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Corso di Laurea Magistrale in Fisica

Tesi di Laurea

Non perturbative instabilities of Anti-de Sitter solutions to M-theory

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Anno Accademico 2017/2018

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Chapter 1

Introduction

There are several reasons why studying the stability properties of non supersymmetric Anti-de Sitter (AdS) spaces deserves special interest. First, AdS geometries naturally emerge in the context of string theory compactifications, here including M-theory. General stability arguments can be drawn in the presence of supersymmetry, but the question is non trivial and still open for the non supersymmetric case. However, it has been recently conjectured by Ooguri and Vafa [1] that all non supersymmetric AdS vacua supported by fluxes are actually unstable. It is hard to find counterexamples to their conjecture, since whenever stability is proved at the perturbative level, by excluding the presence of negative modes that violate the Breitenlohner-Freedman bound in the spectrum, there may still exist non perturbative mechanisms leading to instabilities. As a matter of fact, there have been many proposals of non supersymmetric AdS vacua, but none of them has been demonstrated to be fully stable against any possible decay mechanism.

Secondly, AdS provides the specific background of gravitational theories in the most basic but also best understood examples of the AdS/CFT correspondence, a conjectured duality between theories of quantum gravity in $d+1$ dimensions and quantum field theories without gravity living on the d dimensional boundary. This can be regarded as one of the finest achievements of string theory in the last decade, with consequences possibly going far beyond string theory itself, since on the one hand it allows to compute quantum effects in a strongly coupled field theory using a classical gravitational theory, and on the other hand it allows to tackle hard quantum gravity problems within a more standard quantum field theory framework. However, the conjecture by Ooguri and Vafa seems to put into question the holographic interpretation of non supersymmetric AdS backgrounds as far as ordinary theories of gravity, that is with a finite number of massless fields, are concerned. In other words, even though conformal field theories without supersymmetry are known to exist, they would not admit a holographic dual of this type if the conjecture turned out to be true, thus providing a more fundamental reason to the lack of explicit examples of non supersymmetric AdS holography which is usually attributed to technical difficulties. This would have a big impact on applications of the correspondence where the interesting models are those without supersymmetry, such as those to condensed matter.

At this point, it is natural to ask why such a radical sounding claim about non supersymmetric AdS vacua being unstable should be true. In order to better appreciate the general ideas behind it, let us take a step back and introduce the so-called *swampland* program that has become fairly popular among the string theory community in recent years and still represents an active area of research. At present, we regard string theory as the most promising unifying framework where quantum theory and Einstein's gravity can be reconciled. Furthermore, the remarkably rich web of dualities between the various string theories suggests that the underlying theory is essentially unique. In sharp contrast, but not in contradiction, with this uniqueness is the existence of a vast *landscape* of string vacua, labelled by different choices of the compactification parameters and describing inequivalent physics, possibly very different from that of our observed universe. This is sometimes referred to as the vacuum selection problem. Huge numbers of order 10^{100}, or even 10^{272000}, have appeared in recent attempts to make estimates on the statistics of string vacua. Does it mean that everything is possible in string theory? Or equivalently, that any consistent looking effective field theory of quantum gravity admits a string theory UV completion? It has been argued that the answer to both questions is *no* and that not all of these effective field theories can be consistently completed in the UV. In this picture, the string *landscape* is surrounded by an even vaster sea of consistent looking, but ultimately inconsistent effective field theories which resist an embedding in string theory and are dubbed the *swampland*. In other words, we should expect the UV imprint of quantum gravity to manifest itself at low energies, whence additional contraints on effective field theories are assumed. Up to now, a number of criteria have been proposed to distinguish the *landscape* from the *swampland*. They are mainly motivated by black hole physics and supported by known examples from string theory.

One such *swampland* criterion is the Weak Gravity Conjecture (WGC), which was first formulated in [2] and is based on the observation that gravity must always be the *weakest* force. The basic statement of the WGC postulates the existence of a light charged particle whose mass, in appropriate units, is less or equal than its charge,

$$M \leq Q. \tag{1.1}$$

It is natural to generalize the WGC to extended objects charged under $(p+1)$-form potentials, called p-branes. That is, there should exist branes whose tension (energy per unit volume) is less or equal than their charge density. In appropriate units,

$$T \leq Q. \tag{1.2}$$

In the aforementioned paper by Ooguri and Vafa [1], a stronger version of the WGC is proposed, where the WGC bound is saturated if and only if the theory is supersymmetric and the states are BPS. In particular, the strict inequality holds in the non supersymmetric case. Here comes the connection with AdS instabilities, that can be understood as follows. The standard way in which AdS geometry is obtained in holography is to put a large number of branes next to each other in string theory and then take a near horizon limit. If the branes are BPS, they can be moved around and made closer to each other without any cost in energy, since the gravitational attraction and the gauge repulsion exactly compensate each other. In the non supersymmetric case, due to the stronger WGC there exist branes with tension lower than the charge. The gauge repulsion wins over the gravitational attraction, so the branes repel and fly apart, let alone can be made coincident! It is interesting to observe how the strict WGC bound enters in some earlier descriptions of brane nucleation in AdS [3], [4]. In this process, the nucleated brane expands and reaches the AdS boundary in a finite time, thus reducing the flux of the geometry.

Brane nucleation is an example of a non perturbative mechanism leading to AdS instabilities, but not the only one. Indeed, it can be cast in the more general context of the semi-classical decay of vacua. It can be described as mediated by an instanton, a solution of the Euclidean equations of motion interpolating between the initial and final states. The subsequent evolution can be obtained by the analytic continuation of the instanton and the leading approximation to the decay rate is simply given by e^{-S}, where S is the instanton action. These are all distinctive features of instanton methods, whose application in a field theory context was pioneered by Coleman et al. in a series of papers from the 1970s, namely [5] and [6] for the general theory of false vacuum decay in flat space and [7] for the inclusion of gravity. The simplest example where vacuum decay can be seen at work is in that of a single scalar field ϕ in Minkowski space, with a potential $V(\phi)$ exhibiting two relative minima at different energies. The lower minimum ϕ_T corresponds the the unique ground state of the quantum theory and is called a *true* vacuum, as opposed to the *false* vacuum sitting at the higher minimum ϕ_F, which should be regarded as a metastable state. Suppose to start with a homogeneous configuration where the scalar field is everywhere at ϕ_F. Due to quantum effects, a region of approximate true vacuum, a *bubble* with some finite radius R, can form in the false vacuum background. Then the bubble expands, converting the false vacuum into true. This picture can be not only quantitatively, but also qualitatively different when gravity is included. Indeed, gravitational effects can make the radius R of the bubble at nucleation larger or smaller, and accordingly the decay rate is suppressed or enhanced. In particular, they can stabilize the false vacuum.

An interesting semi-classical decay mode is provided by Witten's *bubble of nothing*, which demonstrates the instability of the Kaluza-Klein vacuum $M_4 \times S^1$ [8]. This space is known to be perturbatively stable against small fluctuations, since there are no tachyonic negative modes in the spectrum. Therefore, it is interesting to look for non perturbative instabilities. In Witten's analysis, an instanton is constructed by analytical continuation of the five dimensional Schwarzschild metric, which displays the correct asymptotically flat behaviour at infinity. Furthermore, the regularity of the solution requires the Schwarzschild radius to coincide with the radius of the Kaluza-Klein circle R. The analytical continuation of the instanton back to Minkowski space describes an expanding bubble. The difference with conventional vacuum decay is that now there is litterally *nothing* inside the bubble. This is possible thanks to the fifth compact direction, which plays a crucial role in the whole construction. Indeed, the Kaluza-Klein circle shrinks to zero at the bubble, thus allowing the geometry to terminate smoothly. However, there is a way to save the Kaluza-Klein vacuum, by including fermions in the theory and choosing periodic boundary conditions for the fermions on the Kaluza-Klein circle. Since a collapsing circle is the boundary of a disk, which only admits anti-periodic fermions, the instanton cannot carry the same spin structure as the Kaluza-Klein vacuum and Witten's decay mode is forbidden. Interestingly enough, the periodic choice is exactly the one which is required by supersymmetry.

We argue that generalizations of Witten's *bubble of nothing* are worth studying in the context of M-theory and string theory compactifications to AdS spaces. They should be regarded as a natural decay mechanism, since in these compactifications there tipically appear non trivial internal geometries, some subspace of which may collapse in analogy with the Kaluza-Klein circle in Witten's example. Therefore, generalized *bubbles of nothing* can describe new interesting decay modes and thus provide further evidence to the conjecture by Ooguri and Vafa. In the present work, we focus on M-theory and its low energy limit, which is eleven dimensional supergravity. We discover that generalized bubble geometries can be remarkably rich and discuss the new ingredients that can possibly come into play. An obvious one is related to the bosonic field content of the theory, which includes a three-form gauge potential A_3 with four-form field strength $F_4 = dA_3$, as well as the metric. In fact, the presence of a four-form flux supporting the vacuum geometry requires some care. We devote special attention to AdS compactifications to four dimensions, of which two main classes are known: Freund-Rubin solutions, where the four-form is taken to be proportional to the volume form of the external space, and Englert-type solutions, where an additional internal flux is turned on. Difficulties connected with fluxes were already pointed out in an earlier attempt to construct generalized bubble geometries by Young [9]. Her no-go argument applies whenever the four-form obeys the Freund-Rubin ansatz and the internal space we want to collapse has the geometry of a N dimensional sphere. If we assume that the geometry terminates smoothly at the bubble, a solution can only exist if $N = 1$, that is a circle. There are at least two reasons why this result is worth discussing. Firstly, a purely external flux is not the most general case, since the four-form can also have components along the internal space, as in Englert solutions. Secondly, the internal geometry can be more complicated, with only a subspace of the compact space, possibly fibered over the rest, that collapses at the bubble. The key role played by a non trivial fibration structure is apparent in the $AdS_5 \times \mathrm{C}P_3$ vacuum geometry, which admits a decay mode mediated by an instanton where a two dimensional sphere rather than a circle shrinks to zero size [10]. Here the flux is purely internal and reorients itself at the bubble in such a way that it has no components along the two dimensional collapsing fiber. To the best of our knowledge, this is the only example of an M-theory *bubble of nothing* discussed in the literature. However, it is also quite peculiar since there is no external flux supporting the vacuum geometry.

Here the most original part of our work begins. Motivated by the analysis of Polchinski et al. who discuss a similar problem in the context of type IIB supergravity [11], we present a general argument that seems to prevent smooth bubble geometries to exist in the presence of external fluxes. It is known that both Freund-Rubin and Englert vacua admit a non zero Page charge, which is defined as an integral over the seven dimensional internal manifold,

$$\int_{M_7} \left(*F_4 + A_3 \wedge F_4\right). \tag{1.3}$$

This quantity is conserved, in the sense that it does not depend on the external coordinates. Now, in a smooth bubble geometry the compact space can be seen as the boundary of an eight dimensional manifold and we can use Stokes' theorem,

$$\int_{M_7=\partial M_8} \left(*F_4 + A_3 \wedge F_4\right) = \int_{M_8} d\left(*F_4 + A_3 \wedge F_4\right) = 0, \tag{1.4}$$

where in the last passage we have used the Maxwell equation for the four-form. In other words, what this equation is telling us is that the Page charge of a smooth bubble geometry must be zero! In particular, such geometry does not have the same boundary conditions as the vacuum, whose Page charge is not zero. This is a serious obstruction indeed. However, a promising way-out is provided by the introduction of M2-brane instantons wrapping the three-sphere in AdS and smeared over some part of the internal space, which can account for the additional flux we need.

As a final step in our exploration of non perturbative AdS instabilities, we focus on a concrete example where all of the ingredients we have introduced, namely the presence of both external and internal fluxes, the possibility of higher dimensional spheres that are non trivially fibered over some base in the internal space to collapse at the bubble and the need to introduce M2-branes instantons, come into play. We consider a specific class of four dimensional AdS compactifications where the seven dimensional internal space is a tri-Sasakian manifold. We regard them as a natural candidate to study, since they are known to admit both Freund-Rubin and Englert vacua. Moreover, tri-Sasakian spaces have the universal fibration structure of a three dimensional sphere S^3 over a four dimensional base, and one could ask whether the S^3 can collapse giving rise to a higher dimensional *bubble of nothing*. They also come equipped with a set of forms, the tri-Sasakian structure, that allows us to write a general ansatz for both the metric and the four-form. We study the near-bubble behaviour of the field equations and verify that a smooth solution which asymptotically matches the vacuum cannot exist, as expected from our general argument based on Page charges. Finally, we discuss the inclusion of branes as a possible way-out.

Outline The work is organized as follows. In Chapter 1, we introduce the *swampland* program and describe the Weak Gravity Conjecture (WGC) as an example of an additional quantum gravity constraint for an effective field theory to admit a consistent UV completion. We also state the stronger WGC by Ooguri and Vafa and discuss its implications regarding the instability of non supersymmetric AdS spaces. In Chapter 2, we first introduce instanton methods in the simpler context of quantum mechanics and then present the theory of false vacuum decay. Finally, we focus on Witten's construction of the *bubble of nothing* and work it out in some detail. In Chapter 3, we are ready to set up our analysis of non perturbative instabilities in the context of M-theory compactifications. The main ingredients for the construction of a generalized *bubble of nothing* are introduced and some known examples are discussed. We also present our general argument based on Page charges. In Chapter 4, we turn to our concrete tri-Sasakian example. In Chapter 5, we draw our conclusions and suggest some avenues for future work.

Chapter 2

Motivation

In this Chapter, we motivate our interest in non supersymmetric AdS vacua, describing the general context where the claim about their instability stems from and discussing some of its implications. First, we introduce the concept of the *swampland*, that is the idea that the *landscape* of string vacua should be thought as an island surrounded by a sea of consistent looking, but ultimately unconsistent, low energy effective theories which cannot be completed in the UV to a full quantum theory of gravity. Identifying the properties any effective field theory of gravity should possess in order to admit a consistent UV completion then becomes a primary task in the exploration of the *swampland*, which presently represents an active, if not exploding area of research.

After that, we specialize to a specific *swampland* criterion which has attracted much attention in the last decade and whose far-reaching consequences have been explored both in particle physics and inflationary cosmology. The basic statement involves a bound on the mass of a particle by its charge. For a pair of such particles, the gravitational attraction is won by the repulsive force associated to the charge, whence the name of Weak Gravity Conjecture (WGC). We present some of the original motivations for the conjecture, expecially those coming from black hole physics, since the existence of these light charged particles is required in order for extremal black holes to decay, and explain how it intertwines with other *swampland* criteria, such as the absence of global symmetries in a consistent quantum theory of gravity.

Finally, we focus on a stronger version of the conjecture, where the WGC bound is allowed to be saturated only by BPS states in a supersymmetric theory. This might seem quite an innocuous extension of the WGC, but turns out to have dramatic consequences for the stability properties of non supersymmetric AdS spaces. If true, this stronger conjecture would imply that all non supersymmetric AdS geometries supported by fluxes are unstable, or equivalently that non supersymmetric AdS/CFT holography belongs to the *swampland*. This might be seen as follows. Holography is usually obtained by putting branes next to each other in string theory and then taking the near horizon limit. If the branes are BPS, there is no problem in doing this, since the repulsive force between the branes is exactly canceled out by the gravitational attraction. If instead the branes are not supersymmetric, the repulsion wins over attraction due to WGC and there is no way to prevent them from flying apart. This obstruction to non supersymmetric AdS/CFT holography would be relevant for applications of the correspondence to many areas of theoretical physics, as far as non supersymmetric models are concerned.

2.1 The landscape and the swampland

In the string theory construction of low dimensional effective field theories there is a very large number of choices to make, from the compactification manifold to background fluxes and branes. This has led to the idea of an extremely vast *landscape* of string vacua, each leading to a different would-be universe. How to pick the right one is then a relevant question, usually referred to as the vacuum selection problem. Some counting techniques are discussed for example in [12] and a nice overall picture of how many vacua string theory has, as well as a statistical description of their properties can be found in [13]. For us, suffice it to say that an often quoted guess is of order 10^{100}. Even more astonishing is the estimate of [14] that the number of possible consistent flux compactifications of F-theory to four dimensions is at least 10^{272000}. Although it is not known wheter all these compactifications are distinct or dual descriptions of the same theory, such a wildly huge number of possibilities clearly suggests that the direct study of all string vacua is a formidable task. Moreover, even if we were able to enumerate all the inequivalent string constructions, a top-down mechanism to prefer one particular choice over another is still missing. Of course, a bottom-up approach is possible: instead of starting from full string theory and studying compactifications to four dimensions, one can study four dimensional effective quantum field theories and try to couple them to gravity. We might naively expect that the string *landscape* is so vast that all consistent looking effective field theories coupled to gravity can arise in some way from a string theory compactification. If this assumption were correct, string theory would lost most of its predictive power and would become pretty useless for phenomenological applications. However, it was first argued in [15] that this is not the case, namely that not all effective field theories can be coupled consistently to gravity with a UV completion, or equivalently that not all of them can arise from a string theory compactification. The set of these ultimately inconsistent theories is called the *swampland*, as opposed to the *landscape*. Where lies the boundary between the two is not known yet, but a number of criteria have been conjectured that allows one to exclude a theory from the landscape, thus rejecting it to the swampland. Some finiteness properties, including finiteness of *vev*s of scalar fields, finiteness of the number of fields and finiteness/restrictions on the rank of the gauge groups, were already studied in [15], but since then other criteria have been proposed, such as those involving the geometry of the moduli space in [16] and the various and inequivalent versions of the Weak Gravity Conjecture, first formulated in [2], then sharpened in [1]. A recent review of *swampland* criteria can also be found in [17]. As a final remark, it is worth observing that, altough general proofs are missing, these criteria are all motivated by general arguments in quantum gravity, coming in particular from black hole physics, and supported by non trivial realizations and known examples in string theory.

2.2 The Weak Gravity Conjecture

The *swampland* criterion which is most relevant for us, providing the main motivation for this thesis, is a sharpened version of the Weak Gravity Conjecture (WGC). This was first formulated in [2] and involves the observation that gravity must always be the *weakest* force, promoting it to a principle which is claimed to be satisfied in all consistent string theory compactifications and therefore has to hold in any consistent effective field theory coupled to gravity.

Weak gauge couplings Before giving a more precise statement, it is convenient to introduce the main ideas and discuss some of their implications. Following [2], we consider a four dimensional theory with gravity and a $U(1)$ abelian gauge field with gauge coupling $g = g_e$. The *electric* WGC states that there exists a light charged particle with mass

$$m_e \lesssim g_{el} M_{\text{Pl}}. \tag{2.1}$$

For such particles, gravity is subdominant, since the gravitational attraction is overwhelmed by the gauge repulsive force. An analogous bound should hold for magnetic monopoles,

$$m_m \lesssim g_m M_{\text{Pl}} \sim \frac{1}{g_e} M_{\text{Pl}}. \tag{2.2}$$

This *magnetic* WGC would have an interesting and rather unexpected consequence. Being proportional to the energy stored in the magnetic field they generate, monopole masses behave as probe of the UV cut off of the theory,

$$m_m \sim \frac{\Lambda}{g_e^2}. \tag{2.3}$$

Therefore, the WGC bound on monopole masses tells us that for small g_e there exists a prematurely low energy cut off where the theory breaks down,

$$\Lambda \lesssim g_e M_{\text{Pl}}. \tag{2.4}$$

This is in contrast with the naive expectation that, as long as the $U(1)$ Landau pole is above the Planck scale, the effective theory breaks down near M_{Pl} when gravity becomes strongly coupled and nothing seems to prevent taking g_e as small as we want.

Black holes and global symmetries Part of the original motivation for the conjecture comes from black hole physics. It is well known that black holes can decay emitting Hawking radiation. When this radiation stops before the black hole has completely evaporated, we are left with a remnant. It is easy to see that the existence of light charged particles whose mass is constrained by the charge as in (2.1) is required to avoid problems involving remnants, allowing extremal black holes to decay. Choosing units where the mass to charge ratio is unit for extremal black holes, i.e. $M = Q$ for extremal black holes, the WGC bound (2.1) for a particle of mass m and charge q can be rephrased as $m < q$. Since the bound for having a charged black hole solution is the Reissner-Nordström bound $M > Q$, extremal black holes can only decay into particles whose mass is less than their charge, whose existence is guaranteed by the WGC. This is schematically shown in Figure 2.1.

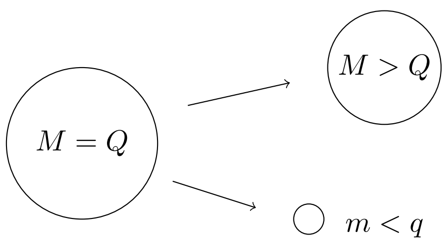

Figure 2.1: Decay of an extremal black hole.

Black hole physics also intertwines the WGC with other *swampland* criteria, such as the absence of global symmetries in a consistent quantum theory of gravity [18]. Imagine sending a particle charged under a global symmetry G inside a black hole. Then, all the information about this global symmetry is lost by the no hair theorem. When the black hole evaporates via Hawking radiation, an equal number of positive an negative charged particle under G is emitted. This process would violate charge conservation in G, since we started with a non zero charge and after the black hole evaporation we are left with a non zero net charge. In order to overcome this inconsistency, it has been conjectured that in a consistent quantum theory of gravity all symmetries must be gauged. This is also true in all known examples from string theory, where global symmetries arise from symmetries of the extra dimensions, which are gauged because diffeomorphisms of the compact space are part of the gauge symmetry of gravity. A review of these arguments can be found for example in [17]. However, in the limit of small gauge coupling, gauge symmetries become physically indistinguishable from continuous global symmetries. This is where the WGC comes into play, since the existence of the cut off Λ prevents this limit to be taken. More explicitly, from (2.4) we see that, as the gauge coupling goes to zero, $g \to 0$, the cut off of the effective theory goes to zero as well, $\Lambda \to 0$, so that this limit cannot be taken smoothly.

The WGC bound As we have seen, the black hole motivation suggests to set the mass to charge ratio equal to one for extremal black holes. Then, in appropriate units, the WGC bound reads

$$\frac{m}{q} \leq 1. \tag{2.5}$$

However, in order to make a more precise statement, we still need to specify which states should satisfy this bound. There are at least three natural possibilities we could think of, that is we might require that the bound is satisfied by

I. the state of minimal charge;

II. the lightest charged particle;

III. the state with the smallest mass to charge ratio.

In order for the statement to be meaningful, such a state must be stable. This is not necessarily the case for the state I. since, for instance, it could decay into lightest particles of higher charge. These might form a stable bound state with the same charge of I. but the mass to charge ratio might be larger than that of the individual particles and, in particular, it might be bigger than one. Therefore, this possibility is certainly incorrect. Of course, the state II. is stable and so is the state III., due to the triangle inequality. Therefore, both possibilities are worth considering. The second one is clearly stronger and implies the third. In particular, it would force the spectrum of the theory to contain a light charged particle. Otherwise, the bound could in principle be satisfied also by a heavy particle with a large charge. As remarked by the author of [2], the weaker statement would reduce the impact of the inequality on low energy physics, but is the one to be supported by most of the evidence they provide.

We formulated the WGC bound for particle states, but it is natural to generalize it to higher dimensional objects, such as p-branes. They are massive objects that extend in p spatial dimensions, plus time and interact with the gravitational and gauge fields through the couplings

$$S_{p-\text{brane}} = T \int d^{p+1}x\sqrt{g} + Q \int A_{p+1}, \tag{2.6}$$

where the integrals are taken over the $(p+1)$-dimensional world-volume of the brane and the parameters T and Q are the tension and the charge of the brane, respectively. The first term tells us the familiar fact that the motion of a massive object in General Relativity is such to minimize the world-volume swept and is a generalization of the integral over the world-line for a massive particle. The second term is the coupling with a $(p+1)$-form gauge potential and generalizes the coupling of a charged particle to the electromagnetic field, which is the integral of the one-form gauge potential on the world-line of the particle. Given a p-form Abelian gauge field in D dimensions, the WGC postulates the existence of electrically and magnetically charged $p-1$ and $D-p-1$ dimensional objects with tensions

$$T_e \leq \left(\frac{g^2}{G_N}\right)^{\frac{1}{2}} \qquad \text{and} \qquad T_m \leq \left(\frac{1}{g^2 G_N}\right)^{\frac{1}{2}}, \tag{2.7}$$

where the gauge coupling g is the charge density with dimension of a mass to the power $p+1-\frac{D}{2}$.

2.3 Sharpening the conjecture

As we have seen, the WGC in its simplest form states that, for a $U(1)$ gauge theory coupled to gravity, there always exists a particle of mass m and charge q satisfying the inequality

$$\frac{m}{M_{\text{Pl}}} \leq q, \tag{2.8}$$

where the Planck mass has been reintroduced. With the black hole motivation in mind, allowing equality might seem unnatural. It should not be included from the requirement that extremal black holes are able to decay, since they only emit particles which satisfy the strict inequality. Moreover, allowing the WGC bound to be saturated might be quite dangerous, as small perturbations to the theory could tip the balance in the wrong direction and violate the WGC.

Supersymmetry and BPS states As already pointed out in the original work [2], the reason for allowing equality comes from supersymmetry. We recall that the supersymmetry algebra involves a set of $2N$ anti-commuting fermionic generators $(Q^I_\alpha,\, \bar{Q}^I_{\dot{\alpha}})$ with $I = 1, \ldots, N$ and $\alpha, \dot{\alpha} = 1, 2$ which obey, among the others, the following anti-commutation relations

$$\left\{Q^I_\alpha, Q^J_\beta\right\} = \varepsilon_{\alpha\beta} Z^{IJ} \qquad \text{and} \qquad \left\{\bar{Q}_{\dot{\alpha}I}, \bar{Q}^J_{\dot{\beta}}\right\} = \varepsilon_{\dot{\alpha}\dot{\beta}} Z^{IJ}. \tag{2.9}$$

The operators Z^{IJ} are a linear combination of the Lorentz scalar generators B_l of an internal symmetry group G and can be proved to be *central charges*, that is they span an invariant abelian subalgebra of G and commute with all other generators. Central charges play a key role in the construction of the massive representations of the supersymmetry algebra, as discussed for example in [19]. The general result is a restriction on the relative magnitude of the mass of a given irrep and the modulus of any central charge eigenvalue, called the BPS bound, which looks like the WGC bound but in the wrong direction,

$$2m \geq |Z_r|, \qquad r = 1, \ldots, \frac{N}{2}. \tag{2.10}$$

According to the number of central charge eigenvalues saturating this bound, we can have multiplets with different lenghts: *long* multiplets, when the bound isn't saturated by any of the eigenvalues, and *short* (or ultrashort) multiplets, also called BPS from the analogous Bogomol'nyi–Prasad–Sommerfield bound for solitons, when the bound is saturated by some (or all) of the eigenvalues. The presence of long multiplets should not worry, since the WGC doesn't state that, for a given charge sector, all particles must have masses less than their charge, but rather that there exists such a particle. Short multiplets are supersymmetry preserving, since they are annihilated by the supersymmetry generators corresponding to the central charge eigenvalues which saturate the bound, and turn out to be more protected against quantum corrections with respect to long multiplets. Thus, the WGC bound can be saturated in a supersymmetric theory with BPS states, with the solid BPS condition preventing the balance to be tipped in the wrong direction.

A natural question to ask is how peculiar is the BPS case, or in other words if there are other cases in which the WGC bound can be saturated. According to the authors of [1], the answer must be negative. The observation that, if the WGC bound is saturated in a non supersymmetric theory, there is no know mechanism analogous to the robustness of the BPS condition which can protect the WGC from being violated when small corrections are taken into account, is the starting point for them to argue for a stronger version of the WGC, where the inequality is saturated if and only if the underlying theory is supersymmetric and the state in question is BPS.

Instability of non supersymmetric AdS Combining this stronger version of the WGC with the analysis of [4] that an AdS geometry supported by a flux is unstable in the presence of branes which are charged under the flux and whose charges are less than their tensions, the authors of [1] were led to the dramatic conclusion that all non supersymmetric AdS geometries supported by fluxes must be unstable.

In order to explain how AdS instabilities can arise through brane nucleation, we are forced to anticipate here some concepts and techniques, such as looking for solutions of the Euclidean equations of motion with finite action and then analitically continue them to Lorentzian signature with respect to a plane of symmetry of the metric, which might sound quite artificial but will be thoroughly accounted for in Chapter 3. We will mainly follow [4], but it is worth mentioning that similar ideas were already present in the earlier work of [3]. Let us consider an AdS_D geometry supported by a constant D-form field strenght. We write the Euclidean AdS metric as

$$ds^2 = L^2 \left(\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\Omega_p^2\right), \tag{2.11}$$

where L is the AdS radius and $d\Omega_p^2$ is the metric of the unit p-sphere. Then the Euclidean action of a spherically symmetric p-brane located at radius $\rho(\tau)$ is from equation (2.6) given by

$$S_E = T\Omega_{p-2} \int d\tau \left[\sinh^{p-2} \rho \sqrt{\cosh^2 \rho + \left(\frac{d\rho}{d\tau}\right)^2} - q \sinh^{p-1} \rho\right], \tag{2.12}$$

where T is the tension of the brane, q is the charge to tension ratio and Ω_{p-2} is the volume of the unit $(p-2)$-sphere. Since this action does not explicitly depend on τ, the Euclidean energy is conserved, in particular it is zero for a spherically symmetric compact surface, yielding

$$\frac{\cosh^2 \rho}{\sqrt{\cosh^2 \rho + \rho'^2}} - q \sinh \rho = 0. \tag{2.13}$$

The qualitative behaviour of the solutions to this equation crucially depends on the value of q. In the BPS case, the forces balance so $q = 1$ and we don't expect to have any instabilities. If the theory is supersymmetric but the branes are non BPS, the BPS bound requires that $q > 1$

and the Euclidean solution is

$$\cosh\rho = \frac{\sinh\tau_{max}}{\sinh\tau}, \tag{2.14}$$

with $\tanh\rho_{max} = q$. Having infinite action, this solution cannot represent an instability. Conversely, in the case $q > 1$, which according to the stronger version of the WGC is not only possible, but inevitable in a non supersymmetric theory, the solution is

$$\cosh\rho = \frac{\cosh\tau_{max}}{\cosh\tau}, \tag{2.15}$$

with $\tanh\rho_{max} = \frac{1}{q}$, which correspond to the radius of the brane at nucleation. This solution has finite action

$$S_E = \frac{2T\Omega_{p-2}}{\sinh\rho_{max}} \int_0^{\rho_{max}} d\rho \frac{\sinh^{p-2}\rho\sqrt{\sinh^2\rho_{max} - \sinh^2\rho}}{\cosh\rho} \tag{2.16}$$

and can be analitically continued to the Lorentzian solution

$$\cosh\rho = \frac{\cosh\rho_{max}}{\cos t}, \tag{2.17}$$

which describes the evolution of the brane after nucleation. In particular, from (2.17) we see that the brane expands, reaching the boundary of AdS at $\rho = \infty$ at the finite time $t = \frac{\pi}{2}$ and thus reducing the flux of the geometry.

The instability of non supersymmetric AdS geometries would imply that there cannot exist non supersymmetric conformal field theories whose holographic duals have such gravity description. As argued in [1], due to the instantaneous nature of the decay, which is a distinctive feature of AdS geometries with respect to, for example, flat or de Sitter space, the dual conformal field theories could not exist even as a metastable state. This can be seen as follows. Recall that an observer sitting at the boundary of AdS can get access to an infinite volume space within an infinitesimal amount of time. Since the decay probability per unit volume associated to brane nucleation is finite, the decay happens instantaneously when seen from the boundary. An alternative explanation relies on holography. The AdS_p geometry can be obtained as the near horizon limit of an extremal $(p-2)$-brane. If the brane is non BPS, according to the stronger version of the WGC, there exist branes with tension less than the charge. Then the extremal brane can decay by emitting such other branes, with the decay rate becoming larger as we measure it closer to the horizon. In the near horizon limit, the decay becomes instantaneous.

