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On the constructions relating moduli of Del Pezzo surfaces,plane curves and configurations of points

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In the memory of Prof. Marco Garuti to whom I owe gratitude.

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Introduction

Del Pezzo surfaces are rational surfaces over a ground field k whose anticanonical divisor $-K$ is ample. The are classified according to their degree which is the self-intersection number of $-K$. Moreover, they are obtained in general by blowing up at most 8 points in general position in the projective plane, except for the product of the projective line with itself. The question that this essay addresses is the well-known classification of Del Pezzo surfaces of any degree. In fact not only do we want to classify them but we want to parametrize isomorphism classes of Del Pezzo surfaces. Indeed the question becomes, what is the description of the coarse moduli space of the Del Pezzo surfaces of any degree if they exist ?

To answer that question we need a prerequisite which is the theory of divisors over algebraic variety. We look at it mostly from the scheme-theoretic point of view inspired by Hartshorne's book ([6]). This is largely addressed in chapter 2.
In fact the concept of divisor is an important tool to study the geometric nature of a variety or scheme. Chapter 2 provides an introduction to divisors, linear equivalence and divisor class group (an abelian group which is a fundamental invariant of varieties). We will start with the Weil divisors (Section 2.1.5) which are more palpable to grasp geometrically speaking but unfortunately are solely defined on noetherian integral and separated regular schemes of codimension one. We go on and extend that notion to a more general concept of divisors, namely the Cartier divisors (Section 2.1.40), which are in connection with the Weil group and the notion of invertible sheaves (Section 2.1.51) (Picard group). Moreover we will see that divisors are also important to study maps from a variety to a projective spaces .
Later on, being familiar with the study of divisors, we give an introduction to the study of algebraic surfaces as in the development of intersection theory, the Riemann-Roch theorem, Künneth formula and the Nakai-Moishezon criterion in section (2.1.102). In fact these notions will be relevant for the study of ampleness of a divisor on Del Pezzo surface in Chapter 5.
In Chapter 3, following the book ([9]) we give a close account on the theory of moduli spaces that arise in con-

nection with the classification of Del Pezzo surfaces in algebraic geometry. Indeed we try to give an idea of what constitutes a moduli problem and to describe the possible solutions we are looking for.

Amongst other things two solutions emerge so far as in the fine moduli space which is pretty rare, and the coarse moduli space which occurs quite often. We proceed by giving examples of coarse moduli spaces as in the moduli space of isomorphism class of n-pointed stable Riemann surfaces of genus g denoted $\mathcal{M}_{g,n}$. In fact when $g = 0$, proposition (5.1.24) states that $\mathcal{M}_{0,n}$ is single point when $n = 3$. All of this because every smooth projective curve of genus 0 is isomorphic to $\mathbb{P}_1(\mathbb{C})$. Moreover the latter is a coarse moduli space. In general $\mathcal{M}_{0,n}$ is a fine moduli space for all $n \geq 3$.

However will see in Chapter 5 that the coarse moduli space of Del Pezzo surfaces of degree $d \leq 8$ does exist (3.1.15) and it is related to the Weyl orbit of a space associated to a projective points set P_n^m for some m integers.

In chapter 4 armed with an n-projective plane, an invertible sheaf $\mathcal{L}$ and a reductive algebraic group $G = \mathrm{GL}(n+1)$, we cook up a projective scheme P_n^m from a graded k algebra of a G-invariant section of $\mathcal{L}$. Moreover we give a precise way to compute it using standard monomials and we go on to show that P_1^4 is in fact the projective line. In the section that follows one of the above, using methods from Geometry Invariant theory [?] we show that the space P_n^m for some integers m can be described as the quotient of the subset of semi-stable points of the product of projective n-planes. From that observation we prove that P_n^m is a rational variety of a certain dimension depending of some condition on m.
Amongst other things we introduce a birational variety to the mth product of the n-projective plane and define its blowing up variety which is obtained from a point set and others points that we call infinitely near point. Furthermore we study the stability of such a variety.

Chapter 5 introduces Del Pezzo surfaces and the generalized Del Pezzo variety (gDP) of type (n, m) which is the blowing up variety of the n-projective space in one point set in "general position". In fact proposition (5.3.12) states that any Del Pezzo surface is isomorphic to a gDP variety of type $(2, m)$. Moreover the similarity between these two does not end there. By looking at the Weyl group $W_{2,m}$ associated to a gDP variety of type $(2, m)$ that act on its set of geometric marking, Theorem (5.3.12) gives the description of the coarse moduli space of Del Pezzo surfaces $\mathcal{M}_{DP}(m)$ to which we provide a proof.

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Divisors

2.1 A brief Overview

We start by recalling some definitions and basic results about the theory of divisors, following mainly [6].

Definition 2.1.1. *Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$. A is regular local ring if $dim_k \mathfrak{m}/\mathfrak{m}^2 = dim A$.*

Lemma 2.1.2. *Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$. The number $dim_k \mathfrak{m}/\mathfrak{m}^2$ is the minimum number of generators of the ideal $\mathfrak{m}$.*

Proof. A being noetherian, the ideal $\mathfrak{m}$ is finitely generated. Let n be the minimum number of generators for $\mathfrak{m}$.

Let $\mathfrak{m} = (a_1, ..., a_n)$. Then $\bar{a}_1, ..., \bar{a}_n$ generate $\mathfrak{m}/\mathfrak{m}^2$ as an $A/\mathfrak{m}$-vector space. Hence $\dim_k \mathfrak{m}/\mathfrak{m}^2 \leq n$. Moreover, if $\bar{a}_1, ..., \bar{a}_n$ were linearly dependent then after re-indexing $\bar{a}_1, ..., \bar{a}_{n-1}$ would still generate $\mathfrak{m}/\mathfrak{m}^2$. However because of Nakayama's lemma consequence as in ([4]) Exercise (6.16) implies that $a_1, ..., a_{n-1}$ generate $\mathfrak{m}$ as an A-module, in contradiction to the minimality of n. Hence $\dim_k \mathfrak{m}/\mathfrak{m}^2 = n$. □

Definition 2.1.3. *We say a scheme X is a regular or non-singular in codimension one if every local ring $\mathcal{O}_x$ of X of dimension one is regular.*

Example 2.1.4.

- *Non singular varieties over a field are examples of schemes regular in codimension one. In fact on a singular variety the local ring of every closed point is regular, hence all the local rings are regular, since they are the localization of the local rings of closed points.*
- *Noetherian normal schemes are regular in codimension one, since any local ring of dimension one is an integrally closed domain, hence regular ([4], proposition 12.14).*

Throughout the scope of this presentation we will always assume unless state otherwise that :

- X is a noetherian, (finiteness condition) ,integral (i.e. irreducible and reduced) and separated (a scheme theoretic analogue of the Hausdorff condition) scheme which is regular in codimension one.

2.1.5 Weil Divisors

Definition 2.1.6. *Let X be scheme as stated above. A prime divisor on X is a closed integral subscheme Y of codimension one. A Weil divisor is an element of the free abelian group DivX generated by the prime divisors. A divisor is* $D = \sum n_i\, Y_i$*, where the* Y_i *are prime divisors, the* n_i *are integers, and only finitely many* n_i *are different from zero. If all the* $n_i \geq 0$*, we say that D is effective.*

Example 2.1.7 (Divisors in $\mathbb{P}^2$)**.** *Let k be a field and* $X = \mathbb{P}^2_k$ *the* $2-$*projective space. Let* $Y \subseteq X$ *a hypersurface. We define* deg *Y to be the degree of the polynomial defining Y.*

Remark 2.1.8. *If Y is a prime divisor on X, let* $\eta \in Y$ *be its generic point i.e.* $\overline{\{\eta\}} = Y$*. As* $\mathcal{O}_{\eta,X}$ *is a local ring whose maximal ideal is generated by one element, then it is easy to see that* $\mathcal{O}_{\eta,X}$ *is a discrete valuation ring with quotient field K. We call its corresponding discrete valuation* $v_Y : K \to \mathbb{Z} \cup \infty$*, the valuation of Y. Since X is separated, Y is uniquely determined by its valuation ([5]).*

Definition 2.1.9. *Now let* $f \in K^*$ *be a any non-zero rational function on X. Then* $v_Y(f) \in \mathbb{Z}$*. If it is positive, we say f has a zero along Y, of that order; if it is negative, we say f has a pole along Y, of order* $-v_Y(f)$*.*

Lemma 2.1.10. *Let X satisfy (*), and let* $f \in K^*$ *be a nonzero function on X. Then* $v_Y(f) = 0$ *for all except finitely many divisors Y.*

Proof. See ([6]) lemma 6.1, chapter 2. ☐

Definition 2.1.11. *Let X satisfy (*) and let* $f \in K^*$*. We define the divisor of f, denoted* (f)*, by*

$$(f) = \sum v_Y(f).Y,$$

where the sum is taken over all prime divisors of X. By the lemma (2.1.10), this is a finite sum, hence it is a divisor. Any divisor which is equal to the divisor of a function is called a principal divisor.

Example 2.1.12. *Let* $X = \mathbb{P}^1$*. And take any rational function* $f(x) := p(x)/q(x)$*; where* $p, q \in \mathbb{C}[x]$*.*

Then the zeroes of $f(x)$ *are exactly the zeroes of the polynomial p and the poles are exactly at the zeroes of q.*

For a particular example then one can take $f(x) = \frac{x}{1-x}$*. This has a zero at* $x = 0$ *and a pole at* $x = 1$*. This is a degree* 0 *rational function whose divisor is* $(f) = 1 \cdot [0] - 1 \cdot [1]$*, which is also of degree* 0*.*

Remark 2.1.13. *If* $f, g \in K^*$*, then* $(f/g) = (f) - (g)$ *because of the properties of valuations. Therefore the map:*

$$\psi\colon K^* \to DivX$$
$$f \mapsto (f)$$

is a group homomorphism and its image consists of the principal divisors of X.

Definition 2.1.14. *Two divisors D and D' on X are said to be linearly equivalent, written $D \sim D'$, if $D - D'$ is a principal divisor. The quotient group $\frac{DivX}{\mathrm{Im}(\psi)} := \mathrm{CL}\,X$ is called the divisor class group of X.*

The divisor class group of a scheme is a very interesting invariant. However, it is not easy to calculate it. Nevertheless in the following propositions and examples, we will determine it in a number of special cases.

Proposition 2.1.15. *Let A noetherian domain. Then A is a unique factorisation domain if and only if $X = \mathrm{Spec}\,A$ is normal and $\mathrm{CL}\,X = 0$.*

Proof. See ([6]) proposition 6.2, Chapter 2. □

Example 2.1.16. *If X is affine $n-$ space $\mathbb{A}_k^n$ over a field k, then $\mathrm{CL}\,X = 0$. In fact, $X = \mathrm{Spec}\,k[x_1, ..., x_n]$, and the polynomial ring is a unique factorization domain.*

Example 2.1.17. *If A is a Dedekind domain, then $\mathrm{CL}(\mathrm{Spec}\,A)$ is just the ideal class group of A, as defined in commutative algebra ([?]). Thus proposition (2.1.15) generalizes the fact that A is a unique factorization domain (UFD) if and only if its ideal class group is 0.*

Proposition 2.1.18. *Let X be a the projective space $\mathbb{P}_k^n$ over a field k. For any divisor $D = \sum n_i Y_i$, define the degree of D by $degD = \sum n_i degY_i$, where $degY_i$ is the degree of hypersurface Y_i. Let H be the hyperplane $x_0 = 0$. Then if D is any divisor of degree d, then $D \sim dH$.*

Proof. Let $S = k[x_1, ..., x_n]$ be the homogeneous coordinate ring of X. If g is a homogeneous element of degree d, we can factor it into irreducible polynomials $g = g_1^{n_1} \dots g_r^{n_r}$. Then g_i defines a hypersurface Y_i of degree $d_i = \deg g_i$, and we can define the divisor of g to be $(g) = \sum n_i Y_i$. Then $\deg(g) = d$.

If D is any divisor of degree d, we can write it as a difference $D_1 - D_2$ of effective divisors of degree d_1, d_2 with $d_1 - d_2 = d$. Let $D_1 = (g_1)$ and $D_2 = (g_2)$. This is possible, because an irreducible hypersurface in $\mathbb{P}^n$ corresponds to a homogeneous prime ideal of height 1 in S, which is principal. Taking products of powers we can write any effective divisor as (g) for some homogeneous g. Now $D - dH = D_1 - D_2 - dH = (g_1) - (g_2) - (x_0^d) = (f)$ where $f = g_1/x_0^d g_2$ is a rational function on X.

□

Lemma 2.1.19. *Let k be a field and K be the function field of $\mathbb{P}_k^n$. Then, if $f \in K^*$, the degree of the principal divisor (f) is zero.*

Proof. A rational function f on X is a quotient g/h of homogeneous polynomials of the same degree. Clearly $(f) = (g) - (h)$, so one sees that $\deg(f) = 0$. □

Proposition 2.1.20. *Let k be a field and K be the function field of $\mathbb{P}_k^n$. The degree function gives an isomorphism $deg : \mathrm{CL}\,X \to \mathbb{Z}$.*

Proof. This follows from (2.1.18), (2.1.19), and the fact that $\deg H = 1$. □

Proposition 2.1.21. *Let X as stated above. Let Z be a proper closed subset of X, and let $U = X - Z$. Then :*

(a) *there is a surjective homomorphism* $\mathrm{CL}\, X \to \mathrm{CL}\, U$ *defined by* $D = \sum n_i Y_i \mapsto \sum n_i(Y_i \cap U)$, *where we ignore those* $Y_i \cap U$ *which are empty;*

(b) *if* $\mathrm{codim}(Z, X) \geq 2$ *then* $\mathrm{CL}\, X \to \mathrm{CL}\, U$ *is an isomorphism;*

(c) *If* Z *is an irreducible subset of codimension* 1, *then there is an exact sequence*

$$\mathbb{Z} \to \mathrm{CL}\, X \to \mathrm{CL}\, U \to 0,$$

where the first map is defined by $1 \mapsto 1.Z$.

Proof. (a) if Y is a prime divisor on X, then $Y \cap U$ is either empty or a prime divisor on U. If $f \in K^*$, and $(f) = \sum n_i Y_i$, then considering f as a rational function on U, we have $(f)_U = \sum n_i(Y_i \cap U)$, so indeed we have a homomorphism $\mathrm{CL}\, X \to \mathrm{CL}\, U$. It is surjective because every prime divisor of U is the restriction of its closure in X.

(b) the groups $\mathrm{Div} X$ and $\mathrm{CL}\, X$ depend only on subsets of codimension 1, so removing a closed subset Z of codimension ≥ 2 does not change anything.

(c) The kernel of $\mathrm{CL}\, X \to \mathrm{CL}\, U$ consists of divisors whose support is contained in Z. If Z is irreducible, the kernel is just the subgroup of $\mathrm{CL}\, X$ generated by $1.Z$.

□

Example 2.1.22. *Let* Y *be an irreducible curve of degree* d *in* $\mathbb{P}^2_k$. *Then* $\mathrm{CL}(\mathbb{P}^2 - Y) = \mathbb{Z}/d\mathbb{Z}$. *This follows immediately from (2.1.18), (2.1.21).*

Proposition 2.1.23. *Let* X *satisfies (*). Then* $X \times \mathbb{A}^1$ *i.e* $X \times_{\mathrm{Spec}\, \mathbb{Z}} \mathrm{Spec}\, \mathbb{Z}[t]$ *also satisfies (*), and* $\mathrm{CL}\, X \cong \mathrm{CL}(X \times \mathbb{A}^1)$.

Proof. See ([6]) Proposition 6.6, chapter 2.

□

Remark 2.1.24. *Let* $\mathbf{Q}$ *be a quadratic surface in* $\mathbb{P}^3$, *and* Y *any irreducible hypersurface of* $\mathbb{P}^3$ *which does not contain* $\mathbf{Q}$. *Then we can assign multiplicities to the irreducible components of* $Y \cap \mathbf{Q}$ *so as to obtain a divisor* $Y.\mathbf{Q}$ *on* $\mathbf{Q}$.

Indeed, on each standard open set U_i *of* $\mathbb{P}^3$, Y *is defined by a single function* f; *we can take the value of this function (restricted to* $\mathbf{Q}$*) for each valuation of a prime divisor of* $\mathbf{Q}$ *to define the divisor* $Y.\mathbf{Q}$.

By linearity we extend this map to define a divisor $D.\mathbf{Q}$ *on* $\mathbf{Q}$, *for each divisor* $D = \sum n_i Y_i$ *on* $\mathbb{P}^3$, *such that no* Y_i *contains* $\mathbf{Q}$.

2.1.25 Divisors on curves

In this section we will pay attention further to the divisor class group of divisors on curves. We will define the degree of a divisor on a curve, and we will show that on a complete non-singular curve, the degree is stable under linear equivalence.

As written above, we use terminologies that look unfamiliar as in ”curves” and ”complete non-singular curves”. Therefore we must present a good dictionary as what those words mean.

Definition 2.1.26. *A morphism $f : X \to Y$ of algebraic varieties or (schemes) is separated if the diagonal of X denoted $\Delta(X)$, is closed in the fiber product $X \times_Y X$ of X over Y where the latter is a variety (scheme) that satisfies the universal property and makes the following diagram commutative:*

$$\begin{tikzcd}
X \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
X \arrow[r, "f"'] & X
\end{tikzcd}$$

.

Definition 2.1.27. *A morphism $f : X \to Y$ of algebraic varieties or (schemes) is called universally closed if its fiber product morphisms $f \times Id : X \times_k Z \to Y \times_k Z$ are closed maps of the underlying topological space for any algebraic variety (scheme) Z.*

Definition 2.1.28. *A morphism $f : X \to Y$ of algebraic varieties or (schemes) that is universally closed, separated and of finite type is proper over Y.*

Definition 2.1.29. *Let k be an algebraic closed field. A curve over k is an integral separated scheme X of finite type over k, of dimension one. If X is proper over k, we say that X is complete. If all the local rings X are regular local rings, we say that X is nonsingular, or regular*

Proposition 2.1.30. *Let X be a nonsingular curve over k with function field K. Then the following conditions are equivalent:*

(a) *X is projective;*

(b) *X is complete;*

(a) *$X \cong t(C_K)$, where C_K is the abstract nonsingular curve of (see 6), and t is a functor from varieties to schemes.*

Proof. See ([6]), Proposition 6.7.

□

Proposition 2.1.31. *Let X be a complete nonsingular curve over k, let Y be any curve over k, and let $f : X \to Y$ be a morphism. then*

(a) *either $f(X) =$ a point,*

(b) *or $f(X) = Y$. In case (b), $K(X)$ is a finite extension field of $K(Y)$, f is a finite morphism, and Y is also complete.*

Proof. Since X is complete, $f(X)$ must be closed in Y, and proper over Spec$k$. On the other hand, $f(X)$ is irreducible. Thus either (a) $f(X) =$ a point, or (b) $f(X) = Y$, and in case (b), Y is also complete.

In case (b), f is dominant, so it induces an inclusion $K(Y) \subseteq K(X)$ of functions fields. Since both fields are finitely generated extension fields of transcendence degree 1 of k, $K(X)$ must be a finite algebraic extension of $K(Y)$. To show that f is a finite morphism, Let $V =$ Spec$B$ in $K(X)$ be any open affine subset of Y. Let A be the integral closure of B in $K(X)$. the A is a finite B module, and Spec$A$ is isomorphic to an open subset $U$of X. Clearly $U = f^{-1}V$, so this shows that f is a finite morphism. □

Definition 2.1.32. *If $f : X \to Y$ is a finite morphism of curves, we define the degree of f to be the degree of the field extension* $[K(X : K(Y))]$.

