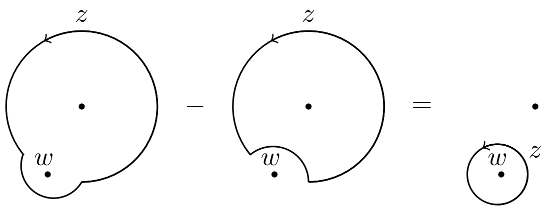

Figure 1.2: Sum on contour integrals.

the signs $\pm$ are respectively for bosonic and fermionic operators. With this definition and the help of Fig.(1.2) we have to interpret the previous integral as:

$$\oint_{\mathcal{C}} dz\,[\phi_1(z), \phi_2(w)] = \oint_{|z|>|w|} dz\phi_1(z)\phi_2(w) - \oint_{|z|<|w|} dz\phi_2(w)\phi_1(z) = \oint_{\mathcal{C}(w)} dz R\left(\phi_1(z)\phi_2(w)\right).$$

Thus we can express the infinitesimal transformation of $\phi(w, \overline{w})$ through:

$$\delta_{\epsilon,\overline{\epsilon}}\phi(w, \overline{w}) = \frac{1}{2\pi i}\oint_{\mathcal{C}(w)} dz\epsilon(z) R\left(T(z)\phi(w, \overline{w})\right) + anti - chiral.$$

Comparig this result with the transformation (1.10) of a primary-field, taking $\overline{\epsilon} = 0$ and employing the Cauchy-Riemann identities we obtain the following relation between the energy-momentum tensor of the theory and this primary-field:

$$R\left(T(z)\phi(w, \overline{w})\right) = \frac{h}{(z-w)^2}\phi(w, \overline{w}) + \frac{1}{z-w}\partial_w\phi(w, \overline{w}) + \text{finite terms} \tag{1.20}$$

This expression is called *Operator Product Expansion* (OPE) between these two fields, which defines an algebraic product structure on the space of quantum fields. Through this new relation we can give an alternative definition of a primary field with conformal dimension $(h, \overline{h})$, like a field whose OPE with the energy-momentum tensor is given by (1.20), and an analogous expression for the anti-holomorphic part.

The *OPE* of the chiral energy-momentum tensor with itself reads:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \text{finite terms}$$

A similar result holds for the anti-chiral part $\overline{T}(\overline{z})$, while the OPE $T(z)\overline{T}(\overline{w})$ contains only non-singular terms. Computing the infinitesimal transformation of the energy-momentum tensor through this latter result we obtain:

$$\delta_\epsilon T(z) = -2\partial\epsilon T(z) - \epsilon(z)\partial T(z) - \frac{c}{12}\partial^3\epsilon.$$

The last term is not present at the classical level, where $c = 0$, but appears when we consider the quantum theory and the quantity $c \neq 0$ represents the so called *Conformal Anomaly.* We conclude that $T(z)$ is a *quasi-primary* field of conformal weight 2.

The exponentiation of this infinitesimal variation to a finite transformation $z \to w(z)$ leads to:

$$T'(w) = \left(\frac{dw}{dz}\right)^{-2}\left[T(z) - \frac{c}{12}\{w; z\}\right], \tag{1.21}$$

where we have introduced the *Schwarzian derivative*:

$$\{w; z\} = \frac{(d^3w/dz^3)}{dw/dz} - \frac{3}{2}\left(\frac{d^2w/dz^2}{dw/dz}\right)^2$$

It is convenient to define the Laurent expansion of energy-momentum tensor in the following way:

$$\begin{aligned} T(z) &= \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \qquad & L_n &= \frac{1}{2\pi i}\oint dz z^{n+1} T(z) \\ \overline{T}(\overline{z}) &= \sum_{n\in\mathbb{Z}} \overline{z}^{-n-2} \overline{L}_n \qquad & \overline{L}_n &= \frac{1}{2\pi i}\oint d\overline{z}\overline{z}^{n+1} \overline{T}(\overline{z}) \end{aligned} \tag{1.22}$$

If we use this expression for the conserved charge and choose a particular conformal transformation $\epsilon(z) = -\sum_n \epsilon_n z^{n+1}$ we obtain:

$$Q = -\sum_{n\in\mathbb{Z}} \epsilon_n L_n$$

namely L_n and $\overline{L}_n$ are the generators of local conformal transformations on the Hilbert space, just like l_n and $\overline{l}_n$ on the function space. They generate the *Virasoro algebra* with central charge c, and the same is true for the anti-holomorphic counterpart. It can be also shown that the holomorphic and anti-holomorphic algebras commute.

$$\left[L_m; \overline{L}_n\right] = 0$$

In particular L_0, L_1, L_{-1} generate $sl(2, \mathcal{C})$, the maximal closed finite subalgebra of Virasoro, and similarly the corresponding anti-holomorphic.

1.4 The CFT Hilbert space

In this section we want to build a representation of the conformal group on the Hilbert space of a CFT.
Let be $|0\rangle$ the vacuum state: it has to be invariant under global conformal transformations. This implies that the state be annihilated by L_{-1}, L_0, L_1 operators and their anti-holomorphic counterpart. This, in turn, can be obtained through the requirement that $T(z)$ and $\overline{T}(\overline{z})$ are well-defined as $z, \overline{z} \to 0$:

$$\begin{aligned} &L_n|0\rangle = 0 \quad and \quad \overline{L}_n|0\rangle = 0 \qquad n \geq -1 \\ &\langle 0|L_m = 0 \quad and \quad \langle 0|\overline{L}_m = 0 \qquad m \leq 1. \end{aligned}$$

On the other hand, the states $L_n|0\rangle$ with $n < -1$ are non-trivial states of Hilbert space.
When primary-fields act on the vacuum state create asymptotic states, labelled by conformal wheigt $(h, \overline{h})$ of field:

$$|h, \overline{h}\rangle = \phi(0, 0)|0\rangle.$$

This state is called *Highest Weight State*. This correspondence $1-1$ between states and operator is called *state-operator correspondence*.

From the OPE between the energy-momentum tensor and the primary field we find the following commutation rules:

$$[L_n, \phi(w,\overline{w})] = \frac{1}{2\pi i}\oint_w dz z^{n+1} T(z)\phi(w,\overline{w}) = h(n+1)w^n\phi(w,\overline{w}) + w^{n+1}\partial\phi(w,\overline{w})$$
$$\left[\overline{L}_n, \phi(w,\overline{w})\right] = \frac{1}{2\pi i}\oint_w dz\overline{z}^{n+1}\overline{T}(\overline{z})\phi(w,\overline{w}) = \overline{h}(n+1)\overline{w}^n\phi(w,\overline{w}) + \overline{w}^{n+1}\overline{\partial}\phi(w,\overline{w})$$

Applying these relations to asymptotic state, we deduce that $|h,\overline{h}\rangle$ is eigenstate of Hamiltonian $H \propto L_0 + \overline{L}_0$ with eigenvalue $(h+\overline{h})$:

$$L_0|h,\overline{h}\rangle = h|h,\overline{h}\rangle \qquad \overline{L}_0|h,\overline{h}\rangle = \overline{h}|h,\overline{h}\rangle,$$

also:

$$L_n|h,\overline{h}\rangle = 0 \quad and \quad \overline{L}_n|h,\overline{h}\rangle = 0 \qquad \forall n > 0.$$

If we expand the holomorphic field in modes, according to (1.15), we obtain:

$$[L_n, \phi_m] = [n(h-1) - m]\,\phi_{n+m}, \tag{1.23}$$

in particular

$$[L_0, \phi_m] = -m\phi_m.$$

Therefore ϕ_m act as raising and lowering operators for the eigenstates of L_0. Each application of ϕ_{-m} $(m > 0)$ increases the conformal dimension of the state by m. Also L_m $(m > 0)$ acts on the same way, i.e.:

$$[L_0, L_{-m}] = mL_{-m}.$$

In order to construct the whole Hibert space, we act with L_{-m} on the Highest Weight State, so we obtain the excited states:

$$L_{-k_1}L_{-k_2}...L_{-k_n}|h,\overline{h}\rangle \qquad with \qquad k_1 \geq k_2 \geq ... \geq k_n$$

called *descendants* of asymptotic state $|h,\overline{h}\rangle$. The order of k_i is conventional. We can label these states by their eigenvalues respect to L_0, in particular the eingenvalue of previous descendant state is:

$$h' = h + k_1 + k_2 + ... + k_n = h + N$$

and N is called the *level of the descendant*. The number of states with level N is simply the partitions number $p(N)$ of integer N.
Since we have to construct the descend states of asymptotic state $|h\rangle$ through application of conformal algebra generators L_m, this set of states is a subset of Hilbert space closed under Virasoro algebra called *Verma module* $V_{h,c}$, thus forms a its representation.
In a Verma module, depending on the combination (h,c), there can be states of vanishing or negative norm. For Unitary theories, the later should be absent and vanishing norm states should be removed.
In order to find the null states we define for each level N a matrix $M_N(h,c)$, where the entries are defined as the product of states in the Verma module at fixed level, and we study its determinant, called *Kac-determinant*. We discover that if the Kac-determinant at any lever is *negative*, then there exist negative norm states at that level and the representation of Virasoro algebra is not unitary, while if the Kac-determinant is greater or equal to zero we find unitary representations under certain conditions on the parameters (h,c).
A Unitary CFT with a finite number of Virasoro algebra representations is called *Rational CFTs*

or *RCFTs*.
Application of CFTs to statistical models don't require the Unitarity as necessary condition. Therefore we can to build theories whit central charges c and a finite set of highest weights: this theories are called *minimal models*.

1.4.1 Correlation functions

At the quantum level, the correlation functions are the main object of study. Thus, let's define *correlation function* as the vacuum expectation values of the R-ordered products of field operators. Since we are considering theories with conformal invariance, whose fields transform suitably under these transformations, the correlation functions of the theory have to be invariant under $SL(2,\mathbb{C})$,i.e. the conformal global group.
Let be given the *n-points* correlation function between n primary-fields expressed in terms of holomorphic and anti-holomorphic coordinates:

$$G^{(n)}(z_i,\overline{z}_1) = \langle R\left(\phi_1(z_1,\overline{z}_1)\phi_2(z_2,\overline{z}_2)...\phi_n(z_n,\overline{z}_n)\right)\rangle.$$

Under the conformal map $z \to w(z)$, the conformal transformations (1.10) for these primary-fields with conformal dimension $(h_i,\overline{h}_i)$, impose that:

$$\langle R\left(\phi_1(w_1,\overline{w}_1)\phi_2(w_2,\overline{w}_2)...\phi_n(w_n,\overline{w}_n)\right)\rangle = \prod_{i=1}^{n}\left(\frac{dw}{dz}\right)_{w=w_i}^{-h_i}\left(\frac{d\overline{w}}{d\overline{z}}\right)_{\overline{w}=\overline{w}_i}^{-\overline{h}_i}\langle R\left(\phi_1(z_1,\overline{z}_1)\phi_2(z_2,\overline{z}_2)...\phi_n(z_n,\overline{z}_n)\right)\rangle.$$

Let's focus our attention on the two-point function. The translations and rotation invariances impose that it must depend on the modulus of the difference of the coordinates:

$$\langle\phi_1(z,\overline{z})\phi_2(w,\overline{w})\rangle = C_{12}\left(z-w\right)^{-(h_1+h_2)}\left(\overline{z}-\overline{w}\right)^{-(\overline{h}_1+\overline{h}_2)}.$$

The invariance under special conformal transformations require that $h = h_1 = h_2$ and $\overline{h} = \overline{h}_1 = \overline{h}_2$:

$$\langle\phi_1(z,\overline{z})\phi_2(w,\overline{w})\rangle = \begin{cases} \frac{C_{12}}{(z-w)^{2h}(\overline{z}-\overline{w})^{2\overline{h}}} & \text{if}(h_1,\overline{h}_1) = (h_2,\overline{h}_2) \\ 0 & \text{if}(h_1,\overline{h}_1) \neq (h_2,\overline{h}_2) \end{cases} \tag{1.24}$$

Therefore $SL(2,\mathbb{C})/\mathbb{Z}_2$ conformal symmetry fixes the two-point function of two quasi-primary field up to a constant that can be rescaled by field redefinition, and tells us that the fields are correlated if and only if they have the same conformal dimension.
Let us now consider the three-point function. The $SL(2,\mathbb{C})/\mathbb{Z}_2$ conformal symmetry fixes its form up to a constant to:

$$\langle\phi_1(z_1,\overline{z}_1)\phi_2(z_2,\overline{z}_2)\phi_3(z_3,\overline{z}_3)\rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3}z_{23}^{h_2+h_3-h_1}z_{13}^{h_1+h_3-h_2}}\cdot\frac{1}{\overline{z}_{12}^{\overline{h}_1+\overline{h}_2-\overline{h}_3}\overline{z}_{23}^{\overline{h}_2+\overline{h}_3-\overline{h}_1}\overline{z}_{13}^{\overline{h}_1+\overline{h}_3-\overline{h}_2}}, \tag{1.25}$$

where $z_{ij} = (z_i - z_j)$ and $\overline{z}_{ij} = (\overline{z}_i - \overline{z}_j)$, and the *structure constants* C_{123} depend of the theory. The n-points correlation functions, with $n \geq 4$, can be written in terms of three independent *anharmonic ratio* η:

$$\eta = \frac{z_{ij}z_{kl}}{z_{ik}z_{jl}} \qquad z_{ij} = z_i - z_j.$$

The explicit form of these dependence is completely fixed by conformal invariance in terms of the 3-points functions, that encode the whole dynamics.
The generic form of the two and three-point functions allows us to extract the general form of OPE between two quasi-primary fields in terms of other quasi-primary fields.
Let us consider a primary-field $\phi(w)$ and its descendant fields obtained by the action of Virasoro generators $L_{-n}\phi(w)$, with $n > 0$. Those are just the fields appearing in the *OPE*:

$$T(z)\phi(w) = \sum_{n\geq 0} (z-w)^{n-2} L_{-n}\phi(w).$$

Let us focus on the holomorphic fields $\phi(w)$ of the theory and let us introduce the *Normal Ordering* prescription on the product of two fields:

$$N(\phi_1\phi_2)(w) =: \phi_1\phi_2 : (w),$$

which places the annihilation operators on the right. If the conformal dimension of $\phi(w)$ is integer for the first values of n we have the following descendants:

$$L_0\phi(w) = h\phi(w) \qquad L_{-1}\phi(w) = \partial\phi(w) \qquad L_{-2}\phi(w) = N(\phi T)(w).$$

Using the *OPE* just defined we can to write the correlator of a descendant field $L_{-n}\phi$ with other primary-fields applying a differential operator in the correlator involving the primary-field $\phi(w)$ in the following way:

$$\langle L_{-n}\phi(w)\phi_1(w_1)...\phi_N(w_N)\rangle = \mathcal{L}_{-n}\langle\phi(w)\phi_1(w_1)...\phi_N(w_N)\rangle$$

with

$$\mathcal{L}_{-n} = \sum_{i=1}^{N}\left(\frac{(n-1)h_i}{(w_i-w)^n} - \frac{1}{(w_i-w)^{n-1}}\partial_{w_i}\right).$$

Another way to transfer the conformal symmetry to quantum level is to apply the path-integral quantization. Due to the fact that the local symmetry makes equivalent infinity integrational path, divergent integrals appear. In order to solve this problem we introduce on the theory a *gauge fixing* that kills the equivalents paths but it breaks action invariance. If the variation of the gauged action is a linear combination of fields, then the classical-quantum symmetry transport theorem assures us conformal symmetry conservation at quantum level whether the Ward identities are satisfied:

$$\begin{aligned}
\partial_\mu\langle T^\mu_\nu X\rangle &= -\sum_i \delta(\mathbf{x}-\mathbf{x}_i)\frac{\partial}{\partial x_i^\nu}\langle X\rangle \\
\langle\left(T^{\rho\nu} - T^{\nu\rho}\right)X\rangle &= -i\sum_i \delta(\mathbf{x}-\mathbf{x}_i)S_i^{\nu\rho}\langle X\rangle \\
\langle T^\mu_\mu X\rangle &= -\sum_i \delta(\mathbf{x}-\mathbf{x}_i)\Delta_i\langle X\rangle
\end{aligned} \tag{1.26}$$

where X is the product of n local fields at the coordinates $\mathbf{x}_i$, $i = 1, ...n$.

1.5 Examples of CFTs

In the previous section we introduced CFTs through operator algebra and OPEs for the fields of theory, now we want to establish a link with the usual Lagrangian formalism of quantum field theory, that naturally appear in string theories. Therefore in this section we will treat some simple examples of CFTs in terms of Lagrangian formalism.

1.5.1 The free Boson

Let's start with a real massless scalar field $\phi(x, y)$ defined in a bidimensional space with Minkowskian signature. The action for such a theory takes the following form:

$$S = \frac{1}{2} g \int d^2 x \partial_\mu \phi \partial^\mu \phi,$$

where g is some normalisation constant. Since in this theory there is no mass term setting a scale, we expect this action to be conformally invariant.
In order to simplify our treatment, we consider the map to complex coordinates $(z, \overline{z})$ previously defined. The action in the new coordinates take the form:

$$S = \frac{1}{2} g \int dz d\overline{z} \partial \phi \cdot \overline{\partial} \phi, \tag{1.27}$$

where we defined $\partial = \partial_z$ and $\overline{\partial} = \partial_{\overline{z}}$. The equation of motion for this action is obtained by varying S respect to the field:

$$\partial \overline{\partial} \phi(z, \overline{z}) = 0.$$

This expression tell us that:

$$\begin{aligned} j(z) &= i\partial\phi(z, \overline{z}) && \text{is a chiral field} \\ \overline{j}(\overline{z}) &= i\overline{\partial}\phi(z, \overline{z}) && \text{is a anti-chiral field.} \end{aligned}$$

Acting with a conformal transformation $z, \overline{z} \to w(z), \overline{w}(\overline{z})$ on the field $\phi(z, \overline{z})$ we notice that the action is invariant iff the conformal dimension of $\phi(z, \overline{z})$ is $(h, \overline{h}) = (0, 0)$, then the fields $j(z), \overline{j}(\overline{z})$ are primary-fields with conformal dimension respectively $(1, 0)$ and $(0, 1)$.
Remembering that the propagator is the Green function of the equation of motion of the theory, we compute it for the previous action of the free boson and we obtain the following solution:

$$K(z, \overline{z}, w, \overline{w}) = \langle \phi(z, \overline{z}) \phi(w, \overline{w}) \rangle = -\frac{1}{4\pi g} ln|z - w|^2 + const \tag{1.28}$$

Now we consider only the holomorphic part of (1.28) and, differentiating it respect to z and w, we find:

$$\langle j(z) j(w) \rangle = \frac{1}{4\pi g} \frac{1}{(z - w)^2}.$$

This can be interpreted as the OPE of field $j(z)$ with itself:

$$j(z) j(w) \sim \frac{1}{4\pi g} \frac{1}{(z - w)^2} + ... \tag{1.29}$$

A similar expression can be found for the anti-chiral field $\overline{j}(\overline{z})$. The (1.29) reflect the bosonic character of field: the exchange of two factors leaves the correlator invariant.
We define the energy-momentum tensor by varying the action (1.27) respect to metric tensor and we find the following result:

$$T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial_\nu \phi - \eta_{\mu\nu} \mathcal{L} = g \left(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \eta_{\mu\nu} \partial_\alpha \phi \partial^\alpha \phi \right)$$

that in complex coordinates has holomorphic part:

$$T(z) := -2\pi T_{zz} = -2\pi g : \partial\phi(z) \partial\phi(z) : .$$

Applying the Wick theorem to OPE between $T(z)$ and $\partial\phi(w)$:

$$
\begin{aligned}
T(z)\partial\phi(w) &= -2\pi g : \partial\phi(z)\partial\phi(z) : \partial\phi(w) = \\
&= -4\pi g : \partial\phi(z)\underline{\partial\phi(z) : \partial\phi(w)} \sim \frac{\partial\phi(z)}{(z-w)^2} + ... \\
&\sim \frac{\partial\phi(w)}{(z-w)^2} + \frac{\partial_w^2\phi(w)}{z-w} + ...
\end{aligned}
\tag{1.30}
$$

where in the last line we have expanded $\partial\phi(z)$ around to w.
The (1.30) shows that $\partial\phi(z)$, hence $j(z)$, is a primary-field having conformal dimension $(1,0)$.
Using the Wick theorem also to compute the OPE of $T(z)$ with itself:

$$
\begin{aligned}
T(z)T(w) = 4\pi^2 g^2 &: \partial\phi(z)\partial\phi(z) :: \partial\phi(w)\partial\phi(w) : \\
&\sim \frac{1/2}{(z-w)^4} - 4\pi g \frac{: \partial\phi(z)\partial\phi(w) :}{(z-w)^2} + ... \\
&\sim \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + ...
\end{aligned}
$$

Then $T(z)$ is not a primary field because of the anomalous first term, moreover this tell us that the CFT of the free boson has central charge $c = 1$, i.e. the modes of $T(z)$, and their correspondent counterpart, generate the Virasoro algebra with this central charge.

1.5.2 Bosonic quantization

The map (1.13) assures us that the surfaces plane and cylinder are related by a conformal transformation. Exploiting this link, we would like quantize the free bosonic system on the cylinder. The bosonic field $\phi(x,t)$ defined on the cylinder of circumference L respect the periodicity propriety:

$$\phi(x,t) = \phi(x+L,t),$$

thus the expansion on Fourier modes of field is:

$$\phi(x,t) = \sum_n e^{2\pi i n x/L}\phi_n(t),$$

where the coefficients are:

$$\phi_n(t) = \frac{1}{L}\int dx e^{-2\pi i n x/L}\phi(x,t).$$

The Lagrangian in terms of these coefficients is:

$$\mathcal{L} = \frac{1}{2} g L \sum_n \left[\dot{\phi}_n \dot{\phi}_{-n} - \left(\frac{2\pi n}{2}\right)^2 \phi_n \phi_{-n} \right]. \tag{1.31}$$

The momentum conjugate to ϕ_n is:

$$\pi_n = gL\dot{\phi}_{-n} \qquad\qquad [\phi_m;\pi_n] = i\delta_{mn}$$

and the Hamiltonian is:

$$H = \frac{1}{2gL}\sum_n \left[\pi_n\pi_{-n} + (2\pi n g)^2\phi_n\phi_{-n}\right]$$

We notice that $\phi_n^\dagger = \phi_{-n}$ and $\pi_n^\dagger = \pi_{-n}$. H is the sum of decoupled harmonic oscillators with frequencies $\omega_n = 2\pi|n|/L$.
Through the usual procedure, we consider the modes of field and momentum as operator and define the usual creation and annihilation operators:

$$a_n = \frac{1}{\sqrt{4\pi g}} \begin{cases} -i\left(2\pi g|n|\phi_n + i\pi_n\right) & n > 0 \\ i\left(2\pi g|n|\phi_n - i\pi_n\right) & n < 0 \end{cases}$$

and

$$\overline{a}_n = \frac{1}{\sqrt{4\pi g}} \begin{cases} -i\left(2\pi g|n|\phi_{-n} + i\pi_{-n}\right) & n > 0 \\ i\left(2\pi g|n|\phi_{-n} - i\pi_{-n}\right) & n < 0 \end{cases}$$

with the following commutation rules:

$$[a_n; a_m] = [\overline{a}_n; \overline{a}_m] = n\delta_{nm} \qquad\qquad [a_n; \overline{a}_m] = 0.$$

The zero mode needs to be treated separately.
The operatorial form of Hamiltonian is:

$$H = \frac{1}{2gL}\pi_0^2 + \frac{2\pi}{L}\sum_{n\neq 0}\left(a_{-n}a_n + \overline{a}_{-n}\overline{a}_{-n}\right).$$

The commutator

$$[H; a_{-m}] = \frac{2\pi}{L} m a_{-m}.$$

tell us that the operator a_{-m} $(m > 0)$ applied on the eigenstates of H with eigenvalue E produce new eigenstates of H with energy $E + 2\pi m/L$. We write the modes expansion of field, making clear the time dependence, in terms of constant operators:

$$\phi(x,t) = \phi_0 + \frac{1}{gL}\pi_0 t + \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}\left(a_n e^{2\pi i n(x-t)/L} - \overline{a}_n e^{2\pi i n(x+t)/L}\right)$$

Performing a Wick rotation $t \to -i\tau$ and using the conformal coordinates $z = e^{2\pi(\tau - ix)/L}$ and $\overline{z} = e^{2\pi(\tau+ix)/L}$ we obtain:

$$\phi(z,\overline{z}) = \phi_0 - \frac{i}{4\pi g}\pi_0 ln(z\overline{z}) + \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}\left(a_n z^{-n} + \overline{a}_n \overline{z}^{-n}\right).$$

$\phi(z,\overline{z})$ is not a primary-field, but $\partial\phi$ and $\overline{\partial\phi}$ are. We write the expansion of holomorphic field:

$$j(z) = i\partial\phi(z) = \frac{1}{\sqrt{4\pi g}}\sum_n a_n z^{-n-1} \tag{1.32}$$

where the zero mode is $a_0 = \overline{a}_0 = \frac{\pi_0}{\sqrt{4\pi g}}$. Another primary-field that can be to build using the non-conformal field $\phi(z,\overline{z})$ is the *vertex operator*:

$$V(z,\overline{z}) =: e^{i\alpha\phi(z,\overline{z})} : .$$

Through the modes expansion of $\phi(z)$, the Fourier expansion of the exponential operators and the Wick theorem we can compute the OPE of the vertex operator with the energy-momentum tensor:

$$
\begin{aligned}
T(z)V_\alpha(w,\overline{w}) = -2\pi g \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!} : \partial\phi(z)\partial\phi(z) :: \phi(w,\overline{w})^n : \\
\sim -\frac{1}{8\pi g}\frac{1}{(z-w)^2}\sum_{n=2}^{\infty} : \phi(w)^{n-2} : + \frac{1}{z-w}\sum_{n=1}^{\infty}\frac{(i\alpha)^n}{n!} n : \partial\phi(z)\phi(w,\overline{w})^{n-1} : \\
\sim \frac{\alpha^2}{8\pi g}\frac{V_{\alpha(w,\overline{w})}}{(z-w)^2} + \frac{\partial_w V_\alpha(w,\overline{w})}{z-w}
\end{aligned}
\tag{1.33}
$$

The OPE with $\overline{T}$ has exactly the same form. This proves that V_α are primary-field with conformal dimension $\left(h_\alpha, \overline{h}_\alpha\right) = \left(\frac{\alpha^2}{8\pi g}, \frac{\alpha^2}{8\pi g}\right)$.
The action (1.27) is invariant under translations $\phi(z,\overline{z}) \to \phi(z,\overline{z}) + A$. This invariance must be respected by correlators $\langle V_\alpha V_\beta \rangle$, which implies the condition $\alpha + \beta = 0$. Remembering the previous discussion about two-point function of primary-fields, we find:

$$
\langle V_\alpha(z,\overline{z}) V_{-\alpha}(w,\overline{w})\rangle = \frac{1}{(z-w)^{\alpha^2/4\pi g}(\overline{z}-\overline{w})^{\alpha^2/4\pi g}}.
$$

We conclude this section building the Hilbert space of the theory. The independence of Hamiltonian on ϕ_0 implies that the eigenvalues of π_0 are a good quantum number which allows us to label sets of eigenstates of H. Since π_0 commute with all a_n and $\overline{a}_n$, these operators cannot change its value, then the Hilbert space will be construct upon a one-parameter family of vacua $|\alpha\rangle$ labelled by their continuous eigenvalues of a_0:

$$
a_n|\alpha\rangle = \overline{a}_n|\alpha\rangle = 0 \quad (n>0) \qquad a_0|\alpha\rangle = \overline{a}_0|\alpha\rangle = \alpha|\alpha\rangle. \tag{1.34}
$$

The energy-momentum tensor in terms of modes of field is:

$$
T(z) = -2\pi g : \partial\phi\partial\phi := \frac{1}{2}\sum_{n,m\in Z} z^{-n-m-2} : a_n a_m :
$$

Using its expansion (1.22), we find the following identification between the modes of field with the modes $L_n, \overline{L}_n$:

$$
L_n = \frac{1}{2}\sum_{m\in Z} : a_{n-m}a_m : \qquad L_0 = \sum_{n>0} a_{-n}a_n + \frac{1}{2}a_0^2 \tag{1.35}
$$

with $n \neq 0$, and similarly solutions for the anti-holomorphic counterparts. We have already explained above that the effect of a_m on the conformal dimension of the states is the same that L_m. The conformal dimension of the state $|\alpha\rangle$ is $\alpha^2/2$ (setting $g = 1/4\pi$), and the states with higher weight are obtained by acting through the creation operators on the vacua:

$$
a_{-1}^{n_1}a_{-2}^{n_2}...\overline{a}_{-1}^{\overline{n}_1}\overline{a}_{-2}^{\overline{n}_2}...|\alpha\rangle, \qquad (n_i, \overline{n}_j \geq 0) \tag{1.36}
$$

and they are eigenstates of $L_0, \overline{L}_0$ with eigenvalues:

$$
h = \frac{1}{2}\alpha^2 + \sum_j j n_j \qquad \overline{h} = \frac{1}{2}\alpha^2 + \sum_j j\overline{n}_j.
$$

The vacua $|\alpha\rangle$ of the theory can be obtained to start by an absolute vacuum $|0\rangle$ and acting on it with the vertex operators $|\alpha\rangle = V_\alpha(0,0)|0\rangle$.

1.5.3 Compactified Boson

The invariance of Hamiltonian under translation $\phi \to \phi + const$ allows us to restrict the domain of field variation to a circle with radius R, i.e. the identification:

$$\phi + 2\pi R = \phi.$$

This brings two modification to our previous analysis. First, the center of mass π_0 cannot longer assume arbitrary value: in order to the vertex operator V_α be well-defined , its eigenvalues must be integer multiples of $1/R$. Second, the boundary condition take the form:

$$\phi(x+L,t) = \phi(x,t) + 2\pi m R$$

where m, called *winding number*, counts the number of times that ϕ turn around the cylinder circumference.
These two considerations lead to following modification of previous mode expansion :

$$\phi(x,t) = \phi_0 + \frac{n}{gRL}t + \frac{2\pi R m}{L}x + \frac{i}{\sqrt{4\pi g}}\sum_{k\neq 0}\frac{1}{k}\left(a_k e^{2\pi i k\frac{(x-t)}{L}} - \overline{a}_{-k}e^{-2\pi i k\frac{(x-t)}{L}}\right)$$

that in terms of complex coordinates becomes:

$$\phi(z,\overline{z}) = \phi_0 - i\left(\frac{n}{4pigR} + \frac{1}{2}mR\right) lnz + \frac{i}{\sqrt{4\pi g}}\sum_{k\neq 0}\frac{1}{k}a_k z^{-k} + anti - holomorphic$$

The holomorphic derivative has the expansion:

$$i\partial\phi(z) = \left(\frac{n}{4\pi gR} + \frac{1}{2}mR\right)\frac{1}{z} + \frac{1}{\sqrt{4\pi g}}\sum_{k\neq 0} a_k z^{-k-1}.$$

Inserting these results into the definition of energy-momentum tensor and writing its mode expansion, we find the expression of holomorphic and anti-holomorphic zero mode in terms of modes of field:

$$\begin{aligned} L_0 &= \sum_{n>0} a_{-n}a_n + 2\pi g\left(\frac{n}{4\pi gR} + \frac{1}{2}mR\right)^2 \\ \overline{L}_0 &= \sum_{n>0} \overline{a}_{-n}\overline{a}_n + 2\pi g\left(\frac{n}{4\pi gR} - \frac{1}{2}mR\right)^2. \end{aligned}$$

The last two terms of the latter results correspond to conformal dimension of possible vacuum state of the theory:

$$h_{m,n} = 2\pi g\left(\frac{n}{4\pi gR} + \frac{1}{2}mR\right)^2.$$

These vacuum state can be created through the action of vertex operators V_k, with momentum quantized on the circle $k = \frac{n}{R}$, upon the absolute vacuum $|0\rangle$. Thus we obtain infinite states of highest weight labelled by the momentum n and winding number m $|n,m\rangle$.

1.5.4 The free Fermions

A second simple example of CFT is given by a free Majorana fermion in two-dimensional Minkowski space:

$$S = \frac{1}{2}g \int d^2x \Psi^+ \gamma^0 \gamma^\mu \partial_\mu \Psi$$

where g is a normalisation constant, $\left\{\gamma^0; \gamma^\mu\right\}$ are the bidimensional γ-matrix that satisfy to Clifford algebra. We perform a Wick rotation and define the new complex coordinates $z, \overline{z}$ in the usual way. The two-components spinor Ψ is written in terms of two real fields:

$$\Psi = \begin{pmatrix} \psi(z,\overline{z}) \\ \overline{\psi}(z,\overline{z}) \end{pmatrix} \qquad \overline{\Psi} = \Psi^+ \gamma^0,$$

in terms of which the action becomes:

$$S = \frac{1}{2}g \int dz d\overline{z} \left(\psi \overline{\partial} \psi + \overline{\psi} \partial \overline{\psi}\right). \tag{1.37}$$

Through usual variational method we find the equation of motion for the real fields:

$$\partial \overline{\psi} = \overline{\partial} \psi = 0$$

thus we can conclude that $\psi(z)$ is a chiral field and $\overline{\psi}$ is an anti-chiral field.
Proceeding as for the example of the free boson, we see that the action (1.37) is invariant under conformal transformations if and only if the fields $\psi(z)$ and $\overline{\psi}(\overline{z})$ are primary with conformal dimensions respectively $\left(\frac{1}{2}, 0\right)$ and $\left(0, \frac{1}{2}\right)$.
The Green function of the equation of motion of the theory give us the form of propagators for the fields:

$$\begin{aligned}\langle \psi(z)\psi(w)\rangle &= \frac{1}{2\pi g}\frac{1}{z-w} \\ \langle \overline{\psi}(\overline{z})\overline{\psi}(\overline{w})\rangle &= \frac{1}{2\pi g}\frac{1}{\overline{z}-\overline{w}}\end{aligned}$$

from which we deduce the *OPE* of chiral fermion with itself:

$$\psi(z)\psi(w) \sim \frac{1}{2\pi g}\frac{1}{z-w}. \tag{1.38}$$

This result reflects the anticommuting character of the field.
The energy-momentum tensor relating to action is:

$$T^{zz} = 2g\overline{\psi}\overline{\partial\psi} \qquad T^{\overline{z}\overline{z}} = 2g\psi\partial\psi \qquad T^{z\overline{z}} = -2g\psi\overline{\partial}\psi.$$

We notice that the energy-momentum tensor is symmetric under the equation of motion. We focus our attention on the holomorphic component:

$$T(z) = -2\pi T_{zz} = -\frac{1}{2}\pi T^{\overline{z}\overline{z}} = -\pi g : \psi(z)\partial\psi(z) : . \tag{1.39}$$

Using the Wick theorem we obtain its *OPE* with the holomorphic fermionic field:

$$\begin{aligned} &T(z)\psi(w) = -\pi g : \psi(z)\partial\psi(z) : \psi(w) \\ &\sim \frac{1}{2}\frac{\partial\psi(z)}{z-w} + \frac{1}{2}\frac{\psi(z)}{(z-w)^2} \sim \frac{1}{2}\frac{\psi(w)}{(z-w)^2} + \frac{\partial\psi(w)}{z-w}\end{aligned}$$

that is consistent with the conformal dimension of field ψ. Now we write the *OPE* of field $T(z)$ with itself:

$$\begin{aligned} T(z)T(w) &= \pi^2 g^2 : \psi(z)\partial\psi(z) :: \psi(w)\partial\psi(w) : \\ &\sim \frac{1}{4}\frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \end{aligned}$$

Again the first terms tell us that the energy-momentum tensor is not a primary-field and the central charge of the CFT given by a real free fermion is $c = \frac{1}{2}$.

1.5.5 Fermionic quantization

As in the previous bosonic case, now we want to quantize the free fermionic system on the cylinder.
Let's consider a cylinder of circumference L, the fermionic character allows us to distinguish between two periodicity conditions compatible with the action (1.37):

$$\begin{aligned} \psi(x+L,t) &= +\psi(x,t) \qquad && \text{Ramond Sector} \\ \psi(x+L,t) &= -\psi(x,t) \qquad && \text{Neveu-Schwarz Sector.} \end{aligned} \tag{1.40}$$

We write the real fermionic field through the canonical quantization in terms of creation and annihilations operators at fixed time $t = 0$:

$$\psi(x) = \sqrt{\frac{2\pi}{L}} \sum_k \psi_k e^{2\pi i k x/L} \tag{1.41}$$

with the anti-commutation rule $\{\psi_k; \psi_q\} = \delta_{k+q,0}$. In the periodic case (R-sector) the label k on the modes expansion takes integer values, whereas in the anti-periodic case (NS-sector) it takes semi-integer values ($k \in \mathbb{Z} + \frac{1}{2}$).
The Hamiltonian can be written in terms of fermionic modes:

$$H = \sum_{k>0} \omega_k \psi_{-k}\psi_k + E_0 \qquad \omega_k = \frac{2\pi|k|}{L}$$

E_0 is the vacuum energy of the system. The Hamiltonian related to anti-holomorphic field has the same form: we have to consider their sum for the complete theory.
Introducing the temporal evolution up the modes ψ_k and performing a Wick rotation to Euclidean time ($t \to -i\tau$), the complete expansion of holomorphic fermionic field is:

$$\psi(x,\tau) = \sqrt{\frac{2\pi}{L}} \sum_k \psi_k e^{-2\pi k(\tau - ix)/L}.$$

In the R-sector there exist the zero mode ψ_0 that does not appear in H, thus it leads to a degeneracy of the vacuum state $|0\rangle$ of the theory. By anti-commutation rules follows $b_0^2 = \frac{1}{2}$.
Now we perform the map (1.13), since the conformal dimension of field ψ is $1/2$, this transformation will affect the field itself:

$$\psi_{cil}(z) = \left(\frac{dz}{dw}\right)^{1/2} \psi_{pl}(z) = \sqrt{\frac{2\pi z}{L}} \psi_{pl}(z).$$

and the new mode expansion is:

$$\psi(z) = \sum_k \psi_k z^{-k-1/2}.$$

The factor $\sqrt{z}$ exchange the periodicity condition (1.40) after a rotation of 2π around the origin:

$$\begin{aligned} \psi(e^{2\pi i}z) &= -\psi(z) \qquad (k \in Z) \qquad \text{Ramond} \\ \psi(e^{2\pi i}z) &= +\psi(x) \qquad (k \in Z + \frac{1}{2}) \quad \text{Neveu-Schwarz.} \end{aligned} \tag{1.42}$$

Let's start computing the two-point function in the NS-sector from the mode expansion:

$$\begin{aligned} \langle \psi(z)\psi(w) \rangle &= \left\langle \sum_{n=1/2}^{\infty} \sum_{m=-1/2}^{-\infty} \psi_n z^{-n-1/2} \psi_m z^{-m-1/2} \right\rangle \\ &= \sum_{n=1/2}^{\infty} \sum_{m=-1/2}^{-\infty} z^{-n-1/2} w^{-m-1/2} \langle \psi_n \psi_m \rangle \\ &= \sum_{n=1/2}^{\infty} z^{-n-1/2} w^{n-1/2} \\ &= \sum_{m=0}^{\infty} \frac{1}{z} \left(\frac{w}{z} \right)^m = \frac{1}{z-w}. \end{aligned} \tag{1.43}$$

The result in the R-sector is different, we find:

$$\begin{aligned} \langle \psi(z)\psi(w) \rangle_A &= \left\langle \sum_{n=0}^{\infty} \sum_{m=0}^{-\infty} \psi_n z^{-n-1/2} \psi_m z^{-m-1/2} \right\rangle_A = \\ &= \frac{1}{2} \frac{1}{\sqrt{zw}} + \sum_{n=1}^{\infty} \sum_{m=-1}^{-\infty} z^{-n-1/2} w^{-m-1/2} \langle \psi_n \psi_m \rangle = \\ &= \frac{1}{2} \frac{1}{\sqrt{zw}} + \sum_{n=1}^{\infty} z^{-n-1/2} w^{n-1/2} = \\ &= \frac{1}{\sqrt{zw}} \left(\frac{1}{2} + \frac{w}{z-w} \right) = \frac{\frac{1}{2} \left(\sqrt{\frac{z}{w}} + \sqrt{\frac{w}{z}} \right)}{z-w}. \end{aligned} \tag{1.44}$$

The fact that the results coincide in the limit $w \to z$ tell us that the theory is independent of the periodicity condition at short distances.
We would like to compute the vacuum energy density in the two sector in the plane. Using the normal-ordering prescription:

$$\langle T(z) \rangle = \frac{1}{2} lim_{\epsilon \to 0} \left[-\langle \psi(z+\epsilon) \partial \psi(z) \rangle + \frac{1}{\epsilon^2} \right]$$

we obtain:

$$\langle T(z) \rangle = \begin{cases} 0 & \text{NS-sector} \\ \frac{1}{16z^2} & \text{R-sector} \end{cases}$$

We can now turn back to the cylinder and compute the vacuum expectation value of the stress-energy tensor also in this space. Through the equation (1.21) of transformation under the map (1.13) we find:

$$T_{Cil}(w) = \left(\frac{2\pi}{L} \right)^2 \left[z^2 T(z) - \frac{c}{24} \right]. \tag{1.45}$$

Now we want to find the expressions for the conformal generators L_n in terms of mode operator for the two type of periodicity conditions on the plane and on the cylinder. The expression for the energy-momentum tensor leads to:

$$
\begin{aligned}
T_{pla}(z) = \sum_n L_n z^{-n-2} &= \frac{1}{2}\sum_{k,q}\left(k+\frac{1}{2}\right) z^{-q-\frac{1}{2}} z^{-k-\frac{3}{2}} : \psi_q \psi_k : \\
&= \frac{1}{2}\sum_{k,n}\left(k+\frac{1}{2}\right) z^{-n-2} : \psi_{n-k}\psi_k :
\end{aligned}
$$

Therefore we have the following expression for the conformal generators on the plane:

$$
L_n = \frac{1}{2}\sum_k \left(k+\frac{1}{2}\right) : \psi_{n-k}\psi_k : . \tag{1.46}
$$

Since L_0 is the coefficient of $1/z^2$ in the mode expansion, the non-zero expectation value in the Ramond-sector implies a constant term on its expression:

$$
\begin{aligned}
L_0 &= \sum_{k>0} k\psi_{-k}\psi_k && \text{NS-sector}\left(k \in \mathbb{Z}+\frac{1}{2}\right) \\
L_0 &= \sum_{k>0} k\psi_{-k}\psi_k + \frac{1}{16} && \text{R-sector}\,(k \in \mathbb{Z})\,.
\end{aligned}
$$

Applying these results to energy-momentum tensor on the cylinder using the (1.45), we obtain the following vacuum expectation values:

$$
\langle T(w)\rangle_{Cyl} = \begin{cases} -\frac{1}{48}\left(\frac{2\pi}{L}\right)^2 & \text{NS-sector} \\ \frac{1}{24}\left(\frac{2\pi}{L}\right)^2 & \text{R-sector} \end{cases}
$$

The mode expansion of energy-momentum tensor becomes:

$$
T_{Cil}(w) = \left(\frac{2\pi}{L}\right)^2 \sum_{n\in\mathbb{Z}} L_n z^{-n} - \frac{c}{24} = \left(\frac{2\pi}{L}\right)^2 \sum_{n\in\mathbb{Z}} \left(L_n - \frac{c}{24}\delta_{n0}\right) e^{-2\pi n w/L}.
$$

In particular the zero modes on the two sectors are shifted by a constant respect to plane:

$$
(L_0)_{Cil} = \left[L_0 - \frac{c}{24}\right] = \begin{cases} \sum_{k>0} k\psi_{-k}\psi_k - \frac{1}{48} & \text{NS-sector} \\ \sum_{k>0} k\psi_{-k}\psi_k + \frac{1}{24} & \text{R-sector} \end{cases}
$$

The vacuum expectation values fix the constants added to the Hamiltonian (E_0). The cylinder Hamiltonian becomes:

$$
H = \left(\frac{2\pi}{L}\right)\left[(L_0)_{Cyl} + (\overline{L}_0)_{Cyl}\right] = \left(\frac{2\pi}{L}\right)\left[(L_0) + (\overline{L}_0) - \frac{c}{12}\right]
$$

Considering only holomorphic part of Hamiltonian, the additive constants in two sectors are:

$$
H_R = \left(\frac{2\pi}{L}\right)\left[(L_0) - \frac{c}{24}\right] = \left(\frac{2\pi}{L}\right)\begin{cases} \sum_{k>0} k\psi_{-k}\psi_k - \frac{1}{48} & \text{NS-sector} \\ \sum_{k>0} k\psi_{-k}\psi_k + \frac{1}{24} & \text{R-sector.} \end{cases}
$$

The Hilbert space of the free fermion theory on the NS-sector on the plane is built to acting with the creation operator ψ_{-k} and $\overline{\psi}_{-k}$, $k \in \mathbb{Z}^+ + \frac{1}{2}$ up a vacuum state $|0\rangle$. In particular, due the Fermi-statistics, each mode $\psi_{-k}, \overline{\psi}_{-k}$ can only appear once. For the chiral sector the states:

$$|0\rangle, \qquad \psi_{-1/2}|0\rangle, \qquad \psi_{-3/2}|0\rangle, \qquad \psi_{-3/2}\psi_{-1/2}|0\rangle, ...$$

are eigenstates of L_0 operator with respective eigenvalues $0, \frac{1}{2}, \frac{3}{2}, 2, ...$
The Hilbert space of the theory for the R-sector deserve a special treatment because there are present fermionic zero modes ψ_0.

1.5.6 The Ghost Systems

After having studied free boson and free fermion CFTs, we will briefly describe a ghost system, which play a fundamental role in the covariant quantisation of bosonic string.
The action of the (b, c) ghost system in two dimensional space is:

$$S = \frac{1}{2}g \int d^x b_{\mu\nu}\partial^\mu c^\nu$$

where $b_{\mu\nu}$ is a symmetric traceless field, and both $b_{\mu\nu}$ and c^μ are anti-commuting fields. The equations of motion are:

$$\partial^\alpha b_{\mu\nu} = 0 \qquad \partial^\alpha c^\beta + \partial^\beta c^\alpha = 0.$$

that in the holomorphic complex coordinates becomes:

$$\begin{array}{cc} \begin{array}{c} \overline{\partial} b = 0 \\ \partial \overline{b} = 0 \end{array} & \qquad \begin{array}{c} \overline{\partial} c = 0 \\ \partial \overline{c} = 0 \\ \partial c = -\overline{\partial}\overline{c}. \end{array} \end{array}$$

After computing the propagator, we find the *OPE* of the fields is:

$$b(z)c(w) \sim \frac{1}{\pi g}\frac{1}{z-w}$$

from which we also derive the following results:

$$\begin{aligned} \langle c(z)b(w)\rangle &= \frac{1}{\pi g}\frac{1}{z-w} \\ \langle b(z)\partial c(w)\rangle &= -\frac{1}{\pi g}\frac{1}{(z-w)^2} \\ \langle \partial b(z)c(w)\rangle &= \frac{1}{\pi g}\frac{1}{(z-w)^2}. \end{aligned}$$

The canonical energy-momentum tensor of the theory is:

$$T^{\mu\nu}_{(c)} = \frac{1}{2}g\left(b^{\mu\alpha}\partial^\nu c_\alpha - \eta^{\mu\nu}b^{\alpha\beta}\partial_\alpha c_\beta\right)$$

which is not symmetric. Through the Belinfante procedure we can define a new identically symmetric traceless form, called *Belinfante tensor*:

$$T^{\mu\nu}_B = \frac{1}{2}g\left\{b^{\mu\alpha}\partial^\nu c_\alpha + b^{\nu\alpha}\partial^\mu c_\alpha + \partial_\alpha b^{\mu\nu}c^\alpha - \eta^{\mu\nu}b^{\alpha\beta}\partial_\alpha c_\beta\right\}. \tag{1.47}$$

The holomorphic component is:

$$T(z) = T^{\overline{zz}} = 4T_{zz} = \pi g : (2\partial c b + c\partial b) : . \tag{1.48}$$

In order to find the conformal dimensions of ghost fields we compute their *OPE* with the energy-momentum tensor applying the Wick theorem:

$$\begin{aligned} T(z)c(w) &= \pi g : (2\partial c b + c\partial b) : c(w) \\ &\sim -\frac{c(z)}{(z-w)^2} + 2\frac{\partial c(z)}{z-w} \\ &\sim -\frac{c(w)}{(z-w)^2} + \frac{\partial c(w)}{z-w} \end{aligned}$$

i.e. c is a primary field with conformal dimension $\left(h;\overline{h}\right) = (-1,0)$.

$$\begin{aligned} T(z)b(w) &= \pi g : (2\partial c b + c\partial b) : b(w) \\ &\sim 2\frac{b(z)}{(z-w)^2} - \frac{\partial_z b(z)}{z-w} \\ &\sim 2\frac{b(w)}{(z-w)^2} + \frac{\partial_w b(w)}{z-w} \end{aligned}$$

thus b is a primary-field with conformal dimension $\left(h;\overline{h}\right) = (2,0)$. Finally we compute the *OPE* of energy-momentum tensor with itself:

$$T(z)T(w) \sim -\frac{13}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}.$$

From this latter result we deduce that the central charge of the CFT given by a ghost system is $c = -26$, which implies that the theory is not unitary. As we will see in the next chapters, the value of the central charge is related to the critical dimension of the space-time on the bosonic string theories. Indeed if we associate this system to a set of 26 bosonic field we could get a total energy-momentum tensor that is a primary-field with conformal dimension $h = 2$.

1.6 CFTs on the torus

In this section we will describe conformal field theories on the torus, and we will introduce the partition function for theories with fermionic and bosonic fields.

1.6.1 The torus

The torus T^2 is the only closed oriented Riemman surface with *genus* $g = 1$.
Let's define two linearly independent vectors identified through two complex numbers on the complex plane $(\omega_1;\omega_2) \in \mathbb{C}$. They generate a *lattice*:

$$\Lambda = \{m\omega_1 + m'\omega_2 | (m,m') \in \mathbb{Z}\},$$

and we call them *periods* of lattice.
We define the torus by identification of lattice points $T^2 = \mathbb{C}/\Lambda$ with $\mathbb{C}$ covering space. Under conformal invariance the only invariant quantity is the ratio:

$$\tau = \frac{\omega_2}{\omega_1} = \tau_1 + i\tau_2$$

called *modular parameter*. We can restrict the domain of τ in $Im\tau > 0$. The upper half-plane in which the values of τ run is called *Teichmüller* space and it is indicated with $\mathbb{H}_+$. There are again residual transformations that change the modular parameter but leaves invariant the conformal structure of torus. They are called *modular transformations* and are defined through following parameters transformations:

$$\begin{cases} \omega_1 \longmapsto c\omega_2 + d\omega_1 \\ \omega_2 \longmapsto a\omega_2 + b\omega_1 \end{cases}$$

These generate the following transformations of modular parameter:

$$\tau \quad \longmapsto \quad \frac{a\tau + b}{c\tau + d} \qquad (ad - bc) = 1$$

the unitary constraint arises from the require that the unit cell of the lattice should have the same area whatever the periods we use. We also notice that changing sign of all parameters (a, b, c, d), we obtain the same transformation of τ. Again, the residual transformations form the *Modular group* $SL(2, \mathbb{Z})/\mathbb{Z}_2 \equiv PSL(2, \mathbb{Z})$. The full group $PSL(2, \mathbb{Z})$ can be generated by the modular transformations:

$$\begin{aligned} \mathcal{T}: &\quad \tau \longmapsto \tau + 1 \\ \mathcal{S}: &\quad \tau \longmapsto -\frac{1}{\tau} \end{aligned}$$

that satisfy the conditions:

$$(\mathcal{S}\mathcal{T})^3 = \mathcal{S}^2 = \mathbb{1}.$$

Their matrix representation on the periods of lattice space is:

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We note that $\mathcal{U} = \mathcal{T}\mathcal{S}\mathcal{T} : \tau \to \frac{\tau}{\tau+1}$, the latter transformation together with $\mathcal{T}$ are called *Dehn twists*.
Let us define *fundamental domain* the set of points of Teichmüller space cannot be obtained from each other through subsequent applications of $\mathcal{T}$ and $\mathcal{S}$:

$$\mathcal{F} = \left\{ Im\tau > 0, -\frac{1}{2} \leq Re\tau \leq 0, |\tau| \geq 1 \quad \cup \quad Im\tau < 0, 0 < Re\tau < \frac{1}{2}, |\tau| > 1 \right\}. \tag{1.49}$$

Applying modular transformations on a Fundamental domain we obtain another Fundamental domain. The *Moduli space* of the torus is the quotient of the Teichmüller space and the modular group:

$$\mathcal{M}_1 = \frac{\textit{Teichmüller space}}{PSL(2, \mathbb{Z})}.$$

We note that the modular group does not act freely on the whole parameters space. Indeed $\tau = i$ is a fixed point of $\mathcal{S}$, and $\tau = e^{2\pi i/3}$ is a fixed point of $\mathcal{S}\mathcal{T}$. Therefore $\mathcal{M}_1$ is not a smooth manifold but is an *orbifold*.

