**University of Padua**

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DEPARTMENT OF MATHEMATICS "TULLIO LEVI-CIVITA"

Bachelor's Degree in Mathematics


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# De Giorgi's Theorem on the Isoperimetric Property of the Hypersphere


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*Academic year 2021/2022*

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# Introduction

In this thesis will present and explain the isoperimetric property of the hypersphere, formulated by Ennio De Giorgi and proved in [4]. Such property asserts that, in arbitrary dimension, the set (in the class of sets with orientable boundary and finite measure) that minimizes its "perimeter" to parity of "volume" is (equivalent to) an hypersfphere.

In order to accomplish this objective we need first to give a precise mathematical definition of "volume". This will be realized in Chapter 1, in particular in Section 1, where we present some elementary definitions and results of measure and integration theory from [7], [8] and [1]. In the same section we will also give an important generalization of the notion of (positive) measure, that is a vector-valued measure. Since many different textbooks give many different definitions of what a vector-valued measure is, we specify that in this work we will be using the definition given in [8]. Finally, in Chapter 1, we will state and prove the Gauss Greens theorem as stated in [9].

Once we have defined the "volume of a set", we will then proceed to define its perimeter. As it will be seen in Chapter 2, there are two distinct definition of perimeter: the *Caccioppoli*'s definition the definition formulated by *Ennio De Giorgi* (see [2]). The first definition introduces the concept of perimeter of a $N$-dimensional Borel set using the theory presented in Section 1 of Chapter 2, in particular it defines the perimeter of a set as the total variation of its characteristic function (see [6]); the latter, on the other hand, in [2] introduces a smoothing operator and this, thanks to the property of convolution exposed in Section 2 of Chapter 1, will allow to approximate "well enough" the characteristic function of a set by a family of $C^\infty$ functions, thereby defining the perimeter of a set as a limit of an integral of the norm of a (non-distributional) gradient. In Section 2 of Chapter 2 it will be proven the equivalence between these definitions, and we will motivate the choice of using the De Giorgi's definition.

Lastly, in Chapter 3, it will be proven the isoperimetric property. As we will see, the proof is a consequence of different results found in [2] and [3], perhaps the most important one in the fact that the perimeter of a general Borel set can be thought as the limit of the perimeter of some appropriate poligonal sets (see Section 2 of Chapter 3), but with some further considerations which are the subject of Section 3 of the same Chapter. In particular a crucial role will be played by the *Steiner symmetrization*, in fact we will prove that the subset of $\mathbb{R}^N$ that (at parity of "volume") minimizes its "perimeter" must be (equivalent to) a rotation of its Steiner symmetrization with respect to an arbitrary hyperplane, hence must be symmetrical with respect to every hyperplane passing through its barycenter.

# Acknowledgments

I would like to thank all the people that supported me during the course of my studies. In particular I wish to extend my special thanks to four persons:

The first person I thank is my supervisor, prof. Monti Roberto, which gave me both the guidelines for writing this thesis and the freedom to make it an exceptional opportunity to challenge myself.

The second person is my girlfriend, Amada, which has patiently endured all of my monologues before every exam and has always stood by me, even in the toughest moments.

The third and fourth persons I thank are my parents, Chiara and Giuliano. They have, in fact, always supported all of my choices and there is no other person I owe my determination and work philosophy more than them.

# Chapter 1: Prelimiary Results and Notation

The role of this chapter is to briefly, but precisely, introduce all the fundamental notions necessary to understand the various concepts that will be introduced in the sequel.

Section 1.1 will present some basic definitions, properties and theorems of measures as well as the definition of integral in a general measure space and of a vector-valued measure on a measurable space.

In Section 1.2 we will discuss some elementary properties of the convolution of two functions, which will be used later on when giving De Giorgi's definition of perimeter of a set.

Lastly, in Section 1.3, it will be presented the notion of Hausdorff measure and it will be stated and proved the Gauss Green's Theorem.

## 1.1. Basic Measure and Integration Theory

**Definition 1.1:** Let $X$ be a nonempty set and $\mathscr{M}$ be a $\sigma$-algebra of subsets of $X$. A function $\mu : \mathscr{M} \to [0, \infty]$ is said to be a (positive) *measure on* $X$ if the following properties are satisfied:

(i) $\mu(\varnothing) = 0$.

(ii) Given a sequence of elements of $\mathscr{M}$, $(M_n)_{n\geq 1}$, such that $M_m \cap M_n = \varnothing$ for each $m \neq n$, we have $\mu\left(\sum_{n=1}^{\infty} M_n\right) = \sum_{n=1}^{\infty} \mu(M_n)$.

In this case we will call $(X, \mathscr{M})$ a *measurable space*, $(X, \mathscr{M}, \mu)$ a *measure space.*

**Definition 1.2:** Given a topological space $(X, \tau)$ we define the *Borel $\sigma$-algebra of* $(X, \tau)$ as the $\sigma$-algebra generated from $\tau$. This $\sigma$-algebra will be denoted as $\mathscr{B}_X$ and its elements will be called *Borel sets of* $X$.

**Definition 1.3:** A *Borel measure on* $(X, \tau)$ is a measure $\mu : \mathscr{M} \to [0, \infty]$ such that $\mathscr{B}_X \subseteq \mathscr{M}$. Moreover, if $\mu : \mathscr{M} \to [0, \infty]$ is a Borel measure on $(X, \tau)$, $\mu$ is said to be: *inner regular* if, for each $B \in \mathscr{B}_X$ we have

$$\mu(B) = \sup\left\{\mu(K) : K \subseteq B, K \text{ is compact}\right\}. \tag{1}$$

**Definition 1.4:** Let $\mu : \mathscr{M} \to [0, \infty]$ be Borel measure on $X$. $\mu$ is said to be a *Radon measure* if

(i) $\mu$ is inner-regular.

(ii) For each $K$ compact set $\mu(K)$ is finite.

**Definition 1.5:** The *Lebesgue measure on* $\mathbb{R}^N$, namely $\mathcal{L}^N$ (or simply $\mathcal{L}$), is the unique Borel measure such that

$$\mathcal{L}^N\left(\bigcup_{n=1}^{m}\left(\prod_{i=1}^{N}]a_n^{(i)},b_n^{(i)}]\right)\right):=\sum_{n=1}^{m}\left(\prod_{i=1}^{N}b_n^{(i)}-a_n^{(i)}\right). \tag{2}$$

Where the union in (2) is disjointed.

Let us now proceed with the construction of the integral on a general measure space.

**Definition 1.6:** Let $(X,\mathscr{M})$ be a measurable space. A function $f:X\to\mathbb{R}$ is said to be *a $\mathscr{M}$-measurable function* (or simply *a measurable function*, if $\mathscr{M}$ is understood) if for every Borel set of $\mathbb{R}$ we have that $f^{-1}(B)$ is a element of $\mathscr{M}$.

**Definition 1.7:** Let $(X,\mathscr{M})$ be a measurable space and $E$ be a measurable set. We define the *characteristic function of $E$* as the map $\chi_E:X\to\{0,1\}$ defined by the position

$$\chi_E(x):=\begin{cases}1, & \text{if } x\in E,\\ 0, & \text{if } x\notin E.\end{cases} \tag{3}$$

**Definition 1.8:** Let $(X,\mathscr{M})$ be a measurable space. A function $\varphi:X\to\mathbb{R}$ is said to be *a $\mathscr{M}$-simple function* (or *simple function*, if $\mathscr{M}$ is understood) if $\varphi$ is a finite linear combination of characteristic functions.

Obviously any characteristic function is measurable, and since sum of to measurable function and multiplication of a measurable function by a scalar (and multiplication of two measurable functions and the inverse of a never null measurable function) is still a measurable function, we have that any simple function is measurable. Furthermore, if $(\varphi_n)_{n\geq 1}$ is a sequence of measurable functions, then $\sup_n\varphi_n$, $\inf_n\varphi_n$, $\limsup_{n\to\infty}\varphi_n$, $\liminf_{n\to\infty}\varphi_n$ and, if there exists, $\lim_{n\to\infty}\varphi_n$ are measurable functions.

It is easy to see that any $\mathscr{M}$-simple function $\varphi$ can be written as linear combination of a finite number of characteristic functions of sets $E_1,\ldots,E_m$ with $E_i\cap E_j=\varnothing$ if $i\neq j$. In the sequel, when we write

$$\varphi=\sum_{n=1}^{m}c_i\chi_{E_n} \tag{4}$$

we always intend that $E_i\cap E_j=\varnothing$ if $i\neq j$.

**Definition 1.9:** Let $(X,\mathscr{M},\mu)$ be a measure space and $\varphi$ a simple function as in (4). We define the *integral of $\varphi$ on the set $X$ with respect to $\mu$* as

$$\int_X\varphi\,d\mu:=\sum_{n=1}^{m}c_i\mu(E_i) \tag{5}$$

If $Y\subseteq X$ is a measurable subset of $X$, then we define the *integral of $\varphi$ on the set $Y$ with respect to $\mu$* as the integral on the set $X$ with respect to $\mu$ of the function $\varphi\chi_Y$.

We can now define the integral for nonnegative measurable functions.

**Definition 1.10:** If $(X,\mathscr{M},\mu)$ is a measure space and $f:X\to[0,\infty]$ is a measurable function. We define the *integral of $f$ on the set $X$ with respect to $\mu$* as

$$\int_X f\,d\mu := \sup\left\{\int_X \varphi\,d\mu : 0 \le \varphi \le f,\ \text{with } \varphi\ \mathscr{M}\text{-simple function}\right\}. \tag{6}$$

Analogously as the previous definition, if $Y \subseteq X$ is measurable, the *integral of $f$ on $Y$ with respect to $\mu$* is the integral of $f\chi_Y$ on $X$ with respect to $\mu$.

Finally we can give the definition of integral in the general case.

**Definition 1.11:** If $(X, \mathscr{M}, \mu)$ is a measure space and $f : X \to \mathbb{R}$ is a measurable function such that the integral on $X$ with respect to $\mu$ of either $f^+$ or $f^-$is finite, where $f^+ := \max\{f, 0\}$ and $f^- := \max\{-f, 0\}$, then we define

$$\int_X f\,d\mu := \int_X f^+\,d\mu - \int_X f^-\,d\mu. \tag{7}$$

Given $Y \subseteq X$ measurable, the definition of the *integral of $f$ on $Y$ with respect to $\mu$* is analogous as in the case of $f$ nonnegative.

**Definition 1.12:** Let $(X, \mathscr{M}, \mu)$ be a measure space and $f, g$ be two measurable function. We say that $f$ is $\mu$-almost everywhere (or a.e. ) equal to $g$ and we write $f \overset{a.e.}{=} g$ if there exists $N \in \mathscr{M}$ such that $\mu(N) = 0$ and $f \equiv g$ on $N^c$. Let now $\sim$ be the equivalence relation on $\mathcal{M}$ defined by

$$f \sim g :\Longleftrightarrow f \overset{a.e.}{=} g, \tag{8}$$

then we denote as $\mathcal{M}'$ the quotient of $\mathcal{M}$ with respect to $\sim$.

**Definition 1.13:** Given a measure space $(X, \mathscr{M}, \mu)$, let $p \in [1, \infty[$. We define $\|\cdot\|_p$ and $L^p(X, \mathscr{M}, \mu)$ respectively as

$$\|f\|_p := \int_X |f|\,d\mu, \quad f \in \mathcal{M}, \quad L^p(X, \mathscr{M}, \mu) := \{[f]_\sim \in \mathcal{M}' : \|f\|_p < \infty\}. \tag{9}$$

We also define $\|\cdot\|_\infty$ and $L^\infty(X, \mathscr{M}, \mu)$ as

$$\|f\|_\infty := \inf\{a \ge 0 : \mu\{x \in X : |f(x)| > a\} = 0\}, \tag{10}$$

$$L^\infty(X, \mathscr{M}, \mu) := \{[f]_\sim \in \mathcal{M}' : \|f\|_\infty < \infty\} \tag{11}$$

With abuse of notation we will always write $f \in L^p(X, \mathscr{M}, \mu)$ instead of $[f]_\sim \in L^p(X, \mathscr{M}, \mu)$, for $p \in [1, \infty]$. If $(X, \mathscr{M}, \mu) = (\mathbb{R}^N, \mathscr{B}_{\mathbb{R}^N}, \mathcal{L}^N)$, we will always write $L^p(\mathbb{R}^N)$, or even $L^p$, instead of $L^p(\mathbb{R}^N, \mathscr{B}_{\mathbb{R}^N}, \mathcal{L}^N)$.

Let us now introduce a first generalization of a measure.

**Definition 1.14:** Let $(X, \mathscr{M})$ be a measurable space. A function $\mu : \mathscr{M} \to [-\infty, \infty]$ is said to be *real-valued measure on $X$* if the following conditions are satisfied:

(i) $\mu(\varnothing) = 0$.

(ii) Given a sequence of elements of $\mathscr{M}$, $(M_n)_{n\ge1}$, such that $M_m \cap M_n = \varnothing$ for each $m \ne n$, we have $\mu\left(\sum_{n=1}^\infty M_n\right) = \sum_{n=1}^\infty \mu(M_n)$.

In the sequel, in order to clarify when we are referring to measures and when to real-valued measures we will say *"(positive) measure"* instead of simply *"measure"*.

**Theorem 1.1:** *Let $(X, \mathscr{M})$ be a measurable space and $\mu$ a real-valued measure.*

*Then there exist two unique (positive) measures* $\mu^+, \mu^- : \mathscr{M} \to [0, \infty]$ *such that* $\mu = \mu^+ - \mu^-$ *and there exist* $E, F \in \mathscr{M}$ *such that* $E \cap F = \varnothing$, $E \cup F = X$, $\mu^+(E) = 0$ *and* $\mu^-(F) = 0$.

*Proof.* See [7, p. 87] □

**Definition 1.15:** With the notation used in Theorem 1.1 we say that $|\mu| := \mu^+ + \mu^-$ is the *total variation* of the real-valued measure $\mu$. Moreover we say that a real-valued measure $\mu$ is a *Radon measure* if its total variation $|\mu|$ is a (positive) Radon measure.

