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Gravitational instability via the Schrödinger equation

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Chapter 1

Introduction

The analysis of cosmological structure formation in the early Universe is currently one of the most interesting activities in modern cosmology.
The local Universe shows a rich hierarchical pattern of galaxy clustering, that involves a considerable range of length scales, from galaxy clusters to super-clusters and filaments, while the early Universe was almost homogeneous with the exception of slight fluctuations seen in the cosmic microwave background radiation.
The present inhomogeneous Universe and the much smoother past Universe are connected by the effect of *gravitational instability*.
The pioneer of this model is Sir James Jeans, whose theory demonstrated that, starting from a homogeneous and isotropic fluid, small fluctuations in the density, $\delta\rho$, and in the velocity, δv, could evolve with time [1]. These density fluctuations growth is an effect of the self-gravitation of density and it leads to an instability which can cause eventually the collapse of the fluctuation to a gravitationally bound object. This is the so-called *gravitational instability*.

One can predict the density fluctuations behaviour by comparing the fluctuation scale length λ with the Jeans length, λ_J. If we suppose to have a spherical inhomogeneity of radius λ containing a small positive density fluctuation $\delta\rho > 0$ of mass M in a background fluid of mean density ρ, the density fluctuation will grow if the stabilizing effect of the force per unit mass arising from the pressure is smaller than the self-gravitational force per unit mass[1]
At the time Jeans worked on this theory, the expansion of the Universe was not

[1]More precisely:

$$F_p \simeq \frac{P\lambda^2}{\rho\lambda^3} \simeq \frac{c_s^2}{\lambda} < F_g \simeq \frac{GM}{\lambda^2} \simeq \frac{G\rho\lambda^3}{\lambda^2} \quad \Rightarrow \quad \lambda > \frac{c_s}{\sqrt{G\rho}} \quad ,$$

with c_s the sound speed and $\lambda_J = \frac{c_s}{\sqrt{G\rho}}$ the Jeans length. If $\lambda < \lambda_J$, the the fluctuation will oscillate like an acoustic wave.

known so he did his calculations assuming a static background fluid. This led to a wrong collapse velocity value but gave an important contribute to the understanding of how structure formed and grew in the early Universe.

Historically, the difficulties found trying to explain the structure formation in a scenario with only baryonic matter opened the way for theories built around the hypothesis that the Universe is dominated by non-baryonic and weakly interacting *dark matter*.
Observations suggest that there is approximately five times more non-baryonic dark matter than baryonic matter, therefore the study of large-scale structures evolution usually involves the non-baryonic dark matter.
In the 1980s there were two different fashionable models:

1. the Hot Dark Matter Scenario, characterized by the assumption that the Universe is dominated by collisionless particles with a very large velocity dispersion. In this theory, the structure formation is not hierarchical but, starting from super-clusters, we have galaxy clusters by fragmentation;

2. the Cold Dark Matter Scenario, also characterized by the assumption that the Universe is dominated by collisionless particles, this time with a very small velocity dispersion. This theory predicts hierarchical structure formation and it is in general agreement with astronomical observations, moreover it is favored by most cosmologist.

The second one will be the scenario we will work in.

Our work will proceed in the following order: in Chapter 2, we will introduce some fundamental concepts that will prove to be useful in the following dissertation.
In Chapter 3, we will enlight the standard basics of gravitational instability in an expanding Universe, from the perturbation theory to the Zel'dovich and adhesion approximation, emphasizing the most remarkable features and issues.
The aim of Chapter 4 is to examine an alternative approach to the study of large-scale structure formation based on the description of Cold Dark Matter as a complex-scalar field.

Chapter 2

The background cosmology

2.1 Friedmann-Robertson-Walker metric

A concept that is worth mentioning for his significance in Cosmology is the *Cosmological Principle*, which states that on large scales the Universe appears homogeneous and isotropic[1], which means that it does not have any priviled position or direction.

A metric which describes a spacetime consistent with the Cosmological Principle is the *Friedmann-Robertson-Walker metric*:

$$ds^2 = c^2dt^2 - a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2sin^2\theta d\phi^2\right) \quad ,$$

where k is the spatial curvature, scaled so as to take the values 0, representing the flat space, or ± 1, which means positive or negative curvature.
The time coordinate is the cosmological proper time, namely the time of a comoving observer.

2.2 Friedmann equations: the Einstein-de Sitter model

The dynamics of a Friedmann-Robertson-Walker universe are determined by the Friedmann equations:

$$3\left(\frac{\dot{a}}{a}\right) = 8\pi G\rho - \frac{3kc^2}{a^2} + \Lambda c^2 \quad , \tag{2.1}$$

[1]As a matter of fact, homogeneity and isotropy are two valid concepts if one observes the Universe on a distance of hundreds of Mpc.

$$\frac{\dot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3\frac{P}{c^2}\right) + 3\frac{\Lambda c^2}{3} \quad , \tag{2.2}$$

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) \quad . \tag{2.3}$$

We are working in the *Einstein-de Sitter model*, which is characterized by both negligible spatial curvature k and cosmological constant Λ.
In order to find a solution to the set of equations, we must specify the kind of fluid we are dealing with. A perfect fluid is characterized only by energy density and isotropic pressure. We assume that the pressure depends only on the density:

$$P = P(\rho) \quad .$$

Therefore, we are treating a barotropic perfect fluid. For simplicity, we set a linear dependence between P and ρ:

$$P = w\rho c^2 \quad ,$$

where w is a dimensionless constant. If we replace it in (2.3), we have:

$$\dot{\rho} = -3(1+w)\frac{\dot{a}}{a}\rho \quad \Rightarrow \quad \frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \quad \Rightarrow \quad \rho \propto a^{-3(1+w)} \quad .$$

For a matter-dominated universe, $w = 0$ namely $P = 0$, so we are dealing with a pressureless (collisionless) fluid, called *dust*. This leads to:

$$\rho \propto a^{-3} \quad ,$$

which is a relation that links the density to the scale factor.

Chapter 3

Cosmological structure formation

In this chapter, we will discuss the gravitational instability in an expanding Universe by means of many approaches, where CDM is described as a fluid.
In the fluid approach we will investigate the behaviour of CDM perturbation in the linear regime, with small fluctuations $\delta \ll 1$. Instead, a Lagrangian approach proves to be more suitable in the quasi-linear regime for collisionless fluids. In particular, we will look for solutions in the Zel'dovich and the adhesion approximations [2].

3.1 The fluid approach

In order to write the dynamical equations that govern the behaviour of the Cold Dark Matter (CDM), described as a fluid, we introduce the coordinates:

$$\vec{r} = a(t)\vec{x} \quad ,$$

which have the property of being inertial with reference to the background Friedmann-Robertson-Walker evolution, with $a(t)$ the *scale factor* of the Universe, a measure of the universal expansion rate, and $\vec{x}$ the comoving coordinates.
Thus, we have the relations:

$$\nabla_{\vec{r}} = \frac{1}{a}\nabla_{\vec{x}} \quad , \tag{3.1}$$

$$\vec{w} = \dot{\vec{r}} = \frac{\dot{a}}{a}\vec{r} + a\frac{d\vec{x}}{dt} = H\vec{r} + \vec{v} \quad , \tag{3.2}$$

where H is the *Hubble constant* and $\vec{v}$ is the *peculiar velocity*, expression of the departure of matter motion from the Hubble flow[1].

[1] The motion of astronomical objects due to the expansion of the Universe.

For a generic function $f(\vec{r},t)$, the convective derivative can be written as:

$$\frac{Df(\vec{r},t)}{Dt} = \left.\frac{\partial f}{\partial t}\right|_{\vec{r}} + (\vec{w}\cdot\nabla_{\vec{r}})f = \left.\frac{\partial f}{\partial t}\right|_{\vec{r}} + H(\vec{r}\cdot\nabla_{\vec{r}})f + (\vec{v}\cdot\nabla_{\vec{r}})f \quad ,$$

where we used (3.1) and (3.2).
If we think f as a function of $(\vec{x},t)$, following the same steps as before, we have:

$$\frac{Df(\vec{x},t)}{Dt} = \left.\frac{\partial f}{\partial t}\right|_{\vec{x}} + (\dot{\vec{x}}\cdot\nabla_{\vec{x}})f = \left.\frac{\partial f}{\partial t}\right|_{\vec{x}} + \frac{1}{a}(\vec{v}\cdot\nabla_{\vec{x}})f \quad .$$

The two convective derivatives have to be equal, therefore:

$$\left.\frac{\partial f}{\partial t}\right|_{\vec{x}} = \left.\frac{\partial f}{\partial t}\right|_{\vec{r}} + H(\vec{r}\cdot\nabla_{\vec{r}})f \quad . \tag{3.3}$$

Keeping in mind these relations, we consider adiabatic perturbation, which means that the entropy is constant, so the dynamics of CDM is described by three equations: the *continuity equation*

$$\left.\frac{\partial \rho}{\partial t}\right|_{\vec{r}} + \nabla_{\vec{r}}(\rho\vec{w}) = 0 \quad , \tag{3.4}$$

the *Euler equation*

$$\left.\frac{\partial \vec{w}}{\partial t}\right|_{\vec{r}} + (\vec{w}\cdot\nabla_{\vec{r}})\vec{w} = -\frac{1}{\rho}\nabla_{\vec{r}}p - \nabla_{\vec{r}}\Phi \quad , \tag{3.5}$$

and the *Poisson equation*

$$\nabla^2_{\vec{r}}\Phi = 4\pi G\rho \quad . \tag{3.6}$$

3.1.1 First-order Eulerian perturbations

We are looking for perturbative solutions, so we will linearize the equations (3.4), (3.5) and (3.6), perturbing physical quantities relative to an unperturbed coordinate system, i.e. the comoving coordinates.
Hence, we introduce the perturbation to the the CDM density field in the homogeneous FRW background, $\delta\rho$, and the *peculiar gravitational potential* $\phi(\vec{x},t)$, expression of the fluctuations in potential with respect to the homogeneous background:

$$\rho = \rho_b + \delta\rho \quad , \quad \Phi = \Phi_b + \phi \quad .$$

We fix $\delta\rho = \rho_b(t)\delta$, with $\delta = \delta(\vec{x},t)$ called the density contrast.
It is appropriate to emphasize the fact that $\delta\rho$ can be negative, in fact in the standard case where there is a Gaussian distribution of initial fluctuations, when the variance of the δ is of order unity then a Gaussian distribution assigns a non-zero probability to regions with $\delta < 1$, i.e. with $\rho < 0$.