1.6.2 Partition function on a torus

A theory defined on a torus can be treat through the path integral formalism. The difference from the plane is that the action of the theory is invariant under translations of the periods. This does not mean that the conformal fields of the theory be invariant but periodic respect to translations of the periods. In the path integral quatization, where the invariance of the action is guaranteed by the periodic boundary conditions of conformal fields, theories, we define the *partition function* as:

$$Z = \int \mathcal{D}X\mathcal{D}c\mathcal{D}b e^{iS[X,h,c,b]}$$

where X indicates all matter fields and (c, b) are the ghost fields, obtained through the Faddeev-Popov procedure so to regularize the integration measure.
We want to find an expression for the partition function Z of the theory in terms of Virasoro generators. We can think of the torus with modulus τ as formed by taking a field theory on a circle, evolving for Euclidean time $2\pi\tau_2$, translating in σ_1 by $2\pi\tau_1$, and then identifying the ends. Since the Hamiltonian and momentum operators are the generators of translation along time and space directions, we can define the partition function as the sum of states propagating around the torus in the τ_2 direction and weights them with a factor $e^{-2\pi\tau_2 H}$. In the light-cone gauge we have:

$$Z(\tau) = Tr_{\mathcal{H}}\left[e^{2\pi(iP\tau_1-\tau_2 H)}\right]. \tag{1.50}$$

Using the form of operators H and P on the cylinder with circumference $L = 2\pi$, we obtain:

$$\begin{aligned} Z(\tau,\overline{\tau}) =& Tr_{\mathcal{H}}\left[e^{2\pi i\tau_1\left[(L_0)_{cyl}-\left(\overline{L}_0\right)_{cyl}\right]}e^{-2\pi\tau_2\left[(L_0)_{cyl}+\left(\overline{L}_0\right)_{cyl}\right]}\right] \\ &= q^{-\frac{c}{24}}\overline{q}^{-\frac{\overline{c}}{24}}Tr_{\mathcal{H}}\left[q^{L_0}\overline{q}^{\overline{L}_0}\right] \end{aligned} \tag{1.51}$$

where we introduced the quantities $q = e^{2\pi i\tau}$ and $\overline{q} = e^{2\pi i\overline{\tau}}$. The domain of variation of modular parameter τ is the moduli space $\mathcal{M}_1$, therefore field theories defined up torus related by modular transformations have the same function $Z(\tau, \overline{\tau})$. In order to explain this propriety we require that le partition function be *modular invariant*. We will just verify that it is invariant under the generators $\mathcal{T}$ and $\mathcal{U}$. In other words, since the modular transformations are conformal transformations and we would expect that the vacuum amplitude be invariant respect to these transformations, then the partition function must be modular invariant.
We notice that this procedure preserve all local symmetries of theory while the global group$SL(2,\mathbb{C})$ with infinitesimal generators $\left\{L_0; L_{\pm 1}; \overline{L}_0; \overline{L}_{\pm 1}\right\}$ is reduced to group $U(1) \times U(1)$ with infinitesimal generators $\left\{L_0; \overline{L}_0\right\}$.

1.6.3 Free fermion

Let us consider a system with one holomorphic free fermion $\left(c = \frac{1}{2}\right)$ and one anti-holomorphic free fermion $\left(\overline{c} = \frac{1}{2}\right)$. Fermionic fields have two possible periodicity conditions along the two circles of the torus; for the holomorphic part we have:

$$\psi(z + \omega_1) = e^{2\pi iu}\psi(z) \qquad \psi(z + \omega_2) = e^{2\pi iv}\psi(z) \tag{1.52}$$

and similar conditions for anti-holomorphic part. We denote with $(u; v)$ the periodicity conditions that are satisfied by holomorphic part. We have four possible sectors corresponding to four different *spin structures* $(0,0)\,;\left(\frac{1}{2},0\right);\left(0,\frac{1}{2}\right);\left(\frac{1}{2},\frac{1}{2}\right)$, or respectively $(R,R)\,;(NS,R)\,;(R,NS)\,;(NS,NS)$

and an analogue for the anti-holomorphic part.
We want to compute the partition function for this theory, using the expression (1.51). In order to study the modular invariance of the partition function we analyze the transformation proprieties of spin structures under modular transformations. We find:

$$\begin{array}{cc}
(0;0) \xrightarrow{\mathcal{T}} (0;0) & (0;0) \xrightarrow{\mathcal{S}} (0;0) \\
\left(0;\frac{1}{2}\right) \xrightarrow{\mathcal{T}} \left(0;\frac{1}{2}\right) & \left(0;\frac{1}{2}\right) \xrightarrow{\mathcal{S}} \left(\frac{1}{2};0\right) \\
\left(\frac{1}{2};0\right) \xrightarrow{\mathcal{T}} \left(\frac{1}{2};\frac{1}{2}\right) & \left(\frac{1}{2};0\right) \xrightarrow{\mathcal{S}} \left(0;\frac{1}{2}\right) \\
\left(\frac{1}{2};\frac{1}{2}\right) \xrightarrow{\mathcal{T}} \left(\frac{1}{2};0\right) & \left(\frac{1}{2};\frac{1}{2}\right) \xrightarrow{\mathcal{S}} \left(\frac{1}{2};\frac{1}{2}\right)
\end{array}$$

The totally periodic condition $(0;0)$ is modular invariant, but the corresponding sector give a trivial contribution to the partition function. Since the total partition function must be modular invariant, if we consider also the anti-holomorphic part and we assume that it takes a diagonal form, then we obtain:

$$Z(\tau,\overline{\tau}) = \sum_{u,v} |Z_{u,v}|^2. \tag{1.53}$$

We proceed with the calculation of quantities $Z_{u,v}$ by (1.51). In the Hamiltonian language $Z_{u,v}$ is the trace on the periodic/anti-periodic sector respectively for $u = 0, \frac{1}{2}$. Since we have a partition function defined on the cylinder and Virasoro zero modes on the plane, we have to implement the periodicity conditions in the time direction inserting an operator that anticommutes with the fermionic field. This operator is:

$$(-1)^F \qquad\qquad F = \sum_{r\in\mathbb{Z}^+ + \nu} \psi_{-r}\psi_r$$

with F *Fermion Number* operator. Then we define:

$$\begin{aligned}
Z_{0,0} &= \frac{1}{\sqrt{2}} q^{-\frac{1}{48}} Tr_P \left(-1\right)^F q^{L_0} = \frac{1}{\sqrt{2}} Tr_P \left(-1\right)^F q^{\sum_{n>0} n\psi_{-n}\psi_n + \frac{1}{24}} = 0 \\
Z_{\frac{1}{2},0} &= q^{-\frac{1}{48}} Tr_A \left(-1\right)^F q^{L_0} = Tr_A \left(-1\right)^F q^{\sum_{k>0} k\psi_{-n}\psi_n - \frac{1}{48}} \\
Z_{0,\frac{1}{2}} &= \frac{1}{\sqrt{2}} q^{-\frac{1}{48}} Tr_P q^{L_0} = \frac{1}{\sqrt{2}} Tr_P q^{\sum_{n>0} n\psi_{-n}\psi_n + \frac{1}{24}} \\
Z_{\frac{1}{2},\frac{1}{2}} &= q^{-\frac{1}{48}} Tr_A q^{L_0} = Tr_A q^{\sum_{k>0} n\psi_{-n}\psi_n - \frac{1}{48}}
\end{aligned}$$

where $n \in \mathbb{Z}$ and $k \in \mathbb{Z} + \frac{1}{2}$.
Using the basis $\{|0\rangle; \psi_{-n}|0\rangle\}$, we split the total *Hilbert* space in terms of two-dimensional subspaces $\mathcal{H} = \otimes_{n>0}\mathcal{H}_n$, and by basic propriety of the trace we find:

$$\begin{aligned}
Z_{\frac{1}{2},0} &= q^{-\frac{1}{48}} \prod_{k>0} \left(Tr_{\mathcal{H}_k} \left(-1\right)^F q^{k\psi_{-k}\psi_k} \right) = q^{-1/48} \prod_{n=0}^{\infty} \left(1 - q^{n+\frac{1}{2}}\right) = \sqrt{\frac{\theta_3}{\eta}} \\
Z_{0,\frac{1}{2}} &= q^{\frac{1}{24}} \prod_{n>0} \left(Tr_{\mathcal{H}_n} q^{n\psi_{-n}\psi_n} \right) = q^{\frac{1}{24}} \prod_{n=0}^{\infty} \left(1 + q^n\right) = \sqrt{\frac{\theta_2}{\eta}} \\
Z_{\frac{1}{2},\frac{1}{2}} &= q^{-\frac{1}{48}} \prod_{k>0} \left(Tr_{\mathcal{H}_k} q^{k\psi_{-k}\psi_k} \right) = q^{-\frac{1}{48}} \prod_{n=0}^{\infty} \left(1 + q^{n+\frac{1}{2}}\right) = \sqrt{\frac{\theta_3}{\eta}}.
\end{aligned}$$

In the last equality we used the *Jacobi Theta* functions $\theta_i = \theta_i(0,\tau)$ and the *Dedekind eta* function $\eta(q)$ defined in Appendix A.
Finally we can write the total partition function for fermions including all periodicity conditions:

$$Z_{ferm}(\tau) = |Z_{0,\frac{1}{2}}|^2 + |Z_{\frac{1}{2},0}|^2 + |Z_{\frac{1}{2},\frac{1}{2}}|^2 = |\frac{\theta_2}{\eta}| + |\frac{\theta_3}{\eta}| + |\frac{\theta_4}{\eta}|. \tag{1.54}$$

Through the proprieties of transformation of the $\theta_i(\tau)$ and $\eta(\tau)$ functions under modular transformations summed in the Appendix, we can easily verify that $Z_{ferm}(\tau)$ is modular invariant.

1.6.4 Free Boson

The next simplest case of CFT on a torus is that of free boson. We want to proceed analogously to previous fermionic case, where we have imposed boundary conditions on a torus and calculated the partition function by an Hamiltonian formalism.
Let us consider a bosonic field compactified on a circle of radius R:

$$X(z,\overline{z}) = X(z,\overline{z}) + 2\pi R.$$

When we calculate the partition function, we need to consider all sectors (n', n) with boundary conditions (for $\omega_1 = 1$):

$$X(z+\tau, \overline{z}+\overline{\tau}) = X(z,\overline{z}) + 2\pi n' R \qquad X(z+1,\overline{z}+1) = X(z,\overline{z}) + 2\pi n R. \tag{1.55}$$

We compute the partition function Z_{bos} using the operatorial formalism. Let us start from the (1.51) and substitute it the Virasoro zero modes of the compactified boson. We split the Hilbert space in sectors labelled by the eigenvalues of H and P:

$$\begin{aligned} H|m,n\rangle &= \left(L_0 + \overline{L}_0\right)|m,n\rangle = \frac{1}{2}\left(p_L^2 + p_R^2\right)|m,n\rangle = \left(\frac{m^2}{4R^2} + n^2R^2\right)|m,n\rangle \\ P|m,n\rangle &= \left(L_0 - \overline{L}_0\right)|m,n\rangle = \frac{1}{2}\left(p_L^2 - p_R^2\right)|m,n\rangle = mn|m,n\rangle \end{aligned}$$

and we construct the excited states acting on these eigenstates by the oscillators. The trace split into an oscillators contribute:

$$Tr_{\mathcal{H}} q^{L_0} = Tr q^{\sum_{n=1}^{\infty} \alpha_{-n}\alpha_n} = \prod_{n=1}^{\infty}\left(1 + q^n + q^{2n} + ...\right) = \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

and a contribute of sum over winding-momentum. If we include also the anti-holomorphic oscillators, we find:

$$Z_{bos}(\tau, R) = \frac{1}{\eta\overline{\eta}} \sum_{n,m=-\infty}^{\infty} q^{\frac{1}{2}p_L^2}\overline{q}^{\frac{1}{2}p_R^2}. \tag{1.56}$$

The partition function (1.56) is modular invariant. This can be understood in a more general framework. Let's consider a two-dimensional space with Lorentzian signature generated by $\left(k, \overline{k}\right)$:

$$(p_L; p_R) = m\underbrace{\left(\frac{1}{2R}; \frac{1}{2R}\right)}_{k} + n\underbrace{(R, -R)}_{\overline{k}}.$$

Then, $\{k, \overline{k}\}$ are generators of an even self-dual Lorentzian integer lattice $\Gamma_{1,1}$[2]. In general partition functions of the form:

$$Z_{\Gamma_{r,s}} = \frac{1}{\eta^r \overline{\eta}^s} \sum_{(p_L, p_R) \in \Gamma_{r,s}} q^{\frac{1}{2}p_L^2} \overline{q}^{\frac{1}{2}p_R^2}$$

are modular covariant provided that $\Gamma_{r,s}$ is an $r+s$-dimensional even self-dual lattice of signature (r, s). The fact that $p_L^2 - p_R^2 \in 2\mathbb{Z}$ ensures the $\mathcal{T}$-invariance (up to a possible phase from $(\eta^r \overline{\eta}^s)^{-1}$ when $r - s \neq 0 \text{mod} 24$), while the self-duality property guarantees invariance under $\mathcal{S}$. Such lattices exist in every $r - s$ mod 8 dimensions, and for $r - s \neq 0$ are unique up $SO(r, s)$. In Euclidean case, there are a finite number of lattice in $d = 0$ mod 8 dimensions, unique up $SO(d)$.

[2]See Appendix B

Chapter 2

Bosonic String theory

String theory is currently the best candidate to give a unified model of Nature's forces (gravity including) in a single quantum-mechanical framework. The basic assumption is to imagine the matter at fundamental level described through tiny one-dimensional strings rather than through point-particles. General relativity, electromagnetism and Yang-Mills gauge theories appear naturally in low-energies limit. However, among its main features, it leads extra spatial dimension to appear.
In this chapter we will give a brief description of the simplest string theory called *Bosonic string*, both at the classical and quantum level [2], [6], [7], [8], [9]. However this theory presents different problems, that will be solved with the introduction of supersymmetry on the next chapter, as the presence of particles, *Tachions*, with negative mass in the spectrum and the absence of fermionic particles. Despite these problems, it is useful to start describing the Bosonic String Theory so to introduce the main concepts common to all String theories.
The String Theory becomes a CFT on two dimensions after the conformal gauge fixing. Before we have a two-dimensional field theory coupled to two dimensions gravity with diffeomorphism-invariance. In *conformal gauge* the diffeomorphism-invariance is reduced to conformal-invariance, which is still infinite dimensional. However the String Theory remembers its diffeomorphism-invariance through reparametrization ghosts, which appear in BRST quantization, the conformal anomaly and the physical state condition, which is the relict of the equation of motion for the world-sheet metric.

2.1 The relativistic string

A point-particle moving in a d-dimensional Minkoswki space-time describes a one-dimensional line called *world-line*. Since a *string* is an one-dimensional object moving in the same space, it will sweep a two-dimensional surface called *world-sheet*.
We start quickly studying the classical action of the point-particle and its quantization, later we will extend the same treatment to the one-dimensional string.

2.1.1 Relativistic point-particle

Let us consider a point-particle with mass m in a d-dimensional Minkowski space-time with metric $\eta_{\mu\nu} = diag\,(-1, 1, 1, ..., 1)$. Its action is simply the length of its world-line parametrized as $x^\mu = x^\mu(\tau)$:

$$S = -m \int d\tau \sqrt{-\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\eta_{\mu\nu}}. \tag{2.1}$$

The action is manifestally *Poincarè-invariant*, moreover it is also invariant under reparameterization of τ by any monotonic function:

$$\tau \quad \to \quad \tilde{\tau} = \tilde{\tau}(\tau).$$

This latter is a *gauge-invariance* of the theory, i.e. in our description there is a non-physical degree of fredoom. Indeed the conjugate momenta:

$$p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}$$

are not all independent, they satisfy the mass-shell condition:

$$p_\mu p^\mu + m^2 = 0.$$

From the world-line point of view, the particle cannot stay still, but it must move in a timelike direction in which $(p_0) \geq -m^2$.

2.1.2 The Nambu-Goto action

The string world-sheet is a bidimensional surface on d-dimensional Minkowski space-time parametrized by a spatial coordinate σ and a temporal one τ, packaged into $\sigma^a = (\sigma, \tau)$. It this chapter we will focus on *closed* string, thus the parameter σ is periodic with range $\sigma \in [0; 2\pi)$. The string evolution can be described as maps from two-dimensional world-sheet to space-time through the coordinates:

$$X^\mu(\tau, \sigma) \qquad \mu = 0, 1, ...d-1$$

with the additional condition for closed string:

$$X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi).$$

We need to generalize the action for the point-particle, proportional to the length of its world-line, to that for the sting, proportional to the area of world-sheet. Let γ_{ab} be the induced metric inherited from the d-dimensional Minkowski space-time to bidimensional surface:

$$\gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial \sigma^a}\frac{\partial X^\nu}{\partial \sigma^b}\eta_{\mu\nu},$$

then the seeked action take the form:

$$S_{NG} = -T \int d^2\sigma \sqrt{-det\gamma_{ab}} = -T \int d^2\sigma \left[-\left(\dot{X}\right)^2 \left(X'\right)^2 + \left(\dot{X}X'\right)^2 \right]^{1/2}, \tag{2.2}$$

where $\dot{X}^\mu = \partial X^\mu / \partial\tau$ is the "time" derivative, and $X'^\mu = \partial X^\mu / \partial \sigma$ is the "spatial" derivative. The (2.2) is called *Nambu-Goto action.* Since the space-time coordinates have the dimension $[X] = M^{-1}$, while the parameters σ^a are dimensionless, then the proportional constant must have the dimension $[T] = M^2$ so the action be dimensionless. T has the physical interpretation of *tension* of string, which behaves as an elastic band and its potential energy increases linearly with length. We introduce the *Regge parameter* $\alpha' = \frac{1}{2\pi T}$, through which we define a string length scale $l_s = \sqrt{\alpha'}$ and a string mass scale $M_s \sim \frac{1}{\sqrt{\alpha'}}$.
The principal symmetries of Nambu-Goto action are:

Poincarè-invariance of space-time. From world-sheet point of view, it is a global symmetry because the parameters of transformation don't depend on sheet-coordinates.

Reparameterization-invariance: $\sigma^a \to \tilde{\sigma}^a(\sigma)$. This is a gauge-symmetry, it reflect the fact that we have a redundancy in our description: the coordinates σ^a are not physical.

In order to derive the equation of motion associate to action (2.2) we introduce the momenta:

$$\begin{aligned}\Pi^\tau_\mu &= \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -T\frac{\left(\dot{X}\cdot X'\right)X'_\mu - \left(X'^2\right)\dot{X}_\mu}{\sqrt{\left(\dot{X}\cdot X'\right)^2 - \dot{X}^2 X'^2}} \\ \Pi^\sigma_\mu &= \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -T\frac{\left(\dot{X}\cdot X'\right)\dot{X}_\mu - \left(X'^2\right)X'_\mu}{\sqrt{\left(\dot{X}\cdot X'\right)^2 - \dot{X}^2 X'^2}}\end{aligned} \tag{2.3}$$

so the equation of motion obtained through the variational principle take the form:

$$\frac{\partial \Pi^\tau_\mu}{\partial \tau} + \frac{\partial \Pi^\sigma_\mu}{\partial \sigma} = 0$$

2.2 The Polyakov action

The square-root in the Nambu-Goto action leads to complicated equation of motion and makes rather difficult to quantize using path integral techniques. However we can define an equivalent action to (2.2) without square-root at the expense of introducing an additional field on the world-sheet. The new field is $h_{\alpha\beta}(\tau,\sigma)$; it is a dynamical metric of the world-sheet with signature $(-;+)$.
Let be given the *Polyakov action*:

$$S_P = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}. \tag{2.4}$$

From the world-sheet point of view this action is a set of scalar fields coupled to 2d gravity.
The equations of motion for the fields $X^\mu(\tau,\sigma)$ are:

$$\partial_\alpha\left(\sqrt{-h}h^{\alpha\beta}\partial_\beta X^\mu\right) = 0.$$

Varying the action with respect to metric field we obtain its equation of motion:

$$\delta S_P = -\frac{T}{2}\int d^2\sigma \delta h^{\alpha\beta}\left(\sqrt{-h}\partial_\alpha X^\mu \partial_\beta X^\nu - \frac{1}{2}\sqrt{-h}h_{\alpha\beta}h^{\rho\sigma}\partial_\rho X^\mu \partial_\sigma X^\nu\right)\eta_{\mu\nu},$$

from which:

$$h_{\alpha\beta} = 2f(\tau,\sigma)\partial_\alpha X \cdot \partial_\beta X$$

where the function $f^{-1} = h^{\rho\sigma}\partial_\rho X \cdot \partial_\sigma X$. By comparison of this result with definition of induced metric $\gamma_{\alpha\beta}$, we discover that they differ by the conformal factor f. Within this difference, the equations of motion for the fields X^μ related to two actions are identical. The presence of this conformal factor reflects the existence of an extra symmetry in the Polyakov action.
Let us summarize the symmetries of the Polyakov action.

Poicarè invariance, is a global symmetry from world-sheet point of view.

$$X^\mu \quad \longmapsto \quad \Lambda^\mu_\nu X^\nu + c^\mu$$

Reparameterization invariance (also *diffeomorphisms-invariance*), is a gauge-symmetry on the world-sheet. We define the change of coordinates on the worldsheet as $\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma)$, under which the fields X^μ transforms as worldsheet scalars, while $h_{\alpha\beta}$ as a 2d metric:

$$\begin{aligned} X^\mu(\sigma) &\quad\longmapsto\quad \tilde{X}^\mu(\tilde{\sigma}) = X^\mu(\sigma) \\ h_{\alpha\beta}(\sigma) &\quad\longmapsto\quad \tilde{h}_{\alpha\beta}(\tilde{\sigma}) = \frac{\partial\sigma^\rho}{\partial\tilde{\sigma}^\alpha}\frac{\partial\sigma^\gamma}{\partial\tilde{\sigma}^\beta} h_{\rho\gamma}(\sigma). \end{aligned}$$

If we consider an infinitesimal transformation $\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma) = \sigma^\alpha - \eta^\alpha(\sigma)$, the corresponding infinitesimal transformations of fields are:

$$\begin{aligned} \delta X^\mu(\sigma) &= \eta^\alpha \partial_\alpha X^\mu(\sigma) \\ \delta h_{\mu\nu}(\sigma) &= \nabla_\alpha \eta_\beta + \nabla_\beta \eta_\alpha, \end{aligned}$$

where the covariant derivative is defined as:

$$\nabla_\alpha \eta_\beta = \partial_\alpha \eta_\beta - \Gamma^\rho_{\alpha\beta}\eta_\rho$$

with Levi-Civita connection associated to worldsheet metric:

$$\Gamma^\rho_{\alpha\beta} = \frac{1}{2} h^{\rho\gamma}\left(\partial_\alpha h_{\gamma\beta} + \partial_\beta h_{\gamma\alpha} - \partial_\gamma h_{\alpha\beta}\right).$$

Weyl-invariance, is a gauge symmetry defined by following fields transformations:

$$\begin{aligned} X^\mu(\sigma) &\quad\longmapsto\quad X^\mu(\sigma) \\ h_{\alpha\beta}(\sigma) &\quad\longmapsto\quad \Omega^2(\sigma) h_{\alpha\beta}(\sigma). \end{aligned}$$

Infinitesimally we can write $\Omega^2(\sigma) = e^{2\phi(\sigma)}$, for small $\phi(\sigma)$, thus:

$$\delta h_{\alpha\beta}(\sigma) = 2\phi(\sigma) h_{\alpha\beta}(\sigma).$$

The Weyl symmetry in not a coordinates change, it is a theory invariance under a local change of scale which preserves the angles between all lines. The string worldsheets in the Fig.(2.1) are equivalent from the Polyakov action point of view.
This propriety is special for two-dimensional theory, however it restricts the kind of interactions that can be added to the action, and these constraints becomes even more stringent in the quantum theory.

The reparametrization-invariance allows us to choose a convenient gauge for the worldsheet metric, called *conformal gauge*:

$$h_{\alpha\beta}(\sigma) = e^{2\phi(\sigma)}\eta_{\alpha\beta}, \tag{2.5}$$

i.e. the two degrees of freedom in the choice of parameters allows us to fix two of the three degrees of freedom of the metric.
Again, using the Weyl-invariance we can remove the last independent component of the metric and set $\phi(\sigma) = 0$ such that $h_{\alpha\beta} = \eta_{\alpha\beta}$. Through this choice the action (2.4) takes the simple form:

$$S_P = -\frac{1}{4\pi\alpha'}\int d^2\sigma \partial_\alpha X \partial_\beta X, \tag{2.6}$$

and the equations of motion become:

$$\partial_\alpha \partial^\alpha X^\mu = 0 \tag{2.7}$$

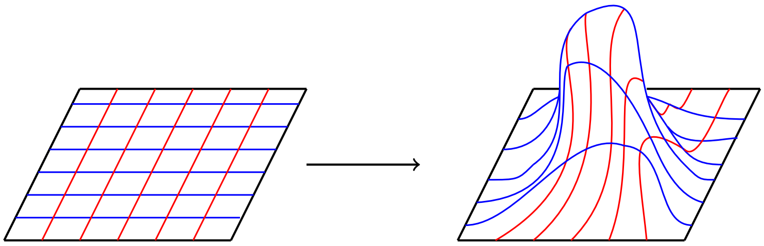

Figure 2.1: Example of Weyl transformation.

We define the energy-momentum tensor to be:

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{-h}}\frac{\delta S_P}{\delta h_{\alpha\beta}} = \partial_\alpha X \partial_\beta X - \frac{1}{2}\eta_{\alpha\beta}\eta^{\rho\gamma}\partial_\rho X \partial_\gamma X, \tag{2.8}$$

where we performed the choice of flat metric.
The (2.7) are subject to two constraints arising from the equation of motion of the metric:

$$T_{\alpha\beta} = 0 \qquad \begin{cases} T_{01} = \dot{X}\cdot X' = 0 \\ T_{00} = T_{11} = \frac{1}{2}\left(\dot{X}^2 + X'^2\right) = 0. \end{cases} \tag{2.9}$$

The first condition tell us that we have to choose a parametrisation such that lines $\sigma = const$ are perpendicular to lines $\tau = const$. Now we use the residual-gauge arising from Weyl transformations to introduce the *statical gauge*:

$$X^0 = R\tau$$

with $R = const$. The previous constraints for the spatial components in this gauge becomes:

$$\dot{\overline{X}} \cdot \overline{X}' = 0$$
$$\dot{\overline{X}}^2 + \overline{X}'^2 = R^2$$

The first condition tell us that the motion of string must be perpendicular to string itself, i.e. the oscillations will be described by transverse modes. The second condition relates R with the length of string when $\dot{\overline{X}} = 0$, i.e. starting from a stretched string at a time with $\dot{\overline{X}} = 0$, the string will contract under its own tension to later times.

2.3 Modes expansions

We introduce the lightcone coordinates on the worldsheet:

$$\begin{aligned} \sigma^+ &= \tau + \sigma \\ \sigma^- &= \tau - \sigma. \end{aligned} \tag{2.10}$$

The Minkowski metric in these coordinates takes the form:

$$\eta_{\alpha\beta} = \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \qquad \eta^{\alpha\beta} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}.$$

The equation of motion of coordinated fields becomes:

$$\partial_+ \partial_- X^\mu = 0$$

whose more general solution separates the *left*- and *right*-moving modes of string:

$$X^\mu(\tau, \sigma) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma_-).$$

Remembering the periodicity condition of closed string, we can perform a Fourier expansion:

$$\begin{aligned} X_L^\mu(\sigma^+) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\tilde{\alpha}_n^\mu e^{-in\sigma^+} \\ X_R^\mu(\sigma^-) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\tilde{\alpha}_n^\mu e^{-in\sigma^-}. \end{aligned} \tag{2.11}$$

The quantities x^μ and p^μ are respectively *position* and *momentum* of the center of mass of the string. This can be easily proven by studying the Noether currents associated to symmetry translation $X^\mu \to X^\mu + c^\mu$. Since the coordinated fields are real, the coefficients of Fourier satisfy:

$$(\alpha_n^\mu) = \left(\alpha_{-n}^\mu\right)^* \qquad (\tilde{\alpha}_n^\mu) = \left(\tilde{\alpha}_{-n}^\mu\right)^*.$$

The constraints (2.9) becomes on the lightcone coordinates:

$$(\partial_+ X)^2 = (\partial_- X)^2 = 0.$$

Let us consider the field:

$$(\partial_- X^\mu) = \partial_- X_R^\mu = \frac{\alpha'}{2}p^\mu + \sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \alpha_n^\mu e^{-in\sigma^-} = \sqrt{\frac{\alpha'}{2}} \sum_{n\in Z} \alpha_n^\mu e^{-in\sigma^-} \tag{2.12}$$

where we have defined the zero mode as $\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu$. Therefore:

$$\begin{aligned} (\partial_- X^\mu)^2 &= \left[\sqrt{\frac{\alpha'}{2}} \sum_{n\in Z} \alpha_n^\mu e^{-in\sigma^-}\right]^2 \\ &= \frac{\alpha'}{2} \sum_{m,p} \alpha_m \cdot \alpha_p e^{-i(m+p)\sigma^-} \\ &= \frac{\alpha'}{2} \sum_{m,n} \alpha_m \cdot \alpha_{n-m} e^{-in\sigma^-} \\ &= \alpha' \sum_n L_n e^{-in\sigma^-} = 0 \end{aligned}$$

where we have defined the sum of oscillators modes:

$$L_n = \frac{1}{2} \sum_m \alpha_{n-m} \cdot \alpha_m. \tag{2.13}$$

The same analysis can be performed for the left-moving modes through an analogous sum:

$$\tilde{L}_n = \frac{1}{2} \sum_m \tilde{\alpha}_{n-m} \cdot \tilde{\alpha}_m, \tag{2.14}$$

with $\alpha_0^\mu = \tilde{\alpha}_0^\mu$. The equations (2.9) impose an infinite number of conditions on the classical string solutions:

$$L_n = \tilde{L}_n = 0 \qquad \forall n \in \mathbb{Z}.$$

In particular, the constraints arising from L_0 and $\tilde{L}_0$ have a rather special interpretation because they include the square of the momentum p^μ that respect the mass-shell condition on the Minkowskian space-time. Thus we have two expressions for the effective mass of string, one in terms of left-oscillators and one in terms of right-oscillators, and they must be equal to each other:

$$M^2 = \frac{4}{\alpha'} \sum_{n>0} \alpha_n \cdot \alpha_{-n} = \frac{4}{\alpha'} \sum_{n>0} \tilde{\alpha}_n \cdot \tilde{\alpha}_{-n}. \tag{2.15}$$

This is called *level matching condition*, and it will play an important role in the string quantization.

2.4 Covariant quantization

The goal of this section is to quantize the string action (2.4) with flat metric through the covariant quantization method. We promote the fields X^μ and their conjugate momenta $\Pi_\mu = \frac{1}{2\pi\alpha'}\dot{X}_\mu$ to operator, and replace their Poisson brackets by commutators $\{,\} \to i^{-1}[,]$:

$$[X^\mu(\sigma,\tau), \Pi_\nu(\sigma',\tau)] = i\delta(\sigma - \sigma')\delta^\mu_\nu$$
$$[X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] = [\Pi^\mu(\sigma,\tau), \Pi^\nu(\sigma',\tau)] = 0$$

The Fourier expansion coefficients of the equation of motion solutions are now operator which satisfy the following commutation rules:

$$[x^\mu, p_\nu] = i\delta^\mu_\nu \qquad [\alpha^\mu_n, \alpha^\nu_m] = [\tilde{\alpha}^\mu_n, \tilde{\alpha}^\nu_m] = n\eta^{\mu\nu}\delta_{n+m,0} \qquad [\alpha^\mu_n, \tilde{\alpha}^\nu_m] = 0.$$

The reality condition on the fields becomes the hermicity conditions on the operators: $(\alpha^\mu_n)^\dagger = \alpha^\mu_{-n}$ and $(\tilde{\alpha}^\mu_n)^\dagger = \tilde{\alpha}^\mu_{-n}$. Rescaling these operators by $a_n = \frac{\alpha_n}{\sqrt{n}}$ and $a_{-n} = \frac{\alpha_{-n}}{\sqrt{n}}$, we write the new commutation rules:

$$[a_n, a_{-m}] = \delta_{nm}.$$

equal to those of harmonic oscillator. So each scalar field gives rise to two infinite towers, one for right-moving modes and one for left-moving modes, of creation and annihilation operators, with α_n acting as a annihilation operator for $n > 0$ and as a creation operator for $n < 0$.
In order to construct the Hilbert space of the theory, we define the vacuum absolute state $|0\rangle$ annihilated by all α_n and $\tilde{\alpha}_n$ modes with $n > 0$. The difference with the QFT is that $|0\rangle$ is not the vacuum state of space-time but those of a single string. In fact the operator x^μ and p^μ add an extra-structure to the vacuum. In momentum representation the vacuum carries another quantum number eigenvalues of momentum operator:

$$\widehat{p}^\mu|0, p^\mu\rangle = p^\mu|0, p^\mu\rangle.$$

The other states are obtained acting on these vacua with the creation operators:

$$\left(\alpha^{\mu_1}_{-1}\right)^{n_{\mu_1}} \left(\alpha^{\mu_2}_{-2}\right)^{n_{\mu_2}} ... \left(\tilde{\alpha}^{\nu_1}_{-1}\right)^{n_{\nu_1}} \left(\tilde{\alpha}^{\nu_2}_{-2}\right)^{n_{\nu_2}} ...|0, p\rangle$$

and represent excited states of the string. Each state is a different species of particles in space-time, and these species are an infinite number.

Exactly as the QED in Lorentz gauge, in our Hilbert space there are negative norm states, called *ghost*, coming from the Minkowski metric on the commutation rules. In order to eliminate these states we would like to impose appropriate gauge-fixing conditions on the operators.
Since L_n and $\bar{L}_n$ are defined in terms of product of operators, in the quantum theory we have to choose an order. In particular for the zero-modes L_0 and $\bar{L}_0$ we discover to have an ambiguity in this definition because the operators on the product do not commutate, i. e. they are not completely determined by classical theory. Since the commutators are constant quantities, the different choices of L_0 and $\bar{L}_0$ are related by a constant shift $L_0 \to L_0 + a$:

$$\begin{aligned} L_0 &= \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n + \frac{1}{2}\alpha_0^2 \\ \bar{L}_0 &= \sum_{n=1}^{\infty} \bar{\alpha}_{-n}\bar{\alpha}_n + \frac{1}{2}\bar{\alpha}_0^2. \end{aligned}$$

In order to fix the constants, we impose the standard condition:

$$\begin{aligned} [L_1, L_{-1}] &= 2L_0 \\ \left[\bar{L}_1, \bar{L}_{-1}\right] &= 2\bar{L}_0, \end{aligned}$$

which are equivalent to require that $\{L_{-1}, L_0, L_1\}$ generate a subalgebra $SL(2,\mathbb{C})$, and also their analogues bar. Under these conditions the algebra of $L_n, \bar{L}_n$ operators take the following form:

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \\ \left[\bar{L}_m, \bar{L}_n\right] &= (m-n)\tilde{L}_{m+n} + \frac{\bar{c}}{12}m(m^2-1)\delta_{m+n,0}. \end{aligned} \tag{2.16}$$

We find one couple of *Virasoro algebra* with central charges c and $\bar{c}$. Physically these terms arises as quantum effect due to the breaking of Weyl invariance at quantum level. In the representation through d free bosons $c = \bar{c} = d$, $d = \eta^{\mu}_{\mu}$ is related to the dimension of the embedding space-time. Thus we can think that each boson contributes to central charge of one unity.
Let us now come back to the Virasoro generators. At classical level the constraints on the Fourier components of energy-momentum tensor are $L_n = \bar{L}_n = 0$ but, because their commutations rules are not trivial, the same conditions cannot be imposed at quantum level. However, we can require that all positive mode-operators annihilate any physical state:

$$\begin{aligned} &L_n|\psi\rangle_{phys} = \bar{L}_n|\psi\rangle_{phys} = 0 \qquad\qquad \forall n > 0 \\ &(L_0 - a)\,|\psi\rangle_{phys} = \left(\bar{L}_0 - a\right)|\psi\rangle_{phys} = 0. \end{aligned} \tag{2.17}$$

The first of these two conditions is called *Virasoro condition.* In second condition of (2.17) the quantity a is some constant.
As we saw, the operators L_0 and $\bar{L}_0$ play an important role in determining the spectrum of the string because they include a term quadratic in the momentum $\alpha_0^{\mu} = \tilde{\alpha}_0^{\mu} = \sqrt{\frac{\alpha'}{2}}p^{\mu}$. Using the redefinitions $L_0 + a$ and $\tilde{L}_0 + a$, the second constraint of (2.17) and the classical condition (2.15) we obtain the *quantum level condition*:

$$M^2 = \frac{4}{\alpha'}\left(-a + \sum_{m=1}^{\infty} \alpha_{-m}\cdot\alpha_m\right) = \frac{4}{\alpha'}\left(-a + \sum_{m=1}^{\infty} \tilde{\alpha}_{-m}\cdot\tilde{\alpha}_m\right). \tag{2.18}$$

We notice that the undetermined constant a has a direct physical effect: it changes the mass spectrum of the string. We can write the (2.18) through the *Number operators*:

$$N = \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m \qquad \tilde{N} = \sum_{m=1}^{\infty} \tilde{\alpha}_{-m} \cdot \tilde{\alpha}_m \tag{2.19}$$

that counts the number of excited modes of string. Then the level condition tell us that the number of left-moving modes must be equal to right-moving modes. Finally we define the Hilbert space of physical states as the quotient:

$$\mathcal{H} = \frac{KerL_n(n \geq 0)}{ImL_{-n}(n > 0)}.$$

2.5 Lightcone Quantization

This different method of quantization for the string theory consist in solving the constraints at classical level, leaving only the physical degrees of freedom, and later performing the quantization of classical solutions.
The gauge symmetries of the theory allow us to choose the *light-cone coordinates* (2.10), where the metric take the form:

$$ds^2 = -d\sigma^+ d\sigma^-$$

However we still have a residual gauge, in fact transformations like $\sigma^+ \to \tilde{\sigma}^+(\sigma^+)$ and $\sigma^- \to \tilde{\sigma}^-(\sigma^-)$ lead an overall factor on the metric that can be remove by a compensating Weyl transformation. Again, if we fixed 3 components of worldsheet metric corresponding to 3 gauge-invariance degrees of freedom, then we wonder why do we still have some gauge symmetry left? The reason is that $\tilde{\sigma}^+, \tilde{\sigma}^-$ are functions of just a single variable, not two.
The physical implication of this residual gauge is to reduce the degrees of freedom of string solutions. In fact the equation of motion solutions (2.11) are $2d$ functions of a single variable, the constraints (2.9) reduces the number of independent functions to $2(d-1)$. The residual choice of parameters $\tilde{\sigma}^{\pm}$ still lowers this number to $2(d-2)$: these are the degrees of freedom of *transverse* fluctuations of string.
In order to remove the remaining invariance we want to impose the *light-cone gauge.* Let's start by choosing the space-time coordinates:

$$\begin{aligned} X^+ &= \sqrt{\frac{1}{2}}\left(X^0 + X^{d-1}\right) \\ X^- &= \sqrt{\frac{1}{2}}\left(X^0 - X^{d-1}\right). \end{aligned} \tag{2.20}$$

We notice that this choice is not manifestally Lorentz-invariant.The new space-time metric is:

$$ds^2 = -2dX^+dX^- + \sum_{i=1}^{d-2} dX^i dX^i.$$

Now we fix the gauge by requiring that the equation of motion solution for X^+ to be:

$$X^+ = X_L^+(\sigma^+) + X_R^+(\sigma^-) = \frac{1}{2}\left(x^+ + \alpha' p^+ \sigma^+\right) + \frac{1}{2}\left(x^+ + \alpha' p^+ \sigma^-\right) = x^+ + \alpha' p^+ \tau. \tag{2.21}$$

This is the light-cone gauge. Notice that X^+ is a null space-time coordinate proportional to timelike worldsheet parameter τ.
The constraint equations for this coordinate are trivial, thus we wonder if there are extra-constraints on the other coordinate-solutions. Let's consider the usual ansatz for X^- to solve its equation of motion:

$$X^- = X_L^-(\sigma^+) + X_R^-(\sigma^-).$$

Now we impose the constraints (2.9) in these new coordinates for this solution:

$$\left(\partial_+ X^-\right)^2 = \left(\partial_- X^-\right)^2 = 0,$$

using the gauge-fixing we find:

$$\begin{aligned}
\partial_+ X_L^- &= \frac{1}{\alpha' p^+} \sum_{i=1}^{d-2} \partial_+ X^i \partial_+ X^i \\
\partial_- X_R^- &= \frac{1}{\alpha' p^+} \sum_{i=1}^{d-2} \partial_- X^i \partial_- X^i .
\end{aligned}$$

So, up to an integration constant, the function $X^-(\sigma^+, \sigma^-)$ is completely determined by other fields $X^i(\sigma^+, \sigma^-)$:

$$\begin{aligned}
X_L^-(\sigma^+) &= \frac{1}{2}\left(x^- + \alpha' p^- \sigma^+\right) + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \tilde{\alpha}_n^- e^{-in\sigma^+} \\
X_R^-(\sigma^-) &= \frac{1}{2}\left(x^- + \alpha' p^- \sigma^-\right) + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \alpha_n^- e^{-in\sigma^-}
\end{aligned}$$

x^- is a integration constant, while $p^-; \bar{\alpha}_n^-, \alpha_n^-$ are fixed by previous conditions in terms of other fields:

$$\alpha_n^- = \sqrt{\frac{1}{2\alpha'}} \frac{1}{p^+} \sum_{m=-\infty}^{+\infty} \sum_{i=1}^{d-2} \alpha_{n-m}^i \alpha_m^i$$

and an equivalent expression for $\bar{\alpha}_n^-$, while for $\alpha_0^- = \sqrt{\frac{\alpha'}{2}} p^-$ we find:

$$\begin{aligned}
\frac{\alpha' p^-}{2} &= \frac{1}{2p^+} \sum_{i=1}^{d-2} \left(\frac{1}{2} \alpha' p^i p^i + \sum_{n\neq 0} \alpha_n^i \alpha_{-n}^i \right) \\
\frac{\alpha' p^-}{2} &= \frac{1}{2p^+} \sum_{i=1}^{d-2} \left(\frac{1}{2} \alpha' p^i p^i + \sum_{n\neq 0} \bar{\alpha}_n^i \bar{\alpha}_{-n}^i \right),
\end{aligned}$$

from which we can reconstruct the classical level matching conditions:

$$M^2 = 2p^+ p^- - \sum_{i=1}^{d-2} p^i p^i = \frac{4}{\alpha'} \sum_{i=1}^{d-2} \sum_{n\neq 0} \alpha_{-n}^i \alpha_n^i = \frac{4}{\alpha'} \sum_{i=1}^{d-2} \sum_{n\neq 0} \bar{\alpha}_{-n}^i \bar{\alpha}_n^i .$$

By comparison with the level condition (2.15), we observe that here the i-sum is only over transverse oscillators. Thus, the general classical solution is written in terms of $2(d-2)$ transverse

oscillators and the zero-modes x^i, p^i, p^+, x^- describing the center of mass and momentum of the string.
Finally, we promote these physical degrees of freedom to operators role, and we impose the following commutation rules:

$$\begin{aligned} \left[x^i, p^j\right] = i\delta^{ij} \qquad \left[x^-, p^+\right] = -i \\ \left[\alpha_n^i, \alpha_m^j\right] = \left[\bar{\alpha}_n^i, \bar{\alpha}_m^j\right] = n\delta^{ij}\delta_{n+m,0}. \end{aligned} \tag{2.22}$$

Now we have all the ingredients to construct the Hilbert space of the theory. Let $|0, p\rangle$ be the vacuum states labelled by eigenvalues of momentum operator:

$$\begin{aligned} &P^\mu|0, p\rangle = p^\mu|0, p\rangle \\ &\alpha_n^i|0, p\rangle = \tilde{\alpha}_n^i|0, p\rangle = 0 \qquad n > 0 \end{aligned}$$

and the other excited states are given by acting with the creation operators $\alpha_{-n}^i, \tilde{\alpha}_{-n}^i$ with $(n > 0)$ on these vacua. The difference with the covariant quantization is that now we are acting only with transverse oscillators, for this reason the Hilbert space is, by construction, positive definite.
Defining the new number operators:

$$N = \sum_{i=1}^{d-2}\sum_{n>0} \alpha_{-n}^i \alpha_n^i \qquad \bar{N} = \sum_{i=1}^{d-2}\sum_{n>0} \bar{\alpha}_{-n}^i \bar{\alpha}_n^i$$

and introducing the normal ordered into the level condition we find:

$$M^2 = \frac{4}{\alpha'}\left(N - a\right) = \frac{4}{\alpha'}\left(\bar{N} - a\right),$$

where a is a normal ordering constant. In order to fix the a value we write the sum:

$$\begin{aligned} \frac{1}{2}\sum_{i=1}^{d-2}\sum_{n\neq 0} \alpha_{-n}^i \alpha_n^i &= \frac{1}{2}\sum_{i=1}^{d-2}\left[\sum_{n>0} \alpha_{-n}^i \alpha_n^i + \sum_{n<0} \alpha_{-n}^i \alpha_n^i\right] \\ &= \frac{1}{2}\sum_{i=1}^{d-2}\left[2\sum_{n>0} \alpha_{-n}^i \alpha_n^i + \sum_{n>0}\left[\alpha_{-n}^i; \alpha_n^i\right]\right] \\ &= \sum_{i=1}^{d-2}\sum_{n>0} \alpha_{-n}^i \alpha_n^i + \frac{d-2}{2}\sum_{n>0} n \end{aligned}$$

where we used the oscillators's commutation rules. We notice that a correspond to latter divergent term of the last line. So, we consider the Riemann zeta-function:

$$\zeta(s) = \sum_{n=1}^{+\infty} n^{-s} \qquad Re(s) > 1 \tag{2.23}$$

that admits a unique analytic continuation to all s values, except one pole in $s = 1$. In particular we find $\zeta(-1) = -1/12$, therefore the quantum level matching condition becomes:

$$M^2 = \frac{4}{\alpha'}\left(\widehat{N} - \frac{d-2}{24}\right) = \frac{4}{\alpha'}\left(\widehat{\bar{N}} - \frac{d-2}{24}\right). \tag{2.24}$$

2.6 The string spectrum

In this section we would like to make a brief analysis of the spectrum of a single free string. Let us start with the *ground state* $|0,p\rangle$, that has not excited oscillators. The condition (2.24) tells us that this state has a negative mass-squared:

$$M^2 = -\frac{1}{6\alpha'}(d-2).$$

The corresponding particles in the space-time are called *tachyons* $T(X^\mu)$. This type of particles are a problem for the bosonic string because correspond to unstable vacuum states. However, there may be a stable vacuum which is not accessible by perturbative theory.
Before starting the analysis of excited states, we remember the Wigner representation of the Poincarè group on a d-dimensional Minkowski space-time $\mathbb{R}^{1,d-1}$:

> *Massive particles* are classified by representations of $SO(d-1)$, little group in which transforms internal index of momentum p^μ with $p_\mu p^\mu = -M^2$.
>
> *Massless particles* are classified by irreducible representations of $SO(d-2)$, little group of momentum transformations with $p_\mu p^\mu = 0$.

We now look at the first excited states $N = \bar{N} = 1$. Since the level matching condition tell us that the number of states left and right are equal, we act on the vacuum state with both operators α^i_{-1} and $\bar{\alpha}^i_{-1}$:

$$\bar{\alpha}^i_{-1}\alpha^i_{-1}|0;p\rangle. \tag{2.25}$$

We obtain $(d-2)^2$ particle states with mass:

$$M^2 = \frac{4}{\alpha'}\left(1 - \frac{d-2}{24}\right).$$

Since there is no way to package these $(d-2)^2$ states into a representation of $SO(d-1)$, i.e. the first excited string state cannot form a massive representation of Lorentz group. However these states can provide irreducible representations of $SO(d-2)$, thus in order to respect the Lorentz-invariance these states must be massless. And this is only the case if the dimension of space-time is:

$$M^2 = \frac{4}{\alpha'}\left(1 - \frac{d-2}{24}\right) = 0 \qquad \longmapsto \qquad d = 26. \tag{2.26}$$

The bosonic string theory on $d = 26$ dimensions contains massless particle on its spectrum: these particles are interesting because they give rise to long range interactions. The states (2.25) transform on the representation $\mathbf{24}\otimes\mathbf{24}$ of $SO(24)$, that can be decomposed into three irreducible representations:

$$(d-2)^2 = \underbrace{\frac{(d-2)(d-3)}{2}}_{\text{Anti-symmetric}B_{ij}} \oplus \underbrace{\frac{(d-2)(d-1)}{2}}_{\text{Traceless symmetric}G_{ij}} \oplus \underbrace{1}_{\text{trace}\Phi}$$

To each of these representations we associate a massless field on space-time, respectively:

$$B_{\mu\nu}(x) \qquad G_{\mu\nu}(X) \qquad \Phi(X).$$

$B_{\mu\nu}$ is the *Kalb-Ramond* field, and it defines a two-form on the space-time. The scalar field $\Phi(X)$ is called *dilaton*. The field $G_{\mu\nu}(X)$ represents a massless particle with spin 2 on the space-time

thus, pushed by the idea of Feynman and Weinberg , that any theory of interacting massless spin two particles must be equivalent to general relativity, we would like to identify this field with the space-time metric.
We analyse higher excited states of the spectrum. The string at level $N = \bar{N} = 2$ has two different states both in left-sector and in right-sector, thus the total set of states at level 2 is:

$$\left(\alpha^i_{-1}\alpha^j_{-1} \oplus \alpha^k_{-2}\right) \otimes \left(\bar{\alpha}^i_{-1}\bar{\alpha}^j_{-1} \oplus \bar{\alpha}^k_{-2}\right) |0, p\rangle.$$

This states have mass $M^2 = 4/\alpha'$. In each sector we have $\frac{1}{2}d(d-1) - 1$ states, that does fit nicely into a representation of $SO(d-1)$, in particular it is the symmetric traceless tensorial representation.
The only consistency requirement that we need for Lorentz invariance is to fix up the first excited states: the space-time dimension must be $d = 26$. In $d = 26$ all higher excited states will be massive, so they will be representations of $SO(25)$.