Chapter 3

False vacuum decay

This Chapter deals with the theory of false vacuum decay, as originally developed by Coleman et al. in a series of papers from the 1970s, namely [5] and [6] for the general theory in flat space and [7] for the inclusion of gravity. This theory is of primary interest for us, since it provides a possible non perturbative mechanism leading to instabilities, based on the existence of a specific solution to the Euclidean equations of motion, dubbed the *bounce*, with finite action. In fact, it paved the way to Witten's analysis of the instability of the Kaluza-Klein vacuum in [8], which represents the starting point for us towards an exploration into AdS instabilities of the same kind.

The basic example where vacuum decay can be seen at work is that of a scalar field theory in four dimensional Minkowski spacetime, described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - V(\phi), \tag{3.1}$$

where the potential $V(\phi)$ has two relative minima with different energies, as shown in Figure 3.1. The lower minimum ϕ_T corresponds to the unique ground state of the quantum theory and is

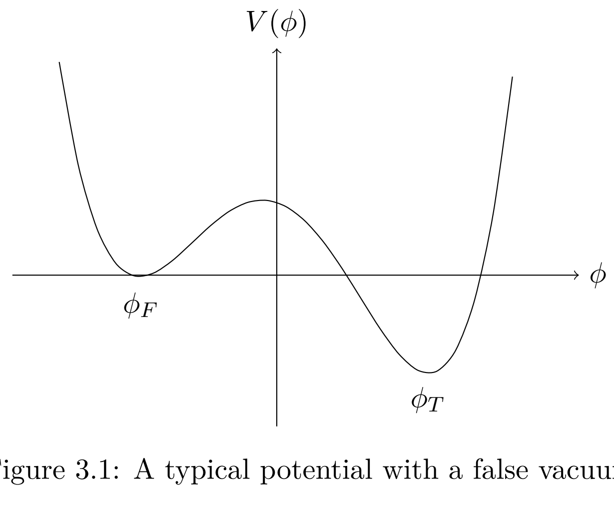

Figure 3.1: A typical potential with a false vacuum.

called a *true* vacuum. The higher one ϕ_F is called a *false* vacuum because, though classically stable, it is rendered unstable by quantum effects and can decay via a tunneling process. If Figure 3.1 were the plot of the potential energy for a quantum mechanical particle in one dimension, the barrier would be just a finite one. However, in the field theory context, barrier penetration requires more care. Remember that $V(\phi)$ is an energy density, which must be integrated to give the potential energy,

$$U[\phi] = \int d^3x \left[\frac{1}{2}(\nabla\phi)^2 + V(\phi)\right]. \tag{3.2}$$

To go from the false to the true vacuum through a series of spatially homogeneous configurations would require traversing an infinite potential energy barrier and thus would have a vanishing tunneling amplitude. Instead, the tunneling process takes the spatially homogeneous configuration where the field is everywhere in the false vacuum to one with a region of approximate true vacuum, a *bubble*, immersed in a false vacuum background. A nice picture for the enucleation of a bubble comes from statistical mechanics. Imagine the boiling of a superheated fluid, where thermodynamic fluctuations cause bubbles of the vapor phase to materialize in the fluid phase. Bubbles can form large enough so that it is energetically favorable for them to grow, converting the fluid to the vapor phase. Here quantum fluctuations replace thermodynamic ones, and the false and true vacuum correspond to the fluid and vapor phase. Bubbles can form anywhere in space, so the relevant quantity to be computed is the decay probability per unit time per unit volume, which can be shown to be of the form

$$\frac{\Gamma}{V} = Ae^{-\frac{B}{\hbar}}\left[1 + \mathcal{O}(\hbar)\right], \tag{3.3}$$

where the exponent B is given by the Euclidean action of the bounce.

The general formalism for the computation of the A and B coefficients developed by Coleman et al. is based on instanton methods. Therefore, before addressing the field theory problem, it is convenient to start reviewing these methods in the simpler context of quantum mechanics. This is what we do in the first part of this Chapter, where we describe instantons in quantum mechanics and introduce the basic ideas, namely the concept of the bounce solution that is connected with the decay of a metastable state. In particular, we derive an expression for the decay rate and find that it is exponentially suppressed, with exponent given by the Euclidean action of the bounce. Then, we explain how these methods can be generalized in a field theory context. We discuss the interpretation of false vacuum decay as a tunneling process and describe the evolution after tunneling, as directly obtained from the analytic continuation of the bounce. We also introduce the thin-wall approximation, which defines a particularly simple regime to study where the bubble nucleation picture becomes apparent. Then, we make another step forward to include the effects of gravity. It turns out that the picture can be qualitatively different from the flat case. Working in the thin-wall approximation we find that the radius of the bubble at nucleation can be made larger or smaller by gravitational effects, and correspondingly the nucleation of the bubble can be less or more likely, respectively. In particular, it can happen that the radius of the bubble at nucleation is infinite, or equivalently the decay probability vanishes, so that a metastable vacuum in flat space might be actually stable in the presence of gravity. This concludes the general discussion and we can eventually turn to a concrete application, namely Witten's construction of a *bubble of nothing* as the bounce solution demonstrating the instability of the Kaluza-Klein vacuum. This decay is somewhat peculiar, if not dramatic with respect to the usual vacuum decay, as we will see.

3.1 Instantons and bounces in quantum mechanics

In this section, we provide some basic notions about instanton methods in quantum mechanics. We will mainly follow [20] and [21], to which we also refer for the field theory generalization of the following sections. By *instantons*, we here mean Euclidean, or imaginary time, solutions to the equations of motion. Why should they be physically interesting? A heuristic motivation is the following. Consider a quantum mechanical particle of unit mass moving in one dimension with Hamiltonian

$$H = \frac{p^2}{2} + V(x). \tag{3.4}$$

If the potential $V(x)$ has a barrier and the energy E of the incident wave associated to the particle is less than the value of the potential at the top of the barrier, as shown in Figure 3.2, there is an exponentially suppressed but non vanishing probability of finding the particle at the opposite side of the barrier. In the semi-classical approximation, the amplitude for transmission

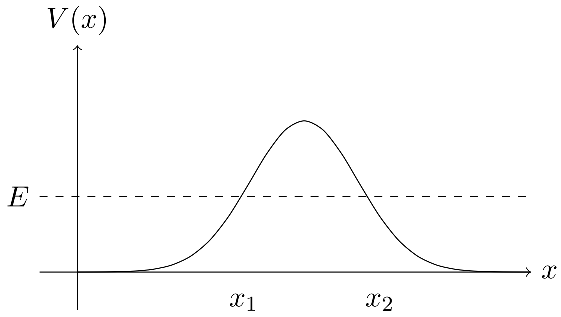

Figure 3.2: A potential barrier in one dimension. The classically forbidden region $x_1 < x < x_2$ is defined by the classical turning points, such that $V(x_1) = V(x_2) = E$.

through the barrier is proportional to $e^{-\frac{B}{\hbar}}$, where B is an integral given by the WKB formula,

$$B = \int_{x_1}^{x_2} dx \sqrt{2\left[V(x) - E\right]}, \tag{3.5}$$

and the integration extremes x_1 and x_2 are the classical turning points at energy E. In the classically forbidden region, the energy of the particle is less than its potential energy, as if its kinetic energy was negative, the velocity being obtained as the derivative with respect to an imaginary time.

According to the form of the potential, this very simple system can already exhibit some distinctive features of instanton effects, namely the mixing of states that would be degenerate in the absence of tunneling, which is connected with *instantons*, and the decay of a metastable state, which is the case we are most interested in and is connected with *bounces*. Our basic tool is the Euclidean path integral, which gives the probability amplitude of finding the particle in x_f at time $\frac{T}{2}$, provided that it initially was in x_i at time $-\frac{T}{2}$, as a sum over all possible paths connecting the initial and final states, each weighted by the exponential of minus the Euclidean action,

$$\langle x_f | e^{-\frac{HT}{\hbar}} | x_i \rangle = \int \mathcal{D}x \; e^{-\frac{S_E}{\hbar}}. \tag{3.6}$$

The energy E_0 of the lowest energy eigenstate can be extracted from the large T behaviour of this matrix element. Indeed, expanding the left hand side in terms of energy eigenvalues and taking the large T limit, only the E_0 term survives in the sum,

$$\langle x_f | e^{-\frac{HT}{\hbar}} | x_i \rangle = \sum_n e^{-\frac{E_n T}{\hbar}} \langle x_f \,|\, n \rangle \langle n \,|\, x_i \rangle \underset{T \to \infty}{\longrightarrow} e^{-\frac{E_0 T}{\hbar}} \langle x_f \,|\, 0 \rangle \langle 0 \,|\, x_i \rangle \,. \tag{3.7}$$

A general procedure to evaluate (3.6) is the saddle point approximation, with the integral dominated by the stationary points of the Euclidean action,

$$\frac{\delta S_E[\bar{x}]}{\delta x} = -\frac{d^2\bar{x}}{d\tau^2} + V''(\bar{x}) = 0. \tag{3.8}$$

Observe that this equation describes the motion of a particle in the inverted potential $-V(x)$. A generic path satisfying the boundary conditions $x(-\frac{T}{2}) = x_i$ and $x(\frac{T}{2}) = x_f$ can be expanded around a solution $\bar{x}$ as

$$x(\tau) = \bar{x}(\tau) + \sum_n c_n \psi_n(\tau), \tag{3.9}$$

where the ψ's are an orthonormal basis of functions vanishing at the boundary $\psi(\pm\frac{T}{2}) = 0$. Thus, the integration measure can be written as

$$\mathcal{D}x = \prod_n \frac{dc_n}{\sqrt{2\pi}}. \tag{3.10}$$

It is convenient to take the ψ's to be eigenfunctions of the second variational derivative of the Euclidean action at $\bar{x}$,

$$\frac{\delta^2 S_E[\bar{x}]}{\delta^2 x} = -\frac{d^2\psi_n}{d\tau^2} + V''(\bar{x})\psi_n = \lambda_n \psi_n \equiv S_E''[\bar{x}], \tag{3.11}$$

where, for the moment, all the eigenvalues λ_n are assumed to be positive. Upon Gaussian integration, the contribution to (3.6) from this stationary point reduces to

$$I = \int \prod_n \frac{dc_n}{\sqrt{2\pi}} e^{-\frac{S_E[\bar{x}] + \frac{1}{2}\sum_k \lambda_k c_k^2 + \cdots}{\hbar}} = e^{-\frac{S_E[\bar{x}]}{\hbar}} \left(\det S_E''[\bar{x}]\right)^{-\frac{1}{2}} \left[1 + \mathcal{O}(\hbar)\right], \tag{3.12}$$

where ellipses denote the higher order terms that we have neglected and the functional determinant is the product of the eigenvalues

$$\det S_E''[\bar{x}] = \prod_n \lambda_n. \tag{3.13}$$

Finally, we are ready to specialize this general procedure to a couple of interesting cases.

The instanton The first case we discuss is the symmetric double-well potential of Figure 3.3a.

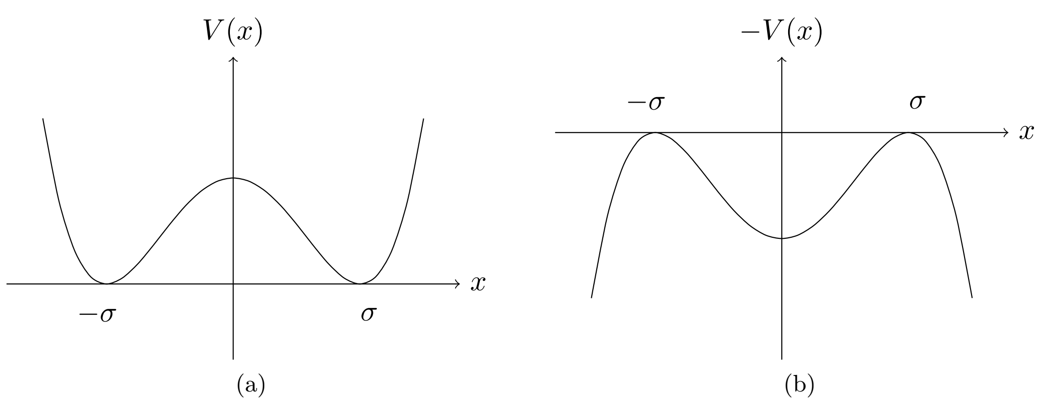

Figure 3.3: A symmetric double-well potential with two degenerate minima.

If the barrier were infinitely high, and hence impenetrable, energy eigenstates would be confined to one side or the other and the two ground states, $|L\rangle$ and $|R\rangle$, would be degenerate, with energy E_0. With a finite barrier, the two lowest eigenstates are given by the symmetric and antisymmetric linear combinations

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|L\rangle \pm |R\rangle), \tag{3.14}$$

with energies

$$E_\pm = E_0 \mp \frac{\Delta}{2}. \tag{3.15}$$

In order to compute the energy splitting, consider the matrix elements

$$\langle \pm\sigma| e^{-\frac{HT}{\hbar}} |\sigma\rangle = \sum_n e^{-\frac{E_n T}{\hbar}} \langle \pm\sigma|n\rangle \langle n|\sigma\rangle \underset{T\to\infty}{\to} |\langle\sigma|+\rangle|^2 e^{-\frac{E_+ T}{\hbar}} \pm |\langle\sigma|-\rangle|^2 e^{-\frac{E_- T}{\hbar}}, \tag{3.16}$$

where we have used $\langle\sigma|\pm\rangle = \pm\langle -\sigma|\pm\rangle$. If the two lowest eigenstates are well separated from the others, we also have $|\langle\sigma|+\rangle| = |\langle\sigma|-\rangle|$ and, in the limit of large T,

$$e^{\frac{(E_- - E_+)T}{\hbar}} = \frac{\langle\sigma|e^{-\frac{HT}{\hbar}}|\sigma\rangle + \langle -\sigma|e^{-\frac{HT}{\hbar}}|\sigma\rangle}{\langle\sigma|e^{-\frac{HT}{\hbar}}|\sigma\rangle - \langle -\sigma|e^{-\frac{HT}{\hbar}}|\sigma\rangle}. \tag{3.17}$$

Our general procedure tells us how to compute these matrix elements: it suffices to find the stationary points of the Euclidean action, i.e. the solutions of the equations of motion (3.8) in the inverted potential, which is shown in Figure 3.3b. For $\langle\sigma|e^{-\frac{HT}{\hbar}}|\sigma\rangle$, we have the trivial constant solution $x_0(\tau) = \sigma$ whose contribution is simply

$$I_0 = \left(\det S''_E[x_0]\right)^{-\frac{1}{2}}, \tag{3.18}$$

as the potential was chosen to vanish at the minima. For $\langle -\sigma|e^{-\frac{HT}{\hbar}}|\sigma\rangle$, we have a more interesting solution $x_1(\tau)$, where the particle starts at the top of the left hill at $\tau = -\frac{T}{2}$ and moves to the top of the right hill at time $\tau = \frac{T}{2}$. This solution is called an *instanton* and is sketched in Figure 3.4 after taking T to ∞.

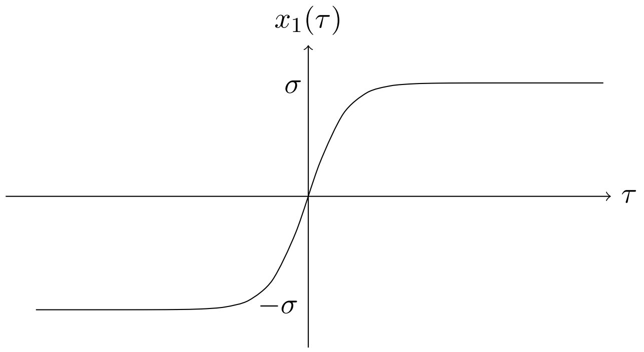

Figure 3.4: An instanton centered in $\tau = 0$.

According to (3.12), it should give a contribution

$$e^{-\frac{S_E[x_1]}{\hbar}} \left(\det S''_E[x_1]\right)^{-\frac{1}{2}}. \tag{3.19}$$

However, there is a problem. The time derivative of the instanton,

$$\psi_0(\tau) = (S_E[x_1])^{-\frac{1}{2}} \frac{dx_1}{d\tau}, \tag{3.20}$$

is an eigenfunction of zero eigenvalue, as it can be easily seen using the equations of motion,

$$\frac{d^3 x_1}{d\tau^3} + V''(x_1)\frac{dx_1}{d\tau} = \frac{d}{d\tau}\left(-\frac{d^2 x_1}{d\tau^2} + V'(x_1)\right), \tag{3.21}$$

and as should be expected from time translation invariance. Were we to integrate over the corresponding coefficient c_0, we would end up with a divergent quantity. The solution is straightforward: we can integrate over the location of the center of the instanton τ_1, instead. The variations due to small changes of c_0 and τ_1 are easy to compare,

$$dx = \frac{dx_1}{d\tau} d\tau_1 = \psi_0 dc_0 \qquad \Rightarrow \qquad dc_0 = (S_E[x_1])^{\frac{1}{2}} d\tau_1. \tag{3.22}$$

Thus, the contribution of the instanton can be more appropriately written as

$$I_1 = e^{-\frac{S_E[x_1]}{\hbar}} \left(\det S''_E[x_0]\right)^{-\frac{1}{2}} KT, \tag{3.23}$$

with

$$K = \left(\frac{S_E[x_1]}{2\pi\hbar}\right)^{\frac{1}{2}} \left(\frac{\det' S''_E[x_1]}{\det S''_E[x_0]}\right)^{-\frac{1}{2}}, \tag{3.24}$$

where $\det'$ means that the zero mode must not be included in the determinant. Finally, there are also approximate stationary points to be considered, configurations $x_n(\tau)$ of n alternating and well separated instantons and anti-instantons, with action $S_E[x_n] = nS_E[x_1]$. Upon integrating over the n centers,

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau_1 \int_{\tau_1}^{\frac{T}{2}} d\tau_2 \cdots \int_{\tau_{n-1}}^{\frac{T}{2}} d\tau_n = \frac{T^n}{n!} \tag{3.25}$$

their contribution is

$$I_n = e^{-n\frac{S_E[x_1]}{\hbar}} \left(\det S''_E[x_0]\right)^{-\frac{1}{2}} K^n \frac{T^n}{n!}. \tag{3.26}$$

Summing over stationary and approximately stationary points, we find

$$\begin{aligned} \langle\sigma|e^{-\frac{HT}{\hbar}}|\sigma\rangle &= \sum_{\text{even } n} I_n = \left(\det S''_E[x_0]\right)^{-\frac{1}{2}} \cosh\left(e^{-\frac{S_E[x_1]}{\hbar}} KT\right), \\ \langle-\sigma|e^{-\frac{HT}{\hbar}}|\sigma\rangle &= \sum_{\text{odd } n} I_n = \left(\det S''_E[x_0]\right)^{-\frac{1}{2}} \sinh\left(e^{-\frac{S_E[x_1]}{\hbar}} KT\right), \end{aligned} \tag{3.27}$$

which inserted into equation (3.17) give

$$\Delta = E_- - E_+ = 2Ke^{-\frac{S_E[x_1]}{\hbar}}. \tag{3.28}$$

Observe that perturbative corrections to the individual energies are much larger than the exponentially small instanton correction we have computed, but they don't contribute to the difference. Now, imagine to start at $t = 0$ with a wavefunction localized on the left-hand well, for example $|\Psi(0)\rangle = |L\rangle$. Time evolution gives

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-i\frac{E_+ t}{\hbar}} (|+\rangle + e^{-i\frac{\Delta t}{\hbar}} |-\rangle) = \frac{1}{2} e^{-i\frac{E_+ t}{\hbar}} \left[(1 + e^{-i\frac{\Delta t}{\hbar}}) |L\rangle + (1 - e^{-i\frac{\Delta t}{\hbar}}) |R\rangle \right], \quad (3.29)$$

so the system oscillates back and forth with a frequency Δ.

The bounce The second example we study is based on the quantum mechanical analog of the potential of Figure 3.1. Imagine to modify it in such a way that the right-hand well with the lower minimum is broader than the left-hand well with the local minimum. Without barrier penetration, there would be a discrete spectrum of energy eigenstates on the left side of the barrier, the lower of which we denote again $|L\rangle$, and a denser spectrum on the other side. With barrier penetration restored, energy eigenstates are mixtures of left and right states, but the contribution of the left states tends to zero as the width of the right-hand well increases. The limiting case in which the width becomes infinite is shown in Figure 3.5a. If we started at $t = 0$ with the wavefunction $|\Psi(0)\rangle = |L\rangle$ and computed the time evolution $|\Psi(t)\rangle$, instead of oscillation we would find that $\langle L|\Psi(t)\rangle$ vanishes exponentially with time. It is a metastable state with a complex energy whose imaginary part is related to the decay rate by

$$\mathrm{Im}\, E_0 = -\frac{\Gamma}{2}. \quad (3.30)$$

As we will see later, this complex energy can be defined as the analytic continuation of the real energy of the stable ground state of a potential where $x = \sigma$ is a global minimum.

The matrix element to be computed is

$$\langle \sigma | e^{-\frac{HT}{\hbar}} |\sigma\rangle = \sum_n e^{-\frac{E_n T}{\hbar}} |\langle \sigma | n \rangle|^2 , \quad (3.31)$$

from which we can extract the energy E_0 of the metastable state. Just as before, we look for

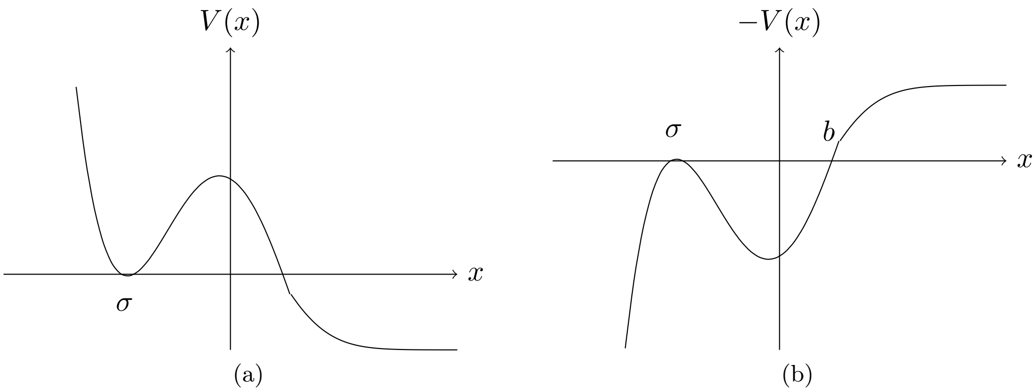

Figure 3.5: The limiting case in which the width of the right-hand well becomes infinite.

solutions to the Euclidean equations of motion in the inverted potential, which is shown in Figure 3.5b. Obviously, there is the trivial configuration $x_0(\tau) = \sigma$, whose contribution is

$$I_0 = \left(\det S_E''[x_0]\right)^{-\frac{1}{2}} , \quad (3.32)$$

as the potential vanishes at the minimum. A more interesting solution is the *bounce*, with the particle starting at the top of the lowest hill at $\tau = -\frac{T}{2}$, bouncing off the classical turning point and coming back to the initial point at $\tau = +\frac{T}{2}$, as shown in Figure 3.6 after taking T to ∞. As for the instanton, there is a zero mode which is proportional to the time derivative of the

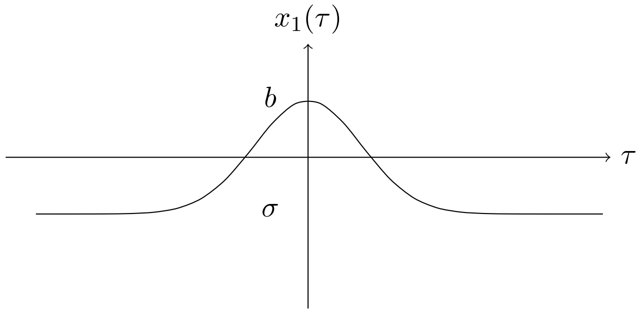

Figure 3.6: A bounce centered in $\tau = 0$.

bounce. The solution is again to introduce a collective coordinate specifying the location of the bounce and to integrate over it, obtaining

$$I_1 = e^{-\frac{S_E[x_1]}{\hbar}} \left(\det S_E''[x_0]\right)^{-\frac{1}{2}} KT, \tag{3.33}$$

with

$$K = \left(\frac{S_E[x_1]}{2\pi\hbar}\right)^{\frac{1}{2}} \left(\frac{\det' S_E''[x_1]}{\det S_E''[x_0]}\right)^{-\frac{1}{2}}, \tag{3.34}$$

This expression for K is almost correct. Since the zero mode, being proportional to the time derivative of the bounce, has a node located at the center of the bounce, a general property of the one dimensional Schrödinger equation implies that there must exist a nodeless eigenfunction corresponding to a lower eigenvalue, i.e. a negative eigenvalue which appears in the square root of the determinant. Thus, non-perturbative corrections give an imaginary part to the energy and the state is metastable. In order to obtain the correct expression for K, consider the set of configurations $x(\tau)$ shown in Figure 3.7, which are parametrized by their maximum value c. In

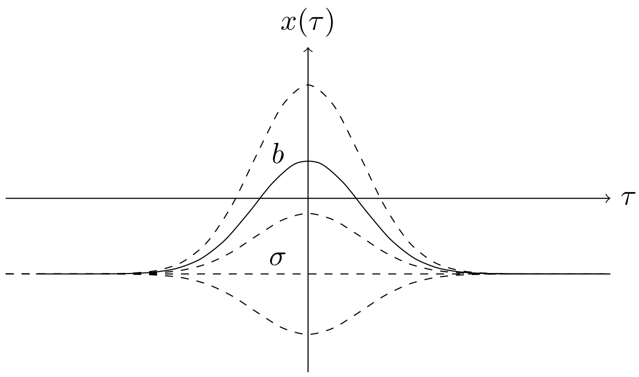

Figure 3.7: A set of configurations parametrized by their maximum value c. The solid line corresponds to the bounce of Figure 3.6, with $c = b$.

particular, $c = \sigma$ corresponds to the constant solution $x_0(\tau)$, which is clearly a local minimum of the action since any small deviation gives a positive contribution both to the kinetic and the potential term. As c increases, both of these terms increase until $c = b$, which is a maximum and corresponds to the bounce, is reached. The situation is sketched in Figure 3.8. As we have seen, problems arise in the integration over the coefficient of the negative mode, which is analogous

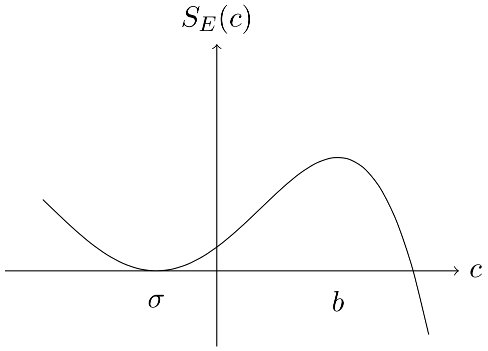

Figure 3.8: The Euclidean action for a set of configurations such as those shown in Figure 3.7.

to

$$J = \int_{-\infty}^{+\infty} \frac{dc}{\sqrt{2\pi}} e^{-\frac{S(c)}{\hbar}}. \tag{3.35}$$

In order to make this integral convergent, we can deform the integration contour as shown in Figure 3.9. Thus, we get an imaginary part which in the saddle point approximation is given by

$$\mathrm{Im}\, J = \mathrm{Im} \int_{b}^{b+i\infty} \frac{dc}{\sqrt{2\pi}} e^{-\frac{\left[S(b)-\frac{1}{2}S''(b)(c-b)^2+\dots\right]}{\hbar}} = \frac{1}{2} e^{-\frac{S(b)}{\hbar}} \left|S''(b)\right|^{-\frac{1}{2}}, \tag{3.36}$$

where the factor $\frac{1}{2}$ comes from integrating over half of the Gaussian function only. Thus, the

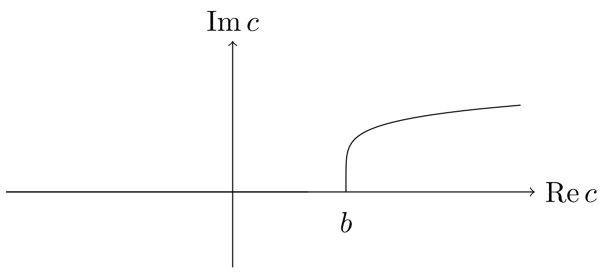

Figure 3.9: The contour of integration in the complex plane that makes the integral J in (3.35) well-defined.

correct expression for K is

$$K = \frac{i}{2} \left(\frac{S_E[x_1]}{2\pi\hbar} \right)^{\frac{1}{2}} \left| \frac{\det' S''_E[x_1]}{\det S''_E[x_0]} \right|^{-\frac{1}{2}}, \tag{3.37}$$

Finally, there is the contribution coming from all possible multibounce solutions. For a configuration of n well-separated bounces, upon integrating over the n centers, such contribution is

$$I_n = e^{-n\frac{S_E[x_1]}{\hbar}} \left(\det S''_E[x_0]\right)^{-\frac{1}{2}} K^n \frac{T^n}{n!}. \tag{3.38}$$

Finally, summing over all the stationary and approximately stationary points we obtain

$$\langle\sigma|e^{-\frac{HT}{\hbar}}|\sigma\rangle = \sum_n I_n = \left(\det S''_E[x_0]\right)^{-\frac{1}{2}} \exp\left[KTe^{-\frac{S_E[x_1]}{\hbar}}\right], \tag{3.39}$$

and the energy of the metastable state can be extracted from the large T behaviour of this expression,

$$E_0 = \lim_{T\to\infty} \frac{1}{2T} \ln\det S''_E[x_0] - Ke^{-S_E[x_1]}, \tag{3.40}$$

whose imaginary part is related to the decay rate through (3.30),

$$\Gamma = -2\,\mathrm{Im}\,E_0 = \left(\frac{S_E[x_1]}{2\pi\hbar}\right)^{\frac{1}{2}} \left|\frac{\det' S''_E[x_1]}{\det S''_E[x_0]}\right|^{-\frac{1}{2}} e^{-\frac{S_E[x_1]}{\hbar}}. \tag{3.41}$$

As we have already observed before, non perturbative corrections are in general exponentially suppressed with respect to perturbative ones. This would be true for the real part of E_0, but the non perturbative term we have computed is indeed the leading contribution to the imaginary part.

