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Chern-Simons theory and its applications

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Abstract

The aim of this thesis is to introduce and study basic general properties and certain mathematical and physical aspects of Chern-Simons field theories, as well as their application to the description of particles with fractional statistics and spin on the plane called anyons.

Contents

Introduction

In the last half-century gauge field theories have been one of the most powerful and successful frameworks used to describe a wide variety of physical phenomena and also to build new theoretical models, one of the most important of them being the Standard Model of fundamental interactions of elementary particles. Basically these are field theories whose dynamics is descibed by a lagrangian density that is invariant under the local action of a Lie group on the field thus possessing some redundant degrees of freedom.
Among these gauge theories stands the 3-dimensional Chern-Simons theory, that has been an object of study since the 80's, called after the mathematicians Shiing-Shen Chern and James Harris Simons because its action is basically the integral of the namesake differential 3-form. It is a topological field theory (i.e. it does not depend on the spacetime metric) which does not carry any independent degree of freedom, which means that the solutions of its equations of motion are pure gauge. Despite the first sight appeareance of a rather trivial theory, it carries lots of interesting aspects that emerge when one considers Chern-Simons theory as a part of a more general theory. Indeed it has many applications both in mathematics (in particular in topology where it can be used to compute some topological manifold invariants) and in physics in many different branches, as for example condensed matter, low dimensional gravity and string theory.
The focus of our work will be mainly on the introduction of the classical Chern-Simons theory, both in the abelian and the non-abelian case (the distinction is based on whether the gauge group is commutative or not), and then on the study of two of its main applications: the possibility of existence of gauge invariant massive vector fields in $(2+1)$-dimensional spacetime, and one possible realization of the anyons, a type of quasi-particles that occurs only in 2-dimensional systems and that play an important role in some condensed matter phenomena, e.g. in the description of the fractional quantum Hall effect.
In chapter 1 we will give some preliminary definitions that are needed in the study of a relativistic theory on the plane: after setting up some conventions, we will define the $(2+1)$-dimensional Poincaré group and introduce its representations (spinors, scalars, vectors and tensors) which are the bases of any field theory. Then we will briefly analyze the 3-dimensional version of Maxwell's electrodynamics, which is the simplest example of an abelian vector theory with gauge freedom that lives on the plane, and we will also explain how its peculiarities lead to a dual formulation in terms of a simple scalar field carrying a single independent degree of freedom.
Chapter 2 is dedicated to the introduction of the free abelian Chern-Simons theory, to the derivation of some of its properties, such as gauge invariance and independence of the spacetime metric, and then to the study of the first interesting application: the combina-

tion of the Chern-Simons term with the Maxwell's lagrangian. The equations of motion will show us that this theory describes a massive vector field governed by Klein-Gordon equation. The stress-energy tensor of the theory is the same as that of the 3-dimensional electrodynamics and the field carries one independent physical degree of freedom. Part of this chapter is devoted to a detailed study of the spin of the field, whose derivation is rather original since we did not find it in the literature on this topic. We will follow two different approaches, one passing through the action of the conserved angular momentum as generator of rotations, the other making use of group theory and the representations of the Poincaré group introduced in chapter 1. In the end, following the article by Deser and Jackiw [3], we will show that also this theory admits a dual formulation which has exactly the same properties, and that both these theories descend from a more general "master" lagrangian.

Non-abelian Chern-Simons theory is discussed in chapter 3. Firstly we will give a general introduction into the construction of gauge theories. Then we will introduce the lagrangian of the non-abelian Chern-Simons theory and focus ourselves on understanding how it changes under a gauge transformation. We will find that if spacetime manifold is topologically non-trivial, the gauge invariance is not present unless we consider the quantized theory.

Finally in chapter 4 we will consider one of the main applications of Chern-Simons theory: the description of anyons. These are quasi-particles that behave very differently from bosons and fermions from the statistical point of view. Experimentally they are a kind of emergent phenomenon that can occur in many physical situations typical of condensed matter. Formally they can be realized in different ways; coupling Chern-Simons vector field to a system of non-relativistic quantum hard-core particles is the one we will analyze. To this end we will follow [9], firstly using the path integral concept and exploiting the topological properties of the configuration space in order to find a general description of anyons, and then showing how the Chern-Simons theory provides a very elegant way to realize a system that fits perfectly into the above introduced framework.

Throughout this work we will use natural units (i.e. $c = \hbar = 1$) and the Einstein notation with greek indeces taking values 0, 1, and 2 and labelling spacetime directions, while latin letters labelling either spatial directions or other kind of quantities (as spinor components) depending on the context.

CHAPTER 1

The (2+1)-dimensional Poincaré group

All of the theories and phenomena that we are going to describe take place in a spacetime with only two spatial dimensions whose symmetries are slighty different from these of the ordinary $(3+1)$-dimensional spacetime. So we start our work by describing the $(2+1)$-dimensional Poincaré group and briefly introducing its representations.

1.1 Definition

We consider $\mathbb{R}^3$ as spacetime with coordinates $(x^0,\, x^1,\, x^2)$, where $x^0 = t$, and the flat metric $\eta_{\mu\nu} = diag(-1, +1, +1)$. The infinitesimal spacetime-invariant interval is defined as:

$$-\, ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu = -\left(dx^0\right)^2 + \sum_{i=1}^{2} \left(dx^i\right)^2 . \tag{1.1}$$

The Poincaré group is the group of coordinate transformations that leaves (1.1) unchanged and, as in 4 dimensions, it is given by the semi-direct product of the spacetime translations group and the Lorentz group, which in our case is SO(1,2). The elements of the latter are the 3×3 matrices Λ satisfing the equation $\eta = \Lambda^T \eta\, \Lambda$ where η is the spacetime metric matrix. So a generic Poincaré transformation is given by:

$$x \longrightarrow x' = \Lambda x + a \qquad\qquad \Lambda \in \mathrm{SO}(1{,}2)\ , \quad a \in \mathbb{R}^3 .$$

We will mainly deal with the connected component of this group that is made up by the restricted Lorentz group i.e. the group of matrices Λ that have the determinant equal to 1 and a non-negative first entry, because this is the actual group of invariance of the physical laws. The restricted Lorentz group will be denoted by $\mathrm{SO}^+(1{,}2)$. The other connected components, which are not subgroups of SO(1,2), are those that include parity transformations and time inversion. In general these kind of symmetries are not respected by physical phenomena, though there are some of fundamental theories that are invariant under them too, as Maxwell's electrodynamics for instance.

1.2 $SO^+(1,2)$ and its representations

1.2.1 Generators and Lie algebra

It is easy to see that the restricted Lorentz group $SO^+(1,2)$ is a 3 parameter Lie group and consequently it has 3 independent generators. They can be found by considering an infinitesimal Lorentz transformation, characterized by the infinitesimal antisymmetric tensor parameter $\delta\omega^{\mu\nu}$; by definition they are given by the antisymmetric tensor generators $\{J_{\mu\nu}\}$ such that:

$$\Lambda(\delta\omega) = \mathbb{I} - \frac{i}{2}\,\delta\omega^{\mu\nu}\,J_{\mu\nu}\,.$$

A finite Lorentz transformation can be written in terms of the exponential map:

$$\Lambda(\omega) = \exp\left(-\frac{i}{2}\,\omega^{\mu\nu}\,J_{\mu\nu}\right)$$

where now $\omega^{\mu\nu}$ is still antisymmetric but not infinitesimal. Only three of the $J_{\mu\nu}$ generators are independent due to the antisymmetry with respect to the covariant indices. In particular defining the following operators makes this statement explicit:

$$K_1 \equiv J_{10} = -J_{01} \qquad K_2 \equiv J_{20} = -J_{02} \qquad J \equiv J_{12} = -J_{21}\,.$$

It is easy to see that K_1 and K_2 are boost generators and J is the generator of rotations in the spatial plane by considering separately each of these transformations and splitting the matrix $\delta\omega^{\mu\nu}$ in the proper way. In the matrix form of the vector representation they are:

$$K_1 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad K_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

and they satisfy the Lie algebra:

$$[K_1,\, K_2] = -i\,J \qquad [K_1,\, J] = -i\,K_2 \qquad [K_2,\, J] = i\,K_1\,.$$

Following straightforward computations one can find a covariant expression for the Lie algebra of $SO^+(1,2)$:

$$[J_{\mu\nu},\, J_{\rho\sigma}] = i\left(\eta_{\mu\sigma}\,J_{\rho\nu} - \eta_{\nu\sigma}\,J_{\rho\mu} + \eta_{\mu\rho}\,J_{\nu\sigma} - \eta_{\nu\rho}\,J_{\mu\sigma}\right)\,. \tag{1.2}$$

1.2.2 Fundamental representation

The fundamental representation of a group is defined as its smallest faithful representation (i.e. the homomorphism defining the representation is injective and so it is an isomorphism), which means that all other representations can be build by taking tensor products of this one. In the case of $SO^+(1,2)$ the fundamental representation is that of two-component spinors. The generators of this representation are the so-called gamma matrices γ_μ which satisfy the anticommutation relations forming the Clifford algebra:

$$\{\gamma^\mu,\, \gamma^\nu\} = -2\,\eta^{\mu\nu}\,\mathbb{I}\,.$$

A possible set of γ^μ is given by:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad \gamma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

They are clearly unitary 2×2 complex matrices and their spacetime vector indeces can be lowered with the metric: $\gamma_\mu = \eta_{\mu\nu}\,\gamma^\nu$. Their commutation relations are:

$$\left[\gamma^0,\,\gamma^1\right] = 2i\,\gamma^2 \qquad \left[\gamma^0,\,\gamma^2\right] = -2i\,\gamma^1 \qquad \left[\gamma^1,\,\gamma^2\right] = -2i\,\gamma^0$$

which can be brought to the covariant expression $[\gamma^\mu,\,\gamma^\nu] = 2i\,\epsilon^{\mu\nu\rho}\,\eta_{\rho\sigma}\,\gamma^\sigma$. We claim that these gamma matrices provide a representation of the Lie algebra of the (2+1)-dimensional Lorentz group; indeed we can build matrices associated to the covariant generators $J_{\mu\nu}$ as follows:

$$J_{\mu\nu} = \frac{1}{4}\,i\,[\gamma_\mu,\,\gamma_\nu] = -\frac{1}{2}\,\epsilon_{\mu\nu\rho}\,\eta^{\rho\sigma}\,\gamma_\sigma\,.$$

Let us now show that the commutation relations between these matrices are the same as (1.2):

$$[J_{\mu\nu},\,J_{\rho\sigma}] = \frac{1}{4}\,\epsilon_{\mu\nu\alpha}\,\eta^{\alpha\lambda}\,\epsilon_{\rho\sigma\beta}\,\eta^{\beta\delta}\,[\gamma_\lambda,\,\gamma_\delta] = -i\,J_{\lambda\delta}\,\eta^{\alpha\lambda}\,\eta^{\beta\delta}\,\epsilon_{\mu\nu\alpha}\,\epsilon_{\rho\sigma\beta}\,.$$

We can now use the identity:

$$\begin{aligned}\epsilon_{\mu\nu\alpha}\,\epsilon_{\rho\sigma\beta} = \eta_{\mu\rho}\,(-\eta_{\nu\sigma}\,\eta_{\alpha\beta} + \eta_{\nu\beta}\,\eta_{\alpha\sigma}) - \eta_{\mu\sigma}\,(-\eta_{\nu\rho}\,\eta_{\alpha\beta} + \eta_{\nu\beta}\,\eta_{\alpha\rho}) +\\ + \eta_{\mu\beta}\,(-\eta_{\nu\rho}\,\eta_{\alpha\sigma} + \eta_{\nu\sigma}\,\eta_{\alpha\rho})\end{aligned}$$

in order to get exactly (1.2), which proves that our guess was right. Now we can build a representation of the Lorentz group through the exponential map. Given an arbitrary Lorentz transformation $\Lambda(\omega)$, its matrix form in our representation will be:

$$\begin{aligned}S(\omega) &= \exp\left(-\frac{i}{2}\,\omega^{\mu\nu}\,J_{\mu\nu}\right) = \exp\left(\frac{i}{4}\,\epsilon^{\mu\nu\rho}\,\omega_\rho\,\epsilon_{\mu\nu\alpha}\,\eta^{\alpha\beta}\,\gamma_\beta\right) = \exp\left(-\frac{i}{2}\,\delta^\rho{}_\alpha\,\omega_\rho\,\eta^{\alpha\beta}\,\gamma_\beta\right)\\ &= \exp\left(-\frac{i}{2}\,\omega_\beta\,\gamma^\beta\right)\end{aligned}$$

where ω_μ is the vector dual of the antisymmetric tensor $\omega^{\mu\nu}$ i.e. $\omega^{\mu\nu} = \epsilon^{\mu\nu\rho}\,\omega_\rho$. So the action of $S(\omega)$ on complex two-component spinors gives us a representation of SO$^+$(1,2). Using $a, b, ...$ as spinorial indices, the spinor ψ_a transforms under a Lorentz transformation as:

$$\psi_a \longrightarrow \psi'_a = [S(\omega)]_a{}^b\,\psi_b\,. \tag{1.3}$$

From now on we will omit the ω dependence of the transformations for simplicity, so that a generic Lorentz transformation is simply Λ and the corresponding matrix in a given representation is S. Also we will suppress spinorial indeces, so that the (1.3) becomes simply $\psi' = S\,\psi$.

1.2.3 Other representations

Now that we have the elements of the fundamental representation of SO$^+$(1,2) and their transformation laws, we can build other representations by combining spinors together in different ways.
First of all, given a spinor ψ it is useful to introduce the Dirac adjoint $\bar\psi$, since it is transformed under the inverse Lorentz transformation S^{-1}. It is defined as:

$$\bar\psi \equiv \psi^\dagger\,\gamma^0 \qquad \text{i.e.} \qquad \bar\psi_a \equiv (\psi^\dagger)^b\,(\gamma^0)_{ba}\,.$$

In order to prove the ansatz on its transformation law, let us consider the following identity (it is easy to get from direct calculations):

$$\gamma^0\,(\gamma^\mu)^\dagger\,\gamma^0 = \gamma^\mu\,.$$

By multiplying both members by the factor $\frac{i}{2}\,\omega_\mu$, using the fact that the parameters ω are real, it follows that:

$$\gamma^0\left(-\frac{i}{2}\,\omega_\mu\,\gamma^\mu\right)^\dagger\gamma^0 = \frac{i}{2}\,\omega_\mu\,\gamma^\mu \qquad\Longrightarrow\qquad \gamma^0\,S^\dagger\gamma^0 = S^{-1}$$

where we have used the facts that $\exp(BAB^{-1}) = B\exp(A)B^{-1}$, $\exp(A^\dagger) = \exp(A)^\dagger$ and $\exp(-A) = \exp(A)^{-1}$. Thanks to this identity and the fact that $\gamma^0\,\gamma^0 = \mathbb{I}$ we can figure out how $\bar\psi$ transforms:

$$\bar\psi \longrightarrow \bar\psi' = (S\psi)^\dagger\gamma^0 = \psi^\dagger\,S^\dagger\,\gamma^0 = \psi^\dagger\,\gamma^0\,\gamma^0\,S^\dagger\,\gamma^0 = \bar\psi\,S^{-1}\,.$$

We are now ready to introduce other representations:

- *Scalar representations.* They are objects of the form $\varphi = \bar\psi\,\psi$ (the contraction of the spinorial indeces is implied) such that they are invariant under SO$^+$(1,2) action:

$$\varphi \longrightarrow \varphi' = \bar\psi'\,\psi' = \bar\psi\,S^{-1}\,S\,\psi = \bar\psi\,\psi = \varphi\,.$$

- *Vector representations.* Let us consider objects of the form $v^\mu = \bar\psi\,\gamma^\mu\,\psi$. Under Lorentz transformations they transform in the following way:

$$v^\mu \longrightarrow v'^\mu = \bar\psi'\,\gamma^\mu\,\psi' = \bar\psi\,S^{-1}\,\gamma^\mu\,S\,\psi\,.$$

  We can say that v^μ is a 3-vector if the equality $S^{-1}\,\gamma^\mu\,S = \Lambda^\mu{}_\nu\,\gamma^\nu$ holds, so that the transformation law of v^μ is exactly that of a contravariant 3-vector: $v'^\mu = \Lambda^\mu{}_\nu\,v^\nu$. Let us prove that this is true at least for infinitesimal transformations. Consider separately the LHS and the RHS at the first order in $\delta\omega_\mu$:

$$\begin{aligned}
\text{LHS} \to S^{-1}\,\gamma^\mu\,S &= \left(\mathbb{I} + \frac{i}{2}\,\delta\omega_\alpha\,\gamma^\alpha\right)\gamma^\mu\left(\mathbb{I} - \frac{i}{2}\,\delta\omega_\alpha\,\gamma^\alpha\right) = \gamma^\mu + \frac{i}{2}\,\delta\omega_\alpha\,[\gamma^\alpha,\,\gamma^\mu]\\
&= \gamma^\mu - \delta\omega_\alpha\,\epsilon^{\alpha\mu\nu}\,\eta_{\nu\rho}\,\gamma^\rho = \gamma^\mu - \delta\omega^\mu{}_\rho\,\gamma^\rho
\end{aligned}$$

$$\begin{aligned}
\text{RHS} \to \Lambda^\mu{}_\nu\,\gamma^\nu &= \left[\mathbb{I} - \frac{i}{2}\,\delta\omega^{\alpha\beta}J_{\alpha\beta}\right]^\mu{}_\nu\gamma^\nu = \gamma^\mu - i\left[\delta\omega^{10}K_1 + \delta\omega^{20}K_2 + \delta\omega^{12}J\right]^\mu{}_\nu\gamma^\nu\\
&= \gamma^\mu + \left[\begin{pmatrix} 0 & \delta\omega^{10} & \delta\omega^{20}\\ \delta\omega^{10} & 0 & -\delta\omega^{12}\\ \delta\omega^{20} & \delta\omega^{12} & 0\end{pmatrix}\right]^\mu{}_\nu\gamma^\nu\\
&= \gamma^\mu - \left[\begin{pmatrix} 0 & \delta\omega^{01} & \delta\omega^{02}\\ \delta\omega^{01} & 0 & \delta\omega^{12}\\ \delta\omega^{02} & -\delta\omega^{12} & 0\end{pmatrix}\right]^\mu{}_\nu\gamma^\nu\\
&= \gamma^\mu - \delta\omega^{\mu\rho}\,\eta_{\rho\nu}\,\gamma^\nu = \gamma^\mu - \delta\omega^\mu{}_\nu\,\gamma^\nu\,.
\end{aligned}$$

  So we have LHS = RHS and thus the equality is true at the first order. With some more effort it can be proved also for finite Lorentz transformations but we will not do it.