Comoving coordinates. First of all, we rewrite these equations in the comoving coordinates. Take the continuity equation, using (3.3) we have:

$$\left.\frac{\partial\rho}{\partial t}\right|_{\vec{x}} - H(\vec{r}\cdot\nabla)\rho + \nabla_{\vec{r}}(\rho\vec{w}) = 0 \quad .$$

If we insert the vectorial identity:

$$\nabla_{\vec{r}}(\rho\vec{w}) = (\nabla_{\vec{r}}\cdot\vec{w})\rho + \nabla_{\vec{r}}\rho\cdot\vec{w} \quad ,$$

the equation can be written as:

$$\left.\frac{\partial\rho}{\partial t}\right|_{\vec{x}} - H(\vec{r}\cdot\nabla)\rho + H(\nabla_{\vec{r}}\cdot\vec{r})\rho + (\nabla_{\vec{r}}\cdot\vec{v})\rho + H(\vec{r}\cdot\nabla_{\vec{r}})\rho + (\vec{v}\cdot\nabla_{\vec{r}})\rho = 0 \quad .$$

We now simplify using:

$$H(\nabla_{\vec{r}}\cdot\vec{r})\rho = 3H\rho \quad ,$$

and

$$(\vec{v}\cdot\nabla_{\vec{r}})\rho + (\nabla_{\vec{r}}\cdot\vec{v})\rho = \nabla_{\vec{r}}(\rho\vec{v}) \quad ,$$

then we obtain the final equation:

$$\left.\frac{\partial\rho}{\partial t}\right|_{\vec{x}} + 3H\rho + \frac{1}{a}\nabla_{\vec{x}}(\rho\vec{v}) = 0 \quad . \tag{3.7}$$

As regards the Euler equation, using (3.2), we have:

$$\left.\frac{\partial(H\vec{r})}{\partial t}\right|_{\vec{r}} + \left.\frac{\partial\vec{v}}{\partial t}\right|_{\vec{r}} + H(\vec{r}\cdot\nabla_{\vec{r}})H\vec{r} + H(\vec{r}\cdot\nabla_{\vec{r}})\vec{v} + (\vec{v}\cdot\nabla_{\vec{r}})H\vec{r} + (\vec{v}\cdot\nabla_{\vec{r}})\vec{v} = -\frac{1}{\rho}\nabla_{\vec{r}}p - \nabla_{\vec{r}}\Phi_b - \nabla_{\vec{r}}\phi \quad .$$

The background terms are those without the peculiar velocity $\vec{v}$. They represent the zero order of our expansion, so they are equal to zero:

$$\left.\frac{\partial(H\vec{r})}{\partial t}\right|_{\vec{r}} + H(\vec{r}\cdot\nabla_{\vec{r}})H\vec{r} + \nabla_{\vec{r}}\Phi_b = 0 \quad .$$

With some calculations, we obtain:

$$(\dot{H} + H^2)\vec{r} = -\frac{4\pi G}{3}\rho_b\vec{r} \quad ,$$

which is satisfied by Friedmann equations (2.1) and (2.3), leading to[2]:

$$\dot{H} = -4\pi G\rho \quad .$$

[2]We are treating collisionless fluid, so the pressure term is negligible.

But we are interested in the terms that concern the perturbation, which gives:

$$\left.\frac{\partial \vec{v}}{\partial t}\right|_{\vec{r}} + H(\vec{r}\cdot\nabla_{\vec{r}})\vec{v} + (\vec{v}\cdot\nabla_{\vec{r}})(H\vec{r}) + (\vec{v}\cdot\nabla_{\vec{r}})\vec{v} = -\frac{1}{\rho}\nabla_{\vec{r}}p - \nabla_{\vec{r}}\phi$$

Keeping in mind that $\nabla_{\vec{r}}\vec{r} = \mathbb{1}$ and $\frac{\partial \vec{v}}{\partial t}|_{\vec{r}} + H(\vec{r}\cdot\nabla_{\vec{r}})\vec{v} = \frac{\partial \vec{v}}{\partial t}|_{\vec{x}}$, then:

$$\left.\frac{\partial \vec{v}}{\partial t}\right|_{\vec{x}} + H\vec{v} + \frac{1}{a}(\vec{v}\cdot\nabla_{\vec{x}})\vec{v} = -\frac{1}{a\rho}\nabla_{\vec{x}}p - \frac{1}{a}\nabla_{\vec{x}}\phi \quad . \tag{3.8}$$

Lastly, we have the Poisson equation:

$$\nabla^2_{\vec{x}}\phi = 4\pi G a^2 \rho \quad . \tag{3.9}$$

Linearization. As a first step, we replace $\rho = \rho_b(1+\delta)$ in (3.7):

$$(1+\delta)\left(\frac{\partial \rho_b}{\partial t} + 3H\rho_b\right) + \rho_b\frac{\partial \delta}{\partial t} + \frac{1}{a}\vec{\nabla}_{\vec{x}}\cdot[\rho_b(1+\delta)\vec{v}] = 0 \quad .$$

The term between brackets, $\frac{\partial \rho_b}{\partial t} + 3H\rho_b$, is equal to zero because of the Friedmann equation without perturbation, (2.3).
Neglecting the non-linear terms, we obtain the *linearized continuity equation*:

$$\dot{\delta} + \frac{1}{a}\vec{\nabla}_{\vec{x}}\cdot\vec{v} = 0 \quad . \tag{3.10}$$

As regards the Euler equation,(3.8), we can just neglect the quadratic term $(\vec{v}\cdot\vec{\nabla}_{\vec{x}})\vec{v}$, being $\vec{v}$ a perturbation, so we have:

$$\frac{\partial \vec{v}}{\partial t} + H\vec{v} = -\frac{1}{a\rho}\nabla_{\vec{x}}p - \frac{1}{a}\nabla_{\vec{x}}\phi \quad . \tag{3.11}$$

If we linearize the cosmological Poisson equation (3.9), we obtain:

$$\nabla^2_{\vec{x}}\phi = 4\pi G a^2 \delta\rho \quad . \tag{3.12}$$

To further simplify the study of the solutions, we Fourier expand (3.10), (3.11) and (3.12) replacing:

$$\delta(\vec{x},t) = \frac{1}{(2\pi)^3}\int e^{i\vec{k}\cdot\vec{x}}\delta(\vec{k},t)\mathrm{d}^3k \quad ,$$
$$\vec{v}(\vec{x},t) = \frac{1}{(2\pi)^3}\int e^{i\vec{k}\cdot\vec{x}}\vec{v}(\vec{k},t)\mathrm{d}^3k \quad ,$$
$$\phi(\vec{x},t) = \frac{1}{(2\pi)^3}\int e^{i\vec{k}\cdot\vec{x}}\phi(\vec{k},t)\mathrm{d}^3k \quad .$$

Then, (3.10) becomes:

$$\frac{1}{(2\pi)^3}\int e^{i\vec{k}\cdot\vec{x}}\dot{\delta}(\vec{k},t) + \frac{1}{a}(\nabla_{\vec{x}} e^{i\vec{k}\cdot\vec{x}})\vec{v}(\vec{k},t)\mathrm{d}^3k = 0$$

$$\frac{1}{(2\pi)^3}\int e^{i\vec{k}\cdot\vec{x}}\left[\dot{\delta}(\vec{k},t) + \frac{i}{a}\vec{k}\cdot\vec{v}(\vec{k},t)\right]\mathrm{d}^3k = 0$$

$$\Rightarrow \quad \dot{\delta}_{\vec{k}} + \frac{i}{a}\vec{k}\cdot\vec{v}_{\vec{k}} = 0 \quad . \tag{3.13}$$

Let us now replace $\vec{v}$ and ρ in the Euler equation (3.11):

$$\frac{1}{(2\pi)^3}\int e^{i\vec{k}\cdot\vec{x}}(\dot{\vec{v}}(\vec{k},t)+H\vec{v}(\vec{k},t))\mathrm{d}^3k = -\frac{\rho_b}{a\rho_b}\left(\frac{\partial p}{\partial \rho}\right)\nabla_{\vec{x}}\delta + \frac{1}{a(2\pi)^3}\int \nabla_{\vec{x}}(e^{i\vec{k}\cdot\vec{x}})\phi(\vec{k},t)\mathrm{d}^3k$$

$$\frac{1}{(2\pi)^3}\int e^{i\vec{k}\cdot\vec{x}}(\dot{\vec{v}}(\vec{k},t)+H\vec{v}(\vec{k},t))\mathrm{d}^3k = -\frac{1}{a}\left(\frac{\partial p}{\partial \rho}\right)\frac{1}{(2\pi)^3}\int i\vec{k}e^{i\vec{k}\cdot\vec{x}}\left[\left(\frac{\partial p}{\partial \rho}\right)\delta(\vec{k},t)+\phi(\vec{k},t)\right]\mathrm{d}^3k$$

$$\Rightarrow \quad \dot{\vec{v}}_{\vec{k}} + H\vec{v}_{\vec{k}} = -\frac{i\vec{k}}{a}(c_s^2\delta_{\vec{k}} + \phi_{\vec{k}}) \quad , \tag{3.14}$$

where $c_s^2 = \frac{\partial p}{\partial \rho}$.
Finally, the cosmological Poisson equation (3.12) can be written as:

$$\frac{1}{(2\pi)^3}\int (-k^2)e^{i\vec{k}\cdot\vec{x}}\phi(\vec{k},t)\mathrm{d}^3k = -\frac{4\pi G\rho_b a^2}{(2\pi)^3}\int e^{i\vec{k}\cdot\vec{x}}\delta(\vec{k},t)\mathrm{d}^3k$$

$$\Rightarrow \quad k^2\phi_{\vec{k}} = -4\pi G a^2 \rho_b \delta_{\vec{k}} \quad . \tag{3.15}$$

We now take advantage of the:

Kelvin circulation theorem. *In a barotropic perfect fluid subject to conservative body forces, the circulation around a closed curve (which encloses the same fluid elements) moving with the fluid remains constant with time:*

$$\frac{d\Gamma}{dt} = \frac{d}{dt}\oint_l \vec{\omega}\cdot\mathrm{d}\vec{l} = 0 \quad ,$$

with $\vec{\omega} = \nabla\times\vec{v}$ the vorticity.