2.7 Open string

We would like to conclude this chapter with a brief introduction on open strings and D-branes. A fundamental feature of open strings is the existence of two *end points*, indeed the spatial coordinate of string runs in $\sigma \in [0; \pi]$. Since the dynamics of a generic point of string is governed by local physics, then we can use to describe the open string dynamics by the Polyakov action (2.4) with the same equation of motion of closed strings implemented with boundary conditions. The request for validity of the variational principle is encoded in the following expression:

$$\partial_\sigma X^\mu \delta X_\mu = 0 \qquad \sigma = 0, \pi.$$

There are two different types of boundary conditions that can be imposed, Fig. (2.2):

Neumann conditions: $\partial_\sigma X^\mu = 0$ at $\sigma = 0, \pi$
Since we don't have restriction on δX for this type of conditions, the end-points of string can move freely.

Dirichlet conditions: $\delta X^\mu = 0$ at $\sigma = 0, \pi$
The end-points of string will remain in a constant position of space-time.

Let us consider an open string in d-dimensional space-time. We impose Neumann conditions for the first $p+1$ coordinates and Dirichlet conditions for the others:

$$\begin{aligned} \partial_\sigma X^a &= 0 \qquad && a = 0, 1, ..., p \\ X^I &= c^I \qquad && I = p+1, ..., d-1. \end{aligned}$$

Thus, the end-points of string can move freely on a $(p+1)$-dimensional hypersurface of space-time called *Dp-brane*, where p is its spatial dimension. The branes are dynamical object in string theory.
However this boundary condition breaks space-time Poincarè invariance, i.e. the Lorentz group is broken to:

$$SO(1, d-1) \quad \mapsto \quad SO(1, p) \times SO(d-p-1).$$

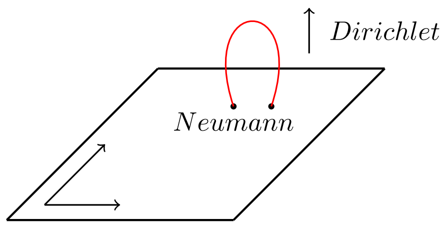

Figure 2.2: Neumann and Dirichlet conditions.

Let us start from the solutions of equation of motion written in terms of Fourier modes:

$$X_L^\mu(\sigma^+) = \frac{1}{2}x^\mu + \alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\tilde{\alpha}_n^\mu e^{-in\sigma^+}$$

$$X_R^\mu(\sigma^-) = \frac{1}{2}x^\mu + \alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu e^{-in\sigma^-},$$

we impose on them the boundary conditions:

$$\begin{array}{llll} \text{Neumann} & \partial_\sigma X^a = 0 & \alpha_n^a = \tilde{\alpha}_n^a & \\ \text{Dirichlet} & X^I = c^I & x^I = c^I, \quad p^I = 0, & \alpha_n^a = -\tilde{\alpha}_n^a \end{array}$$

So for both boundary conditions, we only have one set of independent oscillators, the others are then determined by the boundary conditions.
Now we want to quantize the theory, thus we promote x^a, p^a and α_n^μ to operator, where $a = 0, 1, ...p$ (Neumann directions). So we expect that the open string quantization gives rise to states which are restricted to lie on the brane.
Defining the light-cone coordinates lying on the brane:

$$X^\pm = \sqrt{\frac{1}{2}} \left(X^0 \pm X^p\right)$$

we can proceed to quantization in the same manner as for the closed string. The mass formula for the states is:

$$M^2 = \frac{1}{\alpha'} \left(\sum_{i=1}^{p-1} \sum_{n>0} \alpha_{-n}^i \alpha_n^i + \sum_{i=p+1}^{d-1} \sum_{n>0} \alpha_{-n}^i \alpha_n^i - a \right). \tag{2.27}$$

As in the closed string case we have the normal-ordering constant a, and the require of reduced symmetry $SO(1,p) \times SO(d-p-1)$ leads to same constraints $d = 26$ and $a = 1$. The differences with the closed case are the overall factor (4) and the presence of only α-modes.

2.7.1 The open string spectrum

The ground state is defined through:

$$\alpha_n^i |0, p\rangle = 0 \qquad n > 0, \quad i = 1, 2, ...p-1, p+1, ...d-1, \tag{2.28}$$

and its mass is negative:

$$M^2 = -\frac{1}{\alpha'}.$$

So, also the open string spectrum contains *tachyons* with mass halved compared to those of closed string.
In the first excited state, $N = 1$[1], we must distinguish between two classes of oscillators.
The oscillators *longitudinal* to brane:

$$\alpha^a_{-1}|0,p\rangle \qquad a = 1, ..., p-1$$

where the space-time indices transform under $SO(1,p)$. These states provide a vector representation for $SO(1,p)$, thus are massless particles ($M^2 = 0$) with spin $s = 1$ on the brane, i.e. photons.
The oscillators *transverse* to brane:

$$a^I_{-1}|0,p\rangle \qquad I = p+1, ...d-1.$$

These states are scalar under $SO(1,p)$, therefore they can be thought of as arising from scalar fields $\phi^I(\xi^a)$ living on the brane, where ξ^a are the brane coordinates. They describe transverse movements of the brane, and their coordinates dependence tell us that also deformations are allowed. The fields ϕ^I provide a vectorial representation of $SO(d-p-1)$.
The higher excited states at generic level N are labeled by their masses $M^2 = \frac{1}{\alpha'}(N-1)$ and spins. We have a their representation through discrete points of *Regge trajectories*.

2.7.2 Brane Dynamics

The branes are dynamical objects, indeed the massless fields $\phi^I(\xi)$ have a natural interpretation as transverse fluctuations of the brane. Therefore the brane should have an action which describes how it moves into space-time. This is the *Dirac action*:

$$S_{D_p} = -T_p \int d^{p+1}\xi \sqrt{-det\gamma},$$

where T_p is the tension of Dp-brane, ξ^a with $a = 0, ...p$ are the worldvolume coordinates of brane, and γ is the determinant of pull-back of $\eta_{\mu\nu}$ into the worldvolume:

$$\gamma_{ab} = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b}\eta_{\mu\nu}.$$

Finally, we wonder if it is possible to quantize the brane? The answer, currently, is no, because technical and conceptual problems arise. One of the problem is that the quantization of object with higher dimension of string leads to continuous spectra.

2.8 Path integral Quantization

The Feynmann path integral is a way to represent a quantum theory and it contains, within its definition, the interactions in string theory. The classical action (2.4) does not contain interaction terms, and the interaction is inserted by summing over all possible world-sheets.

[1] N is the number operator, defined separately for transverse and longitudinal modes.

Let us perform the sum over world-sheets through the integration over the Euclidean world-sheet metric $h_{ab}(\sigma_1, \sigma_2)$ and over all embeddings $X^\mu(\sigma_1, \sigma_2)$ of world-sheet in Minkowski space-time:

$$Z\left[\lambda\right] = \int \mathcal{D}X\mathcal{D}h e^{-S[h,X,\lambda]}, \tag{2.29}$$

where $S = S_P + \lambda\chi$, with:

$$\chi\left(\Sigma\right) = \frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{h}R^{(2)} + \frac{1}{2\pi}\int_{\partial\Sigma} dsk.$$

where $R^{(2)}$ is the Ricci-curvature of the world-sheet, ds is the proper time along the boundary in the metric h, and k is the geodesic curvature of the boundary.[2] The quantity χ depends only on the topology of the world-sheet, it is the *Euler number* of surface. In the closed string case the boundary integration is zero, therefore for Ricci-flat surface we can write $\chi\left(\Sigma\right) = 2(1-g)$, where g is the *genus* of two-dimensional surface. Using this definition on the path integral (2.29), we can look it as a sum on all the different topologies of world-sheet weighing by the factor g_s^{2g-2}, with $g_s = e^{-\lambda}$. Therefore g_s can be interpreted as a coupling constant, and the world-sheet sum as a perturbative expansion.
Now the problem is that the path integral (2.29) is not quite right. It contains a huge quantity of equivalent fields configurations (h, X), i.e. they are related to one another by *diff* $\times$ *Weyl* local transformations, thus they represent the same physical configuration. In order to solve the problem we define a new regoularized partition function $Z_R\left[\lambda\right]$ dividing the previous one by the volume of this local symmetry group:

$$Z_R\left[\lambda\right] = \int \frac{\mathcal{D}X\mathcal{D}h}{V_{diff\times Weyl}} e^{-S[h,X,\lambda]}.$$

A way to isolate the divergent part of integral is to use the *Faddev-Popov* method. The result is a regularized function integrated over all the fields $X(\sigma)$, and the measure of integration is weighed with the new quantity $\Delta_{FP}(h)$, called *Faddev-Popov determinant* :

$$Z_R\left[\lambda;\widehat{h}\right] = \int \mathcal{D}X\Delta_{FP}(\widehat{h})e^{-S\left[\widehat{h},X,\lambda\right]}.$$

The symbol $\widehat{h}$ represent a some fiducial metric chosen using the gauge freedom, and Δ_{FP} incorporates all possible inequivalent gauge transformation. It is possible to show that the Faddev-Popov operator can written introducing Grassmann *ghost* fields c^a and b_{ab} through the $S_g(c, b)$ action:

$$\Delta_{FP}(\widehat{h}) = \int \mathcal{D}c\mathcal{D}b e^{-\frac{1}{2\pi}\int d^2\sigma\sqrt{\widehat{h}}b_{ab}\nabla^a c^b}.$$

A key feature of string theory is that it is not consistent in all spacetime backgrounds, but only in those satisfying certain conditions. For Bosonic String the theory is consistent just in $D = 26$ dimensional space-time, and for the Superstring the require will become $D = 10$. In the light-cone analysis, this condition arises by requiring the Lorentz-invariance of space-time. From a world-sheet point of view the problem is related to transport of Weyl symmetry (classical) to quantum level. Classically the Weyl symmetry require that the world-sheet energy-momentum tensor is traceless, while at quantum level it is generally non-vanishing for curved world-sheet:

$$T^a_a(\sigma) = a_1 R^{(2)},$$

[2] $k = \pm t^a n_b \nabla_a t^t$, where t^a is a unit vector tangent to the boundary and n^a is an outward pointing unit vector orthogonal to t^a. The sign $+$ is for Lorentzian world-sheets, while the sign $-$ for Euclidian world-sheets.

where a_1 is a constant (*Weyl anomaly*), while $R^{(2)}$ is the Ricci-scalar of world-sheet. This quantity can be related to the central charge of a CFT on a flat world-sheet. Since a conformal transformation consists of a coordinate transformation plus a Weyl transformation, we obtain the following equalities:

$$c = -12a_1 \qquad\qquad T^a_a(\sigma) = -\frac{c}{12}R.$$

The world-sheet theory contains the fields $X^\mu(\sigma)$, with total central charge $c_X = D$, and the ghost fields (c, b), with total central charge $c_g = -26$, thus the total central charge of the theory is $c = c_X + g_g = D - 26$. The theory is Weyl-invariant only for $D = 26$. When the Weyl anomaly is non-vanishing, different gauge choices are inequivalent, and pathologies appear (non-unitarity, non-covariance). We conclude that a flat world-sheet CFT can be coupled to a curved metric in a Weyl-invariant way if and only if the total central charge c is zero. Moreover the central charge of anti-holomorphic sector must be $\bar{c} = c$, so to couple CFT and two-gravity in a diffeomorphic-invariant way.

2.9 String in curved space-time

In this section we would like to consider an extension of String theory to curved space-time.
The fields $X^\mu(\sigma)$ on the Polyakov action can be thought of as coordinates of world-sheet manifold Σ into a D dimensional space-time. Since these coordinates define a differentiable map:

$$i: \quad \Sigma \quad \longrightarrow \quad \mathcal{M}^D,$$

the final space is called *target space*, and the map is called *immersion*. So far we have considered only *Minkowski* space-time as target, with metric $\eta_{\mu\nu}$. The extension to curved spaces can be implemented by replacing the flat metric $\eta_{\mu\nu}(X)$ in the Polyakov action (2.4) with a general metric $G_{\mu\nu}(X)$:

$$S_G = \frac{1}{4\pi\alpha'}\int_\Sigma d^\sigma \sqrt{h} h^{ab} G_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu.$$

The graviton is a string state, and this new action can be thought of as describing a coherent state of gravitons by exponentiating the graviton vertex operator.
The inclusion of graviton on the world-sheet string action suggests a way to insert other string states. In particular we want to include all massless bosons of spectrum so to fix the background of theory. The immersion map induces a new function, called *pull-back*, that allows us to transfer the 2-form $B = B_{\mu\nu}dx^\mu dx^\nu$ of $\mathcal{M}$ into a 2-form on the worldsheet $i_*(B)$. Then, the new worldsheet string action takes the form:

$$S = -\frac{T}{2}\int_\Sigma d\sigma d\tau \left\{ \left[G_{\mu\nu}(x)\partial_a X^\mu \partial_b X^\nu \right] \eta_{ab} + i_*(B) \right\}$$

For closed surfaces ($\partial\Sigma = 0$) this action is invariant under the gauge-transformation[3]:

$$B \quad \longmapsto \quad B + d\Lambda, \tag{2.30}$$

which allows us to say that the strings are charged with respect to gauge-field B, and the charge is the tension T. Finally the dilaton can be inserted through a term like:

$$S_\Phi = \int_\Sigma d\sigma d\tau \Phi(x)\sqrt{-g}R_g$$

[3]This can be shown using the Stokes theorem on the variation of the action.

that respect the symmetries of initial action.
The total new action becomes:

$$S_\sigma = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{h}\left[\left(h^{ab}G_{\mu\nu}(X) + i\epsilon^{ab}B_{\mu\nu}(X)\right)\partial_a X^\mu \partial_b X^\nu + \alpha' R\Phi(X)\right]. \tag{2.31}$$

We remember that $B_{\mu\nu}(X)$ is an anti-symmetric tensor on the space-time, while the dilaton involves both the scalar field $\Phi(X)$ and the diagonal of $G_{\mu\nu}(X)$. The action (2.31) is space-time invariant under general change of coordinates $X^\mu \longrightarrow X'^\mu(X)$, in particular $G_{\mu\nu}$ and $B_{\mu\nu}$ transform as tensors and $\Phi(X)$ as a scalar. Moreover, the gauge invariance (2.30) is a generalization of the electromagnetic gauge transformation to a potential with two anti-symmetric indices.
A field theory such as (2.31), in which the kinetic term is field-dependent and no-longer quadratic in X, is called *Non Linear Sigma Model* (NLSM).
We expand the path integral around the classical solution as $X^\mu(\sigma) = x_0^\mu + Y^\mu(\sigma)$, and we obtain:

$$G_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu = \left[G_{\mu\nu}(x_0) + \partial_\omega G_{\mu\nu}(x_0)Y^\omega + \frac{1}{2}\partial_\omega\partial_\rho G_{\mu\nu}(x_0)Y^\omega Y^\rho + ...\right]\partial_a Y^\mu \partial_b Y^\nu.$$

For the field Y^μ we find an action with quadratic, cubic,... interactions, with coupling constants given by subsequent derivatives of metric. If R_c is the curvature radius of target space, the derivatives of the metric are of order R_c^{-1}. The effective dimensionless coupling is $\sqrt{\alpha'}R_c^{-1}$. Therefore if R_c is much greater that the string scale l_s, the coupling constant is small and the perturbative theory is valid. Moreover, since we are working with length scales much larger that the string length, we can ignore the internal structure of string and use low energy effective field theory.
Finally, NLSM is a renormalizable theory: the dimension of fields Y^μ is zero and all interactions have dimension two. Nevertheless the couplings are infinite in number.

Chapter 3

Supersymmetry

Supersymmetry (SUSY) is a space-time symmetry that maps particles and fields with integer spins (bosons) into particles and fields with half-integer spins (fermions), and vice versa, through certain G generators. In supersymmetric theories each one-particle state has at least a super-partner, therefore we can organize the states through *super-multiplets* of single particle states.
If SUSY is a *local* symmetry for the theory, then the theory must be diffeomorphic invariant: this tight tie between General Relativity and supersymmetry is holding in *Supergravity* theories. The number N of supersymmetry generators (fermionic) for a local and interacting theory, with maximal spin 2, is limited. In any dimension, if the maximal spin of the theory is 2 then there is a maximal number of 32 supersymmetries. In a $d = 4$ dimensional space-time, where a spinor has 4 real independent components, there are at most $32 = 4 \cdot 8$ supersymmetries ($N = 8$), for analogous reasons in a $d = 10$ space-time there are at most $32 = 16 \cdot 2$ supersymmetries ($N = 2$), while in a $d = 11$ space time there are at most $32 = 32 \cdot 1$ supersymmetries ($N = 1$).
There are several theoretical reasons that lead us to introduce the supersymmetry. On the one hand, the fermionic generators allow us to extend the Coleman-Mandula theorem, that provides a symmetry group of type '*Poincarè*×*Internal symmetries*', where the Poincarè generators commute with the Internal algebra, which generators must be scalars. On the other hand the Supersymmetry is the most plausible candidate to describe an extension of Standard Model.
We are interested in introducing supersymmetry in string theory to solve some failure of bosonic string. The presence of tachyons in the bosonic string spectrum leads to instability of the vacuum. One basic symmetry principle that guarantees the absence of a tachyon in the string spectrum is space-time supersymmetry. In order to define such symmetry we need to introduce fermionic degrees of freedom on the space-time, which come from fermionic degrees of freedom on the worldsheet.
We will discuss in this chapter fermionic string theories with worldsheet supersymmetry ([2], [6], [7], [12]), but that don't necessarily possess space-time supersymmetry, thus they will not all be tachyon-free. The way to introduce supersymmetry into string theory imposing world-sheet supersimmetry is called *Ramond-Neveau-Schwarz procedure*(RNS) which features manifest world-sheet supersymmetry but lacks manifest space-time supersymmetry.
Another way, called *Green-Schwarz formalism*, is to request manifest space-time supersymmetry at the cost of manifest world-sheet supersymmetry. The last procedure is not manifestally Lorentz-invariant. These two procedures are equivalent up to GSO-projection that remove all the unphysical states from the string spectrum, tachyons included.

3.1 Superconformal field theory

Since the world-sheet of a bosonic closed string is described by a two-dimensional Conformal Field Theory,we have previously discussed CFTs with bosonic fields. Now we are interested to provide a description of world-sheet of a closed Superstring that means a two-dimensional Superconformal Field Theory (SCFT). The SCFT is the generalization of a CFT with both bosonic and fermionic fields related by supersymmetry.

3.1.1 $\mathcal{N}=1$ Superconformal Model

Let us start exploring the simplest case of SCFT in two-dimensions containing a free bosonic field $X(z,\overline{z})$ and a free Majorana fermion $\psi(z,\overline{z})$. Using the map from cylindrical coordinates to complex coordinates $z=\tau+i\sigma$, $\overline{z}=\tau-i\sigma$, the action of the theory becomes:

$$S_{\mathcal{N}=1}=\frac{1}{4\pi\alpha'}\int dzd\overline{z}\left[2\partial X\overline{\partial}X-\alpha'\left(\psi\overline{\partial}\psi+\overline{\psi}\partial\overline{\psi}\right)\right].$$

We immediately note that there is an extra local symmetry, beyond the known ones, defined through the chiral fields transformations:

$$\delta X=\epsilon(z)\psi \qquad \delta\psi=\epsilon(z)\partial X \qquad \delta\overline{\psi}=0, \tag{3.1}$$

and anti-chiral fields transformations:

$$\delta X=\overline{\epsilon}(\overline{z})\overline{\psi} \qquad \delta\overline{\psi}=\overline{\epsilon}(\overline{z})\overline{\partial}X \qquad \delta\psi=0. \tag{3.2}$$

If the action is invariant under both chiral and anti-chiral transformations, we say that the theory is $\mathcal{N}=(1,1)$ supersymmetric.
Using the Noether's theorem we compute the *supercurrents* associated to this symmetry:

$$G(z)=i\left(\frac{2}{\alpha'}\right)^{1/2}:\psi\partial X: \qquad \overline{G}(\overline{z})=i\left(\frac{2}{\alpha'}\right)^{1/2}:\overline{\psi}\overline{\partial}X: \tag{3.3}$$

where we have imposed the *Normal Ordering* necessary to quantum level.
Since the bosonic and fermionic free fields are independent of each other, the energy-momentum tensor is simply the sum of the bosonic and the fermionic one:

$$\begin{aligned}T(z)&=-\frac{1}{\alpha'}:\partial X(z)\partial X(z):-\frac{1}{2}:\psi(z)\partial\psi(z):\\ \overline{T}(\overline{z})&=-\frac{1}{\alpha'}:\overline{\partial}X(\overline{z})\overline{\partial}X(\overline{z}):-\frac{1}{2}:\overline{\psi}(\overline{z})\overline{\partial\psi}(\overline{z}):.\end{aligned} \tag{3.4}$$

The periodic identification of cylinder $z\simeq z+2\pi$ plus Lorentz invariance lead to two possible periodicity conditions for ψ:

$$\begin{aligned}\psi(z+2\pi)&=e^{2\pi i\nu}\psi(z)\\ \overline{\psi}(\overline{z}+2\pi)&=e^{-2\pi i\overline{\nu}}\overline{\psi}(\overline{z})\end{aligned}$$

where $\nu,\overline{\nu}=0$ identifies the Ramond sector (periodic) and $\nu,\overline{\nu}=\frac{1}{2}$ the Neveu-Schwarz one (anti-periodic). The supercurrents have the same periodicity as the corresponding field. Thus, there are four different Hilbert space labeled by the periodicity conditions on the holomorphic and anti-holomorphic sectors $(\nu;\overline{\nu})$: $(NS-NS)$, $(NS-R)$, $(R-NS)$, $(R-R)$. Since the holomorphic and anti-holomorphic fermionic fields have conformal dimensions $\left(\frac{1}{2}, 0\right)$ and $\left(0, \frac{1}{2}\right)$, then their Laurent expansion is:

$$\psi(z) = \sum_{r\in\mathbb{Z}+\nu} \psi_r z^{-r-1/2} \qquad \overline{\psi}(\overline{z}) \sum_{r\in\mathbb{Z}+\nu} \overline{\psi}_r \overline{z}^{-r-1/2}. \tag{3.5}$$

Let us also recall the bosonic expansions:

$$\partial X(z) = -i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{r\in\mathbb{Z}} \alpha_r z^{-r-1} \qquad \overline{\partial} X(\overline{z}) = -i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{r\in\mathbb{Z}} \overline{\alpha}_r \overline{z}^{-r-1}. \tag{3.6}$$

We summarize their non-trivial quantization rules:

$$\begin{aligned} \{\psi_r; \psi_s\} &= \{\overline{\psi}_r; \overline{\psi}_s\} = \delta_{r;-s} \\ [\alpha_m; \alpha_n] &= [\overline{\alpha}_m; \overline{\alpha}_n] = m\delta_{m,-n} \end{aligned} \tag{3.7}$$

Using the two-point functions of bosonic and fermionic fields calculated on the first chapter, we obtain the following OPE of the chiral current $G(z)$:

$$\begin{aligned} G(z)G(w) &= \frac{1}{(z-w)^3} + \frac{2T(w)}{z-w} + \text{finite} \\ T(z)G(w) &= \frac{3}{2}\frac{G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w} + \text{finite}. \end{aligned}$$

We immediately notice that $G(z)$ is a primary field of conformal dimension $\left(\frac{3}{2}, 0\right)$, likewise we can obtain that also $\overline{G}(\overline{z})$ is a primary field of conformal dimension $\left(0, \frac{3}{2}\right)$. This suggest us the Laurent modes expansion:

$$G(z) = \sum_{r\in\mathbb{Z}+\nu} G_r z^{-r-\frac{3}{2}} \qquad \text{with} \qquad G_r = \sum_{s\in\mathbb{Z}} \alpha_s \psi_{r-s}. \tag{3.8}$$

We expand the new energy-momentum tensor (3.4) in a Laurent series $T(z) = \sum_{m\in\mathbb{Z}} L_m z^{-m-2}$ in the usual way. This implies $L_m = L_m^{bos} + L_m^{ferm}$ leading to:

$$L_m = \frac{1}{2}\left\{\sum_{r\in\mathbb{Z}} : \alpha_r \alpha_{m-r} : + \sum_{s\in\mathbb{Z}+\nu}\left(s + \frac{1}{2}\right) : \psi_{m-s}\psi_s :\right\}. \tag{3.9}$$

The normal ordering shift the zero mode L_0 by a constant $1/16$ in the Ramond sector.
We know that L_m^{bos} and L_m^{ferm} satisfy the Virasoro algebra with central charge $c = 1$ and $c = \frac{1}{2}$ respectively. Since these algebras are independent of each other, i.e. $\left[L_m^{bos}; L_n^{ferm}\right] = 0$,then we find that the new L_m satisfy the Virasoro algebra with central charge $c = 1 + \frac{1}{2} = \frac{3}{2}$.
Using the commutation rules (1.23) of primary fields $\partial X(z)$ $(h = 1)$ and $\psi(z)$ $h = \frac{1}{2}$ with the modes of stress energy-tensor, we obtain the commutator of L_m with the mode G_r:

$$\begin{aligned} [L_m; G_r] &= \sum_{s\in\mathbb{Z}} \left(\left[L_m^{bos}, \alpha_s\right]\psi_{r-s} + \alpha_s\left[L_m^{ferm}; \psi_{r-s}\right]\right) \\ &= \sum_{s\in\mathbb{Z}} \left[-s\alpha_{m+s}\psi_{r-s} + \alpha_s\left(-\frac{m}{2} - r + s\right)\psi_{m+r-s}\right] \\ &= \left(\frac{m}{2} - r\right) G_{m+r}. \end{aligned}$$

Again, we calculate also the anti-commutator of two Laurent modes G_r and G_s:

$$\begin{aligned}\{G_r;G_s\} &= \sum_{p,q\in\mathbb{Z}}\{\alpha_p\psi_{r-p};\alpha_q\psi_{s-q}\} = \sum_{p,q\in\mathbb{Z}}\left(\alpha_p\alpha_q\{\psi_{r-p};\psi_{s-q}\}+[\alpha_q;\alpha_p]\,\psi_{s-q}\psi_{r-p}\right)\\ &= \sum_{p\in\mathbb{Z}}\alpha_p\alpha_{r+s-p}+\sum_{u\in\mathbb{Z}+\nu}\left(u+\frac{1}{2}\right)\psi_{s+r-u}\psi_u-\sum_{u\in\mathbb{Z}+\nu}\left(r+\frac{1}{2}\right)\psi_{r+s-u}\psi_u\\ &= 2L_{r+s}+\frac{1}{2}\left(r^2-\frac{1}{4}\right)\delta_{r+s,0}.\end{aligned}$$

We have obtained an extension $\mathcal{N}=1$ of Virasoro algebra. The specification $\mathcal{N}=1$ refers to the fact that there is one *superpartner* for each bosonic field of initial theory: the free fermion $\psi(z)$ is the superpartner of the free boson, and $G(z)$ is the superpartner of $T(z)$. Finally, we summarize the rules of $\mathcal{N}=1$ *super Virasoro algebra* with central charge c:

$$\begin{aligned}[][L_m;L_n] &= (m-n)L_{m+n}+\frac{c}{12}(m^3-m)\delta_{m+n,0}\\ [L_m;G_r] &= \left(\frac{m}{2}-r\right)G_{m+r}\\ \{G_r;G_s\} &= 2L_{r+s}+\frac{c}{3}\left(r^2-\frac{1}{4}\right)\delta_{r+s,0}.\end{aligned}\tag{3.10}$$

The anti-holomorphic fields give a second copy of these algebras. The (3.10) algebra is infinite-dimensional, however it contains, on the NS sector, a finite subalgebra generated by $\left\{L_0;L_{\pm1},G_{\pm\frac{1}{2}}\right\}=$ *osp* $(1|2)$. The corresponding super group $OSP\,(1|2)$ plays the same role for SCFTs as $SL(2,\mathbb{C})/\mathbb{Z}_2$ does for usual CFTs and can be used to define super quasi-primary conformal fields.
In analogy to CFT, let us define $\mathcal{N}=1$ *superconformal highest weight state* by the conditions:

$$\begin{aligned}L_n|h\rangle &= 0 \qquad n>0\\ G_r|h\rangle &= 0 \qquad r>0.\end{aligned}$$

Then we can study highest weight representations of the $\mathcal{N}=1$ super Virasoro algebra. Unitary highest weight representations are possible only for discrete values of the central charge:

$$c=\frac{3}{2}\left(1-\frac{8}{(m+2)(m+4)}\right),\tag{3.11}$$

with $0<c<\frac{3}{2}$.

3.1.2 $\mathcal{N} = 2$ Superconformal Model

In this section we deal with an extension of previous model with two superpartners for each field of bosonic theory. The $\mathcal{N} = 2$ *Superconformal* theories have special applications in String Theory.
Let us define a complex free boson $\Phi(z, \overline{z})$ in terms of two real fields $X_{1,2}(z, \overline{z})$:

$$\begin{cases} \Phi(z, \overline{z}) = \frac{1}{\sqrt{2}} \left(X^{(1)}(z, \overline{z}) + iX^{(2)}(z, \overline{z}) \right) \\ \Phi^*(z, \overline{z}) = \frac{1}{\sqrt{2}} \left(X^{(1)}(z, \overline{z}) - iX^{(2)}(z, \overline{z}) \right). \end{cases}$$

and the corresponding currents $j(z) = i\partial\Phi(z, \overline{z})$ and $\overline{j}(\overline{z}) = i\overline{\partial}\Phi(z, \overline{z})$. Similarly we introduce the complex free fermionic fields $\Psi(z)$ and $\overline{\Psi}(\overline{z})$. We consider only holomorphic part and we write the fields through their real components:

$$\begin{cases} (i\partial\Phi)(z) = \frac{i}{\sqrt{2}} \left(\partial X^{(1)} + i\partial X^{(2)} \right) \\ \Psi(z) = \frac{1}{\sqrt{2}} \left(\psi^{(1)}(z) + i\psi^{(2)}(z) \right). \end{cases}$$

Since we are considering the free theory, the energy-momentum tensor will be the sum of a bosonic part with a fermionic part:

$$T(z) = -: \left(\partial\Phi\partial\overline{\Phi} \right) : (z) + \frac{1}{2} : \left(\Psi\partial\overline{\Psi} \right) : (z) + \frac{1}{2} : \left(\overline{\Psi}\partial\Psi \right) : (z), \tag{3.12}$$

where $\overline{\Phi}(z)$ and $\overline{\Psi}(z)$ denote the complex conjugate of bosonic and fermionic fields, they are not the anti-holomorphic fields. The Laurent modes L_m associated to $T(z)$ are written in terms of fields modes and satisfy a Virasoro algebra with central charge:

$$c = 1 + 1 + \frac{1}{2} + \frac{1}{2} = 3.$$

When we deal with complex fermionic fields, there exists another field of conformal dimension $h = 1$ expressed in the following way:

$$J_{U(1)}(z) = -: \left(\Psi\overline{\Psi} \right) : (z) = -i : \psi^{(1)}\psi^{(2)} : (z) \tag{3.13}$$

and associated modes:

$$J_n = +i \left(: \psi^{(1)}\psi^{(2)} : \right)_n = -i \sum_{s \in \mathbb{Z}+\nu} \psi^{(1)}_{n-s}\psi^{(2)}_s.$$

In analogy with the previous section , we can construct two new *supercurrents* fermionic fields:

$$G^+(z) = \sqrt{2}i : \left(\partial\overline{\Phi}\Psi \right) : (z) \qquad\qquad G^-(z) = \sqrt{2}i : \left(\partial\Phi\overline{\Psi} \right) : (z). \tag{3.14}$$

Using the ordinary modes expansion for the bosonic and fermionic fields, and the fact that their corresponding theories commute, we obtain following modes expansions for the supercurrents:

$$G^{\pm}(z) = \sum_{r \in \mathbb{Z}\pm\nu} G^{\pm}_r z^{-r-\frac{3}{2}}$$

$$G^{\pm}_r = \frac{1}{\sqrt{2}} \sum_{s \in \mathbb{Z}} \left(\alpha^{(1)}_s \mp i\alpha^{(2)}_s \right) \left(\psi^{(1)}_{r-s} \pm i\psi^{(2)}_{r-s} \right).$$

Now, we have everything we need to compute the commutators and anti-commutators between modes. The $\mathcal{N}=2$ *superconformal algebra* is expressed in terms of the Laurent modes L_m of the energy-momentum tensor (generators of conformal symmetry), its superpartners $G_r^{(\pm)}$ (generators of supersymmetry) and the modes J_n (generators of $U(1)$ current). For half-integer labels of $G_r^{\pm}$ this algebra is also known as *Neveu-Schwarz* algebra, while for integer labels is called *Ramond* algebra. Summarize the results.

$$
\begin{aligned}
&[L_m; L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n,0} \\
&[L_m; J_n] = -nj_{m+n} \\
&\left[L_m; G_r^{\pm}\right] = \left(\frac{m}{2} - r\right) G_{m+r}^{\pm} \\
&[J_m; J_n] = m\delta_{m+n,0} \\
&\left[J_m; G_r^{\pm}\right] = \pm G_{m+r}^{\pm} \\
&\left\{G_r^{+}; G_s^{+}\right\} = \left\{G_r^{-}; G_s^{-}\right\} = 0 \\
&\left\{G_r^{+}; G_s^{-}\right\} = 2L_{r+s} + (r-s)j_{r+s} + \left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}.
\end{aligned}
\tag{3.15}
$$

The first line is the usual Virasoro algebra with central charge c. The second and the third lines tell us that $G^{\pm}(z)$ and $J(z)$, respectively, are primary fields of conformal dimension $h_J = 1$ and $h_G = \frac{3}{2}$. The next two lines specify the $U(1)$ current algebra, under which $G^{\pm}$ have charge ± 1. The last two relations describe supercurrents algebra.
Since the (3.15) admits a commuting subalgebra of Cartan generates by L_0 and J_0, then we can label the states of Hilbert space through their charges respect these operators:

$$
L_0|h,q\rangle = h|h,q\rangle \qquad\qquad J_0|h,q\rangle = q|h,q\rangle.
$$

The unitarity require the condition $h \geq \frac{|q|}{2}$, which is saturated by the *chiral* and *anti-chiral* states $|h,q\rangle$ defined by the following conditions in the NS sector:

$$
\begin{cases}
G_{-1/2}^{+}|h,q\rangle = 0 & \qquad chiral \\
G_{-1/2}^{-}|h,q\rangle = 0 & \qquad anti-chiral
\end{cases}
$$

and analogue conditions with the $G_0^{\pm}$ operators on the R sector. Now we focus on the holomorphic sector of the theory, and we define *Super primary states* via the following equations:

$$
G_{n+1/2}^{+}|h,q\rangle = G_{n+1/2}^{-}|h,q\rangle = 0 \qquad \forall n \geq 0.
$$

These states have conformal weight $h = \frac{q}{2}$. The operators $G_{-1/2}^{\pm}$ acting on primary fields define a super primary field with the following four components:

$$
\left(|h,q\rangle, G_{-1/2}^{+}|h,q\rangle, G_{-1/2}^{-}|h,q\rangle, G_{-1/2}^{+}G_{-1/2}^{-}|h,q\rangle\right). \tag{3.16}
$$

In the case of a chiral field the (3.16) defines a short supermultiplet BPS. These states are often present in unitary supersymmetric theories.

3.1.3 Spectral Flow

A special characteristic feature of $\mathcal{N}=2$ SCFT is the so-called *Spectral Flow*. There exists a continuous class of automorphisms of the $\mathcal{N}=2$ superVirasoro algebra that acts through deformation of generators modulated by a continuous parameter η:

$$\begin{aligned} L_n &\quad\Longrightarrow\quad L'_n = L_n + \eta j_n + \frac{\eta^2}{6} c\delta_{n,0} \\ j_n &\quad\Longrightarrow\quad j'_n = j_n + \frac{c}{3}\eta\delta_{n,0} \\ G_r^{\pm} &\quad\Longrightarrow\quad G_r'^{\pm} = G_{r\pm\eta}^{\pm}. \end{aligned} \tag{3.17}$$

The new generators still satisfy the super algebra (3.15). We notice that the moding of the generators $G_r^{\pm}$ is changed. In particular, if the parameter $\eta \in \mathbb{Z} + \frac{1}{2}$, then the flow maps the Neveu-Schwarz sector, with half-integer modes for the supercurrents, into Ramond sector with integer modes: there exists a one-to-one mapping between the sectors.
We will investigate the behaviour of corresponding representations under the spectral flow. The one-parametr transformations (3.17) can be written through a unitary one-parameter group that acts on the quantum mechanical operators and on the states as:

$$L'_n = U_\eta L_n U_\eta^+, \qquad j'_m = U_\eta j_m U_\eta^+, \qquad |\phi_\eta\rangle = U_\eta|\phi\rangle.$$

The eigenvalues of the new generators (L'_0, J'_0) are the same of old generators (L_0, J_0). The conformal dimension and the $U(1)$-charge of new states don't change comparated to the original theory:

$$\begin{aligned} L'_0|\phi_\eta\rangle &= U_\eta L_0 U_\eta^+ U_\eta|\phi\rangle = U_\eta h|\phi\rangle = h|\phi_\eta\rangle \\ j'_0|\phi_\eta\rangle &= U_\eta j_0 U_\eta^+ U_\eta|\phi\rangle = U_\eta q|\phi\rangle = q|\phi_\eta\rangle. \end{aligned}$$

Since the Spectral Flow has no effect on the moding of generators L_m and J_m, we can compare the new states $|\phi_n\rangle$ with the old ones. We conclude that the Spectral Flow transforms the original states into the new states through the map:

$$h_\eta = h - \eta q + \frac{\eta^2}{6} c \qquad\qquad q_\eta = q - \frac{c}{3}\eta.$$

Let us consider a Spectral Flow transformation with $\eta = \frac{1}{2}$, that maps the Neveu-Schwarz sector into the Ramond sector, and we write its action on a chiral primary field of weight h_0:

$$\left| h_0 = \frac{q_0}{2}, q_0 \right\rangle_{NS} \qquad\Longrightarrow\qquad \left| h_{1/2} = \frac{c}{24}, q_{1/2} = q_0 - \frac{c}{6} \right\rangle_R.$$

In the Ramond sector there are different ground states with the same weigh $h_{1/2}$ and different many $U(1)$-charge $q_{1/2}$ as there are chiral primaries in the NS sector.
We call *Specral Flow operator* the operator that maps the NS vacuum with $h = q = 0$ to Ramond sector. This field must have conformal weight $h = c/24$ and charge $q = -c/6$.

3.2 $\mathcal{N} = (4, 4)$ superconformal field theories

The matter field on the world-sheet are the coordinates of string on d-dimensional space-time. The treatment in terms of two-dimensional CFT allows us to split our analysis on chiral algebras in two sectors: one purely holomorphic and one anti-holomorphic. These sectors can be to treat separately. We have discussed the $\mathcal{N} = 1, 2$ Superconformal algebras, now we will discuss $\mathcal{N} = 4$ SCA, in the holomorphic sector. In order to complete our Strings description, we must consider

also anti-olomorphic sector. The result is a pair of Superconformal algebras indicated with the supersymmetries numbers on both sectors $\left(\mathcal{N},\overline{\mathcal{N}}\right)$.
Let us consider an extension of $\mathcal{N}=(2,2)$ superconformal algebra by a current $su(2)\oplus su(2)$ with $c=\tilde{c}=6k$, we obtain a $\mathcal{N}=(4,4)$ superconformal algebra at level k. We are interested to the case $k=1$. The holomorphic $\mathcal{N}=4$ algebra is generated by four supercurrents $G_{\pm}(z),G'_{\pm}(z)$ of conformal weight $(3/2,0)$, the stress energy tensor $T(z)$ and a $su(2)$ Kac-Moody algebra whose currents $J^3(z),J^{\pm}(z)$ are contained into fermionic $so(4)$. The commutation relations are summarized below:

$$\begin{array}{cc}
\left[L_m;j_n^3\right]=-nj_{m+n}^3 & \left[L_m;j_n^{\pm}\right]=-nj_{m+n}^{\pm}\\
\left[L_m;G_r^{\pm}\right]=\left(\frac{m}{2}-r\right)G_{m+r}^{\pm} & \left[L_m;G_r^{'\pm}\right]=\left(\frac{m}{2}-r\right)G_{m+r}^{'\pm}\\
\left[2J_m^3;2J_n^3\right]=2m\delta_{m+n,0} & \left[J_m^3;J_n^{\pm}\right]=\pm J_{m+n}^{\pm}\\
\left[J_m^+;J_n^-\right]=m\delta_{m+n,0}+2J_{m+n}^3 & \left[J_m^{\pm};J_n^{\pm}\right]=0\\
\left[j_m^3;G_r^{\pm}\right]=\pm\frac{1}{2}G_{m+r}^{\pm} & \left[j_m^3;G_r^{'\pm}\right]=\mp\frac{1}{2}G_{m+r}^{'\pm}\\
\left[j_m^{\pm};G_r^{\mp}\right]=\pm G_{m+r}^{'\mp} & \left[j_m^{\pm};G_r^{\pm}\right]=0\\
\left[j_m^{\pm};G_r^{'\pm}\right]=\mp G_{m+r}^{\pm} & \left[j_m^{\pm};G_r^{'\pm}\right]=0\\
\left\{G_r^{\pm};G_s^{'\mp}\right\}=2(s-r)J_{s+r}^{\pm} & \left\{G_r^{\pm};G_s^{'\pm}\right\}=0\\
\multicolumn{2}{c}{\left\{G_r^+;G_s^-\right\}=\left\{G_r^{'+};G_s^{'-}\right\}=2L_{r+s}\quad\pm(r-s)J_{r+s}^3+2\left(r^2-\frac{1}{4}\right)\delta_{r+s,0}.}
\end{array}\tag{3.18}$$

The subscripts r,s take half-integer values for the NS sector and integer values for the R sector. If the (anti)holomorphic sector of space of states $\mathcal{H}$ of a field theory is the space of a representation of ($\overline{\mathcal{N}}=4$) $\mathcal{N}=4$ superconformal algebra then the theory is $\mathcal{N}=(4,4)$ supersymmetric.
Unitary representations of $\mathcal{N}=4$ SCA are possible only for values of central charge with positive integer levels. Furthermore, if the highest weight state ($\mathcal{N}=4$ *superconformal primary state*) has weight h and $su(2)_R$ spin $l\in\mathbb{Z}/2$, unitarity imposes also the constraints $h\geq l$ in the NS sector, and $h\geq\frac{k}{4}$ in the R sector.
There are two classes of unitary representations of $\mathcal{N}=4$ SCA labelled by (h,l): the *BPS* (*massless* or *short*) representations, and the *non-BPS* (*massive* or *long*) representations.

$$\text{MASSLESS}\quad\begin{cases}h=l & l=0,\frac{1}{2}\qquad\text{NS sector}\\ h=\frac{1}{4} & l=0,\frac{1}{2}\qquad\text{R sector}\end{cases}$$

$$\text{MASSIVE}\quad\begin{cases}h>0 & l=0\qquad\text{NS sector}\\ h>\frac{1}{4} & l=\frac{1}{2}\qquad\text{R sector}\end{cases}$$

The characters for the BPS and non-BPS representations in the NS sector are tabulated in the appendix B of [11].
In the full $\mathcal{N}=(4,4)$ SCFT, operators wich are BPS on both the left and right sides are called $\frac{1}{2}$-BPS, the operators that are BPS on one side and non-BPS on the other one are called $\frac{1}{4}$-BPS.
A realisation of small $\mathcal{N}=4$ SCA at level $k=1$ is given by a SCFT related to Type II Superstring theories compactified on a K3 surface. In this case the $\frac{1}{2}$ BPS primaries in the $(NS-NS)$ sector are the identity operator $\left(h=\overline{h}=l=\bar{l}=0\right)$ and 20 others operators $\left(h=\overline{h}=l=\bar{l}=\frac{1}{2}\right)$ which correspond to 20 $(1,1)$ harmonic forms on K3.
Under *Spectral Flow* the identity operator is mapped into the ground state of the $(R-R)$ sector labelled with $\left(h=\overline{h}=\frac{1}{4};l=\bar{l}=\frac{1}{2}\right)$. The others 20 operators are mapped into 20 ground states with $\left(h=\overline{h}=\frac{1}{4},l=\bar{l}=0\right)$.
The $K3$ CFT also contains $\frac{1}{4}$BPS primaries of spin $\left(s,\frac{1}{2}\right)$ and $\left(\frac{1}{2},s\right)$ for integer $s\geq1$. They are

captured by the $K3$ elliptic genus in the NS sector decomposed into $\mathcal{N}=4$ characters:

$$Z_{K3}^{NS} = 20ch_{1/2}^{BPS} + ch_0^{BPS} - ch_0^{non-BPS}\left(90q + 462q^2 + 1540q^3 + ...\right) \tag{3.19}$$

where the latter factor counts the $\left(s, \frac{1}{2}\right)$BPS primaries. Here we have assumed the absence of more currents at generic moduli of the $K3$ CFT which may be justified by conformal perturbation theory. In this case the only currents that appear on the non-BPS characters are the $\frac{1}{4}$BPS. Currents with generic spins can appear at special point of moduli space, they can be viewed as limits of non-BPS operators.

3.3 Superstring

Let us consider the bosonic string action (2.4) described through the fields $X^\mu(\tau,\sigma)$ representing their coordinates on the d-dimensional space-time. From a world-sheet point of view these are d scalar fields, but vectors in space-time. In order to require the local world-sheet supersymmetry, through the RNS procedure, we must add a set of fields $\psi_a^\mu(\tau,\sigma)$, where a is a spinorial world-sheet index, and μ a vectorial space-time one. Similarly we must add a fermionic superpartners to world-sheet metric h_{ab}: the *Gravitino* $\chi_a(\tau,\sigma)$, a scalar space-time but with a spinorial world-sheet index. The new full action is:

$$\begin{aligned} S = -\frac{1}{8\pi}\int d^2\sigma\sqrt{-h}\left[\frac{2}{\alpha'}h^{ab}\partial_a X^\mu \partial_b X_\mu + 2i\overline{\psi}^\mu \rho^a \partial_a \psi_\mu + \right. \\ \left. -i\overline{\chi}_a \rho^b \rho^a \psi^\mu \left(\sqrt{\frac{2}{\alpha'}}\partial_b X_\mu - \frac{i}{4}\overline{\chi}_b \psi_\mu\right)\right]. \end{aligned} \tag{3.20}$$

where ρ^a are two *Gamma matrices* in two dimensions. This action is invariant under global space-time *Poincarè* transformations, while the local world-sheet symmetries are:

Supersymmetry

$$\begin{aligned} \delta_\epsilon X^\mu &= \sqrt{\frac{\alpha'}{2}}i\overline{\epsilon}\psi^\mu \\ \delta_\epsilon \psi^\mu &= \frac{1}{2}\rho^a\left(\sqrt{\frac{2}{\alpha'}}\partial_a X^\mu - \frac{i}{2}\overline{\chi}_a \psi^\mu\right)\epsilon \\ \delta_\epsilon \chi_a &= 2D_a \epsilon \end{aligned}$$

where $\epsilon(\tau,\sigma)$ is a Majorana spinor.

Reparametrisations-invariance $(\tau,\sigma) \longmapsto (\tilde{\tau},\tilde{\sigma})$.

Weyl transformations

$$\begin{aligned} \delta_\Lambda X^\mu &= 0 \\ \delta_\Lambda \psi^\mu &= -\frac{1}{2}\Lambda\psi^\mu \\ \delta_\Lambda \chi_a &= \frac{1}{2}\Lambda\chi_a \end{aligned}$$

where $\Lambda(\tau,\sigma)$ is a scalar function.

Super-Weyl transformations

$$\delta_\eta \chi_a = \rho_a \eta$$

where $\eta(\tau, \sigma)$ is a Majorana spinor. The other fields transformations are trivial.

Let us use the light-cone coordinates $\sigma^\pm$ and choose the gauge:

$$h_{ab} = \eta_{ab} \qquad and \qquad \chi^a = 0,$$

then the action (3.20) becomes:

$$S = \frac{1}{\pi\alpha'} \int d^2\sigma \eta_{\mu\nu} \left[\partial_+ X^\mu \partial_- X^\nu + \frac{i}{2} \alpha' \left(\psi_-^\mu \partial_+ \psi_-^\nu + \psi_+^\mu \partial_- \psi_+^\nu \right) \right]. \tag{3.21}$$

There is still one residual gauge, therefore we can choose to fix the light-cone gauge analogous to the bosonic case:

$$X^+ = \beta\alpha' p^+ \tau \qquad \psi_a^+ = 0.$$

Imposing the latter conditions on the equations of motion, we obtain X^- and ψ_a^- in terms of transverse coordinates X^I and ψ_a^I.

3.3.1 Open Superstring

Let us now focus on the fermionic part of action (3.21). The equation of motions of fermionic fields for the transverse directions are:

$$\partial_+ \psi_-^I = 0 \qquad \partial_- \psi_+^I = 0$$

to which we have to add boundary conditions to string extremes. We choose:

$$\psi_+^I(\sigma = 0) = +\psi_-^I(\sigma = 0) \qquad \psi_+^I(\sigma = \pi) = \pm\psi_-^I(\sigma = \pi),$$

where the sign '+' defines the periodic *Ramond* sector, while the opposite sign the anti-periodic *Neveu-Schwarz* one.

