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Primordial Gravitational Wave Propagation through Cosmic Inhomogeneities

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Contents

Introduction

In this Master's thesis we analyze the propagation of primordial gravitational wave background through cosmic inhomogeneities. The outline is as follows.

In chapter one we briefly introduce the concept of stochastic gravitational wave background. Moreover, we discuss its characteristics, the hypothesis regarding its nature and the experimental limits on its observation.

In chapter two, we discuss the cosmological perturbations theory. We define what perturbations in a cosmological context are and discuss the gauge problem. In the final part of the chapter we focus on the dynamics of the perturbations.

In chapter three we study the propagation of gravitational waves in curved space-time. This is the formalism that is needed in order to describe the effect of inhomogeneities on the gravitational wave background. We additionally recall the geometric optics approximation for gravitational waves, that makes clear the analogy between gravitational waves and electromagnetic waves.

Finally, in chapter four we discuss our results on the propagation of gravitational waves through cosmic inhomogeneities. We discover that they suffer both the Sachs-Wolfe and Integrated Sachs-Wolfe effect just as electromagnetic radiation does, and we discuss the correction to the power spectrum of gravitational radiation due to scalar perturbations.

In this section, we introduce the conventions and notations that will be used throughout the thesis. The signature of our metric is $(-,+,+,+)$. We denote the derivative in respect to the cosmic time t with a dot, a prime denotes differentiation in respect to conformal time η defined through

$$d\eta = \frac{dt}{a},$$

where a is the scale factor. The covariant derivative of a generic function f is indicated with the symbol $D_\mu f$, while the ordinary partial derivative is indicated with $\partial_\mu f$. The Hubble rate is defined as

$$H = \frac{\dot{a}}{a}$$

and the conformal Hubble parameter is

$$\mathcal{H} = aH.$$

The convention for the Fourier transform is

$$f(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} e^{\imath \mathbf{k}\cdot\mathbf{x}} \tilde{f}(\eta, \mathbf{k}).$$

Chapter 1

Gravitational waves: a preliminary description of different types of sources

In this section we want to describe, briefly, the different types of sources of gravitational waves. In fact, there are two different kinds of sources: astrophysical sources and cosmological sources. With the adjective cosmological we refer to the mechanism that produced gravitational waves in the early evolution stage of the universe: the inflation epoch. These sources have peculiar properties, differently from the astrophysical ones: the gravitational waves that they produce are spread in a wide range of frequencies, also they are expected be a stochastic background. Both these properties depend on the fact that these gravitational waves are the result, in the standard scenario, of the amplification of the primordial vacuum fluctuations. We will focus on this type of gravitational waves for the rest of the thesis.

1.1 Gravitational waves from astrophysical sources

In the astrophysical category, as the name suggests, all the astrophysical objects that produce with their own motion gravitational waves are grouped. We focus on motion with typical velocity small respect to the speed of light. In this case, by studying the production of gravitational waves we can use the multipole expansion. From [5], the gravitational power radiated from a generic isolated system is

$$\mathcal{W}_{gr} = \frac{G}{5c^5}\dddot{\mathcal{P}}^{ij}\dddot{\mathcal{P}}_{ij}, \tag{1.1.1}$$

where $\mathcal{P}^{ij}$ is the reduced gravitational quadrupole of the system. We do not want to study in detail the generation of gravitational waves, but it is important to compare the formula (1.1.1) with the analogous equation for the electromagnetic power radiated from a charged system

$$\mathcal{W}_{em} = \frac{1}{6\pi c^3}|\ddot{\vec{D}}|^2 + \frac{1}{6\pi c^5}|\ddot{\vec{M}}| + \frac{1}{80\pi c^5}\dddot{\mathcal{P}}^{ij}\dddot{\mathcal{P}}_{ij}, \tag{1.1.2}$$

where $\vec{D}$,$\vec{M}$ and $\mathcal{P}^{ij}$ are respectively the electric dipole, the magnetic dipole and the electric quadrupole [5]. From these formulas we see that the dominant term in the equation (1.1.2) is of order $\frac{1}{c^3}$, while the dominant term of the equation (1.1.1) is of order $\frac{1}{c^5}$. This explains, although the production of gravitational waves and of electromagnetic waves are similar, the fact that the strength of these radiations is quite different.

This argument holds only for the gravitational waves from astrophysical sources, because they are the ones generated by the motion of astrophysical objects. The mechanism that generated the expected stochastic gravitational waves background isn't related to the accelerated motion of some object.

We classify the gravitational waves that come from astrophysical objects by their properties in the frequency domain.

1.1.1 Continuous gravitational waves

Continuous gravitational waves are produced by systems that have fairly constant and well-defined frequency. Examples of these are binary stars or a single star swiftly rotating about its axis, with intrinsic asymmetry from residual crustal deformation. These sources are expected to produce comparatively weak gravitational waves since they evolve over longer periods of time and they are usually less catastrophic than sources producing inspiral or burst gravitational waves. This gravitational waves should produce a "sound" with a continuous tone since the frequency of the gravitational waves is nearly constant, therefore these sources are expected to be nearly monochromatic. The typical frequencies of these waves depend on the properties of the source system, like its mass, revolution duration ecc. For this reason, we have a very large frequency band for these continuous sources, from 10^{-8} Hz to 10^{2} Hz [1].

From this class of gravitational waves, we have obtained the first indirect proof of the existence of this phenomenon. In 1974, R.A. Hulse and J.H. Taylor discovered the binary star PSR 1913+16, and after they observed it for ten years. This pulsar and its companion rotate with a revolution period $T \simeq 7.75$ h, with a distance among them of $d \simeq 1.8 \times 10^{6}$ km. The pulsar has also a very fast proper rotation motion along its axis, with a rotation period $\tau = 59$ ms, and because of this, it emits electromagnetic radiation with the same period. From the observation of this electromagnetic radiation, Hulse and Taylor studied the properties of the system. They observed that the orbital period T was decreasing, due to the fact that the pulsar was emitting gravitational waves and was losing energy. This discovery earned the Noble Prize of Physics in 1993 to R.A. Hulse and J.H. Taylor [5].

1.1.2 Inspiral gravitational waves

Inspiral gravitational waves are generated during the end-of-life stage of binary systems where the two objects merge into one. These systems are usually two neutron stars, two black holes, or a neutron star and a black hole whose orbits have degraded to the point that the two masses are about to coalesce. As the two masses rotate around each other, their orbital distances decrease and their speeds increase. This makes the frequency of the gravitational waves to increase until the moment of coalescence. These sources are expected to give a stronger signal in respect to a binary star system.

Binary star systems are common in our galaxy, but only a thin part of these will end their life with two compact objects in an orbit tight enough to lead to compact binary coalescence in a Hubble time. This is due to several causes. That end result requires both stars to be massive enough to undergo collapse to a compact object without destroying its companion. The frequency band expected for this kind of gravitational waves is from 1 Hz to 10^4 Hz [1]. In order to estimate the number of merges events that occur in a galaxy of the dimensions of ours, there are two distinct approaches, one based on an a priori calculation of binary star evolution. The second estimation method is based largely upon extrapolation from observed double neutron star systems in our local galaxy. The analysis of these methods does not concern this thesis.

In summary, these estimates for the coalescence of two neutron stars yield rates of once every 104 years in a galaxy size of the Milky Way [2]. The corresponding rates for a neutron star and black hole system coalescence are once per 300,000 years. For a system formed by two black holes, the corresponding rates are once per 2.5 million years.

Conversion of coalescence rates into detected coalescence rates depends, of course, on details of frequency-dependent detector sensitivity and on averaging over stellar orientations and sky positions.

The coalescence of two compact massive objects (neutron stars and black holes) into a single final black hole can be divided into three reasonably distinct stages: inspiral, merger and ringdown. The gravitational waves emitted in the inspiral phase can be modeled very accurately. The next phases, when the stars reach the last stable orbit and fall rapidly towards one another are poorly understood. Neither of these final phases will be likely to generate more signal than the inspiral phase, so the detectability of such systems rests on the tracking of their orbital emissions [3].

1.1.3 Burst gravitational waves

Burst gravitational waves come from short-duration unknown sources or unanticipated ones. There are hypotheses that some systems such as supernovae or gamma ray bursts sources may produce burst gravitational waves, but too little is known about the details of these systems to anticipate the form these waves will have.

Burst gravitational waves are the analogue of gamma ray bursts. These one are associated usually to a catastrophic cosmic event, with the release of a lot of energy. Electromagnetic observations provide important, even if indirect, informations on the progenitor. On the other hand, gravitational waves emitted from the central source, carry direct information on its nature.

Gravitational waves signals are expected to be extremely weak when they reach the Earth, so a crucial part of any search is to distinguish a real signal from the “background” of detector noise fluctuations. This is especially important for a burst search since there isn’t a certain expected waveform to compare with and any noise transient might be interpreted as a true gravitational wave signal.

An example of burst gravitational waves source is a core collapse supernovae, which has well known dynamics. Gravitational waves from supernovae carry information about the supernova itself, like neutrinos observation from supernova.

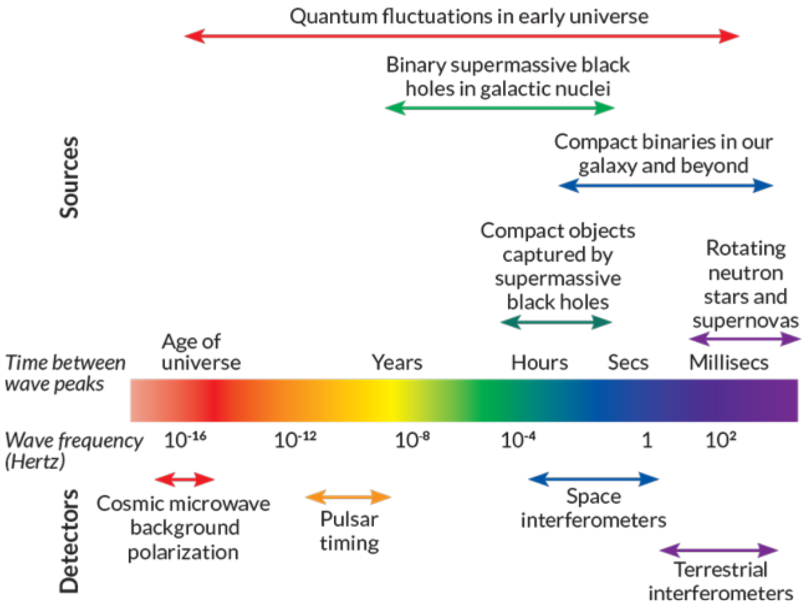

Figure 1.1: Sources and dectors of gravitational waves as a function of the frequency

1.2 Stochastic gravitational wave background

We now turn to analyze the stochastic gravitational waves of cosmological origin. The existence of this background of gravitational waves is predicted by the standard scenario of inflation [26].

A direct detection of these gravitational waves is a step forward in the understanding of the early universe cosmology and of the high energy physics, energy that could not be reached from other ways.

1.2.1 The frequency spectrum

A stochastic background of gravitational radiation is a random gravitational wave signal, that could be thought as the production of a large number of weak, independent, and unresolved gravitational wave sources. In many ways it is analogous to the Cosmic Microwave Background radiation.

It is useful to introduce the dimensionless quantity $\Omega_{gw}(f)$, which characterizes the energy distribution in frequencies of the gravitational wave background in units of ρ_c, the critical energy density of the universe. It is defined as

$$\Omega_{gw}(f) = \frac{1}{\rho_c}\frac{d\rho_{gw}}{d\log(f)}, \tag{1.2.1}$$

where ρ_c is the critical density

$$\rho_c = \frac{3H_0^2}{8\pi G}, \tag{1.2.2}$$

and H_0 is the Hubble parameter evaluated at present time. Usually H_0 is expressed in term of the parameter h as $H_0 = h \times 100$ Km s^{-1}Mpc^{-1}. Since in the determination of h there are several systematic errors that contribute, it is not a good idea to normalize the value of ρ_{gw} with a quantity that depends on h. For this reason, it is useful to consider the quantity $h^2\Omega_{gw}$, which is independent of h, instead of Ω_{gw}. From the Planck collaboration we have $h = 0.678 \pm 0.009$ [9].

We will relate the quantity Ω_{gw} with the power spectrum of the gravitational waves, that for gaussian gravitational waves completely describes their statistical properties. The starting point is the plane wave expansion of the gravitational metric perturbations. In the transverse traceless gauge (which will be defined in the next chapters), this can be written in the form of a plane wave expansion [8, 7]

$$h_{\mu\nu}(t, \mathbf{x}) = \sum_{A=+,\times} \int_{-\infty}^{+\infty} df \int_{S^2} d\Omega h_A(f, \hat{\Omega}) e^{\imath 2\pi f\left(t - \hat{\Omega}\cdot\mathbf{x}\right)} e^A_{\mu\nu}(\hat{\Omega}). \tag{1.2.3}$$

In the previous formula A is a index that labels the two independent polarizations, and $h^*_A(f, \Omega) = h_A(-f, \Omega)$, since $h_{\mu\nu}(t, \mathbf{x})$ is real.

The equation (1.2.3) requires some explanations. In an expanding universe we can perform the Fourier transform only in the spatial variable, because our universe is homogeneous and isotropic only spatially. The integral over the frequencies must not be confusing: we aren't doing the Fourier transform of the temporal component. We have only written the integral over the wavevector $\mathbf{k}$ in polar coordinates, and we have used $\mathbf{k} = 2\pi f \hat{\Omega}$.

The assumptions that the stochastic background produced from inflation is isotropic and unpolarized imply that the ensemble average of the Fourier amplitudes $h^A(f, \Omega)$ could be written as

$$\langle h^A(f, \Omega) h^{A'}(f', \Omega')\rangle = (2\pi)^3 \delta(f + f')\delta(\Omega, \Omega')\delta(A, A')P(f), \tag{1.2.4}$$

where $P(f)$ is the power spectrum, and $\delta\left(\Omega, \Omega'\right) = \delta\left(\cos(\theta) + \cos(\theta')\right)\delta\left(\phi + \phi'\right)$.

The quantity $P(f)$ completely describes the statistical properties of $h^A(f, \Omega)$, if it is a random gaussian field, as we have assumed. We can relate $P(f)$ with the energy density of the gravitational waves. The energy density of the gravitational waves is given by the formula [25]:

$$\rho_{gw} = \frac{1}{32\pi G}\langle \dot{h_{\mu\nu}}(t, \mathbf{x}) \dot{h^{\mu\nu}}(t, \mathbf{x})\rangle, \tag{1.2.5}$$

where the over-dot denotes time derivative. Inserting in the equation (1.2.5) the expansion for $h_{\mu\nu}(t, \mathbf{x})$ given by the equations (1.2.3), we can relate ρ_{gw} with $P(f)$, and we obtain

$$\rho_{gw} = \int_0^{+\infty} df \left(\frac{d\rho_{gw}(f)}{df}\right) = \frac{32\pi^5}{G}\int_0^{\infty} df f^2 P(f). \tag{1.2.6}$$

In order to obtain the previous formula, we used the fact that $P(f) = P(-f)$ and $\sum_{A,A'=+,\times} e^{\mu\nu}_A e^{A'}_{\mu\nu} = 4$. From the definition of $\Omega_{gw}(f)$ we obtain

$$\Omega_{gw}(f) = \frac{256\pi^3}{3H_0^2} f^3 P(f). \tag{1.2.7}$$

$P(f)$ is predicted by the theoretical model that one considers. For the standard inflation scenario of the inflation one has

$$P(f) = \frac{2\pi^2}{f^3} A(f_0)^2 \left(\frac{f}{f_0}\right)^{n_T}, \tag{1.2.8}$$

where n_T is the spectral index, and it is expected to be nearly 0. In this way we see that Ω_{gw} depends slightly on the frequencies f, i.e. we have contributions from all the frequencies to Ω_{gw} [12].

The inflation model predicts that also for the scalar perturbations we have a power spectrum P_s of the form of equation (1.2.7), with spectral index n_s. We could define the quantity r, which is the ratio between the two power spectrums, as

$$r = \frac{P}{P_s}. \tag{1.2.9}$$

The inflation model relates the quantity r and the energy at which inflation takes place with the equation [10]

$$V \simeq \left(1.88 \times 10^{16} GeV\right)^4 \left(\frac{r}{0.10}\right). \tag{1.2.10}$$

In this way with $r \sim 0.10$, we have $E \simeq 10^{16} GeV$, which is the typical scale of Grand Unification theories. The so-called consistency relation holds for slow-roll inflation models, this relation binds the tensor spectral index n_T and r [26]

$$r = -8n_T. \tag{1.2.11}$$

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1.2.2 Observational constraints on the stochastic gravitational waves background

We now report here some constraints on the stochastic gravitational waves background. These constraints are expressed in term of upper limits on the quantity Ω_{gw}, and on the quantity r [10, 16, 13, 15]. These constraints come from different observations. In some cases we have a constraints on Ω_{gw} in a certain frequency band, while in other cases we have a constraint in the integrated value over all frequencies.

Constraint from pulsar timing

A pulsar is a highly magnetized, rotating neutron star that emits a beam of electromagnetic radiation. High precision measurements of millisecond pulsars provide a natural way to study low frequency gravitational waves. A gravitational wave passing between the earth and the pulsar will cause a slight change in the time of arrival of the pulse, leading to a detectable signal.

The European Pulsar Timing Array (EPTA) is a European collaboration to combine five $100m$ class radio telescopes to observe an array of pulsars with the specific goal of detecting gravitational waves. They improved the experimental limit on Ω_{gw} using a six-pulsar data set spanning 18 years of observations from the 2015 European Pulsar Timing Array data release. The result that they have found is [16]

$$h^2\Omega_{gw}(2.8 \times 10^{-9} Hz) \leq 1.1 \times 10^{-9}, \tag{1.2.12}$$

Constraints from interferometers

The Laser Interferometer Gravitational Wave Observatory (LIGO) [4] is a ground based interferometer project operating in the frequency range of 10 Hz $-$ a few kHz. LIGO consists of two Michelson interferometers in Hanford, Washington, H1 with 4 km long arms, and H2 with 2 km long arms, along with a third interferometer in Livingston Parish, Louisiana, L1 with 4 km long arms.

Virgo is a ground based interferometer placed in Santo Stefano a Macerata, Cascina, Tuscany. It has two Michelson interferometers with 3 km long arms. It is designed to operate in the frequency range 10 Hz $-$ 10^5 Hz.

A joint analysis of the data of these two interferometers gives us an upper limit for the stochastic gravitational waves from cosmological origin. We have

$$\Omega_{gw} = (1.8 \pm 4.3) \times 10^{-6} \tag{1.2.13}$$

for the frequencies band $(41.5 - 169.5) Hz$ and

$$\Omega_{gw} = (9.6 \pm 4.3) \times 10^{-5} Hz \tag{1.2.14}$$

for the frequency band $(170 - 600) Hz$ [15].

Constraint from nucleosynthesis

The theory of big bang nucleosynthesis successfully predicts the observed abundances of several light elements in the universe. In doing so, it places constraints on a number of cosmological parameters. This in turn results in an indirect constraint on the energy density in a gravitational wave background: the presence of a significant amount of gravitational radiation at the time of nucleosynthesis changes the total energy density of the universe, which affects the rate of expansion, leading to an overabundance of helium and thus spoiling the predictions of big bang nucleosynthesis. From the nucleosynthesis we obtain [13]

$$\int_{f_1}^{f_2} \Omega_{gw}(f) d\left(\log(f)\right) \leq 1.5 \times 10^{-5}. \tag{1.2.15}$$

We notice that this physical process constrains the integrated value of Ω_{gw}, over the whole frequency spectrum. The lower cutoff frequency f_1 corresponds to the Hubble radius at the time of big bang nucleosynthesis and takes the value $f_1 \sim 10^{-10}$ Hz. In fact, scales bigger than this were outside the horizon at that epoch, so they didn't contribute to the physics of the universe. The upper cutoff frequency f_2 is the ultraviolet cutoff. This is given by the Planck frequency, $f_2 = f_P = 1.86 \times 10^{43}$ Hz.

Constraint from the Cosmic Microwave Background data

Observations of the Cosmic Microwave Background have implications for a wide variety of topics, including constrain inflation, dark matter, dark energy and large scale structure. Gravitational waves induce temperature fluctuations through the Sachs-Wolfe and Integrated Sachs-Wolfe effect, as scalar perturbations do. From the measurement of these effects we can constrain the gravitational waves content in the universe.

Maps of the Cosmic Microwave Background polarization anisotropies are naturally decomposed into curl-free E modes and gradient-free B modes. Primordial gravitational waves can be observed in the polarization of the Cosmic Microwave Background. B modes are not generated at linear order in perturbation theory by the scalar perturbations. Tensor perturbations are expected to produce B-mode polarization on large angular scale. The B-mode polarization signal produced by scalar perturbations is very small and is dominated by the weak lensing of E-mode polarization on small angular scales. Until now, B modes polarization generate unambiguously generated from tensor perturbations have not yet been detected.

The Planck mission was to produce full sky maps of the Cosmic Microwave Background anisotropies, and recently they have released results from four years of data acquisition.

The result for the value of r is

$$r < 0.10, \tag{1.2.16}$$

at 95% of confidence level [10], at which corresponds the value for Ω_{gw} [11]

$$h_0^2 \Omega_{gw} \leq 1 \times 10^{-14} \tag{1.2.17}$$

in the frequency range $10^{-17} \div 10^{-16} Hz$.

Chapter 2

Cosmological perturbations

Like in any other perturbation theory, this approach is based on the existence of a background solution. In this case the background solution is the Standard Hot Big-Bang Model, which is based on the hypothesis of homogeneity and isotropy of the universe. This model excellently describes a lot of observations of our universe: the expansion and cooling of the universe, the abundance of light nuclei from primordial nucleosynthesis and finally the Cosmic Microwave Background isotropy, which was the first proof in '60 of the Standard Hot Big-Bang model. Obviously, this model is only a rough approximation, and there are a lot of observational confirmations of that, like the anisotropies of the Cosmic Microwave Background.

