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A Fisher-KPP model with a fast diffusion line in a periodic medium

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Preface

Modelling population dynamics trough differential equations is an old issue. A basic model is the logistic equation. Thanks to the introducing partial derivatives, one can study not only the variation of the total mass of the population, but also its distribution in the territory. The equations that combines a diffusion process with a variation of the density are called reaction-diffusion equations.

Since their introduction in two celebrated papers by Fisher [10] and by Komogorov, Pertovsky and Piskunov [14], reaction-diffusion equations appear in a vast mathematical literature for their interest both for the modelling potential in a wide range of fields, among which medicine, biology, and ecology, and for realising a new class of PDEs, requiring new techniques and raising questions. The two main behaviours that a solution u of a reaction-diffusion equation may present are the *invasion* and the *extinction*. If at large times the density of population tends to 0, then we have extinction. On the other side, if at large times the population reaches its maximum density all over the domain, then we say that there is invasion. In the last case, many questions regarding the way the population invades the plane arise. When one dimensional solution exists, then one can define the *travelling fronts*, that are solutions propagating as waves. The speed at which the solution converge to its limit is called *asymptotic speed* and it has been computed or estimated in many cases.

One interesting phenomenon to be investigated is how the presence of a transportation system affects the spreading of a population or epidemics. This interest is motivated by several concrete situations. One classic example is the spreading of the “Black Death” plague in Europe during the 14th century [18]. It has been observed that the silk road has speeded up the diffusion of the infection along its path. Then the epidemic illness diffused in the inland, more slowly. Another example of a similar phenomenon is given by the recent spreading of the Processionary caterpillar in Europe. The species is naturally invasive, but the unexpected speed of its diffusion may be caused by the transport of some individuals by vehicles travelling on roads through infested areas. The same situation happens with other species, like the Aedas Albopictus mosquito.

A model for a diffusion in an environment with a faster diffusion line was first introduced by Berestycki, Roquejoffre and Rossi in [6]. The field is modelled with the halfplane $\Omega = \mathbb{R} \times \mathbb{R}_+$ and the line with the x axis; the main idea is to use two different variables for modelling the density of population along the line (u) and on the half plane (v). The system reads

$$\begin{cases} \partial_t u(x,t) - D\partial^2_{xx}u(x,t) &= \nu v(x,0,t) - \mu u(x,t), & x \in \mathbb{R}, t > 0, \\ \partial_t v(x,y,t) - d\Delta v(x,y,t) &= f(v), & (x,y) \in \Omega, t > 0, \\ -d\partial_y v(x,0,t) &= -\nu v(x,0,t) + \mu u(x,t), & x \in \mathbb{R}, t > 0, \end{cases}$$

where the three equations describe, respectively, the dynamic on the line, the dynamic on the half plane and the exchanges of population between the line and the half plane. On the line, the diffusion is faster than in Ω. The authors of [6] prove existence and uniqueness of a positive bounded stationary solution and show that given a reasonable initial datum, the corresponding solution converges to the stationary solution. Moreover, they show that the presence of the line increases the spreading speed. Another version of the model with a reaction term on the line was presented by the same authors in [7].

In this thesis, we present the road-field model with a different reaction term. We take f depending on the density of population v and depending periodically on the variable space y, and we take weaker hypotheses on f. Some similar hypotheses can be found in the model by Berestycki, Hamel and Roques [3], that studies a reaction-diffusion process in a periodic medium, but has no faster diffusion line. The model we investigate combine the interest in the effects of a transportation network with the need to have a heterogeneous medium, reflecting the natural environment [13]. Many of the proofs that we are presenting in this work are combinations and adaptations of the ideas of the three papers [6, 7, 3].

We prove the existence and uniqueness of solutions and we study the asymptotic behaviour of them. The eigenvalue problem corresponding to the second equation of the system, i.e. $-d\phi'' - f_v(y,0)\phi = \lambda_0\phi$ with ϕ periodic in y, has an important role in the investigation of the asymptotic behaviours. By examining the periodic eigenvalue and eigenfunction, we find some sufficient conditions entailing invasion and extinction. In fact, the sign of λ_0 tells whether the solution $(u,v) \equiv (0,0)$ is stable or not. If $(0,0)$ is an unstable solution, then any solution of the system starting from a nonnegative datum, different from 0, invades the domain. This result was also found for the model in [3]. In the same article, Berestycki et al. also show that if $(0,0)$ is unstable then any solution faces extinction. In our model, the presence of the road increases the technical difficulties. We have to introduce a new hypothesis on the eigenfunction in order to provide a sufficient condition for extinction. Without this hypothesis, the question about the asymptotic behaviour is still open. Since the conditions regarding spectral properties are not easy to verify, we provide some handy and natural hypothesis entailing extinction. We show that by assuming f to be symmetric in the space variable y we can have a complete characterisation of the asymptotic behaviour of the solutions.

Contents

Chapter 1

Introduction and main results

The aim of the thesis is to study a system of partial differential equations, modelling a biological process in a periodical environment with a faster diffusion on a line.

The domain of the spatial variables is $\Omega := \mathbb{R} \times \mathbb{R}_+$ and the line where the faster diffusion takes place is the boundary of Ω, thus the x axis. Let us take D, d, μ, ν positive numbers and let $f(y,v) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a function; later we will specify the hypothesis on these quantities. We call $v(x,y,t)$ the density of population in the field and $u(x,t)$ the density of population on the road. The system we consider reads

$$\begin{cases} \partial_t u(x,t) - D\partial^2_{xx}u(x,t) &= \nu v(x,0,t) - \mu u(x,t), & x \in \mathbb{R}, t > 0, \\ \partial_t v(x,y,t) - d\Delta v(x,y,t) &= f(y,v), & (x,y) \in \Omega, t > 0, \\ -d\partial_y v(x,0,t) &= -\nu v(x,0,t) + \mu u(x,t), & x \in \mathbb{R}, t > 0, \end{cases} \tag{M}$$

where the three equations describe, respectively, the population dynamic on the line depending on the population density u, the dynamic on the half plane depending on the population density v and the exchanges between the line and the half plane.

We are interested in the existence and uniqueness and in the behaviour for large time of the solution (u,v) of (M) with initial datum

$$\begin{cases} u|_{t=0} = u_0 & \text{in } \mathbb{R}, \\ v|_{t=0} = v_0 & \text{in } \Omega. \end{cases} \tag{0.1}$$

In this chapter we present in detail the hypotheses on the model, the context in which this work is nested, the main results with their interpretation and finally a glance at the open problems.

1.1 Explanation and hypotheses on the model

Let us focus first on the coefficients appearing in the equations, and then on the reaction function.

The positive quantities ν and μ represent, respectively, the fraction of population of the field moving to the road and the fraction of population of the road relocating in the field. Of course, the movement of the population from the field to the road and viceversa is balanced in the equations. On the road, where no reaction takes place, there is a diffusion of the population and the incoming flow νv and the outgoing flow $-\mu u$ are the only factors that varies the mass of the population. In the field, the exchange with the road appears as mixed boundary condition. The main reason to introduce the road is to have a part of the environment where the individuals can move faster. This is modelled by choosing $D > d$, thus a stronger diffusion coefficient in the road.

The difficulties of this model come largely from the reaction function f, that acts like a birth-death rate in the field. It takes two arguments, the spatial variable y and the density of population v. The function f is Hölder continuous for some $\gamma \in (0,1)$ with respect to y, locally uniformly with respect to v, and Lipschitz with respect to v locally uniformly with respect to y. We want f to model a population dynamics, hence we require f to have a stable state at 0 and to be negative if v is above some saturation level $M > 0$, that is

$$f(y,0) = 0 \qquad \forall\, y \in \mathbb{R},$$
$$\exists \quad M > 0 \quad \text{such that} \quad f(y,M) \leq 0 \qquad \forall\, y \in \mathbb{R}.$$

Notice that this means that the saturation level is not necessarily 1, like in most part of population dynamic model. Also, it possibly depends on the space.

To simulate a birth-death rate one requires f to satisfy the condition

$$\forall\, v_1 < v_2 \quad \frac{f(y,v_1)}{v_1} - \frac{f(y,v_2)}{v_2} > 0 \qquad \forall\, y \in \mathbb{R},$$

that is an hypothesis of type Kolmogorov-Petrovski-Piskunov [14] and it is a strong concavity hypothesis. The new feature we are introducing in this work, that was not present in the road-field model of [6], is the periodic dependence on the space for f, that is

$$f(y+\ell, v) = f(y,v) \qquad \forall\, y \in \mathbb{R}_+,\ \forall\, v \in \mathbb{R}$$

for some fixed $\ell > 0$. We expressly omit a positivity hypothesis on f, which was taken in [6], in order to study deeper aspects of the model.

We spend some words also on the solutions. As u and v are supposed to be densities of population, we want them to be nonnegative and also bounded. A function describing a concrete quantity as a density of population is supposed to be regular; therefore, we want u and v to be derivable twice in the space variables and once in the time variable. So we impose even stronger requirements on u_0 and v_0, asking for bounded and nonnegative functions, Hölder continuous with theirs derivatives up to order 2. The existence of nonnegative bounded solutions is one of the results shown in this thesis and it is an evidence of the good design of the model. Another evidence of for that is the conservation of the total mass in the case $f \equiv 0$, as one can find in [6].

1.2 Interest of the model in the context of the current research

Reaction-diffusion equations of the type

$$\partial_t u - \Delta u = f(u)$$

have first been introduced in two milestones articles by Fisher [10] and Kolmogorov, Petrovsky and Piskunov [14] to study the spreading of an advantaging gene. The applications to the equation were later extended to many other fields like medicine, biology, physics, and ecology. Since the pioneering works, a vast literature was devoted to the investigation on the existence and uniqueness of the solutions and to the study of their properties, among which the most celebrated is the *spreading* or *propagation* on the domains, that is, the locally uniformly convergence of the solution to the saturation level as times goes large. In the basic cases where the level of saturation is $M = 1$, one says that the equation exhibit the spreading property if any solution starting from a compactly supported datum $u_0 \geq 0,\ u_0 \not\equiv 0$ satisfies the limit $u(x,t) \to 1$ as $t \to +\infty$. Deeper spreading properties are studied by comparing the solution to *travelling fronts*, i.e. planar fronts of the form $U(x,t) = \phi(x \cdot e + ct)$, where $e \in \mathbb{R}^N,\ |e| = 1$ is the direction we are considering, $c > 0$ is the speed at which the front is moving and ϕ satisfies

$$-\phi'' - c\phi' = f(\phi), \qquad \phi(-\infty) = 1,\ \phi(+\infty) = 0.$$

One of the most outstanding result of Kologorov, Petrovsky and Piskunov [14] was the existence of an *asymptotic speed*, that is, a minimum value c^* from which on a travelling front exists.

The largest part of the existing papers focus on *homogeneous* equations, that are equations whose reaction function depends only on the solution u. The *heterogeneous* extensions are obtained, for example, putting a space dependence in f, as we will see below, or imposing the problem in domains where the geometry affects the form of the solution [2]. Heterogeneous equations are a better choice for modelling population dynamics; in fact, the literature shows that natural environments are far from being homogeneous and present a mosaic of different habitats, often fragmented by natural or artificial barriers like rivers or roads [13]. However, heterogeneous reaction-diffusion equations raise difficulties with respect to the homogeneous case. Even the definition of spreading and travelling front have to be adapted to the context and the existence and the value of the asymptotic speed have to be discuss case by case. Moreover, the behaviour of the fronts may change with the direction e.

Skipping a great part of what can be said about reaction-diffusion equations, we here presents two recent models from which the work of this thesis was inspired.

The idea of of introducing a line with faster diffusion in the framework of reaction-diffusion equation was first presented by Berestycki, Roquejoffre and Rossi in [6]. In that paper, the reaction function on the field was an homogeneous function $f(v)$ with the *KPP hypothesis*, that is, a function with two stable states in 0 and 1 and a sublinearity property:

$$f(0) = f(1) = 0, \qquad 0 < f(s) \leq f'(0)s \quad \forall s \in (0,1).$$

Notice that in the model of this thesis the hypothesis $0 < f(s)$ is missing, and that is one of the reasons for the interest of the model. This work gives the ideas for a great part of the basic properties of the model, such as the comparison principles and the existence of solutions, but also focuses on the existence of planar front of exponential type and the estimates of the asymptotic speed. The following work [7] by the same authors introduces a reaction term $g(u)$ in the equation for the line. This time, the difficulties impact also the form of the asymptotic solution, that has the form $(U, V(y))$, while in the previous case it was simply $(\frac{1}{\mu}, 1)$. Thank to this, more general techniques to study the asymptotic behaviour of solutions were developed. In both articles, comparing the results to the case with no line, an increase of the asymptotic speed c^* in the direction of the line was found and the dependence of c^* from the parameters of the model was analysed in depth.

The second model that inspired this thesis is the diffusion in a periodically fragmented environment. Berestycki, Hamel and Roques analysed in the papers [3] and [4] the equation

$$u_t - \nabla \cdot (A(x)\nabla u) = f(x, u), \qquad x \in \mathbb{R}^N \tag{2.2}$$

where the functions A and f are periodic in the spatial variable x. On the function f they take the same hypotheses of the system (M); the only difference with the model presented in this thesis is that our reaction function depends only on one spatial variable, namely y. As the positivity requirement for f is missing, different behaviours of the solutions appear. The authors distinguish between *extinction*, when $u(x,t) \to 0$ uniformly in $\mathbb{R}^N$ as $t \to +\infty$, and *invasion*, when $u(x,t) \to p(x)$ uniformly in $\mathbb{R}^N$ as $t \to +\infty$, being $p(x)$ is the (unique) stationary positive solution of the equation (2.2). The most interesting result is the complete characterisation of the asymptotic behaviour with the principal eigenvalue λ_0, solution to

$$\begin{cases} -d\psi'' - f_v(y,0)\psi = \lambda\psi, & y \in \mathbb{R}, \\ \psi > 0, \quad \|\psi\|_{L^\infty[0,\ell]} = 1, & \psi(y) = \psi(y+\ell) \ \forall y \in \mathbb{R}. \end{cases} \tag{2.3}$$

Being more precise, the authors prove invasion when $\lambda_0 < 0$ and extinction when $\lambda_0 \geq 0$. This is due to the tight link between existence of subsolutions or supersolutions and asymptotic limit of the solution of the equation. The astonishing naturalness of this condition opens the door to further research of the predictions that can be made on the solutions just studying the eigenvalue problem associated with the equation. In this work, we look for similar conditions to study the asymptotic behaviour of solutions

1.3 Statements of the main results

This is the first work, to our knowledge, on the system (M). Thus, our investigation start from the very basic properties of the model, such as the existence and uniqueness of the solution, and we also prove some technical tools like the comparison principle, adapting the proofs of the work on the basic road-field model [6]. Following the order of Chapter 2, we show

Theorem 1.3.1 (Comparison Principle)**.** *Let $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ be respectively a sub-solution bounded from above and a super-solution bounded from below of* (M) *such that $\underline{u} \leq \overline{u}$, $\underline{v} \leq \overline{v}$ at $t = 0$. Then $\underline{u} \leq \overline{u}$, $\underline{v} \leq \overline{v}$ for all $t > 0$; moreover, if $\underline{u} = \overline{u}$ or $\underline{v} = \overline{v}$ at $T > 0$, then $(\underline{u}, \underline{v}) = (\overline{u}, \overline{v})$ in $[0, T)$.*

After proving this fundamental result, an immediate consequence is the uniqueness of the solution nonnegative bounded to the Cauchy problem associated with (M) and the datum (0.1). Showing the existence of the solution is instead a non standard result. Many difficulties arise not only because of the non-homogeneous nature of f, but above all for the coupled system and the presence of a mixed border condition. After a long path, we finally show

Theorem 1.3.2. *Let (u_0, v_0) be a couple of bounded and nonnegative functions, Hölder continuous with theirs derivative up to order 2. Suppose also that u_0 and v_0 satisfies*

$$-\,d\partial v_0(x,0) = (\mu u_0(x) - v_0(x,0))\,. \tag{3.4}$$

Then there is a classical solution (u, v) to the Cauchy problem (M) *with initial datum (u_0, v_0). Moreover, the solution (u, v) is bounded and nonnegative.*

The most interesting part of the investigation is the study about the existence of a positive, bounded stationary solution. We analyse this because of modelling reasons: a density of population must be nonnegative, thus a nonzero stationary solution gives the possibility to have some persistence of the species. More specifically, we look for conditions on the elements of the system that entail extinction or invasion similar to the ones studied in [3]. Surprisingly, the presence of the road causes new phenomenons and the sign of the principal eigenvalue of the periodic system is not sufficient to characterise the asymptotic behaviour. Therefore, new hypothesis had to be introduced:

Hypothesis 1.3.3 (H1)**.** The periodic eigenfunction ϕ solution to (2.3) satisfies

$$\partial_y \phi(0) \le 0. \tag{3.5}$$

This is a fine aspect of the spectral properties of (2.3) and but its appearance arise naturally during the proofs of the results. We are able to prove the following:

Theorem 1.3.4. *Let (u, v) be a solution of* (M) *with initial datum* (0.1)*. Then:*

- ***(Invasion)**: Suppose $\lambda_0 < 0$. Then there exists a unique couple (q, p) stationary solution for* (M) *such that both q and p are positive and bounded. Moreover q is constant and p is independent of x and, for every (u, v) solution of the system raised from a nonnegative datum (u_0, v_0), we have $u(x, t) \to q$ and $v(x, y, t) \to p(y)$ locally uniformly as $t \to +\infty$.*

- ***(Extinction)** Suppose that $\lambda_0 \ge 0$ and that the periodic eigenfunction ϕ solution to* (2.3) *satisfies* (H1)*. Then there is no positive stationary solution to* (M) *and moreover for every nonnegative bounded solution (u, v) we have $u(x, t) \to 0$, $v(x, y, t) \to 0$ locally uniformly as $t \to +\infty$.*

One other thing we can show is that for some $\varepsilon > 0$ we have $p(y) > \varepsilon$ for all $y \in \mathbb{R}_+$. Thus, p is separated from 0.