**Definition 1.16:** Let $(X, \mathscr{M})$ be a measurable space and let $N \in \mathbb{N}$ with $N \geq 2$. A function $\mu = (\mu^{(1)}, \dots, \mu^{(N)}) : \mathscr{M} \to \mathbb{R}^N$ is said to be a vector-valued measure on $X$ if $\mu^{(i)}$ is a real-valued measure. A vector-valued measure $\mu$ is said to be a *Radon measure* if each of its components is a Radon measure. Finally, when we write $|\mu|$ we intend the (positive) Radon measure on $\mathscr{B}(X)$ defined by the position

$$|\mu|(B) := \sup \left\{ \sum_{j \geq 1} |\mu(E_j)| : E_j \in \mathscr{B}(X), E = \bigcup_{j \geq 1} E_j, E_i \cap E_j = \varnothing \right\}. \tag{12}$$

$|\mu|$ is said to be *"total variation of* $\mu$*"* and, given $B \in \mathscr{B}(X)$, we say that $|\mu|(B)$ is the *"total variation of* $\mu$ *in* $B$*"*.

In Section 1 of Chatper 2 we will see that if $f : \mathbb{R}^N \to \mathbb{R}$ is a function of bounded variation (we will define what that means in the same section), then there is a naturally associated vector-valued Radon measure $\mu_F = (\mu_F^{(1)}, \dots, \mu_F^{(N)})$ such that we have

$$\int_{\mathbb{R}^N} f \, \nabla \cdot g \, d\mathcal{L}^N = \int_{\mathbb{R}^N} g \cdot d\mu_F \left( := \sum_{n=1}^{N} \int_{\mathbb{R}^N} g_i \, d\mu_F^{(i)} \right) \tag{13}$$

for each $g = (g_1, \dots g_N) \in C_c^1(\mathbb{R}^N; \mathbb{R}^N)$.

## 1.2. Convolution

**Definition 2.1:** Let $f, g$ be two real-valued measurable functions defined on $\mathbb{R}^N$. If both $f$ and $g$ are positive, or both $f$ and $g$ are elements of $L^1(\mathbb{R}^N)$, we define the real-valued function $f * g$ by

$$f * g(x) := \int_{\mathbb{R}^N} f(y) g(x - y) \, d\mathcal{L}(y). \tag{14}$$

The function $f * g$ we be called *convolution product* (or *convolution*) *of* $f$ *and* $g$.

In what follows, we shall need the fact that if $f$ is a measurable function on $\mathbb{R}^N$, then the function $K(x, y) := f(x - y)$ is a measurable function on $\mathbb{R}^N \times \mathbb{R}^N$. We have $K = f \circ s$, where $s(x, y) := x - y$; since $s$ is continuous, $K$ is a measurable function if $f$ is measurable.

Some basic properties of the convolution of two functions are the following.

**Proposition 2.1:** *Let* $f, g$ *and* $h$ *be real-valued measurable functions defined on* $\mathbb{R}^N$. *Assuming that all the convolution involved are defined, it holds:*

*(i)* $f * g = g * f$.
*(ii)* $(f * g) * h = f * (g * h)$.
*(iii)* *Let* $z \in \mathbb{R}^N$*. If we define the function* $\tau_z(x) := x - z$ *for all* $x \in \mathbb{R}^N$*, then* $(f \circ \tau_z) * g = f * (g \circ \tau_z) = (f * g) \circ \tau_z$.
*(iv)* $\mathrm{Supp}(f * g) \subseteq \overline{\{z + y \in \mathbb{R}^N : z \in \mathrm{Supp}(f),\, y \in \mathrm{Supp}(g)\}}$

*Proof.* See [7, p. 240]. □

Let us now see some more properties which link convolution product and $L^p$ spaces on $\mathbb{R}^N$.

**Theorem 2.1:** *Let* $p \in [1, \infty]$, $f \in L^1(\mathbb{R}^N)$ *and* $g \in L^p(\mathbb{R}^N)$*. Then*

*(i)* $f * g(x)$ *is defined for almost every* $x \in \mathbb{R}^N$.
*(ii)* $f * g \in L^p(\mathbb{R}^N)$.
*(iii)* $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.

*Proof.* Since, by definition of convolution and (i) of Proposition 2.1, we have

$$f * g(x) = \int_{\mathbb{R}^N} f(y) g(x - y)\, d\mathcal{L}(y), \tag{15}$$

it suffices to prove that the map

$$y \mapsto f(y) g(x - y) \tag{16}$$

is a $L^1(\mathbb{R}^N)$ function, for almost every $x \in \mathbb{R}^N$.

If $1 \leq p < \infty$ then let $\varphi : \mathbb{R}^N \to [0, \infty]$ be defined as follows

$$\varphi(x) := \int_{\mathbb{R}^N} |f(y) g(x - y)|\, d\mathcal{L}(y). \tag{17}$$

Then we have, using Minkowsky inequality,

$$\begin{aligned} \|\varphi\|_p &= \left( \int_{x \in \mathbb{R}^N} |\varphi(x)|^p \, d\mathcal{L}(x) \right)^{\frac{1}{p}} \\ &\leq \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} |f(y)|^p |g(x - y)|^p \, d\mathcal{L}(x) \right)^{\frac{1}{p}} d\mathcal{L}(y) \\ &= \int_{\mathbb{R}^N} |f(y)| \|g\|_p \, d\mathcal{L}(y) \\ &= \|f\|_1 \|g\|_p < \infty. \end{aligned} \tag{18}$$

Therefore $\varphi$ is a $L^p(\mathbb{R}^N)$ function. From known properties of $L^q(\mathbb{R}^N)$ spaces, we have that $|\varphi(x)| < \infty$ for almost every $x \in \mathbb{R}^N$. Since

$$|f * g(x)| = \left| \int_{\mathbb{R}^N} f(y) g(x - y)\, d\mathcal{L}(y) \right| \leq \varphi(x) \tag{19}$$

for every $x \in \mathbb{R}^N$, (i) is proven. Finally

$$\begin{aligned} \|f * g\|_p &= \left( \int_{\mathbb{R}^N} |f * g(x)|^p \, d\mathcal{L}(x) \right)^{\frac{1}{p}} \\ &\leq \left( \int_{\mathbb{R}^N} |f * g(x)|^p \, d\mathcal{L}(x) \right)^{\frac{1}{p}} = \|\varphi\|_p, \end{aligned} \tag{20}$$

hence, from (18) and (20), (ii) and (iii) follow.

If $p = \infty$, then it suffices to recall the Hölder inequality. □

One of the most remarkable properties of convolution is that, citing [7], *"$f * g$ is at least as smooth as either $f$ or $g$"*, because formally we have

$$\begin{aligned} \partial^\alpha (f * g)(x) &= \partial^\alpha \int_{\mathbb{R}^N} f(x-y)g(y)) \, d\mathcal{L}(y) \\ &= \int_{\mathbb{R}^N} \partial^\alpha f(x-y)g(y) \, d\mathcal{L}(y) \\ &= (\partial^\alpha f) * g, \end{aligned} \tag{21}$$

and similarly $\partial^\alpha (f * g) = f * (\partial^\alpha g)$, where $\alpha = (\alpha_1, \dots, \alpha_N)$ and $\partial^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}$. To make this more precise, one only needs to impose conditions on $f$ and $g$ so that differentiation under the integral sign is legitimate. One such result is the following.

**Proposition 2.2:** *If $f \in C^k(\mathbb{R}^N)$, $g \in L^1(\mathbb{R}^N)$ and $\partial^\alpha f$ is bounded for $|\alpha| < k$, then $f * g \in C^k$ and (21) holds.*

*Proof.* See [7, p. 242]. □

The following theorem underlines many of the important application of convolution on $\mathbb{R}^N$, in particular it is what will allow us to give De Giorgi's definition of perimeter of set in Chapter 2. We firstly introduce some notation: if $\varphi$ is any function on $\mathbb{R}^N$ and $t > 0$, we set

$$\varphi_t(x) := \frac{1}{t^N} \varphi\left(\frac{x}{t}\right) \tag{22}$$

we observe that if $\varphi \in L^1(\mathbb{R}^N)$, then the integral of $\varphi_t$ on $\mathbb{R}^N$ with respect to $\mathcal{L}$ is independent $t$ (one can easily see that by applying the theorem of change of variables in $\mathbb{R}^N$). Moreover, the "mass" of $\varphi_t$ becomes concentrated at the origin as $t \to 0^+$.

**Theorem 2.2:** *Suppose $\varphi \in L^1(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} \varphi \, d\mathcal{L} = a$, then:*

*(i) If $f \in L^p$, with $1 \leq p < \infty$, then $f * \varphi_t \to af$ in $L^p(\mathbb{R}^N)$ as $t \to 0^+$.*

*(ii) If $f$ is bounded and uniformly continuous, then $f * \varphi_t \to af$ uniformly as $t \to 0^+$.*

*Proof.* (i) Setting $y = tz$, we get

$$\begin{aligned} f * \varphi_t(x) - af(x) &= \int_{\mathbb{R}^N} (f(x-y) - f(x))\varphi_t(y)\, d\mathcal{L}(y) \\ &= \int_{\mathbb{R}^N} (f(x-tz) - f(x))\varphi(z)\, d\mathcal{L}(z) \\ &= \int_{\mathbb{R}^N} (f \circ \tau_{tz}(x) - f(x))\varphi(z)\, d\mathcal{L}(z), \end{aligned} \tag{23}$$

where $\tau_{tz}(x) := x - tz$. Applying Minkowsky inequality we get

$$\|f * \varphi_t - af\|_p \le \int_{\mathbb{R}^N} \|f \circ \tau_{tz} - f\|_p |\varphi(z)|\, d\mathcal{L}(z). \tag{24}$$

Now, $\|f \circ \tau_{tz} - f\|_p$ is bounded by $2\|f\|_p$ and tends to 0 as $t \to 0^+$ for each $z \in \mathbb{R}^N$, therefore the proof follows from the dominated convergence theorem.

(ii) The proof is exactly the same, with $\|\cdot\|_p$ replaced with $\|\cdot\|_\infty$. The estimate of $\|f * \varphi_t - af\|_\infty$ is obvious, and $\|f \circ \tau_{tz} - f\|_\infty \to 0$ as $t \to 0^+$ by the uniform continuity of $f$. □

## 1.3. Hausdorff Measures and Gauss Green's Theorem

In this section, when we say that $E$ is a subset of $\mathbb{R}^N$ we always mean that $E$ is a Borel subset of $\mathbb{R}^N$. Let us now introduce some terminology and definitions.

**Definition 3.1:** Let $N, k \in \mathbb{N}$, with $N \ge 2$ and $1 \le k \le N-1$. A bounded open set $A \subseteq \mathbb{R}^k$ and a function $f = (f^{(1)}, \dots, f^{(N)}) \in C^1(\mathbb{R}^k; \mathbb{R}^N)$ define a *$k$-dimensional parametrized surface $f(A)$ in $\mathbb{R}^N$*, provided $f$ is injective on $A$ with $Jf(x) > 0$ for every $x \in A$. Here $Jf(x)$ denotes

$$Jf(x) := \sqrt{\det(\nabla f(x)^T \nabla f(x))}, \tag{25}$$

where $\nabla f(x)$ is the matrix $(a_{i,j})$ such that $a_{i,j} = \frac{\partial}{\partial x_j} f^{(i)}$ for each $i \in \{1, \dots, N\}$, $j \in \{1, \dots, k\}$ and $\nabla f(x)^T$ indicates the transposed of $\nabla f(x)$.

**Definition 3.2:** The $k$-dimensional area of $f(A)$ is defined as

$$k\text{-dimensional area of } f(A) := \int_A Jf(x)\, d\mathcal{L}^k(x). \tag{26}$$

In the study of geometric variational problems we need to extend this definition of $k$-dimensional area to more general sets than $k$-dimensional $C^1$-images. Hausdorff measures are introduced to this end. To avoid the use of parametrizations the definition is based on a converging procedure.

**Definition 3.3:** Given $N, k \in \mathbb{N}$, $\delta > 0$, the *$k$-dimensional Hausdorff measure of step $\delta$ of a set $E \subseteq \mathbb{R}^N$* is defined as

$$\mathcal{H}^k_\delta(E) := \inf\left\{ \sum_{i=1}^\infty \frac{\omega_k}{2^k} \operatorname{diam}(F_i)^k : E \subseteq \bigcup_{i=1}^\infty F_i,\ \operatorname{diam}(F_i) < \delta \right\}, \tag{27}$$

where

$$\omega_k := \frac{\sqrt{\pi^k}}{\Gamma(1 + k/2)}, \tag{28}$$

where $\Gamma : ]0, \infty[ \to [1, \infty[$ is Euler's Gamma function. The *$k$-dimensional Hausdorff measure of $E$* is then

$$\mathcal{H}^k(E) := \sup \left\{ \mathcal{H}^k_\delta(E) : \delta \in ]0, \infty[ \right\} = \lim_{\delta \to 0^+} \mathcal{H}^k_\delta(E). \tag{29}$$

One can easily prove that $\mathcal{H}^k$ is in fact a measure Radon measure, is translation-invariant and it satisfies $\mathcal{H}^k(\lambda E) = \lambda^k \mathcal{H}^k(E)$, for each $\lambda > 0$. Moreover, it can (less easily) be proven that $\mathcal{H}^k(f(A))$ agrees with the classical notion of a $k$-dimensional parameterized surface $f(A)$ as defined in Definition 3.2. We can also extend the definition of $k$-dimensional Hausdorff measure as follows.

**Definition 3.4:** Given $s \in [0, \infty[$, the *$s$-dimensional Hausdorff measures $\mathcal{H}^s_\delta$ and $\mathcal{H}^s$* are defined by simply replacing $k$ with $s$ in Definition 3.3.