In order to describe these observational issues, we add to the homogeneous and isotropic background model small quantities, which allow us to study the problem with the techniques of perturbations theory. The main difference between cosmological perturbation theory, and the ordinary one which is used in other branches of physics is that in cosmology we perturb the metric tensor, so it is the geometry of space-time which results changed. So we have to compare quantities defined in different space-times (manifolds), which requires a prescription for switching from the background manifold to the physical manifold. This is the *Gauge problem*, which follows from the arbitrariness of the map which we can use. Obviously, physical quantities cannot depend on the gauge choice, so it is important to distinguish geometric and gauge modes in the metric perturbations. Einstein equations relate the metric tensor with the energy-momentum tensor of the matter, for this reason we have to perturb also the energy-momentum tensor. As for the metric tensor, we have also gauge modes in the energy-momentum tensor perturbations.

Cosmological perturbations theory is a very well studied argument. There are several reviews in the literature, for example [17] and [18] for linear theory. The gauge problem in cosmology is treated in [22], and a review about the possible gauges is [20]. Second order perturbation theory was reviewed in [23]. For a complete treatment about the gauge problem in cosmology see [22]. A useful reference about gauge-invariant variables is [29]. For a solution of the second-order perturbation equations see [21] and [30].

2.1 Defining perturbations

Any tensor quantities can be separated in to a homogeneous time dependent-part, defined in the background, plus inhomogeneous perturbation, which depends both time and space coordinates

$$T(\eta, \mathbf{x}) = T^{(0)}(\eta) + \delta T(\eta, \mathbf{x}). \tag{2.1.1}$$

The perturbation can be expanded in series as

$$\delta T(\eta, \mathbf{x}) = \sum_{n_1}^{\infty} \lambda^n \frac{\delta T^{(n)}(\eta, \mathbf{x})}{(n!)}, \tag{2.1.2}$$

where the subscript represents the order of the perturbation.

We now define the perturbations of the metric tensor. These definitions are given without specifying the gauge. We will follow in this section [20], which is a complete review of this topic. Our background space-time is described by a spatially flat FRW metric, defined by the line element

$$ds^2 = a^2 \left(-d\eta^2 + x^i x^j \delta_{ij}\right), \tag{2.1.3}$$

where η is the conformal time and $a(\eta)$ is the scale factor. Cosmic time and conformal time are related through the relation $dt = a(\eta) d\eta$. The perturbations to the metric tensor are:

$$\delta g_{00} = -2a^2 \Phi, \tag{2.1.4}$$

$$\delta g_{0i} = a^2 B_i, \tag{2.1.5}$$

$$\delta g_{ij} = 2a^2 C_{ij}. \tag{2.1.6}$$

We can split the δg_{0i} and δg_{ij} in scalar, vector and tensor quantities

$$B_i = B_{,i} - S_i, \tag{2.1.7}$$

$$C_{ij} = -\Psi \delta_{ij} + E_{,ij} + \frac{1}{2}\left(F_{i,j} + F_{j,i}\right) + \frac{1}{2} h_{ij}. \tag{2.1.8}$$

In these formulas Φ, B, Ψ and E are scalar quantities, F_i and S_i are vector quantities and h_{ij} is a tensor perturbation. The adjectives tensor, vector and scalar reflect the transformation property of these quantities under spatial coordinates transformation of the constant η hypersurface.

This decomposition of B_i and C_{ij} is known in Euclidean space as Helmholtz decomposition and implies some properties of these quantities. The pure vector perturbations like F_i and S_i are divergence-free, while any vectors constructed from a scalars like $E_{,i}$ and $B_{,i}$ are curl-free.

The tensor h_{ij} is transverse, ${h_{ij,}}^i = 0$, and trace-free, $h^i{}_i = 0$. We rise and lower spatial index of perturbation with the comoving background metric δ_{ij}.

Note that we have 10 degrees of freedom (d.o.f) in $\delta g_{\mu\nu}$, as we expected, since the metric tensor must be symmetric. These d.o.f are given by four scalars, two vectors (each one with two d.o.f) and one tensor (with two d.o.f). This division of the metric tensor in scalars, vectors and tensor is arbitrary, but it is useful since at linear order in perturbation theory these quantities evolve independently, as we will see later.

The quantities defined in equations (2.1.4)-(2.1.6) include all orders. With the expansion given in the equation (2.1.1), we can write for the metric perturbations, up to second-order,

$$\Phi = \Phi^{(1)} + \frac{1}{2}\Phi^{(2)}, \tag{2.1.9}$$

$$B_i = B_i^{(1)} + \frac{1}{2}B_i^{(2)}, \tag{2.1.10}$$

$$C_{ij} = C_{ij}^{(1)} + \frac{1}{2}C_{ij}^{(2)}. \tag{2.1.11}$$

The contravariant metric tensor follows from the constraint

$$g_{\mu\nu}g^{\nu\rho} = \delta_\mu{}^\rho, \tag{2.1.12}$$

which, at second-order, gives

$$g^{00} = -\frac{1}{a^2}\left(1 - 2\Phi^{(1)} - \Phi^{(2)} + 4\Phi^{(1)2} - B_k^{(1)}B^{(1)k}\right), \tag{2.1.13}$$

$$g^{0i} = \frac{1}{a^2}\left(B^{i(1)} + \frac{1}{2}B^{(2)i} - 2\Phi^{(1)}B^{i(1)} - 2B_k^{(1)}C^{ki(1)}\right), \tag{2.1.14}$$

$$g^{ij} = \frac{1}{a^2}\left(\delta^{ij} - 2C^{ij(1)} - C^{ij(2)} + 4C^{ik(1)}C^{(1)j}{}_k - B^{(1)i}B^{(1)j}\right). \tag{2.1.15}$$

We have written the perturbations division for a rank-two tensor, but clearly this division holds also for a four-vector,

$$V^\mu = \left(V^0, V_{,}{}^i + V^i\right). \tag{2.1.16}$$

As before, V^0 and V are scalars on the spatial hypersurfaces, whereas V^i is a divergence-free vector.

2.2 Geometry of spatial hypersurfaces

With the perturbed metric defined in the previous section, we can define a unit time-like vector, which is a normal vector to constant η hypersurface. This is defined as

$$n_\mu = \alpha\frac{\partial\eta}{\partial x^\mu}, \tag{2.2.1}$$

where α is a normalization factor.

We immediately see that $\frac{\partial\eta}{\partial x^0} = 1$ and $\frac{\partial\eta}{\partial x^i} = 0$; these relations give us that the normalization factor

$$n_\mu n^\mu = \alpha^2\frac{\partial\eta}{\partial x^\mu}\frac{\partial\eta}{\partial x^\nu}g^{\mu\nu} = \alpha^2\left(\frac{\partial\eta}{\partial x^0}\right)g^{00}\alpha^2 = g^{00}\alpha^2 = -1, \tag{2.2.2}$$

which gives

$$\alpha = \pm\left(-g^{00}\right)^{-\frac{1}{2}}. \tag{2.2.3}$$

Inserting in the previous equation the expression for g^{00} given by equation (2.1.4), we get

$$\alpha = \pm \left[a^{-2}\left(1 - 2\Phi^{(1)} - \Phi^{(2)} + 4\Phi^{(1)2} - B_k^{(1)}B^{(1)k}\right)\right]^{-\frac{1}{2}}. \qquad (2.2.4)$$

Since the perturbations are small quantities, we can expand the square root. In doing this, we have to pay attention that the expansions in different perturbations are independent. We have

$$(1 - 2\Phi^{(1)} + 4\Phi^{(1)2})^{-\frac{1}{2}} \simeq 1 + \Phi^{(1)} - \frac{\Phi^{(1)}2}{2}, \qquad (2.2.5)$$

$$(1 - \Phi^{(2)})^{-\frac{1}{2}} \simeq 1 + \frac{\Phi^{(2)}}{2}, \qquad (2.2.6)$$

$$(1 - B^{(1)i}B_i^{(1)})^{-\frac{1}{2}} \simeq 1 + \frac{B^{(1)i}B_i^{(1)}}{2}. \qquad (2.2.7)$$

With this manipulation we get for α

$$\alpha = -a\left(1 + \Phi^{(1)} + \frac{1}{2}\Phi^{(2)} - \frac{1}{2}\Phi^{(1)2} + \frac{1}{2}B_k^{(1)}B^{(1)k}\right) \qquad (2.2.8)$$

where we selected the minus sign for α, in order to have a time-like unit vector, which becomes

$$n_\mu = -a\left[1 + \Phi^{(1)} + \frac{1}{2}\Phi^{(2)} - \frac{1}{2}\Phi^{(1)2} + \frac{1}{2}B_k^{(1)}B^{(1)k}, \mathbf{0}\right], \qquad (2.2.9)$$

From this formula, with the metric given in equations (2.1.13)-(2.1.15), we can calculate the contravariant constant η hypersurface vector,

$$n^0 = \frac{1}{a}\left(1 - \Phi^{(1)} - \frac{1}{2}\Phi^{(2)} + \frac{3}{2}\Phi^{(1)2} - \frac{1}{2}B_k^{(1)}B^{(1)k}\right), \qquad (2.2.10)$$

$$n^i = \frac{1}{a}\left(-B^i - \frac{1}{2}B^{(2)i} + 2B_k^{(1)}C^{(1)ki} + \Phi_1 B^{(1)i}\right). \qquad (2.2.11)$$

The covariant derivative of any time-like vector, can be decomposed following [28, 24], as

$$n_{\mu;\nu} = \frac{1}{3}\theta\mathcal{P}_{\mu\nu} + \sigma_{\mu\nu} - \omega_{\mu\nu} - a_\mu n_\nu \qquad (2.2.12)$$

where $\mathcal{P}_{\mu\nu}$ is a projector orthogonal to n^μ

$$\mathcal{P}_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu, \qquad (2.2.13)$$

the expansion rate θ is

$$\theta = n^\mu{}_{;\mu}, \qquad (2.2.14)$$

the symmetric and trace-free shear is

$$\sigma_{\mu\nu} = \frac{1}{2}\mathcal{P}^\alpha{}_\mu\mathcal{P}^\beta{}_\nu\left(n_{\alpha;\beta} + n_{\beta;\alpha}\right) - \frac{1}{3}\theta\mathcal{P}_{\mu\nu}, \qquad (2.2.15)$$

the antisymmetric vorticity is

$$\omega_{\mu\nu} = \frac{1}{2}\mathcal{P}^{\alpha}{}_{\mu}\mathcal{P}^{\beta}{}_{\nu}\left(n_{\alpha;\beta} - n_{\beta;\alpha}\right), \tag{2.2.16}$$

and the acceleration is

$$a_{\mu} = n_{\mu;\nu}n^{\nu}. \tag{2.2.17}$$

This expansion could be done for every time-like vector, as the fluid four-velocity u_μ is. If we restrict our study to spatial hypersurfaces, these quantities are analogous to those that we define in Newtonian fluid dynamics. The vorticity for n_μ given in the formula (2.2.16) is automatically zero, because the term in parenthesis vanishes.

The scalar part of the shear is given by

$$\sigma_{ij} = \left(\partial_i\partial_j - \frac{1}{3}\nabla^2\delta_{ij}\right)a\sigma, \tag{2.2.18}$$

where σ is the shear potential

$$\sigma = E' - B. \tag{2.2.19}$$

The vector and tensor parts of the shear are given by

$$\sigma_{Vij} = a\left(F'_{(i,j)} - B_{(i,j)}\right), \qquad \sigma_{Tij} = \frac{a}{2}h'_{ij}. \tag{2.2.20}$$

2.3 Gauge transformations

We analyze now how tensorial quantities change under gauge transformations. One basic assumption of perturbation theory is the existence of a one-parameter family of solutions, which depends on one real parameter λ. The family of solutions is $\mathcal{M}_\lambda$; $\lambda = 0$ identifies the background solution, and after the calculations we can set $\lambda = 1$ to recover the physical solution.

Following [21] and [22], we consider two one-to-one parameter maps from $\mathcal{M}_0$ to $\mathcal{M}_\lambda$, and we call these maps ϕ_λ and ψ_λ. The choice of one of these maps is the gauge choice. If we set on $\mathcal{M}_0$ the coordinates system x^μ, these coordinates will be carried on $\mathcal{M}_\lambda$ with one map, ϕ_λ as example.

Consider p a point in the background, with coordinates $x^\mu(p)$, and $O = \psi_\lambda(p)$ the associated point on $\mathcal{M}_\lambda$ with the same coordinates x^μ. Let's assume that if we use the map ϕ_λ, the same point O is the image of the point q. In that way a gauge transformation could be seen as a one-to-one correspondence through different points of the background, keeping the coordinates system fixed. In fact we can construct this map just considering the composition $\Phi_\lambda = \phi_\lambda^{-1}(\psi_\lambda)$, which relates the point p with the point q. This map, which expresses the coordinates of q, $\tilde{x}^\mu(q)$ as a function of those of $x^\mu(p)$, is called "*active coordinates transformation*" because it works keeping fixed the coordinates system over the space-time. Differently from this, a usual coordinate transformation is called "*passive coordinate transformation*", and it changes only "the label" of each point of $\mathcal{M}_0$.

Consider now a generic tensor T defined on $\mathcal{M}_\lambda$. With the two gauges we can define two different tensors on $\mathcal{M}_0$, calling them $T_\lambda(p)$ and $\tilde{T}_\lambda(q)$, which are two representations of T_λ on $\mathcal{M}_0$. These tensors are related to each other by the map Φ_λ. Now we define the perturbation, by comparing these tensors with the background tensor $T^{(0)}$

$$\delta T_\lambda = T_\lambda(p) - T^{(0)}, \tag{2.3.1}$$

$$\delta \tilde{T}_\lambda = \tilde{T}_\lambda(q) - T^{(0)}. \tag{2.3.2}$$

Taking the difference of these two equations we get

$$\delta \tilde{T}_\lambda = \delta T_\lambda + \tilde{T}_\lambda(q) - T_\lambda(p) \tag{2.3.3}$$

In order to obtain the transformation rule for the perturbations we have to calculate the difference between two tensors, but those tensors are calculated at different space-time points, keeping fixed the coordinate system. These two tensors are related through Φ_λ, which could be written at first-order in λ as

$$x^\mu(q) = x^\mu(p) + \lambda \xi^\mu(x(p)), \tag{2.3.4}$$

which is an active coordinate transformation. Starting from (2.3.4), we can define a passive coordinate transformation, defining a new coordinate system, y, in such a way that the y's coordinates of the point q coincide with the x's coordinates of the point p

$$y^\mu(q) := x^\mu(p) = x^\mu(q) - \lambda \xi^\mu(x(p)) \simeq x^\mu(q) - \lambda \xi^\mu(x(q)), \tag{2.3.5}$$

where the last $\simeq$ is justified by the fact that the difference between $x^\mu(p)$ and $x^\mu(q)$ is of first-order in λ, and then ξ term is already of first-order in λ. Considering now as a concrete case a rank-two tensor $T_{\mu\nu}(x)$; the transformation rule under a passive coordinate transformation is

$$\tilde{T}_{\mu\nu}(y) = \frac{\partial x^\rho}{\partial y^\mu}\frac{\partial x^\sigma}{\partial y^\nu} T_{\rho\sigma}(x). \tag{2.3.6}$$

The Jacobian matrix of the transformation to first-order in ξ is

$$\frac{\partial x^\rho}{\partial y^\mu} = \delta^\rho_\mu + \frac{\partial \xi^\rho}{\partial x^\mu}, \tag{2.3.7}$$

where ξ and its derivatives are infinitesimal 4-vectors. Using this equation on (2.3.6) we obtain

$$\tilde{T}_{\mu\nu}(y) = T_{\mu\nu}(x) + \frac{\partial \xi^\rho}{\partial x^\mu} T_{\rho\nu}(x) + \frac{\partial \xi^\sigma}{\partial x^\nu}(x) T_{\mu\sigma}(x) + \mathcal{O}(\xi^2) \tag{2.3.8}$$

In the left head side of the previous equation we have the tensor $\tilde{T}$ evaluated at the point y. If we insert equation (2.3.5), we get

$$\tilde{T}_{\mu\nu}(y) = \tilde{T}_{\mu\nu}(x - \xi) = \tilde{T}_{\mu\nu}(x) - \frac{\partial T_{\mu\nu}}{\partial x^\rho}\xi^\rho(x) + \mathcal{O}(\xi^2), \tag{2.3.9}$$

where in the last term we replace $\tilde{T}$ with T, since the difference between these terms is $\mathcal{O}(\xi)$. Summarizing, we have

$$\tilde{T}_{\mu\nu}(x) = T_{\mu\nu}(x) + \frac{\partial T_{\mu\nu}}{\partial x^\rho}\xi^\rho(x) + \frac{\partial \xi^\rho}{\partial x^\mu} T_{\rho\nu}(x) + \frac{\partial \xi^\sigma}{\partial x^\nu}(x) T_{\mu\sigma}(x) + \mathcal{O}(\xi^2), \quad (2.3.10)$$

which is the expression of the Lie derivative $\mathcal{L}_\xi$ along the vector ξ, for a rank 2 covariant tensor,

$$\tilde{T}_{\mu\nu}(x) = T_{\mu\nu}(x) + \lambda \mathcal{L}_\xi T_{\mu\nu}. \quad (2.3.11)$$

Equation (2.3.9) is only the first-order transformation rule, since we have considered in (2.3.4) only first-order transformations in λ. In general, one can write ξ as a series

$$\xi^\mu = \lambda \xi_1^\mu + \frac{1}{2}\lambda^2 \xi_2^\mu + \mathcal{O}(\xi^3), \quad (2.3.12)$$

and defining a generalization of the equation (2.3.4)

$$\tilde{x}^\mu(q) = x^\mu(p) + \lambda \xi^\mu(x(p)) + \frac{\lambda^2}{2}\left(\xi^\mu_{(1),\nu}\xi^\nu - \xi^\mu_{(2)}\right) + \mathcal{O}(\lambda^2). \quad (2.3.13)$$

With these definitions, one can demonstrate [22] that the equation (2.3.11) could be generalized as

$$\tilde{T}(x) = e^{\lambda \mathcal{L}_\xi} T(x) = T + \lambda \mathcal{L}_{\xi_{(1)}} T + \frac{\lambda^2}{2}\left(\mathcal{L}^2_{\xi_{(1)}} + \mathcal{L}_{\xi_{(2)}}\right) T + \dots \quad (2.3.14)$$

that at first-order reduces to (2.3.11). Looking back at equation (2.3) we read the first-order gauge transformation rule for a generic perturbation, which is

$$\delta \tilde{T}^{(1)}_\lambda = \delta T^{(1)}_\lambda + \mathcal{L}_\xi T_\lambda = \delta T^{(1)}_\lambda + \lambda \mathcal{L}_\xi T^{(0)}, \quad (2.3.15)$$

and the gauge transformation rule for a second-order perturbation

$$\delta \tilde{T}^{(2)}_\lambda = \delta T^{(2)}_\lambda + 2\mathcal{L}_{\xi_1} \delta T^{(1)}_\lambda + \mathcal{L}_{\xi_2} T^{(0)}_\lambda + \mathcal{L}^2_{\xi_1} T^{(0)}_\lambda. \quad (2.3.16)$$

Starting from these formulas,then we can move from one gauge to another, by specifying the components of vectors $\xi^{(1)}$ and $\xi^{(2)}$. The choice of gauge in which to work, is dictated by convenience, in fact there are gauges in which calculations are simpler, or by physical reasons. In fact one could work in a gauge in which some physical properties of the dynamics of perturbations are evident.

According with the split of the perturbations in scalars, vectors and tensors, we also split the vectors ξ^μ, which are the generators of the gauge transformation. We write

$$\xi^{(1)\mu} = \left(\alpha^{(1)}, \beta^{(1)}{}_{,}{}^{i} + \gamma^{(1)i}\right), \quad (2.3.17)$$

and for the second-order transformation generator

$$\xi^{(2)\mu} = \left(\alpha^{(2)}, \beta^{(2)}{}_{,}{}^{i} + \gamma^{(2)i}\right), \quad (2.3.18)$$

γ^i is divergence-free.

We calculate now explicitly how tensorial perturbations change under a gauge transformation. We start from the simplest one, the four-scalar.

- **Four-scalar**

The first-order transformation rule is given in equation (2.3.15). The Lie derivative formula for a scalar is

$$\mathcal{L}_\xi \phi = \phi_{,\lambda}\xi^\lambda, \tag{2.3.19}$$

which gives

$$\tilde{\delta}\rho^{(1)} = \delta\rho^{(1)} + \rho'^{(0)}\alpha^{(1)}. \tag{2.3.20}$$

At second-order, with the same procedure but with the formula (2.3.16), we obtain

$$\begin{aligned}\tilde{\delta}\rho^{(2)} =& \delta\rho^{(2)} + \rho'^{(0)}\alpha^{(2)} + \alpha^{(1)}\left(\rho''^{(0)}\alpha^{(1)} + \rho'^{(0)}\alpha'^{(1)} + 2\delta\rho'^{(1)}\right)\\ &+ \left(2\delta\rho^{(1)} + \rho'^{(0)}\alpha^{(1)}\right)_{,k}\left(\beta^{(1)}{}_{,}{}^{k} + \gamma^{(1)k}\right).\end{aligned} \tag{2.3.21}$$

- **Four-vectors**

We now look to a four-vectors, like the four-velocity of the fluid. The procedure is the same of the four-scalars case, but now we have to use the Lie derivative formula for a four-vector, which is

$$\mathcal{L}_\xi u_\mu = u_{\mu,\alpha}\xi^\alpha + u^\alpha \xi_{,}{}^{\alpha}{}_{\mu}. \tag{2.3.22}$$

Up to first-order we obtain, with equation (2.3.15),

$$\tilde{\delta u}_\mu^{(1)} = \delta u_\mu^{(1)} + u_\mu'^{(0)}\alpha^{(1)} + u_\lambda^{(0)}\xi^{(1)\lambda}{}_{,\mu}. \tag{2.3.23}$$

Up to second-order we obtain, from equation (2.3.16), the transformation rule

$$\begin{aligned}\tilde{\delta u}_\mu^{(2)} =& \delta_\mu^{(2)} + u_\mu'^{(0)}\alpha^{(2)} + u_0^{(0)}\alpha_{,\mu}^{(2)} + u_\mu''^{(0)}\alpha^{(1)2} + u_\mu'^{(0)}\alpha_{,\lambda}^{(1)}\xi^{(1)\lambda}\\ &+ 2u_0'^{(0)}\alpha^{(1)}\alpha_{,\mu}^{(1)} + u_0^{(0)}\left(\xi^{(1)\lambda}\alpha_{,\mu\lambda}^{(1)} + \alpha_{,\lambda}^{(1)}\xi_{,\mu}^{(1)\lambda}\right)\\ &+ 2\left(\delta u_{\mu,\lambda}^{(1)}\xi^{(1)\lambda} + \delta u_\lambda^{(1)}\xi_{,\mu}^{(1)\lambda}\right).\end{aligned} \tag{2.3.24}$$

- **Rank-two tensors**

For a rank-two tensor, we will act in the same ways of the cases presented before. Now the calculation is a bit longer, due to the expression of the Lie derivative for tensors, which is

$$\mathcal{L}_\xi g_{\mu\nu} = g_{\mu\nu,\lambda}\xi^\lambda + g_{\mu\lambda}\xi^\lambda{}_{,\nu} + g_{\nu\lambda}\xi^\lambda{}_{,\mu}. \tag{2.3.25}$$

Let's start from the first transformation rule. Applying this equation to the $0-0$ component of the equation we get,

$$\tilde{\delta g}_{00}^{(1)} = g_{00,0}\xi^{(1)0} + 2g_{00}\xi^{(0)0}{}_{,0} = \delta g_{00} - 2a'a\alpha^{(1)} - 2a^2\alpha'(1) \quad , \tag{2.3.26}$$

where $\delta g_{00}^{(1)} = -2a^2\Phi^{(1)}$, from which we read

$$\tilde{\Phi}^{(1)} = \Phi^{(1)} + \mathcal{H}\alpha^{(1)} + \alpha'^{(1)}. \tag{2.3.27}$$

The previous equation shows us that the lapse function transforms as a four-scalar.