- *Tensor representations.* Similarly to the vector representation one can also consider objects with more than one gamma matrix in their definition:

$$T^{\mu_1 \dots \mu_p} = \bar{\psi}\, \gamma^{\mu_1} \dots \gamma^{\mu_p}\, \psi\,.$$

Using the previously introduced equality $S^{-1}\, \gamma^\mu\, S = \Lambda^\mu{}_\nu\, \gamma^\nu$ we find that the transformation law of $T^{\mu_1 \dots \mu_p}$ is:

$$\begin{aligned} T^{\mu_1 \dots\, \mu_p} \longrightarrow T'^{\mu_1 \dots\, \mu_p} &= \bar{\psi}\, S^{-1}\, \gamma^{\mu_1} \dots \gamma^{\mu_p}\, S\, \psi = \bar{\psi}\, \left(S^{-1} \gamma^{\mu_1} S\right) \dots \left(S^{-1} \gamma^{\mu_p} S\right) \psi \\ &= \Lambda^{\mu_1}{}_{\nu_1} \dots \Lambda^{\mu_p}{}_{\nu_p}\, T^{\nu_1 \dots\, \nu_p} \end{aligned}$$

that is exactly that of a tensor of rank $(p, 0)$. Obiviously from this representations one can also obtain tensors of rank (p, q) by lowering the indeces with the metric, and symmetric (antisymmetric) tensors by taking sums (differences) of the same tensor with exchanged indeces.

The vector representation is the one we will consider for the majority of the following topics.

1.3 Representation of the Poincaré group

In order to study the representations of the full Poincaré group, we have to introduce also the generators of the spacetime translation group. This is a 3 parameter Lie group and thus has 3 independent generators that we denote by P_μ. In particular in the vector representation they are known to be $P_\mu = -i\, \partial_\mu$. The Lie algebra associated to the Poincaré group is given by the following commutators:

$$[P_\mu,\, P_\nu] = 0 \qquad\qquad [J_{\mu\nu},\, P_\rho] = i\, (\eta_{\mu\rho}\, P_\nu - \eta_{\nu\rho}\, P_\mu)$$

together with the commutation relation between Lorentz generators (1.2). It can be proved that the representations of the 2+1 Poincaré group are labelled by the eigenvalues of the two Casimir operators of the group: the generator of translations squared $P_\mu\, P^\mu$ and the Pauli-Lubanski pseudoscalar $W = \frac{1}{2}\, \epsilon_{\mu\nu\rho}\, P^\mu\, J^{\nu\rho}$. In particular the first has a continuous spectrum and its eigenvalue is identified with $-m^2$, while the second has eigenvalues of the form ms, where m is the mass and s the spin of the representation.

1.4 Vector field example: classical electrodynamics

1.4.1 Main features

The most common and at the same time instructive example of a vector theory is given by Maxwell's electromagnetism. In a spacetime with only 2 spatial dimensions this theory is formulated similarly to the ordinary 4-dimensional one, though its physical properties are different, as we will see.
Firstly we give the definition of the electromagnetic potential A_μ and consequently of the field strength tensor $F_{\mu\nu}$:

$$A_\mu \equiv (-\phi,\, A_i)\,, \qquad\qquad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu\,.$$

where ϕ is the electric potential and $\vec{A} \equiv (A^1,\, A^2)$ is the magnetic vector potential. As well known, the potential is defined up to a gauge transformation. In the classical theory the physical entity is $F_{\mu\nu}$ that is invariant under such transformations[1]. In particular, given an arbitrary scalar function of spacetime coordinates Λ, one has:

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda \qquad \Longrightarrow \qquad F'_{\mu\nu} = F_{\mu\nu} + \cancel{\partial_\mu \partial_\nu \Lambda} - \cancel{\partial_\nu \partial_\mu \Lambda} = F_{\mu\nu}\,.$$

Through all the analysis we will assume that $A_\mu(x)$ are differentiable functions of spacetime coordinates that vanish at spacial infinity at any time t, that is physically reasonable. These definitions lead to the expression of field strength tensor written in terms of electric and magnetic fields:

$$\begin{cases} F^{i0} = -E^i \\ F^{ij} = \epsilon^{ij}\, B \end{cases} \qquad \Longrightarrow \qquad F^{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 \\ -E^1 & 0 & B \\ -E^2 & -B & 0 \end{pmatrix}. \tag{1.4}$$

Note that, unlike in 4 dimensions, here only the electric field is a vector while the magnetic field is a scalar. If we define $F_{\mu\nu}$ directly in terms of derivatives of the potential, we immediately obtain the first set of Maxwell's equations as the Bianchi identity[2]:

$$0 = \epsilon_{\mu\nu\rho}\, \partial^\mu \partial^\nu A^\rho - \epsilon_{\mu\nu\rho}\, \partial^\mu \partial^\rho A^\nu = \epsilon_{\mu\nu\rho}\, \partial^\mu F^{\nu\rho}\,.$$

Upon having introduced these objects, we are now ready to write down the lagrangian density that describes free electrodynamics, that is the same for all spacetime dimensions:

$$\mathcal{L}_M = -\frac{1}{4}\, F^{\mu\nu}\, F_{\mu\nu}\,. \tag{1.5}$$

The dynamic variables in (1.5) are clearly the components of the potential A_μ. However the fact that only $F_{\mu\nu}$ appears ensures that the gauge invariance is automatically achieved. The corresponding action takes the form:

$$S_M = -\frac{1}{4} \int_{\Sigma_1}^{\Sigma_2} d^3x\, F^{\mu\nu}(x)\, F_{\mu\nu}(x)\,,$$

where Σ_1 and Σ_2 are surfaces in $\mathbb{R}^3$ at fixed times. From now on we will not indicate the explicit dependence of the fields on x^μ, except when necessary. By taking the infinitesimal variation δS to be zero, one finds the equations of motion for A_μ, which are nothing but the second set of Maxwell's equations:

$$\partial_\mu F^{\mu\nu} = 0\,. \tag{1.6}$$

These are a set of three equations for the potential, but there are some constraints that reduce the number of degrees of freedom of the theory. In particular, gauge invariance identifies solutions that differ by a gauge transformation and thus reduces the number of independent solutions. In order to take this into account, we should fix the gauge freedom

[1] This is not true in the quantum theory as showed by Aharonov-Bohm effect: the wavefunction of an electron is affected by the electromagnetic potential even in regions of space where $F_{\mu\nu}$ is zero.

[2] This statement is true because of the assumption of $\mathbb{R}^3$ as spacetime; infact Bianchi identity is satisfied globally by just one potential if and only if the spacetime is simply connected mathematically speaking.

in such a way that we can count the true physically independent solutions. There are many ways to do it, imposing some constraints to be satisfied by A^μ, for instance:

$\partial_\mu A^\mu = 0$	$\nabla \cdot \vec{A} = 0$	$A^0 = 0$
Lorenz-gauge	Coulomb-gauge	Weyl-gauge

Let us pick the Lorenz-gauge: given any arbitrary potential $\tilde{A}^\mu$ and being Λ the scalar field that defines a gauge transformation, in order to arrive at the Lorenz-gauge, Λ has to satisfy:

$$0 = \partial_\mu A^\mu = \partial_\mu \tilde{A}^\mu + \partial_\mu \partial^\mu \Lambda \qquad \Longleftrightarrow \qquad \Box\Lambda = -\partial_\mu \tilde{A}^\mu \,.$$

This equation determines Λ up to a residual gauge transformation (that preserves the Lorenz-gauge condition), which can be parametrized by another scalar field Λ' satisfying $\Box\Lambda' = 0$. In the Lorenz-gauge, the equations of motion for A_μ reduce to $\Box A_\mu = 0$ and hence one can use the residual gauge freedom to fix one component of the potential, for instance $A^0 = 0$, that imposes the Weyl-gauge. In this way the gauge invariance is completely fixed. Therefore we have two constraints on A^μ given by gauge fixing conditions and so the theory has only one degree of freedom.

An intuitive interpretation of this can be achieved by comparing this theory with the 4-dimensional version. In the latter, the solutions for A^μ are (superposition of) massless plane waves and the two degrees of freedom are expressed through the polarization of the fields in the plane orthogonal to the propagation direction. Instead in the 3-dimensional theory the solutions are still plane waves but the space orthogonal to the propagation direction is just a line, so the fields do not have any polarization freedom.

1.4.2 Dual description

A noteworthy feature of 3-dimensional electrodynamics is that the field strength tensor has three independent components (due to its antisymmetry) so through it we can define a vector that contains the same amount of components:

$$F^\mu \equiv \frac{1}{2}\,\epsilon^{\mu\nu\rho}\,F_{\nu\rho} = \epsilon^{\mu\nu\rho}\,\partial_\nu A_\rho \,.$$

Contracting both sides with $\epsilon_{\mu\alpha\beta}$ and using the identity $\epsilon_{\mu\alpha\beta}\,\epsilon^{\mu\nu\rho} = -\delta_\alpha{}^\nu\,\delta_\beta{}^\rho + \delta_\alpha{}^\rho\,\delta_\beta{}^\nu$, one finds immediately the inverse relation between $F_{\mu\nu}$ and F^μ:

$$F_{\alpha\beta} = -\epsilon_{\mu\alpha\beta}\,F^\mu \,.$$

So we can rewrite the lagrangian density (1.5) in the following way:

$$\mathcal{L}_M = -\frac{1}{4}\,F^{\mu\nu}\,F_{\mu\nu} = -\frac{1}{4}\,\epsilon_{\rho\mu\nu}\,\epsilon^{\sigma\mu\nu}\,F_\sigma\,F^\rho = \frac{1}{2}\,\delta_\rho{}^\sigma\,F_\sigma\,F^\rho = \frac{1}{2}\,F_\rho\,F^\rho \,. \qquad (1.7)$$

Thinking of the (1.7) as a lagrangian density for the potential A_μ as the dynamic variable obviously leads to the same equations of motion (1.6).

Nevertheless there exist a dual theory of free electrodynamics that relies exactly on the peculiar possibility of defining this F^μ in 3 dimensions. Indeed one can directly use F^μ as the dynamic variable of the theory and impose a constraint that takes into account its dependency on A_μ. Noting that $\partial_\mu F^\mu \equiv 0$ as a consequence of its expression in term of the potential, we define the master Maxwell's lagrangian density by adding this constraint through a Lagrange multiplier:

$$\mathcal{L}_{MD} = \frac{1}{2}\, F^{\mu}\, F_{\mu} + \varphi\, \partial_{\mu} F^{\mu} \tag{1.8}$$

where φ is an auxiliary scalar field that has the role of the Lagrange multiplier. Not surprisingly, by varying the action with respect to φ we obtain the constraint $\partial_{\mu} F^{\mu} = 0$ as one of the equations of motion and we are back in the original theory. Whereas if we take the variation with respect to F^{μ} we have:

$$\delta S_{MD} = \int_{\Sigma_1}^{\Sigma_2} d^3x\, (F_{\mu}\, \delta F^{\mu} + \varphi\, \partial_{\mu} \delta F^{\mu}) = \int_{\Sigma_1}^{\Sigma_2} d^3x\, \delta F^{\mu}\, (F_{\mu} - \partial_{\mu}\varphi) + \cancel{\int_{\Sigma_1}^{\Sigma_2} d^3x\, \partial_{\mu}(\varphi\, \delta F^{\mu})}$$

where the last term vanishes thanks to Gauss's theorem, assuming that $\delta F^{\mu}\big|_{\Sigma_{1,2}} = 0$. It follows that the equation of motion for φ is:

$$F_{\mu} = \partial_{\mu}\varphi\, . \tag{1.9}$$

By substituting this expression for F_{μ} into (1.8), using Leibniz's rule and skipping the total derivative, we obtain the dual lagrangian for the field φ:

$$\mathcal{L}_{MD} = -\frac{1}{2}\, \partial^{\mu}\varphi\, \partial_{\mu}\varphi\, . \tag{1.10}$$

In this way we have reached a very interesting result: the dual of the 3-dimensional Maxwell's theory is a scalar theory, whose equation of motion is:

$$\Box\varphi = 0\, . \tag{1.11}$$

However, the claimed duality of the two theoris has still to be proved. We do this by showing that the equations of motion (1.6) for A^{μ} and the (1.11) for φ are interrelated:

$$F^{\alpha\beta} = -\epsilon^{\mu\alpha\beta}\, F_{\mu} = -\epsilon^{\mu\alpha\beta}\, \partial_{\mu}\varphi \quad \Longleftrightarrow \quad \partial_{\alpha}\, F^{\alpha\beta} = -\epsilon^{\mu\alpha\beta}\, \partial_{\alpha}\partial_{\mu}\varphi = 0$$

$$\partial^{\mu}\varphi = F^{\mu} = \epsilon^{\mu\nu\rho}\, \partial_{\nu} A_{\rho} \quad \Longleftrightarrow \quad \partial_{\mu}\partial^{\mu}\varphi = \epsilon^{\mu\nu\rho}\, \partial_{\mu}\partial_{\nu} A_{\rho} = 0\, .$$

Solving $\Box\varphi = 0$ we have completely determined the problem, which means that the theory has one degree of freedom; thus with much less arguing we have just achieved the same result of 1.4.1. Note that the roles of the equations of motion and the Bianchi identity in the dual formulation are interchanged.

Furthermore, discussing about the conserved quantities (see Appendix A for an explaination of how they are derived), it turns out that the elecromagnetic field is spinless. Formally it can be proved computing the Noether's current associated to the field φ under a Lorentz transformation and the corresponding conserved charge. Firstly let us compute the canonical stress-energy tensor:

$$T^{\mu\nu} = -\eta^{\mu\nu}\, \mathcal{L} + \frac{\partial\mathcal{L}}{\partial\partial_{\mu}\varphi}\, \partial^{\nu}\varphi = \frac{1}{2}\, \eta^{\mu\nu}\, \partial^{\rho}\varphi\, \partial_{\rho}\varphi - \partial^{\mu}\varphi\, \partial^{\nu}\varphi\, . \tag{1.12}$$

Now the total variation of the field $\bar{\delta}\varphi$ under Lorentz transformations is 0 since φ is a scalar field, while $\delta x^{\mu} = \omega^{\mu}{}_{\nu}\, x^{\nu}$ (with $\omega^{\mu\nu} = -\omega^{\nu\mu}$ and $\|\omega\| \ll 1$) is the infinitesimal variation of the coordinates. Thus we have:

$$J^{\mu} = -T^{\mu\nu}\, \delta x_{\nu} = -T^{\mu\nu}\, \omega_{\nu\rho}\, x^{\rho} = \frac{1}{2}\, \omega_{\nu\rho}\, (x^{\nu}\, T^{\mu\rho} - x^{\rho}\, T^{\mu\nu})$$

$$\Longrightarrow \qquad J^\mu = \frac{1}{2}\,\omega_{\nu\rho}\,\mathcal{J}^{\mu\nu\rho} \qquad \text{defining} \qquad \mathcal{J}^{\mu\nu\rho} \equiv x^\nu\,T^{\mu\rho} - x^\rho\,T^{\mu\nu}\,.$$

So it follows that the angular momentum tensor is given by:

$$J^{\mu\nu} \equiv \int d^2x\,\mathcal{J}^{0\mu\nu} = \int d^2x\,\left(x^\mu\,T^{0\nu} - x^\nu\,T^{0\mu}\right)$$

which has only the orbital contribution, so φ has spin 0 as one might expect.
The analyzed dual description of electrodynamics has many interesting aspects that simplify the treatment and emerge just from mathematical properties that are unique for the 3-dimensional theory.

CHAPTER 2

Classical Chern-Simons theory

In a $(2+1)$-dimensional spacetime we can construct a classical theory even simpler than electrodynamics, that is still gauge invariant but has no propagating degrees of freedom. This is the Chern-Simons theory, the object of the study. In this chapter we will firstly analyze the free theory and then look at what happens when it is combined with Maxwell's theory, finding that also in this case it admits a dual description.

2.1 Free theory

Given a generic vector field A_μ, the abelian Chern-Simons theory in a $(2+1)$-dimensional spacetime[3] is represented by the lagrangian density:

$$\mathcal{L}_{CS} = \kappa\, \epsilon^{\mu\nu\rho} A_\mu\, \partial_\nu A_\rho \tag{2.1}$$

where κ is a real parameter of an appropriate dimension, depending on that of A_μ, such that the associated action is dimentionless. For instance, if A_μ is Maxwell's potential, we must have[4] $[\kappa] = L^{-1}$. The first important thing to notice is that (2.1) is gauge-invariant up to a divergence:

$$\begin{aligned}\mathcal{L}_{CS} \to \mathcal{L}'_{CS} &= \kappa\, \epsilon^{\mu\nu\rho}\, (A_\mu + \partial_\mu\Lambda)\, \partial_\nu(A_\rho + \partial_\rho\Lambda) = \kappa\, \epsilon^{\mu\nu\rho}\, A_\mu\, \partial_\nu A_\rho + \kappa\, \epsilon^{\mu\nu\rho}\, \partial_\mu\Lambda\, \partial_\nu A_\rho \\ &= \mathcal{L}_{CS} + \partial_\mu(\kappa\, \epsilon^{\mu\nu\rho}\, \Lambda\, \partial_\nu A_\rho)\,.\end{aligned}$$

This is a key feature since gauge-symmetry is fundamental in most of nowadays field theories. Moreover this is a topological field theory because the action does not depend on the spacetime metric; indeed, even though we have assumed a flat spacetime, the theory can obviously be coupled to an exterior non-trivial metric $g_{\mu\nu}(x)$ but in that case we will have:

$$\text{Volume form:} \quad \sqrt{|g|}\, d^3x \qquad\qquad \text{Levi-Civita tensor:} \quad \frac{1}{\sqrt{|g|}}\, \epsilon^{\mu\nu\rho}$$

[3] A generalization to odd higher dimensions can be constructed.

[4] We recall that $[d^3x] = L^3$, $[\partial_\mu] = L^{-1}$, and $[A_\mu] = L^{-\frac{1}{2}}$ (in $(2+1)$-dimensional spacetime).

where g is defined to be the determinant of the metric matrix. From these definitions it follows that the general expression for the action is:

$$\mathcal{S}_{CS} = \int_{\Sigma_1}^{\Sigma_2} d^3x \, \kappa \, \epsilon^{\mu\nu\rho} A_\mu \, \partial_\nu A_\rho \tag{2.2}$$

whether the spacetime metric is flat or not.
Let us now derive the equations of motion:

$$\begin{aligned} \delta S_{CS} &= \kappa \int_{\Sigma_1}^{\Sigma_2} d^3x \, \epsilon^{\mu\nu\rho} \left(\delta A_\mu \, \partial_\nu A_\rho + A_\mu \, \partial_\nu \delta A_\rho \right) \\ &= \kappa \int_{\Sigma_1}^{\Sigma_2} d^3x \, \delta A_\mu \left(\epsilon^{\mu\nu\rho} \, \partial_\nu A_\rho - \epsilon^{\rho\nu\mu} \, \partial_\nu A_\rho \right) + \kappa \cancel{\int_{\Sigma_1}^{\Sigma_2} d^3x \, \partial_\nu \left(\epsilon^{\mu\nu\rho} A_\mu \, \delta A_\rho \right)} \end{aligned}$$

where the last term vanishes thanks to Gauss's theorem, assuming that $\delta A_\mu \big|_{\Sigma_{1,2}} = 0$. It follows that equations of motion are given by:

$$2\kappa \, \epsilon^{\mu\nu\rho} \, \partial_\nu A_\rho = 0 \, . \tag{2.3}$$

In flat space, as the one we are considering, Poincaré lemma states that all the solutions of (2.3) take the form $A_\mu = \partial_\mu \varphi$ for any arbitrary scalar funcion φ and hence A_μ is a pure gauge. This means that the solution is completely trivial once we fix the gauge freedom, which proves that the Chern-Simons free theory has no local physical degrees of freedom. It means that this lagrangian has a little physical interest on its own but it provides us a way by which we can introduce interactions or other extra features in a theory without breaking gauge invariance, such as the mass of A_μ that we will consider in the next section. Lastly it is important to notice that, unlike Maxwell's theory, Chern-Simons theory is not parity invariant due to the fact that $\epsilon^{\mu\nu\rho}$ is a pseudotensor and thus acquires a minus sign under Lorentz transformations with negative determinant, as parity for instance[5]. It means that the invariance group of the theory is only the restricted Lorentz group $SO^+(1,2)$. This is a crucial point in the discussion of the sign of the spin carried by the fields as we will see later.