We choose to work with a unique field defined as:

$$\psi^I(\tau, \sigma) = \begin{cases} \psi_-^I(\tau, \sigma) & \sigma \in [0; \pi] \\ \psi_+^I(\tau, -\sigma) & \sigma \in [-\pi; 0] \end{cases}$$

The periodicity conditions in terms of the new field become:

$$\begin{aligned} \psi^I(\tau, -\pi) &= \psi^I(\tau, \pi) && \text{Ramond sector} \\ \psi^I(\tau, -\pi) &= -\psi^I(\tau, \pi) && \text{Neveu-Schwarz sector.} \end{aligned}$$

We would like to construct the Hilbert space of theory, knowing that the whole Hilbert space will be the sum up two sectors. Let us start with the anti-periodic sector, the mode expansion of the general solution of the equations of motion is:

$$\psi^I(\tau, \sigma) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} b_r e^{-ir(\tau - \sigma)}. \tag{3.22}$$

Now we calculate the momenta conjugated to fields and we compute the common Dirac brackets. Finally, through the usual quantization procedure, we promote the modes b_r to operators and we replace the Dirac brackets with anti-commutator:

$$\left\{ b_r^I, b_s^J \right\} = \delta^{IJ} \delta_{r+s,0}.$$

This relations tell us that the modes b_r with $r < 0$ are the creation operators, and oscillators with $r > 0$ are annihilation operators.
Let us define the vacuum state $|0\rangle_{NS}$ on the Neveu-Schwarz sector as the state annihilated by each positive modes:

$$b_r|0\rangle_{NS} = 0 \qquad if \quad r > 0.$$

The other states are obtained acting with the creation operators on the vacuum:

$$\underbrace{...\left(b^{I_2}_{-r_2}\right)^{n_{r_2}}\left(b^{I_1}_{-r_1}\right)^{n_{r_1}}|0\rangle_{NS}}_{\text{Fermionic part}}\otimes\underbrace{...\left(\alpha^I_{-n}\right)^{\sharp_n}...|0,p\rangle}_{\text{Bosonic part}} \tag{3.23}$$

We notice that the power of fermionic operator is 0 or 1, and the entire Hilbert space is the tensor product between bosonic and fermionic space.
The new *Number Operator* includes a fermionic part and a bosonic part:

$$N^\perp = \sum_{n=1}^{\infty} n\left(\alpha^I_n\right)^+\alpha^I_n + \sum_{r\in\mathbb{N}+\frac{1}{2}} rb^I_{-r}b^I_r.$$

From a space-time point of view the entire state is a bosonic state.
Imposing the constraints, arising from the equations of motion of the metric, we find the mass condition:

$$M^2 = \frac{1}{\alpha'}\left(N^\perp + a\right), \tag{3.24}$$

where $a_B = -\frac{1}{24}$ and $a_F = -\frac{1}{48}$. In order for the theory to be Lorentz-invariant we will show that $a = -\frac{1}{2}$, then the *critical dimension* of Superstring theory is:

$$a = -\frac{d-2}{24}\left(1+\frac{1}{2}\right) \quad \longmapsto \quad d = 10. \tag{3.25}$$

Let us now look at the spectrum of the fermionic string in the *NS*-sector.
Let us start with the ground state $|0\rangle_{NS}\otimes|0,p\rangle$ that has no excited oscillators. The condition (3.24) tells us that this state has a negative mass-squared:

$$M^2 = \frac{a}{\alpha'}$$

The corresponding particle in the space-time is a *tachyonic* scalar field.
The first excited states have $N^\perp = \frac{1}{2}$ and are obtained acting on the fermionic vacuum state:

$$b^I_{-\frac{1}{2}}|0\rangle_{NS}\otimes|0,p\rangle. \tag{3.26}$$

The result are $(d-2)$ states. Since there is not way to package these state into a massive representation of Lorentz group on $(d-2)$ dimensions, they must be massless particles, therefore:

$$M^2 = 0 \qquad a = -\frac{1}{2}.$$

These states are *photons* of space-time. In $d = 10$ all higher excited states will be massive, so they will be representations of $SO(9)$.
Let us now examine the states in the *Ramond* sector of Hilbert space, with periodic boundary conditions. The modes expansion becomes:

$$\psi^I(\tau,\sigma) = \sum_{n\in\mathbb{Z}} d^I_n e^{-in(\tau-\sigma)}. \tag{3.27}$$

Promoting the oscillation modes to operator, their anti-commutation rules are:

$$\left\{d_n^I, d_m^J\right\} = \delta^{IJ}\delta_{n+m,0}, \tag{3.28}$$

where the positive modes are annihilation operators, and the negative ones are creation operators. Since there are also zero modes that satisfy a Clifford algebra $\left\{d_0^I; d_0^J\right\} = \delta^{IJ}$ on the Euclidian space of transverse coordinates, we expect to have space-time spinors. We introduce a new representation of Clifford algebra through the creation operators $\left\{D^i\right\}$, $i = 1,2,3,4$ and the annihilation operators $\left\{\overline{D}^i\right\}$:

$$D^1 = \frac{d_0^1 + id_0^2}{\sqrt{2}} \qquad \overline{D}^1 = \frac{d_0^1 - id_0^2}{\sqrt{2}}$$
$$\left\{D_1; \overline{D}_1\right\} = 1$$

and analogous relations for the other operators.
Let us define the vacuum state in the Ramond sector as the state annihilated by each positive mode and such that:

$$D^i|0\rangle_R = 0 \qquad i = 1,2,3,4.$$

The states:

$$\left(\overline{D}^4\right)^\natural \left(\overline{D}^3\right)^\natural \left(\overline{D}^2\right)^\natural \left(\overline{D}^1\right)^\natural |0\rangle_R \equiv |R\rangle_A$$

with $A = 1,2,3,...16$ and $\natural = 0,1$ are *Ramond vacua* and Dirac fermions on $d-2 = 8$ dimensions. These states form a reducible representation of $SO(8)$:

$$\mathbf{16} = \mathbf{8}_s \oplus \mathbf{8}_{\overline{s}}$$

respectively generated by an even/odd number of vacuum fermionic operators.
The mass condition is (3.24) with $a_B = a_R = \frac{1}{24}$. In this sector the bosonic contribute delete exactly fermionic contribute. Therefore the ground level, i.e. $N^\perp = 0$, are 16 massless states.
The other states are constructed acting with the creation operators on these vacua:

$$... \left(d_{-n}^I\right)^\natural |R\rangle_a \otimes ... \left(\alpha_{-m}^I\right)^{\sharp_m} ...|0, p\rangle.$$

3.3.2 GSO-projection

It can be shown that the fermionic string theory with all the states in both the NS and R sectors is inconsistent. This can be reconciled by making a truncation of the spectrum, called the GSO (Gliozzi-Scherk-Olive) projection, which renders the spectrum tachyon free and space-time supersymmetric.
Let's define a quantum number, eigenvalue of operator $(-1)^F$, where F is the world-sheet fermion number. We impose that the new operator anti-commutes with the fermionic modes in both sectors:

$$\left\{(-1)^F; b_r^I\right\} = 0 \qquad \left\{(-1)^F; d_n\right\} = 0$$

and commute with Hamiltonian, i.e. $(-1)^F$ is a conserved charge.
We assign the charge -1 to vacuum state on NS sector:

$$(-1)^F |0\rangle_{NS} = -|0\rangle_{NS}$$

while on the R sector we have:

$$(-1)^F |R\rangle_A = \pm|R\rangle_A$$

with the sign '+' for states on the even representation $\mathbf{8}_s$ of $SO(8)$ and '$-$' for states on the odd representation $\mathbf{8}_{\overline{s}}$.
The GSO projection amounts to demanding that all states have chirality $(-1)^F = +1$. Now we arrive at a supersymmetric spectrum between two sector. Since the NS sector provides space-time bosons, and the R sector space-time fermions, through GSO-projection we have introduced the space-time supersymmetry in $d = 10$ and we have deleted the tachionic particles from the spectrum.
We can simply see that the number of massless bosons and fermions is equal, it can show for each mass level, we have $\mathcal{N} = 1$ space-time SUSY.

3.3.3 Closed Superstring

We obtain the closed string by formally considering two open strings identified respectively with the *Left* and *Right* sectors. In order to build the closed string spectrum, we must take the tensor product of two open string spectra.
Since we want to eliminate the tachyons, we consider $(-1)^F = +1$ on the NS-sector. We still have the sectors listed in Tab.(3.1).
(NS, NS) and (R, R) sectors lead to space-time bosons, while the two sectors (NS, R) and

Left	Right
NS_+	NS_+
$R_\pm$	$R_\pm$
NS_+	$R_\pm$
$R_\pm$	NS_+

Table 3.1: Oper Supertring Sectors

(R, NS) lead to space-time fermions. On the Ramond sector we have two different chiralities, therefore we can choose between two inequivalent possibilities: to have same chirality in both left and right sectors, or to have opposite chirality. In particular, the theory with $(-1)^{F_R} = (-1)^{F_L} = 1$ is called *Type IIB Superstring*, while the theory with $(-1)^{F_L} = -(-1)^{F_R} = 1$ is called *Type IIA Superstring*. These theories have no tachyons in their spectrum, and they are the only two possible consistent superstring theories.

$$\begin{array}{ll} (NS_+, NS_+)\,, (R_+, R_-)\,, (NS_+, R_-)\,, (R_+, NS_+) & \text{Type IIA} \\ (NS_+, NS_+)\,, (R_+, R_+)\,, (NS_+, R_+)\,, (R_+, NS_+) & \text{Type IIB} \end{array}$$

The massless spectrum in terms of representations of products of little group $SO(8)$ for the *IIB Superstring* is:

$$\begin{array}{l} \text{Bosons:} \quad [(1) + (28) + (35)_v]_{NS-NS} + [(1) + (28) + (35)_s]_{RR} \\ \text{Fermions:} \quad [(8)_c + (56)_c]_{NS-R} + [(8)_c + (56)_c]_{R-NS}\,. \end{array}$$

We find a total of 128 bosonic and 128 fermionic states, indicating a supersymmetric spectrum. The massless spectrum of this theory is that of type IIB supergravity in 10 dimensions. In the NSNS sector we have always a dilaton, a B-field rappresented by $(28)_s$ representation, and one graviton corresponding to $(35)_v$ representation. In the RR sector we have one 1-form, one 3-form and one 5-form self-dual. The fermionic degrees of freedom are those of two on-shell gravitinos $(56)_c$ with spin $\frac{3}{2}$ and of two spin $\frac{1}{2}$ fermions, called dilatinos. The presence of two spacetime

supersymmetries (two gravitinos) is the signature of an $\mathcal{N}=2$ supersymmetric theory.
The massless spectrum in terms of representations of products of little group $SO(8)$ for the *IIA Superstring* is:

$$\begin{aligned}
&\text{Bosons:} \quad [(1)+(28)+(35)_v]_{NS-NS}+[(8)_v+(56)_v]_{RR} \\
&\text{Fermions:} \quad [(8)_c+(56)_c]_{NS-R}+[(8)_s+(56)_s]_{R-NS} \, .
\end{aligned}$$

In the bosonic sector we find the same degrees of freedom of the type IIB in the NSNS sector, while in the RR sector we have one 0-form, one 2-form and one 4-form of a graviton in $(35)_v$, an anti-symmetric rank three tensor $(56)_v$, an anti-symmetric rank two tensor (28), one vector $(8)_v$ and one real scalar (dilaton). In the fermionic sector, there are two gravitinos with spin $\frac{3}{2}$ and two dilatinos of spin $\frac{1}{2}$. Again, we have $\mathcal{N}=2$ spacetime supersymmetry.
Both *Type IIA* and *Type IIB* theories own two supersymmetry generators G^I, with $I=1,2$, that are Majorana-Weyl spinors of $SO(1,9)$.[7] They have $32=16+16$ real components, therefore we say that the theories of *Type II* have 32 supercharges.
When we compactifies on a torus T^4, the result is a four-dimensional theory with $\mathcal{N}=8$ spacetime supersymmetry, then there are still 32 supercharges corresponding to eight fermionic generators (Majorana spinors of $SO(1,3)$). However, other types of compactification reduces the number of supercharges, i.e. the compactification breaks some supersymmetries.

Chapter 4

String compactifications

In *General Relativity* the space-time is dynamic and 4-dimensional, while the Bosonic String and the Superstring theories are consistent only in respectively $D = 26$ and $D = 10$ dimensional spaces-time. The idea is to write the D-dimensional space-time as a product of a d-dimensional Minkowski space (e.g. 4-dimensional 'space-time') and a some $D - d$-dimensional compact Riemannian space:

$$\mathcal{M}^D = \mathcal{M}^d \times K^{D-d}. \tag{4.1}$$

The physics on length scales much larger than the size of K is the same as in a d-dimensional Minkowski space: $D - d$ dimensions have been *compactified.*
In the next sections of this chapter we follow the analysis of compactiofications given in [2], [6], [7].

4.1 Toroidal compactification

Let's consider a $D = d + 1$-dimensional space-time with direction X^d periodic:

$$X^d \sim X^d + 2\pi R.$$

This is called *toroidal compactification* along the X^d direction. The metric tensor can be splitted between compact and non-compact directions:

$$ds^2 = G^D_{MN} dX^M dX^N = G_{\mu\nu} dX^\mu dX^\nu + G_{dd} \left(dX^d + A_\mu dX^\mu\right)^2$$

where indices μ, ν run on non-compact dimensions $0, 1, ..., d-1$. Let's suppose that $G_{\mu\nu}, G_{dd}$ and A_μ depend only on the non-compact coordinates. Then the metric G_{MN} is the most general metric invariant under traslations of X^d. Moreover this form allows us reparametrisation-invariance under the gauge transformations:

$$\begin{cases} X'^d(X^M) = X^d + \lambda(X^\mu) \\ A'_\mu = A_\mu - \partial_\mu \lambda \end{cases}$$

arise as part of the higher-dimensional coordinate group. This mechanism that incorporate gauge transformation through compactification is called *Kaluza-Klein mechanism.*

Now we take a CFT of a single periodic scalar field $X(z,\overline{z}) \equiv X^d$ and set $G_{dd} = 1$, then its worldsheet action is the same as a noncompact theroy:

$$S = \frac{1}{2\pi\alpha'} \int d^2z \partial X \overline{\partial} X.$$

The periodicity of the field has two main effect. One is that string states must be identified under $X \simeq X + 2\pi R$. This requires that the translation operator $exp\,(2\pi i R p)$ leaves the string states invariant, so the center of mass momentum is quantized:

$$p = \frac{n}{R} \qquad\qquad n \in \mathbb{Z}. \tag{4.2}$$

Another feature is special to string theory. Closed string states are labelled by number w of times that winds around the compact direction:

$$X\left(e^{2\pi i} z\right) = X\left(\sigma + 2\pi\right) = X(\sigma) + 2\pi R w. \tag{4.3}$$

The integer w is the *winding number*.
We consider the Laurent expansion of $e.o.m$ solutions:

$$\partial X(z) = -i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{m=-\infty}^{\infty} \frac{\alpha_m}{z^{m+1}}, \qquad \overline{\partial} X(\overline{z}) = -i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{m=-\infty}^{\infty} \frac{\tilde{\alpha}_m}{\overline{z}^{m+1}}. \tag{4.4}$$

The total change in coordinate X around the string is:

$$2\pi R w = \oint \left(dz \partial X + d\overline{z} \overline{\partial} X\right) = 2\pi(\alpha'/2)^{1/2} \left(\alpha_0 - \tilde{\alpha}_0\right), \tag{4.5}$$

the total Noether momentum is:

$$p = \frac{1}{2\pi\alpha'} \oint \left(dz \partial X - d\overline{z} \overline{\partial} X\right) = (2\alpha')^{-1/2} \left(\alpha_0 + \tilde{\alpha}_0\right). \tag{4.6}$$

We set the left and right momentum:

$$\begin{aligned} p_L &\equiv \left(2/\alpha'\right)^{1/2} \alpha_0 = \frac{n}{R} + \frac{wR}{\alpha'} \\ p_R &\equiv \left(2/\alpha'\right)^{1/2} \tilde{\alpha}_0 = \frac{n}{R} - \frac{wR}{\alpha'} \end{aligned} \tag{4.7}$$

so the Virasoro generators are:

$$\begin{aligned} L_0 &= \frac{\alpha' p_L^2}{4} + \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n \\ \tilde{L}_0 &= \frac{\alpha' p_R^2}{4} + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}\tilde{\alpha}_n \end{aligned} \tag{4.8}$$

The CFT on torus are a good representation for toroidal compactified dimensions in String theory, and allows us to describe the two compactification effects on the strings.

4.2 Narain compactification

Let us generalize the analysis to k periodic dimensions:

$$X^m \simeq X^m + 2\pi R \qquad 26-k \leq m \leq 25, \tag{4.9}$$

therefore we have $d-k$ non-compact dimensions and the spacetime is now $\mathcal{M}_d \times T^k$. We display in this section an elegant mathematical description for general toroidal compactifications.
For any given compactification the spectrum of momenta $(p_L; p_R)$ form a lattice in a $2k$ dimensional vector space $\mathbb{R}^{2k}$. Let us work with dimensionless momenta $l_{L/R} = \left(\frac{\alpha'}{2}\right)^{1/2} p_{L/R}$ and be Γ the corresponding lattice. The *OPE* of two vertex operators is:

$$\begin{aligned}&: e^{ip_L\cdot X_L(z)+ip_R\cdot X_R(z)} :: e^{ip'_L\cdot X_L(0)+ip'_R\cdot X_R(0)} : \\ &\sim z^{l_L\cdot l'_L}\overline{z}^{l_R\cdot l'_R} : e^{i(p_L+p'_L)\cdot X_L(0)+i(p_R+p'_R)\cdot X_R(0)} :\end{aligned}$$

i.e. as one vertex operator circles the other, the product take a phase $e^{2\pi i(l_L\cdot l'_L - l_R\cdot l'_R)}$. Then we must require for all momenta $l, l' \in \Gamma$:

$$l \circ l' = l_L \cdot l'_L - l_R \cdot l'_R \quad \in \quad \mathbb{Z},$$

that defines an internal product with signature (k,k) in $\mathbb{R}^{2k}$. For each integer lattice the condition $\Gamma \subset \Gamma^*$ holds the modular invariance. The invariance under $\mathcal{T}: \tau \to \tau + 1$ require that $L_0 - \overline{L}_0$ be an integer for all string states (it suffices to see the (1.51)), from which it follows that Γ is an *even* lattice. Modular invariance under $\mathcal{S}: \tau \to -1/\tau$ requires a more detailed calculation. Let's start from the partition function (1.56) generalised to k compactification dimension, and use the following identity of δ function on the Poisson resummation formula:

$$\sum_{l'\in\Gamma} \delta(l-l') = V_\Gamma^{-1} \sum_{l''\in\Gamma^*} e^{2\pi i l''\circ l},$$

where the volume of a unit cell of lattice Γ fixes the normalization. Therefore we can write:

$$\begin{aligned}Z_\Gamma(\tau) &= V_\Gamma^{-1}(\eta(\tau))^{2k} \sum_{l''\in\Gamma^*} \int d^{2k} l e^{2\pi i l''\circ l + \pi i \tau l_L^2 - \pi i \overline{\tau} l_R^2} \\ &= V_\Gamma^{-1}(\tau\overline{\tau})^{-k/2}(\eta(\tau))^{2k} \sum_{l''\in\Gamma^*} e^{-\pi i l_L^2/\tau + \pi i l_R^2/\overline{\tau}} \\ &= V_\Gamma^{-1} Z_{\Gamma^*}\left(-\frac{1}{\tau}\right)\end{aligned}$$

where we have used the modular transformation $\mathcal{S}$ on the $\eta(\tau)$ function. Then the condition for modular invariance is that $\Gamma = \Gamma^*$. Thus, the consistency conditions require that Γ be an even self-dual lattice of $2k$ dimensions with signature (k,k).
We notice that these conditions depend on the momenta l only through the internal product $\circ$, which is invariant under $O(k,k,\mathbb{R})$ transformations. However $O(k,k,\mathbb{R})$ is not a symmetry of the theory, but the product $O(k,\mathbb{R})_L \times O(k,\mathbb{R})_R$, that changes only the basis of subspaces (l_L, l_R) separately, is a symmetry for the theory. Therefore the space of inequivalent theories corresponding to "inequivalent" lattices is:

$$\frac{O(k,k,\mathbb{R})}{O(k,\mathbb{R})_L \times O(k,\mathbb{R})_R} \tag{4.10}$$

This is equivalent to the earlier description in terms of the backgrounds of $G_{\mu\nu}$ and $B_{\mu\nu}$.

4.2.1 T-duality

Let us consider a bosonic string theory in $D = 26$ dimensions with only X^{25} compactified in a circle of radius R. The mass-shell condition (2.24) becomes:

$$M^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \left(N + \tilde{N} - 2\right) \tag{4.11}$$

the first term comes from the compact momentum, the second term from the potential energy of the winding state, the third term from the oscillators, and the last term from zero-point energy. At a generic value of R there are 24^2 massless states, as in the non-compact case, obtained by $w = n = 0$ and $N = \tilde{N} = 1$. However these states can be divided in four groups according to to whether the oscillations are in the non-compact directions μ or in the internal direction:

$$\begin{array}{cc} \alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|0,p\rangle & \left(\alpha^{\mu}_{-1}\tilde{\alpha}^{25}_{-1} + \alpha^{25}_{-1}\tilde{\alpha}^{\mu}_{-1}\right)|0,p\rangle \\ \left(\alpha^{\mu}_{-1}\tilde{\alpha}^{25}_{-1} - \alpha^{25}_{-1}\tilde{\alpha}^{\mu}_{-1}\right)|0,p\rangle & \alpha^{25}_{-1}\tilde{\alpha}^{25}_{-1}|0,p\rangle. \end{array}$$

The second states are *Kaluza-Klein* vectors on space-time, while the last state is a scalar. This massless spectrum is the same that we find by considering low energy field theory.
In our discussion of massless states we have omitted states that are massless for special value of R. For $R = \alpha'^{1/2}$, also the conditions:

$$\begin{array}{cc} n = w = \pm 1, \quad N = 0, \quad \tilde{N} = 1 & n = -w = \pm 1, \quad N = 1, \quad \tilde{N} = 0 \\ n = \pm 2, \quad w = N = \tilde{N} = 0 & w = \pm 2, \quad n = N = \tilde{N} = 0 \end{array}$$

defines massless states. The first four states includes four new gauge bosons that extend the *Kaluza-Klein* symmetry $U(1) \times U(1)$ into $SU(2) \times SU(2)$.
The relation (4.11) tell us that as $R \to \infty$ winding states becomes infinitely massive, while the compact momenta give a continuous spectrum. Looking the opposite limit we discover that the states carrying compact momentum become infinitely massive, while the winding states become continuous. Therefore the energy cost to warp a string around a small circle is low. Compact theories with small radii have similar spectra with noncompact theories, thus the limits $R \to 0$ and $R \to \infty$ provide the same physical theory. This equivalence is known as *T-duality.* In particular the spectra of String theories is invariant under:

$$R \leftrightarrow \frac{\alpha'}{R} \qquad n \leftrightarrow w,$$

the transformations of background fields under this symmetry is given from *Buscher's rules.* Again, compactifications on very small circles correspond to compactifications on very large circles. This is possible thanks to the existence of a typical length of string $l_s = 2\pi\alpha'^{1/2}$. Under T-duality the type *IIA* String theory is mapped to type *IIB* String theory and viceversa. The T-duality leaves fixed the holomorphic fields $\partial X^\mu, \psi^\mu$, while it changes sign of the anti-holomorphic counterparts $\bar{\partial} X^\mu, \bar{\psi}^\mu$. At quantum level this transformation is implemented by an operator that act on the right RR sector changing chiralities.
The compactification on several dimensions k enlarges the *T-duality* transformations to the automorphisms group $O(k, k, \mathbb{Z})$ of lattice $\Gamma^{k,k}$. Therefore the space of inequivalent lattice, i.e. inequivalent backgrounds, is:

$$\mathcal{M}_{\Gamma_{k,k}} = \frac{O(k,k,\mathbb{R})}{O(k,\mathbb{R})_L \times O(k,\mathbb{R})_R \times O(k,k\mathbb{Z})}. \tag{4.12}$$

4.3 Calabi-Yau compactification

In this section we would like to deal other compactifications beyond the toroidal one. The requirements for compact theories are to resolve the discrepancy between the critical dimension $D = 10$ of Superstring and $d = 4$ observed dimensions, and to provide a suitable low energy description of Standard Model of particle physics.
Let us require that the space-time manifold be given by (4.1) and keep only constant modes on compactified dimensions, so to obtain a dimensional reduction. $\mathcal{N} = 1, 2$ supersymmetric theories in $D = 10$-dimensional manifolds with toroidal compactification $K^{D-d} = T^6$ yield to $\mathcal{N} = 4, 8$ supersymmetric theories in $d = 4$ upon dimensional reduction. This is due to decomposition of Weyl representation of $SO(1, 9)$, where the supercharge Q lives, under $SO(1, 3) \times SO(6)$:

$$\mathbf{16} = (\mathbf{2_L}, \mathbf{4}) + \left(\mathbf{2_R}, \overline{\mathbf{4}}\right). \tag{4.13}$$

The large amount of supersymmetry renders toroidal compactifications of Superstrings unrealistic. The supersymmetric extension of Standard Model requires $\mathcal{N} = 1$ and $d = 4$ dimensions. To obtain such theories we must consider other ways of compactification that preserve just some supersymmetries, in particular we must understand the relation between the conserved supercharges on $d = 4$ dimensions and the manifold K^6.
The requirement of some unbroken supersymmetry under compactification leads to the existence of Killing spinors $\epsilon(x^M)$, which parametrizes the supersymmetry transformations, that satisfies the Killing conditions:

$$\langle 0|\nabla_M \epsilon|0\rangle = 0$$

where ∇_M is the covariant derivative containing the spin connections. This imposes topological and differential restrictions on the manifold. In particular, for $D - d = 6$, K^6 must have *Holonomy group* $\mathcal{H} = SU(3)$ (sufficient) and must be a compact Ricci-flat manifold (necessary). The decomposition of spinor representations $\mathbf{4}, \overline{\mathbf{4}}$ of $SO(6) \simeq SU(4)$ under the Holonomy group $\mathcal{H}$ gives us the number of singlet representations of $\mathcal{H}$ that there are on it. Since $\mathcal{H}$ can be seen as the set of possible transformations of a vector $\mathbf{v}$ transported along a closed curve of manifold K^6, and the Killing spinors must remain unchanged under parallel transport, then they are singlet under $\mathcal{H}$.
We suppose that $\mathcal{H} = SU(3)$, then the spinor representation $\mathbf{4}$ of $SO(6)$ is decomposed into $\mathbf{4}_{SU(4)} = (\mathbf{1} + \mathbf{3})_{SU(3)}$, which contains a singlet and a tripled. Therefore, if $\mathcal{H} = SU(3)$ there is one covariantly constant spinor of positive and one of negative chirality $\epsilon_\pm$. Starting from a $D = 10$ dimensional theory with $\mathcal{N} = 1$ supersymmetry, using the decomposition (4.13):

$$\epsilon = (\epsilon_R \otimes \epsilon_+) + (\epsilon_L \otimes \epsilon_-),$$

since ϵ is a Majorana spinor, hence (ϵ_R, ϵ_L) form a single $SO(1, 3)$ Majorana spinor associated to a single supersymmetry generator in $d = 4$.
If $\mathcal{H} = SU(2)$, the spinor representation $\mathbf{4}$ of $SO(6) \simeq SU(4)$ is decomposed into $\mathbf{4}_{SU(4)} = (\mathbf{1} + \mathbf{1} + \mathbf{2})_{SU(2)}$. There would exist two covariantly constant spinors of each chirality: we obtain $\mathcal{N} = 2$ supersymmetries in $d = 4$.
If we consider Type *IIA* and *IIB* Superstring theories, starting from $D = 10$ and $\mathcal{N} = 2$, the number of supersymmetries on $d = 4$ is doubled.
We define *Calabi-Yau* manifold CY_n as $2n$-dimensional compact Riemann manifolds with Holonomy group $SU(n) \subset SO(2n)$. They admit covariantly constant spinors and they are Ricci-flat. For $n = 1$ there is only a family of CY_1 manifolds: the torus T^2 parametrized by moduli space (4.12) with $k = 2$. There are two classes of CY_2 manifolds that we know: one is the torus T^4, and one are the $K3$ surfaces. For $n \geq 3$ there is a large number of distinct CY_n's classes.

We are interested to consider compactifications from $D = 10$, $\mathcal{N} = 1$ to $d = 6$. The require of unbroken supersymmetry allows us toroidal compactification T^4 or $K3$ compactification. The first case does not reduce the number of real supercharges (16 in $D = 10$ dimensions), therefore we obtain a theory in $d = 6$ with $\mathcal{N} = 2$ supersymmetries, or rather $(1, 1)$, one on the left-handed and the other a right-handed Weyl representations. In order to analyze the second case, we decompose the **16** Weyl representation of $SO(1, 9)$ under $SO(1, 5) \times SO(4)$:

$$\mathbf{16} = (\mathbf{4}_L, \mathbf{2}) + (\mathbf{4}_R, \mathbf{2}) \,,$$

where $\mathbf{4}_{L/R}$ are the Weyl spinors representations of $SO(1, 5)$, and $\mathbf{2}, \mathbf{2}'$ are those of $SO(4)$. The supersymmetry parameter ϵ is a Majorana-Weyl spinor in $SO(1, 9)$ with $16 = 8 + 8$ real component contained into $\mathbf{4}_{L/R}$ representations. Under $SU(2)$ only one $SO(4)$ spinor is covariantly constant,i.e. $\mathbf{2}$, therefore ϵ gives only a $\mathbf{4}_L$ supersymmetry.
Starting from $\mathcal{N} = 2$ in $D = 10$, $K3$ compactification leads to $\mathcal{N} = (1, 1)$ supersymmetries for *Type IIA*, and $\mathcal{N} = (2, 0)$ for *Type IIB*.
When we compactify a higher-dimensional theory D on $\mathcal{M}^D$, we distinguish between *internal* directions, along the compact subspace K^{D-d}, and *external* directions, along the d-dimensional Minkowski space $\mathcal{M}^d$. Now, we would like to know the resulting theory on d dimensions and how this is related to choice of K^D. If we consider toroidal compactifications, the procedure begins with a splitting of total metric between internal and external components:

$$G_{MN} = G_{\mu\nu} + G_{mn}, \tag{4.14}$$

where the greek indices run into external dimensions $\mu, \nu = 0, 1, .., d - 1$, and the latin indices rus into internal dimensions $m, n = d, d + 1, ...D - 1$. We replace the metric (4.14) in the action (2.31), and we Fourier expand the solutions of equations of motion in the internal directions. The result is a set of decoupled equations for all massive modes, and zero modes independent of the internal coordinates. We study the limit of small compactification radii ($R \to 0$), in which only zero modes remain light, while the others become very heavy and can be discarded. The heavy modes correspond to strings carrying momenta along the internal directions (torus). The discarding of massive modes coincide to eliminate the internal directions: we obtain a *dimensional reduction*.
Other types of compactification on curved internal manifolds allows us to discard massive modes on the $\tilde{R} \to 0$[1] limit and to obtain massless field corresponding to zero modes on internal directions. However the massive modes are no longer decoupled from each other, therefore setting massive modes to zero may not be a solution of equations of motion. Therefore, it is not immediate to write the effective action for the zero modes.
The equation of motion of a field $\phi^{mn\dots}_{\mu\nu\dots}(x^R)$ is written through differential operators $\mathcal{O} = \mathcal{O}_d + \mathcal{O}_{int}$ of order p (with $p = 1$ for fermions and $p = 2$ for bosons):

$$(\mathcal{O}_d + \mathcal{O}_{int})\, \phi = 0.$$

The zero modes are eigenstates of $\mathcal{O}_{int}$ operator with zero eigenvalue. For massless scalar fields with K^{D-d} compact manifold, we find $\mathcal{O}_{int} = \nabla$ with only one scalar zero mode, thus a scalar on D dimension produces just one massless scalar on d dimension.
For massless Dirac fields Ψ the operators $\mathcal{O}$ are Dirac operators, $\mathcal{O} = \Gamma^M D_M$, where the covariant derivative depends by background metric and gauge fields. If $D = 10$ and $d = 4$ the zero modes of $\not{D}_{int}$ are spinors in d dimension. The number of zero modes of $\not{D}_{int}$ depend to topological proprieties of compact manifold, for CY_3 compactification we have two zero modes ($\epsilon_\pm$).

[1] $\tilde{R}$ is a typical length scale of manifold K^{D-d}

For massless fields of higher dimensions there are p-form gauge fields $A^{(p)}$ in d dimension with field strength $F^{(p+1)} = dA^{(p)}$. Imposing the gauge fixing $d^*A^{(p)} = 0$ we obtain the following equation of motion:

$$\Delta A^{(p)} = (\Delta_d + \Delta_{int}) A^{(p)} = 0 \qquad \Delta_d = dd^* + d^*d,$$

therefore the number of massless d-dimensional fields is given by number of zero modes of internal Laplacian Δ_{int}. This is a cohomological problem, and the number of zero modes is the *Betti* number b_m. For example, let's consider the two-form B_{MN}, under compactification we can divide internal and external directions in the following form:

$$B_{MN} \to B_{\mu\nu} \oplus B_{\mu m} \oplus B_{mn},$$

where the first term is a scalar respect to internal manifold and, since $b_0 = 0$, it corresponds to a single zero mode on $\mathcal{M}^d$. The second term yields b_1 zero modes that are vectors in $\mathcal{M}^d$. The third term produces b_2 zero modes that are scalars on d dimensions. In general for each p-form in D dimensions, we obtain b_n massless fields, $n = 0, 1, ...p$, that are $(p-n)$-forms in d dimensions. Now we would like to analyze the zero modes of metric $G_{MN}(x)$, which decomposes as:

$$G_{MN} \to G_{\mu\nu} \oplus G_{\mu m} \oplus G_{mn}$$

Again, the first term correspond to a single zero mode, the second term has massless modes if $b_1 \neq 0$ which corresponds to massless gauge bosons in d dimensions. The last term yields massless modes that are scalars in d dimensions and it can written through a fluctuation term h_{mn} around to vacuum expectation value: $G_{mn} = \tilde{g}_{mn} + h_{mn}$. In order to unbroken supersymmetry the internal manifold must preserve Ricci-flatness: $R_{mn}(\tilde{g} + h) = 0$. Therefore the fluctuations h_{mn} are degeneracies of vacuum that preserve the Ricci-flatness. They are celled (metric) *moduli*, and they change the size and shape of compactification manifold, but not its topology.
The torus T^2 has one Kähler modulus (the area of the torus) and one complex structure modulus (the ratio τ). A fundamental propriety of compactification in String theory is that the same theory is obtained through different compactification manifolds, e.g. T-duality on toroidal compactifications.
Another example, coming from Calabi-Yau compactification and involving topologically different manifolds, is the *mirror* symmetry. For each Calabi-Yau manifold $\mathcal{M}$, there exists a mirror manifold $\widehat{\mathcal{M}}$ such that $IIA(\mathcal{M}) = IIB(\widehat{\mathcal{M}})$, including perturbative and non-perturbative effects. The manifolds of mirror pair have opposite Euler numbers. The mirror map $\mathcal{M} \leftrightarrow \widehat{\mathcal{M}}$ exchanges complex structure and Kähler moduli among manifolds.

4.4 The world-sheet perspective

We want to analyze string theories on K3 surfaces from the world-sheet point of view. The world-sheet approach describes perturbative aspects of the string theory.
We remember that on the world-sheet point of view the string theory is a two dimensional theory given by the map from a Riemann surface (the string world-sheet) into a target space X:

$$\xi : \Sigma \quad \to X.$$

In the conformal gauge the action (2.31) is:

$$S = \frac{i}{4\pi\alpha'} \int_\Sigma (G_{\mu\nu} - B_{\mu\nu})\, \partial X^i \overline{\partial} X^j d^2z - 2\pi \int_\Sigma \phi R^{(2)} d^2z + ... \tag{4.15}$$

where we have not considered the fermions, and G_{ij}, B_{ij}, ϕ are respectively the Riemann metric of target space, the components of real 2-form B and the dilaton, while $R^{(2)}$ is the curvature of Σ. This two-dimensional theory is a *non linear sigma model* described in the last section of the second chapter.
In order to obtain a consistent string theory we require that the two dimensional theory be conformally invariant with a specific value of central charge. This determines specifics constraints on the various parameters of the theory. One way of require the conformal invariance is to set ϕ constant, B closed and G_{ij} Ricci-flat.
Imposing supersymmetry on the *non linear sigma models* we obtain an interesting relation between these and the Kähler manifolds. A world-sheet supersymmetry transformation will be of the form:

$$\delta_\epsilon X^i = \bar{\epsilon} l^i_j \psi^j . \tag{4.16}$$

When $\mathcal{N} = 1$ we have necessarily $l^i_j = \delta^i_j$, while when $\mathcal{N} > 1$ each additional $\tilde{l}^i_j$ ($l^i_j = \delta^i_j + \tilde{l}^i_j$) acts as an complex structure and gives to X the structure of a Kähler manifold for $\mathcal{N} = 2$ and of hyperkähler manifold for $\mathcal{N} = 4$.
The conformal invariance imposes a division into the holomorphic and anti-holomorphic parts.
When X is a smooth K3 manifold we have an $\mathcal{N} = (4, 4)$ superconformal field theory, at least to leading order in α'/R^2. On this case, it has been demonstrated that there are no corrections to the Ricci-flat metric after the leading term: the Ricci-flat metric is the exact solution on K3 surfaces.
The goal is to find the moduli space of conformally invariant *non linear sigma models* with K3 target space. We have three fields in the action, their variations span the moduli space. Since a complete analysis is more complicate, we take some assumptions on the parameters variation.
We assume that any generic deformation of the Ricci-flat metric g_{ij} provides another inequivalent Ricci-flat metric with an inequivalent conformal field theory. We have 58 parameters for the metric (the dimension of moduli space of Einstein metrics on K3 surfaces (C.7)). The closed 2-form B accounts 22 parameters because we have negleted its exact part, because it is irrelevant under the map into target space. Finally we consider the dilaton. Since Φ is constant over Σ, we consider that the last term of (4.15) contributes to action by a sum over the genera g of Σ through the fuction $2\Phi(1 - g)$. Since we are considering the perturbative limit of the theory, where we can use the CFT on the fixed Σ surface, we will ignore this sum, thus we ignore the dilaton for the moment.
We have $58 + 22 = 80$ parameters to describe the moduli space.
$\mathcal{N} = (4, 4)$ superconformal field theory contains an $su(2) \oplus su(2) \simeq so(4)$ symmetry, this symmetry acts on the tangent directions to a point in the moduli space and so will be a subgroup of the holonomy group. Therefore we must find a 80 dimensional space that contains $SO(4)$ in the holonomy group.
The Berger and Simons theorem assures us that any smooth neighbourhood of the moduli space of conformally invariant *non linear sigma models* with a K3 target space is isomorphic to an open subset of:

$$\mathcal{T}_\sigma = \frac{O(4, 20)}{O(4) \times O(20)} \tag{4.17}$$

where $\mathcal{T}_\sigma$ is called *Teichmüller space*. Now we want to know the global structure of moduli space. For this we are obliged to make assumptions about our CFT: let's assume that the moduli space be a Hausdorff space, then it is isomorphic to $\mathcal{M}_\sigma \simeq G_\sigma \setminus \mathcal{T}_\sigma$, where G_σ is a some discrete group. The space $\mathcal{T}_\sigma$ is the Grassmannian of space-like 4-planes in $\mathbb{R}^{4,20}$, and must contain the Grassmannian of space-like 3-planes in $\mathbb{R}^{3,19}$ as subspace since this parametrised the Einstein metrics on K3 surfaces that appears in the action S.

Let us introduce the even self-dual lattice $\Gamma_{4,20} \subset \mathbb{R}^{4,20}$: we would like to show that $\Gamma_{4,20}$ play the same role for *non linear sigma models* as $\Gamma_{3,19}$ for Einstein metrics and that $G_\sigma \simeq O(\Gamma_{4,20})$.[2] Indeed, considered the previous theorem and the assumption that the moduli space is Hausdorff, we have that the moduli space of conformally invariant non linear sigma models on a K3 surface is:

$$\mathcal{M}_\sigma = O(\Gamma_{4,20}) \setminus O(4,20)/\left(O(4) \times O(20)\right) \tag{4.18}$$

where the Teichmüller space can be to decomposed as:

$$\mathcal{T}_\sigma = \frac{O(4,20)}{O(4) \times O(20)} \simeq \frac{O(3,19)}{O(3) \times O(19)} \times \mathbb{R}^{22} \times \mathbb{R}_+$$

The first factor is the Teichmüller space for the metrics on the K3 surfaces, the second for the B-field and the last for the volume.

4.5 Elliptic Genus of K3

Let's consider a conformal field theory sigma-moldel with target space K3. Since this theory has $\mathcal{N} = (4,4)$ superconformal symmetry on the world-sheet, his space of states can be decomposed into representations of superconformal algebra $\mathcal{N} = 4$ for left and right movers. We can to write the space of states decomposition:

$$\mathcal{H} = \oplus_{i,j} N_{ij} \mathcal{H}_i \otimes \mathcal{H}_j \tag{4.19}$$

where i and j labelled the different $\mathcal{N} = 4$ algebra representations, and N_{ij} is the multiplicity of $\mathcal{H}_i \otimes \mathcal{H}_j$ representation in the whole space. We remember that these representations are labelled by two quantum number corresponding to Cartan generators L_0 and J_0 (respectively $\overline{L}_0$, $\overline{J}_0$). Let us define the Ramond-Ramond ground states (RR) of a $\mathcal{N} = (4,4)$ SCFT as the set of states of Hilbert space that are annihilated by all the positive modes of the generators of two copy of (3.18) algebra and by zero modes of the fermionic currents. At the moment we focus our attention just to chiral sector of the Hilbert space and we introduce the *Witten index* through the following definition:

$$W(\tau) = Tr_{\mathcal{H}} \left[(-1)^{J_0} q^{L_0 - \frac{c}{24}} \right], \tag{4.20}$$

where $\mathcal{H}$ is just the chiral sector of whole Hilbert space. Using the commutation relation of superconformal algebra, we easily see that only Ramond ground states contribute to this quantity. Therefore the Witten index is independent by τ and count the number of Ramond ground states on $\mathcal{H}$. If we perform the same analysis for a non-chiral theory, with both left- and right moving degree of freedom, the Witten index will count the number of states that are Ramond ground states for both left and right copy of $\mathcal{N} = 4$ superconformal algebra.
[16] [17] Let us consider the partition function for a NLSM on some Calabi-Yau. While this can been calculated explicitly at some special points in the moduli space, the exact expression for a generic Calabi-Yau is not known. However, there exist a new function, independent of the metric and the choice of B-field of target space manifold, but dependent of his topology, called *Elliptic Genus* and defined as:

$$\phi(\tau, z) = Tr_{RR} \left(q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{\overline{c}}{24}} (-1)^{F + \overline{F}} y^{J_0} \right). \tag{4.21}$$

[2] $O(\Gamma_{4,20})$ contains elements that exchange two orthogonal $\Gamma_{2,10}$ sublattices and which they identify a pair of theories on K3 related by *Mirror symmetry*.

This trace is taken on the RR part of spectrum, the bar represents the right-moving modes, $q = e^{2\pi i\tau}$ and $y = e^{2\pi iz}$, F and $\overline{F}$ are the left-right-moving fermionic number operator, and J_0^3 is the zero mode of current J^3 coming from superconformal algebra. Notice that the elliptic genus does not depend on $\overline{\tau}$, because only right-moving ground-states contribute to the function, thus it is an holomorphic function. The other right-moving states that are not annihilated by a supercharge zero mode appear always as a boson-fermion pair: their contribution vanishes because $(-1)^{\overline{F}}$ weighs the two states with opposite signs. The same thing does not happen for left-moving contributions because of the $y^{J_0^3}$ factor.
The partition function count all states of Hilbert space on the left and right sector, while the Witten index counts only RR ground states. The Elliptic genus counts states that are Ramond ground states on the right side and unconstrained on the left-moving of Hilbert space.
Since we are considering superconformal theories with $K3$ target space, in this thesis we will focus on the Elliptic Genus for sigma models on $K3$.
A path integral interpretation shows that the (4.21) function has nice transformation proprieties under modular transformations. Moreover the Spectral Flow automorphism of the algebra leads constraints on the eigenspaces of L_0, J_0 operators. From this facts, it follows that the Elliptic Genus defines a weak Jacobi form of weight zero and index one.
A weak Jacobi form of weight w and index m is a function:

$$\begin{aligned} \phi_{w,m} : \mathbb{H}_+ \times \mathbb{C} \quad &\to\to \quad \mathbb{C} \\ (\tau, z) \quad &\to \quad \phi_{w,m}(\tau, z) \end{aligned}$$

that satisfies the following transformation proprieties:

$$\phi_{w,m}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^w\, e^{2\pi im\frac{cz^2}{c\tau+d}}\phi_{w,m}(\tau, z) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \tag{4.22}$$

$$\phi_{w,m}(\tau, z + l\tau + l') = e^{-2\pi im(l^2\tau+2lz)}\phi_{w,m}(\tau, z) \qquad l, l' \in \mathbb{Z}, \tag{4.23}$$

and admits the following Fourier expansion:

$$\phi(\tau, z) = \sum_{n\geq 0, l\in\mathbb{Z}} c(n,l)\, n^n y^l$$

with $c(n,l) = (-1)^w c(n,-l)$.
Weak Jacobi form have been classified and there is only one weak Jacobi form with $w = 0$ and $m = 1$ up to normalisation. Therefore the (4.21) must agree with this unique form. We can write the form $\phi_{0,1}(\tau, z)$ as

$$\phi_{0,1}(\tau, z) = 8 \sum_{i=2,3,4} \frac{\theta_i(\tau, z)^2}{\theta_i(\tau, 0)^2}$$

where $\theta_i(\tau, z)$ are the Jacobi Theta function defined in the Appendix A. Since the coefficients of the function are integer numbers they cannot change continuously as one moves around in the moduli space of K3 sigma-models: the Elliptic Genus must stay invariant under model deformations.
Therefore we need to define only one topological invariant of target surface to fix the whole elliptic genus. This quantity can be identify with $\phi(\tau, 0)$, that correspond to topological Euler characteristic of target surface. For two topologically distinct surface T^4 and $K3$, that are the unique types of CY_2, we obtain the following result:

$$\phi_{\mathcal{M}}(\tau, z=0) = \begin{cases} 0 & \mathcal{M} = T^4 \\ 24 & \mathcal{M} = K3. \end{cases} \tag{4.24}$$

Let be $\mathcal{H}^{(0)}$ the Hilbert space of states whose right-moving part are the ground state: they are all states that contribute to Elliptic Genus. We can decompose $\mathcal{H}^{(0)}$ on irreducible representations of $\mathcal{N}=4$ superconformal algebra:

$$\mathcal{H}^{(0)} = 20 \cdot \mathcal{H}_{h=\frac{1}{4},j=0} \oplus 2 \cdot \mathcal{H}_{h=\frac{1}{4},j=\frac{1}{2}} \oplus \sum_{n=1}^{\infty} \mathcal{D}_n \mathcal{H}_{h=\frac{1}{4}+n,j=\frac{1}{2}} \tag{4.25}$$

where $\mathcal{H}_{h,j}$ denotes the irreducible representation whose Virasoro primary states have conformal weight h and transforms on spin j representation of $su(2)$. In this language, we can write the (4.21) as: [18]

$$\phi_{K3}(\tau, z) = 20\chi_{h=\frac{1}{4},j=0}(\tau, z) - 2\chi_{h=\frac{1}{4},j=\frac{1}{2}}(\tau, z) + \sum_{n=1}^{\infty} A_n \chi_{h=\frac{1}{4}+n,j=\frac{1}{2}}(\tau, z) \tag{4.26}$$

where $\chi_{\frac{1}{4},j}$ is the character of the massless representation (BPS), while $\chi_{\frac{1}{4}+n,j}$ is the characters of the massive representation (non-BPS). Following Eguchi, Ooguri and Tachikawa observation[19] the coefficients A_n are sums of dimensions irreducible representations of $\mathbb{M}_{24}$.
By analogy with Monstrous Moonshine, in order to study the origin of this coefficients A_n, we introduce a discret symmetry group G of our superconformal angebra and see how his elements g acts on the space $\mathcal{H}^{(0)}$. Let us define the analogues of the McKay Thompson series, called *Elliptic Twining Genus*:

$$\phi_g(\tau, z) = Tr_{\mathcal{H}^{(0)}} \left(g q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{\overline{c}}{24}} (-1)^{F+\overline{F}} y^{J_0} \right). \tag{4.27}$$

This quantity is invariant under deformations of the model as long as g is a symmetry.
It is possible to shown that the Elliptic Twined Genera are also weak Jacobi form of weight zero and a some index, with suitable proprieties of transformation under a modular subgroup of $SL(2\mathbb{Z})$ which depends on the twining symmetry g.
This new function can be seen like ϕ_{K3} replacing dimension $A_n = dim(\mathcal{H}_n)$ by the trace of the g element $Tr_{\mathcal{H}_n}(g)$. Then the character decomposition of this new function has the form:

$$\begin{aligned}\phi_g(\tau, z) = \frac{1}{2} \Big[& Tr_{\mathcal{H}_{\frac{1}{4},0}}(g) \chi_{h=\frac{1}{4},j=0}(\tau, z) - Tr_{\mathcal{H}_{\frac{1}{4},\frac{1}{2}}}(g) \chi_{h=\frac{1}{4},j=\frac{1}{2}}(\tau, z) + \\ & + \sum_{n=1}^{\infty} Tr_{\mathcal{H}_n}(g) \chi_{h=\frac{1}{4}+n,j=\frac{1}{2}}(\tau, z) \Big]\end{aligned} \tag{4.28}$$

The character of g only depends on its conjugaty class.

Chapter 5

Orbifolds

Let us consider a differentiable manifold $\mathcal{M}$ and let G be a finite group that acts on $\mathcal{M}$ through the following map:

$$\begin{array}{rcl} \phi_g : G \times \mathcal{M} & \mapsto & \mathcal{M} \\ (g, x) & \mapsto & \phi_g(g, x) = g(x) \end{array}$$

Let us define *G-orbits* in $\mathcal{M}$ as the equivalence classes respect to the group action:

$$[x] = \{y \in \mathcal{M} | \quad \exists g \in G \quad \text{such that} \quad y = g(x)\}$$

Now we can introduce the *quotient space* of the orbits of G on $\mathcal{M}$:

$$\mathcal{M}/G = \{[x] \,| x \in \mathcal{M}\} \,. \tag{5.1}$$

This space is not always a manifold, in general it defines a mathematical object called *orbifold*. If the group of transformations acts on the coordinates of manifold without fixed points,then the quotient space $\mathcal{M}/G$ is a non-singular smooth manifold. If the group G, or whatever its subgroup, act on the manifold leaving fixed points,then the quotient space is an *orbifold* with a certain discret number of singularities.
From now on we will focus on Riemannian manofolds $\mathcal{M}$, of which G is an isometry group.
We want to give now an example, let's be $\mathcal{M} = \mathbb{R}^2 = \mathbb{C}$ the initial manifold and $\mathbb{Z}_n$ (n-th square roots of complex unit group) the finite group that act on manifold through:

$$gz = e^{2\pi i k/n} z, \qquad \forall z \in \mathbb{C},$$

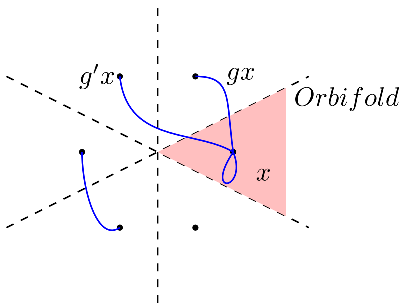

Figure 5.1: $\mathbb{C}/\mathbb{Z}_6$ orbifold.

this action identifies every $z \in \mathbb{C}$ with other $(n-1)$ points in $\mathbb{C}$. Notice in graphic representation that this action identifies a *fundamental domain* under the $\mathbb{Z}_n$ action. In other words, distinct points in the fundamental domain belong to distinct orbits are between them non equivalent and every $\mathbb{C}/\mathbb{Z}_n$ element has a *representative* on the fundamental domain.
The domain ends points (rays) are identified, therefore the geometrical representation of this orbifold is a cone. Let us define *fixed point* on $\mathcal{M}$ respect to $H < G$ action the point $x \in \mathcal{M}$ such that:

$$h(x) = x \qquad \forall h \in H < G.$$

The origin represents a *fixed point* under the whole group action. In general it's possible to own more fixed points or entire subspace of manifold that are invariant under the action group. These points, in this case the point $x = 0$, make the quotient space not everywhere differentiable: $\mathbb{C}/\mathbb{Z}_n$ is not a manifold. The curvature of the geometric object is zero everywhere except in the fixed points, where it is not well defined. For this reason, the quotient space is not a manifold and it falls into the class of orbifolds.
In this brief introduction we have provided a geometrical idea of orbifold through a simple case in which the group G is Abelian, but nothing prevents us to consider non-abelian group.
We seen now how to use this object in a CFT theory.