We turn now on the $0-i$ component of the metric tensor. The Lie derivative is

$$\begin{aligned}\mathcal{L}_\xi g_{0i} =& g_{0i,\lambda}\xi^{(1)\lambda} + g_{0\lambda}\xi^{(1)\lambda}{}_{,i} + g_{\lambda i}\xi^{(1)\lambda}{}_{,0}\\ &- a^2\alpha^{(1)}_{,i} + a^2\xi^{(1)}_{i,0}\end{aligned} \tag{2.3.28}$$

from which we obtain, having in mind that $\delta g^{(1)}_{0i} = a^2 B^{(1)}_i$,

$$\tilde{B_i}^{(1)} = B^{(1)}_i - \alpha^{(1)}_{,i} + \left(\beta'^{(1)}_{,i} + \gamma'^{(1)}_i\right). \tag{2.3.29}$$

If we take the divergence, we get the transformation rule of B

$$\tilde{B}^{(1)} = B^{(1)} + \beta'^{(1)} - \alpha^{(1)}. \tag{2.3.30}$$

Subtracting this part to equation (2.3.29) we have the rule for S_i

$$\tilde{S_i}^{(1)} = \tilde{B_i}^{(1)} - \tilde{B_{,i}}^{(1)} = S^{(1)}_i - \gamma'^{(1)}_i. \tag{2.3.31}$$

We apply again the equation (2.3.25), in order to obtain the transformation rule for $\delta g^{(1)}_{ij} = 2a^2 C^{(1)}_{ij}$,

$$\begin{aligned}\mathcal{L}_{\xi^{(1)}} g_{ij} =& g_{ij,0}\xi^{(1)0} + g_{ik}\xi^{(1)k}{}_{,j} + g_{jk}\xi^{(1)k}{}_{,i}\\ &2aa'\delta_{ij}\alpha^{(1)} + a^2\left(\xi^{(1)}_{i,j} + \xi^{(1)}_{j,i}\right),\end{aligned} \tag{2.3.32}$$

which becomes for $C_{ij^{(1)}}$

$$2\tilde{C_{ij}}^{(1)} = 2C^{(1)}_{ij} + 2\mathcal{H}\alpha^{(1)}\delta_{ij} + \xi^{(1)}_{i,j} + \xi^{(1)}_{j,i}. \tag{2.3.33}$$

We report here (2.1.8), where we split the perturbation $C^{(1)}_{ij}$ in scalars, vectors and tensors: $C^{(1)}_{ij} = -\Psi^{(1)}\delta_{ij^{(1)}} + E^{(1)}_{,ij} + \frac{1}{2}\left(F^{(1)}_{i,j} + F^{(1)}_{j,i}\right) + \frac{1}{2}h^{(1)}_{ij}$.

Taking the trace of the equation (2.3.32) we get

$$-3\tilde{\Psi}^{(1)} + \nabla^2\tilde{E}^{(1)} = -3\Psi^{(1)} + \nabla^2 E^{(1)} + 3\mathcal{H}\alpha^{(1)} + \nabla^2 B^{(1)}, \tag{2.3.34}$$

and applying the operator $\partial_i\partial_j$ to (2.3.32) we obtain

$$-3\nabla^2\tilde{\psi}^{(1)} + \nabla^2\nabla^2\tilde{E}^{(1)} = -3\nabla^2\Psi^{(1)} + \nabla^2\nabla^2 E^{(1)} + 3\mathcal{H}\nabla^2\alpha^{(1)} + \nabla^2\nabla^2\beta^{(1)}. \tag{2.3.35}$$

Now obtaining $\tilde{\Psi}^{(1)}$ from (2.3.34), and inserting it in equation (2.3.35) we read the transformation rule for E:

$$\tilde{E}^{(1)} = E^{(1)} + \beta^{(1)}. \tag{2.3.36}$$

Putting this inside equation (2.3.35), we obtain the transformation law for Ψ :

$$\tilde{\Psi}^{(1)} = \Psi^{(1)} - \mathcal{H}\alpha^{(1)}. \tag{2.3.37}$$

To obtain the transformation law for $F_i^{(1)}$ we have to consider the trace of (2.3.32)

$$2\tilde{C}^{(1)}{}_{ij,}{}^{j} = 2C^{(1)}{}_{ij,}{}^{j} + 2\mathcal{H}\alpha^{(1)}_{,i} + \nabla^2\xi^{(1)}_i + \nabla^2\beta^{(1)}_{,i}. \tag{2.3.38}$$

Using the transformation rules for $E^{(1)}$ and $\Psi^{(1)}$ in this equation we obtain the law for $F_i^{(1)}$:

$$\tilde{F_i}^{(1)} = F_i^{(1)} + \gamma_i^{(1)}. \tag{2.3.39}$$

Finally, if we insert equations (2.3.36), (2.3.37) and (2.3.39) in (2.3.32), we have the transformation rule for tensor perturbations

$$\tilde{h_{ij}}^{(1)} = h^{(1)}_{ij}. \tag{2.3.40}$$

The last equation tells us that tensor perturbations, at first-order, are gauge-invariant. This could be expected since in the generator of the gauge transformation $\xi^{(1)}$ we have only two scalars and one vector. This gauge invariance is useful since we do not have to take care about working with gauge-invariant quantities, since tensors are already invariant.

The procedure for obtaining the second-order transformation rule follows what we have already done for the first-order. The metric tensor transforms, using equation (2.3.16), as

$$\begin{aligned}
\delta\tilde{g_{\mu\nu}}^{(2)} =& \delta g^{(2)}_{\mu\nu} + g^{(0)}_{\mu\nu,\lambda}\xi^{(2)\lambda} + g^{(0)}_{\lambda\nu}\xi^{(2)\lambda}{}_{,\mu} \\
& + 2\left(\delta g^{(1)}_{\mu\nu,\lambda}\xi^{(1)\lambda} + \delta g^{(1)}_{\mu\lambda}\xi^{(1)\lambda}{}_{,\nu} + \delta g^{(1)}_{\lambda\nu}\xi^{(1)\lambda}{}_{,\mu}\right) + g^{(0)}_{\mu\nu,\lambda\alpha}\xi^{(1)\lambda}\xi^{(1)\alpha} \\
& + g^{(0)}_{\mu\nu,\lambda}\xi^{(1)\lambda}{}_{,\alpha}\xi^{(1)\alpha} + 2\left(g^{(0)}_{\mu\lambda,\alpha}\xi^{(1)\alpha}\xi^{(1)\lambda}{}_{,\nu} + g^{(0)}_{\lambda\nu,\alpha}\xi^{(1)\alpha}\xi^{(1)\lambda}{}_{,\mu}\right. \\
& \left. + g^{(0)}_{\lambda\alpha}\xi^{(\lambda)}{}_{,\mu}\xi^{(\alpha)}{}_{,\nu}\right) + g^{(0)}_{\mu\lambda}\left(\xi^{(1)\lambda}{}_{,\nu\alpha}\xi^{(1)\alpha} + \xi^{(1)\lambda}{}_{,\alpha}\xi^{(1)\alpha}{}_{,\nu}\right) \\
& + g^{(0)}_{\lambda\nu}\left(\xi^{(1)\lambda}{}_{,\mu\alpha}\xi^{(1)\alpha} + \xi^{(1)\lambda}{}_{,\alpha}\xi^{(1)\alpha}{}_{,\mu}\right).
\end{aligned} \tag{2.3.41}$$

We get the transformation equation for $\Psi^{(2)}$ by looking at the $0-0$ component of the equation (2.3.41)

$$\begin{aligned}
\tilde{\Phi}^{(2)} =& \Phi^{(2)} + \mathcal{H}\alpha^{(2)} + \alpha'^{(2)} + \alpha^{(1)}\left(\alpha''(1) + 5\mathcal{H}\alpha'^{(1)} + (\mathcal{H}' + 2\mathcal{H}^2)\alpha'^{(1)}\right. \\
& \left. + 4\mathcal{H}\Phi^{(1)} + 2\Phi'^{(1)}\right) + 2\alpha'^{(1)}\left(\alpha'^{(1)} + 2\Phi^{(1)}\right) + \xi^{(1)}_k\left(\alpha'^{(1)} + \mathcal{H}\alpha'^{(1)}\right. \\
& \left. + 2\Phi^{(1)}\right)_{,}^{\;k} + \xi'^{(1)}_k\left(\alpha^{(1)}{}_{,}{}^{k} - 2B^{(1)k} - \xi'^{(1)k}\right).
\end{aligned} \tag{2.3.42}$$

From the $0-i$ component of the equation (2.3.41) we get the transformation formula for $B_i^{(2)}$, which is

$$\tilde{B}^{(2)}_i = B^{(2)}_i + \xi'^{(2)}_i - \alpha^{(2)}_{,i} + \chi_{Bi}, \tag{2.3.43}$$

where χ_{Bi} contains the terms quadratic in the first-order perturbations, and it is

$$
\begin{aligned}
\chi_{Bi} =& 2\left((2\mathcal{H}B_i^{(1)} + B_i'^{(1)})\alpha^{(1)} + B_{i,k}^{(1)}\xi^{(1)k} - 2\Phi^{(1)}\alpha_{,i}^{(1)} + B_k^{(1)}\xi^{(1)k}{}_{,i} + B_i^{(1)}\alpha'^{(1)}\right.\\
&\left.+2C_{ij}^{(1)}\xi'^{(1)j}\right) + 4\mathcal{H}\alpha^{(1)}(\xi_i'^{(1)} - \alpha_{,i}^{(1)}) + \alpha'^{(1)}(\xi_i'^{(1)} - 3\alpha_{,i}^{(1)}) + \alpha^{(1)}(\xi''^{(1)})_i\\
&-\alpha_{,i}'^{(1)}) + \xi'^{(1)k}(\xi_{i,k}^{(1)} + 2\xi_{k,i}^{(1)}) + \xi^{(1)k}(\xi_{i,k}'^{(1)} - \alpha_{,ik}^{(1)}) - \alpha_{,k}^{(1)}\xi'^{(1)k}{}_{,i}.
\end{aligned}
\tag{2.3.44}
$$

To get the transformations of the vector and of the scalar part of $B_i^{(2)}$, we take the divergence of equations (2.3.44) and after that we apply the inverse of the Laplacian. We find for the scalar part

$$
\tilde{B}^{(2)} = B^{(2)} - \alpha^{(2)} + \beta'^{(2)} + \nabla^{-2}\chi_B{}^k{}_{,k},
\tag{2.3.45}
$$

and for the vector part we get

$$
\tilde{S}_i^{(2)} = S_i^{(2)} - \gamma'^{(2)i} - \chi_{Bi} + \nabla^{-2}\chi_B{}^k{}_{,ki}.
\tag{2.3.46}
$$

We now turn to the transformation properties of the spacial part of the metric tensor. The expressions are more complicated, because now we have an inverse gradient that operates on the product of first-order quantities. The perturbed spatial part transforms to second-order as

$$
2\tilde{C}_{ij}^{(2)} = 2C_{ij}^{(2)} + 2\mathcal{H}\alpha^{(2)}\delta_{ij} + \xi_{i,j}^{(2)} + \xi_{j,i}^{(2)} + \chi_{ij},
\tag{2.3.47}
$$

where χ_{ij} contains terms quadratic in the first-order perturbations, and it is given by the expression

$$
\begin{aligned}
\chi_{ij} =& 2\left[\left(\mathcal{H}^2 + \frac{a''}{a}\right)\alpha^{(1)2} + \mathcal{H}(\alpha^{(1)}\alpha'^{(1)} + \alpha_{,k}^{(1)}\xi^{(1)k})\right]\delta_{ij}\\
&4\left[\alpha^{(1)}(C_{ij}'^{(1)} + 2\mathcal{H}C_{ij}^{(1)}) + C_{ij,k}^{(1)}\xi^{(1)k} + C_{ik}^{(1)}\xi^{(1)k}{}_{,j} + C_{jk}^{(1)}\xi^{(1)k}{}_{,i}\right]\\
&+2(B_i^{(1)}\alpha_{,j}^{(1)} + B_j^{(1)}\alpha_{,i}^{(1)}) + 4\mathcal{H}\alpha^{(1)}(\xi_{i,j}^{(1)} + \xi_{j,i}^{(1)}) - 2\alpha_{,i}^{(1)}\alpha_{,j}^{(1)}\\
&+2\xi_{k,i}^{(1)}\xi^{(1)k}{}_{,j} + \alpha^{(1)}(\xi_{i,j}'^{(1)} + \xi_{j,i}'^{(1)}) + (\xi_{i,jk}^{(1)} + \xi_{j,ik}^{(1)})\xi^{(1)k}\\
&+\xi_{i,k}^{(1)}\xi^{(1)k}{}_{,j} + \xi_{j,k}^{(1)}\xi^{(1)k}{}_{,i} + \xi_i'^{(1)}\alpha_{,j}^{(1)} + \xi_j'^{(1)}\alpha_{,i}^{(1)}.
\end{aligned}
\tag{2.3.48}
$$

The spatial part of the metric perturbation is decomposed as we wrote in equation (2.1.8), which, at second-order, gives

$$
2C_{ij}^{(2)} = -2\Psi^{(2)}\delta_{ij} + 2E_{,ij}^{(2)} + 2F_{(i,j)}^{(2)} + h_{ij}^{(2)}.
\tag{2.3.49}
$$

We now follow what we did in the first-order case. The trace of equation (2.3.47) is

$$
-3\tilde{\Psi}^{(2)} + \nabla^2\tilde{E}^{(2)} = -3\Psi^{(2)} + \nabla^2 E^{(2)} + 3\mathcal{H}\alpha^{(2)} + \nabla^2\beta^{(2)} + \frac{1}{2}\chi^k{}_k.
\tag{2.3.50}
$$

Now applying $\partial^i\partial^j$ to equation (2.3.47) we obtain

$$-\nabla^2\tilde{\Psi}^{(2)}+\nabla^2\nabla^2\tilde{E}^{(2)} = -\nabla^2\Psi^{(2)}+\nabla^2\nabla^2 E^{(2)}+\mathcal{H}\nabla^2\alpha^{(2)}+\nabla^2\nabla^2\beta^{(2)}+\frac{1}{2}\chi^{(2)ij}{}_{,ij}. \tag{2.3.51}$$

From equations (2.3.50) and (2.3.51) we get the transformations of $\Psi^{(2)}$ and $E^{(2)}$, which are

$$\tilde{\Psi}^{(2)} = \Psi^{(2)} - \mathcal{H}\alpha^{(2)} - \frac{1}{4}\chi^k{}_k + \frac{1}{4}\nabla^{-2}\chi^{ij}{}_{,ij}, \tag{2.3.52}$$

$$\tilde{E}^{(2)} = E^{(2)} + \beta^{(2)} + \frac{3}{4}\nabla^{-2}\nabla^{-2}\chi^{ij}{}_{,ij} - \frac{1}{4}\nabla^{-2}\chi^k{}_k. \tag{2.3.53}$$

Taking the divergence of equation (2.3.47) we obtain

$$2\tilde{C}^{(2)}{}_{ij,}{}^{j} = 2C^{(2)}{}_{ij,}{}^{j} + 2\mathcal{H}\alpha^{(2)}_{,j} + \nabla^2\xi^{(2)}_i + \nabla^2\beta^{(2)}_{,i} + \chi_{ik,}{}^{k}, \tag{2.3.54}$$

and substituting the results for $\tilde{E}^{(2)}$ and $\tilde{\Psi}^{(2)}$ we get

$$\tilde{F}^{(2)}_i = F^{(2)}_i + \gamma^{(2)}_i + \nabla^{-2}\chi_{ik,}{}^{k} - \nabla^{-2}\nabla^{-2}\chi^{kl}{}_{,kli}. \tag{2.3.55}$$

We are now able to give the transformation rules of the second-order tensor perturbations, by substituting the equations for $\Psi^{(2)}$, $E^{(2)}$ and $F^{(2)}$. We obtain

$$\begin{aligned}\tilde{h}^{(2)}_{ij} =& h^{(2)}_{ij} + \chi_{ij} + \frac{1}{2}(\nabla^{-2}\chi^{ik}{}_{,kl} - \chi^k{}_k)\delta_{ij} + \frac{1}{2}\nabla^{-2}\nabla^{-2}\chi^{kl}{}_{,klij} \\ & \frac{1}{2}\nabla^{-2}\chi^k{}_{k,ij} - \nabla^{-2}(\chi_{ik,}{}^{k}{}_{j} + \chi_{lk,}{}^{k}{}_{i}).\end{aligned} \tag{2.3.56}$$

2.4 Energy-momentum tensor for fluids

General Relativity strictly connects the evolution of the geometry of the space time with its matter and energy content, through the Einstein field equation. It is quite obvious that perturbations in the components of the metric tensor generate perturbations in the stress energy tensor of the system, and vice-versa. This is the reason why, in order to consider the evolution of the perturbations, we must perturb the stress energy tensor.

We define the four-velocity of the matter fluid as

$$u^\mu = \frac{dx^\mu}{d\tau}, \tag{2.4.1}$$

where τ is the proper time comoving with the fluid, subject to the condition

$$u^\mu u_\mu = -1. \tag{2.4.2}$$

From the isotropy requirement of the background universe, we deduce that the spatial part of the four velocity has to be null, since the existence of a peculiar velocity will break this assumption. So the background four-velocity is

$$u^\mu = (1, \mathbf{0}). \tag{2.4.3}$$

The spatial part of the four-velocity is, up to second-order

$$u^i = \frac{dx^i}{d\tau} = \frac{1}{a}\frac{dr^i}{d\tau} = \frac{1}{a}\left(v^{(1)i} + \frac{1}{2}v^{(2)i}\right) \tag{2.4.4}$$

From the relation $u^\mu u^\nu g_{\mu\nu} = -1$, we calculate the time component u_0, which is

$$u^0 = \frac{1}{a}\left(1 - \Phi^{(1)} - \frac{1}{2}\Phi^{(2)} + \frac{3}{2}\Phi^{(1)2} + \frac{1}{2}v_k^{(1)}v^{(1)k} + v_k^{(1)}B^{(1)k}\right). \tag{2.4.5}$$

With the inverse metric $g^{\mu\nu}$ we calculate the covariant four-velocity, which is

$$u_0 = -a\left(1 + \Phi^{(1)} + \frac{1}{2}\Phi^{(2)} - \frac{1}{2}\Phi^{(1)2} + \frac{1}{2}v_k^{(1)}v^{(1)k}\right), \tag{2.4.6}$$

$$u_i = a\left(v_i^{(1)} + B_i^{(1)} + \frac{1}{2}v_i^{(2)} + \frac{1}{2}B_i^{(2)} - \Phi^{(1)}B_i^{(1)} + 2C_{ik}^{(1)}v^{(1)k}\right). \tag{2.4.7}$$

As usual, we split v_i in scalar and vector parts

$$v_i = v_{,i} + v_{(vec)i}, \tag{2.4.8}$$

where $v_{(vec)i}$ is divergence-less.

We consider a single fluid system. The background energy-momentum tensor is defined as [25]

$$T_{\mu\nu}^{(0)} = (\rho^{(0)} + p^{(0)})u_\mu^{(0)}u_\nu^{(0)} + p^{(0)}g_{\mu\nu}^{(0)}; \tag{2.4.9}$$

this is the energy-momentum tensor for a perfect fluid, since as said before, in the background there are no peculiar velocities. As usual the background quantities are function only of η.

Now we turn to the perturbations; we consider two types of perturbations: the first one do not alter the shape of the energy-momentum tensor, so we write

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}, \tag{2.4.10}$$

where all the quantities of this formula have background and perturbation terms

$$\rho = \rho^{(0)} + \delta\rho^{(1)} + \frac{1}{2}\delta\rho^{(2)}, \qquad p = p^{(0)} + \delta p^{(1)} + \frac{1}{2}\delta p^{(2)}, \tag{2.4.11}$$

and u^μ has the expression of formula (2.4.3), while $g_{\mu\nu}$ is given in section 2.1. This type of perturbations do not alter the perfect fluid shape of the energy-momentum tensor.