3.2 The field theory approach

At this point, we are ready to go back to the scalar field theory setup introduced at the beginning of the Chapter and use the results we have derived in the quantum mechanical context to compute the decay rate. The first step is to look for a solution of the Euclidean equations of motion $\phi_b(\tau, \mathbf{x})$ of minimal action, the analog of the bounce, which in this case will be a field configuration interpolating between the true and false vacua, ϕ_T and ϕ_F. Once we have found such a solution, the answer is

$$\frac{\Gamma}{V} = Ae^{-\frac{B}{\hbar}} \tag{3.42}$$

where the tunneling exponent

$$B = S_E[\phi_b] - S_E[\phi_F] \tag{3.43}$$

is the difference between the Euclidean action of the bounce and that of the homogeneous false vacuum $\phi = \phi_F$. We start from the Euclidean action,

$$S_E = \int d\tau d^3x \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi)\right], \tag{3.44}$$

with field equation

$$\frac{\partial^2\phi}{\partial\tau^2} + \nabla^2\phi = V'(\phi). \tag{3.45}$$

Our boundary conditions are

$$\lim_{\tau\to\pm\infty} \phi(\tau, \mathbf{x}) = \phi_F, \qquad \lim_{|\mathbf{x}|\to\infty} \phi(\tau, \mathbf{x}) = \phi_F, \tag{3.46}$$

where the first is a familiar one and the second comes from the requirement that the configurations along the tunneling path all have finite potential energy, measured relative to the false vacuum, so that the B coefficient is finite. Both the field equation and the boundary conditions display an $O(4)$ rotational symmetry in Euclidean space, so a natural guess is to assume the $O(4)$ invariant ansatz $\phi = \phi(\rho)$, with $\rho = \sqrt{\tau^2 + \mathbf{x}^2}$. Thus the field equation reduces to

$$\frac{d^2\phi}{d\rho^2} + \frac{3}{\rho}\frac{d\phi}{d\rho} = V'(\phi), \tag{3.47}$$

If ρ is interpreted as a time and $\phi(\rho)$ as a particle position, this equation describes the motion of a particle in a potential $-V(\phi)$ subject to a decreasing in time viscous damping force. The boundary conditions become

$$\lim_{\rho\to\infty}\phi=\phi_F,\qquad \lim_{\rho\to 0}\frac{d\phi}{d\rho}=0, \tag{3.48}$$

where the second one is needed to have a non singular solution at the origin, because of the friction term. Under mild conditions on the potential V, it was shown in [22] that this guess is right, namely that there always exists an $O(4)$ invariant solution which moreover has strictly lower action than any non $O(4)$ invariant solution. Existence is something we should expect from the nice *undershoot-overshoot* argument by Coleman [20]. From (3.48), we know that the particle is released at rest at time zero. If it is released sufficiently far from ϕ_T, so that the initial energy is lower than the value of the potential at ϕ_F, it obviously undershoots and will never reach ϕ_F. Due to the presence of the friction term, overshoot requires a little more care. Suppose to start very close to ϕ_T and stay there until the friction term has died out and we can neglect it. Then energy is essentially conserved and the particle overshoots, passing ϕ_F at some finite time. By continuity, we expect that the initial position ϕ^* can be properly chosen in such a way that the particle comes to rest at time infinity at ϕ_F.

Assuming $O(4)$ symmetry, the tunneling exponent B in the decay rate can be written as

$$B=2\pi^2\int_0^\infty d\rho\rho^3\left[\frac{1}{2}\left(\frac{d\phi}{d\rho}\right)^2+V(\phi)-V(\phi_F)\right]. \tag{3.49}$$

As for the pre-exponential factor A, the expression (3.41) obtained for a single degree of freedom can be easily generalized to the field theory case. The main difference is now the presence of three additional zero modes corresponding to spatial translations of the bounce, besides the zero mode corresponding to translation in Euclidean time. Integrating over the four collective coordinates, which describe the location of the center of the bounce in Euclidean space, gives the volume factor V, which must be divided out to give the decay rate per unit volume. After these modifications, (3.41) becomes

$$\Gamma=V\left(\frac{B}{2\pi\hbar}\right)^2\left|\frac{\det' S_E''[\phi_b]}{\det S_E''[\phi_F]}\right|^{-\frac{1}{2}}e^{-\frac{B}{\hbar}}. \tag{3.50}$$

The thin wall approximation An interesting limit case is when the two minima are nearly degenerate, i.e. the energy difference between the true and false vacuum

$$\varepsilon=V(\phi_F)-V(\phi_T) \tag{3.51}$$

is small compared to the height of the barrier. To be concrete, let us take a symmetric potential $V_0(\phi)$, with minima at $\pm\sigma$,

$$V_0'(\pm\sigma)=0,\qquad V_0''(\pm\sigma)=\mu^2, \tag{3.52}$$

and add a small term that breaks the symmetry,

$$V(\phi)=V_0(\phi)-\varepsilon\frac{\phi+\sigma}{2\sigma}. \tag{3.53}$$

Actually, the minima ϕ_T and ϕ_F of this new potential differ from $+\sigma$ and $-\sigma$ by terms of order ε, which we will ignore. From the mechanical analogy of the *undershoot-overshoot* argument, we expect that, in order to avoid undershoot, the particle must start very close to ϕ_T and stay there until some very large time $\rho \approx R$, when the damping force has become negligible. Then it moves quickly through the valley and comes to rest at ϕ_F. The form of the solution is sketched in Figure 3.10. Translating back to the field theory language, the bounce looks like a spherical

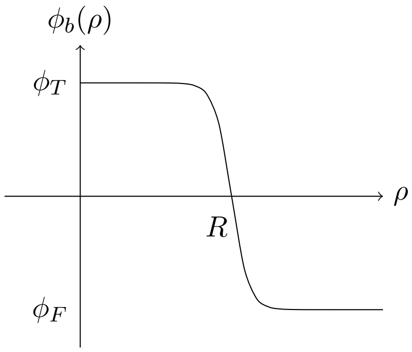

Figure 3.10: The bounce in the thin wall approximation.

bubble of critical radius R, with a thin wall separating the true vacuum inside from a background of false vacuum outside. For ρ near R, we can neglect both the friction term in (3.47) and the ε-dependent term in V, obtaining the soliton equation

$$\frac{d^2\phi}{d\rho^2} = V_0'(\phi). \tag{3.54}$$

A solution $\phi_0(\rho)$ for this equation is defined by

$$\rho = \int_{\phi_T}^{\phi_0} \frac{d\phi}{\sqrt{2\left[V_0(\phi) - V_0(\phi_F)\right]}}, \tag{3.55}$$

with action

$$\sigma = \int d\rho \left[\frac{1}{2}\left(\frac{d\phi_0}{d\rho}\right)^2 + V(\phi) - V(\phi_F)\right] = \int_{\phi_F}^{\phi_T} d\phi \sqrt{2\left[V(\phi) - V(\phi_F)\right]}, \tag{3.56}$$

which can be interpreted as the surface tension of the bubble. Thus, the thin wall approximate description of the bounce is

$$\phi_b(\rho) = \begin{cases} \phi_T, & \rho \ll R, \\ \phi_0(\rho - R), & \rho \approx R, \\ \phi_F, & \rho \gg R. \end{cases} \tag{3.57}$$

In order to compute the B exponent, which is given by equation (3.49), we can divide the integral into three regions: the inside and outside of the bubble, where the field is a constant, and the surface of the bubble, where we can replace ρ by R and V by V_0. Thus, we obtain

$$B = -\varepsilon\frac{\pi^2}{2}R^4 + 2\pi^2 R^3 \sigma, \tag{3.58}$$

where the first term is the volume energy coming from the inside of the bubble and the second term is a surface energy. The critical radius $R = R_0$ can be determined by imposing that the bounce is a stationary point,

$$\frac{dB}{dR} = 2\pi^2 R^2(-\varepsilon R + 3\sigma) = 0 \quad \Rightarrow \quad R_0 = \frac{3\sigma}{\varepsilon}. \tag{3.59}$$

This result allows us to verify the self-consistency of the thin wall approximation, justifying our earlier neglect of the friction term in (3.47). Away from the wall, this term is negligible because ϕ is approximately constant. At the wall, it is negligible because ρ is large or equivalently, from equation (3.59), because ε is small. Finally, we can evaluate B at the stationary point, obtaining

$$B_0 = \frac{27\pi^2\sigma^4}{2\varepsilon^3}. \tag{3.60}$$

Differentiating equation (3.59) with respect to R gives

$$\left.\frac{d^2B}{dR^2}\right|_{R_0} = -\frac{18\pi^2\sigma^2}{\varepsilon} < 0, \tag{3.61}$$

which shows that R_0 is a maximum of the action in the set of configurations described by (3.57) and suggests the presence of a negative mode corresponding to variation of R. Differentiating (3.47) with respect to ρ, we obtain

$$\phi''' + \frac{3}{\rho}\phi'' - \frac{3}{s^2}\phi' = \frac{d^2V}{d\phi^2}\phi', \tag{3.62}$$

or equivalently

$$S''[\phi_b]\phi' = \left[-\frac{d^2}{d\rho^2} - \frac{3}{\rho}\frac{d}{d\rho} + \frac{d^2V}{d\phi^2}\right]\phi' = -\frac{3}{\rho^2}\phi'. \tag{3.63}$$

In the thin wall approximation, where ϕ' is non negligible only for $\rho \approx R$, we can safely replace ρ by R in the right-hand side, thus obtaining a negative eigenvalue $-\frac{3}{R^2}$.

Evolution of the bubble The bounce plays a double role. Not only its action determines the nucleation rate, but its analytic continuation to Minkowski space gives the evolution of the bubble after nucleation, without the need of further calculation. Making the substitution $\tau \to it$ in the $O(4)$ invariant Euclidean bounce, gives the $O(1,3)$ invariant Minkowskian solution

$$\phi(t, \mathbf{x}) = \phi_b(\rho = \sqrt{|\mathbf{x}|^2 - t^2}), \tag{3.64}$$

which of course is valid outside the light cone of the origin. Now the physical interpretation of the bounce is clear. In the thin wall approximation, the bubble at nucleation has a thin wall of radius R_0 separating the true vacuum interior from the false vacuum exterior. Once the bubble has formed, it begins to expand tracing out an hyperboloid in Minkowski space, which is shown in Figure 3.11. The radius of the wall is given by

$$R(t) = \sqrt{R_0^2 + t^2}, \tag{3.65}$$

and increases at a velocity

$$v = \frac{dR}{dt} = \frac{t}{R} = \sqrt{1 - \left(\frac{R_0}{R}\right)^2}, \tag{3.66}$$

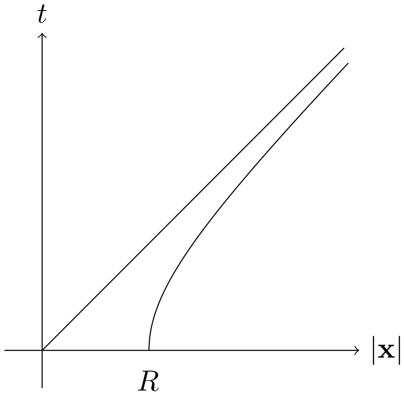

Figure 3.11: Evolution of the bubble in Minkowski space

which asimptotically approaches the speed of light. In particular, if R_0 is a microscopic quantity, this happens almost instantaneously. The energy carried by the wall of the expanding bubble is obtained by Lorentz-boosting the surface tension of the wall at rest,

$$E_{\text{wall}} = 4\pi R^2 \sigma \frac{1}{\sqrt{1-v^2}} = \frac{4\pi}{3} R^3 \varepsilon, \tag{3.67}$$

which is exactly the gain in volume energy coming from the conversion of false vacuum to true. Thus, in the thin wall approximation, all the released energy goes to accelerate the bubble wall.

Including finite temperature effects Our discussion here is based on the review of [21], but more details can be found in the references therein. In the presence of a non zero temperature $T = \frac{1}{\beta}$, some modifications to the previously outlined picture of bubble nucleation are required. Firstly, the zero temperature scalar field potential $V(\phi)$ is replaced by a finite temperature effective potential $V_{\text{eff}}(\phi, T)$. Secondly, new instability mechanisms are available.

An example is provided by thermally assisted quantum tunneling, where instead of tunneling directly from the false vacuum, the system is thermally excited to an higher energy state for which the barrier penetration integral is smaller. Keeping track only of the exponential factors, the thermally averaged decay rate is

$$\Gamma_{\text{tunn}} \sim \int_{E_F}^{E_{\text{top}}} dE e^{-\beta(E-E_F)} e^{-B(E)} \sim e^{-\beta(E^*-E_F)} e^{-B(E^*)}, \tag{3.68}$$

where E^* minimizes the exponent. Minimization is achieved if the bounce solution ϕ_b is also periodic in the Euclidean time τ with period β. Then, we have

$$\Gamma_{\text{tunn}} \sim e^{-(S[\phi_b]-S[\phi_F])}, \tag{3.69}$$

where integration in both actions is over a single period β.

Another example are purely thermal transitions, where the system is thermally excited directly to the top of the barrier without nedd of quantum tunneling at all,

$$\Gamma_{\text{therm}} \sim e^{-\beta(E_{\text{top}}-E_F)}. \tag{3.70}$$

The two possibilities are shown in Figure 3.12.

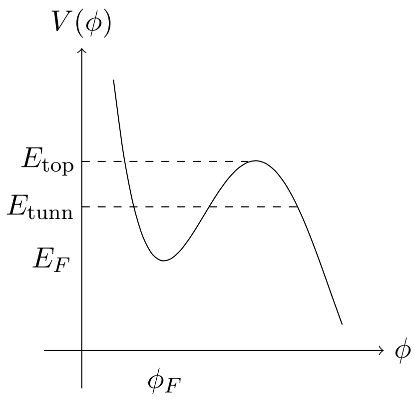

Figure 3.12: The two possible transitions at finite temperature.

3.3 Including gravity

Gravitational effects have been neglected so far, but they can be important when the energies involved in the transition are close to the Planck scale or, even at lower energies, if the radius of the bubble at nucleation is large enough to be sensitive to the curvature of spacetime. Moreover, they can come into play in the evolution of the bubble after nucleation. The extension of the theory of vacuum decay to include gravity is a non trivial matter. One possibility is to argue from analogy, as was first done by Coleman and de Luccia in [7]. With gravitational effects taken into account, the decay rate should still be related to a bounce solution ϕ_b,

$$\frac{\Gamma}{V} = Ae^{-\frac{B}{\hbar}}, \tag{3.71}$$

with

$$B = S_E[\phi_b] - S_E[\phi_F], \tag{3.72}$$

but the Euclidean action now includes also a gravitational contribution given by an Euclidean Einstein-Hilbert term,

$$S_E = \int d^4x\sqrt{g}\left[\frac{1}{2}g^{ab}\partial_a\phi\partial_b\phi + V(\phi) - \frac{1}{2k}R\right], \tag{3.73}$$

where the metric g_{ab} has positive signature, g is the determinant of the metric and $k = 8\pi G$. It is worth observing that now the absolute value of the potential matters: adding a constant to V amounts to introducing a cosmological constant Λ. Thus, the nature of spacetime in the initial false vacuum state depends on the value of the potential in the false vacuum,

$$\Lambda_F = \sqrt{\frac{3}{kV(\phi_F)}}. \tag{3.74}$$

Moreover, since $V(\phi_F) \neq V(\phi_T)$, we expect to have different cosmological constants inside and outside the bubble.

As we have seen, in flat space, i.e. when gravitational effects are negligible, the $O(4)$ invariant bounce has minimum action. Altough a corresponding result in curved space has not been proved, it seems to be a reasonable assumption. If it turns out that there exists another solution with lower action, then the estimate for the decay rate we are going to obtain is only a lower bound. The $O(4)$ symmetry implies that the metric has the form

$$ds^2 = d\xi^2 + \rho^2(\xi) d\Omega_3^2, \tag{3.75}$$

where $d\Omega_3^2$ is the metric of the three-sphere, with Ricci scalar

$$R = -\frac{6}{\rho^2}(\rho\rho'' + \rho'^2 - 1), \tag{3.76}$$

and $\phi = \phi(\xi)$, with diagonal energy momentum tensor

$$T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\partial\left(\sqrt{g}\mathcal{L}\right)}{\partial g^{\mu\nu}} \quad \Rightarrow \quad \begin{cases} T_{\xi\xi} & = \frac{1}{2}\phi'^2 - V(\phi), \\ T_{ij} & = -\rho^2\left(\frac{1}{2}\phi'^2 + V(\phi)\right)\delta_{ij}, \quad (i \neq \xi). \end{cases} \tag{3.77}$$

With this ansatz, the Klein Gordon equation for the scalar field in a curved background,

$$\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu\phi\right) = \frac{dV}{d\phi}, \tag{3.78}$$

becomes

$$\phi'' + 3\frac{\rho'}{\rho}\phi' = \frac{dV}{d\phi}. \tag{3.79}$$

and the Einstein's equations for the metric

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = kT_{\mu\nu} \tag{3.80}$$

have just one indipendent component, e.g. the $\xi\xi$-component,

$$\rho'^2 = 1 + \frac{k}{3}\rho^2\left(\frac{1}{2}\phi'^2 - V(\phi)\right). \tag{3.81}$$

Finally, the action is

$$S_E = 2\pi^2 \int d\xi \left\{\rho^3\left[\frac{1}{2}\phi'^2 + V(\phi)\right] + \frac{3}{k}(\rho^2\rho'' + \rho\rho'^2 - \rho)\right\}. \tag{3.82}$$

Integrating by parts to eliminate the second derivative term and neglecting the surface contribution, as we are only interested in the action difference between two solutions that agree at infinity, we obtain

$$S_E = 4\pi^2 \int d\xi \left\{\rho^3\left[\frac{1}{2}\phi'^2 + V(\phi)\right] - \frac{3}{k}(\rho\rho'^2 + \rho)\right\}, \tag{3.83}$$

which using (3.81) becomes

$$S_E = 4\pi^2 \int d\xi \; \rho\left(\rho^2 V(\phi) - \frac{3}{k}\right). \tag{3.84}$$

The thin wall approximation The only difference between equation (3.79) and its counterpart (3.47) in the flat case is the coefficient of the friction term, which is now proportional to $\frac{\rho'}{\rho}$ instead of $\frac{1}{\rho}$. Since in the thin wall approximation this term is neglected anyway, the analog of equation (3.55) can still be used to implicitly define a solution,

$$\xi - \bar{\xi} = \int_{\frac{\phi_F - \phi_T}{2}}^{\phi} \frac{d\phi}{\sqrt{2\left[V_0(\phi) - V_0(\phi_F)\right]}}, \tag{3.85}$$

where $\bar{\xi}$ is an integration constant. Once we have ϕ, we can solve (3.81) to find ρ. Since this is a first-order differential equation, we also need an integration constant, which can be chosen to be the radius of curvature of the bubble wall,

$$\bar{\rho} = \rho(\bar{\xi}). \tag{3.86}$$

This can be determined by imposing that B is stationary. As in the flat case, we divide the integration region in three parts. From equation (3.81) we have

$$d\xi = d\rho \left(1 - \frac{k}{3}\rho^2 V\right)^{-\frac{1}{2}}, \tag{3.87}$$

so from the inside of the bubble we have

$$\begin{aligned} B_{\text{inside}} &= -\frac{12\pi^2}{k}\int_0^{\bar{\rho}} d\rho\, \rho \left\{\left[1 - \frac{k}{3}\rho^2 V(\phi_T)\right]^{\frac{1}{2}} - (\phi_T \to \phi_F)\right\} \\ &= \frac{12\pi^2}{k^2}\left(V(\phi_T)^{-1}\left\{\left[1 - \frac{k}{3}\rho^2 V(\phi_T)\right]^{\frac{3}{2}} - 1\right\} - (\phi_T \to \phi_F)\right). \end{aligned} \tag{3.88}$$

In the wall, we can replace ρ with $\bar{\rho}$ and V with V_0, whence

$$B_{\text{wall}} = 4\pi^2\bar{\rho}^3 \int d\xi \left[V_0(\phi) - V_0(\phi_F)\right] = 2\pi^2\bar{\rho}^3\sigma. \tag{3.89}$$

Outside the bubble, the bounce and the false vacuum agree and we don't have any contribution,

$$B_{\text{outside}} = 0. \tag{3.90}$$

Now we focus on two interesting cases. The first one is

$$V(\phi_F) = \varepsilon, \qquad V(\phi_T) = 0. \tag{3.91}$$

Then we have a stationary point at

$$\bar{\rho} = \frac{12\sigma}{4\varepsilon + 24\pi G\sigma^2} = \frac{R_0}{1 + \left(\frac{R_0}{2\Lambda_F}\right)^2}, \tag{3.92}$$

where R_0 is the critical radius of the bubble in the flat case. Evaluating B at the stationary point, we get

$$B = \frac{B_0}{\left[1 + \left(\frac{R_0}{2\Lambda_F}\right)^2\right]^2}, \tag{3.93}$$

where B_0 is the expression we get in the flat case. Thus, gravitation makes the critical radius smaller and the nucleation of the bubble more likely. The second case is

$$V(\phi_F) = 0, \qquad V(\phi_T) = \varepsilon. \tag{3.94}$$

Then, the stationary point is

$$\bar{\rho} = \frac{R_0}{1 - \left(\frac{R_0}{2\Lambda_T}\right)^2}, \tag{3.95}$$

and at this point B is

$$B = \frac{B_0}{\left[1 - \left(\frac{R_0}{2\Lambda_T}\right)^2\right]^2}. \tag{3.96}$$

Now, the situation is just the opposite: gravitation makes the critical radius larger and the nucleation of the bubble less likely. Indeed, if $R_0 = 2\Lambda_T$, or equivalently $\varepsilon = \frac{3}{4}k\sigma^2$, the critical radius diverges and the decay probability vanishes: gravitational effects have stabilized the false vacuum.

If one expected that it was always possible to tunnel from a vacuum of higher energy to another vacuum of lower energy, then this result might seem a bit surprising. However, it must be remembered that the bounce does not describe tunneling from false vacuum to true vacuum, but rather from false vacuum to a configuration with a bubble of true vacuum surrounded by a false vacuum background. In the flat case, the energy of a bubble of radius $\bar{\rho}$ at the time of its materialization is

$$E = -\frac{4\pi}{3}\varepsilon\bar{\rho} + 4\pi\sigma\bar{\rho}^2 = \frac{4\pi}{3}\varepsilon\bar{\rho}^2(R_0 - \bar{\rho}), \tag{3.97}$$

and has to be zero from energy conservation. Without gravity, this is achieved by setting $\bar{\rho} = R_0$. With gravity included, a bubble that nucleates in Minkowski spacetime must also have zero energy, since there is a conserved energy if the space is asymptotically flat. Thus, $\bar{\rho}$ must be bigger than R_0 if the gravitational contribution to the energy is positive or smaller if it is negative. In order to compute these corrections, there are two terms to be considered. The first is the Newtonian potential energy of the bubble, which is

$$E_{\text{Newton}} = -\frac{\varepsilon\pi\bar{\rho}^5}{15\Lambda_T^2}. \tag{3.98}$$

The second is a correction to the volume of the bubble. From equation (3.87), the infinitesimal volume element is

$$4\pi\rho^2 d\xi = 4\pi\rho^2 d\rho\left(1 - \frac{1}{2}\frac{\rho^2}{\Lambda_T^2}\right) + \mathcal{O}(G^2), \tag{3.99}$$

and upon integration we obtain

$$E_{\text{geom}} = \frac{2\pi\varepsilon\bar{\rho}^5}{5\Lambda_T^2}. \tag{3.100}$$

The sum of these two terms is positive,

$$E_{\text{grav}} = E_{\text{Newton}} + E_{\text{geom}} = \frac{\pi\varepsilon\bar{\rho}^5}{3\Lambda_T^2}, \tag{3.101}$$

so the bubble is larger in the presence of gravity than in its absence. More explicitly, the energy of the bubble is the sum of a negative volume term and a positive surface term. In the flat case, it is always possible to compensate these two terms by making the bubble large enough, no matter how small ε is. This is no more true when gravity is included, because the negative energy density inside the bubble distorts the geometry in such a way that the volume/surface ratio is diminished and, if ε is small enough, no bubble, no matter how big, can have zero energy. This result can also be seen as a consequence the peculiar feature of Anti-de Sitter spacetime that the volume enclosed by a two-sphere of radius R only grows as R^2 at large radius, rather than as R^3, so the volume/surface ratio is bounded.

The Hawking Moss instanton The Coleman-de Luccia solution is not the only example of an instanton of cosmological interest. It is at least worth mentioning that the existence of another type of solution was pointed out by Hawking and Moss [23]. It is a homogeneous solution where the scalar field sits at a local maximum of the potential $\phi = \phi_{\text{top}}$ and thus is more akin to the high-temperature thermal production of a critical bubble than to a quantum tunneling bounce. The critical radius can be computed as

$$\rho = \Lambda_{\text{top}} \sin\left(\frac{\xi}{\Lambda_{\text{top}}}\right), \tag{3.102}$$

and the Euclidean action is

$$S_{HM} = -\frac{24\pi^2}{k^2 V(\phi_{\text{top}})}, \tag{3.103}$$

leading to a tunneling exponent

$$B_{HM} = S_{HM} - S[\phi_F] = \frac{24\pi^2}{k^2}\left(\frac{1}{V(\phi_F)} - \frac{1}{\phi_{\text{top}}}\right). \tag{3.104}$$

3.4 Witten's bubble of nothing

An interesting and highly non trivial application of instanton methods is Witten's analysis of the non-perturbative instability of the Kaluza-Klein vacuum in [8]. The original Kaluza-Klein theory assumes for the ground state of pure gravity in 5 dimensions a product space $M_4 \times S^1$, where M_4 is the four dimensional Minkowski spacetime and the compact space is a circle of radius R. An analysis of the mass spectrum of the effective four-dimensional theory shows that there aren't any tachyonic modes, so this space is classically stable. The next step is to investigate the possibility of a semiclassical instability, looking for a bounce solution of the Euclidean Einstein's equations of motion with the same asymptotic behaviour of the vacuum state and leading to a negative mode in the functional determinant obtained by expanding the action around the bounce.

Construction of the bounce The Euclidean Kaluza-Klein metric, obtained by analytical continuation (i.e. sending $t \to -i\tau$), is

$$ds^2 = d\tau^2 + dx^2 + dy^2 + dz^2 + d\phi^2, \tag{3.105}$$

where ϕ is a periodic variable with period $2\pi R$. Using polar coordinates in the four dimensional Euclidean space, the metric becomes

$$ds^2 = dr^2 + r^2 d\Omega_3^2 + d\phi^2, \tag{3.106}$$

where $d\Omega_3^2$ is the metric of the three-sphere. Being asymptotically flat, a good candidate for a bounce solution is the five-dimensional Euclidean Schwarzschild metric,

$$ds^2 = \left(1 - \frac{\alpha}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2 + \left(1 - \frac{\alpha}{r^2}\right) d\phi^2. \tag{3.107}$$

In order for this space to be non singular, it is easy to see that ϕ must be a periodic variable with period $2\pi\sqrt{\alpha}$. Changing coordinate $r = \sqrt{\alpha} + \lambda^2$ and expanding near $\lambda = 0$, the two pathological terms in the metric give

$$\left(1 - \frac{\alpha}{r^2}\right)^{-1} dr^2 + \left(1 - \frac{\alpha}{r^2}\right) d\phi^2 \quad \rightarrow \quad 2\sqrt{\alpha}\left(d\lambda^2 + \frac{\lambda^2}{\alpha} d\phi^2\right). \tag{3.108}$$

This expression is nothing but the metric of the plane in polar coordinates, provided that we choose ϕ to be periodic with period $2\pi\sqrt{\alpha}$. Moreover, in order for this solution to asymptotically approach the Kaluza-Klein vacuum described by the metric (3.105), where ϕ is a periodic variable with period $2\pi R$, we must set $\sqrt{\alpha} = R$. Thus, the right expression for the bounce is

$$ds^2 = \left[1 - \left(\frac{R}{r}\right)^2\right]^{-1} dr^2 + r^2 d\Omega_3^2 + \left[1 - \left(\frac{R}{r}\right)^2\right] d\phi^2, \tag{3.109}$$

with r running from R to ∞, because we had $r = \sqrt{\alpha} + \lambda^2$ with λ running from 0 to ∞. Finally, the presence of the negative mode can be argued from the analysis of [24] and is discussed in the appendix of [8].

Interpretation of the bounce In order to understand what kind of space the Kaluza-Klein vacuum decays into, we can study the analytical continuation of the Euclidean bounce to Minkowski space. The metric of the three-sphere in equation (3.109) can be written as

$$d\Omega_3^2 = d\theta^2 + \sin^2\theta d\Omega_2^2, \tag{3.110}$$

where $d\Omega_2^2$ is the metric of the two-sphere and θ is related to the Euclidean time through $\tau = r\cos\theta$. Then, Wick-rotating with respect to the three dimensional surface $\tau = 0$, i.e replacing $\tau \rightarrow it$ corresponds to Wick-rotating with respect to the surface $\theta = \frac{\pi}{2}$, i.e. replacing $\theta \rightarrow \frac{\pi}{2} + i\psi$. Thus, we obtain

$$ds^2 = \left[1 - \left(\frac{R}{r}\right)^2\right]^{-1} dr^2 - r^2 d\psi^2 + \cosh^2\psi d\Omega_2^2 + \left[1 - \left(\frac{R}{r}\right)^2\right] d\phi^2. \tag{3.111}$$

For the moment, let us drop the factors inside the square brackets and focus on the two dimensional space spanned by the variables r and ψ only,

$$ds^2 = dr^2 - r^2 d\psi^2. \tag{3.112}$$

This is just the metric of the two dimensional Minkowski space in a weird coordinate system,

$$x = r\cosh\psi, \qquad t = r\sinh\psi, \tag{3.113}$$

which doesn't cover the whole space, but only the exterior of the light cone $x^2 - t^2 > 0$, which is called the Rindler wedge and is showed in Figure 3.13a. Thus, without the factors inside the square brackets, the metric of equation (3.111) would exactly be the Kaluza-Klein metric, just in a weird coordinate system which doesn't cover the whole space. These factors are unimportant at large r, so the metric asymptotically approaches the Kaluza-Klein metric, as it should be. However, when these factors are included, the variable r runs from R to ∞, so that only the exterior of the hyperboloid $x^2 - t^2 > R^2$ can be described. With no compact dimensions, this

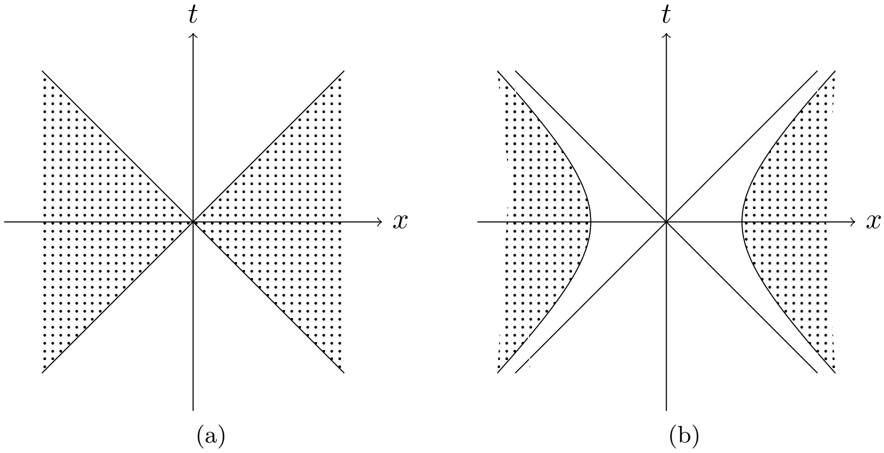

Figure 3.13: The exterior of the light cone and of a hyperboloid in Minkowski space.

space would have a boundary, but this is no more true when the fifth dimension is taken into account. The last term in equation (3.111) tells us that the radius of the Kaluza-Klein circle, while equal to R asymptotically, is $R\sqrt{1-\left(\frac{R}{r}\right)^2}$ and shrinks to zero as r approaches R. Thus there is no boundary and the space is both non singular and geodesically complete.

