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Lusin-type theorem for functions with prescribed gradient

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Introduction

The problem of determining if, given a vector field f, there exists a function whose gradient is f is a common question in Analysis. In $\mathbb{R}$ the Fundamental theorem of calculus states that if f is continuous then the indefinite integral of f is an antiderivative, while in higher dimension the existence of a function whose gradient is f depends on whether or not curl$f$ is zero at the points of a simply connected open set; if this does not happen, there cannot exist a potential of f defined on the whole domain. Nonetheless, there are other results regarding functions with prescribed gradient, stating that a potential can be defined outside a set "small enough", under the right conditions.

In 1990 Giovanni Alberti published the paper [1], where he presented and proved a theorem on functions with prescribed gradient that shared some similarities to Lusin's Theorem, hence the "Lusin-type" definition used by Alberti to describe his result. More specifically, the theorem stated that given a Borel vector field f on a finite measure set and some ε greater than zero, it is always possible to find a set of measure less than ε such that there exists a function whose gradient is exactly f outside this set, and whose p-norm is bounded by the norm of f; in other words, the theorem states that for every vector field there exists a function which is a potential for it outside a small set.

A decade later, in 2003, Zoltan M. Balogh ([2]) studied characteristic sets and found a connection between them and the set where f agrees with the gradient of a function. In Chapter 4 of that work, Balogh proved that if f is a $\mathscr{C}^1$ smooth vector field, estimates similar to those by Alberti can be obtained even imposing higher regularities on the functions, such as asking that the gradient of the function is not only continuous but also Lipschitz or α-Hölder.

The purpose of this work is to study the topic of functions with prescribed gradient analysing the main results contained in those two papers and presenting proofs of some of them; to do that we will use common notions in Real Analysis and Measure Theory, along with some more specific results on Hausdorff measures.

After some preliminar definitions and results in Chapter 1, in Chapter 2 we will discuss and prove Alberti's Lusin-type theorem. The proof, which comes from Alberti's paper, is a constructive one: after splitting the domain of f in cubes, we will construct appropriate functions on the cubes and use them to define a function whose gradient approximates f outside a set of measure ε; then, we will use an iterative construction to describe the function as a series and prove that this series satisfies the desired properties.

Chapter 3 will instead be dedicated to some refinements and applications of Alberti's theorem. We will follow Balogh steps and prove his result, similarly to Alberti's theorem: with an iterative construction we will build a fractal set and we will prove that a function defined on it has the correct regularity and its gradient agrees with f on it; the estimates will follow from those on the measure and the Hausdorff dimension of the set.

Lastly, we will consider a specific vector field and study the set where the gradient agrees with the function; in this example (also by Balogh) we will use specific notions from Geometric Measure Theory to prove that imposing higher regularities, such as the function being of class $\mathscr{C}^2$, strongly bounds the Hausdorff dimension of the set where the gradient agrees with f.

Chapter 1

Preliminar results

In this chapter we will present some useful preliminar results; they will be definitions and well-known results and theorems in Analysis that will be used frequently in the following chapters.

For the whole chapter we will consider functions defined on $\Omega \subseteq \mathbb{R}^N$ open set, unless specified differently.

We begin with common notions regarding differentiability, gradients and the Mean value theorem.

Definition (Gradient and Hessian of a function)**.** Given $f : \Omega \to \mathbb{R}$ differentiable in Ω, we define the *gradient* of f as $\nabla f : \Omega \to \mathbb{R}^N$ with components

$$(\nabla f)_i = \frac{\partial f}{\partial x_i}$$

for all $1 \leq i \leq N$.

Given $f : \Omega \to \mathbb{R}$, if all second-order partial derivatives of f exist we define the *Hessian* of f as the matrix $\nabla^2 f : \Omega \to \mathcal{M}_{n\times n}(\mathbb{R})$ with components

$$(\nabla^2 f)_{i,j} = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

for all $1 \leq i, j \leq n$.

Definition (Functions of class C^k)**.** Let $f : \Omega \to \mathbb{R}$ be a function. Then:

- for $k \in \mathbb{N}$ we say that f is *of class* $\mathscr{C}^k$, and we write $f \in \mathscr{C}^k(\Omega)$, if all k-th order partial derivatives of f are defined and they are continuous functions in Ω;
- we say that f is *of class* $\mathscr{C}^\infty$, and we write $f \in \mathscr{C}^\infty(\Omega)$, if $f \in \mathscr{C}^k(\Omega)$ for every $k \in \mathbb{N}$;
- for $0 < \alpha \leq 1$, we say that f is *of class* $\mathscr{C}^{1,\alpha}$, and we write $f \in \mathscr{C}^{1,\alpha}(\Omega)$, if $f \in \mathscr{C}^1(\Omega)$ and ∇f is α-Hölder, meaning that there exists a constant $C > 0$ such that $|\nabla f(x) - \nabla f(y)| \leq C|x - y|^\alpha$ for all $x, y \in \Omega$; in particular $f \in \mathscr{C}^{1,1}(\Omega)$ if $f \in \mathscr{C}^1(\Omega)$ and ∇f is Lipschitz.

Theorem 1.1 (Mean Value Theorem)**.** *Let $g : [a, b] \to \mathbb{R}$ be a continuous function and differentiable on $]a, b[$, where $a < b$. Then there exists $\tau \in]a, b[$ such that*

$$g(b) - g(a) = g'(\tau) \cdot (b - a).$$

Theorem 1.2 (Differentiability of a function series)**.** *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $\mathscr{C}^1(\Omega)$ and let f be defined as the series $f(x) = \sum_{n=1}^{\infty} f_n(x)$. If*

i. there exists $x_0 \in \Omega$ such that $f(x_0)$ converges,

ii. the series $F(x) = \sum_{n=1}^{\infty} \nabla f_n(x)$ is uniformly convergent in Ω,

then $f(x)$ is uniformly convergent in Ω and $\nabla f = F$ in Ω.

Then, we introduce some important concepts in Measure Theory, such as Lebesgue measure and its properties, L^p norms and Hölder's Inequality and Lusin's Theorem.

Definition (Lebesgue measure)**.** Denoting by $\mathscr{L}(\mathbb{R}^N)$ the σ-algebra of Lebesgue-measurable subsets of $\mathbb{R}^N$, for all $E \in \mathscr{L}(\mathbb{R}^N)$ we denote with $|E|$ the *Lebesgue measure* of E. By definition of measure, the following properties hold true:

- $|\emptyset| = 0$;
- $|E| \geq 0$ for all $E \in \mathscr{L}(\mathbb{R}^N)$;
- $|\cdot|$ is σ-additive, meaning that given $\{E_i\}_{i\in\mathbb{N}}$ collection of pairwise disjoint sets in $\mathscr{L}(\mathbb{R}^N)$, we have
$$\left|\bigcup_{i=1}^{\infty} E_i\right| = \sum_{i=1}^{\infty} |E_i|.$$

Moreover, $|\cdot|$ is subadditive, meaning that given $\{E_i\}_{i\in\mathbb{N}}$ collection of sets in $\mathscr{L}(\mathbb{R}^N)$ we have
$$\left|\bigcup_{i=1}^{\infty} E_i\right| \leq \sum_{i=1}^{\infty} |E_i|.$$

Theorem 1.3 (Continuity of the Lebesgue measure)**.** *If $\{E_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathscr{L}(\mathbb{R}^N)$ such that $E_n \subseteq E_{n+1}$ for all $n \in \mathbb{N}$, then*

$$\left|\bigcup_{n=1}^{\infty} E_n\right| = \lim_{n\to\infty} |E_n|.$$

If $\{F_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathscr{L}(\mathbb{R}^N)$ such that $F_{n+1} \subseteq F_n$ for all $n \in \mathbb{N}$ and $|F_1| < \infty$, then
$$\left|\bigcap_{n=1}^{\infty} F_n\right| = \lim_{n\to\infty} |F_n|.$$

Theorem 1.4 (Regularity of the Lebesgue measure). *For $E \in \mathscr{L}(\mathbb{R}^N)$*

$$|E| = \inf\{|U| : E \subseteq U,\, U \text{ open set}\} = \sup\{|K| : K \subseteq E,\, K \text{ compact set}\}.$$

Definition (Borel sets and Borel functions). The *Borel σ-algebra* $\mathscr{B}(\Omega)$ is the smaller σ-algebra of subsets of Ω that contains all open subsets of Ω with respect to the topology induced on Ω; a set $E \in \mathscr{B}(\Omega)$ is called *Borel set*.

The function $f : \Omega \to \mathbb{R}^M$, is a *Borel function* if for all $A \subseteq \mathbb{R}^M$ open sets, $f^{-1}(A) \in \mathscr{B}(\Omega)$.

Definition (L^p norm). Given $f : \Omega \to \mathbb{R}^M$ measurable function:

- for $1 \leq p < \infty$ we denote $\|f\|_p = \left(\int_\Omega |f(x)|^p dx\right)^{1/p}$, where the integral is with respect to the Lebesgue measure;
- we denote
$$\|f\|_\infty = \inf\Big\{K \geq 0 : \big|\{x \in \Omega : |f(x)| > K\}\big| = 0\Big\}.$$

Definition (Infinity-norm for matrices). Given $F : \Omega \to \mathcal{M}_{N\times N}(\mathbb{R})$ we will use the following notations:

$$\|F(x)\|_\infty = \max\left\{|(F(x))_{i,j}| : 1 \leq i, j \leq N\right\}$$
$$\|F\|_\infty = \inf\Big\{K \geq 0 : \big|\{x \in \Omega : \|F(x)\|_\infty > K\}\big| = 0\Big\}.$$

Theorem 1.5 (Hölder's Inequality). *Suppose $1 \leq p, q \leq \infty$ such that $1/p + 1/q = 1$ (where we consider $1/\infty = 0$). If $f, g : \Omega \to \mathbb{R}$ are measurable functions on Ω then*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Theorem 1.6 (Lusin's Theorem). *Let $\Omega \subseteq \mathbb{R}^N$ be an open set with finite measure and let $f : \Omega \to \mathbb{R}^N$ be a measurable function. Then, for every $\varepsilon > 0$ there exists a compact set $K \subseteq \Omega$ such that $|K^c| < \varepsilon$ and f restricted to K is continuous; moreover, there exists a continuous function $f_1 : \Omega \to \mathbb{R}^N$ that coincides with f on K.*

In Chapter 3 we will also use other Measure Theory notions, such as Hausdorff measures and Hasudorff dimension.

Definition (Hausdorff measures). Let $k \geq 0$, $\delta > 0$ and $E \subseteq \mathbb{R}^N$; we denote

$$\mathcal{H}^k_\delta(E) = \frac{\omega_k}{2^k} \inf\left\{\sum_{n\in\mathbb{N}} (\operatorname{diam}(A_n))^k : \operatorname{diam}(A_n) < \delta,\ E \subseteq \bigcup_{n\in\mathbb{N}} A_n\right\}$$

where

$$\omega_k = \frac{\pi^{k/2}}{\Gamma(1 + k/2)} \quad \text{and} \quad \Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx.$$

We define the *k-dimensional Hausdorff measure* by

$$\mathcal{H}^k(E) = \sup_{\delta>0} \mathcal{H}^k_\delta(E).$$

Lemma 1.7. *Given $k \geq 0$ and $E \subseteq \mathbb{R}^N$, the function $\mathcal{H}^k_\delta(E)$ is decreasing in δ, meaning that*

$$\mathcal{H}^k(E) = \sup_{\delta>0} \mathcal{H}^k_\delta(E) = \lim_{\delta\to 0^+} \mathcal{H}^k_\delta(E).$$

Theorem 1.8 (Properties of the Hausdorff measures)**.** *Considering the Hausdorff measures in $\mathbb{R}^N$ we have:*

- *for every $k \geq 0$, $\mathcal{H}^k$ is a Borel measure, meaning that all Borel sets are $\mathcal{H}^k$-measureable sets;*
- *$\mathcal{H}^0$ is the counting measure in $\mathbb{R}^N$;*
- *for $k > N$, $\mathcal{H}^k(E) = 0$ for all $E \subseteq \mathbb{R}^N$;*
- *if $B \subseteq \mathbb{R}^N$ is a Borel set, then for all $\delta > 0$ we have $|B| = \mathcal{H}^N(B) = \mathcal{H}^N_\delta(B)$.*

Definition (Hausdorff dimension)**.** The *Hausdorff dimension* of $E \subseteq \mathbb{R}^N$ is defined as

$$\dim_{\mathcal{H}}(E) = \inf\{k \geq 0 : \mathcal{H}^k(E) = 0\}.$$

Theorem 1.9 (Hausdorff measure of a set depending on its dimension)**.** *Given $E \subseteq \mathbb{R}^N$ we have that if $0 \leq k' < k$ and $\mathcal{H}^k(E) > 0$, then $\mathcal{H}^{k'}(E) = \infty$. It follows that $\mathcal{H}^k(E) = \infty$ if $k < \dim_{\mathcal{H}}(E)$ and $\mathcal{H}^k(E) = 0$ if $k > \dim_{\mathcal{H}}(E)$.*

Lemma 1.10. *Restricting the coverings of $E \subseteq \mathbb{R}^N$ to coverings with cubes Q_n with faces parallel to the coordinate hyperplanes, we can define*

$$\mathcal{M}^k_\delta(E) = \inf\left\{\sum_{n\in\mathbb{N}} (\ell(Q_n))^k : \mathit{diam}(Q_n) < \delta,\ E \subseteq \bigcup_{n\in\mathbb{N}} Q_n\right\}$$

where $\ell(Q_n)$ is the side length of Q_n, and

$$\mathcal{M}^k(E) = \sup_{\delta>0} \mathcal{M}^k_\delta(E).$$

With these definitions, there exist two constants $K_1, K_2 > 0$ such that for all $E \subseteq \mathbb{R}^N$

$$K_1 \cdot \mathcal{M}^k(E) \leq \mathcal{H}^k(E) \leq K_2 \cdot \mathcal{M}^k(E);$$

this means that evaluating the measures with only cubes gives the same Hausdorff dimension.

