UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Fisica e Astronomia “Galileo Galilei”

Dipartimento di Matematica "Tullio Levi-Civita"

Corso di Laurea in Fisica

Tesi di Laurea

Quantum mechanics in the star-product formalism

Relatore

Prof. Paolo Rossi

Laureando

Silvia Ragni

Anno Accademico 2021/2022

Contents

1 Introduction

Quantum Mechanics is usually described through the Dirac formalism, dealing with Hilbert spaces $\mathscr{H}$, smooth functions $\in \mathscr{H}$ and operators defined on the Hilbert spaces. Hence the mathematical language is the one of complex analysis. In one dimension, physical states are represented as functions $\phi \in L^2(\mathbb{R})$ such that $|\phi(x)|^2$ is the probability distribution associated to the state with respect to the variable $x \in \mathbb{R}$, which can be either the position q or the linear momentum p. There is no probability distribution which depends both on q and p, because of the Heisenberg uncertainty principle, which forbids us to know with arbitrary precision both position and linear momentum. Instead, physical observables are represented as self-adjoint operators acting on physical states; in particular the operators $\hat{Q}$ and $\hat{P}$ associated to q and p fulfil the commutation rule $[\hat{Q}, \hat{P}] := \hat{Q}\hat{P} - \hat{P}\hat{Q} = i\hbar$, where $\hbar$ is the reduced Plank constant.

The aim of this thesis is to describe Quantum Mechanics directly on the phase-space, like in Classical Mechanics. However, differently from Classical Mechanics, we have to deal with the uncertainty principle, which does not allow the construction of probability distributions with respect to both q and p, but just of *quasi-probabily distributions*. Moreover, while in Classical Mechanics the smooth functions representing observables commute with each other, in Quantum Mechanics that should not happen as a consequence of quantization, which can mathematically be seen as a *deformation* of the algebra of the smooth functions on the phase-space, the *Poisson algebra*. In particular the point-wise product of the Poisson algebra must be deformed in a new operation, called *star-product*, which is no more commutative. The formalism coming from such an operation allows an algebraic description of Quantum Mechanics.

The first chapter of this thesis will be dedicated to the mathematical description of the star-product, following Bordermann [2], Cattaneo [3] and Esposito's [4] works: first of all it will be given a definition through properties of this object, followed by a description of what a deformation of an algebra formally means. Then it will be possible to give a definition of star-products as series and show that it fulfils the conditions required in the first definition. In the end, the Kontsevich theorem and formula, which will explicate the terms of the series, will be enunciated.

In the second chapter, based on Blaszak and Domański's paper [1], it will be discussed the physical description in the star-product formalism: it will be defined what physical states and observables are and how it is possible to study physical systems. In particular it will be shown how to deal with eigenvalues problems, time evolution and mean values. At the end two physical systems, a free particle and a harmonic oscillator, will be presented.

First of all it is necessary to introduce basic notions of algebra, Poisson manifolds and the definition of multi-differential operators. Then we will also recall the canonical quantization principle.

1.1 Basic notions of algebra

Definition 1.1. *Let R be a ring with the unit 1; a* left-module *M over R is an abelian group $(M, +_M)$ on which it is defined an operation $R \times M \longrightarrow M$ such that, $\forall r, s \in R$ and $\forall v, w \in M$:*

1. *$r(v +_M w) = rv +_M rw$;*
2. *$(r +_R s)v = rv +_M sv$;*
3. *$(rs)v = r(sv)$;*
4. *$1v = v$.*

Definition 1.2. *Let R be a ring with the unit 1; a* right-module *M over R is an abelian group $(M, +_M)$ on which it is defined an operation $M \times R \longrightarrow M$ such that, $\forall r, s \in R$ and $\forall v, w \in M$:*

1. *$(v +_M w)r = rv +_M rw$;*
2. *$v(r +_R s) = rv +_M sv$;*
3. *$v(rs) = (rv)s$;*
4. *$v1 = v$.*

Note 1. *The difference between definitions 1.1 and 1.2 comes from the third condition. However, if the ring is also commutative, a right-module differs from a left one just in a writing convention. In fact, for $r, s \in R$ and $v \in M$, $(rs)v = (sr)v = s(rv)$.*

Definition 1.3. *Let A be a left-module over the ring R and $\nu : A \times A \longrightarrow A$ an operation which satisfies the R-bi-linearity condition: $\forall x, y, z \in A$ and $\forall r, s \in R$,*

$\nu(rx +_A sy, z) = r\nu(x, z) +_A s\nu(y, z)$ *and* $\nu(z, rx +_A sy) = r\nu(z, x) +_A s\nu(z, y)$.
An algebra over the ring R *is given by the pair* (A, ν)*.*

Definition 1.4. *Let* K *be a field and* $(A, +_A)$ *a vector space over* K*. Let* $\nu : A \times A \longrightarrow A$ *be an operation such that satisfies the R-bi-linearity condition in respect of the* $+_A$ *operation. An* algebra over the field K *is given by the pair* (A, ν)*.*

Definition 1.5. *An algebra is* associative/commutative *if the operation* ν *satisfies the commutative/associative property.*

Definition 1.6. *An* involution of an algebra (A, μ) *is an operation* † *on* A*,* $^\dagger : A \longrightarrow A$*, such that,* $\forall f, g \in A$*:*

- $(f^\dagger)^\dagger = f$*;*
- $(\mu(f, g))^\dagger = \mu(g^\dagger, f^\dagger)$*;*
- $(rx +_A sy)^\dagger = r'x^\dagger +_A s'y^\dagger$.

Note 2. *The most common example of involution of algebra is the complex conjuagtion function* $^* : \mathbb{C} \longrightarrow \mathbb{C}$ *such that,* $\forall z = x + iy \in \mathbb{C}$*, with* $x, y \in \mathbb{R}$*,* $z^* = x - iy$*.*

Definition 1.7. *A* Lie algebra *is an algebra* $(A, [.,.])$ *over a field equipped with the operation* $[.,.]$ *such that,* $\forall f, g, h \in A$*, it satisfies:*

- *bilinearity:* $[f, \alpha g + \beta h] = \alpha[f, g] + \beta[f, h]$ *and* $[\alpha g + \beta h, f] = \alpha[g, f] + \beta[h, f]$
- *Jacobi Identity:* $[f, [g, h]] + [h, [f, g]] + [g, [h, f]] = 0$*;*
- *antisymmetry:* $[f, g] = -[g, f]$*.*

Note 3. *The antisymmetry property implicates the nil-power property:* $[f, f] = -[f, f] = 0 \ \forall f \in A$*.*

Definition 1.8. *The* $[.,.]$ *with the properties described above are called* Lie brackets*. Given a smooth manifold* M *and two vector fields* $X, Y \in \mathfrak{X}(M)$*,* $[X, Y] := \mathcal{L}_X Y = -\mathcal{L}_Y X$*, where* $\mathcal{L}_X Y$ *denotes the Lie-derivation of* Y *with respect to* X*.*

Definition 1.9. *A* Lie Group *is a group* $(G, \cdot)$ *where* G *is a smooth manifold and both the operations* $\cdot : G \times G \longrightarrow G$*,* $a, b \longmapsto a \cdot b$ *and* $^{-1} : G \longrightarrow G$*,* $a \longmapsto a^{-1}$ *are smooth.*

1.2 Poisson structures

Definition 1.10. *Let* L *be a Lie algebra with the brackets* $\{.,.\}$*. The brackets* $\{.,.\}$ *are called* Poisson brackets *if they satisfy the Leibniz rule:* $\forall f, g, h \in L$*,* $\{f, gh\} = \{f, g\}h + g\{f, h\}$*.*

Definition 1.11. *A* Poisson algebra $(P, \{,\})$ *is a commutative associative algebra with a Lie algebra operation* $\{.,.\}$ *satisfying the Leibniz rule.*

Definition 1.12. *A* Poisson manifold $(M, \{.,.\})$ *is a smooth manifold* M *with a structure of Poisson algebra* $\{.,.\}$ *on the commutative associative ring* $C^\infty(M)$*.*

Definition 1.13. *A* Poisson tensor field $\Pi : M \longrightarrow TM \wedge TM$ *is the only tensor field such that,* $\forall f, g \in C^\infty(M)$*,* $\{f, g\} = \Pi(df, dg)$*. Given the atlas* $\{(U_i, \phi_i)\}_{i \in I}$*, in local charts* $(U_i, \phi_i = (x_i^1, ..., x_i^m))$*, using the Einstein notational convention,* $\Pi = \Pi^{jk} \frac{\partial}{\partial x_i^j} \wedge \frac{\partial}{\partial x_i^k}$*,* $df = \frac{\partial f}{\partial x_i^j} dx_i^j$*,* $dg = \frac{\partial g}{\partial x_i^j} dx_i^j$ *and*

$$\Pi(df, dg) = \Pi^{jk} \frac{\partial f}{\partial x_i^j} \frac{\partial g}{\partial x_i^k}.$$

Note 4. *The Poisson tensor field* Π *is* antisymmetric*.*

There is another possible definition for the Poisson tensor field, which does not involve the Poisson brackets; so it is also possible to define a Poisson manifold with its Poisson tensor field and then to define the Poisson brackets from it.

Definition 1.14. *Let* M *be a* m*-dimentional smooth manifold and* $\{(U_i, \phi_i = (x_i^1, ..., x_i^n))\}_{i \in I}$ *an atlas. Let* Π *be an antisymmetric bi-vector field; using the Einstein convention over the indexes, in the chart* (U_i, ϕ_i) *it takes the form* $\Pi = \Pi^{jk} \frac{\partial}{\partial x_i^j} \wedge \frac{\partial}{\partial x_i^k}$*.* Π *is called a* Poisson tensor field *if satisfies the following condition* $\forall \lambda, \mu, \nu$*:*

$$\Pi^{\sigma\lambda} \frac{\partial}{\partial x^\sigma} \Pi^{\mu\nu} + \Pi^{\sigma\mu} \frac{\partial}{\partial x^\sigma} \Pi^{\nu\lambda} + \Pi^{\sigma\nu} \frac{\partial}{\partial x^\sigma} \Pi^{\lambda\mu} = 0. \quad (1)$$

Definition 1.15. *A Poisson Manifold is a pair* (M, Π) *where* M *is a smooth manifold and* Π *a Poisson tensor field.*

Proposition 1.1. $\forall f, g \in C^\infty(M)$*,* $\{f, g\}_\Pi := \Pi(fd, dg)$ *are Poisson brackets. Hence definition 1.13 is equivalent to definition 1.14 and definition 1.12 to 1.15.*

Proof. The only thing that should be proved is that the brackets $\{., .\}_\Pi$ satisfy the Jacobi rule. That comes from the condition 1 in definition 1.14. □

1.3 Multi-differential operators

The discussion will take place in smooth manifolds, so the formulas and the mathematical expressions will be written in local charts.

Definition 1.16. *Let* M *be a smooth* m*-dimentional manifold and let* $\{(U_i,\ \phi_i = (x_i^1, .., x_i^m))\}_{i \in I}$ *be an atlas. A* multi-index $I = (i_1, ..., i_m) \in \mathbb{N}^m$ *is a collection of index* $i_1, ..., i_m$ *such that* $|I| := i_1 + ... + i_m$ *and the* abbreviation for iterated partial derivatives *is denoted by*

$$\partial_I := \frac{\partial^{i_1 + ... + i_m}}{(\partial x_i^1)^{i_1} ... (\partial x_i^m)^{i_m}}.$$

Definition 1.17. *A* differential operator of order n *is a* $\mathbb{C}$*-linear map* $D : C^\infty(M, \mathbb{C}) \to C^\infty(M, \mathbb{C})$ *such that in every local chart* (U_i, ϕ_i) *and* $\forall f \in C^\infty(M, \mathbb{C})$*,* D *takes the local form:*

$$D(f)|_{U_i} := \sum_{I \in \mathbb{N}^n, |I| = n} D^I \partial_I (f|_{U_i}),$$

where $\forall I$ *the function* $D^I : U_i \to \mathbb{C}$ *is smooth.*

Definition 1.18. *A* multi-*differential operator* of rank k *(or a* k-differential operator*) and of order* n D *is a* $\mathbb{C}$*-*k*-multi-linear map:* $\underbrace{C^\infty(M, \mathbb{C}) \times ... \times C^\infty(M, \mathbb{C})}_{k\ times} \longrightarrow C^\infty(M, \mathbb{C})$ *such that in local chart it is written:*

$$D(f_1, ..., f_k)|_{U_i} := \sum_{I_1, ..., I_k \in \mathbb{N}^n, |I_1|, ..., |I_k| = n} D^{I_1, ..., I_k} \partial_{I_1}(f_1|_{U_i}) ... \partial_{I_k}(f_k|_{U_i}),$$

where $\forall I_1, ... I_k$*,* $D^{I_1, ..., I_k} : U \to \mathbb{C}$ *is smooth.*

Note 5. *Poisson brackets* $\{., .\}$ *are an example of bi-differential operator.*

1.4 Canonical quantization

In Quantum Mechanics it is postulated that *physical observables* are mathematically described by *self-adjoint operators* on Hilbert spaces $\mathscr{H}$. Firstly we shortly recall what these objects are.

Definition 1.19. *Given* $\mathscr{H}$*, the common* operators norm *is defined as*

$$\|\hat{A}\| := \inf \left\{ c \geq 0 \ \middle|\ \|\hat{A}\psi\|_{L^2} \leq c \, \|\psi\|_{L^2} \quad \forall \psi \in L^2(\mathbb{R}) \right\}.$$

Definition 1.20. *An operator* $\hat{A}$ *is said to be* bounded *if there exists some* $c \in \mathbb{R}^+$ *such that* $\|\hat{A}\psi\|_{L^2} \leq c \, \|\psi\|_{L^2}$ $\forall \psi \in L^2(\mathbb{R})$.

Definition 1.21. *Given the natural scalar product* $< .|. >$ *on* $\mathscr{H}$*, and a linear bounded operator* $\hat{A}$*, it is possible to define the* adjoint operator $\hat{A}^\dagger$ *as* $< \hat{A}^\dagger \psi_1 | \psi_2 > := < \psi_1 | \hat{A} \psi_2 >$.

Definition 1.22. *Given the natural scalar product* $< .|. >$ *on* $\mathscr{H}$*, and a linear operator* $\hat{A}$ *such that its domain* $\mathcal{D}_A := \left\{ \psi \in \mathscr{H} \ \middle|\ \hat{A}\psi \in \mathscr{H} \right\}$ *is dense in* $\mathscr{H}$*, the* adjoint operator $A^\dagger$ *is defined as* $< \hat{A}^\dagger \psi_1 | \psi_2 > := < \psi_1 | \hat{A} \psi_2 >$*. The* domain $\mathcal{D}_{\hat{A}^\dagger}$ *is defined as* $\mathcal{D}_{\hat{A}^\dagger} := \left\{ \psi_1 \in \mathscr{H} \ \middle|\ < \psi_1 | \hat{A} \psi_2 > = < \phi | \psi_2 >,\ \phi \in \mathscr{H},\ \psi_2 \in \mathcal{D}_{\hat{A}} \right\}$

Note 6. *The definition is well given because of the density of* $\mathcal{D}_{\hat{A}}$ *in* $\mathscr{H}$ *and the Riesz representation theorem, which implicate that any* $\phi \in \mathscr{H}$ *such that* $< \psi_1 | \hat{A} \psi_2 > = < \phi | \psi_2 >$ *with* $\psi_2 \in \mathcal{D}_{\hat{A}}$*,* $\psi_1 \in \mathcal{D}_{\hat{A}^\dagger}$ *is unique.*

Note 7. *Note that* $\mathcal{D}_A \subseteq \mathcal{D}_{\hat{A}^\dagger}$.

Definition 1.23. *A linear operator $\hat{A}$ such that its domain $\mathcal{D}_{\hat{A}}$ is dense in $\mathscr{H}$ is called* hermitian *if* $< \hat{A}^\dagger\psi_1|\psi_2 >=< \hat{A}\psi_1|\psi_2 > \forall\psi_1, \psi_2 \in \mathcal{D}_A$.

Definition 1.24. *A linear operator $\hat{A}$ is* symmetric *if its domain $\mathcal{D}_{\hat{A}}$ is dense in $\mathscr{H}$ and it is hermitian.*

Definition 1.25. *An operator $\hat{A}$ is* self-adjoint *if it is linear, symmetric and such that its domain $\mathcal{D}_{\hat{A}}$ is equal to the domain of the adjoint operator $\mathcal{D}_{\hat{A}^\dagger}$ ($\mathcal{D}_{\hat{A}} = \mathcal{D}_{\hat{A}^\dagger}$).*

The *Heisenberg uncertainty principle* states that, for every self-adjoint operators $\hat{A}, \hat{B}$ defined on $\mathscr{H}$, $\Delta\hat{A}\cdot\Delta\hat{B} \geq \left|\frac{[\hat{A},\hat{B}]}{2}\right|$, where $\Delta\hat{X} := \frac{<\psi|(\hat{X}-x)|\psi>}{<\psi|\psi>} \in \mathbb{R}\ \forall x \in \mathbb{R}$, $\forall$ self-adjoint operator $\hat{X}$ on $\mathscr{H}$ and $\forall$ function $|\psi >\in \mathscr{H}$; so, given the position $\hat{Q}$ and the linear momentum $\hat{P}$ operators, $\Delta\hat{P}\cdot\Delta\hat{Q} \geq \frac{\hbar}{2}$. Hence the Poisson Algebra used to describe Classical Mechanics, where the position q and the linear momentum p commute with each other and can be known with arbitrary precision, in Quantum Mechanics requires to be deformed in order to fulfil the above principle: such deformation is called canonical quantization and is defined by the formula

$$[\![\hat{A}, \hat{B}]\!] = \frac{1}{i\hbar}[\hat{A}, \hat{B}],$$

where $[\hat{A}, \hat{B}] := \hat{A}\hat{B} - \hat{B}\hat{A}$ and the brackets $[\![., .]\!]$ replace the classical Poisson brackets $\{., .\}$.

2 Star-product formalism

2.1 Definition of star-product throughout its properties

In Classical Mechanics we usually work with smooth functions defined on Poisson manifolds $(M, \{., .\})$. The operation which characterizes the Poisson algebra is the point-wise product, which is associative and commutative; in order to fulfill the canonical quantization, we want to deform the above operation into a star-product, which shall be still associative but non-commutative. In such formalism the canonical quantization will take the form

$$[\![f, g]\!] = \frac{1}{i\hbar}[f, g], \text{ where } f, g \in C^\infty(M) \text{ and } [f, g] = f \star g - g \star f.$$

Furthermore, when $\hbar$ is very small with respect to the other physical quantities taken into consideration, which means when we can say that $\hbar \longrightarrow 0$, the classical limit must be respected.