Remark 2.1.33. *If X is a nonsingular curve, then X satisfies the condition (*) used above, so we can talk about divisors on X. In this case, the prime divisor is just a closed point, so an arbitrary divisor can be written $D = \sum n_i P_i$, where the P_i are closed points, and $n_i \in \mathbb{Z}$. We define the degree of D to be $\sum n_i$.*

Definition 2.1.34. *If $f : X \to Y$ is a finite morphism of nonsingular curves, we define a homomorphism*

$$\begin{aligned} f^* : DivY &\to DivX \\ Q &\mapsto \sum_{f(P)=Q} v_P(f^*t).P, \end{aligned}$$

where $t \in \mathcal{O}_Q$ is a local parameter at Q i.e, t is an element of $K(Y)$ with $v_Q(t) = 1$ and v_P is the valuation corresponding to the discrete valuation ring $\mathcal{O}_P$.

Remark 2.1.35. *The map f^* is well defined: f is a finite morphism, thus the pre-image of Q will be finite. Moreover, $f^*(Q)$ is independent of the choice of the local parameter t. Indeed, if t' is another parameter at Q, then $t' = ut$ where u is a unit in $\mathcal{O}_Q$. For any point $P \in X$ with $f(P) = Q$, f^*u will be a unit in $\mathcal{O}_P$, therefore $v_P(f^*t) = v_P(f^*t')$.*

Corollary 2.1.36. *The map f^* can be extended by linearity to all divisors on Y. Hence, one sees easily it preserves linear equivalence. Thus it induces a morphism $f^* : CLY \to CLX$.*

Proposition 2.1.37. *Let $f : X \to Y$ be a finite morphism of nonsingular curves. Then for any divisor D on Y we have $degf^*D = degf.degD$*

Proof. See ([6]), proposition 6.9, chapter 2. □

Corollary 2.1.38. *A principal divisor on a complete nonsingular curve X has degree* 0*. Consequently the degree function induces a surjective homomorphism $deg : CLX \to \mathbb{Z}$.*

Proof. See ([6]), corollary 6.10, chapter 2. □

Remark 2.1.39.
(a) *Let X be the nonsingular cubic curve $y^2z = x^3 - xz^2$ in $\mathbb{P}_k^2$ with char $k \neq 2$. We set $\mathrm{CL}^0 X$ to be the kernel of the degree map $\mathrm{CL}\,X \to \mathbb{Z}$. Indeed there is a natural* $1:1$ *correspondence between the set of closed points of X and the elements of the group $\mathrm{CL}^0 X$.*

(b) *If X is any complete nonsingular curve, then the group $\mathrm{CL}^0 X$ is isomorphic to the group of closed points of an abelian variety called the Jacobian variety of X. The dimension of the latter variety is the genus of the curve.*

(c) *Let X be a nonsingular projective variety of dimension ≥ 2 then $\mathrm{CL}\, X/\mathrm{CL}^0 X$ is the so-called the Néron-Severi group of X and it is finitely generated. Moreover $\mathrm{CL}^0 X$ is isomorphic to the group of closed points of an abelian variety called the Picard variety.*

2.1.40 Cartier Divisors

In this section we want to extend the notion of divisor to an arbitrary scheme. It turns out that using the irreducible subvarieties of codimension one doesn't work very well. So instead, we take as our point of departure the idea that a divisor should be something which locally looks like the divisor of a rational function. However one should know that this is in no way a generalization of the Weil divisors, but it provides a good notion to use on arbitrary schemes [([6])].

Definition 2.1.41. *Let X be a scheme. For each open affine subset $U = \operatorname{Spec} A$, Let S the set of elements of A which are not zero divisors, and let $K(U) := S^{-1}A$ be the localization of A by the multiplicative subset S. We call $K(U)$ the total quotient ring of A. For each open set U, let $S(U)$ denote the set of elements of $\Gamma(U, \mathcal{O}_X)$ which are not zero divisors in each local $\mathcal{O}_x$ for $x \in U$. Then the rings $S(U)^{-1}\Gamma(U, \mathcal{O}_X)$ form a presheaf, whose associated sheaf (sheafification) of rings $\mathcal{K}$ we call the sheaf of total quotient rings of $\mathcal{O}$. On an arbitrary scheme, the sheaf $\mathcal{K}$ replaces the concept of function field of an integral scheme. We denote by $\mathcal{K}^*$ the sheaf (of multiplicative groups) of invertible elements in the sheaf of rings $\mathcal{K}$. Similarly $\mathcal{O}^*$ is the sheaf of invertible elements in $\mathcal{O}$.*

Definition 2.1.42. *A Cartier divisor on a scheme X is a global section of the sheaf $\mathcal{K}^*/\mathcal{O}^*$.*

Proposition 2.1.43. *([6]) A Cartier divisor on X can be described by $\{(U_i, f_i)\}_i$, where $\{U_i\}_i$ is an open cover of X, and $f_i \in \Gamma(U_i, \mathcal{K}^*)$ such that for each i, j, $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}^*)$.*

Proof. See ([6]), Chapter 2. □

Definition 2.1.44. *A Cartier divisor is principal if it is in the image of the natural map $\varphi : \Gamma(X, \mathcal{K}^*) \to \Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$, and two Cartier divisors are linearly equivalent if their difference is principal.*

Remark 2.1.45. *Although the group operation on $\mathcal{K}^*/\mathcal{O}^*$ is multiplication, we will use the language of additive groups when speaking of Cartier divisors, so as to preserve the analogy with Weil divisors.*

Remark 2.1.46. *Let $D = \{(U_i, f_i)\}_i$ and $D' = \{(U_i, f'_i)\}_i$ be two Cartier divisors. In the following, one will illustrate the additive group operation between the latter:*

- $D + D' = \{(U_i, f_i.f'_i)\}_i$;
- $-D = \{(U_i, 1/f_i)\}_i$;
- $D + (-D) = \{(U_i, 1)\}_i$ *which is the zero Cartier divisor.*

Definition 2.1.47. *A Cartier divisor $D = \{(U_i, f_i)\}_i$ is effective if $f_i \in \Gamma(U_i, \mathcal{O}_X)$.*

Proposition 2.1.48. *Let X be an integral, separated noetherian scheme, all of whose local rings are unique factorization domains (in which case we say X is locally factorial). Then the group DivX of Weil divisors on X is isomorphic to the group of Cartier divisors $\Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$, and furthermore, the principal Weil divisors correspond to the principal Cartier divisors under this isomorphism.*

Proof. First note that X is normal, hence satisfies (*), since a UFD is integrally closed. Thus it makes sense to talk about Weil divisors. Since X is integral, the sheaf $\mathcal{K}$ is just the constant sheaf corresponding to the function field K of X. Now let a Cartier divisor be given by $\{(U_i, f_i)\}_i$ where $\{U_i\}_i$ is an open cover of X, and $f_i \in \Gamma(U_i, \mathcal{K}^*) = K^*$. We define the associated Weil divisor as follows. For each prime divisor Y, take the coefficient of Y to be $v_Y(f_i)$, where i is an index for which $Y \cap U_i \neq$. If j is another such index, then f_i/f_j is invertible on $U_i \cap U_j$, so $v_Y(f_i/f_j) = 0$ and $v_Y(f_i) = v_Y(f_j)$. Thus we obtain a well-defined Weil divisor $D = \sum v_Y(f_i) Y$ on X (the sum is finite because X is noetherian).

Conversely, if D is Weil divisor on X, let $x \in X$ be any point. Then D induces a Weil divisor D_x on the local scheme Spec $\mathcal{O}_x$, since $\mathcal{O}_x$ is a *UFD*, D_x is a principal divisor, by (2.1.15), so let $D_x = (f_x)$ for some $f_x \in K$. Now the principal divisor (f_x) on X has the same restriction to Spec $\mathcal{O}_x$ as D, hence they differ only at prime divisors which do not pass through x. There are only finitely many of these which have a non-zero coefficient in D or (f_x), so there is an open neighborhood U_x of x such that D and (f_x) have the same restriction to U_x. Covering X with such open sets U_x, the functions f_x give a Cartier divisor on X. Note that if f, f' give the same Weil divisor on an open set U, then $f/f' \in \Gamma(U, \mathcal{O}^*)$, since X is normal (cf. proof of (2.1.15)). Thus we have a well-defined Cartier divisor.

The two constructions are inverse to each other, so we see that the groups of Weil divisors and Cartier divisors are isomorphic. Furthermore it is clear from the construction that the principal divisors correspond to each other. □

Remark 2.1.49. *Since a local ring is UFD (Matsumura [7][Th.48, p.142]), this proposition applies in particular to any regular integral separated noetherian scheme. A scheme is regular if all of its local rings are regular local rings.*

Example 2.1.50. *Let X be the cuspidal cubic curve $y^2 z = x^3$ in $\mathbb{P}^2_k$, with char$k \neq 0$. In this case X does not satisfy (*), so we cannot talk about Weil divisors on X. However we can talk about* $\mathrm{CaCL}\, X := \Gamma(X, \mathcal{K}^*/\mathcal{O}^*)/ \operatorname{Im} \varphi$, *the Cartier divisor classes modulo principal divisors.*

2.1.51 Invertible sheaves

Definition 2.1.52. *An invertible sheaf on a ringed space X is a locally free $\mathcal{O}_X$- module of rank* 1.

Proposition 2.1.53. *If $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on a ringed space X, so is $\mathcal{L} \otimes \mathcal{M}$. If $\mathcal{L}$ is any invertible sheaf on X, then exists an invertible sheaf $\mathcal{L}^{-1}$ on X such that $\mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathcal{O}_X$.*

Proof. The first statement is clear, since $\mathcal{L}$ and $\mathcal{M}$ are both locally free of rank 1, and $\mathcal{O}_X \otimes \mathcal{O}_X \cong \mathcal{O}_X$. For the second statement, let $\mathcal{L}$ be any invertible sheaf, and take $\mathcal{L}^{-1}$ to be the dual sheaf $\check{\mathcal{L}} = \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$. Then $\check{\mathcal{L}} \otimes \mathcal{L} \cong \mathcal{H}om(\mathcal{L}, \mathcal{L}) = \mathcal{O}_X$. □

Definition 2.1.54. *For any ringed space X, we define the Picard group of X, Pic X, to be the group of isomorphism classes of invertible sheaves on X, under the operation $\otimes$. Proposition (2.1.53) shows that in fact it is a group.*

Remark 2.1.55. *Hartshorne ([6]) shows* $PicX \cong H^1(X, \mathcal{O}_X^*)$.

Definition 2.1.56. *Let D be a Cartier divisor on a scheme X, represented by $\{(U_i, f_i)\}_i$ as above. We define a subsheaf $\mathcal{L}(D)$ of the sheaf of total quotient rings $\mathcal{K}$ by taking the latter as being the sub-$\mathcal{O}_X$-module of $\mathcal{K}$ generated by f_i^{-1} on U_i.*

This is well-defined, since f_i/f_j is invertible on $U_i \cap U_j$, so f_i^{-1} and f_j^{-1} generate the same $\mathcal{O}_X$-module. We call $\mathcal{L}(D)$ the sheaf associated to D.

Proposition 2.1.57. *Let X be a scheme. Then:*

(a) for any Cartier divisor D, $\mathcal{L}(D)$ is an invertible sheaf on X. The map $D \mapsto \mathcal{L}(D)$ gives a $1-1$ correspondence between Cartier divisors on X and invertible subsheaves of $\mathcal{K}$;

(b) $\mathcal{L}(D_1 - D_2) \cong \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}$;

(c) $D_1 \sim D_2$ if and only if $\mathcal{L}(D_1) \cong \mathcal{L}(D_2)$.

Proof. See ([6]), chapter 2, proposition 6.13. □

Corollary 2.1.58. *On any scheme X, the map $D \mapsto \mathcal{L}(D)$ gives an invertible homomorphism of the group $CaCLX$ of Cartier divisors modulo linear equivalence to $PicX$.*

Remark 2.1.59. *The map* $\mathrm{CaCL}\, X \to \mathrm{Pic}\, X$ *may not be surjective, because there may be invertible sheaves on X which are not isomorphic to any invertible subsheaf of $\mathcal{K}$. However the following proposition states that it holds true when X satisfies one condition.*

Proposition 2.1.60. *If X is an integral scheme, the homomorphism* $\mathrm{CaCL}\, X \to \mathrm{Pic}\, X$ *of (2.1.58) is an isomorphism.*

Proof. See ([6]) chapter 2, proposition 6.15. □

Corollary 2.1.61. *If X is a noetherian, integral, separated locally factorial scheme, then there is a natural isomorphism* $\mathrm{CL}\, X \cong \mathrm{Pic}\, X$.

Proof. this follows from (2.1.58) and (2.1.60) above.

□

Definition 2.1.62. *Let Y and X be smooth variety and let $f : X \to Y$ be a morphism. By taking the inverse image of invertible sheaves we get a homomorphism of groups*

$$f^* : PicY \to PicX.$$

If f is surjective we can also define the inverse image of a divisor D on Y. Meaning, that operation satisfies: $f^*(\mathcal{L}(D)) = \mathcal{L}(f^*D)$

Corollary 2.1.63. *If $X = \mathbb{P}_k^n$ for some field k, then every invertible sheaf on X is isomorphic to $\mathcal{O}_{\mathbb{P}_k^n}(l)$ for some $l \in \mathbb{Z}$.*

2.1.64 Projective Morphisms

In this section of we discuss morphisms of a given scheme to projective space. We will prove how the latter morphism is determined by giving an invertible sheaf $\mathcal{L}$ on X and a set of global sections. We will give some criteria for this morphism to be an immersion.

Remark 2.1.65. *Let A be a fixed ring, and consider the projective space $\mathbb{P}^n_A = \operatorname{Proj} A[x_0, ..., x_n]$ over A. On $\mathbb{P}^n_A$ we have the invertible sheaf $\mathcal{O}(1)$, and the homogeneous coordinates $x_0, ..., x_n$ give rise to global sections $x_0, ..., x_n \in \Gamma(\mathbb{P}^n_A, \mathcal{O}(1))$. One sees easily that that $\mathcal{O}(1)$ is generated by the global sections $x_0, ..., x_n$, i.e. the images of these sections generate the stalk $\mathcal{O}(1)_P$ of the sheaf $\mathcal{O}(1)$ as a module over the local ring $\mathcal{O}_P$, for each point $P \in \mathbb{P}^n_A$.*

Now let X be any scheme over A, and let $\varphi : X \to \mathbb{P}^n_A$ be an A-morphism of X to $\mathbb{P}^n_A$. Then $\mathcal{L} = \varphi^*(\mathcal{O}(1))$ is an invertible sheaf on X, and the global sections $s_0, ..., s_n$, where $s_i = \varphi^*(x_i)$, $s_i \in \Gamma(X, \mathcal{L})$, generated the sheaf $\mathcal{L}$.

Conversely, the theorem below shows that $\mathcal{L}$ and the sections s_i determine φ.

Theorem 2.1.66. *Let A be a ring, and let X be a scheme over A.*

(a) if $\varphi : X \to \mathbb{P}^n_A$ is A-morphism, then $\varphi^*(\mathcal{O}(1))$ is an invertible sheaf on X, which is generated by the global sections $s_i = \varphi^*(x_i)$, $i = 0, 1, ..., n$.

(b) Conversely, if $\mathcal{L}$ is an invertible sheaf on X, and if $s_0, ..., s_n \in \Gamma(X, \mathcal{L})$ are global sections which generate $\mathcal{L}$, such that n there exists a unique A-morphism $\varphi : X \to \mathbb{P}^n_A$ such that $\mathcal{L} \cong \varphi^*(\mathcal{O}(1))$ and $s_i = \varphi^*(x_i)$ under this isomorphism.

Proof. See ([6]) chapter 2, page 150. □

Example 2.1.67. *If X is a scheme over A, $\mathcal{L}$ an invertible sheaf, and $s_0, ..., s_n$ any set of global sections, which do not necessarily generate $\mathcal{L}$, we can always consider the open set $U \subseteq X$ (possibly empty) over which the s_i do generate $\mathcal{L}$. Then $\mathcal{L}|_U$ and $s_i|_U$ give a morphism $U \to \mathbb{P}^n_A$. Such is the the case for example, if we take $X = \mathbb{P}^{n+1}_k$, $\mathcal{L} = \mathcal{O}(1)$, and $s_i = x_i$, $i = 0, ..., n$ (omitting x_{n+1}). These sections generate everywhere except at the point $(0, 0, ..., 0, 1) = P_0$. Thus $U = \mathbb{P}^{n+1} - P_0$, and the corresponding morphism $U \to \mathbb{P}^n$ is nothing other than the projection from the point P_0 to $\mathbb{P}^n$.*

We give some criteria for a morphism to a projective space to be a closed immersion.

Proposition 2.1.68. *Let $\varphi : X \to \mathbb{P}^n_A$ be a morphism of schemes over A, corresponding to an invertible sheaf $\mathcal{L}$ on X and sections $s_0, ..., s_n \in \Gamma(X, \mathcal{L})$ as above. Then φ is a closed immersion if and only if*

(a) each open set $X_i = X_{s_i}$ is affine, and

(b) for each i, the map of rings $A[y_0, ..., y_n] \to \Gamma(X_i, \mathcal{O}_{X_i})$ defined by $y_i \mapsto s_j/s_i$ is surjective.

With more hypotheses, we can give a more local criterion.

Proposition 2.1.69. *Let k be an algebraically closed field, let X be a projective scheme over k, and let $\varphi : X \to \mathbb{P}^n_k$ be a morphism (over k) corresponding to $\mathcal{L}$ and $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$ as above. Let $V \subseteq \Gamma(X, \mathcal{L})$ be the subspace spanned by the s_i. Then φ is a closed immersion if and only if*

(a) elements of V separate points, i.e., for any two distinct closed points $P, Q \in X$, there is an $s \in V$ such that $s \in m_P \mathcal{L}_P$ but $s \notin m_Q \mathcal{L}_Q$, or vice versa, and

(b) elements of V separate tangent vectors, i.e., for each closed point $P \in X$, the set $\{s \in V | s_P \in m_P \mathcal{L}_P\}$ spans the k-vector space $m_P \mathcal{L}_P / m_P^2 \mathcal{L}_P$.

Proof. See ([6]), chapter 2, page 152. □

Now that we have seen that a morphism from a scheme X to a projective space can be characterized by giving an invertible sheaf on X and a suitable set of its global sections, we can reduce the study of varieties in projective space to the study of schemes with certain invertible sheaves and given global sections.

2.1.70 Ample and invertible sheaves

Definition 2.1.71. *A sheaf $\mathcal{L}$ on X is very ample relative to Y (where X is a scheme over Y) if there is an immersion $i : X \to \mathbb{P}^n_Y$, where $\mathbb{P}^n_Y$ is a the fiber product $\mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} Y$ for some n such that $\mathcal{L} \cong i^* \mathcal{O}(1)$.*

Remark 2.1.72 (([6]), page 153). *In case $Y = \operatorname{Spec} A$, this is the same thing as saying that $\mathcal{L}$ admits a set of global section $s_0, \ldots, s_n$ such that the corresponding morphism $X \to \mathbb{P}^n_A$ is an immersion.*

Proposition 2.1.73. *If $\mathcal{L}$ is a very ample invertible sheaf on a projective scheme X over a noetherian ring A, then for any coherent sheaf $\mathcal{F}$ on X, there is an integer $n_0 > 0$ such that for all $n_0 \geq n_0$, $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections ($\mathcal{L}^n := \mathcal{L}^{\otimes n}$.)*

We will used this last property of being generated by global sections to define the notion of an ample invertible sheaf.