Definition (Bi-Lipschitz function)**.** A function $f : \mathbb{R}^N \to \mathbb{R}^M$ is *bi-Lipschitz* if there exist two constants $A, B > 0$ such that

$$A|x-y| \leq |f(x) - f(y)| \leq B|x-y| \quad \forall x, y \in \mathbb{R}^N.$$

Theorem 1.11 (Invariance of Hausdorff dimension under bi-Lipschitz functions)**.** *Let* $M \geq N$ *and* $f : \mathbb{R}^N \to \mathbb{R}^M$ *be a bi-Lipschitz function; then for all* $0 \leq k \leq N$ *and* $E \subseteq \mathbb{R}^N$

$$\dim_{\mathcal{H}}(f(E)) = \dim_{\mathcal{H}}(E).$$

Lastly, we present some other definitions and results, including a lemma which will be used often in the following chapters and that we will prove.

Definition (Distance between two sets)**.** Given two sets $A, B \subseteq \mathbb{R}^N$ we define the *distance between* A *and* B as $\text{dist}(A, B) = \inf\{|x - y| : x \in A, y \in B\}$; in particular, given $x_0 \in \mathbb{R}^N$ the *distance between* x_0 *and* B is $\text{dist}(x_0, B) = \inf\{|x_0 - y| : y \in B\}$.

Theorem 1.12 (Distance between closed and compact sets)**.** *If* $A, B \subseteq \mathbb{R}^N$ *are disjoint closed sets and* A *is compact, then* $dist(A, B) > 0$.

Definition (Operator norm)**.** Let V, W be normed $\mathbb{R}$-vector spaces, $V \neq \{0\}$, and let $A : V \to W$ be a linear operator; then the *operator norm* of A is

$$\|A\|_{\text{op}} = \inf\{C \geq 0 : |Av| \leq C|v| \text{ for all } v \in V\} = \sup\left\{\frac{|Av|}{|v|} : v \in V \setminus \{0\}\right\}$$

where we denoted both the norm in V and the norm in W as $|\cdot|$.

Definition (Line integral)**.** Let $f : \Omega \to \mathbb{R}^N$ be a vector field; given a piecewise smooth curve $\gamma : [a, b] \to \mathbb{R}^N$ with $|\gamma'(t)| \neq 0$ for all $t \in [a, b]$ and $\gamma([a, b]) \subseteq \Omega$, the *line integral* of f along γ is

$$\int_\gamma f ds := \int_a^b f(\gamma(t)) \cdot \gamma'(t) dt.$$

Theorem 1.13 (Gradient theorem)**.** *Let* $u : \Omega \to \mathbb{R}$ *be a differentiable function and* $\gamma : [a, b] \to \mathbb{R}^N$ *with* $|\gamma'(t)| \neq 0$ *for all* $t \in [a, b]$ *and* $\gamma([a, b]) \subseteq \Omega$*; if* γ *is a piecewise smooth curve and* Ω *is simply connected, then*

$$\int_\gamma \nabla u \, ds = u(\gamma(b)) - u(\gamma(a));$$

if γ *is a closed curve, meaning that* $\gamma(a) = \gamma(b)$*, then*

$$\int_\gamma \nabla u \, ds = 0.$$

Theorem 1.14 (Tietze Extension Theorem)**.** *If* X *is a normal space,* $A \subseteq X$ *is a closed subset and* $f : A \to \mathbb{R}^N$ *is continuous, then there exists a continuous function* $F : X \to \mathbb{R}^N$ *such that* $F|_A = f$ *and*

$$\sup_{x \in X} |F(x)| = \sup_{x \in A} |f(x)|.$$

Lemma 1.15. *Let $A, B \subseteq \mathbb{R}^N$ be closed cubes with the same center and parallel faces, with side length respectively $2a$ and $2a + 4d$, $a, d > 0$; then there exists a function $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ such that:*

(i) $\varphi \in \mathscr{C}^\infty(\mathbb{R}^N)$;

(ii) $0 \le \varphi(x) \le 1$ *for all* $x \in \mathbb{R}^N$;

(iii) $\varphi(x) = 1$ *for* $x \in A$ *and* $\varphi(x) = 0$ *for* $x \in B^c$;

(iv) $\|\nabla\varphi\|_\infty \le {}^{C_1}/_{d}$ *and* $\left\|\nabla^2\varphi\right\|_\infty \le {}^{C_2}/_{d^2}$ *for some constants* C_1, C_2.

Proof. Without loss of generality we can assume $A = [-a, a]^N$ and $B = [-a-2d, a+2d]^N$, since all wanted properties are invariant for isometries and in these hypotheses A and B are isometric to the aforementioned cubes via the same translation and rotation.

Denoting with B_r the closed ball in $\mathbb{R}^N$ of center 0 and radius r, let $\psi \in \mathscr{C}_c^\infty(\mathbb{R}^N)$ be a function such that $\psi \ge 0$, $\operatorname{supp}(\psi) \subseteq B_1$ and $\int_{\mathbb{R}^N} \psi(x)dx = 1$; these properties can always be obtained eventually normalizing appropriately a generic function in $\mathscr{C}_c^\infty(\mathbb{R}^N)$. We then define ψ_d, $\varphi : \mathbb{R}^N \to \mathbb{R}$ as

$$\psi_d(x) = \frac{1}{d^N}\psi\left(\frac{x}{d}\right) \quad \text{and} \quad \varphi(x) = \left(\mathbb{1}_{[-a-d,a+d]^N} * \psi\right)(x) = \int_{\mathbb{R}^N} \mathbb{1}_{[-a-d,a+d]}(x-y)\psi(y)dy$$

where $\mathbb{1}_I$ is the indicator function of I and $*$ is the convolution product.

Since $\mathbb{1}_{[-a-d,a+d]^N} \in L^1(\mathbb{R}^N)$ and $\psi \in \mathscr{C}^\infty(\mathbb{R}^N)$ it follows that $\varphi \in \mathscr{C}^\infty(\mathbb{R}^N)$ thus proving *(i)*, and for all $i, j \in \{1, \dots, N\}$

$$\frac{\partial\varphi}{\partial x_i} = \mathbb{1}_{[-a-d,a+d]^N} * \frac{\partial\psi_d}{\partial x_i} \quad \text{and} \quad \frac{\partial^2\varphi}{\partial x_j\partial x_i} = \mathbb{1}_{[-a-d,a+d]^N} * \frac{\partial^2\psi_d}{\partial x_j\partial x_i}. \tag{1.1}$$

It can be seen that

$$\int_{\mathbb{R}^N} \psi_d(x)dx = \int_{\mathbb{R}^N} \frac{1}{d^N}\psi\left(\frac{x}{d}\right)dx = \int_{\mathbb{R}^N} \frac{1}{d^N}\psi(y)d^N\,dy = \int_{\mathbb{R}^N} \psi(y)dy = 1;$$

since $0 \le \mathbb{1}_{[-a-d,a+d]^N} \le 1$ and $\psi_d \ge 0$, for all $x \in \mathbb{R}$

$$\varphi(x) = \int_{\mathbb{R}^N} \mathbb{1}_{[-a-d,a+d]^N}(x-y)\psi_d(y)dy \ge 0$$

and

$$\varphi(x) = \int_{\mathbb{R}^N} \mathbb{1}_{[-a-d,a+d]^N}(x-y)\psi_d(y)dy \le \int_{\mathbb{R}^N} \psi_d(y)dy = 1$$

thus proving *(ii)*.

For $x \in A$ and $y \in B_d$ we have $|(x-y)-x| = |y| \le d$ so $x - y \in [-a-d, a+d]^n$, and $\operatorname{supp}(\psi_d) = B_d$, therefore

$$\begin{aligned}\varphi(x) &= \int_{\mathbb{R}^N} \mathbb{1}_{[-a-d,a+d]^N}(x-y)\psi_d(y)dy = \int_{B_d} \mathbb{1}_{[-a-d,a+d]^N}(x-y)\psi_d(y)dy = \\ &= \int_{B_d} \psi_d(y)dy = 1;\end{aligned}$$

while for $z \in B^c$ we have $|(z-y)-x| = |(z-x)-y| \geq |z-x|-|y| > 2d-d = d$ therefore $z-y \notin [-a-d, a+d]^N$ and

$$\varphi(z) = \int_{\mathbb{R}^n} \mathbb{1}_{[-a-d,a+d]^N}(z-y)\psi_d(y)dy = \int_{B_d} \mathbb{1}_{[-a-d,a+d]^N}(z-y)\psi_d(y)dy = 0,$$

which proves *(iii)*.

Lastly, since $\psi \in \mathscr{C}^\infty(\mathbb{R}^N)$, all first-order and second-order derivatives of ψ are continuous on its support, which is a compact set, therefore they have maximum, meaning that

$$\|\nabla\psi\|_\infty < \infty \quad \text{and} \quad \left\|\nabla^2\psi\right\|_\infty < \infty.$$

Since for all $x \in \mathbb{R}^N$

$$\nabla\psi_d(x) = \nabla\left(\frac{1}{d^N}\psi\left(\frac{x}{d}\right)\right) = \frac{1}{d^{N+1}}\nabla\psi\left(\frac{x}{d}\right) \quad \text{and} \quad \nabla^2\psi_d(x) = \frac{1}{d^{N+2}}\nabla^2\psi\left(\frac{x}{d}\right)$$

we have, using **(1.1)**

$$|\nabla\varphi(x)| \leq \int_{\mathbb{R}^N} |\nabla\psi_d(y)|\, dy \leq \int_{B_d} \frac{1}{d^{N+1}} \left|\nabla\psi\left(\frac{y}{d}\right)\right| dy \leq \frac{1}{d^{N+1}} \int_{B_d} \|\nabla\psi\|_\infty \, dy =$$
$$= \frac{Kd^N \|\nabla\psi\|_\infty}{d^{N+1}} = \frac{C_1}{d}$$

and

$$\left\|\nabla^2\varphi(x)\right\|_\infty \leq \int_{\mathbb{R}^N} \|\nabla\psi_d(y)\|_\infty \, dy \leq \int_{B_d} \frac{1}{d^{N+2}} \left\|\nabla^2\psi\left(\frac{y}{d}\right)\right\|_\infty dy \leq$$
$$\leq \frac{1}{d^{N+2}} \int_{B_d} \left\|\nabla^2\psi\right\|_\infty dy = \frac{Kd^N \left\|\nabla^2\psi\right\|_\infty}{d^{N+2}} = \frac{C_2}{d^2}.$$

for some constants K, C_1, C_2.

Then

$$\|\nabla\varphi\|_\infty \leq \frac{C_1}{d} \quad \text{and} \quad \left\|\nabla^2\varphi\right\|_\infty \leq \frac{C_2}{d^2}$$

proving *(iv)*.

□

Chapter 2

Alberti's Lusin-type theorem

The purpose of this chapter is to prove Alberti's Theorem, a result that states that given a Borel vector field f and $\varepsilon > 0$, f agrees with ∇u for some function u of class $\mathscr{C}^1$ outside an open set with measure less than ε.

In general, given a vector field $f : \Omega \to \mathbb{R}^N$ with $\Omega \subseteq \mathbb{R}^N$ simply connected open set, there exists a function whose gradient is f if and only if $\operatorname{curl} f = 0$ on Ω. This means that if $\operatorname{curl} f \neq 0$ at some points of Ω such function whose gradient is f cannot exist; but Alberti's theorem states that if f is a Borel vector field, a weaker property holds true: there exists a function whose gradient is f outside a set of arbitrarily small measure. This is true even if $\operatorname{curl} f \neq 0$ everywhere in Ω: in this case the set where f does not agree with ∇u must be dense in Ω.

Theorem 2.1 (Alberti's Theorem)**.** *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure and let $f : \Omega \to \mathbb{R}^N$ be a Borel function; then $\forall \varepsilon > 0$ there exist an open set $A \subseteq \Omega$ and a function $u \in \mathscr{C}^1(\Omega)$ such that*

$$|A| \leq \varepsilon |\Omega| \tag{2.1a}$$

$$f = \nabla u \quad \text{on } \Omega \setminus A \tag{2.1b}$$

$$\|\nabla u\|_p \leq C \varepsilon^{1/p - 1} \|f\|_p \quad \forall p \in [1, +\infty] \tag{2.1c}$$

where C is a constant which depends on N only.

Remark 2.2. When $p = 1$ Alberti's Theorem holds true even without $|\Omega| < \infty$ and it can be stated as follows:

Let Ω be an open subset of $\mathbb{R}^N$ and let $f : \Omega \to \mathbb{R}^N$ be a Borel function; then $\forall \varepsilon > 0$ there exist a function $u \in \mathscr{C}^1(\Omega)$ such that $f = \nabla u$ outside an open set with measure less than ε and $\|\nabla u\|_1 \leq C \|f\|_1$.