Hence the star-product can be defined as a deformation of the commutative point-wise product which must fulfil the canonical quantization; it should also satisfy the following natural conditions:

1. $f \star (g \star h) = (f \star g) \star h$ (associativity);
2. $f \star g = f \cdot g + O(\hbar)$ and $[\![f, g]\!] = \{f, g\} + O(\hbar)$ (for $\hbar \longrightarrow 0$, it must reproduce the classical limit);
3. $f \star 1 = 1 \star f = f$ where 1 is the unit (1 must remain the unit).

2.2 Deformation of an algebra

As the star-product algebra is nothing but a deformation of the Poisson algebra, in order to describe the star-product formalism, it is necessary to formally define what a deformation of an algebra is.

The discussion will take place in sets of *formal power series*, so we need to introduce some basic notions about them. Their main feature is that, differently from the *power series*, there is no need of any notion of convergence, as they are *formal.*

Let R be a ring and M a left-module over R.

Definition 2.1. *A* formal power series *is a map $a: \mathbb{N} \longrightarrow M$ with coefficients in M such that $a := \sum_{r=0}^\infty \lambda^r a_r$, where $a_r = a(r) \in M$, $r \in \mathbb{N}$ and λ is the* formal parameter*.*

Proposition 2.1. *Let $M[\![\lambda]\!]$ and $R[\![\lambda]\!]$ be the set of all formal series with coefficients in M and R respectively; then $M[\![\lambda]\!]$ is a left-module over the ring $R[\![\lambda]\!]$ defined via the following operations:*

- $a + b := \sum_{r=0}^\infty \lambda^r(a_r + b_r)$;
- $\alpha\beta := \sum_{r=0}^\infty \lambda^r\gamma_r = \sum_{r=0}^\infty \lambda^r(\sum_{s=0}^r \alpha_s\beta_{r-s})$;
- $\alpha a := \sum_{r=0}^\infty \lambda^r(\sum_{s=0}^r \alpha_s a_{r-s})$,

$\forall a = \sum_{r=0}^\infty \lambda^r a_r,\ b = \sum_{r=0}^\infty \lambda^r b_r \in M[\![\lambda]\!]$ *and* $\forall\alpha = \sum_{r=0}^\infty \lambda^r\alpha_r,\ \beta = \sum_{r=0}^\infty \lambda^r\beta_r \in R[\![\lambda]\!]$.

It holds the following lemma, which won't be proven:

Lemma 2.2. *Given $M, M_1, ..., M_k$ left-modules over R and the $R[\![\lambda]\!]$-multi-linear map $\phi : M_1[\![\lambda]\!] \times ... \times M_k[\![\lambda]\!] \longrightarrow M[\![\lambda]\!]$, for each $r \in \mathbb{N}$ the map $\phi_r : M_1 \times ... \times M_k \longrightarrow M$ such that*

$$\phi(a_{(1)}, ..., a_{(k)}) = \sum_{r=0}^{\infty} \lambda^r \left(\sum_{\substack{0 \le s, r_1, ..., r_k) \le r \\ s+r_1+...+r_k = r}}^{r} \phi_s(a_{(1)r_1}, ..., a_{(k)r_k} \right)$$

is unique $\forall a_{(i)} = \sum_{r_i=0}^{\infty} \lambda^{r_i} a_{(i)r_i}$.

Now we have the elements to define what an associative deformation of an algebra is.

Definition 2.2. *Let (A_0, μ_0) be an associative algebra over a ring R with the unit 1 ($\mu_0(a, 1) = \mu_0(1, a) = a \ \forall a \in A_0$); an* associative deformation *of (A_0, μ_0) is given by a sequence of functions $\mu_1, \mu_2, ... : A_0 \times A_0 \longrightarrow A_0$ with the following properties.*

1. $\sum_{s=0}^{r} (\mu_s(\mu_{r-s}(a, b), c) - \mu_s(a, \mu_{r-s}(b, c))) = 0 \ \ \forall a, b, c \in A_0$ *and* $\forall r \in \mathbb{N}$ *(associative condition);*
2. $\mu_r(a, 1) = \mu_r(1, a) = 0 \ \ \forall r \ge 1$ *(which guarantees that 1 remains the unit).*

Proposition 2.3. *Let be $A := A_0[\![\lambda]\!]$ and the $R[\![\lambda]\!]$-bi-linear multiplication $\mu := \sum_{r=0}^{\infty} \mu_r \ \forall a, b \in A_0[\![\lambda]\!]$, $a = \sum_{r=0}^{\infty} a_r$ and $b = \sum_{r=0}^{\infty} b_r$. Then (A, μ), where $\mu(a, b) := \sum_{r=0}^{\infty} \lambda^r (\sum_{s+t+u}^{r} \mu_s(a_t, b_u))$, is an associative algebra over the ring $R[\![\lambda]\!]$.*

Proof. Obvious from the Lemma 2.2 and the Definition 2.2. □

So, given the associative and commutative algebra (A_0, μ_0) over the ring R, its formal associative deformation is $(A := A[\![\lambda]\!],\ \mu = \sum_{r=0}^{\infty} \lambda^r \mu_r)$ defined over the ring $R[\![\lambda]\!]$. It is also possible to define a Poisson bracket because of the following proposition:

Proposition 2.4. *Let (A_0, μ_0) be a commutative algebra; let $f, g \in A_0$; then $\{f, g\} := \mu_1(f, g) - \mu_1(g, f)$ is a Poisson bracket.*

Proof. In order to show that $\{.,.\}$ is a Poisson Bracket, it must be proved that both the Jacobi identity ($\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0,\ h \in A_0$) and the Leibniz rule ($\{f, \mu_0(gh)\} = \mu_0(\{f, g\}, h) + \mu_0(\{f, h\}, g)$) hold.
The first one is a consequence of the linearity of $\mu_r \ \forall r$, of the associative condition at the second order in the Definition 2.2 and of the commutative property of μ_0.
The second one follows from the associative condition at the first order:

$$0 = \mu_0(\mu_1(f, g), h) - \mu_0(f, \mu_1(g, h)) + \mu_1(\mu_0(f, g), h) - \mu_1(f, \mu_0(g, h))$$

and, adding the same expression with f and h interchanged, we obtain:

$$\begin{aligned} 0 + 0 = 0 =& [\mu_0(\mu_1(f, g), h) - \mu_0(f, \mu_1(g, h)) + \mu_1(\mu_0(f, g), h) - \mu_1(f, \mu_0(g, h))] + \\ &+ [\mu_0(\mu_1(h, g), f) - \mu_0(h, \mu_1(g, f)) + \mu_1(\mu_0(h, g), f) - \mu_1(h, \mu_0(g, f))] = \\ =& \mu_0(\{f, g\}, h) - \mu_0(f, \{g, h\}) + \{\mu_0(f, g), h\} - \{f, \mu_0(g, h)\}; \end{aligned} \tag{2}$$

then we add the last expression with g and h interchanged and we subtract the same one with f and g interchanged:

$$\begin{aligned} 0 = 0 + 0 - 0 =& [\mu_0(\{f, g\}, h) - \mu_0(f, \{g, h\}) + \{\mu_0(f, g), h\} - \{f, \mu_0(g, h)\}] + \\ &+ [\mu_0(\{f, h\}, g) - \mu_0(f, \{h, g\}) + \{\mu_0(f, h), g\} - \{f, \mu_0(h, g)\}] + \\ &- [\mu_0(\{g, f\}, h) - \mu_0(g, \{f, h\}) + \{\mu_0(g, f), h\} - \{g, \mu_0(f, h)\}] = \\ =& 2\mu_0(g, \{f, h\}) + 2\mu_0(h, \{f, g\}) - 2\{f, \mu_0(g, h)\} \end{aligned} \tag{3}$$

□

2.3 Formal Poisson structures

Classical Mechanics is studied on Poisson manifolds, where the algebra is the one of the smooth functions. So, given a Poisson manifold M, its Poisson algebra is $(C^\infty(M,\mathbb{C}),\{.,.\})$, which is commutative over the ring $\mathbb{C}$ with the point-wise product. This Poisson algebra is the one to be deformed in order to describe Quantum Mechanics. This is why the main theme of this subsection is the notion of formal Poisson structure. As in the last discussion, we will work on sets of formal series, so there will be no need of any notion of convergence.

Definition 2.3. *Given a Poisson manifold $(M,\{,\})$ with its Poisson tensor field Π_0 and an atlas $\{U_i,\phi_i\}_{i\in I}$, a* formal deformation of the Poisson tensor field Π_0 *is a formal power series*

$$\Pi_\lambda = \sum_{r=0}^{\infty} \lambda^r \Pi_r$$

where $\forall r$ $\Pi_r \in \mathfrak{X}^2(M)$ is antisymmetric and such that, $\forall \lambda, \mu, \nu$, Π in coordinates satisfies:

$$\Pi_\lambda^{\sigma o}\frac{\partial}{\partial x^\sigma}\Pi_\lambda^{\mu\nu} + \Pi_\lambda^{\sigma\mu}\frac{\partial}{\partial x^\sigma}\Pi_\lambda^{\nu o} + \Pi_\lambda^{\sigma\nu}\frac{\partial}{\partial x^\sigma}\Pi_\lambda^{o\mu} = 0. \tag{4}$$

Definition 2.4. *Given the Poisson structure of $(M,\{.,.\})$ with its algebra $(C^\infty(M,\mathbb{C}),\{.,.\})$, a* formal deformation of the Poisson structure *is the structure of $(M,\{.,.\}_\lambda)$ with its algebra $(C^\infty(M,\mathbb{C})[\![\lambda]\!],\{.,.\}_\lambda)$ defined by the deformed Poisson brackets:*

$$\{f,g\}_\lambda := \sum_{r=0}^{\infty}\lambda^r\left(\sum_{\substack{0\le i,j,k,\le r\\ i+j+k=r}}^{r}\Pi_i(df_j,dg_k)\right),$$

where $f=\sum_{r=0}^\infty \lambda^r f_r$, $g=\sum_{r=0}^\infty \lambda^r g_r \in C^\infty(M,\mathbb{C})[\![\lambda]\!]$ and the Π_k's $\in \mathfrak{X}^2(M)$ are such that $\Pi_\lambda = \sum_{k=0}^\infty \lambda^k \Pi_k$ is a formal deformation of the Poisson tensor Π_0.

Note 8. *The brackets $\{,\}_\lambda$ are Lie brackets because they satisfy:*

1. *the Jacobi identity, because of the condition 4 of definition 2.3;*
2. *the antisymmetry, because of the antisymmetry of Π_λ.*

It can be proven that they satisfy also the Leibniz rule, then they are still Poisson brackets. □

Definition 2.5. *A* formal vector field *is a power series $X=\sum_{k=0}^\infty \lambda^k X_k \in \mathfrak{X}(M)[\![\lambda]\!]$ where $X_k \in \mathfrak{X}(M)$ $\forall k$.*

Note 9. *Formal vector fields form a Lie algebra under the common $[.,.]$ extended on $\mathfrak{X}(M)[\![\lambda]\!]$ by bi-linearity. The corresponding Lie group is the set of the symbols $\exp(\lambda X) := \sum_{k=0}^\infty \lambda^k (X)^k$, where $X \in \mathfrak{X}(M)[\![\lambda]\!]$, whose group structure is given by the Baker-Campbell-Hausdorff formula*

$$\exp(\lambda X)\exp(\lambda Y) = \exp\left(\lambda X + \lambda Y + \frac{\lambda^2}{2}[X,Y] + \frac{\lambda^3}{72}([X,[X,Y]] - [Y,[X,Y]]) + ...\right)$$

.

Definition 2.6. *Two* Poisson structures $\Pi_\lambda = \sum_{k=0}^\infty \lambda^k \Pi_k$ and $\Pi'_\lambda = \sum_{k=0}^\infty \lambda^k \Pi'_k$ *are* equivalent *if there exists a formal vector field $X \in \mathfrak{X}(M)[\![\lambda]\!]$ such that*

$$\Pi'_\lambda = \exp(\lambda X)_*\Pi_\lambda := \exp(\lambda \mathcal{L}_X)\Pi_\lambda := \sum_m^\infty \frac{\lambda^m}{m!}\left[\sum_{\substack{0\le i,j,k,\le m\\ i+j+k=m}}^{m}(\mathcal{L}_{X_i})^j\Pi_k\right].$$

$\phi_\lambda := \exp(\lambda X)$ *is called a* formal diffeomorphism.

If the formal parameter of the discussion below is $\hbar$ and the algebra is the one of the smooth functions over a Poisson Manifold $(C^\infty(M,\mathbb{C}),\ \{,\})$ with the point-wise product, then it possible to define the star-product as:

Definition 2.7. *Let $(M,\{.,.\})$ be a Poisson Manifold. A structure of* star-product *on M is defined by the following sequences of $\mathbb{C}$-linear maps $B_r : C^\infty(M,\mathbb{C})\times C^\infty(M,\mathbb{C}) \to C^\infty(M,\mathbb{C})$ such that $\forall r \ge 0$ and $\forall f,g,h \in C^\infty(M,\mathbb{C})$ fulfill the following conditions:*

1. *Every B_r is a bi-differential operator;*

2. $B_0(f,g) = f \cdot g$;

3. $B_1(f,g) - B_1(g,f) = \{f,g\}$;

4. $B_r(1,g) = B_r(f,1) = 0 \ \forall r \geq 1$;

5. $\sum_{s=0}^{r}[B_s(B_{r-s}(f,g),h)] = \sum_{s=0}^{r}[B_s(f,B_{r-s}(g,h))]$.

The formal series

$$\star := \sum_{r=0}^{\infty} \hbar^r B_r$$

is the star-product *on* M.

Lemma 2.5. *The star-product in 2.7 is well defined.*

Proof. The associativity of Definition 2.1 is satisfied by condition 5. The classical limit is implicated by conditions 2 and 3. Finally it holds $f \star 1 = 1 \star f = f$ because of condition 4 □

Note 10. *As differential operator are linear operator, then the star product is a bi-linear operation.*

Theorem 2.6. *Let* $\star$ *be a star-product on the Poisson Manifold* $(M, \{.,.\})$*. Then the* $\mathbb{C}[\![\hbar]\!]$*-left-module with the star-product* $(C^\infty(M,\mathbb{C})[\![\hbar]\!], \star)$ *is an associative algebra over the ring* $\mathbb{C}[\![\hbar]\!]$ *and,* $\forall f = \sum_{r=0}^{\infty} \hbar^r f_r, g = \sum_{r=0}^{\infty} \hbar^r g_r \in C^\infty(M,\mathbb{C})[\![\hbar]\!]$

$$f \star g := \sum_{r=0}^{\infty} \hbar^r \left(\sum_{\substack{0 \leq i,j,k, \leq m \\ i+j+k=m}}^{m} B_i(f_j, g_k) \right).$$

Proof. It follows from the proposition 2.3. □

Lemma 2.7. *The brackets* $[\![.,.]\!]$ *defined as* $[\![f,g]\!] := \frac{1}{i\hbar}(f \star g - g \star f)$ *are Lie brackets which satisfy the Leibniz rule with respect to the star-product.*

Proof. They are Lie brackets as they satisfy:

- bilinearity, because of the bilinearity of the star-product;
- antisymmetry by definition;
- Jacobi identity:

$$\begin{aligned} J =& [\![f,[\![g,h]\!]]\!] + [\![h,[\![f,g]\!]]\!] + [\![g,[\![h,f]\!]]\!] \\ =& [\![[\![g,h]\!],f]\!] + [\![[\![f,g]\!],h]\!] + [\![[\![h,f]\!],g]\!] \\ =& -J = 0. \end{aligned} \tag{5}$$

The Leibniz rule is satisfied, because:

$$\begin{aligned} [\![f, g \star h]\!] =& \frac{1}{i\hbar}(f \star (g \star h) - (g \star h) \star f) \\ (associativity) =& \frac{1}{i\hbar}((f \star g) \star h - g \star (h \star f)) \\ =& \frac{1}{i\hbar}((f \star g) \star h - (g \star f) \star h) + \frac{1}{i\hbar}(g \star (f \star h) - g \star (h \star f)) \\ =& [\![f,g]\!] \star h + g \star [\![f,h]\!]. \end{aligned} \tag{6}$$

□

Note 11. *The algebra* $(M, \star, [\![.,.]\!])$ *is a non commutative deformed Poisson algebra.*

Definition 2.8. *Let* $(M, \{.,.\})$ *be a Poisson Manifold and* $\star$*,* $\star'$ *two star-products. They are* equivalent *if there is an automorphism* $S : C^\infty(M,\mathbb{C})[\![\hbar]\!] \to C^\infty(M,\mathbb{C})[\![\hbar]\!]$ *which can be written as* $S = id + \sum_{r=1}^{\infty} \hbar^r S_r$*, where* $S_r : C^\infty(M,\mathbb{C}) \to C^\infty(M,\mathbb{C})$ *is* $\mathbb{C} - linear$ *and*

$$f \star' g = S(S^{-1}(f) \star S^{-1}(g))$$

Note 12. *It should be checked that $\star'$ maintains all the properties of the star-product. To simplify calculations, only functions $f, g, h \in C^\infty(M, \mathbb{C})$ will be considered:*

1. *associativity, from the associativity of $\star$:*

$$\begin{aligned} f \star' (g \star' h) &= f \star' S(S^{-1}(g) \star S^{-1}(h)) = S(S^{-1}(f) \star [S^{-1}(g) \star S^{-1}(h)]) \\ &= S([S^{-1}(f) \star S^{-1}(g)] \star S^{-1}(h)) = S(S^{-1}(f) \star (S^{-1}(g)) \star' h \\ &= (f \star' g) \star' h; \end{aligned} \tag{7}$$

2. *classical limit:*
   - *at the 0 order $(f \star g)_0 = (f \star' g)_0 = f \cdot g$, by definition of $\star$ and of S;*
   - *at the first order $B_1'(f,g) = B_1(f,g) - f \cdot S_1(g) - g \cdot S_1(f) + S_1(f \cdot g)$, then $B_1'(f,g) - B_1'(g,f) = B_1(f,g) - B_1(g,f) = \{f,g\}$;*

3. *unit 1: $S(1) = id(1) + \sum_{k=0}^{\infty} \hbar^k S_k(0)$ by definition. S_k are $\mathbb{C}$ -linear functions, so $S_k(0) = 0$ and $id(1) = 1$, then $S(1) = 1$.*

□

To make what said above more understandable, it will be useful to give examples of star-products. To simplify the discussion we choose as Poisson manifold M the space $\mathbb{R}^2$, so there is a unique local chart with the coordinates (q, p). The corresponding quantum mechanics operators are indicated as $(\hat{q}, \hat{p})$ and must satisfy the canonical commutation rule. If we apply those operators on a function $\psi(q)$, we obtain:

$$\begin{aligned} (\hat{q}\psi)(q) &:= q\psi(q) \\ (\hat{p}\psi)(q) &:= -\frac{i}{\hbar}\frac{\partial \psi(q)}{\partial q}. \end{aligned} \tag{8}$$

Now let define the star-product with respect to a parameter σ: given two functions $f, g \in C^\infty(M)$, then

$$\begin{aligned} f(q,p) \star_\sigma g(q,p) &:= f e^{i\hbar\sigma \overleftarrow{\partial}_q \overrightarrow{\partial}_p - i\hbar\bar{\sigma} \overleftarrow{\partial}_p \overrightarrow{\partial}_q} g := \sum_{n,m=0}^{\infty} (-1)^m (i\hbar)^{n+m} \frac{\sigma^n \bar{\sigma}^m}{n!m!} \frac{\partial^{n+m} f(q,p)}{\partial q^n \partial p^m} \frac{\partial^{n+m} g(q,p)}{\partial q^m \partial p^n} \\ &= \sum_{k=0}^{\infty} \frac{(i\hbar)^k}{k!} \sum_{m=0}^{k} \binom{k}{m} \sigma^{k-m} (-\bar{\sigma})^m \frac{\partial^k f(q,p)}{\partial q^{k-m} \partial p^m} \frac{\partial^k g(q,p)}{\partial q^m \partial p^{k-m}}, \end{aligned} \tag{9}$$

where $\bar{\sigma} := 1 - \sigma$ and the binomial coefficient $\binom{k}{m} := \frac{k!}{m!(k-m)!}$.