Definition 2.1.74. *An invertible sheaf $\mathcal{L}$ on a noetherian scheme X is said to be ample if for every coherent sheaf $\mathcal{F}$ on X, there is an integer $n_0 > 0$ (depending on $\mathcal{F}$) such that for every $n \geq n_0$, the sheaf $\mathcal{F} \otimes \mathcal{L}^n$ is generated by its global sections.*

Remark 2.1.75. *The notion of "ample " is an absolute notion, i.e., it depends only on the scheme X, whereas " very ample"is a relative notion, depending on a morphism $X \to Y$.*

Example 2.1.76. *If X is affine, then any invertible sheaf is ample, because every coherent sheaf on an affine scheme is generated by its global sections.*

Theorem 2.1.77. *Let A be a noetherian ring and let X be a proper scheme over $\operatorname{Spec} A$. Let $\mathcal{L}$ be an invertible sheaf on X then the following conditions are equivalent:*

(a) $\mathcal{L}$ is ample.

(b) *for each coherent sheaf $\mathcal{F}$ on X, there is an integer n_0, depending on $\mathcal{F}$, such that $i > 0$, and $n \geq n_0$, $H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0$.*

Proof. See ([6]), Chapter 3. □

Proposition 2.1.78. *Let $\mathcal{L}$ be an invertible sheaf on a noetherian scheme X. Then the following conditions are equivalent:*

(a) *$\mathcal{L}$ is ample;*

(b) *$\mathcal{L}^n$ is ample for all $n > 0$;*

(c) *$\mathcal{L}^n$ is ample for some $n > 0$.*

Proof. (a) $\implies$ (b) is immediate from the definition of ample; (b) $\implies$ (c) is trivial. To prove (c) $\implies$ (a), assume that $\mathcal{L}^n$ is ample. Given a coherent sheaf $\mathcal{F}$ on X, there exists an $n_0 > 0$ such that $\mathcal{F} \otimes (\mathcal{L}^n)^m$ is generated by global sections for all $m \geq n_0$. Considering the coherent sheaf $\mathcal{F} \otimes \mathcal{L}$, there exists an $n_1 > 0$ such that $\mathcal{F} \otimes \mathcal{L} \otimes (\mathcal{L}^n)^m$ is generated by global sections for all $m \geq n_1$. Similarly, for each $k = 1, 2, ..., n-1$, there is an $n_k > 0$ such that $\mathcal{F} \otimes \mathcal{L}^k \otimes (\mathcal{L}^n)^m$ is generated by global sections for all $m \geq n_k$. Now if we take $N = \{n\max n_i | i = 0, 1, ..., n-1\}$, then $\mathcal{F} \otimes (\mathcal{L}^m)$ is generated by global sections for all $m \geq N$. Hence $\mathcal{L}$ is ample. □

Theorem 2.1.79. *Let X be a scheme of finite type over a noetherian ring A, and let $\mathcal{L}$ be an invertible sheaf on X. Then $\mathcal{L}$ is ample if and only if $\mathcal{L}^n$ is very ample over* Spec *A for some $n > 0$.*

Proof. See ([6]), Chapter 2, page 154. □

Example 2.1.80. *Let $X = \mathbb{P}^n_k$, where k is a field. Then $\mathcal{O}(1)$ is very ample by definition. For any $d > 0$, $\mathcal{O}(d)$ is also very ample. Hence $\mathcal{O}(d)$ is ample for all $d > 0$. On the other hand, since the sheaf $\mathcal{O}(l)$ has no global sections for $l < 0$, one sees easily that the sheaves $\mathcal{O}(l)$ for $l \leq 0$ cannot be ample. So on $\mathbb{P}^n_k$, we have $\mathcal{O}(l)$ is ample $\Leftrightarrow$ very ample $\Leftrightarrow l > 0$.*

2.1.81 Linear Systems

We will see in the following sections how global sections of an invertible sheaf correspond to effective divisors on a variety. Thus giving an invertible sheaf and and a set of its global sections is the same as giving a certain set of effective divisors, all linearly equivalent to each other. This leads to the notion of linear system, which is historically an older notion.

For simplicity, we will use this terminology only when dealing with nonsingular projective varieties over an algebraically closed field. Over more general schemes the geometrical intuition associated with the concept of linear system may lead one astray, so it is safer to deal with invertible sheaves and their global sections in that case.

Remark 2.1.82. *Let X be a nonsingular projective variety over an algebraically closed field k. In this case the notions of Weil divisor and Cartier divisor are equivalent (2.1.48). Furthermore, we have a one-to-one correspondence between linear equivalent classes of divisors and isomorphism classes of invertible sheaves (2.1.60). Another useful fact in this situation is that for any invertible sheaf $\mathcal{L}$ on X, the global sections $\Gamma(X, \mathcal{L})$ form a finite-dimensional $k-$vector space.*

Definition 2.1.83. *Let $\mathcal{L}$ be an invertible sheaf on X, and let $s \in \Gamma(X, \mathcal{L})$ be a non zero section of $\mathcal{L}$. We define an effective divisor $D = (s)_0$, the divisor of zeros of s as follows. Over any open set $U \subseteq X$ where $\mathcal{L}$ is trivial, let $\varphi : \mathcal{L}_{|U} \to \mathcal{O}_U$ be an isomorphism. Then $\varphi(s) \in \Gamma(U, \mathcal{O}_U)$. As U ranges over a covering of X, the collection $(U, \varphi(s))$ determines an effective Cartier divisor D on X.*

Remark 2.1.84. *Indeed, φ is determined up to multiplication by an element of $\Gamma(U, \mathcal{O}_U^*)$, so we get a well-defined Cartier divisor.*

Proposition 2.1.85. *Let X be a nonsingular projective variety over the algebraically closed field k. Let D_0 be a divisor on X and let $\mathcal{L} \cong \mathcal{L}(D_0)$ be the corresponding invertible sheaf. Then:*

(a) for each non zero $s \in \Gamma(X, \mathcal{L})$, the divisor of zeros $(s)_0$ is an effective divisor linearly equivalent to D_0;

(b) every effective divisor linearly equivalent to D_0 is $(s)_0$ for some $s \in \Gamma(X, \mathcal{L})$;

(c) two sections $s, s' \in \Gamma(X, \mathcal{L})$ have the same divisor zeros if and only if there is a $\lambda \in k^*$ such that $s' = \lambda s$.

Proof. See ([6]),Proposition 7.7, page 157. □

Definition 2.1.86. *A complete linear system on a nonsingular projective variety is defined as the set (may be empty) of all effective divisors linearly equivalent to some given divisor D_0. It is denoted by $|D_0|$.*

Remark 2.1.87. *We see from the proposition above that the set $|D_0|$ is in one to one correspondence with the set $(\Gamma(X, \mathcal{L}) - 0)/k^*$. This gives $|D_0|$ a structure of the set of closed points of projective space over k.*

Definition 2.1.88. *A linear system $\mathfrak{d}$ on X is a subset of a complete linear system $|D_0|$ which is linear subspace for the projective space structure of $|D_0|$. Thus $\mathfrak{d}$ corresponds to a sub-vector space $V \subseteq \Gamma(X, \mathcal{L})$, where $V = s \in \Gamma(X, \mathcal{L}) | (s)_0 \in \mathfrak{d} \cup 0$.*

Remark 2.1.89. *The dimension of the linear system $\mathfrak{d}$ is its dimension as a linear projective variety. Hence* $\dim \mathfrak{d} = dim V - 1$.

Definition 2.1.90. *A point $P \in X$ is a base point of a linear system $\mathfrak{d}$ if $P \in SuppD$ for all $D \in \mathfrak{d}$. Here SuppD means the union of the prime divisors of D.*

Example 2.1.91. *If $X = \mathbb{P}^n$, then the set of all effective divisors of degree $d > 0$ is a complete linear system of dimension $\binom{n+d}{n} - 1$. Indeed, it corresponds to the invertible sheaf $\mathcal{O}(d)$, whose global sections consist exactly of the space of all homogeneous polynomials in $x_0, ..., x_n$ of degree d. This is a vector space of dimension $\binom{n+d}{n}$, so the dimension of the complete linear system is one less.*

Before starting the following section, one assumes that notions such as cotangent sheaf Ω_X, canonical sheaf ω_X, tangent sheaf T_X and cohomology of sheaf are well known.

2.1.92 Riemann-Roch Theorem

In this section, we will use the word curve to mean a complete, nonsingular curve over an algebraically closed field k. In other word, a curve is an integral scheme of dimension 1, proper over k, all of whose local rings are regular. Such a curve is necessarily projective ([6], chapter 2). In case we want to consider a more general kind of curve, we will use the word "scheme," appropriately qualified, e.g, " an integral scheme of dimension 1 of finite type over k". We will use the word point to mean a closed point, unless we specify the generic point.

We begin by reviewing some concepts which will be used for the study of curves.

Definition 2.1.93. *Let X be curve in a projective space, we define its arithmetic genus $P_A(X)$ as the k- dimension of $H^1(X, \mathcal{O}_X)$. Whereas its geometric genus $P_G(X)$ is the k- dimension of $H^0(X, \omega_X)$*

Proposition 2.1.94. *If X is a curve, then its arithmetic genus and geometric genus are the same, i.e, $P_a(X) = P_G(X)$*

Proof. See ([6]), page 294, chapter 3 □

Remark 2.1.95. *A (Weil) divisor on the curve X is an element of the free abelian group generated by the set of points of X (2.1.33). We write a divisor as $D = \sum n_i P_i$ with $n_i \in \mathbb{Z}$. Its degree is $\sum n_i$. Two divisors are linearly equivalent if their difference is the divisor of a rational function. We have seen that the degree of a divisor depends only on its linear equivalence class. Since X is nonsingular, for every divisor D we have an associated invertible sheaf $\mathcal{L}(D)$, and the correspondence*

$$\begin{aligned} \pi\colon \ \mathrm{CaCL}\, X &\to \mathrm{Pic}\, X \\ D &\mapsto \mathcal{L}(D), \end{aligned}$$

*gives an isomorphism of the group (*CL *X) of divisors modulo linear equivalence with the group* Pic *X of invertible sheaves modulo isomorphism (2.1.60).*

A divisor $D = \sum n_i P_i$ on X is effective if all $n_i \geq 0$. the set of all effective divisors linearly equivalent to a given divisor D is called complete linear system and is denoted by $|D|$. The elements of $|D|$ are in one to one correspondence with the space

$$(H^0(X, \mathcal{L}(D))/k^*,$$

so $|D|$ carries the structure of the set of closed points of projective space. We denote $dim_k H^0(X, \mathcal{L}(D))$ by $l(D)$, so that dimension of $|D|$ is $l(D) - 1$. The number $l(D)$ is finite because $\mathcal{L}$ is an invertible sheaf.

As a consequences of this correspondence we have the following elementary, but useful, result.

Lemma 2.1.96. *Let D be a divisor on a curve X. Then if $l(D) \neq 0$, we must have $degD \geq 0$. Furthermore, if $l(D) \neq 0$ and $degD = 0$, we must have $D \sim 0$, i.e., $\mathcal{L}(D) \cong \mathcal{O}_X$.*

Proof. if $l(D) \neq 0$,then the complete linear system $|D|$ is nonempty. Hence D is linearly equivalent to some effective divisor. Since the degree depends only on the linear equivalence class, and the degree of an effective divisor is nonnegative, we find deg$D \geq 0$. If deg $D = 0$, then D is linearly equivalent to an effective divisor of degree 0. But there is only one such, namely the zero divisor. □

Definition 2.1.97. *Let X be a projective variety of dimension 1, we call any divisor in the corresponding linear equivalence class a canonical divisor and denote it by K_X i.e., $\pi(K_X) = \omega_X$ (2.1.95).*

Theorem 2.1.98 (Riemann-Roch.). *Let D be a divisor on a curve X of genus g. Then*

$$l(D) - l(K_X - D) = degD + 1 - g$$

Proof. See ([6]), page 295, chapter 3. □

Example 2.1.99. *On a curve X of genus g, the canonical divisor K_X has degree $2g - 2$. Indeed, we apply Riemann-Roch theorem with $D = K_X$. Since $l(K_X) = dimH^0(X, \mathcal{L}(K_X)) = dimH^0(X, \omega_X) = g$ and $l(0) = 1$, we have*

$$g - 1 = degK_X + 1 - g,$$

hence $degK_X = 2g - 2$.

Lemma 2.1.100. *Let X be a smooth normal variety and ω be a rational n-form. Then the zeros minus the poles of ω determine the divisor, K_X.*

Proposition 2.1.101 ((Adjunction formula.)). *Let X be a smooth variety and let S be a smooth divisor. Then $(K_X + S)|_S = K_S$*

2.1.102 On the Geometry of a surface

Throughout this section, by surface we mean a nonsingular projective surface over an algebraic closed field k. We will assume our surfaces are projective since according to ([6],chapter 2) any complete nonsingular surface is projective over an algebraically closed field k. Moreover, a curve on a surface will simply mean any effective divisor on the surface and by point we will mean a closed point, unless otherwise specified.

Definition 2.1.103. *Let X be a surface. Let C, D be two divisors on X, if $P \in C \cap D$ is a point of intersection of C and D, then we say that C and D meet transversally at P if the local equations f, g of C, D at P generate the maximal ideal $\mathfrak{m}_P$ of $\mathcal{O}_{P,X}$.*

Definition 2.1.104. *Let C and D two nonsingular curves which meet transversally at a finite number of points $P_1, ..., P_r$. We define $C.D$ to be the intersection number r of C and D.*

The following theorem sheds light on the properties of commutativity, associativity and linear equivalence that the intersection between nonsingular curve has.

Theorem 2.1.105. *Let X be a surface, the map:*

$$\begin{aligned} \Phi\colon \operatorname{Div} X \times \operatorname{Div} X &\to \mathbb{Z} \\ (C, D) &\mapsto C.D, \end{aligned}$$

such that if C, D are nonsingular curves meeting transversally, then $\Phi(C, D) = \sharp(C \cap D)$ (which is the number of points of $C \cap D$ and can also be denoted $i(C, D)$).

Remark 2.1.106. *Let C_1, C_2 two nonsingular curves meeting transversally with another nonsingular curve D. Then the map Φ is symmetric ($\Phi(C, D) = \Phi(D, C)$), additive ($\Phi(C_1 + C_2, D) = \Phi(C_1, D) + \Phi(C_1, D)$) and depends only on the linear equivalence classes i.e. if $C_1 \sim C_2$ therefore $\Phi(C_1, D) = \Phi(C_2, D)$.*

Proof. See [[6], page 358]. □

Lemma 2.1.107. *Let C be an irreducible nonsingular curve on X, and let D be any curve meeting C transversally. Then*

$$\sharp(C \cap D) = deg_C(\mathcal{L}(D) \otimes \mathcal{O}_C),$$

where the degree of the invertible sheaf $\mathcal{L}(D) \otimes \mathcal{O}_C$ is the degree of its associate divisor.

Proof. See [[6], page 358]. □

Definition 2.1.108. *Let C and D be curves with no common irreducible component, and if $P \in C \cap D$, then we define the intersection multiplicity $(C.D)_P$ of C and D to be the dimension of the $k-$vector space $\mathcal{O}_{P,X}/(f, g)$ where f, g are local equations of C, D at P.*

Proposition 2.1.109. *Let C and D two curves on X with no common irreducible component, then*

$$C.D = \sum_{P \in C \cap D} (C.D)_P.$$

Proof. See [[6], page 360]. □

Remark 2.1.110. *Let D be a divisor on the surface X, we can define the self-intersection number D.D which we denote D^2. In fact the self-intersection of D^2 cannot be calculated by the direct method of the intersection multiplicity (2.1.109). One uses instead lemma (2.1.107) which is $D^2 = deg_C((\mathcal{L}(D) \otimes \mathcal{O}_D))$.*

Example 2.1.111. *If $X = \mathbb{P}^2$, one knows that $\operatorname{Pic} X \cong \mathbb{Z}$. We take the class l of a line as generator because any two lines are linearly equivalent. Moreover, as they meet in one point, we have $l^2 = 1$. This determines the intersection pairing map Φ on $\mathbb{P}^2$ by linearity.*

Example 2.1.112. *Let $X = \mathbb{P}^2$ and C, D two curves of respective degree n, m. because $\operatorname{Pic} X \cong \mathbb{Z}$ therefore we will have $C \sim nl$ and $D \cong mh$. Hence $C.D = nm$.*

Example 2.1.113. *Let X be the nonsingular quadric surface in $\mathbb{P}^3$. Then $\operatorname{Pic} X \cong \mathbb{Z} \oplus \mathbb{Z}$ ([[6],chapter 2]). One takes as generators lines l of type $(1, 0)$ and h of type $(0, 1)$, one from each family. Then $l^2 = 0$: the quadric surface being isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ (using the Segree embedding), two distinct lines l, l' of same type will automatically imply that that their intersection is empty. Moreover , as they are linearly equivalent will have $l.l' = l^2 = 0$. Whereas, two line l, m of different type will always meet in one point. Hence $l.m = 1$, this determines the intersection pairing on X. So, if C has type (a, b) and D has type (c, d), then $C.D = ad + cb$.*

Example 2.1.114. *The self-intersections, allows us to define a new invariant of a surface. Let $\Omega_{X/k}$ be the cotangent sheaf of X/k, and $\omega_X = \bigwedge^2 \Omega_{X/k}$ be the canonical sheaf. The canonical divisor K_X is a divisor in the linear equivalence class corresponding to ω_X. For instance, if $X = \mathbb{P}^2$, $K_X = -3h$, so $K_X^2 = 9$. In case of X being a quadric surface, then K_X has type $(-2, -2)$, hence $K_X^2 = 8$.*

Proposition 2.1.115 (Adjunction formula for surfaces). *. If C is a nonsingular curve of genus g on the surface X, and if K_X is the canonical divisor on X, then*

$$2g-2 = C.(C+K_X).$$

Proof. See [([6]), page 361]. □

Remark 2.1.116. *The proposition above provide an expeditious method for computing the genus of a curve on a surface. Let us take the curve C of degree $d \in \mathbb{P}^2$, then one has*

$$\begin{aligned} 2g-2 &= C.(C+K_{\mathbb{P}^2}) \\ &= C.C + C.K_{\mathbb{P}^2} \\ &= d^2 - 3d. \qquad\qquad (\textit{because } C \sim dh \textit{ and } K_{\mathbb{P}^2} \sim -(2+1)h) \end{aligned}$$

which implies that $g = \frac{1}{2}(d-1)(d-2)$.

Example 2.1.117. *Let C a curve of type (a,b) on the quadric surface, then it is easy to see that $C+K_X$ has type $(a-2,b-2)$, so*

$$2g-2 = a(b-2) + (a-2)b,$$

Hence $g = ab - a - b + 1$.