2.2 Massive vector fields in $D = 3$

2.2.1 A conventional way to introduce massive fields

It's a legitimate question to ask ourselves what would happen if A_μ were massive fields, and indeed it suggests an interesting theoretical analysis. A direct way to introduce a mass in the free Maxwell's theory is to add a quadratic term in A_μ to (1.5):

$$\mathcal{L}_{MM} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} m^2 A^\mu A_\mu \, . \tag{2.4}$$

The first important thing to notice is that this lagrangian density is no more gauge invariant; infact, under a gauge transformation, the action acquires the term:

$$-\frac{1}{2} m^2 \left(\partial^\mu \Lambda \, A_\mu + A^\mu \, \partial_\mu \Lambda + \partial^\mu \Lambda \, \partial_\mu \Lambda \right) = -\frac{1}{2} m^2 \, \partial^\mu \Lambda \left(2 \, A_\mu + \partial_\mu \Lambda \right) \, .$$

[5]Note that parity transformations in 2 spatial dimensions involve the change of sign of only one coordinate because the change of sign of both of them is equivalent to a half rotation, which does not change the orientation of the space.

This expression vanishes only if $\partial^\mu \Lambda = 0$ or $\partial_\mu \Lambda = -2\, A_\mu$ but none of these two cases is compatible with the arbitrariness of Λ.
It is very easy to derive the equations of motion by varying the action, finding:

$$\partial_\mu F^{\mu\nu} - m^2\, A^\nu = 0\,. \tag{2.5}$$

By applying ∂_ν to (2.5) we find that the potential must satisfy the constraint $\partial_\mu A^\mu = 0$, that is the only condition that reduces the number of degrees of freedom, since there is no gauge invariance anymore. Moreover, thanks to this constraint, we can rewrite the equation (2.5) as the Klein-Gordon equation:

$$\left(\Box - m^2\right) A^\mu = 0\,. \tag{2.6}$$

This equation, as those of classical Maxwell's theory, can be solved by performing a Fourier transformation:

$$\left(-p_\nu\, p^\nu - m^2\right) \widehat{A}^\mu = 0 \qquad \Longrightarrow \qquad \widehat{A}^\mu(p) = a^\mu(p)\, \delta\left(m^2 + p_\nu\, p^\nu\right)$$

where $a^\mu(p)$ are functions whose mathematical expression depends on physical situations. When we return to the coordinate space by integrating in d^3p we obtain that A^μ is different from zero only if $p_\nu\, p^\nu = -m^2$, which shows us explicitely our claim that $\mathcal{L}_{MM}$ describes a field A_μ of mass m. In particular the explicit general solution is:

$$A^\mu(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3p\, a^\mu(p)\, e^{ip_\nu x^\nu}\, \delta\left(m^2 + p_\nu\, p^\nu\right) \tag{2.7}$$

where we have to require that $a^\mu(-p) = (a^\mu)^*(p)$ in order to have a real $A^\mu(x)$. Physically $A^\mu(x)$ is an infinite superposition of functions of the form $a^\mu(p)\, e^{ip_\nu x^\nu}$ with the constraint on the wave vector given by the mass squared relation and the transversality condition $\partial_\mu A^\mu = 0$.
Summarinzing, this way to build a massive electrodynamics leads to a theory which, in contrast to the massless case, has no gauge symmetry and two degrees of freedom, each one associated with a helicity state, $+1$ or -1, of the field A_μ.
The last important observation that has to be done is that this theory is invariant under the full Poincaré group, right as the classical electrodynamics, and particularly it is parity invariant. Indeed this is consistent with the presence of two degrees of freedom, which are related by the parity transformation.

2.2.2 Gauge-invariant massive electrodynamics

As anticipated before, we will now consider the coupling of Chern-Simons theory (2.1) to Maxwell's theory(1.7), that is the model introduced for the first time in [4]. Defining $\kappa = \frac{1}{2}\, m$, the lagrangian of the theory is:

$$\mathcal{L}_{MCS} = \frac{1}{2}\, F^\mu\, F_\mu + \frac{1}{2}\, m\, \epsilon_{\mu\nu\rho}\, A^\mu\, \partial^\nu A^\rho = \frac{1}{2}\, F^\mu\, F_\mu + \frac{1}{2}\, m\, A^\mu\, F_\mu \tag{2.8}$$

whose corresponding action is obviously:

$$S_{MCS} = \int_{\Sigma_1}^{\Sigma_2} d^3x \left(\frac{1}{2}\, F^\mu\, F_\mu + \frac{1}{2}\, m\, A^\mu\, F_\mu\right).$$

S_{MCS} is gauge invariant because, as we showed previously, such are its two addends, and A_μ carries one physical degree of freedom as in the classical Maxwell's theory, since the Chern-Simons term doesn't bring any degree of freedom. But now we will see that F_μ becomes massive. The first thing to notice is that, as showed in the previous section, m must have the dimention of the inverse of a length, that is a mass in natural units, so the claim is dimensionally consistent.
Now to find the equations of motion we consider the variation of S_{MCS}:

$$\begin{aligned}\delta S_{MCS} &= \int_{\Sigma_1}^{\Sigma_2} d^3x \left(F^\mu \,\delta F_\mu + \frac{1}{2}\, m\, A^\mu \,\delta F_\mu + \frac{1}{2}\, m\, F_\mu \,\delta A^\mu \right) \\ &= \int_{\Sigma_1}^{\Sigma_2} d^3x \left[\left(F^\mu + \frac{1}{2}\, m\, A^\mu \right) \epsilon_{\mu\nu\rho}\, \partial^\nu \delta A^\rho + \frac{1}{2}\, m\, F_\rho \,\delta A^\rho \right] \\ &= \int_{\Sigma_1}^{\Sigma_2} d^3x \left(-\epsilon_{\mu\nu\rho}\, \partial^\nu F^\mu - \frac{1}{2}\, m\, \epsilon_{\mu\nu\rho}\, \partial^\nu\, A^\mu + \frac{1}{2}\, m\, F_\rho \right) \delta A^\rho + \\ &\qquad\qquad + \cancel{\int_{\Sigma_1}^{\Sigma_2} d^3x\, \partial^\nu \left(\epsilon_{\mu\nu\rho}\, F^\mu \,\delta A^\rho + \frac{1}{2}\, m \epsilon_{\mu\nu\rho}\, A^\mu \,\delta A^\rho \right)} \\ &= \int_{\Sigma_1}^{\Sigma_2} d^3x \left(-\epsilon_{\mu\nu\rho}\, \partial^\nu F^\mu + m\, F_\rho \right) \delta A^\rho\end{aligned}$$

where we have again assumed that $\delta A^\mu\big|_{\Sigma_{1,2}} = 0$. Then the equations of motion of the theory are given by:

$$m\, F_\rho + \epsilon_{\rho\nu\mu}\, \partial^\nu F^\mu = 0\,. \tag{2.9}$$

In order to see that this equation describes a field of mass m let us contract the LHS of (2.9) with $\epsilon^{\rho\alpha\beta}$ and then apply ∂_α:

$$m\, \epsilon^{\rho\alpha\beta}\, F_\rho + \left(-\delta^\alpha{}_\nu \delta^\beta{}_\mu + \delta^\alpha{}_\mu \delta^\beta{}_\nu \right) \partial^\nu F^\mu = 0 \quad\Longrightarrow\quad m\, \epsilon^{\rho\alpha\beta}\, F_\rho - \partial^\alpha F^\beta + \partial^\beta F^\alpha = 0$$

$$\Longrightarrow\quad \partial_\alpha \partial^\alpha F^\beta - \cancel{\partial^\beta \partial_\alpha F^\alpha} - m\, \epsilon^{\rho\alpha\beta}\, \partial_\alpha F_\rho = 0 \quad\Longrightarrow\quad \Box F^\beta + m\, \epsilon^{\beta\alpha\rho}\, \partial_\alpha F_\rho = 0\,.$$

Now if we use again (2.9) in order to replace the second term of this expression, we get the Klein-Gordon equation for the field F_μ:

$$\left(\Box - m^2 \right) F^\mu = 0\,. \tag{2.10}$$

Therefore, as was shown for A^μ in Section 2.2.1, what we have found is basically a theory in which the field F^μ has become massive and also the gauge invariance is preserved, unlike in the case analized in secton 2.2.1. This is what makes this model interesting to study from the physical point of view.
Let us now show what is a convenient gauge fixing choice. If we replace F_μ with its expression in terms of A_μ in (2.10) and we commute the derivatives, we get:

$$\epsilon^{\mu\nu\rho}\, \partial_\nu \left[\left(\Box - m^2 \right) A_\rho \right] = 0$$

which, thanks to Poincaré lemma, leads to the equation:

$$\left(\Box - m^2 \right) A_\mu = \partial_\mu \varphi$$

where φ is an arbitrary scalar field. Now under a gauge transformation $A_\mu \to A_\mu + \partial_\mu \Lambda$ this equation becomes:

$$\left(\Box - m^2\right) A_\mu = \partial_\mu \left[\varphi - \left(\Box - m^2\right) \Lambda\right]$$

and so with a proper choice of Λ (i.e. a gauge choice) we can make the RHS vanish and we obtain that A_μ satisfies the Klein-Gordon equation too:

$$\left(\Box - m^2\right) A_\mu = 0\,. \tag{2.11}$$

This choice fixes the gauge up to a Λ' satisfying $\left(\Box - m^2\right) \Lambda' = 0$ that can be used to impose also $\partial_\mu A^\mu = 0$. So in this gauge the solutions to the equations of motion are exactly the same as (2.7):

$$A_\mu(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3p\, a_\mu(p)\, e^{ip_\nu x^\nu}\, \delta\left(m^2 + p_\nu\, p^\nu\right) \tag{2.12}$$

where the reality condition on $A_\mu(x)$ implies $a_\mu(-p) = (a_\mu)^*(p)$. (2.12) describes two propagating degrees of freedom but if we visualize A_μ in the second term of (2.9) and use the above gauge fixing conditions, we get an important relation:

$$m\, F_\rho + \epsilon_{\rho\nu\mu}\, \partial^\nu (\epsilon^{\mu\alpha\beta}\, \partial_\alpha A_\beta) = 0 \quad \Longleftrightarrow \quad m\, F_\rho - \cancel{\partial^\nu \partial_\rho A_\nu} + \Box A_\rho = 0 \quad \Longleftrightarrow$$
$$F_\rho = -m\, A_\rho \tag{2.13}$$

which reduces the number of independent propagating modes in A_μ to one.
Now let us study properties of conserved quantities in this theory. Computations bringing to the stress energy tensor of the theory are done in detail in Appendix B; the result is that, on shell, it may be brought to the same form as in the free electrodynamics i.e.:

$$T^{\mu\nu} = \frac{1}{4}\, \eta^{\mu\nu}\, F^{\alpha\beta}\, F_{\alpha\beta} + F^{\mu\alpha}\, F_\alpha{}^\nu$$

that in terms of the dual field can be rewritten as:

$$T^{\mu\nu} = \frac{1}{2}\, \eta^{\mu\nu}\, F^\alpha\, F_\alpha - F^\mu\, F^\nu\,. \tag{2.14}$$

Moreover the field is no more spinless as we can see by computing the angular momentum tensor (see Appendix A for more detailed explaination of its origin). Note that in its expression one must use the canonical stress-energy tensor $T_c^{\mu\nu}$ that differs from (2.14) by a total derivative (the exact one computed in Appendix B). To compute the angular momentum the total variation of the potential is $\bar{\delta} A^\mu = \omega^{\mu\nu}\, A_\nu$, where $\omega^{\mu\nu}$ are the infinitesimal parameters of a Lorentz transfomation, and so the Noether's current is:

$$\begin{aligned}
J^\mu &= -T_c^{\mu\nu}\, \delta x_\nu + \frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu}\, \bar{\delta} A_\nu \\
&= -T_c^{\mu\nu}\, \omega_{\nu\rho}\, x^\rho + \frac{\partial}{\partial \partial_\mu A_\nu} \left(\frac{1}{2}\, F^\alpha\, F_\alpha + \frac{1}{2}\, m\, \epsilon^{\alpha\beta\sigma}\, A_\alpha\, \partial_\beta A_\sigma\right) \omega_{\nu\rho}\, A^\rho \\
&= \frac{1}{2}\, \omega_{\nu\rho}\, (x^\nu\, T_c^{\mu\rho} - x^\rho\, T_c^{\mu\nu}) + \omega_{\nu\rho}\, A^\rho \left(F_\alpha\, \epsilon^{\alpha\beta\sigma}\, \delta^\mu{}_\beta\, \delta^\nu{}_\sigma + \frac{1}{2}\, m\, \epsilon^{\alpha\beta\sigma}\, A_\alpha\, \delta^\mu{}_\beta\, \delta^\nu{}_\sigma\right) \\
&= \frac{1}{2}\, \omega_{\nu\rho}\, (x^\nu\, T_c^{\mu\rho} - x^\rho\, T_c^{\mu\nu}) + \omega_{\nu\rho}\, \epsilon^{\alpha\mu\nu}\, A^\rho \left(F_\alpha + \frac{1}{2}\, m\, A_\alpha\right) \\
&= \frac{1}{2}\, \omega_{\nu\rho} \left[x^\nu\, T_c^{\mu\rho} - x^\rho\, T_c^{\mu\nu} + (\epsilon^{\alpha\mu\nu}\, A^\rho - \epsilon^{\alpha\mu\rho}\, A^\nu) \left(F_\alpha + \frac{1}{2}\, m\, A_\alpha\right)\right]
\end{aligned}$$

$$\Longrightarrow \qquad \mathcal{J}^{\mu\nu\rho} = x^\nu\, T_c^{\mu\rho} - x^\rho\, T_c^{\mu\nu} + (\epsilon^{\alpha\mu\nu}\, A^\rho - \epsilon^{\alpha\mu\rho}\, A^\nu) \left(F_\alpha + \frac{1}{2}\, m\, A_\alpha \right)$$

$$\Longrightarrow \qquad J^{\mu\nu} = \int d^2x \left[x^\mu\, T_c^{0\nu} - x^\nu\, T_c^{0\mu} + \left(\epsilon^{\alpha 0\mu}\, A^\nu - \epsilon^{\alpha 0\nu}\, A^\mu\right) \left(F_\alpha + \frac{1}{2}\, m\, A_\alpha \right) \right] .$$

The non-orbital term of $J^{\mu\nu}$ does not vanish and this means that the field A_μ carries a spin. A way that brings us to the value of this spin is to consider the non-orbital term of $J^{\mu\nu}$, let us call it $S^{\mu\nu}$, and to analyze how it acts on A_μ as an operator. In particular the quantity

$$\begin{aligned} S \equiv \frac{1}{2}\, \epsilon_{ij}\, S^{ij} &= \frac{1}{2}\, \epsilon_{ij} \int d^2x \left(\epsilon^{0ik}\, A^j - \epsilon^{0jk}\, A^i \right) \left(F_k + \frac{1}{2}\, m\, A_k \right) \\ &= \int d^2x\, A^k \left(F_k + \frac{1}{2}\, m\, A_k \right) \end{aligned}$$

should act as the generator of rotations in the plane, and we should figure out the spin value by understanding how it acts on the field A_μ. Firstly we have to compute the canonical momenta conjugate to A_μ:

$$\Pi^\mu \equiv \frac{\partial \mathcal{L}_{MCS}}{\partial \partial_0 A_\mu} = F_\nu\, \epsilon^{\nu\alpha\beta}\, \delta^0{}_\alpha\, \delta^\mu{}_\beta + \frac{1}{2}\, m\, \epsilon^{\nu\alpha\beta}\, A_\nu\, \delta^0{}_\alpha\, \delta^\mu{}_\beta = \epsilon^{\nu 0\mu} \left(F_\nu + \frac{1}{2}\, m\, A_\nu \right)$$

$$\Longrightarrow \qquad \Pi^0 = 0 \qquad\qquad \Pi^i = \epsilon^{ij} \left(F_j + \frac{1}{2}\, m\, A_j \right) .$$

Π^μ and A_ν satisfy the canonical Poisson bracket relations:

$$[A_\mu(t, \vec{x}),\, \Pi^\nu(t, \vec{y})]_P = \delta^\nu{}_\mu\, \delta(\vec{x} - \vec{y}) \qquad \forall t \in \mathbb{R} .$$

We can now write S in terms of Π^i:

$$S = \int d^2x\, A^k\, \epsilon_{ik}\, \Pi^i .$$

We are finally ready to compute the action of S on A_μ via Poisson bracket:

$$\begin{aligned} \left[S,\, A^j(t, \vec{y})\right]_P &= \epsilon_{ik} \int d^2x\, A^k(t, \vec{x}) \left[\Pi^i(t, \vec{x}),\, A^j(t, \vec{y})\right]_P = -\epsilon_{jk} \int d^2x\, A^k(t, \vec{x})\, \delta(\vec{x} - \vec{y}) \\ &= \epsilon_{kj}\, A^k(t, \vec{y}) . \end{aligned}$$

This is the expression for the usual infinitesimal rotation of a vector in $\mathbb{R}^2$ per unit of (small) angle. Hence the spin of the field is 1.