In our case, $\vec{\omega} = \nabla\times\vec{w}$, so:

$$\frac{d}{dt}\oint_l \vec{\omega}\cdot\mathrm{d}\vec{l} = \frac{d}{dt}\oint_l \nabla\times\vec{w}\cdot\mathrm{d}\vec{l} = 0$$

$$\Rightarrow \quad \frac{d}{dt}[\nabla\times(H\vec{r}+\vec{v})] = \nabla\times\left(\dot{H}\vec{r} + H\frac{d}{dt}\vec{r} + \frac{d}{dt}\vec{v}\right) = 0$$

$$\nabla \times \left(\dot{H}\vec{r} + H^2\vec{r} + H\vec{v} + \frac{d}{dt}\vec{v} \right) = 0 \quad \Rightarrow \quad \nabla \times (H\vec{v} + \dot{\vec{v}}) = 0 \quad ,$$

which means that $H\vec{v} + \dot{\vec{v}}$ is irrotational, so it can be described by the gradient of a scalar field:

$$H\vec{v} + \dot{\vec{v}} = \nabla\xi \quad .$$

If we go to the Fourier space we find that $H\vec{v}_{\vec{k}} + \dot{\vec{v}}_{\vec{k}}$ is parallel to $\vec{k}$. Therefore, the component of $\vec{v}(\vec{k}, t)$ orthogonal to $\vec{k}$ obeys the equation:

$$\dot{v}_\perp + Hv_\perp = 0 \quad \Rightarrow \quad v_\perp \propto \frac{1}{a} \quad .$$

As a consequence of the theorem, we can consider only the component of the equations relative to the peculiar velocity parallel to $\vec{k}$, $v_{\vec{k}}$.
By differentiation, (3.13) becomes:

$$\ddot{\delta}_{\vec{k}} + i\frac{k}{a}\dot{v}_{\vec{k}} - i\frac{k}{a}Hv_{\vec{k}} = 0 \quad .$$

If we replace $\dot{v}_{\vec{k}}$ with the expression found by (3.14), and use (3.13) and (3.15), we obtain:

$$\ddot{\delta}_{\vec{k}} + 2H\dot{\delta}_{\vec{k}} + \left(\frac{c_s^2k^2}{a^2} - 4\pi G\rho_b \right)\delta_{\vec{k}} = 0 \quad .$$

We can define a comoving *Jeans wavenumber*:

$$k_J = a\frac{\sqrt{4\pi G\rho_b}}{c_s} \quad .$$

Knowing that for $k \ll k_J$ the fluctuation will grow with time, we have the approximate equation:

$$\ddot{\delta}_{\vec{k}} + 2H\dot{\delta}_{\vec{k}} - 4\pi G\rho_b\delta_{\vec{k}} \simeq 0 \quad . \tag{3.16}$$

For a spatially flat universe dominated by pressureless matter, i.e. the Einstein-de Sitter universe, one has:

$$a \propto t^{\frac{2}{3}}, \quad H = \frac{2}{3t}, \quad \rho_b = \frac{1}{6\pi Gt^2} \quad .$$

If we replace in (3.16) and look for a solution $\delta \propto t^\alpha$:

$$\ddot{\delta}_{\vec{k}} + \frac{4}{3t}\dot{\delta}_{\vec{k}} - \frac{2}{3t^2}\delta_{\vec{k}} = 3\alpha^2 + \alpha - 2 = 0$$

$$\Rightarrow \quad \alpha = \frac{2}{3}, -1 \quad .$$

Then, we find two solution:

1. *growing mode*, $\delta_{\vec{k}} \propto t^{\frac{2}{3}} \propto a$;
2. *decaying mode*, $\delta_{\vec{k}} \propto t^{-1}$.

In particular, we are interested in the growing mode, so we have solutions:

$$\begin{cases} \delta_{\vec{k}} \propto t^{\frac{2}{3}} \propto a \\ v \propto t^{\frac{1}{3}} \\ \phi = const \end{cases} \tag{3.17}$$

Therefore, a density fluctuation which arises at early times, namely for $\delta \ll 1$, will increase with time.

3.2 The Zel'dovich approximation

The evolution of density fluctuations in the so-called 'weakly non-linear regime' can be described using a Lagrangian approach, the Zel'dovich approximation. The Lagrangian approach is an alternative to the Eulerian approach to discuss the dynamic of a collisionless fluid, focusing on the motion of a single 'particle', i.e. an infinitesimal fluid element, instead of focusing on the fields that describe the system properties (such as velocity, density, etc.).
First of all, we rewrite the continuity, Euler and Poisson equations replacing the time variable with $a(t) \propto t^{\frac{2}{3}}$ and let us change variables as follows:

$$a(t) = a_\star \left(\frac{t}{t_\star}\right)^{\frac{2}{3}}, \quad \eta = \frac{\rho}{\rho_b} = 1 + \delta, \quad \vec{u} = \frac{d\vec{x}}{da} = \frac{\vec{v}}{a\dot{a}}, \quad \varphi = \frac{3t_\star^2}{2a_\star^3}\phi \quad . \tag{3.18}$$

Take the continuity equation (3.7). Reminding that in the Einstein-de Sitter model $\rho_b = a^{-3}$, we have:

$$\frac{\partial}{\partial t}(\rho_b \eta) + 3H\rho_b\eta + \frac{a\dot{a}\rho_b}{a}\nabla_{\vec{x}}(\eta\vec{u}) = 0$$

$$-3\frac{\dot{a}}{a^4}\eta + \dot{a}\rho_b\frac{\partial\eta}{\partial a} + 3\frac{\dot{a}}{a}a^{-3}\eta + \dot{a}\rho_b\vec{u}\cdot\nabla_{\vec{x}}\eta + \eta\nabla_{\vec{x}}\cdot\vec{u} = 0$$

$$\Rightarrow \quad \frac{D\eta}{Da} + \eta\nabla_{\vec{x}}\cdot\vec{u} = 0 \quad . \tag{3.19}$$

Consider now the Euler equation (3.8), we are allowed to neglect the pressure term, being the CDM represented as a collisionless fluid. Using (3.18), we have:

$$a\dot{a}^2\frac{\partial\vec{u}}{\partial a} + \dot{a}^2\vec{u} + \ddot{a}a\vec{u} + \frac{\dot{a}}{a}a\dot{a}\vec{u} + a\dot{a}^2(\vec{u}\cdot\nabla_{\vec{x}})\vec{u} = -\frac{2}{3a}\frac{a_\star^3}{t_\star^2}\nabla_{\vec{x}}\varphi \quad .$$

If we replace with the following identities:

$$\dot{a} = \frac{2}{3}\frac{a}{t}, \quad \ddot{a} = -\frac{2}{9}\frac{a}{t^2}, \quad a^3 = a_\star^3\left(\frac{t}{t_\star}\right)^2 \quad ,$$

we obtain the equation:

$$\frac{D\vec{u}}{Da} + \frac{3}{2a}\vec{u} = -\frac{3}{2a}\nabla_{\vec{x}}\varphi \quad . \tag{3.20}$$

Finally, consider (3.12). Reminding that $\delta\rho = \rho_b\delta = (6\pi G a^3)^{-1}\delta$, we obtain:

$$\nabla^2_{\vec{x}}\varphi = \frac{\delta}{a} \quad . \tag{3.21}$$

In the growing mode, $v \propto t^{\frac{1}{3}}$ so $\vec{u} \simeq const$, which means that:

$$\frac{D\vec{u}}{Da} = 0 \quad \text{(linear solution)}. \tag{3.22}$$

Furthermore, as a consequence of the Kelvin circulation theorem, $\vec{u}$ is an irrotational velocity field[3]:

$$\vec{u} = -\nabla\varphi \quad \text{(linear solution)}. \tag{3.23}$$

This also gives us a relation between the velocity field and the peculiar gravitational potential.
The ansatz assumed by Zel'dovich in his theory is that (3.22) and (3.23) are valid even beyond the linear regime. In fact, in the Fourier space:

$$\varphi_{\vec{k}} \propto \frac{\delta_{\vec{k}}}{k^2}, \quad \vec{u}_{\vec{k}} \propto k\varphi_{\vec{k}} \propto \frac{\delta_{\vec{k}}}{k} \quad ,$$

which means that even if the density fluctuation field $\delta_{\vec{k}}$ varies on larger scales than $\vec{u}_{\vec{k}}$ and $\varphi_{\vec{k}}$, the latter keep on a linear level for longer times because of the weights k^{-1} and k^{-2}, respectively.
Once we assume these approximations, we have a new set of equations:

$$\frac{D\vec{u}}{Da} = 0 \quad , \tag{3.24}$$

$$\frac{D\eta}{Da} + \eta(\nabla \cdot \vec{u}) = 0 \quad . \tag{3.25}$$

The Poisson cosmological equation has been decoupled from the others, in fact we will use it only for the initial conditions.

[3]From now on, we will not specify the comoving coordinates, being the ones we will always use.