5.1 Orbifolds in CFT

A general CFT is made up of a set of conformal fields whose product operators generate a closed algebra. For consistency it is necessary to restrict to CFT constructed by mutually local fields that generate subalgebras that make the theory *local*: i.e. such that the operators products are single valued.
The local fields create asymptotic string states, the string scattering amplitudes are written in terms of the correlation functions.
The simplest conformal theories that we have described in the first chapter are those of free bosons and fermions. Using the state-operator cerrespondence, for a boson we have the identity operator and the operators associated to the fields $\partial X(z), \overline{\partial} X(\overline{z})$ and $e^{ik\cdot X}$. If we take a field $X(z,\overline{z})$ compactified on a circle of radius R we obtain only a discrete set of exponential operators. For a fermion, we have again the identity operator, the operators associated to fields $\psi(z), \overline{\psi}(\overline{z})$ and spin fields $\sigma(0)$[1] that change the sign of ψ when carried once around the origin: this set of field is not local. The point $z = 0$ in the local conformal coordinates is a branch singularity point. This forces us to take a cut on the target space and, so the correlators of the theory are not local.
We might construct the twist field $\sigma(w)$ around which the bosonic field $X(z,\overline{z})$ is anti-periodic. In order to obtain a local theory, we must impose several conditions on the geometry of the space-time. Let $X(z,\overline{z})$ be compactified in a circle of radius R ($X \in [0, 2\pi R[$), as this operator turns around the twist operator $\sigma(w)$ the map $X(z)$ takes a minus sign. If we wanted to restore the locality of the theory, we must identify the target space respect to $\mathbb{Z}_2$ symmetry: $X \sim -X$. Now, the twist fields, such that if $X(z,\overline{z})$ turn around it transforms as $X \mapsto -X$ can be included in the theory.The points $X = 0$ and $X = \pi R$ are fixed under the $\mathbb{Z}_2$ action. Tracing X around an infinitesimal contour about w, called punctures, shows that the string wraps once around one of these fixed point. We can define a twist operator for each fixed point. [13]
The Hibert space of the new theory contains new sectors. The *untwisted* sector, containing the

[1] $\sigma(z)$ is the *twist* operator, whose operator product with the fermion operator is defined to have a square-root branch cut. When the field ψ is transported around σ it changes sign and the twist field σ can be used to change the boundary conditions on ψ.

vacuum state, its descendants and the Verma modules of combinations of highest weight fields ∂X and $e^{ik\cdot X}$. The *twisted* sectors, containing the states created by various twisted operators acting on the untwisted fields.
The twist fields provide local boundary condition to the map from a neighbour of w_i, where they are localized, to space-time. They also contain global information needed to determinate the asymptotic behavior of the Green functions on the new theory through their OPE with the bosonic and fermionic fields. These informations are not sufficient to uniquely determine the Green functions: again other terms depending from classical fields having the correct monodromy around the punctures w_i can be added. In order to specify these terms, we need to see the global change of the fields of the theory when they are transported through closed loops that turn around two or more vertex punctures. The change can be understood by looking the string on its world-sheet point of view. The geometry background should contain the global needed information, and is responsible of existence of the twist fields: it is a geometry of orbifolds.
The require of locality imposes conditions on the choice of geometry of the target space-time. Since we want a CFT that be a sigma model on an orbifold target space, we start from the sigma model on the manifold $\mathcal{M}$, we implement its operator space through the G action and we impose the locally conditions. Let us summarize the procedure of construction of a conformal theory on a generic target space $\mathcal{M}/G$, where G is a discrete group. Let us consider a CFT, the group G is a symmetry for our theory if these conditions are respected:

- The *vacuum* is G-invariant ;
- The *energy-momentum tensor* $T(z)$ is G-invariant;
- The *correlation functions* are G-invariant.

From a world-sheet standpoint, a string $X^\mu(z)$ is a map from string world-sheet (z coordinate) to *target space* ($X^\mu, \mu = 0, 1, ..d-1$) defined by the d- dimensional manifold $\mathcal{M}$. Let G be a symmetry of our theory: if G is an *isometry* of $\mathcal{M}$, then we can construct a CFT in the orbifold $\mathcal{M}/G$.
In order to build orbifold CFT, we start from states and operators in the manifold and keep only those invariant under symmetry G (we could have said only operators exploiting the correspondence State-Operator). The projection procedure will be necessary to assure us the locality of the theory. There will be of course invariant operators, since we have supposed that G is a symmetry for our theory: at least the vacuum state and the energy-tensor operator are invariant under symmetry. This projection under invariant states define the *untwisted* sector of Hilbert space of whole theory, closed respect to OPE.
Now we can write the partition function of theory on this untwisted sector: the result is a function *non modular invariant*: the new CFT theory is not consistent. The solution consists in introducing new states in the initial theory: this states are contained in the *twisted* sectors.
The theory equipped with the new Hilbert space is non-local, we must perform a project operation on the G-invariant subspaces in the various sectors.
We said that the operators in the twisted sectors produce cuts in the target space, that make the theory not local. In the other words the non-locality is due to the fact that an operator defined in a point feels the action of another operator (twist operator) defined in another point.
In order to perform the projection we need to define the G action on the twisted operator. Let us consider the G-action on operators in the untwisted sector:

$$\Phi(g \cdot z) = R(g)\Phi(z).$$

In order to find the G-action on the twisted operators, let us define the G-action on the twisted ground state:

$$\begin{aligned} \rho : G &\rightarrow GL(\text{twisted sector}) \\ g &\rightarrow \rho(g)|tw\rangle \end{aligned}$$

and let us consider a non-vanishing correlator of a certain untwisted operator Φ and two twisted states $|tw\rangle, |\tilde{tw}\rangle$:

$$\langle \tilde{tw}|\Phi(z)|tw\rangle \tag{5.2}$$

If G is a symmetry of the theory, then the correlator (5.2) must be G-invariant. This condition provides a connection between the G action on the untwisted operators (that corresponds to an action on the untwisted states, via state-operator correspondence) and the G action on the twisted states:

$$R(g)\Phi(z) = \rho(g)\Phi(z)\rho^{-1}(g).$$

This tell us that the group representation on the twisted sector is defined by the representation on the untwisted sector up a phase. To find the phase we consider the fact that the product of two twisted operators is a untwisted operator:

$$\Phi_{tw}(z)\Phi_{tw}(0) = \Phi_{untw}$$

so:

$$\rho(g)\Phi_{tw}\rho(g)\Phi_{tw}(0) = R(g) \cdot \Phi_{untw}$$

here we have ρ^2, therefore we can define the phase.

5.2 Strings on orbifold

We are interested in considering the string propagation on an orbifold target space.[15] In particular we will focus our attention on spaces constructed by quotient a torus T^d with a discret group G, called *toroidal orbifolds*. First, let us analyse a general method to construct of orbifold spaces.

Let $\mathcal{M}$ be a manifold of Euclidean space and G a group that preserves his metric, then G is necessarily a subgroup of rotations and translations Euclidean group. The torus T^d could be seen like the Euclidean space $\mathbb{R}^d$ quotiented with discrete translations d-dimensional group $\mathbb{Z}^d$.

$$T^d = \mathbb{R}^d/\mathbf{Z}^d. \tag{5.3}$$

We would like to generalize this example, inserting inside the discrete group the rotations or more rotations and translations combinations: this generalisation is called *space group* and indicated with S. A generic element of the space group S can be to indicate with the notation $g = (\theta, v)$, where θ is a rotation and v a translation. The g element acts on the space element x by:

$$g \cdot x = \theta x + v. \tag{5.4}$$

The quotient space $\mathbb{R}^d/S$ is formed by set of orbits through $g \cdot x \sim x$. The following relations are valid in S:

$$\begin{aligned} (i) &\qquad (\theta, v)(\omega, u) = (\theta\omega, v + \theta u) \\ (ii) &\qquad (\theta, v)^{-1} = (\theta^{-1}, -\theta^{-1}v) \\ (iii) &\qquad (\theta, u)(1, v)(\theta, u)^{-1} = (1, \theta v) \end{aligned} \tag{5.5}$$

Let Λ be the S subgroup of pure translations, defined by elements of type $(\mathbb{1}, v)$ that defines the d-dimensional torus $T^d = \mathbb{R}^d/\Lambda$. We can also consider the *point group* P, subgroup of $O(d)$, of the rotations θ represented by an element $g = (\theta, v) \in S$ for a certain v. For each element $\theta \in P$ there is a single v that defines the g element. This involves that P has a well defined action, denoted with $\overline{P} = S/\Lambda$, on T^d. This brings us to two equivalent ways of describing orbifolds: we can start with euclidean space and divide by the space group or equivalently consider the corresponding torus and divide by the point group.

$$\Omega = \mathbb{R}^d/S = \frac{\mathbb{R}^d/\Lambda}{\overline{P}} = \frac{T^d}{\overline{P}}. \tag{5.6}$$

We are interested in building orbifolds for compactification of superstrings on $\mathcal{M}^{1,3} \times T^2 \times \mathcal{M}/G$, where $dim\mathcal{M} = 4$. Therefore we would like to classify the four dimension space groups. Actually we can limit this classification requiring particular proprieties for our theory. If we want to preserve the supersymmetry $\mathcal{N} = 1$ in four dimensions, the point group P should be a discrete subgroup of $SU(2)$, otherwise it could be a discrete subgroup of $SO(4)$.
To describe the string Hilbert space on an orbifold we start by considering string propagation on the manifold before dividing by the group action. To each element g in the group, there is an operator $\overline{g}$ acting on the string Hilbert space. If we want that x and gx are the same point in the orbifold we should consider the subspace of the Hilbert space invariant under the action of $\overline{g}$: $\overline{g}$ should act as the identity operator for the orbifold.
Let us start with $\mathbb{R}^d$ and divide it by the discrete translation group $\mathbb{Z}^d$ to obtain the torus T^d. To each translation $v \in \mathbb{Z}^d$ corresponds an operator e^{iPv} that acts on the string Hilbert space defined on $\mathbb{R}^d$. The projection onto the subspace invariant under these actions amounts to requiring that the eigenvalues of the momentum operator lie on the dual lattice to the lattice defined by $\mathbb{Z}^d$. This subspace defines the string Hilbert space for T^d, where the string states are labelled by discrete values of the momentum.
If we were describing open strings that would be all that is necessary. For closed strings we must increase our requires. The usual boundary condition for closed string is:

$$X(\sigma + 2\pi) = X(\sigma),$$

with minus signs in some fermionic sector. Now this is not the only possibility because on the orbifold the identification $gX \sim X$ leaves that we could request $X(\sigma + 2\pi) = gX(\sigma)$ for some $g \in G$. This implies that for each g element we get a sector $\mathcal{H}_g$ of the Hilbert space in which the boundary conditions are changed to periodicity up to g transformation. These are *twisted sectors*.
In the previous example we should define sector $\mathcal{H}_v$ in which $X(2\pi) = X(0) + v$, where v is a translation in T^d. These sectors are just the *winding sectors* in which the string wraps around the torus. The twisted sectors must be included since they can be pair produced in interactions of untwisted strings. As we already said, they are also necessary to ensure modular invariance.
The projection of twisted sectors onto the group invariant subspaces is more complicated when the group is non abelian because some group elements take one twisted sector to another. Let's consider the sector $\mathcal{H}_g$ and we act through h in a state of this sector. The start point is mapped onto $hX(\sigma)$ and the final point onto $hX(\sigma + 2\pi) = hgX(\sigma) = hgh^{-1}(hX(\sigma))$. Now the string periodicity is give by hgh^{-1}, thus is in the $\mathcal{H}_{hgh^{-1}}$ twisted sector. So to form group invariant states we will take a state in the sector $\mathcal{H}_g$, project it onto invariant subspace of centralizer, or little group, of g, and then take the sum of corresponding states from sectors in the same conjugacy class: we build sectors divided by conjugacy classes.

5.3 NLSM on K3

Let us consider the superstring action with four pairs of bosonic currents $j^a(z) = i\partial X^a(z)$ and $\tilde{j}^b(\bar{z}) = i\bar{\partial}X^b(\bar{z})$, and four pairs of free real fermions $\psi^a(z)$ and $\tilde{\psi}^b(\bar{z})$ with $a, b = 1, 2, 3, 4$ on T^4. The chiral algebra of any T^4 model contains a bosonic $u(1)^4$ algebra, a fermionic $so(4)$ algebra generated by $:\psi^a(z)\psi^b(z):$ with $1 \leq a \leq b \leq 4$, , and 16 fields of weight $(3/2, 0)$ of the form $:\psi_i(z)j_k(z):$. In particular contains a small superconformal algebra $\mathcal{N} = 4$ described in the section 3.2.
It is useful to define the set of complex bosonic fields:

$$\partial Z^1(z) = \frac{1}{\sqrt{2}}\left(j^1(z) + ij^3(z)\right) \qquad \partial Z^{1*}(z) = \frac{1}{\sqrt{2}}\left(j^1(z) - ij^3(z)\right)$$
$$\partial Z^2(z) = \frac{1}{\sqrt{2}}\left(j^2(z) + ij^4(z)\right) \qquad \partial Z^{2*}(z) = \frac{1}{\sqrt{2}}\left(j^2(z) - ij^4(z)\right),$$

and the set of complex fermionic fields:

$$\chi^1(z) = \frac{1}{\sqrt{2}}\left(\psi^1(z) + i\psi^3(z)\right) \qquad \chi^{1*}(z) = \frac{1}{\sqrt{2}}\left(\psi^1(z) - i\psi^3(z)\right)$$
$$\chi^2(z) = \frac{1}{\sqrt{2}}\left(\psi^2(z) + i\psi^4(z)\right) \qquad \chi^{2*}(z) = \frac{1}{\sqrt{2}}\left(\psi^2(z) - i\psi^4(z)\right),$$

and their corresponding anti-chiral fields.
By the new complex fields we can define the $\mathcal{N} = 4$ supercurrents:

$$\begin{aligned}
G^+(z) &= \sqrt{2}i\left(:\chi^{*1}(z)\partial Z^{(1)}(z): + :\chi^{*2}(z)\partial Z^{(2)}(z):\right)\\
G^-(z) &= \sqrt{2}i\left(:\chi^{1}(z)\partial Z^{*(1)}(z): + :\chi^{2}(z)\partial Z^{*(2)}(z):\right)\\
G'^+(z) &= \sqrt{2}\left(-:\chi^{*1}(z)\partial Z^{*(2)}(z): + :\chi^{*2}(z)\partial Z^{*(1)}(z):\right)\\
G'^-(z) &= \sqrt{2}\left(-:\chi^{1}(z)\partial Z^{(2)}(z): + :\chi^{2}(z)\partial Z^{(1)}(z):\right)
\end{aligned} \tag{5.7}$$

Using the propriety $so(4) = su(2) \oplus su(2)$, we define the two $su(2)$ currents:

$$\begin{aligned}
J^3(z) &= \frac{1}{2}\left(:\chi^{*1}(z)\chi^1(z): + :\chi^{*2}(z)\chi^2(z):\right)\\
J^+(z) &= i:\chi^{*1}(z)\chi^{*2}(z): \qquad J^-(z) = i:\chi^1(z)\chi^2(z):,
\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}
A^3(z) &= \frac{1}{2}\left(:\chi^{*1}(z)\chi^1(z): - :\chi^{*2}(z)\chi^2(z):\right)\\
A^+(z) &= i:\chi^{*1}(z)\chi^{2}(z): \qquad A^-(z) = i:\chi^1(z)\chi^{*2}(z):.
\end{aligned} \tag{5.9}$$

The first set of currents, $J_3(z)$ and $J^{\pm}(z)$, are exactly the ones that appear in the definition of the $N = 4$ superconformal algebra.
NLSMs on K3 are two dimensional $\mathcal{N} = (4, 4)$ superconformal fields theories with central charge $c = \bar{c} = 6$. They arise as the worldsheet description of perturbative type IIA string theory on a K3 surface.
In this thesis we want to consider toroidal orbifolds of type T^4/G, where G commutes with

the whole $\mathcal{N} = (4,4)$ superconformal algebra. The resulting orbifold should still be a SCFT with $\mathcal{N} = (4,4)$ superconformal algebra and central charge $c = 6$. Since the only SCFT with $\mathcal{N} = (4,4)$ superconformal algebra are NLSMs on T^4 or $K3$, we expect that the conformal theory on the orbifold target space T^4/G is one of them.
Through the computation of the Witten index (4.20) for the orbifold theory, we can distinguish between these two cases.
In the next sections we are going to consider new orbifold models that had never been studied before.

5.4 Orbifold $T^4/\mathbb{Z}_2.A_5$

In this section we consider a $\mathcal{N} = (4,4)$ superconformal theory with central charge $c = \tilde{c} = 6$ and target space a K3 surface. In particular we study a K3 sigma-model constructed by taking the orbifold $T^4/2.A_5$.
The symmetry group $G = \mathbb{Z}_2.A_5$ is the central extention of alternating group A_5. The order of the group is $|G| = 120$, we report below his character table, where the name of class follows the Atlas notation for the classes of $O^+(2)$:

$[g]$	$1A$	$-1A$	$2B$	$3A$	$-3A$	$5A$	$-5A$	$5A'$	$-5A'$
Order element	1	2	4	3	6	5	10	5	10
$\mid [g] \mid$	1	1	30	20	20	12	12	12	12
Order centralizer	120	120	4	6	6	10	10	10	10

The interesting feature of this group is that it is not Abelian.

5.4.1 Elliptic genus

In order to show that the orbifold theory $T^4/\mathbb{Z}_2.A_5$ corresponds to K3 model we compute the Elliptic genus of the theory. For this we need to project the Hilbert space onto G invariant states. Let us define for each element $h \in G$ the projector $\frac{1}{|G|}(\mathbb{1} + h)$ and insert this quantity on the trace (4.21). As described in previous section an orbifold theory includes, over the Hilbert space of the correspondent theory on T^4 called untwisted, twisted sectors. Then we have to project onto invariant states in each sector. In the g-twisted sector one only needs to project over the centralizer. The result is:

$$\phi_{orbifold}(\tau, z) = \frac{1}{|G|} \sum_{g,h|gh=hg} \phi_{g,h}(\tau; z) \sum_{classes[g]} \frac{1}{|C(g)|} \sum_{h|gh=hg} \phi_{g,h}(\tau; z) \tag{5.10}$$

where $|C(g)|$ is the order of centralizer of the element g, and $\phi_{g,h}$ is the *twisted twining genus*:

$$\phi_{g,h} = Tr_{\mathcal{H}_g} \left(h q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{\bar{c}}{24}} (-1)^{F+\overline{F}} y^{J_0} \right) \tag{5.11}$$

defined as the h-twining trace over the g-twisted sector.
The calculation can be done in two ways: first building the Hilbert space and explicitly computing the traces, or exploiting the construction of the Twining Genus for every conjugation class and applying his modular propriety. [20]
In order to compute the whole Hilbert space let us introduce a suitable representation of the RR states in T^4 NLSM. This can be to build from all possible combination of bosonic and fermionic

oscillators acting upon a suitable ground state. We remember that in the RR sector there are fermionic zero modes that form a non-trivial algebra:

$$\left\{\chi_0^a, \chi_0^{b*}\right\} = \delta^{ab} \qquad\qquad \left\{\tilde{\chi}_0^a, \tilde{\chi}_0^{b*}\right\} = \delta^{ab} \tag{5.12}$$

that has a representation as matrices on $2^{d/2} = 2^4$ dimensional space. Therefore there are 16 RR ground states that transform one into each other up fermionic zero modes. Let be $\chi^{(a)*}, \tilde{\chi}^{(a)*}$, with $a = 1, 2$ the fermionic creation operators and $\chi^{(a)}, \tilde{\chi}^{(a)}$, with $a = 1, 2$, the fermionic annihilation operators. We choose for RR ground states a basis of eigenstates for currents operators $J_0^3, \tilde{J}_0^3, A_0^3, \tilde{A}_0^3$:

$$|k_L, k_R, s_1, s_2, \tilde{s}_1, \tilde{s}_2\rangle \qquad\qquad s_i, \tilde{s}_i = \left\{\pm\frac{1}{2}\right\}. \tag{5.13}$$

k_L and k_R label the winding-momentum contribution deriving from bosonic zero modes, while $s = (s_1, s_2, \tilde{s}_1, \tilde{s}_2)$ runs in the 16-dimensional representation of Clifford algebra. The action of currents on these states is:

$$\begin{aligned} J_0^3|s_1, s_2, \tilde{s}_1, \tilde{s}_2\rangle &= (s_1 + s_2)|s_1, s_2, \tilde{s}_1, \tilde{s}_2\rangle \quad & A_0^3|s_1, s_2, \tilde{s}_1, \tilde{s}_2\rangle &= (s_1 - s_2)|s_1, s_2, \tilde{s}_1, \tilde{s}_2\rangle \\ \tilde{J}_0^3|s_1, s_2, \tilde{s}_1, \tilde{s}_2\rangle &= (\tilde{s}_1 + \tilde{s}_2)|s_1, s_2, \tilde{s}_1, \tilde{s}_2\rangle \quad & \tilde{A}_0^3|s_1, s_2, \tilde{s}_1, \tilde{s}_2\rangle &= (\tilde{s}_1 - \tilde{s}_2)|s_1, s_2, \tilde{s}_1, \tilde{s}_2\rangle. \end{aligned}$$

The action of creation operators on this basis is the following:

$$\begin{aligned} &\chi_0^{(1)*}| -\frac{1}{2}, s_2, \tilde{s}_1, \tilde{s}_2\rangle = |\frac{1}{2}, s_2, \tilde{s}_1, \tilde{s}_2\rangle \\ &\tilde{\chi}_0^{(1)*}|s_1, s_2, -\frac{1}{2}, \tilde{s}_2\rangle = |s_1, s_2, \frac{1}{2}, \tilde{s}_2\rangle \\ &\chi_0^{(2)*}|s_1, -\frac{1}{2}, \tilde{s}_1, \tilde{s}_2\rangle = |s_1, \frac{1}{2}, \tilde{s}_1, \tilde{s}_2\rangle \\ &\tilde{\chi}_0^{(1)*}|s_1, s_2, \tilde{s}_1, -\frac{1}{2}\rangle = |s_1, s_2, \tilde{s}_1, \frac{1}{2}\rangle \\ &\chi_0^{(1)*}|\frac{1}{2}, s_2, \tilde{s}_1, \tilde{s}_2\rangle = 0 \\ &\tilde{\chi}_0^{(1)*}|s_1, s_2, \frac{1}{2}, \tilde{s}_2\rangle = 0 \\ &\chi_0^{(2)*}|s_1, \frac{1}{2}, \tilde{s}_1, \tilde{s}_2\rangle = 0 \\ &\tilde{\chi}_0^{(1)*}|s_1, s_2, \tilde{s}_1, \frac{1}{2}\rangle = 0. \end{aligned}$$

Let us focus our attention on the second term of (5.10): for the untwisted sector we have the contribution of Elliptic Genus and of Twining Elliptic Genus for each element of the group, for any g-twisted sector we have the contribution of Twisted h-Twining Genus for each element $h \in G$ that commute with g. Let us start from the untwisted sector and notice that the function on this sector is given by:

$$\phi_{untw,g}(\tau; z) = \phi_g^{osc}(\tau; z)\phi_g^{gs}(z)\phi_g^{w-m}(\tau) \tag{5.14}$$

the product of the oscillators, winding-momentum, and fermionic ground states contributions.
Now we can explicitly compute the (5.14) for $g = \mathbb{1}$, this quantity corresponds to Elliptic Genus. Let us begin with the ground states contribute. Since the ground states on the RR sector have conformal weight $h = \overline{h} = \frac{c}{24}$ the operators $q^{L_0 - c/24}\overline{q}^{\overline{L}_0 - c/24}$ do not give any contribute to trace.

We would like to build the Hilbert space $\mathcal{H}_{GS}$ of ground states, we take the state annihichilated from any $\chi^{(a)}, \tilde{\chi}^{(a)}$ operators:

$$|GS\rangle = |-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\rangle$$

we act on this state with creation operators in order to generate the whole $\mathcal{H}_{GS}$. Let us call for simplicity $|\chi_0^{(a)}\rangle$ the state obtained by the action of the correspondent creation operator on the ground state. The action of internal trace operator on these ground states is below summarised:

$$\begin{aligned}
(-1)^F y^{J_0^3}|-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\rangle &= y^{-1}|-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\rangle\\
(-1)^F y^{J_0^3}|\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\rangle &= -|\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\rangle\\
(-1)^F y^{J_0^3}|-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\rangle &= -|-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\rangle\\
(-1)^F y^{J_0^3}|-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\rangle &= -y^{-1}|-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\rangle\\
(-1)^F y^{J_0^3}|-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\rangle &= -y^{-1}|-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\rangle\\
(-1)^F y^{J_0^3}|\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\rangle &= y|\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\rangle\\
(-1)^F y^{J_0^3}|-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\rangle &= y^{-1}|-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\rangle\\
(-1)^F y^{J_0^3}|\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\rangle &= |\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\rangle\\
(-1)^F y^{J_0^3}|\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\rangle &= |\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\rangle\\
(-1)^F y^{J_0^3}|-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\rangle &= |-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\rangle\\
(-1)^F y^{J_0^3}|-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\rangle &= |-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\rangle\\
(-1)^F y^{J_0^3}|\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\rangle &= -y|\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\rangle\\
(-1)^F y^{J_0^3}|\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\rangle &= -y|\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\rangle\\
(-1)^F y^{J_0^3}|\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\rangle &= y|\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\rangle\\
(-1)^F y^{J_0^3}|-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\rangle &= -|-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\rangle\\
(-1)^F y^{J_0^3}|\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\rangle &= -|\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\rangle.
\end{aligned}$$

By computing the trace we obtain $\phi_{\mathbb{1}}^{gs}(z) = 0$ therefore $\phi_{\mathbb{1}}(\tau, z) = 0$ as we expected for the Elliptic Genus of torus.
Now we have to calculate the (5.14) for each element of the group, but we have chosen complex fermions such that each element $g \in G$ acts on them by its eigenvalues in the 8-dimensional representation, which is the same as for the bosonic currents because the superconformal algebra must be preserved. Each element in the same conjugacy class acts through the same eigenvalues on the currents, that's why we can just calculate it for one element in each class.

Let us compute explicitly the (5.14) for the class $2B$. In order to compute the ground states contribute to $\phi_{g\in 2B}^{gs}$, let us act with a generic element of the class on the states listed above and

$Class$	ξ_L	ξ_L^{-1}	ξ_R	ξ_R^{-1}
$1A$	1	1	1	1
$-1A$	-1	-1	-1	-1
$2B$	$e\left(\frac{1}{4}\right)$	$e\left(\frac{3}{4}\right)$	$e\left(\frac{1}{4}\right)$	$e\left(\frac{3}{4}\right)$
$3A$	$e\left(\frac{1}{3}\right)$	$e\left(\frac{2}{3}\right)$	$e\left(\frac{1}{3}\right)$	$e\left(\frac{2}{3}\right)$
$-3A$	$e\left(\frac{1}{6}\right)$	$e\left(\frac{5}{6}\right)$	$e\left(\frac{1}{6}\right)$	$e\left(\frac{5}{6}\right)$
$5A$	$e\left(\frac{1}{5}\right)$	$e\left(\frac{4}{5}\right)$	$e\left(\frac{2}{5}\right)$	$e\left(\frac{3}{5}\right)$
$-5A$	$e\left(\frac{3}{10}\right)$	$e\left(\frac{7}{10}\right)$	$e\left(\frac{1}{10}\right)$	$e\left(\frac{9}{10}\right)$
$5A'$	$e\left(\frac{2}{5}\right)$	$e\left(\frac{3}{5}\right)$	$e\left(\frac{1}{5}\right)$	$e\left(\frac{4}{5}\right)$
$-5A'$	$e\left(\frac{1}{10}\right)$	$e\left(\frac{9}{10}\right)$	$e\left(\frac{3}{10}\right)$	$e\left(\frac{7}{10}\right)$

Table 5.1: Eigenvalues of $\mathbb{Z}_2.A_5$ conjugation classes in the ρ_ψ representation of complex fermions

take their trace:

$$\phi^{gs}_{g\in 2B} = y^{-1}\left(1 - e\left(\frac{1}{4}\right)\right)^2\left(1 - e\left(\frac{3}{4}\right)\right)^2,$$

where the notation $e(r)$ indicates $e(r) = e^{2\pi i r}$.
We can easily prove that for each element g of the group that acts on the complex fermions with eigenvalues $\xi_L, \xi_L^{-1}, \xi_R, \xi_R^{-1}$, the ground state contribute is:

$$\begin{aligned}&\phi^{gs}_g(z) = y^{-1} - \xi_L - \xi_L^{-1} - y^{-1}\xi_R - y^{-1}\xi_R^{-1} + y\xi_L\xi_L^{-1} + y^{-1}\xi_R\xi_R^{-1} + \xi_L\xi_R + \xi_L\xi_R^{-1} +\\&\xi_L^{-1}\xi_R + \xi_L^{-1}\xi_R^{-1} - y\xi_L\xi_L^{-1}\xi_R - y\xi_L\xi_L^{-1}\xi_R^{-1} + y\xi_L\xi_L^{-1}\xi_R\xi_R^{-1} - \xi_L^{-1}\xi_R\xi_R^{-1} - \xi_L\xi_R\xi_R^{-1} =\\&y^{-1}\left(1 - y\xi_L\right)\left(1 - y\xi_L^{-1}\right)\left(1 - \xi_R\right)\left(1 - \xi_R^{-1}\right).\end{aligned}$$

The eigenvalues for all conjugacy classes are summarised in Tab.5.1.
Now we would compute the total contribution from the fermionic and bosonic oscillators. Let $|\Omega\rangle$ be a generic ground state and we start by applying to it the left bosonic operators $\left(\partial Z^{(i)}_{-n}\right)^{(*)}$, according to the Tab.5.2:

$$\begin{aligned}&(-1)^F q^{L_0}\overline{q}^{\overline{L}_0} y^{J_0^3} g|\Omega\rangle = (+1)|\Omega\rangle\\&(-1)^F q^{L_0}\overline{q}^{\overline{L}_0} y^{J_0^3} g\partial Z^{(1)}_{-n}|\Omega\rangle = \xi_L q^n \partial Z^{(1)}_{-n}|\Omega\rangle\\&(-1)^F q^{L_0}\overline{q}^{\overline{L}_0} y^{J_0^3} g\left(\partial Z^{(1)}_{-n}\right)^2|\Omega\rangle = \xi_L^2 q^{2n}\left(\partial Z^{(1)}_{-n}\right)^2|\Omega\rangle\\&\quad\quad ... \quad ... \quad ...\end{aligned}$$

We take the trace and we obtain the following contribute:

$$1 + \xi_L q^n + \xi_L^2 q^{2n} + ... = \sum_{k=0}^{+\infty}\left(\xi_L q^n\right)^k = \frac{1}{1 - \xi_L q^n}. \tag{5.15}$$

Let us consider the same quantity for a g^k element of the group:

$$\begin{aligned}&(-1)^F q^{L_0}\overline{q}^{\overline{L}_0} y^{J_0^3} g^k|\Omega\rangle = (+1)|\Omega\rangle\\&(-1)^F q^{L_0}\overline{q}^{\overline{L}_0} y^{J_0^3} g^k\partial Z^{(1)}_{-n}|\Omega\rangle = \xi_L^k q^n \partial Z^{(1)}_{-n}|\Omega\rangle\\&(-1)^F q^{L_0}\overline{q}^{\overline{L}_0} y^{J_0^3} g^k\left(\partial Z^{(1)}_{-n}\right)^2|\Omega\rangle = \xi_L^{2k} q^{2n}\left(\partial Z^{(1)}_{-n}\right)^2|\Omega\rangle\\&\quad\quad ... \quad ... \quad ...\end{aligned}$$

	$\partial Z^{(1)*}$	$\partial Z^{(2)*}$	$\partial Z^{(1)}$	$\partial Z^{(2)}$	$\partial \tilde{Z}^{(1)^*}$	$\partial \tilde{Z}^{(2)^*}$	$\partial \tilde{Z}^{(1)}$	$\partial \tilde{Z}^{(2)}$
g	ξ_L	ξ_L^{-1}	ξ_L^{-1}	ξ_L	ξ_R	ξ_R^{-1}	ξ_R^{-1}	ξ_R
$y^{J_0^3}$	1	1	1	1	1	1	1	1

Table 5.2: Eigenvalues of $g \in \mathbb{Z}_2.A_5$ in the ρ_ψ representation on bosonic fields.

	$\chi^{(1)*}$	$\chi^{(2)*}$	$\chi^{(1)}$	$\chi^{(2)}$	$\tilde{\chi}^{(1)*}$	$\tilde{\chi}^{(2)*}$	$\tilde{\chi}^{(1)}$	$\tilde{\chi}^{(2)}$
g	ξ_L	ξ_L^{-1}	ξ_L^{-1}	ξ_L	ξ_R	ξ_R^{-1}	ξ_R^{-1}	ξ_R
$y^{J_0^3}$	1	y	y	y^{-1}	y^{-1}	1	1	1

Table 5.3: Eigenvalues of $g \in \mathbb{Z}_2.A_5$ in the ρ_ψ representation on fermionic fields.

taking the trace the result is:

$$1 + \xi_L^k q^n + \xi_L^{2k} q^{2n} + ... = \sum_{h=0}^{+\infty} \left(\xi_L^k q^n\right)^h = \frac{1}{1-\xi_L^k q^n}. \tag{5.16}$$

Let us repeat the same calculation applying the bosonic operators $\left(\partial Z^{(2)}\right)$, $\left(\partial Z^{(1)}\right)^*$, $\left(\partial Z^{(2)}\right)^*$ to generic ground state $|\Omega\rangle$. Therefore the total bosonic oscillators left-moving contribute is:

$$\prod_{n=1}^{\infty} \left[\left(1-\xi_L^k q^n\right)^2 \left(1-\xi_L^{-1k} q^n\right)^2\right]^{-1}. \tag{5.17}$$

We must add the fermionic oscillator contribute. by applying to the ground state $|\Omega\rangle$ the left-moving fermionic operators $\left(\chi_{-n}^{(i)}\right)^{(*)}$, according to Tab.5.3, for a generical $g^k \in G$ we obtain:

$$\begin{aligned}
&(-1)^F\, q^{L_0}\overline{q}^{\overline{L}_0} y^{J_0^3} g^k |\Omega\rangle = (+1)\,|\Omega\rangle \\
&(-1)^F\, q^{L_0}\overline{q}^{\overline{L}_0} y^{J_0^3} g^k \chi_{-n}^{(1)*} |\Omega\rangle = -y^{-1}\xi_L^k q^n \chi_{-n}^{(1)} |\Omega\rangle
\end{aligned}$$

Tracing we find:

$$\left(1-y^{-1}\xi_L^k q^n\right). \tag{5.18}$$

So, we repeat the same computation for the other fermionic fields. The total contribute of the bosonic and fermionic oscillators left-moving is:

$$\phi_{g^k}^{osc}(\tau,z) = \prod_{n=1}^{\infty} \frac{\left(1-y^{-1}\xi_L^k q^n\right)\left(1-y^{-1}\xi_L^{-k} q^n\right)\left(1-y\xi_L^k q^n\right)\left(1-y\xi_L^{-k} q^n\right)}{\left(1-\xi_L^k q^n\right)^2\left(1-\xi_L^{-1k} q^n\right)^2}. \tag{5.19}$$

Finally we must find the winding-momentum contribute. According with description on [21], this quantity is:

$$\phi_g^{w-m}(\tau) = \sum_{(k_L,k_R)\in(\Gamma_{w-m}^{4,4})^g} \zeta_g(k_L,k_R)\, q^{\frac{k_L^2}{2}} \overline{q}^{\frac{k_R^2}{2}} \tag{5.20}$$

where $(\Gamma_{w-m}^{4,4})^g$ is the sublattice fixed by g and ζ_g is a suitable phase that depends on the choice of the lift from $\mathbb{Z}_2.A_5$ to total symmetry group $\left(U(1)^4 \times U(1)^4\right).(\mathbb{Z}_2.A_5)$ of the torus. The part

$\left(U(1)^4 \times U(1)^4\right)$, generated by zero modes of bosonic currents, has a non-trivial action only on the states with winding-momentum different to zero. Notice that if $\xi_R = 1$ then $\phi_g^{gs} = 0$, so $\phi_g(\tau, z) = 0$; while if $\xi_R \neq 0$ then ϕ_g^{w-m} must be holomorphic in τ, because $\phi_g(\tau, z)$ is an holomorphic function.
Putting together the three resulting contributions, we can rewrite the (5.14) as:

$$\begin{aligned}\phi_{untw,g^k}(\tau, z) = y^{-1}\left(1 - y\xi_L^k\right)\left(1 - y\xi_L^{-k}\right)\left(1 - \xi_R^k\right)\left(1 - \xi_R^{-k}\right) \cdot \\ \prod_{n=1}^{\infty} \frac{\left(1 - y^{-1}\xi_L^k q^n\right)\left(1 - y^{-1}\xi_L^{-k} q^n\right)\left(1 - y\xi_L^k q^n\right)\left(1 - y\xi_L^{-k} q^n\right)}{\left(1 - \xi_L^k q^n\right)^2\left(1 - \xi_L^{-1k} q^n\right)^2}\end{aligned} \tag{5.21}$$

where the winding-momentum component is trivial because we are considering holomorphic fields. This quantity can be written in terms of theta function, the proof is described in the Appendix A:

$$\phi_{untw,g^k}(\tau, z) = \left(2 - \xi_L^{-k} - \xi_L^k\right)\left(2 - \xi_R^{-k} - \xi_R^k\right) \frac{\theta_1(\tau, z + kr_L)\theta_1(\tau, z - kr_L)}{\theta_1(\tau, kr_L)\theta_1(\tau, -kr_L)}. \tag{5.22}$$

where r_L corresponds to eigenvalues $e^{2\pi i r_L}$.
Now we have to calculate the twisted twining genus (5.11) for each conjugation class. Let us start from the twisted not twining contribute and in particular from oscillators contribute to this quantity. In the g-twisted sector $\mathcal{H}_g$ of the Hilbert space the bosonic oscillators, corresponded to bosons with eigenvalues $\xi_L = e(r_L)$, are moded by integer numbers shifted by exponential argument of correspondent eigenvalue $n \to n + r_L$. We write the action of bosonic operators in this sector:

$$(-1)^F q^{L_0}\overline{q}^{\overline{L}_0} y^{J_0^3}\left(\partial Z_{-r}^{(i)}\right)^h |\Omega\rangle = q^{rh}\left(\partial Z_{-r}^{(i)}\right)^h |\Omega\rangle \tag{5.23}$$

where $r \in \mathbb{N} + k/N$. Therefore the trace over those states is:

$$1 + q^r + q^{2r} + ... = \frac{1}{(1 - q^r)}$$

We take now fermionic operator and we obtain the action:

$$\begin{aligned}(-1)^F q^{L_0}\overline{q}^{\overline{L}_0} y^{J_0^3}|\Omega\rangle &= (+1)|\Omega\rangle \\ (-1)^F q^{L_0}\overline{q}^{\overline{L}_0} y^{J_0^3}\chi_{-r}^{(1)*}|\Omega\rangle &= -y^{-1}q^r\chi_{-r}^{(1)}|\Omega\rangle\end{aligned}$$

Tracing we find:

$$\left(1 - y^{-1}q^r\right). \tag{5.24}$$

We can to compute the contribute for any left fermionic and bosonic operators, the result is:

$$\phi_{g;e}(\tau; z) = \phi^{gs}(\tau, z)\phi^{osc}(\tau, z) = \phi^{gs}(z)\prod_{r\in\mathbb{N}+r_L}\frac{\left(1 - y^{-1}q^r\right)^2(1 - yq^r)^2}{(1 - q^r)^4}.$$

We have to insert an element $h \in C(g)$ in the trace so, repeating the same procedure, we can calculate the oscillators contribute to $\phi_{g,h}$:

$$\phi_{g,h}(\tau, z) = \phi^{gs}(\tau, z)\phi^{osc}(\tau, z) = \phi^{gs}(z)\prod_{r\in\mathbb{N}+r_L}\frac{\left(1 - y^{-1}\xi_L q^r\right)\left(1 - y^{-1}\xi_L^{-1}q^r\right)(1 - y\xi_L q^r)\left(1 - y\xi_L^{-1}q^r\right)}{(1 - \xi_L q^r)^2\left(1 - \xi_L^{-1}q^r\right)^2}.$$

These last two quantities must be computed for each pairs of commuting elements.
We must be careful when we are considering twisted sectors generated by element with centralizer formed by elements that don't belong to the same conjugation class. An example is $g \in 3A$ and $h \in 6A$. Notice that in these cases the ground state of the g-sector, formed by elements of lower order respect to other class, it can be seen as h^2-twisted ground states. Therefore we compute these twisted twining genera starting from twisted ground state of the class with highest order.
There exists another way to calculate the twisted twining genera. Let us consider the following *modular proprieties* of twisted twining genus: [22]

$$\begin{aligned} &\bullet \quad \phi_{g,h}\left(\tau, z+l\tau+l'\right) = e^{-2\pi i(l^2\tau+2lz)}\phi_{g,h}(\tau,z) \qquad l,l' \in \mathbb{Z} \\ &\bullet \quad \phi_{g,h}\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right) = \chi_{g,h}\begin{pmatrix} a & b \\ c & d\end{pmatrix} e^{2\pi i \frac{cz^2}{c\tau+d}}\phi_{h^c g^a, h^d g^b}(\tau,z) \end{aligned} \tag{5.25}$$

where $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL\left(2,\mathbb{Z}\right)$, and for a certain multiplier $\chi_{g,h} : SL\left(2,\mathbb{Z}\right) \to U(1)$. Each $\phi_{g,h}(\tau,z)$ is a weak Jacobi form of weight 0 and index 1 with multiplier $\chi_{g,h}$ under a subgroup $\Gamma_{g,h} \subseteq SL\left(2,\mathbb{Z}\right)$.
Since all commuting pairs are different powers of the same element, the idea is to compute the twining genera on the untwisted sector $\phi_{\mathbb{1},g}$ for each element of group through the formula (5.22), and to obtain all possible $\phi_{g,h}$ through modular by implementing the modular transformation using the formulas (5.25). Let us compute the quantity (5.22) for each element of every class considering the eigenvalues listed in Tab.5.1.

- **Class $-$1A**

$$\phi_{\mathbb{1},-\mathbb{1}}(\tau,z) = 16\frac{\theta_1\left(\tau,z+\frac{1}{2}\right)\theta_1\left(\tau,z-\frac{1}{2}\right)}{\theta_1\left(\tau,+\frac{1}{2}\right)\theta_1\left(\tau,-\frac{1}{2}\right)}.$$

  Through the transformations proprieties of Jacobi Theta functions, the same quantity can written in terms of function $\theta_2(\tau,z)$ in a more compact form:

$$\phi_{\mathbb{1},-\mathbb{1}}(\tau,z) = 16\frac{\theta_2(\tau,z)^2}{\theta_2(\tau,0)^2}.$$

- **Class 2B**

$$\phi_{\mathbb{1},g^k\in 2B}(\tau,z) = \left[e\left(\frac{k}{4}\right)+e\left(\frac{3k}{4}\right)-2\right]^2\frac{\theta_1\left(\tau,z+\frac{k}{4}\right)\theta_1\left(\tau,z-\frac{k}{4}\right)}{\theta_1\left(\tau,+\frac{k}{4}\right)\theta_1\left(\tau,-\frac{k}{4}\right)}.$$

  Where $k=1,3$, in particular in this class we have 15 elements written in form g and 15 in form g^3.

- **Class 3A**

$$\phi_{\mathbb{1},g^k\in 3A}(\tau,z) = \left[e\left(\frac{k}{3}\right)+e\left(\frac{2k}{3}\right)-2\right]^2\frac{\theta_1\left(\tau,z+\frac{k}{3}\right)\theta_1\left(\tau,z-\frac{k}{3}\right)}{\theta_1\left(\tau,+\frac{k}{3}\right)\theta_1\left(\tau,-\frac{k}{3}\right)}.$$

  With $k=1,2$, and 10 elements of type g and 10 of type g^2.

- **Class $-$3A**

$$\phi_{\mathbb{1},g^k\in -3A}(\tau,z) = \left[e\left(\frac{k}{6}\right)+e\left(\frac{5k}{6}\right)-2\right]^2\frac{\theta_1\left(\tau,z+\frac{k}{6}\right)\theta_1\left(\tau,z-\frac{k}{6}\right)}{\theta_1\left(\tau,+\frac{k}{6}\right)\theta_1\left(\tau,-\frac{k}{6}\right)}.$$

  Where $k=1,5$, in particular we have 10 elements written in form g and 10 in form g^5.

- **Class 5A**

$$\phi_{\mathbb{1},g^k\in 5A}(\tau,z)=\left[e\left(\frac{k}{5}\right)+e\left(\frac{4k}{5}\right)-2\right]\left[e\left(\frac{2k}{5}\right)+e\left(\frac{3k}{5}\right)-2\right]\frac{\theta_1\left(\tau,z+\frac{k}{5}\right)\theta_1\left(\tau,z-\frac{k}{5}\right)}{\theta_1\left(\tau,+\frac{k}{5}\right)\theta_1\left(\tau,-\frac{k}{5}\right)}.$$

With $k=1,2,3,4$, and we have 3 elements in the class of each type g^k.

- **Class $-$5A**

$$\phi_{\mathbb{1},g^k\in -5A}(\tau,z)=\left[e\left(\frac{k}{10}\right)+e\left(\frac{9k}{10}\right)-2\right]\left[e\left(\frac{3k}{10}\right)+e\left(\frac{7k}{10}\right)-2\right]\frac{\theta_1\left(\tau,z+\frac{k}{10}\right)\theta_1\left(\tau,z-\frac{k}{10}\right)}{\theta_1\left(\tau,+\frac{k}{10}\right)\theta_1\left(\tau,-\frac{k}{10}\right)}.$$

Where $k=1,3,7,9$, and we have 3 elements for every form g^k.

- **Class 5A$'$**

$$\phi_{\mathbb{1},g^k\in 5A'}(\tau,z)=\left[e\left(\frac{2k}{5}\right)+e\left(\frac{3k}{5}\right)-2\right]\left[e\left(\frac{k}{5}\right)+e\left(\frac{4k}{5}\right)-2\right]\frac{\theta_1\left(\tau,z+\frac{2k}{5}\right)\theta_1\left(\tau,z-\frac{2k}{5}\right)}{\theta_1\left(\tau,+\frac{2k}{5}\right)\theta_1\left(\tau,-\frac{2k}{5}\right)}.$$

Whit $k=1,2,3,4$, and we have 3 elements in the class of each type g^k.

- **Class $-$5A$'$**

$$\phi_{\mathbb{1},g^k\in -5A'}(\tau,z)=\left[e\left(\frac{3k}{10}\right)+e\left(\frac{7k}{10}\right)-2\right]\left[e\left(\frac{k}{10}\right)+e\left(\frac{9k}{10}\right)-2\right]\frac{\theta_1\left(\tau,z+\frac{3k}{10}\right)\theta_1\left(\tau,z-\frac{3k}{10}\right)}{\theta_1\left(\tau,+\frac{3k}{10}\right)\theta_1\left(\tau,-\frac{3k}{10}\right)}.$$

Where $k=1,3,7,9$, and we have 3 elements for every form g^k.

Now we must find the modular transformations that link these twining genera to twisted twining genus associated to each commuting pairs (g,h) in the group. As the (5.10) tells us, we can fix the g-twisted sector choosing a representative g for each conjugation class and write the(5.11) for all h elements commuting with g.