Note that two star-products $\star_\sigma$ and $\star_\rho$ are equivalent, in fact the automorphism $S_{\rho-\sigma}$ such that $f \star_\rho g = S_{\rho-\sigma}(S_{\sigma-\rho}(f) \star_\sigma (S_{\sigma-\rho}(g))$ takes the form

$$S_{\rho-\sigma} = \exp\left[i\hbar(\rho - \sigma)\frac{\partial}{\partial x}\frac{\partial}{\partial p}\right].$$

In particular, $\forall \sigma \in \mathbb{R}$, all the possible $\star_\sigma$ are equivalent.

Proposition 2.8. *The $\star_\sigma$ defined in eq. 9 is a well-defined star-product.*

Proof. It must be proved that $\star_\sigma$ fulfills the conditions in definition 2.1.

1. Associativity:

$$
\begin{aligned}
f \star_\sigma (g \star_\sigma h) =& \sum_{k=0}^{\infty} \frac{(i\hbar)^k}{k!} \sum_{m=0}^{k} \frac{k!}{m!(k-m)!} \sigma^{k-m} (-\bar{\sigma})^m [\partial_q^{k-m} \partial_p^m f] \cdot \\
& \cdot [\partial_q^m \partial_p^{k-m} \left(\sum_{j=0}^{\infty} \frac{(i\hbar)^j}{j!} \sum_{n=0}^{j} \frac{j!}{n!(j-n)!} \sigma^{j-n} (-\bar{\sigma})^n [\partial_q^{j-n} \partial_p^n g][\partial_q^n \partial_p^{j-n} h] \right)] = \\
=& \sum_{k,j=0}^{\infty} \frac{(i\hbar)^{k+j}}{k!j!} \sum_{m=0}^{k} \sum_{n=0}^{j} \frac{k!j!}{m!n!(k-m)!(j-n)!} \sigma^{k+j-m-n} (-\bar{\sigma})^{m+n} [\partial_q^{k-m} \partial_p^m f] \cdot \\
& \cdot \sum_{s=0}^{m} \sum_{t=0}^{k-m} \frac{m!(k-m)!}{s!t!(m-s)!(k-m-t)!} \{ [\partial_q^{j-n+m-s} \partial_p^{n+k-m-t} g][\partial_q^{n+s} \partial_p^{j-n+t} h] \}
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
(f \star_\sigma g) \star_\sigma h =& \sum_{h=0}^{\infty} \frac{(i\hbar)^h}{h!} \sum_{o=0}^{h} \frac{h!}{o!(h-o)!} \sigma^{h-o} (-\bar{\sigma})^o \cdot \\
& \cdot [\partial_q^{h-o} \partial_p^o \left(\sum_{i=0}^{\infty} \frac{(i\hbar)^i}{i!} \sum_{l=0}^{i} \frac{i!}{l!(i-l)!} \sigma^{i-l} (-\bar{\sigma})^l [\partial_q^{i-l} \partial_p^l f][\partial_q^l \partial_p^{i-l} g] \right)] \cdot [\partial_q^o \partial_p^{h-o} h] = \\
=& \sum_{h,i=0}^{\infty} \frac{(i\hbar)^{h+i}}{h!i!} \sum_{o=0}^{h} \sum_{l=0}^{i} \frac{h!i!}{o!n!(h-o)!(i-l)!} \sigma^{h+i-l-o} (-\bar{\sigma})^{o+l} \cdot \sum_{r=0}^{h-o} \frac{(h-o)!}{r!(h-o-r)!} \cdot \\
& \cdot \sum_{u=0}^{0} \frac{o!}{u!(o-u)!} \partial_q^{h-o} \partial_p^o \{ [\partial_q^{i-l+h-o-r} \partial_p^{l+o-u} f][\partial_q^{l+r} \partial_p^{i-l+u} g] \} [\partial_q^o \partial_p^{h-o} h];
\end{aligned}
$$

computing the two expressions, it results that they are equal.

2. It is obvious that at the order 0 $(f \star g)_0 = f \cdot g$; at the first order

$$
(f \star g)_1 = i\hbar \left[\sigma \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - (1-\sigma) \frac{\partial g}{\partial q} \frac{\partial f}{\partial p} \right],
$$

then

$$
\begin{aligned}
[\![f, g]\!] = \frac{1}{i\hbar}[f,g] =& \frac{1}{i\hbar} [(f \star_\sigma g)_1 - (g \star_\sigma f)_1] + O(\hbar) \\
=& \left[\sigma \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - (1-\sigma) \frac{\partial g}{\partial q} \frac{\partial f}{\partial p} \right] - \left[\sigma \frac{\partial g}{\partial q} \frac{\partial f}{\partial p} - (1-\sigma) \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} \right] + \theta(\hbar) \\
=& \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial g}{\partial q} \frac{\partial f}{\partial p} + \theta(\hbar) = \{f, g\} + O(\hbar).
\end{aligned} \tag{11}
$$

3. Taken the second expression in eq. 9, if $k \geq 1$, both $\dfrac{\partial^k 1}{\partial^{k-m} q \partial^m p} = 0$ and $\dfrac{\partial^k 1}{\partial^m q \partial^{k-m} p} = 0$. Then the only non-null term of the summation is the one with $k = 0$, which implicates that $f \star_\sigma 1 = 1 \star_\sigma f = f$.

□

In the cases $\sigma = 0$ and $\sigma = \frac{1}{2}$ the star-product is called *the Kuperschmidt-Manin product* and *the Moyal (or Groenewold) product* respectively.

2.4 Kontsevich's theorem and formula

Now that the star-product has been defined, the problem is to study if and under which conditions such a structure exists. This is why Kontsevich theorem is extremely important.

Theorem 2.9 (Kontsevich 1997)**.** *On every Poisson manifold (M, Π_0) there exists a star-product. In particular, the equivalence classes of star-products $[\star]$ on (M, Π_0) are in bijection with the equivalence classes of formal Poisson structures $[\Pi_\hbar]$ whose zeroth order is equal to Π_0.*

Kontsevich also found an explicit formula for such a bijection. To write the formula we need to introduce some mathematical structures.

Definition 2.9. *A* quiver Γ *is the datum of:*

- *a set V_Γ whose elements are called* vertices*;*
- *a set E_Γ whose elements are called* arrows*;*
- *two maps $t, s : E_\Gamma \longrightarrow V_\Gamma$ associating to an arrow its target (t) and its source (s).*

Definition 2.10. *Given a quiver Γ, a* loop *is a an arrow $a \in E_\Gamma$ such that $t(a) = s(a)$.*

Definition 2.11. *Given a quiver Γ, a* double arrow *is a pair of arrows $a, b \in E_\Gamma$ such that $s(a) = s(b)$ and $t(a) = t(b)$.*

Definition 2.12. *An* admissible quiver (or graph) of order n *is a quiver Γ such that:*

1. *$V_\Gamma = \{1, ..., n\} \cup \{L, R\}$;*
2. *$E_\Gamma = \{a_1, b_1, ..., a_n, b_n\}$;*
3. *$\forall i = 1$, $s(a_i) = s(b_i) = i$;*
4. *Γ has no loops nor double arrows.*

Definition 2.13. *The* set of all the admissible quivers Γ of order n *is called G_n.*

Kontsevich's idea was to associate a particular bidifferential operator $B_{\Gamma,\Pi_\hbar}$ to every admissible graph and to weight it through a constant w_Γ. Then the star-product between two functions is a weighted sum of all the $B_{\Gamma,\Pi_\hbar}$ applied to the two functions.

Definition 2.14. *Given a m-dimentional Poisson Manifold (M, Π_0) with its atlas $\{U_j, \phi_j\}_{j\in I}$ and a graph $\Gamma \in G_n$, let I be a function such that*
$I : E_\Gamma = \{a_1, ..., a_n, b_1, ..., b_n\} \longrightarrow \{1, ..., n\}$; as notation we will call $I(a_i) = i_i$, $I(b_i) = j_i$. A bi-differential operator associated to an admissible graph $B_{\Gamma,\Pi_\hbar}$ *is a bi-differential operator such that, $\forall f, g \in C^\infty(M; \mathbb{C})$, in coordinates takes the form*

$$B_{\Gamma,\Pi_\hbar}(f, g) := \sum_{I:E_\Gamma\to\{1,...,n\}} \left[\prod_{i=1}^{n} \left(\prod_{x\in t^{-1}(i)} \partial_{I(x)}\right) \left(\Pi_\hbar^{I(a_i)I(b_i)}\right)\right] \left(\prod_{x\in t^{-1}(L)} \partial_{I(x)}\right)(f) \left(\prod_{x\in t^{-1}(R)} \partial_{I(x)}\right)(g).$$

Definition 2.15. *The* constant w_Γ associated to an admissible graph $\Gamma \in G_n$ *is defined as:*

$$w_\Gamma := \frac{1}{(2\pi)^{2n}} \int_{\mathcal{H}_n} d\phi_{a_1} \wedge d\phi_{b_1} \wedge ... \wedge d\phi_{a_n} \wedge d\phi_{b_n},$$

where:

- *$\mathcal{H} := \{p \in \mathbb{C} |\ Im(p) > 0\}$;*
- *$\mathcal{H}_n := \{(p_1, ..., p_n) \in \mathcal{H}^n |\ p_i \neq p_j\ \ \forall i \neq j\}$;*
- *$\phi : \mathcal{H}_2 \longrightarrow \mathbb{R}$, $(p, q) \longmapsto arg\left(\frac{q-p}{q-\bar{p}}\right)$;*
- *$\pi_x : \mathcal{H}_n \longrightarrow \mathcal{H}_2$ $(p_1, ..., p_n) \longmapsto (p_{s(x)}, p_{t(x)})$, where $x \in E_\gamma$ and $p_L = 0, p_R = 1$;*
- *$\phi_x : \mathcal{H}_n \longrightarrow \mathbb{R}$, $\phi := \phi \circ \pi_x$.*

Lemma 2.10. *The integral in definition 2.15 converges absolutely.*

Theorem 2.11 (Kontsevich formula)**.** *In local coordinates the bijection of the Kontsevich theorem 2.9 takes the form:*

$$f \star g = \sum_{k=0}^{\infty} \frac{(i\hbar)^k}{k!} \sum_{\Gamma\in G_k} w_\Gamma B_{\Gamma,\Pi_\hbar}(f, g),$$

where $f, g \in C^\infty(M; \mathbb{C})$.

To better clarify the discussion, it will be useful to give a simple example: the Moyal product for the null deformation of a constant Poisson tensor.
Let (M, Π) be a m-dimentional Poisson Manifold such that, in coordinates, $\partial_k \Pi^{ij} = 0$ $\forall k, i, j = 1, ..., m$ and let $\Pi_\hbar$ be $\Pi_\hbar = \Pi$. As $\partial_k \Pi^{ij} = 0$ $\forall k, i, j$, the only contribution to the sum is given by the quivers Γ such that

$t(a_1) = t(a_2) = ... = t(a_n) = L$ and $t(b_1) = t(b_2) = ... = t(b_n) = R$. Then, exchanging a_i with b_i, both w_Γ and $\Pi^{I(a_i}I(b_i)$ take a minus, then the total contribution remains the same. That allows us to write:

$$\sum_{\Gamma \in G_n} w_\Gamma B_{\Gamma,\Pi_\hbar}(f,g) = \frac{2^n}{(2\pi)^{2n}} \left(\int_{\mathcal{H}_n} d\phi_{a_1} \wedge d\phi_{b_1} \wedge ... \wedge d\phi_{a_n} \wedge d\phi_{b_n} \right) \left(\prod_{k=1}^{n} \Pi^{i_k j_k} \right) (\partial_{i_1}...\partial_{i_n} f)(\partial_{j_1}...\partial_{j_n} g)$$

$$\text{it can be proven that } \int_{\mathcal{H}_n} d\phi_{a_1} \wedge d\phi_{b_1} \wedge ... \wedge d\phi_{a_n} \wedge d\phi_{b_n} = \left[\frac{(2\pi)}{2} \right]^n \tag{12}$$

$$= \left(\prod_{k=1}^{n} \Pi^{i_k j_k} \right) (\partial_{i_1}...\partial_{i_n} f)(\partial_{j_1}...\partial_{j_n} g);$$

Then the Moyal product between two functions f, g takes the form:

$$f \star g = \sum_{n=0}^{\infty} \frac{(i\hbar)^n}{n!} \left(\prod_{k=1}^{n} \Pi^{i_k j_k} \right) (\partial_{i_1}...\partial_{i_n} f)(\partial_{j_1}...\partial_{j_n} g) =: f e^{i\hbar \Pi^{ij} \overleftarrow{\partial_i} \overrightarrow{\partial_j}} g.$$

3 Applications to Quantum Mechanics

Now that the star-product formalism has been described, we can study Quantum Mechanics using this formalism.

3.1 Observables, pure and mixed states

First of all we need to discuss which space we are dealing with.

Classical Mechanic is studied on phase spaces; in particular observables are smooth functions defined on a Poisson manifold M and the admissible states of a physical system are described as Dirac δ functions (pure states) or probabilistic distributions (mixed states) on M.

By contrast Quantum Mechanics is typically developed on Hilbert spaces $\mathscr{H}$, where pure states consist in normalized vectors $\psi \in \mathscr{H}$ and physical observables are represented by self-adjoint operators acting on the $\psi \in \mathscr{H}$.

The aim of defining the star-product is to describe Quantum Mechanics directly on the space of classical observables (the ring of functions on the phase space M), starting with the algebra of observables and postponing the construction of the Hilbert space $\mathscr{H}$. Then it must be postulated that physical states are functions in the Hilbert space $L^2(M)$, the space of all square integral functions on $M = \mathbb{R}^{2N}$ with respect to the Lebesgue measure; hence physical states are represented as *pseudo-probabilistic distributions*, which means normalized functions with respect to the L^2-norm that could be also negative. This is why we need to construct the algebra $(L^2(M), \star_\sigma)$; so we must check that, $\forall f, g \in L^2(M)$, $f \star_\sigma g \in L^2(M)$. In order to simplify the discussion, from now on $\mathbb{R}^2$ will be taken as M.

First we need the following theorem, which won't be proven:

Theorem 3.1. *The $\star_\sigma$ defined as in formula 9 can be written also in the following integral form:*

$$\begin{aligned}(f \star_\sigma g)(q,p) =& \frac{1}{2\pi\hbar} \iint \mathcal{F}f[q,p](\xi,\eta) g(q - \bar{\sigma}\eta, p - \sigma\xi) e^{\frac{i}{\hbar}(\xi q - \eta p)}\, d\xi\, d\eta \\ =& \frac{1}{2\pi\hbar} \iint f(q + \sigma\eta, p + \bar{\sigma}\xi) \mathcal{F}g[q,p](\xi,\eta) e^{\frac{i}{\hbar}(\xi q - \eta p)}\, d\xi\, d\eta,\end{aligned} \tag{13}$$

where $Ff[q,p](\xi,\eta) := \frac{1}{2\pi\hbar} \iint f(q,p) e^{-\frac{i}{\hbar}(\xi q - \eta p)}\, dq\, dp$ is the Fourier transform.