From (3.24) it is clear that the set of collisionless particles moves under the effect of their inertia, not subject to any force (speaking of Newtonian forces), and the dynamics preserves the mass conservation, as specified by (3.25).
Let the initial (Lagrangian) coordinate of a particle in the unperturbed distribution be $\vec{q}$, namely the Eulerian position $\vec{x}$ at time $a = a_0$.
The solution to the first equation is:

$$\vec{u}(\vec{x}, a) = \vec{u}_0(\vec{q}) \quad ,$$

with $\vec{u}_0(\vec{q})$ the initial velocity in the Lagrangian position $\vec{q}$. We can further integrate the latter equation to find the particle's trajectory:

$$\int_{a_0}^{a(t)} \frac{d}{da'}\vec{x}(\vec{q}, a')\mathrm{da}' = \int_{a_0}^{a(t)} \vec{u}_0(\vec{q})\mathrm{da}'$$
$$\Rightarrow \quad \vec{x}(\vec{q}, a) - \vec{q} = (a - a_0)\vec{u}_0(\vec{q}) \quad . \tag{3.26}$$

Clearly, each particle follows a straight-line trajectory, being subject to a displacement corresponding to a density perturbation.
The ansatz discussed previously assures that, near the initial conditions, the system is on a linear regime, than we have $\vec{u}_0(\vec{q}) = -\nabla_{\vec{q}}\varphi_0(\vec{q})$.
Therefore, setting $a_0 = 0$ for simplicity:

$$\vec{x}(\vec{q}, a) = \vec{q} - a\nabla_{\vec{q}}\varphi_0(\vec{q}) \quad \Rightarrow \quad \vec{u}(\vec{x}(\vec{q}, a), a) = \frac{\vec{x} - \vec{q}}{a} \quad .$$

We can now solve the continuity equation integrating by separation of variables:

$$\eta(\vec{x}, a) = \eta_0 \cdot e^{-\int_{a_0}^{a} \nabla\cdot\vec{u}(\vec{x}(\vec{q},a'),a')} \quad .$$

It is convenient to use the mass conservation law: consider a particle, our infinitesimal fluid element, then:

$$\eta(\vec{x}, a)\mathrm{d}^3\mathrm{x} = \eta_0(\vec{q})\mathrm{d}^3\mathrm{q} \quad .$$

This equality could have been obtained highlighting the fact that (3.26) defines a unique mapping between the Eulerian and Lagrangian coordinates. Therefore:

$$\eta(\vec{x}, a) = \frac{(1 + \delta_0(\vec{q}))}{|J(\vec{r}, t)|} \quad ,$$

with $|J(\vec{r}, t)| = |\frac{\partial\vec{x}}{\partial\vec{q}}|$ the Jacobian determinant of the mapping.
At early times, for $a_0 \to 0$, the system is in the linear regime, where $\varphi \simeq const$, so (3.21) leads to $\delta_0 \to 0$. Hence:

$$\eta(\vec{x}, a) = \left|\frac{\partial\vec{x}}{\partial\vec{q}}\right|^{-1} \quad . \tag{3.27}$$

We are now interested in studying the physical meaning of this equation.
Using the Einstein notation:

$$x^i = q^i - a\frac{\partial \varphi_0}{\partial q^i} \quad \Rightarrow \quad \frac{\partial x^i}{\partial q^j} = \delta^i_j - a\frac{\partial^2 \varphi_0}{\partial q^i \partial q^j} \quad .$$

The second term on the right side is the *deformation tensor* $D_{0,ij}(\vec{q})$, which is evidently symmetrical because of the equality of mixed partials.
Therefore, it can be locally diagonalized by going to principal axes Q_1, Q_2, Q_3 with eigenvalues $\lambda_1(\vec{q})$, $\lambda_2(\vec{q})$, $\lambda_3(\vec{q})$. Depending on the sign of λ_i, we can have stretching or compression of the fluid element. As regards (3.27):

$$\eta(\vec{x}, a) = \frac{1}{(1 - a\lambda_1(\vec{q}))(1 - a\lambda_2(\vec{q}))(1 - a\lambda_3(\vec{q}))} \quad .$$

At the time $a_{sc} = 1/\lambda_i(\vec{q})$, where $\lambda_i(\vec{q})$ is supposed to be the largest of the positive eigenvalues, in the hypotesis that it exists, a singularity appears and the density becomes locally infinite. This event is called *shell-crossing*, and the region where it occurs is called *caustic*.
It corresponds to the situation where two points with different Lagrangian position $\vec{q}$ end up at the same Eulerian coordinate $\vec{x}$, therefore the map $\vec{q} \to \vec{x}$ is not invertible anymore. If there is more than one positive eigenvalue, then the collapse will occur first along the axis relative to the larger positive eigenvalue. Thus, the collapse is expected to be generically one-dimensional with oblate elipsoids (called *pancakes*) as preferential form of local fluid element.

3.2.1 Problems related to the Zel'dovich approximation

The Zel'dovich approximation becomes exact in one dimension, when φ_0 depends only on one coordinate q, until the moment of shell-crossing. In the caustic, the strong gravitational forces should make particles to be attracted. However, since this approximation is kinematical, particles pass through the caustic and the newborn structures, the pancakes, simply disappear. Therefore, we don't expect to have stable structure.

3.3 The adhesion approximation

The adhesion approximation represents an attempt to describe the dynamics of CDM beyond shell-crossing.
In this model, we add an artificial viscosity term to the Euler equation which makes particles stick to each other as they enter the caustic region. This term is

meant to simulate the action of the gravitational interaction between neighbouring particles, effect that was neglected by the Zel'dovich approximation. As a result, the adhesion approximation predicts the formation of stable large-structures.
We define $\tau = a(t)$. The equations of interest are:

$$\frac{D\vec{u}}{D\tau} = \nu\nabla^2\vec{u} \quad , \tag{3.28}$$

$$\frac{D\eta}{D\tau} = -\eta(\nabla\cdot\vec{u}) \quad . \tag{3.29}$$

The first equation is called *Burgers' equation*. On the right side, there is the kinematical viscosity coefficient ν, with dimensions L^2/T, and it guarantees that its effect is limited to the regions of a possible shell-crossing and can be neglected outside. First of all, we assume an irrotational velocity field:

$$\vec{u} = \nabla\Phi \quad ,$$

and replace it in (3.28):

$$\frac{\partial(\nabla\Phi)}{\partial\tau} + (\vec{u}\cdot\nabla)(\nabla\Phi) = \nu\nabla^2(\nabla\Phi) \quad .$$

Taking advantage of the Einstein notation and swapping places to partials, we write:

$$\partial_i\left(\frac{\partial\Phi}{\partial\tau}\right) + \frac{1}{2}\partial_i(\partial_j\Phi\partial^j\Phi) = \nu\partial_j(\partial_i\partial^i\Phi) \quad .$$

For the second term on the left side, we used:

$$\frac{1}{2}\partial_i(\partial_j\Phi\partial^j\Phi) = \partial_j\Phi\partial_i\partial_j\Phi = u^j\partial_i u^j = [(\vec{u}\cdot\nabla)\vec{u}]_i \quad .$$

Now, by simple integration (using the divergence theorem), we obtain the *Bernoulli equation*:

$$\frac{\partial\Phi}{\partial\tau} + \frac{1}{2}(\nabla\Phi)^2 = \nu\nabla^2\Phi \quad . \tag{3.30}$$

The utility of the Burgers' equation lies in the fact that it possesses an analytic solution. In order to find it, we use the non-linear Hopf-Cole transformation:

$$\Phi = -2\nu\ln\mathcal{U} \quad ,$$

(with $\mathcal{U}$ called velocity expotential) and replace it in the Bernoulli equation:

$$-\frac{2\nu}{\mathcal{U}}\frac{\partial\mathcal{U}}{\partial\tau} + 2\nu^2\frac{(\nabla\mathcal{U})^2}{\mathcal{U}^2} = -2\nu^2\frac{\nabla^2\mathcal{U}}{\mathcal{U}} + 2\nu^2\frac{(\nabla\mathcal{U})^2}{\mathcal{U}^2}$$

$$\Rightarrow \quad \frac{\partial \mathcal{U}}{\partial \tau} = \nu \nabla^2 \mathcal{U} \quad . \tag{3.31}$$

We obtained the *Fokker-Planck equation* (or linear diffusion equation), which is a parabolic linear differential equation with both initial and boundary conditions. If we consider a trial solution of the form:

$$\mathcal{U} = f(\tau) g(\vec{x}) \quad ,$$

and we solve separating the variables, we get:

$$f(\tau) = e^{\mathcal{E}\tau} \quad \Rightarrow \quad \mathcal{E} g(\vec{x}) = \nu \nabla^2 g(\vec{x}) \quad .$$

Going to the Fourier space, we have:

$$\mathcal{E} g_{\vec{k}} = -\nu k^2 g_{\vec{k}} \quad \Rightarrow \quad \mathcal{E} = -\nu k^2 \quad .$$

For a general solution, we employ the superposition principle:

$$\mathcal{U}(\vec{x}, \tau) = \frac{1}{(2\pi)^3} \int e^{-\nu k^2 \tau} g_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} \mathrm{d}^3\mathrm{k} \quad , \tag{3.32}$$

with $g_{\vec{k}}$ a generic function which have to be fixed by the initial and boundary conditions.
We will now proceed looking for the *kernel* of the solution and use the *Chapman-Kolmogorov equation* to acquire the structure of our $\mathcal{U}(\vec{x}, \tau)$.
In the diffusion process, the kernel can be interpreted as the probability that a particle is in the position $\vec{x}$ at the time $\tau \to 0$, given that it was in $\vec{q}$ at $\tau = 0$. This corresponds to the *Dirac delta*:

$$\mathcal{K}(\vec{x}, \tau | \vec{q}, 0) \to \delta(\vec{x} - \vec{q}) \quad .$$

Since the $g_{\vec{k}}$ is arbitrary, we can fix it in order to have the relation written previously:

$$g_{\vec{k}} = e^{-i\vec{k}\cdot\vec{q}} \quad \Rightarrow \quad \mathcal{K}(\vec{x}, \tau | \vec{q}, 0) = \frac{1}{(2\pi)^3} \int e^{-\nu k^2 \tau} e^{i\vec{k}\cdot(\vec{x}-\vec{q})} \mathrm{d}^3\mathrm{k} \quad .$$

It is evident that for $\tau \to 0$ the kernel gives a Dirac delta. The integral can be solved multiplying by a factor $e^{\frac{(\vec{x}-\vec{q})^2}{4\nu\tau}} e^{-\frac{(\vec{x}-\vec{q})^2}{4\nu\tau}}$, then we have:

$$\mathcal{K}(\vec{x}, \tau | \vec{q}, 0) = \frac{1}{(4\pi\nu\tau)^{-\frac{3}{2}}} e^{-\frac{(\vec{x}-\vec{q})^2}{4\nu\tau}} \quad .$$

This solution is a Gaussian distribution centered in $\vec{q}$, with dispersion $2\nu\tau$ which increases with time.