In a large part of the possible scenarios the theorem shows not only if the stationary solution is positive or not, but also the convergence of every solution to it. This is a quite strong result and the presence of further cases make the model more intriguing.

Since H1 is difficult to verify, we investigate some more natural hypotheses that entail $H1$. We managed to have a complete characterisation of the asymptotic behaviour of solutions when the following hypothesis is satisfied:

Hypothesis 1.3.5 (H2)**.** For every fixed $v \in \mathbb{R}$, f is symmetric in $[0, \ell]$ with respect to the centre $y = \frac{1}{2}\ell$, that is

$$f(y, v) = f\left(\frac{1}{2}\ell - y, v\right) \quad \forall y \in \left[0, \frac{1}{2}\ell\right]. \tag{H2}$$

In fact, this condition implies the symmetry of the principal eigenfunction ϕ with centre 0, thus, the condition H1 is satisfied. Hence, in the symmetric case, the sign of the eigenvalue λ_0 completely characterise the asymptotic behaviour of the solutions, as for the model with no road.

1.4 Explanation and interpretation of the results

By studying the connections between the periodic eigenvalue λ_0 and the eigenvalues on balls of radius R, which we call λ_R, we have a deep understanding of the fundamental dynamic in the model. In fact, it is well known that the sign of the eigenvalue determines if the corresponding eigenfunction behaves as a supersolution or a subsolution. This mechanism is at the basis of the techniques used in [6], [3] and in many other works. In this thesis, it was expected to have an overview on the asymptotic behaviours by studying the sign of λ_0. In the case $\lambda_0 < 0$, as it was expected, invasion happens. In fact, we are able to build some subsolutions with the help of the eigenfunctions. Instead, in the case $\lambda_0 \geq 0$, the traditional methods were not sufficient to show the extinction. We can build a supersolution, that is a key tool to show extinction, only by adding a new hypothesis, thus H1. This shows that the presence of a road is a challenging new feature for diffusion models. The presence of the road may not be just a technical difficulty, but it is possible that it impacts profoundly the dynamic with respect to the case with no faster diffusion lines.

1.5 Perspectives

One unexpected case appeared during the investigation, the case $\lambda_0 \geq 0$, $\phi'(0) > 0$. The standard ideas do not apply to this case, so new techniques have to be tried or developed, but this goes beyond the purposes of this thesis. At the moment, there is no reason to think that in this case only one behaviour is possible. We would like to find an example where a positive stationary solution exists. Some fine spectral properties are possibly involved in the determination of the asymptotic behaviour of solutions. This aspect is surprising and deserves further investigation.

One other classical feature that is studied in diffusion models is the propagation speed. Considered the richness of the possibilities for the asymptotic behaviour, we expect that the study of speed of propagation in different directions may enlight some interesting and unknown phenomenon.

Chapter 2

Basic properties

The properties of the model we want to highlight in this work are, from one side, the mathematical consistency and interest and from the other side, the behaviours that can be recognized in natural phenomenons. This chapter deals with both tasks. One of the main goals we show here is the existence and uniqueness of the nonnegative bounded solution to the Cauchy problem related to (M) but, considering the following theorems, the most useful and deep result in this chapter is the Comparison Principle. It establishes the ordering of solutions from the ordering of initial data but, beyond being a classic result required to every reasonable model, the comparison principle will be a key tool in the proofs of many of the theorems that follow all along this thesis. Moreover, the comparison theorem directly implies the uniqueness of a nonnegative bounded solution to the Cauchy problem.

In the first section we state the basic version of the Comparison Principle and we prove it; then we apply it for showing the uniqueness of the nonnegative and bounded solution to (M). In the second section, we present another version of the Comparison Principle which is more adapted to the geometry of the model. The last section is devoted to the proof of the existence of solutions to (M); since the model is extremely nonstandard, a long construction is needed. The proofs of the comparison principles, the Liouville-type result and the existence are adaptations from the proofs of the equivalent result you can find in [6].

2.1 Classic comparison theorem and uniqueness of the solution

The first version of the Comparison principle concerns the ordering of a subsolution and a supersolution starting from two different initial data, which respect an inequality on all the domain. Notice that it works as a strong maximum principle.

Theorem 2.1.1. *Let* $(\underline{u}, \underline{v})$ *and* $(\overline{u}, \overline{v})$ *be respectively a subsolution bounded from above and a super-solution bounded from below of* (M) *such that* $\underline{u} \le \overline{u}$, $\underline{v} \le \overline{v}$ *at* $t = 0$*. Then* $\underline{u} \le \overline{u}$, $\underline{v} \le \overline{v}$ *for all* $t > 0$*; moreover, if* $\underline{u} = \overline{u}$ *or* $\underline{v} = \overline{v}$ *at* $T > 0$*, then* $(\underline{u}, \underline{v}) = (\overline{u}, \overline{v})$ *in* $[0, T)$*.*

Proof. In order to prove the result we first modify the problem and its solutions. Take $\chi : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{gathered}\chi(0) = 0,\\ \chi(z)' = 0, \quad z \in [0,1],\\ \lim_{z\to+\infty} \chi(z) = +\infty,\\ (d+D)|\chi'| + (2d+D)|\chi''| \leq 0 \quad \text{in } \mathbb{R}.\end{gathered}$$

Fix $\varepsilon > 0$ and let $k > 0$, then set

$$\begin{aligned}(\check{u}, \check{v}) &= e^{-kt}(\underline{u}, \underline{v})\\ (\hat{u}, \hat{v}) &= e^{-kt}(\overline{u} + \nu\varepsilon(\chi(|x|) + t + 1), \overline{v} + \mu\varepsilon(\chi(|x|) + \chi(y) + t + 1))\\ h(y,t,v) &= e^{-kt} f(y, v \cdot e^{kt}) - kv.\end{aligned} \tag{1.1}$$

Note that we can choose k such that the function h is nonincreasing in v thanks to the Lipschitz hypothesis on f. In addiction, for every fixed time $t \geq 0$ the funcions $\hat{u}, \hat{v}$ go to infinity as the space variables go to infinity. As one can easily check, the couples $(\check{u}, \check{v})$ and $(\hat{u}, \hat{v})$ are respectively a sub- and a supersolution to the system

$$\begin{cases} \partial_t u - D\partial_{xx} u &= \nu v(x,0,t) - (\mu+k)u, & x \in \mathbb{R},\ t > 0,\\ \partial_t v - d\Delta v &= h(y,t,v), & (x,y) \in \Omega,\ t > 0,\\ -d\partial_y v(x,0,t) &= -\nu v(x,0,t) + (\mu+k)u, & x \in \mathbb{R},\ t > 0,\end{cases} \tag{1.2}$$

and moreover $\check{u} < \hat{u}$ and $\check{v} < \hat{v}$ at $t = 0$.

Assume by contradiction that the inequality does not hold for all the times, so that there is

$$T := \sup\{\tau > 0 \mid \check{u} < \hat{u} \text{ in } \mathbb{R} \times [0,\tau],\ \check{v} < \hat{v} \text{ in } \Omega \times [0,\tau]\} < \infty.$$

Hence we have $\check{u} < \hat{u}$ and $\check{v} < \hat{v}$ at $t < T$.

Thanks to the transformations (1.1), at a distance from the origin R large enough we have $\hat{u} - \check{u} > 1$ and $\hat{v} - \check{v} > 1$. Also, at $t = 0$ we have the strict inequalities $\check{u} < \hat{u}$ in $[-R, R]$ and $\check{v} < \hat{v}$ in the closed set $[-R, R] \times [0, \mathbb{R}]$, so for t small the strict inequalities still hold because of the continuity of the functions. Therefore $T > 0$. At $t = T$ there must be a contact point $\tilde{x} \in \mathbb{R}$ such that $\check{u}(\tilde{x}, T) = \hat{u}(\tilde{x}, T)$ or $(\tilde{x}, \tilde{y}) \in \Omega$ such that $\check{v}(\tilde{x}, \tilde{y}, T) = \hat{v}(\tilde{x}, \tilde{y}, T)$; in fact, $\hat{u}$ and $\hat{v}$ diverge at infinity while $\check{u}$ and $\check{v}$ are bounded.

If the first case subsists, then

$$\partial_t(\hat{u} - \check{u}) - D\partial_{xx}(\hat{u} - \check{u}) + (\mu + k)(\hat{u} - \check{u}) \geq \nu(\hat{v} - \check{v})(x, 0, t) \geq 0$$

and $\hat{u} > \check{u}$ at $t = 0$, hence for the strong maximum principle if $\hat{u}(\tilde{x}, T) = \check{u}(\tilde{x}, T)$ then $\hat{u} = \check{u}$ for all $0 \leq t \leq T$. This is impossible because of the strict inequality at $t = 0$. So it must be $\hat{u} > \check{u}$ and $\check{v}(\tilde{x}, \tilde{y}, T) = \hat{v}(\tilde{x}, \tilde{y}, T)$. Then

$$\partial_t(\hat{v} - \check{v}) - d\Delta(\hat{v} - \check{v}) \geq h(\hat{v}, y, t) - h(\check{v}, y, t)$$

and since h is Lipschitz-continuous, we can apply the maximum principle in the Plancharèl-Lindelöf version [19]. Since $\hat{v} \geq \check{v}$ in $\Omega \times [0, T]$, the condition

$$\limsup_{R\to+\infty} \{e^{-\alpha R^2} \min_{|(x,y)|=R} (\hat{v} - \check{v})\} = 0$$

is satisfied. Thus, the minimum of $\hat{v} - \check{v}$ must be attained on the parabolic boundary of $\Omega \times [0, T]$; considering also the conditions at $t = 0$, we have necessarily $(\tilde{x}, \tilde{y}, T) = (\eta, 0, T)$ for some $\eta \in \mathbb{R}$. But then

$$-d\partial_y \hat{v}(\eta, 0, T) \geq -\nu \hat{v}(\eta, 0, T) + (\mu + k)\hat{u} \geq -\nu \check{v}(\eta, 0, T) + (\mu + k)\check{u} \geq -d\partial_y \check{v}(\eta, 0, T)$$

and this is absurd because $\hat{v}(\eta, 0, T) = \check{v}(\eta, 0, T)$ and $\hat{v} \geq \check{v}$ in $\Omega \times [0, T]$.

Now, due to the arbitrariness of ε, $\overline{u} \geq \underline{u}$ and $\overline{v} \geq \underline{v}$ in $\Omega \times [0, \infty)$. Suppose there is $(x, y) \in \Omega$ a point of contact between the two at some $t = T$. Applying the strong maximum principle in the same way as before we have $\underline{u} = \overline{u}$ or $\underline{v} = \overline{v}$ for $t \leq T$. Also, if $\underline{u} < \overline{u}$ for all $t > 0$, then like before we have $\underline{v} < \overline{v}$. □

Remark 2.1.2. We can easily extend this result to the case of generalized sub- and supersolutions, recalling that a generalized subsolution is the supremum of a set of subsolutions and a generalized supersolution is the inferior of a set of supersolutions. Since the comparison principle holds for every couple of sub- and supersolutions, it holds for the generalized couple.

Now we can apply the theorem we just proved to show a uniqueness result.

Theorem 2.1.3. *There is at most one nonnegative bounded solution to the Cauchy problem* (M) *with initial datum* (u_0, v_0).

Proof. Suppose by contradiction there are two nonnegative bounded solutions (u_1, v_1) and (u_2, v_2) with the same initial data (u_0, v_0). Then by the comparison principle $u_1 \leq u_2$ and $v_1 \leq v_2$; exchanging the roles of (u_1, v_1) and (u_2, v_2) we find the equality. □

2.2 Generalized Comparison Principle

Even though Theorem 2.1.1 applies also to generalized sub- and supersolutions, the geometry of the domain may require the use of more general comparison principles.

Theorem 2.2.1 (Generalized Comparison Principle)**.** *Let* $E \subset \mathbb{R} \times (0, \infty)$*,* $F \subset \Omega \times (0, \infty)$ *be connected open sets. Take* (u_1, v_1) *and* (u_2, v_2) *be two subsolutions of* (M) *bounded from above and such that*

$$\begin{aligned} u_1 &\leq u_2, && \text{on } (\partial E) \cap (\mathbb{R} \times (0, \infty)), \\ v_1 &\leq v_2, && \text{on } (\partial F) \cap (\Omega \times (0, \infty)). \end{aligned}$$

We define

$$\underline{u}(x, t) = \begin{cases} \max(u_1, u_2), & \text{if } (x, t) \in \overline{E}, \\ u_2, & \text{otherwise,} \end{cases}$$

$$\underline{v}(x, y, t) = \begin{cases} \max(v_1, v_2), & \text{if } (x, y, t) \in \overline{F}, \\ v_2, & \text{otherwise,} \end{cases}$$

and we impose

$$\underline{u}(x, t) > u_2(x, t) \Rightarrow \underline{v}(x, 0, t) \geq v_1(x, 0, t), \tag{2.3}$$

$$\underline{v}(x, 0, t) > v_2(x, 0, t) \Rightarrow \underline{u}(x, t) \geq u_2(x, t). \tag{2.4}$$

Then for every $(\overline{u}, \overline{v})$ *supersolution of* (M) *bounded from below such that* $\underline{u} \leq \overline{u}$ *and* $\underline{v} \leq \overline{v}$ *at* $t = 0$*, we have* $\underline{u} \leq \overline{u}$*,* $\underline{v} \leq \overline{v}$ *for all* $t > 0$*.*

Remark 2.2.2. The hypotesis (2.3) and (2.4) can be relaxed; however, for simplicity of notation we require it everywhere.