Theorem 3.2. *Given the Schwartz space $\mathscr{S}(M)$, $\forall \psi, \phi \in \mathscr{S}(M)$, there holds $\|\psi \star_\sigma \phi\|_{L^2} \leq \frac{1}{\sqrt{2\pi\hbar}} \|\psi\|_{L^2} \|\phi\|_{L^2}$. Furthermore there is an unique extension of the $\star_\sigma$ from $\mathscr{S}(M)$ on the whole $L^2(M)$ such that the inequality holds.*

Lemma 3.3 (Jensen's inequality)**.** *Let μ be a positive measure on a σ-algebra $\mathfrak{M}$ in a set Ω such that $\mu(\Omega) = 1$. If f is a real function in $L^1(\Omega, \mu)$ and $\phi : \mathbb{R} \to \mathbb{R}$ is convex, then:*

$$\phi\left(\int_\Omega f\, d\mu \right) \leq \int_\Omega (\phi \circ f)\, d\mu.$$

Corollary 3.3.1. *Let $f, g \in \mathcal{S}(\mathbb{R}^2)$, there holds:*

$$\left| \iint f(q,p) g(q,p)\, dq\, dp \right|^2 \leq \iint |g(q,p)|\, dq\, dp \iint |f(q,p)|^2 |g(q,p)|\, dq\, dp.$$

Proof. We choose $\Omega = \mathbb{R}^2$, $\mathfrak{M} = \mathcal{B}(\mathbb{R}^2)$ (which is the σ-algebra generated by the open spaces of $\mathbb{R}^2$), $\phi(x) = x^2$ and $d\mu(q,p) = \left(\iint |g(q,p)|\, dq\, dp\right)^{-1} |g(q,p)| dq dp$.

$$\begin{aligned} \left|\iint f(q,p)g(q,p)\,dq\,dp\right|^2 &\leq \left(\iint |f(q,p)|\cdot|g(q,p)|\,dq\,dp\right)^2 = \\ &= \left(\iint |g(q,p)|\,dq\,dp\right)^2 \left(\iint |f(q,p)|\,dq\,dp\right)^2 \leq \\ \text{(Jensen's inequality)} &\leq \left(\iint |g(q,p)|\,dq\,dp\right)^2 \iint |f(q,p)|^2\,dq\,dp = \\ &= \iint |g(q,p)|\,dq\,dp \iint |g(q,p)|\cdot|f(q,p)|^2\,dq\,dp. \end{aligned} \tag{14}$$

□

Proof. (theorem 3.2). First we need to prove that $\star_\sigma$ on $\mathscr{S}(M)$ is continuous with respect to the first and the second argument separately.

$$\begin{aligned} \|\psi \star_\sigma \phi\|^2_{L^2} &= \iint |\psi \star_\sigma \phi|^2\,dq\,dp = \\ \text{(eq. 13)} &= \frac{1}{(2\pi\hbar)^2} \iint \left|\iint \mathcal{F}\psi[q,p](\xi,\eta)\phi(q-\bar{\sigma}\eta, p-\sigma\xi)e^{\frac{i}{\hbar}(\xi q - \eta p)}\,d\xi\,d\eta\right|^2 dq\,dp \leq \\ \text{(corollary 3.3.1)} &\leq \frac{1}{(2\pi\hbar)^2} \iint \left[\|\mathcal{F}\psi\|_{L^1} \iint |\mathcal{F}\psi(\xi,\eta)|\cdot|\phi(q-\bar{\sigma}\eta,p-\sigma\xi)|^2\,d\xi\,d\eta\right] dq\,dp = \\ &= \frac{1}{(2\pi\hbar)^2} \|\mathcal{F}\psi\|^2_{L^1} \|\phi\|^2_{L^2}. \end{aligned} \tag{15}$$

Similarly $\|\psi \star_\sigma \phi\|^2_{L^2} \leq \frac{1}{(2\pi\hbar)^2} \|\psi\|^2_{L^2} \|\mathcal{F}\phi\|^2_{L^1}$.
Now we choose an orthonormal basis $\{\chi_{ij} \in \mathcal{S}(M)\}_{i,j\in I}$ of $L^2(M)$ such that $\chi_{ij} \star_\sigma \chi_{kl} = \frac{1}{\sqrt{2\pi\hbar}}\delta_{il}\chi_{jk}$.The existence and the properties of such a basis will be discussed in lemma 3.9, in theorem 3.10 and in its corollary 3.10.1.

Let $\psi, \phi \in \mathcal{S}^2(M)$ be two functions; they can be written as $\psi = \sum_{i,j=0}^{\infty} a_{ij}\chi_{ij}$ and $\phi = \sum_{i,j=0}^{\infty} b_{ij}\chi_{ij}$. Thanks to the continuity proved above, the star-product between the two functions takes the form:

$$\psi \star_\sigma \phi = \left(\sum_{i,j=0}^{\infty} a_{ij}\chi_{ij}\right) \star_\sigma \left(\sum_{k,l=0}^{\infty} b_{kl}\chi_{kl}\right) = \frac{1}{\sqrt{2\pi\hbar}} \sum_{k,j=0}^{\infty}\left(\sum_{i=0}^{\infty} c_{ij}b_{ki}\right)\chi_{kj}.$$

If the two functions $\psi, \phi \in L^2(M)$ instead of $\mathscr{S}(M)$, we can define the star-product between them with the same formula, as $\{\chi_{i,j}\}_{i,j\in I}$ is a basis for $L^2(M)$. So the star-product can be extended on the whole $L^2(M)$ and the extension is unique because $\mathscr{S}(M)$ is dense in $L^2(M)$. Thanks to the inequality of Schwartz, there holds also the following relation:

$$\|\psi \star_\sigma \phi\|^2_{L^2} = \frac{1}{\sqrt{2\pi\hbar}} \sum_{k,j=0}^{\infty}\left(\sum_{i=0}^{\infty} c_{ij}b_{ki}\right)^2 \leq \frac{1}{\sqrt{2\pi\hbar}} \sum_{k,j=0}^{\infty}\sum_{i=0}^{\infty} |c_{ij}|^2|b_{ki}|^2 = \frac{1}{\sqrt{2\pi\hbar}} \|\psi\|^2_{L^2} \|\phi\|^2_{L^2}.$$

□

Now that we have proven that $(L^2(M), \star_\sigma)$ is an algebra, we can start dealing with operators, which will turn to be essential in the definitions of observables and states. The algebra of the operators $\hat{A}_Q$ comes from the deformed Poisson algebra $A_Q = (C^\infty(M), \star_\sigma)$ and its elements are of the form $\hat{A} = A\star_\sigma$ or $\hat{A} = \star_\sigma A$, $A \in A_Q$. It will be seen that a particular kind of these operators acts on $L^2(M)$.

Definition 3.1. *Given a function $A \in A_Q$, an* operator function *associated to A is given by:*

$$\begin{aligned} A_\sigma(\hat{q},\hat{p}) &:= \frac{1}{2\pi\hbar} \iint \mathcal{F}A[q,p](\xi,\eta)e^{\frac{i}{\hbar}(\xi\hat{q}-\eta\hat{p})}e^{\frac{i}{\hbar}(\frac{1}{2}-\sigma)\xi\eta}\,d\eta\,d\xi \\ \text{(Baker-Campbell-Hausdorff formula)} &= \frac{1}{2\pi\hbar} \iint \mathcal{F}A[q,p](\xi,\eta)e^{\frac{i}{\hbar}\xi\hat{q}}e^{-\frac{i}{\hbar}\eta\hat{p}}e^{-\frac{i}{\hbar}\sigma\xi\eta}\,d\eta\,d\xi, \end{aligned} \tag{16}$$

where $\hat{q}, \hat{p}$ are any operators such that they respect $[\hat{q},\hat{p}] = i\hbar = \hat{q}\hat{p} - \hat{p}\hat{q}$.

Note 13. *The formula 16 depends on the parameter σ, which determines the type of ordering: standard ($\sigma = 0$), anti-standard ($\sigma = 1$) and Weyl ($\sigma = \frac{1}{2}$).*

Together with the deformation of the point-wise product in the star-product, it is necessary to deform also the complex-conjugation involution of the algebra.

Definition 3.2. *Given the the algebra A_Q, it is possible to define the operation $^\dagger : A_Q \longrightarrow A_Q$, $f \longmapsto f^\dagger := S_{\sigma-\bar{\sigma}} f^*$.*

Lemma 3.4. *† is an involution of the algebra A_Q.*

Proof. It follows from calcutions. □

Definition 3.3. *A function $f \in A_Q$ is called* hermitian *if $f^\dagger(x) = f(-x)$.*

Definition 3.4. *Given an operator $A_\sigma \in \hat{A}_Q$ associated to the function $A \in A_Q$, its* adjoint operator *$A_\sigma^\dagger$ is defined as*

$$A_\sigma^\dagger := \frac{1}{2\pi\hbar} \iint \mathcal{F}A^*(-\xi, -\eta) e^{-\frac{i}{\hbar}[\xi\hat{q} - \eta\hat{p} + (\frac{1}{2} - \sigma)]} \, d\xi \, d\eta.$$

Theorem 3.5. *Given a function $A \in A_Q$, $A_\sigma^\dagger = A_{\bar{\sigma}}^*$*

Proof. Applying the change of coordinate $(\xi, \eta) \to (-\xi, -\eta)$, the expression in the above integral gets:

$$A_\sigma^\dagger = \frac{1}{2\pi\hbar} \iint \mathcal{F}A^*(\xi, \eta) e^{\frac{i}{\hbar}[\xi\hat{q} - \eta\hat{p} + (\frac{1}{2} - \bar{\sigma})\xi\eta]} \, d\xi \, d\eta = A_{\bar{\sigma}}^*.$$

□

For further use it will be useful to introduce the operators $\underline{\hat{q}}, \underline{\hat{p}}$, such that $[\underline{\hat{q}}, \underline{\hat{p}}] = -i\hbar$; operators $A(\underline{\hat{q}}, \underline{\hat{p}})$ are defined as:

$$A(\underline{\hat{q}}, \underline{\hat{p}}) := \frac{1}{2\pi\hbar} \iint \mathcal{F}A(\xi, \eta) e^{\frac{i}{\hbar}\xi\underline{\hat{q}}} e^{-\frac{i}{\hbar}\eta\underline{\hat{p}}} e^{\frac{i}{\hbar}\bar{\sigma}\xi\eta} \, d\eta \, d\xi. \tag{17}$$

Defined the operator associated to a given function, we want to study how it acts on a function $\psi \in L^2(M)$.

Definition 3.5. *Given two functions $A \in A_Q$ and $\psi \in L^2(M)$,*

$$A_L \star_\sigma \psi := A \star_\sigma \psi,$$

$$A_R \star_\sigma \psi := \psi \star_\sigma A.$$

Theorem 3.6. *Let us define $\hat{q}_\sigma$, $\hat{p}_\sigma$ and $\underline{\hat{q}}_\sigma$, $\underline{\hat{p}}_\sigma$ as*

$$\begin{aligned} \hat{q}_\sigma &:= q + i\hbar\sigma\partial_p, \quad \hat{p}_\sigma := p - i\hbar\bar{\sigma}\partial_q, \\ \underline{\hat{q}}_\sigma &:= \hat{q}_{\bar{\sigma}}^* = q - i\hbar\bar{\sigma}\partial_p, \quad \underline{\hat{p}}_\sigma := \hat{p}_{\bar{\sigma}}^* = p + i\hbar\sigma\partial_q. \end{aligned} \tag{18}$$

For any function $A \in A_Q$,

$$A_L \star_\sigma = A_\sigma(\hat{q}_\sigma, \hat{p}_\sigma),$$

$$A_R \star_\sigma = A_\sigma(\underline{\hat{q}}_\sigma, \underline{\hat{p}}_\sigma).$$

Proof. It is trivial to check that $\hat{q}_\sigma$, $\hat{p}_\sigma$ and $\underline{\hat{q}}_\sigma$, $\underline{\hat{p}}_\sigma$ are well defined operators of the type $\hat{q}$, $\hat{p}$ and $\underline{\hat{q}}$, $\underline{\hat{p}}$, which means that they fulfil the commutation rules in definition 16 and above formula 3.1. Then we must note that, expanding a smooth function $f(x + a)$ in Taylor series, we obtain

$$f(x_0 + a) = \sum_{j=0}^{\infty} \frac{\partial_x^j f(x_0)}{j!} a^j =: f(x_0) e^{a\overleftarrow{\partial_x}} = e^{a\overrightarrow{\partial_x}} f(x_0) \tag{19}$$

Hence the star-product can be formally written in the form

$$A_L \star_\sigma \psi = A \star_\sigma \psi = A(q + i\hbar\sigma\partial_p, p - i\hbar\bar{\sigma}\partial_q)\psi = A(\hat{q}_\sigma, \hat{p}_\sigma)\psi,$$

$$A_R \star_\sigma \psi = \psi \star_\sigma A = A(q - i\hbar\bar{\sigma}\partial_p, p + i\hbar\sigma\partial_q)\psi = A(\underline{\hat{q}}_\sigma, \underline{\hat{p}}_\sigma)\psi.$$

Now, using identity 19,

$$\begin{aligned} e^{\frac{i}{\hbar}\xi\hat{q}}e^{-\frac{i}{\hbar}\eta\hat{p}}\psi(q,p) =& e^{\frac{i}{\hbar}(\xi q+i\hbar\sigma\partial_p)}e^{-\frac{i}{\hbar}(\eta p-i\hbar\bar{\sigma}\partial_q)}\psi(q,p) \\ =& e^{\frac{i}{\hbar}\xi q}e^{-\sigma\partial_p}e^{-\frac{i}{\hbar}\eta p}e^{-\bar{\sigma}\partial_q}\psi(q,p) \\ \text{(Baker-Campbell-Hausdorff formula)} =& e^{\frac{i}{\hbar}\xi q}e^{-\frac{i}{\hbar}\eta p}e^{\frac{i}{\hbar}\sigma\xi\eta}e^{-\sigma\partial_p}e^{-\bar{\sigma}\partial_q}\psi(q,p) \\ =& e^{\frac{i}{\hbar}\xi q}e^{-\frac{i}{\hbar}\eta p}e^{\frac{i}{\hbar}\sigma\xi\eta}e^{-\sigma\partial_p}\psi(q-\bar{\sigma}\eta,p-\sigma\xi). \end{aligned} \tag{20}$$

Then, using formula 16 and the above equation,

$$[A_\sigma(\hat{q}_\sigma,\hat{p}_\sigma)\psi](q,p) = \frac{1}{2\pi\hbar}\iint \mathcal{F}A(\xi,\eta)\psi(q-\bar{\sigma}\eta,p-\sigma\xi)e^{\frac{i}{\hbar}(\xi q-\eta p)}\,d\xi\,d\eta = (A\star_\sigma\psi)(q,p),$$

because of equation 13.

Similarly we obtain

$$[A_\sigma(\underline{\hat{q}}_\sigma,\underline{\hat{p}}_\sigma)\psi](q,p) = (\psi\star_\sigma A)(q,p).$$

□

According to the above theorem, given a smooth function A and a set $\mathcal{D}_{A_L\star_\sigma} := \{f\in L^2(M)\ |A\star_\sigma f\in L^2(M)\}$, then the star-product between A and any function $\psi\in\mathcal{D}_{A_L\star_\sigma}$ can be seen as an operator $A_L\star_\sigma:\mathcal{D}_{A_L\star_\sigma}\longrightarrow C^\infty(M)$, $\psi\longmapsto A\star_\sigma\psi$; similarly, given $\mathcal{D}_{A_R\star_\sigma} := \{f\in L^2(M)\ |f\star_\sigma A\in L^2(M)\}$, we can define the operator $A_R\star_\sigma:\mathcal{D}_{A_R\star_\sigma}\longrightarrow C^\infty(M)$, $\psi\longmapsto\psi\star_\sigma A$.

Note 14. *Note that operators of the form $A_L\star_\sigma$ and $A_R\star_\sigma$ are also linear by definition of star-product 2.7.*

Proposition 3.7. *Given a function $A\in A_Q$, the adjoint operators of its corresponding operators $A_L\star_\sigma$ and $A_R\star_\sigma$ are equal to the adjoint-operators of its involution $A^\dagger$.*

Proof.

$$\begin{aligned}(A_L\star_\sigma)^\dagger =& A^\dagger_\sigma(\hat{q}_\sigma,\hat{p}_\sigma) = A^*_{\bar{\sigma}}(\hat{q},\hat{p}) = (S_{\sigma-\bar{\sigma}}A^*)_{\sigma-(\sigma-\bar{\sigma})}(\hat{q}_\sigma,\hat{p}_\sigma) \\ =& (A^\dagger)_{\bar{\sigma}}(\hat{q}_\sigma,\hat{p}_\sigma) = (A^\dagger)_L\star_\sigma =: A^\dagger_L\star_\sigma \end{aligned} \tag{21}$$

$$\begin{aligned}(A_R\star_\sigma)^\dagger =& A^\dagger_\sigma(\underline{\hat{q}}_\sigma,\underline{\hat{p}}_\sigma) = A^*_{\bar{\sigma}}(\underline{\hat{q}}_\sigma,\underline{\hat{p}}_\sigma) = (S_{\sigma-\bar{\sigma}}A^*)_{\sigma-(\sigma-\bar{\sigma})}(\underline{\hat{q}}_\sigma,\underline{\hat{p}}_\sigma) \\ =& (A^\dagger)_{\bar{\sigma}}(\underline{\hat{q}}_\sigma,\underline{\hat{p}}_\sigma) = (A^\dagger)_R\star_\sigma =: A^\dagger_R\star_\sigma \end{aligned} \tag{22}$$

□

Lemma 3.8. *Both the operators $(A_L\star_\sigma)$ and $(A_R\star_\sigma)$ are hermitian if and only if the function $A\in A_Q$ is hermitian.*

Proof. It follows from the definitions of hermitian operators and functions, from identity 19 and from definition 3.4. □

Now we can define what an observable is in the star-product formalism.

Definition 3.6. *An* observable *is a hermitian function $A\in A_Q$ such that, both $\mathcal{D}_{A_L\star_\sigma}$ and $\mathcal{D}_{A_R\star_\sigma}$ are dense in $L^2(M)$ and $\mathcal{D}_{A_L\star_\sigma}=\mathcal{D}_{(A^\dagger_L\star_\sigma)}$, $\mathcal{D}_{A_R\star_\sigma}=\mathcal{D}_{A^\dagger_R\star_\sigma}$.*

We can also define pure and mixed states in the discussed formalism.

Definition 3.7. *A* pure state *is a function $\psi_{pure}\in L^2(M)$ such that:*

- *ψ_{pure} is hermitian;*
- *$\psi_{pure}\star_\sigma\psi_{pure}=\frac{1}{\sqrt{2\pi\hbar}}\psi_{pure}$;*
- *$\|\psi\|_{L^2}=1$.*

Definition 3.8. *A* mixed state *is a linear combination of pure states, each one weighted with its own probability:*

$$\psi_{mixed}=\sum_r p_r\psi_r \text{ where } 0\le p_r\le 1,\ \sum_r p_r=1,\ \psi_r \text{ is a pure state.}$$

Definition 3.9. *Mixed and pure states can be generically called* states, physical states *or* admissible states.

Definition 3.10. *Given an admissible state $\chi \in L^2(M)$, its* quantum distribution function *is $\rho = \frac{1}{\sqrt{2\pi\hbar}}\chi$.*

Later it will be proved that ρ is a proper quasi-probabilistic distribution function, which is equivalent to say that it is normalized, but does not require to be non-negative. This last property means that ρ is not a distribution function, so cannot describe the probability of finding a particle in the generalized coordinates (q,p): this is a result of the Heisenberg uncertainty principle. By contrast, it is possible to define density probability distributions in only one variable, called *marginal distributions*: $P(q) := \int \rho(q,p)\,dp$, $P(p) := \int \rho(q,p)\,dq$.