We now prove Alberti's Theorem. Before doing that, we prove some auxiliary lemmas needed to prove the Theorem.

This first lemma gives a property similar to uniform continuity on compact sets.

Lemma 2.3. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set with finite measure, let $K \subset \Omega$ be a compact set and let $f : \Omega \to \mathbb{R}^N$ be a continuous function. Then $\forall \eta > 0$ there exists $\delta > 0$ such that for all $x \in K, y \in \Omega$ if $|x-y| < \delta$ then $|f(x) - f(y)| < \eta$, and $Q(x, 4\delta) \subseteq \Omega$ where $Q(x, 4\delta)$ is the cube with center x and side 4δ.*

Proof. Since $K \subset \Omega$ and Ω is an open set, Ω^c and K are disjoint closed sets and K is compact, so $0 < d = \operatorname{dist}(\Omega^c, K) < \infty$ by **(1.12)**. It can be seen that $Q(x, 4\delta') \subseteq B(x, 2\delta'\sqrt{N})$; if $2\delta'\sqrt{N} < d$, i.e. $\delta' < d/2\sqrt{N}$, then $Q(x, 4\delta') \subseteq \Omega$.

Let then $K' = \{x \in \mathbb{R}^N : \operatorname{dist}(K, x) \leq d/2\sqrt{N} = \delta_1\}$; K' is compact, so by Heine-Cantor theorem f is uniformly continuous on K', meaning there exists $\delta_2 > 0$ such that $\forall x, y \in K'$ if $|x - y| < \delta_2$ then $|f(x) - f(y)| < \eta$.

Then, choosing $\delta < \min\{\delta_1, \delta_2\}$ it can be seen that $\forall x \in K \subseteq K', y \in \Omega$ if $|x-y| < \delta$, then $y \in K'$ because $\delta < \delta_1$ and $|f(x) - f(y)| < \eta$ because $\delta < \delta_2$; moreover, from $\delta < \delta_1$ it follows $Q(x, 4\delta) \subseteq \Omega$, which proves the lemma.

□

We then prove another auxiliary lemma.

Lemma 2.4. *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure, let $f : \Omega \to \mathbb{R}^N$ and let η and ε be positive real numbers. Then there exist a compact set $K \subset \Omega$ and a function $u \in \mathscr{C}^1_c(\Omega)$ such that*

$$|\Omega \setminus K| \leq \varepsilon |\Omega| \tag{2.2a}$$

$$|f - \nabla u| \leq \eta \quad \text{on } K \tag{2.2b}$$

$$\|\nabla u\|_p \leq C' \varepsilon^{1/p - 1} \|f\|_p \quad \forall p \in [1, +\infty] \tag{2.2c}$$

where C' is a constant which depends on N only.

Proof. We can assume that $0 < \varepsilon < 1$.

Since Ω has finite measure, by the regularity of the Lebesgue measure **(1.4)** there exists a compact set $K' \subset \Omega$ such that $|\Omega \setminus K'| < \frac{\varepsilon}{2}|\Omega|$. By **Lemma 2.3** there exists $\delta > 0$ such that

$$\forall x \in K', y \in \Omega\,, \text{ if } |x - y| < \delta \text{ then } |f(x) - f(y)| < \eta\,, \text{ and } Q(x, 4\delta) \subset \Omega. \tag{2.3}$$

Let $\{T_i\}_{i \in I}$ be the family of the closed cubes with side δ and centers y_i on the lattice $(\delta\mathbb{Z})^N$ that intersect K'; K' is a compact set so the family $\{T_i\}$ is finite, i.e. $|I| < \infty$, and by choice of δ it follows that $T_i \subseteq \Omega \; \forall i \in I$.

For all $i \in I$:

- let Q_i be the closed cube with center y_i and side $(1 - \frac{\varepsilon}{2N})\delta$, meaning that $Q_i \subset T_i$;
- let $a_i \in \mathbb{R}^N$ be the mean value of f on T_i, i.e. $a_i = \frac{1}{|T_i|} \int_{T_i} f(x)dx$;
- let $\varphi_i \in \mathscr{C}^1(\Omega)$ be such that $\varphi_i(x) = 1$ for $x \in Q_i$, $\varphi_i(x) = 0$ for $x \in T_i^c$ and $\|\nabla \varphi_i\|_\infty \leq 8C_1N/\delta\varepsilon$; such functions exist thanks to **Lemma 1.15** on cubes Q_i, T_i with $d = \delta\varepsilon/8N$.

Eventually, let $u\colon \Omega \to \mathbb{R}$ be $u(x) = \sum_{i\in I} \varphi_i(x)\langle a_i, x - y_i\rangle$ and $K = \bigcup_{i\in I} Q_i$; K is a compact set, because it is a finite union of compact sets, while for u the following properties hold true:

- $\operatorname{supp}(u) \subseteq \bigcup_{i\in I} T_i$ because $\operatorname{supp}(\varphi_i) \subseteq T_i$ for all $i \in I$ by definition;
- u is a sum of functions of class $\mathscr{C}^1$, thus $u \in \mathscr{C}^1(\Omega)$;
- for $i \neq j$ we have $Q_i \cap T_j = \emptyset$, $\varphi_j|_{Q_i} = 0$ and $\varphi_i|_{Q_i} = 1$, therefore

  $$u|_{Q_i} = \sum_{j\in I} \varphi_j|_{Q_i}(x)\langle a_j, x - y_j\rangle = \langle a_i, x - y_i\rangle$$

  so $\nabla u|_{Q_i} = a_i$.

It is now necessary to prove that u and K satisfy **(2.2a)**, **(2.2b)** and **(2.2c)**.

The property **(2.2a)** follows from the inequality

$$1 - \left(1 - \frac{\varepsilon}{2N}\right)^N \leq \frac{\varepsilon}{2} \ \text{ for } \ 0 \leq \varepsilon < 1$$

which is Bernoulli's inequality

$$(1 + x)^n \geq 1 + nx \quad \forall n \in \mathbb{N} \quad \forall x > -1$$

evaluated at $n = N$ and $x = -\varepsilon/2N > -1$.

Using this inequality

$$|T_i \smallsetminus Q_i| = |T_i| - |Q_i| = \left[1 - \left(1 - \frac{\varepsilon}{2N}\right)^N\right] |T_i| \leq \frac{\varepsilon}{2}|T_i|;$$

since $K' \subseteq \bigcup_{i\in I} T_i \subseteq \Omega$ and $K = \bigcup_{i\in I} Q_i \subseteq \bigcup_{i\in I} T_i$,

$$\Omega \smallsetminus K \subseteq (\Omega \smallsetminus K') \cup \left(\bigcup_{i\in I}(T_i \smallsetminus Q_i)\right)$$

and thus

$$|\Omega \smallsetminus K| \leq |\Omega \smallsetminus K'| + \sum_{i\in I} |T_i \smallsetminus Q_i| \leq \frac{\varepsilon}{2}|\Omega| + \sum_{i\in I} \frac{\varepsilon}{2}|T_i| \leq \frac{\varepsilon}{2}|\Omega| + \frac{\varepsilon}{2}|\Omega| = \varepsilon|\Omega|.$$

As for **(2.2b)**, from $\nabla u|_{Q_i} = a_i$ it follows that for $x \in Q_i$

$$\begin{aligned}|\nabla u(x) - f(x)| = |a_i - f(x)| = \left|\frac{1}{|T_i|}\int_{T_i} f(y)dy - \frac{1}{|T_i|}\int_{T_i} f(x)dy\right| \leq \\ \leq \frac{1}{|T_i|}\int_{T_i} |f(x) - f(y)|dy \leq \frac{1}{|T_i|}\int_{T_i} \eta\, dy = \frac{1}{|T_i|}\cdot \eta|T_i| = \eta\end{aligned}$$

where the last inequality follows from **(2.3)** since $x \in K$ and for $y \in T_i \subseteq \Omega, |x-y| < \delta$ as T_i has side δ.

Concerning **(2.2c)**, by the Leibniz rule

$$\nabla u(x) = \nabla \left(\sum_{i \in I} \varphi_i(x) \langle a_i, x - y_i \rangle \right) = \sum_{i \in I} \nabla \varphi_i(x) \langle a_i, x - y_i \rangle + \sum_{i \in I} a_i \varphi_i(x)$$

and so

$$\|\nabla u\|_p \leq \left\| \sum_{i \in I} \nabla \varphi_i \langle a_i, \cdot - y_i \rangle \right\|_p + \left\| \sum_{i \in I} a_i \varphi_i \right\|_p$$

In order to evaluate the p-norms the following facts are useful:

- **a.** $\mathrm{supp}(\varphi_i)$ are disjoint;
- **b.** $\nabla \varphi_i(x) = 0$ for $x \in (T_i \smallsetminus Q_i)^c$;
- **c.** $|\nabla \varphi_i(x)| \leq \|\nabla \varphi_i\|_\infty \leq {}^{8C_1N}/_{\delta\varepsilon}$;
- **d.** for $x \in T_i$ $|\langle a_i, x - y_i \rangle| \leq |a_i||x - y_i| \leq \sqrt{N}\delta|a_i|$ by the Cauchy-Schwarz inequality and the fact that T_i is a cube of side δ;
- **e.** $|T_i \smallsetminus Q_i| \leq \frac{\varepsilon}{2}|T_i| < \varepsilon|T_i|$;
- **f.** $\int_\Omega |\varphi_i(x)|^p dx = \int_{T_i} \varphi_i(x)^p dx \leq |T_i|$ using the facts that $\varphi_i(x) = 0$ for $x \in T_i^c$ and $0 \leq \varphi_i \leq 1$.

For $1 \leq p < \infty$

$$\begin{aligned}
\left\| \sum_{i \in I} \nabla \varphi_i \langle a_i, \cdot - y_i \rangle \right\|_p &= \left(\int_\Omega \left| \sum_{i \in I} \nabla \varphi_i(x) \langle a_i, x - y_i \rangle \right|^p dx \right)^{1/p} = \\
&\overset{\mathbf{a.}}{=} \left(\int_\Omega \sum_{i \in I} |\nabla \varphi_i(x) \langle a_i, x - y_i \rangle|^p \, dx \right)^{1/p} = \\
&\overset{\mathbf{b.}}{=} \left[\sum_{i \in I} \left(\int_{T_i \smallsetminus Q_i} |\nabla \varphi_i(x)|^p |\langle a_i, x - y_i \rangle|^p dx \right) \right]^{1/p} \leq \\
&\overset{\mathbf{c.\,\&\,d.}}{\leq} \left[\sum_{i \in I} \left(\int_{T_i \smallsetminus Q_i} \left(\frac{8C_1N}{\delta\varepsilon} \right)^p (\sqrt{N}\delta|a_i|)^p dx \right) \right]^{1/p} = \\
&= \left[\sum_{i \in I} \left(\frac{8C_1N\sqrt{N}}{\varepsilon} \right)^p |a_i|^p |T_i \smallsetminus Q_i| \right]^{1/p} \leq \\
&\overset{\mathbf{e.}}{\leq} \frac{8C_1N\sqrt{N}}{\varepsilon} \left[\sum_{i \in I} \varepsilon |a_i|^p |T_i| \right]^{1/p} = 8C_1N\sqrt{N}\varepsilon^{1/p-1} \left[\sum_{i \in I} |a_i|^p |T_i| \right]^{1/p}
\end{aligned}$$

and

$$\left\|\sum_{i\in I} a_i\varphi_i\right\|_p = \left(\int_\Omega \left|\sum_{i\in I} a_i\varphi_i(x)\right|^p dx\right)^{1/p} \overset{\mathbf{a.}}{=} \left[\sum_{i\in I}\left(|a_i|^p\int_\Omega |\varphi_i(x)|^p dx\right)\right]^{1/p} \overset{\mathbf{f.}}{\leq} \left[\sum_{i\in I}|a_i|^p|T_i|\right]^{1/p}.$$

Using Holder's inequality **(1.5)**

$$\left|\int_{T_i} f(x)dx\right|^p \leq \left(\int_{T_i}|f(x)|dx\right)^p \leq \left(\int_{T_i}|f(x)|^p dx\right)\left(\int_{T_i} 1^{\frac{p}{p-1}}dx\right)^{p-1} = |T_i|^{p-1}\int_{T_i}|f(x)|^p dx$$

which means that

$$\int_{T_i}|f(x)|^p dx \geq |T_i|^{1-p}\left|\int_{T_i} f(x)dx\right|^p = |T_i|\left|\frac{1}{|T_i|}\int_{T_i} f(x)dx\right|^p = |T_i||a_i|^p$$

and thus

$$\int_\Omega |f(x)|^p dx \geq \sum_{i\in I}\int_{T_i}|f(x)|^p dx \geq \sum_{i\in I}|T_i||a_i|^p$$

It follows that

$$\|\nabla u\|_p \leq 8C_1N\sqrt{N}\varepsilon^{1/p-1}\left(\sum_{i\in I}|a_i|^p|T_i|\right)^{1/p} + \left(\sum_{i\in I}|a_i|^p|T_i|\right)^{1/p} \leq$$

$$\leq C'\varepsilon^{1/p-1}\left(\sum_{i\in I}|a_i|^p|T_i|\right)^{1/p} \leq C'\varepsilon^{1/p-1}\left(\int_\Omega |f(x)|^p dx\right)^{1/p} = C'\varepsilon^{1/p-1}\|f\|_p$$

with $C' = 8C_1N\sqrt{N}+1$, where $C'\varepsilon^{1/p-1} \geq 8C_1N\sqrt{N}\varepsilon^{1/p-1}+1$ because $\varepsilon \leq 1$ and $1/p \leq 1$ and so $\varepsilon^{1/p-1} \geq 1$.