Lastly, one can prove that, if $s = k \in \mathbb{N}$, then the $k$-dimensional Hausdorff measure coincides with the $k$-dimensional Lebesgue measure, that is $\mathcal{H}^k(E) = \mathcal{L}^k(E)$, for each $E \in \mathscr{B}(\mathbb{R}^k)$. Since the proof of this fact is not relevant to the scope of this thesis, we will simply invite the reader to see [9].

Let us now introduce the following notation: as usual we will denote by $x$ the general point of $\mathbb{R}^N$ and by $x'$ the general point of $\mathbb{R}^{N-1}$; with a little abuse of notation we will write $x = (x', x_N)$; if $F : \mathbb{R}^{N-1} \to \mathbb{R}$ and $G : \mathbb{R}^N \to \mathbb{R}$ are functions, then we will use $F(x')$ and $G(x', x_N)$ instead of $F(x_1, \ldots, x_{N-1})$ and $G(x_1, \ldots, x_N)$ respectively; if $\Phi : \mathbb{R}^{N-1} \to \mathbb{R}$ is a function, and $\nabla\Phi$ is its gradient then we write $(\nabla\Phi)^2$ instead of $(\frac{\partial}{\partial x_1}\Phi)^2 + \cdots + (\frac{\partial}{\partial x_{N-1}}\Phi)^2$; finally, we define the *cylinder of center $x \in \mathbb{R}^N$ and radius* $r > 0$, as

$$\mathscr{C}(x, r) := \{y \in \mathbb{R}^N : |y' - x'| < r, |y_N - x_N| < r\}, \tag{30}$$

and the $N-1$-dimensional disc of center $x'$ and radius $r > 0$,

$$\mathscr{D}(x', r) := \{y' \in \mathbb{R}^{N-1} : |y' - x'| < r\}. \tag{31}$$

**Definition 3.5:** Let $E$ be an open subset of $\mathbb{R}^N$ and let $k \in \mathbb{N} \cup \{\infty\}$, $k \geq 1$. We say that $E$ has *$C^k$-boundary* (or *smooth boundary, if $k = \infty$*) if for every $x \in \partial E$ there exist $r > 0$ and $\psi \in C^k(B(x, r))$, where $B(x, r)$ denotes the open ball of $\mathbb{R}^N$ centered in $x$ with radius $r$, with $\nabla\psi(y) \neq 0$ for every $y \in B(x, r)$ and

$$B(x, r) \cap E = \{y \in B(x, r) : \psi(y) < 0\}, \tag{32}$$

$$B(x, r) \cap \partial E = \{y \in B(x, r) : \psi(y) = 0\}. \tag{33}$$

The *outer unit normal $\nu_E$* to $E$ is then defined locally as

$$\nu_E(y) := \frac{\nabla\psi(y)}{|\nabla\psi(y)|}, \quad \forall y \in B(x, r) \cap \partial E. \tag{34}$$

**Definition 3.6:** Given $f : \mathbb{R}^{N-1} \to \mathbb{R}$ and $G \subseteq \mathbb{R}^{N-1}$, we define the *graph of $f$*

*over* $G$ as

$$\mathscr{G}(f;G) := \left\{x \in \mathbb{R}^N : x_N = f(x'), x' \in G\right\}, \tag{35}$$

and we set for brevity $\mathscr{G}(f) := \mathscr{G}(f, \mathbb{R}^{N-1})$.

We are now stating a fundamental result for proving the Gauss-Green theorem.

**Lemma 3.1:** *If $f : \mathbb{R}^{N-1} \to \mathbb{R}$ is a Lipschitz function, then for every subset $G$ of $\mathbb{R}^{N-1}$,*

$$\mathcal{H}^{N-1}\left(\mathscr{G}(f,G)\right) = \int_G \sqrt{1 + (\nabla f)^2}\, d\mathcal{L}^{N-1}. \tag{36}$$

*In fact $\mathcal{H}^{N-1}_{|\mathscr{G}(f)}$ is a Radon measure on $\mathbb{R}^N$ and*

$$\int_{\mathscr{G}(f)} \varphi\, d\mathcal{H}^{N-1} = \int_{\mathbb{R}^{N-1}} \varphi(x', f(x'))\sqrt{1 + (\nabla f(x'))^2}\, d\mathcal{L}^{N-1}(x'), \tag{37}$$

*for every $\varphi \in C^0_c(\mathbb{R}^N)$.*

*Proof.* See [9, p. 89]. □

We observe that if $E$ is an open set of $\mathbb{R}^N$ with $C^1$-boundary, then the restriction of the $N-1$-dimensional Hausdorff measure to the boundary of $E$, namely $\mathcal{H}^{N-1}_{|\partial E}$, is a Radon measure on $\mathbb{R}^N$. Indeed, by the implicit function theorem, if $x \in \partial E$ and $r > 0$ is the same as in Definition 3.5, then there exist $s > 0$ and $f \in C^1(\mathscr{D}(x', s))$ such that $\mathscr{C}(x, s) \subseteq B(x, r)$ and, up to rotation,

$$\mathscr{C}(x, s) \cap E = \{y \in \mathbb{R}^N : y_N > f(x')\}, \tag{38}$$

$$\mathscr{C}(x, s) \cap \partial E = \{y \in \mathscr{C}(x, s) : y_N = f(x')\}. \tag{39}$$

Hence

$$\mathcal{H}^{N-1}_{|\mathscr{C}(x,s)\cap\partial E} = \mathcal{H}^{N-1}_{|\mathscr{G}(f,\mathscr{D}(x',s))}\ , \tag{40}$$

where the right-hand side defines a measure on $\mathbb{R}^N$ by Lemma 3.1. Starting from these considerations it is easily seen that $\mathcal{H}^{N-1}_{|\partial E}$ is a Radon measure on $\mathbb{R}^N$. Let us also notice that, having expressed $\mathscr{C}(x, s) \cap E$ as the epigraph of $f$ over $\mathscr{D}(x', s)$, by the chain rule we infer the following formula for the outer unit normal $\nu_E$ of $E$:

$$\nu_E(x) = \frac{(\nabla f(x'), -1)}{\sqrt{1 + (\nabla f(x'))^2}}, \tag{41}$$

for every $x \in \mathscr{C}(x, s) \cap \partial E$.

We can finally state the Gauss-Green theorem. We want to specify that, in the last part of the proof, we will use a standard argument based on partitions of unity. Since in this thesis we have not defined them, and neither we have stated nor proved their properties, we invite the reader to see [8].

**Theorem 3.1 (Gauss-Green theorem):** *If $E$ is an open subset of $\mathbb{R}^N$ with*

*$C^1$-boundary, then*

$$\int_E \nabla\varphi \, d\mathcal{L}^N = \int_{\partial E} \varphi\nu_E \, d\mathcal{H}^{N-1}, \tag{42}$$

*for every $\varphi \in C_c^1(\mathbb{R}^N)$. Equivalently it holds true*

$$\int_E \nabla \cdot g \, d\mathcal{L}^N = \int_{\partial E} g \cdot \nu_E \, d\mathcal{H}^{N-1}, \tag{43}$$

*for every $g \in C_c^1(\mathbb{R}^N; \mathbb{R}^N)$.*

*Proof.* The equivalence between the two statements is obvious.

Given $\tilde{x} \in \partial E$, up to rotation, we may consider $r, s > 0$ and $f$ as in the previous observations. We claim that

$$\int_E \nabla\varphi \, d\mathcal{L}^N = \int_{\partial E} \varphi\nu_E \, d\mathcal{H}^{N-1}, \quad \forall \varphi \in C_c^1(\mathscr{C}(\tilde{x}, s)). \tag{44}$$

Indeed, given $\varphi \in C_c^1(\mathscr{C}(\tilde{x}, s))$ and $\delta > 0$, we define the Lipschitz function $F_\delta : \mathscr{C}(x, s) \to \mathbb{R}$ by setting

$$F_\delta(x) := \begin{cases} 1, & \text{if } x_N > f(x') + \delta, \\ 0, & \text{if } x_N < f(x') - \delta, \\ \frac{x_N - f(x') + \delta}{2\delta}, & \text{if } |f(x') - x_N| < \delta. \end{cases} \tag{45}$$

Since $F_\delta \to \chi_{\mathscr{C}(x,s)}$ in $L^1(\mathscr{C}(\tilde{x}, s))$ as $\delta \to 0^+$, using the fact that a Lipschitz function admits weak gradient, we have

$$\begin{aligned} \int_E \nabla\varphi \, d\mathcal{L}^N &= \int_{E \cap \mathscr{C}(\tilde{x},s)} \nabla\varphi \, d\mathcal{L}^N \\ &= \lim_{\delta \to 0^+} \int_{\mathscr{C}(\tilde{x},s)} F_\delta \nabla\varphi \, d\mathcal{L}^N \\ &= -\lim_{\delta \to 0^+} \int_{\mathscr{C}(\tilde{x},s)} \varphi \nabla F_\delta \, d\mathcal{L}^N. \end{aligned} \tag{46}$$

Let us now set

$$\Phi_\delta := \{x \in \mathscr{C}(\tilde{x}, s) : |x_N - f(x')| < \delta\}, \tag{47}$$

And notice that $\Phi_\delta = \{x \in \mathscr{C}(\tilde{x}, s) : \nabla F_\delta(x) \neq 0\}$, with

$$\nabla F_\delta(x) = \frac{1}{2\delta} \left(-\nabla f(x'), 1\right), \quad \forall x \in \Phi_\delta. \tag{48}$$

By Fubini's theorem we have

$$\begin{aligned} &\int_{\mathscr{C}(\tilde{x},s)} \varphi \nabla F_\delta d\mathcal{L}^N = \int_{\Phi_\delta} \varphi(x) \nabla F_\delta(x) \, d\mathcal{L}^N(x) \\ &= \int_{\mathscr{D}(\tilde{x}',s)} (-\nabla f(x'), 1) \left( \frac{1}{2\delta} \int_{[f(x')-\delta, f(x')+\delta]} \varphi(x', t) \, d\mathcal{L}(t) \right) d\mathcal{L}^{N-1}(x'). \end{aligned} \tag{49}$$

By continuity, for every $x' \in \mathscr{D}(\tilde{x}', s)$,

$$\lim_{\delta \to 0^+} \frac{1}{2\delta} \int_{[f(x')-\delta, f(x')+\delta]} \varphi(x', t)\, d\mathcal{L}(t) = \varphi(x', f(x')). \tag{50}$$

Finally, by dominated convergence, (41) and Lemma 3.1

$$\begin{aligned}
-\lim_{\delta \to 0^+} \int_{\mathscr{C}(\tilde{x}, s)} \varphi \nabla F_\delta d\mathcal{L}^N &= -\int_{\mathscr{D}(\tilde{x}', s)} \varphi(x', f(x'))\, (-\nabla f(x'), 1)\, d\mathcal{L}^{N-1}(x') \\
&= \int_{\mathscr{D}(\tilde{x}', s)} \varphi(x', f(x'))\nu_E(x', f(x'))\sqrt{1 + (\nabla f(x'))^2}\, d\mathcal{L}^{N-1}(x') \\
&= \int_{\mathscr{C}(\tilde{x}, s)\cap \partial E} \varphi \nu_E\, d\mathcal{H}^{N-1} \\
&= \int_{\partial E} \varphi \nu_E\, d\mathcal{H}^{N-1}.
\end{aligned} \tag{51}$$

From (46), (51) and arbitrariness of $\varphi \in C^1_c(\mathscr{C}(\tilde{x}, s))$, (44) follows.

Let now $\varphi \in C^1_c(\mathbb{R}^N)$ be give, and let $A$ be an open subset of $\mathbb{R}^N$ such that

$$\operatorname{Supp}(\varphi) \cap \partial E \subseteq A. \tag{52}$$

By compactness, and thanks to what we have already proved, there exist finitely many points $x_1, \dots, x_M \in A \cap \partial E$ and finitely many open balls $B(x_1, s_1), \dots, B(x_M, s_M) \subseteq A$ such that, for every $\zeta \in C^1_c(B(x_k, s_k))$, with $1 \le k \le M$,

$$\operatorname{Supp}(\varphi) \cap \partial E \subseteq \bigcup_{k=1}^{M} B(x_k, s_k), \qquad \int_E \nabla(\varphi\zeta)\, d\mathcal{L}^N = \int_{\partial E} (\zeta\varphi)\nu_E\, d\mathcal{H}^{N-1}. \tag{53}$$

We now consider $\zeta_1, \dots, \zeta_M$ with $\zeta_k \in \mathscr{C}^1_c(B(x_k, s_k); [0,1])$ and

$$\sum_{k=1}^{M} \zeta_k(x) = 1, \quad \forall z \in A. \tag{54}$$

By known facts about partitions of unity, we can construct $\zeta_0 \in C^1_c(E; [0,1])$ such that

$$\sum_{k=0}^{M} \zeta_k(x) = 1, \quad \forall x \in E \cup A. \tag{55}$$

Since $\zeta_0 \varphi \in C^1_c(E)$, we have

$$0 = \int_{\mathbb{R}^N} \nabla(\zeta_0\varphi)\, d\mathcal{L}^N = \int_E \nabla(\zeta_0\varphi)\, d\mathcal{L}^N. \tag{56}$$

Hence, from (44), the last of (53) (54), (55) and (56), we obtain

$$\begin{aligned}\int_E \nabla\varphi \, d\mathcal{L}^N &= \sum_{k=0}^{M} \int_E \nabla(\zeta_k \varphi) \, d\mathcal{L}^N \\ &= \sum_{k=1}^{M} \int_{\partial E} (\zeta_k \varphi)\nu_E \, d\mathcal{H}^{N-1} \\ &= \int_{\partial E} \varphi \nu_E \, d\mathcal{H}^{N-1},\end{aligned} \tag{57}$$

therefore the theorem is proved. □

# Chapter 2: BV Functions and Perimeter of Sets

In this chapter we begin by giving the definition of a function of bounded variation defined in $\mathbb{R}^N$ and a characterization of those; then we will discuss about the definition of the perimeter of a given Borel set. In particular we will use the notion of $BV$ functions in order to give two distinct definitions: Caccioppoli's definition and one given from De Giorgi. We will also see how these are equivalent, but the latter allows to establish various hidden results concerning the notion of perimeter of a set.