To further study states properties, we need to better discuss the form of the space $L^2(M)$ and its basis. $L^2(\mathbb{R}^2)$ is isomorphic to the tensor product between $L^2(\mathbb{R})^*$ and $L^2(\mathbb{R})$, where $L^2(\mathbb{R})^*$ is $L^2(\mathbb{R})$-dual space, which can be identified taking the complex-conjugation of functions as duality map $^* : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})^*$. So, given the functions $\psi, \psi_1, \psi_2, \phi, \phi_1, \phi_2 \in L^2(\mathbb{R})$, the tensor product is defined as:

$$(\phi^* \otimes \psi)(q,r) = \phi^*(q)\psi(r)$$

and the scalar product induced on $L^2(\mathbb{R}^2)$ is determined by the identity:

$$< \phi_1^* \otimes \psi_1 | \phi_2^* \otimes \psi_2 >_{L^2} = < \phi_2 | \phi_1 >_{L^2} < \psi_1 | \psi_2 >_{L^2} .$$

As we want to study physics on the phase space $M = \mathbb{R}^2$ using some generalized coordinates (q,p) such that $q \star_\sigma p - p \star_\sigma q = i\hbar$, we need to find an isomorphism between $L^2(\mathbb{R}^2)$ and $L^2(M)$ depending on the particular $\star_\sigma$ chosen. Such transformation can be constructed by composing the two following isomorphisms: given $\chi \in L^2(\mathbb{R}^2)$,

1. $\mathcal{F}_r\chi[r](q,p) := \frac{1}{2\pi\hbar}\int \chi(q,r)e^{-\frac{i}{\hbar}pr}\,dr$;
2. $T_\sigma\chi(q,r) := \chi(q - \bar{\sigma}r, q + \sigma r)$.

Hence, the Hilbert space $L^2(M)$ can be defined as $L^2(M) := \mathcal{F}_r T_\sigma[L^2(\mathbb{R})^* \otimes L^2(\mathbb{R})] = L^2(\mathbb{R})^* \otimes_\sigma L^2(\mathbb{R})$. The generators of $L^2(M)$ take the form:

$$\chi^\sigma(q,p) = (\phi^* \otimes_\sigma \psi)(q,p) := \frac{1}{\sqrt{2\pi\hbar}}\int e^{-\frac{i}{\hbar}pr}\phi^*(q - \bar{\sigma}r)\psi(q + \sigma r)\,dr$$

and the star-product is determined by:

$$< \phi_1^* \otimes_\sigma \psi_1 | \phi_2^* \otimes_\sigma \psi_2 >_{L^2} = < \phi_2 | \phi_1 >_{L^2} < \psi_1 | \psi_2 >_{L^2},$$

where $\phi_1, \phi_2, \psi_1, \psi_2 \in L^2(M)$. It is also possible to induce a basis on $L^2(M)$:

Lemma 3.9. *Given the orthonormal basis $\{\phi_i\}_{i\in I}$ of $L^2(M)$, $\{\chi_{ij}\}_{i,j\in I}$, $\chi_{ij} := \phi_i^* \otimes_\sigma \phi_j$ is a orthonormal basis for $L^2(M)$.*

Corollary 3.9.1. *Given $\chi = \phi^* \otimes_\sigma \psi \in L^2(M)$, with $\phi = \sum_i a_i\phi_i$ and $\psi = \sum_i b_i\phi_i$. Then $\chi = \sum_{i,j} c_{ij}\chi_{ij} = \sum_{i,j} a_i^* b_j \chi_{ij}$.*

Proof. It follows from the definition of χ_{ij} and the $\mathbb{C}$-linearity of the tensor product. □

The basis $\{\chi_{ij}\}_{i,j\in I}$ has got an interesting property:

Theorem 3.10. $\chi_{ij} \star_\sigma \chi_{kl} = \dfrac{1}{\sqrt{2\pi\hbar}}\delta_{il}\chi_{kj}$

Corollary 3.10.1. *Given two functions $\chi_1 = \phi_1^* \otimes_\sigma \psi_1$, $\chi_2 = \phi_2^* \otimes_\sigma \psi_2$, then*

$$\begin{aligned} \chi_1 \star_\sigma \chi_2 &= \frac{1}{\sqrt{2\pi\hbar}} < \phi_1 | \psi_2 >_{L^2} (\phi_2^* \otimes_\sigma \psi_1); \\ \chi_2 \star_\sigma \chi_1 &= \frac{1}{\sqrt{2\pi\hbar}} < \phi_2 | \psi_1 >_{L^2} (\phi_1^* \otimes_\sigma \psi_2) \end{aligned} \tag{23}$$

Proof. It comes from the $\mathbb{C}$-(anti)bilinearity of both the scalar and the tensor products. □

We can now prove some useful properties of physical states.

Theorem 3.11. *Every pure state $\psi_{pure} \in L^2(M)$ takes the form $\psi_{pure} = \phi^* \otimes_\sigma \phi$, for some normalized $\phi \in L^2(\mathbb{R})$. Conversely, every function $\psi \in L^2(M)$ of the above form is a pure state.*

Proof. It is obvious that every function ψ of the above form in hermitian and normalized; also

$$\begin{aligned}\psi_{pure} \star_\sigma \psi_{pure} =& \frac{1}{\sqrt{2\pi\hbar}} < \phi|\phi > (\phi^* \otimes_\sigma \phi)\\ =& \frac{1}{\sqrt{2\pi\hbar}}(\phi^* \otimes_\sigma \phi) = \frac{1}{\sqrt{2\pi\hbar}}\psi_{pure}.\end{aligned} \tag{24}$$

So ψ_{pure} is a pure state.

Now we will prove that every pure state is in the above form. ψ_{pure} can be written as $\psi_{pure} = \sum_{i,j} c_{ij}\chi_{ij}$; the c_{ij}'s are the ij-components of a matrix $\hat{c}$. As ψ_{pure} is hermitian, idempotent and normalized, then the matrix $\hat{c}$ must fulfill the same properties: $\hat{c}^\dagger = \hat{c}$, $\hat{c}^2 = \hat{c}$, $\mathrm{tr}(\hat{c}) = 1$. From the spectral theorem follows that, if a matrix is hermitian, there exists a unitary matrix T such that $\hat{a} := \hat{T}^\dagger\hat{c}\hat{T}$ is diagonal and real, which is equivalent to say that $c_{ij} = \sum_{k,l} T^\dagger_{i,k}(a_k\delta_{k,l})T_{lj} = \sum_k T^*_{k,i}a_kT_{kj}$ with $a_k \in \mathbb{R}\ \forall k$. Hence

$$\psi_{pure} = \sum_{i,j,k} T^*_{ki}a_kT_{kj}(\phi^*_i \otimes_\sigma \phi_j) = \sum_k a_k \left[\left(\sum_i T_{Ki}\phi_i\right)^* \otimes_\sigma \left(\sum_j T_{kj}\phi_j\right)\right] = \sum_k a_k(\psi^*_k \otimes_\sigma \psi_k),$$

with $\psi_k = \sum_i T_{ki}\phi_i$. Condition $\hat{c}^2 = \hat{c}$ implies that, $\forall k$, $a_k^2 = a_k$ and $\sum_k a_k = 1$, which is true if and only if $a_k = \delta_{k,\hat{k}}$ for some $\hat{k}$; so the function $\psi_{pure} = \psi^*_{\hat{k}} \otimes_\sigma \psi_{\hat{k}}$. □

The above theorem states, in other words, that there is a one to one correspondence between pure states of the phase space Quantum Mechanics and the vectors in $L^2(\mathbb{R})$, commonly used in the classical formalism.

Theorem 3.12. *Every quantum distribution function ρ associated to an admissible state $\chi \in L^2(M)$ is a quasi-probabilistic distribution.*

Proof. Firstly it will be proved for pure states:

$$\begin{aligned}\frac{1}{\sqrt{2\pi\hbar}}\iint \psi_{pure}(q,p)\,dq\,dp =& \frac{1}{\sqrt{2\pi\hbar}}\iint (\phi^* \otimes_\sigma \phi)(q,p)\,dq\,dp\\ (\text{specifying } \mathcal{F}_rT_\sigma\ ) =& \frac{1}{2\pi\hbar}\iiint e^{-\frac{i}{\hbar}pr}\phi^*(q-\bar{\sigma}r)\phi(q+\sigma r)\,dq\,dp\,dr\\ (\text{using } \frac{1}{2\pi\hbar}\int e^{-\frac{i}{\hbar}}\,dp = \delta(-r) = \delta(r)\ ) =& \iint \delta(r)\phi^*(q-\bar{\sigma}r)\phi(q+\sigma r)\,dq\,dr\\ (\text{because } \int f(x)\delta(x-a)\,dx = f(a)\ ) =& \int \phi^*(q)\phi(q)\,dq = 1\end{aligned} \tag{25}$$

For mixed states there holds:

$$\begin{aligned}\frac{1}{\sqrt{2\pi\hbar}}\iint \psi_{mixed}(q,p)\,dq\,dp =& \frac{1}{\sqrt{2\pi\hbar}}\iint \left(\sum_r p_r\psi_r(q,p)\right)\,dq\,dp\\ =& \frac{1}{\sqrt{2\pi\hbar}}\sum_r p_r\left(\iint \psi_r(q,p)\,dq\,dp\right)\\ (\psi_r \text{ is a pure state } \forall r) =& \sum_r p_r = 1\end{aligned} \tag{26}$$

□

Theorem 3.13. *Every admissible state $\chi = \sum_k p_k(\phi^*_k \otimes_\sigma \phi_k)$ satisfies:*

$$\begin{aligned}\frac{1}{\sqrt{2\pi\hbar}}\int \chi(q,p)\,dp =& \sum_k p_k|\phi_k(q)|^2\\ \frac{1}{\sqrt{2\pi\hbar}}\int \chi(q,p)\,dq =& \sum_k p_k|\mathcal{F}\phi_k[x](p)|^2.\end{aligned} \tag{27}$$

Proof. In the proof we will consider only the pure states, because the extension to mixed states is trivial and similar to the last one.

$$\begin{aligned}\frac{1}{\sqrt{2\pi\hbar}}\int \chi(q,p)\,dp =& \frac{1}{2\pi\hbar}\iint \phi^*(q-\bar{\sigma}r)\phi(q+\sigma r)e^{-\frac{i}{\hbar}pr}\,dr\,dp\\ \text{(using } \delta \text{ properties)} =& \int \delta(r)\phi^*(q-\bar{\sigma}r)\phi(q+\sigma r)\,dr = |\phi(q)|^2;\end{aligned}$$

$$\begin{aligned}\frac{1}{\sqrt{2\pi\hbar}}\int \chi(q,p)\,dq =& \frac{1}{2\pi\hbar}\iint \phi^*(q-\bar{\sigma}r)\phi(q+\sigma r)e^{-\frac{i}{\hbar}pr}\,dq\,dp\\ & (x_1 = q-\bar{\sigma}r,\ \ x_2 = q+\sigma r \implies r = x_1+x_2)\\ =& \frac{1}{2\pi\hbar}\iint \phi^*(x_1)e^{\frac{i}{\hbar}px_1}\phi(x_2)e^{-\frac{i}{\hbar}px_2}\,dx_1\,dx_2\\ =& \left(\frac{1}{\sqrt{2\pi\hbar}}\int \phi(x)e^{-\frac{i}{\hbar}px}\,dx\right)^*\left(\frac{1}{\sqrt{2\pi\hbar}}\int \phi(x)e^{-\frac{i}{\hbar}px}\,dx\right)\\ =& |\mathcal{F}\phi[x](p)|^2\end{aligned} \tag{28}$$

□

Like in the classical formalism, to states functions it is possible to associate *density operators*, but a further construction is required. First of all note that to every function $\chi \in L^2(M)$ it is possible to associate an operator $\hat{\chi} \in \hat{L}^2(M)$ defined as $\hat{\chi} := \sqrt{2\pi\hbar}\chi\star_\sigma$. $\hat{L}^2(M)$ also inherits a scalar product from $L^2(M)$: given $\chi_1, \chi_2 \in L^2(M)$, then $< \hat{\chi}_1|\hat{\chi}_2 >_{\hat{L}^2} := < \chi_1|\chi_2 >_{L^2}$; there holds also the Cauchy-Schwartz inequality: $\|\hat{\chi}_1\hat{\chi}_2\|_{\hat{L}^2(M)} \leq \|\hat{\chi}_1\|_{\hat{L}^2(M)}\|\hat{\chi}_2\|_{\hat{L}^2(M)}$. It will be proved that $\hat{L}^2(M)$ is isomorphic to another well known space.

Definition 3.11. *Given the space of the bounded operators acting on $L^2(\mathbb{R})$, $\mathcal{B}(L^2(\mathbb{R}))$, and two operators $\hat{A}, \hat{B} \in \mathcal{B}(L^2(\mathbb{R}))$, the* Schmidt scalar product *is defined as $< \hat{A}|\hat{B} >_{S^2} := tr(\hat{A}^\dagger\hat{B})$.*

Definition 3.12. *The* space of Hilbert-Schmidt operators acting on $L^2(\mathbb{R})$, $\mathcal{S}^2(L^2(\mathbb{R}))$, *is the space of all the operators $\hat{A}$ such that $\|\hat{A}\|_{\mathcal{S}^2} := < \hat{A}|\hat{A} >_{\mathcal{S}^2} < +\infty$.*

Lemma 3.14. *Given any operator $\hat{A} \in \mathcal{S}^2(L^2(\mathbb{R}))$, $\|\hat{A}\| \leq \|\hat{A}\|_{\mathcal{S}^2}$. Then $\mathcal{S}^2(L^2(\mathbb{R})) \subseteq \mathcal{B}(L^2(\mathbb{R})$*

Definition 3.13. *Given $\chi = \phi^* \otimes_\sigma \psi \in L^2(M)$, the* density operator *associated to χ is an operator defined as $\hat{\rho} = < \phi|. >_{L^2} \psi$ acting on $L^2(\mathbb{R})$.*

Lemma 3.15. *$\hat{\rho}$ defined as above (definition 3.13) is a Hilbert-Schmidt operator. Conversely, every $\hat{\rho} \in \mathcal{S}^2(L^2(\mathbb{R}))$ can be written as $\hat{\rho} = < \phi|. >_{L^2} \psi$, where $\psi, \phi \in L^2(\mathbb{R})$.*

The following theorem proves that $\hat{L}^2(\mathbb{R})$ can be naturally identified with $\mathcal{S}^2(L^2(\mathbb{R}))$.

Theorem 3.16. *For every $\hat{\chi} \in \hat{L}^2(M)$, $\hat{\chi} = \mathbb{1} \otimes_\sigma \hat{\rho}$, where $\hat{\rho} \in \mathcal{S}^2(L^2(\mathbb{R}))$. Conversely, $\forall$ density operator $\hat{\rho} \in \mathcal{S}^2(L^2(\mathbb{R}))$, $\mathbb{1} \otimes_\sigma \hat{\rho} \in \hat{L}^2(M)$.*

Moreover, given $\hat{\chi}_1 = \mathbb{1} \otimes_\sigma \hat{\rho}_1$, then $\hat{\chi}_2 = \mathbb{1} \otimes_\sigma \hat{\rho}_2$, $< \hat{\chi}_1\hat{\chi}_2 >_{\hat{L}^2} = < \hat{\rho}_1\hat{\rho}_2 >_{\mathcal{S}^2}$.

Proof. Let χ be $\chi = \phi^* \otimes_\sigma \psi$ and the basis functions $\psi_{ij} = \phi_i^* \otimes_\sigma \phi_j$. Then

$$\begin{aligned}\hat{\chi}\chi_{ij} =& \sqrt{2\pi\hbar}\chi \star_\sigma \chi_{ij} = \sqrt{2\pi\hbar}(\phi^* \otimes_\sigma \psi) \star_\sigma (\phi_i^* \otimes_\sigma \phi_j)\\ =& < \phi|\phi_j >_{L^2} (\phi_i^* \otimes_\sigma \psi) = \phi_i^* \otimes_\sigma (\hat{\rho}\phi_j) = (\mathbb{1} \otimes_\sigma \hat{\rho})\chi_{ij}\end{aligned} \tag{29}$$

What proved until now implies that to every χ_{ij} corresponds the operator $\hat{\chi_{ij}} = \mathbb{1} \otimes_\sigma \hat{\rho}_{ij} = < \phi_i|. >_{L^2} (\mathbb{1} \otimes_\sigma \phi_j)$. Then $< \hat{\chi}_{ij}|\hat{\chi}_{kl} >_{\hat{L}^2} = \delta_{ik}\delta_{jl} = < \hat{\rho}_{ij}|\hat{\rho}_{kl} >_{\mathcal{S}^2}$, which implies the equality $< \chi_1|\chi_2 >_{\hat{L}^2} = < \rho_1|\rho_2 >_{\mathcal{S}^2}$. □

Hence all the density operators associated to pure states $\psi_{pure} = \phi^* \otimes_\sigma \phi$ take the form $\hat{\rho}_{pure} = < \phi|. >_{L^2} \phi$; as the mixed states can be written as $\psi_{mixed} = \sum_r p_r\psi_r$ with ψ_r pure states, then also $\hat{\rho}_{mixed} = \sum_r p_r\hat{\rho}_r$.

Note 15. *The discussion has been developed on $= L^2(M)$, but can be extended to a larger family of Hilbert spaces, such as $L^2(M, \mu)$ where μ is a positive measure, throughout an isomorphism S which acts in the same way of the isomorphism in definition 2.8.*

3.2 Eigenvalues problem and time evolution in the star-product formalism

Now that we have studied the form that physical states and observables take in the star-product formalism, we will consider the eigenvalues problem and the time evolution.

In the above section it has been shown that functions in A_Q can be seen as operators $A_L\star_\sigma, A_R\star_\sigma \in \hat{A}_Q$ acting on $L^2(M)$ such that their domains can be defined as $\mathcal{D}_{A_L\star_\sigma} := \{f \in L^2(M) \; |A \star_\sigma f \in L^2(M)\}$, $\mathcal{D}_{A_R\star_\sigma} := \{f \in L^2(M) \; |f \star_\sigma A \in L^2(M)\}$; it will be proved in the below theorem, which is a key-result for the Quantum Mechanics description, that in fact, given a function $\chi = \phi^* \otimes \psi \in L^2(M)$, $A_L\star_\sigma$ acts on ψ and $A_R\star_\sigma$ on ϕ. So $A_L\star_\sigma$ and $A_R\star_\sigma$ can be see also as operators acting on $L^2(\mathbb{R})$; then the domains $\mathcal{D}_{A_L\star_\sigma}$ and $\mathcal{D}_{A_R\star_\sigma}$ defined as above are isomorphic to $\mathcal{D}_{A_\sigma(\hat{q},\hat{p})} := \{\phi \in L^2(\mathbb{R}) \; |A_\sigma(\hat{q},\hat{p})\phi \in L^2(\mathbb{R})\}$ and $\mathcal{D}_{A^\dagger_\sigma(\hat{q},\hat{p})} := \{\phi \in L^2(\mathbb{R}) \; |A_\sigma(\hat{q},\hat{p})\phi \in L^2(\mathbb{R})\}$, respectively.