Secondly, we add to the energy-momentum tensor of equation (2.4.9) the anisotropic stress $\pi^{\mu\nu}$. The tensor, if we put together these contributions, is

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} + \pi_{\mu\nu}, \tag{2.4.12}$$

where $\pi_{\mu\nu}$ has first-order and second-order parts

$$\pi_{\mu\nu} = \pi_{\mu\nu}^{(1)} + \frac{1}{2}\pi_{\mu\nu}^{(1)}. \tag{2.4.13}$$

$pi_{\mu\nu}$ is subject to the condition

$$\pi_{\mu\nu}u^{\nu} = 0, \qquad \pi^{\mu}{}_{\mu} = 0. \tag{2.4.14}$$

The anisotropic stress vanishes for a perfect fluid, and also for a minimally coupled scalar field. Equation (2.4.14) permits to constrain the energy-momentum tensor. At first-order we get

$$\begin{aligned}
&\pi^{(0)}_{\mu\nu}u^{\nu(1)} + \pi^{(1)}_{\mu\nu}u^{\nu(0)} = 0\\
&\pi^{(1)}_{\mu\nu}u^{\nu(0)} \to \pi^{(1)}_{\mu 0} = 0,\\
&\pi^{\mu}{}_{\mu} = 0 \to \pi^{(1)0}{}_{0} = -\pi^{(1)i}{}_{i} = 0.
\end{aligned} \tag{2.4.15}$$

At second-order (2.4.14) gives

$$\begin{aligned}
&\pi^{(2)}_{\mu\nu}u^{\nu(0)} + \pi^{(1)}_{\mu\nu}u^{\nu(1)} + \pi^{(0)}_{\mu\nu}u^{\nu(2)} = \pi^{(2)}_{\mu 0}u^{0}_{(0)} + \pi^{(1)}_{\mu i}u^{i}_{(1)}\\
&\pi^{(1)}_{0i}v^{i}_{(1)} + \frac{1}{2}\pi^{(2)}_{00} \to \pi^{(2)}_{00} = 0\\
&\pi^{\mu}{}_{\mu} = 0 \to \pi^{(2)0}{}_{0} = -\pi^{(2)i}{}_{i} = 0.
\end{aligned} \tag{2.4.16}$$

We split the anisotropic stress in scalar, vector and tensor contributions,

$$\pi_{ij} = a^2\left(\Pi_{,ij} - \frac{1}{3}\nabla^2\Pi\delta_{ij}\Pi_{(i,j)} + \Pi_{ij}\right). \tag{2.4.17}$$

We follow [17] in defining the proper energy density as the eigenvalue of the energy-momentum tensor, and the four-velocity as the corresponding eigenvector,

$$T^{\mu}{}_{\nu}u^{\nu} = -\rho u_{\mu}. \tag{2.4.18}$$

We obtain for the zero order component

$$\begin{aligned}
T^{(0)0}{}_{0} &= -\rho^{(0)}\\
T^{(0)0}{}_{i} &= 0\\
T^{(0)i}{}_{j} &= -p^{(0)}\delta^{i}{}_{j},
\end{aligned} \tag{2.4.19}$$

and for the first-order

$$\begin{aligned}
\delta T^{(1)0}{}_{0} &= -\delta\rho^{(1)}\\
\delta T^{(1)0}{}_{i} &= (\rho_0 + P_0)(v^{(1)}_{i} + B^{(1)}_{i})\\
\delta T^{(1)i}{}_{j} &= \delta^{i}{}_{j}p^{(1)} + \frac{1}{a^2}\pi^{(1)i}{}_{j}
\end{aligned} \tag{2.4.20}$$

and at second-order

$$\begin{aligned}
\delta T^{(2)0}{}_{0} &= -\delta\rho^{(2)} - 2(\rho^{(0)} + p^{(0)})v^{(1)}_{k}(v^{(1)k} + B^{k(1)}),\\
\delta T^{(2)0}{}_{i} &= (\rho^{(0)} + p^{(0)})\left(v^{(2)}_{i} + B^{(2)}_{i} + 4C^{(1)}_{ik}v^{(1)k} - 2\Phi^{(1)}(v^{(1)}_{i} + 2B^{(1)}_{i})\right)\\
&\quad + 2(\delta\rho^{(1)}) + \delta p^{(1)})(v^{(1)}_{i} + B^{(1)}_{i}) + \frac{2}{a^2}(B^{(1)k} + v^{(1)k}\pi^{(1)}{}_{ik}),
\end{aligned} \tag{2.4.21}$$

$$\delta T^{(2)i}{}_j = \delta p^{(2)}\delta^i{}_j + \frac{1}{a^2}\pi^{(2)i}{}_j - \frac{4}{a^2}C^{(1)ik}\pi^{(1)}{}_{jk} + 2(\rho^{(0)} + p^{(0)})v^{(1)i}(v_j^{(1)} + B_j^{(1)}) \tag{2.4.22}$$

We haven't split the quantities in scalar, vector an tensor parts for simplicity. Quantities as the energy density, and the three-velocity v_i are gauge-dependent, and they change along with the choice of the gauge. On the other hand, the anisotropic stress is gauge-independent at first-order, as shown in [20].

2.5 Possible gauge choices and gauge-invariant quantities

As seen in section 2.3, we have to deal with the problem of the gauge. This can also be seen as an asset, because it allows us to simplify the calculations, because we can cancel some perturbations.

How to make a gauge choice? There are no preferred gauges, but different gauges emphasize different features of the dynamical equations. Here we present some common gauges that are often used in the literature. The gauge choice are made by the set up the quantities $\xi_\mu^{(n)}$. So at first-order, we can fix two scalar and one vector perturbations. At second-order we have another four-vector, so we can fix two other scalars, and another vector perturbation.

The gauges that we treat in this section are defined and analyzed at first-order. We consider only the Poisson gauge up to second-order, since it is the one that we will use in the next chapters of the thesis.

2.5.1 Poisson gauge

First order

The Poisson gauge is defined by the choice

- $B_p = 0$,
- $E_p = 0$,
- $F_p^i = 0$.

This gauge generalizes the so-called longitudinal [18] or Newtonian gauge [19], in which vector and tensor perturbations are not considered. By looking at the transformation properties of B, E and F_i in formulas (2.3.30), (2.3.36) and (2.3.39), we find that in this gauge

$$\alpha_p^{(1)} = B^{(1)} - E'^{(1)} \tag{2.5.1}$$

$$\beta_p^{(1)} = -E^{(1)} \tag{2.5.2}$$

$$\gamma_p^{(1)i} = -F^{(1)i}, \tag{2.5.3}$$

where the subscript p stands for Poisson. With these formulas we can calculate the expression in this gauge for the scalar perturbations $\Phi_p^{(1)}$ and $\Psi_p^{(1)}$, from equations (2.3.27) and (2.3.37) we obtain

$$\Phi_p^{(1)} = \Phi^{(1)} + \mathcal{H}(B_p^{(1)} - E_p'^{(1)}) + (B_p^{(1)} - E_p'^{(1)})', \tag{2.5.4}$$

$$\Psi_p^{(1)} = \Psi^{(1)} - \mathcal{H}(B_p^{(1)} - E_p'^{(1)}). \tag{2.5.5}$$

In this gauge the energy density and scalar velocity are

$$\delta\rho_p^{(1)} = \delta_\rho^{(1)} + \rho_0'(B^{(1)} - E^{(1)})) \tag{2.5.6}$$

$$v_p^{(1)} = v^{(1)} + E'^{(1)} \tag{2.5.7}$$

The name Newtonian for this gauge derives from the fact that in many physical situations we have $\Phi = \Psi$, and this variable satisfies a generalization of the Poisson equation.

Second order

In order to extend the Poisson gauge up to second-order, we work in the same way as in the first-order case. The Poisson gauge at second-order is defined by the conditions:

- $B_p^{(2)} = 0$,
- $E_p^{(2)} = 0$,
- $F_p^{(2)i} = 0$.

Requiring $E_p^{(2)} = 0$, from equation (2.3.53), we can determine $\beta_p^{(2)}$

$$\beta_p^{(2)} = -E_p^{(2)} - \frac{3}{4}\nabla^{-2}\nabla^{-2}\chi^{ij}{}_{,ij} + \frac{1}{4}\nabla^{-2}\chi^k{}_k. \tag{2.5.8}$$

$\alpha^{(1)}$, $\beta^{(1)}$ and $\gamma_i^{(1)}$ are fixed because we imposed the Poisson gauge at first-order.

From the condition $B_p^{(2)} = 0$, with the transformation equation (2.3.45), we obtain the expression for $\alpha_p^{(2)}$

$$\alpha_p^{(2)} = B_2 + \beta_p'^{(2)} + \nabla^{-2}\chi_B{}^k{}_{,k}, \tag{2.5.9}$$

with $\beta_p^{(2)}$ given by the equation (2.5.8).

Requiring $F_p^{(2)i} = 0$, from equation (2.3.55), we can set $F_p^{(2)i} = 0$

$$\gamma_{ip}^{(2)} = -F_i^{(2)} - \nabla^{-2}\chi_{ik,}{}^k + \nabla^{-2}\nabla{-2}\chi^{kl}{}_{,kli}. \tag{2.5.10}$$

We now have fixed $\alpha^{(2)}$, $\beta(2)$ and $\gamma_i^{(2)}$, then from equations (2.3.42) and (2.3.52) we can read the expressions in this gauge for $\Phi_p^{(2)}$ and $\Psi_p^{(2)}$. We obtain

$$\begin{aligned}\Phi_p^{(2)} =& \Phi^{(2)} + \mathcal{H}\alpha_p^{(2)} + \alpha_p'^{(2)} + \alpha_p^{(1)}\left(\alpha''(1)_p + 5\mathcal{H}\alpha_p'^{(1)} + (\mathcal{H}' + 2\mathcal{H}^2)\alpha_p'^{(1)}\right.\\ &\left.+4\mathcal{H}\Phi_p^{(1)} + 2\Phi_p'^{(1)}\right) + 2\alpha_p'^{(1)}\left(\alpha_p'^{(1)} + 2\Phi_p^{(1)}\right) + \xi_{pk}^{(1)}\left(\alpha_p'^{(1)} + \mathcal{H}\alpha_p'^{(1)}\right.\\ &\left.+2\Phi_p^{(1)}\right)_,^k + \xi_{kp}'^{(1)}\left(\alpha^{(1)}{}_{,\ p}^{\ k} - 2B_{kp}^{(1)} - \xi_p'^{(1)k}\right),\end{aligned} \tag{2.5.11}$$

and for $\Psi_p^{(2)}$

$$\tilde{\Psi}_p^{(2)} = \Psi_p^{(2)} - \mathcal{H}\alpha_p^{(2)} - \frac{1}{4}\chi^k{}_{kp} + \frac{1}{4}\nabla^{-2}\chi^{ij}{}_{,ijp}. \tag{2.5.12}$$

From equation (2.3.56), we can read the second-order tensor perturbations in the Poisson gauge, which are

$$\begin{aligned} h^{(2)}_{ijp} =& h^{(2)}_{ij} + \chi_{ij_p} + \frac{1}{2}(\nabla^{-2}{\chi_p}^{ik}{}_{,kl} - {\chi_p}^{k}{}_{k})\delta_{ij} + \frac{1}{2}\nabla^{-2}\nabla^{-2}{\chi_p}^{kl}{}_{,klij} \\ & \frac{1}{2}\nabla^{-2}{\chi_p}^{k}{}_{k,ij} - \nabla^{-2}(\chi_{pik,}{}^{k}{}_{j} + \chi_{plk,}{}^{k}{}_{i}). \end{aligned} \tag{2.5.13}$$

2.5.2 Synchronous gauge

The Synchronous gauge is defined by the conditions [19]

- $\Phi^{(1)}_{syn} = 0$,
- $B^{(1)}_{syn} = 0$,
- $S^{(1)i}_{syn} = 0$.

In this gauge the proper time of an observer at fixed spatial coordinates, coincides with cosmic time in the FRW background. From the transformation properties of B, Φ and S_i in formulas (2.3.27),(2.3.30) and (2.3.31) we get

$$\alpha_{syn^{(1)}} = -\frac{1}{a}\left(\int a\Phi^{(1)}d\eta - \mathcal{C}(x^i)\right), \tag{2.5.14}$$

$$\beta^{(1)}_{syn} = \int \left(\alpha^{(1)}_{syn} - B^{(1)}\right) d\eta - \hat{\mathcal{C}}(x^i), \tag{2.5.15}$$

$$\gamma^{(1)i}_{syn} = \int S^{(1)i}d\eta + \hat{\mathcal{C}}(x^i). \tag{2.5.16}$$

The matter variables are

$$\delta\rho^{(1)}_{syn} = \delta\rho^{(1)} - \frac{\rho'^{(0)}}{a}\left(\int a\Phi^{(1)}d\eta - \mathcal{C}(x^i)\right), \tag{2.5.17}$$

$$v^{(1)}_{syn} = v^{(1)} + B^{(1)} - \alpha^{(1)}_{syn}. \tag{2.5.18}$$

Equations (2.5.15) and (2.5.16) do not determine the time slicing unambiguously and we are left with two arbitrary scalar functions of the spatial coordinates, $\mathcal{C}(x^i)$ and $\hat{\mathcal{C}}(x^i)$. We are thus left with a residual gauge-freedom [20].

2.5.3 Comoving orthogonal gauge

The comoving gauge is defined by choosing spatial coordinates such that the three-velocity of the matter fluid vanishes [20]. Orthogonality of the constant-η hypersurfaces to the four-velocity, u^μ, then require $\tilde{v}^i + \tilde{B}^i = 0$. Summarizing, we have

- $v^{(1)i}_{com} = 0$,
- $B^{(1)i}_{com} = 0$.

From the transformation properties of $B^{(1)}$ and $v^{(1)}$ in formulas (2.3.23) and (2.3.30) we have

$$\alpha^{(1)}_{com} = v^{(1)} + B^{(1)} \tag{2.5.19}$$

$$\beta^{(1)}_{com} = \int v^{(1)} d\eta + \mathcal{C}(x^i), \tag{2.5.20}$$

Density perturbations in this gauge follows from equation (2.3.20)

$$\delta\rho^{(1)}_{com} = \delta\rho + \rho'^{(0)}(v^{(1)} + B^{(1)}). \tag{2.5.21}$$

2.5.4 Spatially flat gauge

In this gauge we select spatial hypersurfaces in which the induced three-metric is flat [17, 20]. This corresponds, considering only vectors and scalars, to the conditions

- $\Psi^{(1)}_{flat} = 0,$
- $E^{(1)}_{flat} = 0,$
- $F^{(1)i}_{flat} = 0.$

By looking at the transformation properties of E, Ψ and F_i in formulas (2.3.36), (2.3.37) and (2.3.39) we obtain

$$\alpha^{(1)}_{flat} = \frac{\Psi^{(1)}}{\mathcal{H}}, \tag{2.5.22}$$

$$\beta^{(1)}_{flat} = -E^{(1)}, \tag{2.5.23}$$

$$\gamma^{(1)i}_{flat} = F^{(1)i}. \tag{2.5.24}$$

The energy-density perturbation in this gauge is

$$\delta\rho^{(1)}_{flat} = \delta\rho^{(1)} + \rho^{(0)}\frac{\Psi^{(1)}}{\mathcal{H}}, \tag{2.5.25}$$

and the scalar velocity is

$$v^{(1)}_{flat} = v^{(1)} + E'^{(1)}. \tag{2.5.26}$$

2.5.5 Uniform density gauge

This gauge, as the name suggests, is defined by setting to zero the density perturbation [20]

- $\delta^{(1)}\rho_{un} = 0.$

This, fixes the value of $\alpha^{(1)}$ from equation (2.3.20) as

$$\alpha^{(1)}_{un} = -\frac{\delta\rho^{(1)}}{\rho'^{(0)}}. \tag{2.5.27}$$

This condition does not completely exhaust gauge fixing.

2.5.6 Gauge-invariant variables

From this not exhaustive list of possible gauges, one would think that the solution of our perturbed equations of motion depends on the gauge in which we write down the equations. This is not true at all, because the physical quantities, which are the only ones measurable through experiments, have to be gauge-invariant. We expect to have two scalars and one vector gauge-invariant quantities, by looking at the d.o.f of ξ.

Bardeen, by studying the transformations of metric perturbations in [29], constructed two gauge-independent variables, looking only at geometrical perturbations

$$\Phi_B^{(1)} = \Phi^{(1)} + \mathcal{H}(B^{(1)} - E'^{(1)}) + (B^{(1)} - E'^{(1)})', \tag{2.5.28}$$

$$\Psi_B^{(1)} = \Psi^{(1)} - \mathcal{H}(B^{(1)} - E'^{(1)}), \tag{2.5.29}$$

which are called Bardeen gauge-invariant gravitational potentials. It is important to notice that in the longitudinal gauge, the Bardeen potentials coincide with the scalar perturbations $\Phi^{(1)}$ and $\Psi^{(1)}$.

The only gauge-invariant vector variable, that we can build from metric perturbations, is

$$\Sigma_i^{(1)} = S_i^{(1)} + F_i'^{(1)}. \tag{2.5.30}$$

This quantity represents the amplitude of the shear due to vector perturbations.

With matter quantities we can construct other gauge-invariant variables. The scalars are

$$E_m^{(1)} = \delta\rho^{(1)} + \rho^{(0)}(v^{(1)} + B^{(1)}), \tag{2.5.31}$$

$$E_g^{(1)} = 2\delta\rho^{(1)} + \rho'^{(0)}(2v^{(1)} - E'^{(1)}). \tag{2.5.32}$$

These could be seen as energy density from different points of view. The first one is the energy density from the matter point of view, because $E_m^{(1)} = \delta\rho^{(1)}$ when $v^{(1)} = -B^{(1)}$ which is the condition for which matter world-lines are orthogonal to the constant τ hypersurface. One can also see that $E_g^{(1)}$ measures the energy density relative to hypersurfaces where a normal vector has zero shear.

With matter vector perturbations we can construct the gauge-invariant quantity

$$V_i^{(1)} = v_{(vec)i}^{(1)} + F_i'^{(1)}. \tag{2.5.33}$$

2.6 Dynamics

In general relativity the geometry of space time is related with the matter content through the Einstein equations

$$G^\mu{}_\nu = 8\pi G T_{\mu\nu}, \tag{2.6.1}$$

where the Einstein tensor is defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{2.6.2}$$

The covariant derivative D_μ of the Einstein tensor is zero. This thing guarantees the continuity equation for the energy-momentum tensor, $D_\mu T_{\mu\nu} = 0$. The background equations that we calculate with the FRW metric are the Friedmann equations,

$$\mathcal{H}^2 = \frac{8\pi G}{3a^2}\rho, \tag{2.6.3}$$

$$\mathcal{H}' = -\frac{4\pi G}{3}a^2\left(\rho + 3p\right). \tag{2.6.4}$$

The continuity equation for the energy density is

$$\rho' = -3\mathcal{H}(\rho + p). \tag{2.6.5}$$

The next step is to perturb the Einstein equations, stopping at first-order in perturbations theory. This is quite a tedious operation. From the perturbed metric we calculate $\delta\Gamma^{\mu}{}_{\nu\rho}$, and with this we can then calculate the perturbation of Riemann tensor $\delta R_{\mu\nu\rho\sigma}$, Ricci tensor $R_{\mu\nu}$ and curvature scalar R [20].

2.6.1 Scalar perturbations

The $0-0$ component of Einstein equations, without specifying the gauge, is

$$3\mathcal{H}(\Psi'^{(1)} + \mathcal{H}\Phi^{(1)}) - \nabla^2(\Psi^{(1)} + \mathcal{H}\sigma^{(1)}) = -4\pi g a^2 \delta\rho^{(1)}, \tag{2.6.6}$$

where $\sigma^{(1)}$ is given by the formula (2.2.19). The $0-i$ components give

$$\Psi'^{(1)} + \mathcal{H}\Phi^{(1)} = -4\pi G a^2(\rho^{(0)} + p^{(0)})V^{(1)}, \tag{2.6.7}$$

where the total velocity $V^{(1)}$ is

$$V^{(1)} = v^{(1)} + B^{(1)}. \tag{2.6.8}$$

The perturbed spatial Einstein equations at first-order also gives two evolution equations for scalar metric perturbations

$$\Psi''^{(1)} + 2\mathcal{H}\Psi'^{(1)} + \mathcal{H}\Phi'^{(1)} + (2\mathcal{H}' + \mathcal{H}^2)\Phi^{(1)} = 4\pi G a^2(\delta p^{(1)} + \frac{2}{3}\nabla^2\Pi^{(1)}), \tag{2.6.9}$$

$$\sigma'^{(1)} + 2\mathcal{H}\sigma^{(1)} + \Psi^{(1)} - \Phi^{(1)} = 8\pi G a^2 \Pi^{(1)}. \tag{2.6.10}$$

These equations are general, since the gauge is not specified. If we use the longitudinal gauge, the last equation becomes

$$\Phi_B - \Psi_B = 8\pi G a^2 \Pi^{(1)}, \tag{2.6.11}$$

because the scalar shear potential (2.2.19) vanishes in the longitudinal gauge, and also the Bardeen scalar potential coincides with the scalar perturbations $\Psi^{(1)}$ and $\Phi^{(1)}$.