Proof. Applying Theorem 2.1.1, it is easy to see that $u_2 \leq \overline{u}$, $v_2 \leq \overline{v}$ for all $t > 0$. Assume by contradiction that $(\overline{u}, \overline{v})$ is below $(\underline{u}, \underline{v})$ at some point. Setting χ, ε, h, k as in the precedent proof and making the same transformations, namely setting

$$
\begin{aligned}
(\breve{u}_i, \breve{v}_i) &= e^{-kt}(u_i, v_i) \\
(\breve{u}, \breve{v}) &= e^{-kt}(\underline{u}, \underline{v}) \\
(\hat{u}, \hat{v}) &= e^{-kt}(\overline{u} + \nu\varepsilon(\chi(|x|) + t + 1), \overline{v} + \mu\varepsilon(\chi(|x|) + \chi(y) + t + 1))
\end{aligned}
$$

we find two subsolutions and one supersolution for the problem (1.2), as checked before. Moreover, $(\hat{u}, \hat{v})$ is strictly above $(\breve{u}_2, \breve{v}_2)$ for all $t > 0$. By continuity of the solutions we can say there exists $T < \infty$ such that $\breve{u} < \hat{u}$, $\breve{v} < \hat{v}$ for all $t < T$ and $\breve{u}(\tilde{x}, T) = \hat{u}(\tilde{x}, T)$ for some $\tilde{x} \in \mathbb{R}$ or $\breve{v}(\tilde{x}, \tilde{y}, T) = \hat{v}(\tilde{x}, \tilde{y}, T)$ for some $(\tilde{x}, \tilde{y}) \in \Omega$. Observe that it must be $(\tilde{x}, T) \in E$ or $(\tilde{x}, \tilde{y}, T) \in F$, and the points cannot be at the border thanks to the condition (2.3).

Consider first the case when $\breve{u}(\tilde{x}, T) = \hat{u}(\tilde{x}, T)$ for some $\tilde{x} \in \mathbb{R}$. As known, $\breve{u}_2(\tilde{x}, T) < \hat{u}(\tilde{x}, T) = \breve{u}(\tilde{x}, T) = \breve{u}_1(\tilde{x}, T)$ and $(\tilde{x}, T) \in E$. Thus, since the functions are continuous and E is an open set, there exists $\delta > 0$ such that $\breve{u}_1 > \breve{u}_2$ in $(\tilde{x} - \delta, \tilde{x} + \delta) \times (T - \delta, T + \delta) \subset E$. Set $Q = (\tilde{x} - \delta, \tilde{x} + \delta) \times (T - \delta, T)$. By hypothesis (2.3) in Q at $y = 0$ we have $\breve{v}_1 \leq \breve{v} < \hat{v}$. Then

$$
\partial_t \breve{u} - D\partial_{xx}\breve{u} + (\mu + k)\breve{u} \leq \breve{v}(x, 0, T) < \hat{v}(x, 0, T)
$$

and by the strong maximum principle, as $\breve{u}_1 \leq \hat{u}$ on the parabolic boundary of Q, since there is a contact point between the two then $\breve{u}_1 \equiv \hat{u}$ in all Q, contradiction to the fact that the first contact is at $t = T$.

So we take the case where $\breve{u}_1 < \hat{u}$ for all $t \leq T$ and $\breve{v}(\tilde{x}, \tilde{y}, T) = \hat{v}(\tilde{x}, \tilde{y}, T)$ for some $(\tilde{x}, \tilde{y}, T) \in F$. There exists a neighbourhood $A \subset F$ of $(\tilde{x}, \tilde{y})$ open in the topology of Ω such that in $Q = A \times (T - \rho, T) \subset F$ we have $\breve{v}_2 < \breve{v}_1$. If $\tilde{y} \neq 0$ then $(\tilde{x}, \tilde{y}, T)$ is not in the parabolic boundary of Q and applying the strong maximum principle we have $\breve{v}_1 = \hat{v}$ in Q, which is impossible. If indeed $\tilde{y} = 0$, by hypothesis (2.4) we get $\breve{u}_1(\tilde{x}, T) \leq \breve{u}(\tilde{x}, T)$; thus

$$
\nu(\hat{v} - \breve{v}_1)(\tilde{x}, 0, T) - d\partial_y(\hat{v} - \breve{v}_1)(\tilde{x}, 0, T) \geq \mu(\hat{u} - \breve{u}_1)(\tilde{x}, T) \geq (\mu + k)(\hat{u} - \breve{u})(\tilde{x}, T) > 0
$$

and the lefthandside is negative since $(\hat{v} - \breve{v}_1)(\tilde{x}, 0, T) = 0$ because $(\tilde{x}, T)$ is the contact point, $\partial_y(\hat{v} - \breve{v}_1)(\tilde{x}, 0, T) \geq 0$ because for no $y > 0$ there is contact. Hence we reach a contradiction. □

2.3 Existence of a solution

It not standard to derive the existence of a solution to the Cauchy problem associated with system (M) because of its peculiar form. In fact, the system is coupled, it has a nonlinear reaction term that depends also on the space and the system also has a mixed border condition type equation. Taking inspiration from Appendix A of [6], we build a classical solution as a limit of subsolutions through different steps.

Theorem 2.3.1. *Let* (u_0, v_0) *be a couple of bounded and nonnegative functions, Hölder continuous with theirs derivative up to order 2. Suppose also that* u_0 *and* v_0 *satisfies*

$$-d\partial v_0(x,0) = (\mu u_0(x) - v_0(x,0)). \tag{3.5}$$

Then there is a classical solution (u, v) *to the Cauchy problem* (M) *with initial datum* (u_0, v_0). *Moreover, the solution* (u, v) *is bounded and nonnegative.*

The condition (3.5) is just needed for mathematical reasons. Instead, for modelling reasons, we are only interested in nonnegative and bounded initial data: it is not admissible for a population density to be negative or infinitely growing. Similarly, we want u and v to stay nonnegative and bounded. This theorem completely fulfil the modelling requests.

The rest of this section is devoted to the proof of the existence theorem.

Proof. We begin the construction of the solution in five steps. First we set the initial couple $(u_1, v_1) = (u_0, 0)$, where u_0 is the initial datum. Then we set the couple (u_n, v_n) as solution to the systems

$$\begin{cases} \partial_t u_n(x,t) - Du_n''(x,t) = \nu v_{n-1}(x,0,t) - \mu u_n(x,t), & x \in \mathbb{R},\ t > 0, \\ u_n(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{3.6}$$

and

$$\begin{cases} \partial_t v_n(x,y,t) - d\Delta v_n(x,y,t) = f(y, v_n), & (x,y) \in \Omega,\ t > 0, \\ \nu v_n(x,0,t) - d\partial_y v_n(x,0,t) = \mu u_n(x,t), & x \in \mathbb{R},\ t > 0, \\ v_n(x,y,0) = v_0(x,y), & (x,y) \in \Omega. \end{cases} \tag{3.7}$$

Our aim is to show that the sequence (u_n, v_n) converges to the solution of the problem (M) with initial datum (0.1).

Step 1. We show the existence of u_n and v_n solutions to the systems (3.6) and (3.7).

We first solve (3.6), finding u_n from a given v_{n-1}. With the change of variables

$$z_n(x,t) = u_n(\sqrt{D}x, t)e^{\beta t}$$

we obtain the equivalent system

$$\begin{cases} \partial_t z_n - z_n'' = \nu v_{n-1}(\sqrt{D}x, 0, t)e^{\beta t}, & x \in \mathbb{R},\ t > 0, \\ z_n|_{t=0} = u_0(\sqrt{D}x), & x \in \mathbb{R}, \end{cases} \tag{3.8}$$

if $\beta = -\mu$. Hence, it is sufficient to solve (3.8), that is a simple parabolic equation with fixed potential and initial datum. This is a classical problem and, provided that v_{n-1} is Hölder continuous for some $\gamma \in (0,1)$ and u_0 is bounded and continuous, then there is a classic solution to it and its construction is described by Ladyzhenskaya, Solonnikov and Uraltseva in Chapter 4, Section 14 of [17]. The hypothesis are easy to verify: the function u_0 is bounded and continuous by hypothesis; regarding v_{n-1}, for $n = 2$ the properties are true and for larger n they will be shown during the further steps. Operating the backward change of variable we can solve (3.6). The obtained solution u_n is Hölder continuous and, since $u_0 \in C^{2,\gamma}(\mathbb{R})$, then

$$\|u_n\|_{C^{1+\gamma,2+\gamma}(\mathbb{R}\times(0,+\infty))} \leq c(\|v_{n-1}\|_{C^{\gamma}(\mathbb{R}\times(0,+\infty))} + \|u_0\|_{C^{2+\gamma}(\mathbb{R})})$$

where c is a positive constant not depending on v_{n-1} and u_0.

Now we solve (3.7) given v_0 and u_n. Once again, the system can be led to a simpler one by a change of variables, namely

$$w_n := v_n - v_0 - \frac{\mu}{\nu}(u_n - u_0).$$

Doing the maths, w_n is the solution to the problem

$$\begin{cases} \partial_t w_n - d\Delta w_n = \tilde{f}(x,y,t,w_n), & (x,y) \in \Omega,\ t > 0, \\ \nu w_n(x,0,t) - d\partial_y w_n(x,0,t) = 0, & x \in \mathbb{R},\ t > 0, \\ w_n|_{t=0} = 0, & (x,y) \in \Omega, \end{cases}$$

where

$$\tilde{f}(x,y,t,w_n) = f(y, w_n + v_0 + \frac{\mu}{\nu}(u_n - u_0)) - \frac{\mu}{\nu}(\partial_t u_n - d(u_n - u_0)'').$$

However this system is still more complicated because of the dependence on w_n of $\tilde{f}$. We can apply Theorem 13.24 in [16] that guarantees the existence of a classical solution v_n in the weighted Hölder space $H_{2+\alpha}^{(-1-\delta)}(\Omega)$ under some conditions, that we are now verifying. The operator $\partial_t w - d\Delta w$ is uniformly parabolic and, as $-d$ and $\tilde{f}$ do not depend on ∇w, it is easy to say they are $O(|\nabla w|^2)$. Moreover $\tilde{f}$ is Hölder continuous in x, y, t, and w, because all its components are so. Slitly more complicated is to show that there exists $b_0,\ b_1, M_0 \in \mathbb{R}$ such that

$$w\tilde{f}(x,y,t,w) \leq b_0 d|\nabla w|^2 + b_1 w^2$$

for all $w > M_0$. But it holds

$$\begin{aligned} \frac{\tilde{f}(x,y,t,w)}{w} &\leq \frac{f(y, w + v_0 + \frac{\mu}{\nu}(u_n - u_0))}{w + v_0 + \frac{\mu}{\nu}(u_n - u_0)} \frac{w + v_0 + \frac{\mu}{\nu}(u_n - u_0)}{w} + \frac{\frac{\mu}{\nu}(\partial_t u_n - d(u_n - u_0)'')}{w} \\ &\leq f_v(y,0)\frac{w + v_0 + \frac{\mu}{\nu}(u_n - u_0)}{w} + \frac{\frac{\mu}{\nu}(\partial_t u_n - d(u_n - u_0)'')}{w} \\ &\leq b_1 \end{aligned}$$

so for w sufficiently large there is a constant b_1 for which the inequality is satisfied.

Thus we found for all $n \in \mathbb{N}$ a couple of classical solutions (u_n, v_n).

Step 2. We want uniform L^∞ estimates for u_n and v_n in the whole set of definition of them.

We already have that for each $n \in \mathbb{N}$ the functions u_n and v_n are bounded, but we want to have a uniform bound for all $n \in \mathbb{N}$. We are showing by induction that

$$0 \leq u_n \leq \frac{H}{\mu}, \qquad 0 \leq v_n \leq \frac{H}{\nu},$$

where

$$H := \max\{\nu M,\ \nu \|v_0\|_{\mathrm{L}^\infty(\Omega)},\ \mu \|u_0\|_{\mathrm{L}^\infty(\mathbb{R})}\}.$$

The case $n = 0$ is trivial, since the initial couple is $(0, v_0)$. Given that the result is true for $n-1$, we show that it is true for n. The function $\underline{u} = 0$ is clearly a subsolution for the system (3.6) and the function $\overline{u} = \frac{H}{\mu}$ is a supersolution for the same system. We now want to apply the Phragmèn-Lindelöf principle, which is a version of the maximum principle that applies for parabolic operators in unbounded domains [19]. It requires the conditions

$$\limsup_{R\to+\infty}\{e^{-\alpha R^2} \min_{|x|=R}(u_n - \underline{u})\} = 0, \qquad \limsup_{R\to+\infty}\{e^{-\beta R^2} \min_{|x|=R}(\overline{u} - u_n)\} = 0$$

for suitable $\alpha > 0,\ \beta > 0$. Since by the first step the function u_n is bounded, the limit is 0 and the hypotheses are satisfied. Thus, the principle gives that that $u_n - \underline{u} \geq 0$ and $\overline{u} - u_n \geq 0$, so $\frac{H}{\mu} \geq u_n \geq 0$.

For the study of v_n a supplementary passage is needed because of the presence of the nonlinear term f. There exists $\sigma > 0$ large enough to guarantee that the function $g(y,t,v) = f(y,v)e^{-\sigma t} - \sigma v$ is decreasing in v; in fact, taking $v > w$ one has

$$g(y,t,v) - g(y,t,w) = \frac{f(y,v) - f(y,w)}{v - w} \cdot e^{-\sigma t}(v - w) - \sigma(v - w)$$

and the fraction is bounded because f is Lipschitz. Then the function $s_n = v_n e^{-\sigma t}$ is a solution to the system

$$\begin{cases} \partial_t s_n(x,y,t) - d\Delta s_n(x,y,t) = g(y,t,s_n), & (x,y) \in \Omega,\ t > 0, \\ \nu s_n(x,0,t) - d\partial_y s_n(x,0,t) = \mu u_n(x,t)e^{-\sigma t}, & x \in \mathbb{R},\ t > 0, \\ s_n(x,y,0) = v_0(x,y), & (x,y) \in \Omega. \end{cases} \tag{3.9}$$

The function $\underline{s} = 0$ is a subsolution to the system; the function $\overline{s} = \frac{H}{\nu}$ is a supersolution, since $f(y,s) < 0$ for $s > M$. Once again, $s_n - \underline{s}$ and $\overline{s} - s_n$ are bounded, but to apply the Pharagmèn-Lindelöf principle it is also required to have

$$\partial_t(s_n - \underline{s}) - d\Delta(s_n - \underline{s}) = g(y,t,s_n) - g(y,t,s) \geq 0$$

and the analogue for $\overline{s}s_n$. The last inequality is strict at t=0; then, if there is a point $(\tilde{x}, \tilde{y}, \tilde{t}) \in \Omega \times (0, +\infty)$ where the equality holds and $\tilde{t}$ is the first time for which this happens, we can reach an absurd using the strong maximum principle in a restricted cylinder of $\Omega \times (0, \tilde{t})$ having $(\tilde{x}, \tilde{y}, \tilde{t})$ on the top face. Hence $s_n - \underline{s} \geq 0$ and $\overline{s} - s_n \geq 0$, thus $\frac{H}{\nu} \geq v_n \geq 0$.

Step 3. We find some $W_p^{2,1}$ estimates on compact sets for u_n and v_n, uniformly in $n \in \mathbb{N}$.