Class $-$1A:

The centralizer of this class contains 120 elements according to Tab.5.1, therefore the element $-\mathbb{1}$ commute with each element of the group and we have to consider all following pairs:

$$(-\mathbb{1},\mathbb{1}),(-\mathbb{1},-\mathbb{1}),(-\mathbb{1},g\in 2B),(-\mathbb{1},g\in 3A),(-\mathbb{1},g\in -3A),$$
$$(-\mathbb{1},g\in 5A),(-\mathbb{1},g\in -5A),(-\mathbb{1},g\in 5A'),(-\mathbb{1},g\in -5A').$$

In order to compute the twisted twining genus corresponding to first pair, let's start from twining genus labelled by pair $(\mathbb{1},-\mathbb{1})$. Keeping in mind the modular transformation of Theta function described in the Appendix A, the modular propriety transformation of twisted twining genus (5.25) relating to $SL(2,\mathbb{Z})$ transformation:

$$(\mathbb{1},-\mathbb{1})\quad\xrightarrow[\begin{pmatrix}0&-1\\1&0\end{pmatrix}]{}\quad(-\mathbb{1},\mathbb{1})$$

we obtain:

$$\phi_{(-\mathbb{1},\mathbb{1})}=16\frac{\theta_4(\tau,z)^2}{\theta_4(\tau,0)^2}. \tag{5.26}$$

Analogously we obtain the following results:

$$\begin{aligned}
\phi_{(-\mathbb{1},-\mathbb{1})} &= 16\frac{\theta_3(\tau,z)^2}{\theta_3(\tau,0)^2} && \text{with} && TS\phi_{(\mathbb{1},-\mathbb{1})}\\
\phi_{(-\mathbb{1},g^1\in 2B)} &= 4\frac{\theta_1(\tau,z+\frac{\tau}{2}+\frac{1}{4})\theta_1(\tau,z-\frac{\tau}{2}-\frac{1}{4})}{\theta_1(\tau,+\frac{\tau}{2}+\frac{1}{4})\theta_1(\tau,-\frac{\tau}{2}-\frac{1}{4})} && \text{with} && ST^{-2}S\phi_{(\mathbb{1},g\in 2B)}\\
\phi_{(-\mathbb{1},g^3\in 2B)} &= 4\frac{\theta_1(\tau,z+\frac{\tau}{2}+\frac{3}{4})\theta_1(\tau,z-\frac{\tau}{2}-\frac{3}{4})}{\theta_1(\tau,+\frac{\tau}{2}+\frac{3}{4})\theta_1(\tau,-\frac{\tau}{2}-\frac{3}{4})} && \text{with} && T^2ST^2ST\phi_{(\mathbb{1},g\in 2B)}\\
\phi_{(-\mathbb{1},g^2\in 3A)} &= \frac{\theta_1(\tau,z+\frac{\tau}{2}+\frac{1}{3})\theta_1(\tau,z-\frac{\tau}{2}-\frac{1}{3})}{\theta_1(\tau,+\frac{\tau}{2}+\frac{1}{3})\theta_1(\tau,-\frac{\tau}{2}-\frac{1}{3})} && \text{with} && TST^3ST\phi_{(\mathbb{1},g\in -3A)}\\
\phi_{(-\mathbb{1},g^4\in 3A)} &= \frac{\theta_1(\tau,z+\frac{\tau}{2}+\frac{2}{3})\theta_1(\tau,z-\frac{\tau}{2}-\frac{2}{3})}{\theta_1(\tau,+\frac{\tau}{2}+\frac{2}{3})\theta_1(\tau,-\frac{\tau}{2}-\frac{2}{3})} && \text{with} && T^2STST^{-2}S\phi_{(\mathbb{1},g\in -3A)}\\
\phi_{(-\mathbb{1},g\in -3A)} &= \frac{\theta_1(\tau,z+\frac{\tau}{2}+\frac{1}{6})\theta_1(\tau,z-\frac{\tau}{2}-\frac{1}{6})}{\theta_1(\tau,+\frac{\tau}{2}+\frac{1}{6})\theta_1(\tau,-\frac{\tau}{2}-\frac{1}{6})} && \text{with} && \begin{pmatrix}1&0\\3&1\end{pmatrix}\phi_{(\mathbb{1},g\in -3A)}\\
\phi_{(-\mathbb{1},g^5\in -3A)} &= \frac{\theta_1(\tau,z+\frac{\tau}{2}+\frac{5}{6})\theta_1(\tau,z-\frac{\tau}{2}-\frac{5}{6})}{\theta_1(\tau,+\frac{\tau}{2}+\frac{5}{6})\theta_1(\tau,-\frac{\tau}{2}-\frac{5}{6})} && \text{with} && \begin{pmatrix}-1&-2\\3&5\end{pmatrix}\phi_{(\mathbb{1},g\in -3A)}\\
\phi_{(-\mathbb{1},g^2\in 5A)} &= \frac{\theta_1(\tau,z+\frac{\tau}{2}+\frac{1}{5})\theta_1(\tau,z-\frac{\tau}{2}-\frac{1}{5})}{\theta_1(\tau,+\frac{\tau}{2}+\frac{1}{5})\theta_1(\tau,-\frac{\tau}{2}-\frac{1}{5})} && \text{with} && \begin{pmatrix}3&1\\5&2\end{pmatrix}\phi_{(\mathbb{1},g\in -5A)}\\
\phi_{(-\mathbb{1},g^4\in 5A)} &= \frac{\theta_1(\tau,z+\frac{\tau}{2}+\frac{2}{5})\theta_1(\tau,z-\frac{\tau}{2}-\frac{2}{5})}{\theta_1(\tau,+\frac{\tau}{2}+\frac{2}{5})\theta_1(\tau,-\frac{\tau}{2}-\frac{2}{5})} && \text{with} && \begin{pmatrix}-1&-1\\5&4\end{pmatrix}\phi_{(\mathbb{1},g\in -5A)}\\
\phi_{(-\mathbb{1},g^6\in 5A)} &= \frac{\theta_1(\tau,z+\frac{\tau}{2}+\frac{3}{5})\theta_1(\tau,z-\frac{\tau}{2}-\frac{3}{5})}{\theta_1(\tau,+\frac{\tau}{2}+\frac{3}{5})\theta_1(\tau,-\frac{\tau}{2}-\frac{3}{5})} && \text{with} && \begin{pmatrix}1&1\\5&6\end{pmatrix}\phi_{(\mathbb{1},g\in -5A)}\\
\phi_{(-\mathbb{1},g^8\in 5A)} &= \frac{\theta_1(\tau,z+\frac{\tau}{2}+\frac{4}{5})\theta_1(\tau,z-\frac{\tau}{2}-\frac{4}{5})}{\theta_1(\tau,+\frac{\tau}{2}+\frac{4}{5})\theta_1(\tau,-\frac{\tau}{2}-\frac{4}{5})} && \text{with} && \begin{pmatrix}2&3\\5&8\end{pmatrix}\phi_{(\mathbb{1},g\in -5A)}\\
\phi_{(-\mathbb{1},g\in -5A)} &= \frac{\theta_1(\tau,z+\frac{\tau}{2}+\frac{1}{10})\theta_1(\tau,z-\frac{\tau}{2}-\frac{1}{10})}{\theta_1(\tau,+\frac{\tau}{2}+\frac{1}{10})\theta_1(\tau,-\frac{\tau}{2}-\frac{1}{10})} && \text{with} && \begin{pmatrix}1&0\\5&1\end{pmatrix}\phi_{(\mathbb{1},g\in -5A)}\\
\phi_{(-\mathbb{1},g^3\in -5A)} &= \frac{\theta_1(\tau,z+\frac{\tau}{2}+\frac{3}{10})\theta_1(\tau,z-\frac{\tau}{2}-\frac{3}{10})}{\theta_1(\tau,+\frac{\tau}{2}+\frac{3}{10})\theta_1(\tau,-\frac{\tau}{2}-\frac{3}{10})} && \text{with} && \begin{pmatrix}2&1\\5&3\end{pmatrix}\phi_{(\mathbb{1},g\in -5A)}\\
\phi_{(-\mathbb{1},g^7\in -5A)} &= \frac{\theta_1(\tau,z+\frac{\tau}{2}+\frac{7}{10})\theta_1(\tau,z-\frac{\tau}{2}-\frac{7}{10})}{\theta_1(\tau,+\frac{\tau}{2}+\frac{7}{10})\theta_1(\tau,-\frac{\tau}{2}-\frac{7}{10})} && \text{with} && \begin{pmatrix}-2&-3\\5&7\end{pmatrix}\phi_{(\mathbb{1},g\in -5A)}\\
\phi_{(-\mathbb{1},g^9\in -5A)} &= \frac{\theta_1(\tau,z+\frac{\tau}{2}+\frac{9}{10})\theta_1(\tau,z-\frac{\tau}{2}-\frac{9}{10})}{\theta_1(\tau,+\frac{\tau}{2}+\frac{9}{10})\theta_1(\tau,-\frac{\tau}{2}-\frac{9}{10})} && \text{with} && \begin{pmatrix}4&7\\5&9\end{pmatrix}\phi_{(\mathbb{1},g\in -5A)}\\
\phi_{(-\mathbb{1},g^2\in 5A')} &= \frac{\theta_1(\tau,z+\frac{\tau}{2}+\frac{3}{5})\theta_1(\tau,z-\frac{\tau}{2}-\frac{3}{5})}{\theta_1(\tau,+\frac{\tau}{2}+\frac{3}{5})\theta_1(\tau,-\frac{\tau}{2}-\frac{3}{5})} && \text{with} && \begin{pmatrix}3&1\\5&2\end{pmatrix}\phi_{(\mathbb{1},g\in -5A')}\\
\phi_{(-\mathbb{1},g^4\in 5A')} &= \frac{\theta_1(\tau,z+\frac{\tau}{2}+\frac{6}{5})\theta_1(\tau,z-\frac{\tau}{2}-\frac{6}{5})}{\theta_1(\tau,+\frac{\tau}{2}+\frac{6}{5})\theta_1(\tau,-\frac{\tau}{2}-\frac{6}{5})} && \text{with} && \begin{pmatrix}-1&-1\\5&4\end{pmatrix}\phi_{(\mathbb{1},g\in -5A')}\\
\phi_{(-\mathbb{1},g^6\in 5A')} &= \frac{\theta_1(\tau,z+\frac{\tau}{2}+\frac{9}{5})\theta_1(\tau,z-\frac{\tau}{2}-\frac{9}{5})}{\theta_1(\tau,+\frac{\tau}{2}+\frac{9}{5})\theta_1(\tau,-\frac{\tau}{2}-\frac{9}{5})} && \text{with} && \begin{pmatrix}1&1\\5&6\end{pmatrix}\phi_{(\mathbb{1},g\in -5A')}
\end{aligned}$$

$$\phi_{(-\mathbb{1},g^8\in 5A')} = \frac{\theta_1(\tau, z+\frac{\tau}{2}+\frac{12}{5})\theta_1(\tau, z-\frac{\tau}{2}-\frac{12}{5})}{\theta_1(\tau, +\frac{\tau}{2}+\frac{12}{5})\theta_1(\tau, -\frac{\tau}{2}-\frac{12}{5})} \quad \text{with} \quad \begin{pmatrix} 2 & 3 \\ 5 & 8 \end{pmatrix} \phi_{(\mathbb{1},g\in -5A')}$$

$$\phi_{(-\mathbb{1},g\in -5A')} = \frac{\theta_1(\tau, z+\frac{\tau}{2}+\frac{3}{10})\theta_1(\tau, z-\frac{\tau}{2}-\frac{3}{10})}{\theta_1(\tau, +\frac{\tau}{2}+\frac{3}{10})\theta_1(\tau, -\frac{\tau}{2}-\frac{3}{10})} \quad \text{with} \quad \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} \phi_{(\mathbb{1},g\in -5A')}$$

$$\phi_{(-\mathbb{1},g^3\in -5A')} = \frac{\theta_1(\tau, z+\frac{\tau}{2}+\frac{9}{10})\theta_1(\tau, z-\frac{\tau}{2}-\frac{9}{10})}{\theta_1(\tau, +\frac{\tau}{2}+\frac{9}{10})\theta_1(\tau, -\frac{\tau}{2}-\frac{9}{10})} \quad \text{with} \quad \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \phi_{(\mathbb{1},g\in -5A')}$$

$$\phi_{(-\mathbb{1},g^7\in -5A')} = \frac{\theta_1(\tau, z+\frac{\tau}{2}+\frac{21}{10})\theta_1(\tau, z-\frac{\tau}{2}-\frac{21}{10})}{\theta_1(\tau, +\frac{\tau}{2}+\frac{21}{10})\theta_1(\tau, -\frac{\tau}{2}-\frac{21}{10})} \quad \text{with} \quad \begin{pmatrix} -2 & -3 \\ 5 & 7 \end{pmatrix} \phi_{(\mathbb{1},g\in -5A')}$$

$$\phi_{(-\mathbb{1},g^9\in -5A')} = \frac{\theta_1(\tau, z+\frac{\tau}{2}+\frac{27}{10})\theta_1(\tau, z-\frac{\tau}{2}-\frac{27}{10})}{\theta_1(\tau, +\frac{\tau}{2}+\frac{27}{10})\theta_1(\tau, -\frac{\tau}{2}-\frac{27}{10})} \quad \text{with} \quad \begin{pmatrix} 4 & 7 \\ 5 & 9 \end{pmatrix} \phi_{(\mathbb{1},g\in -5A')}$$

Let us repeat this procedure for any conjugation class of the group and summing the results, we obtain the Elliptic Genus of the orbifold $T^4/\mathbb{Z}_2.A_5$:

$$\phi(\tau;z) = \sum_{classes[g]} \frac{1}{|C(g)|} \sum_{h|gh=hg} \phi_{g,h}(\tau;z) =$$

$$\frac{1}{120}\sum_{h\in G}\phi_{\mathbb{1},g} + \frac{1}{120}\sum_{g\in G}\phi_{-\mathbb{1},g}(\tau,z) + \frac{1}{4}\sum_{h|hg=gh}\phi_{g,h}(\tau,z)\Big|_{g\in 2B} + \frac{1}{6}\sum_{h|hg=gh}\phi_{g,h}(\tau,z)\Big|_{g\in 3A} +$$

$$\frac{1}{6}\sum_{h|hg=gh}\phi_{g,h}(\tau,z)\Big|_{g\in -3A} + \frac{1}{10}\sum_{h|hg=gh}\phi_{g,h}(\tau,z)\Big|_{g\in 5A} + \frac{1}{10}\sum_{h|hg=gh}\phi_{g,h}(\tau,z)\Big|_{g\in -5A} +$$

$$\frac{1}{10}\sum_{h|hg=gh}\phi_{g,h}(\tau,z)\Big|_{g\in 5A'} + \frac{1}{10}\sum_{h|hg=gh}\phi_{g,h}(\tau,z)\Big|_{g\in -5A'} =$$

$$+\frac{1}{120}\Bigg\{16\frac{\theta_2(\tau,z)^2}{\theta_2(\tau,0)^2} + 15\left[e\left(\frac{1}{4}\right)+e\left(\frac{3}{4}\right)-2\right]^2 \sum_{k=0}^{1}\frac{\theta_1(\tau,z+\frac{2k+1}{4})\theta_1(\tau,z-\frac{2k+1}{4})}{\theta_1(\tau,+\frac{2k+1}{4})\theta_1(\tau,-\frac{2k+1}{4})} +$$

$$10\left[e\left(\frac{1}{3}\right)+e\left(\frac{2}{3}\right)-2\right]^2 \sum_{k=1}^{2}\frac{\theta_1(\tau,z+\frac{k}{3})\theta_1(\tau,z-\frac{k}{3})}{\theta_1(\tau,+\frac{k}{3})\theta_1(\tau,-\frac{k}{3})} +$$

$$10\left[e\left(\frac{1}{6}\right)+e\left(\frac{5}{6}\right)-2\right]^2 \sum_{k=0,2}\frac{\theta_1(\tau,z+\frac{2k+1}{6})\theta_1(\tau,z-\frac{2k+1}{6})}{\theta_1(\tau,+\frac{2k+1}{6})\theta_1(\tau,-\frac{2k+1}{6})} +$$

$$3\left[e\left(\frac{1}{5}\right)+e\left(\frac{4}{5}\right)-2\right]\left[e\left(\frac{2}{5}\right)+e\left(\frac{3}{5}\right)-2\right]\left[\sum_{k=1}^{4}\frac{\theta_1(\tau,z+\frac{k}{5})\theta_1(\tau,z-\frac{k}{5})}{\theta_1(\tau,+\frac{k}{5})\theta_1(\tau,-\frac{k}{5})} + \sum_{k=1}^{4}\frac{\theta_1(\tau,z+\frac{2k}{5})\theta_1(\tau,z-\frac{2k}{5})}{\theta_1(\tau,+\frac{2k}{5})\theta_1(\tau,-\frac{2k}{5})}\right] +$$

$$3\left[e\left(\frac{1}{10}\right)+e\left(\frac{9}{10}\right)-2\right]\left[e\left(\frac{3}{10}\right)+e\left(\frac{7}{10}\right)-2\right]\Bigg[\sum_{k=0/\{2\}}^{4}\frac{\theta_1(\tau,z+\frac{2k+1}{10})\theta_1(\tau,z-\frac{2k+1}{10})}{\theta_1(\tau,+\frac{2k+1}{10})\theta_1(\tau,-\frac{2k+1}{10})} +$$

$$\sum_{k=0/\{2\}}^{4}\frac{\theta_1(\tau,z+\frac{3(2k+1)}{10})\theta_1(\tau,z-\frac{3(2k+1)}{10})}{\theta_1(\tau,+\frac{3(2k+1)}{10})\theta_1(\tau,-\frac{3(2k+1)}{10})}\Bigg]\Bigg\}_{1A} +$$

$$+\frac{1}{120}\Bigg\{16\frac{\theta_3(\tau,z)^2}{\theta_3(\tau,0)^2} + 16\frac{\theta_4(\tau,z)^2}{\theta_4(\tau,0)^2} + 15\left[e\left(\frac{1}{4}\right)+e\left(\frac{3}{4}\right)-2\right]^2 \sum_{k=0}^{1}\frac{\theta_1(\tau,z+\frac{\tau}{2}+\frac{2k+1}{4})\theta_1(\tau,z-\frac{\tau}{2}-\frac{2k+1}{4})}{\theta_1(\tau,+\frac{\tau}{2}+\frac{2k+1}{4})\theta_1(\tau,-\frac{\tau}{2}-\frac{2k+1}{4})} +$$

$$10\left[e\left(\frac{1}{6}\right)+e\left(\frac{5}{6}\right)-2\right]^2 \sum_{k=1/(3)}^{5}\frac{\theta_1(\tau,z+\frac{3\tau}{6}+\frac{k}{6})\theta_1(\tau,z-\frac{3\tau}{6}-\frac{k}{6})}{\theta_1(\tau,+\frac{3\tau}{6}+\frac{k}{6})\theta_1(\tau,-\frac{3\tau}{6}-\frac{k}{6})} +$$

$$3\left[e\left(\frac{1}{10}\right)+e\left(\frac{9}{10}\right)-2\right]\left[e\left(\frac{3}{10}\right)+e\left(\frac{7}{10}\right)-2\right]\Bigg[\sum_{k=1/5}^{9}\frac{\theta_1(\tau,z+\frac{5\tau}{10}+\frac{k}{10})\theta_1(\tau,z-\frac{5\tau}{10}-\frac{k}{10})}{\theta_1(\tau,+\frac{5\tau}{10}+\frac{k}{10})\theta_1(\tau,-\frac{5\tau}{10}-\frac{k}{10})}$$

$$\sum_{k=1/5}^{9}\frac{\theta_1(\tau,z+\frac{5\tau}{10}+\frac{3k}{10})\theta_1(\tau,z-\frac{5\tau}{10}-\frac{3k}{10})}{\theta_1(\tau,+\frac{5\tau}{10}+\frac{3k}{10})\theta_1(\tau,-\frac{5\tau}{10}-\frac{3k}{10})}\Bigg]\Bigg\}_{-1A} +$$

$$\frac{1}{4}\Bigg\{\left[e\left(\frac{1}{4}\right)+e\left(\frac{3}{4}\right)-2\right]^2\left[\sum_{k=1}^{4}\frac{\theta_1(\tau,z+\frac{\tau}{4}+\frac{k}{4})\theta_1(\tau,z-\frac{\tau}{4}-\frac{k}{4})}{\theta_1(\tau,+\frac{\tau}{4}+\frac{k}{4})\theta_1(\tau,-\frac{\tau}{4}-\frac{k}{4})}\right]\Bigg\}_{2B} +$$

$$\frac{1}{6}\left\{\left[e\left(\frac{1}{3}\right)+e\left(\frac{2}{3}\right)-2\right]^2\left[\sum_{k=1}^{3}\frac{\theta_1(\tau,z+\frac{\tau}{3}+\frac{k}{3})\theta_1(\tau,z-\frac{\tau}{3}-\frac{k}{3})}{\theta_1(\tau,+\frac{\tau}{3}+\frac{k}{3})\theta_1(\tau,-\frac{\tau}{3}-\frac{k}{3})}\right]+\right.$$

$$\left.\left[e\left(\frac{1}{6}\right)+e\left(\frac{5}{6}\right)-2\right]^2\left[\sum_{k=0}^{2}\frac{\theta_1(\tau,z+\frac{2\tau}{6}+\frac{2k+1}{6})\theta_1(\tau,z-\frac{2\tau}{6}-\frac{2k+1}{6})}{\theta_1(\tau,+\frac{2\tau}{6}+\frac{2k+1}{6})\theta_1(\tau,-\frac{2\tau}{6}-\frac{2k+1}{6})}\right]\right\}_{3A}+$$

$$\frac{1}{6}\left\{\left[e\left(\frac{1}{6}\right)+e\left(\frac{5}{6}\right)-2\right]^2\left[\sum_{k=1}^{6}\frac{\theta_1(\tau,z+\frac{\tau}{6}+\frac{k}{6})\theta_1(\tau,z-\frac{\tau}{6}-\frac{k}{6})}{\theta_1(\tau,+\frac{\tau}{6}+\frac{k}{6})\theta_1(\tau,-\frac{\tau}{6}-\frac{k}{6})}\right]\right\}_{-3A}+$$

$$\frac{1}{10}\left\{\left[e\left(\frac{1}{5}\right)+e\left(\frac{4}{5}\right)-2\right]\left[e\left(\frac{2}{5}\right)+e\left(\frac{3}{5}\right)-2\right]\left[\sum_{k=1}^{5}\frac{\theta_1(\tau,z+\frac{\tau}{5}+\frac{k}{5})\theta_1(\tau,z-\frac{\tau}{5}-\frac{k}{5})}{\theta_1(\tau,+\frac{\tau}{5}+\frac{k}{5})\theta_1(\tau,-\frac{\tau}{5}-\frac{k}{5})}+\right]+\right.$$

$$\left.\left[e\left(\frac{1}{10}\right)+e\left(\frac{9}{10}\right)-2\right]\left[e\left(\frac{3}{10}\right)+e\left(\frac{7}{10}\right)-2\right]\left[\sum_{k=0}^{4}\frac{\theta_1(\tau,z+\frac{2\tau}{10}+\frac{2k+1}{10})\theta_1(\tau,z-\frac{2\tau}{10}-\frac{2k+1}{10})}{\theta_1(\tau,+\frac{2\tau}{10}+\frac{2k+1}{10})\theta_1(\tau,-\frac{2\tau}{10}-\frac{2k+1}{10})}\right]\right\}_{5A}+$$

$$\frac{1}{10}\left\{\left[e\left(\frac{1}{10}\right)+e\left(\frac{9}{10}\right)-2\right]\left[e\left(\frac{3}{10}\right)+e\left(\frac{7}{10}\right)-2\right]\left[\sum_{k=1}^{10}\frac{\theta_1(\tau,z+\frac{\tau}{10}+\frac{k}{10})\theta_1(\tau,z-\frac{\tau}{10}-\frac{k}{10})}{\theta_1(\tau,+\frac{\tau}{10}+\frac{k}{10})\theta_1(\tau,-\frac{\tau}{10}-\frac{k}{10})}\right]\right\}_{-5A}+$$

$$\frac{1}{10}\left\{\left[e\left(\frac{2}{5}\right)+e\left(\frac{3}{5}\right)-2\right]\left[e\left(\frac{1}{5}\right)+e\left(\frac{4}{5}\right)-2\right]\left[\sum_{k=1}^{5}\frac{\theta_1(\tau,z+\frac{\tau}{5}+\frac{3k}{5})\theta_1(\tau,z-\frac{\tau}{5}-\frac{3k}{5})}{\theta_1(\tau,+\frac{\tau}{5}+\frac{3k}{5})\theta_1(\tau,-\frac{\tau}{5}-\frac{3k}{5})}+\right]+\right.$$

$$\left.\left[e\left(\frac{3}{10}\right)+e\left(\frac{7}{10}\right)-2\right]\left[e\left(\frac{1}{10}\right)+e\left(\frac{9}{10}\right)-2\right]\left[\sum_{k=0}^{4}\frac{\theta_1(\tau,z+\frac{2\tau}{10}+\frac{3(2k+1)}{10})\theta_1(\tau,z-\frac{2\tau}{10}-\frac{3(2k+1)}{10})}{\theta_1(\tau,+\frac{2\tau}{10}+\frac{3(2k+1)}{10})\theta_1(\tau,-\frac{2\tau}{10}-\frac{3(2k+1)}{10})}\right]\right\}_{5A'}+$$

$$\frac{1}{10}\left[e\left(\frac{3}{10}\right)+e\left(\frac{7}{10}\right)-2\right]\left\{\left[e\left(\frac{1}{10}\right)+e\left(\frac{9}{10}\right)-2\right]\left[\sum_{k=1}^{10}\frac{\theta_1(\tau,z+\frac{\tau}{10}+\frac{3k}{10})\theta_1(\tau,z-\frac{\tau}{10}-\frac{3k}{10})}{\theta_1(\tau,+\frac{\tau}{10}+\frac{3k}{10})\theta_1(\tau,-\frac{\tau}{10}-\frac{3k}{10})}\right]\right\}_{-5A'}$$

In order to distinguish between a torus and a K3 model we compute the Witten index $\phi(\tau, z = 0)$ which is the Euler number of target space:

$$\begin{aligned}&\phi(\tau,0)=\left[\frac{480}{120}\right]_{1A}+\left[\frac{240}{120}\right]_{-1A}+[4]_{2B}+\left[\frac{30}{6}\right]_{3A}+[1]_{-3A}+\\&\left[\frac{30}{10}\right]_{5A}+[1]_{-5A}+\left[\frac{30}{10}\right]_{5A'}+[1]_{-5A'}=\\&4+2+4+5+1+3+1+3+1=24\end{aligned} \tag{5.27}$$

From the latter result, using non-trivial identities of the *theta* functions, we deduce that the elliptic genus $\phi(\tau, z)$ obtained can be simplify and it is exactly the elliptic genus of $K3$.

5.5 Holomorphic fields in the $T^4/\mathbb{Z}_2.A_5$ model

We want to discuss the string compactification on the $T^4/2.A_5$ orbifold through the representation theory of $\mathcal{N} = (4, 4)$ superconformal algebra.
Highest weight states are labelled by conformal dimension h and isospin l. When $c = 6$, as in our case, unitary requires $h \geq l$ in the NS sector, and $h \geq \frac{1}{4}$ in the R sector. There exist two differents representation classes of $\mathcal{N} = 4$ algebra: massless and massive representations tabulated on the section 3.2.

In order to find the holomorphic fields contained in the spectrum of SCFT with target space T^4/G, we can study the partition function of the theory on the NS-NS sector and write the latter in terms of unitary representations of $\mathcal{N}=4$ algebra.
The presence of holomorphic field of conformal weight h enlarge the $\mathcal{N}=(4,4)$ superconformal algebra with a certain $\mathcal{W}$ algebra generated by these fields.
The spectrum of SCFT on T^4 contains infinite holomorphic fields with conformal dimensions labelled by positive integer values. We expect that the orbifold procedure eliminates some of these holomorphic fields from the spectrum. The result will be a SCFT closer to the more generic model of conformal theory on $K3$. We would expect that this generic model will contain only the basic $\mathcal{N}=(4,4)$ superconformal algebra, with no extra holomorphic fields.
Exactly as in the RR sector, in order for the partition function to be modular invariant, we must to add up each twisted sector of Hilbert space and to insert the projector on the trace so to project under invariant states. All information on the holomorphic component of the algebra of the theory is contained on the Untwisted sector of Hilbert space. Indeed the conformal weights (h_L, h_R) of the ground states on the $\mathcal{H}_g$ sector depend to eigenvalues ξ_L and ξ_R of the g action on the bosonic and fermionic left- and right-moving. In particular $h_R=0$ if and only if $\xi_R=0$. For the group $\mathbb{Z}_2.A_5$ this only occurs for the $g=\mathbb{1}$. Therefore states corresponding to holomorphic fields ($h_R=0$) are only contained on the Untwisted sector.
Let us focus our attention to oscillator contribute to the function on the Untwisted sector:

$$Z^{osc}_{NSNS} = \frac{1}{|2.A_5|}\sum_{g\in G} Tr_{NS,Un}\left[g q^{L_0-\frac{c}{24}}\right].$$

The torus model with $\mathbb{Z}_2.A_5$ symmetry does not have non-vanishing winding-momenta purely left-moving $(k_L, k_R=0)$. Therefore all holomorphic fields, having necessary $k_R=0$, must have $k_L=0$. For this reason the only non-trivial contribute to the function comes from the oscillators contribute. Let us start to compute the bosonic contribute of the trace. Let to be $|\alpha\rangle$ the highest weight state of the NS sector with eigenvalue $|\alpha|^2=h$ respect to L_0 Virasoro operator and we consider only the holomorphic part. For a generic element $g^k\in G$:

$$q^{L_0}g^k|\alpha\rangle = q^{L_0}\xi_L^k|\alpha\rangle = q^{|\alpha|^2}g^k|\alpha\rangle.$$

We have supposed that the element g^k acts on the highest weight state through their eigenvalues and does not change its conformal weight. Now we act on these states with the creation operators that transform in the adjoint representation of the group through the same eigenvalues of the ground states:

$$\begin{aligned}
&q^{L_0}g^k\left(\partial Z^{(i)}_{-n}\right)|\alpha\rangle = q^{L_0}g^k\left(\partial Z^{(i)}_{-n}\right)g^{-k}g^k|\alpha\rangle = q^{|\alpha|^2+n}\xi_L^k\left(\partial Z^{(i)}_{-n}\right)g^k|\alpha\rangle\\
&q^{L_0}g^k\left(\partial Z^{(i)}_{-n}\right)^2|\alpha\rangle = q^{L_0}g^k\left(\partial Z^{(i)}_{-n}\right)g^{-k}g^k\left(\partial Z^{(i)}_{-n}\right)g^{-k}g^k|\alpha\rangle = q^{|\alpha|^2+2n}\xi_L^{2k}\left(\partial Z^{(i)}_{-n}\right)^2 g^k|\alpha\rangle\\
&\qquad\quad \dots\quad \dots
\end{aligned}$$

We take the trace paying attention to the orthogonality condition between the states. We obtain, factorizing also the complex conjugate part, the following results:

- $\alpha = 0$

$$Z^{osc}_{(\mathbb{1},g^k),bos}(\tau) = q^{-\frac{c}{24}} \prod_{n=1}^{\infty} \frac{1}{\left(1 - q^n \xi_L^k\right)\left(1 - q^n \xi_L^{-k}\right)}$$

- $\alpha \neq 0$

$$Z^{osc}_{(\mathbb{1},\mathbb{1}),bos}(\tau) = q^{|\alpha|^2 - \frac{c}{24}} \prod_{n=1}^{\infty} \frac{1}{\left(1 - q^n\right)^2}$$

we have considered a single complex boson.
With a similar calculation we can compute the fermionic oscillator contribute. Let us denote with $|0\rangle$ the fermionic ground state and we act on this states with the creation operator labelled by semi-integer numbers ($r \in \mathbb{Z}_0^+ + \frac{1}{2}$):

$$\begin{aligned} &q^{L_0} g^k |0\rangle = g^k |0\rangle \\ &q^{L_0} g^k \chi^*_{-r} |0\rangle = q^r g^k \chi^*_{-r} g^{-k} g^k |0\rangle = q^r \xi_L^k \chi^*_{-r} g^k |0\rangle \end{aligned}$$

Pauli Exclusion Principle assures us that each fermionic operator act only once and makes a contribute to the trace of $(1 + q^r)$. In order to consider every possible contribute we multiply this result for all r values and, factoring also the part corresponding to complex conjugation field, we obtain:

$$Z^{osc}_{(\mathbb{1},g^k)} = q^{-\frac{c}{24}} \prod_{n=1}^{\infty} \left(1 + \xi_L^k q^{n-\frac{1}{2}}\right)\left(1 + \xi_L^{-k} q^{n-\frac{1}{2}}\right)$$

where $c = 1$ because we have to consider a single complex fermion.
In order to project onto invariant states, we must sum under all elements of the group after putting bosonic and fermionic contribute together:

$$Z^{osc}_{NSNS}(\tau)_{Un} = q^{-\frac{c}{24}} \frac{1}{120} \sum_{g^k \in 2.A_5} \prod_{n=1}^{\infty} \frac{\left(1 + \xi_L^k q^{n-\frac{1}{2}}\right)^2 \left(1 + \xi_L^{-k} q^{n-\frac{1}{2}}\right)^2}{\left(1 - q^n \xi_L^k\right)^2 \left(1 - q^n \xi_L^{-k}\right)^2}. \tag{5.28}$$

In this latter expression $c = 6$ since we have to consider two complex bosons and two complex fermions.
Exactly like the partition function on the RR sector, we have to consider also the contribute come from the Twisted Sector, therefore we must calculate the trace (5.5) on the sectors $\mathcal{H}_{\tilde{g}}$ of the Hilbert space summing up the elements $g \in 2.A_5$ such that $g\tilde{g} = \tilde{g}g$.
The space $\mathcal{H}_{\tilde{g}}$ is generated by bosonic and fermionic oscillators labelled through integer number shifted by exponential argument of eigenvalues corresponding to $\tilde{g}$ element according with the conjugation class of belonging, as summary in Table(5.1):

$$n \quad \rightarrow \quad n + r_L.$$

Let $|\alpha\rangle$ be the highest weight state of the NS $\tilde{g}$-Twisted sector with eigenvalues $|\alpha|^2 = h$ with respect to Virasoro operator L_0 and $g^k \in 2.A_5$ an element such that $\left[g^k; \tilde{g}\right] = 0$:

- $q^{L_0} g^k |\alpha\rangle = q^{L_0} |g^k \alpha\rangle = q^{|\alpha|^2} g^k |\alpha\rangle,$

since the action of g^k on $|\alpha\rangle$ does not change the conformal weight of the state. Now we act on this state with the bosonic creation operators $\left(\partial Z^{(i)}_{-n}\right)$ that transform on the adjoint representation of the group through eigenvalue ξ_L as refered on Table(5.2):

- $q^{L_0} g^k \left(\partial Z^{(i)}_{-n-\tilde{r}_L}\right)|\alpha\rangle = q^{L_0} g^k \left(\partial Z^{(i)}_{-n}\right) g^{-k} g^k |\alpha\rangle = q^{|\alpha|^2+(n+\tilde{r}_L)} \xi_L^k \left(\partial Z^{(i)}_{-n}\right) g^k |\alpha\rangle$
- $q^{L_0} g^k \left(\partial Z^{(i)}_{-n-\tilde{r}_L}\right)^2 |\alpha\rangle = q^{L_0} g^k \left(\partial Z^{(i)}_{-n}\right) g^{-k} g^k \left(\partial Z^{(i)}_{-n}\right) g^{-k} g^k |\alpha\rangle =$
  $= q^{|\alpha|^2+2(n+\tilde{r}_L)} \xi_L^{2k} \left(\partial Z^{(i)}_{-n}\right)^2 g^k |\alpha\rangle$

  ... ...
- $q^{L_0} g^k \left(\partial Z^{(i)}_{-n-\tilde{r}_L}\right)^m |\alpha\rangle = q^{L_0} g^k \left(\partial Z^{(i)}_{-n}\right) g^{-k} ... g^k \left(\partial Z^{(i)}_{-n}\right) g^{-k} g^k |\alpha\rangle =$
  $= q^{|\alpha|^2+m(n+\tilde{r}_L)} \xi_L^{mk} \left(\partial Z^{(i)}_{-n}\right)^m g^k |\alpha\rangle$

We take the trace employing the appropriate orthogonality conditions:

$$q^{|\alpha|^2} + q^{|\alpha|^2+(n+\tilde{r}_L)}\xi_L^k + q^{|\alpha|^2+2(n+\tilde{r}_L)}\xi^{2k} + ... =$$
$$= q^{|\alpha|^2} \sum_{m=0}^{+\infty} q^{m(n+\tilde{r}_L)}\xi_L^{mk} = q^{|\alpha|^2} \frac{1}{\left(1 - q^{(n+\tilde{r}_L)}\xi_L^k\right)}.$$

Now we write the fermionic contribute on the $\tilde{g}$-twisted sector, for which the oscillators $\chi^{(i)}_r$ are labelled by the shifted numbers $r = \frac{1}{2} + n + \tilde{r}_L$ and act on the vacuum state $|0\rangle$:

- $q^{L_0} g^k |0\rangle = g^k |0\rangle$
- $q^{L_0} g^k \chi_{-r} |0\rangle = q^{L_0} g^k \chi_{-r} g^{-k} g^k |0\rangle = q^r \xi_L^k g^k |0\rangle$

The Pauli principle assure us that there aren't other contribute to the trace for this oscillators, then we have:

$$\left(1 + q^r \xi_L^k\right).$$

In order to consider every contribute to the partition function we multiply the bosonic and fermionic results for all n values and we factorize the other holomorphic components:

$$Z^{osc}_{NSNS(\tilde{g},g^k)}(\tau) = q^{-\frac{c}{24}} \prod_{n=1}^{\infty} \frac{\left(1 + \xi_L^k q^{n+\frac{1}{2}+\tilde{r}_L)}\right)^2 \left(1 + \xi_L^k q^{n+\frac{1}{2}+\tilde{r}_L)}\right)^2}{\left(1 - \xi_L^{-k} q^{n+\tilde{r}_L)}\right)^2 \left(1 + \xi_L^{-k} q^{n+\tilde{r}_L)}\right)^2} \tag{5.29}$$

According to the result for the Elliptic Genus on the previous section the total partition function for the orbifold theory on the NS-NS sector is obtained summing up all sectors twisted of Hilbert space projected under invariat states:

$$Z^{osc}_{NSNS}(\tau) = \sum_{[\tilde{g}]\in 2.A_5} \frac{1}{|C(\tilde{g})|} \sum_{h|\tilde{g}h=h\tilde{g}} Z^{osc}_{NSNS(\tilde{g},h)}(\tau).$$

Now we want to isolate on this function the characters associate to different massless and massive representations of superalgebra $\mathcal{N} = 4$. In order to perform that, we can calculate all contributes of Untwisted sector, and expand their sum in series until a certain order.

The resulting series, up to order q^3, for oscillator contribute on the Untwisted sector to holomorphic part of partition function on $(NS-NS)$ sector of conformal theory on $T^4/2.A_5$ is:

$$Z^{osc}_{NS,Un}(q) = 1 + 3q + 4q^{3/2} + 6q^2 + 12q^{5/2} + 21q^3 + o\left(q^{7/2}\right). \tag{5.30}$$

Comparing this result with the character formulas of unitary representations of $\mathcal{N}=4$ algebra summarized on the Appendix B of [11], we can see that there is only the massless character with $h=l=0$ containing the whole short representation on the superconformal algebra. Under Spectral Flow, this representation is mapped into the $h=\frac{1}{4}, l=\frac{1}{2}$ representation of R sector. These operators correspond to 4 RR ground states on Untwisted sector of the sigma model with $T^4/2.A_5$ target space.

After subtracting from (5.30) the expansion of massless representation labelled by $h=l=0$, we can see that the first holomorphic field of extended algebra has conformal weight $h=2$. The holomorphic fields with conformal weight $h=1$ and $h=3$ do not belong to $\mathcal{W}$ algebra.

The orbifold procedure raised by one unity the weight of the lighest holomorphic field outside of the $\mathcal{N}=4$ algebra, and eliminated the holomorphic field with conformal weight $h=3$. This suggest that we might have eliminated from the spectrum the holomorphic fields with odd conformal dimension. We will not try to disprove this conjecture in this thesis. Certainly we have reduced the whole symmetry algebra with respect to T^4.

Chapter 6

Boundary states

The D-branes in string theory can be described in two different ways. First, as an extended object in space-time that can wrap around certain cycles in the target space geometry. Another way is to think about them as boundary conditions imposed at the end-points of open strings. In this case the D-branes correspond to the different open string sector that can be added consistently to a given closed string theory. In terms of the 'world-sheet' approach, D-branes are therefore described by (boundary) conformal field theory.
Boundary CFT provides an exact description in α' of the D-branes and it is independent by the geometric interpretation of CFT in terms of the NLSM. This method is useful only in some case, where the CFT is known explicitly; for NLSM on Kalabi-Yau surfaces this happens only in some point of moduli space, such as orbifold points.
We remember that Boundary states provide a branes' description in term of closed strings. Indeed, the idea is to start from a CFT that describes the closed strings, and owns a certain chiral algebra $\mathcal{A} \times \bar{\mathcal{A}}$ containing the Virasoro algebra, and to extend the description on world-sheet with boundary and to study their boundary constraints. The Boundary conditions, with the request that at least a diagonal pair of the initial algebra is conserved, are represented through *Boundary States*. From a string point of view this construction corresponds to introduce in the theory open strings, that are boundary operators that interpolate two boundary states.
The consistent request for the description, in term of open string, stresses conditions on the possible D-branes to insert on the theory. For RCFT these requests are the *Cardy's conditions*.
In the introductory sections of this chapter we follow the analysis of boundary states given in [26], [27], [28], [29].

6.1 Cardy's contruction

In this section we describe the Boundary States construction for rational theories.
Let $\mathcal{H}$ be the close string space and $\mathcal{A}$ the algebra preserved by Boundary States.
We decompose the space of states of closed string in terms of irreducible representations of $\mathcal{A}\otimes\overline{\mathcal{A}}$:

$$\mathcal{H} = \oplus_{i;j}\mathcal{H}_i \otimes \overline{\mathcal{H}}_j \mathcal{N}_{ij}, \tag{6.1}$$

where $\mathcal{N}_{ij}$ describes the multiplicity with which the irreducible representation $\mathcal{H}_i \otimes \mathcal{H}_j$ appears in $\mathcal{H}$, if this are always finite the theory is said to be *rational* and the sum (6.1) is finite. Conventionally $\mathcal{H}_0$ is the vacuum representation, and $\mathcal{N}_{00} = 1$ for uniqueness of vacuum.
Boundary states that describe Boundary conditions preserving a certain symmetry provide a link

between the left- and right-moving fields of the symmetry, called *gluing condition*.
We can write a *gluing condition* separately for each summand on (6.1). We can find a non-trivial solution when $\mathcal{H}_i$ is the conjugate representation of $\mathcal{H}_j$. In this case there is only one coherent state that satisfies the gluing condition, this is called *Ishibashi state*:

$$\left[W_n - (-1)^{h_s}\rho(\overline{W}_n)\right]|i\rangle\rangle = 0 \qquad \forall n \in \mathbb{Z}, \quad W \in \mathcal{A} \tag{6.2}$$

where W and $\overline{W}$ are the algebra's generators preserved by boundary, h is their conformal weight and ρ denotes an automorphism of the algebra of fields that leaves the stress-energy tensor invariant. In this section we assume for simplicity that ρ is the trivial automorphism.
Let us consider the Virasoro characters:

$$\chi_i(q) = Tr_{\mathcal{H}_i} q^{L_0 - \frac{c}{24}} \tag{6.3}$$

Let us consider that the partition function of the string theory on the torus is diagonal.
Consider the cylinder amplitude with boundary conditions labelled by α and β. In the loop channel (open string):

$$Z_{\alpha\beta} = \sum_i n^i_{\alpha\beta}\chi_i(q) \qquad q = e^{2\pi i\tau} \tag{6.4}$$

where $n^i_{\alpha\beta}$ are the multiplicities of the i-th representation in the loop.
In the tree channel (closed string) the amplitude is:

$$Z_{\alpha\beta} = \langle\langle\alpha|\tilde{q}^{\frac{1}{2}(L_0+\tilde{L}_0-\frac{c}{12})}|\beta\rangle\rangle \qquad \tilde{q} = e^{-2\pi i\frac{1}{\tau}} \tag{6.5}$$

where the boundary $|\alpha/\beta\rangle\rangle$ respects the gluing conditions for the whole algebra.
We choose the Ishibashi states as a basis for the gluing conditions solutions, therefore every consistent Boundary State will be a linear combination of these states:

$$|\alpha\rangle\rangle = \sum_j B^j_\alpha|j\rangle\rangle. \tag{6.6}$$

We can now rewrite the closed string amplitude as [1]

$$Z_{\alpha\beta} = \sum_j \left(B^j_\alpha\right)^* \left(B^j_\beta\right)\chi_j(\tilde{q}). \tag{6.7}$$

We know also that is possible to transform the amplitude from the closed string channel to the open string channel through the modular transformation $S : \tau \to -1/\tau$:

$$Z_{\alpha\beta} = \sum_{ij} n^i_{\alpha\beta} S^i_j \chi_j(\tilde{q}), \tag{6.8}$$

where S is the matrix representation of the transformation $S \in SL(2,\mathbb{Z})$ in the characters space.
Demanding equality of (6.7) and (6.8) yields to:

$$\sum_i S^j_i n^i_{\alpha\beta} = \left(B^j_\alpha\right)^* \left(B^j_\beta\right), \tag{6.9}$$

known as *Cardy's condition*. The requirement that the multiplicities $n^i_{\alpha\beta}$ be non-negative integer number is a strong condition on the coefficients $B^i_{\alpha\beta}$. Moreover, every set of consistent boundary

[1] We have used the orthogonal condition on the Ishibashi state: $\ll j|\tilde{q}^{\frac{1}{2}(L_0+\overline{L}_0-\frac{c}{12})}|j' \gg = \delta_{j,j'}\chi_j(q)$.

states gives rise to a matrix family $\mathcal{N}^{\alpha}_{\beta \mathrm{i}}$ (NIM), one for each representation i.
The set of solutions at Cardy's condition form a lattice. Let us suppose that the following set:

$$M = \{|\alpha_1\rangle\rangle, |\alpha_2\rangle\rangle, ...|\alpha_n\rangle\rangle\} \tag{6.10}$$

satisfies Cardy's condition, *i.e.* the overlap between any two element of M leads to non negative integer number, then so does the set:

$$M = \left\{|\alpha_1 \gg, |\alpha_2\rangle\rangle, ... \sum_{l=1}^{n} m_l |\alpha_l\rangle\rangle\right\}, \qquad m_l \in \mathbb{N}_0. \tag{6.11}$$

What we therefore want to find are the *fundamental* boundary conditions that generate all other boundary conditions upon taking positive integer linear combinations as above. These fundamental boundary conditions are believed to be characterised by the condition that the B^i_α actually form a unitary matrix.
We want to search solutions to (6.9). The first consistent boundary is the vacuum $|0\rangle\rangle$, for which one requires $n^i_{00} = \delta^i_0$ (this condition ensures uniqueness of the open string vacuum). Replacing:

$$S^j_0 = |B^j_0|^2,$$

in the (6.9), we obtain:

$$|0\rangle\rangle = \sum_j \sqrt{S^j_0} |j\rangle\rangle \qquad S^j_0 > 0.$$

We can also define boundary $|l\rangle$ such that $n^i_{0j} = \delta^i_j$, then [2]

$$|l\rangle = \sum_j \frac{S^j_l}{\sqrt{S^j_0}} |j\rangle\rangle. \tag{6.12}$$

6.2 D-branes in flat space

6.2.1 Bosons

Let us consider a 26-dimensional bosonic string theory, where the target space is the non-compact variety $\mathcal{M}^{1,25}$ endowed with the metric $\eta_{\mu\nu}$. From the point of view of the world-sheet theory, the bosonic string consists of 26 free bosonic fields $X^\mu(\tau, \sigma), \mu = 0, 1, ..., 25$ that describe the embedding of the string world- sheet in the target space.
As we previously said the modes α^μ_n and $\tilde{\alpha}^\mu_n$ associated with conformal fields $\partial_z X^\mu_L$ and $\partial_{\bar{z}} X^\mu_R$ satisfy the algebra $\hat{u}(1)^{26} \oplus \hat{u}(1)^{26}$. The algebra representations are labelled by the momenta k of the ground state. Since k can be any vector, there are infinitely many highest weight representations, and the theory is therefore not rational. Therefore the Cardy construction cannot be applied in this case, but it is still possible to construct the Boundary States.
If we want to build boundary states that preserve the full symmetry $\hat{u}(1)^{26} \oplus \hat{u}(1)^{26}$, the gluing condition is:

$$\left(\alpha^\mu_l + \rho(\tilde{\alpha^\mu_{-l}})\right)|\alpha \gg = 0 \qquad \forall l \in \mathbb{Z}.$$

For each field μ we have two possibility:

$$(i) \quad \rho(\tilde{\alpha}^\mu_{-l}) = \tilde{\alpha}^\mu_{-l}$$
$$(ii) \quad \rho(\tilde{\alpha}^\mu_{-l}) = -\tilde{\alpha}^\mu_{-l}.$$

[2]This solves the Cardy's condition thanks to the Verlinde formula

In the first case the gluing condition is the *Neumann* boundary condition:

$$\left(\alpha_l^\mu + \alpha_{-l}^{\tilde{\mu}}\right)|N\rangle\rangle = 0 \qquad \forall l \in \mathbb{Z}, \tag{6.13}$$

in terms of the fields:

$$\frac{1}{2}\sum_l e^{-il\sigma}\left(\alpha_l^\mu + \tilde{\alpha}_{-l}^\mu\right)|N\rangle\rangle = \partial_\tau X^\mu(\tau,\sigma)|_{\tau=0}|N\rangle\rangle = 0,$$

from the closed string point of view the boundary is at $\tau = 0$. The other choice for ρ leads to *Dirichlet* condition:

$$\left(\alpha_l^\mu - \alpha_{-l}^{\tilde{\mu}}\right)|D\rangle\rangle = 0 \qquad \forall l \in \mathbb{Z}, \tag{6.14}$$

together with the zero-mode condition

$$x^\mu|D\rangle\rangle = a^\mu|D\rangle\rangle,$$

where a^μ is a constant, corresponds to boundary condition:

$$X^\mu(\tau,\sigma)|_{\tau=0}|D\rangle\rangle = a^\mu|D\rangle\rangle.$$

In general we can choose Neumann or Dirichlet conditions in every direction, respectively correspondents to longitudinal and transverse directions. Since the theory must respect $SO(1,25)$ symmetry, we assume thet the first $p+1$ components are Neumann components, the other $25-p$ are Dirichlet components. The resulting boundary condition is called *Dp-brane*: it is a $p+1$-dimensional hypersurface with $25-p$ fixed directions.
In the following we shall always work in light-cone gauge. To this end we introduce the light-cone fields:

$$X^\pm(\tau,\sigma) = \frac{1}{\sqrt{2}}\left(X^0(\tau,\sigma) \pm X^{25}(\tau,\sigma)\right).$$

We also fix the world-sheet reparametrisation invariance through $X^+ = 2\pi\alpha' p^+\tau$, this leads to Dirichlet condition on the $X^\pm$ directions: the D-branes are istantons.
Let $|Bp,a\rangle\rangle$ be the Boundary States corresponding to a Dp-brane localized in the position $\overline{a}$. Let us assume that the boundary state satisfies:

$$\begin{aligned} &\left(\alpha_l^\mu + \alpha_{-l}^{\tilde{\mu}}\right)|Bp,\overline{a} \gg = 0 \qquad && \mu = 1,...,p+1 \\ &\left(\alpha_l^\mu - \alpha_{-l}^{\tilde{\mu}}\right)|Bp,\overline{a} \gg = 0 \qquad && \mu = p+2,...,24 \end{aligned} \tag{6.15}$$

where the first condition with $l = 0$ implies that only the representations with $k^\mu = 0$ for $\mu = 1,...p+1$ contribute. For these gluing conditions the Ishibashi state in the representation of momentum k is:

$$|Bp,\overline{k}\rangle\rangle = exp\left\{\sum_{n>0}\left[-\frac{1}{n}\sum_{\mu=1}^{p+1}\alpha_{-n}^\mu\tilde{\alpha}_{-n}^\mu + \frac{1}{n}\sum_{\nu=p+2}^{24}\alpha_{-n}^\nu\tilde{\alpha}_{-n}^\nu\right]\right\}|k\rangle\rangle. \tag{6.16}$$

The whole boundary state is the Fourier transformation:

$$|Bp,\overline{a}\rangle\rangle = \mathcal{N}\int\prod_{\nu=p+2,\ldots,24} dk^\nu e^{ik^\nu a^\nu}|Bp,\overline{k}\rangle\rangle, \tag{6.17}$$

where the normalisation is determined by the analogous of Cardy's condition.
Let us consider the closed string overlap between two boundary states (6.5):

$$\mathcal{A} = \langle\langle Bp, \overline{a}_1 || e^{-tH_c} || Bp, \overline{a}_2 \rangle\rangle, \tag{6.18}$$

where H_c is the closed string Hamiltonian in light-cone gauge and we have performed a Wick rotation. Given the explicit form of boundary states and through the modular transformation $t = 1/\tilde{t}$:

$$\mathcal{A} = \mathcal{N}^2 2^{\frac{23-p}{2}} \tilde{t}^{-\frac{p+1}{2}} e^{-(\overline{a}_1 - \overline{a}_2)^2 \frac{\tilde{t}}{2\pi}} \frac{1}{f_1(\tilde{q})^{24}} \tag{6.19}$$

where $\tilde{q} = e^{2\pi\tilde{t}} = e^{-2\pi\frac{1}{t}}$ and f_1 is the Jacobi Theta function $\theta_1(\tau, z = 0)$ defined in the Appendix A.
This should now be interpreted as the open-string trace:

$$\mathcal{Z} = Tr_{\mathcal{H}_{Dp,Dp}} e^{-2\tilde{t}H_0}, \tag{6.20}$$

where H_0 is the open string Hamiltonian in light-cone gauge and the trace includes an integral over momentum for Neumann directions:

$$\mathcal{Z} = (2\tilde{t})^{-\frac{p+1}{2}} e^{-(\overline{a}_1 - \overline{a}_2)^2 \frac{\tilde{t}}{2\pi}} \frac{1}{f_1(\tilde{q})^{24}}. \tag{6.21}$$

The equality between (6.19) and (6.21) holds the condition:

$$\mathcal{N} = 2^{-6}.$$

6.2.2 Fermions

Let us consider Boundary States in ten-dimensional superstring theory. Now the bosonic degrees of freedom are described by ten fields X^μ, which give rise of eight transvers degrees of freedom in light-cone gauge. In addition we have eight left and right-moving fermion fields with conformal weight $h = \overline{h} = \frac{1}{2}$.
Now we want to introduce D-branes that preserve the full symmetry: free fermion and free boson symmetries separately, but also the superconformal symmetry of the world-sheet theory.
Since the fermion fields conformal weight is $\frac{1}{2}$, by (6.2) the gluing conditions on fermion fields are labelled by $\eta = \pm 1$:

$$\left(\psi_r^\mu + i\eta\rho(\tilde{\psi}_{-r}^\mu)\right) ||D, \eta\rangle\rangle = 0. \tag{6.22}$$