In the following, by "a subset" we always mean "a Borel subset with orientable boundary", and by "a function" we always mean "a Borel measurable function". Furthermore, when we write "$g \in C_c^1(\mathbb{R}^N)$" we mean that $g$ is infinitesimal, together with its first order derivatives, of order not smaller than $|x|^{-(N+1)}$ as $|x| \to \infty$; analogously, writing "$g \in C_c^1(\mathbb{R}^N; \mathbb{R}^N)$" means that $g = (g^{(1)}, \dots, g^{(N)})$ with $g \in C_c^1(\mathbb{R}^N)$; on the other hand, if $E$ is a bounded subset of $\mathbb{R}^N$, then $C_c^1(E)$ is the space of real valued functions $g$ having compact support in $E$; analogously, writing "$g \in C_c^1(E; \mathbb{R}^N)$", with $E$ bounded, means that $g = (g^{(1)}, \dots, g^{(N)})$ with $g \in C_c^1(E)$.

## 2.1. BV Funtions of Several Variables

**Definition 1.1:** Let $U \subseteq \mathbb{R}^N$ be an open set and $f \in L^1(U)$. We say that $f$ has *bounded variation in $U$* if

$$V(f, U) := \sup \left\{ \int_U f \, \nabla \cdot g \, d\mathcal{L}^N \, : g \in C_c^1(U; \mathbb{R}^N), |g(x)| \leq 1, \forall x \in U \right\} < \infty \quad (1)$$

And we say that $V(f, U)$ is the total variation of $f$ in $U$. We write $BV(U)$ to denote the space of functions of bounded variation in $U$.

**Definition 1.2:** Let $U \subseteq \mathbb{R}^N$ be an open set and $f \in L^1_{\text{loc}}(U)$. We say that $f$ has *locally bounded variation in $U$* if for each open set $V \subset U$,

$$V(f, V) := \sup \left\{ \int_V f \, \nabla \cdot g \, d\mathcal{L}^N \, : g \in C_c^1(V; \mathbb{R}^N), |g(x)| \leq 1, \forall x \in V \right\} < \infty. \quad (2)$$

And we say that $V(f, U)$ is the total variation of $f$ in $V$. We write $BV_{\text{loc}}(U)$ to denote the space of functions of locally bounded variation in $U$.

From now on $U$ will denote an open set of $\mathbb{R}^N$ and $\mathscr{B}(U)$ will denote the Borel $\sigma$-algebra of $U$.

**Theorem 1.1:** *Let $f : U \to \mathbb{R}$ be a funtion. Then $f \in BV(U)$ if and only if there* *exists a (unique) finite vector-valued Radon measure* $\mu_f = (\mu_f^{(1)}, \dots, \mu_f^{(N)})$ *on* $\mathscr{B}(U)$ *such that*

$$\int_U f\,\nabla g\,d\mathcal{L}^N = -\int_U g\,d\mu_f \tag{3}$$

*for all* $g \in C_c^1(U)$.

*Proof.* See [6, 168]. □

Furthermore one can prove, using the Riesz's Representation theorem for measures that $|\mu_f|$ is a finite (positive) Radon measure and that $|\mu_f|(U)$ coincide with the total variation of $f$ in $U$, that is

$$\begin{aligned}|\mu_f|(U) &= \sup\left\{\int_U f\,\nabla\cdot g\,d\mathcal{L}^N : g \in C_c^1(U;\mathbb{R}^N), |g(x)| \le 1, \forall x \in U\right\}\\ &= V(f,U).\end{aligned} \tag{4}$$

## 2.2. Definition(s) for Perimeter of Sets

In this section we give two definitions for the $(N-1)$-dimensional measure of the oriented boundary of a Borel subset $E$ od $\mathbb{R}^N$.

We recall that in what follows, when we say that $E$ is a subset of $\mathbb{R}^N$, then we will always mean that $E \in \mathscr{B}(\mathbb{R}^N)$ and $\partial E$ will always be orientable; by $\chi_E$ we denote the characteristic function of $E$, that is the function defined on $\mathbb{R}^N$ defined by $x \mapsto 1$ if $x \in E$ and $x \mapsto 0$ if $x \notin E$; also, as in the previous section, $U$ will always denote an open set of $\mathbb{R}^N$.

First we give the Caccioppoli's (and most common in the textbooks) definition of perimeter, which will be called "C-perimeter", in order to distinguish it form the De Giorgi's one, simply denoted as "perimeter". At the end of the section we will show the equivalence between the two definitions and the reason why we will adopt the latter rather than the first one.

**Definition 2.1:** Let $E \subseteq \mathbb{R}^N$. The C-perimeter of $E$ in $U$ is defined as

$$\begin{aligned}P^{(C)}(E,U) &:= V(\chi_E,U)\\ &= \sup\left\{\int_E \nabla\cdot g\,d\mathcal{L}^N : g \in C_c^1(U;\mathbb{R}^N), |g(x)| \le 1, \forall x \in U\right\}\end{aligned} \tag{5}$$

If $P^{(C)}(E,U) < \infty$ we say that $E$ has *finite C-perimeter in* $U$. If $U = \mathbb{R}^N$ we write $P^{(C)}(E)$ instead of $P^{(C)}(E,\mathbb{R}^N)$ and, if $P^{(C)}(E) < \infty$, $E$ is said to have *finite C-perimeter*.

One can easily observe that if $E \subset \mathbb{R}^N$ has finite C-perimeter and $C^1$-class (piecewise) and oriented boundary, then, by the Gauss-Green's theorem (see Section 3 of Chapter 3) we have, for each $g \in C_c^1(\mathbb{R}^N;\mathbb{R}^N)$ with $|g(x)| \le 1$ for every $x \in \mathbb{R}^N$,

$$\int_E \nabla\cdot g\,d\mathcal{L}^N = \int_{\partial E} g\cdot\nu_e\,d\mathcal{H}^{N-1}, \tag{6}$$

where $\nu_e : \partial E \to \mathbb{R}^N$ indicates the outward-pointing normal vector field for each point of $\partial E$. Since $E$ has finite perimeter, then the member on the left-hand side of

(6) is finite, then every integral is well defined and, by Cauchy-Schwarz inequality ($v \cdot w \leq |x||y|$) we get

$$\int_E \nabla \cdot g \, d\mathcal{L}^N \leq \int_{\partial E} 1 \, d\mathcal{H}^{N-1} = \mathcal{H}^{N-1}(\partial E). \tag{7}$$

On the other hand, if $\psi = \nu_e$, the inequality in (7) becomes an equality. By well known theorems on the density, it follows that for every $\varepsilon > 0$ we can choose $g_\varepsilon \in C_c^1(\mathbb{R}^N; \mathbb{R}^N)$ such that $|g_\varepsilon(x)| \leq 1$ for each $x \in \mathbb{R}^N$ and

$$\left| \int_{\partial E} g_\varepsilon \cdot d\mathcal{H}^{N-1} - \int_{\partial E} \psi \cdot \nu_e \, d\mathcal{H}^{N-1} \right| < \varepsilon, \tag{8}$$

therefore

$$\left| \int_{\partial E} g_\varepsilon \cdot d\mathcal{H}^{N-1} - \mathcal{H}^{N-1}(\partial E) \right| < \varepsilon. \tag{9}$$

This proves that the just-given definition of C-perimeter of $E$ coincides with the $N-1$ dimensional measure of its boundary: exactly what its reasonable to expect from a “perimeter”.

Let’s now give De Griogi’s definition and analytical expression of the perimeter of a set in $\mathbb{R}^N$. First and foremost, for any bounded function $f : \mathbb{R}^N \to \mathbb{R}$ and $\lambda > 0$, we define $W_\lambda f : \mathbb{R}^N \to \mathbb{R}$ by

$$W_\lambda f(x) := \frac{1}{\pi^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-|\xi|^2} f(x + \lambda\xi) \, d\mathcal{L}^N(\xi). \tag{10}$$

From (10), by the dominated convergence theorem, we get

$$\lim_{\lambda \to 0^+} W_\lambda f(x) = f(x) \quad , \quad \forall x \in \mathbb{R}^N. \tag{11}$$

Then, by the theorem on the change of variables, one can easily see that

$$\begin{aligned} W_\lambda f(x) &= \frac{1}{\pi^{\frac{N}{2}}} \int_{\mathbb{R}^N} \frac{e^{-\frac{|x-\xi|^2}{\lambda^2}}}{\lambda^N} f(\xi) \, d\mathcal{L}^N(\xi) \\ &= \frac{1}{\pi^{\frac{N}{2}}} \int_{\mathbb{R}^N} \gamma_\lambda(x - \xi) f(\xi)) \, d\mathcal{L}^N(\xi) \\ &= \frac{1}{\pi^{\frac{N}{2}}} \gamma_\lambda * f(x), \end{aligned} \tag{12}$$

Where $\gamma(\eta) := e^{-|\eta|^2}$ and, if $\psi : \mathbb{R}^N \to \mathbb{R}$ is a function, then as in Section 2 of Chapter 1, $\psi_\lambda$ is defined by the position $x \mapsto \frac{1}{\lambda^N}\psi(x/\lambda)$. From what we have already proven in Chapter 1, noticing that $\int_{\mathbb{R}^N} \gamma \, d\mathcal{L}^N = \pi^{\frac{N}{2}}$, we deduce that

$$\int_{\mathbb{R}^N} |W_\lambda f| \, d\mathcal{L}^N \leq \int_{\mathbb{R}^N} |f| \, d\mathcal{L}^N, \tag{13}$$

$$\lim_{\lambda \to 0^+} \int_{\mathbb{R}^N} |W_\lambda f| \, d\mathcal{L}^N = \int_{\mathbb{R}^N} |f| \, d\mathcal{L}^N. \tag{14}$$

Moreover, if $E \subseteq \mathbb{R}^N$ is a bounded set, then

$$\lim_{\lambda \to 0^+} \int_E |W_\lambda f - f| \, d\mathcal{L}^N = 0 \tag{15}$$

In other words $W_\lambda f$ is to be thought as a $C^\infty(\mathbb{R}^N)$ approximation of $f$ which is bounded, as well as its derivatives of any order. If $f$ is continuous, bounded, differentiable with its first order partial derivatives, then

$$\frac{\partial}{\partial x_i} W_\lambda f = W_\lambda \frac{\partial}{\partial x_i} f, \quad i = 1, \ldots, N \tag{16}$$

Furthermore its easy to prove that the function $\lambda \mapsto \int_{\mathbb{R}^N} |\nabla W_\lambda \chi_E| \, d\mathcal{L}^N$ is monotonic. We are now ready to give the following.

**Definition 2.2:** Let $E \subseteq \mathbb{R}^N$. The perimeter of $E$ is defined as

$$P(E) := \lim_{\lambda \to 0^+} \int_{\mathbb{R}^N} |\, \nabla W_\lambda \chi_E| \, d\mathcal{L}^N. \tag{17}$$

We will say that $E$ has *finite perimeter* if $P(E) < \infty$.

**Definition 2.3:** Given a Radon measure $\mu_{(\lambda)}$ depending on a parameter $\lambda$, we will say that $\mu_{(\lambda)}$ *weakly converges* to a Radon measure $\mu$ for $\lambda \to \lambda_0$, if the equality

$$\lim_{\lambda \to \lambda_0} \int_{\mathbb{R}^N} g \, d\mu_{(\lambda)} = \int_{\mathbb{R}^N} g \, d\mu \tag{18}$$

holds for any $g \in C_c^0(\mathbb{R}^N)$. In that case we will write $\mu_{(\lambda)} \rightharpoonup \mu$, for $\lambda \to \lambda_0$.

The theorem that follows will demonstrate the equivalence between Definition 2.1 and Definition 2.2. In order to prove such theorem we need some auxiliary result

**Lemma 2.1:** *Any sequence $(\mu_n)_{n \geq 1}$ of vector-valued Radon measures which have equibounded total variation, admits a subsequence that weakly converges to a vector-valued Radon measure $\mu$ of bounded total variation.*

*Proof.* See [5, p. 61]. □

With this lemma we can prove

**Theorem 2.1:** *Given $E \subseteq \mathbb{R}^N$, if the perimeter $P(E)$ of $E$ is finite then there exists a (unique) vector-valued Radon measure $\mu = (\mu^{(1)}, \ldots, \mu^{(N)})$ of finite total variation such that*

$$\int_{\mathbb{R}^N} \chi_E \, \nabla g \, d\mathcal{L}^N = \int_{\mathbb{R}^N} g \, d\mu \tag{19}$$

*for every $g \in C_c^1(\mathbb{R}^N)$.*

*Viceversa, if there exists a vector-valued Radon measure $\mu$ of finite total variation such that* (19) *holds for every $g \in C_c^1(\mathbb{R}^N)$, then $P(E)$ is finite and*

$$P(E) = |\mu|(\mathbb{R}^N). \tag{20}$$

*Proof.* ( $\Longrightarrow$ ) Let us pick an arbitrary infinitesimal sequence $(\lambda_n)_{n \geq 1}$ with $\lambda_n > 0$ for every $n \geq 1$ and consider the sequence of vector-valued set functions $(\mu_n)_{n \geq 1}$

defined by

$$\mu_n(B) := -\int_B \nabla W_{\lambda_n}\chi_E \, d\mathcal{L}^N \quad , \quad \forall B \subseteq \mathscr{B}(\mathbb{R}^N), \forall n \geq 1. \tag{21}$$

By definition of $P(E)$, and from (13), we know that

$$|\mu_n|(B) \leq \int_{\mathbb{R}^N} |\nabla W_{\lambda_n}\chi_E| \, d\mathcal{L}^N \leq P(E) \quad , \quad \forall B \in \mathscr{B}(\mathbb{R}^N), \forall n \geq 1, \tag{22}$$

Therefore $(\mu_n)_{n\geq 1}$ is a sequence of vector-valued Radon measures which have equibounded total variation. Therefore there exist a subsequence $(\mu_{n_k})_{k\geq 1}$ of $(\mu_n)_{n\geq 1}$ and a vector-valued Radon measure $\mu$ of bounded total variation such that $\mu_{n_k} \rightharpoonup \mu$. Hence we get

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} d|\mu_n| \geq \int_{\mathbb{R}^N} d|\mu|. \tag{23}$$

Thus, as all the measures in $(\mu_n)_{n\geq 1}$ have total variation not larger than $P(E)$, it follows

$$|\mu|(\mathbb{R}^N) \leq P(E). \tag{24}$$

Let us now pick an arbitrary function $g \in C_c^1(\mathbb{R}^N;\mathbb{R}^N)$. We have

$$\int_{\mathbb{R}^N} g \, d\mu = \lim_{k\to\infty} \int_{\mathbb{R}^N} g \, d\mu_{n_k}. \tag{25}$$

Since $\chi_E$ is a bounded function, by the definition of the operator $W_\lambda$, it follows that the sequence $(W_{\lambda_n}\chi_E)_{n\geq 1}$ is bounded; recalling (15) and the fact that $\lambda_n \to 0^+$ as $n \to \infty$, we have

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} \left(W_{\lambda_{n_k}}\chi_E\right) \nabla g \, d\mathcal{L}^N = \int_{\mathbb{R}^N} \chi_E \nabla g \, d\mathcal{L}^N. \tag{26}$$

On the other hand, form (21) it follows, performing integration by parts,

$$\int_{\mathbb{R}^N} \left(W_{\lambda_{n_k}}\chi_E\right) \nabla g \, d\mathcal{L}^N = -\int_{\mathbb{R}^N} g \nabla W_{\lambda_{n_k}}\chi_E \, d\mathcal{L}^N = \int_{\mathbb{R}^N} g \, d\mu_{n_k}. \tag{27}$$

Finally, form (25), (26) and (27) it follows that

$$\int_{\mathbb{R}^N} \chi_E \nabla g \, d\mathcal{L}^N = \int_{\mathbb{R}^N} g \, d\mu, \tag{28}$$

therefore the proof of ( $\Longrightarrow$ ) is complete.