Theorem 3.17. *Let A be a function $\in A_Q$ and $\chi \in L^2(M)$ such that $\chi = \phi^* \otimes_\sigma \psi$, with $\phi, \psi \in L^2(\mathbb{R})$.*

- *If $\psi \in \mathcal{D}_{A_\sigma(\hat{q},\hat{p})}$, then $A_L \star_\sigma \chi = \phi^* \otimes_\sigma A_\sigma(\hat{q},\hat{p})\psi$;*
- *If $\phi \in \mathcal{D}_{A^\dagger_\sigma(\hat{q},\hat{p})}$, then $A_R \star_\sigma \chi = \left(A^\dagger_\sigma(\hat{q},\hat{p})\phi\right)^* \otimes_\sigma \psi$,*

where $\hat{q} := q$ and $\hat{p} := -i\hbar\partial_q$ are the canonical operator of position and linear momentum.

Proof. Given $\chi \in L^2(M)$ and $A \in A_Q$, then we have

$$A_L \star_\sigma \chi = A_\sigma(\hat{q}_\sigma, \hat{p}_\sigma)\chi = \phi_L,$$

$$A_R \star_\sigma \chi = A_\sigma(\hat{q}^*_{\bar{\sigma}}, \hat{p}^*_{\bar{\sigma}})\chi = \phi_R.$$

χ, ϕ_L and ϕ_R can be rewritten as

$$\chi(q,p) = e^{c_L qp}\chi'_L(q,p), \qquad \phi_L(q,p) = e^{c_L qp}\phi'_L(q,p),$$

$$\chi(q,p) = e^{c_R qp}\chi'_R(q,p), \qquad \phi_R(q,p) = e^{c_R qp}\phi'_R(q,p).$$

Using the Baker-Campbell-Hausdorff formula, we obtain

$$e^{\frac{i}{\hbar}\xi\hat{q}_\sigma}e^{-\frac{i}{\hbar}\eta\hat{p}_\sigma}e^{c_L qp} = e^{c_L qp}e^{\frac{i}{\hbar}\xi\hat{Q}_\sigma}e^{-\frac{i}{\hbar}\eta\hat{P}_\sigma},$$

$$e^{\frac{i}{\hbar}\xi\hat{q}^*_{\bar{\sigma}}}e^{-\frac{i}{\hbar}\eta\hat{p}^*_{\bar{\sigma}}}e^{c_R qp} = e^{c_R qp}e^{\frac{i}{\hbar}\xi\hat{Q}^*_{\bar{\sigma}}}e^{-\frac{i}{\hbar}\eta\hat{P}^*_{\bar{\sigma}}},$$

with

$$\hat{Q}_\sigma = \hat{q}_\sigma + i\hbar\sigma c_L q = q + i\hbar\sigma c_L q + i\hbar\sigma\partial_p, \;\; \hat{P}_\sigma = \hat{p}_\sigma - i\hbar\bar{\sigma}c_L p = p - i\hbar\bar{\sigma}c_L p - i\hbar\bar{\sigma}\partial_q,$$

$$\hat{Q}^*_{\bar{\sigma}} = \hat{q}^*_{\bar{\sigma}} - i\hbar\bar{\sigma}c_R q = q - i\hbar\bar{\sigma}c_R q - i\hbar\bar{\sigma}\partial_p, \;\; \hat{P}^*_{\bar{\sigma}} = \hat{p}^*_{\bar{\sigma}} + i\hbar\sigma c_R p = p + i\hbar\sigma c_R p + i\hbar\sigma\partial_q,$$

such that $[\hat{Q}_\sigma, \hat{P}_\sigma] = i\hbar$ and $[\hat{Q}^*_{\bar{\sigma}}, \hat{P}^*_{\bar{\sigma}}] = -i\hbar$.

Then, from the first equation, we have:

$$A_\sigma(\hat{Q}_\sigma, \hat{P}_\sigma)\chi'_L = \phi'_L,$$

$$A_\sigma(\hat{Q}^*_{\bar{\sigma}}, \hat{P}^*_{\bar{\sigma}})\chi'_R = \phi'_R,$$

which get, taking $c_L = -\frac{i}{\hbar}\bar{\sigma}^{-1}$ and $c_R = \frac{i}{\hbar}\sigma^{-1}$:

$$A_\sigma(\bar{\sigma}^{-1}q + i\hbar\sigma\partial_p, -i\hbar\bar{\sigma}\partial_q)\chi'_L = \phi'_L,$$

$$A_\sigma(\sigma^{-1}q - i\hbar\bar{\sigma}\partial_p, i\hbar\sigma\partial_q)\chi'_R = \phi'_R.$$

Then, applying the anti-Fourier transform to both the sides with respect to the p-variable:

$$A_\sigma(\bar{\sigma}^{-1}q + \sigma r, -i\hbar\bar{\sigma}\partial_q)\mathcal{F}^{-1}_p\chi'_L(q,r) = \mathcal{F}^{-1}_p\phi'_L(q,r),$$

$$A_\sigma(\bar{\sigma}^{-1}q - \bar{\sigma}r, i\hbar\sigma\partial_q)\mathcal{F}^{-1}_p\chi'_R(q,r) = \mathcal{F}^{-1}_p\phi'_R(q,r).$$

Now we apply the following change of coordinates:

$$(\xi_L = \bar{\sigma}^{-1}q + \sigma r, r) \implies \partial_q = \bar{\sigma}^{-1}\partial_{\xi_L},$$

$$(\xi_R = \sigma^{-1}q - \bar{\sigma}r, r) \implies \partial_q = \sigma^{-1}\partial_{\xi_R}$$

and the equations get the simpler form:

$$A_\sigma(\xi_L, -i\hbar\partial_{\xi_L})\mathcal{F}^{-1}_p\chi'_L(\xi_L, r) = \mathcal{F}^{-1}_p\phi'_L(\xi_L, r),$$

$$A_\sigma(\xi_R, i\hbar\partial_{\xi_R})\mathcal{F}_p^{-1}\chi'_R(\xi_R, r) = \mathcal{F}_p^{-1}\phi'_R(\xi_R, r),$$

where ξ_L and ξ_R represent the operators associated to the position, while $-i\hbar\partial_{\xi_L}$ and $-i\hbar\partial_{\xi_R}$ the ones associated to the linear momenta. In these coordinates it is easy to see that, if we chose χ such that

$$F_p^{-1}\chi'_L(\xi_L, r) = \zeta_L(\xi_L)\kappa_L(r) \text{ and } \mathcal{F}_p^{-1}\chi'_R(\xi_R, r) = \zeta_R^*(\xi_R)\kappa_R(r),$$

with $\zeta_L, \zeta_R, \kappa_L, \kappa_R \in L^2(\mathbb{R})$, as it happens with physical states before applying the two isomorphisms discussed above (3.1), then

$$F_p^{-1}\phi'_L(\xi_L, r) = \psi_L(\xi_L)\kappa_L(r) \text{ and } F_p^{-1}\phi'_R(\xi_R, r) = \psi_R^*(\xi_R)\kappa_R(r),$$

for some $\psi_L, \psi_R \in L^2(\mathbb{R})$, which means that the operators act only on one function:

$$A_\sigma(\xi_L, -i\hbar\partial_{\xi_L})\zeta_L(\xi_L) = \psi_L(\xi_L),$$
$$A_\sigma^\dagger(\xi_R, -i\hbar\partial_{\xi_R})\zeta_R(\xi_R) = \psi_R(\xi_R),$$

because in $L^2(\mathbb{R})$ $(A_\sigma(\xi_R, i\hbar\partial_{\xi_R}))^* = A_\sigma^\dagger(\xi_R, -i\hbar\partial_{\xi_R})$. Now we can find χ both from $e^{c_L qp}\chi'_L$ and $e^{c_R qp}\chi'_R$:

$$\begin{aligned}
\chi =& \frac{1}{\sqrt{2\pi\hbar}}\int \zeta_L(\bar\sigma^{-1}q + \sigma r)\kappa_L(r)e^{-\frac{i}{\hbar}(r+\bar\sigma^{-1}q)p}\,dr \\
(y = r + \bar\sigma^{-1}q) =& \frac{1}{\sqrt{2\pi\hbar}}\int \zeta_L(q + \sigma y)\kappa_L(y - \bar\sigma^{-1}q)e^{-\frac{i}{\hbar}yp}\,dy \\
\\
\chi =& \frac{1}{\sqrt{2\pi\hbar}}\int \zeta_R^*(\sigma^{-1}q - \bar\sigma r)\kappa_R(r)e^{-\frac{i}{\hbar}(r-\sigma^{-1}q)p}\,dr \\
(y = r - \sigma^{-1}q) =& \frac{1}{\sqrt{2\pi\hbar}}\int \zeta_R^*(q - \bar\sigma y)\kappa_R(y + \sigma^{-1}q)e^{-\frac{i}{\hbar}yp}\,dy,
\end{aligned} \tag{30}$$

which implies that

$$\kappa_L(y - \bar\sigma^{-1}q) = \zeta_R^*(q - \bar\sigma y), \qquad \kappa_R(y + \sigma^{-1}q) = \zeta_L(q + \sigma y).$$

We explicit also $\phi_L = e^{c_L qp}\phi'_L$ and $\phi_R = e^{c_R qp}\phi'_R$ using the above relations for κ_L and κ_R:

$$\begin{aligned}
A_L \star_\sigma \chi = \phi_L =& \frac{1}{\sqrt{2\pi\hbar}}\int \psi_L(\bar\sigma^{-1}q + \sigma r)\kappa_L(r)e^{-\frac{i}{\hbar}(r+\bar\sigma^{-1}q)p}\,dr \\
=& \frac{1}{\sqrt{2\pi\hbar}}\int \psi_L(q + \sigma y)\zeta_R^*(q - \bar\sigma y)e^{-\frac{i}{\hbar}yp}\,dy \\
=& (\zeta_R^* \otimes_\sigma \psi_L)(q, p) = (\zeta_R^* \otimes_\sigma A_\sigma(\hat q, \hat p)\zeta_L)\,(q, p) \\
\\
A_R \star_\sigma \chi = \phi_R =& \frac{1}{\sqrt{2\pi\hbar}}\int \psi_R^*(\sigma^{-1}q - \bar\sigma r)\kappa_R(r)e^{-\frac{i}{\hbar}(r-\sigma^{-1}q)p}\,dr \\
=& \frac{1}{\sqrt{2\pi\hbar}}\int \psi_R^*(q - \bar\sigma y)\zeta_L(q + \sigma y)e^{-\frac{i}{\hbar}yp}\,dy \\
=& (\psi_R^* \otimes_\sigma \zeta_L)(q, p) = \left[\left(A_\sigma^\dagger(\hat q, \hat p)\zeta_R\right)^* \otimes_\sigma \zeta_L\right](q, p)
\end{aligned} \tag{31}$$

□

Corollary 3.17.1 (eigenvalues problem)**.** *Given $A \in A_Q$, every solution of the $\star_\sigma$-genvalue equation $A \star_\sigma \chi = a\chi$, with $a \in \mathbb{C}$, is of the form: $\sum_i^n \phi_i^* \otimes_\sigma \psi_i$, where the ϕ_i's are arbitrary functions $\in L^2(\mathbb{R})$, the $\psi'_i s \in L^2(\mathbb{R})$ are the a-eigenvector of the problem $A_\sigma(\hat q, \hat p)\psi_i = a\psi_i$ and n is the degeneracy of the a-eigenspace.*

Similarly, given $B \in A_Q$, every solution of the $\star_\sigma$-genvalue equation $\chi \star_\sigma B = b\chi$, with $b \in \mathbb{C}$, is of the form: $\sum_i^n \phi_i^* \otimes_\sigma \psi_i$, where the ψ_i's are arbitrary functions $\in L^2(\mathbb{R})$, the $\phi'_i s \in L^2(\mathbb{R})$ are the b^*-eigenvector of the problem $B_\sigma^\dagger(\hat q, \hat p)\psi_i = b^*\psi_i$ and n is the degeneracy of the b^*-eigenspace.

Proof. It is sufficient to prove the first part of the theorem, because the second one can be shown similarly. The eigenvalues problem in the statement is equivalent to

$$A_\sigma(\xi_L, -i\hbar\partial_{\xi_L})\mathcal{F}_p^{-1}\chi'_L(\xi_L, r) = a\mathcal{F}_p^{-1}\chi'_L(\xi_L, r) = a\sum_i^n \kappa_{L,i}(r)\psi_i(\xi_L),$$

where (ξ_L, r) are the ones defined in the proof of theorem 3.17, $\{\psi_i\}_{i\in I}$ is a basis for the a-eigenspace and the $\kappa'_{L,i}$s $\in L^2(\mathbb{R})$ as $\mathcal{F}_p^{-1}\chi'_L \in L^2(\mathbb{R}^2)$; through the same manipulations used in theorem 3.17 it results that $\kappa'_{L,i}(y - \bar\sigma q) = \phi_i^*(q - \bar\sigma y)$. □

Corollary 3.17.2. *Given* $\chi_1 = \phi_1^* \otimes_\sigma \psi_1$ *and* $\chi_2 = \phi_2^* \otimes_\sigma \psi_2$*, with* $\psi_1, \psi_2, \phi_1, \phi_2 \in L^2(\mathbb{R})$ *and a function* $A \in A_Q$*, then*

$$< \chi_1 | A_L \star_\sigma \chi_2 >_{L^2} = < \phi_2 | \phi_1 >_{L^2} < \psi_1 | A_\sigma(\hat{q}, \hat{p}) \psi_2 >_{L^2}$$

$$< \chi_1 | A_R \star_\sigma \chi_2 >_{L^2} = < A_\sigma^\dagger(\hat{q}, \hat{p}) \phi_2 | \phi_1 >_{L^2} < \psi_1 | \psi_2 >_{L^2} = < \phi_2 | A_\sigma(\hat{q}, \hat{p}) \phi_1 >_{L^2} < \psi_1 | \psi_2 >_{L^2} .$$

Proof. It is an obvious consequence of the theorem 3.17. □

Corollary 3.17.3. *Given* $\chi = \phi^* \otimes_\sigma \psi \in L^2(M)$ *and* $A \in A_Q$*, then*

$$(A_L \star_\sigma)^\dagger \chi = \phi^* \otimes_\sigma A_\sigma^\dagger(\hat{q}, \hat{p}) \psi$$

$$(A_R \star_\sigma)^\dagger \chi = (A_\sigma(\hat{q}, \hat{p}) \phi)^* \otimes_\sigma \psi$$

Proof. It is an immediate consequence of corollary 3.17.2 and of proposition 3.7. □

Corollary 3.17.4. *Given* $A \in A_Q$ *and a pure state* $\psi_{pure} = \phi^* \otimes_\sigma \phi$*, with* $\phi \in L^2(\mathbb{R})$ *such that* $A_\sigma(\hat{q}, \hat{p}) \phi = a\phi$*, then*

$$A_L \star_\sigma \psi_{pure} = a \psi_{pure}, \qquad (A_R \star_\sigma)^\dagger \psi_{pure} = a^* \psi_{pure}.$$

In particular, if A *is an observable,*

$$(A_L \star_\sigma)^\dagger \psi_{pure} = A_L \star_\sigma \psi_{pure} = a \psi_{pure}, \qquad (A_R \star_\sigma)^\dagger \psi_{pure} = A_R \star_\sigma \psi_{pure} = a \psi_{pure},$$

with $a \in \mathbb{R}$*.*

Proof. It is an immediate consequence of corollary 3.17.3 and of definition of observables. □

Another important consequence of theorem 3.17 is that an operator $A_\sigma(\hat{q}_\sigma, \hat{p}_\sigma)$ can be written as $A_\sigma(\hat{q}_\sigma, \hat{p}_\sigma) = \mathbb{1} \otimes_\sigma A_\sigma(\hat{q}, \hat{p})$. From theorem 3.17 and 3.16 follows also that observables A with respect to $\chi = \phi^* \otimes_\sigma \psi \in L^2(M)$ can be seen both as:

- operators of the form $A_\sigma(\hat{q}_\sigma, \hat{p}_\sigma)$ acting on $L^2(M)$-function operators of the form $\hat{\chi} = \sqrt{2\pi\hbar}\chi \star_\sigma$;
- operators of the form $A_\sigma(\hat{q}, \hat{p})$ acting on density operators defined as $< \phi | . >_{L^2} \psi$. In this case the composition between the two operators is defined as:

$$A_\sigma(\hat{q}, \hat{p}) \hat{\rho} = < \phi | . > [A_\sigma(\hat{q}, \hat{p}) \psi] \qquad \hat{\rho} A_\sigma(\hat{q}, \hat{p}) = < A_\sigma^\dagger(\hat{q}, \hat{p}) \phi | . > \psi.$$

The equivalence between the two formulations is given by:

$$A_\sigma(\hat{q}_\sigma, \hat{p}_\sigma) \hat{\psi} = \mathbb{1} \otimes_\sigma A_\sigma(\hat{q}, \hat{p}) \hat{\rho} \qquad \hat{\psi} A_\sigma(\hat{q}_\sigma, \hat{p}_\sigma) = \mathbb{1} \otimes_\sigma \hat{\rho} A_\sigma(\hat{q}, \hat{p}).$$

Definition 3.14. *Let* $\hat{A} \in \hat{A}_Q$ *be an observable and let* $\chi \in L^2(M)$ *a state with its quantum distribution function* $\rho := \frac{1}{\sqrt{2\pi\hbar}} \chi$*. The mean value of the observable* $\hat{A}$ *with respect to the state* χ *is given by*

$$< \hat{A} >_\chi := \iint (A \star_\sigma \rho)(q, p) \, dq \, dp.$$

Note 16. *Note that, given* $f, g \in C^\infty(M)$*, from theorem 13 it follows the equality*

$$\iint (f \star_\sigma g)(q, p) \, dq \, dp = \iint (g \star_\sigma f)(q, p) \, dq \, dp;$$

then the above definition can be given also as $< A >_\chi := \iint (\rho \star_\sigma A)(q, p) \, dq \, dp$*.*

Theorem 3.18. *Let* $\hat{A} \in \hat{A}_Q$ *be an observable and let* $\chi = \sum_r p_r \psi_r = \sum_r p_r \phi_r^* \otimes_\sigma \phi_r$*, with the* $\phi_r' s \in L^2(\mathbb{R})$*, be a mixed state with its corresponding density operator* $\hat{\rho} = \sum_r p_r < \phi_r | . >_{L^2} \phi_r$*. Then*

$$< \hat{A} >_\chi = \mathrm{tr}(\hat{\rho} A_\sigma(\hat{q}, \hat{p})).$$

Proof.