For $p = \infty$, since $\operatorname{supp}(\nabla\varphi_i) \subseteq T_i$ are disjoint, and so are $\operatorname{supp}(\varphi_i) \subseteq T_i$

$$\left\|\sum_{i\in I}\nabla\varphi_i\langle a_i, \cdot - y_i\rangle\right\|_\infty \leq \sup_{i\in I}\|\nabla\varphi_i\langle a_i, \cdot - y_i\rangle\|_\infty \leq$$

$$\overset{\mathbf{c.\,\&\,d.}}{\leq} \sup_{i\in I}\left\{\frac{8C_1N}{\delta\varepsilon}\sqrt{N}\delta|a_i|\right\} = \frac{8C_1N\sqrt{N}}{\varepsilon}\sup_{i\in I}|a_i|$$

and

$$\left\|\sum_{i\in I}a_i\varphi_i\right\|_\infty \leq \sup_{i\in I}\|a_i\varphi_i\|_\infty \leq \sup_{i\in I}|a_i|\,\|\varphi_i\|_\infty \leq \sup_{i\in I}|a_i|$$

using that $\|\varphi_i\|_\infty = 1$.

Since

$$|a_i| = \frac{1}{|T_i|}\int_{T_i}|f(x)|dx \leq \frac{1}{|T_i|}\int_{T_i}\|f\|_\infty\,dx \leq \|f\|_\infty$$

it follows that

$$\|\nabla u\|_\infty \leq \frac{8C_1 N\sqrt{N}}{\varepsilon} \sup_{i\in I} |a_i| + \sup_{i\in I} |a_i| \leq C'\varepsilon^{-1} \sup_{i\in I} |a_i| \leq C'\varepsilon^{-1} \|f\|_\infty$$

with the same C' as before.

□

Lastly, we prove the Theorem.

Proof of Theorem 2.1. We may suppose $\varepsilon < 1$ and f not almost everywhere zero. We discuss separately the following two cases.

First case. f is a continuous bounded function.

Let $\{\eta_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers.

We build the sequences $\{u_n\}_{n\in\mathbb{N}}$, $\{K_n\}_{n\in\mathbb{N}}$ and $\{f_n\}_{n\in\mathbb{N}}$ by induction. We define $u_0 = 0$, $K_0 = \Omega$ and $f_0 = f$; then, given u_{n-1}, K_{n-1}, f_{n-1} we apply **Lemma 2.4** to obtain compact set K_n and $u_n \in \mathscr{C}^1_c(\Omega)$ such that

$$|\Omega \smallsetminus K_n| \leq 2^{-n}\varepsilon|\Omega| \tag{2.4a}$$

$$|f_{n-1} - \nabla u_n| \leq \eta_n \quad \text{on } K_n \tag{2.4b}$$

$$\|\nabla u_n\|_p \leq C'(2^{-n}\varepsilon)^{1/p-1} \|f_{n-1}\|_p \quad \forall p \in [1, +\infty] \tag{2.4c}$$

Then we define $f_n(x) = f_{n-1}(x) - \nabla u_n(x) \quad \forall x \in K_n$, which is continuous because f_{n-1} is continuous by induction and $u_n \in \mathscr{C}^1_c(\Omega)$, and apply Tietze Extension Theorem **(1.14)** to f_n continuous on K_n to extend it to the whole of Ω so that

$$\sup_{x\in\Omega} |f_n(x)| = \sup_{x\in K_n} |f_n(x)| \leq \eta_n. \tag{2.5}$$

Set $A = \Omega \smallsetminus \bigcap_n K_n$ and $u = \sum_n u_n$; we will choose $\{\eta_n\}_{n\in\mathbb{N}}$ such that **(2.1a)**,**(2.1b)** and **(2.1c)** are satisfied, as shown later.

Since $A = \Omega \smallsetminus \bigcap_n K_n = \Omega \cap (\bigcup_n K_n^c) = \bigcup_n (\Omega \cap K_n^c) = \bigcup_n (\Omega \smallsetminus K_n)$, A is an open set because it is union of open sets $\Omega \smallsetminus K_n$ and by **(2.4a)** and subadditivity of the measure, $|A| = |\bigcup_n (\Omega \smallsetminus K_n)| \leq \sum_n |\Omega \smallsetminus K_n| \leq \sum_n 2^{-n}\varepsilon|\Omega| = \varepsilon|\Omega|$, proving **(2.1a)**.

For $1 \leq p \leq \infty$, defining $1/\infty = 0$, we have $(2^{-n})^{1/p-1} = 2^{-n/p} \cdot 2^n \leq 2^n$ because $2^{-n/p} \leq 1$, and

$$\|f_n\|_p = \left(\int_\Omega |f_n|^p\right)^{1/p} \leq \left(\int_\Omega \|f_n\|_\infty^p\right)^{1/p} = |\Omega|^{1/p} \cdot \|f_n\|_\infty;$$

therefore

$$\sum_{n=1}^{\infty} \|\nabla u_n\|_p \overset{\textbf{(2.4c)}}{\leq} \sum_{n=1}^{\infty} C'(2^{-n}\varepsilon)^{1/p-1} \|f_{n-1}\|_p \leq C'\varepsilon^{1/p-1} \sum_{n=1}^{\infty} 2^n \|f_{n-1}\|_p \leq$$
$$\leq 2C'\varepsilon^{1/p-1} \left(\|f_0\|_p + \sum_{n=1}^{\infty} 2^n \|f_n\|_p \right) \leq$$
$$\leq 2C'\varepsilon^{1/p-1} \left(\|f\|_p + \sum_{n=1}^{\infty} 2^n \|f_n\|_\infty |\Omega|^{1/p} \right) \leq$$
$$\overset{\textbf{(2.5)}}{\leq} 2C'\varepsilon^{1/p-1} \|f\|_p \left(1 + \frac{|\Omega|^{1/p}}{\|f\|_p} \sum_{n=1}^{\infty} 2^n \eta_n \right).$$

As f is bounded and not almost everywhere 0, $p \mapsto \frac{|\Omega|^{1/p}}{\|f\|_p}$ is continuous and positive in $[1, \infty]$ hence it has a positive upper bound $M < \infty$; $\{\eta_n\}_{n\in\mathbb{N}}$ can be chosen such that $\sum_{n=1}^{\infty} 2^n \eta_n \leq 1/M$. Then

$$\|\nabla u\|_p = \left\| \sum_{n=1}^{\infty} \nabla u_n \right\|_p \leq \sum_{n=1}^{\infty} \|\nabla u_n\|_p \leq 2C'\varepsilon^{1/p-1} \|f\|_p \left(1 + M \cdot \frac{1}{M} \right) = 4C'\varepsilon^{1/p-1} \|f\|_p$$

which proves **(2.1c)** with $C = 4C'$.

Lastly, if $x \in \Omega \smallsetminus A = \bigcap_n K_n$ then $x \in K_n$ for every $n \in \mathbb{N}$ and thus by definition

$$f_n(x) = f_{n-1}(x) - \nabla u_n(x) = (f_{n-2}(x) - \nabla u_{n-1}(x)) - \nabla u_n(x) = \cdots =$$
$$= f_{n-m}(x) - \sum_{i=n-m+1}^{n} \nabla u_i(x)$$

and for $m = n$

$$f_n(x) = f_0(x) - \sum_{i=1}^{n} \nabla u_i(x) = f(x) - \nabla u(x) + \sum_{i=n+1}^{\infty} \nabla u_i(x);$$

from this it follows that

$$|f(x) - \nabla u(x)| \leq |f_n(x)| + \left| \sum_{i=n+1}^{\infty} \nabla u_i(x) \right| \leq \eta_n + \sum_{i=n+1}^{\infty} |\nabla u_i(x)| \leq \eta_n + \sum_{i=n+1}^{\infty} \|\nabla u_i\|_\infty .$$

We already proved that $\sum_{n=1}^{\infty} \|\nabla u_n\|_\infty \leq 4C'\varepsilon^{-1} \|f\|_\infty < \infty$ because f is bounded, which means that $\sum_{i=n+1}^{\infty} \|\nabla u_i\|_\infty$ converges to 0 as $n \to \infty$; similarly, $\sum_{n=1}^{\infty} 2^n \eta_n \leq 1/M < \infty$ and thus $2^n \eta_n$ converges to 0 for $n \to \infty$.

Then $|f(x) - \nabla u(x)| \leq \eta_n + \sum_{i=n+1}^{\infty} \|\nabla u_i\|_\infty \to 0$ for $n \to \infty$, so $|f(x) - \nabla u(x)| = 0$ which means $f = \nabla u$ on $\Omega \smallsetminus A$, proving **(2.1b)**.

Second case. f is a Borel function.

Let $\varepsilon > 0$ be fixed. Then, there exists $r > 0$ such that, with $B = \{x \in \Omega : |f(x)| > r\}$, $|B| < \varepsilon/4$. This follows from the fact that, denoting $E_n = \{x \in \Omega : |f(x)| > n\}$, $E_n \subseteq E_{n+1}$ and $\bigcap_n E_n = \emptyset$ and $|E_1| \leq |\Omega| < \infty$ so by continuity from above **(1.3)** we have $\lim_{n\to\infty} |E_n| = |\bigcap_n E_n| = 0$ and thus there exists $\bar{n}$ such that $|E_{\bar{n}}| < \varepsilon/4$.

By Lusin's Theorem **(1.6)** there exist f_1 continuous on Ω and a Borel set C such that $|C| \leq |B|$ and f_1 agrees with f outside C. We then define

$$f_2(x) = \begin{cases} f_1(x) & \text{if } |f_1(x)| \leq r \\ \frac{rf_1(x)}{|f_1(x)|} & \text{if } |f_1(x)| > r \end{cases}$$

Since $f_1 = f$ outside C and $|f(x)| \leq r$ outside B, f_2 agrees with f outside $C \cup B$. We have that f_2 is continuous, because f_1 is, and bounded, because $|f_2(x)| \leq r \;\; \forall x \in \Omega$; moreover $|C \cup B| \leq |C| + |B| \leq 2|B| < \varepsilon/2$ and thus, by regularity of the Lebesgue measure **(1.4)**, there exists $A_1 \supseteq C \cup B$ open set such that $|A_1| < \varepsilon/2$ and f_2 agrees with f outside A_1.

It can be seen that $\forall p \in [1, \infty[$

$$\int_{C\cup B} r^p dx = r^p |C \cup B| \leq r^p(|C| + |B|) \leq 2r^p|B| = 2\int_B r^p dx \leq 2\int_B |f(x)|^p dx$$

since $|f| > r$ on B.

Then using $|f_2| \leq r$ and $f_2 = f$ on $\Omega \smallsetminus (C \cup B)$, $\forall p \in [1, \infty[$

$$\begin{aligned} \int_\Omega |f_2(x)|^p dx &= \int_{\Omega\smallsetminus(C\cup B)} |f_2(x)|^p dx + \int_{C\cup B} |f_2(x)|^p dx \leq \\ &\leq \int_{\Omega\smallsetminus(C\cup B)} |f(x)|^p dx + \int_{C\cup B} |r|^p dx \leq \\ &\leq \int_{\Omega\smallsetminus(C\cup B)} |f(x)|^p dx + 2\int_B |f(x)|^p dx \leq \\ &\leq \int_{(\Omega\smallsetminus(C\cup B))\cup B} |f(x)|^p dx + \int_B |f(x)|^p dx \leq 2\int_\Omega |f(x)|^p dx \end{aligned}$$

which means that $\|f_2\|_p \leq 2^{1/p} \|f\|_p \leq 2 \|f\|_p$.

This inequality holds true also for $p = \infty$: if $|B| > 0$, then $\|f\|_\infty > r \geq \|f_2\|_\infty$ whereas if $|B| = 0$, $|C \cup B| \leq 2|B| = 0$ and thus $|C \cup B| = 0$ and $f_2 = f$ outside $C \cup B$ which means that $\|f\|_\infty = \|f_2\|_\infty$.

Since f_2 is bounded, from the first case there exist A_2 open set with $|A_2| < \varepsilon/2$ and $u \in \mathscr{C}^1(\Omega)$ such that $\nabla u = f_2$ outside A_2 and $\|\nabla u\|_p \leq 4C'(\varepsilon/2)^{1/p-1} \|f_2\|_p \quad \forall p \in [1, +\infty]$.

Then: $|A_1 \cup A_2| \leq |A_1| + |A_2| < \varepsilon$, proving **(2.1a)**; for $A = A_1 \cup A_2$, $\nabla u = f$ outside A since $f_2 = f$ outside A_1, proving **(2.1b)**; and $\forall p \in [1, +\infty]$

$$\|\nabla u\|_p \leq 4C'\left(\frac{\varepsilon}{2}\right)^{1/p-1} \|f_2\|_p \leq 4C'(2\varepsilon^{1/p-1})(2\|f\|_p) \leq 16C'\varepsilon^{1/p-1} \|f\|_p$$

proving **(2.1c)** and thus the theorem.