We fix $\rho > 0$, $\delta > 0$, $T > 0$ and $1 < p < \infty$. The Agmon-Douglis-Nirenberg interior estimates we have

$$\begin{aligned}
\|u_n\|_{\mathrm{W}_p^{1,2}([-\rho,\rho]\times[\delta,T-\delta])} &\le c\left(\|v_{n-1}(x,0,t)\|_{\mathrm{L}^p([-\rho-1,\rho+1]\times[0,T])} + \|u_0\|_{\mathrm{L}^\infty([-\rho-1,\rho+1])}\right)\\
&\le c\left(\|v_{n-1}\|_{\mathrm{L}^\infty(\Omega)} + \|u_0\|_{\mathrm{L}^\infty(\mathbb{R})}\right)\\
&\le cH
\end{aligned}$$

where c is a constant depending on D, ν, μ, ρ, δ, p, T and H was defined in Step 2. Consider $Q_\rho := (-\rho,\rho)\times(0,\rho) \subset \Omega$. Using the same estimates for v_n in $Q_\rho \times [\delta, T-\delta]$ we have

$$\begin{aligned}
\|v_n\|_{\mathrm{W}_p^{2,1}(Q_\rho\times[\delta,T-\delta])} &\le c\big(\|f(v_n)\|_{\mathrm{L}^p(Q_\rho\times[\frac{1}{2}\delta,T-\frac{1}{2}\delta])} + \|v_0\|_{\mathrm{L}^\infty(Q_{\rho+1})} +\\
&\quad + \|u_n\|_{\mathrm{W}_p^{1,2}([-\rho-1,\rho+1]\times[\frac{1}{2}\delta,T-\frac{1}{2}\delta])}\big)\\
&\le cH
\end{aligned}$$

where c depends on ν, μ, d, f, ρ, δ, T, ε. Hence, on compact sets the functions u_n and v_n have bounded $W_p^{2,1}$ norms, uniformly on $n \in \mathbb{N}$.

Step 4. We show the existence of a subsequence of $(u_n, v_n)_{n\in\mathbb{N}}$ that is converging to a couple (u, v) in $C^{2,1}$ on compact sets.

We chose $p > 3$; then we can use the general Sobolev inequalities for $k > \frac{n}{p}$ where n is the dimension and k is the order of the derivative (see for examples Theorem 6 in Chapter 5, Section 6 of [9]). We have

$$\begin{aligned}
\|u_n\|_{C^{0,\gamma}([-\rho,\rho]\times[\delta,T-\delta])} &\le c\|u_n\|_{\mathrm{W}_p^{2,1}([-\rho,\rho]\times[\delta,T-\delta])} \le cH,\\
\|v_n\|_{C^{0,\gamma}(Q_\rho\times[\delta,T-\delta])} &\le c\|v_n\|_{\mathrm{W}_p^{2,1}(Q_\rho\times[\delta,T-\delta])} \le cH
\end{aligned}$$

where c is a constant not depending on n and

$$\gamma = \left\lfloor \frac{2}{p} \right\rfloor + 1 - \frac{2}{p}.$$

Also, for $p \to \infty$ we have $\gamma \to 1$.

Let us show that the function $f_n(x,y,t) := f(y, v_n)$ is in $C^{0,\gamma}$ on the bounded set where we have estimates on v_n, thus

$$\begin{aligned}
&\|f_n(x,y,t)\|_{C^{0,\gamma}(Q_\rho\times[\delta,T-\delta])}\\
&\quad := \sup_{Q_\rho\times[\delta,T-\delta]} \frac{|f(y,v_n(x,y,t)) - f(\overline{y}, v_n(\overline{x},\overline{y},\overline{t}))|}{|(x,y,t)-(\overline{x},\overline{y},\overline{t})|^\gamma}\\
&\quad \le \sup_{Q_\rho\times[\delta,T-\delta]} \frac{|f(y,v_n(x,y,t)) \mp f(\overline{y}), v_n(x,y,t) - f(\overline{y}, v_n(\overline{x},\overline{y},\overline{t}))|}{|(x,y,t)-(\overline{xy},\overline{t})|^\gamma}\\
&\quad \le \sup_{[0,\rho]} \frac{|f(y,v_n(x,y,t)) - f(\overline{y}, v_n(x,y,t))|}{|y-\overline{y}|^\gamma}\\
&\qquad\qquad + \mathrm{Lip}_v f \sup_{Q_\rho\times[\delta,T-\delta]} \frac{|v_n(x,y,t) - v_n(\overline{x},\overline{y},\overline{t})|}{|(x,y,t)-(\overline{x},\overline{y},\overline{t})|^\gamma}\\
&\quad \le \mathrm{Lip}_y f + \mathrm{Lip}_v f \cdot \|v_n\|_{C^{0,\gamma}(Q_\rho\times(\delta,T-\delta))}.
\end{aligned} \tag{3.10}$$

Now we use Schauder's Theorem (see for example Theorem 10.1 in Chapter IV of [17]) to estimate the $C^{2,\gamma}$ norm of u_n and v_n in some restriction of the domains. It holds that

$$\begin{aligned}
\|u_n\|_{C^{2,1,\gamma}([-\rho,\rho]\times[\delta,T-\delta])} &\leq c\big(\|v_{n-1}\|_{C^{0,\gamma}(Q_\rho\times[\delta,T-\delta])} \\
&\qquad + \|u_0\|_{C^{2,\gamma}[-\rho,+\rho]} + \|u_n\|_{C^{0,\gamma}([-\rho,\rho]\times[\delta,T-\delta])}\big) \\
&\leq cH \\
\|v_n\|_{C^{2,1,\gamma}(Q_\rho\times[\delta,T-\delta])} &\leq c\big(\|f_n\|_{C^{0,\gamma}(Q_\rho\times[\delta,T-\delta])} \\
&\qquad + \|v_0\|_{C^{2,\gamma}(Q_\rho)} + \|v_n\|_{C^{0,\gamma}(Q_\rho\times[\delta,T-\delta])}\big) \\
&\leq cH.
\end{aligned}$$

Hence, we show that the $C^{2,1,\gamma}$ norms of u_n and v_n are uniformly bounded on every compact subset of $\Omega \times (0, +\infty)$. By the compact injection $C^{1,2,\gamma} \hookrightarrow C^{1,2}$, we have that the sequence $\{(u_n, v_n)\}_{n\in\mathbb{N}}$ has a subsequence $\{(u_{n_k}, v_{n_k})\}_{k\in\mathbb{N}}$ converging in $C^{1,2}$ on every compact subset of $\Omega \times (0, +\infty)$; thanks to the arbitrariness of the domain, the limit holds in $C^{1,2}_{loc}$. Let us call (u, v) the limit of the subsequence.

Step 5. We show that (u, v) is a solution to (M).

If we show that the sequences $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ are increasing, then the whole sequence $\{(u_n, v_n)\}_{n\in\mathbb{N}}$ converges to (u, v). Hence, substituting u and v in (3.6) and (3.7), it is clear that (u, v) is a solution of (M).

So the only thing we have to do is to show that $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ are increasing. First we prove the initial step; the starting function is $v_1 \equiv 0$, and we have

$$\partial(u_3 - u_2) - D(u_3 - u_2)'' + \mu(u_3 - u_2) \geq \nu(v_2 - v_1)(x, 0, t) \geq 0$$

because of v_2 is nonnegative. As in Step 2, we use the parabolic maximum principle on unbounded domains and we have that $u_3 \geq u_2$. The induction step for u_n is identical, given the inequality for v_n. The initial step for v_n is trivial, since $v_1 \equiv 0$. Then, as in Step 2, we can apply the transformation $g(y, t, v) = f(y, v)e^{-\sigma t} - \sigma v$ and $s_n = v_n e^{-\sigma t}$, that preserves the order of v_{n-1} and v_n. Then we can write

$$(s_n - s_{n-1})h(x, y, t) = g(y, t, s_n) - g(y, t, s_{n-1}) = \frac{g(y, t, s_n) - g(y, t, s_{n-1})}{s_n - s_{n-1}} \cdot (s_n - s_{n-1})$$

and, as we showed above, h is bounded. Hence we have

$$\partial_t(s_n - s_{n-1})(x, y, t) - d\Delta(s_n - s_{n-1})(x, y, t) + h(x, y, t)(s_n - s_{n-1}) = 0.$$

but on the boundary $s_n(x, 0, t) \geq s_{n-1}(x, 0, t)$. Hence by the maximum principle $s_n \geq s_{n-1}$ and the prove is completed. $\square$

Chapter 3

Asymptotic behaviour of solutions

The most urgent question that arises studying a model in population dynamics is if the species will survive or not. We want to investigate such question for model we defined in the introduction. Our aim is to find necessary and sufficient conditions which guarantee the densities of population not to converge to 0 as time goes to infinity. We are able to prove a complete result only in some special cases that we will specify later, but, under the general assumptions we made on the model, still some significant results on the asymptotic behaviour of solutions hold. This chapter is devoted to two theorems for invasion and extinction, as defined in Chapter 1, and gives some ideas about the open cases.

Section 3.1 is an introduction to the tools that are used in the statement of the theorems. Then in Section 3.2 we give a sufficient condition for survival and invasion. A sufficient condition for extinction is given in Section 3.3. The conclusive section concerns a case when the conditions are also sufficient as well as further comments on the hypotheses.

3.1 Principal eigenvalues in bounded domains and in unbounded periodic domains

Taking inspiration from the work of Berestycki, Hamel and Roques [3], we would like to describe the long time behaviour of solutions to our problem (M) in terms of fine properties of eigenvalues and eigenfunctions related to single equations of the system. We start with giving a short introduction to the subject.

First of all, we recall the definition of eigenvalue and eigenfunction in a ball; the same holds also for any bold smooth domain (compare [19]). Let $B_R := B_R(0,R)$ be the ball in $\mathbb{R}^2$ of radius $R > 0$ and centre $(0,R)$, so that $(0,0)$ is on the boundary of B_R. Let $f(y,v) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the extension by periodicity on the y variable of the reaction term of the model. The eigenvalue problem related to the equation $-d\Delta v = f(y,v)$ is

$$\begin{cases} -d\Delta \phi_R - f_v(y,0)\phi_R = \lambda \phi_R, & \text{in } B_R(0,R), \\ \phi_R|_{\partial B_R(0,R)} = 0, \quad \|\phi_R\|_{\mathrm{L}^\infty(B_R(0,R))} = 1, \end{cases} \tag{1.1}$$

and its solution is a couple of eigenvalue and eigenfunction (λ, ϕ_R) where $\lambda \in \mathbb{R}$

and $\phi_R \not\equiv 0$ is a real function defined on B_R. Krein-Rutman theory [15] provides existence and uniqueness of an eigenfunction φ_R, positive in B_R, related to a simple real eigenvalue λ_R, that also satisfies $\lambda_R \leq \Re(\lambda)$ for all eigenvalue λ, where $\Re(\lambda)$ is the real part of λ, which is possibly complex. We are showing these results in the next subsection. The value λ_R and the positive function φ_R are called respectively *principal eigenvalue* and *principal eigenfunction* for the problem $-d\Delta v = f(y,v)$ in the ball B_R with Dirichlet boundary condition.

It is also well known that the principal eigenvalue λ_R decreases as R increases; this is also shown in the proof of Lemma 3.1.7 of Subsection 3.1.2. Thus, the quantity

$$\lim_{R\to+\infty} \lambda_R$$

is well defined but it may be equal to $-\infty$. The limit is called *generalized principal eigenvalue* [5]. Remark that $\lim_{R\to+\infty} \lambda_R < 0$ if and only if $\lambda_R < 0$ for some R.

Beyond that, we will also need to generalize the concept of eigenvalue in the case of a periodic medium in dimension 1. A couple (λ, ψ) with $\psi \not\equiv 0$ is a solution to the periodic eigenvalue problem if it solves

$$\begin{cases} -d\psi'' - f_v(y,0)\psi = \lambda\psi, & y \in \mathbb{R}; \\ \|\psi\|_{L^\infty[0,\ell]} = 1, \quad \psi(y) = \psi(y+\ell) \ \forall y \in \mathbb{R}. \end{cases} \tag{1.2}$$

Krein-Rutman theory still applies thanks to the periodicity of f in the variable y, as shown in next the subsection. We call λ_0 and ϕ respectively *principal eigenvalue* and *principal eigenfunction to the periodic problem.*

The theorem presented in Subsection 3.1.1 assures the existence and uniqueness of a couple (λ_0, ϕ) solution to (1.2) such that $\phi > 0$ and $\lambda_0 \leq \Re(\lambda)$ for every eigenvalue of (1.2) λ. The same result can be derived for (λ_R, ϕ_R) in the case of a bounded domain with only few changes in the proofs.

In [3] it is shown an interesting and non obvious property that connects the eigenvalues in the bounded domain and in the unbounded domain:

Proposition 3.1.1. *Using the notation from above,* $\lim_{R\to+\infty} \lambda_R = \lambda_0$.

Even if the proposition seems natural and predictable, it may not hold true for all the eigenvalues problems. Indeed, the arguments we use are strictly related to the form of the operator $-\Delta v$, which is elliptic and self-adjoint.

Proposition 3.1.1 is a key point of this thesis and a significant step toward a deep comprehension of the behaviour of solutions. In fact, this permits to connect the two main tools that are used in the theorems explaining the asymptotic behaviours of solutions: the principal eigenfunction in the ball, that plays a role in showing the existence of a positive solution, and the principal eigenfunction in the unbounded domain, which is related to the non-existence of a positive solution. Therefore, in further theorems we will use λ_0 as a threshold value between the two possibilities.

We are now presenting the existence and uniqueness of λ_0 and λ_R in the first subsection. Then, we will prove Proposition 3.1.1 in the second subsection.

3.1.1 Principal eigenvalue and eigenfunction for a periodic problem

In order to be complete and self-contained, we show the existence and uniqueness of the principal eigenvalue and eigenfunction of the problems (1.1) and (1.2). The system (1.2) has the peculiarity of being defined in a periodic medium, hence the eigenfunctions are required to be periodic. As for the classic notion of principal eigenvalue, the general approach given by Krein-Rutman theorem applies to our differential operator. We start presenting the theory of Krein-Rutman and a consequent statement that applies to nonlinear partial differential operators. Then we will show the precise result for the two cases we are interested in.

We start by giving some notation. Let X be a Banach space. A *cone* $K \subset X$ is a closed convex set such that:

1. for all $\lambda \geq 0$ one has $\lambda K \subset K$;

2. $K \cap (-K) = \{0\}$.

A cone can induce a partial ordering on X in the following way: if $v - u \in K$ then we say $u \leq v$. If X is taken with this partial order, then K is called the *positive cone* of X. If $\overline{K - K} = X$, i.e. if the set $\{v - u \ \mid \ v,\ u \in K\}$ is dense in X, then K is a *total cone*. If the interior $\mathring{K}$ is non empty, then it is called a *solid cone*. It easy to see that a solid cone is also a total cone: let the ball $B_r(x_0)$ be contained in $\mathring{K}$. For all $x \in X,\ x \neq 0$ we have that there exists $\tilde{x} \in B_r(x_0)$ and $r' < r$ such that

$$\tilde{x} - x_0 = \frac{r'}{\|x\|_X} x$$

where $\|\cdot\|_X$ is the norm that make X a Banach space. But for all $\lambda \geq 0$ we have $\lambda K \subset K$. Choosing $\lambda = \dfrac{\|x\|_X}{r'}$ we obtain $\lambda\tilde{x},\ \lambda x_0 \in K$ and

$$\lambda\tilde{x} - \lambda x_0 = x.$$

Let Ω be a bounded domain in $\mathbb{R}^N$. For example, the cone K of the nonnegative functions in $X = \mathrm{L}^p(\Omega)$ is a cone satisfying $K - K = X$, but $\mathring{K}$ is empty. Instead the cone C of nonnegative functions in $Y = W^{k,p}$ is a solid cone for $k \geq 2,\ p > 1$.

In the dual space X^* we are able to define the *dual cone* of K as the set $K^* := \{l \in X^* \,|\, l(x) \geq 0\ \forall x \in K\}$. K^* is not necessarily a cone. It is indeed a closed and convex set, and $\lambda K^* \subset K^*$ for any $\lambda \geq 0$; but it is not generally true that $K^* \cap (-K)^* = \{0\}$. However if K is a total cone then the last condition is verified and K^* is a cone in X^*, as we are showing now. Take $l \in K^* \cap (-K^*)$, then $l(x) \geq 0$ and $-l(x) \geq 0$ for all $x \in K$, so $l(x) = 0$ for all $x \in K$. But K being total means $\overline{K - K} = X$, so $l(x) = 0$ for all $x \in X$. It must be $l \equiv 0$ and the condition is satisfied.