As we have seen, the conventional decay of a false vacuum is characterized by the nucleation of a bubble of true vacuum, whose wall is expanding with time, converting false vacuum to true. Here, the decay of the Kaluza-Klein vacuum is even more dramatic. Rather than a bubble of true vacuum, from the point of view of un observer who is unable to detect the compact dimension, there is bubble of *nothing*, a hole forming in the spacetime whose boundary, just as the wall of the bubble, rapidly expands asymptotically approaching the speed of light,

$$r(t) = \sqrt{R^2 + t^2}. \tag{3.114}$$

We can also compute the B coefficient in the decay rate for the Kaluza-Klein vacuum. We consider the five dimensional Euclidean action

$$S_E = -\frac{1}{32\pi^2 GR}\int d^5x\sqrt{g}R + S_{GH}, \tag{3.115}$$

where the S_{GH} is the Gibbons-Hawking boundary term needed to cancel second derivative terms in the action and $2\pi RG$ is the gravitational constant in five dimensions. The bounce is Ricci flat and the only contribution comes from the boundary term, yielding

$$B = \frac{\pi R^2}{4G}. \tag{3.116}$$

Including fermions Is there a way to save the Kaluza-Klein vacuum? As suggested by Witten, stability can be achieved by introducing fermions. We start observing that an angle α enters in the definition of fermions in a Kaluza-Klein theory,

$$\psi(x,\phi) = \sum_n \psi_n(x) e^{i\left(n-\frac{\alpha}{2\pi}\right)\frac{\phi}{R}}, \tag{3.117}$$

which can be arbitratily chosen as long as the lagrangian of the theory is invariant under $\psi \to e^{i\alpha}\psi$. The value $\alpha = 0$ (*periodic* boundary conditions) is trivially allowed, and so is

$\alpha = \pi$ (*anti-periodic* boundary conditions), since $\psi \to -\psi$ is a symmetry of every Lorentz invariant Lagrangian. Moreover, in the presence of a $U(1)$ symmetry, the value of α would be completely arbitrary.

This ambiguity in the definition of fermions turns out to play a key role in the existence of the bounce. As we have seen, a vacuum can only decay into a state with the same asymptotic behaviour, which implies that, as soon as fermions are included in the theory, the bounce must carry the same spin structure as the vacuum. However, since the bounce might differ from the vacuum in topology and not just in geometry, this is not guaranteed to be the case. Such a topological obstruction due to a topology changing bounce can be easily seen to arise in the decay of the Kaluza-Klein vacuum. Around the Kaluza-Klein circle, fermions can be both periodic and antiperiodic. It is worth observing that only the first possibility is admitted in a supersymmetric context, since supersymmetry transforms periodic bosonic fields into periodic fermionic fields. As for the bounce, the collapsing circle can be seen as the boundary of a disc. Since this space is simply connected, its spin structure is unique and only *anti-periodic* fermions are admitted. Combining these observations with the requirement that the spin structure of the vacuum and that of the bounce must be the same, the conclusion is straightforward. If we choose $\alpha = \pi$ for the Kaluza-Klein vacumm, the analysis of the previous paragraphs is perfectly valid and the bounce still represents an instability. Conversely, if we choose $\alpha = 0$ for the vacuum as required by supersymmetry, and since only $\alpha = \pi$ is admitted for the bounce, the decay channel we have described becomes forbidden and the supersymmetric Kaluza-Klein vacuum is safe.

Fermions also provide an interesting link with the positive energy theorem. According to the general theory of vacuum decay, a false vacuum can only decay to a state with the same energy and the same asymptotic behaviour at spatial infinity. Given these basic features, Minkowski space is guaranteed to be stable against a semiclassical decay process by the positive energy theorem, first proved by Schoen and Yau in [25] and [26] or, using an alternative argument based on spinors, by Witten in [27]. The theorem states that all asymptotically flat solutions of the Einstein's equations have strictly positive energy, except for Minkowski spacetime itself which has zero energy. An analogous statement is not valid for the Kaluza-Klein vacuum, an explicit counterexample being given by the metric (3.109), which is both asymptotically flat and has zero energy. This is possible because the bounce differs from the vacuum in topology, as well as in geometry. Only periodic spinors can be used in the proof of [27], which fails in the presence of antiperiodic spinors and thus cannot apply in our case [8].

Chapter 4

M-theory bubbles of nothing

In this Chapter, we study generalizations of Witten's *bubble of nothing* as a possible non-perturbative mechanism leading to instabilities in the context of non supersymmetric Anti-de Sitter compactifications. The presence of such instabilities would provide further evidence for the conjecture of [16] that all non-supersymmetric AdS geometries are unstable, especially in those cases where stability has already been proven at the perturbative level. In principle, this analysis could have been done in a generic string theory setup. However, in the present work we focus on M-theory and its low energy limit, which is eleven dimensional supergravity.

The Chapter is structured as follows. We begin with some preliminary remarks to set out our notation. First, we introduce the field content of eleven dimensional supergravity, that is the metric g_{MN}, a Majorana spinor Ψ_M and a three-form potential A_{MNP}, with four-form field strength $F_4 = dA_3$. We focus on the bosonic degrees of freedom and write down their equations of motion in a background where the gravitino has been set to zero. Then we provide some basic notions about compactifications *à la* Kaluza-Klein, namely the existence of solutions of the equations of motion of the $D = 11$ theory in which the metric describes the product of a n dimensional spacetime (typically $n = 4$, but other dimensions are also possible) times a $D - n$ compact space. After that, we are ready to embark on our exploration.

Our starting point are AdS compactifications to four dimensions, known as Freund-Rubin solutions, where F_4 is proportional to the volume form of the external space. As shown by a no-go theorem due to Young [9], the presence of an external flux seems to prevent a simple generalization of Witten's *bubble of nothing*, where a higher dimensional sphere rather than a circle shrinks to zero size. A general argument is provided which allows one to argue that, in the presence of an external flux, the Einstein equations for the bubble can't be solved unless the dimension of the collapsing sphere that carries the instability is one, that is a circle. However, there are at least a couple of aspects of this result that are worth discussing. First, a purely external flux is not the most general case, since there also exist AdS compactifications to four dimensions, known as Englert solutions, where an additional internal flux is turned on. Secondly, it is not obvious how Young's argument generalizes to more complicate bubble geometries where the collapsing sphere does not correspond to the entire internal space, since the most interesting case is when it is only a subspace of the internal space, possibly fibrated over the rest, that shrinks to zero size.

The key role played by a non trivial fibration structure becomes evident in the $AdS_5 \times \mathrm{C}P^3$ geometry, where the $\mathrm{C}P^3$ is realized as a fibration of S^2 over a S^4 base. Following [10], we will construct a *bubble of nothing* where the S^2 fiber rather than a circle shrinks to zero size, thus evading the difficulties pointed out by Young. However, this example is in some ways peculiar, since there is no external flux supporting the geometry. Instead, the flux is purely internal and reorients itself at the bubble in such a way that it evaporates from the fiber and the Einstein equations can be solved.

External and internal form fluxes, non trivial fibration structures of the internal space and higher dimensional collapsing spheres would suffice to render the bubble geometry quite elaborate with respect to the original construction by Witten. However, there is still another element that has to be brought in. Motivated by the analysis of [11] which deals with a similar problem in the context of ten dimensional type IIB supergravity, we argue that the Page charge of a bubble geometry must be zero. This poses a serious problem for the construction of the bubble whenever the Page charge of the vacuum is different from zero, which is the case for AdS compactifications of both the Freund-Rubin and Englert type, since two solutions with different Page charges cannot coexist. As a possible way out, we propose to introduce M2 brane instantons as sources of additional flux in the bubble geometry which can account for the mismatching with the vacuum. This completes the general scenario and we can eventually turn to a concrete example.

4.1 Eleven dimensional supergravity

This section is not meant to be an exhaustive introduction to the topic but only a preliminary set up of notation. In particular, spinors and supersymmetry will make only a passing appeareance, since our focus is on supergravity solutions that are both bosonic and non supersymmetric. We refer to [28] for a comprehensive introduction to supergravity theories and to [29] for a review of Kaluza-Klein compactifications in the context of eleven dimensional supergravity.

An overview There are at least two reasons why eleven dimensional supergravity is important. First, $D = 11$ is the maximum allowed spacetime dimension where a supergravity theory can be consistently formulated, since any higher dimension would require the presence of fields of spin higher than two. Secondly, it can be regarded as the low energy limit of a theory of extended objects called M-theory, which is considered to be the master theory containing the various string theories.

Let us spend a few lines on the higher spin consistency problem. We recall that the supersymmetry algebra contains spin $\frac{1}{2}$ fermionic generators Q^I, $I = 1, \dots, \mathcal{N}$ commuting with translations but not with Lorentz generators. Therefore, instead of single particle states, in a supersymmetric theory we deal with supermultiplets of particle states, with particles belonging to the same supermultiplet having the same mass but different spin. In particular, the spin range of the particles in a supermultiplet depends on the number of supersymmetry generators $\mathcal{N}$, which cannot be arbitrary large if the theory has not to contain particles of spin higher than two. According to the dimension of the spinor representation of the Lorentz group and thus on the spacetime dimension, we find for example that maximal supersymmetry corresponds to $\mathcal{N} = 8$ in $D = 4$ and $\mathcal{N} = 2$ in $D = 10$. An equivalent but dimension independent statement is that a theory with maximum spin two can accomodate no more than 32 supercharges, each associated with some component of a spinor of the associated Lorentz group. In $D = 11$, a Majorana spinor has precisely 32 components, so that precisely $\mathcal{N} = 1$ spinor worth of supercharges is allowed.

This maximally supersymmetric eleven dimensional theory was constructed in 1978 by Cremmer, Julia and Scherk [30], but it was only in the mid-1990s that it entered the string theory map in its own right, appearing as a part of the intricate web of dualities that relates the various ten dimensional superstring theories between each other [31]. In fact, two of the superstring theories, namely the type IIA and the heterotic $E_8 \times E_8$, can be seen to exhibit an eleventh dimension at strong coupling. This suggested the existence of a unique underlying fundamental theory, called M-theory, with eleven dimensional supergravity as its low energy limit. However, despite the evidence of its existence is so compelling and the duality aspects are pretty well understood, a precise formulation of M-theory is still lacking.

The field content We are now ready to introduce the field content of eleven dimensional supergravity, whose structure, compared to other lower dimensional supergravity theories, is relatively simple. Obviously, it must contain the gravitational field g_{MN}, that is a graviton. It transforms under the symmetric traceless tensor representation of $SO(D-2)$, the little group of a massless particle, and corresponds to $\frac{(D-1)(D-2)}{2} - 1 = 44$ bosonic degrees of freedom. In the vielbein formalism, which is needed in order to formulate gravity in the presence of fermions, it is described by the vielbein field ${e_A}^M$. Here $M, N, \dots$ are *curved* indices, transforming non trivially under general coordinate transformations, and $A, B, \dots$ are *flat* indices, transforming non trivially under Lorentz transformations. Then, we also have the gravitino field Ψ_M where, in addition to the explicit vector index, an implicit spinor index is understood. For each value of M, we have a 32 component Majorana spinor. If we properly take into account local symmetries, we find that Ψ_M contains 128 real fermionic degrees of freedom. However, supersymmetry requires that the theory must contain an equal number of bosonic and fermionic degrees of freedom, so there are still 84 bosonic degrees of freedom missing. In $D = 11$, this is exactly the number of states coming from a three-form gauge potential A_{MNP}. Thus, the idea of Cremmer, Julia and Scherk, who first formulated eleven dimensional supergravity in [30], was that the theory should contain the metric tensor g_{MN}, a three-form A_{MNP} and the vector spinor Ψ_M.

The Lagrangian and the equations of motion The bosonic part of the action of eleven dimensional supergravity (where the fermions have been set to zero, $\Psi_M = 0$) is given with Minkowski signature by

$$
\begin{aligned}
S = \int d^{11}x \, e \bigg(& \frac{1}{4} {e_B}^N {e_A}^M {R_{MN}}^{AB} - \frac{1}{48} F^{MNPQ} F_{MNPQ} \\
& + \frac{2}{(12)^4} \varepsilon^{M_1 \dots M_{11}} F_{M_1 \dots M_4} F_{M_5 \dots M_8} A_{M_9 \dots M_{11}} \bigg) .
\end{aligned}
\tag{4.1}
$$

Here $e = \det {e_M}^A$, $\varepsilon^{M_1 \dots M_{11}}$ is the Levi-Civita tensor with $\varepsilon_{1 \dots 11} = e$ and $F_{MNPQ} = 4 \partial_{[M} A_{NPQ]}$ is a four-index anti-symmetric tensor field strenght satisfying the Bianchi identity,

$$
\partial_{[M} F_{NPQR]} = 0. \tag{4.2}
$$

In form notation, we have $F_4 = dA_3$ with $dF_4 = 0$. The last term in the action is called a Chern-Simon or topological term and can be rewritten in the form

$$
S_{CS} = -\frac{1}{3} \int F_4 \wedge F_4 \wedge A_3. \tag{4.3}
$$

Varying the action (4.1) with respect to ${e_M}^A$ and A_{MNP}, we obtain the field equations, namely the Einstein equation

$$
R_{MN} - \frac{1}{2} g_{MN} R = \frac{1}{3} \left({F_M}^{PQR} F_{NPQR} - \frac{1}{8} g_{MN} F^{PQRS} F_{PQRS} \right), \tag{4.4}
$$

which is usually rewritten in the trace-reversed form

$$R_{MN} = \frac{1}{3}\left(F_M{}^{PQR}F_{NPQR} - \frac{1}{12}g_{MN}F^{PQRS}F_{PQRS}\right), \tag{4.5}$$

and the Maxwell equation

$$\nabla_M F^{MPQR} = -\frac{1}{576}\varepsilon^{M_1\ldots M_8 PQR}F_{M_1M_2M_3M_4}F_{M_5M_6M_7M_8}, \tag{4.6}$$

where ∇_M is the covariant derivative. It can be usefully rewritten in the compact form

$$d * F_4 = -F_4 \wedge F_4, \tag{4.7}$$

where $*$ means the Hodge dual.

We will be mainly interested in solutions of the Euclidean equations of motion. These are obtained from the analytic continuation of the action, that is upon the substitution $t \to -i\tau$ in (4.1). Then, it is easy to see that the Euclidean Einstein equation is formally identical to its Minkowskian counterpart (4.4). Conversely, this is not true for the Euclidean Maxwell equation where, due to the presence of the Chern-Simon term in the action, an extra i appears,

$$d * F_4 = -iF_4 \wedge F_4. \tag{4.8}$$

The Kaluza-Klein recipe The idea of compactification dates back to 1920s, when Kaluza [32] and Klein [33] proposed a way to unify the gravitational and electromagnetic interactions by postulating a fifth extra dimension for spacetime. They showed how starting from a theory of pure gravity in five dimensions and compactifying it on a circle, a four dimensional Einstein-Maxwell-scalar theory is obtained. This was the first example of dimensional reduction as a mechanism to recover lower dimensional physics from a higher dimensional theory and, altough there have been many advances and developments since then, the procedure still bears their names.

Before getting into AdS compactifications in $D = 11$ supergravity, it is useful to outline which are the main steps in the procedure, without too much entering into technical details. For concreteness, and for the sake of our four dimensional world, we now refer to dimensional reduction to four dimensions, but in general the dimension of the compactified theory can be different than four.

- The starting point is a D dimensional theory of gravity g_{MN} plus matter fields, which we collectively denote as Φ, with action

$$S = \int d^D z \frac{1}{4}\sqrt{-g}R + \ldots. \tag{4.9}$$

- The next step is to look for a spontaneous compactification, that is a stable ground state solution $\langle g_{MN}\rangle$ and $\langle\Phi\rangle$ of the field equations such that the metric $\langle g_{MN}\rangle$ describes a (possibly warped) product space $M_4 \times M_{D-4}$, where M_4 is a maximally symmetric spacetime with coordinates x^μ and M_{D-4} is an internal compact space with coordinates y^m. The requirement of compactness of the internal space guarantees the discreteness of the four dimensional spectrum.

- The spectrum of the four dimensional theory is to be found by considering fluctuations of the fields about their ground state values,

$$\begin{cases} g_{MN} = \langle g_{MN} \rangle + h_{MN}, \\ \Phi = \langle \Phi \rangle + \phi, \end{cases} \tag{4.10}$$

inserting them into the equations of motion and keeping only linear terms. Then each fluctuation generically denoted as $\phi^{mn\dots}_{\mu\nu\dots}$ is decomposed as a sum of terms of the form

$$\varphi_{\mu\nu\dots}(x) Y^{mn\dots}(y), \tag{4.11}$$

where $Y^{mn\dots}(y)$ are eigenfunctions of the mass operator M^2 coming from the kinetic term acting on the internal space,

$$M^2 Y_{(n)} = m_n^2 Y_{(n)}. \tag{4.12}$$

In the end, we obtain a four dimensional effective theory with an infinite tower of massive states with masses m_n quantized in units of a fundamental mass $m \sim R^{-1}$, where R is some characteristic lenght of the compact space, plus a finite number of massless states, including the graviton, corresponding to the zero modes in (4.12).

A vacuum solution is said to be perturbatively stable if all states in the tower have positive energy. In Minkowski spacetime, this is equivalent to say that there are no negative modes (tachyons) in the spectrum, that is $m_n^2 \geq 0$ for each n. However, the question is more subtle for Anti-de Sitter spacetime where, because of the coupling to the curvature, mass is rather an ambiguous concept. The curvature gives indeed a positive contribution to the energy of a field propagating in AdS and masses can be negative, provided that they are not too negative,

$$m_n^2 \geq m_{BF}^2. \tag{4.13}$$

The lower bound, first found by Breitenlohner and Freedman in [34] and [35], is given by

$$m_{BF}^2 = -\frac{D^2}{4L^2}, \tag{4.14}$$

where L is the radius of the D dimensional Anti-de Sitter spacetime. A derivation of their result can be found in the following section.

4.2 Anti-de Sitter geometry

In this section, we provide some basic notions about the geometry of Anti-de Sitter spaces. First, we introduce some of the most commonly used coordinate systems, such as the *embedding* coordinates, the *global* coordinates and the *Poincaré* coordinates. We also describe the Euclideanization of the AdS geometry, that is the analytic continuation of the metric to Euclidean signature. For this part we refer for example to [36]. Then we move on to derive some standard stability properties of AdS spaces, namely the stabilizing role played by supersymmetry and the Breitenlohner-Freedman bound on the mass of a scalar field propagating in AdS. Our discussion mainly follows the review of [29], as well as some of the original articles.

Some AdS coordinates As it is well known, Anti-de Sitter spacetime is the maximally symmetric solution of the Einstein equations with negative cosmological constant. From the D

dimensional Einstein-Hilbert action

$$S = \frac{1}{2k^2}\int d^D x\sqrt{-g}(R-\Lambda), \tag{4.15}$$

we derive the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{\Lambda}{2}g_{\mu\nu}, \tag{4.16}$$

yielding for the Ricci scalar $R = \frac{D}{D-2}\Lambda$. Therefore, the Ricci tensor is proportional to the metric,

$$R_{\mu\nu} = \frac{\Lambda}{D-2}g_{\mu\nu}. \tag{4.17}$$

This relation defines an Einstein space. If we also require that

$$R_{\mu\nu\rho\sigma} = \frac{\Lambda}{(D-1)(D-2)}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\tau}\right), \tag{4.18}$$

the space is said to be maximally symmetric.

A convenient way to describe the D dimensional Anti-de Sitter space is through its embedding in a $(D+1)$ dimensional flat ambient space of appropriate signature, namely as the hyperboloid in $\mathrm{R}^{2,D-1}$ defined by the quadratic equation

$$\eta_{ab}x^a x^b = -(x_0)^2 + \sum_{i=1}^{D-1}(x_i)^2 - (x_D)^2 = L^2, \tag{4.19}$$

with the metric induced by the ambient space,

$$ds^2 = -(dx_0)^2 + \sum_{i=1}^{D-1}(dx_i)^2 - (dx_D)^2. \tag{4.20}$$

Here L is the AdS radius related to the cosmological constant via

$$\frac{1}{L^2} = -\frac{\Lambda}{(D-1)(D-2)}. \tag{4.21}$$

The easiest way to identify the symmetries of AdS is to observe that both (4.19) and (4.20) are invariant under linear coordinate transformations of the form $x'^a = M^a{}_b x^b$, provided that the matrix $M^a{}_b$ belongs to $SO(2,D-1)$. So the isometry group of AdS_D is also $SO(2,D-1)$, with generators

$$L^{ab} = x^a\frac{\partial}{\partial x_b} - x^b\frac{\partial}{\partial x_a}, \tag{4.22}$$

satisfying the $SO(2,D-1)$ algebra

$$\left[L^{ab}, L^{cd}\right] = \eta^{bc}L^{ad} + \eta^{ad}L^{bc} - \eta^{ac}L^{bd} - \eta^{bd}L^{ac}. \tag{4.23}$$

Any two points on the hyperboloid (4.19) are related by a transformation of the isometry group, which means that the space is homogeneous. A set of coordinates is given by

$$\begin{aligned} x_0 &= L\cosh\rho\cos\tau, \\ x_i &= L\sinh\rho\hat{x}_i, \qquad \text{with } \sum_{i=1}^{D-1}\hat{x}_i^2 = 1, \\ x_D &= L\cosh\rho\sin\tau. \end{aligned} \tag{4.24}$$

In these coordinates the metric reads

$$ds^2 = L^2\left(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_{D-2}^2\right), \tag{4.25}$$

where $d\Omega_n^2$ is the metric of the unit n-sphere. Here $\rho \in \mathrm{R}^+$ and $\tau \in [0, 2\pi]$. These coordinates are called *global* since they cover the hyperboloid defined by (4.19) exactly once. In order to avoid time-like closed curves, we can take the universal cover where $\tau \in \mathrm{R}$. Another set of coordinates, which do not cover the whole spacetime but only the so-called Poincaré patch, is given by

$$\begin{aligned} x_0 &= \frac{1}{2u}(1 + u^2(L^2 + \mathbf{y}^2 - t^2)), \\ x_i &= Luy_i, \qquad i = 1, \dots, D-2, \\ x_{D-1} &= \frac{1}{2u}(1 - u^2(L^2 - \mathbf{y}^2 + t^2)), \\ x_D &= Lut \end{aligned} \tag{4.26}$$

Here $u > 0$ and $y^\mu = (t, \mathbf{y})$ is a Lorentz vector. Then metric takes the form

$$ds^2 = L^2\left(\frac{du^2}{u^2} + u^2 dy^\mu dy_\mu\right), \tag{4.27}$$

describing a foliation of $D-1$ dimensional Minkowski spacetime over u, whence the name *Poincaré* coordinates. The warp factor u^2 means that an observer living on a Minkowski slice measures all lengths rescaled by a factor of u according to its position in the u direction. The plane $u = \infty$ is referred as the AdS conformal boundary, since the conformally equivalent metric $\frac{ds^2}{u^2}$ has boundary $\mathrm{R}^{1,D-2}$ at $u = \infty$. The plane $u = 0$ is a Killing horizon, with null Killing vector $\frac{\partial}{\partial t}$. It is a coordinate singularity, since there is no difficulty in extending the metric beyond the horizon, for example using global coordinates. Redefining $u = \frac{1}{z}$, here is another common form of the metric in Poincaré coordinates,

$$ds^2 = L^2\left(\frac{dz^2 + dy^\mu dy_\mu}{z^2}\right). \tag{4.28}$$

Here the conformal boundary is at $z = 0$ and the Killing horizon at $z = \infty$.

Euclidean AdS In this Chapter, we will be mainly concerned with the Euclidean continuation of the metric, that is we send $x_D \to -ix_D$ in (4.19) and $\tau \to -i\tau$, $t \to -it$ in each set of coordinates (4.24), (4.26). Then the Euclidean metric is

$$ds_E^2 = L^2\left(\cosh^2\rho d\tau_E^2 + d\rho^2 + \sinh^2\rho d\Omega_{D-2}^2\right) = L^2\left[\frac{du^2}{u^2} + u^2\left(dt_E^2 + d\mathbf{y}^2\right)\right]. \tag{4.29}$$

Here the $u = 0$ plane is the Euclidean flat space R^{D-1} and the $u = 0$ plane shrinks to a point. It can be convenient to compactify the boundary R^{D-1} to S^{D-1} by adding the point $u = 0$. Then the Euclidean AdS_D looks like a D dimensional ball in R^D.

Setting $L = 1$ and using the freedom of coordinate redefinition by fixing the coefficient near dr^2 to be 1, we can also write the metric in the form

$$ds_E^2 = dr^2 + \sinh^2 r d\Omega_{D-1}^2, \tag{4.30}$$

which is the one we will most often use in the rest of the Chapter.

Supersymmetry and positive energy We now discuss an important implication of AdS supersymmetry. Withouth entering into technical details and focusing on the four dimensional case for simplicity, we simply state that, in a supergravity theory, the $SO(2,3)$ symmetry group can be enhanced to the larger $OSp(4|N)$ group, by including N spinorial generators Q^A ($A = 1, \dots, N$) and the $\frac{N(N-1)}{2}$ generators T^{AB} of $SO(N)$ which rotates the Q^A into themselves. We can define Killing vectors $K^{ab}_\mu(x)$ associated to the $SO(2,3)$ generators L^{ab}, satisfying

$$\nabla_{(\mu} K^{ab}_{\nu)} = 0. \tag{4.31}$$

We can also define Killing spinors $\varepsilon^A(x)$ associated to the spinorial generators Q^A, satisfying

$$\tilde{D}_\mu \varepsilon^A = 0, \tag{4.32}$$

where $\tilde{D}_\mu = D_\mu + m\gamma_\mu\gamma_5$ is a covariant derivative. In the presence of a Killing spinor, that is of an unbroken supersymmetry, a nice algebraic argument can be used to prove the positivity of energy and, thus, AdS stability. Prior to this, however, we must pause a moment on what we mean by both energy and stability.

A possible definition of energy in asymptotically AdS spacetimes parallels the ADM (Arnowitt-Deser-Misner) definition of energy in asymptotically flat spacetimes based on Hamiltonian formalism [37] and was first discussed by Abbott and Deser in [38]. There it is shown how to each generator L^{ab} we can associate a surface integral that can be thought as a generalization of momentum or angular momentum. In particular, the time translation generator L^{04} can be interpreted as the total energy. More explicitly, let us consider a theory of gravity $g_{\mu\nu}(x)$ coupled to scalar fields $\phi^i(x)$, since they are the only non zero fields that are admitted in a maximally symmetric background configuration. Then the equations of motion are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 2T_{\mu\nu}, \tag{4.33}$$

where

$$T_{\mu\nu} = -\partial_\mu\phi^i\partial_\nu\phi^i + \frac{1}{2} g_{\mu\nu}\left[\partial_\sigma\phi^i\partial^\sigma\phi^i - 2V(\phi)\right], \tag{4.34a}$$

$$\Box\phi^i = -\frac{\partial V}{\partial\phi^i}. \tag{4.34b}$$

We consider background solutions where $g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x)$ is the AdS metric and the scalar fields are constant, $\phi^i(x) = \bar{\phi}^i$. From equation (4.34b) we see that we are at a critical point of the scalar potential. Moreover, equation (4.34a) tells us that the cosmological constant is

$\Lambda = 2V(\bar{\phi})$. Then we consider fluctuations around the background values,

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu} + h_{\mu\nu}(x), \qquad \phi^i(x) = \bar{\phi}^i + s^i(x). \tag{4.35}$$

Substituting into equation (4.33) yields

$$\tilde{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\tilde{R} - \Lambda h_{\mu\nu} = 2\left(J_{\mu\nu} + T_{\mu\nu} + \frac{1}{2}\Lambda\bar{g}_{\mu\nu}\right), \tag{4.36}$$

where the tilde means that only terms that are linear in $h_{\mu\nu}$ are retained and $J_{\mu\nu}$ contains terms of higher order in $h_{\mu\nu}$. The important point is that the left-hand side of this equation obeys the Bianchi identity, whence the conservation law

$$\nabla^\mu\left(J_{\mu\nu} + T_{\mu\nu} + \frac{1}{2}\Lambda\bar{g}_{\mu\nu}\right) = 0. \tag{4.37}$$

Finally, the Abbott-Deser conserved energy functional is given by

$$E = \int d^3x\sqrt{-\bar{g}}\left(J^{0\nu} + T^{0\nu} + \frac{1}{2}\Lambda\bar{g}^{0\nu}\right)K_\nu, \tag{4.38}$$

with K_ν being the global time-like Killing vector associated to time translations. We will refer to it as the Killing energy. A configuration is stable if the associated functional is positive for fluctuations vanishing sufficiently fast at spatial infinity, so that the integral converges. Abbott and Deser showed that this is the case if the matrix

$$V_{ij} = \frac{\partial^2 V}{\partial\phi^i\partial\phi^j}. \tag{4.39}$$

has positive eigenvalues, that is $\phi(x) = \bar{\phi}^i$ is a local minimum of the potential. However, in [34] and [35] Breitenlohner and Freedaman were able to prove that this can also happen for maximum or saddle points, and even when the potential is unbounded from below, provided that the eigenvalues are not too negative. We will come back to their result in a moment.

Now we are ready to provide the algebraic argument. We consider the anti-commutation relation for the spinorial supersymmetry generators,

$$\left\{Q^A, \bar{Q}^B\right\} = 2\delta^{AB}\gamma_{ab}L^{ab} + T^{AB}, \tag{4.40}$$

where

$$\gamma_{ab} = \begin{pmatrix} \gamma_{ab} & -\gamma_a \\ \gamma_b & 0 \end{pmatrix}, \qquad \gamma_{ab} = \gamma_{[a}\gamma_{b]}. \tag{4.41}$$

Multiplying by γ^0 and taking the trace yield

$$L^{04} = \mathrm{Tr}\, Q^2 \geq 0, \tag{4.42}$$

so the Killing energy is positive. A more direct proof remaining at the classical level can be found in [39] and the connection with the quantum theory is discussed in [40].

The Breitenlohner-Freedman bound As we have anticipated, a scalar field with a negative mass in AdS_D does not necessarily lead to any instabilities, provided that the mass is not too negative and satisfies the Breitenlohner-Freedman bound $m^2 > m_{BF}^2$, with

$$m_{BF}^2 = -\frac{(D-1)^2}{4L^2}. \tag{4.43}$$

Following [41], we now provide a nice derivation of their result based on a Schrödinger formalism. Let us consider the Klein-Gordon equation for a massive scalar

$$\frac{1}{\sqrt{g}}\partial_\mu \left[\sqrt{g} g^{\mu\nu}\partial_\nu \phi(z,t,\mathbf{y})\right] = m^2\phi(z,t,\mathbf{y}), \tag{4.44}$$

in the Poincaré patch of AdS_{D+1} described by the metric (4.28), that is

$$\left(\partial_z^2 - \partial_t^2 + \partial_i^2 - \frac{D}{z}\partial_z - \frac{m^2L^2}{z^2}\right)\phi(z,t,\mathbf{y}) = 0. \tag{4.45}$$

Assuming the spacetime dependence $\phi(z,t,\mathbf{y}) = \tilde{\phi}(z)e^{ik\cdot y}$, with momentum $k^2 = -\omega^2 + \mathbf{k}^2$, this equation becomes

$$\tilde{\phi}''(z) - \frac{D}{z}\tilde{\phi}'(z) - \left(\frac{m^2L^2}{z^2} + k^2\right)\tilde{\phi} = 0. \tag{4.46}$$

We would like to rewrite it as a Schrödinger equation. In order to do so, we have to get rid of the disturbing first derivative term. So we write $\tilde{\phi}(z) = B(z)\psi(z)$, yielding

$$B\psi'' + \left(2B' - \frac{D}{z}B\right)\psi' + \left[B'' - \frac{D}{z}B' - B\left(\frac{m^2L^2}{z^2} + k^2\right)\right] = 0. \tag{4.47}$$

The term multiplying ψ' can be made to vanish if

$$\frac{B'}{B} = \frac{D}{2z} \quad \Rightarrow \quad B = cz^{\frac{D}{2}}, \tag{4.48}$$

where we set the integration constant $c = 1$. Then equation (4.47) becomes

$$-\psi'' + \left[\mathbf{k}^2 + \frac{1}{z^2}\left(m^2L^2 + \frac{D(D-2)}{4}\right)\right]\psi = \omega^2\psi, \tag{4.49}$$

which can be interpreted as a Schrödinger equation with potential $V(z)$ given by the quantity in square brackets and energy ω^2. In particular, negative energy amounts to imaginary ω, whence a solution that is exponentially growing in time. This is the kind of instability we are looking for.