( $\Longleftarrow$ ) Since clearly

$$\nabla_x e^{-\frac{|x-\xi|^2}{\lambda^2}} = -\nabla_\xi e^{-\frac{|x-\xi|^2}{\lambda^2}}, \tag{29}$$

where $\nabla_x$ and $\nabla_\xi$ denote the gradients with respect to $x$ and $\xi$ respectively, we have

$$
\begin{aligned}
|\nabla W_\lambda \chi_E(\xi)| &= \frac{1}{\pi^{\frac{N}{2}}\lambda^N}\left|\int_{\mathbb{R}^N}\left(\nabla_\xi e^{-\frac{|x-\xi|^2}{\lambda^2}}\right)\chi_E(x)\,d\mathcal{L}^N(x)\right| \\
&= \frac{1}{\pi^{\frac{N}{2}}\lambda^N}\left|\int_{\mathbb{R}^N}\left(\nabla_x e^{-\frac{|x-\xi|^2}{\lambda^2}}\right)\chi_E(x)\,d\mathcal{L}^N(x)\right| \\
&= \frac{1}{\pi^{\frac{N}{2}}\lambda^N}\left|\int_{\mathbb{R}^N} e^{-\frac{|x-\xi|^2}{\lambda^2}}\,d\mu(x)\right| \le \frac{1}{\pi^{\frac{N}{2}}\lambda^N}\int_{\mathbb{R}^N} e^{-\frac{|x-\xi|^2}{\lambda^2}}\,d|\mu(x)|.
\end{aligned} \tag{30}
$$

From (30) it follows

$$
\begin{aligned}
\int_{\mathbb{R}^N}|\nabla W_\lambda\chi_E(\xi)|\,d\mathcal{L}^N(\xi) &\le \frac{1}{\pi^{\frac{N}{2}}\lambda^N}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} e^{-\frac{|x-\xi|^2}{\lambda^2}}\,d|\mu(x)|\right)d\mathcal{L}^N(\xi) \\
&= \frac{1}{\pi^{\frac{N}{2}}\lambda^N}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} e^{-\frac{|x-\xi|^2}{\lambda^2}}\,d\mathcal{L}^N(\xi)\right)d\mu(x)| \\
&= \int_{\mathbb{R}^N} d|\mu(x)| = |\mu|(\mathbb{R}^N),
\end{aligned} \tag{31}
$$

and therefore, by definition of $P(E)$, we have

$$
P(E) \le |\mu|(\mathbb{R}^N). \tag{32}
$$

Hence $P(E)$ is finite and the inequality (24) holds. From (24) and (32) we deduce (20). The proof of the theorem is complete. □

From Theorem 1.1 and Theorem 2.1 we deduce the previously announced equivalence of the two definitions. The main reason why we decide to adopt the De Giorgi's definition is because it provides an analytical expression of the perimeter of a set $E$ through a limit of a volume integral contaning a parameter. In particular the analytical expression of the perimeter allows to obtain, in a rather simple way, some fundamental results which seem not easy to get starting from the Caccioppoli's definition. Perhaps the most important one, that will we more precisely described in Chapter 3, is the following: *the perimeter of $E$ is the lower limit of the perimeter of the poliedral domains approximating $E$.*

# Chapter 3: Isoperimetric Property of the Hypersphere

In this Chapter we will prove the isoperimetric property of the hypersphere in arbitrary dimension, that is: *if $C$ is an hypersphere on $\mathbb{R}^N$, then for every $A \in \mathscr{B}(\mathbb{R}^N)$ satisfying the condition $\mathcal{L}^N(C) = \mathcal{L}^N(A)$, we have that $P(C) \leq P(A)$. In particular we have $P(C) = P(A)$ if and only if $A$ is (equivalent to) a hypersphere.*

The theorem will be proved to be a consequence of the results of Section 1 of this Chapter, where the notion of perimeter is as defined in Section 2 of Chapter 2.

Throughout this Chapter when we mention a set in $\mathbb{R}^N$ and a function defined on $\mathbb{R}^N$, we always mean a Borel set of $\mathbb{R}^N$ and a Borel-measurable function. Furthermore, when we write "$g \in C_c^1(\mathbb{R}^N)$" we mean that $g$ is infinitesimal, together with its first order derivatives, of order not smaller than $|x|^{-(N+1)}$ as $|x| \to \infty$; analogously, writing "$g \in C_c^1(\mathbb{R}^N; \mathbb{R}^N)$" means that $g = (g^{(1)}, \dots, g^{(N)})$ with $g \in C_c^1(\mathbb{R}^N)$; on the other hand, if $E$ is a bounded subset of $\mathbb{R}^N$, then $C_c^1(E)$ is the space of the real valued functions $g$ having compact support in $E$; analogously, writing "$g \in C_c^1(E; \mathbb{R}^N)$", with $E$ bounded, means that $g = (g^{(1)}, \dots, g^{(N)})$ with $g \in C_c^1(E)$. We denote by $\Sigma$ the metric space whose elements are the subsets of $\mathbb{R}^N$ and the distance between two subsets $E_1, E_2$ of $\mathbb{R}^N$ is given by $\mathcal{L}^N(E_1 \triangle E_2)$, where $E_1 \triangle E_2$ is the symmetric difference between $E_1$ and $E_2$; therefore, if $(E_n)_{n \geq 1}$ is a sequence of subsets of $\mathbb{R}^N$, whenever we say that $E_n$ converges to a set $E \subseteq \mathbb{R}^N$, we mean that for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that for every $n \geq N_\varepsilon$ we have $\mathcal{L}^N(E \triangle E_n) < \varepsilon$. Two sets $E_1$, $E_2$ are said to be equivalent if $\mathcal{L}^N(E_1 \triangle E_2) = 0$.

## 3.1. Sets of Finite Perimeter

**Definition 1.1:** Given a set $E \subseteq \mathbb{R}^N$ we define the Gauss-Green function (or measure) corresponding to the set $E$ the unique vector-valued Borel measure $\mu_E$ such that

$$\int_E \nabla g \, d\mathcal{L}^N = \int_{\mathbb{R}^N} g \, d\mu_E \tag{1}$$

for every $g \in C_c^1(\mathbb{R}^N; \mathbb{R}^N)$ and with total variation on $\mathbb{R}^N$ equal to the perimeter of $E$, that is

$$P(E) = |\mu_E|(\mathbb{R}^N) \left( =: \int_{\mathbb{R}^N} d|\mu_E| \right) \tag{2}$$

The existence (and uniqueness) of such $\mu_E$ is assured by Theorem 2.1 of Chapter 2.

**Lemma 1.1:** *Given a sequence $(E_n)_{n\geq 1}$ of subsets of $\mathbb{R}^N$ converging to a set $E \subseteq \mathbb{R}^N$, we have*

*(i)* $\liminf_{n\to\infty} P(E_n) \geq P(E)$.
*(ii) if $(P(E_n))_{n\geq 1}$ is bounded, the sequence of the Gauss-Green functions $(\mu_{E_n})_{n\geq 1}$ corresponding to the sets $(E_n)_{n\geq 1}$ weakly converges to the Gauss-Green function $\mu_E$ corresponding to $E$.*

*Proof.* By definition of distance between subsets, from the relation

$$\lim_{n\to\infty} E_n = E, \tag{3}$$

we get

$$\lim_{n\to\infty} \int_{E_n \triangle E} d\mathcal{L}^N = \lim_{n\to\infty} \int_{\mathbb{R}^N} |\chi_{E_n} - \chi_E| \, d\mathcal{L}^N = 0. \tag{4}$$

Therefore, recalling the definition of the operator $W_\lambda$, we have

$$\lim_{n\to\infty} \nabla W_\lambda \chi_{E_n}(x) = \nabla W_\lambda \chi_E(x) \tag{5}$$

at any point $x \in \mathbb{R}^N$ and for any value of $\lambda > 0$. From (5) it follows, recalling the definition of perimeter of a set, Fatou's lemma and the properties of $W_\lambda$ seen in Chapter 2,

$$\begin{aligned} \liminf_{n\to\infty} P(E_n) &\geq \liminf_{n\to\infty} \int_{\mathbb{R}^N} |\,\nabla W_\lambda \chi_{E_n}|\, d\mathcal{L}^N \\ &\geq \int_{\mathbb{R}^N} \lim_{n\to\infty} |\,\nabla W_\lambda \chi_{E_n}|\, d\mathcal{L}^N \\ &= \int_{\mathbb{R}^N} |\,\nabla W_\lambda \chi_E|\, d\mathcal{L}^N \end{aligned} \tag{6}$$

for any positive value of $\lambda$. Passing to the limit as $\lambda \to 0$ we obtain (i).

From (i) one can see that if $(P(E_n))_{n\geq 1}$ is bounded, then $P(E)$ is finite. In this case, by Theorem 2.1 of Chapter 2, if we denote by $M$ the supremum of $(P(E_n))_{n\geq 1}$, we have

$$|\mu_E|(\mathbb{R}^N) \leq M \quad , \quad |\mu_{E_n}|(\mathbb{R}^N) = P(E_n) \leq M \quad , \quad \forall n \geq 1. \tag{7}$$

Given $g \in C_c^0(\mathbb{R}^N;\mathbb{R}^N)$ and $\varepsilon > 0$ we can find $g_\varepsilon \in C_c^1(\mathbb{R}^N;\mathbb{R}^N)$ such that

$$\|g_\varepsilon - g\|_\infty < \varepsilon. \tag{8}$$

By Theorem 2.1 of Chapter 2, recalling (4),

$$\begin{aligned} \lim_{n\to\infty} \int_{\mathbb{R}^N} g_\varepsilon \, d\mu_{E_n} &= \lim_{n\to\infty} \int_{\mathbb{R}^N} \chi_{E_n} \, \nabla g_\varepsilon \, d\mathcal{L}^N \\ &= \int_{\mathbb{R}^N} \chi_E \, \nabla g_\varepsilon \, d\mathcal{L}^N \\ &= \int_{\mathbb{R}^N} g_\varepsilon \, d\mu_E. \end{aligned} \tag{9}$$

Therefore, taking into account (7), (8) and (9), we get

$$\limsup_{n\to\infty}\left|\int_{\mathbb{R}^N} g\,d\mu_E - \int_{\mathbb{R}^N} g\,d\mu_{E_n}\right| < 2M\varepsilon. \tag{10}$$

Since $\varepsilon$ is arbitrary, it follows

$$\lim_{n\to\infty}\left|\int_{\mathbb{R}^N} g\,d\mu_E - \int_{\mathbb{R}^N} g\,d\mu_{E_n}\right| = 0. \tag{11}$$

This proves (ii) and concludes. □

**Theorem 1.1:** *Let $(E_n)_{n\geq 1}$ be a sequence of subsets of $\mathbb{R}^N$ having equi-bounded perimeter and converging to a set $E$ satisfying*

$$\lim_{n\to\infty} P(E_n) = P(E). \tag{12}$$

*Let $\mu_E$ be the Gauss-Green function corresponding to $E$ and $\mu_{E_n}$ the Gauss-Green function corresponding to $E_n$ for each $n \geq 1$. Let us denote, as usual, $|\mu_E|$ and $|\mu_{E_n}|$ the total variation of $\mu_E$ and $\mu_{E_n}$ for $n \geq 1$ respectively. Then*

$$\lim_{n\to\infty} |\mu_{E_n}|(B) = |\mu_E|(B) \tag{13}$$

*for any set $B \subseteq \mathbb{R}^N$ such that $|\mu_E|(\partial B) = 0$.*

*Proof.* Given a set $B$ such that $|\mu_E|(\partial B) = 0$ and a number $\varepsilon > 0$, from known theorems we can find a function $g = (g^{(1)}, \dots, g^{(N)}) \in C_c^0(\mathbb{R}^N;\mathbb{R}^N)$ satisfying the following conditions:

$$|g(x)| \leq 1 \quad \forall x \in B; \quad |g(x)| = 0 \quad \forall x \notin B; \quad \int_{\mathbb{R}^N} g\cdot d\mu_E \geq |\mu_E|(B) - \varepsilon. \tag{14}$$

On the other and, by Lemma 1.1, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} g\cdot d\mu_{E_n} = \int_{\mathbb{R}^N} g\cdot d\mu_E. \tag{15}$$

Hence, since by $|g(x)| = 0$ for $x \notin B$ and by the very definition of $|\mu_{E_n}|$ it follows

$$\int_{\mathbb{R}^N} g\cdot d\mu_{E_n} \leq |\mu_{E_n}|(B), \tag{16}$$

we have

$$\liminf_{n\to\infty} |\mu_{E_n}|(B) \geq |\mu_E|(B) - \varepsilon. \tag{17}$$

As $\varepsilon$ is arbitrary, (17) can be replaced by

$$\liminf_{n\to\infty} |\mu_{E_n}|(B) \geq |\mu_E|(B). \tag{18}$$

Since $\partial B^c = \partial B$, by replacing $B$ with $B^c$ in (18) we also get

$$\liminf_{n\to\infty} |\mu_{E_n}|(B^c) \geq |\mu_E|(B^c). \tag{19}$$

From the additivity of $|\mu_E|$ and $|\mu_{E_n}|$, using (12), we get

$$\begin{aligned}\lim_{n\to\infty}(|\mu_{E_n}|(B)+|\mu_{E_n}|(B^c)) = \lim_{n\to\infty}|\mu_{E_n}|(\mathbb{R}^N) = \lim_{n\to\infty}P(E_n)\\ = P(E) = |\mu_E|(\mathbb{R}^N) = |\mu_E|(B)+|\mu_E|(B^c).\end{aligned} \tag{20}$$

From (18), (19) and (20), (13) follows. □

Given $E$, $B$ and $|\mu_E|$ as usual, we can intuitively interpret $|\mu_E|(B)$ as the "perimeter of $E$ in $B$"; indeed, for $E$ with $C^1$ boundary $|\mu_E|(B) = \mathcal{H}^{N-1}(\partial E\cap B)$. With this interpretation one can see easily the idea behind both the previous and the following theorem.