We recall that the *spectral radius* of T is the largest absolute value of an eigenvalue of T. Last, T^* denotes the dual operator of T. We are now ready to give the Krein-Rutman theorem, taken by [15].

Theorem 3.1.2 (Krein-Rutman). *Let X be a Banach space, $K \subset X$ a total cone and $T : X \to X$ a compact linear operator strongly positive with respect to the ordering induced by K (i.e., $T(K) \subset K$); let the spectral radius $r(T)$ be positive. Then $r(T)$ is an eigenvalue with an eigenvector $u \in K \setminus \{0\}$ such that $T(u) = r(T)u$. Moreover $r(T^*) = r(T)$ is an eigenvalue of T^* with eigenvector $v^* \in K^*$.*

This widely known result apply, among the others, to the theory of principal eigenvalues. The link between the two of them is the following theorem. Let us first recall that an eigenvalue r of T is *simple* if there exits $v \neq 0$ such that $Tv = rv$ and if $(T - rI)^n w = 0$ then $w \in < v >$.

Theorem 3.1.3. *Let X be a Banach space, $K \subset X$ a solid cone, $T : X \to X$ a compact linear operator which is strongly positive, that is $T(K \setminus \{0\}) \subset K \setminus \{0\}$. Then*

- *$r(T) > 0$, and $r(T)$ is a simple eigenvalue with an eigenvector $v \in \mathring{K}$; there is no other eigenvalue with a positive eigenvector.*
- *$\Re(\lambda) < r(T)$ for all eigenvalues $\lambda \neq r(T)$.*

Applying the latter to the objects involved in (1.2), we can show the existence and uniqueness of the principal eigenvalue and eigenvector satisfying the two usual properties. The presented proof is an adaptation of the proof for the eigenvalue problem in a bounded domain given by Rafael de la Llave (University of Texas at Austin) in some lecture notes.

Theorem 3.1.4. *There exists a unique solution (λ_0, ϕ) to the eigenvalue problem (1.2) such that $\phi > 0$ and for any λ eigenvalue, possibly complex, we have*

$$\lambda_0 \leq \Re(\lambda)$$

and equality holds only if $\lambda_0 = \lambda$.

Proof. Let $\mathcal{S}$ be the circle of length ℓ, then $X = C^{1,\alpha}(\mathcal{S})$ is the set of function that are periodic in $\mathbb{R}$ with period ℓ. Remark that the natural norm on this set is the same norm of $C^{1,\alpha}[0, \ell]$, thus

$$\|u\|_{C^{1,\alpha}(\mathcal{S})} = \sup_{x\in[0,\ell]} |u(x)| + \sup_{x\in[0,\ell]} |u'(x)| + \sup_{x,y\in[0,\ell],x\neq y} \frac{|u'(x) - u'(y)|}{|x - y|^\alpha}.$$

We set $K = \{v \in C^{1,\alpha}(\mathcal{S}) \,|\, v \geq 0\}$ and we claim that K is a solid cone. Trivially K is a cone: it is closed and convex; also, $v \geq 0$ implies $\theta v \geq 0$ for all $\theta \geq 0$; then $v \geq 0$ and $v \leq 0$ implies $v = 0$. If K contains a ball, then it is also a solid cone. Take $u_0 \in \mathring{K}$, thus u_0 periodic on $\mathbb{R}$ with period ℓ and $u > 0$. On the compact set $[0, \ell]$ we have $u > 2\varepsilon$ for some $\varepsilon > 0$. Hence each function v in the ball with centre u_0 and radius ε, $B_\varepsilon(u_0)$, satisfies $v > \varepsilon$ in $\mathcal{S}$, thus $B_\varepsilon(u_0) \subset \mathring{K}$.

By the hypothesis we took on our model, the function $f_v(y, 0)$ is bounded, hence there is $\xi \in \mathbb{R}$ such that $f_v(y, 0) + \xi < 0$. Now consider the operator $T_\xi : X \to X$, where $T_\xi(u) = v$ if

$$-d\, v'' - (f_v(y, 0) + \xi)v = u.$$

This operator is well-posed if for all $u \in X$ there is $v \in X$ satisfying the previous equation. Take the Sobolev space $H^1(\mathcal{S})$. Putting the problem in a weak formulation, we can use Lax-Milgram theorem for coercive operators. In fact, taking $\varphi \in H^1(\mathcal{S})$, we have

$$< T_\xi v, \varphi >= \int_0^\ell d\, v'\varphi' - (f_v(y,0)+\xi)v\varphi\, dy.$$

The operator we obtain is coercive and continuous with respect to the norm of $H^1(\mathcal{S})$. The problem is thus

$$\int_0^\ell d\, v'\varphi' - (f_v(y,0)+\xi)v\varphi\, dy = \int_0^\ell u\varphi\, dy.$$

Also $\int_0^\ell u\varphi\, dy$ a linear and continuous function of φ for any $u \in H^1(\mathcal{S})$ fixed. This completes the hypotheses of Lax-Milgram theorem, hence for all $u \in H^1(\mathcal{S})$ there exists $v \in H^1(\mathcal{S})$ solving $T_\xi(u) = v$. Having by Sobolev injection (Theorem 6 in Chapter 5, Section 6 of [9]) $H^1(\mathcal{S}) \hookrightarrow C^{0,\alpha}(\mathcal{S})$, we can use Schauder estimates for elliptic equations (compare [12], Chapter VI), having that the problem $T_\xi(u) = v$ is satisfied in classical sense; on $[\ell, 2\ell] \subset [\ell-1, 2\ell+1]$ Schauder estimates gives

$$\|v\|_{C^{2,\alpha}[\ell,2\ell]} \le k(\|v\|_{C^{0,\alpha}[\ell-1,2\ell+1]} + \|u\|_{C^{0,\alpha}[\ell-1,2\ell+1]}).$$

But $\|v\|_{C^{0,\alpha}[\ell-1,2\ell+1]} = \|v\|_{C^{0,\alpha}[\ell,2\ell]}$ and the latter is bounded by $\|u\|_{H^1(\mathcal{S})}$ as a consequence of Lax-Milgram theorem [8]. Last, the compact injection $C^{2,\alpha}(\mathcal{S}) \hookrightarrow C^{1,\alpha}(\mathcal{S})$ implies that T_ξ is a compact operator.

Take $u > 0$, then

$$-d\, v'' - (f_v(y,0)+\xi)v > 0.$$

Suppose by the absurd that $\min v \le 0$, so there exists a minimum point $y_0 \in [0,\ell]$ such that $v(y_0) \le 0$. Then, $v''(y_0) \ge 0$ and $(f_v(y_0,0)+\xi)v(y_0) \ge 0$. But so we get

$$0 \ge -d\, v'' - (f_v(y,0)+\xi)v = u > 0$$

and this is impossible, so it must be $\min v > 0$, hence $v > 0$. So T_ξ is strictly positive.

Being all hypotheses satisfied we can apply Theorem 3.1.3, which guarantees the existence and uniqueness of a positive number $\mu := r(T_\xi)$ and of $v \in \mathring{K}$ that satisfy $T_\xi(v) = \mu v$, that is

$$-d\, v'' - (f_v(y,0)+\xi)v = \frac{1}{\mu}v.$$

Hence the couple $\left(\frac{1}{\mu} + \xi, v\right)$ is the unique solution to the eigenvalue problem (1.2) satisfying $v > 0$ and moreover we have

$$\frac{1}{\mu} + \xi > \Re\Big(\frac{1}{\nu} + \xi\Big)$$

for any other ν eigenvalue to L_ξ, that means $\lambda_0 > \Re(\lambda)$ for any λ first term of a solution to (1.2). Thus the proof is complete. □

Now we prove the same result for the problem in the ball. The proof is taken from the lecture notes cited above.

Theorem 3.1.5. *There exists a unique solution* (λ_R, φ_R) *to the eigenvalue problem* (1.1) *such that* $\varphi_R > 0$ *and for any* λ *eigenvalue, possibly complex, we have*

$$\lambda_R \leq \Re(\lambda)$$

and equality holds only if $\lambda_0 = \lambda$.

Proof. Take $X := C_0^{1,\alpha}(B_R(0,R))$ with the usual norm. We have to show that the set $K := \{v \in X \mid v \geq 0 \text{ in } B_R(0,R)\}$ is a solid cone. As in the previous proof, the only nontrivial passage is to show that K has nonempty interior, but this time notice that any function $v \in X$ vanishes on $\partial B_R(0,R)$. Take $u_0 \in K$ such that for every unitary vector $\zeta \in \mathbb{R}^2$ normal to $\partial B_R(0,R)$ in the point P and pointing in an outward direction; we have

$$\frac{\partial u_0}{\partial \zeta}(P) < 0.$$

Since $\partial B_R(0,R)$ is a compact set, there exists $\varepsilon > 0$ such that $\frac{\partial u_0}{\partial \zeta} > 2\varepsilon$ in $\partial B_R(0,R)$. Fix $R' < R$ such that $R - R' << \varepsilon$. There exists $\delta > 0$ such that $u_0 > 2\delta$ in $B_{R'}(0,R)$. Let $\varrho := \min\{\varepsilon,\ \delta\}$. Hence taking $B_\varrho(u_0) \subset X$, for all $u \in B_\varrho(u_0)$ one has that $u > \delta$ in $B_{R'}(0,R)$ and $\frac{\partial u}{\partial \zeta} > \varepsilon$. Using an elementary application of MacLaurin expansion one can see that in the annulus of thickness $R - R'$ in $B_R(0,R)$ the function u is still positive, hence $u \in K$. Then $u_0 \in \mathring{K}$ and K is a solid cone.

Let $\xi \in \mathbb{R}$ be such that $f_v(y,0) + \xi < 0$ in $B_R(0,R)$. Consider the operator $T_\xi : X \to X$ defined by $T_\xi(u) = v$ if

$$-d\Delta v - (f_v(y,0) + \xi)v = u.$$

We show that for any $u \in X$ we have that $T_\xi(u)$ is well defined. Take the Sobolev space $H_0^1(B_R(0,R)) \supset X$. For any $\varphi \in H_0^1(B_R(0,R))$ we have

$$< T_\xi v, \varphi >= \int_{B_R(0,R)} d\nabla v \nabla \varphi - f_v(y,0) v\varphi \, dxdy$$

and the problem is

$$\int_{B_R(0,R)} d\nabla v \nabla \varphi - f_v(y,0) v\varphi \, dxdy = \int_{B_R(0,R)} u\varphi \, dxdy.$$

The hypotheses of Lax-Milgram theorem are easily verified, thus for all $u \in X$ there exists $v \in H_0^1(B_R(0,R))$ such that $T_\xi(u) = v$ in the weak sense. Now by Sobolev injection $H_0^1(B_R(0,R)) \hookrightarrow C_0^{0,\alpha}(B_R(0,R))$ ([9], Theorem 6 in Chapter 5, Section 6) we have

$$\|u\|_{C_0^{0,\alpha}(B_R(0,R))} \leq c\|u\|_{H_0^1(B_R(0,R))}$$

and using Schauder theory ([12]) one has that v is a classical solution and

$$\|v\|_{C_0^{2,\alpha}(B_R(0,R))} \leq c\left(\|u\|_{C_0^{0,\alpha}(B_R(0,R))} + \|v\|_{C_0^{0,\alpha}(B_R(0,R))}\right)$$

Using again that $\|v\|_{C_0^{0,\alpha}(B_R(0,R))}$ is bounded by $\|u\|_{H_0^1(B_R(0,R))}$ and the compact injection $C_0^{2,\alpha}(B_R(0,R)) \hookrightarrow C_0^{0,\alpha}(B_R(0,R))$, the operator T_ξ is compact.

We now prove that T_ξ is strongly positive. Take $u \in \mathring{K}$, then $T_\xi(u) = v$ satisfies

$$-d\Delta v - (f_v(y,0) + \xi)v = u.$$

We have to show that $v \in \mathring{K}$ and by what we said at the beginning of the proof it is sufficient to show that $v > 0$ in $B_R(0,R)$ and $\frac{\partial v}{\partial \zeta} < 0$, being ζ the normal unitary vector pointing outward. Since $u \geq 0$ and $f_v(y,0)+\xi < 0$, by the maximum principle, the first holds. Since $v = 0$ on $\partial B_R(0,R)$, on any point at the boundary we can use Hopf's principle (see [19]) and have $\frac{\partial v}{\partial \zeta} < 0$. Thus T_ξ is strongly positive.

The conclusion follows exactly as in the previous proof. $\square$

3.1.2 Proof of Proposition 3.1.1

We first prove two lemmas and then present the proof of Proposition 3.1.1, showing that $\lim\limits_{R\to+\infty} \lambda_R = \lambda_0$. Actually, we will prove a slightly stronger result. The proof of this subsection are all taken from [3].

Take $z \in \mathbb{R}$ and denote with $B_R(0,z)$ the ball with radius $R > 0$ and centre $(0,z)$. Let λ_R^z and φ_R^z be respectively the principal eigenvalue and the principal eigenfunction, solution to the problem

$$\begin{cases} -d\Delta\varphi_R^z - f_v(y,0)\varphi_R^z = \lambda_R^z\varphi_R^z, & \text{in } B_R(0,z), \\ \varphi_R^z|_{\partial B_R(0,z)} = 0, \quad \|\varphi_R^z\|_{L^\infty(B_R(0,z))} = 1, \quad \varphi_R^z > 0 & \text{in } B_R(0,z). \end{cases}$$

For all the discuss that follows, we continue to denote the principal eigenfunction and eigenvalue of the periodic problem with ϕ and λ_0, as defined at the beginning of this section. The following lemma presents an intuitive fact:

Lemma 3.1.6. *For all $z \in \mathbb{R}$ and $R > 0$, one has $\lambda_R^z > \lambda_0$.*

Proof. Suppose $\lambda_R^z - \lambda_0 \leq 0$. The function φ_R^z is positive in $B_R(0,z)$, vanishes on $\partial B_R(0,z)$ and solves

$$-d\Delta\varphi_R^z - f_v(y,0)\varphi_R^z - \lambda_0\varphi_R^z = (\lambda_R^z - \lambda_0)\varphi_R^z \leq 0 \tag{1.3}$$

in the ball $B_R(0,z)$. The function ϕ is positive on the closed ball $\overline{B_R(0,z)}$, hence for all $\kappa > 0$ small enough we have $\kappa\varphi_R^z < \phi$. Take

$$\kappa^* = \sup\{\kappa > 0 \mid \kappa\varphi_R^z < \phi \text{ in } B_R(0,z)\}.$$

It is trivial that $\kappa^* > 0$ and also, by continuity of the two functions, $\kappa^*\varphi_R^z \leq \phi$ in $\overline{B_R(0,z)}$, and there is $\overline{y}$ such that $\kappa^*\varphi_R^z(\overline{y}) = \phi(\overline{y})$. Recalling that $\varphi_R^z|_{\partial B_R(0,z)} = 0$ while ϕ is always positive, it must be $\overline{y} \in B_R(0,z)$. We have also

$$-d\Delta\phi - f_v(y,0)\phi = \lambda_0\phi$$

in $B_R(0,z)$, while in the same set the inequality (1.3) holds, thus φ^z is a subsolution of the equation satisfied by ϕ. Hence, since the nonnegative function $\phi - \kappa^*\varphi_R^z$ touches its minimum 0 inside $B_R(0,z)$, for the strong maximum principle we have $\phi \equiv \kappa^*\varphi_R^z$ in $B_R(0,z)$. But this is impossible because of the boundary conditions on φ_R^z. Hence it must be $\lambda_R^z > \lambda_0$. $\square$

Lemma 3.1.7. *For all $z \in \mathbb{R}$, the function $R \mapsto \lambda_R^z$ is decreasing.*

Proof. Let $0 < R_1 < R_2$ be two positive numbers and $z \in \mathbb{R}$. Suppose by the absurd that $\lambda_{R_2}^z \geq \lambda_{R_1}^z$. Then we have $\varphi_{R_2}^z > 0$ in the closed ball $\overline{B_{R_1}}$. Applying the same procedure of the previous lemma with $\varphi_{R_2}^z$ at the place of ϕ and $\varphi_{R_1}^z$ at the place of φ_R^z, we find an absurd using the strong maximum principle. Hence it must be $\lambda_{R_2}^z < \lambda_{R_1}^z$. $\square$

We end the introduction about principal eigenvalues with the proof of the Proposition 3.1.1, but we actually prove that $\underset{R\to+\infty}{\lambda_R^z} = \lambda_0$.