We earlier enlighted the meaning of the kernel as a conditional probability. Now, we recall the *Bayes' theorem*, which states that the conditional probability of A, given B, is the ratio between the joint probability of A and B and the probability of B:

$$\mathcal{P}(A|B) = \frac{\mathcal{P}(A,B)}{\mathcal{P}(B)} \quad .$$

Given:

$$\mathcal{P}(A) = \int \mathcal{P}(A,B)\mathrm{dB} \quad ,$$

than we have:

$$\mathcal{P}(A) = \int \mathcal{P}(B)\mathcal{P}(A|B)\mathrm{dB} \quad .$$

A consequence of this theorem is the Chapman-Kolmogorov equation, with $\mathcal{U}_0(\vec{q}) = e^{-\frac{\Phi_0(\vec{q})}{2\nu}} = e^{\frac{\varphi_0(\vec{q})}{2\nu}}$ [4] the initial condition and $\mathcal{K}(\vec{x},\tau/\vec{q},0)$ the conditional probability:

$$\mathcal{U}(\vec{x},\tau) = \int \mathcal{U}_0(\vec{q})\mathcal{K}(\vec{x},\tau|\vec{q},0)\mathrm{d}^3\mathrm{q}$$

$$\Rightarrow \quad \mathcal{U}(\vec{x},\tau) = \frac{1}{(4\pi\nu\tau)^{\frac{3}{2}}}\int e^{-\frac{S(\vec{x},\vec{q},\tau)}{2\nu}}\mathrm{d}^3\mathrm{q} \quad , \tag{3.33}$$

where

$$S(\vec{x},\vec{q},\tau) = \frac{(\vec{x}-\vec{q})^2}{2\tau} - \varphi_0(\vec{q}) \quad ,$$

is the *action*, solution of the *free Hamilton-Jacobi equation* with $\nu = 0$:

$$\frac{\partial S}{\partial \tau} + \frac{1}{2}(\nabla S)^2 = 0 \quad .$$

We finally have a solution of the adhesion approximation. Reminding that $\vec{u}(\vec{x},\tau) = -2\nu\nabla\mathcal{U}/\mathcal{U}$, the result is:

$$\vec{u}(\vec{x},\tau) = \frac{\int \frac{\vec{x}-\vec{q}}{\tau} e^{-\frac{S(\vec{x},\vec{q},\tau)}{2\nu}}\mathrm{d}^3\mathrm{q}}{\int e^{-\frac{S(\vec{x},\vec{q},\tau)}{2\nu}}\mathrm{d}^3\mathrm{q}} \quad ,$$

$$\vec{x} = \vec{q} + \int_0^\tau \vec{u}(\vec{x}(\vec{q},\tau'),\tau')\mathrm{d}\tau' \quad .$$

We are interested in small values of ν, which corresponds to infinitely thin structures. In such a limit, we can solve the integral using the saddle-point method, based on the idea that for a sharply peaked function, such as $e^{-\frac{S(\vec{x},\vec{q},\tau)}{2\nu}}$, the largest contribution to the integral comes from the absolute minima $\vec{q}_s$ of $S(\vec{x},\vec{q},\tau)$, for a

[4] Remember that $\vec{u}_0(\vec{q}) = -\nabla_{\vec{q}}\varphi_0(\vec{q})$.

given $\vec{x}$ and τ.
We, therefore, expand $S(\vec{x},\vec{q},\tau)$ to second order around the minima:

$$\mathcal{U}(\vec{x},\tau) = \frac{1}{(4\pi\nu\tau)^{-\frac{3}{2}}} \int \exp\left[-\frac{S(\vec{x},\vec{q}_s,\tau)}{2\nu} - \frac{1}{4\nu}\sum_{i,j=1}^{3} \frac{\partial^2 S}{\partial q_i \partial q_j}\bigg|_{\vec{q}_s} \delta q_i \delta q_j \right] \mathrm{d}^3\mathrm{q} \quad .$$

If we define

$$j_s(\vec{x},\vec{q}_s,\tau) = \left(\det\left[\frac{\partial^2 S}{\partial q_i \partial q_j}\right]_{\vec{q}_s} \right)^{-\frac{1}{2}} \quad ,$$

then we have:

$$\mathcal{U}(\vec{x},\tau) = e^{-\frac{S(\vec{x},\vec{q}_s,\tau)}{2\nu}} \sum_s j_s(\vec{x},\vec{q}_s,\tau) \quad .$$

Since $\vec{q}_s$ of $S(\vec{x},\vec{q},\tau)$ is the absolute minima of S, it satisfies:

$$\nabla_{\vec{q}} S(\vec{x},\vec{q},\tau)\bigg|_{\vec{q}_s} = 0 \quad . \tag{3.34}$$

Then, we obtain:

$$\vec{u}(\vec{x},\tau) = \sum_s \frac{\vec{x}-\vec{q}_s}{\tau} w_s(\vec{x},\vec{q}_s,\tau) \quad ,$$

with $w_s = j_s/(\sum_s j_s)$ weights associated to different initial positions $\vec{q}_s$.
We have found that the velocity of a particle takes contribution from different initial positions, not just one as seen in the Zel'dovich approximation.
This solution can be used to determine the 'skeleton' of the large-scale structure present at any given time. Outside the mass concentration, the adhesion approximation reduces to the Zel'dovich approximation and the particles follow the trajectory found in (3.26).

The most interesting feature of the adhesion approximation is that we can find the absolute minima of S by means of a geometrical technique, based on the construction of parabolas tangential to the initial velocity potential $\varphi_0(\vec{q})$.
which at a given t and x is gradually elevated by changing the value of H from $H = -\infty$ to some value of H where the parabola $p(\vec{x},\vec{q},\tau)$ touches the initial velocity potential. Consider the parabola:

$$p(\vec{x},\vec{q},\tau) = -\frac{(\vec{x}-\vec{q})^2}{2\tau} + H$$

The coordinate of the contact point indicates the Lagrangian coordinate of the chosen particles, while the apex of the parabola shows the Eulerian coordinate of the particle. In figure 3.1 ([3]) are shown three different situation with three different value of τ:

1. in figure (a), for small value of τ the parabola is narrow and it is tangential to $\varphi_0(\vec{q})$ in one point, so we have a unique map between the Lagrangian and Eulerian coordinates;

2. in figure (b), the parabola becomes wider and
touches $\varphi_0(\vec{q})$ in two different points simultaneously. This means that all the particles in between have stuck together in the top point x;

3. in figure (c), at larger values of τ, the parabola becomes so wide that it can touch $\varphi_0(\vec{q})$ only in the vicinities of the deepest minima.

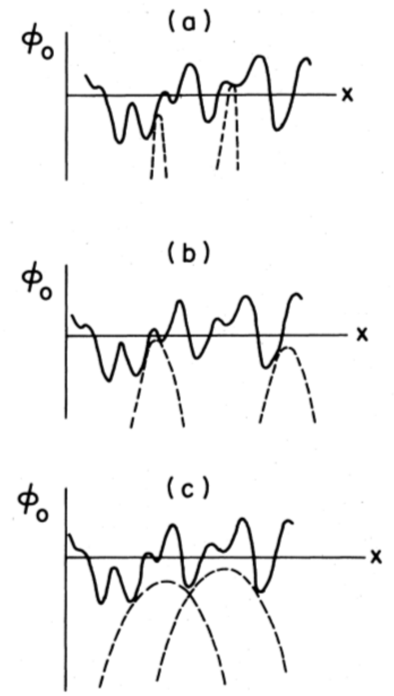


Figure 3.1: Picture taken from [3]

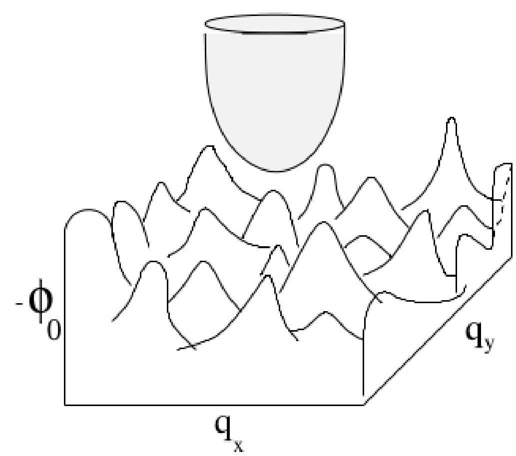


Figure 3.2: A three-dimensional representation of the paraboloid tangential to $\varphi_0(\vec{q})$, [4]

3.3.1 Visual comparison between simulations

We here expose the work of B. S. Sathyaprakash et al. ([5]). In figure 3.3 and 3.4

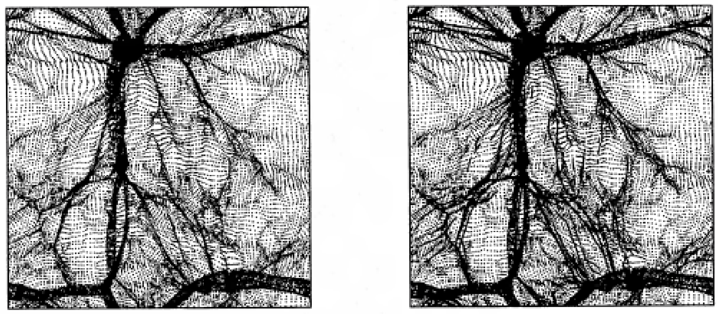


Figure 3.3: N-body and adhesion approximation ([5]).

we have the trajectories followed by particles: it is immediately evident the similarity between the N-body simulation and the adhesion approximation, which have the formation of clumps in the same positions, while the Zel'dovich approximation shows many differences both in particles' trajectories and in clumps's formation.

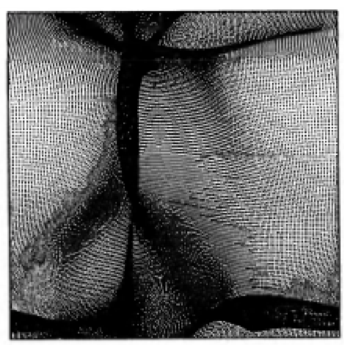


Figure 3.4: Zel'dovich approximation ([5])

In these simulations we can appreciate how the adhesion approximation is a good extension of the Zel'dovich and it also represents an excellent method to predict the 'skeleton' of large-scale structures.