$$
\begin{aligned}
<\hat{A}>_{\chi} &= \frac{1}{\sqrt{2\pi\hbar}} \sum_k p_k \iint (A \star_\sigma \psi_k)(q,p)\, dq\, dp \\
\text{(theorem 3.17)} &= \frac{1}{\sqrt{2\pi\hbar}} \sum_k p_k \iint \left[\phi_k^* \otimes_\sigma (A_\sigma(\hat{q},\hat{p})\phi_k)\right](q,p)\, dq\, dp \\
&= \frac{1}{2\pi\hbar} \sum_k p_k \iiint \phi_k^*(q-\bar{\sigma}r)(A_\sigma(\hat{q},\hat{p})\phi_k)(q+\sigma r) e^{-\frac{i}{\hbar}pr}\, dr\, dq\, dp \\
\text{(using } \delta \text{ properties)} &= \sum_k p_k \iint \phi_k^*(q-\bar{\sigma}r)(A_\sigma(\hat{q},\hat{p})\phi_k)(q+\sigma r)\delta(r)\, dr\, dq \\
&= \sum_k p_k \int \phi_k^*(q)(A_\sigma(\hat{q},\hat{p})\phi_k)(q)\, dq \\
&= \sum_k p_k < \phi_k | A_\sigma(\hat{q},\hat{p})\phi_k >_{L^2} = \mathrm{tr}(\hat{\rho} A_\sigma(\hat{q},\hat{p}))
\end{aligned}
\tag{32}
$$

□

Corollary 3.18.1. *Given the same hypothesis theorem 3.18,*

$$
\begin{aligned}
<\hat{A}>_{\chi} &= \sum_r p_r < \psi_r | A_L \star_\sigma \psi_r >_{\hat{L}^2} \\
&= \sum_r p_r < \psi_r | A_R \star_\sigma \psi_r >_{\hat{L}^2}
\end{aligned}
\tag{33}
$$

Proof.

$$
\begin{aligned}
<\hat{A}>_{\chi} &= \sum_k p_k < \phi_k | A_\sigma(\hat{q},\hat{p})\phi_k >_{L^2} \\
(\phi_k\text{'s are normalized}) &= \sum_k p_k < \phi_k | \phi_k >_{L^2} < \phi_k | A_\sigma(\hat{q},\hat{p})\phi_k >_{L^2} \\
\text{(by definition)} &= \sum_r p_r < \psi_r | A_L \star_\sigma \psi_r >_{\hat{L}^2}
\end{aligned}
\tag{34}
$$

and similarly,

$$
\begin{aligned}
<\hat{A}>_{\chi} &= \sum_k p_k < \phi_k | A_\sigma(\hat{q},\hat{p})\phi_k >_{L^2} \\
&= \sum_k p_k < \phi_k A_\sigma^\dagger(\hat{q},\hat{p}) | \phi_k >_{L^2} < \phi_k | \phi_k >_{L^2} \\
&= \sum_r p_r < \psi_r | A_R \star_\sigma \psi_r >_{\hat{L}^2}
\end{aligned}
\tag{35}
$$

□

Note 17. *As observables correspond to operators which are in particular hermitian, the mean value is always a real number.*

It is now possible to derive the following importan physical result:

Theorem 3.19. *Let $A \in A_Q$ be an observable and let $\psi_{pure} \in L^2(M)$ be a pure state. Then the commutator $[A, \chi] = A \star_\sigma \psi_{pure} - \psi_{pure} \star_\sigma A = 0$ if and only if $A \star_\sigma \psi_{pure} = \psi_{pure} \star_\sigma A = a\psi_{pure}$, for some $a \in \mathbb{R}$.*

Proof. Obviously if $A \star_\sigma \psi_{pure} = \psi_{pure} \star_\sigma A = a\psi_{pure}$, then $[A, \psi_{pure}] = a\psi_{pure} - a\psi_{pure} = 0$.

Conversely, if $[A, \psi_{pure}] = 0$, then

$$A \star_\sigma \psi_{pure} \star_\sigma \psi_{pure} = \psi_{pure} \star_\sigma A \star_\sigma \psi_{pure}.$$

The first side is equal to

$$A \star_\sigma \left(\frac{1}{\sqrt{2\pi\hbar}}\psi_{pure}\right) = \frac{1}{\sqrt{2\pi\hbar}} A \star_\sigma \psi_{pure} = \frac{1}{\sqrt{2\pi\hbar}} \phi^* \otimes_\sigma A_\sigma(\hat{q},\hat{p})\phi.$$

Then

$$
\begin{aligned}
A \star_\sigma \psi_{pure} =& \frac{1}{\sqrt{2\pi\hbar}} \psi_{pure} \star_\sigma A \star_\sigma \psi_{pure} \\
=& \frac{1}{\sqrt{2\pi\hbar}} (\phi^* \otimes_\sigma \phi) \star_\sigma [\phi^* \otimes_\sigma (A_\sigma(\hat{q},\hat{p})\phi)] \\
\text{(from theorem 3.10.1)} =& \frac{1}{2\pi\hbar} < \phi | A_\sigma(\hat{q},\hat{p})\phi >_{L^2} (\phi^* \otimes_\sigma \phi) \\
=& a\psi.
\end{aligned}
\tag{36}
$$

□

In Quantum Mechanics, time evolution is governed by an Hamiltonian H: given the state $\chi \in L^2(M)$, the quantum counterpart of the *Liouville's theorem* takes the form:

$$
i\hbar \frac{\partial \chi}{\partial t} - [H, \chi] = 0.
$$

Moreover H requires to be an observable and physically can be seen as the energy associated to the system. The equation can be expressed also using the *time evolution function* $U(t)$:

$$
\begin{gathered}
i\hbar \frac{\partial [U(t) \star_\sigma \chi(0) \star_\sigma (U(t))^{-1}]}{\partial t} = H \star_\sigma [U(t) \star_\sigma \chi(0) \star_\sigma (U(t))^{-1}] - [U(t) \star_\sigma \chi(0) \star_\sigma (U(t))^{-1}] \star_\sigma H \\
\Longrightarrow \\
\left(i\hbar \frac{\partial U(t)}{\partial t} \right) \star_\sigma [\chi(0) \star_\sigma (U(t))^{-1}] + [U(t) \star_\sigma \chi(0)] \star_\sigma \left(\frac{\partial (U(t))^{-1}}{\partial t} \right) = \\
= (H \star_\sigma U(t))[\star_\sigma \chi(0) \star_\sigma (U(t))^{-1}] - [U(t) \star_\sigma \chi(0)] \star_\sigma ((U(t))^{-1} \star_\sigma H)
\end{gathered}
$$

The relation must be true $\forall \chi(0) \in L^2(M)$, which implies:

$$
\begin{cases}
i\hbar \frac{d(U(t)}{dt} = H_L \star_\sigma U(t) \\
[H, U(t)] = 0
\end{cases}
$$

The differential equation is solved by

$$
U(t) = e_{\star_\sigma}^{-\frac{i}{\hbar}Ht} := \sum_{K=0}^{\infty} \frac{1}{k!} \left(-\frac{i}{\hbar} t \right)^k \underbrace{H \star_\sigma \ldots \star_\sigma H}_{k \text{ times}},
$$

which also commutes with H; the functions $U(t)$ form a one parameter (t) group; the inverse $U(t)^{-1}$ is given by $U(-t) = U(t)^\dagger$, which means that $U(t)$ is a unitary function.

Definition 3.15. *A state χ is called* stationary *if*

$$
\frac{\partial \chi}{\partial t} = -\frac{i}{\hbar}[H, \chi] = 0
$$

Theorem 3.20. *Ae pure state $\psi_{pure} \in L^2(M)$ is stationary if and only if ψ_{pure} is an eigenvector of the Hamiltonian H, which is equivalent to say that $H_L \star_\sigma \psi_{pure} = H_R \star_\sigma \psi_{pure} = E\psi_{pure}$ for some $E \in \mathbb{R}$, corresponding to the energy of the system in the state ψ_{pure}.*

Proof. ψ_{pure} is stationary if and only if $[H, \psi_{pure}] = 0$, if and only if (theorem 3.19) $H_L \star_\sigma \psi_{pure} = H_R \star_\sigma \psi_{pure} = E\psi_{pure}$ for some $E \in \mathbb{R}$ □

Working with density operators $\hat{\rho} \in \mathcal{S}^2(L^2(\mathbb{R}))$, there holds the *Neumann equation*

$$
i\hbar \frac{\partial \hat{\rho}}{\partial t} - [H_\sigma(\hat{q},\hat{p}), \hat{\rho}] = 0.
$$

Stationary states are the ones such that $i\hbar \frac{\partial \hat{\rho}}{\partial t} = [H_\sigma(\hat{q},\hat{p}), \hat{\rho}] = 0$. If the state is pure, then $\hat{\rho} = < \phi | . > \phi$ for some $\phi \in L^2(\mathbb{R})$ and the Neumann equation equation takes the form:

$$
\begin{aligned}
i\hbar \frac{\partial \hat{\rho}}{\partial t} =& [H_\sigma(\hat{q},\hat{p}), \hat{\rho}] \\
< -i\hbar \frac{\partial \phi}{\partial t} | . > \phi + < \phi | . > i\hbar \frac{\partial \phi}{\partial t} =& < -H_\sigma(\hat{q},\hat{p})\phi | . > \phi + < \phi | . > H_\sigma(\hat{q},\hat{p})\phi,
\end{aligned}
\tag{37}
$$

which is equivalent to the Schrödinger equation

$$i\hbar\frac{\partial\phi}{\partial t} = H_\sigma(\hat{q},\hat{p})\phi$$

and the state is stationary if and only if $H_\sigma(\hat{q},\hat{p})\phi = E\phi$. The time evolution of density operators is described by unitary operators defined as

$$U_\sigma(\hat{q},\hat{p},t) = e^{-\frac{i}{\hbar}H_\sigma(\hat{q},\hat{p})t},$$

hence $\hat{\rho}(t)$ takes the form:

$$\hat{\rho}(t) = U_\sigma(\hat{q},\hat{p},t)\hat{\rho}(0)U_\sigma(\hat{q},\hat{p},-t);$$

in fact

$$\begin{aligned}\frac{\partial\hat{\rho}}{\partial t} &= \left(-\frac{i}{\hbar}H_\sigma(\hat{q},\hat{p})\right)U_\sigma(\hat{q},\hat{p},t)\hat{\rho}(0)U_\sigma(\hat{q},\hat{p},-t)+\\ &\quad + U_\sigma(\hat{q},\hat{p},t)\hat{\rho}(0)\left(\frac{i}{\hbar}H_\sigma(\hat{q},\hat{p})\right)U_\sigma(\hat{q},\hat{p},-t)\\ ([U_\sigma(\hat{q},\hat{p},-t),H_\sigma(\hat{q},\hat{p})]=0) &= -\frac{i}{\hbar}\left[H_\sigma(\hat{q},\hat{p})\hat{\rho}(t) - \rho H_\sigma(\hat{q},\hat{p})\right] \qquad \square\end{aligned}$$

Given the mean value $< A >_{\chi(0)}$ of an observable $\hat{A} \in \hat{A}_Q$, we want to find its time evolution;

$$\begin{aligned}\frac{i\hbar}{\sqrt{2\pi\hbar}}\iint\left(A\star_\sigma\frac{\partial\chi(t)}{\partial t}\right)(q,p)\,dq\,dp =& i\hbar\left[\frac{d}{dt}\iint(A\star_\sigma\chi(t))(q,p)\,dq\,dp - \iint\left(\frac{\partial A}{\partial t}\star_\sigma\chi(t)\right)(q,p)\,dq\,dp\right]\\ =& i\hbar\left[\frac{d < \hat{A} >_{\chi(t)}}{dt} - < \frac{\partial\hat{A}}{\partial t} >_{\chi(t)}\right],\end{aligned} \tag{38}$$

where $\frac{\partial\hat{A}}{\partial t} \neq 0$ if and only if A explicitly depends on time t. On the other side, using the Liouville equation together with note 16, we obtain:

$$\begin{aligned}\frac{i\hbar}{\sqrt{2\pi\hbar}}\iint\left(A\star_\sigma\frac{\partial\chi(t)}{\partial t}\right)(q,p)\,dq\,dp =& \frac{1}{\sqrt{2\pi\hbar}}\iint\left[A\star_\sigma(H\star_\sigma\chi(t)-\chi(t)\star_\sigma H)](q,p)\right]dq\,dp\\ =& \frac{1}{\sqrt{2\pi\hbar}}\iint\left\{(A\star_\sigma H\star_\sigma\chi(t))(q,p) - [(A\star_\sigma\chi(t))\star_\sigma H](q,p)\right\}dq\,dp\\ =& \frac{1}{\sqrt{2\pi\hbar}}\iint\left\{(A\star_\sigma H\star_\sigma\chi(t))(q,p) - [H\star_\sigma(A\star_\sigma\chi(t))](q,p)\right\}dq\,dp\\ =& \frac{1}{\sqrt{2\pi\hbar}}\iint\left[(A\star_\sigma H - H\star_\sigma A)\star_\sigma\chi(t)](q,p)\right]dq\,dp\\ =& < [\hat{A},\hat{H}] >_{\chi(t)}\end{aligned} \tag{39}$$

Then

$$i\hbar\frac{d < \hat{A} >_{\chi(t)}}{dt} = i\hbar < \frac{\partial\hat{A}}{\partial t} >_{\chi(t)} + < [\hat{A},\hat{H}] >_{\chi(t)}\,. \tag{40}$$

In the above description states depend on time, while observable don't, a part from some explicit dependence; such choise is called the *Schrödinger picture*. There is an equivalent picture, the *Heisenberg* one, in which states do not depend on time, while observables do. Then equation 40 takes the form:

$$i\hbar\frac{d < \hat{A}(t) >_{\chi(0)}}{dt} = i\hbar < \frac{\partial\hat{A}(t)}{\partial t} >_{\chi(0)} + < [\hat{A}(t),\hat{H}] >_{\chi(0)}, \tag{41}$$

from which follows that

$$i\hbar\frac{d\hat{A}(t)}{dt} = i\hbar\frac{\partial\hat{A}(t)}{\partial t} + [\hat{A}(t),\hat{H}]. \tag{42}$$

Again $\frac{\partial\hat{A}(t)}{\partial t} \neq 0$ if and only if $A(t)$ explicitly depends on time t. If we assume that $\hat{A}(t)$ has no explicit dependence on time t, then the differential equation $i\hbar\frac{d\hat{A}(t)}{dt} = [\hat{A}(t),\hat{H}]$ is solved by some $\hat{A}(t)$ of the form

$\hat{A}(t) = [U(-t) \star_\sigma A \star_\sigma U(t)]_L \star_\sigma$ or $\hat{A}(t) = [U(-t) \star_\sigma A \star_\sigma U(t)]_R \star_\sigma$, where $U(t) = e_{\star_\sigma}^{-\frac{i}{\hbar}Ht}$. Indeed

$$\begin{aligned}
< A(0) >_{\chi(t)} &= \iint \{A(0) \star_\sigma [U(t) \star_\sigma \chi(0) \star_\sigma U(-t)]\}(q,p)\, dq\, dp \\
&= \iint \{[A(0) \star_\sigma U(t) \star_\sigma \chi(0)] \star_\sigma U(-t)\}(q,p)\, dq\, dp \\
&= \iint \{[U(-t) \star_\sigma A(0) \star_\sigma U(t)] \star_\sigma \chi(0)\}(q,p)\, dq\, dp \\
&= < A(t) >_{\chi(0)} .
\end{aligned} \tag{43}$$

Similarly, operators of the type $A_\sigma(\hat{q}, \hat{p}, t)$ undergo a similar equation:

$$i\hbar \frac{dA_\sigma(\hat{q},\hat{p},t)}{dt} = i\hbar \frac{\partial A_\sigma(\hat{q},\hat{p},t)}{\partial t} + [A_\sigma(\hat{q},\hat{p},t), H_\sigma(\hat{q},\hat{p})], \tag{44}$$

which is solved by

$$A_\sigma(\hat{q},\hat{p},t) = U(\hat{q},\hat{p},-t) A_\sigma(\hat{q},\hat{p}) U(\hat{q},\hat{p},t),$$

with $U(\hat{q},\hat{p},t) = e^{-\frac{i}{\hbar}H_\sigma(\hat{q},\hat{p})t}$.

3.3 Physical example: free particle and simple harmonic oscillator

Once that the star-product formalism has been studied, it is possible to apply it on some physical examples. In particular this thesis will treat a free particle and a simple harmonic oscillator.