Proof of the Proposition 3.1.1. Let us call $\mathcal{L}$ the elliptic operator defined by $\mathcal{L}v = -d\Delta v - f_v(y,0)v$. Notice that the weak formulation of the problem $\mathcal{L}v = f$ gives a self-adjoint operator in $H_0^1(B_R(0,z))$: for $v,\ w \in H_0^1(B_R(0,z))$ we have that

$$< \mathcal{L}v, w >= \int_{B_R(0,z)} [d\nabla v \nabla w - f_v(y,0)v\ w]dxdy =< v, \mathcal{L}w > .$$

Hence, using Rayleigh Quotient's formula (given for example in [12], Chapter 8 Section 12) we infer

$$\lambda_R^z = \min_{\psi \in H_0^1(B_R(0,z)),\ \psi \not\equiv 0} Q_R^z(\psi),$$

where

$$Q_R^z(\psi) = \frac{\int_{B_R(0,z)} \left[d|\nabla\psi|^2 - f_v(y,0)\psi^2\right]dx}{\int_{B_R(0,z)} \psi^2}.$$

We call $D_{R-1}(0,z)$ the closed ball with centre $(0,z)$, as $B_R(0,z)$, and radius $R-1$. Then we take a family of functions $\{\chi_R\}_{R\geq 2}$, bounded in $C^2(\mathbb{R}^2)$ and such that

$$\begin{cases} \chi_R(x,y) = 1 & \text{if } (x,y) \in D_{R-1}(0,z), \\ \chi_R(x,y) = 0 & \text{if } (x,y) \notin B_R(0,z), \\ 0 \leq \chi_R \leq 1. \end{cases}$$

Now set $\psi_R = \phi\chi_R$, then $\psi_R \in C_0^2(B_R(0,z))$. We rewrite $Q_R^z(\psi_R)$ in the following lines in order to get an upper bound for its value. Notice that, being ψ_R regular and vanishing at the border, we can take the problem in the strong sense, thus

$$Q_R^z(\psi_R) = \frac{\int_{B_R(0,z)} \left[-d(\Delta\psi_R)\psi_R - f_v(y,0)\psi_R^2\right]dx}{\int_{B_R} \psi_R^2}.$$

The numerator is

$$\int_{B_R(0,z)} \left[-d(\Delta\psi_R)\psi_R - f_v(y,0)\psi_R^2\right]dxdy =$$

$$= \int_{D_{R-1}(0,z)} \left[-d(\Delta\phi)\phi - f_v(y,0)\phi^2\right]dxdy +$$

$$+ \int_{B_R(0,z)\setminus D_{R-1}(0,z)} \left[-d(\Delta(\phi\chi_R))\phi\chi_R - f_v(y,0)(\phi\chi_R)^2\right]dxdy.$$

We now look separately at the two addenda. By the definition of ϕ, for the first we have

$$\int_{D_{R-1}(0,z)} \big[-d(\Delta\phi)\phi - f_v(y,0)\phi^2\big]dxdy = \int_{D_{R-1}(0,z)} \big[\lambda_0\phi^2\big]dxdy. \qquad (1.4)$$

Then we look for an estimate for the second term. Recalling the definition of $\psi_R = \phi\chi_R$, with ϕ and χ_R bounded in $C^2(\mathbb{R}^2)$, we can say that $(\Delta\psi_R)\psi_R$ and $f_v(y,0)\psi_R^2$ are bounded. Also,

$$\mathrm{Vol}(B_R(0,z)\backslash D_{R-1}(0,z)) = c(R^2-(R-1)^2) = cR$$

for some constant c, not depending on R. So it is trivial that

$$\Big|\int_{B_R(0,z)\backslash D_{R-1}(0,z)} \big[-d(\Delta(\phi\chi_R))\phi\chi_R - f_v(y,0)(\phi\chi_R)^2\big]dxdy\Big| \le cR$$

for some c, possibly different, and in the same way

$$\Big|\int_{B_R(0,z)} \psi_R^2\ dxdy - \int_{D_{R-1}(0,z)} \phi^2\ dxdy\Big| \le cR.$$

As the function ϕ is positive, does not depend on x and depends periodically on y, there exists $\alpha > 0$ such that $\phi(y) \ge \alpha$ for all $y \in \mathbb{R}$ and for all $z \in \mathbb{R}$. Thus

$$1 \le \frac{\int_{B_R(0,z)} \psi_R^2\ dxdy}{\int_{D_{R-1}(0,z)} \phi^2 dxdy} = \frac{\int_{B_R(0,z)} \psi_R^2\ dxdy + \int_{D_{R-1}(0,z)} (\phi)^2 - (\phi)^2 dxdy}{\int_{D_{R-1}(0,z)} \phi^2\ dxdy}$$
$$\le c\frac{R}{\alpha(R-1)^2} + 1$$

where $c > 0$ is a suitable constant, independent of R. As $R \to +\infty$ we have

$$\frac{\int_{B_R(0,z)} \psi_R^2\ dxdy}{\int_{D_{R-1}(0,z)} \phi^2 dxdy} \to 1. \qquad (1.5)$$

Likewise we can estimate:

$$\frac{\int_{B_R(0,z)\backslash D_{R-1}(0,z)} \big[-d(\Delta\psi_R)\psi_R - f_v(y+z_2,0)(\psi_R)^2\big]dxdy}{\int_{D_{R-1}(0,z)} \phi^2 dxdy} \le c\frac{R}{(R-1)^2} \to 0. \qquad (1.6)$$

Thus combining equations (1.4), (1.5) and (1.6) we obtain

$$Q_R^z(\psi_R) = \Bigg(\frac{\int_{B_R(0,z)\backslash D_{R-1}(0,z)} \big[-d(\Delta(\phi\chi_R))\phi\chi_R - f_v(y,0)(\phi\chi_R)^2\big]dxdy}{\int_{D_{R-1}(0,z)} \phi^2 dxdy} + \\ + \frac{\int_{D_{R-1}(0,z)} \big[\lambda_0\phi^2\big]dxdy}{\int_{D_{R-1}(0,z)} \phi^2 dxdy}\Bigg) \cdot \frac{\int_{D_{R-1}(0,z)} \phi^2 dxdy}{\int_{B_R(0,z)} \psi_R^2\ dxdy},$$

hence

$$Q_R^z(\psi_R) \le \Big(\lambda_0 + c\frac{R}{(R-1)^2}\Big) \cdot \Big(c\frac{R}{(R-1)^2} + 1\Big) \to \lambda_0.$$

Thus $Q_R^z(\psi_R) \to \lambda_0$ as $R \to +\infty$. So it is clear that $\lim\limits_{R\to+\infty} \lambda_R^z \le \lambda_0$. But from Lemma 3.1.6 we have $\lim\limits_{R\to+\infty} \lambda_R^z \ge \lambda_0$. So the two values are equal. □

3.2 Invasion

We here present a sufficient condition for the existence and uniqueness of a stationary solution that furthermore provides the convergence of any solution raised from a nonnegative datum to the stationary solution. Using the notation of the previous section, we are now ready to give the following:

Theorem 3.2.1. *Suppose $\lambda_0 < 0$. Then there exists a unique (q,p) stationary solution for* (M) *such that both q and p are positive and bounded. Moreover q is constant and p is independent of x and, for every (u,v) solution of the system raised from a nonnegative datum (u_0, v_0), $u(x,t) \to q$ and $v(x,y,t) \to p(y)$ as $t \to +\infty$.*

In a modelling point of view, survival is certain when the solution (u,v), raised from a reasonable initial datum, remains positive and distant from 0. In this context, a nonnegative bounded stationary solution (q,p) represents an admissible equilibrium situation for the population densities. What is said in the theorem is that a negative λ_0 permits the existence of an equilibrium solution, that this solution is positive and bounded (thus the theorem is also a Liouville-type result), and that every solution converges to the equilibrium. These are some remarkable facts, similar to the ones shown in [3] and [7] for comparable models.

This section is devoted to the proof of Theorem 3.2.1. The principal idea is contained in the uniqueness lemma of Subsection 3.2.2. Another lemma, presented in the first subsection, will be useful in order to make the main body the proof in Subsection 3.2.3 more readable.

3.2.1 A convergence lemma

During the proof of Theorem 3.2.1 we will find some solutions monotone in time. Thanks to this lemma we can see that their limits are stationary solutions for (M). The proof is adapted from [7].

Lemma 3.2.2. *Let $(\underline{u}, \underline{v})$ be a bounded solution of* (M) *such that $\underline{u}$ and $\underline{v}$ are monotone in time. It holds that $\underline{u} \to \underline{U}$, $\underline{v} \to \underline{V}$ as $t \to +\infty$. Moreover $(\underline{U}, \underline{V})$ is a stationary solution of* (M).

Proof. Let us first notice that taken an increasing diverging sequence $\{t_n\}_{n\in\mathbb{N}}$, for any fixed $(x,y) \in \Omega$, we have

$$\lim_{n\to+\infty} \underline{u}(x,t_n) = U(x), \qquad \lim_{n\to+\infty} \underline{v}(x,y,t_n) = V(x,y)$$

converging pointwise because u and v are monotone and bounded. Moreover U, V are not depending on t.

Fix $T > 0$, we take the sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $t_n - t_{n-1} < T$ for all $n \in \mathbb{N}$. Define $u_n(x,t) := \underline{u}(x,t_n)$, $v_n(x,y,t) := \underline{v}(x,y,t_n)$ and consider the sequence $\{(u_n, v_n)\}_{n\in\mathbb{N}}$. We want to find a subsequence converging in $C^2_{loc}(\Omega)$, hence we use the well known classic inequalities. Take $\delta > 0$ small and $\rho > 0$ and $1 < p < +\infty$;

then by the Agmon-Douglis-Nirenberg interior estimates (cite? Questa stima me l'hai detta a voce e non riesco a trovarla da nessuna parte)

$$\begin{aligned}\|u_n\|_{W_p^{2,1}([-\rho,\rho]\times[\delta,T-\delta])} &\leq c\left(\|v_{n-1}(x,0,t)\|_{L^p([-\rho-1,\rho+1]\times[0,T])} + \|u_0\|_{L^\infty([-\rho-1,\rho+1])}\right)\\ &\leq c\left(\|v_{n-1}\|_{L^\infty(\Omega)} + \|u_0\|_{L^\infty(\mathbb{R})}\right)\\ &\leq cH\end{aligned}$$

where c depends on $D,\ \nu,\ \mu,\ \rho,\ \delta,\ p,\ T$ and H depends on $\|v_n\|_{L^\infty(\Omega)},\ \|u_0\|_{L^\infty(\mathbb{R})}$ that are finite because $u_0,\ v_0,\ \underline{u},\ \underline{v}$ are bounded. Set now $Q_\rho := (-\rho,\rho)\times(0,\rho)\subset\Omega$, for v_n we obtain

$$\begin{aligned}\|v_n\|_{W_p^{2,1}(Q_\rho\times[\delta,T-\delta])} &\leq c\big(\|f(v_n)\|_{L^p Q_{\rho+1}\times[0,T])} + \|v_0\|_{L^\infty(Q_{\rho+1})}+\\ &\qquad + \|u_n\|_{W_p^{2,1}([-\rho-1,\rho+1]\times[\frac{1}{2}\delta,T-\frac{1}{2}\delta])}\big)\\ &\leq cH,\end{aligned}$$

where c depends on $d,\ f,\ \rho,\ \delta,\ T$ and H is still bounded since $f,\ \underline{u},\ \underline{v}$ are bounded. We apply the general Sobolev inequalities for $k>\frac{n}{p}$ where n is the dimension and k is the order of the derivative (see for examples Theorem 6 in Chapter 5, Section 6 of [9]). We have

$$\begin{aligned}\|u_n\|_{C^{0,\gamma}([-\rho,\rho]\times[\delta,T-\delta])} &\leq c\|u_n\|_{W_p^{2,1}([-\rho,\rho]\times[\delta,T-\delta])} \leq cH,\\ \|v_n\|_{C^{0,\gamma}(Q_\rho\times[\delta,T-\delta])} &\leq c\|v_n\|_{W_p^{2,1}(Q_\rho\times[\delta,T-\delta])} \leq cH,\end{aligned}$$

where c is a constant not depending on n and

$$\gamma = \left\lfloor\frac{2}{p}\right\rfloor + 1 - \frac{2}{p}.$$

Since $t_n - t_{n-1} < T$, we can say that $\|u_n\|_{C^{0,\gamma}([-\rho,\rho]\times(\delta,+\infty))}$ and $\|v_n\|_{C^{0,\gamma}(Q_\rho\times(\delta,+\infty)}$ are uniformly bounded for all $n\in\mathbb{N}$.

Last step uses Schauder theory. We set $g_n(x,y,t) := f(y,v_n(x,y,t))$, in (3.10) in Chapter 2 we show that $g_n\in C^{0,\gamma}(Q_\rho\times[\delta,T-\delta])$. Now, for each $n\in N$. Hence each v_n now is solution to

$$\partial_t v_n - d\Delta v_n = g_n.$$

By Schauder theorem in parabolic case (Theorem 10.1 in Chapter IV of [17]) we have,

$$\begin{aligned}\|u_n\|_{C^{2,1,\gamma}([-\rho,\rho]\times[\delta,T-\delta])} &\leq c\big(\|v_{n-1}\|_{C^{0,\gamma}(Q_\rho\times[\delta,T-\delta])}\\ &\qquad + \|u_0\|_{C^{2,\gamma}[-\rho,+\rho]} + \|u_n\|_{C^{0,\gamma}([-\rho,\rho]\times[\delta,T-\delta])}\big)\\ &\leq cH,\\ \|v_n\|_{C^{2,1,\gamma}(Q_\rho\times[\delta,T-\delta])} &\leq c\big(\|f_n\|_{C^{0,\gamma}(Q_\rho\times[\delta,T-\delta])}\\ &\qquad + \|v_0\|_{C^{2,\gamma}(Q_\rho)} + \|v_n\|_{C^{0,\gamma}(Q_\rho\times[\delta,T-\delta])}\big)\\ &\leq cH.\end{aligned}$$

As ρ and δ are arbitrary, the estimates are valid on every compact subset of $\mathbb{R}$ or Ω. By the compact injection $C^{(1,2),\gamma}\hookrightarrow C^{1,2}$ we can finally say that there is a subsequence $n_k,\ k\in\mathbb{N}$ such that $\{(u_{n_k},v_{n_k})\}_{k\in\mathbb{N}}$ is converging to some $(\underline{U},\underline{V})$ in $C^{1,2}_{loc}$. This also means, $(\underline{U},\underline{V})$ solves (M). But we showed at the beginning of this proof that the pointwise limit of $\{(u_n,\ v_n)\}_{n\in\mathbb{N}}$ is (U,V), independent of t. Thus, $U\equiv\underline{U},\ V\equiv\underline{V}$ and (U,V) is a stationary solution of (M). $\square$

3.2.2 Uniqueness of positive bounded stationary solution independent of x

As it will be clear at the end of Subsection 3.2.3, the uniqueness of the solution to the stationary system

$$\begin{cases} U \equiv \text{constant}, V \equiv V(y) \\ -dV'' = f(y,V), \qquad y \in (0,+\infty) \\ \nu V(0) = \mu U \\ V'(0) = 0. \end{cases} \tag{2.7}$$

implies the uniqueness of the stationary solution to the main system (M). Moreover this lemma will be a central point of the proof of the convergence of a solution issued from a nonnegative datum to the stationary solution.