A more formal and somehow easier way to compute its value, is achieved through the use of group theory tools. We know that the dynamic field of the theory, both in coordinate space and in momenta space, is associated with a representation (reducible in general) of the (2+ 1)-dimensional Poincaré group. As we discussed in Section 1.3 every such representation is characterized by a particular eigenvalue of each of the two Casimir operators of the group. Firstly let us search for the eigenvalues of the generator P^μ, whose realization in the vector representation is $-i\, \partial^\mu$ as we said. Considering the single plane wave that makes up (2.12), it acts on the field A_μ as:

$$P^\mu\, (A_\nu) = -i\, \partial^\mu \left(a_\mu(p)\, e^{ip^\alpha x_\alpha} \right) = p^\mu\, A_\nu .$$

Hence the momentum p^μ carried by the plane wave is an eigenvalue of P^μ. So, as a consequence of Klein-Gordon equation (2.11), the eigenvalue of $P_\mu P^\mu$ associated with the single plane wave field is simply $p_\mu p^\mu = -m^2$.
The fact that W is a pseudoscalar allows us to evaluate it in a clever frame. In particular, if we choose the rest-frame, the eigenvalue of P^μ is:

$$p^\mu = (|m|,\, 0,\, 0)$$

where we take the module of m in order to have a positive energy. Correspondingly the Pauli-Lubanski operator becomes:

$$W = -\frac{1}{2}\, \epsilon_{ij}\, p^0\, J^{ij} = -|m|\, J$$

where $J \equiv \frac{1}{2}\, \epsilon_{ij}\, J^{ij}$ is clearly identified with the generator of rotations on the plane i.e. the generator of SO(2). Now we have to find the eigenvalues of the operator J when it acts on an element A_μ of the representation of interest. Clearly the exponential factor of the plane wave solution is not affected by a rotation since it is a scalar. So we are interested only in the trasformation law of $a_\mu(p) = a_\mu$[6], and in particular in that of its spatial components because the residual gauge fixing condition $\partial_\mu A^\mu = 0$ implies that, in momentum space, $p_\mu\, a^\mu = 0$ which in the rest-frame is satisfied if and only if $a^0 = 0$. This tells us that the a^0 component is non-physical and thus we can restrict to the study of the two dimensional representation of SO(2) where a_i live. Under a spatial rotation by an angle θ, a_i transform in the following way:

$$a'_i = R(\theta)_i^{\ j}\, a_j \qquad\qquad \text{where} \quad R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} .$$

By definition the generator J is the operator such that an infinitesimal rotation can be written as the operator $\mathbb{I} - i\,\delta\theta\, J$. With this information, we can easily obtain the form that J acquires in our representation by expanding $R(\theta)$ in Taylor series at the first order:

$$R(d\theta) = \begin{pmatrix} 1 & -\delta\theta \\ \delta\theta & 1 \end{pmatrix} \quad \Longrightarrow \quad J = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} .$$

It is now easy to compute the eigenvalues of J by diagonalizing this matrix. It turns out they are ± 1, each one corresponding to a different physical situation. In order to check which eigenvalue suits to our system, we should write the condition (2.13) in momentum space in the rest frame:

$$a_\mu(p) = \frac{i}{m}\, \epsilon_{\mu\nu\rho}\, p^\nu\, a^\rho(p) \quad \Longleftrightarrow \quad a_i = \frac{i}{m}\, \epsilon_{i0j}\, p^0\, a^j = i\, \frac{|m|}{m}\, \epsilon_{ij}\, a^j .$$

Now if $m > 0$ the condition tells us that $a_2 = -i\, a_1$, which corresponds to the eigenvalue -1 of J, while if $m < 0$ then $a_2 = i\, a_1$ and the corresponding eigenvalue of J is $+1$. So the only possible eigenvalue of W is m. From literature it is known that in 3-dimensional space the Pauli-Lubanski operator has eigenvalues of the form ms where s is the intrinsic spin of the representation; this means that we found that the absolute value of the spin of A_μ is $s = 1$. Its sign is either $+1$ or -1 depending on whether the parameter m is positive or negative. Here is where the parity non-invariance comes: it forbids the field to carry a ± 1 spin at the same time.

[6] We can forget about the dependence of a_μ on p^μ since we have fixed it with the choice of the rest-frame.

2.2.3 Dual massive electrodynamics

As in the case of the massless theory, there exist a dual way to handle the theory defined by (2.8), that is achieved thanks to Chern-Simons term too, as showed in [3]. Let us consider the following lagrangian density for a dynamic variable f_μ:

$$\mathcal{L}_{MCSD} = -\frac{1}{2}\, f^\mu\, f_\mu - \frac{1}{2m}\, \epsilon^{\mu\nu\rho}\, f_\mu\, \partial_\nu f_\rho\,. \qquad (2.15)$$

We will refer to (2.15) as topologically massive theory, because of the metric independence of the massive term of the lagrangian (that is the Chern-Simons term). Deriving equations of motion is straightforward as in the Section 2.2.2 and leads to:

$$f_\rho + \frac{1}{m}\, \epsilon_{\rho\nu\mu}\, \partial^\nu f^\mu = 0\,. \qquad (2.16)$$

By following a procedure very similar to the one used in Section 2.2.2, we can derive the equations for f_μ in a form that is the same as (2.10) for F_μ:

$$\left(\Box - m^2\right) f^\mu = 0\,, \qquad (2.17)$$

In order to check the equivalence between the two theories, we can look at their number of degrees of freedom. From the (2.16) one can easily see that f_μ should obey the constraint $\partial_\mu f^\mu = 0$ which removes one degree of freedom. If we consider (2.16) in the momentum space, we have:

$$f_\rho - \frac{i}{m}\, \epsilon_{\rho\nu\mu}\, p^\nu\, f^\mu = 0$$

and the constraint becomes $p_\mu\, f^\mu = 0$. Now let us go to the rest-frame, so that $p_0 = |m|$ and $p_i = 0$. Then $p_0\, f^0 = 0$ and so $f^0 = 0$. Thus the equations of motion become:

$$f_j - \frac{i}{m}\, \epsilon_{j0i}\, p^0\, f^i = 0 \qquad \Longrightarrow \qquad f_j + i\,\frac{|m|}{m}\, \epsilon_{ij}\, f^i = 0$$

which means that the spatial components of f^μ are interrelated. Therefore in this theory there is one degree of freedom, just as in the Maxwell-Chern-Simons theory. Despite that, gauge invariance is lost because of the first term in (2.15).

The stress energy tensor of this theory, modulo the equations of motion, is given by (see Appendix B for complete derivation):

$$T^{\mu\nu} = \frac{1}{2}\, \eta^{\mu\nu}\, f^\alpha\, f_\alpha - f^\mu\, f^\nu \qquad (2.18)$$

which coincides with (2.14) if the equality $f_\mu = F_\mu$ holds.

The field f_μ is formally equivalent to the field A_μ of the Maxwell-Chern-Simons theory (once we fix its gauge freedom in the previously discussed way) since both of them satisfy the same equations of motion and the same constraints. Thus we can conclude, without performing an almost identical analysis, that also f_μ carries a spin 1 or -1.

2.2.4 The master lagrangian

All these similarities between the two massive theories just described are actually a consequence of a more general structure. In fact both of them can be derived from a single lagrangian density:

$$\mathcal{L}_T = -\frac{1}{2}\, f^\mu\, f_\mu + f_\mu\, F^\mu + \frac{1}{2}\, m\, F^\mu\, A_\mu\,. \qquad (2.19)$$

If we vary the action with respect to f_μ, we trivially obtain the identification $f_\mu = F_\mu$ as an equation of motion and, substituing this into (2.19), we get the lagrangian (2.8) of the Maxwell-Chern-Simons theory. Instead let us derive the equations of motion of A_μ:

$$\begin{aligned}\delta_{A_\mu} S_T &= \int_{\Sigma_1}^{\Sigma_2} d^3x \left[\left(f_\mu + \frac{1}{2}\, m\, A_\mu \right) \epsilon^{\mu\nu\rho}\, \partial_\nu \delta A_\rho + \frac{1}{2}\, m\, F^\mu\, \delta A_\mu \right] \\ &= \int_{\Sigma_1}^{\Sigma_2} d^3x\, \delta A_\rho \left[-\epsilon^{\mu\nu\rho}\, \partial_\nu \left(f_\mu + \frac{1}{2}\, m\, A_\mu \right) + \frac{1}{2}\, m\, F^\rho \right] + \\ &\qquad + \int_{\Sigma_1}^{\Sigma_2} d^3x\, \partial_\nu \left[\epsilon^{\mu\nu\rho}\, \delta A_\rho \left(f_\mu + \frac{1}{2}\, m\, A_\mu \right) \right]\end{aligned}$$

so that, with the usual boundary conditions, we have:

$$m\, F^\rho - \epsilon^{\mu\nu\rho}\, \partial_\nu f_\mu = 0\,. \tag{2.20}$$

If we use this equation in the last term of (2.19), it becomes:

$$\frac{1}{2}\, \epsilon^{\alpha\beta\mu}\, \partial_\beta f_\alpha\, A_\mu = \partial_\beta \left(\frac{1}{2}\, \epsilon^{\alpha\beta\mu}\, f_\alpha\, A_\mu \right) - \frac{1}{2} f_\alpha\, F^\alpha\,.$$

By substituing this into the total lagrangian and then using again (2.20), we get (up to a total derivative):

$$\mathcal{L}_T = -\frac{1}{2}\, f^\mu\, f_\mu + \frac{1}{2}\, f_\mu\, F^\mu = -\frac{1}{2}\, f^\mu\, f_\mu + \frac{1}{2m}\, f_\mu\, \epsilon^{\alpha\beta\mu}\, \partial_\beta f_\alpha = \mathcal{L}_{MCSD}$$

This proves that $\mathcal{L}_T$ produces also (2.15). Therefore the two theories, the Maxwell-Chern-Simons and the topologically massive one, are dual to each other.

CHAPTER 3

Non-abelian Chern-Simons theory

So far we have considered the abelian Chern-Simons theory where the group of the gauge transformations is U(1). Now we will study its non-abelian generalization. Firstly we will briefly introduce the general construction of a gauge theory and describe the common example given by the Yang-Mills lagrangian. Then we introduce the non abelian Chern-Simons theory and focus ourselves on the analysis of its gauge invariance.

3.1 Local symmetries and gauge fields

3.1.1 The covariant derivative

The approach that we are going to follow starts from some lagrangian density which, in most cases, is known to describe some kind of matter in its kinematical behaviour. Usually this lagrangian is invariant under the action of a continuous symmetry group on the dynamic field. For instance let us consider a spinorial field ψ described by:

$$\mathcal{L}_F = \frac{1}{2}\,\bar{\psi}\,\gamma^\mu\,\partial_\mu\psi\,. \tag{3.1}$$

Obvioulsy a constant element of the group U(1), that is associated with a complex phase factor, acts on ψ leaving (3.1) unchanged:

$$\begin{cases}\psi \longrightarrow \psi' = e^{i\alpha}\,\psi \\ \psi^\dagger \longrightarrow \psi'^\dagger = e^{-i\alpha}\,\psi^\dagger\end{cases} \qquad \Longrightarrow \qquad \mathcal{L}_F \longrightarrow \mathcal{L}'_F = \mathcal{L}_F \qquad \forall\alpha\in\mathbb{R}\,.$$

So U(1) is a rigid symmetry group for the lagrangian density (3.1).
In place of global transformations we can consider gauge transformations, whose group elements that act on the fields are functions of the point of spacetime. The problem is that in general the lagrangian density (3.1) is not invariant under such transformations. In our example we should take $\alpha = \alpha(x)$, so that the transformations are still in U(1) but they are different for different x^μ, which implies:

$$\mathcal{L}_F \longrightarrow \mathcal{L}'_F = \frac{1}{2}\,e^{-i\alpha(x)}\,\bar{\psi}\,\gamma^\mu\partial_\mu\left(e^{i\alpha(x)}\,\psi\right) = \mathcal{L}_F + \frac{i}{2}\,\partial_\mu\alpha(x)\,\bar{\psi}\,\gamma^\mu\,\psi \neq \mathcal{L}_F\,.$$

In order to build an invariant lagrangian density we have to introduce a new operator D_μ called covariant derivative such that:

$$D_\mu\psi \longrightarrow e^{i\alpha(x)}\,D_\mu\psi \qquad \Longleftrightarrow \qquad D_\mu \longrightarrow D'_\mu = e^{i\alpha(x)}\,D_\mu\,e^{-i\alpha(x)}\,.$$

Clearly (3.1) becomes invariant under gauge transformations if we replace ∂_μ with D_μ. The covariant derivative can be constructed with the use of a field A_μ writing $D_\mu = \partial_\mu + ie\,A_\mu$, where e is a non-zero real constant. From the above expression for the transformation law of D_μ we can derive that of A_μ, finding:

$$A_\mu \longrightarrow A'_\mu = A_\mu + e^{-1}\partial_\mu\alpha\,.$$

Let us now consider a more general case in which the spinorial field carries N internal indeces r:

$$\mathcal{L}_F = \frac{1}{2}\,\bar\Psi\,\gamma^\mu\,\partial_\mu\Psi = \frac{1}{2}\bar\Psi^r\,\gamma^\mu\,\partial_\mu\Psi_r \tag{3.2}$$

where Ψ_r is a spacetime spinor which transforms under the fundamental representation of $SO^+(1{,}2)$ for $r = 1, ..., N$. Let G be its gauge group whose generic element $g \in G$ is identified with a unitary $N \times N$ matrix, which means that the gauge group of this theory is $U(N)$ or one of its subgroups. When considering spacetime dependent g, the core of the local invariance problem is that in general $(\partial_\mu\Psi)' = \partial_\mu(g\,\Psi) \neq g\,(\partial_\mu\Psi)$. In order to recover the gauge invariance we introduce the covariant derivative, which is defined by requiring the following identity be held:

$$(D_\mu\Psi)' = g\,(D_\mu\Psi) \qquad \Longleftrightarrow \qquad D'_\mu = g\,D_\mu\,g^{-1}\,. \tag{3.3}$$

Clearly if (3.3) holds, the lagrangian density:

$$\mathcal{L}_F = \frac{1}{2}\,\bar\Psi\,\gamma^\mu\,D_\mu\Psi \tag{3.4}$$

is invariant also under local gauge transformations. Now we have to figure out what form takes this D_μ; clearly it has to be a $N \times N$ matrix and since it has to deal with local behaviour of the gauge group it is natural to write it in terms of the "local elements" of the group, which are the generators of the corresponding Lie algebra $\mathfrak{g}$[7]. Let T^a be the N^2 generators of $\mathfrak{u}(N)$, whose commutation relations and normalization are given by:

$$[T^a,\,T^b] = i f^{abc}\,T^c \qquad\qquad \mathrm{Tr}(T^a\,T^b) = \frac{1}{2}\,\delta^{ab} \tag{3.5}$$

where f^{abc} are real totally antisymmetric coefficients called structure constants (note that if the gauge group is commutative, as in the case of $U(1)$, $f^{abc} = 0$). Let also A^a_μ be a set of N^2 covariant vector fields. In analogy to the $U(1)$ case, we define:

$$D_\mu = \partial_\mu + ie\,A_\mu \qquad\qquad \text{where} \quad A_\mu \equiv A^a_\mu\,T^a\,, \quad e \in \mathbb{R}\,. \tag{3.6}$$

Let us check whether the (3.3) can be satisfied or not:

$$(D_\mu\Psi)' = g\,(D_\mu\Psi) \qquad \Longleftrightarrow \qquad (\partial_\mu + ie\,A'_\mu)(g\,\Psi) = g(\partial_\mu + ie\,A_\mu)\Psi$$

[7]This concept could also be formulated by means of differential geometry language: Lie groups are differential manifolds whose tangent space over the identity element is identified with their Lie algebras. Then the covariant derivative descends from the connection of the manifold which in local coordinates is written as a linear combination of the base of the tangent space i.e. of the Lie algebra generators.

$$\Longleftrightarrow \qquad (\partial_\mu g)\Psi + \cancel{g\,\partial_\mu \Psi} + ie\, A'_\mu\, g\, \Psi = \cancel{g\,\partial_\mu \Psi} + ie\, g\, A_\mu\, \Psi$$

$$\Longleftrightarrow \qquad A'_\mu\, g\, \Psi = g\, A_\mu\, \Psi + ie^{-1}\,(\partial_\mu g)\Psi \qquad \Longleftrightarrow \qquad A'_\mu = g\, A_\mu\, g^{-1} + ie^{-1}(\partial_\mu g)\, g^{-1}\,.$$

So if under a gauge transformation the fields A_μ varies as above, the covariant derivative has the desirable properties. Moreover what we just found tells us that if we consider a constant element of the gauge group the second term vanishes and hence the transformation law of A_μ is that of an element of the adjoint representation of $\mathrm{U}(N)$[8].
The physical interpretation of this construction is that the local invariance of the theory is achieved thanks to the introduction of the fields A_μ, called gauge fields, which mediate some kind of interaction between the matter fields. The strength of the interaction is characterized by the constant e which is the coupling constant of the interacting theory. A concrete example is that of QED: the fermionic lagrangian describes the physics of electrons on their own but when requiring the local invariance under the gauge group U(1) the electromagnetic potential field A_μ appears and every electron starts to interact with each other through it.

3.1.2 Yang-Mills theory

So far we have considered gauge fields as external fields. However a physical theory should also describe their propagation. This result is achieved by including a kinetic term for A_μ in the lagrangian density, and the Yang-Mills lagrangian features a common choice for it, ensuring the gauge invariance.
Firstly let us give a general definition of the field strength tensor (that now is an operator):

$$F_{\mu\nu} \equiv -ie^{-1}\,[D_\mu,\, D_\nu] \tag{3.7}$$

which obviously transforms covariantly under the gauge transformations, since D_μ does:

$$F_{\mu\nu} \longrightarrow F'_{\mu\nu} = -ie^{-1}\left[g\, D_\mu\, g^{-1},\, g\, D_\nu\, g^{-1}\right] = -ie^{-1}\, g\,[D_\mu,\, D_\nu]\, g^{-1} = g\, F_{\mu\nu}\, g^{-1}\,.$$

Also $F_{\mu\nu}$ is antisymmetric in μ and ν. We can write it in another form:

$$F_{\mu\nu} = -ie^{-1}\,[\partial_\mu + ie\, A_\mu,\, \partial_\nu + ie\, A_\nu] = [\partial_\mu,\, A_\nu] - [\partial_\nu,\, A_\mu] + ie\,[A_\mu,\, A_\nu]\ .$$

We can simplify the first two commutators by observing how they act on a generic matter field:

$$[\partial_\mu,\, A_\nu]\,\Psi = \partial_\mu(A_\nu\,\Psi) - A_\nu\,\partial_\mu\Psi = (\partial_\mu A_\nu)\Psi + \cancel{A_\nu\partial_\mu\Psi} - \cancel{A_\nu\partial_\mu\Psi}$$

$$\Longrightarrow\; F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie\,[A_\mu,\, A_\nu]\ . \tag{3.8}$$

It is the presence of the commutator between field operators in (3.8) that motivates referring to Yang-mills theory as a non-abelian theory. Also this expression tells us that $F_{\mu\nu}$ is an element of the adjoint representation of G too. Recalling the relations (3.5), we can also expand the expression (3.8) and write it in terms of the vector fields A^a_μ:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - e\, f^{bca}\, A^b_\mu\, A^c_\nu\,.$$

[8] The adjoint representation of G is definend as that whose elements of the vector space are those of the Lie algebra associated with G and the homomorphism is given by $X' = g\, X\, g^{-1}$ with $g \in G$ and $X \in \mathfrak{g}$. The dimension of the adjoint representation is the same of G

Note that $F_{\mu\nu}$ satisfies the Bianchi identity as a consequence of the Jacobi identity[9]:

$$D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} + D_\rho F_{\mu\nu} = 0\,. \tag{3.9}$$

where the action of the covariant derivative on $F_{\mu\nu}$ is expressed through the adjoint action of A_μ on $F_{\mu\nu}$ (accordingly with the fact that both of them are elements of the adjoint representation):

$$D_\mu F_{\nu\rho} = \partial_\mu F_{\nu\rho} + ie\,[A_\mu,\, F_{\nu\rho}]\ . \tag{3.10}$$