Chapter 4

The wave-mechanical approach

4.1 Introduction

First-order Eulerian perturbations theory allows to examine the linear regime in the large-scale structure formation. The Zel'dovich approximation gives an exact solution in a one-dimensional system until shell-crossing, while the adhesion approximation overcomes the problem of the singularity by adding a fictitious viscous term.
In spite of the success of these theories in the establishment of a standard framework for the investigation and the understanding of large-scale structure formation, there are two amongst many obstacles to a fuller analytical description:

1. The perturbation theory does not guarantee a positive value of the density field everywhere;

2. The Zel'dovich approximation predicts a singularity in the caustic region, where the density becomes infinite.

An alternative approach to the study of collisionless matter was suggested by Widrow and Kaiser [6], based on the formal equivalence between fluid equation and the wave-mechanical formalism. They proposed a wave-mechanical description of CDM represented by a complex scalar field $\psi(\vec{x}, t)$.
Starting from Erwin Madelung's work, which is at the basis of this new approach, in this chapter we will derive the Schrödinger equation from the fluid equations and, with appropriate approximations, we will find a solution in the case of a Cold Dark Matter fluid.

4.1.1 The Madelung-Bohm derivation of the hydrodynamic equations

Right after the pubblication of Schrödinger's equation, Madelung [7] observed that there was a formal equivalence between fluid equations and quantum mechanics equations, indeed you can infer one set from the other.
It was not immediately clear its application in the cosmological context, in fact this equivalence was originally used to find a fluid interpretation of quantum mechanical effects.
We want now to follow the steps of Madelung's work [8], starting from the Schrödinger equation in order to obtain the fluid equations.
Madelung began by writing the complex-valued time-dependant wave function $\psi(x,t)$ (for simplicity, one-dimensional) in polar form[1]:

$$\psi(x,t) = A(x,t) + iB(x,t) = R(x,t)e^{\frac{iS(x,t)}{\hbar}} \quad ,$$

with $R(x,t)$ the amplitude, defined nonnegative at every point, and $S(x,t)$ the action, both real-valued functions. The probability density associated with this wave function is:

$$\rho(x,t) = \psi(x,t)^*\psi(x,t) = R(x,t)^2 \quad .$$

Consider now the Schrödinger's equation:

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\psi(x,t) = i\hbar\frac{\partial}{\partial t}\psi(x,t) \quad . \tag{4.1}$$

If we insert the wave function written before, the resulting equation is split in two equations for the real and the imaginary parts, which lead to a system of two coupled partial differential equations.
In fact:

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2 R}{\partial x^2} - \frac{i\hbar}{2m}\frac{\partial^2 S}{\partial x^2}R - \frac{i\hbar}{m}\frac{\partial R}{\partial x}\frac{\partial S}{\partial x} + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 R + V(x)R(x,t)\right]e^{\frac{iS(x,t)}{\hbar}} =$$

$$= \left[i\hbar\frac{\partial R}{\partial t} - R\frac{\partial S}{\partial t}\right]e^{\frac{iS(x,t)}{\hbar}} \quad .$$

We now rearrange separately real and imaginary parts.

[1] It means that it can be seen as a vector in a two-dimensional space with a "real part" axis and a "imaginary part" one.

Imaginary part. We multiply both sides by $R(x,t)$:

$$-\frac{2R}{m}\frac{\partial R}{\partial x}\frac{\partial S}{\partial x} - \frac{R^2}{m}\frac{\partial^2 S}{\partial x^2} = 2R\frac{\partial R}{\partial t} \quad ,$$

and, by rearrangements, we obtain the first equation of our interest:

$$-\frac{\partial}{\partial x}\left[\frac{R^2}{m}\frac{\partial S}{\partial x}\right] = \frac{\partial R^2}{\partial t} \quad . \tag{4.2}$$

Reminding that $\rho(x,t) = R(x,t)^2$ is the probability density, this equation can be recast into the form of a continuity equation.
Firstly, we have to find the *probability flux* associated to $\psi(x,t)$, defined as:

$$j(x,t) = \frac{\hbar}{2mi}\left[\psi(x,t)^*\frac{\partial}{\partial x}\psi(x,t) - \psi(x,t)\frac{\partial}{\partial x}\psi(x,t)^*\right] = \frac{R^2}{m}\frac{\partial S}{\partial x} \quad .$$

In classical fluid flow, the flux is given by $j(x,t) = \rho(x,t)v(x,t)$, where $v(x,t)$ is the flow velocity. By comparison, we refer to the flow velocity of the probability fluid as a function:

$$v(x,t) = \frac{1}{m}\frac{\partial S}{\partial x} \quad .$$

Returning to (4.2), the term in the brackets on the right side is the probability flux, so we obtain the continuity equation:

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}[\rho(x,t)v(x,t)] = 0 \quad .$$

Real part. We now take the real part:

$$\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 R - \frac{\hbar^2}{2m}\frac{\partial^2 R}{\partial x^2} + VR = -R\frac{\partial S}{\partial t}$$

$$\Rightarrow \quad \frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 = -V(x) - Q(x,t) \quad , \tag{4.3}$$

where $\partial S/\partial x$ is the *flow momentum* and

$$Q(x,t) = -\frac{\hbar^2}{2m}\frac{1}{R}\frac{\partial^2 R}{\partial x^2}$$

is the *Bohm quantum potential*, which has explicit dependence on $\hbar$. Equation (4.3) represents the quantum version of the Hamilton-Jacobi equation and has a remarkable similarity with the Bernoulli equation (3.30) seen in the previous chapter.

4.2 Derivation of the Schrödinger's equation from fluid dynamics

The purpose of this section is to further demonstrate the equivalence between fluid equations and Schrödinger's equation, this time deriving the second one from the fluid dynamics [9].

4.2.1 The Madelung transformation

Let us take the continuity and Euler equation, (3.4), (3.5), with the assumption of an irrotational velocity field $\vec{v} = \nabla\phi$ and a pressureless fluid:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \nabla \phi) = 0 \quad , \tag{4.4}$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 = -V \quad , \tag{4.5}$$

with V a general potential.
We make a transformation known as *Madelung transformation*:

$$\begin{cases} \psi(\vec{x},t) = R(\vec{x},t) e^{\frac{i\phi(\vec{x},t)}{\nu}} \\ \rho = \psi^* \psi = R^2 \end{cases} \quad , \tag{4.6}$$

where we introduced a new parameter ν with dimension L^2/T.
By simple derivation we obtain the following relations:

$$\nabla\psi = \left(\nabla R + i\frac{R}{\nu}\nabla\phi\right) e^{\frac{i\phi}{\nu}} \quad ,$$

$$\nabla^2\psi = \left[\nabla^2 R + \frac{i}{\nu}(R\nabla^2\phi + 2\nabla R \cdot \nabla\phi) - \frac{R}{\nu^2}(\nabla\phi)^2\right] e^{\frac{i\phi}{\nu}} \quad . \tag{4.7}$$

The second relation can be simplified using the continuity equation:

$$\frac{\partial R^2}{\partial t} + \nabla(R^2\nabla\phi) = 0 \quad \Rightarrow \quad 2R\frac{\partial R}{\partial t} + 2R(\nabla R \cdot \nabla\phi) + R^2\nabla^2\phi = 0$$

$$\Rightarrow \quad 2\frac{\partial R}{\partial t} = -\left(2\nabla R \cdot \nabla\phi + R\nabla^2\phi\right) \quad .$$

We replace this relation in (4.7) and obtain:

$$\nabla^2\psi = \left[\nabla^2 R - \frac{i2}{\nu}\frac{\partial R}{\partial t} - \frac{R}{\nu^2}(\nabla\phi)^2\right] e^{\frac{i\phi}{\nu}}$$

$$\Rightarrow \quad (\nabla\phi)^2 = -\frac{\nu^2}{R}\nabla^2\psi e^{-\frac{i\phi}{\nu}} + \frac{\nu^2}{R}\nabla^2 R - \frac{i2\nu}{R}\frac{\partial R}{\partial t} \quad . \tag{4.8}$$

Let us consider the time derivative of $\psi(\vec{x}, t)$:

$$\frac{\partial\psi}{\partial t} = \left(\frac{\partial R}{\partial t} + \frac{iR}{\nu}\frac{\partial\phi}{\partial t}\right)e^{\frac{i\phi}{\nu}} \quad ,$$

we obtain:

$$\frac{\partial\phi}{\partial t} = \frac{i\nu}{R}\left(\frac{\partial R}{\partial t} - e^{-\frac{i\phi}{\nu}}\frac{\partial\psi}{\partial t}\right) \quad . \tag{4.9}$$

If we replace (4.8) and (4.9) in (4.5) we have:

$$\frac{i\nu}{R}\frac{\partial R}{\partial t} - \frac{i\nu}{R}e^{-\frac{i\phi}{\nu}}\frac{\partial\psi}{\partial t} - \frac{\nu^2}{2R}\nabla^2\psi e^{-\frac{i\phi}{\nu}} + \frac{\nu^2}{2R}\nabla^2 R - \frac{i\nu}{R}\frac{\partial R}{\partial t} = -V \quad ,$$

and after simple rearrangements of the terms, we arrive at the Schrödinger's equation:

$$i\nu\frac{\partial\psi}{\partial t} = -\frac{\nu^2}{2}\nabla^2\psi + \left(V + \frac{\nu^2}{2}\frac{\nabla^2 R}{R}\right)\psi \quad . \tag{4.10}$$

By comparison with the Schrödinger's equation of quantum mechanics, we notice a new term, $P = \frac{\nu^2}{2}\frac{\nabla^2 R}{R}$, the so-called *quantum pressure*, already seen in the previous section under the name of Bohm quantum potential. This term resembles a pressure gradient in the interpretation of quantum phenomena in terms of classical fluid behaviour and it represents a source of non-linearity .
It should be clear how this approach overcomes the obtacles discussed in Section 4.1, in fact:

1. The wave function requires $\rho = \psi^*\psi = R^2$ which guarantees an always positive density;

2. No singularity occurs in the wave function.

Equation (4.10) provides an elegant way to include both density and velocity fields in a single complex function.