**Theorem 1.2:** *Given two finite perimeter subsets $E$, $F$ of $\mathbb{R}^N$, let $\mu_E$ and $\mu_F$ be their Gauss-Green functions, respectively. Let $A$ be an open subsets of $\mathbb{R}^N$ satisfying the condition*

$$E\cap A = F\cap A. \tag{21}$$

*Then for any $B\subseteq A$ we have*

$$\mu_E(B) = \mu_F(B). \tag{22}$$

*Proof.* Let $g$ be a function in $C^1_c(A)$. By (21) and the definition of Gauss-Green functions we have

$$\int_{\mathbb{R}^N} g\,d\mu_E = \int_E \nabla g\,d\mathcal{L}^N = \int_F \nabla g\,d\mathcal{L}^N = \int_{\mathbb{R}^N} g\,d\mu_F. \tag{23}$$

The equalities in (23) hold for every $g\in C^1_c(A)$. By known theorems on linear approximation of functions, it follows that the first term equals the last also under the assumption that $g$ is a bounded function identically 0 on $A^c$, in particular $g$ can be the characteristic function of $B$ and therefore the theorem is proved. □

We now give, without proving it, the following.

**Theorem 1.3:** *Given a subset $E$ of $\mathbb{R}^N$ (with $N\geq 2$), one of the following inequalities is always satisfied*
*(i)* $(\mathcal{L}^N(E))^{N-1}\leq (P(E))^N$.
*(ii)* $(\mathcal{L}^N(E^c))^{N-1}\leq (P(E))^N$.

*Proof.* See [5, p. 68:70] □

In particular, from Theorem 1.3, it follows that if $E$ has finite perimeter, then either $E$ or $E^c$ has finite measure.

## 3.2. Approximation by Polygonal Domains

**Definition 2.1:** A domain $\Pi\subseteq\mathbb{R}^N$ will be called *polygonal domain* if $\partial\Pi$ is contained in the union of a finite number of hyperplane of $\mathbb{R}^N$. We will denote by $\mathrm{Pol}(\mathbb{R}^N)$ (or simply Pol) the set of polygonal domains on $\mathbb{R}^N$.

If $N=2$ then the polygonal domains will be polygons, if $N=3$ they will be the polyhedra. If we now consider Pol as a subset of the space $\Sigma$, it is clear that it is dense in $\Sigma$ and, by Lemma 1.1, we have, given an arbitrary set $E \subseteq \mathbb{R}^N$,

$$\liminf_{\mathrm{Pol}\ni\Pi\to E} P(\Pi) \geq P(E). \tag{24}$$

The following result is more precise than the one given in (24).

**Theorem 2.1:** *Let $E$ be a subset of $\mathbb{R}^N$ (with $N \geq 2$). Then $P(E)$ equals the lower limit of the polygonal domains $\Pi$ approximating $E$, that is we have*

$$\liminf_{\mathrm{Pol}\ni\Pi\to E} P(\Pi) = P(E). \tag{25}$$

*Proof.* If $P(E)$ is infinite, from (24) we can conclude. On the other hand, if $P(E)$ is finite, by Theorem 1.3, it follows that either $E$ or $E^c$ has finite measure. Without loss of generality we can suppose $\mathcal{L}^N(E)$ to be finite; in this case $\chi_E$ is an integrable function and therefore, by the properties of the operator $W_\lambda$ we have

$$\lim_{\lambda\to 0^+} \int_{\mathbb{R}^N} |W_\lambda \chi_E - \chi_E| \, d\mathcal{L}^N = 0. \tag{26}$$

Hence, given $\varepsilon > 0$ we can certainly find $\lambda > 0$ such that

$$\int_{\mathbb{R}^N} |W_\lambda \chi_E - \chi_E| \, d\mathcal{L}^N < \varepsilon. \tag{27}$$

By the definition of $W_\lambda$, the function $| \nabla W_\lambda \chi_E|$ turns out to be bounded in $\mathbb{R}^N$ and therefore its supremum, denoted by $M$, is finite. Given $\eta$ such that $0 < \eta < \frac{1}{4}$, let us consider the set $L$ defined by

$$L := \{x \in \mathbb{R}^N : W_\lambda \chi_E(x) > \eta\}. \tag{28}$$

We will now prove that $L$ is a bounded subset of $\mathbb{R}^N$. For any $\rho > 0$ we define the set $I_\rho(L)$ as

$$I_\rho(L) := \{x \in \mathbb{R}^N : \mathrm{dist}(x, L) < \rho\}. \tag{29}$$

Let us set $\bar{\rho} := \frac{\eta}{2M}$. Clearly $L$ is contained in $I_{\bar{\rho}}(L)$ and, since

$$\| \nabla W_\lambda \chi_E\|_\infty < M, \tag{30}$$

at any point $x \in I_{\bar{\rho}}(L)$ we have

$$W_\lambda \chi_E(x) \geq \eta - M\bar{\rho} = \eta - \frac{M\eta}{2M} = \frac{\eta}{2}. \tag{31}$$

From (31) we deduce

$$\frac{\eta}{2}\mathcal{L}^N(I_{\bar{\rho}}(L)) \leq \int_{I_{\bar{\rho}}(L)} |W_\lambda \chi_E| \, d\mathcal{L}^N \leq \int_{\mathbb{R}^N} |W_\lambda \chi_E| \, d\mathcal{L}^N < \infty. \tag{32}$$

Since $\chi_E$ is integrable and (27) holds, we deduce that $\mathcal{L}^N(I_{\bar{\rho}}(L))$ is finite and therefore $L$ is bounded; hence we can always find a number $\alpha$ sufficiently large such that, if $T_\alpha$ denotes the domain

$$T_\alpha := \{x = (x^{(1)}, \dots, x^{(N)}) \in \mathbb{R}^N : |x^{(i)}| \leq \alpha, \ \forall i = 1, \dots, N\}, \tag{33}$$

the following formulas are all satisfied:

$$\int_{T_\alpha^c} \chi_E \, d\mathcal{L}^N < \varepsilon, \tag{34}$$

$$W_\lambda \chi_E(x) < \eta, \quad \forall x \in T_\alpha^c. \tag{35}$$

In the space $\mathbb{R}^{N+1}$, whose generic point is denoted by $(x^{(1)}, \dots, x^{(N)}, y) =: (x, y)$, let us consider the regular hypersurface $\Gamma_1$ defined by the equation

$$y = W_\lambda \chi_E(x), \quad x \in T_\alpha. \tag{36}$$

Since $W_\lambda \chi_E$ is a $C^1$ class function, we can certainly approximate the hypersurface $\Gamma_1$ with a hypersurface $\Gamma_2$ which is contained in the union of a finite number of hyperplanes and its represented by the equations

$$\begin{cases} y = g(x), \\ x \in T_\alpha \end{cases} \tag{37}$$

where $g$ is a continuous function satisfying the following conditions

$$0 < g(x) - W_\lambda \chi_E(x) < \eta, \quad \forall x \in T_\alpha, \tag{38}$$

$$\int_{T_\alpha} |g - W_\lambda \chi_E| \, d\mathcal{L}^N < \varepsilon, \tag{39}$$

$$\int_{T_\alpha} |\nabla g| \, d\mathcal{L}^N < \int_{T_\alpha} |\nabla W_\lambda \chi_E| \, d\mathcal{L}^N + \eta \quad (\leq P(E) + \eta). \tag{40}$$

Since clearly $W_\lambda \chi_E$ is always nonnegative, by (38) the function $g$ will be always positive, and therefore the set $D$ defined by

$$D := \{(x, y) \in \mathbb{R}^{N+1} : 0 \leq y \leq g(x)\} \tag{41}$$

will be a polygonal domain of $\mathbb{R}^{N+1}$. Taking into account (35) and (38), we see that $g(x) < 2\eta$ for any $x \in \partial T_\alpha$. It follows that for any real number $\theta \geq 2\eta$, the hyperplane $y = \theta$ intersects $\partial D$ only at points belonging to $\Gamma_2$. Let us denote by $\rho(\theta)$ the $(N-1)$-dimensional measure of the section of $\Gamma_2$ with the hyperplanw $y = \theta$ and let us indicate by $\Gamma_2^*$ the portion of $\Gamma_2$ contained in the half space $y \geq 2\eta$. Using elemetary theorems on the measure of the sections of a set (theorems that we can certainly apply to the hypersurface $\Gamma_2^*$ which, bein contained in $\Gamma_2$, is in turn contained in the union of a finite number of hyperplanes) we will have

$$\int_{\Gamma_2^*} |n_y| \, d\mathcal{H}^N = \int_{[2\eta, \infty[} \rho(\theta) \, d\mathcal{L}(\theta), \tag{42}$$

where $|n_y|$ indicates the length of the orthogonal projection of the unit normal vector to the hypersurface $y = \theta$. Recalling that $\Gamma_2$ has equations (37), we have

$$\int_{\Gamma_2^*} |v_y| \, d\mathcal{H}^N \leq \int_{\Gamma_2} |v_y| \, d\mathcal{H}^N = \int_{T_\alpha} |\nabla g(x)| \, d\mathcal{L}^N(x). \tag{43}$$

From (40), (42) and (43) it follows

$$\int_{[2\eta,\infty[} \rho(\theta)\, d\mathcal{L}(\theta) < P(E) + \eta \tag{44}$$

and therefore we have

$$\int_{[2\eta,1-\eta]} \rho(\theta)\, d\mathcal{L}(\theta) < P(E) + \eta. \tag{45}$$

Let us now consider, for any $\theta \geq 2\eta$, the section of the domain $D$ with the hyperplane $y = \theta$, which will be denoted by $\Pi(\theta)$. If we identify the hyperplane $y = \theta$ with the space $\mathbb{R}^N$ and therefore the generic point $(x, \theta)$ of such hyperplane with the point $x$, we find that, for almost every value of $\theta$, the set $\Pi(\theta)$ is a poligonal domain, provided it is nonempty. Clearly, we have

$$\begin{cases} g(x) \geq \theta, & \text{for } x \in \Pi(\theta) \\ g(x) < \theta, & \text{for } x \notin \Pi(\theta). \end{cases} \tag{46}$$

Since, for $\theta \geq 2\eta$, the hyperplane $y = \theta$ intersects $\partial D$ only at points belonging to $\Gamma_2$, for almost every value of $\theta$ between $2\eta$ and $1 - \eta$ (since we assumed $\eta < \frac{1}{4}$ these are an interval) the perimeter $P(\Pi(\theta))$ equals $\rho(\theta)$. Therefore there exists, by (45), a value $\bar{\theta}$ between $2\eta$ and $1 - \eta$, such that

$$P(\Pi(\bar{\theta})) = \rho(\bar{\theta}) < \frac{P(E) + \eta}{1 - 3\eta}, \tag{47}$$

being also $\Pi(\theta)$ a polygonal domain of $\mathbb{R}^N$. On the other hand, by (46) we have

$$\begin{cases} g(x) - \chi_E(x) = g(x) \geq \bar{\theta} > 2\eta > \eta, & \text{for } x \in \Pi(\bar{\theta}) \backslash (E \cap \Pi(\bar{\theta})) \\ \chi_E(x) - g(x) = 1 - g(x) \geq 1 - \bar{\theta} > \eta, & \text{for } x \in (E \cap T_\alpha) \backslash (E \cap \Pi(\bar{\theta})), \end{cases} \tag{48}$$

while, from (27) and (39), it follows

$$\int_{T_\alpha} |g - \chi_E|\, d\mathcal{L}^N \leq \int_{T_\alpha} |g - W_\lambda \chi_E|\, d\mathcal{L}^N + \int_{T_\alpha} |W_\lambda \chi_E - \chi_E|\, d\mathcal{L}^N < 2\varepsilon. \tag{49}$$

Hence, from (48) and (49), we get

$$\mathcal{L}^N((E \cap T_\alpha) \vartriangle \Pi(\theta)) < \frac{2\varepsilon}{\eta}. \tag{50}$$

Since by (34)

$$\mathcal{L}^N(E \backslash (T_\alpha \cap E)) < \varepsilon \tag{51}$$

we finally deduce

$$\mathcal{L}^N(E \vartriangle \Pi(\theta)) < \varepsilon + \frac{2\varepsilon}{\eta}. \tag{52}$$

Since the two numbers $\varepsilon$ and $\eta$ are arbitrary, formulas (47) and (52) ensure that

$$\liminf_{\mathrm{Pol}\ni\Pi\to E} P(\Pi) \leq P(E). \tag{53}$$

Therefore, recalling (24) we obtain (25) and the proof of the theorem is complete. □

## 3.3. De Giorgi's Theorem

($\star$) **Theorem (Isoperimetric Property of the Hypersphere):** *Let $C$ be a hypersphere on $\mathbb{R}^N$ (with $N \geq 2$), for every $A \in \mathscr{B}(\mathbb{R}^N)$ satisfying the condition*

$$\mathcal{L}^N(C) = \mathcal{L}^N(A), \tag{54}$$

*the following isoperimetric relation holds*

$$P(C) \leq P(A). \tag{55}$$

*In particular in* (55) *we have the equality if and only if $A$ is (equivalent to) a hypersphere.*

In order to prove ($\star$) we need the following results.