Having introduced the generalized field strength tensor, we are now ready to write down the Yang-Mills lagrangian density:

$$\mathcal{L}_{YM} = -\frac{1}{2}\,\mathrm{Tr}\,(F^{\mu\nu}\,F_{\mu\nu})\,. \tag{3.11}$$

The gauge invariance of this lagrangian density is trivial due to trace properties and the transformation law of $F_{\mu\nu}$. Moreover thanks to the trace identity (3.5), $\mathcal{L}_{YM}$ can also be written as:

$$\mathcal{L}_{YM} = -\frac{1}{2}\,F^{a\mu\nu}\,F^b_{\mu\nu}\,\mathrm{Tr}(T^a\,T^b) = -\frac{1}{4}\,F^{a\mu\nu}\,F^a_{\mu\nu}\,.$$

In order to derive the equations of motion it is convenient to work with the matrix expression (3.11). The variation of the action is:

$$\begin{aligned}
\delta S_{YM} &= -\int_{\Sigma_1}^{\Sigma_2} d^3x\,\mathrm{Tr}\,(F^{\mu\nu}\,\delta F_{\mu\nu})\\
&= -\int_{\Sigma_1}^{\Sigma_2} d^3x\,\mathrm{Tr}\,[F^{\mu\nu}\,(\partial_\mu\delta A_\nu + ie\,\delta A_\mu\,A_\nu + ie\,A_\mu\,\delta A_\nu - (\mu\leftrightarrow\nu))]\\
&= -2\int_{\Sigma_1}^{\Sigma_2} d^3x\,[\mathrm{Tr}\,(F^{\mu\nu}\,\partial_\mu\delta A_\nu) + \mathrm{Tr}\,(ie\,F^{\nu\mu}\,\delta A_\nu\,A_\mu) + \mathrm{Tr}\,(ie\,F^{\mu\nu}\,A_\mu\,\delta A_\nu)]\\
&= -2\int_{\Sigma_1}^{\Sigma 2} d^3x\,[\mathrm{Tr}\,(F^{\mu\nu}\,\partial_\mu\delta A_\nu) - \mathrm{Tr}\,(ie\,A_\mu\,F^{\mu\nu}\,\delta A_\nu) + \mathrm{Tr}\,(ie\,F^{\mu\nu}A_\mu\,\delta A_\nu)]\\
&= 2\int_{\Sigma_1}^{\Sigma 2} d^3x\,[\mathrm{Tr}\,(\partial_\mu F^{\mu\nu}\,\delta A_\nu) + \mathrm{Tr}\,(ie\,[A_\mu,\,F^{\mu\nu}]\,\delta A_\nu)]\\
&= 2\int_{\Sigma_1}^{\Sigma_2} d^3x\ \mathrm{Tr}\,[(\partial_\mu\,F^{\mu\nu} + ie\,[A_\mu,\,F^{\mu\nu}])\,\delta A_\nu]
\end{aligned}$$

where we have used the antisymmetry of $F^{\mu\nu}$, the linearity and the cyclic permutation invariance of the trace, and boundary conditions such that the surface integral vanishes when integrating by parts. Clearly δS_{YM} is zero for every arbitrary matrix δA_ν if and only if the following equations holds:

$$\partial_\mu\,F^{\mu\nu} + ie\,[A_\mu,\,F^{\mu\nu}] = 0 \quad\Longleftrightarrow\quad D_\mu F^{\mu\nu} = 0\,. \tag{3.12}$$

The (3.12) are exactly the equations of motion for a generic pure Yang-Mills theory i.e. they describe the behaviour of the gauge field decoupled from any interaction with matter.

[9]For any three elements A, B, C of a Lie algebra, the identity $[A,[B,\,C]]+[B,[C,\,A]]+[C,[A,\,B]]=0$ holds; in our case we consider three covariant derivatives.

3.2 Non-abelian Chern-Simons theory

Just as we discussed for the Yang-Mills lagrangian, we can introduce the non-abelian Chern-Simons theory. In view of the form of the covariant derivative, and so of the gauge fields introduced in (3.6), the non-abelian Chern-Simons lagrangian is:

$$\mathcal{L}_{CS} = \kappa\, \epsilon^{\mu\nu\rho}\, \mathrm{Tr}\left(A_\mu\, \partial_\nu A_\rho + \frac{2}{3}\, ie\, A_\mu\, A_\nu\, A_\rho \right) . \tag{3.13}$$

Clearly this reduces to the previously described theory (2.1) if the gauge fields are commutative, since the extra term vanishes because of the contraction with the totally antisymmetryc Levi-Civita tensor. By exploiting the trace properties and cyclic permutation invariance, we bring the variation of the action to the following form:

$$\begin{aligned}
\delta S_{CS} &= \int_{\Sigma_1}^{\Sigma_2} d^3x\, \kappa\, \epsilon^{\mu\nu\rho}\, \mathrm{Tr}\left[\delta A_\mu\, \partial_\nu A_\rho + A_\mu\, \partial_\nu \delta A_\rho + \frac{2}{3}\, ie\, (\delta A_\mu\, A_\nu\, A_\rho + A_\mu\, \delta A_\nu\, A_\rho + A_\mu\, A_\nu\, \delta A_\rho)\right] \\
&= \int_{\Sigma_1}^{\Sigma_2} d^3x\, \kappa\, \epsilon^{\mu\nu\rho}\, \mathrm{Tr}\left[\partial_\nu A_\rho\, \delta A_\mu - \partial_\nu A_\mu\, \delta A_\rho + \frac{2}{3}\, ie\, (A_\nu\, A_\rho\, \delta A_\mu + A_\rho\, A_\mu\, \delta A_\nu + A_\mu\, A_\nu\, \delta A_\rho)\right] \\
&= \int_{\Sigma_1}^{\Sigma_2} d^3x\, \kappa\, \mathrm{Tr}\left[\left(\epsilon^{\mu\nu\rho}\, \partial_\nu A_\rho - \epsilon^{\rho\nu\mu}\, \partial_\nu A_\rho + \frac{2}{3}\, ie\, (\epsilon^{\mu\nu\rho}\, A_\nu\, A_\rho + \epsilon^{\nu\mu\rho}\, A_\rho\, A_\nu + \epsilon^{\rho\nu\mu}\, A_\rho\, A_\nu)\right) \delta A_\mu\right] \\
&= \int_{\Sigma_1}^{\Sigma_2} d^3x\, \kappa\, \mathrm{Tr}\left[(2\, \epsilon^{\mu\nu\rho}\, \partial_\nu A_\rho + 2\, ie\, \epsilon^{\mu\nu\rho}\, A_\nu\, A_\rho)\, \delta A_\mu\right]
\end{aligned}$$

where we have obviously taken boundary conditions such that the surface integral vanishes. δS_{CS} vanishes for an arbitrary matrix δA_μ if and only if:

$$\begin{aligned}
2\kappa\, \epsilon^{\mu\nu\rho}\, (\partial_\nu A_\rho + ie\, A_\nu\, A_\rho) = 0 \quad &\Longleftrightarrow \quad \kappa\, \epsilon^{\mu\nu\rho}\, (\partial_\nu A_\rho - \partial_\rho A_\nu + ie\, A_\nu A_\rho - ie\, A_\rho A_\nu) = 0 \\
&\Longleftrightarrow \quad \kappa\, \epsilon^{\mu\nu\rho}\, F_{\nu\rho} = 0 \,.
\end{aligned}$$

By contracting both members with $\epsilon_{\mu\alpha\beta}$ and exploiting the antisymmetry of $F_{\nu\rho}$, we get:

$$F_{\alpha\beta} = 0 \tag{3.14}$$

which means that the solutions for A^a_μ are pure gauge as in the free abelian theory.
The gauge invariance requirement of the action brings us to very important observations. In Appendix C we go through all the necessary computations, and the result is that the variation of the lagrangian density under a generic gauge transformation is given by:

$$\delta\mathcal{L}_{CS} = ie^{-1}\kappa\, \epsilon^{\mu\nu\rho}\, \partial_\nu\, \mathrm{Tr}\left[(\partial_\mu g)\, A_\rho\, g^{-1}\right] - \frac{e^{-2}\kappa}{3}\, \epsilon^{\mu\nu\rho}\, \mathrm{Tr}\left[(\partial_\mu g)\, g^{-1}\, (\partial_\nu g)\, g^{-1}\, (\partial_\rho g)\, g^{-1}\right] .$$

The first term is a total derivative and hence, when integrated to get the action, vanishes for properly chosen boundary conditions on A_μ; instead the second one is not so easy to get rid of. The variation of the action under a gauge transformation can be written as:

$$\delta S_{CS} = -8\pi^2\, e^{-2}\kappa \left(\frac{1}{24\pi^2} \int_{\Sigma_1}^{\Sigma_2} d^3x\, \epsilon^{\mu\nu\rho}\, \mathrm{Tr}\left[(\partial_\mu g)\, g^{-1}\, (\partial_\nu g)\, g^{-1}\, (\partial_\rho g)\, g^{-1}\right] \right) . \tag{3.15}$$

Let us denote the expression in round brackets in (3.15) with $w(g)$. In order to figure out what this integral represents, we have to introduce some mathematical concepts and tools.

Definition 1. *Let X and Y be two topological manifolds of dimension m and $f\colon X \to Y$ a continuous map. Then the degree of f is an integer number defined by the equality:*

$$\deg f \int_Y \omega = \int_X f^*\omega \tag{3.16}$$

where ω is any m-form and f^* denotes the pullback operator.

Intuitively the degree of a continuous map is an integer that counts how many times the domain of the map "wraps" around the codomain and thus labels the homotopy class to which the function belongs. In this sense it is a generalization of the winding number of a function.

Definition 2. *Let G be a compact Lie group. There exist a unique invariant measure $d\mu(g)$ on G satisfing the following properties:*

- $d\mu(g) = d\mu(f \cdot g) = d\mu(g \cdot f) \quad \forall g, f \in G$ *(left- and right-invariance)*
- $\int_G d\mu(g) = 1$ *(normalization).*

In the following we will denote it simply with $d\mu$.

Now we will show that $w(g)$ is the degree of the map $g(x)$ of spacetime into the group manifold G.
Firstly let us assume that the time interval $[t_1, t_2]$ is much bigger than the characteristic temporal scales of the interactions in the system, so that we can integrate the lagrangian density over the entire spacetime manifold, sending $t_1 \to -\infty$ and $t_2 \to +\infty$. Then it is convenient to restrict the integral in (3.15) to elements $g(x) \in \mathrm{SU}(2)\ \forall x$ (more generally considering an SU(2) subgroup of U(N)), because the group SU(2) is homeomorphic (i.e. topologically equivalent) to the sphere S^3 and also because, according to [2], interpreting g as a parametrization of the SU(2) manifold, the expression of $w(g)$ is exactly the integral of the pullback onto the spacetime of the Haar measure $d\mu$ of SU(2). Now the only problem of using (3.16) to compute $\deg g$ is that the function g as defined so far does not fit exactly into the definition of a continuous map between topological manifolds since $\mathbb{R}^3$ is non-compact while $\mathrm{SU}(2) \sim S^3$ is compact. To improve this, we can transform the integral in $w(g)$ in an integral over an Euclidean $\mathbb{R}^3$ by performing a Wick rotation and then compactifying it by idetifying all points at infinity.
Let us explain the procedure in more details. Consider the analytic continuation of the function $g\colon \mathbb{R}^3 \to \mathrm{SU}(2)$ onto $\mathbb{C} \times \mathbb{R}^2$ extending the first coordinate x^0 to a complex variable z^0 such that $\Re(z^0) = x^0$. Then let us consider the two closed paths of integration in the complex plane represented in figure 3.1. Clearly the integrals over these closed paths vanish because the integrand does not have any poles. Also we should restrict to the study of phenomena that are local in spacetime i.e. we require gauge transformation to be the identity $\mathbb{I}_G$ in the limit of $x \to \pm\infty$, which means that also the integrals over the circular arcs vanish when sending the radius to infinity. These considerations imply that the integral of the given function over $\mathbb{R}^3$ is exactly equal to the integral of its analytic continuation over $i\mathbb{R} \times \mathbb{R}^2$. The result of all these manipulations is that we can turn the integral in $w(g)$ over the Minkowski space into an integral over the Euclidean space by redefing x^0 as ix^0 and then changing the path of integration as described above.

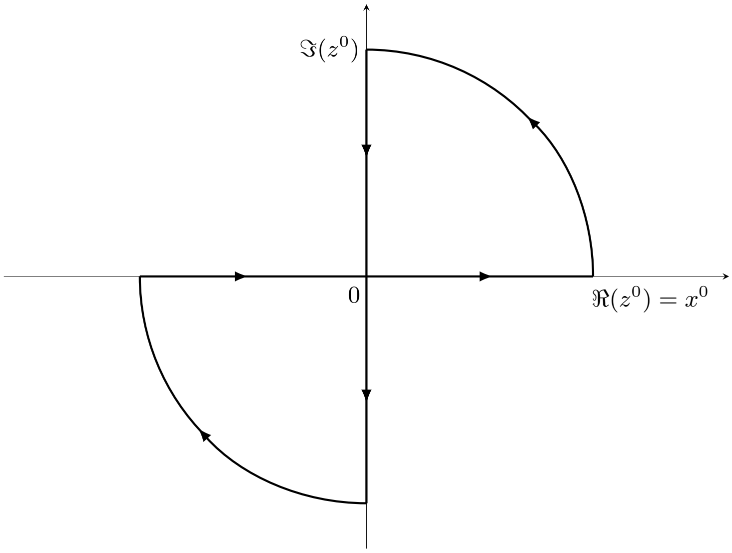


Figure 3.1: *Representation of a Wick rotation in the complex plane.*

Now we still have to ensure that the topology on the euclidean $\mathbb{R}^3$ is compatible with that on S^3, which easily descends from the requirement that $\lim_{x\to\pm\infty} g(x) = \mathbb{I}_G$. Indeed this operation is formally an Alexandroff compactification of $\mathbb{R}^3$, that is adding a point at its boundary or, equivalently, identifying all points at infinity, which by definition leads to a space that is homeomorphic to the sphere S^3.
So we found that g induces a map between two 3-spheres that satisfies the requirements for defining its degree:

$$\deg g \int_{\mathrm{SU}(2)} \omega = \int_{\mathbb{R}^3} g^*\omega$$

where ω is any 3-form defined on SU(2). If we pick the Haar measure of SU(2) as ω, the RHS is exactly $w(g)$ and the integral of LHS gives 1 by definition, so that $\deg g = w(g)$.
Summarizing, we proved that $w(g)$ is an integer that defines the homotopy class of the gauge transformation, an thus the variation of the action (3.15) becomes:

$$\delta S_{CS} = -8\pi^2\, e^{-2}\kappa\, w(g) \qquad\qquad w(g) \in \mathbb{N}\,.$$

Since $w(g)$ does not vanish unless the gauge transformation is continuously deformable to the identity $\mathbb{I}_G$ the non-abelian Chern-Simons theory cannot be gauge invariant, at least in the classical context.
However the gauge invariance can be recovered in the quantum theory. Using the path integral formulation the expectation value of a generic gauge invariant operator $\mathcal{O}$ is given by:

$$\langle\mathcal{O}\rangle = \frac{1}{Z}\int \mathcal{D}A_\mu\, \mathcal{O}[A_\mu]\, e^{iS[A_\mu]}$$

where Z is a normalization constant. It follows that $\langle\mathcal{O}\rangle$ changes under a gauge transformation as:

$$\langle\mathcal{O}\rangle \longrightarrow \langle\mathcal{O}\rangle' = e^{-8i\pi^2\, e^{-2}\kappa\, w(g)}\, \langle\mathcal{O}\rangle\,.$$

Obviously we want $\langle\mathcal{O}\rangle' = \langle\mathcal{O}\rangle$ and this can be achieved if the exponential is equal to 1 whatever the winding number of the transformation is. Hence, recalling that $w(g) \in \mathbb{N}$, the parameter κ should be restricted to:

$$- 8\pi^2 \, e^{-2}\kappa = 2m\pi \qquad \Longleftrightarrow \qquad \kappa = -\frac{m}{4\pi \, e^{-2}} \qquad\qquad m \in \mathbb{Z} \, . \tag{3.17}$$

So we conclude that the non-abelian quantum Chern-Simons theory is gauge invariant if and only if the parameter κ takes the discrete values expressed by (3.17). This condition on κ is called level quantization.

CHAPTER 4

Non-relativistic anyons

Topological properties of 2-dimensional spaces are radically different from those of higher dimensional spaces. One of the consequences is that in two dimensions there exist quantum particles with different properties from those of bosons and fermions. In particular these particles have a spin that is not quantized in integer or half-integer values and also they can satisfy any possible statistics; therefore they are called anyons. We will see that the anyonic behaviour of a system of particles emerges in a very natural and elegant way when they are coupled to the Chern-Simons vector field.