□

Chapter 3

Functions with higher regularity

In this chapter, the term "cube" means a closed cube of $\mathbb{R}^N$ with faces parallel to the coordinate hyperplanes; this means that a "cube of side length $d > 0$ and center $y \in \mathbb{R}^N$" is a set $\tau([-d/2, d/2]^N)$ where $\tau : \mathbb{R}^N \to \mathbb{R}^N$ is the translation $\tau(x) = x + y$.

We now refine Alberti's result. Specifically, we want some estimates on the Lebesgue measure and the Hausdorff dimension of the set where ∇u agrees with f when we impose higher regularities on the function u.

We write $A_{u,f}$ to denote the set where ∇u and f agree, meaning that

$$A_{u,f} = \{x \in \Omega : \nabla u(x) = f(x)\}.$$

With this notation, Alberti's Theorem states that given a Borel vector field $f : \Omega \to \mathbb{R}^N$ and $\varepsilon > 0$, there exists $u_\varepsilon \in \mathscr{C}^1(\Omega)$ such that $|A_{u_\varepsilon,f}| \geq (1-\varepsilon)|\Omega|$.

The following result shows that if f is a smooth $\mathscr{C}^1$ vector field, a result similar to Alberti's theorem can be obtained with functions u whose gradient ∇u is α-Hölder for all $0 < \alpha < 1$; moreover, the Hausdorff dimension of the set $A_{u,f}$ can be arbitrarily near N taking an appropriate function u with Lipschitz gradient.

Theorem 3.1. *Let $Q \subseteq \mathbb{R}^N$ be the unit cube and let $f : Q \to \mathbb{R}^N$ be a $\mathscr{C}^1$ smooth vector field; then:*

i. for any $\alpha > 0$ there exists a $\mathscr{C}^{1,1}$ smooth function $u_\alpha : Q \to \mathbb{R}$ such that $\dim_{\mathcal{H}} A_{u_\alpha,f} \geq N - \alpha$;

ii. for any $\varepsilon > 0$ there exists $u_\varepsilon : Q \to \mathbb{R}$, $u_\varepsilon \in \mathscr{C}^{1,\beta}(Q)$ for all $0 < \beta < 1$ such that $|A_{u_\varepsilon,f}| \geq 1 - \varepsilon$.

Proof. For the proofs of both statements we will construct Cantor-type sets and functions with prescribed gradient on them, in a way similar to that used to prove **Lemma 2.4** and **Theorem 2.1**.

First statement. For every $k > 0$, given Q cube of side length l, we define kQ as the cube with the same center as Q but side length $k \cdot l$; we also define $I = \{1, 2, \ldots, 2^N\}$.

Let $\delta = 2^{-N/N-\alpha}$; since $N/N-\alpha > 1$ it follows that $\delta = 2^{-N/N-\alpha} < 1/2$.

We start by dividing Q in 2^N cubes $\widetilde{Q}_{i_1}, i_1 \in I$ of side length $1/2$ and then we take the cubes $Q_{i_1} = 2\delta \widetilde{Q}_{i_1}$, which have side length $2\delta \cdot 1/2 = \delta < 1/2$; in the second step, each Q_{i_1} is divided in 2^N cubes $\widetilde{Q}_{i_1 i_2}, i_2 \in I$ and as before $Q_{i_1 i_2} = 2\delta \widetilde{Q}_{i_1 i_2}$, with side length δ^2.

In general, given the cubes $Q_{i_1 \dots i_n}$ with $i_m \in I$ for $1 \le m \le n$, the $(n+1)$-th step consists in dividing each of them in 2^N cubes $\widetilde{Q}_{i_1 \dots i_n i_{n+1}}$ and taking the cubes $Q_{i_1 \dots i_n i_{n+1}} = 2\delta \widetilde{Q}_{i_1 \dots i_n i_{n+1}}$, which have side length δ^{n+1} by induction.

From now on we will use the following notation:

- we define $I^n = \{1, 2, \dots, 2^N\}^n = \{(i_1, \dots, i_n) : i_m \in I \text{ for } 1 \le m \le n\}$ and $S = \bigcup_{m=1}^{\infty} I^m$; moreover, if $i = (i_1, \dots, i_n) \in I^n$ then for $1 \le k \le n$ we define $i(k)$ as $i(k) = (i_1, \dots, i_k) \in I^k$;
- for $i = (i_1, \dots, i_n) \in S$ we denote $Q_i = Q_{i_1 \dots i_n}$ and $Q_{i j_1 \dots j_k} = Q_{i_1 \dots i_n j_1 \dots j_k}$ with $j_m \in I$ for $1 \le m \le k$; the set S can be seen as the free semigroup on I with the operation of concatenation and Q_i defined coherently with this operation;
- we define $Q^n = \bigcup_{i \in I^n} Q_i$ and $C = \bigcap_{n=1}^{\infty} Q^n$.

The sets Q^n are closed sets because they are finite union of closed sets Q_i, and $Q^{n+1} \subseteq Q^n$ because by definition $Q_{ij} \subseteq Q_i$ for every $i \in S, j \in I$; therefore C is a Cantor-type set and thus, by well known results [3, page 130], $\dim_{\mathscr{H}} C = N - \alpha$.

We will now define an appropriate function u_α on C whose gradient will be f, from which it will follow that $C \subseteq A_{u_\alpha, f}$ and thus $\dim_{\mathscr{H}} A_{u_\alpha, f} \ge \dim_{\mathscr{H}} C = N - \alpha$.

To do that:

- for all $n \in \mathbb{N}$ and $i \in I^n$ let $\varphi_i \in \mathscr{C}^2(Q)$ be a function such that

$$0 \le \varphi_i \le 1 \quad \text{and} \quad \varphi_i(x) = \begin{cases} 1 & \text{if } x \in \left(1 + \frac{1/2 - \delta}{3}\right) Q_i \\ 0 & \text{if } x \notin \left(1 + \frac{1/2 - \delta}{2}\right) Q_i \end{cases}$$

  and

$$\|\nabla \varphi_i\|_\infty \le \frac{K_1}{\delta^n (1/2 - \delta)} \quad \text{and} \quad \left\|\nabla^2 \varphi_i\right\|_\infty \le \frac{K_2}{\delta^{2n} (1/2 - \delta)^2} \tag{3.1}$$

  for some constants K_1, K_2; this functions exist by **Lemma 1.15** applied on those cubes with

$$d = \frac{1}{4} \left[\left(1 + \frac{1/2 - \delta}{2}\right) \delta^n - \left(1 + \frac{1/2 - \delta}{3}\right) \delta^n \right] = \frac{(1/2 - \delta)\delta^n}{24};$$

- for $i \in S,\ j \in I$ let

$$a_j = \frac{1}{|Q_j|} \int_{Q_j} f(x) dx \quad \text{and} \quad a_{ij} = \frac{1}{|Q_{ij}|} \int_{Q_{ij}} f(x) dx - \frac{1}{|Q_i|} \int_{Q_i} f(x) dx$$

  with the aforementioned notations;

- let $u_n(x) = \sum_{i\in I^n} \varphi_i(x)\langle a_i, x - y_i\rangle$ where $y_i \in Q$ is the center of Q_i and let $u_\alpha(x) = \sum_{n=1}^{\infty} u_n(x)$.

With $i \in I^n$, from the definition of a_i it follows that

$$\sum_{k=1}^{n} a_{i(k)} = \frac{1}{|Q_i|}\int_{Q_i} f(x)dx \tag{3.2}$$

because the sum is telescoping.

We also have that $|a_i| \leq K_3\,\delta^n$ for some constant K_3; it is enough to prove this for $n \geq 2$ since the constant K_2 can always be increased to include the cases with $n = 1$ too.

Since f is of class $\mathscr{C}^1$ on Q compact set, f is Lipschitz with Lipschitz constant M'; this means that for $i \in I^n$ and $x, y \in Q_i$ we have

$$|f(x) - f(y)| \leq M'|x - y| \leq M' \cdot \sqrt{N} \cdot \delta^n = M\delta^n.$$

Then, for $i \in I^n, j \in I$ and y_{ij} center of $Q_{ij} \subseteq Q_i$

$$\begin{aligned}
|a_{ij}| &= \left| \frac{1}{|Q_{ij}|}\int_{Q_{ij}} f(x)dx - \frac{1}{|Q_i|}\int_{Q_i} f(x)dx \right| \leq \\
&\leq \left| \frac{1}{|Q_{ij}|}\int_{Q_{ij}} f(x)dx - f(y_{ij}) \right| + \left| \frac{1}{|Q_i|}\int_{Q_i} f(x)dx - f(y_{ij}) \right| \leq \\
&\leq \frac{1}{|Q_{ij}|}\int_{Q_{ij}} |f(x) - f(y_{ij})|dx + \frac{1}{|Q_i|}\int_{Q_i} |f(x) - f(y_{ij})|dx \leq \\
&\leq \frac{1}{|Q_{ij}|}|Q_{ij}| \cdot M\delta^{n+1} + \frac{1}{|Q_i|}|Q_i| \cdot M\delta^n \leq M(\delta + 1)\delta^n \leq \frac{3M}{2}\delta^n = K_3\,\delta^n,
\end{aligned}$$

proving the estimate.

The functions u_n are $\mathscr{C}^1(Q)$ by construction, therefore one can define the series $F(x) = \sum_{n=1}^{\infty} \nabla u_n(x)$, possibly divergent for some $x \in Q$; by **(1.2)**, if the series u_α converges at least at one point, the uniform convergence of F gives that $u_\alpha(x) = \sum_{n=1}^{\infty} u_n(x)$ is convergent and of class $\mathscr{C}^1$ on Q and

$$\nabla u_\alpha(x) = F(x) = \sum_{n=1}^{\infty} \nabla u_n(x).$$

The convergence of u_α at one point can be easily seen choosing $x \notin C$: x is inside finitely many cubes Q_i therefore only finitely many functions u_n are non zero at x and thus the series $u_\alpha(x)$ is a finite sum. Then, we only need to prove that F is uniformly convergent, meaning that

$$\left\| F - \sum_{m=1}^{n} \nabla u_m \right\|_\infty = \left\| \sum_{m=n+1}^{\infty} \nabla u_m \right\|_\infty$$

converges to 0 for $n \to \infty$. To do it, it is enough to prove

$$\left\| \sum_{m=n+1}^{\infty} |\nabla u_m| \right\|_{\infty} \leq \frac{K}{1/2 - \delta}\delta^n \to 0 \tag{3.3}$$

because the second term converges to 0 for $n \to \infty$.

It can be seen that for every $m \geq n+1$ $u_m = 0$ outside Q^n by definition of φ_i therefore it is enough to evaluate the norm on Q^n.

For $x \in Q^n \setminus C$ there exists $j \in I^k$ with $k \geq n$ such that x is inside the cubes $Q_{j(m)}$ with $1 \leq m \leq k$ but $x \notin Q^{k+1}$, because if x were inside infinitely many cubes Q_i, it would be in C.

We now remember that:

a. x is outside Q^{k+1} therefore $u_m(x) = 0$ for $m \geq k+2$;

b. for $y \in Q$

$$\nabla u_m(y) = \nabla\left(\sum_{i\in I^m} \varphi_i(y)\langle a_i, y - y_i\rangle\right) = \sum_{i\in I^m} \varphi_i(y) a_i + \sum_{i\in I^m} \nabla\varphi_i(y)\langle a_i, y-y_i\rangle;$$

since all φ_i with $i \in I^n$ have disjoint supports we have

$$\nabla u_m(x) = \varphi_{j(m)}(x)\, a_{j(m)} + \nabla\varphi_{j(m)}(x)\langle a_{j(m)}, x - y_{j(m)}\rangle;$$

c. $0 \leq \varphi_i \leq 1$ and by Cauchy-Schwarz inequality $|\langle a_i, x - y_i\rangle| \leq |a_i| \cdot |x - y_i|$;

d. for $m \geq n+1$ we have $|a_{j(m)}| \leq K_3\delta^m \leq K_3\delta^n$ by the previous result and $|x - y_{j(m)}| \leq K_4\delta^m \leq K_4\delta^n$ where $K_4 = \sqrt{N}/2$, because $x \in Q_{j(m)}$.

Then

$$\begin{aligned}\sum_{m=n+1}^{\infty} |\nabla u_m(x)| &\overset{\textbf{a.}\,\&\,\textbf{b.}}{=} \sum_{m=n+1}^{k+1} |\varphi_{j(m)}(x)\, a_{j(m)} + \nabla\varphi_{j(m)}(x)\langle a_{j(m)}, x - y_{j(m)}\rangle| \leq \\ &\overset{\textbf{c.}}{\leq} \sum_{m=n+1}^{k+1} |a_{j(m)}| + |\nabla\varphi_{j(m)}(x)| \cdot |a_{j(m)}| \cdot |x - y_{j(m)}| \leq \\ &\overset{\textbf{d.}\,\&\,\textbf{(3.1)}}{\leq} \sum_{m=n+1}^{k+1} K_3\delta^n + \frac{K_1}{(1/2-\delta)\delta^n} \cdot K_3\delta^n \cdot K_4\delta^n = \frac{K}{1/2-\delta}\delta^n;\end{aligned}$$

this proves the estimate for $x \in Q^n \setminus C$, which is dense in Q^n, therefore the estimate holds in the whole Q^n and thus in Q, proving **(3.3)**.

It follows that $u_\alpha(x) = \sum_{n=1}^{\infty} u_n(x) \in \mathscr{C}^1(Q)$ and $\nabla u_\alpha(x) = \sum_{n=1}^{\infty} \nabla u_n(x)$.