The idea that the uniqueness of the solution to helps in showing the convergence result comes from [7]. However, the differences in the systems require also some original contribution; we have to show that a solution V must be separate from 0 using some sliding methods.

Lemma 3.2.3. *Let $\lambda_0 < 0$. System (3.2.2) admits at most one positive bounded solution (U,V).*

Proof. We prove the lemma by contradiction. Let (U_1,V_1), (U_2,V_2) be two positive, bounded solutions of (3.2.2). Without loss of generality we can say $U_1 \leq U_2$, hence $V_1(0) \leq V_2(0)$. Notice that if they are equal, then V_1 and V_2 coincide in $(0, +\infty)$ because of Cauchy-Lipschitz uniqueness theorem. Thus we can assume $U_1 < U_2$, $V_1(0) < V_2(0)$.

Let us first show that there exists $\varepsilon > 0$ such that $V_1 > \varepsilon$ for any $y \in \mathbb{R}_+$. Since $\lambda_0 < 0$, there is $R > 0$ such that $\lambda_R < 0$. We can construct a subsolution to (3.2.2) in the following way. Take R large enough to have $\lambda_R < 0$ and also $R > 3\ell$, take φ_R solving the eigenvalue problem (1.1). Thanks to the continuity of f in v and the derivability in $v = 0$, we can use MacLaurin expansion to say there exists $\kappa > 0$ small enough to have

$$f(y,\ \kappa\varphi_R) > f_v(y,\ 0)\kappa\varphi_R + \lambda_R\kappa\varphi.$$

Then, extending $\kappa\varphi_R$ to 0 outside B_R, we obtain a generalised subsolution Z for the first equation of (3.2.2). Notice that φ_R and consequently Z are functions in two variables, while V_1 and V_2 are defined on $\mathbb{R}$. However, we can extend V_1 and V_2 to functions of all Ω just taking them constant with respect to the x variable. Since the support of Z is compact, decreasing κ if necessary, we can suppose $V_1 > Z$. Take a point $c \in B_{R-2\ell}(0,R)$, then there exists a ball U, centred in c and with radius $r > \ell$ and $\eta' > 0$, such that $Z|_U > \eta'$. Consider now for $n \in \mathbb{N}$ the translated function $Z_n(x,y) := Z(x, y - \ell n)$. Suppose by the absurd that for some n we have $V_1 \not> Z_n$, so

$$m = \inf\{n \in \mathbb{N} \mid V_1 > Z_n\} < +\infty.$$

By decreasing κ if necessary we can have $V_1 \geq Z_m$ with at least one contact point ξ. It has to be $\xi \in B_R(0, R + \ell m)$ because $Z_m \equiv 0$ outside $B_R(0, R + \ell m)$ and V_1 is nonzero for hypothesis. In $B_R(0, R+\ell m)$ we have

$$-d\Delta(V_1 - Z_m) \geq f(y,\ V_1) - f(y,\ Z_m) = h(y)(V_1 - Z_m)$$

where h is a bounded function because f is Lipschitz, and moreover $V_1 - Z_m \geq 0$, $V_1 - Z_m(\xi) = 0$. Hence we can apply the strong positivity property (taken from lecture notes by Henri Berestycki) and obtain $V_1 - Z_m \equiv 0$, that is impossible. Then for any $n \in \mathbb{N}$ we have $V_1 > Z_n$. Since for $y > R - \ell$ there exists $x \in \mathbb{R}$ such that the point (x, y) belongs to a translation of the ball U, then for $y > R - \ell$ it holds $V_1(y) > \eta'$. Defining

$$0 < \varepsilon < \min \left\{ \min_{[0,\ R-\ell]} V_1,\ \eta' \right\}$$

we get $V_1 > \varepsilon$ for all $y \in (0, +\infty)$.

We take the function $\rho(y) = \left(\frac{V_2}{V_1}\right)$. Consider $V_1^2\rho'$ and suppose by the absurd that there exists $\delta > 0$ such that

$$V_1^2\rho' > \delta \quad \forall\ y \in \mathbb{R}_+.$$

Since V_1 is bounded then this is equivalent to say $\rho' > \sigma$ for suitable $\sigma > 0$. This means that ρ is an unbounded function since it is $C^2(\mathbb{R}_+)$ and has an always positive derivative. But $V_1 > \varepsilon$, hence it must be V_2 unbounded. By the hypothesis on V_2 we reached an absurd.

Hence there exists a sequence $\{\eta_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} V_1^2(\eta_n)\rho'(\eta_n) = 0$. We take the main equations of (3.2.2) for V_1 and V_2; integrating a combination of the two we have

$$\begin{aligned}
\lim_{n\to+\infty} \frac{1}{d} \int_0^{\eta_n} V_1 V_2 \left(\frac{f(y, V_1)}{V_1} - \frac{f(y, V_2)}{V_2} \right) &= \lim_{n\to+\infty} \int_0^{\eta_n} (-V_1'' V_2 + V_1 V_2'')dy \\
&= \lim_{n\to+\infty} - V_1' V_2(\eta_n) + V_1 V_2'(\eta_n) \\
&= \lim_{n\to+} V_1^2(\eta_n)\rho'(\eta_n).
\end{aligned}$$

But by hypothesis $V_1,\ V_2 > 0$ and by an hypothesis we take on f in the model we have

$$\frac{f(y, V_1)}{V_1} > \frac{f(y, V_2)}{V_2}$$

for all $y \in (0, +\infty)$. Hence the left-hand side is positive while the right-hand side goes to 0. This is absurd.

□

We would like to make some comments about the properties of the stationary solution V, that matters to us because we are going to show that it is the limit of each solution in the case $\lambda_0 < 0$.

Remark 3.2.4. As observed in the proof, there exists $\varepsilon > 0$ such that $V > \varepsilon$ everywhere. Hence, for every point $(x, y) \in \Omega$ we will have $v(x, y, t) > \frac{\varepsilon}{2}$ for t large enough; thus the density of population will be distant from 0 and invasion will happen.

Remark 3.2.5. We clarify here that V is not as easy to compute as one may wish. First of all, V is different from the principal periodic eigenfunction ϕ because

$$-dV'' = f(y,V) \neq f_v(y,0)V + \lambda_0 V$$

for f a general function with the hypotheses required in the model. Then, notice that the function V is not necessarily periodic in y. In fact, there are some boundary condition in 0 but not on other points, so the system giving V is not periodic.

3.2.3 Proof of theorem (3.2.1)

By hypothesis $\lambda_0 < 0$, thus for $R \in \mathbb{R}$ large enough we have $\lambda_R < 0$. On the ball $B_R(0,R)$ we can solve uniquely the problem

$$\begin{cases} -d\Delta\varphi_R - f_v(y,0) = \lambda_R\varphi_R & \text{in } B_R(0,R), \\ \varphi_R|_{\partial B_R(0,R)} = 0, \quad \|\varphi_R\|_{\mathrm{L}^\infty(B_R)(0,R)}, & \varphi_R > 0 \text{ in } B_R(0,R). \end{cases}$$

As shown in the precedent proof, the function $\kappa\varphi_R$ extended by 0 is a generalised subsolution in Ω for the equation $\partial_t v - d\Delta v = f(y,v)$.

The couple $(0,\kappa\varphi_R)$ is a subsolution to (M), because

$$\begin{cases} -d\Delta\kappa\varphi_R \leq f(y,\kappa\varphi_R) & (x,y) \in \Omega, \\ -D\partial^2_{xx} 0 \leq 0 & x \in \mathbb{R}, \\ -\partial_y \kappa\varphi_R \leq 0 & x \in \mathbb{R}, \end{cases}$$

where the latter is implied by Hopf lemma for φ_R, which attains his minimum at the boundary of B_R.

Consider the solution (u,v) issued from the nonnegative datum $(u_0,v_0) \neq (0,0)$. For the strong maximum principle, at $t = 1$ (u,v) is positive at all points. Thus, since $\kappa\varphi_R$ has compact support, possibly decreasing κ, at $t = 1$ we have $0 < u,\ \kappa\varphi_R < v$. Let $(\underline{u},\underline{v})$ be the solution issued from $(0,\kappa\varphi_R)$ starting at time $t = 1$. Since the initial datum is a subsolution but not a solution for the stationary system, at $t = 1$ $(\underline{u},\underline{v})$ is increasing. Afterwards it keeps increasing: take the solution issued from the same datum starting at time $t = 1+\varepsilon$, since at the initial time it is smaller than $(\underline{u},\underline{v})$ by the comparison principle it must always be strictly smaller than $(\underline{u},\underline{v})$, so for all times $(\underline{u},\underline{v})$ is increasing. Here we can apply Lemma 3.2.2 and find that $\underline{u} \to U_1$, $\underline{v} \to V_1$ as $t \to +\infty$ and (U_1,V_1) is solution of (3.2.2). The solution (u,v) is always above $(\underline{u},\underline{v})$, thus

$$\begin{aligned} 0 < U_1(x) &\leq \liminf_{t\to+\infty} u(x,t), \\ \underline{V}(x,y) < V_1(x,y) &\leq \liminf_{t\to+\infty} v(x,y,t). \end{aligned}$$

We can also prove that U_1, V_1 do not depend on x. Since $(0,\kappa\varphi_R)$ is not a solution, at $t = 2$ the solution (U_1,V_1) will be strictly above it; hence, there exists $h_0 > 0$ such that for all $h_0 > h > 0$ both $\kappa\varphi_R^+ = \kappa\varphi_R(x+h,y) < V_1(x,y,2)$ and $\kappa\varphi_R^- =$

$\kappa\varphi_R(x-h,y) < V_1(x,y,2)$ hold. Hence, since the limit of the solutions issued from $(0,\kappa\varphi_R^{\pm})$ at $t=2$ is $(U_1(x\pm h), V_1(x\pm h,y))$, by the strong maximum principle we get

$$U_1(x\pm h) \le U_1(x),$$
$$V_1(x\pm h,y) \le V_1(x,y).$$

But this implies that equality must hold for all $0 < h < h_0$, so U_1 and V_1 are independent of x .

It is very easy to find a stationary supersolution for (M). Consider

$$V := \max\{\|v_0\|_\infty,\ \frac{\mu}{\nu}\|u_0\|_\infty,\ M\}$$

reminding that M is the quantity such that for $v \ge M$ we have $f(y,v) \le 0$ for all $y \in \mathbb{R}_+$. Now setting $U := \dfrac{\nu}{\mu}V$ the couple (U,V) is a supersolution because

$$\begin{cases} 0 \ge f(y,V) & (x,y) \in \Omega, \\ 0 \ge \nu V - \mu U = 0 & x \in \mathbb{R}, \\ 0 \ge \mu U - \nu V = 0 & x \in \mathbb{R}. \end{cases}$$

With a procedure similar to the one before, we take the solution $(\overline{u},\overline{v})$ issued from (U,V). Since at $t=0$ the solution (u,v) is below $(\overline{u},\overline{v})$, by the comparison principle we have that the order is maintained for all $t \in (0,+\infty)$. In addition, since the initial datum (U,V) is a supersolution but not a solution, at the initial time $(\overline{u},\overline{v})$ is decreasing and it keeps decreasing, as we can easily see taking the solution starting at $t=\varepsilon$ from the datum (U,V) and applying the comparison principle. Notice that $(\overline{u},\overline{v})$ must always be positive because they are bounded by (u,v). Lemma 3.2.2 applies to $(\overline{u},\overline{v})$, giving a stationary solution (U_2,V_2) with the property

$$\limsup_{t\to+\infty} u(x,t) \le U_2(x) \le U$$
$$\liminf_{t\to+\infty} v(x,y,t) \le V_2(x,y) \le V.$$

Since the initial datum (U,V) does not depend on x, also U_2 and V_2 are independent of x. More precisely, we can repeat in an easier case the argument we used for the subsolution: translating U, V in the x variable we find the translations of U_2, V_2, but as the initial datum is invariant by translation, by unicity we have that U_2 and V_2 coincide with all theirs translations.

At this point we have two stationary bounded positive solutions (U_1,V_1) and (U_2,V_2) which are also independents of x. Lemma 3.2.3 says that they must coincide, so $U_1 \equiv U_2 =: q$ and $V_1 \equiv V_2 =: p$, and $(q,p(y))$ is a stationary positive bounded solution for (M). Additionally,

$$q \le \liminf_{t\to+\infty} u(x,t) \le \limsup_{t\to+\infty} u(x,t) \le q$$
$$p(y) \le \liminf_{t\to+\infty} v(x,y,t) \le \liminf_{t\to+\infty} v(x,y,t) \le p(y).$$

This means that every solution (u,v) issued from a nonnegative datum converges pointwise to (q,p) as $t\to+\infty$. Notice that this implies that (q,p) is the unique stationary positive solution. By the absurd let (q',p') be another nonnegative stationary solution, then the previous observation says that the solution issued from (q',p') must converge to (q,p), in contradiction to the fact that (q',p') is a stationary solution.

3.3 Extinction

Since we saw that $\lambda_0 < 0$ implies, among the others, that the population survives, we would like to say that for $\lambda_0 \geq 0$ the population reaches the extinction, as was found in [3]. But the scenario is more complex than expected. Surprisingly, a new hypothesis have to be added in order to guarantee an extinction theorem similar to the ones that we find in the literature. The new condition to be added is the following:

Hypothesis 3.3.1 (H1)**.** The periodic eigenfunction ϕ solution to (1.2) satisfies

$$\partial_y \phi(0) \leq 0. \tag{H1}$$

The condition H1 is a fine property of the eigenfunction ϕ and we have no easy way to verify for a system whatever. We will later discuss about some more natural and handy hypotheses that implies H1.

Now we will complete our results on asymptotic behaviour of solutions by giving the following theorem. The proof is an adaptation of the analogue in [3], but the presence of the road raised many more cases to verify.

Theorem 3.3.2. *Suppose that $\lambda_0 \geq 0$ and that the periodic eigenfunction ϕ solution to* (1.2) *satisfies H1. Then there is no positive stationary solution to* (M) *and moreover for every nonnegative bounded solution (u, v) we have $u \to 0,\ v \to 0$ locally uniformly.*

Proof. Taking $k \in \mathbb{R}$ such that $\frac{\nu}{\mu}\phi(0) - \frac{d}{\mu}\partial_y \phi(0) \geq k \geq \frac{\nu}{\mu}\phi(0)$, the couple (k, ϕ) is a supersolution to the system (M), extending $\phi(y)$ in $\mathbb{R}^2$ simply taking $\phi(x, y) = \phi(y)$; in fact,

$$\begin{cases} -d\Delta\phi - f(y, \phi) & > -d\Delta\phi - f_v(y, 0)\phi \geq \lambda_0 \phi \geq 0, \qquad y \in \mathbb{R}^+ \\ 0 & \geq \nu\phi(0) - \mu k, \\ -d\partial_y \phi(0) & \geq -\nu\phi(0) + \mu k. \end{cases}$$

Nevertheless, (k, ϕ) and all its multiples are not solution to (M) because the first inequality is strict.