We remember that the $N = 1$ supercharge is:

$$G_r = \sum_{n\in\mathbb{Z}} \eta_{\mu\nu} \psi_{r-n}^\mu \alpha_n^\nu$$

and similarly for $\tilde{G}$, therefore boundary states must satisfy:

$$\left(G_r + i\eta\tilde{G}_{-r}\right) ||D, \eta\rangle\rangle = 0. \tag{6.23}$$

In the fermionic condition (6.22) we choose:

$$\begin{aligned} \rho(\tilde{\psi}_{-r}^\mu) &= +\tilde{\psi}_{-r}^\mu \qquad && \text{Neumann direction} \\ \rho(\tilde{\psi}_{-r}^\mu) &= -\tilde{\psi}_{-r}^\mu \qquad && \text{Dirichlet direction.} \end{aligned}$$

so to satisfy the (6.23). We note that the Ishibashi states exist only in the $(NS-NS)$ and $(R-R)$ sectors, otherwise the condition (6.22) does not make any sense. Let us write the total Boundary State in the following form:

$$\begin{aligned}|Bp,\overline{k},\eta\rangle\rangle = exp\Bigg\{\sum_{n>0}\left[-\frac{1}{n}\sum_{\mu=1}^{p+1}\alpha^{\mu}_{-n}\tilde{\alpha}^{\mu}_{-n}+\frac{1}{n}\sum_{\nu=p+2}^{8}\alpha^{\nu}_{-n}\tilde{\alpha}^{\nu}_{-n}\right]+\\ i\eta\sum_{r>0}\left[-\sum_{\mu=1}^{p+1}\psi^{\mu}_{-r}\tilde{\psi}^{\mu}_{-r}+\sum_{\mu=p+1}^{8}\psi^{\mu}_{-r}\tilde{\psi}^{\mu}_{-r}\right]\Bigg\}|\overline{k}\eta\rangle^{(0)},\end{aligned} \tag{6.24}$$

where $|\overline{k}\eta\rangle^{(0)}$ is the $(NS-NS)$ ground state with momentum $\overline{k}$, while in the $(R-R)$ sector is determined by the condition (6.22) for $r=0$.
The boundary state should be an element of the closed string spectrum of the theory , but the actual closed string spectrum is not just the sum over the different sectors NS-NS, NS-R, R-NS and R-R. As it is well known, the actual spectrum of the closed string theory only consists of the states that are GSO invariant: we want boundary states to be GSO invariant.
In order to write the GSO projector, we introduce left and right moving fermion number generators $(-1)^F$ (anticommutes with left modes and commutes with all other modes) and $(-1)^{\tilde{F}}$ (anticommutes with right modes and commutes with all other modes). Furthermore, both have eigenvalue (-1) on the $(NS-NS)$ ground state, and we can choose some suitable convention on the $(R-R)$ ground states. In the $(NS-NS)$ sector we impose the GSO projection:

$$P_{NS-NS}=\frac{1}{4}\left(1+(-1)^F\right)\left(1+(-1)^{\tilde{F}}\right). \tag{6.25}$$

While in the $(R-R)$ sector we can make two choices:

$$P_{R-R}=\frac{1}{4}\left(1+(-1)^F\right)\left(1-(-1)^{\tilde{F}}\right)\qquad IIA \tag{6.26}$$

that defines type IIA string theory, and:

$$P_{R-R}=\frac{1}{4}\left(1+(-1)^F\right)\left(1+(-1)^{\tilde{F}}\right)\qquad IIB \tag{6.27}$$

that defines type IIB string theory.
After applying GSO projectors on Ishibashi states in the $NS-NS$ sector, it is easy to get the following GSO invariant state:

$$|Bp,\overline{k}\rangle\rangle_{NS-NS}=\frac{1}{\sqrt{2}}\left[|Bp,\overline{k},+\rangle\rangle_{NS-NS}-|Bp,\overline{k},-\rangle\rangle_{NS-NS}\right]. \tag{6.28}$$

In the $(R-R)$ sector the analysis is more complex because there are fermionic zero modes. Let us define:

$$\psi^{\mu}_{\pm}=\frac{1}{\sqrt{2}}\left(\psi^{\mu}_{0}\pm i\tilde{\psi}^{\mu}_{0}\right) \tag{6.29}$$

which satisfy:

$$\left\{\psi^{\mu}_{\pm};\psi^{\nu}_{\pm}\right\}=0\qquad\left\{\psi^{\mu}_{+};\psi^{\nu}_{-}\right\}=\delta^{\mu\nu}.$$

For zero modes the (6.22) becomes:

$$\begin{aligned}\psi^{\mu}_{\eta}|Bp,\overline{k},\eta\gg^{(0)}_{RR}=0\qquad \mu=1,....,p+1\\ \psi^{\nu}_{-\eta}|Bp.\overline{k},\eta\gg^{(0)}_{RR}=0\qquad \nu=p+2,...,8.\end{aligned}$$

Let us define the states:

$$
\begin{aligned}
|Bp,\overline{k},+\rangle\rangle^{(0)}_{RR} &= \prod_{\mu=1}^{p+1}\psi^{\mu}_{+}\prod_{\nu=p+2}^{8}\psi^{\nu}_{-}|Bp,\overline{k},-\rangle\rangle^{(0)}_{RR} \\
|Bp,\overline{k},-\rangle\rangle^{(0)}_{RR} &= \prod_{\mu=1}^{p+1}\psi^{\mu}_{-}\prod_{\nu=p+2}^{8}\psi^{\nu}_{+}|Bp,\overline{k},-\rangle\rangle^{(0)}_{RR}
\end{aligned}
\tag{6.30}
$$

and GSO operator on the ground states:

$$
\begin{aligned}
(-1)^F &= \prod_{\mu=1}^{8}\left(\psi^{\mu}_{+}+\psi^{\mu}_{-}\right) \\
(-1)^{\tilde{F}} &= \prod_{\mu=1}^{8}\left(\psi^{\mu}_{+}-\psi^{\mu}_{-}\right).
\end{aligned}
\tag{6.31}
$$

Finally we summarize the action of GSO operators on the whole Ishibashi states:

$$
\begin{aligned}
(-1)^F|Bp,\overline{k},\eta\rangle\rangle_{RR} &= |Bp,\overline{k},-\eta\rangle\rangle_{RR} \\
(-1)^{\tilde{F}}|Bp,\overline{k},\eta\rangle\rangle_{RR} &= (-1)^{p+1}|Bp,\overline{k},-\eta\rangle\rangle_{RR}
\end{aligned}
\tag{6.32}
$$

therefore the only GSO invariant Ishibashi state is:

$$
|Bp,\overline{k}\rangle\rangle_{RR} = \frac{1}{\sqrt{2}}\left(|Bp,\overline{k},+\rangle\rangle_{RR}+|Bp,\overline{k},-\rangle\rangle_{RR}\right), \tag{6.33}
$$

and the second equation on (6.32) tells us that this is GSO invariant if:

$$
p = \begin{cases} \text{even} & \text{type IIA} \\ \text{odd} & \text{type IIB} \end{cases} \tag{6.34}
$$

In order for the D-branes to be stable objects, we need to apply the GSO projection on the open string spectrum, so to exclude the tachionic open string state. Stable BPS branes (that preserve $\frac{1}{2}$ of the whole space-time supersymmetry) are obtained by linear combinations of $(NS-NS)$ and $(R-R)$ boundary states:

$$
|Bp,k\rangle\rangle = \frac{1}{\sqrt{2}}\left(|Bp\rangle\rangle_{NSNS}\pm i|Bp\rangle\rangle_{RR}\right). \tag{6.35}
$$

The BPS D-branes must be charged states respect to $R-R$ fields. Indeed the corresponding boundary states always have a non-vanishing component along the $R-R$ states.
We can write the boundary states in the coordinates representation by Fourier transformation:

$$
||Bp,\overline{a}\rangle\rangle = \mathcal{N}\int\prod_{\nu=p+2}^{8}dk^{\nu}e^{ik^{\nu}a_{\nu}}|Bp,\overline{k}\rangle\rangle \tag{6.36}
$$

where $\mathcal{N}$ is a suitable normalisation obtained by the analogous of Cardy's conditions:

$$
\mathcal{N}_{NSNS}=1 \qquad \mathcal{N}_{RR}=1.
$$

6.3 The compactified case

Let's consider a target space compactified on some torus. The simplest case is when different directions are decoupled, therefore we start with a theory whose target space is a circle of radius R. The main effect of this compactification is to restrict the possible values of momentum at discrete values, thus we will replace the intergral over momenta by a infinite sum. The spectrum is:

$$\mathcal{H} = \oplus_{m,n}\mathcal{H}_{m,n}, \tag{6.37}$$

where $\mathcal{H}_{m,n}$ is the space of states generated from the ground state:

$$\begin{aligned} \alpha_0|\,(p_L, p_R)\rangle &= p_L|\,(p_L, p_R)\rangle \\ \tilde{\alpha}_0|\,(p_L, p_R)\rangle &= p_R|\,(p_L, p_R)\rangle \end{aligned} \tag{6.38}$$

with $(p_L, p_R) = \left(\frac{m}{R} + nR, \frac{m}{R} - nR\right)$.
As before, we want that boundary states to respect Neumann or Dirichlet conditions. In particular the Neumann condition (6.13) for $l = 0$ implies that Neumann Ishibashi state in $\mathcal{H}_{mn}$ must satisfy at $p_L = -p_R$ (left and right representations of preserved symmetry algebra are conjugate). This condition for generic radius implies $m = 0$: we have Ishibashi states labelled by integer n:

$$|\,(nR, -nR)\rangle\rangle^N \in \mathcal{H}_{(0,n)}. \tag{6.39}$$

Similarly, Dirichlet Ishibashi states can be constructed in $\mathcal{H}_{(m,n)}$ provided that $p_L = p_R$, that for generic radius implies $n = 0$

$$|\left(\frac{m}{R}, \frac{m}{R}\right)\rangle\rangle^D \in \mathcal{H}_{(m,0)} \qquad m \in \mathbb{Z}. \tag{6.40}$$

Therefore the Ishibashi states are:

$$\begin{aligned} |\,(nR, -nR)\rangle\rangle^N &= exp\left(-\sum_{l=1}^{\infty}\frac{1}{l}\alpha_{-l}\overline{\alpha}_{-l}\right)|\,(nR, -nR)\rangle \\ |\left(\frac{m}{R}, \frac{m}{R}\right)\rangle\rangle^D &= exp\left(\sum_{l=1}^{\infty}\frac{1}{l}\alpha_{-l}\overline{\alpha}_{-l}\right)|\left(\frac{m}{R}, \frac{m}{R}\right)\rangle. \end{aligned} \tag{6.41}$$

The D-branes are linear combinations of these states:

$$||w\rangle\rangle = \frac{R^{1/2}}{2^{1/4}}\sum_{n\in\mathbb{Z}} e^{iwnR}|nR, -nR\rangle\rangle^N \tag{6.42}$$

that describes a Neumann brane with Wilson line w, and:

$$||a\rangle\rangle = \frac{R^{-1/2}}{2^{1/4}}\sum_{m\in\mathbb{Z}} e^{ima/R}|\frac{m}{R}, \frac{m}{R}\rangle\rangle^D \tag{6.43}$$

that describes a Dirichlet brane at the $\overline{a}$ position.
At low energy the massless states of open strings are Lorentz vectors in space-time that correspond to fields defined in the corresponding D-branes. The transverse components to D-brane (Dirichlet directions) are interpreted as quantum fluctuations of the D-brane position. The longitudinal components to D-brane (Neumann directions) are seen as a gauge field $U(1)$ defined on the D-brane. If the D-brane wraps a circle, we can introduce a non-trivial background for this gauge field: this background is the Wilson line.
Let us consider now the conformal D-branes that satisfy gluing conditions for conformal symmetry but not necessarily the other symmetries.

6.4 Symmetries of NLSM

NLSM on K3 are two dimensional $\mathcal{N} = (4, 4)$ superconformal fields theories with central charge $c = \bar{c} = 6$. They arise as the worldsheet description of perturbative type IIA string theory on a K3 surface.
As we described in the Appendix C the 80-dimensional moduli-space of sigma-models on K3 has the form:

$$\mathcal{M}_{K3} = O\left(\Gamma^{4,20}\right) \setminus O\left(4,20\right) / \left(O(4) \times O(20)\right)$$

where $\Gamma^{4,20}$ is unique, up isomorphisms $O\left(\Gamma^{4,20}\right)$, even self-duale lattice with signature $(4, 20)$, we can think to it as the integral homology lattice of the K3 manifold (or orbifold).
$O\left(4,20\right) / \left(O(4) \times O(20)\right)$ parametrise the choice of a positive definite four-dimensional real subspace $\Pi \subset \mathbb{R}^{4,20}$.
The position of Π respect to $\Gamma^{4,20}$ determines a point in moduli space to which it corresponds a particular string theory specified by choice of a Ricci-flat metric and a B-field on K3 manifold (orbifold). However, not every choice of Π corresponds to a suitable CFT, for examples when Π is orthogonal to roots of $\Gamma^{4,20}$ the corresponding non-linear σ-model is not well defined.
Let $G = G_\Pi$ be the subgroup of $O\left(\Gamma^{4,20}\right)$ that leaves Π pointwise fixed. We denote by L^G the G-invariant sublattice of $\Gamma^{4,20}$, and with L_G his orthogonal complement. By construction $\Pi = L^G \otimes \mathbb{R}$, and since it has positive signature $(4, 0)$, L_G is a negative lattice of rank at most 20.
In [23] are classified the symmetries of NLSMs on K3 that commute with the $\mathcal{N} = (4, 4)$ algebra and the Spectral Flow operators. This classification can be seen as an extention of Mukai theorem (that classifies the geometric symmetries) [36] [37]. The groups G are subgroups of Conway C_{O_0} group of automorphisms of Leech lattice that fix pointwise a sublattice of at least 4 dimensions. [31]
The symmetries of the worldsheet theory should have an interpretation as lattice symmetries. The lattice $\Gamma^{4,20}$ can be identified with the D-brane charge lattice, and Π describes the four left and right moving supercharges. Therefore the symmetries that preserve the automorphisms of CFT should be in one-to-one correspondence with the symmetries of the D-brane charge lattice that leave Π pointwise invariant. In this picture, the real space $\mathbb{R}^{4,20}$ is identified with the space of 24 (anti-)chiral RR ground states with $h = \overline{h} = \frac{1}{4}$. Under the action of $SU(2)_L \times SU(2)_R$ (part of superconformal symmetry) this space splits into a four dimensional representation $(\mathbf{2}, \mathbf{2})$ with space-basis Π, and 20 singlets. Therefore, since a K3 $\sigma-$model is characterized by $\Pi \in \mathbb{R}^{4,20}$, the group G_Π that leave the superconformal algebra invariant and preserve the spectral flow (then is a CFT symmetry), is the subgroup of $O(\Gamma^{4,20})$ that leaves Π pointwise fixed. This assumption excludes some very interesting symmetries of the theory.
If we identify the boundary states of BPS D-branes of a sigma model on K3, and their corresponding $R - R$ charges, we obtain the exact point of moduli space corresponding to the theory and its symmetry group G_Π.
In the next sections we are going to consider boundary states for some K3 models obtained as torus orbifolds.

6.5 Boundary states in torus orbifolds

Let us consider a type IIA superstring theory compactified on $\mathcal{M}^{1,3} \times T^2 \times T^4/G$. We will restrict our attention to D-branes that extend along the orbifold directions. First of all we would like to write BPS boundary states as superpositions of Ishibashi states satisfying either Dirichlet or Neumann conditions and combined so that the closed string spectrum satisfies the

GSO invariance. As we previously said only boundary states in the (R-R) and (NS-NS) sectors are admitted, moreover, since we are considering a type IIA theory solely Dp-branes with p even are accepted. The boundary states on $T^4 = R^4/L$ are:

$$|D4, x\rangle\rangle \qquad |D2(l_i, l_j), x\rangle\rangle \qquad |D0, x\rangle\rangle.$$

where x denote generically the four real moduli of brane.
The *D0-brane* satisfies zero Neumann conditions and four Dirichlet conditions which fix uniquely the position of the brane in T^4: x represents the position in this four compactified dimensions.
The *D4-brane* satisfies four Neumann conditions and zero Dirichlet conditions, therefore the brane does not have a fixed position in T^4, but wraps along the four compactified directions: x denotes the value of Wilson lines along the four circle of torus.
The *D2-brane* satisfies two Neumann conditions and two Dirichlet conditions. x represents two fixed positions along the Dirichlet directions and two Wilson lines along the Neumann directions $(l_i, l_j) \in L$.
Every boundary state is normalized so that in a open string loop in the NS-sector $Tr_{NS\alpha\alpha} q^{\mathcal{H}_{open}}$ the vacuum character appears once. D-branes that satisfy this propriety are called *fundamental branes*. Notice that for open strings related between two different Dp-brane, the vacuum character on the loop only appears if the moduli are the same, otherwise the lightest open string has energy proportional to $|x - x'|^2$.
Let's define *Witten index* or *intersection number* the 1-loop open string amplitude on the $R(-1)^F$ sector:

$$\langle\langle\alpha|\beta\rangle\rangle = \sum_{i,j} \overline{q_i}(\alpha) q_j(\beta) \langle\chi_i|(-1)^{F+\tilde{F}}|\chi_j\rangle$$

where $\chi_1, \chi_2 ...$ form an orthonormal basis of Ramond-Ramond ground states. This quantity don't depends on moduli but only on the Ramond-Ramond charges of boundary states:

$$q_i(\alpha) = \langle\chi_i|\alpha\rangle\rangle.$$

The non-zero intersection between D-branes on T^4 are:

$$\begin{aligned} \langle\langle D0, x|D4, x'\rangle\rangle &= 1 \\ \langle\langle D2(l_i, l_j), x|D2(l_a, l_b), x'\rangle\rangle &= -\epsilon_{ijab}. \end{aligned}$$

Let us consider the torus orbifold T^4/G, where G is a finite subgroup of $SU(2)$. Let us suppose that G acts geometrically on the space.
G acts on moduli x of the branes, then every modulus' orbit will generically count $|G|$ distinct point. Let us define *bulk D-branes* as the sum over this orbit points:

$$|Dp, x\rangle\rangle_{T^4/G, bulk} = \frac{1}{\sqrt{|G|}} \sum_{g\in G} |Dp, g(x)\rangle\rangle,$$

the result is a G-invariant brane. The choice of normalization is such that the branes are fundamental. Notice that for D2-branes the g element in the sum acts also in the Neumann direction (l_i, l_j). In this calculation, it is important that in the overlap between $|D0, g(x)\rangle\rangle$ and $|D0, g'(x)\rangle\rangle$ the vacuum character appears only if $g(x) = g'(x)$, which for generic x implies $g = g'$.
On the other hand, the intersection number is independent of the moduli and one has:

$$_{T^4/G, bulk}\langle\langle D0, x|D4, x'\rangle\rangle_{T^4/G, bulk} = \frac{1}{|G|} \sum_{g,h\in G} \langle\langle D0, g(x)|D4, h(x')\rangle\rangle = |G|. \qquad (6.44)$$

Bulk D-branes are only charged under Ramond-Ramond ground fields in the untwisted sector.

6.5.1 Fractional branes

A natural starting point are the D-branes on the covering space T^4 that are G-invariant and are called *regular branes.* These regular branes (bulk D-branes) give rise to G *Regular representation* $\mathcal{R}$, of $|G|$ dimension:

$$\mathcal{R} = \sum_I d_I \mathcal{D}^I.$$

This representation is not irreducible, therefore we want to find a more fundamental set of branes such that the open strings attached to the latter carry indices transforming in an irreducible representation. This set exists and such D-branes are called *fractional branes.*
A fractional Dp-brane is a BPS object: it carries only a fractional charge respect to regular Dp-brane under untwisted RR($p+1$)-forms , but it is charged respect to some twisted RR($p+1$)-forms. This branes are positioned at orbifold fixed points.
Let x be a fixed point for the whole group G or under one of its subgroup H: $g(x) = x$ for $\forall g \in H$. Let us consider the bulk D0-brane $|D0, x\rangle\rangle_{T^4/G,bulk}$ positioned at x fixed point: open strings stretched between x and $g(x)$ become massless and the vacuum characters appear more that once: this branes are not fundamental branes.
For each H-fixed point x, there is one fundamental D-brane for each irreducible representation I of H. When $H = G$ is the full group of symmetries, the formula for a fractional D0-brane is given by:

$$|D0, x, I, G\rangle\rangle_{T^4/G,f} = \frac{dim(I)}{\sqrt{|G|}}|D0, x\rangle\rangle + \frac{1}{\sqrt{|G|}} \sum_{[g]\neq[e]} \sqrt{|\,[h]\,|}\,(2sin(\pi\nu_h))\,Tr_I(h)||T0, [h]\,, x\rangle\rangle \tag{6.45}$$

where, $dimI$ is the dimension of representation I; the sum is on conjugation classes of G; $Tr_I(h)$ is the trace of one representative element $h \in [h]$; $\nu_h|0 < \nu_h < 1$ is such that the eigenvalues of h when acting on the fields ∂X^i be $e^{\pm 2\pi i \nu_h}$, and $||T0, [h]\,, x\rangle\rangle$ is a symmetrisation on the whole conjugation class:

$$||T0, [h]\,, x\rangle\rangle = \frac{1}{\sqrt{|\,[h]\,|}} \sum_{h\in[h]} ||T0, h - twisted, x\rangle\rangle$$

where $||T0, h - twisted, x\rangle\rangle$ is a Ishibashi state in the h-twisted sector representation.
For every $h \in G$ the number (not necessarily G-invariant) of h-twisted ground state is the number of h-fixed points, and for every state there is a Ishibashi state $|T0, h - tw, x\rangle\rangle$.
If $H < G$ then the x orbit counts $|G/H|$ distinct points and each of these $g(x)$ is fixed by a subgroup $gHg^{-1} \sim H \in G$. The fractional D0-brane localized at x fixed point is:

$$\begin{aligned} &|D0, x, H, I\rangle\rangle_{T^4/G,f} = \frac{1}{\sqrt{|G/H|}} \sum_{[g]\in G/H} |D0, g(x), I\rangle\rangle_f = \\ &\frac{dimI}{\sqrt{|G|}} \sum_{[g]\in G/H} |D0, g(x)\rangle\rangle + \frac{1}{\sqrt{|G|}} \sum_{[g]\in G/H} \sum_{[h]\neq[e]} (2sin(\pi\nu_h))\,Tr_I(h)|T0, [h]\,, g(x)\rangle\rangle, \end{aligned} \tag{6.46}$$

where I is a irreducible representation of H.
There are a fractional D0-brane for every irrep of H.
The intersection index between fractional D0-branes can be easily calculated through the normalisation condition of Ishibashi states:

$$\langle\langle T0, [h']\,, x'||T0, [h]\,, x\rangle\rangle = \delta_{[h],[h']}\delta_{x,x'}.$$

Since the twisted sector Ishibashi states are orthogonal to the ones in the untwisted sector, so the only nonzero intersection is with bulk D4-branes:

$$ {}_{T^4/G,b}\langle\langle D4, x'|D0, x, H, I\rangle\rangle_{T^4/G,f} = \frac{dimI}{|G|} \sum_{g'\in G} \sum_{g\in G/H} \langle\langle D4, g'(x')|D0, g(x)\rangle\rangle = dimI|G/H|. $$

A completely similar formulation can be to do for fractional D2 and D4 branes when the moduli x are fixed by a subgroup $H \in G$. For D4-branes the twisted ground states will be labelled by fixed Wilson lines, therefore we will have different basis for this space respect the basis labelled by fixed-position for D0-branes. In order to calculate intersection index between fractional D0 and D4 branes one needs to know precisely how these different bases are related. In this thesis we will not try to solve this problem and only consider fractional D0-branes.

6.5.2 Fixed point

In order to construct the fractional D0-brane, let us calculate the fixed points on $T^4 = R^4/L$ under the action of G.
Let $\{l_1, l_2, l_3, l_4\}$ be a L basis, and G_{ij} be the matrix representation of $g \in G$ with respect to this basis. Since the lattice points are given by vectors $(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4$, the matrix G (lattice authomorphism) has integer entries and $\det G = 1$. The G action on the lattice is:

$$ g(x) = \sum_i l_i G_{ij} x_j = \begin{pmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & G_{24} \\ G_{31} & G_{32} & G_{33} & G_{34} \\ G_{41} & G_{42} & G_{43} & G_{44} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, $$

then the fixed point satisfies:

$$ (\mathbb{1} - G) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \in \quad \mathbb{Z}^4. $$

This last expression admits solution for $det\,(\mathbb{1} - G) \neq 0$, then $(\mathbb{1} - G)$ is invertible. Generally $det\,(\mathbb{1} - G) \neq 1$ and the inverse is a rational.
The fixed points are identified by the subset:

$$ \frac{(\mathbb{1} - G)^{-1}\,\mathbb{Z}^4}{\mathbb{Z}^4}. \tag{6.47} $$

The number of fixed points is the number of elementary cells of $(\mathbb{1} - G)^{-1}\,\mathbb{Z}^4$ inside one elementary cell of $\mathbb{Z}^4$, namely $det\,(\mathbb{1} - G)$.
Since $(\mathbb{1} - G)$ is invertible and G has finite order N, we can write:

$$ \begin{aligned} &\left(\mathbb{1} - G^N\right) = 0 \quad \to \quad (\mathbb{1} - G)\left(\mathbb{1} + G + G^2 + ... + G^{N-1}\right) = 0 \\ &\to \left(\mathbb{1} + G + G^2 + ... + G^{N-1}\right) = 0 \\ &G + G^2 + ... + G^{N-1} = -1, \end{aligned} $$

so that

$$ (\mathbb{1} - G)\left(G + 2G^2 + ... + (N-1)G^{N-1}\right) = G + G^2 + ... + G^{N-1} - (N-1)\mathbb{1} = -N, $$

then

$$(\mathbb{1}-G)^{-1}=-\frac{1}{N}\left(G+2G^2+......+(N-1)G^{N-1}\right).$$

This shows that the denominators in the rational entries of $(\mathbb{1}-G)^{-1}$ are divisors of N. The fixed points are always contained in $\frac{1}{N}L/L$. Notice that the origin $x=0$ is always fixed by every $g\in G$.
After a similar reasoning, we find that the number of fixed Wilson lines $x\in\mathbb{R}^4/L^*$ is the same as the number of fixed points. If the eigenvalues of G are $e^{\pm 2i\pi\nu_g}$ on the RR ground states, then this number is:

$$det\,(\mathbb{1}-G)=\left[\left(1-e^{2i\pi\nu_g}\right)\left(1-e^{-2i\pi\nu_g}\right)\right]^2=\left(2sin(\pi\nu_g)\right)^4. \tag{6.48}$$

6.6 $T^4/2.A_4$

Let us consider a torus model with symmetry $2.A_5$. The group $2.A_5$ can be generated by elements of order $3,4$ and 5 that we indicate respectively with g_3,g_4 and g_5. Let $\{\lambda_1,\lambda_2,....,\lambda_8\}$ be a suitable basis for winding-moment lattice $\Lambda^{4,4}$, such that:

$$(\lambda_i,\lambda_j)=\delta_{i,j+4}+\delta_{i,j-4}. \tag{6.49}$$

Let us define the action of the generators g_3,g_4,g_5 on this basis by the following matrices:

$$g_3=\begin{pmatrix}0&1&0&0&0&0&0&0\\-1&-1&0&0&0&0&0&0\\0&0&-1&-1&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&0&-1&1&0&0\\0&0&0&0&-1&0&0&0\\0&0&0&0&0&0&0&-1\\0&0&0&0&0&0&1&-1\end{pmatrix}\qquad g_4=\begin{pmatrix}0&1&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\-1&0&0&-1&0&0&0&0\\0&-1&0&0&0&0&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&-1&0&0&1\\0&0&0&0&-1&0&0&0\\0&0&0&0&0&-1&1&0\end{pmatrix} \tag{6.50}$$

$$g_5=\begin{pmatrix}0&0&0&0&1&0&0&0\\0&0&0&-1&-1&1&-1&0\\0&0&1&1&1&0&1&-1\\0&0&-1&0&-1&0&0&1\\1&1&0&1&1&-1&0&0\\0&1&0&0&1&0&0&0\\0&1&1&0&1&0&0&0\\0&1&1&1&1&0&0&0\end{pmatrix} \tag{6.51}$$

We notice that the first two generators preserve the maximal isotropic sublattices $\Gamma_w=span\,(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ and $\Gamma_m=span\,(\lambda_5,\lambda_6,\lambda_7,\lambda_8)$, while g_5 mix the components of two sublattices between them. The subgroup of $2.A_5$ generated by g_3 and g_4, isomorphic to $2.A_4$ of order 24, acts geometrically on the model. The action of $2.A_5$ on the target space is not geometric. Indeed the g_5 element has different eigenvalues when act on the left- and right-moving bosonic fields, therefore its action cannot be induced by isometries of target space: it is a *non-geometric* symmetry.
Let us focus our attention on $2.A_4$ subgroup: it acts on target space $T^4=\mathbb{R}^4/L$ as well as on

the lattice Λ_w through:

$$g_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad g_4 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{6.52}$$

Before starting with D-branes calculation, we compute how many RR ground states there are in any twisted sector:

$$\Phi(\tau, 0) = [5 + 3 + 4 + 5 + 1 + 5 + 1] = 24. \tag{6.53}$$

Notice that the total elliptic genus is 24, therefore the orbifold theory $T^4/2.A_4$ is a K3 model. We have respectively 5 RR-ground states in the untwisted sector, 3 in $2A$-twisted sector, 4 in $4A$-twisted sector, 5 in $3A$-twisted sector, 1 in $6A$-twisted sector, 5 in $3B$-twisted sector and 1 in $6B$-twisted sector.
In order to determinate the bulk D-branes, as previously described, let us symmetrise the D-branes of torus theory:

$$\begin{aligned} |D0, x\rangle\rangle_{T^4/2.A_4,bulk} &= \frac{1}{\sqrt{24}} \sum_{g \in 2.A_4} |D0, g(x)\rangle\rangle \\ |D4, x\rangle\rangle_{T^4/2.A_4,bulk} &= \frac{1}{\sqrt{24}} \sum_{g \in 2.A_4} |D4, g(x)\rangle\rangle. \end{aligned} \tag{6.54}$$

Since intersection number is moduli independent, we have the following results between D0 and D4-branes:

$${}_{T^4/2.A_4,bulk}\langle\langle D0, x|D4, x'\rangle\rangle_{T^4/2.A_4,bulk} = \frac{1}{24} \sum_{g,h} \langle\langle D0, g(x)|D4, h(x')\rangle\rangle = 24. \tag{6.55}$$

Let us focus on D2-branes and let's see haw $2.A_4$ acts on the two cycle wrapped by the brane:

$$|D2(l_i, l_j), x\rangle\rangle_{T^4/2.A_5,bulk} = \frac{1}{\sqrt{24}} \sum_{g \in 2.A_4} |\left(g(l_i), g(l_j)\right), g(x)\rangle\rangle. \tag{6.56}$$

By choosing a suitable basis we expect that only 3 D2-branes spanned a non trivially charge vector space.
Since we know that the group acts on the pairs of basis (l_i, l_j) through the previous matrix representation, we can to compute the six D2-branes labelled by $\{(l_1, l_2), (l_1, l_3), (l_1, l_4), (l_2, l_3), (l_2, l_4), (l_3, l_4)\}$ and using the intersection rule:

$${}_{T^4/2.A_5,bulk}\langle\langle D2(l_i, l_j), x|D2(l_a, l_b), x'\rangle\rangle = -det\mathcal{B}\epsilon_{ijab} \tag{6.57}$$

where the intersection number is positive (respectively negative) if $\{l_i, l_j, l_a, f_b\}$ forms a negatively (positively) oriented base of L, while $\mathcal{B}$ is the matrix having these vectors as rows . The result is the following matrix:

$$\mathcal{I}_{D2} = -6 \begin{pmatrix} 3 & 2 & 1 & 1 & -1 & 1 \\ 2 & 4 & 2 & 2 & -2 & -2 \\ 1 & 2 & 3 & 3 & 1 & -1 \\ 1 & 2 & 3 & 3 & 1 & -1 \\ -1 & -2 & 1 & 1 & 3 & 1 \\ 1 & -2 & -1 & -1 & 1 & 3 \end{pmatrix}. \tag{6.58}$$

As we expected the rank of $\mathcal{I}_{D2}$ is 3, in particular we can choose the D2-branes labelled by $\{(l_1,l_2),(l_1,l_3),(l_1,l_4)\}$ as basis for the three dimensional lattice of charges.
The other non-zero intersection number is:

$$\langle\langle D0,x|D4,x'\rangle\rangle = 24$$

while all the rest vanish. Summing the result into the bulk Dp-branes intersection matrix:

$$\mathcal{I}_{bulk} = 6 \begin{array}{c} \begin{matrix} D0 & D2(l_1,l_2) & D2(l_1,l_3) & D2(l_1,l_4) & D4 \end{matrix} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 4 \\ 0 & -3 & -2 & -1 & 0 \\ 0 & -2 & -4 & -2 & 0 \\ 0 & -1 & -2 & -3 & 0 \\ 4 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \tag{6.59}$$

We want to diagonalize this matrix since the signs of the eigenvalues should tell us which branes span the sublattice of charges that transforms under $(\mathbf{2},\mathbf{2})$ representation of $SU(2)_L \times SU(2)_R$. Through this sublattice, in particular its orientation respect to $\Gamma^{4,20}$, one can identify, within the moduli space of NLSM on K3, the family of models whose symmetry group contain $2.A_4$. The ordered eigenvalues are:

$$12(-2-\sqrt{2}) \qquad -24 \qquad 24 \qquad -12 \qquad 12(-2+\sqrt{2}),$$

and their corresponding eigenvectors are:

$$\begin{aligned} v_1 &= D2(l_1,l_2) + \sqrt{2}D2(l_1,l_3) + D2(l_1,l_4) \\ v_2 &= -D0 + D4 \\ v_3 &= D0 + D4 \\ v_4 &= -D2(l_1,l_2) + D2(l_1,l_4) \\ v_5 &= D2(l_1,l_2) - \sqrt{2}D2(l_1,l_3) + D2(l_1,l_4) \end{aligned}$$

Now we want to compute the fractional D0-branes, we expect that there are $24-5$ D0-branes independent of each other. In order to reach our goal, we need to compute the fixed point on $T^4 = \mathbb{R}^4/L$ under action of $2.A_4$. The points fixed by both generators g_3 and g_4 will be fixed by the whole group. By (6.47), the fixed points are given by:

$$\frac{(\mathbb{1}-g_3)^{-1}\,\mathbb{Z}^4}{\mathbb{Z}^4} = \left[\begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}\mathbb{Z}^4\right]/\mathbb{Z}^4$$

and

$$\frac{(\mathbb{1}-g_4)^{-1}\,\mathbb{Z}^4}{\mathbb{Z}^4} = \left[\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}\mathbb{Z}^4\right]/\mathbb{Z}^4.$$

Only $x=0$ is fixed by the whole group and we have seven fractional D0-branes for this point, one for each irreducible representation of the group:

$\lvert [g] \rvert$	1	4	4	4	4	6	1
	1A	3A	6A	3B	6B	4A	2A
I_1	1	1	1	1	1	1	1
I_2	1	ω	$\overline{\omega}$	$\overline{\omega}$	ω	1	1
I_3	1	$\overline{\omega}$	ω	ω	$\overline{\omega}$	1	1
I_4	2	-1	1	-1	1	0	-2
I_5	2	$-\overline{\omega}$	ω	$-\omega$	$\overline{\omega}$	0	-2
I_6	2	$-\omega$	$\overline{\omega}$	$-\overline{\omega}$	ω	0	-2
I_7	3	0	0	0	0	-1	3

By (6.45) we obtain the D0-branes at $x = 0$:

$$|D0,0,I_k,2.A_4\rangle\rangle_{T^4/2.A_4,fractional} = \frac{dimI_k}{\sqrt{|G|}}|D0,0\rangle\rangle + \frac{1}{\sqrt{|G|}}\sum_{[g]\in G}\sqrt{|\,[g]\,|}2sin(\pi\nu_g)Tr_{I_k}(g)||T0,[g]\,,0\rangle\rangle, \tag{6.60}$$

therefore:

- $|D0,0,I_1,2.A_4\rangle\rangle_{T^4/2.A_4,fractional} = \frac{1}{\sqrt{24}}|D0,0\rangle\rangle + \frac{1}{\sqrt{24}}\Big[2\sqrt{3}||T0,3A,0\rangle\rangle + 2||T0,6A,0\rangle\rangle + 2\sqrt{3}||T0,3B,0\rangle\rangle + 2||T0,6B,0\rangle\rangle + 2\sqrt{3}||T0,4A,0\rangle\rangle + 2||T0,2A,0\rangle\rangle\Big]$

- $|D0,0,I_2,2.A_4\rangle\rangle_{T^4/2.A_4,fractional} = \frac{1}{\sqrt{24}}|D0,0\rangle\rangle + \frac{1}{\sqrt{24}}\Big[2\sqrt{3}\omega||T0,3A,0\rangle\rangle + 2\overline{\omega}||T0,6A,0\rangle\rangle + 2\sqrt{3}\overline{\omega}||T0,3B,0\rangle\rangle + 2\omega||T0,6B,0\rangle\rangle + 2\sqrt{3}||T0,4A,0\rangle\rangle + 2||T0,2A,0\rangle\rangle\Big]$

- $|D0,0,I_3,2.A_4\rangle\rangle_{T^4/2.A_4,fractional} = \frac{1}{\sqrt{24}}|D0,0\rangle\rangle + \frac{1}{\sqrt{24}}\Big[2\sqrt{3}\overline{\omega}||T0,3A,0\rangle\rangle + 2\omega||T0,6A,0\rangle\rangle + 2\sqrt{3}\omega||T0,3B,0\rangle\rangle + 2\overline{\omega}||T0,6B,0\rangle\rangle + 2\sqrt{3}||T0,4A,0\rangle\rangle + 2||T0,2A,0\rangle\rangle\Big]$

- $|D0,0,I_4,2.A_4\rangle\rangle_{T^4/2.A_4,fractional} = \frac{1}{\sqrt{24}}|D0,0\rangle\rangle + \frac{2}{\sqrt{24}}\Big[-2\sqrt{3}||T0,3A,0\rangle\rangle + 2||T0,6A,0\rangle\rangle - 2\sqrt{3}||T0,3B,0\rangle\rangle + 2||T0,6B,0\rangle\rangle - 4||T0,2A,0\rangle\rangle\Big]$

- $|D0,0,I_5,2.A_4\rangle\rangle_{T^4/2.A_4,fractional} = \frac{2}{\sqrt{24}}|D0,0\rangle\rangle + \frac{1}{\sqrt{24}}\Big[-2\sqrt{3}\overline{\omega}||T0,3A,0\rangle\rangle + 2\omega||T0,6A,0\rangle\rangle - 2\sqrt{3}\omega||T0,3B,0\rangle\rangle + 2\overline{\omega}||T0,6B,0\rangle\rangle - 4||T0,2A,0\rangle\rangle\Big]$

- $|D0,0,I_6,2.A_4\rangle\rangle_{T^4/2.A_4,fractional} = \frac{2}{\sqrt{24}}|D0,0\rangle\rangle + \frac{1}{\sqrt{24}}\Big[-2\sqrt{3}\omega||T0,3A,0\rangle\rangle + 2\overline{\omega}||T0,6A,0\rangle\rangle - 2\sqrt{3}\overline{\omega}||T0,3B,0\rangle\rangle + 2\omega||T0,6B,0\rangle\rangle - 4||T0,2A,0\rangle\rangle\Big]$

- $|D0,0,I_7,2.A_4\rangle\rangle_{T^4/2.A_4,fractional} = \frac{1}{\sqrt{24}}|D0,0\rangle\rangle + \frac{3}{\sqrt{24}}\Big[-2\sqrt{3}||T0,4A,0\rangle\rangle + 6||T0,2A,0\rangle\rangle\Big]$

The sum of these seven fractional branes corresponds to a bulk D0-brane located at $x = 0$, so these boundary states give six new linearly independent charge vectors: we can take the boundary states related to $I_1, I_2, I_3, I_4, I_5, I_6$ independent representations.
In order to study the other fractional D0-branes, we need to consider fixed points under non-trivial subgroups of $2.A_4$. The element of class 2A forms a subgroup isomorphic to $\mathbb{Z}_2$, that acts

on lattice by -1. The elements of class 6A and 6B generate subgroups $\mathbb{Z}_6$, those of class 3A and 3B generate subgroups $\mathbb{Z}_3$, while those of class 4A subgroups $\mathcal{D}_8$. Therefore we have subgroups $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathcal{D}_8$ and $\mathbb{Z}_6$ with respectively $\left(2sin(\pi\frac{1}{2})\right)^4 = 16$, $\left(2sin(\pi\frac{1}{3})\right)^4 = 9$, $\left(2sin(\pi\frac{1}{4})\right)^4 = 4$ and $\left(2sin(\pi\frac{1}{6})\right)^4 = 1$ fixed points on $T^4/2.A_4$.
Let us start with subgroup $\mathbb{Z}_6$: it fixes one point $x = 0$ that was considered above.
Let us now consider the points fixed by the $H = \mathcal{D}_8$ subgroup: there are 4 of these points, of which one is the origin:

$$x_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad x_2 = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{pmatrix} \quad x_3 = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \\ 1/2 \end{pmatrix} \quad x_4 = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}, \tag{6.61}$$

all these points are fixed by everyone element of the group.
The fractional D0-branes at origin has been treated above, the remaining fixed points are organised in a single orbit $\mathcal{O}_{x_1}$ under the $\mathbb{Z}_2.A_4$ action, since $|\mathbb{Z}_2.A_4/\mathcal{D}_8| = 3$. Since there are five inequivalent irreducible representations of $\mathcal{D}_8$ labelled by A_k with $k = 0, 1, 2, 3, 4$, we have five fractional D0-branes for the single orbit $\mathcal{O}_{x_1}$. The character table of $\mathcal{D}_8$ is:

$\mid[g]\mid$	1	1	2	2	2
	1A	2A	4A	4B	4C
I_1	1	1	1	1	1
I_2	1	1	1	-1	-1
I_3	1	1	-1	1	-1
I_4	1	1	-1	-1	1
I_5	2	-2	0	0	0

where we have the first four representations one dimensional and the last of two dimension. Keeping in mind that $2.A_4/\mathcal{D}_8 \simeq \mathbb{Z}_3$, the fractional branes localized at $\mathcal{O}_{x_1}$ are:

$$\bullet|D0, \mathcal{O}_{x_1}, \mathcal{D}_8, I_1\rangle\rangle_{T^4/\mathbb{Z}_2.A_4,f} = \frac{1}{\sqrt{24}} \sum_{[g]\in 2.A_4/\mathcal{D}_8} |D0, g(x_1)\rangle\rangle + \frac{1}{\sqrt{24}} \sum_{[g]\in 2.A_4/\mathcal{D}_8} \Big[2|T0, 2A - tw, g(x_1)\rangle\rangle +$$
$$+\sqrt{2}|T0, 4A - tw, g(x_1)\rangle\rangle + \sqrt{2}|T0, 4B - tw, g(x_1)\rangle\rangle + \sqrt{2}|T0, 4C - tw, g(x_1)\rangle\rangle\Big]$$

$$\bullet|D0, \mathcal{O}_{x_1}, \mathcal{D}_8, I_2\rangle\rangle_{T^4/\mathbb{Z}_2.A_4,f} = \frac{1}{\sqrt{24}} \sum_{[g]\in 2.A_4/\mathcal{D}_8} |D0, g(x_1)\rangle\rangle + \frac{1}{\sqrt{24}} \sum_{[g]\in 2.A_4/\mathcal{D}_8} \Big[2|T0, 2A - tw, g(x_1)\rangle\rangle +$$
$$+\sqrt{2}|T0, 4A - tw, g(x_1)\rangle\rangle - \sqrt{2}|T0, 4B - tw, g(x_1)\rangle\rangle - \sqrt{2}|T0, 4C - tw, g(x_1)\rangle\rangle\Big]$$

$$\bullet|D0, \mathcal{O}_{x_1}, \mathcal{D}_8, I_3\rangle\rangle_{T^4/\mathbb{Z}_2.A_4,f} = \frac{1}{\sqrt{24}} \sum_{[g]\in 2.A_4/\mathcal{D}_8} |D0, g(x_1)\rangle\rangle + \frac{1}{\sqrt{24}} \sum_{[g]\in 2.A_4/\mathcal{D}_8} \Big[2|T0, 2A - tw, g(x_1)\rangle\rangle +$$
$$-\sqrt{2}|T0, 4A - tw, g(x_1)\rangle\rangle + \sqrt{2}|T0, 4B - tw, g(x_1)\rangle\rangle - \sqrt{2}|T0, 4C - tw, g(x_1)\rangle\rangle\Big]$$

$$\bullet|D0, \mathcal{O}_{x_1}, \mathcal{D}_8, I_4\rangle\rangle_{T^4/\mathbb{Z}_2.A_4,f} = \frac{1}{\sqrt{24}} \sum_{[g]\in 2.A_4/\mathcal{D}_8} |D0, g(x_1)\rangle\rangle + \frac{1}{\sqrt{24}} \sum_{[g]\in 2.A_4/\mathcal{D}_8} \Big[2|T0, 2A - tw, g(x_1)\rangle\rangle +$$
$$-\sqrt{2}|T0, 4A - tw, g(x_1)\rangle\rangle - \sqrt{2}|T0, 4B - tw, g(x_1)\rangle\rangle + \sqrt{2}|T0, 4C - tw, g(x_1)\rangle\rangle\Big]$$

$$\bullet|D0,\mathcal{O}_{x_1},\mathcal{D}_8,I_5\rangle\rangle_{T^4/\mathbb{Z}_2.A_4,f}=\frac{2}{\sqrt{24}}\sum_{[g]\in 2.A_4/\mathcal{D}_8}|D0,g(x_1)\rangle\rangle-\frac{2}{\sqrt{24}}\sum_{[g]\in 2.A_4/\mathcal{D}_8}2|T0,2A-tw,g(x_1)\rangle\rangle$$

The sum of these four fractional branes is the bulk D0-brane localized at the points of $\mathcal{O}_{x_1}$, therefore this orbit provides four linearly independent charge vectors: we choose the branes related to representations $k=2,3,4,5$.

In order to write the fractional D0-branes related to $\mathbb{Z}_3$ subgroup, we list neatly the fixed sublattice bases by set of element $\{g_3,g_3^2\}$, $\{g_3g_4,(g_3g_4)^2\}$, $\{g_4g_3,(g_4g_3)^2\}$ and $\{-g_4g_3^2g_4,(-g_4g_3^2g_4)^2\}$:

$$x_1'=\begin{pmatrix}2/3\\-1/3\\0\\0\end{pmatrix}\quad x_2''=\begin{pmatrix}1/3\\1/3\\0\\0\end{pmatrix}\quad x_3''=\begin{pmatrix}0\\0\\1/3\\1/3\end{pmatrix}\quad x_4''=\begin{pmatrix}0\\0\\2/3\\-1/3\end{pmatrix}$$

$$x_1''=\begin{pmatrix}2/3\\0\\1/3\\-1/3\end{pmatrix}\quad x_2''=\begin{pmatrix}0\\1/3\\1/3\\0\end{pmatrix}\quad x_3''=\begin{pmatrix}0\\-1/3\\2/3\\0\end{pmatrix}\quad x_4''=\begin{pmatrix}1/3\\-1/3\\1/3\\1/3\end{pmatrix}$$

$$x_1'''=\begin{pmatrix}1/3\\0\\0\\1/3\end{pmatrix}\quad x_2'''=\begin{pmatrix}-1/3\\2/3\\-1/3\\1/3\end{pmatrix}\quad x_3'''=\begin{pmatrix}-1/3\\1/3\\1/3\\0\end{pmatrix}\quad x_4'''=\begin{pmatrix}-1/3\\0\\0\\2/3\end{pmatrix}$$

$$x_1''''=\begin{pmatrix}2/3\\-1/3\\1/3\\0\end{pmatrix}\quad x_2''''=\begin{pmatrix}-1/3\\0\\1/3\\1/3\end{pmatrix}\quad x_3''''=\begin{pmatrix}0\\-1/3\\0\\2/3\end{pmatrix}\quad x_4''''=\begin{pmatrix}2/3\\0\\1/3\\1/3\end{pmatrix}.$$

For each subgroup of order 3, there are 9 fixed points, one of which is always the origin $x=0$. Since there are 4 different subgroups of order 3, we obtain 32 fixed points organized on 4 orbits $\mathcal{O}_{x_1'},\mathcal{O}_{x_1''},\mathcal{O}_{x_1'''},\mathcal{O}_{x_1''''}$ with 8 points, since $|2.A_4/\mathbb{Z}_3|=8$. For each of these 4 orbits $\mathcal{O}_{x_1'},\mathcal{O}_{x_1''},\mathcal{O}_{x_1'''},\mathcal{O}_{x_1''''}$ we have three D0-branes, one for each inequivalent irreducible representations of $\mathbb{Z}_3$, of which only two lead to linearly independent charge vectors.

$$\bullet\ |D0,\mathcal{O}_i,\mathbb{Z}_3,A_k'\rangle\rangle=\frac{1}{\sqrt{24}}\sum_{[\tilde{g}]\in 2.A_4/\mathbb{Z}_3}|D0,g(x_i)\rangle\rangle+\frac{1}{\sqrt{24}}\sum_{[\tilde{g}]\in\mathbb{Z}_2A_4/\mathbb{Z}_3}\sum_{a=1}^{2}\sqrt{3}e^{2\pi i\frac{ka}{3}}||T0,3A^a-tw,\tilde{g}(x_i)\rangle\rangle$$

In this latter equation A_k' with $k=1,2,3$ labelled the inequivalent irreducible representations of $\mathbb{Z}_3$ and we choose $k=1,2$ for the D0-branes linearly independent. The vector x_i represents the fixed points $x_i',x_i'',x_i''',x_i''''$. Moreover, notice that $3A^a$ labelled the conjugation classes 3A and 3B of $2.A_4$ respectively for $a=1,2$, and since $2.A_4/\mathbb{Z}_3\simeq\mathcal{D}_8$ the sum under $[\tilde{g}]\in 2.A_4/\mathbb{Z}_3$ counts four elements.

The last subgroup is $\mathbb{Z}_2$ that fixes 16 points of the torus of which one is the origin and three have already been considered as fixed points of $\mathcal{D}_8$. The remaining 12 points are organized in a single orbit because $|\mathbb{Z}_2.A_4/\mathbb{Z}_2|=12$. The basis vectors of the fixed sublattice are:

$$x_1'''''=\begin{pmatrix}1/2\\0\\0\\0\end{pmatrix}\quad x_2'''''=\begin{pmatrix}0\\1/2\\0\\0\end{pmatrix}\quad x_3'''''=\begin{pmatrix}0\\0\\1/2\\0\end{pmatrix}\quad x_4'''''=\begin{pmatrix}0\\0\\0\\1/2\end{pmatrix}\tag{6.62}$$

Therefore we have two D0 branes for this orbit $\mathcal{O}_{x_1'''''}$, one for each irreducible representation of $\mathbb{Z}_2$

$$\bullet|D0, \mathcal{O}_{x_1'''''}, \mathbb{Z}_2, I_k\rangle\rangle = \frac{1}{\sqrt{24}} \sum_{g \in A_4} |D0, g(x_i)\rangle\rangle + 2\frac{1}{\sqrt{24}} \sum_{g \in A_4} \epsilon_k ||T0, -\mathbb{1} - tw, g(x_i)\rangle\rangle$$

where $\epsilon_k = \pm 1$. The sum of D0-branes related to inequivalent irreducible representations leads to a bulk D0-brane. Therefore we have only one new linear indipendet change vector for this orbit.