Equation (2.6.11) shows that $\Phi_B - \Psi_B$ is driven by the anisotropic stress. The anisotropic stress vanishes for a perfect fluid or for a minimally coupled scalar field, but it is non-zero to first-order in the presence of free-streaming neutrinos or of a non-minimally coupled scalar field [20, 26]. In longitudinal gauge, and in the absence of anisotropic stress, equation (2.6.9) provides a second-order evolution equations where the source term is proportional to the isotropic pressure

$$\Psi_B''^{(1)} + 3\mathcal{H}\Psi_B'^{(1)} + (2\mathcal{H}' + \mathcal{H}^2)\Psi_B^{(1)} = 4\pi G a^2 \delta p^{(1)} \tag{2.6.12}$$

We can relate the pressure perturbation with the density perturbation in the case of adiabatic perturbations. In this case we have

$$\delta p^{(1)} = c_s^2 \delta\rho^{(1)}, \tag{2.6.13}$$

where c_s^2 is the speed of sound, defined as

$$c_s^2 = \frac{p'}{\rho'}. \tag{2.6.14}$$

In this case, equations (2.6.9) and (2.6.6) yield an evolution equation for Ψ [18]

$$\Psi''^{(1)} + 3(1 + c_s^2)\mathcal{H}\Psi'^{(1)} + \left[2\mathcal{H}' + (1 + 3c_s^2)\mathcal{H}^2 - c_s^2\nabla^2\right]\Psi^{(1)} = 0. \tag{2.6.15}$$

The continuity equation gives the evolution equation for the perturbed energy-density and velocity

$$\delta\rho'^{(1)} + 3\mathcal{H}(\delta\rho^{(1)} + \delta p^{(1)}) - 3(\rho + p)\Psi'^{(1)} + (\rho + p)\nabla^2(V^{(1)} + \sigma^{(1)}), \tag{2.6.16}$$

$$V'^{(1)} + (1 - 3c_s^2)\mathcal{H}V^{(1)} + \Phi^{(1)} + \frac{1}{\rho + p}(\delta p + \frac{2}{3}\nabla^2\Pi) = 0. \tag{2.6.17}$$

2.6.2 Vector perturbations

The divergence-free part of δT_{0i} is

$$\delta q_i^{(1)} = (\rho + p)(v_{(vec)i}^{(1)} - S_i^{(1)}), \tag{2.6.18}$$

and obeys the momentum conservation equation

$$\delta q_i^{(1)} + 4\mathcal{H}\delta q_i^{(1)} = -\nabla^2\Pi_i^{(1)}. \tag{2.6.19}$$

In the absence of anisotropic stress, this equation tells us that the vector perturbations are red-shifted away by the Hubble expansion. The Einstein equations for the gauge-invariant vector perturbations is

$$\nabla^2(F_i'^{(1)} + S_i^{(1)}) = -16\pi G a^2 \delta q_i^{(1)}. \tag{2.6.20}$$

This equation tells us that the vector metric perturbation could be supported only by divergence-free momentum perturbations, so we conclude that in the absence of anisotropic stress, vector perturbations are negligible. This could be seen as a consequence of Kelvin circulation theorem, which says that in the absence of dissipative effects, vorticity is conserved along the fluid trajectories.

2.6.3 Tensor perturbations

The trace-free part of the $i-j$ Einstein equations gives the wave equation

$$h''_{ij^{(1)}}+2\mathcal{H}h^{(1)}_{ij}-\nabla^2 h^{(1)}_{ij}=8\pi Ga^2\Pi^{(1)}_{ij}. \qquad (2.6.21)$$

Again, in the absence of anisotropic stress, we have

$$h''^{(1)}_{ij}+2\mathcal{H}h^{(1)}_{ij}-\nabla^2 h^{(1)}_{ij}=0 \qquad (2.6.22)$$

This equation has oscillating solutions, which correspond to the propagation of gravitational waves. It is important to note that at first-order gravitational waves are not coupled with density perturbations. The coupling between scalar perturbations (which at first -rder depend on density perturbations) and gravitational waves is a second-order effect, that we will consider in the next chapters.

We decompose tensor modes in a scalar amplitude and in a polarization tensor

$$h_{ij}=h(\eta)e_{ij}(\mathbf{x}), \qquad (2.6.23)$$

and going to Fourier space, we find that $h(\eta)$ satisfies the equation

$$h(\eta)''+2\mathcal{H}h'(\eta)+k^2h(\eta)=0. \qquad (2.6.24)$$

This is the same equation of motion of a massless scalar field ϕ in an unperturbed FRW metric, which is

$$\Box\phi=\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right). \qquad (2.6.25)$$

From these considerations we conclude that the two polarization modes behave as two massless scalar fields.

The solution of equation (2.6.22) for a generic fluid is [21, 31]

$$h^{(1)}_{ij}(\eta,\mathbf{x})=\frac{1}{(2\pi)^3}\int d^3\mathbf{k}e^{i\mathbf{kx}}h^{(1)}_\sigma(\eta,\mathbf{k})e^\sigma_{ij}(\hat{\mathbf{k}}), \qquad (2.6.26)$$

where e^σ_{ij} is the polarization tensor, and σ is the polarization index. $h^{(1)}_\sigma$ is the scalar amplitude of the gravitational waves, and it has the following expression

$$h^{(1)}_\sigma(\eta,\mathbf{k})=A(k)a(\eta)\frac{j_1(k\eta)}{k\eta}T(k), \qquad (2.6.27)$$

where j_1 is the first-order spherical Bessel Function. $T(k)$ is the transfer function for gravitational waves; which a numerical fit is given in [31] . The quantity $A(k)$ depends on the mechanism that generated gravitational waves; for the standard inflation scenario it is nearly scale invariant.

We will use this equation as a zero-order solution in chapter (4), in order to calculate the correction due to matter inhomogeneities to the evolution equation of primordial gravitational waves.

In this section we have treated only the dynamics of first-order perturbation theory. In the literature there are several works that study the dynamics of second-order perturbations; see for example [21] and [30].

Chapter 3

Gravitational wave propagation in curved space-time

In this section we will present some basic results about propagation of gravitational waves in curved space-time. We will consider general gravitational waves, in fact we won't refer to the cosmic gravitational wave background, still the results that we will obtain are valid also for gravitational waves from astrophysical sources.

When we consider gravitational waves in Minkowsky flat space-time we have a clear distinction of what is the background metric and what is the "ripple", since the background is independent of the space-time point, and gravitational waves are the only quantities that depend on the coordinates. Instead, when we consider gravitational waves on a curved background, the distinction between perturbations and background is more subtle, because both background metric and gravitational waves depend on the coordinates. As in any other application of general relativity, our equations of motion are covariant under a generic coordinate transformation. We will use this tool to simplify the equation of motion for the gravitational waves.

Gravitational waves in flat space-time are treated in any books about General Relativity, as examples [24, 25]. For propagation of gravitational waves in curved space time we refer to [35, 36, 38]. The geometric optics approximation for gravitational radiation is discussed in [35, 38, 39]

3.1 Gravitational waves in flat space-time

We briefly review the propagation of gravitational waves in a flat background, since a generalization of that is the most interesting case of gravitational waves on curved space-time.

As starting point, we assume that the metric could be written as

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x), \tag{3.1.1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric, and h satisfies $\mid h_{\mu\nu} \mid \ll 1$. With this metric, we can expand the Einstein equations, to obtain the linear theory in $h_{\mu\nu}$ [25]. Since the numerical values of the components of the metric tensor depend on the reference frame in which we calculate them, we assume that exist a coordinate system in which equation (3.1.1) holds. Even after setting the coordinate system, we still remain with a residual coordinate transformation freedom . Consider now the transformation

$$x^{\mu} \to x^{'\mu} = x^{\mu} + \xi^{\mu}. \tag{3.1.2}$$

The metric transforms as a rank-two tensor so we obtain, for the perturbation $h_{\mu\nu}$, the transformation

$$h'_{\mu\nu}(x') = h_{\mu\nu}(x) - (\partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu}) \,. \tag{3.1.3}$$

If we impose that the components of ξ satisfy $\mid \xi \mid \ll 1$, also $h'_{\mu\nu}$ will satisfy $\mid h'_{\mu\nu} \mid \ll 1$.

The equations of motion are the Einstein equations, linearized in $h_{\mu\nu}$. In order to compute the equation, we have to calculate Riemann tensor at linear order in h, which is

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\partial_{\nu}\partial_{\rho}h_{\mu\sigma} + \partial_{\mu}\partial_{\sigma}h_{\nu\rho} - \partial_{\mu}\partial_{\rho}h_{\nu\sigma} - \partial_{\nu}\partial_{\sigma}h_{\mu\rho}\right). \tag{3.1.4}$$

The Riemann tensor is invariant under residual coordinate transformations, as we can easily see inserting equation (3.1.3) in the previous equation [38].

Instead of $h_{\mu\nu}$, it is useful to consider the quantity

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \tag{3.1.5}$$

where

$$h = h_{\mu\nu}\eta^{\mu\nu}. \tag{3.1.6}$$

The Riemann tensor expressed in term of $\bar{h}_{\mu\nu}$, becomes

$$\begin{aligned} R_{\mu\nu\rho\sigma} =& \frac{1}{2}\left(\partial_{\nu}\partial_{\rho}\bar{h}_{\mu\sigma} + \partial_{\mu}\partial_{\sigma}\bar{h}_{\nu\rho} - \partial_{\mu}\partial_{\rho}\bar{h}_{\nu\sigma} - \partial_{\nu}\partial_{\sigma}\bar{h}_{\mu\rho}\right. \\ &\left. - \frac{1}{2}\eta_{\mu\sigma}\partial_{\nu}\partial_{\rho}\bar{h} - \frac{1}{2}\eta_{\nu\rho}\partial_{\mu}\partial_{\sigma}\bar{h} + \frac{1}{2}\eta_{\nu\sigma}\partial_{\mu}\partial_{\rho}\bar{h} + \frac{1}{2}\eta_{\mu\rho}\partial_{\nu}\partial_{\sigma}\bar{h}\ \right). \end{aligned} \tag{3.1.7}$$

After calculating the Ricci tensor and the curvature scalar, the Einstein equations can be written as

$$\Box \bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^{\rho}\partial^{\sigma}\bar{h}_{\rho\sigma} - \partial^{\rho}\partial_{\mu}\bar{h}_{\nu\rho} - \partial^{\rho}\partial_{\nu}\bar{h}_{\mu\rho} = -16\pi G T_{\mu\nu}. \tag{3.1.8}$$

We use now the invariance of the Riemann tensor in transformation like that of equation (3.1.3), in order to set at zero the divergence of $\bar{h}_{\mu\nu}$,

$$\partial^{\mu}\bar{h}'_{\mu\nu} = 0. \tag{3.1.9}$$

This is possible, because if we start from

$$\partial^{\nu}\bar{h}_{\mu\nu} = f_{\nu}, \tag{3.1.10}$$

we will end up, after the transformation, with

$$\partial^\mu \bar{h}'_{\mu\nu} = \partial^\mu \bar{h}_{\mu\nu} - \Box \xi_\nu. \tag{3.1.11}$$

It is sufficient to set $\Box \xi_\nu = f_\nu$ (which has always a solutions since the $\Box$ operator is invertible), and we obtain condition (3.1.9).

With this assumption, the Einstein equations become

$$\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}. \tag{3.1.12}$$

This equation, together with equation (3.1.9), implies

$$\partial_\mu T^{\mu\nu} = 0, \tag{3.1.13}$$

which is the conservation of the energy-momentum tensor in the linearized theory.

In order to study the propagation of the gravitational waves outside the source, the Einstein equations become

$$\Box \bar{h}_{\mu\nu} = 0. \tag{3.1.14}$$

In this case, we can further simplify equation (3.1.9). In fact, we can perform another transformation of the type of (3.1.3). Now we set $\Box \xi_\mu = 0$, in such a way that the condition given in equation (3.1.9) still holds. If $\Box \xi_\mu = 0$, also $\Box \xi_{\mu\nu} = 0$, with

$$\xi_{\mu\nu} = \partial_\nu \xi_\mu + \partial_\mu \xi_\nu - \eta_{\mu\nu} \partial^\rho \xi_\rho, \tag{3.1.15}$$

because the D'Alembert operator and partial derivatives commute. We can use the four functions ξ_μ to constrain equation (3.1.14). We can set the condition $\bar{h} = 0$ with ξ_0; in this case we get $\bar{h}_{\mu\nu} = h_{\mu\nu}$. With the remaining functions ξ_i, we set $h_{0i} = 0$. Condition (3.1.9) for $\mu = 0$, gives

$$\partial_0 h_{00} + \partial^i h_{0i} = 0, \tag{3.1.16}$$

which implies that

$$\partial_0 h_{00} = 0. \tag{3.1.17}$$

Since the gravitational waves are, by definition, time dependent quantities, the previous condition implies $h_{00} = 0$. Summarizing, fixing all the symmetry transformations allowed from our equations, we have the following conditions:

$$h_{0\nu} = 0, \quad h = 0, \quad \partial^j h_{ij} = 0. \tag{3.1.18}$$

These conditions define the transverse trace-less gauge [25]. Note that the assumption of being in the vacuum, where $T_{\mu\nu} = 0$, is necessary to set $h_{00} = 0$.

The procedure that we adopted here is analogous of what we do in classical electrodynamics. Here the equation of motion in the vacuum is $\partial_\mu F^{\mu\nu} = 0$, which is invariant under the transformation $A_\mu \to A_\mu + \partial_\mu \theta$. We impose the gauge condition $\partial_\mu A^\mu = 0$, and we obtain the equation $\Box A^\mu = 0$. We have the residual gauge invariance and if we perform a gauge transformation, with a function θ' that satisfies the condition $\Box\theta' = 0$, the four-potential stays the same. We fix the residual gauge invariance by setting $A^0 = 0$ and $\partial^i A_i = 0$.

Equation (3.1.14) now describes the propagation of two degrees of freedom and, as a solution, it admits a superposition of plane waves

$$h_{ij}(t, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \left(\mathcal{A}(\mathbf{k})_{ij} e^{\imath kx} + \mathcal{A}(\mathbf{k})^*_{ij} e^{-\imath kx} \right), \tag{3.1.19}$$

where

$$\mathcal{A}_{ij}(\mathbf{k}) = \mathcal{A} e_{ij}(\mathbf{k}). \tag{3.1.20}$$

$\mathcal{A}$ is the amplitude of the fluctuation and e_{ij} is the polarization tensor [25, 38].

3.2 Gravitational waves in curved spacetime

Until now we have considered gravitational waves in flat background. In such a case, since the background does not depend on the coordinates, while the fluctuations do, there is a clear distinction between the background and the fluctuations. Now, instead, we have in addition a background which is a function of the space-time point. The question is: how can we separate the background from the fluctuations? As we have done before, we split the metric as [38]

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu} + h_{\mu\nu}(x), \tag{3.2.1}$$

with the condition

$$\mid h_{\mu\nu} \mid \ll 1. \tag{3.2.2}$$

When we have a clear separation in the scale (spatial or in time) of variation of the two terms on the right side of the equation (3.2.1), the separation is well defined. In particular, we assume that, in some coordinate systems, $\bar{g}_{\mu\nu}$ varies on a scale L_b, while $h_{\mu\nu}$ varies on a scale λ. In this case the separation performed in equation (3.2.1) is well defined only if we have

$$\lambda \ll L_b. \tag{3.2.3}$$

The same argument can be used in the frequency domain. In fact, calling f the typical frequency of $h_{\mu\nu}$, and f_b the typical frequency of variation of $\bar{g}_{\mu\nu}$, we also have

$$f \gg f_b. \tag{3.2.4}$$

Note that the conditions in equations (3.2.3) and (3.2.4) are independent, because L_b and f_b are not related in principle [38].

Now, we study how the perturbation $h_{\mu\nu}$ propagates over the background metric, and how it modifies it. The first step is to expand the Einstein equations in $h_{\mu\nu}$. Differently from the flat background case, now we have two parameters that control this expansion [38]. One of the parameters is given by the typical amplitude of the perturbations

$$h = \mid h_{\mu\nu} \mid, \tag{3.2.5}$$

and the other is

$$\epsilon = \frac{\lambda}{L_b}. \tag{3.2.6}$$

It's convenient to write Einstein equations in the form

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right), \tag{3.2.7}$$

where T is the trace of the energy-momentum tensor. Now we expand the Einstein equations in powers of h. The Ricci tensor is

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + \cdots \tag{3.2.8}$$

Now we use the second parameter of the expansion, $\frac{\lambda}{L_b}$. $\bar{R}_{\mu\nu}$ contains only low frequencies modes, since it is constructed only with $\bar{g}_{\mu\nu}$. Instead, $R^{(1)}_{\mu\nu}$ contains only high frequency modes, because it is linear in $h_{\mu\nu}$. Finally, $R^{(2)}_{\mu\nu}$, which is quadratic in $h_{\mu\nu}$, contains both high and low frequency modes. That happens because, in terms like $h_{\mu\nu} h_{\rho\sigma}$, we can have a mode with high $\mathbf{k}$, and another one with $\mathbf{k}' = -\mathbf{k}$, which combine themselves in a low frequencies mode.

We can now separate the equation (3.2.7) in high and low parts [35, 36, 37],

$$\bar{R}_{\mu\nu} = -R^{(2)low}_{\mu\nu} + 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)^{low}, \tag{3.2.9}$$

$$R^{(1)}_{\mu\nu} = -R^{(2)high}_{\mu\nu} + 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)^{high}, \tag{3.2.10}$$

where the superscript "low" denotes the projection on the low wavelengths, while the superscript "high" denotes the projection on the high wavelengths.

The first equation describes how gravitational waves modify the background curvature, while the second one could be seen as a dynamical propagation equation of the gravitational waves [38].

Even if our goal is to study the propagation of gravitational waves over a curved space-time, for the sake of completeness we will now concentrate on equation (3.2.9).

3.2.1 Gravitational wave effects on the background space-time

When there is a clear separation between the scale λ and L_b, the low frequency equation can be treated with an average technique. This method has been introduced for the first time in the astrophysical context by Brill and Hartle in [37]

It is based on the definition of a new scale l, which respects the condition $\lambda \ll l \ll L_b$. The projection over the long wavelength modes is performed by computing the average of the equation (3.2.9) over a scale with typical size l. This average leaves unchanged the long wavelength modes, since they are constant over a scale l, because $l \ll L_b$. High wavelength modes, instead, oscillate very fast, so, when they are mediated over a scale l the result is zero [38].

This average technique was first applied in statistical physics to integrate out fluctuations that take place over a scale smaller than l. That was done to study the dynamics of the system on long scales, how L_b is in our case. Now this technique is widely used in several branches of physics. We can write the equation (3.2.9) as

$$\bar{R}_{\mu\nu} = \langle R^{(2)}_{\mu\nu} \rangle + 8\pi G \langle T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \rangle. \tag{3.2.11}$$

We define an effective matter energy-momentum tensor, $\bar{T}_{\mu\nu}$, as

$$\langle T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \rangle = \bar{T}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{T}, \tag{3.2.12}$$

where $\bar{T} = \bar{g}^{\mu\nu} \bar{T}_{\mu\nu}$ is the trace. From the average definition, we know that $\bar{T}_{\mu\nu}$ has only low frequency modes, and it could be seen as a macroscopic version of $T_{\mu\nu}$. We also define the quantity

$$t_{\mu\nu} = +\frac{1}{8\pi G} \langle R^{(2)}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} R^{(2)} \rangle, \tag{3.2.13}$$

where $R^{(2)} = \bar{g}^{\mu\nu} R^{(2)}_{\mu\nu}$. We define the trace of $t_{\mu\nu}$

$$t = \bar{g}^{\mu\nu} t_{\mu\nu} = \frac{1}{8\pi G} \langle R^{(2)} \rangle. \tag{3.2.14}$$

To obtain this equation we bring $\bar{g}^{\mu\nu}$ inside the average parenthesis, because it contains only low frequency modes [38]. Inserting equation (3.2.14) in (3.2.13), we see that

$$-\langle R^{(2)}_{\mu\nu} \rangle = 8\pi G \left(t_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} t \right). \tag{3.2.15}$$

With this formula, we can rewrite the equation (3.2.9) as

$$\bar{R}_{\mu\nu} = 8\pi G \left(\bar{T}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \right) + 8\pi G \left(t_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} t \right), \tag{3.2.16}$$

or, equivalently as

$$\bar{R}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{R} = 8\pi G \left(\bar{T}_{\mu\nu} + t_{\mu\nu} \right). \tag{3.2.17}$$

These are the "coarse grained" Einstein equations [38]. They determine the evolution of the background metric $\bar{g}_{\mu\nu}$ (which is the long wavelength component of the metric) as a function of the long wavelength modes of the matter energy tensor $\bar{T}_{\mu\nu}$, and $t_{\mu\nu}$ (which depends only on the gravitational wave field $h_{\mu\nu}$). From equation (3.2.17) we see that the effects on the background of $h_{\mu\nu}$ are formally identical to those of matter with energy-momentum tensor $t_{\mu\nu}$. Therefore, we can identify $t_{\mu\nu}$ with the energy-momentum tensor of the gravitational waves [38, 36]. We also note that the Bianchi identities hold for the background metric,

$$D^{\mu} \left(\bar{R}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{R} \right) = 0, \tag{3.2.18}$$

which implies

$$D^{\mu}\left(\bar{T}_{\mu\nu}+t_{\mu\nu}\right)=0. \tag{3.2.19}$$

The conservation of the sum of the background and gravitational wave energy-momentum tensors tells us that, in general, matter sources and gravitational waves exchange energy and momentum.

3.2.2 Gravitational wave propagation equation

We will analyze now equation (3.2.10), because our goal is to study the evolution of gravitational waves in a curved background. We are not interested in how the curvature of the background is modified by gravitational waves.

The explicit expression of $R_{\mu\nu}$ is obtained from the Levi Civita connection, which can be written as

$$\Gamma^{\mu}{}_{\nu\rho}=\bar{\Gamma}^{\mu}{}_{\nu\rho}+\Gamma^{(1)\mu}{}_{\nu\rho}, \tag{3.2.20}$$

where $\Gamma^{(1)\mu}{}_{\nu\rho}$ is

$$\Gamma^{(1)\mu}{}_{\nu\rho}=\frac{1}{2}\bar{g}^{\mu\sigma}\left(D_{\nu}h_{\rho\sigma}+D_{\rho}h_{\nu\sigma}-D_{\sigma}h_{\nu\rho}\right), \tag{3.2.21}$$

and the covariant derivatives are calculated with the background metric $\bar{g}_{\mu\nu}$ [27, 38]. The formula (3.2.21) is easy to obtain in a locally inertial frame corresponding to $\bar{g}_{\mu\nu}$, converting the ordinary derivatives to covariant derivatives. That can be done because the difference between two affine connections corresponding to two metrics, $\bar{g}_{\mu\nu}$ and $g_{\mu\nu}$, is a tensor [27].