Let us recall the notation used in Section 3 of Chapter 1, that will make easier to state and prove the next theorem: as usual we will denote by $x$ the general point of $\mathbb{R}^N$ and by $x'$ the general point of $\mathbb{R}^{N-1}$; with a little abuse of notation we will write $x = (x', x_N)$; if $F : \mathbb{R}^{N-1} \to \mathbb{R}$ and $G : \mathbb{R}^N \to \mathbb{R}$ are functions, then we will use $F(x')$ and $G(x', x_N)$ instead of $F(x_1, \dots, x_{N-1})$ and $G(x_1, \dots, x_N)$ respectively. Finally, if $\Phi : \mathbb{R}^{N-1} \to \mathbb{R}$ is a function, and $\nabla\Phi$ is its gradient then we write $(\nabla\Phi)^2$ instead of $(\frac{\partial}{\partial x_1}\Phi)^2 + \cdots + (\frac{\partial}{\partial x_{N-1}}\Phi)^2$.

**Theorem 3.1:** *If $k$ is a real number and $f : T \subseteq \mathbb{R}^{N-1} \to \mathbb{R}$ $(N \geq 2)$ is a function defined on a bounded and connected domain $T$ of $\mathbb{R}^{N-1}$ which is uniformly Lipschitz continuous and $f(x') > k$ for every $x' \in \mathbb{R}^{N-1}$, then the domain $D \subseteq \mathbb{R}^N$ defined by*

$$D := \{(x', x_N) \in \mathbb{R}^N : x' \in T, k \leq x_N \leq f(x')\} \tag{56}$$

*has finite perimeter. Moreover, if $H$ is a set contained in $(T \backslash \partial T)$, $H^*$ is the subset of $\mathbb{R}^N$ defined by*

$$H^* := \{(x', x_N) \in \mathbb{R}^N : x' \in H, x_N = f(x')\}, \tag{57}$$

*and $|\mu_D|$ is the Gauss-Green function corresponding to the set $D$, we have*

$$|\mu_D|(H^*) = \int_H \sqrt{1 + (\nabla f)^2}\, d\mathcal{L}^{N-1}. \tag{58}$$

*Proof.* From Theorem 2.1 follows that $D$ can be approximated by a sequence $(D_n)_{n\geq 1}$ of polygonal domains, such that $D_n$ is defined by

$$D_n := \{x \in \mathbb{R}^N : x' \in T, k \leq x_N \leq f_n(x')\}, \quad \forall n \geq 1, \tag{59}$$

and where the functions $f_n$ are continuous in $T$, $f_n(x') > k$ for each $x' \in T$, for

$n \geq 1$, and satisfy the relations

$$f_n \to f \quad \text{uniformly}, \tag{60}$$

$$\lim_{n\to\infty} \int_T \left| \frac{\partial}{\partial x_h} f_n - \frac{\partial}{\partial x_h} f \right| d\mathcal{L}^{N-1} = 0, \quad h = 1, \dots, N-1. \tag{61}$$

Recalling that the perimeter of a polygonal domain coincides with the $(N-1)$-dimensional measure of its boundary and taking into account (60) and (61), its easy to see that $(P(D_n))_{n\geq 1}$ is bounded. Indeed we have

$$P(D_n) = \mathcal{L}^{N-1}(T) + \int_{\partial T} (f_n - k)\, d\mathcal{L}^{N-2} + \int_T \sqrt{1 + (\nabla f_n)^2}\, d\mathcal{L}^{N-1}. \tag{62}$$

Therefore, by Lemma 1.1, also $D$ has finite perimeter. Let us now consider a $C^1$-class function $g : \mathbb{R}^N \to \mathbb{R}$ such that $g(x', x_N) = 0$ for every $x' \in \partial T$. Since

$$\begin{aligned}\int_{D_n} \frac{\partial}{\partial x_N} g(x)\, d\mathcal{L}^N(x) \\ &= \int_T \left( \int_{\mathbb{R}} \frac{\partial}{\partial x_N} g(x', x_N) \chi_{[k, f_n(x')]}(x_N)\, d\mathcal{L}(x_N) \right) d\mathcal{L}^{N-1}(x') \\ &= \int_T (g(x', f_n(x')) - g(x', k))\, d\mathcal{L}^{N-1}(x')\end{aligned} \tag{63}$$

and

$$\begin{aligned}\int_{D_n} \frac{\partial}{\partial x_h} g(x)\, d\mathcal{L}^N(x) &= \int_{\partial D_n} g \nu_e^{(h)}\, d\mathcal{H}^{N-1} \\ &= \int_T g(x', f_n(x')) \frac{\partial}{\partial x_h} f_n(x')\, d\mathcal{L}^{N-1}(x'),\end{aligned} \tag{64}$$

for $h = 1, \dots, N-1$. Passing to the limit as $n \to \infty$ in (63) and (64), we get

$$\int_{\mathbb{R}^N} \frac{\partial}{\partial x_N} g(x) \chi_D(x)\, d\mathcal{L}^N(x) = \int_T (g(x', f_n(x')) - g(x', k))\, d\mathcal{L}^{N-1}(x') \tag{65}$$

$$\int_{\mathbb{R}^N} \frac{\partial}{\partial x_h} g(x) \chi_D(x)\, d\mathcal{L}^N(x) = \int_T g(x', f(x')) \frac{\partial}{\partial x_h} f(x')\, d\mathcal{L}^{N-1}(x'), \tag{66}$$

for $h = 1, \dots, N-1$. Therefore, if we denote by $\mu_D = (\mu_D^{(1)}, \dots, \mu_D^{(N)})$ the Gauss-Green function corresponding to $D$, by $V$ a subset of $H$, by $V^*$ the set

$$V^* := \{x \in \mathbb{R}^N : x' \in V, x_N = f(x')\}, \tag{67}$$

from (65) and (66) we deduce, taking into account the arbitrariness of $g$ and the definition of the Gauss-Green function,

$$\mu_D^{(N)}(V^*) = \int_V d\mathcal{L}^{N-1}(x') \tag{68}$$

$$\mu_D^{(h)}(V^*) = \int_V \frac{\partial}{\partial x_h} f(x')\, d\mathcal{L}^{N-1}(x'), \tag{69}$$

for $i = 1, \dots, N$. Recalling the definition the function of total variation of $\mu_D$, namely $|\mu_D|$, the formula is valid (58) for every $V \subseteq H$. This suffices to prove the theorem. □

In what follows we will prove a theorem on symmetric normal sets with respect to a hyperplane which, together with Theorem 2.1, has a crucial role in the proof of the isoperimetric property of the hypersphere stated in ($\star$). We begin with the definition of *"normal sets with respect to a hyperplane"*.

**Definition 3.1:** Given a set $E \subseteq \mathbb{R}^N$ (with $N \geq 2$) and a hyperplane $I$ of $\mathbb{R}^N$, we will say that $E$ is *pointwise normal with respect to $I$* if, given any orthogonal line to $I$, the intersection of such a line with $E$ is either a segment, or a point, or the empty set. We will say that $E$ is *normal in mean* (or simply *normal*) *with respect to $I$*, if $E$ is equivalent to a pointwise normal set.

Before proving the previously announced theorem on symmetric normal sets, it will be useful the following lemma.

**Lemma 3.1:** *Let $\alpha$ and $\gamma$ be two Radon measures defined on $\mathscr{B}(\mathbb{R}^N)$ with finite total variation. If*

$$\alpha(B) \cdot (\gamma(B) - \alpha(B)) \geq 0 \tag{70}$$

*for any set $B \subseteq \mathbb{R}^N$, then it also holds*

$$2|\gamma|(B)(|\gamma|(B) - |\alpha|(B)) \geq (|\gamma - \alpha|(B))^2 \tag{71}$$

*for any set $B \subseteq \mathbb{R}^N$.*

*Proof.* From (70) it follows

$$|\alpha(B)|^2 - \gamma(B) \cdot \alpha(B) \leq 0, \tag{72}$$

therefore we get

$$|\alpha(B)|^2 + |\gamma(B) - \alpha(B)|^2 = 2(|\alpha(B)|^2 - \gamma(B) \cdot \alpha(B)) + |\gamma(B)|^2 \leq |\gamma(B)|^2. \tag{73}$$

Let us define the functions $\varphi$ and $\psi$ as follows

$$\varphi := \frac{d\gamma}{d|\gamma|}, \quad \psi := \frac{d\alpha}{d|\gamma|}, \tag{74}$$

that is

$$\gamma(B) = \int_B \varphi \, d|\gamma|, \quad \alpha(B) = \int_B \psi \, d|\gamma| \tag{75}$$

for any set $B \subseteq \mathbb{R}^N$. Form (73), and known theorems on differentiation of Radon measures, we have

$$|\varphi(x)| = 1, \quad 1 - |\psi(x)|^2 = |\varphi(x)|^2 - |\psi(x)|^2 \geq |\varphi - \psi|^2, \quad |\psi(x)| \leq 1 \tag{76}$$

for any $x \in \mathbb{R}^N$, also

$$|\alpha|(B) = \int_B |\psi| \, d|\gamma|, \quad |\gamma - \alpha|(B) = \int_B |\varphi - \psi| \, d|\gamma|. \tag{77}$$

From (76) it follows

$$
\begin{aligned}
2(|\gamma|(B) - |\alpha|(B)) &= 2\int_B (1-|\psi|)\, d|\gamma| \\
&\geq \int_B (1-|\psi|^2)\, d|\gamma| \\
&\geq \int_B |\varphi-\psi|^2\, d|\gamma|,
\end{aligned} \tag{78}
$$

and by Schwarz inequality we have

$$
\left(\int_B |\varphi-\psi|^2\, d|\gamma|\right)\int_B d|\gamma| \geq \left(\int_B |\varphi-\psi|\, d|\gamma|\right)^2 = (|\gamma-\alpha|(B))^2. \tag{79}
$$

Recalling the definition of $|\gamma|(B)$, form (78) and (79) we get (71). □

We are now ready for proving the stated theorem concerning symmetric normal sets with respect to a hyperplane. We recall the notation given before Theorem 3.1.

**Theorem 3.2:** *Let $E$ be a subsets of $\mathbb{R}^N$ (with $N \geq 2$) having finite perimeter and finite measure. For any point $y := (y_1, \ldots, y_{N-1}) \in \mathbb{R}^{N-1}$ we denote by $f(y) := f(y_1, \ldots, y_{N-1})$ the linear measure of the intersection of $E$ with the line $R(y)$ of $\mathbb{R}^N$ (whose generic point will be denoted by $x$) having equations $x' = y$. Let $L$ be the set defined by*

$$
L := \{(x', x_N) \in \mathbb{R}^N : -f(x') < 2x_N < f(x')\}. \tag{80}
$$

*Then*

$$
P(L) \leq P(E). \tag{81}
$$

*If equality holds in* (81)*, then $E$ is normal with respect to the hyperplane $x_N = 0$ and, denoting by $|\mu_E|$ and by $|\mu_L|$ the sets function of total variation of the Gauss-Green functions corresponding to $E$ and $L$ respectively, we have, for any $M \subseteq \mathbb{R}^{N-1}$,*

$$
|\mu_E|(M \times \mathbb{R}) = |\mu_L|(M \times \mathbb{R}). \tag{82}
$$

*Proof.* By Theorem 2.1 there exists a sequence $(E_n)_{n\geq 1}$ of polygonal domains satisfying the conditions

$$
\lim_{n\to\infty} \mathcal{L}^N(E_n \triangle E) = 0, \quad \lim_{n\to\infty} P(E_n) = P(E). \tag{83}
$$

Moreover, we can suppose that, for any integer $n \geq 1$, the normal to the boundary of $E_n$ is never parallel to the hyperplane $x_N = 0$. This is surely possible since, if for some value of $n$ this assumption is not satisfied, it can always be achieved by performing an arbitrarily small rotation of the domain $E_n$.

Let us introduce the following notation: $D_n$ is the polygonal domain of the space $\mathbb{R}^{N-1}$ consisting of all the points $y$ such that $R(y)$ has nonempty intersetion with $E_n$, $f_n(y)$ is the linear measure of the intersection, and $L_n$ is the polygonal domain of $\mathbb{R}^N$ defined by

$$
L_n := \{(x', x_N) \in \mathbb{R}^N : x' \in D_n, -f_n(x') \leq 2x_N \leq f_n(x')\}. \tag{84}
$$

Taking into account the first condition in (31) and the definition of $L$ and $L_n$, one can immediately verify that

$$\lim_{n\to\infty} \mathcal{L}^N(L_n \triangle L) = 0. \tag{85}$$

Let us now fix a value of the index $n$ and let us denote by $\mu_{E_n} := (\mu_{E_n}^{(1)}, \dots, \mu_{E_n}^{(N)})$ the Gauss-Green function corresponding to $E_n$, by $\mu_{L_n} := (\mu_{L_n}^{(1)}, \dots, \mu_{L_n}^{(N)})$ the Gauss-Green function corresponding to $L_n$, by $|\mu_{E_n}|$ the total variation function of $\mu_{E_n}$ and by $|\mu_{L_n}|$ the total variation function of $\mu_{L_n}$. In addition, for any set $M \subseteq \mathbb{R}^{N-1}$, let $\gamma := (\gamma^{(1)}, \dots, \gamma^{(N)})$, where $\gamma^{(h)}(M)$ is the total variation of $\mu_{E_n}^{(h)}(M \times \mathbb{R})$, and $\alpha := (\alpha^{(1)}, \dots, \alpha^{(N)})$, where $\alpha^{(h)}(M)$ is the total variation of $\mu_{L_n}^{(h)}(M \times \mathbb{R})$.