Now, let $x \in C$. There exists a sequence of cubes $\{Q_{j_n}\}_{n\in N}$, with $j_n \in I^n$ uniquely determined by the property $x \in Q_{j_n}$ for all $n \in \mathbb{N}$; by construction $Q_{j_{n+1}} \subseteq Q_{j_n}$.

By definition, for every $n \in N$ we have $\varphi_{j_n}(x) = 1$ thus $u_n(x) = \langle a_{j_n}, x - y_{j_n}\rangle$ and $\nabla u_n(x) = a_{j_n}$.

Then from $\nabla u_\alpha = \sum_{n=1}^{\infty} \nabla u_n$ it follows that

$$\nabla u_\alpha(x) = \sum_{n=1}^{\infty} \nabla u_n(x) = \sum_{n=1}^{\infty} a_{j_n} = \lim_{m\to\infty} \sum_{n=1}^{m} a_{j_n} \overset{\textbf{(3.2)}}{=} \lim_{m\to\infty} \frac{1}{|Q_{j_m}|}\int_{Q_{j_m}} f(y)dy = f(x),$$

by continuity of f.

This proves that $C \subseteq A_{u_\alpha,f}$ and therefore $\dim_{\mathscr{H}} A_{u_\alpha,f} \geq \dim_{\mathscr{H}} C = N - \alpha$.

Lastly, we need to prove the higher regularity of u_α, and in particular that ∇u_α is Lipschitz. We want to prove the estimate

$$|\nabla u_\alpha(x) - \nabla u_\alpha(y)| \leq \frac{K}{(1/2 - \delta)^2}|x - y|$$

for all $x, y \in Q$ and for some constant K.

With the same notations as before for a fixed $x \in Q \setminus C$, we have for every $m \in \mathbb{N}$:

$$\begin{aligned}
\left\|\nabla^2 u_m(x)\right\|_\infty &\overset{\textbf{b.}}{=} \left\|\nabla\left(\varphi_{j(m)}(x)a_{j(m)} + \nabla\varphi_{j(m)}(x)\langle a_{j(m)}, x - y_{j(m)}\rangle\right)\right\|_\infty = \\
&= \left\|\nabla^2\varphi_{j(m)}(x)\langle a_{j(m)}, x - y_{j(m)}\rangle + 2\cdot\nabla\varphi_{j(m)}(x)\otimes a_{j(m)}\right\|_\infty \leq \\
&\overset{\textbf{c.}}{\leq} \left\|\nabla^2\varphi_{j(m)}\right\|_\infty\cdot|a_{j(m)}|\cdot|x - y_{j(m)}| + 2\left\|\nabla\varphi_{j(m)}\right\|_\infty\cdot|a_{j(m)}| \leq \\
&\overset{\textbf{d.}\,\&\,\textbf{(3.1)}}{\leq} \frac{K_2}{(1/2-\delta)^2\delta^{2m}}\cdot K_3\delta^m\cdot K_4\delta^m + 2\cdot\frac{K_1}{(1/2-\delta)\delta^m}\cdot K_3\delta^m \leq \frac{K}{(1/2-\delta)^2}
\end{aligned}$$

for some constant K.

Since $\nabla\varphi_i$ have disjoint supports by definition, so do $\nabla^2\varphi_i$ and thus $\nabla^2 u_m$ have disjoint supports for $m \in \mathbb{N}$, which means that

$$\nabla^2\left(\sum_{m=1}^{n} u_m(x)\right) = \sum_{m=1}^{n} \nabla^2 u_m(x);$$

for every $x \in Q \setminus C$ either $\nabla^2 u_m(x) = 0\ \forall m \in \mathbb{N}$ or there exists a unique $\overline{m} \in \mathbb{N}$ such that $\nabla^2 u_{\overline{m}}(x) \neq 0$, it follows that

$$\left\|\nabla^2\left(\sum_{m=1}^{n} u_m(x)\right)\right\|_\infty = \left\|\sum_{m=1}^{n}\nabla^2 u_m(x)\right\|_\infty \leq \left\|\nabla^2 u_{\overline{m}}(x)\right\|_\infty \leq \frac{K}{(1/2-\delta)^2}.$$

The uniform estimate for $x \in Q \setminus C$ and the facts that $Q \setminus C$ is dense in Q and $u_\alpha \in \mathscr{C}^2(Q)$ give that

$$\left\|\nabla^2 u_\alpha\right\|_\infty \leq \frac{K}{(1/2-\delta)^2}.$$

Then the Mean value theorem **(1.1)** applied to

$$g : [0,1] \to R^N,\ \ g(t) = \nabla u_\alpha(tx + (1-t)y)$$

gives

$$|\nabla u_\alpha(x) - \nabla u_\alpha(y)| = |g(1) - g(0)| \le (1-0)|g'(t)| = |\nabla^2 u_\alpha(tx + (1-t)y)(x-y)| \le$$
$$\le \left\|\nabla^2 u_\alpha\right\|_\infty |x-y| \le \frac{K}{(1/2 - \delta)^2}|x-y|$$

which proves that ∇u_α is a Lipschitz function.

Second statement. Let $\varepsilon > 0$ and for every $k \in \mathbb{N}$ let $\delta_k = 2^{-1-c/k^2} < 1/2$ for some constant $c = c(\varepsilon) > 0$ chosen appropriately; then define for every $n \in \mathbb{N}$

$$\Delta_n = \prod_{k=1}^n \delta_k = 2^{-n-\sum_{k=1}^n c/k^2} \le 2^{-n}.$$

We then construct the set C similarly to that in the first statement. The first step consists in dividing Q in 2^N cubes $\widetilde{Q}_{i_1}, i_1 \in I$ of side length $1/2$ and then taking the cubes $Q_{i_1} = 2\delta_1 \widetilde{Q}_{i_1}$, with side length $\Delta_1 = \delta_1$; then in general, given the cubes $Q_i, i \in I^n$, in the $(n+1)$-th step we divide each of them in 2^N cubes $\widetilde{Q}_{ii_{n+1}}$ and take the cubes $Q_{ii_{n+1}} = 2\delta_{n+1}\widetilde{Q}_{ii_{n+1}}$, with side length Δ_{n+1} by induction. Then, $Q^n = \bigcup_{i\in I^n} Q_i$ and $C = \bigcap_{n=1}^\infty Q^n$.

Since $Q^{n+1} \subseteq Q^n$ for every $n \in \mathbb{N}$, by continuity from above **(1.3)** we have $|C| = \lim_{n\to\infty} |Q^n|$. Since all Q_i with $i \in I^n$ are disjoint, we have

$$|Q^n| = \left|\bigcup_{i\in I^n} Q_i\right| = \sum_{i\in I^n} |Q_i| = 2^{Nn}\Delta_n^N = 2^{Nn} \cdot 2^{-Nn-N\sum_{k=1}^n c/k^2} = 2^{-Nc\sum_{k=1}^n 1/k^2} \to 2^{-Kc}$$

for $n \to \infty$, because $\sum_{k=1}^\infty 1/k^2$ converges; we then choose

$$c = c(\varepsilon) \le -\frac{1}{K}\log_2(1-\varepsilon)$$

so that

$$|C| = \lim_{n\to\infty} |Q^n| = 2^{-Kc} \ge 1 - \varepsilon.$$

Like before, we will construct a function u_ε with gradient on C equal to f, so $C \subseteq A_{u_\varepsilon,f}$ and thus $|A_{u_\varepsilon,f}| \ge |C| \ge 1-\varepsilon$.

For all $n \in \mathbb{N}$ and $i \in I^n$ we define $\varphi_i \in \mathscr{C}^2\left(\mathbb{R}^N\right)$ as before applying **Lemma 1.15** to cubes

$$A = \left(1 + \frac{1/2 - \delta_n}{3}\right) Q_i \quad \text{and} \quad B = \left(1 + \frac{1/2 - \delta_n}{2}\right) Q_i$$

such that instead of **(3.1)** we have

$$\|\nabla\varphi_i\|_\infty \le \frac{k_1}{\Delta_n(1/2 - \delta_n)} \quad \text{and} \quad \left\|\nabla^2\varphi_i\right\|_\infty \le \frac{k_2}{\Delta_n^2(1/2-\delta_n)^2}; \tag{3.4}$$

then, we define a_i and u_n as before and $u_\varepsilon = \sum_{n=1}^\infty u_n$.

From the inequality $e^{-x} \leq 1 - x/2$ which holds for $0 \leq x \leq \pi/2$, for n large enough we have

$$2^{-c/n^2} = e^{-c\ln(2)/n^2} \leq 1 - \frac{c\ln(2)}{2n^2} = 1 - \frac{k_3}{n^2}$$

and therefore

$$1 - 2^{-c/n^2} \geq \frac{k_3}{n^2} \quad \text{so} \quad \frac{1}{2} - \delta_n = \frac{1}{2}(1 - 2^{-c/n^2}) \geq \frac{k_3}{2n^2}$$

Then from **(3.4)** we obtain

$$\|\nabla\varphi_i\|_\infty \leq \frac{k_1}{\Delta_n(1/2 - \delta_n)} \leq \frac{2k_1 n^2}{\Delta_n k_3} = \frac{K_1 n^2}{\Delta_n} \quad \text{and} \quad \left\|\nabla^2\varphi_i\right\|_\infty \leq \frac{K_2 n^4}{\Delta_n^2} \tag{3.5}$$

for some constants K_1, K_2.

Again, with the same notations as before and using the inequalities $\Delta_n \leq 2^{-n}$, $|a_{j(m)}| \leq K_3\Delta_n$ and $|x - y_{j(m)}| \leq K_4\Delta_n$ for $m \geq n+1$, for $x \in Q^n \setminus C$ we have:

$$\begin{aligned}\sum_{m=n+1}^{\infty} |\nabla u_m(x)| &\leq \sum_{m=n+1}^{k+1} |a_{j(m)}| + |\nabla\varphi_{j(m)}(x)| \cdot |a_{j(m)}| \cdot |x - y_{j(m)}| \leq \\ &\leq \sum_{m=n+1}^{k+1} K_3\Delta_n + \frac{K_1 n^2}{\Delta_n} \cdot K_3\Delta_n \cdot K_4\Delta_n \leq K_5 n^2 \Delta_n \leq \frac{K_5 n^2}{2^n}\end{aligned}$$

which extends to all $x \in Q$ and therefore by **(1.2)** it gives the uniform convergence of $\sum_{n=1}^{\infty} \nabla u_n = \nabla u_\varepsilon$.

Regarding the regularity of u_ε, we will prove a weaker estimate than the one in the proof of the first statement.

With the same calculations as before, for $x \in Q \setminus C$ and $m \in \mathbb{N}$

$$\begin{aligned}\left\|\nabla^2 u_m(x)\right\|_\infty &\leq \left\|\nabla^2\varphi_{j(m)}\right\|_\infty \cdot |a_{j(m)}| \cdot |x - y_{j(m)}| + 2\left\|\nabla\varphi_{j(m)}\right\|_\infty \cdot |a_{j(m)}| \leq \\ &\leq \frac{K_2 m^4}{\Delta_m^2} \cdot K_3\Delta_m \cdot K_4\Delta_m + 2 \cdot \frac{K_1 m^2}{\Delta_m} \cdot K_3\Delta_m \leq K_6 m^4\end{aligned}$$

which means that $\mathrm{Lip}(\nabla u_m) = \left\|\nabla^2 u_m\right\|_\infty \leq K_6 m^4$.

For any $x, y \in Q$, chosen $n \in \mathbb{N}$ such that $2^{-n-1} \le |x-y| \le 2^{-n}$, we have

$$
\begin{aligned}
|\nabla u_\varepsilon(x) - \nabla u_\varepsilon(y)| &= \left| \sum_{m=1}^{n} \nabla u_k(x) - \sum_{m=1}^{n} \nabla u_k(y) + \sum_{m=n+1}^{\infty} \nabla u_k(x) - \sum_{m=n+1}^{\infty} \nabla u_k(y) \right| \le \\
&\le \left| \sum_{m=1}^{n} (\nabla u_k(x) - \nabla u_k(y)) \right| + \left| \sum_{m=n+1}^{\infty} \nabla u_k(x) \right| + \left| \sum_{m=n+1}^{\infty} \nabla u_k(y) \right| \le \\
&\le \operatorname{Lip}\left(\sum_{m=1}^{n} \nabla u_m \right) \cdot |x-y| + 2 \left\| \sum_{m=n+1}^{\infty} |\nabla u_k| \right\|_\infty \le \\
&\le K_6 \left(\sum_{m=1}^{n} m^4 \right) |x-y| + 2K_5 n^2 \cdot 2^{-n} \le \\
&\le K_6 n^5 |x-y| + 2K_5 n^2 \cdot 2^{-n} \le K_6 n^5 |x-y| + 2K_5 n^5 \cdot 2|x-y| = \\
&= K n^5 |x-y| \le K \left| \log_2 |x-y| \right|^5 |x-y|
\end{aligned}
$$

where the last inequality follows from $-n-1 \le \log_2 |x-y| \le -n$ and thus $|\log_2 |x-y|| \ge n$.