In the same way, we perform the calculation of the Riemann tensor, to first-order in $h_{\mu\nu}$. Then, we perform the calculation in the frame where $\bar{\Gamma}^{\mu}{}_{\nu\rho}=0$. From (3.2.21) we see that $\Gamma^{\mu}{}_{\nu\rho}=\mathcal{O}(h)$, so we deduce that the term $\Gamma\Gamma$ doesn't contribute. In this way we have $R^{\mu}{}_{\nu\rho\sigma}=\partial_{\rho}\Gamma^{\mu}{}_{\nu\sigma}-\partial_{\sigma}\Gamma^{\mu}{}_{\nu\rho}+\mathcal{O}(h^2)$. Replacing this formula in equation (3.2.21), and writing covariant derivative instead of ordinary derivative (since $\bar{\Gamma}^{\mu}{}_{\nu\rho}=0$) we obtain a covariant expression for the Riemann tensor

$$\begin{aligned}R_{\mu\nu\rho\sigma}=&\bar{R}_{\mu\nu\rho\sigma}+\frac{1}{2}\left(D_{\rho}D_{\nu}h_{\mu\sigma}+D_{\sigma}D_{\mu}h_{\nu\rho}-D_{\rho}D_{\mu}h_{\nu\sigma}-D_{\sigma}D_{\nu}h_{\mu\rho}\right.\\&\left.+h_{\mu}{}^{\tau}\bar{R}_{\tau\nu\rho\sigma}-h_{\nu}{}^{\tau}\bar{R}_{\tau\mu\rho\sigma}\right)\end{aligned} \tag{3.2.22}$$

To obtain the linearization of the Ricci tensor, we start from the definition $R_{\mu\nu}=g^{\alpha\beta}R_{\alpha\mu\beta\nu}$. The linear contributions are due to $\bar{g}^{\alpha\beta}R^{(1)}{}_{\alpha\mu\beta\nu}-h^{\alpha\beta}\bar{R}_{\alpha\mu\beta\nu}$. Finally, the linear Ricci tensor is

$$R^{(1)}_{\mu\nu}=\frac{1}{2}\left(D^{\alpha}D_{\mu}h_{\nu\alpha}+D^{\alpha}D_{\nu}h_{\mu\alpha}-D^{\alpha}D_{\alpha}h_{\mu\nu}-D_{\nu}D_{\mu}h\right), \tag{3.2.23}$$

where $h=\bar{g}^{\alpha\beta}h_{\alpha\beta}$ [27, 38].

In the equations (3.2.9) and (3.2.10) we apparently have terms with different power dependence on h, because $R^{(1)}_{\mu\nu}$ is linear in h, while $R^{(2)high}_{\mu\nu}$ is quadratic in h. This is not an inconsistency, because we have a second small parameter, ϵ which can compensate the different power dependencies on h of the terms in equations (3.2.9) and (3.2.10) [38].

Let's consider first the case of no external matter, $T_{\mu\nu} = 0$. Now the relative strength of ϵ and h are fixed by equation (3.2.9). The terms involved in these equations are

$$\bar{R}_{\mu\nu} \sim (\partial g)^2 + \partial\partial g, \tag{3.2.24}$$

and $\partial g \sim \frac{1}{L_b}$, from the definition of the scale L_b. The other term involved is $R^{(2)low}_{\mu\nu}$, which gets contributions of the type

$$R^{(2)low}_{\mu\nu} \sim (\partial h)^2 + h\partial\partial h, \tag{3.2.25}$$

and again we can estimate the order of magnitude of the derivative of h by $\partial h \sim \frac{h}{\lambda}$. We can conclude from equation (3.2.9), in the absence of matter, that

$$\frac{1}{L_b^2} \sim \left(\frac{h}{\lambda}\right)^2, \tag{3.2.26}$$

which gives

$$h \sim \frac{\lambda}{L_b} = \epsilon. \tag{3.2.27}$$

Consider now the opposite limit, where $T_{\mu\nu} \neq 0$ and the contribution of the gravitational waves to the curvature is much smaller than the contribution of matter. In this case we have $\frac{1}{L_b^2} \sim \left(\frac{h}{\lambda}\right)^2 +$ (matter contribution)$\ll \left(\frac{h}{\lambda}\right)^2$, so in this case we have

$$h \ll \left(\frac{\lambda}{L_b}\right)^2. \tag{3.2.28}$$

From these conditions, given in equations (3.2.27) and (3.2.28), we understand why we can't extend the linearization procedure, given in the previous section, beyond linear order. In fact, if we force the background metric to be $\eta_{\mu\nu}$, this corresponds to send L_b to 0, or equivalently $\frac{1}{L_b} = +\infty$. In this situation, any small perturbation h violates the condition $h \lesssim \frac{\lambda}{L_b}$, and the expansion in powers of h has no validity.

We can also understand from equations (3.2.27) and (3.2.28) that the gravitational waves are well defined only for small fluctuations, because if $h \sim 1$, also $\epsilon = \frac{\lambda}{L_b} \sim 1$. Also the separation between the scales λ and L_b is the basis of the definition of the gravitational waves on curved backgrounds, so if this condition lacks, the gravitational waves are not well defined. [38, 35]

We will study now equation (3.2.10) with external matter presence. In this case we know from the previous argumentation that the expansions in the parameters h and ϵ are different, because $h \ll \epsilon$. We keep only terms linear in h. Then we expand the result in powers of ϵ, and we obtain

$$R^{(1)}_{\mu\nu} = 0. \tag{3.2.29}$$

$R^{(2)high}_{\mu\nu}$ is negligible because it is quadratic in h, while $R^{(1)}_{\mu\nu}$ is linear in h. $\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{high}$ has a high frequency component. $T_{\mu\nu}$ is smooth, because it is the energy-momentum tensor of the matter source, but it has a high frequency part because, in general, it depends on the metric $g_{\mu\nu}$. Also the term $g_{\mu\nu}T$ has high frequency parts. The first one is given by the product of $h_{\mu\nu}$, with the low frequency mode of T. The second one is due to the product of $\bar{g}_{\mu\nu}$, with the high frequency parts of T.

Summarizing we have

$$\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{high} \sim \mathcal{O}(\frac{h}{L_b^2}). \tag{3.2.30}$$

In addiction, we know that

$$R^{(1)}_{\mu\nu} \sim \partial^2 h \sim \mathcal{O}(\frac{h}{\lambda}). \tag{3.2.31}$$

So we see, by comparing the formulas (3.2.30) and (3.2.31) that $\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{high}$ is smaller, by a factor ϵ^2, than $R^{(1)}_{\mu\nu}$ [38]. So it doesn't contribute to first-order in ϵ.

Using the expression given in formula (3.2.10) for $R^{(1)}_{\mu\nu}$, equation (3.2.29) becomes

$$\bar{g}^{\rho\sigma}\left(D_\rho D_\nu h_{\mu\sigma} + D_\rho D_\mu h_{\nu\sigma} - D_\nu D_\mu h_{\rho\sigma} - D_\rho D_\sigma h_{\mu\nu}\right) = 0. \tag{3.2.32}$$

This equation could be simplified following what we have done in section 3.1, using now $\bar{g}_{\mu\nu}$ instead of $\eta_{\mu\nu}$ [38]. We can use the transverse trace-less gauge, which is identified by the conditions

$$D^\nu h_{\mu\nu} = 0, \quad h = 0. \tag{3.2.33}$$

After switching the covariant derivatives of the first and second terms, equation (3.2.32) becomes,

$$D^\rho D_\rho h_{\mu\nu} + 2R_{\mu\rho\nu\sigma}h^{\rho\sigma} - R_{\mu\rho}h^\rho{}_\nu - R_{\nu\rho}h^\rho{}_\mu = 0. \tag{3.2.34}$$

We can further simplify this equation, outside the matter source, where $\bar{T}_{\mu\nu} = 0$. In this case, we know from equation (3.2.9) that $\bar{R}_{\mu\nu}$ depends only on $R^{(2)low}_{\mu\nu}$, so $\bar{R}_{\mu\nu} = \mathcal{O}\left(\frac{h^2}{\lambda^2}\right)$. If we restrict our treatment to linear order in h, we can drop $\bar{R}_{\mu\nu}$ because it is quadratic in h. Also, since $R_{\mu\rho\nu\sigma}h^{\rho\sigma} = \mathcal{O}\left(\frac{h}{L_b^2}\right)$ we neglect the term $R_{\mu\rho\nu\sigma}h^{\rho\sigma}$. This term is linear in h and hence it is allowed from this point of view, still we restrict further our calculation to first-order in ϵ. From the same arguments we have $D^\rho D_\rho h_{\mu\nu} = \mathcal{O}\left(\frac{h}{\lambda^2}\right)$, therefore $R_{\mu\rho\nu\sigma}h^{\rho\sigma}$ is negligible in this approximation [38].

Thus, restricting our treatment to first-order in h and in ϵ we have

$$D^\rho D_\rho h_{\mu\nu} = 0. \tag{3.2.35}$$

This is the propagation equation for gravitational waves in curved space time, in the limit $\lambda \ll L_b$. So we find that, after separating the Einstein equations in high and low frequencies modes, the low mode equation describes the effect of gravitational waves on the background space-time, while the high mode equation describes the propagation of gravitational waves on the curved space-time. The next step is to treat the equation with the geometric optics approximation [35, 38, 39].

3.3 Geometric optics approximation

We have seen that if we consider the gravitational waves in flat background, we find that these satisfy a wave equation. Therefore a generic solution is given by a superposition of elementary plane waves like $h \sim Ae^{\imath kx}$, where k satisfies the null ray condition $k^2 = 0$ [25]. This solution is in very close analogy with the geometric optics approximation or eikonal approximation of the Maxwell equations for light rays. The eikonal approximation was applied to gravitational waves in curved space-time for the first time by Isaacson in his two papers [35, 36].

In curved space-time we still have a wave equation, but now the equations depend on the background metric through the covariant D'Alembertian operator, $\Box_{\bar{g}} = D_\mu D_\nu \bar{g}^{\mu\nu}$. The eikonal approximation is valid when λ is much smaller than the other length scales considered in the problem. So we set $\lambda \ll L_b$, as we did several times in the previous section. L_b is the typical scale of variation of the background metric. Under these conditions we can use the eikonal approximation. This approximation consists in considering a phase ϕ of the gravitational waves, phase that varies rapidly with respect to the scale L_b. We define the expansion for $h_{\mu\nu}$ as [38, 39]

$$h_{\mu\nu}(x) = A_{\mu\nu} e^{\imath \frac{\phi(x)}{\epsilon}}, \tag{3.3.1}$$

where ϵ is an expansion parameter equal to $\frac{\lambda}{L_b}$, which is set to unit after the calculation. Equation (3.3.1) is only an ansatz, which we have to insert in the equation (3.2.35). We define the gravitational waves wave-vector as

$$k_\mu = \partial_\mu \phi, \tag{3.3.2}$$

and inserting the ansatz (3.3.1) in equation (3.2.35) we obtain

$$\begin{aligned} \epsilon^{-2}\left[-(D_\rho\phi)(D^\rho\phi)A_{\mu\nu}\right] + \imath\epsilon^{-1}\left[(D_\rho D^\rho\phi)A_{\mu\nu} + 2(D^\rho\phi)(D_\rho A_{\mu\nu})\right] \\ + (D_\rho D^\rho\phi)A_{\mu\nu} = 0. \end{aligned} \tag{3.3.3}$$

To leading order (ϵ^{-2}), we see that the wave-vector of the gravitational waves has to satisfy the condition

$$\bar{g}^{\mu\nu} k_\mu k_\nu = 0. \tag{3.3.4}$$

This condition is the equivalent in curved space-time of the condition $k^2 = 0$, condition that light rays and gravitational waves satisfy in flat space-time [35]. Equation (3.3.4) tells us that gravitational wave wave-vector is a null type vector, and is called "eikonal equation". From equation (3.3.4) it follows that $D_\nu (k^\mu k_\mu) = 2k^\mu D_\nu k^\mu = 0$. From the definition of the wave-vector, $k_\mu = \partial_\mu \phi = D_\mu \phi$ (ϕ is a scalar quantity), we can interchange the μ and ν indices because the covariant derivatives of a scalar quantity commute, $D_\nu k_\mu = D_\mu k_\mu$. With this consideration, we obtain the equation

$$k^\mu D_\mu k_\nu = 0. \tag{3.3.5}$$

Equation (3.3.5) is the geodesic equation in the background metric $\bar{g}_{\mu\nu}$. We can consider the line with k^μ as tangent vector, $k^\mu = \frac{dx^\mu}{dl}$, where l is the affine parameter that parametrizes $x^\mu(l)$ [35]. With this definition we can rewrite the equation (3.3.5) as

$$\frac{d^2x^\mu}{dl^2} + \Gamma^\mu{}_{\rho\sigma}\frac{dx^\rho}{dl}\frac{dx^\sigma}{dl} = 0. \tag{3.3.6}$$

To write equation (3.3.6), we use the fact that $k^\mu D_\mu = \frac{d}{d\lambda}$. Equation (3.3.6) states that the curves orthogonal to the surfaces with constant phase travel along null geodesics of the background. This statement is the analogue of what we have for light rays, where the direction of propagation is orthogonal to the constant-phase plane.

The next order(ϵ^{-1}) gives the equation

$$(D_\rho D^\rho\phi)A_{\mu\nu} + 2(D^\rho\phi)(D_\rho A_{\mu\nu}). \tag{3.3.7}$$

We write the quantity $A_{\mu\nu}$ as

$$A_{\mu\nu} = \mathcal{A}e_{\mu\nu}, \tag{3.3.8}$$

where $\mathcal{A}$ is the scalar amplitude, and $e_{\mu\nu}$ is the polarization tensor, which is normalized to unit: $e_{\mu\nu}e^{\mu\nu} = 1$. When we insert this decomposition in equation (3.3.7), we get

$$2k^\rho(D_\rho\mathcal{A})e_{\mu\nu} + 2k^\rho(D_\rho e_{\mu\nu})\mathcal{A} + \mathcal{A}e_{\mu\nu}D_\rho k^\rho = 0. \tag{3.3.9}$$

Multiplying this equation by $e^{\mu\nu}$ we see that the second term of the equation (3.3.9) is the derivative of the normalization factor of the polarization tensor,

$$2k^\rho(D_\rho e_{\mu\nu})e^{\mu\nu}\mathcal{A} = \frac{d}{dl}\left(e_{\mu\nu}e^{\mu\nu}\right)\mathcal{A} = 0, \tag{3.3.10}$$

which gives zero [35]. This tells us that the polarization tensor is parallel transported along the null geodesic. We haven't imposed the gauge condition until now. The Lorentz gauge conditions are $D^\nu h_{\mu\nu} = 0, \quad h = 0$, and they constrain the polarization tensor which satisfies the conditions

$$k^\nu e_{\mu\nu} = 0 \qquad e_{\mu\nu}\bar{g}^{\mu\nu} = 0. \tag{3.3.11}$$

Summarizing, the polarization tensor is transverse and parallel transported. Because of this consideration, the equation (3.3.9) becomes

$$2k^\rho(D_\rho\mathcal{A}) + \mathcal{A}D_\rho k^\rho = 0, \tag{3.3.12}$$

and using the fact that $k^\mu D_\mu = \frac{d}{dl}$, we have

$$\frac{d\ln(\mathcal{A})}{dl} = -\frac{1}{2}D_\mu k^\mu, \tag{3.3.13}$$

which is the equation that governs the evolution of the amplitude of the gravitational wave [35].

We have seen that gravitational waves are in a close relation with light rays, since they share the same features. Gravitons, like photons, propagate along null geodesics, and suffer the same physical effects of photons. When they pass near massive objects they are deviated, with the same deflection angle, and they are redshifted, just as photons are.

In the study of the anisotropies of the Cosmic Microwave Background, the electromagnetic radiation that passes through inhomogeneities, experimented also the Sachs Wolfe effect and integrated Sachs Wolfe effect [33, 34]. Since gravitational radiation behaves like electromagnetic radiation in the geometric optics limit, we expect also gravitational radiation to have the same effects [39]. If we focus on the primordial gravitational wave stochastic background, we will find that the analogy with the Cosmic Microwave Background is very strong.

Chapter 4

Primordial gravitational wave propagation through cosmic inhomogeneities

In chapter 2 we have defined the perturbations in the cosmological context, and in chapter 3 we have reviewed how gravitational radiation propagates through a curved space-time. The aim of this thesis is to apply this formalism to the stochastic gravitation radiation originated by the early evolution of the universe, having in mind the standard slow-roll inflation scenario [26].

This gravitational radiation is represented in this scenario by the tensor perturbations modes in the division of the metric that we have performed in section 2.1. Also, in section 2.6, we have reported the solution of the first-order equation of motion for tensor perturbations. At first-order, the propagation equation for tensor perturbations is simply the equation of motion for a massless scalar field.

In chapter 3 we have studied the propagation of gravitational waves in a generic background. We have to stress once again, that the gravitational waves that we considered in that section are general. They could be the primordial gravitational waves, just as they could be gravitational waves originated from astrophysical process. In section 3.3 we have introduced the geometric optics approximation for gravitational radiation, and we have seen that, in this limit, gravitational radiation behaves like electromagnetic radiation.

In the first part of this chapter we will apply the formalism developed in chapter 3 to the primordial gravitational waves described in chapter 2. In this way, and with the tools of the geometric optics approximation, we will calculate the correction to amplitude, frequencies and phase. These corrections are analogous to those of electromagnetic radiation [39]. Since we are interested in primordial gravitational waves, the parallel with the Cosmic Microwave Background is straightforward. We will find that there is an analog for gravitational radiation of the Integrate Sachs-Wolfe effect for the Cosmic Microwave Background.

In the second part of the chapter we will analyze the problem from a different point of view, without geometric optics approximation. We will do this because with this approach we can apply the Green method, and therefore we can calculate the correction to the power spectrum expected from the standard inflation scenario.

4.1 Metric in the longitudinal gauge

Before moving forward it is necessary to specify the background metric in which the gravitational waves are moving through. The background metric that we consider is given by the FRW metric, plus the scalar perturbations. We will work in the longitudinal gauge, where we have only the scalar perturbations Φ and Ψ. To consider only scalar perturbations and to neglect all the vector modes, results in more than the mere choice of the gauge. In fact, we can't neglect all the vector modes present in the perturbed metric only with the choice of the gauge. By considering only these two perturbations, we fix the dynamical statement not to consider vector perturbations. We are allowed to do that, because, in the standard inflation scenario, we don't have the production of vector perturbations [21]. The background metric $\bar{g}_{\mu\nu}(\eta, \mathbf{x})$ is

$$\bar{g}_{\mu\nu}(\eta, \mathbf{x}) = \begin{pmatrix} -a^2(\eta)e^{-2\Phi(\eta,\mathbf{x})} & 0 & 0 & 0 \\ 0 & a^2(\eta)e^{-2\Psi(\eta,\mathbf{x})} & 0 & 0 \\ 0 & 0 & a^2(\eta)e^{-2\Psi(\eta,\mathbf{x})} & 0 \\ 0 & 0 & 0 & a^2(\eta)e^{-2\Psi(\eta,\mathbf{x})} \end{pmatrix}. \tag{4.1.1}$$

We write the metric in this way, with the exponential factor, because it is easier to compute the inverse metric, since the metric is diagonal and we have only to change the sign at the exponent. The inverse metric is

$$\bar{g}^{\mu\nu}(\eta, \mathbf{x}) = \begin{pmatrix} -a^{-2}(\eta)e^{+2\Phi(\eta,\mathbf{x})} & 0 & 0 & 0 \\ 0 & a^{-2}(\eta)e^{+2\Psi(\eta,\mathbf{x})} & 0 & 0 \\ 0 & 0 & a^{-2}(\eta)e^{+2\Psi(\eta,\mathbf{x})} & 0 \\ 0 & 0 & 0 & a^{-2}(\eta)e^{+2\Psi(\eta,\mathbf{x})} \end{pmatrix}. \tag{4.1.2}$$

If we expand the exponential factors in the metric up to first-order in Φ and Ψ, we find that the metric calculated in the longitudinal gauge becomes :

$$\bar{g}_{\mu\nu}(\eta, \mathbf{x}) = a^2(\eta)\begin{pmatrix} -1+2\Phi(\eta,\mathbf{x}) & 0 & 0 & 0 \\ 0 & 1-2\Psi(\eta,\mathbf{x}) & 0 & 0 \\ 0 & 0 & 1-2\Psi(\eta,\mathbf{x}) & 0 \\ 0 & 0 & 0 & 1-2\Psi(\eta,\mathbf{x}) \end{pmatrix}. \tag{4.1.3}$$

The total metric is the sum of background metric and tensor perturbations

$$g_{\mu\nu}(\eta, \mathbf{x}) = \bar{g}_{\mu\nu}(\eta, \mathbf{x}) + h_{\mu\nu}(\eta, \mathbf{x}). \tag{4.1.4}$$

4.2 Integrated Sachs-Wolfe effect for gravitational radiation

Now, we apply the equations of the geometric optics obtained in section 3.3 together with the metric form given in equation (4.1.3). These equations constrain the amplitude, the wavevector and the frequencies of the gravitational waves, as we will see. The gravitational wave wave-vector have to satisfy the condition $k^\mu D_\mu k_\nu = 0$, which implies that the curve x^μ with tangent vector k^μ satisfies the equation

$$\frac{d^2 x^\mu}{dl} + \Gamma^\mu{}_{\rho\sigma}\frac{dx^\rho}{dl}\frac{dx^\sigma}{dl} = 0. \tag{4.2.1}$$

Since the gravitational wave-vector satisfies the null geodesic equation, one can essentially follow the derivation of the temperature anisotropies of the Cosmic Microwave Background [32]. It will be convenient to separate the dependence on the scale-factor by working in the conformal background metric $\tilde{g}_{\mu\nu} = a^{-2}\bar{g}_{\mu\nu}$. The wave-vector k^μ of a light ray in the physical metric is related to the wave-vector $\tilde{k}^\mu$ in the conformally transformed metric by $\tilde{k}^\mu = a^{-2}k^\mu$; the affine parameters are related by $dl = a d\lambda$. The null geodesics $x^\mu(l)$ with affine parameter l in the background metric $\bar{g}_{\mu\nu}$ are the same as the null geodesic $\tilde{x}^\mu(\lambda)$ with affine parameter λ, in the perturbed metric $\tilde{g}_{\mu\nu}$ [32, 39].