In the next section will be interested in the Riemann-Roch theorem version for surfaces, which plays an important role to prove the Hodge index theorem and the Nakai's criterion for an ample divisor.

One recalls that for any divisor D on a surface X, we set $l(D)$ being the $\dim_k H^0(X, \mathcal{L}(D))$, which is also equal to $\dim |D| + 1$ where, $|D|$ is in fact the complete linear system of D. We also recall that the arithmetic genus $P_a(X)$ of X is defined by $P_a = \chi(\mathcal{O}_X) - 1$ ([4], page 131).

Theorem 2.1.118. *Let D a divisor on the surface X, then*

$$l(D) - s(D) + l(K_X - D) = \frac{1}{2} D.(D - K_X) + P_a(X) + 1,$$

where $s(D)$ is the dimension of $H^1(X, \mathcal{L}(D))$ and is called the superabundance of D.

Proof. See, [[6], page 362] □

Definition 2.1.119. *Let S be a surface. The intersection form on $Cl(S)$ is the symmetric bilinear form*

$$\begin{aligned} i\colon\ & Cl(S) \to Cl(S) \\ & (C,D) \mapsto i(C,D) = \chi(\mathcal{O}_S) - \chi(\mathcal{L}(-C)) - \chi(\mathcal{L}(-D)) + \chi(\mathcal{L}(-C-D)). \end{aligned}$$

where $\chi(\mathcal{O}_S) := \sum_{i=0}^{\infty} dim H^i(S, \mathcal{O}_S)$ the Euler characteristic of S.

Corollary 2.1.120 (The Rieman-Roch theorem for surfaces.). *Let X be a surface and D be a divisor on X. Then*

$$\chi(\mathcal{L}(D)) = \chi(\mathcal{O}_X) + \frac{1}{2}i(D, D - K_X).$$

Proof. We start by computing $i(-D, D - K_X)$. By definition we have

$$i(-D, D - K_X) = \chi(\mathcal{O}_X) - \chi(\mathcal{L}(D)) - \chi(\omega_X \otimes \mathcal{L}(-D)) + \chi(\omega_X).$$

By Serre duality we have $\chi(\omega_X \otimes \mathcal{L}(-D)) = \chi(\mathcal{L}(D))$ and $\chi(\omega_X) = \chi(\mathcal{O}_X)$.
From this we see that

$$i(-D, D - K_X) = 2(\chi(\mathcal{O}_X) - \chi(\mathcal{L}(D))).$$

On the other hand we have

$$i(-D, D - K_X) = -i(D, D) + i(D, K_X)$$

it now remains to rearrange the terms in the equation. □

Proposition 2.1.121 (The genus formula.). *Let C be an irreducible curve on a surface X. Then*

$$P_a(C) = 1 + \frac{1}{2}i(C, C + K_X).$$

Proof. Recall that

$$g_a(C) = \dim H^1(C, \mathcal{O}_C) = 1 - \chi(\mathcal{O}_C) \quad \text{we use that } C \text{ is connected so } H^0(C, \mathcal{O}_C) = 1.$$

We consider the ideal sheaf $\mathcal{L}(-C)$ of C and short exact sequence

$$0 \to \mathcal{L}(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0.$$

By taking Euler characteristics we get

$$\chi(\mathcal{O}_C) = \chi(\mathcal{O}_X) - \chi(\mathcal{L}(-C)).$$

It follows that

$$1 - g_C = \chi(\mathcal{O}_X) - \chi(\mathcal{L}(-C)).$$

On the other hand, the Rieamann-Roch theorem gives that

$$\chi(\mathcal{L}(-C)) = \chi(\mathcal{O}_X) + \frac{1}{2}i(-C, -C - K_X).$$

Substituting this into the above relation gives

$$1 - g_C = \chi(\mathcal{O}_X) - (\chi(\mathcal{O}_X) + \frac{1}{2}i(-C, -C - K_X)) = -\frac{1}{2}i(C, C + K_X).$$

□

Theorem 2.1.122 (The Künneth formula). *. Let X and Y be compact and separated schemes over a field k and $\mathcal{F}$ be a sheaf on X and $\mathcal{C}$ a sheaf on Y. There is an isomorphism*

$$H^r(X \times_k Y, p_X^*\mathcal{F} \otimes p_Y^*\mathcal{C}) \cong \bigoplus_{i+j=r} H^i(X, \mathcal{F}) \otimes_k H^j(Y, \mathcal{C}).$$

Proof. See ([6]). □

Theorem 2.1.123 (The Nakai-Moishezon criterion.). *A divisor D on a surface X is ample if and only if $i(D, D) > 0$ and $i(D, C) > 0$ for every irreducible curve C on X.*

Proof. See ([6]) □

Definition 2.1.124. *A divisor D on a surface X is numerically equivalent to zero, written*

$$D \equiv 0,$$

if $i(D, E) = 0$ for all divisors E.

Remark 2.1.125. *We say D and E are numerically equivalent, written $D \equiv E$, if*

$$D - E \equiv 0.$$

3
Moduli problems

Moduli arise in connection with classification problems in algebraic geometry. The main ingredients of a classification problem are a collection of objects $\mathcal{M}$ and an equivalence relation $\sim$ on $\mathcal{M}$. The problem is the following: describe the set of equivalence classes $\mathcal{M}/\sim$. One can often find some discrete invariants which partition $\mathcal{M}/\sim$ into a countable number of subsets, but in algebraic geometry this rarely gives a complete solution of the problem. Almost always there exist continuous families of objects of $\mathcal{M}$, and we would like to give $\mathcal{M}/\sim$ some algebro-geometric structure to reflect this fact. This is the aim of the theory of moduli.

Our main purpose in this chapter is to try to give some idea of what constitutes a moduli problem and to describe the sort of solution for which one should look for.Moreover, throughout this chapter we will follow the book of P.E. Newstead ([9]). However, before getting to the main definitions and propositions of what a moduli problem is we will start by recalling some important concepts that will be relevant for this section. Concept such as vector bundle which are equivalent to quasi-coherent sheaves of finite rank as we will see in 3.0.9.

Definition 3.0.1. *A vector bundle of rank n over X consists of a variety E, a morphism $p : E \to X$, and a structure of n-dimensional vector space of each fibre $E_x = p^{-1}(x)$ such that, for all $x \in X$,*

(i) the vector space structure on E_x is compatible with the structure of variety induced from that of E;

(ii) there exists a neighbourhood U of x and an isomorphism

$$\psi : U \times k^n \to p^{-1}(U)$$

of p over U such that $p \circ \psi = p_U$ with p_U the restriction of p to $p^{-1}(U)$ besides, the map

$$\begin{aligned} \tau\colon k^n &\to E_y \\ v &\mapsto \psi(y, v), \end{aligned}$$

should be linear for all $y \in U$.

Remark 3.0.2 ([9], page 18). *A vector bundle of rank* 1 *is called a line bundle. Moreover if X is a variety and E is a vector bundle over X then E is necessary a variety.*

Definition 3.0.3. *Let $E \xrightarrow{\pi} X$ and $F \xrightarrow{\chi} X$ be vector bundles over X. A homomorphism from E to F is a morphism $h : E \to F$ such that:*

(i) the diagram

$$\begin{tikzcd} & E \arrow[dl, "h"'] \arrow[dr, "\pi"] & \\ F \arrow[rr, "\chi"] & & X \end{tikzcd}$$

is commutative, and

(ii) for each $x \in X$ the restricted map $h_x : E_x \to F_x$ is linear.

Remark 3.0.4. *In addition, $h : E \to F$ is an isomorphism of vector bundles if it is bijective and and its inverse is a homomorphism from F to E.*

Definition 3.0.5. *A section of vector bundle E over X is a morphism $s : X \to E$ such that $p \circ s = 1_X$.*

Remark 3.0.6. *There is an obvious structure of vector space on the set of all sections of E. There is a bijective correspondence between sections of E and homomorphisms from $E \times k$ to E, with the homomorphism corresponding to a section s being given by*

$$(x, \lambda) \mapsto \lambda.s(x) \quad \textit{for all} \quad x \in X, \quad \lambda \in k.$$

Definition 3.0.7. *Let $\varphi : Y \to X$ be morphism of varieties and E a vector bundle over X, one can define an induced vector bundle $\varphi^\star E$ over Y by :*

$$\varphi^\star E = \{(y, v) : \quad y \in Y, \quad v \in E, \quad \varphi(y) = p(v)\} \subset Y \times E.$$

Definition 3.0.8. *Let s a section of E, we can define an induced section $\varphi^\star s$ of $\varphi^\star E$ by the formula :*

$$\varphi^\star s(y) = \big(y, s(\varphi(y))\big) \quad \textit{for all} \quad y \in Y.$$

similarly, for any homomorphism $h : E_1 \to E_2$ of vector bundles over X, we have a homomorphism

$$\begin{aligned} \varphi^\star h \colon \varphi^\star E_1 &\to \varphi^\star E_2 \\ (y, v) &\mapsto (y, h(v)). \end{aligned}$$

Remark 3.0.9. *Let X be a variety.The set of sections of a vector bundle E will be written $\mathcal{L}(E)$. Moreover, the set $\mathcal{L}(E)$ is a module over the ring $\mathcal{O}_X(X)$ and we associate with any open set $U \subset X$ the set $\mathcal{L}(E, U)$ of sections of the bundle E restricted to U. Hence from ([11]), we obtain a sheaf that we denote $\mathcal{L}_E$ which is a sheaf of modules over the structure sheaf $\mathcal{O}_X$.*

The following theorem establishes a correspondence between vector bundles and locally free sheaves.

Theorem 3.0.10 ([11]). *The correspondence $E \to \mathcal{L}_E$ establishes a one-to-one correspondence between vector bundles and locally free sheaves of finite rank.*

Proof. See ([11], page 58) □

To have an insight of what a moduli problem is we begin with a basic example

Example 3.0.11. *Let X be a fixed variety, let A consist of all vector bundles over X, and let $\sim$ be given by isomorphism of bundles. A family of objects of A parameterised by S is a vector bundle E over $S \times X$. The object E_s corresponding to a point s of S is the vector bundle over X obtained by restricting E to $\{s\} \times X$.*

For any morphism $\varphi : S \to S'$, the induced family is simply $(\varphi \times 1_X)^\star E$.

According to example(3.0.11), the basic ingredients for a moduli problem are then the collection A, the equivalence relation $\sim$ and the concept of family. As is suggested by the example above, the precise definition of family depends on the particular problem.

However, in all cases, we require families to satisfy the following properties:

Definition 3.0.12 (Properties of moduli problem).

(a) A family parameterised by the variety $\{pt\}$ is a single object of A.

(b) There is a notion of equivalence of families parameterised by any given variety S, which reduces to the given equivalence relation on A when $S = \{pt\}$; We shall denote this relation by $\sim$.

(c) For any morphism $\varphi : S' \to S$ and any family X parameterised by S, there is an induced family $\varphi^\star X$ parameterised by S'. Moreover this operation satisfies the functorial properties

$$(\varphi \circ \varphi')^\star = \varphi'^{\star} \circ \varphi^\star, \quad 1_S^\star = identity,$$

and is compatible with $\sim$, i.e.,

$$X \sim X \implies \varphi^\star X \sim \varphi^\star X'.$$

Remark 3.0.13. *Let X be a family parameterised by a variety S. For any point s of S, we denote by X_s the object of A induced by the inclusion morphism $i_s : \{s\} \to S$, meaning $X_s := i_s^\star X$.*

The conditions above complete the description of a moduli problem. In the next section we will consider what sort of solution it is reasonable to expect.

3.1 Moduli Spaces

Given a moduli problem as described above, we would like to define on the set $A/\sim$ a structure of variety which reflects the structure of families of objects of A. The main purpose of this section is to suggest some ways in which this idea can be made precise.

Suppose then that M is a variety whose underlying set is $A/\sim$. For any family X parameterised by S, we have a map

$$\begin{aligned} \nu_X \colon S &\to M \\ s &\mapsto \nu_X(s) = [X_s], \end{aligned}$$

where $[X_s]$ denotes the equivalence class of the object X_s. It seems reasonable to ask that this map be a morphism; and the best we could hope for would be that ν should define a bijective correspondence between equivalence classes of families parameterised by S and morphisms $S \to M$.

This idea can conveniently be expressed in categorical terms. For this, we write $\mathcal{F}(S)$ for the set of equivalence classes of families parameterised by S. By the conditions stated above $\mathcal{F}$ is a contravariant functor from the category of algebraic varieties to the category of sets. Moreover, if we denote by $\mathrm{Hom}(S, M)$ the set of morphisms from S to M, we have natural maps

$$\begin{aligned} \phi(S) \colon \mathcal{F}(S) &\to \mathrm{Hom}(S, M) \\ [X] &\mapsto \phi(S)([X]) = \nu_X, \end{aligned}$$

and these maps determine a natural transformation

$$\phi : \mathcal{F} \to \mathrm{Hom}(-, M).$$

What we are asking is that ϕ should be an isomorphism of functors, or in the usual language of categories that $\mathcal{F}$ should be represented by (M, ϕ). From this the definition of fine moduli space results.

Definition 3.1.1. *A fine moduli space for a given moduli problem is a pair (M, ϕ) which represents the functor $\mathcal{F}$.*

Remark 3.1.2. *Note that in this definition we did not insist a priori that $M = A/\sim$. For if (M, ϕ) represents $\mathcal{F}$, we have a natural bijection*

$$\phi(pt) : A/\sim = \mathcal{F}(pt) \to Hom(pt, M) = M.$$

Moreover, for any variety S and any s in S, the inclusion of $\{s\}$ in S induces a commutative diagram

$$\begin{array}{ccc} \mathcal{F}(S) & \xrightarrow{\phi(S)} & Hom(S, M) \\ \Big\downarrow{\scriptstyle X \mapsto X_s} & & \Big\downarrow{\scriptstyle \phi \mapsto \phi(s)} \\ \mathcal{F}(pt) & \xrightarrow{\phi(pt)} & M \end{array}$$

Unfortunately there are very few problems in which one can hope for a a fine moduli space. It is therefore necessary to find some weaker condition which nevertheless determines a unique structure of variety on M. The solution is to drop the requirement that M satisfy a universal property for families, and ask instead that ϕ should have a universal property for natural transformation $\mathcal{F} \to \mathrm{Hom}(-, N)$.

Definition 3.1.3. *A coarce moduli space for a given moduli problem is a variety M together with a natural transformation*

$$\phi : \mathcal{F} \to Hom(-, M)$$

such that

(a) *$\phi(pt)$ is bijective;*

(b) *for any variety N and any natural transformation $\psi : \mathcal{F} \to Hom(-, N)$, there exists a unique natural transformation*

$$\Omega : Hom(-, M) \to Hom(-, N)$$

such that $\psi = \Omega \circ \phi$.

Proposition 3.1.4. *A coarse moduli space (M, ϕ) is fine moduli space if and only if and only if*

(a) *there exists a family U parameterised by M such that, for all $m \in M$, U_m belongs to the equivalence class $\phi(pt)^{-1}(m)$, and*

(b) *for any families X, X' parameterised by a variety S,*

$$\nu_X = \nu_{X'} \Leftrightarrow X \sim X'.$$

Proof. This follows easily from Definitions (3.0.11) and (3.1.1.) In fact

- (i) $\Leftrightarrow \phi$ is surjective,
- (ii) $\Leftrightarrow \phi$ is injective.

□

3.1.5 Examples of Moduli spaces

In this section we are interested in a very simple moduli problem, namely that of classifying the endomorphisms of finite dimensional vector spaces following ([9]). The question that we are asking is whether it is possible to put a natural structure of algebraic variety on the set of all endomorphisms of finite dimensional vectors spaces.

Definition 3.1.6. *One considers the pair (V, T), where V is an n-dimensional vector space over k and T is an endomorphism of V. An isomorphism from (V, T) to (V', T') is an isomorphism of vector spaces $f : V \to V'$ such that*

$$f \circ T = f \circ T'.$$

Remark 3.1.7. *We are particularly interested in the classification of the pairs (V, T) up to isomorphism. By the basis theorem of linear algebra (two basis of the same vector space have the same cardinal), this amounts to the classification of $n \times n$ matrices up to similarity.*

Definition 3.1.8. *A family of endomorphisms of $n-$dimensional vector spaces parameterised by a variety S is a pair (B, T), where B is a vector bundle of rank n over S and T is an endomorphism of E.*

Remark 3.1.9. *Let two families (B, T) and (B', T') parameterised by S. Then an isomorphism from (B, T) to (B', T') is an isomorphism $h : B \to B'$ such that $h \circ T = T' \circ h$. Moreover, for any family (B, T) parameterised by S and any morphism $\phi : S' \to S$, we have an induced family $\phi^\star(B, T) = (\phi^\star B, \phi^\star T)$ parameterised by S'.*

It is easy to see that these remarks and definition satisfy conditions (3.0.12). Therefore we have the appropriate setting for a moduli problem, which we denote by (End_n). As in the previous section above we write $\mathcal{F}(S)$ for the set of isomorphism classes of families parameterised by S.

Proposition 3.1.10. *There is no fine moduli space for (End_n).*

Proof. We take any complete variety S such that there exists a non-trivial line bundle L over S (for example $S = \mathbb{P}^1$, $L = H$). Then, for any endomorphism T of the trivial bundle $I_n := X \times k^n$, we have non-isomorphic families (I_n, T) and $(I_n \otimes L, T \otimes 1_L)$, which would determine the same morphism from S to any moduli space (See claim below) □

Claim. Let X be a complete variety, and L a line bundle over X such that $I_n \otimes L$ is trivial for some $n \geq 1$. Then L is trivial.

Proof. See [[9],page 12] □

Proposition 3.1.11. *For $n > 1$, there is no coarse moduli space for (End_n).*

In fact, if M is a variety and $\mathcal{F} \to \mathrm{Hom}(-, M)$ a natural transformation, then any two endomorphisms with the same characteristic polynomial are represented by the same point of M.

Proof. Suppose that $n = 2$, let $\lambda \in k$, and consider the morphism

$$\begin{aligned} \eta \colon k &\to M_{2\times 2} \\ t &\mapsto B_t = \begin{pmatrix} \lambda & t \\ 0 & \lambda \end{pmatrix}. \end{aligned}$$

The morphism defines an element of $\mathcal{F}(k)$ and hence determines a morphism $\phi : k \to M$. If $t \neq 0$, the matrix B_t is similar to B_1. It follows that $\phi(t) = \phi(1)$ for all $t \neq 0$ and hence also for $t = 0$. Thus B_0 and B_1 are represented by the same point of M (although, of course these two matrices are not similar). For $n > 2$ the general proof unfolds the same.