4.1 Fractional statistics: general aspects

We start by discussing general aspects of anyons, regardless of the way they are practically realized. The key of the topic is that in a 2-dimensional space the exchange of two identical particles can follow an infinite number of homotopically different paths and thus the argument of the invariance of the wavefunction when a certain exchange is performed twice, that would have led to bosons and fermions, is no more applicable. Instead in this case the phase factor acquired by the wavefunction can be arbitrary.
Let us see how it happens. Suppose we have a system made up by N identical hard-core particles that live in the euclidean $\mathbb{R}^d$, where d is a natural number. We will denote the configuration space with M_N^d. The dynamic of these particles is uniquely specified by the propagator:

$$\langle q', t' | q, t \rangle = \int_q^{q'} \mathcal{D}q \, e^{i \int_t^{t'} d\tau \, L(q(\tau), \dot{q}(\tau))} \qquad q, q' \in M_N^d \tag{4.1}$$

where L is intended to be the classical lagrangian of the system. Now let us suppose $q' = q$ i.e. consider paths that are loops in M_N^d. Such paths can be partitioned in homotopy equivalence classes, and the quotient space, together with a composition law given by the concatenation of loops, is the fundamental group $\pi_1(M_N^d)$. Also it is true that, given two fixed and not coincident endpoints, we can associate to every element of the fundamental group an element of the set of homotopy classes of paths connecting them. More formally, being $x \in M_N^d$ the base point used to compute the fundamental group, $\pi(q_1, q_2)$ the set of homotopy classes of paths going from q_1 to q_2, and $\gamma(q)$ an arbitrary chosen path going from x to q, there exist a bijection $f_{q_1q_2}\colon \pi_1(M_N^d) \to \pi(q_1, q_2)$ given by:

$$f_{q_1q_2}(\alpha) = [\gamma^{-1}(q_1)] \cdot \alpha \cdot [\gamma(q_2)] \qquad \forall \alpha \in \pi_1(M_N^d)\,.$$

This allows us to label (not uniquely because of the arbitrariety of the choice of γ) each homotopy class in $\pi(q_1, q_2)$ with an element of the fundamental group and so with a little abuse of language we will say that also open paths with fixed endpoints belong to a certain element of the fundamental group. In this sense we can then split the path integral (4.1) into a sum of path integrals $\langle q', t'|q, t\rangle_\alpha$, each of them over all possible paths in a single homotopy class:

$$\langle q', t'|q, t\rangle = \sum_{\alpha\in\pi_1(M_N^d)} \langle q', t'|q, t\rangle_\alpha = \sum_{\alpha\in\pi_1(M_N^d)} \int_q^{q'} \mathcal{D}q_\alpha\, e^{i\int_t^{t'} d\tau\, L(q_\alpha(\tau),\, \dot{q}_\alpha(\tau))}\,.$$

We are going to show that a statistical interaction between particles can be implemented by assigning different weights $\chi(\alpha)$ to each addend of the sum over homotopic classes of paths defining the propagator:

$$\langle q', t'|q, t\rangle = \sum_{\alpha\in\pi_1(M_N^d)} \chi(\alpha)\ \langle q', t'|q, t\rangle_\alpha = \sum_{\alpha\in\pi_1(M_N^d)} \chi(\alpha) \int_q^{q'} \mathcal{D}q_\alpha\, e^{i\int_t^{t'} d\tau\, L(q_\alpha(\tau),\, \dot{q}_\alpha(\tau))}\,. \tag{4.2}$$

The requirement for the propagator to be a probability amplitude imposes a constraint on the weights. In fact we can write the propagator from q to q'' as a convolution of propagators with a fixed intermediate point:

$$\begin{aligned}
\langle q'', t''|q, t\rangle &= \int_{M_N^d} dq'\ \langle q'', t''|q', t'\rangle\, \langle q', t'|q, t\rangle \\
&= \sum_{\alpha_1,\alpha_2\in\pi_1(M_N^d)} \chi(\alpha_1)\,\chi(\alpha_2) \int_{M_N^d} dq'\ \langle q'', t''|q', t'\rangle_{\alpha_2}\ \langle q', t'|q, t\rangle_{\alpha_1} \\
&= \sum_{\alpha\in\pi_1(M_N^d)}\ \sum_{\substack{\alpha_1,\alpha_2\in\pi_1(M_N^d)\\ \alpha_1\cdot\alpha_2=\alpha}} \chi(\alpha_1)\,\chi(\alpha_2) \int_{M_N^d} dq'\ \langle q'', t''|q', t'\rangle_{\alpha_2}\ \langle q', t'|q, t\rangle_{\alpha_1}\,.
\end{aligned} \tag{4.3}$$

On the other hand, the propagator $\langle q'', t''|q, t\rangle_\alpha$ can be written as a sum of the convolution of two propagators over paths whose composition is α:

$$\langle q'', t''|q, t\rangle_\alpha = \sum_{\substack{\alpha_1,\alpha_2\in\pi_1(M_N^d)\\ \alpha_1\cdot\alpha_2=\alpha}} \int_{M_N^d} dq'\ \langle q'', t''|q', t'\rangle_{\alpha_2}\ \langle q', t'|q, t\rangle_{\alpha_1}\,.$$

Now if we substitute this expression into (4.2) and compare the result with (4.3), we obtain that the weights have to satisfy:

$$\chi(\alpha_1)\,\chi(\alpha_2) = \chi(\alpha_1\cdot\alpha_2)\,. \tag{4.4}$$

In particular (4.4) tells us that the weights have to be a one dimensional representation of the fundamental group $\pi_1(M_N^d)$. Also the representation has to be unitary in order to not mess up the normalization. The physical meaning of the weight $\chi(\alpha)$ is that it is the phase factor that the wavefunction of the system acquires under an exchange of particles following a certain path in the homotopy class α at a fixed time t. Indeed this operation in terms of the path integral can be regarded as a nonphysical abstract process where the wavefunction $\psi(q,t)$ evolves with the propagators $\langle q',t|q,t\rangle_\beta = \delta_{\alpha\beta}\,\delta(q-q')$[10]. Thus the wavefunction $\psi'(q,t)$ after the exchange will be given by:

$$\psi'(q,t) = \int_{M_N^d} dq'\,\langle q,t|q',t\rangle\,\psi(q',t) = \sum_{\beta\in\pi_1(M_N^d)} \chi(\beta)\int_{M_N^d} dq'\,\langle q,t|q',t\rangle_\beta\,\psi(q',t)$$
$$= \sum_{\beta\in\pi_1(M_N^d)} \chi(\beta)\,\delta_{\alpha\beta}\int_{M_N^d} dq'\,\delta(q'-q)\,\psi(q',t) = \chi(\alpha)\,\psi(q,t)$$

that is exactly the result we were looking for.
So we have to find out what is the fundamental group associated to our configuration space. At first sight one could think that $M_N^d = (\mathbb{R}^d)^N$ but it is clear that the hypothesis of hard-core particles requires to remove configurations such that two or more particles occupy the same position. Moreover we also have to identify configurations that differ in the ordering of the particles, which means that we have to quotient by the permutation group S_N. It is clear that there is a significant difference between the case where the space is 2-dimensional and that where it has a higher dimension. Indeed if we remove a point from $\mathbb{R}^d$ we obtain a space that is not simply connected only if $d = 2$, which means that only in this case paths going from one point to another may not be homotopic to each other. Thus it is not surprising that the fundamental group of M_N^d is different in the two above distinguished cases (see e.g. [9]):

$$\pi_1(M_N^d) = \begin{cases} S_N & \text{if } d \geq 3 \\ B_N & \text{if } d = 2 \end{cases} \tag{4.5}$$

where B_N is the braid group with N objects. Let us now remind the definition of these two finite groups.

Definition 3. *The permutation (or symmetric) group S_N is defined as the group with $N-1$ generators $\{\sigma_1, ..., \sigma_{N-1}\}$ satisfing the relations:*

- $\sigma_i^2 = \mathbb{I} \quad \forall i = 1, ..., N-1$
- $\sigma_i\,\sigma_j = \sigma_j\,\sigma_i$ *if* $j \neq i \pm 1$
- $\sigma_i\,\sigma_{i+1}\,\sigma_i = \sigma_{i+1}\,\sigma_i\,\sigma_{i+1} \quad \forall i = 1, ..., N-2.$

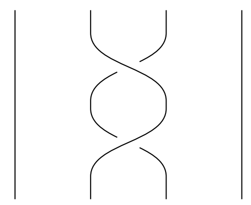
More informally the permutation group is simply the group where the elements are the operations that permute a collection of N objects, regardless of the way that has been used. The generator σ_i is the operation that exchanges objects i and $i+1$.

Definition 4. *The braid group B_N is defined as the group with $N-1$ generators $\{\sigma_1, ..., \sigma_{N-1}\}$ satisfing the relations:*

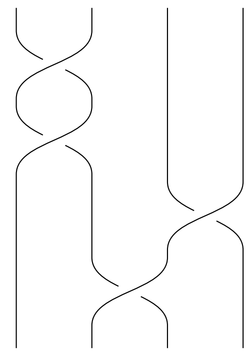
- $\sigma_i\,\sigma_j = \sigma_j\,\sigma_i$ *if* $j \neq i \pm 1$
- $\sigma_i\,\sigma_{i+1}\,\sigma_i = \sigma_{i+1}\,\sigma_i\,\sigma_{i+1} \quad \forall i = 1, ..., N-2.$

[10] Intuitively initial and final configurations have to be the same and only the propagator of the a priori given homotopy class of the exchange must contribute.

The braid group is conceptually very similar to the permutation group but the fact that in general $\sigma_i^2 \neq \mathbb{I}$ means that the way we are exchanging elements is meaningful. A pictorial way to represent the braid group uses a set of N vertical strings where the operation σ_i acts by crossing strings i and $i+1$ with the former passing over the latter, if we suppose the bottom configuration to be the starting one and the top configuration to be the ending one. Some examples are represented in the pictures below.



The move $\sigma_i^{-1}\,\sigma_i$



The move $\sigma_1^2\,\sigma_3\,\sigma_2$

These kinds of diagrams can also be interpreted as describing the world lines of the N particles in the system and the paths they follow when exchanged.
We are looking for a representation $\chi\colon \pi_1(M_N^d) \to \mathrm{U}(1)$ where, as we said, the fundamental group is either S_N or B_N. In both cases we can exploit the property $\sigma_i\,\sigma_{i+1}\,\sigma_i = \sigma_{i+1}\,\sigma_i\,\sigma_{i+1}$ together with the commutativity of U(1):

$$\chi(\sigma_i)\,\chi(\sigma_{i+1})\,\chi(\sigma_i) = \chi(\sigma_{i+1})\,\chi(\sigma_i)\,\chi(\sigma_{i+1})$$

$$\iff \quad \chi(\sigma_{i+1})\,\chi(\sigma_i)\,\chi(\sigma_i) = \chi(\sigma_{i+1})\,\chi(\sigma_{i+1})\,\chi(\sigma_i) \quad \iff \quad \chi(\sigma_i) = \chi(\sigma_{i+1})\,.$$

Hence χ maps every generator of $\pi_1(M_N^d)$ to the same element of U(1), which we denote with z. Now if $d \geq 3$, and so $\pi_1(M_N^d) = S_N$, we also have the property $\sigma_i^2 = \mathbb{I}$ which implies that $z^2 = 1$ i.e. that $z = 1$ or $z = -1$. Indeed the only one dimensional unitary representations of S_N are the identical one and the alternating one, corresponding exactly to bosons and fermions respectively. Instead in the case $d = 2$ i.e. $\pi_1(M_N^d) = B_N$ we have $z = e^{-i\nu\pi}$, where ν is a real parameter that we will call statistics which depends on the system. Of course it is possible that $\nu = 0, 1$ which corresponds to a system of bosons or fermions.
Now let us consider a certain configuration of the given particles in the $d = 2$ space. We can describe it by means of the azimutal angles formed by the straight line connecting each pair of particles and an arbitrary chosen axis, for example the x^1 one. In formulae the angle associated to the couple made up by particle I and particle J, for $I < J$, is:

$$\theta_{I,J} = \tan^{-1}\left(\frac{x_I^2 - x_J^2}{x_I^1 - x_J^1}\right) \qquad \forall I, J = 1, ..., N\,. \tag{4.6}$$

We define also the angle with exchanged indeces to be $\theta_{J,I} = \theta_{I,J} + \pi$. Under an elementary exchange operation $\sigma_k \in B_N$ the angle $\theta_{k,k+1}$ obviously goes into $\theta'_{k,k+1} = \theta_{k,k+1} + \pi$ while for all the other angles the indeces k and $k+1$ are exchanged i.e. $\theta'_{I,k} = \theta_{I,k+1}$ and viceversa. Thus defining the quantity $\Delta\theta^{(\alpha)}_{I,J} \equiv \theta'_{I,J} - \theta_{I,J}$, where now $\theta'_{I,J}$ are the angles obtained with a generic permutation $\alpha \in B_N$, we get the following identity:

$$\sum_{I<J} \Delta\theta^{(\sigma_k)}_{I,J} = \pi \qquad \forall k = 1, ..., N-1\,.$$

It follows that the above representation of B_N can be written in the following form:

$$\chi(\sigma_k) = e^{-i\,\nu\pi} = \exp\left[-\,i\,\nu \sum_{I<J} \Delta\theta^{(\sigma_k)}_{I,J}\right].$$

This expression can be extended to every element $\alpha = \sigma_{k_1} ... \sigma_{k_p}$ of the braid group, using the fact that $\Delta\theta^{(\sigma_k)}_{I,J} + \Delta\theta^{(\sigma_l)}_{I,J} = \Delta\theta^{(\sigma_k\sigma_l)}_{I,J}$:

$$\begin{aligned}\chi(\alpha) = \chi(\sigma_{k_1} ... \sigma_{k_p}) = \chi(\sigma_{k_1}) ... \chi(\sigma_{k_p}) &= \exp\left[-\,i\,\nu \sum_{I<J} \Delta\theta^{(\sigma_{k_1})}_{I,J}\right] ... \exp\left[-\,i\,\nu \sum_{I<J} \Delta\theta^{(\sigma_{k_p})}_{I,J}\right] \\ = \exp\left[-\,i\,\nu \sum_{I<J} \left(\Delta\theta^{(\sigma_{k_1})}_{I,J} + ... + \Delta\theta^{(\sigma_{k_p})}_{I,J}\right)\right] &= \exp\left[-\,i\,\nu \sum_{I<J} \Delta\theta^{(\sigma_{k_1} ... \sigma_{k_p})}_{I,J}\right] \\ = \exp\left[-\,i\,\nu \sum_{I<J} \Delta\theta^{(\alpha)}_{I,J}\right].\end{aligned}$$

We can now think about the angles $\theta_{I,J}$ as functions of time (determined by the dynamics of the particles and in general very complicated) and so write their variation as an integral between two fixed times of their time derivative:

$$\chi(\alpha) = \exp\left[-\,i\,\nu \sum_{I<J} \int_t^{t'} d\tau\, \frac{d}{d\tau}\theta^{(\alpha)}_{I,J}(\tau)\right]. \tag{4.7}$$

Though expression (4.7) may look rather formal, it brings us to a relevant general result. In fact substituting (4.7) in (4.2), we get:

$$\langle q', t'|q, t\rangle = \sum_{\alpha\in\pi_1(M^d_N)} \int_q^{q'} \mathcal{D}q_\alpha\, e^{\,i\int_t^{t'} d\tau \left[L\left(q_\alpha(\tau),\, \dot q_\alpha(\tau)\right) - \nu \sum_{I<J} \frac{d}{d\tau}\theta^{(\alpha)}_{I,J}(\tau)\right]}.$$

Therefore we can regard our system of anyons described by the lagrangian L as a system of bosons with an additional statistical interaction described by the lagrangian:

$$L' = L - \nu \sum_{I<J} \frac{d}{d\tau}\theta^{(\alpha)}_{I,J}(\tau)\,. \tag{4.8}$$

Notice that this additional interaction is relevant only at quantum level since its term is a total derivative that does not change the classical equations of motion. Indeed it depends mainly on the topological properties of the configuration space which are clearly non-local an thus not relevant for the classical system.

4.2 N particles coupled to the Chern-Simons field

A practical realization of the above general framework is achieved with the use of Chern-Simons theory. The mechanism that leads to particles acquiring arbitrary phase factors when exchanged is conceptually similar to that of Aharonov-Bohm effect: the Chern-Simons field attaches to every charged particle a magnetic flux and thus every particle feels the statistical effect of the vector potential generated by the others.
Let us consider a non-relativistic quantum system of N particles of mass m and charge e with respect to the Chern-Simons field A_μ in two spatial dimensions. Let $\vec{r}_I(t)$ be the position of the I-th particle in $\mathbb{R}^2$ and $v_I^\alpha(t) = (1, \vec{v}_I(t))$ its 3-velocity. Charged particles provide a matter current that satisfies the continuity equation:

$$j^\mu(x) = \sum_{I=1}^{N} e\, v_I^\mu(t)\, \delta^{(2)}(\vec{x} - \vec{r}_I(t)) \qquad \Longrightarrow \qquad \partial_\mu j^\mu = 0\,. \tag{4.9}$$

Obiviously we identify $j^0 = \rho$ as the charge density and $\vec{j}$ as the current density. So we can couple the kinetic lagrangian of the particles to the vector field A_μ described by the abelian Chern-Simons term (2.1) in the standard minimal way. Then the resulting total lagrangian of the system is:

$$L = \sum_{I=1}^{N} \frac{1}{2}\, m\, v_I^2 + \int_{\mathbb{R}^2} dx^1 dx^2\, \left(\kappa\, \epsilon^{\mu\nu\rho} A_\mu\, \partial_\nu A_\rho + j^\mu A_\mu\right)\,. \tag{4.10}$$

Note that if we substitute the current expression (4.9) in (4.10) and exploit the Dirac delta function properties we obtain the following equivalent expression for the lagrangian:

$$L = \sum_{I=1}^{N} \left[\frac{1}{2}\, m\, v_I^2 - e\left(-A_0(t, \vec{r}_I(t)) - \vec{v}_I(t) \cdot \vec{A}(t, \vec{r}_I(t))\right)\right] + \kappa \int_{\mathbb{R}^2} dx^1 dx^2\, \epsilon^{\mu\nu\rho} A_\mu\, \partial_\nu A_\rho\,. \tag{4.11}$$

Being $\vec{a}_I(t)$ the acceleration of the I-th particle, the equation of motion for each variable $\vec{r}_I$ is then the Lorentz equation:

$$m\, a_I^i(t) = e\left(E^i(t, \vec{r}_I(t)) + \epsilon^{ij}\, v_I^j(t)\, B(t, \vec{r}_I(t))\right) \tag{4.12}$$

where obviously $\vec{E}$ and B are defined in terms of the Chern-Simons vector field in the same way as electric and magnetic fields (see equations (1.4)).
Then we easily find the remaining equations of motion by taking the variation of the action $S = \int dt\, L$ with respect to A_μ:

$$j^\mu = -2\kappa\, \epsilon^{\mu\nu\rho}\, \partial_\nu A_\rho \qquad \Longleftrightarrow \qquad j^\mu = -\kappa\, \epsilon^{\mu\nu\rho}\, F_{\nu\rho}\,. \tag{4.13}$$

The fact that the Chern-Simons theory does not carry any independent degree of freedom should suggests that (4.13) just express the fields $\vec{E}$ and B through the matter current. This can be seen more clearly if we separate the temporal and the spatial equations:

$$\rho = -\kappa\, \epsilon^{0ij}\, F_{ij} = -\kappa\, \epsilon^{ij}\, \epsilon_{ij}\, B = -2\kappa\, B$$

$$j^i = -\kappa\, \epsilon^{ij0}\, F_{j0} - \kappa\, \epsilon^{i0j}\, F_{0j} = 2\kappa\, \epsilon^{ij}\, F_{0j} = -2\kappa\, \epsilon^{ij}\, E^j$$

$$\Longrightarrow \quad \begin{cases} B(x) = -\dfrac{\rho}{2\kappa} = -\dfrac{e}{2\kappa}\displaystyle\sum_{I=1}^{N} \delta^{(2)}(\vec{x} - \vec{r}_I(t)) \\ E^i(x) = \dfrac{1}{2\kappa}\,\epsilon^{ij}\, j^j = \dfrac{e}{2\kappa}\,\epsilon^{ij}\displaystyle\sum_{I=1}^{N} v_I^j(t)\,\delta^{(2)}(\vec{x} - \vec{r}_I(t)) \quad . \end{cases} \tag{4.14}$$

Physically these equations attach an electric and a magnetic field to every particle. These fields are non-zero only along the particle trajectories. Indeed the Lorentz equation for the I-th particle becomes:

$$m\, a_I^i(t) = \frac{e^2}{2\kappa}\,\epsilon^{ij}\sum_{J=1}^{N}\left[\left(v_J^j(t) - v_I^j(t)\right)\delta^{(2)}(\vec{r}_J(t) - \vec{r}_I(t))\right]$$

which means that a particle feels a non-vanishing force only if it occupies the same position of another one, which is impossible in the hypothesis of hard-core particles. Hence the resulting Lorentz force on each particle vanishes.