To summarize, we have 5 linearly independent bulk D-branes charged only under the untwisted RR ground fields, 6 fractional D0-branes at the origin charged under both untwisted and twisted RR ground fields; 4 fractional D0-branes from the single orbit of $\mathcal{D}_8$-fixed points, carrying charge of untwisted RR fields and of 2A and 4A-twisted sector fields; 8 fractional D0-branes from the four inequivalent orbits $\mathcal{O}_{x_1'}, \mathcal{O}_{x_1''}, \mathcal{O}_{x_1'''}, \mathcal{O}_{x_1''''}$ of $\mathbb{Z}_3$-fixed points each with two independent branes, carrying charge of untwisted RR fields and of 3A and 3B-twisted sector fields; 1 fractional D0-brane from the single orbit $\mathcal{O}_{x_1'''''}$ of $\mathbb{Z}_2$-fixed points charged under untwisted and 2A-twisted sector fields.

The corresponding charge vectors span a lattice of maximal dimension 24 but the lattice is not unimodular. Therefore the set of bulk and fractional D-branes that we have calculated in this section does not constitute a basis for the boundary states of the model. We must still consider fractional $D2$ and $D4$ branes.

6.7 $T^4/2.A_5$

The geometric interpretation of the toroidal CFT as a NLSM on a certain torus T^4, allows us to identify the winding and momentum lattices as the maximal isotropic sublattices of $\Lambda^{4,4}$ generated by $\Gamma_w = span\,(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and $\Gamma_m = span\,(\lambda_5, \lambda_6, \lambda_7, \lambda_8)$ respectively. From the action (6.50) and (6.51) of the generators of $2.A_5$ on the lattice $\Lambda^{4,4}$, it can be shown that the g_3 and g_4 transformations, induced by isometries of target space, preserve these maximal isotropic sublattices. While the g_5 action mixes the components of two sublattices among them: it is a non-geometric symmetry of the theory. This fact makes the method described in the previous sections to compute the bulk and the fractional branes on orbifolds unusable for the orbifold $T^4/2.A_5$. In this section we derive the bulk D-branes for this orbifold model to symmetrize the bulk D-branes obtained for the $T^4/2.A_4$ model respect to the action of $g_5 \in 2.A_5$. The result is the bulk D-branes of the $T^4/2.A_5$ model, written in terms of the bulk D-branes of the $T^4/2.A_4$ model.

The RR ground states generate a 16-dimensional vector space V as a representation space of the algebra of fermionic zero modes. We want to describe the D-branes in terms of their charges with respect to these RR ground states. Let us define the RR-charge of the D-brane respect to a certain RR field, as the overlap between the boundary state representing the D-brane and the RR ground state corresponding to the field. With this definition, the D-branes charges can be seen as vectors in the dual space of V, and they actually span a $\Gamma^{8,8}$ lattice. [21] The RR charges are completely determined by the condition (6.22) for $r = 0$.

The chirality operator splits the whole lattice in two sublattices with opposite chirality: $\Gamma^{4,4}_{even} \oplus_\perp \Gamma^{4,4}_{odd}$. The GSO projection imposes even dimension for the branes of the type IIA string theory, therefore they have non-trivial components only in the $\Gamma^{4,4}_{even}$ sublattice.

Let us write the Dirichlet conditions (6.22) that define the $D0$-brane in terms of operators $c(\lambda)$, written in terms of fermionic zero modes:

$$c\,(\lambda) = \overline{\lambda}_L \cdot \psi_0 + i\overline{\lambda}_R \cdot \tilde{\psi}_0 \qquad \forall \lambda \in \Gamma^{4,4}_{even}, \tag{6.63}$$

that respect the Clifford algebra:

$$\{c(\lambda), c(\mu)\} = 2\,(\lambda, \mu)\,,$$

with $\lambda, \mu \in \Gamma^{4,4}$ and internal product (λ, μ) defined by (6.49).
If $\lambda \in \Gamma^{4,4}$ is a null vector, i.e. $(\lambda, \lambda) = 0$, then the corresponding operator $c(\lambda)$ is nilpotent of degree 2, and the kernel of this operator in not empty.
As in the previous section, let us choose the vector null basis $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8\}$ of the lattice $\Gamma^{4,4}$, and let us identify the pure momentum lattice as the maximal isotropic sublattice $\Gamma_m = \{\lambda_5, \lambda_6, \lambda_7, \lambda_8\}$, and the pure winding lattice as the maximal isotropic sublattice $\Gamma_w = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$. Now, we can write the Dirichlet conditions (6.22), for $r = 0$, of the $D0$-brane as:

$$c(\lambda_i)|D0\rangle = 0 \qquad\qquad i = 5, 6, 7, 8. \tag{6.64}$$

There is a unique unidimensional subspace of V that it satisfy at the same time the four conditions (6.64). The RR charge of the $D0$-brane is directs along this direction: it is defined by the (6.64) conditions up to an arbitrary normalization constant.
Similarly, the other D-branes can be constructed acting on the $D0$-brane by $c(\lambda)$ operators on the following order:

$$\begin{aligned}
&|D2(12)\rangle = c(\lambda_1)c(\lambda_2)|D0\rangle\\
&|D2(13)\rangle = c(\lambda_1)c(\lambda_3)|D0\rangle\\
&|D2(14)\rangle = c(\lambda_1)c(\lambda_4)|D0\rangle\\
&|D2(23)\rangle = c(\lambda_2)c(\lambda_3)|D0\rangle\\
&|D2(24)\rangle = c(\lambda_2)c(\lambda_4)|D0\rangle\\
&|D2(34)\rangle = c(\lambda_3)c(\lambda_4)|D0\rangle\\
&|D4\rangle = c(\lambda_1)c(\lambda_2)c(\lambda_3)c(\lambda_4)|D0\rangle.
\end{aligned} \tag{6.65}$$

Let us use the same procedure to define the D-branes transformed by the g_5 element. The new $D0$-brane, transformed under g_5, is labelled by the "new winding" vectors:

$$\tilde{\lambda}_5 = g_5\lambda_5 = \begin{pmatrix} -1\\ 1\\ -1\\ 1\\ -1\\ -1\\ -1\\ -1 \end{pmatrix} \tilde{\lambda}_6 = g_5\lambda_5 = \begin{pmatrix} 0\\ -1\\ 0\\ 0\\ 1\\ 0\\ 0\\ 0 \end{pmatrix} \tilde{\lambda}_7 = g_5\lambda_5 = \begin{pmatrix} 0\\ 1\\ -1\\ 0\\ 0\\ 0\\ 0\\ 0 \end{pmatrix} \tilde{\lambda}_8 = g_5\lambda_5 = \begin{pmatrix} 0\\ 0\\ 1\\ -1\\ 0\\ 0\\ 0\\ 0 \end{pmatrix},$$

that are again null vectors but they do not belong to $\Gamma_w \in \Gamma^{4,4}$.
Let $|D0'\rangle$ be the new $D0$-brane that respects the Dirichlet conditions with respect to the new winding directions:

$$c\left(\tilde{\lambda}_5\right)|D0'\rangle = c\left(\tilde{\lambda}_6\right)|D0'\rangle = c\left(\tilde{\lambda}_7\right)|D0'\rangle = c\left(\tilde{\lambda}_8\right)|D0'\rangle = 0. \tag{6.66}$$

These conditions determine $D0$ up to normalization constant.
Since the element g_5 act non-geometrically on the winding-momentum lattice, we expect that the new $D0$-brane be a linear combination of the various D-branes of T^4 with different dimensions. Let us consider the following ansatz:

$$|D0'\rangle = a|D0\rangle + b|D2(12)\rangle + c|D2(13)\rangle + d|D2(14)\rangle + e|D2(23)\rangle + f|D2(24)\rangle + g|D2(34)\rangle + h|D4\rangle, \tag{6.67}$$

where a, b, c, d, e, f, g, h are integer constants. Imposing the (6.66) conditions on the ansatz, we obtain:

$$|D0'\rangle = \mathcal{N}\left(D2(23) - D2(24) + D2(34) + D4\right), \tag{6.68}$$

where $\mathcal{N} \in \mathbb{C}$ carries all the uncertainty on the normalization and on the phase.
Now, we build the other D-branes acting on with the operators $c\left(\tilde{\lambda_1}\right), c\left(\tilde{\lambda_2}\right), c\left(\tilde{\lambda_3}\right), c\left(\tilde{\lambda_4}\right)$. By (6.51), the action of g_5 on the momentum vectors $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \Gamma_m$ is:

$$\tilde{\lambda_1} = g_5\lambda_1 = \begin{pmatrix} 0\\0\\0\\0\\-1\\0\\0\\0 \end{pmatrix} \tilde{\lambda_2} = g_5\lambda_2 = \begin{pmatrix} 0\\0\\0\\0\\-1\\-1\\-1\\-1 \end{pmatrix} \tilde{\lambda_3} = g_5\lambda_3 = \begin{pmatrix} 0\\0\\-1\\1\\0\\0\\-1\\-1 \end{pmatrix} \tilde{\lambda_4} = g_5\lambda_4 = \begin{pmatrix} 0\\1\\-1\\0\\-1\\0\\0\\-1 \end{pmatrix}.$$

Let us define the other branes as:

$$\begin{aligned}
|D2'(12)\rangle &= c(\tilde{\lambda}_1)c(\tilde{\lambda}_2)|D0\rangle\\
|D2'(13)\rangle &= c(\tilde{\lambda}_1)c(\tilde{\lambda}_3)|D0\rangle\\
|D2'(14)\rangle &= c(\tilde{\lambda}_1)c(\tilde{\lambda}_4)|D0\rangle\\
|D2'(23)\rangle &= c(\tilde{\lambda}_2)c(\tilde{\lambda}_3)|D0\rangle\\
|D2'(24)\rangle &= c(\tilde{\lambda}_2)c(\tilde{\lambda}_4)|D0\rangle\\
|D2'(34)\rangle &= c(\tilde{\lambda}_3)c(\tilde{\lambda}_4)|D0\rangle\\
|D4'\rangle &= c(\tilde{\lambda}_1)c(\tilde{\lambda}_2)c(\tilde{\lambda}_3)c(\tilde{\lambda}_4)|D0\rangle.
\end{aligned}$$

Using the Clifford algebra relations and the conditions (6.64), we obtain the new D-branes, transformed under g_5, in terms of fundamental D-branes of the torus:

$$\begin{aligned}
&|D2'(12)\rangle = \mathcal{N}\left(-|D2(23)\rangle - |D2(24)\rangle - |D2(34)\rangle\right)\\
&|D2'(13)\rangle = \mathcal{N}\left(-|D2(23)\rangle + |D2(24)\rangle\right)\\
&|D2'(14)\rangle = \mathcal{N}\left(-|D2(23)\rangle\right)\\
&|D2'(23)\rangle = \mathcal{N}\left(|D2(13)\rangle - |D2(14)\rangle - |D2(23)\rangle + |D2(24)\rangle\right)\\
&|D2'(24)\rangle = \mathcal{N}\left(-|D2(12)\rangle + |D2(13)\rangle - |D2(24)\rangle + |D2(34)\rangle\right)\\
&|D2'(34)\rangle = \mathcal{N}\left(-|D2(12)\rangle - |D2(34)\rangle - |D4\rangle - |D0\rangle\right)\\
&|D4'\rangle = \mathcal{N}\left(+|D2(23)\rangle - |D2(24)\rangle + |D2(34)\rangle + |D0\rangle\right).
\end{aligned}$$

In order to fix the normalization constant $\mathcal{N}$, we require that the g_5 action be a rotation of $2\pi/5$ around a fixed 4-dimensional sublattice of $\Gamma^{4,4}$. We obtain $\mathcal{N} = -1$. Let us write the matrix action of g_5 on the transformed branes:

$$g_5|_{D-b} = \begin{pmatrix} & D0 & D2(12) & D2(13) & D2(14) & D2(23) & D2(24) & D2(34) & D4 \\ D0' & 0 & 0 & 0 & 0 & -1 & 1 & -1 & -1 \\ D2'(12) & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ D2'(13) & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ D2'(14) & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ D2'(23) & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 \\ D2'(24) & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 \\ D2'(34) & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ D4' & -1 & 0 & 0 & 0 & -1 & +1 & -1 & 0 \end{pmatrix},$$

and it is easy to verify that $\left(g_5|_{D-branes}\right)^5 = \mathbf{1}$.
Now, we would like to write the bulk D-branes of the theory on $T^4/2.A_5$ target space. It is convenient to start from the bulk D-branes of the theory for the $T^4/2.A_4$ orbifold, and symmetrize them respect to the action of $\left(\mathbf{1}, g_5, g_5^2, g_5^3, g_5^4\right)$.
Let us compute explicitly the result for the $D0$-brane:

- $$|D0,x\rangle\rangle_{T^4/2.A_5,Bulk} = \frac{1}{\sqrt{120}} \sum_{g\in 2.A_4} \left[\mathbb{1} + g_5 + g_5^2 + g_5^3 + g_5^4\right] |D0, g(x)\rangle\rangle =$$

$$\frac{1}{\sqrt{120}} \sum_{g\in 2.A_4} [(|D0, g(x)\rangle\rangle) +$$
$$(-|D2(g(\lambda_2), g(\lambda_3)), g(x)\rangle\rangle + |D2(g(\lambda_2), g(\lambda_4)), g(x)\rangle\rangle - |D2(g(\lambda_3)g(\lambda_4)), g(x)\rangle\rangle - |D4, g(x)\rangle\rangle) +$$
$$(-|D2(g(\lambda_1), g(\lambda_4)), g(x)\rangle\rangle + |D2(g(\lambda_2), g(\lambda_4)), g(x)\rangle\rangle - |D2(g(\lambda_3)g(\lambda_4)), g(x)\rangle\rangle - |D4, g(x)\rangle\rangle) +$$
$$(-|D2(g(\lambda_1), g(\lambda_3)), g(x)\rangle\rangle - |D2(g(\lambda_3), g(\lambda_4)), g(x)\rangle\rangle - |D4, g(x)\rangle\rangle)$$
$$(-|D2(g(\lambda_1), g(\lambda_3)), g(x)\rangle\rangle - |D4, g(x)\rangle\rangle)] =$$
$$\frac{1}{\sqrt{120}} \sum_{g\in 2.A_4} [|D0, g(x)\rangle\rangle - |D2(g(\lambda_1), g(\lambda_2)), g(x)\rangle\rangle - |D2(g(\lambda_1), g(\lambda_3)), g(x)\rangle\rangle +$$
$$-|D2(g(\lambda_1)g(\lambda_4)), g(x)\rangle\rangle - |D2(g(\lambda_2)g(\lambda_3)), g(x)\rangle\rangle + 2|D2(g(\lambda_2)g(\lambda_4)), g(x)\rangle\rangle +$$
$$-3|D2(g(\lambda_3)g(\lambda_4)), g(x)\rangle\rangle - 4|D4, g(x)\rangle\rangle]\,.$$

Similarly, we obtain the following results for the other D-branes:

- $$|D2(12),x\rangle\rangle_{T^4/2.A_5,Bulk} = \frac{1}{\sqrt{120}} \sum_{g\in 2.A_4} [3|D0, g(x)\rangle\rangle + 2|D2(g(\lambda_1), g(\lambda_2)), g(x)\rangle\rangle +$$
$$2|D2(g(\lambda_1), g(\lambda_3)), g(x)\rangle\rangle + 2|D2(g(\lambda_1)g(\lambda_4)), g(x)\rangle\rangle + 2|D2(g(\lambda_2)g(\lambda_3)), g(x)\rangle\rangle +$$
$$-4|D2(g(\lambda_2)g(\lambda_4)), g(x)\rangle\rangle + 6|D2(g(\lambda_3)g(\lambda_4)), g(x)\rangle\rangle + 3|D4, g(x)\rangle\rangle]\,;$$

- $$|D2(13),x\rangle\rangle_{T^4/2.A_5,Bulk} = \frac{1}{\sqrt{120}} \sum_{g\in 2.A_4} [2|D0, g(x)\rangle\rangle - 2|D2(g(\lambda_1), g(\lambda_2)), g(x)\rangle\rangle +$$
$$3|D2(g(\lambda_1), g(\lambda_3)), g(x)\rangle\rangle + 3|D2(g(\lambda_1)g(\lambda_4)), g(x)\rangle\rangle + 3|D2(g(\lambda_2)g(\lambda_3)), g(x)\rangle\rangle +$$
$$-6|D2(g(\lambda_2)g(\lambda_4)), g(x)\rangle\rangle + 4|D2(g(\lambda_3)g(\lambda_4)), g(x)\rangle\rangle + 2|D4, g(x)\rangle\rangle]\,;$$

- $$|D2(14),x\rangle\rangle_{T^4/2.A_5,Bulk} = \frac{1}{\sqrt{120}} \sum_{g\in 2.A_4} [|D0, g(x)\rangle\rangle - |D2(g(\lambda_1), g(\lambda_2)), g(x)\rangle\rangle +$$
$$-|D2(g(\lambda_1), g(\lambda_3)), g(x)\rangle\rangle + 4|D2(g(\lambda_1)g(\lambda_4)), g(x)\rangle\rangle + 4|D2(g(\lambda_2)g(\lambda_3)), g(x)\rangle\rangle +$$
$$-3|D2(g(\lambda_2)g(\lambda_4)), g(x)\rangle\rangle + 2|D2(g(\lambda_3)g(\lambda_4)), g(x)\rangle\rangle + |D4, g(x)\rangle\rangle]\,;$$

• $|D2(23),x\rangle\rangle_{T^4/2.A_5,Bulk} = \dfrac{1}{\sqrt{120}}\sum_{g\in 2.A_4}\left[|D0,g(x)\rangle\rangle - |D2(g(\lambda_1),g(\lambda_2)),g(x)\rangle\rangle +\right.$
$-|D2(g(\lambda_1),g(\lambda_3)),g(x)\rangle\rangle + 4|D2(g(\lambda_1)g(\lambda_4)),g(x)\rangle\rangle + 4|D2(g(\lambda_2)g(\lambda_3)),g(x)\rangle\rangle +$
$\left.-3|D2(g(\lambda_2)g(\lambda_4)),g(x)\rangle\rangle + 2|D2(g(\lambda_3)g(\lambda_4)),g(x)\rangle\rangle + |D4,g(x)\rangle\rangle\right];$

• $|D2(24),x\rangle\rangle_{T^4/2.A_5,Bulk} = \dfrac{1}{\sqrt{120}}\sum_{g\in 2.A_4}\left[-|D0,g(x)\rangle\rangle + |D2(g(\lambda_1),g(\lambda_2)),g(x)\rangle\rangle +\right.$
$-4|D2(g(\lambda_1),g(\lambda_3)),g(x)\rangle\rangle + |D2(g(\lambda_1)g(\lambda_4)),g(x)\rangle\rangle + |D2(g(\lambda_2)g(\lambda_3)),g(x)\rangle\rangle +$
$\left.+3|D2(g(\lambda_2)g(\lambda_4)),g(x)\rangle\rangle - 2|D2(g(\lambda_3)g(\lambda_4)),g(x)\rangle\rangle - |D4,g(x)\rangle\rangle\right];$

• $|D2(34),x\rangle\rangle_{T^4/2.A_5,Bulk} = \dfrac{1}{\sqrt{120}}\sum_{g\in 2.A_4}\left[|D0,g(x)\rangle\rangle + 4|D2(g(\lambda_1),g(\lambda_2)),g(x)\rangle\rangle +\right.$
$-|D2(g(\lambda_1),g(\lambda_3)),g(x)\rangle\rangle - |D2(g(\lambda_1)g(\lambda_4)),g(x)\rangle\rangle - |D2(g(\lambda_2)g(\lambda_3)),g(x)\rangle\rangle +$
$\left.2|D2(g(\lambda_2)g(\lambda_4)),g(x)\rangle\rangle + 2|D2(g(\lambda_3)g(\lambda_4)),g(x)\rangle\rangle + |D4,g(x)\rangle\rangle\right];$

• $|D4,x\rangle\rangle_{T^4/2.A_5,Bulk} = \dfrac{1}{\sqrt{120}}\sum_{g\in 2.A_4}\left[-4|D0,g(x)\rangle\rangle - |D2(g(\lambda_1),g(\lambda_2)),g(x)\rangle\rangle +\right.$
$-|D2(g(\lambda_1),g(\lambda_3)),g(x)\rangle\rangle - |D2(g(\lambda_1)g(\lambda_4)),g(x)\rangle\rangle - |D2(g(\lambda_2)g(\lambda_3)),g(x)\rangle\rangle +$
$\left.2|D2(g(\lambda_2)g(\lambda_4)),g(x)\rangle\rangle - 3|D2(g(\lambda_3)g(\lambda_4)),g(x)\rangle\rangle + |D4,g(x)\rangle\rangle\right].$

Using the intersection matrices (6.59) and (6.58) of the bulk D-branes of $T^4/2.A_4$, we compute the intersection matrix of the bulk D-branes of $T^4/2.A_5$:

$$\tilde{\mathcal{I}}_{2.A_5} = 24\begin{pmatrix} -4 & 3 & 2 & 1 & 1 & -1 & 1 & 1 \\ 3 & -6 & -4 & -2 & -2 & 2 & -2 & 3 \\ 2 & -4 & -6 & -3 & -3 & 3 & 2 & 2 \\ 1 & -2 & -3 & -4 & -4 & -1 & 1 & 1 \\ 1 & -2 & -3 & -4 & -4 & -1 & 1 & 1 \\ -1 & 2 & 3 & -1 & -1 & -4 & -1 & -1 \\ 1 & -2 & 2 & 1 & 1 & -1 & -4 & 1 \\ 1 & 3 & 2 & 1 & 1 & -1 & 1 & -4 \end{pmatrix} \tag{6.69}$$

As we would have expected from the calculus of the Elliptic Genus of the orbifold $T^4/2.A_5$, which assure us that we have 4 RR ground states in the Untwisted sector, the rank of $\tilde{\mathcal{I}}_{2.A_5}$ is 4. Therefore, we can choose only 4 bulk D-branes of $T^4/2.A_5$ that span a non-trivial charge vector space.
Let $\left\{|\tilde{D0}\rangle\rangle, |D2\tilde{(12)}\rangle\rangle, |D2\tilde{(13)}\rangle\rangle, |D2\tilde{(14)}\rangle\rangle\right\}$ be the no-trivial basis of bulk D-branes of $T^4/2.A_5$ that spans the charge vector space, with intersections defined by the following matrix:

$$\mathcal{I}_{2.A_5} = 24\begin{pmatrix} -4 & 3 & 2 & 1 \\ 3 & -6 & -4 & -2 \\ 2 & -4 & -6 & -3 \\ 1 & -2 & -3 & -4 \end{pmatrix}. \tag{6.70}$$

This concludes the analysis of the bulk D-branes for the $T^4/2.A_5$ orbifold.

Conclusions and outlooks

In this thesis we have presented the construction and the main features of two dimensional Conformal Field Theories (CFT). We reviewed the Bosonic String and the Superstring Theories, mainly focusing on the world-sheet picture, where the string dynamics is entirely described by a two dimensional CFT. We used the compactification idea, that aims to solve the discrepancy between the critical dimension of String Theory and the four observed dimension of space-time, to construct new space-time backgrounds to describe the development of strings. In particular we have focused our attention on *Type IIA* Superstring compactification on orbifold surfaces, specially on toroidal orbifolds T^4/G, where G is a finite group of discete symmetries. Torus orbifolds T^4/G can be interpreted as singular limits of Calabi-Yau manifolds of complex dimension two ($K3$). The corresponding String theory is a NLSM with $\mathcal{N} = (4, 4)$ superconformal algebra. We studied the main proprieties of orbifolds T^4/G theories, as the spectrum, the current algebra and boundary states, using CFT methods.
One of the original features of this thesis is contained on the choice of group G: while most of the research has been done using cyclic groups, in this work we have generalised these conditions and we considered *non-abelian* groups which do not admit a geometric description as isometries of torus T^4.
We have performed explicitly the computation for the group $2.A_5$ and we have obtained a SCFT with a symmetry algebra reduced with respect to T^4 algebra. In particular the orbifold procedure has eliminated some of the holomorphic fields from the spectrum. The expectation is to construct new torus orbifold models, choosing G more and more generic, that are getting closer to the more generic model of conformal theory on $K3$. We would expect that this generic model contains only the basic $\mathcal{N} = (4, 4)$ superconformal algebra with no extra holomorphic fields.
In the last chapter we have presented the description of the D-branes on the compact directions of the orbifold T^4/G with the method of boundary states. The construction studied so far cannot be applied if we choose G as a non-geometric symmetry group. Therefore, we have computed the bulk and fractional D-branes for the orbifold model $T^4/2.A_4$, where $2.A_4 \subset 2.A_5$ contains only elements of $2.A_5$ with geometric action on the torus. Next, we have derived the bulk D-branes of $T^4/2.A_5$ model symmetrizing the bulk D-branes of $T^4/2.A_4$ with respect to the action of the non-geometric elements of $2.A_5$.
Future perspectives of this analysis could be the computation of fractional D-branes of the $T^4/2.A_5$ orbifold model, and the formalization of a new general method for the study of the D-branes on compact surfaces as T^4/G, or T^6/G, where G be the most general group as possible.

Appendix A

Theta functions

Let us give the definition of Jacobi theta functions:

$$
\begin{aligned}
\theta_1(\tau,z) &= -iq^{1/8}y^{-1/2}\,(y-1)\prod_{n=1}^{\infty}(1-q^n)\,(1-yq^n)\left(1-y^{-1}q^n\right)\\
\theta_2(\tau,z) &= q^{1/8}\left(y^{1/2}-y^{-1/2}\right)\prod_{n=1}^{\infty}(1-q^n)\,(1+yq^n)\left(1+y^{-1}q^n\right)\\
\theta_3(\tau,z) &= \left(y^{1/2}-y^{-1/2}\right)\prod_{n=1}^{\infty}(1-q^n)\left(1+yq^{n-1/2}\right)\left(1+y^{-1}q^{n-1/2}\right)\\
\theta_4(\tau,z) &= (1-q^n)\left(1-yq^{n-1/2}\right)\left(1-y^{-1}q^{n-1/2}\right),
\end{aligned}
$$

where $y=e^{2\pi iz}$ and $q=e^{2\pi i\tau}$.
Under *Modular transformations* they transform as:

$$
\begin{aligned}
\theta_1(\tau+1,z) &= e^{\frac{2\pi i}{8}}\theta_1(\tau,z) & \theta_1(-\frac{1}{\tau},\frac{z}{\tau}) &= -(-i\tau)^{1/2}e^{\frac{i\pi z^2}{\tau}}\theta_1(\tau,z)\\
\theta_2(\tau+1,z) &= e^{\frac{2\pi i}{8}}\theta_2(\tau,z) & \theta_1(-\frac{1}{\tau},\frac{z}{\tau}) &= (-i\tau)^{1/2}e^{\frac{i\pi z^2}{\tau}}\theta_4(\tau,z)\\
\theta_3(\tau+1,z) &= \theta_4(\tau,z) & \theta_3(-\frac{1}{\tau},\frac{z}{\tau}) &= (-i\tau)^{1/2}e^{\frac{i\pi z^2}{\tau}}\theta_3(\tau,z)\\
\theta_4(\tau+1,z) &= \theta_3(\tau,z) & \theta_4(-\frac{1}{\tau},\frac{z}{\tau}) &= (-i\tau)^{1/2}e^{\frac{i\pi z^2}{\tau}}\theta_2(\tau,z).
\end{aligned}
$$

We prove now the equivalence between the expressions (5.21) and (5.22). Let us start from the last equation:

$$
\begin{aligned}
\phi_{un,g^k}(\tau,z) &= \left(2-\xi_L^{-k}-\xi_L^{k}\right)\left(2-\xi_R^{-k}-\xi_R^{k}\right)\frac{\theta_1(\tau,z+kr_L)\theta_1(\tau,z-kr_L)}{\theta_1(\tau,kr_L)\theta_1(\tau,-kr_L)} =\\
&\left(2-\xi_L^{-k}-\xi_L^{k}\right)\left(2-\xi_R^{-k}-\xi_R^{k}\right)y^{-1}\frac{\left(\xi_L^{k}y-1\right)\left(\xi_L^{-k}y-1\right)}{\left(\xi_L^{k}-1\right)\left(\xi_L^{-k}-1\right)}\cdot\\
&\prod_{n=1}^{\infty}\frac{\left(1-y^{-1}\xi_L^{k}q^n\right)\left(1-y^{-1}\xi_L^{-k}q^n\right)\left(1-y\xi_L^{k}q^n\right)\left(1-y\xi_L^{-k}q^n\right)}{\left(1-\xi_L^{k}q^n\right)^2\left(1-\xi_L^{-1k}q^n\right)^2} =\\
\left(1-y\xi_L^{k}\right)\left(1-y\xi_L^{-k}\right)&\left(1-\xi_R^{k}\right)\left(1-\xi_R^{-k}\right)y^{-1}\prod_{n=1}^{\infty}\frac{\left(1-y^{-1}\xi_L^{k}q^n\right)\left(1-y^{-1}\xi_L^{-k}q^n\right)\left(1-y\xi_L^{k}q^n\right)\left(1-y\xi_L^{-k}q^n\right)}{\left(1-\xi_L^{k}q^n\right)^2\left(1-\xi_L^{-1k}q^n\right)^2}.
\end{aligned}
$$

Appendix B

Lattices

A *lattice* is the set of points of the real space $\mathbb{R}^n$ such that:

$$L = \left\{ \sum_{i=1}^{n} n_i \overline{\lambda}_i | n_i \in \mathbb{Z} \right\}. \tag{B.1}$$

The set $\mathcal{B} = \{\lambda_1, \lambda_2, ...\lambda_n\}$ of linearly indipendent vectors that span the lattice L is a basis for this space. We can define a *inner product* through the following bilinear form:

$$q: \quad L \quad \Longrightarrow \quad \mathbb{R}, \tag{B.2}$$

the matrix Q associated at this form respect a given basis is called Gram matrix.
The *dual* L^* of a lattice L is the lattice of vectors having integral inner products with all vectors in L. The *determinant* or *discriminant* of lattice is $detL = |detQ|$, and $Vol(L) = \sqrt{detL}$ is the volume of unitary cell of lattice. We can define the Gram matrix also for the dual lattice $Q^* = Q^{-1}$ and the volume of unitary cell of L^* is $(Vol(L))^{-1}$.
The lattice L is called *integral* if $\overline{v} \cdot \overline{w} \in \mathbb{Z}$ for $\forall \overline{v}, \overline{w} \in L$, this is true if and only if $L \subset L^*$, i.e. the bilinear form takes value in $\mathbb{Z}$. Through the (B.2), we can to define a *quadratic form*:

$$Q: \quad L \quad \Longrightarrow \quad \mathbb{R} \tag{B.3}$$

and extend it by linearity to dual:

$$Q^*: \quad L^* \quad \Longrightarrow \quad \mathbb{R}. \tag{B.4}$$

The lattice is *even* if the quadratic form Q takes values in $2\mathbb{Z}$ for all $\overline{v} \in L$, otherwise it is called *odd*.
Let's consider an even lattice L and its dual lattice L^*:

$$L^* = \left\{ x \in L \bigotimes \mathbb{Q} | (x, y) \in \mathbb{Z}, \forall y \in L \right\}.$$

The *discriminant group* of L is:

$$A_L := L^*/L \tag{B.5}$$

is a finite abelian group equipped with a quadratic form:

$$\begin{aligned} q_L: \quad \frac{L^*}{L} \quad &\Longrightarrow \quad \mathbb{Q}/2\mathbb{Z} \\ x + L \quad &\Longrightarrow \quad (x, x)(mod 2\mathbb{Z}), \end{aligned} \tag{B.6}$$

the group with its quadratic form define the *discriminant space* $A_L = (L^*/L, q_L)$. We denote by $O(L)$ the group of automorphisms (or *isometries*) of L (notice that $O(L) \simeq O(L^*)$), and similarly $O(A_L)$ as the group of automorphisms of the discriminant group.
A sublattice $K \subseteq L$ is *primitive* in L if L/K is a free abelian group. We set:

$$K^\perp = \{x \in L|(x,y) = 0, \forall y \in K\}.$$

Assume now that L is an even unimodular ($L = L^*$). If K is primitive in L then there is a isomorphism:

$$i: \quad A_K \quad \Longrightarrow \quad A_{K^\perp}$$

such that

$$q_{K^\perp}(i(a)) = -q_K(a) \qquad \forall a \in A_K.$$

We can recover L from $K \bigoplus K^\perp$ by adjoining the cosets:

$$C = \{(a, i(a))\,|a \in A_K\} \subseteq A_K \bigoplus A_{K^\perp}.$$

We take $G \subseteq O(L)$, the invariant and co-invariant lattice for G are respectively:

$$\begin{aligned} L^G &= \{x \in L|g(x) = x, \forall g \in G\} \\ L_G &= \left(L^G\right)^\perp, \end{aligned} \tag{B.7}$$

and are primitive sublattices of L. The restriction of the G-action to L_G induces an embedding $G \subseteq O(L_G)$, and we denote by $\tilde{G}$ the pointwise stabilizer of L^G in $O(L)$, then we have $G \subseteq \tilde{G}$ and $L^G = L^{\tilde{G}}$.
A *root* of L is a primitive vector $v \in L$ such that reflection in $(\mathbb{Z}v)^\perp$ is an isometry of L. The root sublattice of L is the sublattice spanned by all roots. The root vectors respect the condition $(r,r) = 2$.
The Λ *Leech lattice* is the unique positive-define even unimodular lattice of rank 24 without roots.
The isometries's group of Λ is the *Conway* group C_{O_0}. For each subgroup $H \in C_{O_0}$, we define a fixed-points sublattice:

$$\Lambda^H = \{v \in \Lambda|hv = v, \forall h \in H\}.$$

The set of all fixed sublattices of Λ are called $\mathcal{F}$. The Conway group acts by translation on $\mathcal{F}$:

$$g \in C_{O_0} \quad \to \quad g\Lambda^G = \Lambda^{gHg^{-1}}.$$

Under the action of C_{O_0} there are 290 orbits on the set of fixed-point's sublattices of Λ, this orbits are classified in [31].
For each even lattice S with signature (t^+, t^-), the theorem 1.12.4 of [32] assure us that there exist a primitive embedding of S into an even unimodular lattice L of signature (l^+, l^-) such that :

$$\begin{cases} l^+ - l^- = 0 \\ l^+ \geq t^+ \quad l^- \geq t^- \\ t^+ + t^- \leq \frac{1}{2}(l^+ + l^-). \end{cases}$$

Alternatively, the theorem 1.12.2 assure us an embedding into a lattice K of signature $(l^+ - t^+, l^- - t^-)$ and associated discriminant form $q_K = -q_S$. In other words one can construct an even unimodular lattice L such that $K \bigoplus S \subseteq L$ and their embedding are primitive.

In our case, since Γ^G has signature $(20,4)$, it can be embedding into an even unimodular lattice $\Gamma^{20,8+20}$, and its orthogonal complement N of signature $(0,24)$ with associate discriminant form: $q_S = -q_{\Lambda^G} = q_{\Lambda_G}$. Finally the theorem 1 in [33] proves so that there exists an even unimodular lattice N of signature $(0,24)$ in which the lattice Γ_G can be primitively embedding. If N does not contain roots then N is the Leech lattice Λ, otherwise is a Niemeier lattice.
There exists other 23 Niemeier lattices (beyond the Leech lattice) that classify the 24 equivalence classes of 24-dimensional negative-definite even unimodular lattices.
Moreover the symmetry group G is isomorphic to a subgroup $\tilde{G}$ of automorphisms group $O(N)$ of the Niemeier lattice N that fixes a sublattice $N^{\tilde{G}}$ of rank at least 4 and the converse is also true. Through the theorem 2 in [33] we make sure that there exists a positive four-plane Π such that $Stab(\Pi)$ contains $\tilde{G}$ as a subgroup.
It was conjectured [34] that for each of the 24 Niemeier lattices N there exists a non-algeabric K3-surface X whose the Picard lattice $P(X)$ [1](related to the curves in X) can be primitively embedded only in N. It is possible to find a particular choice of B-field such that the Picard lattice be a sublattice of Γ_G. Then we expect that the Niemeier lattice tell us informations about the discret symmetries of string theory on K3.
Let be $G \in O(\Gamma^{4,20})$ the group fixing the four-plane Π, but extending its action in the whole lattice, the group G can preserve at larger subspace Γ^G. We want to parametrize all possible four-plane fixed by G:

$$\mathcal{F}_G = \left\{ \Pi \subseteq \Lambda^G \bigotimes \mathbb{R}, sign(\Pi) = (4,0) \right\} / \mathcal{N}_{O^+(\Gamma^{4,20})(G)} \tag{B.8}$$

where $\mathcal{N}_{O^+(\Gamma^{4,20})(G)}$ is the normalizer of G in $O^+(\Gamma^{4,20})$ that fixes the lattice Γ^G setwise.
Physically a positive four-plane defines a NLSM on K3, while $\mathcal{F}_G$ defines a set of NLSMs on K3 with symmetry groups which contain G.
A lattice $L \in \Gamma^{4,20}$ with quadratic form q associated with the matrix Q respect to base $\mathcal{B}$, under a change $\mathcal{B} \to \mathcal{B}'$, the matrix transforms as $Q' = A^T Q A$ with $A \in O^+(\Gamma)$. The equivalent lattices have matrices associated to correspondent quadratic forms related by similitude transformations.Therefore in order to classify the inequivalent lattices we have to list all possible conjugacy classes in $O^+(\Gamma^{4,20})$ and their corresponding twining genera. Let us consider the eigenvalues of group in the 24-dimensional representation $\rho_{24} : O^+(\Gamma^{4,20}) \to End(\Gamma^{4,20} \bigotimes_{\mathbb{Z}} \mathbb{R})$ (this is also 24-dimensional representation of RR ground states in the NLSM corresponding). We can encode this information into the *Frame shape*, *i.e.*:

$$\pi_g \prod_l^N l^{k_l}, \tag{B.9}$$

where N is the order of g, while the k_l are defined through the characteristic polynomial :

$$det\left(\lambda \mathbb{1}_{24} - \rho_{24}(g)\right) = \prod_l^N \left(t^l - 1\right)^{k_l}. \tag{B.10}$$

A Frame shape correspond to a four-plane on $\Gamma^{4,20} \bigotimes_{\mathbb{Z}} \mathbb{R}$ that is preserved by a subgroup of $O^+(\Gamma^{4,20})$ that contain the element g that defined the same Frame shape. It can be demostrated that the four-plane preserving Frame shapes of $O^+(\Gamma^{4,20})$ are the 42 four-plane preserving conjugation classes of C_{O_0}.
If $g, g' \in O^+(\Gamma^{4,20})$ have the same Frame shape, then the co-invariant lattices Γ_g and $\Gamma_{g'}$ are

[1]The Picard group is the group of isomorphism classes of invertible sheaves.

isomorphic. Indeed the element $\widehat{g}, \widehat{g'}$ are conjugate in C_{O_0}, but g and g' can not be conjugate in $O^+(\Gamma^{4,20})$. The problem is to classify the conjugation classes of $O^+(\Gamma^{4,20})$ for a given Frame shape: this is equivalent to classify different primitive embedding of corresponding lattice $\Lambda_{\widehat{g}}$ on $\Gamma^{4,20}$. Only for one of 42 four-plane preserving Frame shapes of $O^+(\Gamma^{4,20})$ is determined the number of its classes. The remaining 41 conjugation classes of C_{O_0} give rise to 80 distinct $O^+(\Gamma^{4,20})$ conjugation classes.

Appendix C

Calabi-Yau manifold

In the fourth chapter we have introduced the concept of Calabi-Yau manifolds CY_n through their holonomy group $SU(n)$. In terms of differential geometry the technical definition tell us that a *Calabi-Yau* manifold is a *Kähler* manifold with vanishing first Chern class $c_1(\mathcal{M} = CY_n) = 0$.[1]
What $c_1(CY_n) = 0$ assures us of our requires of compactification? Given any Kähler metric g, with Kähler form $K|dK = 0$, we can find a unique (Calabi) Ricci-flat metric g' associated to Kähler form K' belonging to the same Kähler class of K ($[K] = [K']$). If $\mathcal{M}$ is compact, it is always possible (Yau). Since the first Chern class is represented by the Ricci form $\mathcal{R}$ (in particular it is a $(1,1)$-form for the complex structure of manifold), and the latter changes under change of metric by an exact form:

$$g \longrightarrow g' : \qquad \mathcal{R}(g') = \mathcal{R}(g) + d\alpha$$

then $c_1(\mathcal{M}) = 0$ is a necessary condition to Ricci-flatness, while the fact that be also sufficient it is hard to prove. However, we always consider compact Kähler manifolds with $c_1(\mathcal{M}) = 0$ and vanishing Ricci curvature, i.e. with holonomy group $\mathcal{H} \subset SU(n)$.
How is the holonomy group related to Ricci-flatness of manifold? Let's start to give a formal definition of *holonomy* group $\mathcal{H}$. Let be $\mathcal{M}$ a manifold and $\pi : \mathcal{E} \to \mathcal{M}$ the projection that define the *fiber bundle* $\mathcal{E}$. Let's consider a point $P \in \mathcal{M}$ and be $e_1 \in \pi^{-1}(P)$ a vector in the fibre. We choose a closed path Γ in $\mathcal{M}$ passing by P, and we transport the vector e_1 through Γ in according to $\mathcal{E}$'s connection. The result is a new vector $e_2 = g_\Gamma(e_1) \in \pi^{-1}(P)$. The set of transformations g_Γ related to all paths starting and ending at P form the holonomy group of $\mathcal{M}$. For a generic manifold of real dimension m, the holonomy group is a subgroup of $O(m)$, if the manifold is oriented then $\mathcal{H} \subset SO(m)$. The complex structure of Kähler manifolds, with $dim_{\mathbb{C}\mathcal{M}=n}$, assures us that elements of holomorphic fiber $T^{1,0}(\mathcal{M})$ do not mix with elements of anti-olomorphic fiber $T^{0,1}(\mathcal{M})$ under parallel transport, therefore the holonomy group will be a subgroup of $U(n)$. If we also require the Ricci-flatness the holonomy group boils down to a subgroup of $SU(n)$.
It can be proven that on a manifold, of n complex dimension, with $\mathcal{H} \subset SU(n)$ there exist a unique nowhere vanishing holomorphic $(n,0)$-form and covariantly constant. Since $\mathcal{M}$ is a Kähler Ricci-flat manifold, the form is harmonic, therefore we have:

$$h^{n,0} = h^{0,n} = 1.$$

Starting from an holomorphic, and hence harmonic, $(p,0)$ form we can buid, by contraction, an harmonic $(0, n-p)$ form, in particular it can be shown the following relation among Hodge

[1]The mathematical preliminaries of this section are summarized in the Appendix.

numbers:

$$h^{p,0} = h^{0,p} = h^{0,n-p} = h^{n-p,0}.$$

The harmonic form transforms as a singlet under holonomy group if the manifold is compact and Ricci-flat.
We assume that the holonomy group of our manifold be exactly $\mathcal{H} = SU(n)$. One $(p,0)$ form on the manifold transforms in the $\wedge^p \mathbf{n}$ representation of $SU(n)$. Decomposing this product, the singlet representation only appears if $p = 0$ or $p = n$, therefore we obtain that for CY_n:

$$h^{p,0} = h^{0,p} = 0 \qquad 0 < p < n. \tag{C.1}$$

In particular $h^{(1,0)} = 0$ implies that there are no continuous isometries on CY_n.[2]

C.0.1 K3 surfaces

We would like to focus our attention on the K3 surfaces and to describe their *Moduli Space*. A K3 surface is a compact complex Kähler manifold of complex dimension two, i.e. a surface S such that:

$$\begin{aligned} &h^{1,0}(S) = dim\left(H^{1,0}(S)\right) = 0 \\ &c_1(S) = 0 \end{aligned}$$

where $h^{1,0}(S)$ is the dimension of the coohomology group $H^{1,0}(S)$. An interesting fact about K3 surfaces is the following theorem:

Theorem 1 *Two K3 surfaces are always diffeomorphic.*

Therefore if we take a particular K3 surface, its topological invariants are the same for each K3 surface. The latter allows us to take a simple $K3$ surface and to compute for it some topological invariants, in particular the Euler characteristic: [3]

$$\chi(K3) = \sum_{p=0}^{n} (-1)^p b_p = 24.$$

Through this result, the definition relations of K3 surfaces and the Poincarè duality we can to deduce the Hodge numbers:

$$h^{0,0} = 1 \quad h^{1,0} = h^{0,1} = 0 \quad h^{2,0} = h^{0,2} = 1 \quad h^{1,1} = 20 \quad h^{2,1} = h^{1,2} = 0 \quad h^{2,2} = 1.$$

Let's start to consider the K3 surface purely as an object in algebraic geometry and to compute the moduli space of complex structures for K3 surfaces. In order to measure the complex structure we need to find some simple quantities that depend on complex structure. They are the *periods*, which are integral of the holomorphic 2-form Ω, over integral 2-cycles in K3. Since the dimension of the space $H_2(K3,\mathbb{Z})$ of equivalence classes of 2-cycles modulo boundaries is $b_2(K3) = 22$, then is true the isomorphism $H_2(K3,\mathbb{Z}) \simeq \mathbb{Z}^{22}$ as group. It becomes necessary to implement the group structure with an inner product:

$$\alpha_1 \cdot \alpha_2 = \sharp\left(\alpha_1 \cap \alpha_2\right), \qquad \forall \alpha_i \in H_2(K3,\mathbb{Z}) \tag{C.2}$$

[2] We remember that an isometry means a non-trivial solution of Killing equation $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$.
[3] The Euler characteristic of $\mathcal{M}$ is the alternating sum of Betti numbers: $\chi(\mathcal{M}) = \sum_{p=0}^{n} (-1)^p b_p$, where $n = dim\mathcal{M}$.

where $\sharp$ counts the oriented intersections between cycles. This abelian group structure, with inner product (C.2), gives to $H_2(K3,\mathbb{Z})$ the structure of lattice. The signature of this lattice (determined from the index theorem of complex signature) is $(3,19)$. Moreover the *Poincarè duality* assure us that $H_2(K3,\mathbb{Z})$ is a self-dual (unimodular) lattice, then the lattice of integral cohomology $H^2(K3,\mathbb{Z})$ (forms) is isomorphic to lattice of integral homology $H_2(K3,\mathbb{Z})$ (cycles). We require that the lattice $H_2(K3,\mathbb{Z})$ be *even*. The classification of even self-dual lattice is known. In particular $\Gamma_{3,19}$ is unique up to isometries, and we could choose a basis of elements such that their inner products form the matrix:

$$\left(\begin{array}{c|c|c|c|c} -E_8 & & & & \\ \hline & -E_8 & & & \\ \hline & & U & & \\ \hline & & & U & \\ \hline & & & & U \end{array}\right)$$

where E_8 is the Cartan matrix associate to lattice E_8, while U represents the hyperbolic plane:

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In order to describe the periods $\varpi_i = \int_{e_i} \Omega$ of 2-forms over 2-cycles in terms of lattice $\Gamma_{3,19}$ we must embody the basis $\{e_i\}$ of 2-cycles on $\Gamma_{3,19}$: the choice is called *marking* of K3 surface. A natural embedding is:

$$\Gamma_{3,19} \simeq H^2(K3,\mathbb{Z}) \subset H^2(K3,\mathbb{R}) \simeq \mathbb{R}^{3,19}.$$

The two-forms admitted in $\mathbb{R}^{3,19}$ span a plane Ω with space-like vectors. The choice of a complex structure on a K3 surfaces corresponds to determine a vector space $\mathbb{R}_{3,19}$, which contains an even self-dual lattice $\Gamma_{3,19}$ and an oriented 2-plane Ω. A change of complex structure on K3 determines a rotation of Ω respect to $\Gamma_{3,19}$. The moduli space of complex structures on a marked K3 surface is the space of all possible oriented 2-planes in $\mathbb{R}^{3,19}$ respect to fixed lattice $\Gamma_{3,19}$. This space is called *Grassmannian*:

$$Gr^+\left(\Omega,\mathbb{R}^{3,19}\right) \simeq \frac{O^+(3,19)}{\left(O(2)\times O(1,19)\right)^+}, \tag{C.3}$$

the sign $+$ denotes that we are using the subgroup which preserves orientation on the space-like directions.

$K3$ surfaces defined by 2-plane Ω, which contains a light-like direction, will be near the boundary of our moduli space. As we approach the boundary, we expect that the $K3$ surfaces to degenerate in some way. However, there will be points of moduli space, away from this boundary, corresponding to degenerate $K3$ surfaces. They are *orbifolds*. These points must necessarily be included in our Moduli space.

Now we want to eliminate the marking effect, and we obtain that the Moduli space of complex structures on K3 (including orbifold points) is the quotient:

$$\mathcal{M}_c \simeq \frac{Gr^+\left(\Omega,\mathbb{R}^{3,19}\right)}{O^+\left(\Gamma_{3,19}\right)}. \tag{C.4}$$

$\mathcal{M}_c$ is not actually a Hausdorff space, and this fact is seen in string theory in fairly pathological circumstances. Then we try to modify the previous result, and we choice to consider the moduli space of Einstein metrics on a K3 surface. We will assume that the metric is Kähler, and we define *Einstein metric* a metric on a Riemann manifold whose Ricci curvature is proportional to the metric. Then for a K3 surface this imply that the metric is Ricci-flat.

In a K3 surface, respect to the Hdge dual we may decompose the cohomology $H_2(\mathcal{M},\mathbb{R})$ into the cohomology of the the self-dual and antiself-dual 2-forms $\mathcal{H}^{\pm}$:

$$H_2(\mathcal{M},\mathbb{R}) = \mathcal{H}^+ \oplus \mathcal{H}^-, \tag{C.5}$$

where $dim\mathcal{H}^+ = 3$ and $dim\mathcal{H}^- = 19$.
We focus our attention on the space $\mathcal{H}^+$ viewed as a subspace of $H_2(\mathcal{M},\mathbb{R})$. It is possible to show that $\mathcal{H}^+$ is spanned by the previous plane Ω and by the Kähler forms direction (space-like) that we can represent through the form K. The Yau's theorem assures us that once we have fixed Ω and K, a unique Einstein metric exists on the K3 surface.
Moreover an important aspect of this theorem is that rotations within the space $\mathcal{H}^+$ may affects what we consider to be a Kähler form, and consequently the complex structure of K3 surface, but not affects the Riemannian metric. Therefore the whole family of complex structures on K3 is parametrized by the sphere S_2 of ways in which $\mathcal{H}^+$ is divided into Ω and K.

Theorem 2 *The moduli space of Einstein metrics $\mathcal{M}_E$ for a K3 surface (including orbifold points) is given by the Grassmannian of oriented 3-planes within the space $\mathbb{R}_{3,19}$ modulo the effects of diffeomorphisms acting on the lattice $H_2(K3,\mathbb{Z})$:*

$$\mathcal{M}_E \simeq O^+(\Gamma_{3,19}) \setminus O^+(3,19)/(SO(3)\times O(19))\times \mathbb{R}_+ \tag{C.6}$$

where $\mathbb{R}_+$ denote the volume of the K3 surface defined through:

$$V(K3) = \int_{\mathcal{M}} K\wedge K > 0.$$

This is isomorphic to the space:

$$\mathcal{M}_E \simeq O(\Gamma_{3,19}) \setminus O(3,19)/(O(3)\times O(19))\times \mathbb{R}_+ \tag{C.7}$$

which is a Hausdorff space.

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