However, the connection between this Schrödinger equation and the AdS Klein-Gordon equation (4.45) requires a further inspection. It can be shown that the condition of conservation for the AdS Killing energy associated to the time translation Killing vector ∂_t, corresponding to a vanishing energy flux out of the AdS conformal boundary at $z = 0$, is equivalent to the condition of Hermiticity for the Schrödinger operator in (4.49). Moreover, the conserved energy for a scalar field configuration in AdS is also finite if the Schrödinger wavefunction $\psi(z)$ has finite norm. Therefore, the problem of finding an instability of the AdS equation (4.45) has been translated into the problem of finding normalizable negative energy states on which the Schrödinger operator in (4.49) is Hermitian. Now that our quantum mechanical problem is well-posed, we can set $\mathbf{k} = 0$ and rewrite (4.49) as

$$-\psi'' + \frac{\alpha}{z^2}\psi = -\beta^2\psi, \tag{4.50}$$

where $\alpha = \left(m^2L^2 + \frac{D(D-2)}{4}\right)$ and negative energy means that β is real. We recognize it as a Bessel equation,

$$\psi'' - \left(\beta^2 + \frac{\nu^2 - \frac{1}{4}}{z^2}\right)\psi = 0, \tag{4.51}$$

with $\nu = \frac{1}{2}\sqrt{1+4\alpha}$. The solutions to this equation can be written in terms of Hankel functions,

$$\psi(z) = \sqrt{z}\left[C_1 H_\nu^{(1)}(i\beta z) + C_2 H_\nu^{(2)}(i\beta z)\right], \tag{4.52}$$

with the large z asymptotic behaviour

$$H_\nu^{(1)}(i\beta z) \sim \frac{2}{i\beta z}\cosh(\beta z), \qquad H_\nu^{(2)}(i\beta z) \sim -i\frac{2}{i\beta z}\sinh(\beta z). \tag{4.53}$$

In order for the solution to vanish as $z \to \infty$, we must take the linear combination

$$\psi(z) = C\sqrt{z}\left[H_\nu^{(1)}(i\beta z) + iH_\nu^{(2)}(i\beta z)\right]. \tag{4.54}$$

As for the small z behaviour, we must distinguish between real or imaginary ν, that is $\alpha > -\frac{1}{4}$ or $\alpha < -\frac{1}{4}$. For real ν, and as $z \to 0$, this expression becomes

$$\psi(z) = C\sqrt{z}\left[\frac{1}{\Gamma(\nu+1)}\left(\frac{i\beta z}{2}\right)^\nu - i\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{i\beta z}\right)^\nu\right]. \tag{4.55}$$

Imposing the Hermiticity condition (which corresponds to an appropriate boundary condition in the AdS picture, as we have already discussed) on two solutions (4.55) with different energies β_1, β_2, we obtain

$$\left(\frac{\beta_2}{\beta_1}\right)^\nu + \left(\frac{\beta_1}{\beta_2}\right)^\nu = 0, \tag{4.56}$$

which cannot be solved for any values of β_1, β_2 real. Thus, there can never be a negative energy solution satisfying the Hermiticity condition, which translated back into the AdS language means that no instabilities can arise, provided that $\alpha > -\frac{1}{4}$. The situation is different for imaginary ν. In this case, the small z limit gives

$$\psi(z) \sim \sqrt{z}\cos\left[(\operatorname{Im}\nu)\ln\left(\frac{z\beta}{2}\right) + \delta\right], \tag{4.57}$$

where δ is a ν-dependent phase and the Hermiticity condition can be solved. Recalling the definition of α, this result is exactly what we would have expected from the Breitenlohner-Freedman bound (4.43).

4.3 Young's no-go argument

In this section, we introduce AdS compactifications of eleven dimensional supergravity to four dimensions, of which two main classes are known: Freund-Rubin solutions, in which the only non zero components of the four-form lie in spacetime, and Englert-type solutions, in which the four-form is also non zero when its indices all lie in the internal space. With the exception of the round seven-sphere, Freund-Rubin solutions can preserve or not supersymmetry, according to the orientation of the internal manifold. Englert solutions do not preserve any supersymmetry. They can be obtained from a supersymmetric solution by switching on a four-form that is constructed as a bilinear of a Killing spinor and flipping the orientation. Then we comment on the failure of the earlier attempt of [9] to generalize Witten's original construction of a *bubble of nothing*, prodiving a general argument which seems to prevent higher dimensional spheres to collapse in the presence of a non zero flux in the bubble geometry.

The Freund-Rubin ansatz A simple class of solutions to eleven dimensional supergravity was found by Freund and Rubin in [42]. They describe ground state configurations of the form $AdS_4 \times M_7$, where AdS_4 is the four dimensional Anti-de Sitter spacetime and M_7 is a compact seven dimensional Einstein space. As the authors emphasize, the dimension four for spacetime is not *ad hoc*, but it is a consequence of the field equations through the presence of a *four-form*, which in turn is dictated by supersymmetry. Indeed, the fact that the correct spacetime dimension was an output, rather than an input, was regarded as an attractive feature of the theory. The presence of an external four-form flux is also interesting for us, since it can be shown to represent an obstruction to the construction of a bounce geometry, that is a bubble of nothing, as it will become clear in the following.

To begin, we observe that the requirement of maximal spacetime simmetry implies that the v.e.v. of any fermion field should vanish, so we set (from now on, we will omit the angle brackets which denoted a ground state in the previous section)

$$\Psi_M = 0, \tag{4.58}$$

and focus on the bosonic equations (4.5) and (4.6) for g_{MN} and A_{MNP}. Then, we assume a spontaneous compactification for the eleven dimensional space where the metric describes a direct product $M_{11} = M_4 \times M_7$, that is

$$g_{MN}(x,y) = \begin{pmatrix} g_{\mu\nu}(x) & 0 \\ 0 & g_{mn}(y) \end{pmatrix}, \tag{4.59}$$

where $g_{\mu\nu}(x)$ is the metric of the four dimensional spacetime M_4 with coordinates x^μ and $g_{mn}(y)$ is the metric of the seven dimensional internal space M_7 with coordinates y^m. We also set

$$F_{\mu\nu\rho\sigma} = F_{\mu\nu\rho\sigma}(x), \qquad F_{mnpq} = F_{mnpq}(y), \tag{4.60a}$$

$$F_{\mu npq} = F_{\mu\nu pq} = F_{\mu\nu\rho q} = 0. \tag{4.60b}$$

Here the y-independence of $F_{\mu\nu\rho\sigma}$ and the x-independence of F_{mnpq} are a consequence of the Bianchi identity (4.2), together with (4.60b). Then, the ansatz of Freund and Rubin is to set

$$F_{\mu\nu\rho\sigma} = \xi\varepsilon_{\mu\nu\rho\sigma}, \qquad F_{mnpq} = 0, \tag{4.61}$$

with ξ a real constant. Equivalently, the four-form is set proportional to the volume form of the external space,

$$F_4 = \xi\, \mathrm{vol}_4\,. \tag{4.62}$$

With this ansatz, the Bianchi identity and the Maxwell equation are trivially satisfied and the

Einstein equations yield the product of two Einstein spaces,

$$R_{\mu\nu} = -\frac{4}{3}\xi^2 g_{\mu\nu}, \tag{4.63a}$$

$$R_{mn} = \frac{2}{3}\xi^2 g_{mn}. \tag{4.63b}$$

The maximally simmetric solution to (4.63) is AdS_4, since the curvature is negative. Moreover, having positive curvature and Euclidean signature, M_7 is automatically compact.

Skew-whiffing An interesting feature of Freund-Rubin vacua is *skew-whiffing*, i.e. orientation reversal of M_7. Since the four-form appears quadratically in the Einstein equations, for each Freund-Rubin solution (4.61), with $\xi \neq 0$, we can obtain another solution by a reversal of orientation of M_7, for example by sending ${e_m}^a$ to $-{e_m}^a$, or equivalently by keeping the orientation fixed but inserting a minus sign in front of ξ in (4.61). With the exception of the round seven-sphere, where both orientations admit maximal supersymmetry, it can be shown that at most one orientation can be compatible with supersymmetry. From the general argument of section 4.2, we already know that supersymmetric solutions are automatically stable. What about their non supersymmetric skew-whiffed counterparts? It is maybe a bit surprising to find that they are also perturbatively stable, since the stability properties of Freund-Rubin vacua can be shown to be insensitive to orientation at the perturbative level. Therefore, it is meaningful to look for non perturbative instabilities in a Freund-Rubin setup.

Of course these statements require a bit of explanation, which we now provide. To begin, we review the criterion of unbroken supersymmetry in the effective four dimensional theory, since it is also relevant to construct Englert-type solutions. We recall from (4.58) that fermions have been set to zero. Therefore, for a supersymmetric vacuum we require that they stay zero under a supersymmetry transformation, that is

$$\delta\Psi_M = \tilde{D}_M \varepsilon = 0, \tag{4.64}$$

where the supercovariant derivative $\tilde{D}_M$ takes the remarkably simple structure

$$\tilde{D}_\mu = D_\mu + \frac{\xi}{3}\gamma_\mu\gamma_5, \qquad \tilde{D}_m = D_m + \frac{\xi}{6}\Gamma_m. \tag{4.65}$$

Here, $\gamma_\mu = {e_\mu}^\alpha \gamma_\alpha$ and $\Gamma_m = {e_m}^a \Gamma_a$, with

$$\{\gamma_\alpha, \gamma_\beta\} = -2\eta_{\alpha\beta}, \qquad \{\Gamma_a, \Gamma_b\} = -2\delta_{ab}. \tag{4.66}$$

To solve equation (4.64), we look for solutions of the form $\varepsilon(x,y) = \varepsilon(x)\eta(y)$, where $\varepsilon(x)$ is an anticommuting four component spinor in $D = 4$ and $\eta(y)$ a commuting eight component spinor in $D = 7$. They must satisfy

$$\tilde{D}_\mu \varepsilon(x) = 0, \qquad \tilde{D}_m \eta(y) = 0. \tag{4.67}$$

Since AdS admits the maximum number as far as spacetime is concerned, the number $\mathcal{N}$ of unbroken generators is given by the number of Killing spinors on M_7, i.e. the number of solutions to

$$\tilde{D}_m \eta = \left(\partial_m - \frac{1}{4}{\omega_m}^{ab}\Gamma_{ab} - \frac{\xi}{6}{e_m}^a\Gamma_a\right)\eta = 0. \tag{4.68}$$

We are now ready to state the skew-whiffing theorem. Let $\eta_\pm$ be Killing spinors for two Freund-

Rubin vacua related by skew-whiffing,

$$\left(D_m \mp \frac{\xi}{6}\Gamma_m\right)\eta_\pm = 0. \tag{4.69}$$

Then the scalar field defined by $\phi \equiv \bar{\eta}_+\eta_-$ satisfies

$$\Box\phi = -\frac{7}{9}\xi^2\phi, \tag{4.70}$$

yielding

$$E_{mn} = \nabla_m\nabla_n\phi + \frac{\xi^2}{9}g_{mn}\phi = 0. \tag{4.71}$$

This is the equation for conformal Killing vectors $\nabla_m\phi$ on M_7, which is known to have solutions only for the round seven-sphere. This argument would fail if $\phi = 0$, but it is not difficult to show that this cannot be the case. Therefore, there can exist at most one orientation that preserves supersymmetry.

The second ingredient we need is the necessary and sufficient condition for a Freund-Rubin solution to be perturbatively stable. We state it without derivation and refer to [43] for details. There, an analysis of the spectrum of the four dimensional theory reveals that classical instabilities can only arise from the scalar sector. Moreover, the criterion for stability can be expressed as a certain bound on the spectrum of the Lichnerowicz operator on M_7,

$$\Delta_L \geq \frac{\xi^2}{3}, \tag{4.72}$$

where ξ is the Freund-Rubin parameter appearing in (4.61). What is interesting about this result is that ξ appears *quadratically* in the bound. Thus, it is not affected by an orientation reversal, that is by sending ξ to $-\xi$. In particular, the general stability argument in the presence of supersymmetry which we discussed in section 4.2 can be used to argue that Freund-Rubin solutions with an unbroken supersymmetry satisfy (4.72). But then, this is also true for the corresponding skew-whiffed solution, which is not supersymmetric but still turns out to be perturbatively stable.

Englert solutions In the Freund-Rubin ansatz, the four-form is taken to have legs in the external space only. This assumption can be relaxed, allowing for a non vanishing internal flux,

$$F_4 = F_4^e + F_4^i, \tag{4.73}$$

where the first and second terms live in the external and internal space, respectively. Solutions of this kind were first discovered by Englert in [44]. In components, we set

$$F_{\mu\nu\rho\sigma} = \frac{2}{3}\xi\varepsilon_{\mu\nu\rho\sigma}, \qquad F_{mnpq} \neq 0, \tag{4.74}$$

where a numerical factor in front of ξ is introduced for convenience. Then, the Maxwell equation (4.7) reads

$$d * F_4^i = d *_7 F_4^i \wedge \mathrm{vol}_4 = -2F_4^i \wedge F_4^e = -\frac{2}{3}\xi F_4^i \wedge \mathrm{vol}_4, \tag{4.75}$$

yielding

$$d *_7 F_4^i = -\frac{1}{3}\xi F_4^i. \tag{4.76}$$

Equivalently, in components,

$$\nabla_m F^{mnpq} = \frac{1}{18}\xi\varepsilon^{npqrstu}F_{rstu}. \tag{4.77}$$

The Einstein equation (4.5) splits into

$$R_{\mu\nu} = \frac{1}{3}\left(-\frac{16}{9}\xi^2 - \frac{1}{12}F^{mnpq}F_{mnpq}\right)g_{\mu\nu}, \tag{4.78}$$

$$R_{mn} = \frac{1}{3}\left(F_m{}^{pqr}F_{npqr} - \frac{1}{12}g_{mn}F^{pqrs}F_{pqrs} + \frac{8}{9}\xi^2 g_{mn}\right). \tag{4.79}$$

The requirement of maximal spacetime symmetry $R_{\mu\nu} = \Lambda g_{\mu\nu}$ implies that $F^{mnpq}F_{mnpq}$ is a (positive) constant, leading to $\Lambda < 0$, i.e. Anti-de Sitter spacetime.

A possible way to construct the internal flux is to start from a Freund-Rubin solution admitting at least one Killing spinor η, which satisfies equation (4.68). Then we set

$$A_{mnp} = c\bar{\eta}\Gamma_{mnp}\eta, \tag{4.80}$$

where c is a real constant. Using the Killing spinor equation (4.68), we can check that the field strenght $F_{mnpq} = 4\partial_{[m}A_{npq]}$ satisfies

$$\nabla_m F^{mnpq} = -\frac{1}{18}\xi\varepsilon^{npqrstu}F_{rstu}. \tag{4.81}$$

Were it not for the sign difference, this would be a good candidate for a solution of the Maxwell equation (4.77). However, this can be easily remedied by skew-whiffing, sending ξ in $-\xi$ in (4.74). Normalizing $\bar{\eta}\eta = 1$ and using the Fierz identity, we can also compute

$$F_m{}^{pqr}F_{npqr} = \frac{128}{3}c^2\xi^2 g_{mn}. \tag{4.82}$$

Since the Killing spinor equation (4.68) requires that the metric on M_7 satisfies equation (4.63b), we find that this construction solves the Einstein equations (4.78) with

$$R_{\mu\nu} = -\frac{10}{9}\xi^2 g_{\mu\nu}, \tag{4.83a}$$

$$R_{mn} = \frac{2}{3}\xi^2 g_{mn}, \tag{4.83b}$$

provided that $c = \frac{1}{4}$.

We conclude with some remarks on what is known about the stability properties of these solutions [45]. Some, but not all of them, have been proven to be classically unstable. For example, Englert solutions constructed from supersymmetric solutions are unstable if the supersymmetric solution preserves two or more supersymmetries [46]. In addition, the Englert solution using the squashed S^7 was demonstrated to be unstable in [47], while a subset of the solutions supported by the $N(k,l)$ coset manifolds have also shown to be unstable [46], [48].

The no-go argument It was shown in [9] that the presence of a four-form flux prevents a simple generalization of Witten's bubble of nothing where a higher dimensional sphere rather than a circle shrinks to zero size. The argument works as follows.

For convenience, let us go back for a moment to Witten's original construction as discussed in Chapter 3. There we wrote the Euclidean continuation of the $M_4 \times S^1$ Kaluza-Klein metric as

$$ds^2 = dr^2 + r^2 d\Omega_3^2 + d\psi^2, \tag{4.84}$$

where r is the radial coordinate, running from 0 to ∞, $d\Omega_3^2$ is the metric of the three-sphere and ψ is the coordinate of the Kaluza-Klein circle of radius R. Then we showed that there exists another non singular and geodesically complete solution of the Euclidean Einstein equations with the same asymptotic behaviour. Such a solution is the Euclidean five dimensional Schwarzschild metric,

$$ds^2 = \frac{dr^2}{1-\left(\frac{R}{r}\right)^2} + r^2 d\Omega_3^2 + \left(1-\left(\frac{R}{r}\right)^2\right) d\psi^2, \tag{4.85}$$

where now r runs from R to ∞.

In a higher dimensional theory, the Kaluza-Klein circle is replaced by a compact space, which for example can be taken to be an N-sphere. The Euclidean $M_4 \times S^N$ metric can be written as

$$ds^2 = dr^2 + r^2 d\Omega_3^2 + \sum_{i=1}^{N} \frac{R^2}{f_s^2}(dy^i)^2, \tag{4.86}$$

with

$$f_s = 1 + \frac{1}{4}\sum_{i=1}^{N}(y^i)^2, \tag{4.87}$$

and where y^i are the stereographic coordinates on the N-sphere. In order to find another solution with the same asymptotic behaviour of (4.86), a generalization of (4.85) could be considered,

$$ds^2 = dr^2 + f(r) d\Omega_3^2 + h(r)\sum_{i=1}^{N} \frac{R^2}{f_s^2}(dy^i)^2, \tag{4.88}$$

where the coefficient in front of dr^2 can be set to one by coordinate redefinition. Suppose that the N-sphere collapses at some finite radius $r = r_0$, where $f(r)$ stays constants. In order for the geometry to terminate smoothly, we require that $h(r) \sim (r - r_0)^2$ as $r \to r_0$. However, such a solution for the metric is forbidden if the four-form is taken to be proportional to the volume form of the external space, as in the Freund-Rubin ansatz. Allowing an r-dependence for the Freund-Rubin parameter $\xi = \xi(r)$ in (4.62), we set

$$F_4 = \xi(r)\,\mathrm{vol}_4\,. \tag{4.89}$$

Thus, the Bianchi identity is trivially satisfied and from the Maxwell equation

$$d(*F_4) = \xi'(r) dr \wedge \mathrm{vol}_N + \xi(r) d\,(\mathrm{vol}_N) = \left(\xi'(r) + \xi(r)\frac{N h'(r)}{2h(r)}\right) dr \wedge \mathrm{vol}_N = 0, \tag{4.90}$$

the function $\xi(r)$ is determined up to an integration constant,

$$\xi(r) = \frac{c}{h(r)^{\frac{N}{2}}}. \tag{4.91}$$

However, this solution cannot solve the Einstein equations, as can be easily seen by examining the behaviour at the bubble. It suffices to consider the trace of the Einstein equation,

$$R = \frac{1}{36} F^{PQRS} F_{PQRS}. \tag{4.92}$$

Using the ansatz (4.88) for the metric and (4.89) for the four-form, we obtain

$$-\frac{3N}{2}\frac{f'h'}{fh}+\frac{6}{f}-\frac{3f''}{f}+\frac{N(N-3)}{4}\frac{h'^2}{h^2}+\frac{N(N-1)}{h}-\frac{Nh''}{h}=\frac{2}{3}\frac{c_1^2}{h^N}. \tag{4.93}$$

We now take the limit $r\to r_0$. The curvature contribution from the left hand side of this equation is of order $\frac{1}{(r-r_0)^2}$ but the flux contribution from the right hand side, which is quadratic in the four form and thus proportional to $\xi(r)^2=\frac{c^2}{h(r)^N}$, is of order $\frac{1}{(r-r_0)^{2N}}$. If $N>1$, there is no way in which the divergence coming from the flux contribution can be compensated. Therefore, a solution can only exist if $N=1$, which is a circle.

4.4 Re-orienting the flux

Spontaneous compactification can occur in more dimensions than just four. An example is given by compactifications to five spacetime dimensions of the form $AdS_5\times K_6$, where the six dimensional compact space K_6 is a Kähler manifold [49]. This means that there exists a covariantly conserved complex structure $\omega_m{}^n$ satisfying

$$\omega_m{}^n\omega_n{}^p=-\delta_m^p,\qquad \nabla_m\omega_n{}^p=0. \tag{4.94}$$

The corresponding Kähler form ω_{mn} is obtained by lowering the upper index and satisfies

$$d\omega=0,\qquad \frac{1}{6}\omega\wedge\omega\wedge\omega\neq 0. \tag{4.95}$$

In the ground state, the metric g_{MN} is assumed to take the direct product form, where $g_{\mu\nu}$ is the metric on AdS_5 and g_{mn} is the metric on K_6. The ansatz for the four form is

$$F=\frac{c}{2}\omega\wedge\omega, \tag{4.96}$$

where c is a real constant. Substituting (4.96) into the Einstein equation, we find that the Ricci tensors of the external and internal space respectively satisfy

$$R_{\mu\nu}=-2c^2g_{\mu\nu},\qquad R_{mn}=2c^2g_{mn}. \tag{4.97}$$

In order for the radius of AdS to be normalized to unit, we must choose $c=\sqrt{2}$. Thus we have a solution where the eleven dimensional spacetime is the product of five dimensional Anti-de Sitter spacetime and a six dimensional positive Kähler-Einstein manifold. Some examples of such manifolds are $\mathbb{C}P^3$, $\mathbb{C}P^2\times S^2$, $S^2\times S^2\times S^2$. All these solutions break supersymmetry, as follows from the general analysis of [50], but $AdS_5\times\mathbb{C}P^3$ is the only example of this type that is known to be stable at the perturbative level [51], so it is interesting to look for non perturbative instabilities. This was recently done in [10], where an explicit instanton solution is found with a S^2 instead of a circle shrinking to zero size. This is possible thanks to the non trivial fibration structure of $\mathbb{C}P^3$, which can be realized as an S^2 fibration over an S^4 base. Following [10], we now review the construction of the instanton.

Introducing two arbitrary functions $h(r)$ and $g(r)$ that control the size of the S^2 fiber and the S^4 base as we vary the AdS radial coordinate r, we define the vielbein for the CP^3 as

$$\begin{aligned}
e^1 &= g^{\frac{1}{2}}(r)d\mu, \\
e^i &= \frac{g^{\frac{1}{2}}(r)}{2}\sin\mu\,\Sigma_{i-1} \qquad \text{for } i=2,3,4, \\
e^5 &= h^{\frac{1}{2}}(r)\left(d\theta - A_1\sin\phi + A_2\cos\phi\right), \\
e^6 &= h^{\frac{1}{2}}(r)\sin\theta\left(d\phi - \cot\theta\left(A_1\cos\phi + A_2\sin\phi\right) + A_3\right),
\end{aligned} \tag{4.98}$$

where

$$\begin{aligned}
\Sigma_1 &= \cos\gamma\,d\alpha + \sin\gamma\,\sin\alpha\,d\beta, \\
\Sigma_2 &= -\sin\gamma\,d\alpha + \cos\gamma\,\sin\alpha\,d\beta, \\
\Sigma_3 &= d\gamma + \cos\alpha\,d\beta, \\
A_i &= \cos\left(\frac{\mu}{2}\right)^2\Sigma_i.
\end{aligned} \tag{4.99}$$

For the round CP^3, $g(r) = h(r) = \frac{1}{2}$.We also take the vielbein for the Euclidean AdS to be,

$$\begin{aligned}
e^7 &= dr, \\
e^8 &= f^{\frac{1}{2}}(r)dx_1, \\
e^9 &= f^{\frac{1}{2}}(r)\sin x_1 dx_2, \\
e^{10} &= f^{\frac{1}{2}}(r)\sin x_1\sin x_2 dx_3, \\
e^{11} &= f^{\frac{1}{2}}(r)\sin x_1\sin x_2\sin x_3 dx_4.
\end{aligned} \tag{4.100}$$

Then the Euclidean metric can be written as

$$\begin{aligned}
ds^2 = g(r)&\left(d\mu^2 + \frac{1}{4}\sin^2\mu\sum_{i=1}^{3}\Sigma_i^2\right) + h(r)\left(d\theta - A_1\sin\phi + A_2\cos\phi\right)^2 \\
&+ h(r)\sin^2\theta\left[d\phi - \cot\theta\left(A_1\cos\phi + A_2\sin\phi\right) + A_3\right]^2 + dr^2 + f(r)d\Omega_4^2,
\end{aligned} \tag{4.101}$$

where $d\Omega_4^2$ is the metric of the unit four-sphere. The next step is to write an ansatz for the four-form, using the $SU(3)$-structure of the squashed CP^3.

SU(3)-structure of CP^3 In order to introduce the language of G-structures, and in particular of $SU(3)$-structures, we now recall some basic theory of fiber bundles. Our discussion will mainly follow [52]. Given a smooth manifold M of dimension d, a bundle E with base M and fiber F is a manifold itself, equipped with a smooth projection π to the base and locally looking like a product of the base with the fiber. Such description is called a local trivialization and is valid only locally, that is on a patch U_α of M. Furthermore, there are transition functions $t_{\alpha\beta}$, describing how the fiber transforms between two intersecting patches U_α and U_β, so that globally we have not, in general, a trivial product. They must satisfy the consistency conditions

$$t_{\alpha\beta}t_{\beta\alpha} = 1, \tag{4.102}$$

and on the triple overlap $U_\alpha \cap U_\beta \cap U_\gamma$,

$$t_{\alpha\beta}t_{\beta\gamma} = t_{\alpha\gamma}, \tag{4.103}$$

which give them the properties of a group, called the *structure group*. The bundles of interest of us are the tangent bundle TM, with fiber in each point $p \in M$ the tangent space T_pM, and the associated frame bundle FM, with fiber in p the set of ordered bases of T_pM. We now consider two patches U_α and U_β with local trivializations (p, e_a) and (p, e'_a), respectively. Here $e_a = e^i_a \frac{\partial}{\partial x^i}$ for $a = 1, \dots, d$ is a set of d indipendent vectors forming a base of the tangent space, that is a local frame. On the overlap of these patches we have the relation

$$e'^i_a = \frac{\partial x'^i}{\partial x^j} e^j_a, \tag{4.104}$$

which can also be written in terms of transitions functions as the action from the right of the structure group, which in this case is the group $GL(d, \mathrm{R})$ of invertible $d \times d$ real matrices,

$$e'^i_a = e^i_b \left(t_{\beta\alpha}\right)^b_{\ a}. \tag{4.105}$$

Then M is said to have a G-structure with $G \subset GL(d, \mathrm{R})$ if it is possible to reduce the frame bundle such that it has structure group G. Whether this is possible depends on the topological properties of the manifold itself.

A G-structure can be conveniently defined as follows. Suppose to have a set of globally defined, non-degenerate G-invariant tensors. Being globally defined, they can be made to take exactly the same form in all patches by appropriately choosing the frames e_a in each patch. Then, only those transition functions that leave these objects invariant are allowed and the structure group reduces to G. An extreme example is the case where one is able to find a global section of FM. Then the only transition function that preserves this form is the identity and the structure group is trivial. An interesting intermediate case is that of coset spaces G/H, which is also the one that is relevant for us.