Let's consider the geodesic equation for $\tilde{x}^\mu(\lambda)$ in the conformally transformed metric

$$\frac{d\tilde{k}^\mu(\lambda)}{d\lambda} + \tilde{\Gamma}^\mu{}_{\rho\sigma}\tilde{k}^\rho\tilde{k}^\sigma = 0 \tag{4.2.2}$$

Following [39], we write the perturbed geodesic as

$$\tilde{x}^\mu(\lambda) = \tilde{x}^{(0)\mu}(\lambda) + \tilde{x}^{(1)\mu}(\lambda) \tag{4.2.3}$$

and the perturbed wavevector as

$$\tilde{k}^\mu(\lambda) = \tilde{k}^{(0)\mu}(\lambda) + \tilde{k}^{(1)\mu}(\lambda). \tag{4.2.4}$$

The zero-order geodesic is

$$\tilde{x}^{(0)\mu}(\lambda) = (\lambda, (\lambda_r - \lambda)n^i) \tag{4.2.5}$$

and the unperturbed wavevector is

$$\tilde{k}^{(0)\mu(\lambda)} = (1, -n^i). \tag{4.2.6}$$

We set the observer at the coordinates point $x^\mu = (\eta_r, \vec{0})$ and n^i is a unit vector that points in the direction of arrival of the gravitational waves. Equation (4.2.2) becomes, to first-order in perturbations

$$\frac{\tilde{k}^{(1)\mu}(\lambda)}{d\lambda} + \tilde{\Gamma}^{(1)\mu}{}_{\rho\sigma}\tilde{k}^{(0)\rho}(\lambda)\tilde{k}^{(0)\sigma}(\lambda) = 0, \tag{4.2.7}$$

where we used $\tilde{\Gamma}^{(0)\mu}{}_{\rho\sigma} = 0$. We now report the affine connection coefficients:

$$\tilde{\Gamma}^0{}_{00} = \Phi', \tag{4.2.8}$$

$$\tilde{\Gamma}^0{}_{i0} = \Phi_{,i}, \tag{4.2.9}$$

$$\tilde{\Gamma}^0{}_{ij} = -\Psi'\delta_{ij}, \tag{4.2.10}$$

$$\tilde{\Gamma}^i{}_{00} = \Phi_{,i} \tag{4.2.11}$$

$$\tilde{\Gamma}^i{}_{j0} = -\Psi'\delta^i{}_j, \tag{4.2.12}$$

$$\tilde{\Gamma}^i{}_{jk} = -\Psi_{,k}\delta^i_j - \Psi_{,j}\delta^i_k + \Psi^i_{,}\delta_{jk}. \tag{4.2.13}$$

The temporal component of the wavevector obeys the following equation

$$\frac{d\tilde{k}^{(1)0}}{d\lambda} = \partial_\eta(\Phi + \Psi) - 2\frac{d\Phi}{d\lambda}, \tag{4.2.14}$$

where we have used the expression given in equations (4.2.8)-(4.2.13), and also we used $\frac{d\Phi}{d\lambda} = \partial_\eta\Phi - \Phi_{,i}n^i$. As far as the spatial component is concerned it's convenient to separate the vector in parallel and perpendicular components to the vector n^i. We define the projector

$$\perp^i{}_j = \delta^i{}_j - n^i n_j \tag{4.2.15}$$

this operator, acting on a vector, gives a vector orthogonal to n^i. To obtain the component parallel to n^i of a given vector, we simply project the vector on n^i. One finds for $\tilde{k}^{(1)i}_{\parallel}$

$$\frac{d\tilde{k}^{(1)i}_{\parallel}}{d\lambda} = -n^i\left(\frac{d(\Phi - \Psi)}{d\lambda} + \partial_\eta(\Phi + \Psi)\right), \tag{4.2.16}$$

and for $\tilde{k}^{(1)i}_{\perp}$

$$\frac{d\tilde{k}^{(1)i}_{\perp}}{d\lambda} = -(\delta_{ij} - n^i n^j)\partial_j(\Phi + \Psi), \tag{4.2.17}$$

where we have defined

$$\tilde{k}^{(1)i}_{\parallel} = n^i n_j \tilde{k}^{(1)j} \tag{4.2.18}$$

and

$$\tilde{k}^{(1)i}_{\perp} = \perp^i{}_j \tilde{k}^{(1)j}, \tag{4.2.19}$$

which satisfies

$$\tilde{k}^{(1)i} = \tilde{k}^{(1)i}_{\parallel} + \tilde{k}^{(1)i}_{\perp} \tag{4.2.20}$$

We can integrate the equations (4.2.14) and (4.2.16) from the emission to the observer position. We call λ_e the affine parameter at the beginning of the geodesic. We have imposed as initial condition that $\tilde{k}^{(1)\mu}$ is a null type vector

$$\tilde{k}^\mu\tilde{k}^\nu\bar{g}_{\mu\nu}(\lambda_e) = 0. \tag{4.2.21}$$

If we open the previous formula we have three contributions

$$\begin{aligned}
\tilde{k}^\mu\tilde{k}^\nu g_{\mu\nu}(\lambda_e) =& \tilde{k}^{(1)\mu}\tilde{k}^{(0)\nu}g^{(0)}_{\mu\nu} + \tilde{k}^{(0)\mu}\tilde{k}^{(1)\nu}g^{(0)}_{\mu\nu} + \tilde{k}^{(0)\mu}\tilde{k}^{(0)\nu}g^{(1)}_{\mu\nu} \\
&= 2\left(\tilde{k}^{(1)0}(\lambda_e)(-1) + \tilde{k}^{(1)i}(\lambda_e)n_i\right) + (\Phi(\lambda_e) + \Psi(\lambda_e)) \qquad (4.2.22) \\
&= 0
\end{aligned}$$

For the sake of simplicity we can set $\tilde{k}^{(1)i}(\lambda_e) = 0$, and we obtain from the previous formula

$$\tilde{k}^{(1)i}(\lambda_e) = 0, \quad \tilde{k}^{(1)0}(\lambda_e) = -\left(\Phi + \Psi\right)|_{\lambda_e} \tag{4.2.23}$$

With this initial condition, we integrate (4.2.14) and (4.2.16), and we obtain

$$\tilde{k}^{(1)0}(\lambda) = -\left(\Phi + \Psi\right)|_{\lambda_e} - 2\Phi\,|_{\lambda_e}^{\lambda} + \int_{\lambda_e}^{\lambda} \partial_\eta \left(\Phi + \Psi\right) d\lambda' \tag{4.2.24}$$

$$\begin{aligned} \tilde{k}_{\parallel}^{(1)i} &= \tilde{k}^{(1)i}\left(\left(\Psi - \Phi\right)|_{\lambda_e}^{\lambda} + \int_{\lambda_e}^{\lambda} \partial_\eta \left(\Phi + \Psi\right) d\lambda'\right) \\ \tilde{k}_{\perp}^{(1)i} &= -\perp^{ij} \int_{\lambda_e}^{\lambda} \partial_j (\Phi + \Psi) d\lambda' \end{aligned} \tag{4.2.25}$$

The integral in the previous formula is the integrated Sachs-Wolfe effect for gravitational radiation analogous to that of the Cosmic Microwave Background. We can see this effect by looking at the frequency of the gravitational waves, that is defined in the reference frame of the cosmological fluid, which is defined as

$$\omega = -u_\mu k^\mu, \tag{4.2.26}$$

where u_μ is the covariant four-velocity of the fluid, given by the formulas (2.4.6) and (2.4.7), and $k^\mu = a^{-2}\tilde{k}^\mu$. We obtain for the frequency

$$\omega = \frac{1}{a}\left(1 - \Psi(\lambda_e) - \Phi\,|_{\lambda_e}^{\lambda} + \vec{n}\cdot\vec{v} + I_{iSW}\right), \tag{4.2.27}$$

where we have defined

$$I_{iSW} = \int_{\lambda_e}^{\lambda} \partial_\eta \left(\Phi + \Psi\right) d\lambda' \tag{4.2.28}$$

From equation (4.2.27) we can calculate the ratio between receiving and emitting frequencies, and we find

$$\frac{\omega_r}{\omega_e} = \frac{\nu_r}{\nu_e} = \frac{1 - \Phi\,|_e^r + \vec{n}\cdot\vec{v}\,|_e^r + I_{iSW}(\lambda_r)}{1+z}, \tag{4.2.29}$$

where z is the redshift, defined as usual

$$1 + z = \frac{a_r}{a_e}. \tag{4.2.30}$$

From equation (4.2.29) we see that the presence of the inhomogeneities causes corrections to the usual redshift effect that occurs in the FRW background. We have a non-ntegrated effect, which is proportional to the value of the gravitational potential at emission and reception, and an integrated Sachs-Wolfe correction. The formula (4.2.27) is analogous to what we find for the temperature anisotropies of the Cosmic Microwave Background [33, 34].

We can calculate the gravitational wave phase fluctuations due to inhomogeneities from the formula

$$\frac{d\phi}{d\lambda} = \tilde{k}^{(0)\mu} D_\mu \phi = -\tilde{k}^{(0)\mu}\tilde{k}^{(1)}_\mu = -\tilde{k}^{(1)0}, +\tilde{k}^{(0)}_i \tilde{k}^{(1)i}_\parallel, \tag{4.2.31}$$

which gives

$$\frac{d\phi}{d\lambda} = \Psi + \Phi \to \phi = \int_{\lambda_e}^{\lambda} (\Psi + \Phi)\, d\lambda'. \tag{4.2.32}$$

This is the Shapiro time delay, analogous to that associated to the electromagnetic radiation moving through a gravitational field.

Now, we look at the gravitational waves amplitude, which is regulated by equation (3.3.13). We rewrite here the equation:

$$\frac{d\ln(\mathcal{A})}{dl} = -\frac{1}{2} D_\mu k^\mu \tag{4.2.33}$$

We write the amplitude as

$$\mathcal{A} = \mathcal{A}^{(0)}(1+\xi). \tag{4.2.34}$$

The amplitude equation becomes

$$\frac{1}{1+\xi}\frac{d\xi}{d\lambda} = -\frac{1}{2}\left(\partial_\eta \tilde{k}^{(1)0} + \partial_i \tilde{k}^{(1)i}_\parallel + \partial_i \tilde{k}^{(1)i}_\perp + \tilde{\Gamma}^{\mu}{}_{\mu\nu}\tilde{k}^{(0)\nu}\right), \tag{4.2.35}$$

where we have expanded the denominator

$$\frac{1}{1+\xi} = 1 - \xi, \tag{4.2.36}$$

since the correction is small.

$$-2\frac{d\xi}{d\lambda} = \partial_\eta \tilde{k}^{(1)0} + \partial_i \tilde{k}^{(1)i}_\parallel + \partial_i \tilde{k}^{(1)i}_\perp + \tilde{\Gamma}^{\mu(1)}{}_{\mu\nu}\tilde{k}^{(0)\nu}. \tag{4.2.37}$$

The terms on the right hand side of the previous equation are

$$\partial_\eta \tilde{k}^{(1)0} = \partial_\eta\left(-2\Phi + I_{iSW}\right) \tag{4.2.38}$$

$$\partial_i \tilde{k}^{(1)i}_\parallel = \tilde{k}^{(0)i}\partial_i\left(\Psi - \Phi - I_{SW}\right) = \frac{d}{d\lambda}\left(\Psi - \Phi + I_{SW}\right) - \partial_\eta\left(\Psi - \Phi + I_{SW}\right) \tag{4.2.39}$$

$$\partial_i \tilde{k}^{(1)i}_\perp = -\perp^{ij}\int_{\lambda_e}^{\lambda} \partial_j\partial_i\left(\Phi + \Psi\right) d\lambda' \tag{4.2.40}$$

$$\tilde{\Gamma}^{\mu(1)}{}_{\mu\nu}\tilde{k}^{(0)\nu} = \frac{d}{d\lambda}\left(\Phi - 3\Psi\right) \tag{4.2.41}$$

Summing all these terms, we find for the amplitude perturbation ξ the equation

$$-2\frac{d\xi}{d\lambda} = -\partial_\eta\left(\Psi + \Phi\right) + \frac{d}{d\lambda}\left(-2\Psi + I_{IS}\right) - \perp^{ij}\int_{\lambda_e}^{\lambda}\partial_j\partial_i\left(\Phi + \Psi\right) d\lambda' \tag{4.2.42}$$

which, after integration, gives

$$\xi = -\Psi\mid_{\lambda_e}^{\lambda} - \frac{1}{2}\perp^{ij}\int_{\lambda_e}^{\lambda}\int_{\lambda_e}^{\lambda'}\partial_j\partial_i\left(\Phi+\Psi\right)d\lambda' d\lambda''. \tag{4.2.43}$$

We note that the I_{iSW} term has been canceled. The remaining contribution is the magnification due to gravitational lensing [41].

Summarizing the results of this section, we have :

$$h_{\mu\nu} = \mathcal{A}e_{\mu\nu}e^{\imath\frac{\phi}{\epsilon}}, \tag{4.2.44}$$

$$\phi = \int_{\lambda_e}^{\lambda}\left(\Psi+\Phi\right)d\lambda', \tag{4.2.45}$$

$$\mathcal{A} = \mathcal{A}_{(0)}\left(1+\xi\right) = \mathcal{A}_0\left(1-\Psi\mid_{\lambda_e}^{\lambda} - \frac{1}{2}\perp^{ij}\int_{\lambda_e}^{\lambda}\int_{\lambda_e}^{\lambda'}\partial_j\partial_i\left(\Phi+\Psi\right)d\lambda' d\lambda''\right), \tag{4.2.46}$$

$$\frac{\omega_r}{\omega_e} = \frac{\nu_r}{\nu_e} = \frac{1-\Phi\mid_e^r + \vec{n}\cdot\vec{v}\mid_e^r + I_{iSW}(\lambda_r)}{1+z} \tag{4.2.47}$$

Inserting all these results in equation (4.2.44), we get

$$h_{\mu\nu} = \mathcal{A}^{(0)}e_{\mu\nu}\left(1-\Psi\mid_{\lambda_e}^{\lambda} - \frac{1}{2}\perp^{ij}\int_{\lambda_e}^{\lambda}\int_{\lambda_e}^{\lambda}\partial_j\partial_i\left(\Phi+\Psi\right)d\lambda' d\lambda''\right)e^{\int_{\lambda_e}^{\lambda}(\Phi+\Psi)d\lambda'}. \tag{4.2.48}$$

This result is general. In fact gravitational waves with different origin could be distinguished by the zero-order amplitude $\mathcal{A}_{(0)}$. The unperturbed amplitude in the case of primordial gravitational waves is given by equation (2.6.25), which is the solution of the equation of the first-order tensor perturbations.

$$\mathcal{A}^{(0)}(\mathbf{x},\eta) = \int\frac{d^3k}{(2\pi)^3}3\chi(\mathbf{k},0)\frac{j_1(k\eta)}{k\eta}T(k)e^{\imath\mathbf{k}\cdot\mathbf{x}}. \tag{4.2.49}$$

4.3 Primordial gravitational wave power spectrum corrections

In the previous section, we used the geometric optics approximation, and we obtained some useful information about the behavior of the gravitational waves moving through cosmic inhomogeneities, which are described by the scalar perturbations Φ and Ψ.

Now we won't use the geometric optics approximation. We will expand equation (3.2.35), which is the equation of motion for the gravitational waves in curved background. We will integrate this equation using the Green method for a partial differential equation. The zero order solution that we will use is given by equation (2.6.25), since, once again we are interested in primordial gravitational waves originated from inflation.

The equation of motion for gravitational waves in the limit of $\frac{\lambda}{\epsilon} \ll 1$ is

$$\Box_{\bar{g}}h_{\mu\nu} = 0. \tag{4.3.1}$$

It is known that gravitational waves satisfy the same equation of a massless scalar field. So, we can use the expansion for the D'Alembertian operator for scalar quantities, which is

$$\Box_{\bar{g}}\chi = \frac{1}{\sqrt{-\bar{g}}}\partial_\mu(\sqrt{-\bar{g}}\bar{g}^{\mu\nu}\partial_\nu\chi), \tag{4.3.2}$$

where $\bar{g}$ stands for the determinant of the background metric $\bar{g}_{\mu\nu}$. Using the metric with the exponential factor, we have

$$\sqrt{-\bar{g}} = a^4 e^{\Phi-3\Psi}. \tag{4.3.3}$$

Expanding equation (4.3.2), with the metric given from equation (4.1.1), we get

$$\begin{aligned}\Box_{\bar{g}}\chi &= \frac{1}{\sqrt{-\bar{g}}}\left[\partial_0(-a^2e^{-3\Psi}e^{-\Phi}\partial^0\chi) + \partial_i a^2 e^{\Phi}e^{-\Psi}\partial^i\chi)\right] \\ &= \frac{1}{\sqrt{-\bar{g}}}\Big[-2a'ae^{-\Phi}e^{-3\psi}\chi' - a^2(e^{-3\Psi-\Phi})'\chi' \\ &-a^2e^{-3\Psi-\Phi}\chi'' + a^2\partial_i(e^{\Phi-\Psi})\partial^i\chi + a^2e^{\Phi-\Psi}\nabla^2\chi\Big] \\ &= 0.\end{aligned} \tag{4.3.4}$$

We said previously that we won't use the geometric optics approximation in this section. But, according to what we said in chapter 3, in the study of gravitational waves on curved background, it is fundamental the existence of a well defined separation of the scales f_b, which represents the typical time scale of variation of the background quantities, and f, which represents the time scale of variation of the gravitational waves. According to this assumption, we have neglected the time derivatives of scalar perturbations.

From equation (2.6.11) we could see that the difference between the scalar potentials Φ and Ψ, in the longitudinal gauge, is proportional to the anisotropic stress. But, the anisotropic stress is zero to first-order if we consider cold dark matter and baryon. Free-streaming neutrinos contribute to anisotropic stress with a first-order effect [20]. We do not consider the contribution of the free-streaming neutrinos, since their amount is small in the energy budget of the universe. In this way we can argue that the difference between Φ and Ψ is a second-order effect, and according to our approximations we can neglect $\partial_i(e^{\Phi-\Psi})$.

With this approximation, the equation becomes

$$D_\mu D^\mu\chi = -2\frac{a'}{a^3}e^{-2\Phi}\chi' - \frac{e^{-2\Phi}}{a^2}\chi'' + \frac{e^{2\Psi}}{a^2}\nabla^2\chi = 0, \tag{4.3.5}$$

which could be written finaly as

$$\chi'' + 2\mathcal{H}\chi' - e^{2\Psi+2\Phi}\nabla^2\chi = 0. \tag{4.3.6}$$

Equation (4.3.6) holds also for the gravitational waves, and we can write

$$h''_{\mu\nu} + 2\mathcal{H}h_{\mu\nu} - e^{2\Psi+2\Phi}\nabla^2 h_{\mu\nu} = 0. \tag{4.3.7}$$

To solve equation (4.3.7) it is convenient to do the Fourier transform. In our convention we have

$$h_{\mu\nu}(\mathbf{x},\eta) = \int \frac{d^3k}{(2\pi)^3} h_{\mu\nu}(\mathbf{k},\eta) e^{\imath\mathbf{k}\cdot\mathbf{x}}. \tag{4.3.8}$$

The Fourier transform of equation (4.3.7) is

$$\begin{aligned} &\int \frac{d^3k}{(2\pi)^3} h''_{\mu\nu}(\mathbf{k},\eta) e^{\imath\mathbf{k}\cdot\mathbf{x}} + 2\mathcal{H} \int \frac{d^3k}{(2\pi)^3} h_{\mu\nu}(\mathbf{k},\eta) e^{\imath\mathbf{k}\cdot\mathbf{x}} \\ &- e^{2\Psi+2\Phi}\nabla^2 \int \frac{d^3k_1}{(2\pi)^3} h_{\mu\nu}(\mathbf{k}_1,\eta) e^{\imath\mathbf{k}_1\cdot\mathbf{x}} = 0. \end{aligned} \tag{4.3.9}$$

Now, we expand the exponential factor to first-order in the scalar perturbations and, also, we neglect the difference between Φ and Ψ, $e^{2\Phi+2\Psi} \approx 1 + 2\Phi + 2\Psi \approx 1 + 4\Phi$. The third term of equation (4.3.9) becomes

$$\begin{aligned} &- e^{2\Psi+2\Phi}\nabla^2 \int \frac{d^3k_1}{(2\pi)^3} h_{\mu\nu}(\mathbf{k}_1,\eta) e^{\imath\mathbf{k}_1\cdot\mathbf{x}} = (1+4\Phi) \int \frac{d^3k_1}{(2\pi)^3} k_1^2 h_{\mu\nu}(\mathbf{k}_1,\eta) e^{\imath\mathbf{k}_1\cdot\mathbf{x}} \\ &= \left(1 + \int \frac{d^3k}{(2\pi)^3} \Phi(\mathbf{k}-\mathbf{k}_1,\eta) e^{\imath(\mathbf{k}-\mathbf{k}_1)\cdot\mathbf{x}}\right) \int \frac{d^3k_1}{(2\pi)^3} k_1^2 h_{\mu\nu}(\mathbf{k}_1,\eta) e^{\imath\mathbf{k}_1\cdot\mathbf{x}} \\ &= \int \frac{d^3k_1}{(2\pi)^3} k_1^2 h_{\mu\nu}(\mathbf{k}_1,\eta) e^{\imath\mathbf{k}_1\cdot\mathbf{x}} + 4 \int \frac{d^3k_1}{(2\pi)^3} \Phi(\mathbf{k}-\mathbf{k}_1,\eta) k_1^2 h_{\mu\nu}(\mathbf{k}_1,\eta). \end{aligned} \tag{4.3.10}$$

With this manipulation, equation (4.3.9) becomes

$$h''_{\mu\nu}(\mathbf{k},\eta) + 2\mathcal{H} h_{\mu\nu}(\mathbf{k},\eta) + k^2 h_{\mu\nu}(\mathbf{k},\eta) + 4 \int \frac{d^3k_1}{(2\pi)^3} \Phi(\mathbf{k}-\mathbf{k}_1,\eta) k_1^2 h_{\mu\nu}(\mathbf{k}_1,\eta) = 0. \tag{4.3.11}$$

Now, we recall that in our gauge we have $h_{0\mu} = 0$ and $h = 0$, so we can write the indices in the previous formula as spatial indices. We decompose as usual the gravitational waves as $h_{ij}(\mathbf{k},\eta) = h(\mathbf{k},\eta)_{(\alpha)} e^{(\alpha)}_{ij}(\mathbf{k})$, where h is the scalar amplitude and e_{ij} is the polarization tensor. (α) is an index that assumes the value $+$ and $\times$, and labels the two independent polarization components. We will consider polarization tensors given by

$$e^{(+)}_{ik}(\mathbf{k}) = \frac{1}{\sqrt{2}} \left(e_{1i}(\mathbf{k}) e_{1j}(\mathbf{k}) - e_{2i}(\mathbf{k}) e_{2j}(\mathbf{k})\right) \tag{4.3.12}$$

$$e^{(\times)}_{ik}(\mathbf{k}) = \frac{1}{\sqrt{2}} \left(e_{1i}(\mathbf{k}) e_{2j}(\mathbf{k}) + e_{2i}(\mathbf{k}) e_{1j}(\mathbf{k})\right), \tag{4.3.13}$$

where e_1 and e_2 are unit vectors orthogonal to $\mathbf{k}$. By multiplying equation (4.3.9) by $e^{ij}_{(\alpha)}$ we get

$$\begin{aligned} &h''_{(\alpha)}(\mathbf{k},\eta) + 2\mathcal{H} h_{(\alpha)}(\mathbf{k},\eta) + k^2 h_{(\alpha)}(\mathbf{k},\eta) \\ &+ \int \frac{d^3k_1}{(2\pi)^3} \Phi(\mathbf{k}-\mathbf{k}_1,\eta) k_1^2 h_{ij}(\mathbf{k}_1,\eta) e^{ij}_{(\alpha)} = 0 \end{aligned} \tag{4.3.14}$$

Equation (4.3.14) is the main equation of this section, and it is the starting point to calculate the corrections to the power spectrum arising from the presence of inhomogeneities. In fact, the last term of the equation (4.3.14) is the one responsible for the coupling between gravitational waves and scalar perturbations. In this term, we have the presence of k_1^2, which could be dominant on small scales.