Given $0 < \alpha < 1$, the function $g : \mathbb{R}_{>0} \to \mathbb{R}$ defined as $g(x) = |\log_2(x)|^5 x^{1-\alpha}$ can be extended by continuity at $x = 0$ as $g(0) = 0$; with this extension, g is continuous in $[0, \sqrt{N}]$ and thus it has maximum over it, meaning that $g(x) \le M$ for all $x \in [0, \sqrt{N}]$. Since for $x, y \in Q$, $|x-y| \in [0, \sqrt{N}]$ we have that

$$
\begin{aligned}
|\nabla u_\varepsilon(x) - \nabla u_\varepsilon(y)| &\le K \left| \log_2 |x-y| \right|^5 |x-y| = \\
&= K \left(\left| \log_2 |x-y| \right|^5 |x-y|^{1-\alpha} \right) |x-y|^\alpha \le KM|x-y|^\alpha
\end{aligned}
$$

which proves that ∇u_ε is α-Hölder for all $0 < \alpha < 1$. □

Despite the theorem we just proved, not for all $\mathscr{C}^1$ smooth vector fields the set $A_{u,f}$ is guaranteed to have maximal Hausdorff dimension when u has high regularity. In fact, the following result shows that, at least for some vector fields, if u has Lipschitz gradient then the set $A_{u,f}$ cannot have maximal Hausdorff dimension, while if u is taken to be $\mathscr{C}^2$ then the Hausdorff dimension of $A_{u,f}$ is smaller than half of that of the whole space.

Theorem 3.2. *Let $\widetilde{Q} \subseteq \mathbb{R}^N$ and $Q = \widetilde{Q} \times \widetilde{Q} \subseteq \mathbb{R}^{2N} = \mathbb{R}^N \times \mathbb{R}^N$ be respectively the unit cubes in $\mathbb{R}^N$ and $\mathbb{R}^{2N}$ and let $f_0 : Q \to \mathbb{R}^{2N}$ be the vector field defined as $f_0(x, y) = (2y, -2x)$ for all $x, y \in \widetilde{Q} \subseteq \mathbb{R}^n$. Then:*

i. if $u : Q \to \mathbb{R}$ is a $\mathscr{C}^{1,1}$ smooth function, then $\dim_{\mathcal{H}} A_{u,f_0} \le 2N - \lambda$ where $\lambda > 0$ depends on the Lipschitz constant for ∇u, therefore there cannot exist a $\mathscr{C}^{1,1}$ smooth function u such that $\dim_{\mathcal{H}} A_{u,f_0} = 2N$;

ii. if $u : Q \to \mathbb{R}$ is a $\mathscr{C}^2$ smooth function, then $\mathcal{H}^N(A_{u,f_0}) < \infty$, meaning that $\dim_{\mathcal{H}} A_{u,f_0} \le N$.

Proof. First statement. Let $u : Q \to \mathbb{R}$ be of class $\mathscr{C}^{1,1}$ and let $l > 0$ be the Lipschitz constant of ∇u.

Given an arbitrary cube $Q' \subseteq Q$ of side length $d > 0$, we denote by r the supremum of radii of balls all contained in $Q' \setminus A_{u,f_0}$. We want to prove that there exists a constant α which depends only on l such that $r > \alpha d$; from this kind of porosity it will follow that the set A_{u,f_0} cannot have Hausdorff dimension $2N$.

We now suppose $r \leq d/4$ otherwise $r > d/4$ and we have the porosity estimate with $\alpha = 1/4$.

Denoting by $x_1, x_2, \dots, x_N, y_1, \dots, y_N$ the coordinates in $\mathbb{R}^{2N}$, if the center of Q' is $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_N, \widetilde{y}_1, \dots, \widetilde{y}_N)$ we define P as the plane of points in $\mathbb{R}^{2N}$ such that $x_i = \widetilde{x}_i, y_i = \widetilde{y}_i$ for $2 \leq i \leq N$. Points on P can be described by the local coordinates $z_1 = (x_1, y_1)$ and $\widetilde{x} \in P$ has local coordinates $\widetilde{z}_1 = (\widetilde{x}_1, \widetilde{y}_1)$. Then, we take γ a closed cycle on P in the shape of a square of side length $d/2$ with center in $\widetilde{z}_1$ and sides parallel to the coordinate axes of P.

By definition of r it follows that for all $p \in \gamma$ the ball $B(p, 2r)$ intersects A_{u,f_0} in at least one point. It can be easily seen that given $a, b \in Q$ we have $|f_0(a) - f_0(b)| = 2|a - b|$; then, fixed $p \in \gamma$ and taken $q \in B(p, 2r) \cap A_{u,f_0}$, we have $f_0(q) = \nabla u(q)$ and therefore

$$\begin{aligned} |f_0(p) - \nabla u(p)| &= |f_0(p) - f_0(q) + \nabla u(q) - \nabla u(p)| \leq \\ &\leq |f_0(p) - f_0(q)| + |\nabla u(q) - \nabla u(p)| \leq 2|p - q| + l|p - q| < \\ &< 2(2 + l)r \end{aligned}$$

using the fact that ∇u is a Lipschitz function with Lipschitz constant l.

By the gradient theorem **(1.13)** $\int_\gamma \nabla u \, ds = 0$ because γ is a closed curve; therefore we obtain

$$\left| \int_\gamma f_0 \, ds \right| = \left| \int_\gamma (f_0 - \nabla u) \, ds \right| \leq \int_\gamma |f_0 - \nabla u| \, ds < \int_\gamma 2(2 + l) r \, ds = 2(2 + l) r \cdot 2d$$

because the length of γ is $2d$.

We now evaluate explicitly the integral along the curve γ.

We denote by $(\cdot)_{z_1}$ the restriction of functions in the local coordinates z_1 on P: in particular, given $x = (x_1, \dots, x_N, y_1, \dots, y_N)$ we have $(x)_{z_1} = (x_1, y_1)$, while $(f_0(x))_{z_1} = (2y_1, \dots, 2y_N, -2x_1, \dots, -2x_n)_{z_1} = (2y_1, -2x_1)$.

We can parametrize the cycle as $\gamma : [0, 1] \to P \subseteq \mathbb{R}^{2N}$ such that $\gamma([i-1/4, i/4])$ for $i = 1, 2, 3, 4$ are the four sides of the square of the cycle γ. We then suppose $(\gamma(0))_{z_1} = (\widetilde{x}_1 + d/4, \widetilde{y}_1 + d/4)$ and γ going counterclockwise.

We now evaluate the integral along $\gamma([0, 1/4])$; the integrals along the other three sides are analogous. For $t \in [0, 1/4]$ we have

$$(\gamma(t))_{z_1} = (\gamma(0))_{z_1} + (-2dt, 0) = \left(\widetilde{x}_1 + \frac{d}{4} - 2dt, \widetilde{y}_1 + \frac{d}{4} \right)$$

$$(\gamma'(t))_{z_1} = (-2d, 0)$$

$$(f_0(\gamma(t)))_{z_1} = \left(2\widetilde{y}_1 + \frac{d}{2}, -2\widetilde{x}_1 - \frac{d}{2} + 4dt\right)$$

$$(f_0(\gamma(t)))_{z_1} \cdot (\gamma'(t))_{z_1} = -2d \cdot \left(2\widetilde{y}_1 + \frac{d}{2}\right) = -4d\widetilde{y}_1 - d^2$$

from which it follows that

$$\int_{\gamma([0,1/4])} f_0 \, ds = \int_0^{1/4} f_0(\gamma(t)) \cdot \gamma'(t) \, dt = \int_0^{1/4} (-4d\widetilde{y}_1 - d^2) \, dt = -d\widetilde{y}_1 - \frac{d^2}{4}.$$

Splitting the integral over the intervals $\gamma([{}^{i-1}/4, {}^{i}/4])$ we obtain

$$\int_\gamma f_0 \, ds = \left(-d\widetilde{y}_1 - \frac{d^2}{4}\right) + \left(-d\widetilde{x}_1 - \frac{d^2}{4}\right) + \left(d\widetilde{y}_1 - \frac{d^2}{4}\right) + \left(d\widetilde{x}_1 - \frac{d^2}{4}\right) = -d^2;$$

from

$$d^2 = \left|\int_\gamma f_0 \, ds\right| < 4(2+l)rd$$

it follows

$$r > \frac{d}{4(2+l)}$$

which gives the porosity estimate with $\alpha = 1/4(2+l)$.

We now prove that from the porosity estimate it follows that $\dim_{\mathcal{H}} A_{u,f_0} < 2N - \lambda$ where λ depends only on l.

Firstly, the estimate is true also for the supremum of side lengths of open cubes all contained in $Q' \setminus A_{u,f_0}$. In fact, if $B \subseteq Q' \setminus A_{u,f_0}$ is a ball of radius r_B, then there exists an open cube $Q_B \subseteq B \subseteq Q' \setminus A_{u,f_0}$ of side length $d_B = 2r_B/\sqrt{N}$ and therefore the supremum of side lengths of open cubes all contained in $Q' \setminus A_{u,f_0}$ is

$$\overline{r} \geq \sup_{B \subseteq Q' \setminus A_{u,f_0}} d_B = \frac{2}{\sqrt{N}} \sup_{B \subseteq Q' \setminus A_{u,f_0}} r_B > \frac{2}{\sqrt{N}} \alpha d = \overline{\alpha} d.$$

From now on, we will denote the new porosity estimate on cubes like the one on balls, with r and α in place of $\overline{r}$ and $\overline{\alpha}$.

Let then Q' be a cube of side length d that covers a portion $Q' \cap A_{u,f_0}$ of A_{u,f_0}; by the new porosity estimate there exists a cube $\widetilde{Q} \subseteq Q' \setminus A_{u,f_0}$ of side length $\widetilde{r} > \alpha d$. With $k \in \mathbb{N}$ such that $2^{-k} d < \widetilde{r}/2$, we can divide each side of Q' in 2^k parts, thus dividing Q' in 2^{2Nk} cubes of side length $2^{-k}d$; then, there is at least one of these cubes all contained in $\widetilde{Q}$, meaning that to cover $Q' \cap A_{u,f_0}$ it is enough to use $2^{2Nk} - 1$ of these cubes.

Fix $0 \leq D \leq 2N$, $\delta > 0$ and $\varepsilon > 0$; by definition of Hausdorff measure, there exists a sequence $\{Q_i\}_{i \in \mathbb{N}}$ of cubes such that $\operatorname{diam}(Q_i) < \delta$ for all $i \in \mathbb{N}$, $A_{u,f_0} \subseteq \bigcup_{i \in \mathbb{N}} Q_i$ and

$$\mathcal{H}^D_\delta(A_{u,f_0}) \geq C \sum_{i=0}^{\infty} \ell(Q_i)^D - \varepsilon.$$

As said previously, each cube Q_i can be decomposed in 2^{2Nk} cubes $Q_{i,j}$ with only the first $2^{2Nk}-1$ necessary to cover $Q_i \cap A_{u,f_0}$; each $Q_{i,j}$ has $\operatorname{diam}(Q_{i,j}) < \delta/2^k$ and $A_{u,f_0} \subseteq \bigcup_{i,j} Q_{i,j}$ so it follows that

$$\mathcal{H}^D_{\delta/2^k}(A_{u,f_0}) \leq C \sum_{i=0}^{\infty} \sum_{j=1}^{2^{2Nk}-1} \ell(Q_{i,j})^D = C \sum_{i=0}^{\infty} (2^{2Nk}-1) \left(\frac{\ell(Q_i)}{2^k} \right)^D =$$
$$= \frac{2^{2Nk}-1}{2^{kD}} \left(C \sum_{i=0}^{\infty} \ell(Q_i)^D \right) \leq \frac{2^{2Nk}-1}{2^{kD}} \left(\mathcal{H}^D_\delta(A_{u,f_0}) + \varepsilon \right).$$

Since this is true for all $\varepsilon > 0$ we obtain

$$\mathcal{H}^D_{\delta/2^k}(A_{u,f_0}) \leq \beta_D \cdot \mathcal{H}^D_\delta(A_{u,f_0}) \quad \text{with} \quad \beta_D = \frac{2^{2Nk}-1}{2^{kD}};$$

then, by induction we have for $n \in \mathbb{N}$

$$\mathcal{H}^D_{\delta/2^{nk}}(A_{u,f_0}) \leq \beta_D^n \cdot \mathcal{H}^D_\delta(A_{u,f_0}).$$

We have $\beta_D < 1$ if $2^{2Nk}-1 < 2^{kD}$ meaning that

$$D > \frac{1}{k} \log_2(2^{2Nk}-1) =: 2N - \lambda \quad \text{with} \quad \lambda > 0.$$

Then for $D \in \left]2N-\lambda, 2N\right]$ we have that

$$\mathcal{H}^D(A_{u,f_0}) = \lim_{\eta\to\infty} \mathcal{H}^D_\eta(A_{u,f_0}) = \lim_{n\to\infty} \mathcal{H}^D_{\delta/2^{nk}}(A_{u,f_0}) \leq \lim_{n\to\infty} \beta_D^n \cdot \mathcal{H}^D_\delta(A_{u,f_0}) = 0$$

because $\beta_D < 1$.

Therefore $\dim_{\mathcal{H}} A_{u,f_0} \leq 2N - \lambda$, where λ depends only on α which depends only on l, proving the result.

Second statement. Let $u : Q \to \mathbb{R}$ be of class $\mathscr{C}^2$. Denoting by J the matrix $J = \begin{pmatrix} 0 & \mathbb{1}_N \\ -\mathbb{1}_N & 0 \end{pmatrix}$, for $x \in A_{u,f_0}$ we can write $\nabla u(x) = f_0(x) = 2Jx$ thinking of x as a column vector in $\mathbb{R}^{2N}$.