4.2.2 The Schrödinger's equation for Cold Dark Matter

In order to find a Schrödinger version of the fluid equations that govern the dynamics of CDM [9], we consider the following set of equations in the limit $\nu \to 0$:

$$\frac{\partial\eta}{\partial a} + \nabla\cdot(\eta\nabla\Phi) = 0 \quad , \tag{4.11}$$

$$\frac{\partial\Phi}{\partial a} + \frac{1}{2}(\nabla\Phi)^2 = -\frac{3}{2a}(\Phi + \varphi) \quad , \tag{4.12}$$

seen in Subsection 3.1.3. The Bernoulli equation (4.12) has been derived from (3.20) with the assumption of irrotational velocity field.
The gravitational potential φ is related to the peculiar gravitational potential[2] ϕ through:

$$\varphi = \frac{2}{3a^3H^2}\phi \quad .$$

Consider now the Madelung transformation:

$$\begin{cases} \psi = Re^{\frac{i\Phi}{\nu}} = (1+\delta)^{-1/2}e^{\frac{i\Phi}{\nu}} \\ \eta = \psi^*\psi = R^2 \end{cases} \quad .$$

Following the same steps as in the previous Subsection, but with the time variable $a(t)$, we obtain Schrödinger's equation:

$$i\nu\frac{\partial\psi}{\partial a} = -\frac{\nu^2}{2}\nabla^2\psi + \left(V + \frac{\nu^2}{2}\frac{\nabla^2 R}{R}\right)\psi \quad , \tag{4.13}$$

where we defined $V = \frac{3}{2a}(\Phi + \varphi)$, which satisfies the modified Poisson equation:

$$\nabla^2\left(V + \frac{3i\nu}{4a}\ln(\psi/\psi*)\right) = -\frac{3}{2a^2}(|\psi| - 1) \quad , \tag{4.14}$$

derived from (3.21). If we neglect the quantum pressure term, we are left with the more familiar linear Schrödinger equation:

$$i\nu\frac{\partial\psi}{\partial a} = -\frac{\nu^2}{2}\nabla^2\psi + V\psi \quad .$$

We now insert the Madelung transformation in this equation:

$$1. \qquad i\nu\frac{\partial\psi}{\partial a} = i\nu\left(\frac{\partial R}{\partial a} + R\frac{i}{\nu}\frac{\partial\Phi}{\partial a}\right)e^{\frac{i\Phi}{\nu}} \quad , \tag{4.15}$$

$$2. \qquad -\frac{\nu^2}{2}\nabla^2\psi = \left(-\frac{\nu^2}{2}\nabla^2 R - i\nu\nabla R\cdot\nabla\Phi - \frac{i\nu}{2}R\nabla^2\Phi + \frac{1}{2}(\nabla\Phi)^2 + VR\right)e^{\frac{i\Phi}{\nu}} \tag{4.16}$$

By simple rearrangements of the terms, we arrive to the equation:

$$\frac{\partial\Phi}{\partial a} + \frac{1}{2}(\nabla\Phi)^2 = -V - \frac{\nu^2}{2}\frac{\nabla^2 R}{R} \quad , \tag{4.17}$$

which is the Bernoulli equation with a new term, the quantum pressure.
This implies that we are allowed to neglect the quantum pressure term in the Schrödinger equation and consider it in the fluid equations instead [10].

[2] Knowing that:

$$H = \frac{\dot{a}}{a} = \frac{2}{3t}, \quad \varphi = \frac{3t_\star^2}{2a_\star^3}\phi = \frac{3t^2}{2a^3}\phi \quad ,$$

we can write:

$$\varphi = \frac{2}{3a^3H^2}\phi \quad .$$

4.3 The free-particle approximation

To first-order in Lagrangian perturbation theory one can use the Zel'dovich approximation and put $V = 0$. The Schrödinger equation reduces to the free-particle Schrödinger equation:

$$i\nu\frac{\partial\psi}{\partial a} + \frac{\nu^2}{2}\nabla^2\psi = 0 \quad ,$$

and the Bernoulli equation:

$$\frac{\partial\Phi}{\partial a} + \frac{1}{2}(\nabla\Phi)^2 = -\frac{\nu^2}{2}\frac{\nabla^2 R}{R} \quad ,$$

resembles the equation (3.30) except that the term $\nu\nabla^2\Phi$ has been replaced by the quantum pressure. This new approximation, an alternative to the adhesion approximation, is called the *free-particle approximation* [10].
In order to find a solution, we will exploit the similarity with the adhesion approximation making a Wick rotation, which connects (3.31) with the free-particle Schrödinger equation by going to the imaginary time $a(t) = \tau \rightarrow i\tau$.
Hence, the solution $\psi(\vec{x},\tau)$ can be written in the form:

$$\psi(\vec{x},\tau) = \int \mathcal{G}(\vec{x},\tau|\vec{q},0)\psi_i(\vec{q})\mathrm{d}^3\mathrm{q} \quad ,$$

where $\mathcal{G}(\vec{x},\tau|\vec{q},0)$ is the *free-particle propagator*, which in quantum mechanics involves a sum over all possible spacetime paths connecting the points $(\vec{q},0)$ and $(\vec{x},\tau)$. $\psi_i(\vec{q})$ is some initial wave function.
The path integral for the free-particle propagator is a standard result [11]:

$$\mathcal{G}(\vec{x},\tau|\vec{q},0) = \frac{1}{(2i\pi\nu\tau)^{\frac{3}{2}}}e^{\frac{i(\vec{x}-\vec{q})^2}{2\nu\tau}}$$

$$\Rightarrow \quad \psi(\vec{x},\tau) = \frac{1}{(2i\pi\nu\tau)^{\frac{3}{2}}}\int \psi_i(\vec{q})e^{\frac{i(\vec{x}-\vec{q})^2}{2\nu\tau}}\mathrm{d}^3\mathrm{q} \quad .$$

Inserting the Madelung transformation, we have:

$$\psi(\vec{x},\tau) = \frac{1}{(2i\pi\nu\tau)^{\frac{3}{2}}}\int (1+\delta)^{\frac{1}{2}}\exp\left[\frac{i(\vec{x}-\vec{q})^2}{2\nu\tau} + \Phi_i(\vec{q})\right]\mathrm{d}^3\mathrm{q} \quad . \tag{4.18}$$

In the limit $\nu \rightarrow 0$ the integrand is an oscillating function of $\vec{q}$, whereas in the adhesion approximation it was a highly peaked function. Another difference is that this time the integral gives information about both velocity potential and density, instead of the only velocity expotential $\mathcal{U}$.

The dominant contribution to the integral will be from the points where the phase varies least rapidly with $\vec{q}$, namely:

$$\nabla_{\vec{q}} S(\vec{x},\tau|\vec{q},0) = \nabla_{\vec{q}}\left[\frac{i(\vec{x}-\vec{q})^2}{2\nu\tau} + \Phi_i(\vec{q})\right]\Bigg|_{\vec{q}_s} = 0 \quad \Rightarrow \quad \vec{x} = \vec{q}_s - \tau\nabla_{\vec{q}}\Phi_i(\vec{q}) \quad ,$$

with $\vec{q}_s$ turning points (minima, maxima or saddle points). We arrive to the trajectory found in the Zel'dovich approximation.
Now, if we expand the integrand of (4.18) around $\vec{q}_s$, we obtain:

$$\psi(\vec{x},\tau) = \frac{(1+\delta)^{\frac{1}{2}}}{(2i\pi\nu\tau)^{\frac{3}{2}}}\int \exp\left[\frac{iS(\vec{x},\vec{q}_s,\tau)}{\nu} + \frac{i}{\nu}\sum_{i,j=1}^{3}\frac{\partial^2 S}{\partial q_i \partial q_j}\bigg|_{\vec{q}_s}\delta q_i \delta q_j\right]\mathrm{d}^3\mathrm{q}$$

$$\Rightarrow \quad \psi(\vec{x},\tau) \simeq \sqrt{\eta(\vec{q}_s)}e^{\frac{iS(\vec{x},\vec{q}_s,\tau)}{\nu}}\left[\det\left(\delta_{ij} - \frac{\partial^2\Phi_i}{\partial q_i \partial q_j}\right)\bigg|_{\vec{q}_s}\right]^{-\frac{1}{2}} \quad .$$

We notice that:

$$\frac{\eta(\vec{q}_s)}{\zeta_s} = \eta(\vec{x},\tau) \quad ,$$

with $\zeta_s = \det\left(\delta_{ij} - \frac{\partial^2\Phi_i}{\partial q_i\partial q_j}\right)\Big|_{\vec{q}_s}$, is equivalent to the density found in the Zel'dovich approximation. Hence:

$$\psi(\vec{x},\tau) \simeq \sqrt{\eta(\vec{x},\tau)}e^{\frac{iS(\vec{x},\vec{q}_s,\tau)}{\nu}} \quad . \tag{4.19}$$

In the multistreaming regime, there is more than one saddle point so one needs to sum over all of them, hence:

$$\psi(\vec{x},\tau) \simeq \sum_{\vec{q}_s}\sqrt{\eta(\vec{q}_s)}e^{\frac{i(S(\vec{x},\vec{q}_s,\tau)}{\nu}}\left[\det\left(\delta_{ij} - \frac{\partial^2\Phi_i}{\partial q_i \partial q_j}\right)\bigg|_{\vec{q}_s}\right]^{-\frac{1}{2}} \quad . \tag{4.20}$$

The turning points can be found using the same geometrical technique described in the adhesion approximation, except for the fact that only the particles at $\vec{q}_1$ and $\vec{q}_2$ are at $\vec{x}_1$ at time τ and the solution is obtained by summing over all the turning points which touch the paraboloid. Another difference between these approximations lies in the oscillatory nature of the esponent in the free-particle approximation, thus in (4.20) we sum all over the stationary points.

4.3.1 Visual comparison between approximations

We will now cite the works of P. Coles and K. Spencer, [12], and C. J. Short and P. Coles, [13], who used simulations to test the quantum mechanical formalism in

the context of structure formation.
In one-dimension, we know that the Zel'dovich approximation gives exact results before shell-crossing. Therefore, it is useful to compare the density contrast evolution in one-dimensional Zel'dovich and free-particle approximation.
In the following picture ([12]), we have the density contrast using both of these methods (the Zel'dovich approximation is given by dashed line while the free-particle solution in solid line) at three different times (given by the time-variable b) and for different values of ν.