Our strategy is to solve equation (4.3.14) with the Green method. In this way we will have a zero order solution, and a correction that will depend on the potential Φ.

We now define the new quantity

$$\chi(\mathbf{k}, \eta)^{(\alpha)} = a(\eta) h^{(\alpha)}(\mathbf{k}, \eta). \tag{4.3.15}$$

With this new variable, equation (4.3.14) becomes

$$\chi''^{(\alpha)} + \left(k^2 - \frac{a''}{a}\right) \chi^{(\alpha)} + 4 \int \frac{d^3 k_1}{(2\pi^3)} \Phi(\mathbf{k} - \mathbf{k}_1) k_1^2 \chi_{ij}(\mathbf{k}_1, \eta) e^{ij(\alpha)}(\mathbf{k}) = 0 \tag{4.3.16}$$

The zero-order equation is obtained by setting the scalar potential Φ to zero in equation (4.3.16)

$$\chi''^{(\alpha)} + \left(k^2 - \frac{a''}{a}\right) = 0. \tag{4.3.17}$$

This is the equation (2.6.22) expressed using the quantity χ. We already have written the solution of this equation in the formula (2.6.25). For $\chi(\mathbf{k}, \eta)$ we obtain

$$\chi(\mathbf{k}, \eta) = 3\chi(\mathbf{k}, 0) \frac{\mathcal{J}_1(k\eta)}{k\eta} T(k), \tag{4.3.18}$$

where $T(k)$ is the linear transfert function for gravitational waves [31], and j_1 is the first-order spherical Bessel function.

Since we are interested in small scales, and for $k\eta \gg 0$ we have $j_1 \simeq \frac{\cos(k\eta)}{k\eta}$, and equation (4.3.18) becomes

$$\chi(\mathbf{k}, \eta) = 3\chi(\mathbf{k}, 0) \frac{\cos(k\eta)}{(k\eta_0)^2} T(k) = A(\mathbf{k}) \cos(k\eta), \tag{4.3.19}$$

where in the denominator we have replaced η with η_0, where the subscript 0 stands for quantities evaluated at the present time. We have done this because $k\eta \gg 0$. In this way we have written the zero order solution, given in equation (4.3.19), as an oscillating solution, with an amplitude proportional to the wavenumber; in fact we have

$$A(\mathbf{k}) = 3 \frac{\chi(\mathbf{k}, 0)}{(k\eta_0)^2} T(k). \tag{4.3.20}$$

We can now compute the first-order solution of equation (4.3.14). We use the two independent solutions $\chi = A\cos(k\eta)$ and $\bar{\chi} = B\sin(k\eta)$, which are obtained from equation (4.3.17) by neglecting the expansion term in the equation. We find

$$\chi^{(\alpha)}(k,\eta) = \chi_0^{\alpha}(k,\eta)$$
$$+ 4\int \frac{d^3k_1}{(2\pi^3)}\Phi(\mathbf{k}-\mathbf{k}_1)k_1^2\Sigma^{(\alpha)}(\mathbf{k},\mathbf{k}_1)\int_0^{\eta} d\eta' \frac{\chi(k,\eta)\bar{\chi}(k,\eta') - \bar{\eta}(k,\eta)\chi(k,\eta')}{W(\eta')}$$
$$\times \chi_0(k_1,\eta'), \tag{4.3.21}$$

the Wronskian of χ and $\bar{\chi}$ is $W = -ABk$, and $\chi_0(k_1,\eta')$ is given by equation (4.3.19), and

$$\Sigma^{(\alpha)}(\mathbf{k},\mathbf{k}_1) = e^{(+)}{}_{ij}(\mathbf{k}_1)e^{(\alpha)ij}(\mathbf{k}) + e^{(\times)}{}_{ij}(\mathbf{k}_1)\epsilon^{(\alpha)ij}(\mathbf{k}) \tag{4.3.22}$$

is due to the combination $\chi_{ij}(k_1)e^{(\alpha)ij}(\mathbf{k})$ in equation (4.3.21). We can easily perform the integration over η' and we get

$$\begin{aligned}
&\int_0^{\eta} d\eta' \frac{\chi(k,\eta)\bar{\chi}(k,\eta') - \bar{\chi}(k,\eta)\chi(k,\eta')}{ABk} \\
&= A(\mathbf{k})\int_0^{\eta} \frac{\cos(k\eta)\sin(k\eta')\cos(k_1\eta') - \cos(k\eta')\sin(k\eta)\cos(k_1\eta')}{-k} \\
&= A(\mathbf{k})\left[\frac{\cos(k_1\eta) - \cos(k\eta)}{k^2 - k_1^2}\right].
\end{aligned} \tag{4.3.23}$$

With this result, the equation (4.3.16) becomes

$$\delta\chi^{(\alpha)}(\mathbf{k},\eta) = 4\int \frac{d^3k_1}{(2\pi^3)}\Phi(\mathbf{k}-\mathbf{k}_1)k_1^2\Sigma^{(\alpha)}(\mathbf{k},\mathbf{k}_1)A(\mathbf{k})\left[\frac{\cos(k_1\eta) - \cos(k\eta)}{k^2 - k_1^2}\right]. \tag{4.3.24}$$

The term $\delta\chi^{\alpha}(\mathbf{k},\eta)$ quantifies the correction to the stochastic gravitational waves background due to the presence of inhomogeneities.

To compute $\Sigma^{(\alpha)}(\mathbf{k},\mathbf{k}_1)$, we choose the vector $\mathbf{k}$ along the z axis. We can thus take the vectors $\mathbf{e}_1(k)$ and $\mathbf{e}_2(k)$, present in equations (4.3.12) and (4.3.13), as the unit vector of the x and y axes. The three vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{k}$ form an orthogonal complete set of vectors, and they are

$$\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ k \end{pmatrix}, \quad \mathbf{e}_1(k_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2(k_1) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{4.3.25}$$

We now define θ and ϕ as the polar and azimuthal angles of $\mathbf{k}_1$, in the coordinates system given by $\mathbf{e}_1(k)$,$\mathbf{e}_2(k)$ and $\hat{\mathbf{k}}$. Then we define the two vectors $\mathbf{e}_1(k_1)'$ and $\mathbf{e}_2(k_1)'$, which are orthogonal to $\mathbf{k}_1$. These vectors are given by

$$\begin{aligned}
k_1 &= \begin{pmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix}, \quad \mathbf{e}_1'(k_1) = \begin{pmatrix} \sin(\phi) \\ -\cos(\phi) \\ 0 \end{pmatrix} \\
,\mathbf{e}_2'(k_1) &= \begin{pmatrix} -\cos(\theta)\cos(\phi) \\ -\cos(\theta)\sin(\phi) \\ \sin(\theta) \end{pmatrix}.
\end{aligned} \tag{4.3.26}$$

We can compute $\Sigma^{(+)}(k, k_1)$ and $\Sigma^{(\times)}(k, k_1)$, by using equations (4.3.12) and (4.3.13).

$$2\Sigma^{(+)}(k, k_1) = (1+\cos^2(\theta))(\sin^2(\phi)-\cos^2(\phi))-4\cos(\theta)\sin(\phi)\cos(\phi) \quad (4.3.27)$$

and

$$2\Sigma^{(\times)}(k, k_1) = -2(1+\cos^2(\theta))\sin(\phi)\cos(\phi) - 2\cos(\theta)(\sin^2(\phi) - \cos^2(\phi)). \quad (4.3.28)$$

We can compute the power spectrum of the primordial gravitational waves. Since we have the correction $\delta\chi^{\alpha}$ to the solution χ^{α} obtained in the case without inhomogeneities, we will obtain a correction term to the power spectrum. We define the power spectrum of the quantity $\chi(k)^{(\alpha)}$ as

$$\langle \chi^{(\alpha)}(\mathbf{k}, \eta)\chi^{(\alpha)}(\mathbf{k}', \eta)\rangle = (2\pi)^3\delta^3(\mathbf{k}+\mathbf{k}')P^{(\alpha)}(k). \quad (4.3.29)$$

Now $\chi^{(\alpha)} = \chi^{(\alpha)}_{(0)} + \delta\chi^{(\alpha)}$, so we can define the quantity $\Delta P^{(\alpha)}(k)$ from the following equation:

$$\langle \delta\chi^{(\alpha)}(\mathbf{k}, \eta)\delta\chi^{(\alpha)}(\mathbf{k}', \eta)\rangle = (2\pi)^3\delta^3(\mathbf{k}+\mathbf{k}')\Delta P^{(\alpha)}(k), \quad (4.3.30)$$

where $\Delta P^{(\alpha)}(k)$ represents the correction to the power spectrum. In the previous formula we have assumed that when we calculate the quantity $\langle \chi^{(\alpha)}_{(0)}\delta\chi^{(\alpha)}\rangle$ we obtain 0, since as an assumption they belong to different statistical ensembles. By inserting in equation (4.3.30) the expression for $\delta\chi^{(\alpha)}$ given by the equation (4.3.24), we get

$$\begin{aligned}\langle \delta\chi^{(\alpha)}(\mathbf{k}, \eta)\delta\chi^{(\alpha)}(\mathbf{k}', \eta)\rangle &= \langle \int \frac{d^3k_1}{(2\pi)^3}\int \frac{d^3k_1'}{(2\pi)^3}\Phi(\mathbf{k}-\mathbf{k}_1)\Phi(\mathbf{k}'-\mathbf{k}_1') \\ &\times \Sigma^{(\alpha)}(\mathbf{k}, \mathbf{k}_1)\Sigma^{(\alpha)}(\mathbf{k}', \mathbf{k}_1') \times A(k)A(k')f(k, k_1)f(k', k_1')\rangle,\end{aligned} \quad (4.3.31)$$

with $f(k, k_1)$ defined as

$$f(k, k_1) = 4k_1^2\left[\frac{\cos(k_1\eta) - \cos(k\eta)}{k^2 - k_1^2}\right]. \quad (4.3.32)$$

Keeping the average parenthesis inside the integral, and introducing the power spectrum of the scalar perturbations $P_\Phi(k)$ defined as

$$\langle \Phi(\mathbf{k})\Phi(\mathbf{k}')\rangle = (2\pi)^3\delta^3(\mathbf{k}+\mathbf{k}')P_\Phi(k) \quad (4.3.33)$$

and the power spectrum of the quantity $A(k)$ defined as

$$\langle A(\mathbf{k})A(\mathbf{k}')\rangle = (2\pi)^3\delta^3(\mathbf{k}+\mathbf{k}')P_A(k), \quad (4.3.34)$$

equation (4.3.31) becomes

$$\Delta^{(\alpha)}P(k) = \int \frac{d^3k_1}{(2\pi)^3}f(k, k_1)^2\Sigma^{(\alpha)}(\mathbf{k}, \mathbf{k}_1)^2P_\Phi(\mid \mathbf{k}-\mathbf{k}_1 \mid)P_A(k_1). \quad (4.3.35)$$

$P_A(k)$ is related with $P_{\chi(0)}$, the power spectrum of primordial gravitational waves generated from inflation through equation (4.3.20), and we have

$$P_A(k) = 9\frac{T_g(k)^2}{(k\eta_0)^4} P_{\chi(0)}, \tag{4.3.36}$$

where

$$P_{\chi(0)} = \frac{2\pi^2}{k^3} A_T^2(k_0)(\frac{k}{k_0})^{n_T} \tag{4.3.37}$$

is the power spectrum of the primordial gravitational wave generated from inflation, with $A_T(k_0)$ amplitude to a certain scale of reference and n_T as spectral index. It's important to stress that in order to obtain equation (4.3.35), we had made a double average. One is related to the stochastic properties of gravitational waves, and the other one is related to the stochastic properties of the scalar perturbations Φ. These averages are independent, since we have averaged quantities which are defined as statistically independent. Another approach could have been computing only the average of the gravitational waves, and considering a well-defined Φ profile characteristic of a cosmic structure. An interesting case could be considering the profile originated from a dark matter halo.

We can relate the power spectrum of the scalar potential Φ with the matter perturbations power-spectrum P_δ through the Poisson equation, which in Fourier space is

$$\delta(\mathbf{k}, a) = -\frac{2}{3}k^2 \frac{\Phi(\mathbf{k})(a)a(\eta)}{\Omega_m H_0^2}. \tag{4.3.38}$$

Introducing the dimensionless power spectrum $\Delta_\delta(k) = \frac{2\pi^2}{k^3}P_\delta(k)$ (that doesn't depend on the convention choice of Fourier transform) we get

$$P_\Phi(k) = \frac{9}{4}\frac{\Omega_m^2 H_0^2}{k^4 a(\eta)^2}\frac{2\pi^2}{k^3}\Delta_\delta(k). \tag{4.3.39}$$

Equation (4.3.35) becomes

$$\Delta^{(\alpha)}P(k) = \int \frac{d^3k_1}{(2\pi)^3} 16k_1^4 \left[\frac{\cos(k_1\eta) - \cos(k\eta)}{k^2 - k_1^2}\right]^2 \mid \Sigma^{(\alpha)}(\mathbf{k}_1) \mid^2$$
$$\times \left[\frac{9}{4}\frac{\Omega_m^2 H_0^4}{\mid \mathbf{k} - \mathbf{k}_1 \mid^4 a(\eta)^2} \frac{2\pi^2}{\mid \mathbf{k} - \mathbf{k}_1 \mid^3} \Delta_\delta(\mid \mathbf{k} - \mathbf{k}_1 \mid)\right] \left[9\frac{T_g(k_1)^2}{(k_1\eta_0)^4}\frac{2\pi^2}{k_1^3}A_T^2(k_0)(\frac{k_1}{k_0})^{n_T}\right]. \tag{4.3.40}$$

It is useful to consider the integral over d^3k_1 in spherical coordinates, because we can easily do the integral over ϕ. In fact the only quantity that depends on ϕ is $\mid \Sigma^{(\alpha)} \mid^2$. Having squared equations (4.3.12) and (4.3.13), and after having integrated them over ϕ, we obtain the following result for $\Delta P(k)$:

$$\Delta P(k) = \frac{9}{4}\frac{\Omega_m^2 H_0^4}{a^2(\eta)} \int_{-1}^{1} d(\cos(\theta))(4\cos^2(\theta) + (1+\cos^2(\theta))^2)$$
$$\times \int_0^\infty \frac{dk_1}{k_1} \frac{k_1^3}{\mid \mathbf{k} - \mathbf{k}_1 \mid^3} \left[\frac{\cos(k_1\eta) - \cos(k\eta)}{k^2 - k_1^2}\right]^2 \frac{k_1^4}{\mid \mathbf{k} - \mathbf{k}_1 \mid^4} \Delta_\delta(\mid \mathbf{k} - \mathbf{k}_1 \mid)$$
$$\times \left[9\frac{T_g(k_1)^2}{(k_1\eta_0)^4}\frac{2\pi^2}{k_1^3} A_T^2(k_0)(\frac{k_1}{k_0})^{n_T}\right]. \tag{4.3.41}$$

In the previous formula we dropped the polarization index α. We have done this because the integral over ϕ of $\mid \Sigma^{(+)} \mid^2$ and $\mid \Sigma^{(\times)} \mid^2$ gives the same result. The polarization components of the gravitational waves are affected by cosmic inhomogeneities just as we expected. We expected that since the power spectrum is a quantity that doesn't depend by the polarization, and we considered scalar perturbations. If we would have considered vector perturbations instead of scalar perturbations, we would have not expected the integrals to give the same results. In fact, in this case we would have had a privileged direction, and the result would have depended on the relative position of the direction of observation of the gravitational waves and the mean value of the vector perturbations.

We can simplify equation (4.3.41). We define the new variables

$$y = \frac{k_1}{k}, \quad x = \frac{\mid \mathbf{k} - \mathbf{k}_1 \mid}{k}. \tag{4.3.42}$$

With these variables we have that $\cos(\theta) = \frac{1+y-2x^2}{2y}$, and the integral becomes

$$\Delta P(k) = \sqrt{2}\frac{9}{4}\frac{\Omega_m^2 H_0^4}{a^2(\eta)} \left[\frac{2\pi^2}{k^3} A_T^2(k_0)\left(\frac{k}{k_0}\right)^{n_T}\right] \int_0^{+\infty} dy y^2 \left[9 y^{n_T} \frac{Tg^2(ky)}{(k^2y^2\eta^2)}\right]$$
$$\times \left[\frac{\cos(ky\eta) - \cos(k\eta)}{1-y^2}\right]^2 \int_{|1+y|}^{|1-y|} \frac{dx}{x^6} \Delta_\delta(kx)$$
$$\times \left[4\left(\frac{1+y-2x^2}{2y}\right)^2 + \left(1 + \left(\frac{1+y-2x^2}{2y}\right)^2\right)^2\right], \tag{4.3.43}$$

where we added a $\sqrt{2}$ that comes from the square sum of the two polarization contributions.

In the last formula we have indicated the dimensionless power spectrum as Δ_δ. At linear order, we can relate the density fluctuation $\delta(\mathbf{k}, \eta)$ at late time with the primordial potential fluctuations $\Phi(\mathbf{k}, 0)$, using the linear transfer functions $T(k)$

$$\delta(\mathbf{k}, \eta) = -\frac{3}{5}\frac{k^2}{H_0^2 \Omega_{0m}} \Phi(\mathbf{k}, 0) T(\mathbf{k}) D(a), \tag{4.3.44}$$

where $D(a)$ is the growing mode.

However, the structures are highly non linear in this epoch, and it is better to leave indicated Δ_δ. Another option could be to use a numerical fit [40] for the non linear power spectrum.

These non-linearities are present starting from scales of order $L \simeq \frac{2.5}{h}$ Mpc, corresponding to a frequency of $f_b \simeq \frac{c}{L} \simeq 10^{-15} hHz$, (at the present epoch). We are thus interested in frequencies f of the gravitational waves, that are higher than the scale f_b.

This frequency band is reachable with the proposed detectors eLISA (10^{-5}Hz $\div 0.1$)Hz [14] and BBO (optimal frequencies around 0.1Hz with $\Omega_{gw} \sim 10^{-17}$). The goal is to see what is the resulting power spectrum at these frequencies, which is due to the presence of high over-density on scales less than a few mega parsec, entering through equation (4.3.11). The approximation that we have done in section 4.3, when we have neglected the time derivative of the scalar potentials, results, retrospectively, to be valid since we have $f \gg f_b$. When this condition holds, we are in the region in which geometric optics apply, and the results obtained in section 4.2 are valid for these frequencies. In this region also the wave effects (diffraction) discussed in [41] are negligible.

Conclusions and future outlook

In this thesis we studied the propagation of stochastic gravitational wave background through cosmic inhomogeneities. The aim of our work was to characterize the effects suffered by gravitational waves during that propagation. In doing so, we followed two different approaches.

First, we studied the how gravitational radiation propagates via the geometric optics approximation. By means of this approximation, valid only for frequencies higher than the inverse of the typical time-scale variation of scalar perturbations, the analogies between the Cosmic Microwave Background and the stochastic gravitational wave background became immediately clear. In fact, we found that both the amplitude (equation (4.2.43)) and the frequency of the gravitational waves (equation (4.2.27)) experience the same effect as those of the Cosmic Microwave Background: the Sachs-Wolfe and Integrated Sachs-Wolfe effect.

After the characterization of these effects, we adopted another approach. In particular, we didn't use the geometric optics approximation anymore, even if we maintained the fact that the frequencies of interest are higher than the time-scale variation of scalar perturbations. We solved the equation of motion for gravitational waves with the Green method and employing this solution we calculated the correction for the power spectrum of gravitational waves. In doing so, we averaged on both the tensor perturbations and the scalar perturbations, even if these averages in principle are separated. As a result, we found a correction of the power spectrum that depends on the matter perturbation power spectrum through the double integral of equation (4.3.43).

Further extensions of these two approaches are possible. In fact, we have to consider that our theoretical results relate the quantities describing the gravitational waves with scalar perturbations: a possible improvement of this could be the explicit calculation of the integral present in Eq. (4.2.27) and Eq. (4.2.43) with the potential profile of dark matter halos given e.g. by the Navarro Frenk White profile [42].

Equation (4.3.30) could also be further investigated, since we left the integral written in an implicit form. A possibility would be to try to solve the integral with numerical or analytical techniques and then to study the behavior of the correction at different frequencies.

Another topic that we didn't explore is the wave effects that occur when the wavelength of the gravitational waves reaches the dimensions of the structure that originated the potential perturbations. Note that in this limit we cannot apply the geometric optics approximation: the problem has to be studied with the tools from scattering theory.

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