The Hamiltonian of a free particle is $H = \frac{\hat{p}^2}{2m}$, where m is the particle mass and can be taken as unitary ($m = 1$). Given a pure state $\chi = \phi^* \otimes_\sigma \phi$, the Liouville equation is: $i\hbar \dfrac{\partial \chi}{\partial t} = [H, \chi]$, which is equivalent to the Schrödinger equation: $i\hbar \dfrac{\partial \phi}{\partial t} = \dfrac{\partial^2 \phi}{\partial q^2}$. A solution can be of the form of a plain wave:

$$e^{-\frac{i}{\hbar}\frac{p^2}{2}t} e^{\frac{i}{\hbar}pq},$$

which is not in $L^2(\mathbb{R})$; then the physical solution will be a linear combination of wave functions:

$$\phi(q,t) = \frac{1}{\sqrt{2\pi\hbar}} \int f(p) e^{-\frac{i}{\hbar}\frac{p^2}{2}t} e^{\frac{i}{\hbar}pq}\, dp = \frac{1}{\sqrt{2\pi\hbar}} \int g(p,t) e^{\frac{i}{\hbar}pq}\, dp, \quad f \in L^2(\mathbb{R}).$$

Then the state χ takes the form

$$\chi(q,p,t) = (\phi^* \otimes_\sigma \phi)(q,p,t) = \frac{1}{(2\pi\hbar)^{3/2}} \int e^{-\frac{i}{\hbar}pr} \left(\int g(p',t) e^{\frac{i}{\hbar}p'(q-\bar{\sigma}r)}\, dp' \right)^* \left(\int g(p',t) e^{\frac{i}{\hbar}p'(q+\sigma r)}\, dp' \right) dr.$$

Taking the function $f(p)$ of a Gaussian-like form:

$$f(p) = \frac{e^{-\frac{(p-p_0)^2}{4(\Delta p)^2}}}{(2\pi)^{1/4}(\Delta p)^{1/2}},$$

$\phi(q,t)$ gets

$$\begin{aligned}
\phi(q,t) &= \frac{1}{\sqrt{2\pi\hbar}} \int \frac{e^{-\frac{(p-p_0)^2}{4(\Delta p)^2}}}{(2\pi)^{1/4}(\Delta p)^{1/2}} e^{-\frac{i}{\hbar}\frac{p^2}{2}t} e^{\frac{i}{\hbar}pq}\, dp \\
(\Delta q \Delta p = \frac{\hbar}{2}) &= \frac{e^{-\frac{p^2}{4(\Delta p)^2}} e^{-\frac{(q - i\frac{\Delta q}{\Delta p}p_0)^2}{4\Delta q(\Delta q + i\Delta p t)}}}{\sqrt{2\pi\hbar}(2\pi)^{1/4}(\Delta p)^{1/2}} \int e^{\left[\sqrt{\frac{1}{4\Delta p^2} + \frac{it}{2\hbar}}p - \frac{q - i\frac{\Delta q}{\Delta p}p_0}{2\sqrt{\Delta q}\sqrt{\Delta q + i\Delta p t}}\right]^2} dp \\
y = \sqrt{\frac{1}{4\Delta p^2} + \frac{it}{2\hbar}}p &- \frac{q - i\frac{\Delta q}{\Delta p}p_0}{2\sqrt{\Delta q}\sqrt{\Delta q + i\Delta p t}}, \quad \text{then } dp = \frac{2\Delta p\sqrt{\Delta q}}{\sqrt{\Delta q + i\Delta p t}} \implies \\
&= \frac{e^{-\frac{p_0^2}{4(\Delta p)^2}}}{(2\pi)^{1/4}\sqrt{\Delta q + i\Delta p t}} \exp\left(\frac{(q - i\frac{\Delta x}{\Delta p}p_0)^2}{4(\Delta q)^2 + 4i\Delta q \Delta p t} \right).
\end{aligned} \tag{45}$$

Hence, choosing $\sigma = \frac{1}{2}$, $\chi(q,p,t)$ results to be:

$$
\begin{aligned}
\chi(q,p,t) =& \frac{e^{-\frac{p_0^2}{2\Delta p^2}}}{2\pi\sqrt{\hbar(\Delta q^2+\Delta p^2t^2)}} \int e^{-\frac{i}{\hbar}pr} e^{\frac{\left[(q-\frac{1}{2}r)+i\frac{\Delta q}{\Delta p}p_0\right]^2(\Delta q+i\Delta pt)+\left[(q+\frac{1}{2}r)-i\frac{\Delta q}{\Delta p}p_0\right]^2(\Delta q-i\Delta pt)}{4\Delta q(\Delta q^2+\Delta p^2t^2)}}\,dr \\
=& \frac{1}{2\pi\sqrt{\hbar(\Delta q^2+\Delta p^2t^2)}} \int e^{-\frac{\frac{\Delta q}{2}r^2-2i\left(\Delta p\cdot qt+\frac{\Delta q^2}{\Delta p}p_0-\frac{\Delta q^2}{\Delta p}p-\Delta p\cdot pt^2\right)r+\Delta x\cdot 2q^2+2\Delta q\cdot p_0t^2-4\Delta q\cdot p_0qt}{4\Delta q(\Delta q^2+\Delta p^2t^2)}}\,dr \\
=& \frac{e^{-\frac{1}{2}\left(\frac{p_0-p}{\Delta p}\right)^2-\frac{1}{2}\left(\frac{q-pt}{\Delta q}\right)^2}}{2\pi\sqrt{\hbar(\Delta q^2+\Delta p^2t^2)}} \int e^{-\frac{\left[\sqrt{\frac{\Delta q}{2}}r-\frac{\sqrt{2}i\left(\Delta p\cdot qt+\frac{\Delta q^2}{\Delta p}p_0-\frac{\Delta q^2}{\Delta p}p-\Delta p\cdot pt^2\right)}{\sqrt{\Delta q}}\right]^2}{4\Delta q(\Delta q^2+\Delta p^2t^2)}}\,dr \\
y =& \frac{\sqrt{\frac{\Delta q}{2}}r-\frac{\sqrt{2}i\left(\Delta p\cdot qt+\frac{\Delta q^2}{\Delta p}p_0-\frac{\Delta q^2}{\Delta p}p-\Delta p\cdot pt^2\right)}{\sqrt{\Delta q}}}{2\sqrt{\Delta q(\Delta^2+\Delta p^2t^2)}},\ \text{then } dr=\sqrt{8(\Delta^2+\Delta p^2t^2)} \implies \\
=& \frac{e^{-\frac{1}{2}\left(\frac{p_0-p}{\Delta p}\right)^2-\frac{1}{2}\left(\frac{q-pt}{\Delta q}\right)^2}}{\sqrt{\pi\Delta q\Delta p}}.
\end{aligned}
\tag{46}
$$

Then we can calculate the expectation values as:

$$
\begin{aligned}
< q >_{\chi(t)} =& \frac{1}{\sqrt{2\pi\hbar}}\iint q \star_{1/2} \chi(q,p,t)\,dq\,dp \qquad < p >_{\chi(t)} = \frac{1}{\sqrt{2\pi\hbar}}\iint p \star_{1/2} \chi(q,p,t)\,dq\,dp \\
< q^2 >_{\chi(t)} =& \frac{1}{\sqrt{2\pi\hbar}}\iint q^2 \star_{1/2} \chi(q,p,t)\,dq\,dp \qquad < p^2 >_{\chi(t)} = \frac{1}{\sqrt{2\pi\hbar}}\iint p^2 \star_{1/2} \chi(q,p,t)\,dq\,dp \\
< \Delta q >_{\chi(t)} =& \sqrt{< q^2 >_{\chi(t)} - < q >_{\chi(t)}^2} \qquad < \Delta p >_{\chi(t)} = \sqrt{< p^2 >_{\chi(t)} - < p >_{\chi(t)}^2}
\end{aligned}
\tag{47}
$$

It will be explicitly calculated only the first mean value just to give an example:

$$
\begin{aligned}
< q >_{\chi(t)} =& \frac{1}{\sqrt{2\pi\hbar}}\iint q \star_{1/2} \chi(q,p,t)\,dq\,dp \\
=& \frac{1}{\sqrt{2\pi\hbar}}\iint \left[\frac{1}{2\pi\hbar}\iint \mathcal{F}q(\xi,\eta)\chi(q-\frac{1}{2}\eta, p+\frac{1}{2}\xi, t)e^{\frac{i}{\hbar}(\xi q-\eta p)}\,d\xi\,d\eta\right]dq\,dp \\
=& \frac{1}{\sqrt{2\pi\hbar}}\iint \left[\frac{1}{2\pi\hbar}\iint 2\pi\hbar\cdot i\hbar\delta'(\xi)\delta(\eta)\chi(q-\frac{1}{2}\eta, p+\frac{1}{2}\xi, t)e^{\frac{i}{\hbar}(\xi q-\eta p)}\,d\xi\,d\eta\right]dq\,dp \\
=& \frac{1}{\sqrt{2\pi\hbar}}\iint \left[-i\hbar\iint \delta(\xi)\delta(\eta)\frac{\partial}{\partial\xi}\left(\chi(q-\frac{1}{2}\eta, p+\frac{1}{2}\xi, t)e^{\frac{i}{\hbar}(\xi q-\eta p)}\right)d\xi\,d\eta\right]dq\,dp \\
=& \frac{-i\hbar}{\sqrt{2\pi\hbar}}\iint \frac{\partial}{\partial\xi}\left(\chi(q-\frac{1}{2}\eta, p+\frac{1}{2}\xi, t)e^{\frac{i}{\hbar}(\xi q-\eta p)}\right)\Big|_{\xi=0,\eta=0}dq\,dp \\
=& \frac{-i\hbar}{\sqrt{2\pi\hbar}}\iint \left[\frac{iq}{\hbar}-\frac{t}{2\Delta q}\left(\frac{q-pt}{\Delta q}\right)-\frac{1}{2\Delta p}\left(\frac{p_0-p}{\Delta p}\right)\right]\frac{e^{-\frac{1}{2}\left(\frac{p_0-p}{\Delta p}\right)^2-\frac{1}{2}\left(\frac{q-pt}{\Delta q}\right)^2}}{\sqrt{\pi\Delta q\Delta p}}\,dq\,dp \\
=& \frac{1}{\sqrt{2\pi\Delta p^2}}\int e^{-\frac{1}{2}\left(\frac{p_0-p}{\Delta p}\right)^2}\left[\frac{1}{\sqrt{2\pi\Delta q^2}}\int qe^{-\frac{1}{2}\left(\frac{q-pt}{\Delta q}\right)^2}dq\right]dp - \frac{i\hbar}{\sqrt{2\pi\hbar}}\iint -\frac{1}{2}\frac{\partial}{\partial p}\chi(q,p,t)\,dp\,dq \\
=& \frac{t}{\sqrt{2\pi\Delta p^2}}\int pe^{-\frac{1}{2}\left(\frac{p_0-p}{\Delta p}\right)^2}dp - \frac{i\hbar}{\sqrt{2\pi\hbar}}\int 0\,dq = p_0t.
\end{aligned}
\tag{48}
$$

Similarly we obtain:

$$
< q >_{\chi(t)} = p_0t \quad < q^2 >_{\chi(t)} = \Delta q^2+\Delta p^2t^2+p_0^2t^2 \quad < \Delta q >_{\chi(t)} = \sqrt{\Delta q^2+\Delta p^2t^2}
$$

$$
< p >_{\chi(t)} = p_0 \quad < p^2 >_{\chi(t)} = \Delta p^2+p_0^2 \quad < \Delta p >_{\chi(t)} = \Delta p
$$

It is interesting to note that $< \Delta q >_{\chi(t)} \cdot < \Delta p >_{\chi(t)} \geq \Delta q\cdot\Delta p = \frac{\hbar}{2}$ $\forall t\in\mathbb{R}$ and the equality is true if and only if $t=0$.

The Hamiltonian of the harmonic oscillator is $H = \frac{p^2}{2m}+\frac{1}{2}\omega mq^2$, where the mass m can be taken again as $m=1$ to simplify calculations. Then the Hamiltonian reads $H=\frac{1}{2}(p^2+\omega^2q^2)$ and the eigenvalues problem takes the form $H\star_\sigma\chi = E_L\chi$, $\chi\star_\sigma H = E_R\chi$. Like in the common formalism, it is useful to introduce the *destruction* and *construction functions*:

$$
a(q,p) = \frac{\omega q+ip}{\sqrt{2\hbar\omega}}, \qquad \bar{a} = \frac{\omega q-ip}{\sqrt{2\hbar\omega}},
$$

such that $a\star_\sigma = (\bar{a}\star_\sigma)^\dagger$, $\bar{a}\star_\sigma = (a\star_\sigma)^\dagger$ and $[a,\bar{a}] = 1$. Using these functions as coordinates, the Hamiltonian takes the form: $H = \hbar\omega(\bar{a}\star_\sigma a + \frac{1}{2}) =: \hbar\omega(N + \frac{1}{2})$. The function $N := \bar{a}\star_\sigma a$ is hermitian as $N^\dagger = (\bar{a}\star_\sigma a)^\dagger = a^\dagger \star_\sigma \bar{a}^\dagger = \bar{a}\star_\sigma a = N$, hence it is an observable; then the eigenvalues problem gets: $N \star_\sigma \chi_{nm} = n\chi_{nm}$, $\chi_{nm}\star_\sigma N = m\chi_{nm}$, which brings to:

$$H \star_\sigma \chi_{nm} = E_n\chi_{nm} := \hbar\omega(n + \frac{1}{2})\chi_{nm}, \qquad \chi_{nm}\star_\sigma H = E_m\chi_{nm} =: \hbar\omega(m + \frac{1}{2})\chi_{nm}.$$

It is possible to show that the eigenvalues of N are integer and non negative and the operators associated to a and $\bar{a}$ act in the following way:

$$a \star_\sigma \chi_{nm} = \sqrt{n}\chi_{(n-1)m}, \qquad \bar{a}\star_\sigma \chi_{nm} = \sqrt{n+1}\chi_{(n+1)m},$$
$$\chi_{nm}\star_\sigma a = \sqrt{m+1}\chi_{n(m+1)}, \qquad \chi_{nm}\star_\sigma \bar{a} = \sqrt{m-1}\chi_{n(m-1)}.$$

Hence

$$\chi_{nm} = \frac{1}{\sqrt{n!m!}} \underbrace{\bar{a}\star_\sigma \dots \star_\sigma \bar{a}}_{n \text{ times}} \star_\sigma \chi_{00} \star_\sigma \underbrace{a \star_\sigma \dots \star_\sigma a}_{m \text{ times}},$$

which, for $\sigma = \frac{1}{2}$, becomes:

$$\chi_{nm}(a,\bar{a}) = \frac{1}{\sqrt{n!m!}} \sum_{k=0}^{max\{m,n\}} (-1)^k k! \binom{n}{k}\binom{m}{k} (2)^{n+m} \bar{a}^{m-k} a^{n-k} \chi_{00}(a,\bar{a}).$$

To find $\chi_{00}(a,\bar{a})$ we need to solve the equations system

$$\begin{cases} a \star_\sigma \chi_{00} = 0 \\ \chi_{00}\star_\sigma \bar{a} = 0 \end{cases}$$

As formula 13 works only if the coordinates (q,p) fulfill the commutation rule $[q,p] = i\hbar$, while $[a,\bar{a}] = 1$, we need to introduce the coordinate $\bar{a}' := i\hbar\cdot\bar{a}$ such that $[a,\bar{a}'] = i\hbar$; then $\tilde{\chi}_{00}(a,\bar{a}') := \chi_{00}(a,\bar{a}(\bar{a}'))$; also we must note that $\tilde{\chi}_{00}\star_\sigma \bar{a}' = i\hbar\cdot\tilde{\chi}_{00}\star_\sigma\bar{a} = 0$. Then, taking $\sigma = \frac{1}{2}$:

$$\begin{aligned}
a \star_{1/2} \tilde{\chi}_{00}(a,\bar{a}') =& \frac{1}{2\pi\hbar}\iint \mathcal{F}a[a,\bar{a}'](\xi,\eta)\tilde{\chi}_{00}(a - \frac{1}{2}\eta, \bar{a}' - \frac{1}{2}\xi) e^{\frac{i}{\hbar}(\xi a - \eta\bar{a}')}\, d\xi\, d\eta \\
=& \frac{1}{2\pi\hbar}\iint 2\pi\hbar\cdot i\hbar\cdot\delta'(\xi)\delta(\eta)\tilde{\chi}_{00}(a - \frac{1}{2}\eta, \bar{a}' - \frac{1}{2}\xi) e^{\frac{i}{\hbar}(\xi a - \eta\bar{a}')}\, d\xi\, d\eta \\
=& -i\hbar\iint \delta(\xi)\delta(\eta)\left[-\frac{1}{2}\frac{\partial\tilde{\chi}_{00}(a - \frac{1}{2}\eta, \bar{a}' - \frac{1}{2}\xi)}{\partial(\bar{a}' - \frac{1}{2}\xi)} + \frac{i}{\hbar}a\tilde{\chi}_{00}(a - \frac{1}{2}\eta, \bar{a}' - \frac{1}{2}\xi)\right] e^{\frac{i}{\hbar}(\xi a - \eta\bar{a}')}\, d\xi\, d\eta \\
=& -i\hbar\left[-\frac{1}{2}\frac{\partial\bar{a}'}{\partial\bar{a}}\frac{\partial\chi_{00}(a,\bar{a})}{\partial\bar{a}} + \frac{i}{\hbar}a\chi_{00}(a,\bar{a})\right] \\
=& \frac{1}{2}\frac{\partial\chi_{00}(a,\bar{a})}{\partial\bar{a}} + a\chi_{00}(a,\bar{a}) = 0 \\
\Longrightarrow& \ \chi_{00}(a,\bar{a}) = f(a)e^{-2a\bar{a}}.
\end{aligned}$$

Similarly:

$$\begin{aligned}
\tilde{\chi}_{00}(a,\bar{a}') \star_{1/2} \bar{a} =& i\hbar\left[\frac{1}{2}\frac{\partial\chi_{00}(a,\bar{a})}{\partial\bar{a}} + \bar{a}\chi_{00}(a,\bar{a})\right] = 0 \\
\Longrightarrow& \ \chi_{00}(a,\bar{a}) = \mathcal{C}e^{-2a\bar{a}}, \text{ where } \mathcal{C}\in\mathbb{C} \text{ is the normalization constant.}
\end{aligned} \tag{49}$$

The associated quasi-probability distribution function $\rho_{00}(q,p) = \frac{1}{\sqrt{2\pi\hbar}}\chi_{00}(a(q,p),\bar{a}(q,p)) = \frac{\mathcal{C}}{\sqrt{2\pi\hbar}} e^{-\frac{1}{\hbar\omega}(\omega^2 q^2 + p^2)}$, hence $\mathcal{C} = \sqrt{\frac{2}{\pi\hbar}}$. Then

$$\rho_{nm}(q,p) = \rho_{nm}(q,p,t=0) = \frac{1}{\sqrt{n!m!}}\sum_{k=0}^{m}(-1)^k k!\binom{n}{k}\binom{m}{k}(2)^{n+m}\bar{a}^{m-k}a^{n-k}\left[\frac{1}{\pi\hbar}e^{-\frac{1}{\hbar\omega}(\omega" q^2 + p^2)}\right].$$

Time evolution is described by

$$\rho_{nm}(q,p,t) = e_{\star_{1/2}}^{-\frac{i}{\hbar}Ht} \star_\sigma \rho_{nm}(q,p)\star_\sigma e_{\star_{1/2}}^{\frac{i}{\hbar}Ht} = e^{-\frac{i}{\hbar}(E_n - E_m)t}\rho_{nm}(q,p) = e^{-i\omega(n-m)t}\rho_{nm}(q,p);$$

if $m = n$, ρ_{nn} represents a pure stationary state, then $\rho_{nn}(p,q,t) = \rho_{nn}(p,q)\ \forall t$.

References

[1] M. Blaszak, Z. Domański, "Phase space quantum mechanics", *Faculty of Physics, Adam Mickiewicz University*, 2011.

[2] M. Bordermann, "Deformation Quantization: a survey", *International Conference on Noncommutative Geometry and physics*, volume 103 of *Journal of Physics: Conference Series*, 2008.

[3] A. S. Cattaneo, D. Indelicato, "Formality and Star Products", *Institut für Mathematik,Universität Zürich–Irchel*, 2004.

[4] C. Esposito, "Formality Theory: From Poisson structures to Deformation Quantization", *Springer Cham*, 2015.