□

Definition 3.1.12. *Let g and n be two integers. A stable n-pointed Riemann surface of genus g is a compact Riemann surface C of genus g with n distinct points* $x_1, \ldots, x_n \in C$*, such that set of automorphisms of C which fix all* x_i*, is finite, the only singularity of C are simple nodes and the marked points are distinct and do not coincide with the nodes.*

Definition 3.1.13. $(C, x_1, \ldots, x_n)$ *and* $(D, y_1, \ldots, y_n)$ *be two stables n-pointed Riemann surfaces are equivalents, if there is exist an isomorphism of Riemann surface* $f : C \to D$ *such that* $f(x_i) = y_i$ *for all* $1 \leq i <\leq n$*.*

Remark 3.1.14. *Let* $2g - 2 + n > 0$*. We denote by* $\mathcal{M}_{g,n}$ *the set of isomorphism classes of Riemann surfaces of genus g with n-marked points.*

Proposition 3.1.15. *If* $g = 0, n = 3$*, a single point is a coarse moduli space of* $\mathcal{M}_{0,3}$*.*

Proof. If $g = 0, n = 3$. Every rational curve (C, x_1, x_2, x_3) with three marked points can be identified with $(\mathbb{P}^1(\mathbb{C}), 0, 1, \infty)$ in a unique way. Thus $\mathcal{M}_{0,3} =$ point. □

Proposition 3.1.16.

$$\{z \in \mathbb{C}, \quad Im(z) > 0\}/SL(\mathbb{Z}, 2) \cong \mathcal{M}_{1,1}$$

with $SL(\mathbb{Z}, 2)$ *the set of* 2×2 *matrices of determinant equal to* 1 *and entries in* $\mathbb{Z}$*.*

Proof. Every elliptic curve E is isomorphic to the quotient of $\mathbb{C}$ by a rank 2 lattice L. The image of $0 \in \mathbb{C}$ is a natural marked point on E. Thus

$$\mathcal{M}_{1,1} = \{\text{Lattices}\}/\mathbb{C}^*.$$

Consider a direct basis (z_1, z_2) of a lattice L. Multiplying L by $1/z_1$ we obtain a lattice with basis $(1, \tau)$, where $\tau \in \{z \in \mathbb{C}, \quad \text{Im}(z) > 0\} := \mathbb{H}$ the upper half-plane. Choosing another basis of the same lattice we obtain another point $\tau' \in \mathbb{H}$. Thus the group $\text{SL}(\mathbb{Z}, 2)$ of direct base changes in a lattice acts on $\mathbb{H}$. This action is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}.$$

We have

$$\mathcal{M}_{1,1} = \mathbb{H}/\text{SL}(\mathbb{Z}, 2).$$

□

4

Points sets in projective space

The purpose of this section is to construct a projective scheme induced by the subalgebra of G-invariant elements of a graded algebra. Later on, using tools from Geometry invariant theory we will see how the point of the latter projective scheme can be interpreted as isomorphism classes of Del Pezzo surface. This Chapter is based from chapter 1 and 2 of the book ([3])

Before we dive into that, it is important to bear in mind the following notation we will use throughout this section :

Notation 4.0.1. *k: an algebraically closed field of characteristic $p \geq 0$;*
$\mathbb{P}_n$ *: the $n-$dimensional projective space over k;*
$\mathbb{P}_n^m = (\mathbb{P}_n)^m = \mathbb{P}_n \times \cdots \times \mathbb{P}_n$*, m times;*
$\pi_i : \mathbb{P}_n^m \to \mathbb{P}_n =$ *the $i-$th projection;*
$G = Aut(\mathbb{P}_n)$*;*

$\sigma : G \times \mathbb{P}_n^m \to \mathbb{P}_n^m$ *: the morphism of diagonal action:*

$$\sigma(g, (x^1, ..., x^m)) = (g(x^1), \dots, g(x^m)), \quad g \in G, \quad , (x^1, ..., x^m) \in \mathbb{P}_n^m;$$

$p_1 : G \times \mathbb{P}_n^m \to G, \quad p_2 : G \times \mathbb{P}_n^m \to \mathbb{P}_n^m =$ *the projections;*

$\mathcal{L} = \bigotimes_{i=1}^m \pi_i^\star(\mathcal{O}_{\mathbb{P}_n}(l))$*, where l is the smallest positive integer satisfying the equality*

$$lm = w(n+1)$$

for some integer w.

Definition 4.0.2. *A linear algebraic group H over a field k is a smooth closed subgroup scheme of $GL(n+1)$ over k, for some positive integer n.*

Definition 4.0.3. *A connected linear algebraic group H over an algebraically closed field k is called reductive if every smooth connected solvable normal subgroup of H is trivial.*

Example 4.0.4. *The group $G = Aut(\mathbb{P}_n)$ is a reductive algebraic group.*

Except stated otherwise, Throughout this section we fix the integers n and m and the invertible sheaf $\mathcal{L}$ of $\mathbb{P}_n^m$ and G as described above.

Definition 4.0.5. *A G-linearization of $\mathcal{F}$ a sheaf on an algebraic variety X with an action $\sigma : G \times X \to X$ is an isomorphism:*

$$\sigma^\star(\mathcal{F}) \simeq p_2^\star(\mathcal{F}),$$

where $p_2 : G \times X \to X$ is the second projection.

Proposition 4.0.6. *$\mathcal{L}$ admits a unique G-linearization.*

Proof. According to ([8],chapter 1, proposition 1.4) G does not admits non-trivial characters, therefore it is enough to construct one $G-$linearization of $\mathcal{L}$. Moreover, one can view G as an open subset of $\mathbb{P}_{n^2+2n}$, the complement of which is the determinantal hypersurface of degree $n+1$. This will imply that

$$\mathcal{O}_G(1)^{\otimes n+1} \simeq \mathcal{O}_G(n+1) \simeq \mathcal{O}_G.$$

Since G acts linearly on each factor $\mathbb{P}_n$ of $\mathbb{P}_n^m$, we have a natural isomorphism

$$(\pi_i \circ \sigma)^\star(\mathcal{O}_{\mathbb{P}_n}(l)) \simeq p_1^\star \mathcal{O}_G(l) \otimes (\pi_i \circ p_2)^\star \mathcal{O}_{\mathbb{P}_n}(l).$$

Thus

$$\begin{aligned}
\sigma^*(\mathcal{L}) = \sigma^*(\otimes_{i=1}^m \pi_i^*(\mathcal{O}_{\mathbb{P}_n}(l))) &= \otimes_{i=1}^m (\pi_i \circ \sigma)^*(\mathcal{O}_{\mathbb{P}_n}(l))) \\
&\simeq \otimes_{i=1}^m p_1^* \mathcal{O}_G(l) \otimes (\pi_i \circ p_2)^*(\mathcal{O}_{\mathbb{P}_n(l)}) \\
&\simeq p_1^* \mathcal{O}_G(ml) \otimes p_2^* \mathcal{L} = p_1^* \mathcal{O}_G(w(n+1)) \otimes p_2^* \mathcal{L} \\
&\simeq p_1^* \mathcal{O}_G \otimes p_2^* \mathcal{L} \simeq p_2^* \mathcal{L}.
\end{aligned}$$

□

Remark 4.0.7. *For every G-linearized sheaf $\mathcal{F}$ on a G-variety X there is a natural linear representation of G in the space $\Gamma(X, \mathcal{F})$. In fact it comes from the fact that we have the following composition:*

$$\Gamma(X, \mathcal{F}) \xrightarrow{\sigma^*} \Gamma(G \times X, \sigma^* \mathcal{F}) \to \Gamma(G \times X, p_2^* \mathcal{F}) \to \Gamma(G, \mathcal{O}_G) \otimes \Gamma(X, \mathcal{F}),$$

In fact the second arrow is defined by the linearization of $\mathcal{F}$ and the last arrow is defined by the Künneth formula (2.1.122). Moreover we can regard $g \in G$ as a homomorphism of k-algebras

$$\begin{aligned} g\colon \Gamma(G, \mathcal{O}_G) &\to k \\ t &\mapsto g(t) := t(g). \end{aligned}$$

and we set the action of G on $\Gamma(X, \mathcal{F})$ to be as follow : For $g \in G$ we have

$$\begin{aligned} \rho(g)\colon \Gamma(X, \mathcal{F}) &\to \Gamma(X, \mathcal{F}) \\ s &\mapsto \rho(g)(s) := (g \otimes 1)(\sigma^*(s)). \end{aligned}$$

As a matter of fact for every $g \in G$ and $s \in \Gamma(X, \mathcal{F})$ we have :

$$\begin{aligned} \rho(g)(s) &= (g \otimes 1)(\sigma^*(s)), \\ &= (g \otimes 1)(s_G \otimes s_X) && (\textit{where } \sigma^*(s) = s_G \otimes s_X \in \Gamma(G, \mathcal{O}_G) \otimes \Gamma(X, \mathcal{F}),) \\ &= g(s_G) \otimes 1(s_X), \\ &= s_G(g) \otimes s_X \in \Gamma(X, \mathcal{F}) && \textit{with } s_G(g) \in k\,. \end{aligned}$$

Definition 4.0.8. *We will denote the subspace of G-invariant sections $\Gamma(X, \mathcal{F})^G$ i.e the subspace of $\Gamma(X, \mathcal{F})$ whose element s are such that $\rho(g)(s) = s$ for every $g \in G$.*

Coming back to our situation, by setting $X = \mathbb{P}_n^m$ and $\mathcal{F} = \mathcal{L}$, the following proposition holds

Proposition 4.0.9. *The graded k-algebra $R_n^m = \bigoplus_{k=0}^{\infty} \Gamma(\mathbb{P}_n^m, \mathcal{L}^{\otimes k})^G$ is of finite type, where $\mathcal{L}^{\otimes k}$ is also a G-linearization that is the $k-th$ tensor product of the G-linearization of $\mathcal{L}$.*

Proof. Since $\mathcal{L}$ is an ample invertible sheaf on $\mathbb{P}_n^m$, the graded k-algebra

$$\bigoplus_{k=0}^{\infty} \Gamma(\mathbb{P}_n^m, \mathcal{L}^{\otimes k})$$

is of finite type. Furthermore the group G acts on this algebra by automorphisms of graded algebras, and R_n^m is the subalgebra of G-invariant elements graded by

$$(R_n^m)_k = \Gamma(\mathbb{P}_n^m, \mathcal{L}^{\otimes k})^G.$$

However, since G a is reductive algebraic group, R_n^m is of finite type over k according to ([?]). □

Remark 4.0.10. *One can set*

$$P_n^m = Proj(R_n^m)$$

to obtain a projective algebraic variety over k. This projective variety will be precisely the object of our study.

The following section will give us some tools in order to be able to compute R_n^m as a space of graded homogeneous polynomials.

4.1 Standard Monomials

Let $\mathbb{P}_n = \mathbb{P}(V)$ the projectivization of a $n+1$-dimensional space V over k. One has the following isomorphism

$$\Gamma(\mathbb{P}_n, \mathcal{O}_{\mathbb{P}_n}(k)) \cong \mathrm{Sym}^k(V^*),$$

where

$$\mathrm{Sym}(V^*) = \bigoplus_{k=0}^{\infty} \mathrm{Sym}^k(V^*).$$

is the graded symmetric algebra of the dual vector space V^*. Furthermore, by the Künneth formula,

$$(R_n^m)_k = \Gamma(\mathbb{P}_n^m, \mathcal{L}^{\otimes k})^G = \Gamma(\mathbb{P}_n^m, \otimes_{i=1}^m \pi_i^* \mathcal{O}_{\mathbb{P}_n}(kl))^G \cong ((\mathrm{Sym}^{kl}(V^*))^{\otimes m})^G.$$

Remark 4.1.1. *The linear representation of G in $\Gamma(\mathbb{P}_n^m, \mathcal{L}^{\otimes k})$ is the m-th tensor product of its natural representation in the space of homogeneous polynomial function on V of degree kl.*

Proposition 4.1.2. *Consider an element of R_n^m as a function $\mu(v^1, ..., v^m)$ on V^m which is a homogeneous polynomial of degree kl in each variable. Then the functions*

$$\mu_\tau(v^1, ..., v^m) = \prod_{1 \leq wk} det(v^{\tau_{i1}}, \dots, v^{\tau_{in+1}})$$

span $(R_n^m)_k$, where $\tau_{ij} \in \{1, ..., m\}$ and each of the value in $\{1, ..., m\}$ occurs exactly kl times.

Definition 4.1.3. *A $wk \times (n+1)$ matrix*

$$\tau = \begin{bmatrix} \tau_{11} & \cdots & \tau_{1n+1} \\ \vdots & \ddots & \vdots \\ \tau_{wk1} & \cdots & \tau_{wkn+1} \end{bmatrix}$$

is said to be tableau if

(i) all its entries belongs in $\{1, ..., m\}$,

(ii) the set

$$\{(i,j) : \quad \tau_{ij} = a\}$$

has cardinality kl for all $a \in \{1, ..., m\}$

(iii)

$$\sum_{j=1}^{wk} \tau_{ij} = wk$$

.

and the corresponding function μ_τ is said to be the monomial that belong to τ. The number wk is weight of τ or μ_τ and the number kl is its degree.

Remark 4.1.4. *We assume that $\tau_{ij} \neq \tau_{ij'}$ for $j \neq j'$ and each i. Otherwise the corresponding monomial is zero.*

Definition 4.1.5. *A tableau τ is said to be standard if*

$$\begin{aligned} &\tau_{ij} < \tau_{ij+1} && \text{for each } i = 1, ..., wk, \quad j = 1, ..., n, \\ &\tau_{ij} \leq \tau_{i+1j} && \text{for each } i = 1, ..., wk-1, \quad j = 1, ..., n+1. \end{aligned}$$

Definition 4.1.6. *A standard monomial is a monomial that belongs to a standard tableau.*

Theorem 4.1.7. *The standard monomials of degree kl and weight wk form a basis of $(R_n^m)_k$.*

Proof. To prove the linear independence of the set that spans the space $(R_n^m)_k$ we refer to ([2]) for its proof. We will prove instead that any mononial is a linear combination of standard ones. We present the following algorithm for the proof which is iterative:

- We suppose that μ_τ is not standard. We then permute the entries of each row of τ so that in the new tableau τ' all rows are in strictly increasing order. Then we will have :

  $$\mu_\tau = \pm \mu_{\tau'}.$$

  Next we permute the rows of τ' so that in the obtained tableau τ''

  $$\tau''_{i1} \leq \tau''_{i+11} \quad \text{for each } i = 1, ..., wk.$$

  Continue permuting the rows so that if $\tau_{ij} = \tau_{i+11}$ then $\tau_{ij+1} \leq \tau_{i+1j+1}$. Note that these permutations do not change the monomial. We call the monomial obtained so far semi-standard.

- The rest of the algorithm proceeds by induction on the lexicographic order of tableaux defined by the setting $\tau < \tau'$ if

  $$(\tau_{11}, \ldots, \tau_{wkn+1}) < (\tau'_{11}, \ldots, \tau'_{wkn+1})$$

  with respect to the lexicographic order.

  Furthermore we suppose that μ_τ is not yet standard and let i_0 be such that

  $$\tau_{i_0 j_0} > \tau_{i_0+1 j_0}$$

  for some j_0. Consider the increasing sequences:

  $$S_1 = (s_1, ..., s_{j_0}), \quad S_2 = (s_{j_0+1}, ..., s_{n+2}), \quad S = (S_1, S_2),$$

  Where

  $$\begin{aligned} s_k &= \tau_{i_0+1k} && \text{if } k \leq j_0, \\ &= \tau_{i_0 k-1} && \text{if } k > j_0. \end{aligned}$$

  Let $A \subset \sum_{n+2}$ be the subset of the permutation group $\sum_{n+2}$ such that $\sigma \in A$ if and only if $(s_{\sigma(1)}, ..., s_{\sigma(j_0)})$ and $(s_{\sigma(j_0+1)}, ..., s_{\sigma(n+2)})$ are increasing subsequences of S. One set for every $\sigma \in A$

$$\tau_{\sigma'} = (\tau_{i_0,1}, \ldots, \tau_{i_0,j_0-1}, s_{\sigma(j_0+1)}, \ldots, s_{\sigma(n+2)}),$$
$$\tau_{\sigma''} = (s_{\sigma(1)}, \ldots, s_{\sigma(j_0)}, \tau_{i_0+1,j_0+1}, \ldots, \tau_{i_0+1,n+1}).$$

On the other hand for every sequence $\gamma = (i_1, \ldots, i_{n+1})$ of numbers from $\{1, \ldots, m\}$ one will consider the determinant

$$(\gamma) = (i_1, \ldots, i_{n+1}) = \det(v^{i_1}, \ldots, v^{i_{n+1}}) \in (V^*)^{\otimes n+1}$$

as a section of $\pi_{i_1}^* \mathcal{O}_{\mathbb{P}_n}(1) \otimes \ldots \pi_{i_{n+1}}^* \mathcal{O}_{\mathbb{P}_n}(1)$. For example,

$$\mu_\tau = \prod_{i=1}^{wk} (\tau_i),$$

where τ_i is the i-th row of τ. Then we have

$$\sum_{\sigma \in A} \operatorname{sgn}(\sigma)(\tau'_\sigma \tau''_\sigma) \in \Gamma(\mathbb{P}_n^m, \pi_{s_1^*}(\mathcal{O}_{\mathbb{P}_n}(1)) \otimes \ldots \otimes \pi_{s_{n+2}^*}(\mathcal{O}_{\mathbb{P}_n}(1))) \cong (V^*)^{\otimes n+2}$$

is skew-symmetric and dim $V = n+1$ hence the above sum is identically zero.
therefore we can write

$$(\tau_{i_0})(\tau_{i_0+1}) = -\sum_{\sigma \in A'} \operatorname{sgn}(\sigma)(\tau'_\sigma \tau''_\sigma)$$

where $A' = A - \{\mathrm{id}\}$.
Let $\tau(\sigma)'$ denote the tableau that is obtained from τ by replacing τ_{i_0} with τ_σ, and replacing τ_{i_0+1} with τ''_σ.
Let $\tau(\sigma)$ be obtained from $\tau(\sigma)'$ by rearranging the rows in increasing order. then

$$\mu_\tau = -\sum_{\sigma \in A'} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau'(\sigma))\mu_{\tau(\sigma)}$$

Where $\mu_{\tau(\sigma)} = \operatorname{sgn}(\tau'(\sigma))\mu_{\tau(\sigma)'}$. It is obvious that

$$\tau(\sigma) < \tau$$

for every $\sigma \in A'$. Thus we can continue our algorithm until we express μ_τ as a linear combination of standard monomials.

$\square$

Remark 4.1.8. *Since all standard monomials are equal to zero if $m \leq n$, we see that for such m and n, all the spaces P_n^m are empty. Similarly, if $m = n+1$ then all standard monomials are powers of $\mu_{1,\ldots,n+1}$. Hence P_n^m is a one-point set.*

Example 4.1.9 ($n = 1, m = 4$). *We take $w = 2, \quad l = 1$. A standard tableau τ of degree k must look like*

$$\tau = \begin{bmatrix} a_1^1 & a_2^2 \\ a_2^1 & a_3^2 \\ a_3^1 & a_4^2 \end{bmatrix}$$

where a_i^j is a column vector that consists entirely of the integer i in the $j-$th column of τ. Let us denote by $|a_i^j|$ the height of a_i^j, meaning the number of times that the integer i is realized in the j-th column of the matrix τ. It follows from the definition of standard tableau that we have the following equalities:

$$|a_1^1| = |a_4^2| = k, \quad |a_2^1| + |a_2^2| = |a_3^1| + |a_3^2| = k$$
$$|a_1^1| + |a_2^1| + |a_3^1| = |a_2^2| + |a_3^2| + |a_4^2| = 2k.$$

This shows that $(R_1^4)_k$ is completely determined by standard tableau τ with $|a_2^1| = a$ that satisfies

$$0 \leq a \leq k.$$

Therefore we will have

$$dim((R_1^4)_k) = k + 1.$$

Moreover if we set the following standard tableau

$$t_0 = \mu \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad t_1 = \mu \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Then by rearranging the determinants in the factorization of t_0^i and t_1^{k-i} we will have :

$$t_0^i t_1^{k-i} = \mu \begin{bmatrix} a_1^1 & a_2^2 \\ a_2^1 & a_3^2 \\ a_3^1 & a_4^2 \end{bmatrix}$$

where $a = |a_2^1| = k - i$.