Even though the total force acting on the particles is zero, the quantum dynamics of the system is non-trivial because of the presence of statistical interactions arising from the Aharonov-Bohm mechanism. Indeed let us consider the first equation in (4.14) and integrate both sides along an arbitrary little circle C_I that contains only the I-th particle. Then the LHS results in the magnetic flux Φ_I through the circle, while for the RHS we have:

$$-\frac{e}{2\kappa}\int_{C_I} dx^1 dx^2 \sum_{J=1}^{N}\delta^{(2)}(\vec{x} - \vec{r}_J(t)) = -\frac{e}{2\kappa}\sum_{I=1}^{N}\delta_{IJ} = -\frac{e}{2\kappa}\,.$$

Therefore, whenever a particle has a charge e, the Chern-Simons dynamics makes it also have a magnetic flux $\Phi = -\frac{e}{2\kappa}$ attached. The standard Aharonov-Bohm effect appears when we have a magnetic flux confined in a solenoid that affects the wavefunction of a particle making it acquire a phase factor depending on the line integral of $\vec{A}$ over a certain path. Here instead each particle takes also the role of the solenoid for all the other particles, leading to the same kind of phenomenon.

Let us now figure out the behaviour of this system from the statistical point of view introduced in Section 4.1. Firstly let us write the expression $B = -\frac{\rho}{2\kappa}$ in terms of A^i:

$$B = \frac{1}{2}\,\epsilon^{ij}\,F^{ij} = \epsilon^{ij}\,\partial^i A^j = -\frac{e}{2\kappa}\sum_{I=1}^{N}\delta^{(2)}(\vec{x} - \vec{r}_I(t))\,.$$

This equation is clearly linear and thus we can solve it for one term of the sum at a time and then sum together the solutions. Now we notice that a single term equation for the vector potential, at least in the rest-frame of the particle in consideration, is analytically analogous to the case of a constant uniform electric current in an infinite solenoid. So the general solution, with adjusted constants and in the Weyl gauge $A^0 = 0$, is given by (see [5]):

$$\widetilde{A}_I^i(x) = -\frac{e}{4\pi\kappa}\,\epsilon^{ji}\,\frac{x^j - r_I^j(t)}{|\vec{x} - \vec{r}_I(t)|^2} \tag{4.15}$$

where tilde indicates that this is the field generated by just one particle. However, when computing the value $\vec{A}_I$ of the vector potential that the I-th particle feels, we should pay attention to neglect the field generated by itself, since from the Lorentz equation it is clear that there is no self-interaction:

$$A_I^i = \frac{e}{4\pi\kappa}\,\epsilon^{ij}\sum_{J\neq I}\frac{r_I^j(t)-r_J^j(t)}{|\vec{r}_I(t)-\vec{r}_J(t)|^2}\,. \tag{4.16}$$

So, once we have determined the expression for the Chern-Simons field A_μ through its field equations and thus eliminated its kinetic term from the lagrangian, we are left with a system of N particles moving in the effective vector potential given by (4.16).
Now if we introduce the azimutal angles of the particles as in (4.6) and think about the expression (4.16) as a function of the N position vectors $\vec{r}_I$, we have the following equality (we omit time dependence in order to simplify the notation):

$$\begin{aligned}
-\frac{\partial}{\partial r_I^i}\,\theta_{I,J} &= -\frac{\partial}{\partial r_I^i}\left[\tan^{-1}\left(\frac{r_I^2-r_J^2}{r_I^1-r_J^1}\right)\right] = -\frac{1}{1+\left(\frac{r_I^2-r_J^2}{r_I^1-r_J^1}\right)^2}\frac{\partial}{\partial r_I^i}\left(\frac{r_I^2-r_J^2}{r_I^1-r_J^1}\right)\\
&= -\frac{(r_I^1-r_J^1)^2}{|\vec{r}_I-\vec{r}_J|^2}\left[-\delta^{i1}\,\frac{r_I^2-r_J^2}{(r_I^1-r_J^1)^2}+\delta^{i2}\,\frac{1}{r_I^1-r_I^1}\right] = \epsilon^{ij}\,\frac{r_I^j-r_J^j}{|\vec{r}_I-\vec{r}_J|^2}\,.
\end{aligned}$$

It follows that (4.16) can be written in the form:

$$A_I^i = -\frac{e}{4\pi\kappa}\,\frac{\partial}{\partial r_I^i}\sum_{J\neq I}\theta_{I,J}\,. \tag{4.17}$$

Now that we have determined the effective vector potential experienced by the particles, we can insert it in the lagrangian, obtaining the effective lagrangian describing the system of particles:

$$L' = \sum_{I=1}^{N}\left(\frac{1}{2}\,m\,v_I^2+e\,v_I^i\,A_I^i\right) = \sum_{I=1}^{N}\left(\frac{1}{2}\,m\,v_I^2\right) - \frac{e^2}{4\pi\kappa}\sum_{I=1}^{N}\sum_{J\neq I}v_I^i\,\frac{\partial}{\partial r_I^i}\,\theta_{I,J}\,.$$

The double sum in the second term can be simplified:

$$\begin{aligned}
\sum_{I=1}^{N}\sum_{J\neq I}v_I^i\,\frac{\partial}{\partial r_I^i}\,\theta_{I,J} &= \sum_{I=1}^{N}\left[\sum_{J=1}^{I-1}v_I^i\,\frac{\partial}{\partial r_I^i}\,\theta_{I,J}+\sum_{J=I+1}^{N}v_I^i\,\frac{\partial}{\partial r_I^i}\,\theta_{I,J}\right]\\
&= \sum_{J=1}^{N}\sum_{I=1}^{J-1}v_J^i\,\frac{\partial}{\partial r_J^i}\,\theta_{J,I}+\sum_{I<J}v_I^i\,\frac{\partial}{\partial r_I^i}\,\theta_{I,J} = -\sum_{I<J}v_J^i\,\frac{\partial}{\partial r_I^i}\,\theta_{I,J}+\sum_{I<J}v_I^i\,\frac{\partial}{\partial r_I^i}\,\theta_{I,J}\\
&= \sum_{I<J}\left(v_I^i-v_J^i\right)\frac{\partial}{\partial r_I^i}\,\theta_{I,J}
\end{aligned}$$

where we have used the fact that $\frac{\partial}{\partial r_J^i}\,\theta_{I,J} = -\frac{\partial}{\partial r_I^i}\,\theta_{I,J}$ and $\frac{\partial}{\partial r_I^i}\,\theta_{J,I} = \frac{\partial}{\partial r_I^i}\,\theta_{I,J}$, which are simple consequences of the definition of $\theta_{I,J}$. Moreover the single addend of the sum can be written as a time derivative:

$$\frac{d}{dt}\,\theta_{I,J}\left(\vec{r}_I(t),\,\vec{r}_J(t)\right) = \left(\frac{\partial}{\partial r_I^i}\,\theta_{I,J}\right)\frac{dr_I^i}{dt}+\left(\frac{\partial}{\partial r_J^i}\,\theta_{I,J}\right)\frac{dr_J^i}{dt} = v_I^i\,\frac{\partial}{\partial r_I^i}\,\theta_{I,J}-v_J^i\,\frac{\partial}{\partial r_I^i}\,\theta_{I,J}\,.$$

Therefore the expression for the lagrangian L' becomes:

$$L' = \sum_{I=1}^{N} \left(\frac{1}{2}\, m\, v_I^2 \right) - \frac{e^2}{4\pi\kappa} \sum_{I<J} \frac{d}{dt}\, \theta_{I,J} \, . \tag{4.18}$$

We have thus managed to write down a lagrangian describing our N particles in the form (4.8) and thus, according to the general framework described in Section 4.1, our particles can be treated as anyons with statistics:

$$\nu = \frac{e^2}{4\pi\kappa} = \frac{\kappa\, \Phi}{4\pi} \, . \tag{4.19}$$

So we can describe quantum particles with any possible statistics by adjusting the constant κ of the Chern-Simons term.

Conclusion

In the present work we have reviewed the main features of the classical Chern-Simons theory as a topological gauge theory on the plane. We proved its gauge invariance, its metric independence, and the fact that it does not carry any local physical degree of freedom.
We also explored a couple of its important applications. We found that a theory whose dynamics is governed by a sum of the Maxwell lagrangian and the Chern-Simons one describes a massive gauge-invariant vector field which is somehow similar to the classical electrodynamics vector potential yet satisfying Klein-Gordon equation and carring a non-zero spin. We have showed that the same dynamics is described by a dual theory which is realized thanks to Chern-Simons lagrangian too, and both the theories actually arise from a third more general theory.
Also we studied the unusual statistical dynamics of anyons, quasi-particles on the plane which are neither bosons nor fermions. Topological properties of 2-dimensional spaces allows their existance and so we explained how Chern-Simons theory provides a way for modelling a concrete system of N anyons moving freely on the plane.
A further study of this interesting gauge theory could deal with some phenomenological applications, such as the description of fractional quantum hall effect and quantum vortices, as well as other more formal aspects, for instance its role in topological gravity, supergravity and string theory.

APPENDIX A

Noether's current in classical field theories

In any theory described by a lagrangian the Noether's theorem states that to every symmetry of this lagrangian (i.e. to any transformation that leaves the action unchanged) corresponds a conserved quantity. In this appendix we recall the general expression of the corresponding current, the Noether's current, associated with a generic spacetime transformation, and therefore the definition of the stress-energy tensor and the angular momentum density tensor. We will work in 2+1 dimension (it is enough for the purposes of the thesis), in the setting of a general field theory with φ_I as dynamic variables, where I is a generic index that labels the fields, so that φ_I can be a scalar, a vector, a higher order tensor, or whatever else. The infinitesimal transformation acts on the coordinates and on the fields in the following way:

$$\begin{cases} x'^{\mu} = x^{\mu} + \delta x^{\mu} \\ \varphi'_I(x') = \varphi_I(x) + \bar{\delta}\varphi_I(x) \quad . \end{cases} \tag{A.1}$$

It's now convenient to define $\delta\varphi_I(x) \equiv \varphi'_I(x) - \varphi_I(x)$. Note that it doesn't matter whether the spacetime point in which we evaluate either $\delta\varphi_I$ or $\bar{\delta}\varphi_I$ is x or x' since the differences between the two cases is negligible (because it is of higher order in δx^{μ}) and the transformations we are talking about are infinitesimal; this allows us to leave the spacetime point dependence of the field variations implicit.
We want to find a quantity J^{μ} such that $\partial_{\mu} J^{\mu} = 0$ and that depends on the particular symmetry considered. We start from requiring the action to be invariant under the transformation (A.1) i.e. $\mathcal{L}'\, d^3x' = \mathcal{L}\, d^3x$ (we hide the integrals because the boundaries transform accordingly with the coordinates). The relation between the two volume forms is:

$$d^3x' = \det\left(\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right) d^3x = \det\left(\delta^{\mu}{}_{\nu} + \partial_{\nu}\delta x^{\mu}\right) d^3x = \left(1 + \partial_{\mu}\delta x^{\mu}\right) d^3x\,.$$

Now let us handle the equality between the actions before and after the symmetry transformations, that must be true at the first order:

$$\mathcal{L}\left(\varphi'_I(x'),\, \partial_{\alpha}\varphi'_I(x')\right) d^3x' = \mathcal{L}\left(\varphi_I(x),\, \partial_{\alpha}\varphi_I(x)\right) d^3x$$

$$\Longleftrightarrow$$

$$\mathcal{L}\left(\varphi_I'(x'),\, \partial_\alpha\varphi_I'(x')\right) + \mathcal{L}\left(\varphi_I'(x'),\, \partial_\alpha\varphi_I'(x')\right)\partial_\mu\delta x^\mu - \mathcal{L}\left(\varphi_I(x),\, \partial_\alpha\varphi_I(x)\right) = 0$$

$$\Longleftrightarrow$$

$$\mathcal{L}\left(\varphi_I'(x'),\, \partial_\alpha\varphi_I'(x')\right) - \mathcal{L}\left(\varphi_I'(x),\, \partial_\alpha\varphi_I'(x)\right) + \mathcal{L}\left(\varphi_I'(x),\, \partial_\alpha\varphi_I'(x)\right)\partial_\mu\delta x^\mu + \\ + \mathcal{L}\left(\varphi_I'(x),\, \partial_\alpha\varphi_I'(x)\right) - \mathcal{L}\left(\varphi_I(x),\, \partial_\alpha\varphi_I(x)\right) = 0$$

$$\Longleftrightarrow$$

$$\partial_\mu\mathcal{L}\left(\varphi_I'(x),\, \partial_\alpha\varphi_I'(x)\right)\,\delta x^\mu + \mathcal{L}\left(\varphi_I'(x),\, \partial_\alpha\varphi_I'(x)\right)\partial_\mu\delta x^\mu + \frac{\partial\mathcal{L}}{\partial\varphi_I}\,\delta\varphi_I + \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi_I}\,\partial_\mu\delta\varphi_I = 0$$

$$\Longleftrightarrow$$

$$\partial_\mu\left[\mathcal{L}\,\delta x^\mu + \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi_I}\,\delta\varphi_I\right] + \frac{\partial\mathcal{L}}{\partial\varphi_I}\,\delta\varphi_I - \partial_\mu\,\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi_I}\,\delta\varphi_I = 0\,.$$

In this equality the terms that are out of the square brackets vanish if the Euler-Lagrange equations of the theory are satisfied so we find exactly the Noether's current J^μ we were looking for:

$$J^\mu \equiv \mathcal{L}\,\delta x^\mu + \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi_I}\,\delta\varphi_I \qquad \Longrightarrow \qquad \partial_\mu J^\mu = 0\,.$$

Let us now express $\delta\varphi_I$ as $\delta\varphi_I = \bar{\delta}\varphi_I - \partial_\mu\varphi_I\,\delta x^\mu$. It follows:

$$\begin{aligned} J^\mu &= \mathcal{L}\,\delta x^\mu - \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi_I}\,\partial_\nu\varphi_I\,\delta x^\nu + \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi_I}\,\bar{\delta}\varphi_I \\ &= \left(\mathcal{L}\,\eta^{\mu\nu} - \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi_I}\,\partial^\nu\varphi_I\right)\delta x_\nu + \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi_I}\,\bar{\delta}\varphi_I\,. \end{aligned}$$

The term enclosed in parenthesis is exactly what defines the canonical stress-energy tensor of the theory:

$$T^{\mu\nu} \equiv -\eta^{\mu\nu}\,\mathcal{L} + \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi_I}\,\partial^\nu\varphi_I \qquad \Longrightarrow \qquad J^\mu = -T^{\mu\nu}\,\delta x_\nu + \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi_I}\,\bar{\delta}\varphi_I\,.$$

Note that it is proportional to the Noether's current associated to spacetime translations; indeed this symmetry is given by $\delta x^\mu = a^\mu$, with $a^\mu \in \mathbb{R}^3$, and $\bar{\delta}\varphi_I = 0$, so that Noether's current is:

$$J^\mu = -a_\nu\,T^{\mu\nu}\,.$$

The conserved quantity obtained by integrating $T^{0\mu}$ over spatial dimensions is the total momentum, and so if the theory is invariant under Poincaré spacetime translations, it directly follows that the momentum is conserved.
The other particularly important symmetry of a relativistic invariant action is the one under Lorentz transformations, which leds to the definition of the angular momentum density tensor as a Noether's current. Writing it in general terms is a little bit tricky and not so much useful, so it will be computed in specific cases; however we can state that it is the tensor $\mathcal{J}^{\mu\nu\rho}$ such that Noether's current associated to Lorentz transformations takes the form:

$$J^\mu = \frac{1}{2}\,\omega_{\nu\rho}\,\mathcal{J}^{\mu\nu\rho}$$

where $\omega_{\nu\rho}$ is the antisymmetryc parameter tensor of the infinitesimal Lorentz transformation. $\mathcal{J}^{\mu\nu\rho}$ is significant for understanding whether the field is carrying a spin or not.

APPENDIX B

Stress-energy tensors derivation

Maxwell-Chern-Simons theory

According to the general result reviewed in Appendix A, the canonical stress-energy tensor associated to the lagrangian density (2.8) is:

$$
\begin{aligned}
T_c^{\mu\nu} &= -\eta^{\mu\nu}\,\mathcal{L} + \frac{\partial \mathcal{L}}{\partial\partial_\mu A_\rho}\,\partial^\nu A_\rho \\
&= -\eta^{\mu\nu}\left(\frac{1}{2}\,F^\alpha\,F_\alpha + \frac{1}{2}\,m\,A^\alpha\,F_\alpha\right) + \frac{\partial}{\partial\partial_\mu A_\rho}\left(\frac{1}{2}\,F^\alpha\,F_\alpha + \frac{1}{2}\,m\,\epsilon^{\alpha\beta\sigma}\,A_\alpha\,\partial_\beta A_\sigma\right)\partial^\nu A_\rho \\
&= -\eta^{\mu\nu}\left(\frac{1}{2}\,F^\alpha\,F_\alpha + \frac{1}{2}\,m\,A^\alpha\,F_\alpha\right) + \left(F_\alpha\,\epsilon^{\alpha\beta\sigma}\,\delta^\mu{}_\beta\,\delta^\rho{}_\sigma + \frac{1}{2}\,m\,\epsilon^{\alpha\beta\sigma}\,A_\alpha\,\delta^\mu{}_\beta\,\delta^\rho{}_\sigma\right)\partial^\nu A_\rho \\
&= -\eta^{\mu\nu}\left(\frac{1}{2}\,F^\alpha\,F_\alpha + \frac{1}{2}\,m\,A^\alpha\,F_\alpha\right) + \left(F_\alpha\,\epsilon^{\alpha\mu\rho} + \frac{1}{2}\,m\,\epsilon^{\alpha\mu\rho}\,A_\alpha\right)\partial^\nu A_\rho \\
&= -\frac{1}{2}\,\eta^{\mu\nu}\,F^\alpha\,F_\alpha - \frac{1}{2}\,m\,\eta^{\mu\nu}\,A^\alpha\,F_\alpha + \epsilon^{\alpha\mu\rho}\,F_\alpha\,F^\nu{}_\rho + \epsilon^{\alpha\mu\rho}\,F_\alpha\,\partial_\rho A^\nu + \frac{1}{2}\,m\,\epsilon^{\alpha\mu\rho}\,A_\alpha\,\partial^\nu A_\rho \\
&= \left(-\frac{1}{2}\,\eta^{\mu\nu}\,F^\alpha\,F_\alpha - F^{\mu\rho}\,F^\nu{}_\rho\right) - \frac{1}{2}\,m\,\eta^{\mu\nu}\,A^\alpha\,F_\alpha + \epsilon^{\alpha\mu\rho}\,F_\alpha\,\partial_\rho A^\nu + \frac{1}{2}\,m\,\epsilon^{\alpha\mu\rho}\,A_\alpha\,\partial^\nu A_\rho\,.
\end{aligned}
$$