Now assume by contradiction that the couple (q, p) is a nonnegative stationary solution to our problem. As k and ϕ are positive and bounded, we can define

$$\gamma^* := \inf\{\gamma \in \mathbb{R} | \gamma k > q(x)\ \forall x \in \mathbb{R}, \quad \gamma\phi(x, y) > p(x, y)\ \forall (x, y) \in \overline{\Omega}\} < \infty.$$

Suppose $\gamma^* > 0$, otherwise $p = 0$ and $q = 0$. So we have $\inf\limits_{x\in\mathbb{R}}(\gamma^* k - q) = 0$ or $\inf\limits_{(x,y)\in\Omega}(\gamma^*\phi - p) = 0$.

Suppose the second does not hold, so the first must hold. Then there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that $\lim\limits_{n\to+\infty}(\gamma^* k - q)(x_n) = 0$. Possibly passing to a subsequence, $x_n \to \overline{x}$ or $x_n \to \pm\infty$.

If $x_n \to \overline{x}$, then the minimum of the function $\gamma^* k - q$ is 0 and it is reached at $\overline{x} \in \mathbb{R}$. We set $\Gamma \subset\subset \mathbb{R}$ an open connected set such that $\overline{x} \in \Gamma$. In Γ we have

$$-D\Delta(\gamma^* k - q) \geq \nu\phi(0) - \mu k - \nu p(x, 0) - \nu q.$$

Since $\gamma^* k - q \geq 0$ on Γ, the strong positivity property (taken from some lecture notes by Henri Berestycki) gives $\gamma^* k - q \equiv 0$ in Γ and, taking a sequence of sets invading $\mathbb{R}$, we have $\gamma^* k - q \equiv 0$ in $\mathbb{R}$. The latter is absurd because $(\gamma^* k, p)$ is not a solution, in fact

$$\gamma^* k \geq \frac{\nu}{\mu} \gamma^* \phi(0) > \frac{\nu}{\mu} p(x, 0),$$

where the last inequality holds for the supplementary hypothesis $\inf_{(x,y)\in\overline{\Omega}} (\gamma^* \phi - p) > 0$.

Then assume $x_n \to +\infty$, the case with $-\infty$ being analogous. We take the sequence of functions

$$\begin{aligned} q_n(x) &= q(x + x_n), \\ p_n(x, y) &= p(x + x_n, y). \end{aligned}$$

Our aim is to build a convergent subsequence of $\{(q_n, p_n)\}_{n\in\mathbb{N}}$ and then apply the maximum principle to the limit (q_∞, p_∞). Fix $1 < p < +\infty$ and $\rho > 0$, set $Q_\rho := (-\rho, \rho) \times (0, \rho)$, then we can use the Agmon-Douglis-Nirenberg esitimates and find

$$\begin{aligned} \|q_n\|_{W^{2,p}(-\rho,\rho)} &\leq c(\|q_n\|_{L^\infty(-1-\rho,1+\rho)} + \|p_n\|_{L^p(\Omega)}) \leq cH, \\ \|p_n\|_{W^{2,p}(Q_\rho)} &\leq c(\|f(p_n)\|_{\mathrm{L}^p(Q_{\rho+1})} + \|q_n\|_{W^{2,p}(-1-\rho,1+\rho)}) \leq cH \end{aligned}$$

for c depending only on f, D, μ, ν, ρ, L and H depending on $\sup q$, $\sup p$. Then, by Sobolev injection ([9])

$$\begin{aligned} \|q_n\|_{C^{0,\alpha}(-\rho,\rho)} &\leq c\|q_n\|_{W^{2,p}(-\rho,\rho)}, \\ \|p_n\|_{C^{0,\alpha}(Q_\rho)} &\leq c\|p_n\|_{W^{2,p}(Q_\rho)} \end{aligned}$$

for some $\alpha \in (0, 1)$. To complete last passage, we first have to show that $g_n := f(p_n)$ belongs to $C^{0,\alpha}(Q_\rho)$. This was also shown in (3.10) in Chapter 2. Then by Schauder esitimates (see [12]) we have

$$\begin{aligned} \|q_n\|_{C^{2,\alpha}(-\rho,\rho)} &\leq c\|p_n\|_{C^{0,\alpha}(Q_\rho)} \leq cH \\ \|p_n\|_{C^{2,\alpha}(Q_\rho)} &\leq c(\|g_n\|_{C^{0,\alpha}(Q_\rho)} + \|q_n\|_{C^{2,\alpha}(-\rho,\rho)}) \leq cH. \end{aligned}$$

Being ρ arbitrary, the sequence $\{(q_n, p_n)\}_{n\in\mathbb{N}}$ is bounded in $C^{2,\alpha}_{loc}$ and moreover since $C^{2,\alpha}_{loc} \hookrightarrow C^2_{loc}$ we can take a subsequence converging in C^2_{loc} to a couple (q_∞, p_∞). Notice that (q_∞, p_∞) is still a solution to (M). By construction, $q_\infty(0) = \lim_{n\to\infty} q(x_n) = \gamma^* k$. Applying the previous part of the case $\inf_{x\in\mathbb{R}} (\gamma^* k - q) = 0$ we reach an absurd. Hence it must be $\inf_{(x,t)\in\Omega} (\gamma^* \phi - p) = 0$. Then there exists a sequence (x_{n_k}, y_{n_k}) such that $\lim_{n\to\mathbb{N}} (\gamma^* \phi - p)(x_n, y_n) = 0$ and $\gamma^* k - q > 0$ in $\mathbb{R}$. Suppose there is a subsequence (x_{n_k}, y_{n_k}) converging to some $(\overline{x}, \overline{y}) \in \Omega$. We have

$$-d\Delta(\gamma^* \phi - p) \geq f(y, \gamma^* \phi) - f(y, p) =: h(x, y)(\gamma^* \phi - p)$$

in Ω with $h(x, y)$ bounded and $\gamma^* \phi - p \geq 0$. Then we can apply the strong positivity property to $\gamma^* \phi - p$ in a neighbourhood of $(\overline{x}, \overline{y})$ in Ω. Hence we have $\gamma^* \phi - p \equiv 0$, but this is absurd since $\gamma^*\phi$ cannot satisfies the equality on the first equation of the system.

The cases where $x_n \to \pm\infty$, $y_n \to \overline{y}$ with $\overline{y} > 0$ are treated with the same arguments of convergence used for the case of $\inf_{x\in\mathbb{R}}(\gamma^* k - q) = 0$. The case $y_n \to +\infty$ is even easier, since q is not affected by translation.

We have left the cases where $y_n \to 0$. Suppose $(x_{n_k}, y_{n_k}) \to (\overline{x}, 0)$ with $\overline{x} \in \mathbb{R}$. If $x_n \to \pm\infty$, using the same arguments as before we can reduce to the case of $x_n \to \overline{x}$. As before, $\gamma^*\phi - p$ is a nonnegative function that satisfies $-d\Delta(\gamma^*\phi - p) > 0$ in Ω and $-d\partial_y(\gamma^*\phi - p) \geq 0$ and we also know that it reaches its minimum at the boundary point $(\overline{x}, 0)$. But Hopf's Lemma says $-\partial_y(\gamma^*\phi - p) < 0$. This is a contradiction, so it cannot be neither $\inf_{x\in\mathbb{R}}(\gamma^* k - q) = 0$ nor $\inf_{(x,t)\in\Omega}(\gamma^*\phi - p) = 0$. Hence $\gamma^* = 0$ and the only bounded nonnegative stationary solution is $(0, 0)$.

We now want to show that every solution (u, v) raised from a nonnegative bounded datum converges to $(0, 0)$. Take $n \in \mathbb{R}$ such that $nk > \sup_{x\in\mathbb{R}} u_0(x)$ and $n\phi > \sup_{(x,y)\in\Omega} v_0(x, y)$. Consider the solution $(\overline{u}, \overline{v})$ starting from the initial datum $(nk, n\phi)$. As was shown in 3.2.3, since $(nk, n\phi)$ is a supersolution, then $(\overline{u}, \overline{v})$ is decreasing at $t = 0$ and it keeps decreasing. Hence we can apply Lemma 3.2.2 to $(\overline{u}, \overline{v})$ and we find that $\overline{u} \to U$, $\overline{v} \to V$ as $t \to +\infty$ with (U, V) stationary nonnegative bounded solution of (M). But we have just proved that $(0, 0)$ is the only stationary nonnegative bounded solution, hence $U \equiv 0$ and $V \equiv 0$.

By the comparison principle 2.1.1, we have that being $0 \leq u_0 < kn$ and $0 \leq v_0 < n\phi$, for all $t \geq 0$ one has $0 \leq u \leq \overline{u}$ and $0 \leq v \leq \overline{v}$. From what was said before, we have $u \to 0$ and $v \to 0$ pointwise as $t \to +\infty$. □

3.4 The symmetric case and further comments on the hypothesis H1

Hypothesis H1 is quite difficult to verify. Thus, we aim to find some other properties, more easy to verify, entailing H1. We have the following:

Hypothesis 3.4.1 (H2)**.** For every fixed $v \in \mathbb{R}$, f is symmetric in $[0, \ell]$ with respect to the centre $y = \frac{1}{2}\ell$, that is

$$f(y, v) = f\left(\frac{1}{2}\ell - y, v\right) \quad \forall y \in \left[0, \frac{1}{2}\ell\right]$$

From this symmetry we can infer many interesting properties for f. For example,

Proposition 3.4.2. *If f periodic in y with period ℓ satisfies hypothesis* (3.4.1)*, then f is symmetric on the interval $\left[\frac{1}{2}\ell, \frac{3}{2}\ell\right]$ with respect to the point $y = \ell$.*

Proof. Let $v \in \mathbb{R}$ be fixed. The thesis of the proposition is equivalent to show that for every $z \in \left[\ell, \frac{3}{2}\ell\right]$ we have $f(z, v) = f(\ell - z, v)$. Directly by hypothesis (3.4.1) we have that for all $y \in \left[\frac{1}{2}\ell, \ell\right]$ we have $f(y, v) = f\left(\frac{1}{2}\ell - y, v\right)$ and by the periodicity of f we obtain $f\Big(\frac{1}{2}\ell - y, v\Big) = f\Big(\frac{3}{2}\ell - y, v\Big)$. Now call $z = \frac{3}{2}\ell - y$. Then $z \in \Big[\ell, \frac{3}{2}\ell\Big]$ and $\frac{1}{2}\ell - y = \frac{1}{2}\ell - z + \frac{3}{2}\ell = 2\ell - z$. So the previous equality reads now and $f(2\ell - z, v) = f(z, v)$. But using f periodicity $f(\ell - z, v) = f(z, v)$, as desired. $\square$

So f has several points that work as centre of symmetry: $y = \frac{1}{2}\ell$, $y = \ell$, and all the points $y \in \mathbb{R}_+$ whose distance from one of these is an entire multiple of the period ℓ.

Now we look at the principal eigenfunction ϕ, solution to periodic eigenvalue problem, and its derivative in 0, $\phi'(0)$.

Proposition 3.4.3. *Let f satisfies hypothesis* (3.4.1)*. Then the principal eigenfunction ϕ is symmetric in the interval $\left[\frac{1}{2}\ell, \frac{3}{2}\ell\right]$ with respect to the point $y = \ell$. Thus, $\phi'(0) = 0$.*

Proof. Recall the eigenvalue problem related to the equation $-d\Delta v = f(y, v)$. The couple (λ_0, ϕ) where λ_0 is the principal eigenvalue and ϕ is the principal eigenfunction is the only solution to the system

$$\begin{cases} -d\Delta\phi - f_v(y, 0)\phi = \lambda_0\phi, & \text{in } \mathbb{R}_+; \\ \phi > 0 \text{ in } \mathbb{R}_+, \quad \|\phi\|_\infty = 1, & \phi(y) = \phi(y+\ell)\ \forall y \in \mathbb{R}_+. \end{cases}$$

Extend $f(y, v)$ by periodicity on the first variable so that f is defined in the whole $\mathbb{R}^2$. Then, by hypothesis (3.4.1) and by Proposition 3.4.2, f is symmetric with respect to 0 in the variable y, that is $f(y, v) = f(-y, v)$ for all $y \in \mathbb{R}$, $v \in \mathbb{R}$. Hence $f_v(y, 0) = f_v(-y, 0)$ for all $y \in \mathbb{R}$. So exchanging y with $-y$ in the system, we obtain the system

$$\begin{cases} -d\Delta\phi - f_v(-y, 0)\phi = \lambda_0\phi, & \text{in } \mathbb{R}_+; \\ \phi > 0 \text{ in } \mathbb{R}_+, \quad \|\phi\|_\infty = 1, & \phi(y) = \phi(y+\ell)\ \forall y \in \mathbb{R}_+. \end{cases}$$

whose unique solution must be $\phi(-y)$. But, being the two systems identical, we have $\phi(y) \equiv \phi(-y)$. So ϕ is symmetric with respect to the point $y = 0$, and this implies $\phi'(0) = 0$. $\square$

The last proposition tells that the condition (3.4.1) on f implies $\phi'(0) = 0$, so in particular ϕ satisfies hypothesis H1 from the previous section and we don't have the nasty problem of studying the behaviour of ϕ'. Briefly,

Theorem 3.4.4. *If f satisfies* (3.4.1)*, then the sign of the principal eigenvalue λ_0 characterise the asymptotic behaviour of the solution (u, v) to the system* (M)*.*

The function f needs to satisfy a bunch of properties, but it is still possible to provide examples.

Example 3.4.5. Consider the function $f(y, v) = d(\cos(y) + \cos^2(y))v - v^2$ with $d > 0$. It verifies all the hypotheses required in the model: we have the modelling features

$f(y,0) = 0$ and, since $d(\cos(y) + \cos^2(y))$ is bounded, there exists $M > 0$ such that $f(y,M) < 0$ for all $y \in \mathbb{R}_+$. Then for all $0 \leq v_1 < v_2$ it holds

$$\begin{aligned}\frac{f(y,v_1)}{v_1} - \frac{f(y,v_2)}{v_2} &= d(\cos(y) + \cos^2(y)) - v_1 - d(\cos(y) + \cos^2(y)) + v_2 \\ &= v_2 - v_1 \geq 0,\end{aligned}$$

so also the concave hypothesis is satisfied. Trivially f is periodic in the variable y with period $\ell = \pi$. Moreover, f fulfils condition (3.4.1) since it is symmetric with respect to the origin for any fixed $v \in \mathbb{R}$. This is a simple case where it is possible for us to show also that $\phi'(0) = 0$. In fact, the system

$$\begin{cases} -d\Delta\phi - d(\cos(y) + \cos^2(y))\phi = \lambda_0\phi, & \text{in } \mathbb{R}_+; \\ \phi > 0 \text{ in } \mathbb{R}_+, \quad \|\phi\|_\infty = 1, \quad \phi(y) = \phi(y+\ell)\ \forall y \in \mathbb{R}_+, \end{cases}$$

has the easy solution $(-d, e^{\cos(y)-1})$. Let us verify that ϕ is the solution: the second derivative reads

$$\phi''(y) = (-\cos(y) + \sin^2(y))e^{\cos(y)-1} = (-\cos(y) + 1 - \cos^2(y))e^{\cos(y)-1}$$

and substituting ϕ'' in the equation we have

$$d\cos(y) - d + d\cos^2(y) - d\cos(y) - d\cos^2(y) = -d$$

so $(-d, e^{\cos(y)-1})$ is a solution. We have that $\phi(y) = e^{\cos(y)-1}$ is periodic with period π, positive and its maximum is $\phi(0) = e^0 = 1$. As expected, $\phi'(0) = -\sin(0)e^{\cos(0)-1} = 0$. However $\lambda_0 = -d < 0$, hence the first case holds and the population spreads all over the domain Ω.

There are several situation where $\phi'(0) > 0$ and Theorem (3.3.2) does not applies. In these cases, the behaviour is not clear yet. We are convinced that one can construct a case with $\lambda_0 \geq 0$, $\phi'(0) > 0$ but the population do not reaches the extinction. However the construction of such an example is beyond the purposes of this thesis and could be done in further research works.

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