This implies that

$$R_1^4 \cong k[t_0, t_1]$$

Thus $P_1^4 \cong \mathbb{P}_1$

4.2 Points sets in projective space from a Geometric Invariant Theory point of view

In this section we will show that the spaces P_n^m from the previous section are quotients of some open subset of $\mathbb{P}_n^m$ by the action of $\mathrm{PGL}(n+1)$. This approach is quite palpable as the set $\mathbb{P}_n^m$ is directly linked to the set of semi-stable points of n-projective space modulo out by a reductive algebraic group.

Before getting into that let us revisit some useful notions.

Definition 4.2.1. *Let G be a reductive algebraic group that acts regularly on an algebraic variety X and $\mathcal{L}$ be a G-linearized ample invertible sheaf on X. A point $x \in X$ is semi-stable with respect to $\mathcal{L}$ if there exists a G-invariant section $s \in \Gamma(X, \mathcal{L}^{\otimes k})$ for some positive k such that*

$$s(x) \neq 0$$

Definition 4.2.2. *A semi-stable point $x \in X$ is said to be stable if there exists a G-invariant sections $s \in \Gamma(X, \mathcal{L}^{\otimes k})$ with $s(x) \neq 0$, such that G acts with closed orbits in the set*

$$\{y \in X \quad | s(y) \neq 0\}$$

and the stabilizer group

$$G_x = \{g \in G \quad | g.x = x\}$$

is finite.

Definition 4.2.3. *We denote respectively by $X^{ss}(\mathcal{L})$ and $X^{s}(\mathcal{L})$ the subsets of semi-stable and stable points of X.*

Remark 4.2.4. *Both of the above subsets are open G-invariant subsets of X.*

Definition 4.2.5. *A categorical quotient X/G is an algebraic variety together with a surjective morphism $\pi : X \to X/G$ which is G-equivariant, where G acts identically on X/G, and is universal with respect to this property.*

Definition 4.2.6. *A geometric quotient is a categorical quotient the fibres of which are the orbits of G in X*

Theorem 4.2.7. *Assume that X is proper. Then the categorical quotient $X^{ss}(\mathcal{L})/G$ exists and there is an isomorphism*

$$X^{ss}(\mathcal{L})/G \cong \mathit{Proj}(\bigoplus_{k=0}^{\infty} \Gamma(X, \mathcal{L}^{\otimes k})^G).$$

Moreover, the open subset $X^{s}(\mathcal{L})/G$ of $X^{ss}(\mathcal{L})/G$ is a geometric quotient of $X^{s}(\mathcal{L})$.

Proof. We first recall that

$$X \cong \mathrm{Proj}(\bigoplus_{k=0}^{\infty} \Gamma(X, \mathcal{L}^{\otimes k}))$$

because X is proper and $\mathcal{L}$ is ample. Let

$$A_X = \bigoplus_{k=0}^{\infty} \Gamma(X, \mathcal{L}^{\otimes k}), \quad C_X = \mathrm{Spec} \quad A_X.$$

The group G acts on A_X and on C_X and therefore according to ([8], Theorem 1.1) the space

$$(\mathrm{Spec} A_X)^G \cong C_X/G.$$

Let $m \in C_X/G$ be the point defined by the maximal ideal $\bigoplus_{k>0}^{\infty} \Gamma(X, \mathcal{L}^{\otimes k})$ of $(A_X)^G$. Then its pre-image in C_X is the set of all points which define non-semi-stable points in X. Thus the projection $C_X \to C_X/G$ induces a morphism

$$X^{ss}(\mathcal{L}) \to \mathrm{Proj}((A_X)^G).$$

It will be easy to check that it is indeed a categorical quotient of $X^{ss}(\mathcal{L})$ by G.

□

We denote by

$$\Phi : (\mathbb{P}_n^m)^{ss} \to P_n^m$$

the canonical projection of the categorial quotient. We set

$$\mathcal{D} = (\mathbb{P}_n^m)^{ss} - (\mathbb{P}_n^m)^s, \quad \mathcal{D}' = \Phi(\mathcal{D}).$$

The projection

$$\Phi : (\mathbb{P}_n^m)^s \to P_n^m - \mathcal{D}'$$

is the geometric quotient.

Corollary 4.2.8.

$$P_n^m \cong (\mathbb{P}_n^m)^{ss}(\mathcal{L})/G.$$

4.2.9 Semi-stability criterion

The Hilbert-Mumford numerical criterion allows us to describe the set of semi-stable point sets.

Definition 4.2.10. *A one-parameter subgroup of G, $\lambda : k^* \to G$ is a group homomorphism.*

Proposition 4.2.11. *Let x be a closed point of X and $\lambda : k^* \to G$ the one-parameter subgroup, we define the map:*

$$\begin{aligned} \mu_x : k^* &\to X \\ \alpha &\mapsto \lambda(\alpha).x. \end{aligned}$$

Then μ_x extends uniquely to a morphism:

$$\mu'_x : \mathbb{A}_1 \to X$$

and defines the point

$$\mu'_x(0) := lim_{\alpha \to o} \lambda(\alpha).x$$

Remark 4.2.12. *$\mu'_x(0)$ is fixed under the action of the image of k^* by λ and the restriction of $\mathcal{L}$ to it defines a k^*-linearized invertible sheaf on it. It is therefore completely determined by a character*

$$\chi(\lambda, x) \colon k^* \to k^*$$
$$\alpha \mapsto \alpha^{r(\lambda,x)} \quad \textit{where } r(\lambda, x) \textit{ is an integer.}$$

Proposition 4.2.13. *Let X a proper algebraic variety. Then we have*

$$x \in X^{ss}(\mathcal{L}) \quad \textit{iff} \quad r(\lambda, x) \le 0 \quad \textit{for all } \lambda : k^* \to G$$

$$x \in X^{s}(\mathcal{L}) \quad \textit{iff} \quad r(\lambda, x) < 0 \quad \textit{for all } \lambda : k^* \to G$$

Theorem 4.2.14. *Let $x = (x^1, ..., x^m) \in \mathbb{P}_n^m$. Then $x \in (\mathbb{P}_n^m)^{ss}(\mathcal{L})$ if and only if for any proper subset $\{i_1, ..., i_k\}$ of $\{1, ..., m\}$*

$$dim\, span(x^{i_1}, ..., x^{i_k}) + 1 \ge \frac{k(n+1)}{m}.$$

Moreover, x is stable if and only if the strict inequalities hold.

Proof. Let $\lambda : k^* \to G$ be a one parameter subgroup of G. We then choose homogeneous coordinates in $\mathbb{P}_n$ in such a way that the action of $\lambda(k^*)$ is diagonalized. this means that

$$\lambda(\alpha)(t_0, ..., t_1) = (\alpha^{r_0} t_0, ..., \alpha^{r_n} t_n)$$

for some integers r_i. We may also assume that:

$$r_0 \ge r_1 \ge ... \ge r_n, \quad \sum_{i=1}^{n} r_i = 0 \quad r_0 > 0$$

Let

$$X = \begin{bmatrix} t_0^{(1)} & \dots & t_0^{(m)} \\ \dots & \dots & \dots \\ t_n^{(1)} & \dots & t_n^{(m)} \end{bmatrix}$$

be the matrix whose columns are the homogeneous coordinates of the points $x^1, ..., x^m$. For every ordered m-tuple $I = (i_1, ..., i_m), \quad i_j \in \{0, ..., n\}$ we denote by X_I the monomial $t_{i_1}^{(1)} \dots t_{i_m}^{(m)}$.

The monomials X_I are coordinates of points of $\mathbb{P}_n^m$ in the Segre embedding given by the sheaf $\otimes_{i=1}^m \mathcal{O}_{\mathbb{P}_n}(1)$. For every ordered l-tuple $L = (L(1), ..., L(l))$ the products

$$X_L = \prod_{i=1}^{l} X_{L(i)}$$

are the coordinates of points of $\mathbb{P}_n^m$ in the Segre-Veronese embedding given by the sheaf $\otimes_{i=1}^m \mathcal{O}_{\mathbb{P}_n}(1)$. A one parameter subgroup $\lambda : k^* \to G$ acts on these coordinates via :

$$\lambda(\alpha)(X_L) = \alpha^{N(L)} X_L,$$

where

$$N(L) = \sum_{i=0}^{n} n_i r_i,$$

and where n_i is the number of times that i appears in $L(1), ..., L(l)$. By proposition (4.2.13) we have to look for the points $(x^1, ..., x^m)$ such that

$$\min_L \{N(L) : \quad X_L \neq 0\} \leq 0 \quad (\text{resp.}<0). \tag{4.0}$$

Permuting the points $x^1, ..., x^m$ we may assume that the matrix X of their coordinates has the following form:

$$\begin{bmatrix} * & \dots & * & * & \dots & * & \dots & * & \dots \\ 0 & \dots & 0 & * & \dots & * & \dots & * & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & * & \dots \end{bmatrix},$$
$$\qquad \underbrace{\quad}_{K_0} \qquad\qquad \underbrace{\quad}_{K_1} \qquad\qquad\qquad \underbrace{\quad}_{K_n}$$

where the bottom most entry in each column that is indicated by a $''*''$ is non-zero. Obviously the minimum $N(L)$ occurs when

$$L(i) = \{\underbrace{0, ..., 0}_{K_0}, \underbrace{1, ..., 1}_{K_1}, \dots, \underbrace{n, ..., n}_{K_n}\}$$

ordered sequence for all $i = 1, ..., l$, and it is equal to $l \sum_{i=0}^{n}$. Now note that every vector $r = (r_0, ..., r_n)$ satisfying (4.2.9) can be written as a linear combination of the vectors

$$r_d = (r_{d,0}, ..., r_{d,n}) = (\underbrace{n-d, ..., n-d}_{d+1}, \underbrace{-(d+1), ..., -(d+1)}_{n-d})$$

$d = 0, ..., n-1$, with positive coefficients. This shows that it is enough to check (4.2.9) for each λ defined by

$r = r_d$ for some $d < n$. We find that :

$$\begin{aligned}
N(L)/l &= \sum_{i=0}^{n} r_{d,i} K_i = (n-d) \sum_{i=0}^{d} K_i - (d+1) \sum_{i=d+1}^{n} K_i \\
&= (n-d) \sum_{i=0}^{d} K_i - (d+1)(m - \sum_{i=0}^{d} K_i) \\
&= (n+1) \sum_{i=0}^{d} K_i - m(d+1).
\end{aligned}$$

Therefore (4.2.9) holds if and only if

$$\sum_{i=0}^{d} K_i \leq \frac{m(d+1)}{(n+1)} \quad \text{for } d = 0, ..., n-1.$$

Having said that, we observe that the maximal number of points among the x^i's which span a projective subspace of dimension at most d is equal to $\sum_{i=0}^{d} K_i$. Thus (4.2.9) holds if and only if the condition of the theorem is satisfied.

$\square$

Corollary 4.2.15.

$$(\mathbb{P}_n^m)^{ss} = (\mathbb{P}_n^m)^s \Leftrightarrow m \quad \textit{and} \quad n+1 \quad \textit{are coprime}$$

In particular, P_n^m is nonsingular in this case.

Corollary 4.2.16. *Let $m \leq n$, then*

$$(\mathbb{P}_n^m)^s = \emptyset.$$

Proof.

$$\dim < x_1, ...x_{m-1} > +1 \leq m - 1 < \frac{(m-1)(m+1)}{m} \leq \frac{(m-1)(n+1)}{m} < n+1.$$

$\square$

Remark 4.2.17. *Similarly, we see that if $m = n+1$*

$$(\mathbb{P}_n^m)^{ss} = \emptyset.$$

Definition 4.2.18. *Let $d = (d_1, \ldots, d_j)$ be a partition of $n+1$. A partition d with respect to m is admissible if $k_i = \frac{d_i m}{(n+1)}$ is an integer for each $i = 1, ..., j$.*

Remark 4.2.19. *The partition $d = (n+1)$ is admissible and is called trivial.*

Definition 4.2.20. *Let d be an admissible partition of $n+1$ with respect to m and $L_1, ..., L_j$ be disjoint subspaces of $\mathbb{P}_n$ of dimension $d_1 - 1, ..., d_j - 1$ respectively. We consider the natural map :*

$$\Theta : (L_1^{K_1})^s \times ... \times (L_j^{K_j})^s \rightarrow \mathbb{P}_n^m$$

We set $U_d(L_1, ..., L_j) = Im(\Theta)$ in $\mathbb{P}_n^m$ and U_d the union of the subsets of $U_d(L_1, ..., L_j)$ for all possible choices of disjoint subspaces $L_1, ..., L_j$ and their images under permutation of factors.

Proposition 4.2.21. $U_{(n+1)} = (\mathbb{P}_n^m)^s, \quad \mathcal{D} = \cup_{d \neq (n+1)} U_d$ *and*

$$dim\Phi(U_d) = \sum_{i=1}^{j} (d_i - 1)(K_i - d_i - 1).$$

Proof. The proof goes as follow:

$$\begin{aligned}\dim\Phi(U_d) &= \sum_{i=1}^{j} \dim((L_i^{K_i})^s/\mathrm{PGL}(d_i)) \\ &= \sum_{i=1}^{j} (d_i - 1)(K_i - d_i - 1).\end{aligned}$$

□

Theorem 4.2.22. *P_n^m is a normal rational variety of dimension $n(m-n-2)$ if $m \geq n+2$ and dimension zero if $m = n+1$. Its singular locus is contained in $\mathcal{D}'$.*

Proof. The ring $\bigoplus_{k=0}^{\infty} \Gamma(X, \mathcal{L}^{\otimes k})$ being normal follow from the fact that the Segre and Veronese varieties are projectively normal. Hence, this implies that R_n^m is normal and therefore

$$P_n^m = \mathrm{Proj}(R_n^m)$$

is normal. We know that, if $m \geq n+2$,

$$\dim P_n^m = \dim\Phi((\mathbb{P}_n^m)^{ss}) = \dim\Phi(U_{n+1}) = n(m-n-2),$$

and P_n^m is a point if $m = n+1$. Moreover, the singularities of P_n^m and its rationality will follow from the much stronger result that $P_n^m - \mathcal{D}'$ is covered by open subsets each of which is isomorphic to an open $U \subset \mathbb{A}_{n(m-n-2)}$. In order to see that we remark that a point set $(x^1, ..., x^m) \in (\mathbb{P}_n^m)^s$cannot be separated by two disjoint linear subspaces. Indeed, there do not exist disjoint linear projective subspaces L' and L'' of $\mathbb{P}_n$ such that every x_i lies in either L' or L''. In fact, if this happens, after a permutation of the points, the coordinate matrix of x look like

$$\begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}.$$

This will imply that $\dim G_x > 0$, and hence x is not stable. Thus we can choose $n+1$ point x^i which are not in one hyperplane, say $x^1, ..., x^{n+1}$. Without loss of generality we may assume that the coordinate matrix of $x^1, ..., x^{n+1}$ is equal to the identity matrix I_{n+1}.

Moreover, for each k between 2 and $m-n$ we let S_k^i be the set of integers i such that the points $x^1, ..., \hat{x}^i, ..., x^{n+1}, x^{k+n}$ span $\mathbb{P}_n$. In other words,

$$S'_k = \{i \in \{0, ..., n\} \quad | x_i^{n+k} \neq 0\},$$

Where $x^{n+k} = (x_0^{n+k}, ..., x_n^{n+k})$. However, as the set $\{0, ..., n\}$ cannot be separated into two disjoint subsets I' and I'' such that every S'_k is contained in I' or I''. Thus there exits a suitable set of subsets $S_k \subset S'_k$ such that

(i) $\cup S_k = \{0, ..., n\}$;

(ii) $S_i \cup (S_{i-1} \cup ... \cup S_2)$ consists of one integer, for $3 \leq i \leq m - n$.

Let U be the open subset of $\mathbb{P}_n^m$ defined by

$$\begin{aligned}
\mathbb{P}_n &= \mathrm{Span}(x^1, ..., \hat{x}^i, ..., x^{n+1}, x^{n+k}) \qquad \text{for all } i \in S_i, \quad k = 2, ..., m - n, \\
\mathbb{P}_n &= \mathrm{Span}(x^1, ..., x^{n+1}).
\end{aligned}$$

There exists a unique $g \in G$ such that for every $x \in U$ the coordinate matrix of $g.x$ has the following form:

$$[I_{n+1} \quad X],$$

where for each K, the K-th column of X has 1 as the entries in he rows whose indices are from S_K.

For example, if $\{S_2, ..., S^{m-n}\} = \{\{0, ..., n\}, \{n\}, ..., \{n\}\}$, the coordinate matrix of $g.x$ must look like

$$\begin{pmatrix}
1 & 0 & 0 & \dots & 0 & 1 & * & \dots & * \\
0 & 1 & 0 & \dots & 0 & 1 & * & \dots & * \\
\dots & \dots & \dots & \dots & \dots & \dots & * & \dots & \dots \\
0 & 0 & \dots & 0 & 1 & 1 & 1 & \dots & 1
\end{pmatrix}$$

To see this, we observe that after reducing the points $x^1, ..., x^{n+1}$ to the points $(1, 0, ..., 0), ..., (0, ..., 0, 1)$ by a suitable $g \in G$, there are still non-trivial transformations left in G which fix $x^1, ..., x^{n+1}$. They are the homotheties

$$(t_0, ..., t_n) \to (\lambda_0 t_0, ..., \lambda_n t_n), \quad \lambda_0 ... \lambda_n = 1.$$

Thus we may use them to normalize the j-th coordinate of $x_{n+2}, j \in S_2$. Then we normalize the j-th coordinate of x^{n+3} for $j \in S_2 \cap S_3$ by a projective factor. Next one uses again the homotheties to normalize the remaining i-th coordinate of x^{n+3} for $i \in S_3$, and so on. Clearly this defines g uniquely and defines a G-equivariant isomorphism

$$G \times \mathbb{A}_{n(m-n-2)} \cong U,$$

where G acts on G by left multiplication and identically on the affine space. However, the affine space $\mathbb{A}_{n(m-n-2)}$ is the space of all non-normalized coordinates of the points from $g.x$. This shows that $(\mathbb{P}_n^m)^s$ is covered by the invariant open subsets $U \cap (\mathbb{P}_n^m)^s$ whose quotients are open in $\mathbb{A}_{n(m-n-2)}$. And this will prove the above statement.

Besides, it also shows that the projection

$$\Phi : (\mathbb{P}_n^m)^s \to P_n^m - \mathcal{D}'$$

is a principal fibre bundle of G over $P_n^m - \mathcal{D}'$ in the sense of ([**?**]), Definition 0.10.

$\square$

Example 4.2.23. *If $n = 1$ and m is odd then we have*

$$(\mathbb{P}_1^m)^s = (\mathbb{P}_1^m)^{ss},$$

moreover P_1^m is a nonsingular rational variety of dimension $m - 3$. Besides that, when $m = 5$, P_1^5 is isomorphic to a Del Pezzo surface of degree 5 which will be studied in chapter 5.