In fact, different realizations of $\mathrm{C}P^3$ as a coset space are associated to different $SU(3)$-structures. A convenient one is $\mathrm{C}P^3 = \frac{Sp(2)}{S(U(2)\times U(1))}$, which is homogeneous and can be used even after we change the relative sizes of the base S^4 and the fiber S^2. The associated $SU(3)$-structure is given by a real two-form J and a complex three-form Ω,

$$\begin{aligned}
J &= -\sin\theta\,\cos\phi\left(e^{12}+e^{34}\right) - \sin\theta\,\sin\phi\left(e^{13}+e^{42}\right) - \cos\theta\left(e^{14}+e^{23}\right) + e^{56},\\
\mathrm{Re}\,\Omega &= \cos\theta\,\cos\phi\left(e^{126}+e^{346}\right) + \cos\theta\,\sin\phi\left(e^{136}+e^{426}\right) + \sin\phi\left(e^{125}+e^{345}\right)\\
&\quad - \cos\phi\left(e^{135}+e^{425}\right) - \sin\theta\left(e^{146}+e^{236}\right),\\
\mathrm{Im}\,\Omega &= -\cos\theta\,\cos\phi\left(e^{125}+e^{345}\right) - \cos\theta\,\sin\phi\left(e^{135}+e^{425}\right) + \sin\phi\left(e^{126}+e^{346}\right)\\
&\quad - cos\phi\left(e^{136}+e^{426}\right) + \sin\theta\left(e^{145}+e^{235}\right),
\end{aligned}$$

where $e^{12} = e^1 \wedge e^2$ etc. They satisfy the compatibility condition

$$J \wedge \mathrm{Re}\,\Omega = 0, \qquad J \wedge \mathrm{Im}\,\Omega = 0. \tag{4.106}$$

The action of the external derivative d_6 and the Hodge star $*_6$ of the $\mathrm{C}P^3$ is given by

$$\begin{aligned}
d_6 J &= \frac{3}{2}\mathcal{W}_1\,\mathrm{Im}\,\Omega, & *_6\left(J\wedge e^{56}\right) &= J - e^{56},\\
d_6\,\mathrm{Im}\,\Omega &= 0, & *_6\left(J\wedge J\right) &= 2J,\\
d_6\,\mathrm{Re}\,\Omega &= \mathcal{W}_1 J\wedge J + \mathcal{W}_2\wedge J, & *_6\Omega &= i\Omega.
\end{aligned} \tag{4.107}$$

Here $\mathcal{W}_1$ and $\mathcal{W}_2$ are torsion classes of the $SU(3)$-structure,

$$\begin{aligned} \mathcal{W}_1 &= \frac{2}{3}\frac{g+h}{gh^{\frac{1}{2}}}, \\ \mathcal{W}_2 &= \frac{2h-g}{gh^{\frac{1}{2}}}\left(\frac{2}{3}J - 2e^{56}\right). \end{aligned} \tag{4.108}$$

Then it is straightforward to generalize these relations to the whole eleven dimensional space, as

$$\begin{aligned} d\left(J - e^{56}\right) &= \frac{1}{h^{\frac{1}{2}}}\operatorname{Im}\Omega + \frac{g'}{g}\left(J - e^{56}\right)\wedge e^7, \\ d\left(e^{56}\right) &= \frac{h^{\frac{1}{2}}}{g}\operatorname{Im}\Omega + \frac{h'}{h}e^{567}, \\ d\operatorname{Re}\Omega &= \frac{2h^{\frac{1}{2}}}{g}J\wedge J - 2\frac{2h-g}{gh^{\frac{1}{2}}}J\wedge e^{56} - \left(\frac{g'}{g} + \frac{h'}{2h}\right)\operatorname{Re}\Omega\wedge e^7, \\ d\operatorname{Im}\Omega &= -\left(\frac{g'}{g} + \frac{h'}{2h}\right)\operatorname{Im}\Omega\wedge e^7, \end{aligned} \tag{4.109}$$

and

$$\begin{aligned} &*\left(J\wedge e^{56}\right) = \left(J - e^{56}\right)\wedge e^{7\,8\,9\,10\,11}, \qquad &*\left(J\wedge J\right) = 2J\wedge e^{7\,8\,9\,10\,11}, \\ &*\left(e^7\wedge\operatorname{Re}\Omega\right) = \operatorname{Im}\Omega\wedge e^{8\,9\,10\,11}, \qquad &*\left(e^7\wedge\operatorname{Im}\Omega\right) = -\operatorname{Re}\Omega\wedge e^{8\,9\,10\,11}. \end{aligned} \tag{4.110}$$

Ansatz for the four form Now we can use the $SU(3)$-structure forms to write an ansatz for the four form,

$$F_4 = \xi_1(r)J\wedge J + \xi_2(r)J\wedge e^{56} + d\left(\xi_3(r)\operatorname{Im}\Omega\right) + d\left(\xi_4(r)\operatorname{Re}\Omega\right) + \xi_5(r)e^{8\,9\,10\,11}. \tag{4.111}$$

Using (4.109), the Bianchi identity gives

$$\begin{aligned} dF_4 = &\left(\xi_5' + 2\xi_5\frac{f'}{f}\right)e^{7\,8\,9\,10\,11} + \left(\xi_1' + 2\xi_1\frac{g'}{g}\right)J\wedge J\wedge e^7 \\ &+ \left[-2\xi_1\left(\frac{g'}{g} - \frac{h'}{h}\right) + \xi_2' + \xi_2\left(\frac{g'}{g} - \frac{h'}{h}\right)\right]J\wedge e^{567} = 0. \end{aligned} \tag{4.112}$$

The three coefficients must vanish separately, yielding

$$\xi_1(r) = \frac{c_1}{g^2(r)}, \qquad \xi_2(r) = -2\frac{c_1}{g^2(r)} + \frac{c_2}{g(r)h(r)}, \qquad \xi_5(r) = \frac{c_5}{f^2(r)}. \tag{4.113}$$

In order to impose the Maxwell equation (4.7), the differentials in (4.111) have to be explicited,

$$F_4 = a_1(r)J\wedge J + a_2(r)J\wedge e^{56} + a_3(r)\operatorname{Im}\Omega\wedge e^7 + a_4(r)\operatorname{Re}\Omega\wedge e^7 + \xi_5(r)e^{8\,9\,10\,11}, \tag{4.114}$$

where

$$\begin{aligned}
a_1(r) &= \xi_1(r) + 2\xi_4(r)\frac{\sqrt{h(r)}}{g(r)},\\
a_2(r) &= \xi_2(r) - 2\xi_4(r)\frac{2h(r)-g(r)}{g(r)\sqrt{h(r)}},\\
a_3(r) &= -\left[\xi_3'(r) - \xi_3(r)\left(\frac{g'(r)}{g(r)} + \frac{h'(r)}{2h(r)}\right)\right],\\
a_4(r) &= -\left[\xi_4'(r) - \xi_4(r)\left(\frac{g'(r)}{g(r)} + \frac{h'(r)}{2h(r)}\right)\right].
\end{aligned} \tag{4.115}$$

The Hodge dual can be easily computed using (4.110),

$$* F_4 = \Big[(2a_1 + a_2)\left(J - e^{56}\right) \wedge e^7 + 2a_1 e^{567} + a_3 \operatorname{Re}\Omega - a_4 \operatorname{Im}\Omega\Big] \wedge e^{8\,9\,10\,11} + \xi_5 e^{1234567} \tag{4.116}$$

and upon differentiation becomes

$$\begin{aligned}
d * F_4 = \Bigg\{ & a_3 \frac{2\sqrt{h}}{g} J \wedge J + 2a_3 \frac{g-2h}{g\sqrt{h}} J \wedge e^{56} - \left[a_3' + a_3\left(\frac{2f'}{f} + \frac{g'}{g} + \frac{h'}{2h}\right)\right] \operatorname{Re}\Omega \wedge e^7 \\
&+ \left[\frac{2a_1 + a_2}{\sqrt{h}} + 2a_1 \frac{\sqrt{h}}{g} + a_4' + a_4\left(\frac{2f'}{f} + \frac{g'}{g} + \frac{h'}{2h}\right)\right] \operatorname{Re}\Omega \wedge e^7 \Bigg\} \wedge e^{8\,9\,10\,11}.
\end{aligned} \tag{4.117}$$

The wedge product is

$$F_4 \wedge F_4 = \xi_5(r)\left(a_1(r) J \wedge J + a_2(r) J \wedge e^{56} + a_3(r) \operatorname{Im}\Omega \wedge e^7 + a_4(r) \operatorname{Re}\Omega \wedge e^7\right) \wedge e^{8\,9\,10\,11}. \tag{4.118}$$

Inserting (4.117) and (4.118) into (4.7), observing that the various coefficients must vanish separately and using the previous results in (4.113), one finds a system of four equations,

$$\begin{aligned}
\xi_5 a_1 - a_3 \frac{2\sqrt{h}}{g} &= 0,\\
\xi_5 a_2 + 2a_3 \frac{2h-g}{g\sqrt{h}} &= 0,\\
-\xi_5 a_3 + (2a_1 + a_2)\frac{1}{\sqrt{h}} + 2a_1 \frac{\sqrt{h}}{g} - a_4' - a_4\left(\frac{g'}{g} + \frac{h'}{2h} + 2\frac{f'}{f}\right) &= 0,\\
-\xi_5 a_4 + a_3' + a_3\left(\frac{g'}{g} + \frac{h'}{2h} + 2\frac{f'}{f}\right) &= 0,
\end{aligned} \tag{4.119}$$

which is solved by

$$c_1 = c_2, \qquad \xi_3(r) = \frac{c_3}{g(r)\sqrt{h(r)}}, \qquad \xi_4(r) = -\frac{c_1}{2g(r)\sqrt{h(r)}}, \tag{4.120}$$

which does not allow for a *bubble of nothing* solution, and

$$\xi_3(r) = \frac{c_3}{g(r)\sqrt{h(r)}}, \qquad \xi_4(r) = -\frac{\xi(r)}{4\sqrt{2}g(r)\sqrt{h(r)}}, \qquad \xi_5 = 0, \tag{4.121}$$

where c_3 can be set to zero without loss of generality because it gives no contribution to the four form and the function $\xi(r)$ satisfies the differential equation

$$\xi'' + \frac{2f'\xi'}{f} - \frac{4h\left(\xi - 2\sqrt{2}c_1\right)}{g^2} - \frac{2\left(\xi - 2\sqrt{2}c_2\right)}{h} = 0. \tag{4.122}$$

We still have to impose that the ansatz obeys the boundary conditions of the vacuum. We observe that the Kähler form of the round $\mathrm{C}P^3$ can be expressed in terms of the $SU(3)$-structure forms as

$$\omega = 2e^{56} - J, \tag{4.123}$$

which inserted in (4.96) with $c = \sqrt{2}$ gives

$$F_4 = \frac{1}{\sqrt{2}}\left(J \wedge J - 4J \wedge e^{56}\right). \tag{4.124}$$

Then, in order for (4.111) to converge to (4.124) as $r \to \infty$, we must set $c_1 = \frac{3}{4\sqrt{2}}$, $c_2 = 0$ and $\xi(\infty) = 1$. With this choice, equation (4.122) becomes

$$\xi'' + \frac{2f'\xi'}{f} - \frac{4h\left(\xi - \frac{3}{2}\right)}{g^2} - \frac{2\xi}{h} = 0 \tag{4.125}$$

and the four-form can be written as[1]

$$F_4 = \frac{3}{4\sqrt{2}g^2(r)}J \wedge J - \frac{3}{2\sqrt{2}g^2(r)}J \wedge e^{56} - d\left(\frac{\xi(r)}{4\sqrt{2}g(r)\sqrt{h(r)}}\,\mathrm{Re}\,\Omega\right). \tag{4.126}$$

Finally the Einstein equations are

$$-\frac{g''}{2g} - \frac{f'g'}{fg} - \frac{g'h'}{2gh} - \frac{g'^2}{2g^2} - \frac{h}{g^2} + \frac{3}{g} - \frac{\xi'^2}{24g^2h} - \frac{\xi^2}{12g^2h^2} - \frac{2\left(\xi - \frac{3}{2}\right)^2}{3g^4} = 0, \tag{4.127a}$$

$$-\frac{h''}{2h} - \frac{f'h'}{fh} - \frac{g'h'}{gh} + \frac{h}{g^2} + \frac{1}{h} - \frac{\xi'^2}{24g^2h} - \frac{\xi^2}{3g^2h^2} + \frac{\left(\xi - \frac{3}{2}\right)^2}{3g^4} = 0, \tag{4.127b}$$

$$-\frac{f''}{2f} - \frac{f'g'}{fg} - \frac{f'h'}{2fh} - \frac{f'^2}{2f^2} + \frac{3}{f} + \frac{\xi'^2}{12g^2h} + \frac{\xi^2}{6g^2h^2} + \frac{\left(\xi - \frac{3}{2}\right)^2}{3g^4} = 0 \tag{4.127c}$$

$$\begin{gathered}\frac{8f'g'}{fg} + \frac{4f'h'}{fh} + \frac{3f'^2}{f^2} + \frac{h'^2}{2h^2} + \frac{4g'h'}{gh} + \frac{3g'^2}{g^2} + \frac{2h}{g^2} + \\ -\frac{\xi'^2}{4g^2h} + \frac{\xi^2}{2g^2h^2} + \frac{\left(\xi - \frac{3}{2}\right)^2}{g^4} - \frac{12}{g} - \frac{2}{h} - \frac{12}{f} = 0.\end{gathered} \tag{4.127d}$$

Due to the Bianchi identities, only three out of these four equations are independent. As a consistency check, it is easy to verify that the Maxwell equation (4.125) and the Einstein equations (4.127) are solved by the $AdS_5 \times \mathrm{C}P^3$ vacuum,

$$f(r) = \sinh^2(r), \qquad g(r) = h(r) = \frac{1}{2}, \qquad \xi(r) = 1. \tag{4.128}$$

[1]Our expression for the four-form differs from that in equation (15) of [10] by a factor 24, which would lead to an overall coefficient in front of the flux contribution in the Einstein equations. However, since our Einstein equations are exactly the same as theirs, this is probably just a typo.

Near-bubble expansion We now look for a solution where the S^2 fiber shrinks to zero size at some finite AdS radius $r = r_0$. That is, we assume the following expansions at the bubble,

$$\begin{aligned} \xi(r) &= \xi_0 + \xi_1(r-r_0) + \xi_2(r-r_0)^2 + \dots, \\ f(r) &= f_0 + f_1(r-r_0) + f_2(r-r_0)^2 + \dots, \\ g(r) &= g_0 + g_1(r-r_0) + g_2(r-r_0)^2 + \dots, \\ h(r) &= h_2(r-r_0)^2 + h_3(r-r_0)^3 + \dots. \end{aligned} \tag{4.129}$$

Plugging into the field equations, we find that $\xi_0 = \xi_1 = \xi_3 = 0$, $f_1 = g_1 = 0$, $h_2 = 1$, $h_3 = 0$ so that the correct expansions are

$$\begin{aligned} \xi(r) &= \xi_2(r-r_0)^2 + \mathcal{O}((r-r_0)^4), \\ f(r) &= f_0 + \mathcal{O}((r-r_0)^2), \\ g(r) &= g_0 + \mathcal{O}((r-r_0)^2), \\ h(r) &= (r-r_0)^2 + \mathcal{O}((r-r_0)^4). \end{aligned} \tag{4.130}$$

Here some comments are in order. We observe that the smoothness condition $h_2 = 1$ is implied by the field equations, rather than being imposed as an additional boundary condition as in Witten's example. Moreover, the fact that $\xi_0 = \xi_1 = 0$ allows for a nice interpetation in terms of flux conservation: in order for the S^2 fiber to collapse ar $r = r_0$, the four-form must reorient itself at the bubble in such a way that it does not have components on the S^2. This can be seen as follows. It is convenient to rewrite the expression (4.126) for the four-form as

$$\frac{1}{3\sqrt{2}} F_4 = \frac{4\left(\frac{3}{2} - \xi\right)}{g^2} e^{1234} - \frac{2\xi}{gh} J \wedge e^{56} + \frac{\xi'}{gh^{\frac{1}{2}}} \operatorname{Re}\Omega \wedge dr. \tag{4.131}$$

Since J and $\operatorname{Re}\Omega$ are proportional to the same powers of h which appear in the denominators, the last two terms in the right-hand side of this expression can be made vanish precisely by setting $\xi = \xi' = 0$ at the bubble. In such a way, only the first term, which is proportional to the volume form of the S^4 base, survives and flux conservation is preserved. The fact that both functions ξ and h vanish at the bubble is crucial to evade Young's argument. In general, if a N dimensional sphere of radius $h^{\frac{1}{2}}(r)$ in the internal space is supported by a flux, the flux contribution to the Einstein equations is at most proportional to $\frac{1}{h^N(r)}$. If the sphere has to collapse at the bubble, smoothness implies that $h(r) \sim (r - r_0)^2$ and such term cannot be compensated by the curvature contribution, which is of order $\frac{1}{(r-r_0)^2}$, unless $N = 1$. Looking for example at the Einstein equation (4.127b) for h, we see that the presence of a flux term with two powers of h at the denominator is not pathological, since this term is proportional to $\frac{\xi^2}{h^2}$ and the function $\xi(r) \to 0$ as $h(r) \to 0$. Therefore this equation can be satisfied even if $N = 2$.

A solution with this behaviour can indeed be found by a combination of algebraic relations and numerical integration. We refer to the original article [10] for further details.

4.5 M2-brane instantons

In this section, a new ingredient to the construction of the bounce is introduced. As we have already pointed out, the usual Kaluza-Klein bubble has the remarkable feature that the geometry is smooth at the radius r_0 where the Kaluza-Klein circle pinches off. As described in Chapter 2, this has to be imposed as a boundary condition in Witten's original construction and is achieved by setting the analog of the Schwarzschild radius of the five dimensional Euclidean bounce equal to the radius of the Kaluza-Klein circle. This is also true in the CP^3 example of section 4.4, where a smoothness condition for the collapsing 2-sphere, namely $h(r) \sim (r - r_0)^2$ at the bubble, turns out to be implied by the Einstein equations. However, in the presence of a non vanishing flux supporting the geometry it can happen that the requirement of smoothness has to be relaxed. This was first argued for an $AdS \times S^5/Z_k$ geometry in [11], using an argument based on the flux quantization condition. There it is shown that the geometry cannot terminate smoothly, since there must be brane instantons wrapping the bubble and accounting for the singularity.

Motivated by the analysis of [11], it is worth examining how p-branes can enter in the game. We start with a working definition of p-branes as massive extended objects which are charged under gauge potentials and with a derivation of the charge quantization condition. Then, we explain how this condition can introduce a complication in the geometry of the bounce, both for Freund-Rubin solutions, where only an external flux is present, and Englert solutions, where an additional internal flux is turned on. Allowing the presence of singularities, if somewhat unexpected, turns out to enrich the structure of the bounce, possibly giving rise to more complicated dynamics. In particular, Young's no go theorem might need to be reconsiderd in this light.

p-branes and flux quantization Antisymmetric tensor fields in string and M-theory are naturally described as differential forms. Let us consider in D dimensions a $(p+1)$-form gauge potential, denoted as A_{p+1}, with gauge transformations

$$A_{p+1} \to A_{p+1} + d\Lambda_p, \tag{4.132}$$

with Λ_p a $(p+1)$-form gauge parameter. Then the field strength tensor $F_{p+2} = dA_{p+1}$ is gauge invariant. The objects electrically charged under A_{p+1} are called p-branes. They are extended objects with p spatial dimensions sweeping out a $p+1$ dimensional world-volume W_{p+1} as they evolve in time. The electric coupling is expressed as the integral of the gauge potential on the world-volume of the brane

$$S_e = Q \int_{W_{p+1}} A_{p+1} \tag{4.133}$$

and generalizes the coupling of a charged particle with the electromagnetic field, expressed as the integral of the one form electromagnetic gauge potential A_1 on the world-line W_1 of the particle,

$$S_e = Q \int_{W_1} A_1. \tag{4.134}$$

In D dimensions we can also define the dual field strength as the Hodge dual $F_{D-p-2} = *F_{p+2}$. Then the dual gauge potential is defined locally as $F_{D-p-2} = dA_{D-p-3}$. A p-brane is said to be electrically charged under A_{p+1} and magnetically charged under A_{D-p-3}, and vice versa for a $(D-p-4)$-brane.

Using Gauss' theorem, we can measure the electric charge carried by a p-brane by the flux on a sphere surrounding the source in the transverse $(D-p-1)$ dimensional space,

$$Q_e = \int_{S^{D-p-2}} *F_{p+2}, \tag{4.135}$$

and analogously for the magnetic charge,

$$Q_m = \int_{S^{p+2}} F_{p+2}. \tag{4.136}$$

These numbers are integers by a general flux quantization condition, as follows from a standard argument which can be found for example in [53]. This quantization is simple to understand: the (suitably normalized) flux through a cycle surrounding the branes is the number of enclosed branes, which is an integer.

At this point, it is useful to remark that it is also possible to have solutions with no brane sources and still non trivial fluxes, an example being given by the Freund-Rubin solutions described in section 4.3. However, we will eventually show that, as soon as we try to deform the Freund-Rubin vacuum geometry at some finite radius to construct a bubble solution, brane sources are needed for this deformation to be consistent.

The $AdS_5 \times S^5/\mathrm{Z}_k$ example In order to understand how the flux quantization condition can enter in the construction of the bounce, we refer to the analysis of [11] and work out the relevant aspects in some detail. The first observation is that $AdS \times S^5/\mathrm{Z}_k$ can be realized as a fibration of S^1 over $\mathrm{C}P^2$, with Euclidean metric

$$ds^2 = \frac{dr^2}{1+\frac{r^2}{R^2}} + r^2 d\Omega_4^2 + R^2 \left[ds^2_{\mathrm{C}P^2} + (d\chi + \Lambda)^2 \right], \tag{4.137}$$

where the coordinate χ of the fiber has periodicity $\frac{2\pi}{k}$ due to the Z_k orbifolding and Λ encloses the fibration structure. If we want the S^1 fiber to play the role of the Kaluza-Klein circle in Witten's original construction, we can consider for the bounce a metric of the form

$$ds^2 = \rho(r) dr^2 + f(r) d\Omega_4^2 + g(r) ds^2_{\mathrm{C}P^2} + h(r) (d\chi + \Lambda)^2, \tag{4.138}$$

which asymptotically matches with (4.137) and with $h(r)$ going to zero at a finite radius r_0 where the other functions are still non zero. Moreover, in order for the geometry to terminate smoothly, we should require that $h(r)$ vanish quadratically. However, the following argument shows that this cannot be the case. The five dimensional internal manifold supports a non zero five form flux,

$$\frac{1}{(2\pi)^3 \alpha'^2} \int_{S^5/\mathrm{Z}_k} F_5 = \frac{2\pi N}{k}, \tag{4.139}$$

where we must include a factor $\frac{1}{\alpha'^2}$ for dimensional reasons (the string scale $\alpha' = l_s^2$ is a *lenght*^{2} and F_5 is a *lenght*^{4}, so the left-hand side is dimensionless) and a factor $\frac{1}{(2\pi)^{d-1}}$, with d the dimensions of the integral, in order for the flux to be quantized in integer units. Since in the bounce geometry the collapsing circle is the boundary of a disk, the internal manifold can be seen as the boundary of a six dimensional manifold M_6 and we can use Stokes' theorem and the Bianchi identity to argue that

$$\int_{S^5/\mathrm{Z}_k} F_5 = \int_{M_6} dF_5 = 0, \tag{4.140}$$

which is in conflict with equation (4.139). In other words, what the flux quantization condition is telling us is that the geometry cannot be smooth at the bubble. The way out proposed in [11] is that the solution has D3-brane instantons wrapped on the S^4 at $r = r_0$ and smeared on the $\mathrm{C}P^2$. In particular, this implies that some metric functions must be singular at the bubble, with the singularities of smeared D3-branes. More details can be found in Appendix **??**.

Page charge and brane instantons In our M-theory setup, the relevant objects which can possibly appear are M2 and M5-branes, since only the four form F_4 and its dual, defined as

$$F_7 = *F_4 + A_3 \wedge F_4, \tag{4.141}$$

are present. By duality, we mean that the Maxwell equation for F_4 corresponds to the Bianchi identity for F_7. Moreover, since we are concerned with Euclidean solutions, they will enter as M2-brane instantons, that is extended objects with three spatial dimensions, rather than two spatial dimensions, plus time.

Let $AdS_4 \times M_7$ be a compactification of the Freund-Rubin type, with external flux $F_4 = dA_3$. For example, we can take $A_3 = \zeta(r)\,\mathrm{vol}_{S^3}$, with the function ζ chosen such that $F_4 = dA_3$ is proportional to the volume of AdS_4. Then $A_3 \wedge F_4 = 0$ and $F_7 = *F_4$. We have a non zero flux on the seven dimensional internal manifold,

$$\frac{1}{(2\pi)^5 \alpha'^2} \int_{M_7} *F_4 = N. \tag{4.142}$$

Now suppose to deform the vacuum geometry in such a way that there is a bubble of arbitrary dimension at some finite radius $r = r_0$ in AdS. Provided that the bubble is smooth, the internal manifold can be seen as the boundary of an eight dimensional manifold, $M_7 = \partial M_8$ and we can use Stokes' theorem,

$$\int_{M_7} *F_4 = \int_{M_8} d * F_4 = 0. \tag{4.143}$$

Comparing with (4.142), we conclude that the bounce solution does not satisfy the $AdS_4 \times M_7$ boundary conditions set by flux quantization, unless one adds an extra source of flux. In analogy with the argument of [11], a solution is to have M2-brane instantons wrapping the S^3 at $r = r_0$ and smeared over some part of the internal space.

We expect that this argument still applies when an additional internal flux is turned on, as in Englert solutions. That is, $F_4 = F_4^e + F_4^i$, with $F_4^e = dA_3^e$ and $F_4^i = dA_3^i$. The Maxwell equations in Euclidean signature yield

$$d(*F_4^e + iA_3^i \wedge F_4^i) = 0, \tag{4.144a}$$
$$d(*F_4^i + iA_3^e \wedge F_4^i + iA_3^i \wedge F_4^e) = 0. \tag{4.144b}$$

Focusing on the first one, we can argue exactly as before. There is a non zero flux on the internal seven dimensional manifold,

$$\frac{1}{(2\pi)^5 \alpha'^2} \int_{M_7} (*F_4^e + iA_3^i \wedge F_4^i) = N, \tag{4.145}$$

but, in the presence of a smooth bubble of arbitrary dimension in the bounce geometry, $M_7 = \partial M_8$ and Stokes'theorem implies that this flux is zero,

$$\int_{M_7} (*F_4^e + iA_3^i \wedge F_4^i) = \int_{M_8} d(*F_4^e + iA_3^i \wedge F_4^i) = 0. \tag{4.146}$$

Therefore, in order to account for the flux we need, we have to introduce M2-brane instantons.

What about the other equation? We don't expect this term to give rise to a further complication. The reason is the following. Suppose to integrate equation (4.144b),

$$\int \left(*F_4^i + A_3^e \wedge F_4^i + A_3^i \wedge F_4^e \right) . \tag{4.147}$$

Then Stokes' theorem and Maxwell equations tell us that this integral is zero, but no flux quantization condition can enter this time, since we are integrating over vol_4, that is over the external space, which is non compact and therefore has trivial cohomology group. We will eventually check that this is the case in the explicit example of Chapter 5 where, even forgetting about non compactness of the external space, the internal geometry will not contain any homologically non trivial cycle allowing for quantization.

We can also rephrase the flux quantization argument in the language of Page charges. It was first observed by Page in [54] that when the $D = 11$ spacetime has the $M_4 \times M_7$ topology, the integral

$$P = \frac{1}{\pi^4} \int_{M_7} (*F_4 + A_3 \wedge F_4) \tag{4.148}$$

gives a conserved charge P which is independent of the point x of spacetime. This is non zero both in the case of Freund-Rubin and Englert solutions. However, as we have seen, Stokes' theorem tells us that the Page charge of a smooth bubble of arbitrary dimension must be zero. Therefore a smooth bubble geometry cannot have the same asymptotic behaviour of the vacuum, which is the basic requirement for an acceptable bounce solution.

To conclude, it is interesting to compare this result with Young's no-go theorem, which was discussed in section 4.3. According to our argument based on Page charges, it seems that the case $n = 1$, that is a collapsing circle, is not admitted as a bounce solution as long as the geometry is smooth. Notice that this statement goes beyond Young's no-go argument. It also suggests a possible way-out, which is the inclusion of brane instantons as additional sources of flux in the bubble geometry. This possibility is further investigated in the following Chapter.

Chapter 5

A tri-Sasakian bubble geometry?

At this point, we have all the technical machinery we need to attack a concrete example. This is provided by a class of AdS compactifications on tri-Sasakian manifolds, which are a natural candidate to study for several reasons. Firstly, they are known to admit both Freund-Rubin and Englert vacua. Secondly, the tri-Sasakian geometry allows us to single out three Kaluza-Klein directions in the internal manifold, corresponding to the three-sphere S^3 that is fibered over a four dimensional base in the universal tri-Sasakian structure, and one could ask whether the S^3 can shrinks to zero at some finite value of the AdS radius, giving rise to a higher dimensional *bubble of nothing*. Thirdly, although from our general discussion in Chapter 4 some certainly are, is not known if *all* Englert solutions supported by an internal tri-Sasakian manifold are classically unstable.

The Chapter is structured as follows. To begin, we introduce the tri-Sasakian structure, a set of differential forms which can always be defined on tri-Sasakian manifolds and will be used as building blocks in making an ansatz for the four-form. We also write down the metric, introducing some arbitrary functions $f(r)$, $g(r)$ and $h(r)$ which are the analog of those defined in section 4.4. Then we impose the Bianchi identity and the Maxwell equation on the ansatz, ending up with two differential equations for two arbitrary functions $\alpha(r)$ and $\gamma(r)$ which are the analog of the function $\xi(r)$ defined in section 4.4. As a preliminary step towards the construction of a bounce solution, we focus on the two Maxwell equations plus the trace of the Einstein equation and study their near-bubble expansion. What we find is that a solution of these equations which is both smooth and obeys the boundary conditions of the Englert vacuum does not exist. However, this was only to be expected from our analysis of section 4.5, since a smooth bubble solution and the Englert vacuum have different Page charges, so they cannot coexist. Finally we discuss the introduction of M2-branes in the bubble geometry as a possible way-out.

Tri-Sasakian geometry Following [55], we provide here some basic notions about tri-Sasakian manifolds. Let us start with some definitions. A *Sasaki-Einstein manifold* is a Riemannian manifold (M, g) of dimension $2n+1$ such that its cone is Calabi-Yau, that is it has reduced holonomy $SU(n+1)$. An equivalent characterization is that it carries a Sasaki-Einstein structure (ξ, η, J, Ω), where ξ is a unit Killing vector, η the dual one-form given by $\eta(Y) = g(\xi, Y)$, J a real two-form and Ω a complex n-form. They satisfy

$$\begin{aligned} d\eta &= 2J, \qquad d\Omega = (n+1)i\eta \wedge \Omega, \\ J \wedge \Omega &= 0, \qquad \iota_\xi J = \iota_\xi \Omega = 0. \end{aligned} \tag{5.1}$$

A *tri-Sasakian manifold* is a Riemannian manifold of dimension $4m+3$, with $m \geq 1$, such that its cone is hyper-Kähler. In particular, since all Calabi-Yau manifolds are hyper-Kähler, a tri-Sasakian manifold is also a Sasaki-Einstein manifold, while the converse is not true. An equivalent characterization, which is more useful for us, is that a tri-Sasakian manifold admits three Killing vectors ξ^I $(I = 1, 2, 3)$, generating the $so(3)$ algebra,

$$\left[\xi^I, \xi^J, =\right] 2\varepsilon^{IJK}\xi^K, \tag{5.2}$$

and each making it Sasakian, with the dual forms η^I, defined by $\iota_{\xi^I}\eta^J = \delta^{IJ}$, satisfying

$$d\eta^I = 2J^I - \varepsilon^{IJK}\eta^J \wedge \eta^K, \tag{5.3}$$

with $\iota_{\xi^I} J^J = 0$. From this equation, it follows that

$$dJ^I = 2\varepsilon^{IJK} J^J \wedge \eta^K. \tag{5.4}$$

The vectors ξ^I define a three dimensional foliation, where the leaves are either $SU(2)$ or $SO(3)$ and the space of leaves is a quaternionic Kähler manifold (or orbifold), with the three almost complex structures given by $g^{-1}J^I$. The unit tri-Sasakian metric is

$$ds^2_{3S} = ds^2_{\text{QK}} + \eta_1^2 + \eta_2^2 + \eta_3^2. \tag{5.5}$$

If $m = 1$, which is the relevant case for us, we also have

$$J^I \wedge J^J = 2\delta^{IJ}\,\text{vol}_{QK}, \tag{5.6}$$

and the tri-Sasakian volume form is given by

$$\text{vol}_{3S} = \text{vol}_{QK} \wedge \eta^1 \wedge \eta^2 \wedge \eta^3 = \frac{1}{2} J^1 \wedge J^1 \wedge \eta^1 \wedge \eta^2 \wedge \eta^3. \tag{5.7}$$

Our ansatz Now we are ready to engage into the construction of the bounce. In doing so, we have two different but equivalent possibilities. One is to solve the equations of motion in Lorentzian signature and make the substitution $t \to -i\tau$ only at the end. The other is to solve the equations directly written in Euclidean signature. We choose the second one, where Euclidean signature is assumed from the beginning. The Euclidean metric is taken to be

$$ds^2_{11} = dr^2 + f(r)d\Omega_3^2 + g(r)ds^2_{\text{QK}} + h(r)\left(\eta_1^2 + \eta_2^2 + \eta_3^2\right), \tag{5.8}$$

where $d\Omega_3^2$ is the metric of the unit three-sphere and $f(r)$, $g(r)$, $h(r)$ are arbitrary functions of the AdS radius. Then the vielbein can be defined as

$$\begin{aligned} e^{1234} &= \frac{1}{2}g^2(r)J^1 \wedge J^1, \qquad e^5 = h^{\frac{1}{2}}(r)\,\eta^1, \qquad e^6 = h^{\frac{1}{2}}(r)\,\eta^2, \\ e^7 &= h^{\frac{1}{2}}(r)\,\eta^3, \qquad e^8 = dr, \qquad e^{9\,10\,11} = f^{\frac{3}{2}}(r)\,\text{vol}_{S^3}, \end{aligned} \tag{5.9}$$

and the volume form of the eleven dimensional space is

$$\text{vol}_{11} = g^2 h^{\frac{3}{2}}\,\text{vol}_{3S} \wedge \text{vol}_4, \tag{5.10}$$

where

$$\mathrm{vol}_4 = dr \wedge \mathrm{vol}_{S^3}\,. \tag{5.11}$$

Our ansatz for the four form is $F_4 = F_4^e + F_4^i$, with the external and internal fluxes given by

$$F_4^e = \xi(r)\,\mathrm{vol}_4, \tag{5.12a}$$

$$\begin{aligned} F_4^i &= \alpha(r) J^1 \wedge J^1 + \beta(r)\varepsilon^{IJK} dr \wedge \eta^I \wedge \eta^J \wedge \eta^K \\ &\quad + \gamma(r)\varepsilon^{IJK} J^I \wedge \eta^J \wedge \eta^K + \delta(r) dr \wedge J^I \wedge \eta^I, \end{aligned} \tag{5.12b}$$

where $\xi(r)$, $\alpha(r)$, $\beta(r)$, $\gamma(r)$, $\delta(r)$ are arbitrary functions of the AdS radius. In particular, equation (5.12b) is the most general ansatz that can be written in terms of the tri-Sasakian structure (η^I, J^I). The next step is to write down the Bianchi identity and the Maxwell equation for the ansatz. This is straightforward, since the set (η^I, J^I) is closed under the action of wedge product, Hodge star and exterior derivative, which are all the operations we need to perform. We will use the differentials

$$d\eta^I = 2J^I - \varepsilon^{IJK}\eta^J \wedge \eta^K, \qquad dJ^I = 2\varepsilon^{IJK} J^J \wedge \eta^K. \tag{5.13}$$