In order to obtain some properties of the functions $\alpha_n$ and $\gamma_n$, which will be useful in the sequel, let us begin by observing that, since $E_n$ is a polygonal domain of $\mathbb{R}^N$ and since the normal to $\partial E_n$ is never parallel to the hyperplane $x_N = 0$, $f_n$ is continuous in $D_n$, $f_n(\xi) = 0$ for any $\xi \in \partial D_n$ and $D_n$ can be decomposed in a finite number of polygonal domains $G_1, \dots, G_m$ having the properties:

a) For any positive integer $k \le m$, the number of points where the line $R(y)$ intersects $\partial E_n$ is constant with respect to $y$ in the interior of $G_k$.
b) Denoting by $p(k)$ such a number (which is even and $\ge 2$) and letting $g_{k,1}(y)$, ... ,$g_{k,p(k)}(y)$ be the $N$-th coordinate of the $p(k)$ points belonging to $R(y) \cap \partial E_n$ (considered in increasing order), the functions $g_{k,1}, \dots, g_{k,p(k)}$ are Liptschitz contnuous in the interior of $G_k$.

Using properties a), b) and recalling the definition of $f_n$, one can immediatly see the following inequalities

$$\left|\frac{\partial}{\partial y_h} f_n(y)\right| \le \sum_{l=1}^{p(k)} \left|\frac{\partial}{\partial y_h} g_{k,l}(y)\right|, \quad \text{for } y \in G_k \backslash \partial G_k, \quad h = 1, \dots, N-1 \tag{86}$$

holds. Recalling that if $P$ is a polygonal domain in $\mathbb{R}^N$ and $\mu_P := (\mu_P^{(1)}, \dots, \mu_P^{(N)})$ is the Gauss-Green function corresponding to $P$, then

$$\mu_P^{(h)}(B) = \int_{B\cap\partial P} C^{(h)}\, d\mathcal{H}^{N-1}, \quad \text{for } h = 1, \dots, N, \tag{87}$$

where $C^{(h)}$ is the cosine formed by the outer normal to $\partial P$ with the $x_h$-th axis; and taking into account the fact that the (outer) normal to $\partial E_n$ is never parallel to the hyperplane $x_N = 0$, we find the following expressions for the functions $\alpha^{(h)}$, $\gamma^{(h)}$, $|\mu_{E_n}|$ and $|\mu_{L_n}|$:

$$\gamma^{(h)}(M) = \sum_{k=1}^{m} \int_{M\cap G_k} \sum_{l=1}^{p(k)} \left|\frac{\partial}{\partial y_h} g_{k,l}\right| d\mathcal{L}^{N-1} \quad \text{for } h = 1, \dots, N-1, \tag{88}$$

$$\alpha^{(h)}(M) = \sum_{k=1}^{m} \int_{M\cap G_k} \left|\frac{\partial}{\partial y_h} f_n\right| d\mathcal{L}^{N-1} \quad \text{for } h = 1, \dots, N-1, \tag{89}$$

$$\gamma^{(N)} = \sum_{k=1}^{m} p(k)\mathcal{L}^{N-1}(M \cap G_k), \quad \alpha^{(N)} = \sum_{k=1}^{m} 2\mathcal{L}^{N-1}(M \cap G_k), \tag{90}$$

$$|\mu_{E_n}|(M \times \mathbb{R}) = \sum_{k=1}^{m} \int_{M \cap G_k} \sum_{l=1}^{p(k)} \sqrt{1 + (\nabla g_{k,l})^2}\, d\mathcal{L}^{N-1} \tag{91}$$

$$|\mu_{L_n}|(M \times \mathbb{R}) = \sum_{k=1}^{m} \int_{M \cap G_k} \sqrt{4 + (\nabla f_n)^2}\, d\mathcal{L}^{N-1}. \tag{92}$$

If we denote (as usual) $|\alpha|$ and $|\gamma|$ as the total variation functions of the vector-valued Radon measures $\alpha$ and $\gamma$ respectively, from (88), (90) and (91) it follows

$$|\mu_{E_n}|(M \times \mathbb{R}) \geq |\gamma|(M), \tag{93}$$

and from (89), (90) and (92) it follows

$$|\mu_{L_n}|(M \times \mathbb{R}) = |\alpha|(M). \tag{94}$$

From (86), (88), (89) and (90) we deduce the relations

$$0 \leq \alpha^{(h)}(M) \leq \gamma^{(h)}(M) \quad \text{for } h = 1, \dots, N, \quad M \subseteq \mathbb{R}^{N-1}, \tag{95}$$

and we see that, denoting by $H_n$ the set of all the points $y \in \mathbb{R}^{N-1}$ such that the line $R(y)$ meets $\partial E_n$ at more than two points, we always have

$$0 \leq \alpha^{(N)}(M) \leq \gamma^{(N)}(M) - 2\mathcal{L}^{N-1}(H_n \cap M). \tag{96}$$

From (95) we deduce the relation

$$\alpha(M) \cdot (\gamma(M) - \alpha(M)) = \sum_{h=1}^{N} \alpha^{(h)}(M)(\gamma^{(h)}(M) - \alpha^{(h)}(M)) \geq 0, \tag{97}$$

while from (96) it follows

$$|\gamma - \alpha|(M) \geq |\gamma(M) - \alpha(M)| \geq 2\mathcal{L}^{N-1}(H_n \cap M). \tag{98}$$

From Lemma 3.1, (93), (94) and (97) we have

$$|\mu_{E_n}|(M \times \mathbb{R}) \geq |\mu_{L_n}|(M \times \mathbb{R}) \tag{99}$$

and, recalling (98),

$$|\mu_{E_n}|(M \times \mathbb{R}) \left(|\mu_{E_n}|(M \times \mathbb{R}) - |\mu_{L_n}|(M \times \mathbb{R})\right) \geq 2 \left(\mathcal{L}^{N-1}(H_n \cap M)\right)^2. \tag{100}$$

In particular, if $M = \mathbb{R}^{N-1}$, from the definition of perimeter, inequalities (99) and (100) become

$$P(E_n) = |\mu_{E_n}|(\mathbb{R}^N) \geq P(L_n) = |\mu_{L_n}|(\mathbb{R}^N) = P(L_n) \tag{101}$$

$$P(E_n)(P(E_n) - P(L_n)) \geq 2 \left(\mathcal{L}^{N-1}(H_n)\right)^2. \tag{102}$$

From (101), passing to the limit as $n \to \infty$ and taking (83), (85) and Lemma 1.1 into account, inequality (81) follows. Moreover, from (102) we see that, in order that equality holds in (81), the following relations must be simultaneously satisfied:

$$\lim_{n\to\infty} \mathcal{L}^{N-1}(H_n) = 0, \quad \lim_{n\to\infty} P(L_n) = P(L). \tag{103}$$

Recalling the definition of $H_n$, from (83) and the first of (103) we see that $E$ must be normal with respect to the hyperplane $x_N = 0$. On the other hand, from (83), (85) and the first of (103) we deduce, recalling also Theorem 1.1,

$$\lim_{n\to\infty} |\mu_{E_n}|(M \times \mathbb{R}) = |\mu_E|(M \times \mathbb{R}), \quad \lim_{n\to\infty} |\mu_{L_n}|(M \times \mathbb{R}) = |\mu_L|(M \times \mathbb{R}), \tag{104}$$

for any set $M \subseteq \mathbb{R}^{N-1}$ which satisfies the conditions:

$$|\mu_E|(\partial(M \times \mathbb{R})) = 0, \quad |\mu_L|(\partial(M \times \mathbb{R})) = 0. \tag{105}$$

Therefore, recalling (99),

$$|\mu_E|(M \times \mathbb{R}) \geq |\mu_L|(M \times \mathbb{R}). \tag{106}$$

Since (106) is valid for any $M \subseteq \mathbb{R}^{N-1}$ satisfying (105), it is also verified for any $M \subseteq \mathbb{R}^{N-1}$.

On the other hand, if equality holds in (81), by Lemma 1.1, we get

$$|\mu_E|(\mathbb{R}^N) = |\mu_L|(\mathbb{R}^N), \tag{107}$$

and therefore, by additivity of the functions $|\mu_E|$, $|\mu_L|$, inequality in (106) can be identically verified only if (82) is satisfied. □

We are finally ready to prove ($\star$)

*Proof of* ($\star$). Let $C$ be a hypersphere of $\mathbb{R}^N$ and $A$ a subset of $\mathbb{R}^N$ with

$$\mathcal{L}^N(C) = \mathcal{L}^N(A). \tag{108}$$

In view of Theorem 2.1, in order to prove that

$$P(C) \leq P(A) \tag{109}$$

it is enough to show that, given an arbitrary polygonal domain $\Pi$ of finite measure, and a hypersphere satisfying the condition

$$\mathcal{L}^N(C) = \mathcal{L}^N(\Pi), \tag{110}$$

it results

$$P(C) \leq P(\Pi). \tag{111}$$

To this aim let us observe that, as a polygonal domain $\Pi$ has finite measure, it is necessarily bounded and therefore there exists a hypersphere $C^*$ centered at the origin of the coordinates and of radius large enough which contains $\Pi$. If we denote by $\Gamma$ the class

$$\Gamma := \left\{ B \subseteq \mathbb{R}^N : \mathcal{L}^N(B) = \mathcal{L}^N(\Pi),\ P(B) \leq P(\Pi),\ B \subseteq C^* \right\}, \tag{112}$$

from Lemma 1.1 and the fact that given a bounded set $L$ and a positive number $l$, the class of all subsets of $L$ having perimeter less or equal to $l$ is compact in

$\big(\mathscr{B}(\mathbb{R}^N), \mathcal{L}^N(\cdot \vartriangle \cdot)\big)$ (see [4]), it follows that the functional $P(\cdot)$ has a minimizer in the class $\Gamma$. Let us indicate by $E$ one of these sets of minimal perimeter, and let us compare it with the set $L$, defined starting from the set $E$ and whose construction is described in Theorem 3.2. It is easy to realize that $L$ still belongs to the class $\Gamma$ and therefore, taking into account the minimality of $E$ and Theorem 3.2 we get

$$P(L) = P(E). \tag{113}$$

From (113), using again Theorem 3.2, we deduce that $E$ is normal with respect to the hyperplane $x_N = 0$. On the other hand, the minimality property of $E$ is still valid for any set obtained by rotating $E$ around the origin, hence $E$ is normal with respect to any hyperplane of $\mathbb{R}^N$, that is $E$ is (equivalent to) a convex set.

Since perimeters of two equivalent sets coincide, we can also suppose $E$ is a convex which can be represented as

$$E := \left\{(x', x_N) \in \mathbb{R}^N : f_1(x') \le x_N \le f_2(x'),\ x' \in D\right\}, \tag{114}$$

where, by well known properties of convex sets, $D$ is a convex domain of $\mathbb{R}^{N-1}$, and $f_1$, $f_2$ are two functions defined in $D$, which are uniformly Lipshitz continuous in any connected set $T \subseteq D \backslash \partial D$.

If $E$ is represented by (114), the set $L$ constructed via the procedure in Theorem 3.2 will be represented as

$$L := \left\{(x', x_N) \in \mathbb{R}^N : 2|x_N| < f_2(x') - f_1(x'),\ x' \in D\right\}. \tag{115}$$

Let us denote by $|\mu_E|$ and $|\mu_L|$ the total variations of the Gauss-Green functions corresponding to $E$ and $L$ respectively. Let $T \subseteq D \backslash \partial D$ be a connected subset, and let $M \subseteq T \backslash \partial T$; by Theorem 1.1 and Theorem 3.1, we have

$$|\mu_E|(M \times \mathbb{R}) = \int_M \left(\sqrt{1 + (\nabla f_1)^2} + \sqrt{1 + (\nabla f_2)^2}\right) d\mathcal{L}^{N-1}, \tag{116}$$

$$|\mu_L|(M \times \mathbb{R}) = \int_M \sqrt{4 + (\nabla (f_2 - f_1))^2}\, d\mathcal{L}^{N-1}. \tag{117}$$

From Theorem 3.2 and using (113) it follows

$$|\mu_E|(M \times \mathbb{R}) = |\mu_L|(M \times \mathbb{R}). \tag{118}$$

Equalities in (116), (117) and (118) can be simultaneously satisfied only if at almost every point of $M$ we have

$$\frac{\partial}{\partial y_1} f_1 + \frac{\partial}{\partial y_1} f_2 = \ldots = \frac{\partial}{\partial y_{N-1}} f_1 + \frac{\partial}{\partial y_{N-1}} f_2 = 0. \tag{119}$$

From arbitrariness of $T$ and $M$, formula (119) are then satisfied at almost every point of the convex domain $D$ and therefore the sum $f_1 + f_2$ is constant in $D$, that is the domain $E$ is symmetric with respect to the hyperplane $x_N = 0$.

Taking into account once again that the minimality property of $E$ holds for any set obtained from $E$ through a rotation around the origin, we conclude that $E$ is normal and symmetric with respect to all hyperplanes passing through the barycenter, and therefore $E$ is a hypersphere. As $E$ has minimal perimeter in the class $\Gamma$ and since $\Pi$ belongs to $\Gamma$, (111) is proved.

From (111), as we have already observed, (109) follows. To prove that if the equality holds in (109) then $A$ is (equivalent to) a hypersphere, it is enough to observe that, in this case, $A$ has minimal perimeter in the class of all sets having the same measure of $C$ and hence the argument considered above for $E$ can be repeated for $A$. □

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