The above example amplifies the connection that point sets and and Del Pezzo surfaces have in common. The following section will provide us with concepts and notions that will enable us to see those connection in more details. However before we get into that it is important to recall the following definition :

Definition 4.2.24. *A point set $x \in \mathbb{P}_n^m$ is said to be general if any subset of $K \leq n + 1$ points spans a $K - 1$-dimensional linear projective space. The set of all general points will be denoted*

$$(\mathbb{P}_n^m)^{gen}$$

Remark 4.2.25. *Moreover Theorem (4.2.14) implies:*

$$(\mathbb{P}_n^m)^{gen} \subset (\mathbb{P}_n^m)^s$$

4.3 Blowing-ups of point sets

We will see that there is a natural variety associated to each projective point set x in $\mathbb{P}_n^m$. Among other things we will introduce an important notion namely,the blowing-up variety $V(x)$ which is defined by a general point set x and by other point sets that will be called infinitely near point sets. In this section we will provide a dictionary for some of the concepts mentionned above and lay out the blueprint for the most important notion of this chapter: the Generalized Del Pezzo varieties (GDP).

Definition 4.3.1. *Let Z be a smooth algebraic variety of dimension $n > 1$, $z \in Z$ be a closed point. A variety $Z' = Z(z)$ is the the blow-up of Z at z if it satisfies the following properties:*

(i) there exists a proper birational morphism $\pi : Z(z) \to Z$ that is an isomorphism over $Z - \{z\}$;

(ii) there is a natural isomorphism
$$\pi^{-1}(z) \cong \mathbb{P}(T(Z)_z) \cong \mathbb{P}_{n-1},$$
where $T(Z)_z$ is tangent space of Z at z.

Remark 4.3.2. *The blow-up of Z at z is defined uniquely up to isomorphism.*

Definition 4.3.3. *Let Z be a smooth algebraic variety of dimension $n > 1$, and $zin \in Z$. An infinitely near point of order 1 to Z is a closed point $z' \in Z(z)$ lying in $E = \pi^{-1}(z)$. It will be denoted by*

$$z' \to z$$

Remark 4.3.4. *An infinitely near point of order k to z is defined by induction as an infinitely near point of order 1 to an infinitely near point of order $k-1$ to z. It is denoted*

$$z^{(k)} \to \ldots \to z^{(1)} \to z.$$

Let Z^m denote the Cartesian product of m copies of Z. For every subset L of $\{1, \ldots, m\}$ with $|L| \geq 2$ and with $0 \leq k \leq m$, we denote

$$\Delta(m)_L = \{(z^1, \ldots, z^m) \in Z^m : \quad : z_i = z_j \quad \text{for all } i, j \in L\},$$

$$\Delta(m)_k = \cup_{|L|=k} \Delta(m)_L, \quad \Delta(m) = \Delta(m)_2,$$

$$U(m)_k = Z^m - \Delta(m)_k. \quad U(m) = U(m)_2,$$

$$\pi_i : Z^m \to Z, \quad \text{i-th projection,}$$

$$\pi^m = \pi_1 \times \ldots \times \pi_{m-1} : Z^m \to Z^{m-1}$$

Theorem 4.3.5. *For every $m \geq 1$ there exists a proper birational morphism of smooth varieties*

$$b_m : \hat{Z}^m \to Z^m$$

satisfying the following properties:

(a) the restriction of b_m over $U(m)$ is an isomorphism;

(b) b_m is a composition of blowing-ups with smooth centers,

(c) If $m \geq 2$ there exists a smooth proper morphism

$$\hat{\pi}^m : \hat{Z}^m \to \hat{Z}^{m-1}$$

such that the fibre $(\hat{\pi}^m)^{-1}(z)$ over $z \in \hat{Z}^{m-1}$ is isomorphic to the blowing-up of z considered as a closed point on the fibre $(\hat{\pi}^{m-1})^{-1}(\hat{\pi}^{m-1})(z))$;

(d) the diagram :

$$\begin{array}{ccc} \hat{Z}^m & \xrightarrow{b_m} & Z^m \\ \downarrow{\scriptstyle \hat{\pi}^m} & & \downarrow{\scriptstyle \pi^m} \\ \hat{Z}^{m-1} & \xrightarrow{b_{m-1}} & Z^{m-1} \end{array}$$

commutes;

(e) *If* $m = 1$

$$\hat{Z}^1 = Z^1 = Z,$$
$$(\hat{\pi}^2)^{-1}(z) = Z(z) = \quad \textit{blowing-up of} \quad z \in Z$$

Proof. Let $\hat{Z}^0$ be a single point, $\hat{Z}^1 = Z$, $\hat{\pi}^1 : \hat{Z}^1 \to \hat{Z}^0$. Then for each $i > 1$ define inductively a $Y = \hat{Z}^{j-1}$-variety $\hat{\pi}^j : \hat{Z}^j \to \hat{Z}^{j-1}$ as follows. By assumption, $\hat{Z}^{j-1}$ is a $V = \hat{Z}^{j-2}$-variety. Define $\hat{Z}^j$ as the blowing-up of the diagonal of $Y \times_V Y$, and the morphism

$$\hat{\pi}^j : \hat{Z}^j \to \hat{Z}^{j-1}$$

as the composition of the blowing-up morphism with the projection of the fibred product to he first factor. Define the projections $b_i : \hat{Z}^j \to Z^i$ by induction as follows. Let b_1 be the identity. Assume that $b_{i-1} : \hat{Z}^{j-1} \to Z^{i-1}$ is defined. The composition of the two projections

$$Y \times_V Y \to Y = \hat{Z}^{j-1}$$

with b_{i-1} define two projections to Z^{i-1}, hence $2i-2$ projections p_K and q_K to Z, $K = 1, ..., i-1$. Since $p_K = q_K$ for $K = 1, ..., i-2$, we obtain i projections $p_1, ..., p_{i-1}, q_{i-1}$ to Z. Let b_1 be the composition of the blowing-up morphism $\hat{Z}^j \to Y \times_V Y$ with the product $Y \times_V Y \to Z^i$ of the these projections. Since we only blow up smooth projective varieties along smooth centers, all the varieties $\hat{Z}^j$ are smooth and the morphisms $\hat{\pi}_i$ are proper and birational. Proporties (a),(d),(e) follows immediately from the construction. Meanwhile to (c) we use the definition of the tangent space of a variety Z at a point $z \in Z$ as the fibre of the inverse transform of the normal sheaf of the diagonal of $Z \times Z$ under the diagonal map $Z \to Z \times Z$. On the other hand to see (b) we use induction on m. By construction $b_2 : \hat{Z}^2 \to Z^2$ is the blowing-up of Δ_{12}. Assume $b_{m-1} : \hat{Z}^{m-1} \to Z^{m-1}$ is the composition of blowing-ups with smooth centers.

The morphism

$$\varphi_0 = \hat{\pi}^{m-1} \times 1 : X_0 = \hat{Z}^{m-1} \times Z \to Z^m = Z^{m-1} \times Z$$

is a composition of blowing-ups with smooth centers. Thus, one easily checks that the morphism $b_m : \hat{Z}^m \to Z^m$ is equal to the composition:

$$\hat{Z}^m = X_m \xrightarrow{\varphi_m} X_{m-1} \to \cdots \to X_1 \xrightarrow{\varphi_1} X_0 \xrightarrow{\varphi_0} Z^m,$$

where $\varphi_1 : X_1 \to X_0$ is the blowing-up of $\varphi_0^{m-1}(\Delta_{1m}(m))$, $\varphi_2 : X_2 \to X_1$ is the blowing-up $(\varphi_0 \circ \varphi_1)^{-1}(\Delta_{2m}(m))$, and so on. It is easy to see that

$$X_{m-1} \cong \hat{Z}^{m-1} \times_{\hat{Z}^{m-2}} \hat{Z}^{m-1},$$

and φ_m is the blowing-up to the diagonal isomorphic to $(\varphi_0 \circ \ldots \circ \varphi_{m-1})^{-1}((\Delta)_{m-1m}(m))$. □

Remark 4.3.6. *It is convenient to see every closed point of $\hat{Z}^m$ as an m-tuple $\hat{z} = (\hat{z}^1, \ldots, \hat{z}^m)$, where each point $\hat{z}^i$ is either a point of Z or an infinitely near point to some $\hat{z}^j$ with $j < i$. We will drop the hat over a point from Z. In this notation the morphism $b_m : \hat{Z}^m \to Z^m$ sends $\hat{z} = (\hat{z}^1, \ldots, \hat{z}^m)$ to $z = (z^1, \ldots, z^m)$, where $z^i \in Z$ and $\hat{z}^i$ is either equal to z^i or is infinitely near some to some $z^j, j < i$. Moreover the projection $\hat{\pi}^m$ is the map*

$$(\hat{z}^1, \ldots, \hat{z}^m) \to (\hat{z}^1, \ldots, \hat{z}^{m-1})$$

Definition 4.3.7. *The blowing-up variety $Z(\hat{z})$ of $\hat{z} = (\hat{z}^1, \ldots, \hat{z}^m) \in \hat{Z}^m$ is defined by:*

$$Z(\hat{z}) = (\hat{\pi}^{m+1})^{-1}(\hat{z}) \subset \hat{Z}^{m+1}.$$

Remark 4.3.8. *The blowing-up variety $Z(\hat{z})$ comes with a natural birational morphism*

$$\sigma(\hat{z}) : Z(\hat{z}) \to Z.$$

Moreover, it is the composition of the morphisms

$$Z((\hat{z}^1, \ldots, \hat{z}^m)) \xrightarrow{\sigma(\hat{z})_m} Z((\hat{z}^1, \ldots, \hat{z}^{m-1})) \xrightarrow{\sigma(\hat{z})_{m-1}} \ldots \xrightarrow{\sigma(\hat{z})_2} Z(\hat{z}^1) \xrightarrow{\sigma(\hat{z})_1} Z$$

Where

$$\sigma(\hat{z})_i = Z((\hat{z}^1, \ldots, \hat{z}^i)) \to Z((\hat{z}^1, \ldots, \hat{z}^{i-1}))$$

is the blowing up of the point $(\hat{z}^1, \ldots, \hat{z}^i) \in Z((\hat{z}^1, \ldots, \hat{z}^{i-1})) \subset \hat{Z}^i$.

Coming back to our situation where $Z = \mathbb{P}_n$, we define the space

$$\hat{Z}^m = \hat{\mathbb{P}}_n^m,$$

The birational morphism

$$b_m : \hat{\mathbb{P}}_n^m \to \mathbb{P}_n^m,$$

and the projection

$$\hat{\pi}^m : \hat{\mathbb{P}}_n^m \to \hat{\mathbb{P}}_n^{m-1},$$

satisfying the properties stated in the above theorem (4.3.5).

4.4 Stability in $\hat{Z}^m$

Let Z be a smooth algebraic variety and G an algebraic group that acts on Z. Hence G acts on Z^m for every positive integer m. In this section we will see a way to extend the action of G to $\hat{Z}^m$. However, we will see that, this can be done in an iterative manner by using the fact that each $\hat{Z}^{i+1}$ is obtained from $\hat{Z}^i$ by blowing up the diagonal of $\hat{Z}^i \times_{\hat{Z}^{i-1}} \hat{Z}^i$ and that the extension of the action of G in $\hat{Z}^i$ to the fibred product leaves the diagonal invariant.

Definition 4.4.1. *Let $i : X \to Y$ be a morphism of schemes such that all its fibers are smooth. The relative tangent bundle of i is the dual of the relative cotangent sheaf $\Omega_{X/Y} = i^\star(\mathcal{J}/\mathcal{J}^2)$, where $\mathcal{J}$ is the ideal sheaf of the fiber product $X \times_Y X$.*

Remark 4.4.2. *Let $\mu : G \times Z \to Z$ be an action of an algebraic group on Z, and for all $m \geq 0$ let us denote by $\mu^m : G \times Z^m \to Z^m$ be the induced diagonal action of G on Z^m. We want to extend this action to an action*

$$\hat{\mu}^m : G \times \hat{Z}^m \to \hat{Z}^m$$

such that

$$\hat{\mu}^{m-1} \circ (1 \times \hat{\pi}^m) = \hat{\pi}^m \circ \hat{\mu}^{m-1},$$
$$\text{and} \quad \mu^m \circ (1 \times b_m) = b_m \circ \hat{\mu}^m.$$

To do so, let us denote by T_m the relative tangent bundle of the morphism

$$\hat{\pi}^m : \hat{Z}^m \to \hat{Z}^{m-1}$$
$$(\hat{z}^1, ..., \hat{z}^m) \mapsto (\hat{z}^1, ..., \hat{z}^{m-1}).$$

If $\hat{z}^m$ is infinitely near of order 1 to some $\hat{z}^i$, then it belongs to some fibre of

$$\mathbb{P}(T_m|(\hat{\pi}^m)^{-1}((\hat{z}^1, ..., \hat{z}^{m-1}))) \cong \mathbb{P}(T(Z((\hat{z}^1, ..., \hat{z}^{m-1}))))$$

and it suffices to set

$$\hat{\mu}^m(g, ((\hat{z}^1, ..., \hat{z}^m))) = (\hat{\mu}^{m-1}(g, ((\hat{z}^1, ..., \hat{z}^{m-1}))), dg(\hat{z}^m))$$

where dg is the differential of the map

$$\hat{\mu}^m(g) : (\hat{\pi}^m)^{-1}(((\hat{z}^1, ..., \hat{z}^{m-1}))) \to (\hat{\pi}^m)^{-1}(\hat{\mu}^{m-1}(g, ((\hat{z}^1, ..., \hat{z}^{m-1})))).$$

Moreover if $\hat{z}^m = z^m \in Z$ then we have :

$$\hat{\mu}^m(g, (z^1, ..., z^m)) = (\hat{\mu}^{m-1}(g, (z^1, ..., z^{m-1})), g(z^m)).$$

Remark 4.4.3. *Finding the stable points in $\hat{Z}^m$ requires to study the functorial behavior of stability under G-equivariant maps. The followings definitions and propositions are one step towards that problem.*

Definition 4.4.4. *The scheme X is quasicompact if it has a Zariski cover by finitely many open affine subscheme.*

Example 4.4.5. *Any affine scheme is a quasicompact.*

Definition 4.4.6. *Let $f : X \to Y$ be a morphism of schemes. We say that f is quasicompact morphism if the inverse image of any quasicompact Zariski open subset of Y is quasicompact.*

Definition 4.4.7. *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$ module. We say that $\mathcal{L}$ is f-ample if $f : X \to Y$ is quasicompact and if for every affine open $V \subset Y$ the restriction of $\mathcal{L}$ to the open subscheme $f^{-1}(V)$ of X is ample.*

Proposition 4.4.8. *let G be a reductive group acting on algebraic varieties X and Y, and $f : X \to Y$ be a G-equivariant morphism. Let $\mathcal{L}$ (resp. m) be a G-linearized invertible sheaf on Y (resp. on X). Assume that $\mathcal{L}$ is ample and m is f-ample. Then for sufficiently large n the sheaf $f^*(\mathcal{L}^{\otimes n}) \otimes m$ is ample, and*

$$f^{-1}(Y^s(\mathcal{L})) \subset X^s(f^*(\mathcal{L}^{\otimes n}) \otimes m).$$

Proof. See ([8], Proposition 2.18). □

We now state the result obtains by Z.Reichstein ([10]).

Proposition 4.4.9. *Following the notation from the previous proposition we have*

$$f^{-1}(Y^{ss}(\mathcal{L})) \subset X^{ss}(f^*(\mathcal{L}^{\otimes n}) \otimes m),$$

for sufficiently large n.

Proof. See ([10] □

Lemma 4.4.10. *Let $f : X \to Y$ be the blowing up of a G-invariant closed subscheme C of Y. We denote the exceptional divisor of f by E. We fix a very ample G-linearized invertible sheaf $\mathcal{L}$ on Y, and set*

$$\hat{\mathcal{L}}_K = f^*(\mathcal{L}^{\otimes K}) \otimes \mathcal{O}_X(-E).$$

Then $\hat{\mathcal{L}}_K$ is very ample G-linearized sheaf on X if K is sufficiently large. Moreover The action of G on Y extends naturally to an action on X.

Remark 4.4.11. *Let $p : Y^{ss} \to Y^{ss}/G$ be the quotien map and let*

$$\tilde{C} = p^{-1}(p(C \cap Y^{ss})).$$

For every subvariety Z of Y we denote by Z' its proper inverse transform under f, that is, the closure of $f^{-1}(Z - (C \cap Z))$ in X.

Proposition 4.4.12. *Assume X and C are smooth. then, for sufficiently large K, the open subset $X(\hat{\mathcal{L}}_K)^{ss}$ and $X(\hat{\mathcal{L}}_K)^s$ are independent of K and*

$$X(\hat{\mathcal{L}}_K)^{ss} = f^{-1}(Y(\mathcal{L})^{ss}) - \tilde{C}',$$

$$X(\hat{\mathcal{L}}_K)^s = X(\hat{\mathcal{L}}_K)^{ss} - (Y(\mathcal{L})^{ss} - Y(\mathcal{L})^s)'.$$

Remark 4.4.13. *In fact we want to apply these result to the case where $f : X \to Y$ is the map*

$$b_m : \hat{Z}^m \to Z^m.$$

By Theorem (4.3.5), the morphism b_m is a composition

$$Y_K = \hat{Z}^m \xrightarrow{f_k} Y_{k-1} \to \cdots \to Y_2 \xrightarrow{f_2} Y_1 \xrightarrow{f_1} Y_0 = Z^m$$

of blowing-ups with smooth centers.

The following algorithm help us to determine a sequence of G-linearized ample invertible sheaves $\mathcal{L}_i$ at each Y_i. It goes as follow:

(i) Each blowing-up above is G-equivariant, and its center is isomorphic to a certain proper inverse transform of some $\Delta(m)_{ij}$. We choose a G-linearized ample invertible sheaf $\mathcal{L}_0$ on $Y_0 = Z^m$.

(ii) Then from Lemma (4.4.10) define a similar sheaf $\mathcal{L}_1 = f_1^*(\mathcal{L}_0^{\otimes n}) \otimes \mathcal{O}_{Y_1}(-E_1)$, where n is sufficiently large and E_1 is the exceptional divisor of f_1.

(iii) Proceed this way until we obtain a sequence of G-linearized ample invertible sheaves $\mathcal{L}_i$ at each Y_i. Moreover each of the $\mathcal{L}_i$ defines the subset Y_i^{ss} (resp. Y_i^s) of semi-stable (resp. stable) points.

Corollary 4.4.14. *Let $\Phi_i : Y_i^{ss} \to Y_i^{ss}/G$ be the corresponding quotient projection. We set*

$$\mathcal{D}_i = Y_i^{ss} - Y_i^s,$$

$$\tilde{C}_i = \Phi_i^{-1}(\Phi_i(C_i \cap Y_i^{ss})),$$

where C_i is the center of the blowing-up f_{i+1}. Then we have

$$Y_{i+1}^{ss} = f_i^{-1}(Y_i^{ss}) - \tilde{C}_i',$$

$$Y_{i+1}^s = Y_{i+1}^{ss} - \mathcal{D}_i',$$

where the "prime " means proper inverse transform.