As for the Hodge star, we have

$$\begin{aligned} *_7\left(J^1 \wedge J^1\right) &= \frac{2h^{\frac{3}{2}}}{g^2}\eta^1 \wedge \eta^2 \wedge \eta^3, \qquad & *_7\left(\eta^1 \wedge \eta^2 \wedge \eta^3\right) &= \frac{g^2}{2h^{\frac{3}{2}}} J^1 \wedge J^1, \\ *_7\left(J^I \wedge \eta^I\right) &= \frac{h^{\frac{1}{2}}}{2}\varepsilon^{IJK} J^I \wedge \eta^J \wedge \eta^K, \qquad & *_7\left(J^I \wedge \eta^J \wedge \eta^K\right) &= \frac{1}{h^{\frac{1}{2}}} J^I \wedge \varepsilon^{JKM}\eta^M. \end{aligned} \tag{5.14}$$

From the Bianchi identity, we have $dF_4^i = 0$. We compute the differentials

$$\begin{aligned} \varepsilon^{IJK} d\left(\eta^I \wedge \eta^J \wedge \eta^K\right) &= 6\varepsilon^{IJK} J^I \wedge \eta^J \wedge \eta^K \\ \varepsilon^{IJK} d\left(J^I \wedge \eta^J \wedge \eta^K\right) &= 0 \\ d\left(J^I \wedge \eta^I\right) &= 6J^1 \wedge J^1 + \varepsilon^{IJK} J^I \wedge \eta^J \wedge \eta^K. \end{aligned} \tag{5.15}$$

Whence,

$$dF_4^i = (\alpha' - 6\delta)\alpha' dr \wedge J^1 \wedge J^1 + (-6\beta + \gamma' - \delta) dr \wedge \varepsilon^{IJK} J^I \wedge \eta^J \wedge \eta^K. \tag{5.16}$$

In order for this expression to vanish, we have

$$\delta = \frac{\alpha'}{6}, \qquad \beta = \frac{\gamma'}{6} - \frac{\alpha'}{36}. \tag{5.17}$$

The Maxwell equation (4.8) splits into

$$d * F_4^e = -iF_4^i \wedge F_4^i, \tag{5.18a}$$

$$d * F_4^i = -2iF_4^e \wedge F_4^i. \tag{5.18b}$$

We can easily compute

$$\begin{aligned} *F_4^e = \xi g^2 h^{\frac{3}{2}}\,\mathrm{vol}_{3S}, \qquad d * F_4^e &= \left(\xi g^2 h^{\frac{3}{2}}\right)' dr \wedge \mathrm{vol}_{3S}, \\ F_4^i \wedge F_4^i &= 24(\alpha\beta + \gamma\delta) dr \wedge \mathrm{vol}_{3S}\,. \end{aligned} \tag{5.19}$$

Then equation (5.18a) yields

$$\left(\xi g^2 h^{\frac{3}{2}}\right)' = i(\frac{\alpha^2}{3} - 4\alpha\gamma)', \tag{5.20}$$

which is trivial to integrate,

$$\xi = \frac{i}{g^2 h^{\frac{3}{2}}}\left(\frac{\alpha^2}{3} - 4\alpha\gamma + c\right). \tag{5.21}$$

The integration constant c is fixed by the boundary conditions at $r \to \infty$ to be $c = 3k$. Here k is an arbitrary positive parameter that appears in the Englert vacuum. Since k always factorizes in the equations, it is not relevant for our discussion and we are free to set $k = 1$. We also compute

$$\begin{aligned}
*F_4^i &= \frac{2\alpha h^{\frac{3}{2}}}{g^2}\eta^1 \wedge \eta^2 \wedge \eta^3 \wedge \mathrm{vol}_4 - \frac{3\beta g^2}{h^{\frac{3}{2}}} J^1 \wedge J^1 \wedge e^{9\,10\,11} \\
&+ \frac{2\gamma}{h^{\frac{1}{2}}} J^I \wedge \eta^I \wedge \mathrm{vol}_4 - \frac{\delta h^{\frac{1}{2}}}{2}\varepsilon^{IJK} J^I \wedge \eta^J \wedge \eta^K \wedge e^{9\,10\,11}, \\
d * F_4^i &= \left[-\left(3\beta\frac{g^2}{h^{\frac{3}{2}}}\right)' - \frac{9\beta f'}{2f}\frac{g^2}{h^{\frac{3}{2}}} + \frac{12\gamma}{h^{\frac{1}{2}}}\right] J^1 \wedge J^1 \wedge \mathrm{vol}_4 \\
&+ \left[\frac{2\alpha h^{\frac{3}{2}}}{g^2} + \frac{2\gamma}{h^{\frac{1}{2}}} - \left(\frac{\delta h^{\frac{1}{2}}}{2}\right)' - \frac{3f'\delta h^{\frac{1}{2}}}{4f}\right] \varepsilon^{IJK} J^I \wedge \eta^J \wedge \eta^K \wedge \mathrm{vol}_4, \\
F_4^e \wedge F_4^i &= \left(\alpha\xi J^1 \wedge J^1 + \gamma\xi\varepsilon^{IJK} J^I \wedge \eta^J \wedge \eta^K\right) \wedge \mathrm{vol}_4 .
\end{aligned} \tag{5.22}$$

Then equation (5.18b) yields

$$\begin{aligned}
-\left(3\beta\frac{g^2}{h^{\frac{3}{2}}}\right)' - \frac{9\beta f'}{2f}\frac{g^2}{h^{\frac{3}{2}}} + \frac{12\gamma}{h^{\frac{1}{2}}} + 2i\alpha\xi = 0, \\
\frac{2\alpha h^{\frac{3}{2}}}{g^2} + \frac{2\gamma}{h^{\frac{1}{2}}} - \left(\frac{\delta h^{\frac{1}{2}}}{2}\right)' - \frac{3f'\delta h^{\frac{1}{2}}}{4f} + 2i\gamma\xi = 0,
\end{aligned} \tag{5.23}$$

which substituting β, δ and ξ from equations (5.17) and (5.21) become

$$\frac{2\alpha h^{\frac{3}{2}}}{g^2} + \frac{2\gamma}{h^{\frac{1}{2}}} + \frac{2\gamma\left(-9 - \alpha^2 + 12\alpha\gamma\right)}{3g^2 h^{\frac{3}{2}}} - \frac{h^{\frac{1}{2}} f'\alpha'}{8f} - \frac{h'\alpha'}{24h^{\frac{1}{2}}} - \frac{h^{\frac{1}{2}}\alpha''}{12} = 0 \tag{5.24a}$$

$$\begin{aligned}
\frac{12\gamma}{h^{\frac{1}{2}}} + \frac{2\alpha\left(-9 - \alpha^2 + 12\alpha\gamma\right)}{3g^2 h^{\frac{3}{2}}} + \frac{g^2 f'\left(\alpha' - 6\gamma'\right)}{8fh^{\frac{3}{2}}} + \frac{gg'\left(\alpha' - 6\gamma'\right)}{6h^{\frac{3}{2}}} \\
- \frac{g^2 h'\left(\alpha' - 6\gamma'\right)}{8h^{\frac{5}{2}}} + \frac{g^2\left(\alpha'' - 6\gamma''\right)}{12h^{\frac{3}{2}}} = 0
\end{aligned} \tag{5.24b}$$

Now we turn to the trace of the Einstein equation, which in our conventions is

$$R = \frac{1}{36} F^{PQRS} F_{PQRS}. \tag{5.25}$$

The Ricci scalar for the metric (5.8) can be computed from the expression given in [55] for the Weyl-rescaled metric

$$ds_{11}^2 = e^{2\phi} ds_4^2 + e^{2U} ds_{\mathrm{QK}}^2 + e^{2V}\left(\eta_1^2 + \eta_2^2 + \eta_3^2\right), \tag{5.26}$$

where in our notation $g = e^{2U}$, $h = e^{2V}$ and $\phi = -2U - \frac{3}{2}V$, that is

$$R = e^{-2\phi}R_4 + R_7 - 4e^{-2\phi}g^{\mu\nu}\partial_\mu U\partial_\nu U - 3e^{-2\phi}g^{\mu\nu}\partial_\mu V\partial_\nu V - 2e^{-2\phi}\nabla^2\phi - 2e^{-2\phi}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi, \quad (5.27)$$

with

$$R_7 = 48e^{-2U} + 6e^{-2V} - 12e^{-4U+2V}. \quad (5.28)$$

If we now define

$$d\tilde{s}_4^2 = e^{2\phi}ds_4^2, \quad (5.29)$$

the Weyl-rescaled four dimensional Ricci tensor is given by

$$\tilde{R}_4 = e^{-2\phi}\left(R_4 - 6\nabla^2\phi - 6g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right), \quad (5.30)$$

whence

$$R = \tilde{R}_4 + R_7 - 4\tilde{g}^{\mu\nu}\partial_\mu U\partial_\nu U - 3\tilde{g}^{\mu\nu}\partial_\mu V\partial_\nu V + 4e^{-2\phi}\nabla^2\phi + 4\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \quad (5.31)$$

Here

$$\nabla^2\phi = \frac{1}{g_4}\partial_\mu\left(\sqrt{g_4}g^{\mu\nu}\partial_\nu\phi\right) = \frac{e^{4\phi}}{\sqrt{\tilde{g}_4}}\partial_\mu\left(e^{-2\phi}\sqrt{\tilde{g}_4}\tilde{g}^{\mu\nu}\partial_\nu\phi\right) = e^{2\phi}\tilde{\nabla}^2\phi - 2e^{2\phi}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \quad (5.32)$$

Given the four dimensional metric $d\tilde{s}_4^2 = dr^2 + fd\Omega_3^2$ and using $U' = \frac{g'}{2g}$, $V' = \frac{h'}{2h}$, we find

$$R = \tilde{R}_4 + R_7 - \frac{g'^2}{g^2} - 6\frac{g'h'}{gh} - 6\frac{f'g'}{fg} - \frac{9}{2}\frac{f'h'}{fh} - 4\frac{g''}{g} - 3\frac{h''}{h}, \quad (5.33)$$

with

$$\tilde{R}_4 = \frac{6-3f''}{f}, \qquad R_7 = \frac{48}{g} + \frac{6}{h} - \frac{12h}{g^2}. \quad (5.34)$$

As for the contribution of the four-form, we have

$$F_4 \wedge *F_4 = \frac{1}{24}F^{PQRS}F_{PQRS} = \left(\xi^2 + \frac{4\alpha^2}{g^4} + \frac{36\beta^2}{h^3} + \frac{24\gamma^2}{g^2h^2} + \frac{6\delta^2}{g^2h}\right)\mathrm{vol}_{11}. \quad (5.35)$$

Substituting once again β, δ and ξ from equations (5.17) and (5.21), the trace of the Einstein equation is given by

$$\begin{aligned} &\frac{6-3f''}{f} + \frac{48}{g} + \frac{6}{h} - \frac{12h}{g^2} - \frac{g'^2}{g^2} - 6\frac{g'h'}{gh} - 6\frac{f'g'}{fg} - \frac{9}{2}\frac{f'h'}{fh} - 4\frac{g''}{g} - 3\frac{h''}{h} \\ &\quad - \frac{8\alpha^2}{3g^4} - \frac{16\gamma^2}{g^2h^2} + \frac{2\left(9+\alpha^2-12\alpha\gamma\right)^2}{27g^4h^3} - \frac{\alpha'^2}{9g^2h} - \frac{\left(\alpha'-6\gamma'\right)^2}{54h^3} = 0. \end{aligned} \quad (5.36)$$

As a consistency check, it is easy to verify that these equations (where we have reintroduced the parameter k) are solved by the following known solutions [55], here given in Euclidean signature:

1. $\mathcal{N} = 3$ supersymmetric tri-Sasakian solution ($k > 0$) and skew-whiffing ($k < 0$):

$$\begin{aligned} &g = h = |k|^{\frac{1}{3}}, \qquad f = L^2\sinh^2\left(\frac{r}{L}\right), \qquad \Lambda = -\frac{6}{L^2} = -\frac{24}{|k|^{\frac{1}{3}}}, \\ &\alpha = \beta = \gamma = \delta = 0, \qquad \xi = 3i\,\mathrm{sgn}\,(k)\,|k|^{-\frac{1}{6}}; \end{aligned} \quad (5.37)$$

2. $\mathcal{N}=1$ supersymmetric squashed solution ($k>0$) and skew-whiffing ($k<0$):

$$\begin{gathered} g = 5\left(\frac{|k|}{15}\right)^{\frac{1}{3}}, \quad h = \left(\frac{|k|}{15}\right)^{\frac{1}{3}}, \quad f = L^2 \sinh^2\left(\frac{r}{L}\right), \quad \Lambda = -\frac{216}{5}\left(\frac{3}{25\,|k|}\right)^{\frac{1}{3}}, \\ \alpha=\beta=\delta=\gamma=0, \qquad \xi = -\operatorname{sgn}(k)\,\frac{9}{5}\left(\frac{15}{|k|}\right)^{\frac{1}{6}}; \end{gathered} \tag{5.38}$$

3. Englert solution corresponding to solution 2 ($k>0$):

$$\begin{gathered} g = 5\left(\frac{4k}{75}\right)^{\frac{1}{3}} \quad h = \left(\frac{4k}{75}\right)^{\frac{1}{3}} \quad f = L^2 \sinh^2\left(\frac{r}{L}\right) \quad \Lambda = -\frac{18}{k^{\frac{1}{3}}}\left(\frac{6}{5}\right)^{\frac{1}{3}}, \\ \alpha = \pm(3k)^{\frac{1}{2}}, \qquad \beta=\delta=0, \qquad \gamma = \pm\frac{(3k)^{\frac{1}{2}}}{5}, \qquad \xi = 3i\left(\frac{2}{5}\right)^{\frac{2}{3}}\left(\frac{3}{k}\right)^{\frac{1}{6}}. \end{gathered} \tag{5.39}$$

Near-bubble expansion From the Maxwell equations plus the trace of the Einstein equation we found a system of three differential equations in five unknown functions, which we now reproduce for further reference,

$$\frac{2\alpha h^{\frac{3}{2}}}{g^2} + \frac{2\gamma}{h^{\frac{1}{2}}} + \frac{2\gamma\left(-9-\alpha^2+12\alpha\gamma\right)}{3g^2h^{\frac{3}{2}}} - \frac{h^{\frac{1}{2}}f'\alpha'}{8f} - \frac{h'\alpha'}{24h^{\frac{1}{2}}} - \frac{h^{\frac{1}{2}}\alpha''}{12} = 0 \tag{5.40a}$$

$$\begin{aligned} \frac{12\gamma}{h^{\frac{1}{2}}} + \frac{2\alpha\left(-9-\alpha^2+12\alpha\gamma\right)}{3g^2h^{\frac{3}{2}}} + \frac{g^2f'\left(\alpha'-6\gamma'\right)}{8fh^{\frac{3}{2}}} + \frac{gg'\left(\alpha'-6\gamma'\right)}{6h^{\frac{3}{2}}} + \\ -\frac{g^2h'\left(\alpha'-6\gamma'\right)}{8h^{\frac{5}{2}}} + \frac{g^2\left(\alpha''-6\gamma''\right)}{12h^{\frac{3}{2}}} = 0 \end{aligned} \tag{5.40b}$$

$$\begin{aligned} \frac{6-3f''}{f} + \frac{48}{g} + \frac{6}{h} - \frac{12h}{g^2} - \frac{g'^2}{g^2} - 6\frac{g'h'}{gh} - 6\frac{f'g'}{fg} - \frac{9}{2}\frac{f'h'}{fh} - 4\frac{g''}{g} - 3\frac{h''}{h} + \\ -\frac{8\alpha^2}{3g^4} - \frac{16\gamma^2}{g^2h^2} + \frac{2\left(9+\alpha^2-12\alpha\gamma\right)^2}{27g^4h^3} - \frac{\alpha'^2}{9g^2h} - \frac{\left(\alpha'-6\gamma'\right)^2}{54h^3} = 0 \end{aligned} \tag{5.40c}$$

Let r_0 be the radius of the bubble where the geometry pinches off. We also define $\tilde{r} = r - r_0$, so that the near-bubble limit corresponds to taking $\tilde{r} \to 0$. What about the boundary conditions at $\tilde{r}=0$? In order for the geometry to terminate smoothly, we must require that $h(r) \sim \tilde{r}^2$. So, from the geometry side we have the following expansions at the bubble,

$$\begin{aligned} g(r) &= g_0 + g_1\tilde{r} + g_2\tilde{r}^2 + \ldots, \\ h(r) &= \tilde{r}^2 + h_3\tilde{r}^3 + h_4\tilde{r}^4 + \ldots, \\ f(r) &= f_0 + f_1\tilde{r} + f_2\tilde{r}^2 + \ldots. \end{aligned} \tag{5.41}$$

As for the four form coefficients in equation (5.12b), from the $\mathrm{C}P^3$ example we learnt that the flux must re-orient itself at the bubble in such a way that it has no legs along the collapsing fiber and we expect to have vanishing β, γ and δ at the bubble. How fast these functions should go to zero is *a priori* not obvious, so for the moment we simply assume

$$\begin{aligned} \alpha(r) &= \alpha_0 + \alpha_1\tilde{r} + \alpha_2\tilde{r}^2 + \ldots, \\ \gamma(r) &= \gamma_0 + \gamma_1\tilde{r} + \gamma_2\tilde{r}^2 + \ldots. \end{aligned} \tag{5.42}$$

Now the idea is to plug these expansions in (5.40), isolate the divergent terms and try to set them to zero by appropriately choosing the expansion coefficients. The divergences are at most cubic in (5.40a), quartic in (5.40b) and of the sixth order in (5.40c), whence thirteen conditions on the expansions coefficients come,

$$\frac{m_3}{r^3} + \frac{m_2}{r^2} + \frac{m_1}{r} + \mathcal{O}(1) = 0 \tag{5.43a}$$

$$\frac{n_4}{r^4} + \frac{n_3}{r^3} + \frac{n_2}{r^2} + \frac{n_1}{r} + \mathcal{O}(1) = 0 \tag{5.43b}$$

$$\frac{t_6}{r^6} + \frac{t_5}{r^5} + \frac{t_4}{r^4} + \frac{t_3}{r^3} + \frac{t_2}{r^2} + \frac{t_1}{r} + \mathcal{O}(1) = 0. \tag{5.43c}$$

Here the ordering of the three equations is the same as in (5.40). To begin, we observe that

$$n_4 = -\frac{1}{4} g_0^2 \left(\alpha_1 - 6\gamma_1\right) \overset{!}{=} 0 \quad \Rightarrow \quad \gamma_1 = \frac{\alpha_1}{6}. \tag{5.44}$$

Next, we have

$$m_3 = -\frac{2\gamma_0 \left(9 + \alpha_0^2 - 12\alpha_0\gamma_0\right)}{3g_0^2} \overset{!}{=} 0 \quad \Rightarrow \quad \gamma_0 = 0 \quad \text{or} \quad \gamma_0 = \frac{9+\alpha_0^2}{12\alpha_0}. \tag{5.45}$$

whence there are two possibilities to consider. The first one, $\gamma_0 = 0$, is easy to exclude, since it would imply

$$t_6 = \frac{2\left(9 + \alpha_0^2\right)^2}{27 g_0^4}, \tag{5.46}$$

which cannot be set to zero. As for $\gamma_0 = \frac{9+\alpha_0^2}{12\alpha_0}$, a similar conclusion holds. We find that

$$m_2 = -\frac{\left(9 + a_0^2\right)^2 a_1}{18\alpha_0^2 g_0^2} \overset{!}{=} 0 \quad \Rightarrow \quad \alpha_1 = 0. \tag{5.47}$$

Then, $t_6 = t_5 = 0$, automatically. Moreover,

$$n_3 = -\frac{1}{3} g_0^2 \left(\alpha_2 - 6\gamma_2\right) \overset{!}{=} 0 \quad \Rightarrow \quad \gamma_2 = \frac{\alpha_2}{6}, \tag{5.48}$$

implying that

$$t_4 = \frac{\left(9 + \alpha_0^2\right)^2}{9 g_0^2 \alpha_0^2}, \tag{5.49}$$

which cannot be set to zero. Therefore, there is no choice of the expansion coefficients such that the divergences cancel out. It is worth observing that problems are encountered when dealing with a term proportional to $\left(9 + a_0^2\right)^2$, which cannot be set to zero. Here the choice of the integration constant c in equation (5.21) is crucial, since for arbitrary c this term would have been $\left(3c + a_0^2\right)^2$ which can be set to zero. In other words, what our analysis is telling us is that there cannot exist a solution of (5.40) which is both smooth and has $c = 3$. Unfortunately, the value of $c = 3$ is set by the boundary condition of the Englert vacuum and there is nothing we can do about it. However, this is something we should have expected from our argument in section 4.5 based on Page charges.

Let us go back for a moment to the integral in equation (4.145) and evaluate it in our concrete tri-Sasakian example, where

$$A_3^i = \frac{\alpha}{6} J^I \wedge \eta^I + \frac{1}{6}\left(\gamma - \frac{\alpha}{6}\right) \varepsilon^{IJK} \eta^I \wedge \eta^J \wedge \eta^K, \tag{5.50a}$$

$$A_3^e = \zeta e^{9\,10\,11}, \quad \text{with } \xi = \left(\zeta + \ln f^{\frac{3}{2}}\right)'. \tag{5.50b}$$

Upon differentiation, these are the right expressions to reproduce our ansatz (5.12b) for the four-form. Then we have

$$* F_4^e + i A_3^i \wedge F_4^i = \left[\xi g^2 h^{\frac{3}{2}} + i\left(4\alpha\gamma + \frac{\alpha^2}{3}\right)\right] \mathrm{vol}_{3S}\,. \tag{5.51}$$

We observe that the quantity into square brackets is exactly the integration constant c defined in equation (5.21). This is not surprising, since it comes from integrating a total derivative coming from that part of Maxwell equation which is proportional to $dr \wedge \mathrm{vol}_{3S}$, that is the same as computing the integral in (4.145). In other words, c is a Page charge, whose value is $c = 3$ for the Englert vacuum. Therefore, the impossibity to found a solution in our near-bubble expansion beacuse of the value of c can be conveniently rephrased as the non existence of a smooth solution with the same Page charge as the vacuum, as argued in section 4.5.

We can also check what happens to the integral in equation (4.147). We have to integrate

$$\begin{aligned} *F_4^i + A_3^e \wedge F_4^i + A_3^i \wedge F_4^e = &\left[\left(\alpha\zeta - \frac{3\beta g^2}{h^{\frac{3}{2}}}\right) J^1 \wedge J^1 \wedge + \left(-\frac{\delta h^{\frac{1}{2}}}{2} + \gamma\zeta\right) \varepsilon^{IJK} J^I \wedge \eta^J \wedge \eta^K\right] e^{9\,10\,11} \\ &+ \left\{\left[\frac{2\alpha h^{\frac{3}{2}}}{g^2} + \xi\left(\gamma - \frac{\alpha}{6}\right) - \zeta\beta\right] \eta^1 \wedge \eta^2 \wedge \eta^3 + \left(\frac{2\gamma}{h^{\frac{1}{2}}} + \frac{\alpha\xi}{6} + \zeta\delta\right) J^I \wedge \eta^I\right\} \wedge \mathrm{vol}_4, \end{aligned} \tag{5.52}$$

which is not zero. In fact, it is only its derivative that vanishes. However, no quantization condition can arise this time since there is no non trivial cycle in the internal manifold allowing for quantization and, as already discussed in section 4.5, no further complication come from this term.

Including branes As suggested in section 4.5, one possible way-out is that the bubble geometry is not smooth, but has the singularity of M2-brane instantons wrapping the S^3 in AdS_4 and smeared over the quaternionic Kähler subspace of the internal manifold. To begin, it is convenient to rewrite the ansatz (5.8) for the metric as

$$ds_{11}^2 = \rho(r) dr^2 + f(r) d\Omega_3^2 + g(r) ds_{\mathrm{QK}}^2 + h(r)\left(\eta_1^2 + \eta_2^2 + \eta_3^2\right). \tag{5.53}$$

where we have introduced an arbitrary function $\rho(r)$. Then, the equations in (5.40) have to be modified. However, running through our calculations again, it is easy to see how the function ρ

enters and the result is

$$\begin{aligned}\frac{2\alpha h^{\frac{3}{2}}\rho^{\frac{1}{2}}}{g^2}+\frac{2\gamma\rho^{\frac{1}{2}}}{h^{\frac{1}{2}}}+\frac{2\gamma\left(-9-\alpha^2+12\alpha\gamma\right)\rho^{\frac{1}{2}}}{3g^2h^{\frac{3}{2}}}-\frac{h^{\frac{1}{2}}f'\alpha'}{8f\rho^{\frac{1}{2}}}-\frac{h'\alpha'}{24h^{\frac{1}{2}}\rho^{\frac{1}{2}}}+&\\ +\frac{h^{\frac{1}{2}}\alpha'\rho'}{24\rho^{\frac{3}{2}}}-\frac{h^{\frac{1}{2}}\alpha''}{12\rho^{\frac{1}{2}}}&=0\end{aligned}\tag{5.54a}$$

$$\begin{aligned}\frac{12\gamma\rho^{\frac{1}{2}}}{h^{\frac{1}{2}}}+\frac{2\alpha\left(-9-\alpha^2+12\alpha\gamma\right)\rho^{\frac{1}{2}}}{3g^2h^{\frac{3}{2}}}+\frac{g^2f'\left(\alpha'-6\gamma'\right)}{8fh^{\frac{3}{2}}\rho^{\frac{1}{2}}}+\frac{gg'\left(\alpha'-6\gamma'\right)}{6h^{\frac{3}{2}}\rho^{\frac{1}{2}}}+&\\ -\frac{g^2h'\left(\alpha'-6\gamma'\right)}{8h^{\frac{5}{2}}\rho^{\frac{1}{2}}}-\frac{g^2\left(\alpha'-6\gamma'\right)\rho'}{24h^{\frac{3}{2}}\rho^{\frac{3}{2}}}+\frac{g^2\left(\alpha''-6\gamma''\right)}{12h^{\frac{3}{2}}\rho^{\frac{1}{2}}}&=0\end{aligned}\tag{5.54b}$$

$$\begin{aligned}\frac{6}{f}+\frac{3f'\rho'}{2f\rho^2}-\frac{3f''}{f\rho}+\frac{48}{g}+\frac{6}{h}-\frac{12h}{g^2}-\frac{g'^2}{g^2\rho}-\frac{6g'h'}{gh\rho}-\frac{6f'g'}{fg\rho}+\frac{2g'\rho'}{g\rho^2}-\frac{4g''}{g\rho}-\frac{3h''}{h\rho}+&\\ -\frac{9}{2}\frac{f'h'}{fh\rho}+\frac{3h'\rho'}{2h\rho^2}-\frac{8\alpha^2}{3g^4}-\frac{16\gamma^2}{g^2h^2}+\frac{2\left(9+\alpha^2-12\alpha\gamma\right)^2}{27g^4h^3}-\frac{\alpha'^2}{9g^2h\rho}-\frac{\left(\alpha'-6\gamma'\right)^2}{54h^3\rho}&=0\end{aligned}\ .\tag{5.54c}$$

The smooth near-bubble expansions in (5.41) and (5.42) have also to be modified. In order to make a reasonable ansatz, we start recalling the flat brane result from Appendix A. There it is shown that a flat M2-brane solution can be written in the form

$$ds^2 = H(r)^{-\frac{2}{3}}ds^2_{\parallel} + H(r)^{\frac{1}{3}}ds^2_{\perp},\tag{5.55a}$$

$$F_4 = dH^{-1}(r)\wedge \mathrm{vol}_{\parallel},\tag{5.55b}$$

where $ds^2_{\parallel}$ and $ds^2_{\perp}$ are the metric on the parallel and transverse directions to the brane, respectively. Here we consider M2-brane instantons wrapping the S^3 in AdS and smeared over the four dimensional base of the tri-Sasakian manifold. If we stay very close to the branes, we expect the effects of curvature to be negligible, so that we can treat the branes as being in a locally flat spacetime and use the result above. Then the geometry in the near brane regime, labelled "M2", is

$$\begin{aligned}\rho_{M2}&\sim H(r)^{\frac{1}{3}}, & f_{M2}&\sim H(r)^{-\frac{2}{3}},\\ g_{M2}&\sim H(r)^{\frac{1}{3}}, & h_{M2}&\sim H(r)^{\frac{1}{3}},\end{aligned}\tag{5.56}$$

with the harmonic function $H(r)$ given by

$$H(r) = 1+\frac{\alpha}{r^2},\tag{5.57}$$

since there are only four transverse directions, due to the smearing. However, since we are not in flat space, we expect to have corrections to the behaviour displayed in (5.56) and assume the following expansions at the bubble,

$$\begin{aligned}\rho(r)&=\frac{1}{\tilde r^{\frac{2}{3}}}\left(\rho_0+\rho_1 r^2+\rho_2 r^2+\dots\right), & f(r)&=\tilde r^{\frac{4}{3}}\left(f_0+f_1\tilde r+f_2\tilde r^2+\dots\right),\\ g(r)&=\frac{1}{\tilde r^{\frac{2}{3}}}\left(g_0+g_1\tilde r+g_2\tilde r^2+\dots\right), & h(r)&=\frac{1}{\tilde r^{\frac{2}{3}}}\left(\rho_0\tilde r^2+h_3\tilde r^3+h_4\tilde r^4+\dots\right),\\ \alpha(r)&=\alpha_0+\alpha_1\tilde r+\alpha_2\tilde r^2+\dots, & \gamma(r)&=\gamma_0+\gamma_1\tilde r+\gamma_2\tilde r^2+\dots.\end{aligned}\tag{5.58}$$

Now we can proceed as before, plugging the expansions (5.58) into each of the three equations in (5.54). We obtain

$$\frac{m_1}{r} + \mathcal{O}(1) = 0 \tag{5.59a}$$
$$\frac{n_4}{r^4} + \frac{n_3}{r^3} + \frac{n_2}{r^2} + \frac{n_1}{r} + \mathcal{O}(1) = 0 \tag{5.59b}$$
$$\frac{t_{\frac{10}{3}}}{r^{\frac{10}{3}}} + \frac{t_{\frac{7}{3}}}{r^{\frac{7}{3}}} + \frac{t_{\frac{4}{3}}}{r^{\frac{4}{3}}} + \frac{t_{\frac{1}{3}}}{r^{\frac{1}{3}}} + \mathcal{O}(1) = 0. \tag{5.59c}$$

The situation looks somewhat better than before, and indeed a solution is easy to find. If we choose for example $\gamma_0 = 0$, $\gamma_{1,2,3,4} = \frac{\alpha_{1,2,3,4}}{6}$ all of the coefficients of the divergent terms in (5.59) cancel out except for $t_{\frac{4}{3}}$ and $t_{\frac{1}{3}}$. However, they can also be set to zero, for example by appropriately choosing f_0 and f_1.