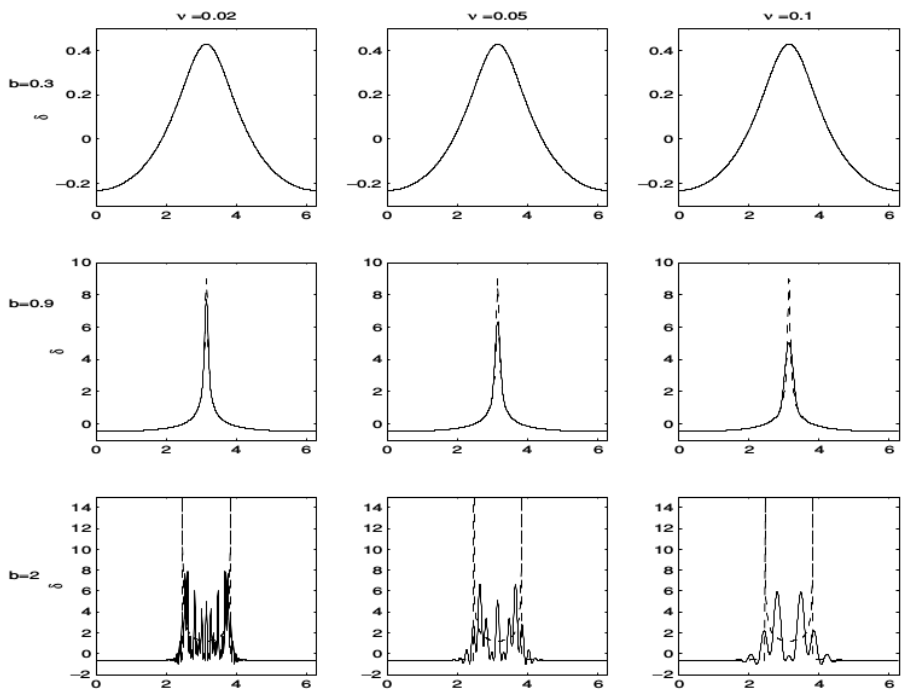


Figure 4.1: Picture taken from [12]

We can see that, particularly for small value of ν, the results are similar: when $b = 0.3$, we see that the density growth is approximately linear and that density contrasts are almost identical. When $b = 0.9$, just before shell-crossing, the density growth becomes highly non-linear and we notice very high density peaks in the Zel'dovich approximation.
Lastly, after shell-crossing, when $b = 2.0$, the two approximations give qualitatively different behaviour, which however is not physical.

Picture 4.2 ([13]) shows slices arbitrarily taken at a z-coordinate $100h^{-1}$ Mpc (at $a = a_0 = 1$) through the final N-body and free-particle density fields for the cases

$\Gamma = 0.12$, $\Gamma = 0.6$ and $\Gamma = 20$, where Γ is a dimensionless parameter linked to ν. We notice that for $\Gamma = 0.12$, the global morphology of the free-particle density field

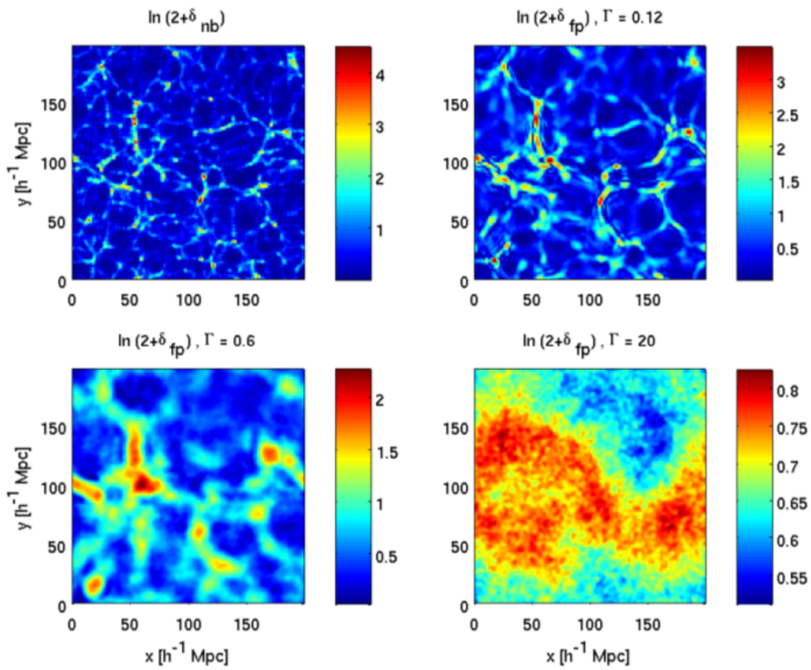


Figure 4.2: Picture taken from [13]

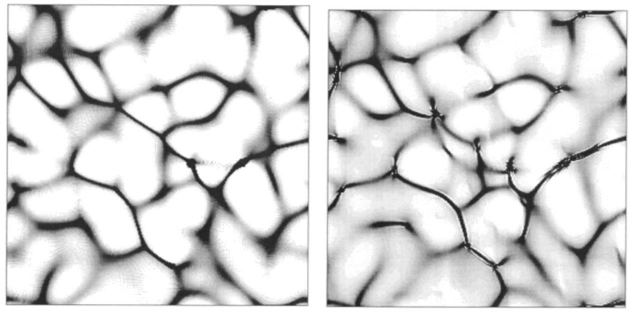
agrees with that of the N-body density field, with the peaks in the same position. Instead, for larger values of Γ, the free-particle density field looks smoother than the N-body one, because of the effect of the quantum pressure which inhibits the the collapse of density perturbations over a greater distance. When $\Gamma = 20$, the quantum pressure dominates the convective term and there is no growth of density fluctuations at all.

Chapter 5

Conclusion

In this dissertation, we presented an alternative method to study the formation of large-scale structures by means of the gravitational instability, the *wave mechanical approach*.
We saw that, in the linear regime, this phenomenon is well understood. Indeed, we obtained a pertubative solution where the density fluctuation, arisen in early times, could grow with time.
The quasi-linear and the non-linear regime are much more complicated and generally non inclined to analytic solution. In this case, we studied the Zel'dovich and the adhesion approximations. The first one gives an excellent comprehension of the behaviour of our cosmic fluid particles until shell-crossing, where the approximation fall into a singularity and we have infinite density. The adhesion approximation overcomes the singularity problem by adding an artificial viscosity term and it gives stable structures.

Widrow and Kaiser were the first to realize the usefulness of the quantum mechanical formalism in the problem of cosmological structure formation.
In [6], they develop a numerical technique alternative to the N-body simulations. We bring as example figure 5.1, which compares two-dimensional CDM universe simulations done with the Schrödinger method (b) and the N-body method (a).
Their work encouraged the study of structure formation using the wave-mechanical approach: we recall Coles and Spencer [12], who test this approach in a one-dimensional collapse and study the solutions of the Schrödinger equation with a time-independant potential and a time-dependant one.
We also mentioned the work of Short and Coles [10], who developed the free-particle approximation with successful results in the weakly non-linear regime.
To conclude, one can also find interesting in the developing of this new approach the works of: Johnston, Lasenby and Hobson [14], where they use techniques from



(a) (b)

Figure 5.1: N-body simulation (a) and Schrödinger method (b), ([6]).

multi-particles quantum mechanics in order to find numerical solutions in the context of multiple fluids, and Short and Coles [13], where they test the free-particle approximation through simulations and compare the results with those obtained by the linearized fluid approach and the Zel'dovich approximation, finding that it provides another useful analytical tool for studying large-scale structure formation.

Bibliography

[1] P. P. Coles and F. Lucchin, *Cosmology.* Wiley, 2002.

[2] S. Matarrese, “Notes on gravitational instability,” 2002–2005.

[3] S. F. Shandarin and Y. B. Zeldovich, “The large-scale structure of the universe:turbulence, intermittency, structures in a self-gravitating medium,” *Reviews on Modern Physics*, vol. 61, pp. 185–217, apr 1989.

[4] S. Matarrese and R. Mohayaee, “The growth of structure in the intergalactic medium,” *Monthly Notices of the Royal Astronomical Society*, vol. 329, pp. 37–60, jan 2002.

[5] B. S. Sathyaprakash, V. Sahni, D. Munshi, D. Pogosyan, and A. L. Melott, “Gravitational instability in the strongly non-linear regime: a study of various approximations,” *Monthly Notices of the Royal Astronomical Society*, vol. 275, pp. 463–482, jul 1995.

[6] L. M. Widrow and N. Kaiser, “Using the Schrödinger equation to simulate collisionless matter,” *The Astrophysical Journal*, vol. 416, pp. L71–L74, oct 1993.

[7] E. A. Spiegel, “Fluid dynamical form od the linear and nonlineare Schrödinger equations,” vol. 1980, pp. 236–240, oct 1979.

[8] R. E. Wyatt, *Quantum Dynamics with Trajectories: Introduction to Quantum Hydrodynamics: 28 (Interdisciplinary Applied Mathematics).* Springer, 2006.

[9] S. Matarrese and R. Mohayaee, “Schrödinger equation’s equation as a model of cosmological structure formation,” in preparation.

[10] C. J. Short and P. Coles, “Gravitational instability via the Schrödinger equation,” *Journal of Cosmology and Astroparticle Physics*, vol. 2006, pp. 012–012, dec 2006.

[11] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals: Emended Edition (Dover Books on Physics).* Dover Publications, 2010.

[12] P. Coles and K. Spencer, "A wave-mechanical approach to cosmic structure formation," *Monthly Notices of the Royal Astronomical Society*, vol. 342, pp. 176–184, jun 2003.

[13] C. J. Short and P. Coles, "Wave mechanics and the adhesion approximation," *J. Cosmol. Astropart. Phys.*, vol. 2006, pp. 016–016, dec 2006.

[14] R. Johnston, A. N. Lasenby, and M. P. Hobson, "Cosmological fluid dynamics in the Schrödinger formalism," *Monthly Notices of the Royal Astronomical Society*, vol. 402, pp. 2491–2502, mar 2010.