Let $x \in A_{u,f_0}$ and $v, w \in \mathbb{R}^{2N}$ such that $x+v, x+w \in A_{u,f_0}$ too. Applying the Mean value theorem **(1.1)** to the function $g : [0,1] \to \mathbb{R}, g(t) = \langle \nabla u(x+tv), w \rangle$, since $g'(t) = \langle \nabla^2 u(x+tv)v, w \rangle$ we have

$$\langle \nabla u(x+v), w \rangle - \langle \nabla u(x), w \rangle = g(1) - g(0) = g'(\tau_1) = \langle \nabla^2 u(x+\tau_1 v)v, w \rangle$$

for $\tau_1 \in]0,1[$; therefore

$$\langle \nabla^2 u(x+\tau_1 v)v, w \rangle = \langle \nabla u(x+v), w \rangle - \langle \nabla u(x), w \rangle = \langle 2J(x+v), w \rangle - \langle 2Jx, w \rangle = \langle 2Jv, w \rangle.$$

Similarly, for some $\tau_2 \in]0,1[$

$$\langle \nabla^2 u(x+\tau_2 w)w, v \rangle = \langle 2Jw, v \rangle;$$

since J is antisymmetric and $\nabla^2 u$ is symmetric because $u \in \mathscr{C}^2$, then

$$\langle \nabla^2 u(x+\tau_2 w)v, w\rangle = -\langle 2Jv, w\rangle.$$

Putting together the two equations and using Cauchy-Schwarz inequality, we have

$$\begin{aligned}|\langle 4Jv, w\rangle| &= \left|\left\langle \left(\nabla^2 u(x+\tau_1 v) - \nabla^2 u(x+\tau_2 w)\right) v, w\right\rangle\right| \leq \\ &\leq \left\|\nabla^2 u(x+\tau_1 v) - \nabla^2 u(x+\tau_2 w)\right\|_\infty \cdot |v| \cdot |w|.\end{aligned}$$

We have $u \in \mathscr{C}^2$, so $\nabla^2 u$ is continuous on Q compact set and therefore it is uniformly continuous. Then $\nabla^2 u(x+\tau_1 v) \to \nabla^2 u(x)$ for $|v| \to 0$ and $\nabla^2 u(x+\tau_2 w) \to \nabla^2 u(x)$ for $|w| \to 0$; therefore

$$\begin{aligned}&\left\|\nabla^2 u(x+\tau_1 v) - \nabla^2 u(x+\tau_2 w)\right\|_\infty \leq \\ &\qquad \leq \left\|\nabla^2 u(x+\tau_1 v) - \nabla^2 u(x)\right\|_\infty + \left\|\nabla^2 u(x) - \nabla^2 u(x+\tau_2 w)\right\|_\infty \to 0\end{aligned}$$

for $|v|, |w| \to 0$.

This means that for every $\varepsilon > 0$ there exists δ_ε such that if $|v|, |w| < \delta_\varepsilon$ then $|\langle Jv, w\rangle| \leq \varepsilon \cdot |v| \cdot |w|$; by uniform continuity of $\nabla^2 u$, this δ_ε is independent of x.

We now prove the following result: given $\eta_0 > 0$ there exists $\varepsilon_0 > 0$ such that if $K \subseteq \mathbb{S}^{2N-1}$ has the property that $|\langle v, Jw\rangle| \leq \varepsilon_0$ for all $v, w \in K$, then there exists a N-vector space W in $\mathbb{R}^{2N}$ such that $\operatorname{dist}(k, W) \leq \eta_0$ for all $k \in K$.

Given $N+1$ independent vectors $x_1, \ldots, x_{N+1}$ we denote by V_i the N-vector space generated by all of them except for x_i, meaning that $V_i = \operatorname{span}\{x_j : j \neq i\}$. The N-vector space W in the result exists if and only if there do not exist $x_1, \ldots, x_{N+1} \in K$ linear independent vectors such that for every $1 \leq i \leq N+1$, $\operatorname{dist}(x_i, V_i) \geq \alpha\eta_0$ with $v_i \in V_i$ and α some geometric constant.

This means that to prove the result it is enough to show that given $x_1, \ldots, x_{N+1} \in K$ linear independent vectors such that for every $1 \leq i \leq N+1$, $\operatorname{dist}(x_i, V_i) \geq \alpha\eta_0 =: r_0$ then there exist $1 \leq \bar{i}, \bar{j} \leq N+1$ such that $|\langle x_{\bar{i}}, Jx_{\bar{j}}\rangle| \geq \varepsilon_0$ where ε_0 only depends on r_0 and not on the vectors x_i taken.

A preliminar result states that: given $a_1, \ldots, a_n$ basis of unit vectors of $\mathbb{R}^n$ such that for all i we have $\operatorname{dist}(a_i, \widetilde{V}_i) \geq r_0$ with $\widetilde{V}_i := \operatorname{span}\{a_j : j \neq i\}$, if $v = \sum_{i=1}^n \beta_i a_i$ is a unit vector then $|\beta_i| \leq r_0^{-1}$ for all i. We fix $1 \leq i \leq n$ and we take b_i unit vector orthogonal to $\widetilde{V}_i$; since $|\langle a_i, b_i\rangle|$ is the distance of a_i from $\widetilde{V}_i$, then $|\langle a_i, b_i\rangle| \geq r_0$. By Cauchy-Schwarz inequality we have

$$|\langle v, b_i\rangle| \leq |v||b_i| = 1$$

and thus

$$1 \geq |\langle v, b_i\rangle| \geq \left|\left\langle \sum_{j=1}^n \beta_j a_j, b_i\right\rangle\right| = \left|\sum_{j=1}^n \beta_j \langle a_j, b_i\rangle\right| = |\beta_i \langle a_i, b_i\rangle| \geq r_0|\beta_i|$$

because for all $j \neq i$ we have $a_j \in \widetilde{V}_i$ so $\langle a_j, b_i\rangle = 0$; this gives $|\beta_i| \leq r_0^{-1}$.

The spaces $V = \text{span}\{x_i : 1 \le i \le N+1\}$ and $W = \text{span}\{Jx_i : 1 \le i \le N\}$ have dimension $\dim(V) = N+1$ and $\dim(W) = N$, so necessarily $V \cap W \neq \emptyset$; let then $v \in V \cap W$ be a unit vector with

$$v = \sum_{i=1}^{N+1} \alpha_i x_i = \sum_{i=1}^{N} \beta_i J x_i.$$

We can apply twice the preliminar result on v: $V \cong \mathbb{R}^{N+1}$ with basis $x_1, \ldots, x_{N+1}$ and the condition on distances follows from the hypothesis so we obtain $|\alpha_i| \le r_0^{-1}$; similarly $W \cong \mathbb{R}^N$ with basis $Jx_1, \ldots, Jx_N$ and the estimate on distances follows the same way because J is an isometry, so $|\beta_i| \le r_0^{-1}$.

Then

$$1 = |v|^2 = \langle v, v \rangle = \left\langle \sum_{i=1}^{N+1} \alpha_i x_i, \sum_{j=1}^{N} \beta_j J x_j \right\rangle = \sum_{i=1}^{N+1} \sum_{j=1}^{N} \alpha_i \beta_i \langle x_i, Jx_j \rangle \le$$

$$\le \sum_{i=1}^{N+1} \sum_{j=1}^{N} |\alpha_i||\beta_i||\langle x_i, Jx_j \rangle| \le r_0^{-2} \sum_{i=1}^{N+1} \sum_{j=1}^{N} |\langle x_i, Jx_j \rangle|$$

which means, by Pigeonhole Principle, that there exist $\bar{i}, \bar{j}$ such that

$$|\langle x_{\bar{i}}, Jx_{\bar{j}} \rangle| \ge \frac{r_0^2}{N(N+1)} =: \varepsilon_0,$$

which is the result.

Let now $\bar{\varepsilon}$ be the ε_0 from this result with $\eta_0 = 1/4$. We fix $x \in A_{u,f_0}$; from what we proved before, there exists $\delta_{\bar{\varepsilon}}$ such that if $v, w \in \mathbb{R}^{2N}$ are such that $x + v, x + w \in A_{u,f_0}$ and $|v|, |w| < \delta_{\bar{\varepsilon}}$ then $|\langle Jv, w \rangle| \le \bar{\varepsilon} \cdot |v| \cdot |w|$.

We define

$$K_x = \left\{ \frac{y - x}{|y - x|} : y \in B(x, \delta_{\bar{\varepsilon}}) \cap A_{u,f_0} \right\};$$

if $y \in B(x, \delta_{\bar{\varepsilon}}) \cap A_{u,f_0}$ the previous result holds, so for $v_1, v_2 \in K_x$ with

$$v_1 = \frac{y_1 - x}{|y_1 - x|} \quad \text{and} \quad v_2 = \frac{y_2 - x}{|y_2 - x|}$$

we have

$$|\langle Jv_1, v_2 \rangle| = \frac{|\langle J(y_1 - x), (y_2 - x) \rangle|}{|y_1 - x||y_2 - x|} \le \frac{\bar{\varepsilon}|y_1 - x||y_2 - x|}{|y_1 - x||y_2 - x|} = \bar{\varepsilon}$$

which, by the result just proved, means that there exists a N-vector space W_x in $\mathbb{R}^{2N}$ such that $\text{dist}(k, W_x) \le 1/4$ for all $k \in K_x$.

This means that given $y \in B(x, \delta_{\bar{\varepsilon}}) \cap A_{u,f_0}$, we have $\text{dist}(y - x, W_x) \le 1/4|y - x|$; denoting by $P_V(v)$ the orthogonal projection of vector v on the subspace V, we have in general that

$$|P_V(v)|^2 + \text{dist}(v, V)^2 = |v|^2 \quad \text{so} \quad |P_V(v)|^2 = |v|^2 - \text{dist}(v, V)^2$$

therefore with $v = y - x$ and $V = W_x$ we obtain

$$|P_{W_x}(y-x)|^2 = |y-x|^2 - \operatorname{dist}(y-x, W_x)^2 \geq |y-x|^2 - \frac{1}{4}|y-x|^2 = \frac{3}{4}|y-x|^2$$

which means

$$|P_{W_x}(y-x)| \geq \frac{3}{4}|y-x|.$$

Consider now the Grassmannian $G(\mathbb{R}^{2N}, N)$, that is, the set of all N-dimensional linear subspaces of $\mathbb{R}^{2N}$, equipped with the metric $d(X,Y) = \|P_X - P_Y\|_{\text{op}}$; with this metric topology $G(\mathbb{R}^{2N}, N)$ is compact, so it follows that there exist $W_1, \ldots, W_m \in G(\mathbb{R}^{2N}, N)$ N-vector spaces such that $G(\mathbb{R}^{2N}, N) = \bigcup_{i=1}^m B_d(W_i, 1/4)$, where m depends only on N.

For all $x \in A_{u,f_0}$ we have $W_x \in G(\mathbb{R}^{2N}, N)$ so there exists W_i such that

$$\frac{1}{4} \geq d(W_i, W_x) = \|P_{W_i} - P_{W_x}\|_{\text{op}} = \sup_{\substack{v \in \mathbb{R}^{2N} \\ v \neq 0}} \frac{|(P_{W_i} - P_{W_x})(v)|}{|v|}$$

using the definition of operator norm; then for $y \in B(x, \delta_{\bar{\varepsilon}}) \cap A_{u,f_0}$ we have

$$|P_{W_i}(y-x)| \geq |P_{W_x}(y-x)| - |(P_{W_i} - P_{W_x})(y-x)| \geq \frac{3}{4}|y-x| - \frac{1}{4}|y-x| = \frac{1}{2}|y-x|.$$

Let now H be a subset of A_{u,f_0} with $\operatorname{diam}(H) \leq \delta_{\bar{\varepsilon}}$; since $H \subseteq B(x, \delta_{\bar{\varepsilon}}) \cap A_{u,f_0}$ for any $x \in H$, the projection $P_{W_i}(y-x)$ is well defined for all $y \in H$ and has the previous property. We then denote by H_i the subset of point of H whose associated N-vector space is W_i, with $1 \leq i \leq m$.

Given $x \in H_i$ and two points $x', x'' \in H_i$ such that $P_{W_i}(x'-x) = P_{W_i}(x''-x)$, then

$$0 = |P_{W_i}(x'-x) - P_{W_i}(x''-x)| = |P_{W_i}(x'-x'')| \geq \frac{1}{2}|x'-x''|$$

because $x'' \in H_i$, which gives $x' = x''$; this proves that the projection of points in H_i on the affine N-space $x + W_i$ is injective, and thus H_i is part of a single graph over W_i.

Specifically, H_i is the graph of the inverse of the projection on $x + W_i$; this is a bi-Lipschitz function because for any $y, x \in H_i$ we have

$$2|P_{W_i}(y-x)| \geq |y-x| \geq |P_{W_i}(y-x)|$$

so it follows by **(1.11)** that the graph containing H_i has Hausdorff dimension equal to $\dim_{\mathcal{H}} W_i = N$.

In conclusion, A_{u,f_0} can be split into at most $K \cdot \delta_{\bar{\varepsilon}}^{-2N}$ subsets, each of which can be split into at most m pieces H_i with $\dim_{\mathcal{H}} H_i \leq N$; therefore, A_{u,f_0} is union of finitely many sets of Hausdorff dimension smaller than N so $\dim_{\mathcal{H}} A_{u,f_0} \leq N$.

□

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