Having passed this preliminary check, we are encouraged to write down the full set of the Einstein equations and see if they can be solved. To begin, let us compute the right-hand side of (4.5), that is the energy-momentum tensor of the four-form. The term where the indices of the four-form are all contracted has already been computed in (5.35), so we focus on the matrix $F^2_{AB} = F_A{}^{CDE} F_{BCDE}$ with two free indices. Using the properties of the three almost complex structures J^I, namely $J^{Ia}_b J^{Ib}_c = -\delta^a_c$ and $J^{Iab} J^I_{ab} = 4\delta^{IJ}$, with a, b labeling the directions $1, 2, 3, 4$, we are spared to give an explicit coordinate parametrization to the vielbein, at least for the moment. Let us define for example

$$H_4 = J^I \wedge e^{58} \quad \Rightarrow \quad H_{ABCD} = 12 J_{[AB}\delta^5_C \delta^8_{D]}. \tag{5.60}$$

Then the relevant contractions to compute are

$$H_{ABC8} = 3J_{[AB}\delta^5_{C]} \quad \Rightarrow \quad H_{8ABC}H_8^{ABC} = 9J_{[AB}\delta^5_{C]} J^{AB}\delta^C_5 = 3J_{ab}J^{ab} = 12, \tag{5.61}$$

those where both indices of J are contracted, and

$$H_{aBCD} = 6J_{a[B}\delta^5_C \delta^8_{D]} \quad \Rightarrow \quad H_{aCDE}H^{bCDE} = 36 J_{a[C}\delta^5_D \delta^8_{E]} J^{bC}\delta^D_5 \delta^E_8 = 6J_{ac}J^{bc} = 6\delta^b_a, \tag{5.62}$$

where one single index of J is contracted. Then, we have a diagonal matrix with four indipendent components,

$$\begin{aligned} F^2_{11} &= 24\frac{\alpha^2}{g^4} + 72\frac{\gamma^2}{g^2h^2} + 18\frac{\delta^2}{g^2h\rho} \\ F^2_{55} &= 216\frac{\beta^2}{h^3\rho} + 96\frac{\gamma^2}{g^2h^2} + 12\frac{\delta^2}{g^2h\rho} \\ F^2_{88} &= 6\frac{\xi^2}{\rho} + 216\frac{\beta^2}{h^3} + 36\frac{\delta^2}{g^2h\rho} \\ F^2_{99} &= 6\frac{\xi^2}{\rho}. \end{aligned} \tag{5.63}$$

The needed Ricci tensor can be computed by modeling the tri-Sasakian geometry on coset manifolds, see [55] for details. We eventually find that it is also diagonal, with only independent components

$$\begin{aligned}
R_{11} &= \frac{12}{g} - \frac{6h}{g^2} - \frac{g'^2}{2g^2\rho} - \frac{3g'h'}{4g\rho} - \frac{3f'g'}{4fg\rho} + \frac{g'\rho'}{4g\rho^2} - \frac{g''}{2g\rho},\\
R_{55} &= \frac{2}{h} + \frac{4h}{g^2} - \frac{h'^2}{4h^2\rho} - \frac{3f'h'}{4fh\rho} - \frac{g'h'}{gh\rho} + \frac{h'\rho'}{4h\rho^2} + \frac{h'\rho'}{4h\rho^2} - \frac{h''}{2h\rho},\\
R_{88} &= \frac{3f'^2}{4f^2\rho} + \frac{g'^2}{g^2\rho} + \frac{3h'^2}{4h^2\rho} + \frac{3f'\rho'}{4f\rho^2} + \frac{g'\rho'}{g\rho^2} + \frac{3h'\rho'}{4h\rho^2} - \frac{3f''}{2f\rho} - \frac{2g''}{g\rho} - \frac{3h''}{2h\rho},\\
R_{99} &= \frac{2}{f} - \frac{f'^2}{4f^2\rho} - \frac{f'g'}{fg\rho} - \frac{3f'h'}{4fh\rho} + \frac{f'\rho'}{4f\rho^2} - \frac{f''}{2f\rho}.
\end{aligned} \tag{5.64}$$

Then the four Einstein equations are

$$\begin{aligned}
&\frac{12}{g} - \frac{6h}{g^2} - \frac{g'^2}{2g^2\rho} - \frac{3g'h'}{4g\rho} - \frac{3f'g'}{4fg\rho} + \frac{g'\rho'}{4g\rho^2} - \frac{g''}{2g\rho}\\
&-\frac{16\alpha^2}{3g^4} - \frac{8\gamma^2}{g^2h^2} - \frac{\alpha'^2}{18g^2h\rho} + \frac{2\left(3+\frac{\alpha^2}{3}-4\alpha\gamma\right)^2}{3g^4h^3} + \frac{24\left(-\frac{\alpha'}{36}+\frac{\gamma'}{6}\right)^2}{h^3\rho} = 0,
\end{aligned} \tag{5.65a}$$

$$\begin{aligned}
&\frac{2}{h} + \frac{4h}{g^2} - \frac{h'^2}{4h^2\rho} - \frac{3f'h'}{4fh\rho} - \frac{g'h'}{gh\rho} + \frac{h'\rho'}{4h\rho^2} + \frac{h'\rho'}{4h\rho^2} - \frac{h''}{2h\rho}\\
&\frac{8\alpha^2}{3g^4} - \frac{16\gamma^2}{g^2h^2} + \frac{2\left(3+\frac{\alpha^2}{3}-4\alpha\gamma\right)^2}{3g^4h^3} - \frac{48\left(-\frac{\alpha'}{36}+\frac{\gamma'}{6}\right)^2}{h^3\rho} = 0,
\end{aligned} \tag{5.65b}$$

$$\begin{aligned}
&\frac{3f'^2}{4f^2\rho} + \frac{g'^2}{g^2\rho} + \frac{3h'^2}{4h^2\rho} + \frac{3f'\rho'}{4f\rho^2} + \frac{g'\rho'}{g\rho^2} + \frac{3h'\rho'}{4h\rho^2} - \frac{3f''}{2f\rho} - \frac{2g''}{g\rho} - \frac{3h''}{2h\rho}\\
&\frac{8\alpha^2}{3g^4} + \frac{16\gamma^2}{g^2h^2} - \frac{2\alpha'^2}{9g^2h\rho} - \frac{4\left(3+\frac{\alpha^2}{3}-4\alpha\gamma\right)^2}{3g^4h^3} - \frac{48\left(-\frac{\alpha'}{36}+\frac{\gamma'}{6}\right)^2}{h^3\rho} = 0,
\end{aligned} \tag{5.65c}$$

$$\begin{aligned}
&\frac{2}{f} - \frac{f'^2}{4f^2\rho} - \frac{f'g'}{fg\rho} - \frac{3f'h'}{4fh\rho} + \frac{f'\rho'}{4f\rho^2} - \frac{f''}{2f\rho}\\
&\frac{8\alpha^2}{3g^4} + \frac{16\gamma^2}{g^2h^2} - \frac{\alpha'^2}{9g^2h\rho} - \frac{4\left(3+\frac{\alpha^2}{3}-4\alpha\gamma\right)^2}{3g^4h^3} + \frac{24\left(-\frac{\alpha'}{36}+\frac{\gamma'}{6}\right)^2}{h^3\rho} = 0.
\end{aligned} \tag{5.65d}$$

Now we plug in the expansions (5.58) and try to solve them at least at $\mathcal{O}(1)$. We observe that the choice

$$\gamma_1 = \frac{\alpha_1}{6}, \tag{5.66}$$

which is required by the Maxwell equations, is sufficient to cancel the divergences at leading and next-to-leading order in the Einstein equations. Then problems come. Subtracting (5.65c) from (5.65d), we obtain, at first non trivial order,

$$\frac{2}{f_0} + \frac{2\left(\alpha_2 - 6\gamma_2\right)^2}{9h_2^3\rho_0}, \tag{5.67}$$

which is the sum of two positive terms and cannot be set to zero!

Chapter 6

Conclusions

In this final Chapter, we summarize what we have learnt about generalized *bubbles of nothing* as a possible non perturbative mechanism leading to instabilities in M-theory compactifications to AdS spaces. We will put emphasis on the most interesting aspects of our general discussion in Chapter 4 and also make some comments on the concrete tri-Sasakian example of Chapter 5.

Higher dimensional *bubbles of nothing* seem to be a natural mechanism to consider, since compactifications tipically involve non trivial internal spaces, a subspace of which may collapse in analogy with the Kaluza-Klein circle of Witten's original construction. The most serious obstruction to the existence of such solutions seems to come from the presence of fluxes supporting the vacuum geometry. Various aspects regarding flux conservation were already discussed in [9], [10] and [11]. Moreover, we manage to provide a general no-go argument based on flux quantization which applies to four dimensional AdS compactifications of both the Freund-Rubin and the Englert type. It forbids the existence of smooth bubble geometries with the same boundary conditions as the vacuum, since the Page charge of these vacua is always non zero but it would vanish for a smooth bubble. A promising way-out is provided by the inclusion of M2-brane instantons, wrapping the S^3 in AdS and possibly smeared over some part of the internal space, which might account for the additional flux we need.

All of these aspects turn out to be relevant in our attempt to construct a tri-Sasakian bubble geometry, where the three-dimensional fiber should play the role of the Kaluza-Klein circle in Witten's example. We verify that a smooth bubble cannot exist, as expected from our argument based on Page charges. Then we consider the inclusion of M2-branes in the bubble geometry and set up a perturbative expansion of the field equations in the near-brane regime, where we should be able to use the flat brane result. The leading order divergences indeed cancel, but an obstruction is found at the level of the Einstein equations, which may be interpreted geometrically as the impossibility to compensate the contribution of the flux with that of the Ricci tensor, since they both have the same sign. However, this is not our final word about it since, unless we are able to get a deeper understanding of this geometric obstruction, we cannot still exclude that a more general ansatz for the bubble solution could work and, in that case, a numerical analysis might be worth doing.

Appendix A

Brane solutions in supergravity

String theories contain p-branes as non-perturbative states, whose tension (mass per unit world-volume) is proportional to inverse powers of the string coupling g_s, while the string oscillation modes in the perturbative spectrum have masses independent of g_s. As explained in section 4.5, a p-brane in D dimensions couples electrically to a $(p+1)$-form and magnetically to the dual $(D-p-3)$-form. Therefore, the allowed values of p can be simply determined by inspection of the bosonic field content of the theory. From what we know about non perturbative states in quantum field theory, it should be possible to construct them as collective excitations of the spacetime fields, described by classical solutions of the spacetime equations of motion. Since we don't have a spacetime action for the full string theory, the strategy is to look for solutions of the classical equations of motion of its low energy approximation, that is supergravity. In this Appendix, we provide some basic notions about brane solutions in supergravity theories, focusing on D3-branes in $D=10$ type IIB and M2-branes in $D=11$ supergravity, which are the cases of interest for us in Chapter 4 and 5. We will mainly follow [56].

The backreacted brane geometry Let us consider a low energy effective action of the form

$$S=\frac{1}{2k_d^2}\int d^Dx\sqrt{-g}\left(R-\frac{1}{2}g^{MN}\partial_M\phi\partial_N\phi-\frac{1}{2(p+2)!}e^{a_p\phi}F_{p+2}^2\right), \tag{A.1}$$

where we have dropped the Chern-Simons term. Here $F_{p+2}^2=F_{M_0\ldots M_p+1}F^{M_0\ldots M_p+1}\equiv F^2$. For string theory, $D=10$ and $2k_{10}^2=(2\pi)^7\alpha'^4g_s^2=\frac{1}{2\pi}l_s^8g_s^2$. The parameter a_p controls the dilaton coupling, taking the values $a_0=-1$ for the NS-NS two-form and $a_p=\frac{3-p}{2}$ with $a_p+a_{p'}=0$ and $p+p'=6$ for the R-R sector. Dropping the dilaton and setting $D=11$, $p=2$ and $a_p=0$ give the action of eleven dimensional supergravity. The Planck lenght is defined by $2k_{11}^2=\frac{1}{2\pi}(2\pi l_P)^9$. The equations of motion derived from this action are

$$\begin{aligned}
&\Box\phi=\frac{a_p}{2(p+2)!}e^{a_p\phi}F^2,\\
&\nabla_M\left(e^{a_p\phi}F^{MM_1\ldots M_p+1}\right)=0,\\
&R_{MN}=\frac{1}{2}\partial_M\phi\partial_N\phi+\frac{e^{a_p\phi}}{2(p+1)!}\left(F_{MN}^2-\frac{p+1}{(D-2)(p+2)}g_{MN}F^2\right),
\end{aligned} \tag{A.2}$$

where $F_{MN}^2=F^{MM_1\ldots M_p+1}F_N{}^{M_1\ldots M_p+1}$. We observe that for $p=3$ the action (A.1) is not valid since $F_5^2=0$ for self-dual F_5. However, the correct equations are obtained if we multiply the F^2 term in (A.1) by a factor $\frac{1}{2}$ and impose the self-duality condition separately.

To begin, let us discuss the symmetries of the solutions we are interested in. We denote by

$n = p+1$ the dimension of the world-sheet of a p-brane and $\tilde{n} = \tilde{p}+1 = D-n-2 = D-p-3$ the dimension of the world-sheet of the dual object. Our solutions should be Poincaré invariant in the n directions along the brane and rotation invariant in the $D-n$ directions transverse to the brane, thus exhibiting a $SO(1,n-1)\times SO(D-n)$ symmetry. We can conveniently split the spacetime coordinates as $x^M = (x^\mu, y^m)$, with $\mu = 0,\dots,n-1$ and $m = 1,\dots,D-n$. Then the most general ansatz for the metric and dilaton of a p-brane sitting at $y=0$ that is compatible with the symmetries is

$$\begin{aligned} ds^2 &= e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{2B(r)}\delta_{mn}dy^m dy^n, \\ e^\phi &= e^\phi(r), \end{aligned} \tag{A.3}$$

where $r = (\delta_{mn}y^m y^n)^{\frac{1}{2}}$ is the radial distance from the brane. A reasonable boundary condition is that this metric is asympotically flat i.e. it reduces to Minkowski spacetime as $r\to\infty$, since we expect the deformation of spacetime due to the presence of the brane to vanish far from the brane. We also require that $\phi\to 0$ in this limit. As for the gauge field, we have the two possibilities

$$F^{\text{el}}_{p+2} = P_e * \left(e^{-a_p\phi}\,\text{vol}_{S^{D-p-2}}\right), \qquad F^{\text{mag}}_{p+2} = P_m\,\text{vol}_{S^{p+2}}, \tag{A.4}$$

where vol_{S^n} is the volume form of the unit n-sphere. They couple electrically with a p-brane and magnetically with a $\tilde{p}$-brane, respectively. Inserting the ansatz (A.3) and (A.4) into the equations of motion (A.2) gives a system of coupled non linear ordinary differential equations for $A(r)$, $B(r)$ and $\phi(r)$. However, imposing the condition

$$nA + \tilde{n}B = 0, \tag{A.5}$$

leads to considerable semplifications. This relation follows from the requirement that the solution preserves half of the supercharges, whence the brane is called a $\frac{1}{2}$-BPS state, and from the flat space boundary condition. Then the solution can be expressed in terms of a single harmonic function $H(r)$ as

$$\begin{aligned} &ds^2 = H^{-\frac{\tilde{n}}{D-2}}(r)dx^\mu dx^\nu\eta_{\mu\nu} + H^{\frac{n}{D-2}}(r)\left(dr^2 + r^2 d\Omega_{D-n-1}\right), \\ &e^\phi = H^{\frac{a_p}{2\zeta}}, \qquad \text{with} \qquad \zeta = \begin{cases} +1 & \text{electric brane} \\ -1 & \text{magnetic brane,} \end{cases} \\ &F^{\text{el}}_{p+2} = dH^{-1}(r)\wedge \text{vol}_{\mathrm{R}^{1,p}} \qquad \text{and} \qquad F^{\text{mag}}_{p+2} = N\alpha n\,\text{vol}_{S^{p+2}}, \quad N\in \mathrm{Z}. \end{aligned} \tag{A.6}$$

The sign in front of the field strenght is arbitrary and distinguishes brane from opposite-charged anti-branes. We observe that the solution for F^{el} is of the form $F = dA$ and could have been obtained with the ansatz $A^{\text{el}}_{\mu_0\dots\mu_p} = -\varepsilon_{\mu_0\dots\mu_p}\left(e^{C(r)}-1\right)$ leading to $e^C = H^{-1}$. As for $H(r)$, it is the Green function of the transverse Laplace operator, that is the solution of the Laplace equation in the space transverse to the brane with source given by a δ function localized at the origin,

$$\Box H(r) = -N\alpha\tilde{n}\Omega_{\tilde{n}+1}\delta^{(\tilde{n}+2)}(y), \tag{A.7}$$

where Ω_n is the volume of the n-sphere. The solution of this equation depends on the dimension of the transverse space as

$$H(r) = \begin{cases} -\alpha\,|\mathbf{y}| & \text{if } \tilde{n} = -1, \\ -\alpha \ln r & \text{if } \tilde{n} = 0, \\ 1 + \frac{N\alpha}{r^{\tilde{n}}} & \text{if } \tilde{n} > 0. \end{cases} \tag{A.8}$$

Here N is an integer corresponding to the number of coincident branes and α is the only integration constant which is left after imposing the boundary conditions $g_{MN} \to \eta_{MN}$ and $\phi \to 0$ when $r \to \infty$. It can be related to the tension τ_p of the brane via

$$\alpha = \frac{2k_D^2 \tau_p}{\tilde{n}\Omega_{\tilde{n}+1}}. \tag{A.9}$$

From (A.8) we observe that the solutions with $\tilde{n} = -1$ and $\tilde{n} = -1$ diverge as $r \to \infty$ and deserve special treatment.

The electric and magnetic charge densities of an electric p-brane and a magnetic $\tilde{p}$-brane are

$$Nq_e = \frac{(-1)^p}{\sqrt{2}k_D} \int_{S^{\tilde{n}+1}} e^{a_p\phi} * F^{\text{el}}_{p+2} = \sqrt{2}k_D \tau_p N, \tag{A.10a}$$

$$Nq_m = \frac{1}{\sqrt{2}k_D} \int_{S^{n+1}} e^{a_p\phi} F^{\text{mag}}_{p+2} = \sqrt{2}k_D \tau_p N, \tag{A.10b}$$

and using the Dirac quantization condition we find a relation between the tensions of an electric p-brane and its magnetic dual,

$$\tau_p \tau_{\tilde{p}} = \frac{\pi n}{k_D^2}, \qquad n \in \mathrm{Z}. \tag{A.11}$$

From the two equations in (A.10) we see that, in appropriate units, the brane tension is equal to the (electric or magnetic) charge. These branes are called *extremal* and saturate the BPS bound, as implied by the BPS condition (A.5). One important consequence of this equality is the zero-force condition. Usually, when we put two massive charged objects at a certain distance, they will attract or repulse themselves by a combination of gravitational and gauge interactions but, if the BPS bound is saturated, the gravitational attraction is exactly compensated by the gauge repulsion. Therefore extremal branes do not exert force on each other and can be separated or moved around in spacetime with no cost in energy. The zero-force condition allows us to generalize the harmonic function in (A.8) for $\tilde{n} > 0$, which corresponds to a stack of N coincident p-branes sitting at $y = 0$, to

$$H(r) = 1 + \sum_{i=1}^{N} \frac{\alpha}{|\mathbf{y} - \mathbf{y}_i|^{\tilde{n}}}, \tag{A.12}$$

which describes N branes sitting at the points $\mathbf{y} = \mathbf{y}_i$. We can verify the stability of this configuration by putting a probe brane into the background given by (A.12) at the transverse position $Y^m(\xi)$. In static gauge $X^\mu(\xi) = \xi^\mu$ the action of this probe brane is

$$-\tau_p \int_{W_{p+1}} d^{p+1}\xi \left(e^{-\frac{a_p\Phi}{2}} \sqrt{-\det\left(e^{2A}\eta_{\mu\nu} + e^{2B}\partial_\mu Y^m \partial_\nu Y^n \delta_{mn}\right)} - H^{-1} \right), \tag{A.13}$$

yielding the potential

$$V = \tau_p \left(e^{-\frac{a_p\Phi}{2} + (p+1)A} \right) = 0, \tag{A.14}$$

which is zero from (A.6).

For our reference in section 4.5, we now specialize (A.6) to the two cases of interest for us, which are D3-branes in $D = 10$ type IIB and M2-branes in $D = 11$ supergravity. In order to reproduce the correct results, we have to introduce the concept of smearing [57]. This is a common technique in brane engineering, which allows one to construct new brane solutions from old. Using a delta function source gives a solution which preserves some set of translational symmetries (in the directions parallel to the brane) and breaks another set (in the directions transverse to the brane). However, a solution can be obtained that preserves more translational symmetries by using a more symmetric source, e.g. one supported on a line, on a plane or on a higher dimensional surface. Thus, smeared branes can be imagined as an array of branes in which a large number of unsmeared basic branes are placed in the spacetime with a small spacing between themselves. The metric in (A.6) distinguishes between *wrapped* and *localized* directions spanning the parallel and transverse space to the brane, respectively. We can also introduce some *smeared* directions in the transverse space, that is we take a p-brane and smear it uniformly along some directions in the transverse space. Then the metric is still given by (A.6), but the harmonic function $H(r)$ is different, since instead of $\tilde{n}$ in (A.8) we take $\tilde{n}$ minus the number of smeared directions. In section 4.5 and Chapter 5, we considered bubble geometries for

- $AdS_5 \times S^5/\mathrm{Z}_k$, with D3-branes wrapping the S^4 in AdS_5 and smeared over the $\mathrm{C}P_2$. In this case, due to the smearing there are effectively only two transverse directions, given by the radial direction r in AdS_5 and the internal Kaluza-Klein direction corresponding to the collapsing S^1. Then the relevant harmonic function is the Green function of the Laplace equation in two dimensions, whence the dependence on r is logarithmic. As discussed in [11], in the near-brane regime (here labelled as "D3") we can treat the source as being in a locally flat spacetime and we expect the various functions appearing in the metric (4.138) to take the form

$$\begin{aligned} \rho_{D3}(r) &\sim H^{\frac{1}{2}}(r), & f_{D3}(r) &\sim H^{-\frac{1}{2}}(r), \\ g_{D3}(r) &\sim H^{\frac{1}{2}}(r), & h_{D3}(r) &\sim \tilde{r}^2 H^{\frac{1}{2}}(r), \end{aligned} \tag{A.15}$$

  with $H(r)$ from (A.8) given by

$$H(r) = -\ln r. \tag{A.16}$$

- $AdS_4 \times M_{3S}$, with M2-branes wrapping the S^3 in AdS_4 and smeared over the quaternionic Kähler subspace of the tri-Sasakian internal manifold. Due to the smearing, we have only four transverse directions, corresponding to the radial distance r in AdS_4 and the three dimensional collapsing subspace in the internal manifold. Then the relevant harmonic function is the Green function of the Laplace equation in four dimensions, that is

$$H(r) = 1 + \frac{\alpha}{r^2}, \tag{A.17}$$

  where we have included the number of branes in the definition of α. Then it is reasonable to assume for the various functions appearing in the metric (5.53) the following near-brane behaviour (whence the labelling "M2"),

$$\begin{aligned} \rho_{M2}(r) &\sim H^{\frac{1}{3}}(r), & f_{M2}(r) &\sim H^{-\frac{2}{3}}(r), \\ g_{M2}(r) &\sim H^{\frac{1}{3}}(r), & h_{M2}(r) &\sim \tilde{r}^2 H^{\frac{1}{3}}(r), \end{aligned} \tag{A.18}$$

Appendix B

Anti-de Sitter from near-horizon limits

In this Appendix, we explain how Anti-de Sitter space naturally emerges in the near-horizon limit of certain configuration of branes, which is relevant for our discussion in Chapter 2. This property is familiar from the Reissner-Nordström metric that describes the spacetime outside an extremal charged black hole, whose near-horizon region looks like $AdS_2 \times S^2$. The observation that an analogous result holds for brane metrics played a role in Maldacena's original argument for the statement of the AdS/CFT correspondence in [58] and clarify our discussion in Chapter 2.

The Reissner-Nordström black hole To begin, we briefly describe how Anti-de Sitter space appears in the near-horizon limit of the Reissner-Nordström metric for an extremal black hole. This is a standard result which can be found for example in [59]. Setting the Newton's constant $G = 1$, the Reissner-Nordström metric in four dimensions is given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2, \qquad f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \tag{B.1}$$

with M and Q the mass and electric charge of the black hole, respectively. If $M < Q$, the metric coefficient $f(r)$ is always positive and the true curvature singularity at $r = 0$ is a naked singularity, thus violating the cosmic censorship conjecture. If instead $M > Q$, $f(r)$ is positive at large r and small r and negative in the intermediate region inside the two zeros at $r = r_\pm = M + \sqrt{M^2 - Q^2}$, which describe event horizons. The near-horizon geometry can be shown to be approximately Rindler $\times\, S^2$. Finally, in the extremal case $M = Q$ the inner and outer horizons collapse into a single horizon located at $r = M$. In order to study the near-horizon geometry, we define new coordinates

$$r = Q\left(1 + \frac{\lambda}{z}\right), \qquad t = \frac{QT}{\lambda}, \tag{B.2}$$

where λ is an arbitrary parameter. Plugging into the metric (B.1) and taking the limit $\lambda \to 0$ with all the coordinates held fixed, which describes the near-horizon region since $r \to Q$ in this limit, yield

$$ds^2 = \frac{Q^2}{z^2}\left(dz^2 - dT^2\right) + Q^2 d\Omega_2^2. \tag{B.3}$$

This metric is nothing but $AdS_2 \times S^2$ in the Poincaré coordinates (4.28). It is worth observing that in the subextremal case the approximate Rindler $\times\, S^2$ near-horizon geometry does not solve by itself the Einstein-Maxwell equations, while the $AdS_2 \times S^2$ geometry actually does.

The AdS/CFT correspondence We now argue that a similar result holds for brane metrics and explain why this is important in the context of the AdS/CFT correspondence, where string theory and M-theory on certain supergravity geometries that include AdS factors are conjectured to be equivalent to certain conformally invariant quantum field theories. Even though we won't be able to enter into technical details, since it would go beyond the scope of this thesis, it is useful to present here at least the basic ideas, since they allow us to somewhat broader the picture outlined at the end of Chapter 2. In particular, we present Maldacena's original $AdS_5 \times S^5$ example and also discuss some explicit realizations in M-theory.

Following [36], we now provide some generalities about the correspondence. To begin, we observe that there are two apparently very different ways of realizing theories with $O(2, D)$ symmetry, namely comformal field theories in D dimensions and gravitational theories in AdS_{D+1}. Since they live in spacetimes with different dimensions, it might sound weird that they can be related. However, the holographic idea that the number of degrees of freedom of a region of spacetime grows with the area, rather than the volume, is not a novelty when dealing with gravity. For example, it is well known from the Bekenstein-Hawking formula that the entropy of a black hole is proportional to the area of the horizon.

In order to set a correspondence between the two theories, we need the following ingredients:

- on the gravity side, we have *bulk* fields in $D+1$ dimensions whose interactions are described by an effective action (for example, that of a supergravity theory) with AdS_{D+1} vacuum, S_{AdS};
- on the gauge side, we have *boundary* fields in D dimensions with Lagrangian L_{CFT}, the CFT spectrum being specified by a complete set of primary operators.

The idea is to identify the off shell background fields $h(x)$ introduced as sources to compute the correlation functions of the CFT operators with the boundary values of on shell bulk fields $\hat{h}(x, x_{D+1})$, as schematically shown in Figure B.1. More explicitly, let O be a CFT operator.

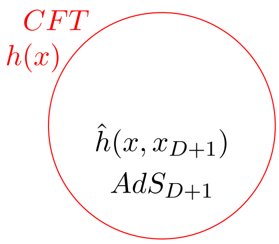

Figure B.1: Formulation of the correspondence in the Euclidean picture of AdS.

In order to compute correlation functions, we consctruct the functional generator of connected correlation functions of O as

$$e^{W(h)} = \langle e^{\int hO} \rangle_{CFT}, \tag{B.4}$$

where $h(x)$ is a background field. Then,

$$\langle O \dots O \rangle = \frac{\delta^n W}{\delta h^n}. \tag{B.5}$$

To each source configuration we can associate a unique bulk field configuration

$$h(z) \to \hat{h}(x, x_{D+1}), \tag{B.6}$$

by demanding that $\hat{h}$ solves the $D+1$ dimensional equations of motion and obeys appropriate boundary conditions. Then, the prescription for the correspondence is

$$e^{W(h)} = e^{S_{AdS}(\hat{h})}, \tag{B.7}$$

that is we can obtain the CFT functional generator $W(h)$, depending on the arbitrary (off shell) configuration $h(x)$, by evaluating (on shell) the $D+1$ dimensional action on the solution of the equations of motion $\hat{h}(x, x_{D+1})$ that reduces to $h(x)$ at the boundary. Since the knowledge of $W(h)$ completely determines the CFT, equation (B.7) states the required equivalence between the CFT and the gravitational theory. However, we have not explained how to determine the field h that couples to the operator O. This can often been done using symmetries, since they must have the same $O(2, D)$ quantum numbers. In particular, global symmetries in the CFT correspond to gauge symmetries in AdS.

Some explicit realizations The first, and still best understood, realization of the AdS/CFT correspondence is Maldacena's original conjecture in [58] that the $\mathcal{N} = 4$ super Yang-Mills (SYM) theory with gauge group $U(N)$ is dual to the type IIB string background $AdS_5 \times S_5$. The statement was motivated by the observation that the two theories can be obtained by the same decoupling limit $\alpha' \to 0$ performed on the world-volume theory and on the back-reacted metric in spacetime, respectively. On the gauge side, $\mathcal{N} = 4$ SYM can be realized on the world-volume of a stack of N parallel D3-branes, which interact with the bulk fields that live in $D = 10$ Type IIB string theory. However, in the limit $\alpha' \to 0$, these interactions can be turned off and the brane theory decouples from the bulk, thus reducing to $\mathcal{N} = 4$ SYM. Moreover, in order to keep all physical quantities finite, the appropriate limit to perform is

$$\begin{aligned} \alpha' &\to 0, \\ g_s = \frac{g_{YM}^2}{4\pi} &\quad \text{fixed}, \\ N &\quad \text{fixed}, \\ \phi^i = \frac{r_i}{\alpha'} &\quad \text{fixed}. \end{aligned} \tag{B.8}$$

That is, we are zooming on the region where the branes sit.

On the gravity side, as we have discussed in Appendix A, a D3-brane can be seen as a solution of the equations of motion of type IIB supergravity. From (A.6), the back-reacted brane geometry is

$$ds^2 = H^{-\frac{1}{2}}(r)\eta_{\mu\nu}dx^\mu dx^\nu + H^{\frac{1}{2}}(r)\left(dr^2 + r^2 d\Omega_5\right), \tag{B.9}$$

with the harmonic function $H(r)$ given by

$$H(r) = 1 + \frac{g_{YM}^2 N \alpha'^2}{r^4} = 1 + \frac{g_{YM}^2 N}{\alpha'^2 \phi^4}. \tag{B.10}$$

Taking the limit (B.8), we obtain

$$ds^2 = \alpha' \left[R^2 \frac{d\phi^2}{\phi^2} + \frac{\phi^2}{R^2}(dx)^2 + R^2 \right] \tag{B.11}$$

Perturbative quantum field theory is a good description only when the AdS radius L is small, while supergravity is a good approximation only when L is large, so the two different regimes of validity don't overlap.

In the case of a flat M2-brane, an analogous near-horizon limit produces $AdS_4 \times S^7$. It is conjectured that M-theory on $AdS_4 \times S^7$ with N units of flux on the S^7 is dual to $\mathcal{N} = 8$ SYM with gauge group $SU(N)$ in three dimensions. It is easy to verify that the symmetries on the two sides match: the $SO(2,3) \times SO(8)$ isometries of $AdS_4 \times S^7$ correspond to the conformal invariance and the R-symmetry of the dual theory. An interesting generalization is to replace the flat space transverse to the brane with a cone over a base X,

$$dr^2 + r^2 ds^2(X). \tag{B.12}$$

Apart from the special case when X is the round sphere, these spaces have a conical singularity at $r = 0$. Then this construction can be interpreted as N coincident M2-branes sitting at the conical singularity. By taking a near-horizon limit, one finds that it is now $AdS_4 \times X$ and this provides a rich class of new AdS/CFT examples.

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