Now let us manipulate a bit the last three terms:

$$
\begin{aligned}
&-\frac{1}{2}\,m\,\eta^{\mu\nu}\,A^\alpha\,F_\alpha + \epsilon^{\alpha\mu\rho}\,F_\alpha\,\partial_\rho A^\nu + \frac{1}{2}\,m\,\epsilon^{\alpha\mu\rho}\,A_\alpha\,\partial^\nu A_\rho \\
&\quad= -\frac{1}{2}\,m\,\eta^{\mu\nu}\,A^\alpha\,F_\alpha + \epsilon^{\alpha\mu\rho}\,F_\alpha\,\partial_\rho A^\nu + \frac{1}{2}\,m\,\epsilon^{\alpha\mu\rho}\,A_\alpha\,F^\nu{}_\rho + \frac{1}{2}\,m\,\epsilon^{\alpha\mu\rho}\,A_\alpha\,\partial_\rho A^\nu \\
&\quad= -\frac{1}{2}\,m\,\eta^{\mu\nu}\,A^\alpha\,F_\alpha + \epsilon^{\alpha\mu\rho}\,F_\alpha\,\partial_\rho A^\nu - \frac{1}{2}\,m\,\eta^{\nu\sigma}\,\epsilon^{\alpha\mu\rho}\,A_\alpha\,\epsilon_{\lambda\sigma\rho}\,F^\lambda + \frac{1}{2}\,m\,\epsilon^{\alpha\mu\rho}\,A_\alpha\,\partial_\rho A^\nu \\
&\quad= \cancel{-\frac{1}{2}\,m\,\eta^{\mu\nu}\,A^\alpha\,F_\alpha} + \epsilon^{\alpha\mu\rho}\,F_\alpha\,\partial_\rho A^\nu + \cancel{\frac{1}{2}\,m\,\eta^{\nu\mu}\,A_\alpha\,F^\alpha} - \frac{1}{2}\,m\,\eta^{\nu\alpha}\,A_\alpha\,F^\mu + \frac{1}{2}\,m\,\epsilon^{\alpha\mu\rho}\,A_\alpha\,\partial_\rho A^\nu \\
&\quad= -\frac{1}{2}\,m\,A^\nu\,F^\mu - \epsilon^{\alpha\mu\rho}\,A^\nu\,\partial_\rho F_\alpha - \frac{1}{2}\,m\,\epsilon^{\alpha\mu\rho}\,A^\nu\,\partial_\rho A_\alpha + \partial_\rho\left[\epsilon^{\alpha\mu\rho}\,A^\nu\left(F_\alpha + \frac{1}{2}\,m\,A_\alpha\right)\right] \\
&\quad= -A^\nu\left(m\,F^\mu + \epsilon^{\alpha\mu\rho}\,\partial_\rho F_\alpha\right) + \partial_\rho\left[\epsilon^{\alpha\mu\rho}\,A^\nu\left(F_\alpha + \frac{1}{2}\,m\,A_\alpha\right)\right]\,.
\end{aligned}
$$

The first term vanishes because of the equations of motion and other one is of the form $\partial_\rho X^{[\mu\rho]\nu}$ and thus it does not contribute to the stress-energy tensor. Therefore, modulo this term, $T^{\mu\nu}$ turns out to be equivalent to:

$$T^{\mu\nu} = -\frac{1}{2}\,\eta^{\mu\nu}\,F^{\alpha}\,F_{\alpha} - F^{\mu\rho}\,F^{\nu}{}_{\rho} = \frac{1}{4}\,\eta^{\mu\nu}\,F^{\alpha\beta}\,F_{\alpha\beta} + F^{\mu\rho}\,F_{\rho}{}^{\nu}\,.$$

Note that it can be written in terms of the dual field as follows:

$$\begin{aligned} T^{\mu\nu} &= -\frac{1}{2}\,\eta^{\mu\nu}\,F^{\alpha}\,F_{\alpha} - \eta^{\nu\sigma}\,\epsilon^{\alpha\mu\rho}\,F_{\alpha}\,\epsilon_{\beta\sigma\rho}\,F^{\beta} = -\frac{1}{2}\,\eta^{\mu\nu}\,F^{\alpha}\,F_{\alpha} + \eta^{\nu\mu}\,F_{\alpha}\,F^{\alpha} - \eta^{\nu\alpha}\,F_{\alpha}\,F^{\mu} \\ &= \frac{1}{2}\,\eta^{\mu\nu}\,F^{\alpha}\,F_{\alpha} - F^{\mu}\,F^{\nu}\,. \end{aligned}$$

Dual massive electrodynamics

According to the general expression of Appendix A, the canonical stress-energy tensor associated to the lagrangian density (2.15) is:

$$\begin{aligned} T_c^{\mu\nu} &= -\eta^{\mu\nu}\,\mathcal{L} + \frac{\partial\mathcal{L}}{\partial\partial_\mu f_\rho}\,\partial^\nu f_\rho = -\eta^{\mu\nu}\left(-\frac{1}{2}\,f^\rho\,f_\rho - \frac{1}{2m}\,\epsilon^{\rho\alpha\beta}\,f_\rho\,\partial_\alpha f_\beta\right) - \frac{1}{2m}\,\epsilon^{\sigma\alpha\beta}\,f_\sigma\,\delta^\mu{}_\alpha\,\delta^\rho{}_\beta\,\partial^\nu f_\rho \\ &= \frac{1}{2}\,\eta^{\mu\nu} f^\rho\,f_\rho + \frac{1}{2m}\,\eta^{\mu\nu}\,\epsilon^{\rho\alpha\beta}\,f_\rho\,\partial_\alpha f_\beta - \frac{1}{2m}\,\epsilon^{\sigma\mu\rho}\,f_\sigma\,\partial^\nu f_\rho \\ &= \frac{1}{2}\,\eta^{\mu\nu} f^\rho\,f_\rho + \frac{1}{2m}\,\eta^{\mu\nu}\,\epsilon^{\rho\alpha\beta}\,f_\rho\,\partial_\alpha f_\beta - \frac{1}{2m}\,\epsilon^{\sigma\mu\rho}\,f_\sigma\,\eta^{\nu\lambda}\,(\partial_\lambda f_\rho - \partial_\rho f_\lambda) - \frac{1}{2m}\,\epsilon^{\sigma\mu\rho}\,f_\sigma\,\partial_\rho f^\nu\,. \end{aligned}$$

Now note that contracting (2.16) with $\epsilon^{\rho\alpha\beta}$ we obtain:

$$\epsilon^{\rho\alpha\beta}\,f_\rho + \frac{1}{m}\,\epsilon^{\rho\alpha\beta}\,\epsilon_{\rho\nu\mu}\,\partial^\nu f^\mu = 0 \qquad\Longleftrightarrow\qquad \epsilon^{\rho\alpha\beta}\,f_\rho = \frac{1}{m}\left(\partial^\alpha f^\beta - \partial^\beta f^\alpha\right)\,. \tag{B.1}$$

So if we evaluate the stress-energy tensor on the mass shell, we can use the equation (B.1) to replace the antisymmetryc round bracket term. Let us manipulate also the other terms:

$$\begin{aligned} T_c^{\mu\nu} &= \frac{1}{2}\,\eta^{\mu\nu} f^\rho\,f_\rho + \frac{1}{2m}\,\eta^{\mu\nu}\,\epsilon^{\rho\alpha\beta}\,f_\rho\,\partial_\alpha f_\beta - \frac{1}{2}\,\eta^{\nu\lambda}\,\epsilon^{\sigma\mu\rho}\,f_\sigma\,\epsilon_{\alpha\lambda\rho}\,f^\alpha - \frac{1}{2m}\,\epsilon^{\sigma\mu\rho}\,f_\sigma\,\partial_\rho f^\nu \\ &= \frac{1}{2}\,\eta^{\mu\nu} f_\rho\cancel{\left(f^\rho + \frac{1}{m}\,\epsilon^{\rho\alpha\beta}\,\partial_\alpha f_\beta\right)} + \frac{1}{2}\,\eta^{\nu\mu}\,f_\alpha\,f^\alpha - \frac{1}{2}\,\eta^{\nu\lambda}\,f_\lambda\,f^\mu - \frac{1}{2m}\,\epsilon^{\sigma\mu\rho}\,f_\sigma\,\partial_\rho f^\nu \\ &= \frac{1}{2}\,\eta^{\nu\mu}\,f_\alpha\,f^\alpha - \frac{1}{2}\,f^\nu\,f^\mu + \frac{1}{2m}\,\epsilon^{\sigma\mu\rho}\,f^\nu\,\partial_\rho f_\sigma - \partial_\rho\left(\frac{1}{2m}\,\epsilon^{\sigma\mu\rho}\,f_\sigma\,f^\nu\right) \\ &= \frac{1}{2}\,\eta^{\nu\mu}\,f_\alpha\,f^\alpha - f^\nu\,f^\mu - \partial_\rho\left(\frac{1}{2m}\,\epsilon^{\sigma\mu\rho}\,f_\sigma\,f^\nu\right)\,. \end{aligned}$$

The last term is of the form $\partial_\rho X^{[\mu\rho]\nu}$ and thus does not contribute to the stress-energy tensor. Therefore $T^{\mu\nu}$ turns out to be equivalent to:

$$T^{\mu\nu} = \frac{1}{2}\,\eta^{\mu\nu}\,f^\alpha\,f_\alpha - f^\mu\,f^\nu\,.$$

APPENDIX C

Gauge transformation of non-abelian $\mathcal{L}_{CS}$

Let us compute how does (3.13) varies under a gauge transformation g. For simplicity we initially consider separately the two terms $A_\mu\,\partial_\nu A_\rho$ and $A_\mu\,A_\nu\,A_\rho$. In the following computations we will use the fact that both of them are arguments of the trace and so we can cyclic permute the composition order of every addend; moreover terms with symmetric indeces will vanish when contracted with the Levi-Civita tensor and so we will drop them. For the first term we have:

$$\begin{aligned}
&\left(gA_\mu g^{-1} + ie^{-1}(\partial_\mu g)g^{-1}\right)\partial_\nu\left(gA_\rho g^{-1} + ie^{-1}(\partial_\rho g)g^{-1}\right) = \\
&\quad = \left(gA_\mu g^{-1} + ie^{-1}(\partial_\mu g)g^{-1}\right)\left[(\partial_\nu g)A_\rho g^{-1} + g(\partial_\nu A_\rho)g^{-1} + gA_\rho(\partial_\nu g^{-1}) + \cancel{ie^{-1}(\partial_\nu\partial_\rho g)g^{-1}} + \right.\\
&\qquad\qquad \left. + ie^{-1}(\partial_\rho g)(\partial_\nu g^{-1})\right] \\
&\quad = gA_\mu g^{-1}(\partial_\nu g)A_\rho g^{-1} + gA_\mu(\partial_\nu A_\rho)g^{-1} + gA_\mu A_\rho(\partial_\nu g^{-1}) + ie^{-1}gA_\mu g^{-1}(\partial_\rho g)(\partial_\nu g^{-1}) + \\
&\qquad + ie^{-1}(\partial_\mu g)g^{-1}(\partial_\nu g)A_\rho g^{-1} + ie^{-1}(\partial_\mu g)(\partial_\nu A_\rho)g^{-1} + ie^{-1}(\partial_\mu g)A_\rho(\partial_\nu g^{-1}) + \\
&\qquad - e^{-2}(\partial_\mu g)g^{-1}(\partial_\rho g)(\partial_\nu g^{-1}) \\
&\quad = A_\mu\partial_\nu A_\rho + A_\rho A_\mu g^{-1}(\partial_\nu g) + A_\mu A_\rho(\partial_\nu g^{-1})g + ie^{-1}A_\mu g^{-1}(\partial_\rho g)(\partial_\nu g^{-1})g + \\
&\qquad + ie^{-1}A_\rho g^{-1}(\partial_\mu g)g^{-1}(\partial_\nu g) + ie^{-1}(\partial_\mu g)(\partial_\nu A_\rho)g^{-1} + ie^{-1}(\partial_\mu g)A_\rho(\partial_\nu g^{-1}) + \\
&\qquad - e^{-2}(\partial_\mu g)g^{-1}(\partial_\rho g)(\partial_\nu g^{-1})\,.
\end{aligned} \tag{C.1}$$

In the same way, for the second term we have:

$$\begin{aligned}
&\left(gA_\mu g^{-1} + ie^{-1}(\partial_\mu g)g^{-1}\right)\left(gA_\nu g^{-1} + ie^{-1}(\partial_\nu g)g^{-1}\right)\left(gA_\rho g^{-1} + ie^{-1}(\partial_\rho g)g^{-1}\right) = \\
&\quad = \left(gA_\mu A_\nu g^{-1} + ie^{-1}gA_\mu g^{-1}(\partial_\nu g)g^{-1} + ie^{-1}(\partial_\mu g)A_\nu g^{-1} - e^{-2}(\partial_\mu g)g^{-1}(\partial_\nu g)g^{-1}\right)\left(gA_\rho g^{-1} + \right.\\
&\qquad \left. + ie^{-1}(\partial_\rho g)g^{-1}\right) \\
&\quad = gA_\mu A_\nu A_\rho g^{-1} + ie^{-1}gA_\mu A_\nu g^{-1}(\partial_\rho g)g^{-1} + ie^{-1}gA_\mu g^{-1}(\partial_\nu g)A_\rho g^{-1} - e^{-2}gA_\mu g^{-1}(\partial_\nu g)g^{-1}(\partial_\rho g)g^{-1} + \\
&\qquad + ie^{-1}(\partial_\mu g)A_\nu A_\rho g^{-1} - e^{-2}(\partial_\mu g)A_\nu g^{-1}(\partial_\rho g)g^{-1} - e^{-2}(\partial_\mu g)g^{-1}(\partial_\nu g)A_\rho g^{-1} + \\
&\qquad - ie^{-3}(\partial_\mu g)g^{-1}(\partial_\nu g)g^{-1}(\partial_\rho g)g^{-1} \\
&\quad = A_\mu A_\nu A_\rho + ie^{-1}\left[A_\mu A_\nu g^{-1}(\partial_\rho g) + A_\rho A_\mu g^{-1}(\partial_\nu g) + A_\nu A_\rho g^{-1}(\partial_\mu g)\right] - e^{-2}\left[A_\mu g^{-1}(\partial_\nu g)g^{-1}(\partial_\rho g) + \right.\\
&\qquad \left. + A_\nu g^{-1}(\partial_\rho g)g^{-1}(\partial_\mu g) + A_\rho g^{-1}(\partial_\mu g)g^{-1}(\partial_\nu g)\right] - ie^{-3}(\partial_\mu g)g^{-1}(\partial_\nu g)g^{-1}(\partial_\rho g)g^{-1}\,.
\end{aligned} \tag{C.2}$$

The first term of (C.1) and the first of (C.2) make up the starting lagrangian so they will not appear in the variation. Now in (C.2) we can use the fact that all the terms are contracted with $\epsilon^{\mu\nu\rho}$ in order to exchange indeces and simplify the expression, so (C.2) modulo the first term becomes:

$$3ie^{-1}A_\mu A_\nu g^{-1}(\partial_\rho g) - 3e^{-2}A_\mu g^{-1}(\partial_\nu g)g^{-1}(\partial_\rho g) - ie^{-3}(\partial_\mu g)g^{-1}(\partial_\nu g)g^{-1}(\partial_\rho g)g^{-1}\,. \tag{C.3}$$

Now we put together with the correct coefficients the variation due to the first term (C.1) and that due to the second (C.3). For simplicity we consider separately the terms with quadratic, linear and no dependence on A_μ. Quadratic terms:

$$\begin{aligned}&\kappa\,\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left[A_\rho A_\mu g^{-1}(\partial_\nu g) + A_\mu A_\rho(\partial_\nu g^{-1})g - 2A_\mu A_\nu g^{-1}(\partial_\rho g)\right] = \\ &\quad = \kappa\,\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left[-A_\mu A_\nu g^{-1}(\partial_\rho g) - A_\mu A_\nu(\partial_\rho g^{-1})g\right] = -\kappa\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left[A_\mu A_\nu\,\partial_\rho(g^{-1}g)\right] = 0\,.\end{aligned} \tag{C.4}$$

Terms linear in A_μ:

$$\begin{aligned}&\kappa\,\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\big[ie^{-1}A_\mu g^{-1}(\partial_\rho g)(\partial_\nu g^{-1})g + ie^{-1}A_\rho g^{-1}(\partial_\mu g)g^{-1}(\partial_\nu g) + ie^{-1}(\partial_\mu g)(\partial_\nu A_\rho)g^{-1} + \\ &\qquad\qquad + ie^{-1}(\partial_\mu g)A_\rho(\partial_\nu g^{-1}) - 2ie^{-1}A_\mu g^{-1}(\partial_\nu g)g^{-1}(\partial_\rho g)\big] \\ &= ie^{-1}\kappa\,\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\big[A_\mu g^{-1}(\partial_\rho g)(\partial_\nu g^{-1})g - A_\mu g^{-1}(\partial_\nu g)g^{-1}(\partial_\rho g) + \partial_\nu\big((\partial_\mu g)A_\rho g^{-1}\big) - \partial_\nu\big(g^{-1}(\partial_\mu g)\big)A_\rho + \\ &\qquad\qquad + (\partial_\mu g)A_\rho(\partial_\nu g^{-1})\big] \\ &= ie^{-1}\kappa\,\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\big[A_\mu g^{-1}(\partial_\rho g)\big(\partial_\nu(g^{-1}g)\big) + \partial_\nu\big((\partial_\mu g)A_\rho g^{-1}\big) - \cancel{(\partial_\nu g^{-1})(\partial_\mu g)A_\rho} + \cancel{(\partial_\mu g)A_\rho(\partial_\nu g^{-1})}\big] \\ &= ie^{-1}\kappa\,\epsilon^{\mu\nu\rho}\,\partial_\nu\,\mathrm{Tr}\left[(\partial_\mu g)A_\rho\,g^{-1}\right]\,.\end{aligned} \tag{C.5}$$

Terms not depending on A_μ:

$$\begin{aligned}&\kappa\,\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left[-e^{-2}(\partial_\mu g)g^{-1}(\partial_\rho g)(\partial_\nu g^{-1}) + \frac{2}{3}\,e^{-2}(\partial_\mu g)g^{-1}(\partial_\nu g)g^{-1}(\partial_\rho g)g^{-1}\right] = \\ &\qquad = e^{-2}\kappa\,\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left[(\partial_\mu g)g^{-1}(\partial_\nu g)\left((\partial_\rho g^{-1}) + \frac{2}{3}\,g^{-1}(\partial_\rho g)g^{-1}\right)\right] \\ &\qquad = e^{-2}\kappa\,\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left[(\partial_\mu g)g^{-1}(\partial_\nu g)g^{-1}\left(g(\partial_\rho g^{-1}) + \frac{2}{3}\,(\partial_\rho g)g^{-1}\right)\right] \\ &\qquad = -\frac{1}{3}e^{-2}\kappa\,\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left[(\partial_\mu g)g^{-1}(\partial_\nu g)g^{-1}(\partial_\rho g)g^{-1}\right]\,.\end{aligned} \tag{C.6}$$

Putting together all the pieces we find the total variation of the lagrangian:

$$\delta\mathcal{L}_{CS} = ie^{-1}\kappa\,\epsilon^{\mu\nu\rho}\,\partial_\nu\,\mathrm{Tr}\left[(\partial_\mu g)\,A_\rho\,g^{-1}\right] - \frac{e^{-2}\kappa}{3}\,\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left[(\partial_\mu g)\,g^{-1}\,(\partial_\nu g)\,g^{-1}\,(\partial_\rho g)\,g^{-1}\right]\,.$$

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