Proof. Applying Proposition (4.4.9) and (4.4.12) to each f_i. □

Remark 4.4.15. *From that observation we have*

$$Y_{i+1}^{ss} \subset f_i^{-1}(Y_i^{ss}),$$

$$f_i^{-1}(Y_i^s) \subset Y_{i+1}^{ss}.$$

Moreover, since

$$\tilde{C}_{i+1} \cap \tilde{C}_i' = \varnothing,$$

we have

$$Y^{ss}_{i+2} = f^{-1}_{i+1}(f^{-1}_i(Y^{ss}_i)) - (\tilde{C}'_{i+1} - \tilde{C}''_i),$$

where $\tilde{C}''_i$ is the proper inverse transform of $\tilde{C}_i$ under $f_i \circ f_{i+1}$.

Starting with $i = 0$ up to $i = K$, we use the above Remark to obtain the following Theorem.

Theorem 4.4.16. *Let $\tilde{C}'$ (resp. $\mathcal{D}'_0$) be the proper inverse transform of $\tilde{C} = \Phi_0^{-1}(\Phi_0(\partial(m)))$ (resp. of $\mathcal{D}_0$) under b_m. There exist a G-linearized ampple invertible sheaf $\hat{\mathcal{L}}_0$ on $\hat{Z}^m$ such that*

$$\hat{Z}^m(\hat{\mathcal{L}}_0)^{ss} = b_m^{-1}(Z^m(\mathcal{L}_0)^{ss}) - \tilde{C}',$$

$$\hat{Z}^m(\hat{\mathcal{L}}_0)^s = \hat{Z}^m(\hat{\mathcal{L}}_0)^{ss} - \mathcal{D}'_0.$$

Remark 4.4.17. *In particular we have*

$$\hat{Z}^m(\hat{\mathcal{L}}_0)^{ss} \subset b_m^{-1}(Z^m(\mathcal{L}_0)^{ss}),$$
$$b_m^{-1}(Z^m(\mathcal{L}_0)^s) \subset \hat{Z}^m(\hat{\mathcal{L}}_0)^s$$

Definition 4.4.18. *Let $Z = \mathbb{P}_n$and $G = PGL(n+1)$, we take for $\mathcal{L}_0$ our standard sheaf $\mathcal{L}$ and obtain a G-linearized ample invertible sheaf $\hat{\mathcal{L}}$. We can define the open subsets of $\hat{\mathbb{P}^m_n}$:*

$$(\hat{\mathbb{P}^m_n})^{ss}, (\hat{\mathbb{P}^m_n})^s,$$

and the quotient

$$\hat{P}^m_n = (\hat{\mathbb{P}^m_n})^{ss}/G,$$

the projection

$$\hat{\Phi} : (\hat{\mathbb{P}^m_n})^{ss} \to \hat{P}^m_n,$$

and the morphism

$$\bar{b}_m : \hat{P}^m_n \to P^m_n$$

such that the following diagram is commutative:

$$\begin{tikzcd}
(\hat{\mathbb{P}^m_n})^{ss} \arrow[r, "b_m"] \arrow[d, "\hat{\Phi}"'] & (\mathbb{P}^m_n)^{ss} \arrow[d, "\Phi"] \\
\hat{P}^m_n \arrow[r, "\bar{b}_m"] & P^m_n
\end{tikzcd}$$

Proposition 4.4.19. *$\hat{\Delta}(m)$ is a hypersurface in $\hat{Z}^m$. Its irreducible components are the hypersurfaces $\hat{\Delta}(m)_l$, with $|l| = 2$.*

Proof. See ([3]) page 56. $\square$

5

Del Pezzo surfaces and Generalized Del Pezzo varieties

5.1 Introduction to Del Pezzo surfaces

In this section we will introduce Del Pezzo surfaces and provide notions and concepts that will be useful for the final part of this essay. We follow closely the note of "Commutative algebra and algebraic geometry " by Olof Bergvall (here) [1].

Definition 5.1.1 (Rational Maps)**.** *Let X and Y be varieties. A rational map $f : X \dashrightarrow Y$ is a morphism*

$$f : U \to Y,$$

with U a non-empty open subset of X.

Remark 5.1.2. *We say that $g : U \to Y$ and $h : V \to Y$ with U, V open subsets of X are equivalent if*

$$g_{U\cap V} = h_{U\cap V}.$$

Definition 5.1.3. *A rational map $f : U \subset X \dashrightarrow Y$ is dominant if $f(U)$ contains a dense open subset of Y.*

Definition 5.1.4. *A birational map is a rational map $f : X \dashrightarrow Y$ with rational inverse, meaning that f is dominant and there exists a rational map*

$$g : Y \dashrightarrow X$$

such that

$$g \circ f = id_X \quad \textit{and} \quad f \circ g = id_Y.$$

Remark 5.1.5. *We say that two varieties X and Y are birationally equivalent if there is a birational map*

$$f : X \dashrightarrow Y.$$

Definition 5.1.6. *A variety X which is birationally equivalent to $\mathbb{P}_n$ for some n is called rational.*

5.1.7 Blow-ups

Blow-ups are a class of especially simple birational maps. However, in the following we will only need the special case of blowing up a surface in a finite set of points.

Definition 5.1.8. *Let us consider a surface S and $\varepsilon : \hat{S} \to S$ its blow-up at a regular point P (See 4.2.9).*

Remark 5.1.9. *Let C be an irreducible curve in S which passes through P with multiplicity m, then the closure $\hat{C}$ of $\varepsilon^{-1}(S - \{P\})$ is an irreducible curve on $\hat{S}$ called the strict transform of C. The curve $\varepsilon^* C$ is called the total transform of C.*

Lemma 5.1.10. *Let S be a surface and let C be an irreducible curve on S passing through the point $P \in S$ with multiplicity m. Then*

$$\varepsilon^* C = \hat{C} + mE.$$

Proof. This is a local calculation. In fact, where $E = \varepsilon^{-1}(P)$ is the class of exceptional curve of the blow-up. It is clear that $\varepsilon^* C = \hat{C} + kE$ for some non-negative integer k so what we want to show is that $k = m$. As in the construction of the blow-up, we choose local coordinates x and y so that $y = 0$ is not tangent to C at P. Then

$$f = f_m(x, y) + f_{m+1}(x, y) + \dots$$

where $f_k(x, y)$ is the homogeneous part of degree k of f. Since m is the multiplicity of C at P, we have that $f_m(x, y)$ is not identically zero. We construct $\hat{U}$ as above and choose the coordinates x and $t = y/x$ around $(P, [1 : 0])$. Then

$$\varepsilon^* f = f(x, tx) = x^m(f_m(1, t) + x f_{m+1}(1, t) + \dots),$$

from which we conclude that $k = m$.

□

Proposition 5.1.11. *Let S be a surface, let $\varepsilon : \hat{S} \to S$ be the blow-up of a regular point $P \in S$ and let $E \subset \hat{S}$ be the exceptional curve. Then*

(i) there is an isomorphism $Cl(S) \bigoplus \mathbb{Z} \to Cl(\hat{S})$ defined by

$$(D, n) \mapsto \varepsilon^* D + nE.$$

(ii) Let D_1 and D_2 be divisors on S. Then

$$i(\varepsilon^* D_1, \varepsilon^* D_2) = i(D_1, D_2),$$

$$i(\varepsilon^* D_1, E) = 0,$$
$$i(E, E) = -1$$

(iii) the canonical class of $\hat{S}$ is

$$K_{\hat{S}} = \varepsilon^* K_S + E.$$

Lemma 5.1.12. *Let S be a surface, let C be an irreducible curve in S, let $P \in C$ be a point of multiplicity m and let $\varepsilon : \hat{S} \to S$ be the blow-up of S at P. Then the arithmetic genus of $\hat{C}$ is*

$$P_a(\hat{C}) = P_a(C) - \frac{1}{2}m(m-1)$$

Proof. Let E be the exceptional divisor of the blow-up. By lemma (5.1.10) we have

$$\hat{C} = \varepsilon^* C - mE.$$

By the genus formula we have

$$\begin{aligned} P_a(\hat{C}) &= 1 + \frac{1}{2} i(\hat{C}, \hat{C} + K_{\hat{S}}) \\ &= 1 + \frac{1}{2} i(\varepsilon^* C - mE, \varepsilon^* C - mE + K_{\hat{S}}). \end{aligned}$$

By Proposition (5.1.11) we have

$$K_{\hat{S}} = \varepsilon^* K_S + E$$

so,

$$P_a(\hat{C}) = 1 + \frac{1}{2} i(\varepsilon^* C - mE, \varepsilon^*(C + K_S) - (m-1)E)$$

By Proposition (5.1.11) we have

$$i(\varepsilon^* C, \varepsilon^*(C + K_S)) = i(C, C + K_S),$$
$$i(\varepsilon^* C, E) = i(\varepsilon^*(C + K_S), E) = 0$$
$$i(E, E) = -1.$$

We now see that

$$P_a(\hat{C}) = 1 + \frac{1}{2} i(C, C + K_S) - \frac{1}{2}m(m-1) = P_a(C) - \frac{1}{2}m(m-1).$$

□

Remark 5.1.13. *If P is a multiple point of C, then*

$$P_a(\hat{C}) < P_a(C)$$

Theorem 5.1.14 (Resolution of singularities of curves in surfaces.). *Let S be a surface and let C be an irreducible curve in S. Then there is a finite sequence of blow-ups*

$$\hat{S} = S_n \to S_{n-1} \to \ldots \to S_0 = S$$

such that if $f : \hat{S} \to S$ is their composition, then the strict transform $\hat{C}$ is smooth.

Proof. If C is nonsingular we have nothing to prove. Otherwise we let $P \in C$ be a point of multiplicity $m \geq 2$ and let

$$\varepsilon_1 : S_1 \to S$$

be the blow-up of S in P. If C_1 is the strict transform of C, then $P_a(C_1) < P_a(C)$. If C_1 is smooth we stop, otherwise we repeat the above and thus obtain a sequence $S_1, S_2, \ldots$ of surfaces containing curves $C_1, C_2, \ldots$ such that

$$P_a(C_{i+1}) < P_a(C_i).$$

We need to show that the procedure stops. To see this, It is enough to show that the arithmetic genus are bounded from below. But this is clear: since the curve C_i are connected we have

$$\dim \mathrm{H}^0(C_i, \mathcal{O}_{C_i}) = 1$$

so

$$P_a(C_i) = (-1)(\mathrm{H}^0(C_i, \mathcal{O}_{C_i}) - \mathrm{H}^1(C_i, \mathcal{O}_{C_i})) - 1 = \dim \mathrm{H}^1(C_i, \mathcal{O}_{C_i}) \geq 0.$$

$\square$

Definition 5.1.15 (Minimal surfaces.). *Let S be a surface and let $B(S)$ denote the set of isomorphism classes of surfaces which are birationally equivalent to S. Recall that a surface S' dominates S if there is a birational morphism*

$$S' \to S.$$

Remark 5.1.16. *The set $B(S)$ is partially ordered by domination.*

Definition 5.1.17. *A surface S is minimal if its isomorphism class in $B(S)$ is minimal.*

Remark 5.1.18. *Furthermore, S is minimal if and only if every birational morphism $f : S \to Y$ to a surface Y is an isomorphism.*

Proposition 5.1.19. *Let S be surface. The group $\mathcal{C}\ell(S)$ contains a subgroup $\mathcal{C}\ell^0(S)$ consisting of divisor classes of degree 0. the quotient group*

$$NS(S) = Cl(S)/Cl^0(S)$$

is called the Néron-Severi group of S. The Néron-Severi group is a finitely generated abelian group an its rank $\rho(S)$ is called the Picard number of S. If $f : S \to Y$ is a birational morphism, then

$$\rho(S) \geq \rho(Y)$$

with equality if and only if f is an isomorphism.

Proposition 5.1.20. *Every surface dominates a minimal surface.*

Proof. Let S be a surface. If S is not minimal, there is $f_1 : S \to S_1$ such which is not an isomorphism. Then $\rho(S) > \rho(S_1)$. If S_1 is not minimal we repeat the above. Continuing in this way we get a sequence of birational morphisms

$$f_i : S_{i-1} \to S_i$$

which are not isomorphisms and the Picard ranks are thus strictly decreasing. Since the Picard rank is non-negative, this must terminate eventually. $\square$

We recall that S is a surface, P is a point of S and $\varepsilon : \hat{S} \to S$ is the blow-up of S in P, then the exceptional curve E is siomorphic to $\mathbb{P}^1$ and satisfies

$$i(E, E) = -1.$$

However the Following theorem of castelnuovo states that the converse is also true. Meaning that any curve E on a surface which is isomorphic to $\mathbb{P}^1$ and satisfies $i(E, E) = -1$ is the exceptional curve of a blow-up.

Theorem 5.1.21 (Castelnuovo's contractibility criterion)**.** *Let S be a surface and let C be a curve on S which is isomorphic to $\mathbb{P}^1$ and such that $i(C, C) = -1$. Then there is a surface Y and a point P on Y such that there is an isomorphism φ from S to the blow-up $\varepsilon : \hat{Y} \to Y$ such that C is identified with E under φ.*

Remark 5.1.22. *Castelnuovo's criterion is useful to detect exceptional curves because a surface is minimal if and only if it contains no exceptional curves.*

Example 5.1.23. *By the Bezout's theorem we have $i(C, C) > 0$ for all curves C on $\mathbb{P}^2$ so Castelnuovo's contractibility criterion tell us that $\mathbb{P}^2$ is minimal. Furthermore the surfaces*

$$\Sigma_n := Proj(\mathcal{O}_{\mathbb{P}^1} \bigoplus \mathcal{O}_{\mathbb{P}^1}(-n)) \quad for\ n \geq 0\ ,$$

called the Hirzebruch surfaces are also minimal with

$$\Sigma_0 \cong \mathbb{P}^1 \times \mathbb{P}^1.$$

Proposition 5.1.24. *Any rational surface can be obtained from a projective plane or a Hirzebruch surface by blowing up a finite number of points.*

Definition 5.1.25. *A rational surface S with degree d such that $-K_S$ is ample and $i(K_S, K_S) = d$ is called Del Pezzo surface.*

Remark 5.1.26. *As stated in proposition (5.1.24), Del Pezzo surfaces can be obtained from a projective plane or Hizerbruch surface by blowing up a finite number of points. However, we must assume for minimality reasons that these operations are carried out in the projective plane and that we blow up points outside the exceptional locus in each step.*

For instance let P_1 and P_2 be points in $\mathbb{P}^2$ and let L be the line through the two points. Let X be the blow-up of $\mathbb{P}^2$ at P_1 and P_2 and let E_1 and E_2 be the corresponding exceptional curves. Then lemma (5.1.10) implies

$$\hat{L} = \varepsilon^* L - E_1 - E_2,$$

so

$$i(\hat{L}, \hat{L}) = i(\varepsilon^* L, \varepsilon^* L) + i(E_1, E_1) + i(E_2, E_2) = 1 - 1 - 1 = -1.$$

Moreover, since $K_{\mathbb{P}^2} = -3L$ it follows from proposition (5.1.11) that

$$K_S = -3\varepsilon^* L + E_1 + E_2$$

so, $i(\hat{L}, K_S) = -3 + 1 + 1 = -1$. Thus by the genus formula we have

$$P_a(\hat{L}) = 1 + \frac{1}{2} i(\hat{L}, K_X + \hat{L}) = 1 + \frac{1}{2}(-1 - 1) = 0.$$

Which means that $\hat{L} \cong \mathbb{P}^1$ and by the Castelnuovo's contractibility criterion we conclude that $\hat{L}$ is exceptional. Thus if we blow up further, we want to choose a point outside L to comply with our previous assumptions.

Definition 5.1.27. *A set of points $P_1, \ldots, P_n$ with $n \leq 8$ in the projective plane $\mathbb{P}^2$ are said to be in general position if:*

(i) *No three points should lie on a line,*

(ii) *No six points should lie on a conic,*

(iii) *and no eight points should lie on a nodal cubic such that the nodes is one of the points.*

Proposition 5.1.28. *Let S be a Del Pezzo surface. Its degree d is equal to $9 - n$ with $n \leq 8$ the numbers of points in general position in $\mathbb{P}^2$ that we blow up.*

Proof. The proof is rather simple in fact, from proposition (5.1.11) we have that

$$-K_S = 3\varepsilon^* L - E_1 - E_2 - \ldots - E_n$$

which implies that:

$$\begin{aligned} i(-K_S, -K_S) &= i(3\varepsilon^* L, 3\varepsilon^* L) - i(E_1, E_1) - \ldots - i(E_n, E_n) \\ &= 9 - n. \end{aligned}$$

□

Remark 5.1.29. *It turns out that the only Del Pezzo surface not obtained as a blow-up of $\mathbb{P}^2$ is $\mathbb{P}^1 \times \mathbb{P}^1$. Its degree is 8.*

5.2 Generalized Del Pezzo Varieties

In this section we study the rational varieties obtained by blowing up a point set $\hat{x} \in \hat{\mathbb{P}}_n^m$.

Definition 5.2.1. *A generalized Del Pezzo (gDP-variety) of type* (n, m) *is an algebraic variety* V *isomorphic to a blowing up* $V(\hat{x})$ *of some point set* $\hat{x} \in \hat{\mathbb{P}}_n^m$.

Definition 5.2.2. *A blowing-down structure of type* (n, m) *is a pair* (V, σ) *where* V *is a gDP-variety of type* (n, m) *and* σ *is a sequence of birational morphisms*

$$V = V_m \xrightarrow{\sigma_m} V_{m-1} \xrightarrow{\sigma_{m-1}} V_{m-2} \dots \xrightarrow{\sigma_2} V_1 \xrightarrow{\sigma_1} V_0 = \mathbb{P}^n$$

where $\sigma_i : V_i \to V_{i-1}$ *is a blowing up of a closed point.*

Remark 5.2.3. *Let* (V, σ) *and* (V', σ') *be two blowing-down structures. They are isomorphic if there exist isomorphisms*

$$\varphi_i : V_i \to V_i'$$

such that

$$\sigma' \circ \varphi_i = \varphi_{i-1} \circ \sigma_i, \quad i = 0, ..., m.$$

Proposition 5.2.4. *Let* Σ_m *the permutation group action on* $\hat{\mathbb{P}_n^m}$, *and if* $n > 2$ *the blowing-down structure of gDP-variety is defined uniquely up to isomorphism and up to* Σ_m*-action.*

Proof. See ([3]) page 64. □

Definition 5.2.5. *Let* X *and* Y *be two smooth algebraic varieties. The morphism*

$$f : X \dashrightarrow Y$$

is said to be a pseudo-isomorphism if it induces an isomorphism in codimension 1, *meaning it is an isomorphism of open subset whose complements have codimension* ≥ 2.

Remark 5.2.6. *Any pseudo-isomorphism of surface is an isomorphism.*

Definition 5.2.7. *Let* X *be a smooth algebraic variety of dimension* n. *A cycle* Y *of codimension* r *on* X *is an element of the free abelian group generated by the closed irreducible subvarieties of* X *of codimension* r *i.e*

$$Y = \sum n_i Y_i,$$

where the Y_i *are subvarieties, and* $n_i \in \mathbb{Z}$.

Remark 5.2.8. *We set* $A^r(X)$ *the group of algebraic cycles of codimension* r *modulo algebraic equivalence on* X. *Moreover*

$$A(X) = \bigoplus_{i=0}^{n} A^i(X)$$

is its Chow ring. Hence we set

$$N^1(X) = A^1(X)/\equiv,$$

$$N_1(X) = A^{n-1}(X)/\equiv,$$

where $\equiv$ *denotes numerical equivalence as in (2.1.124). If* $dimX \quad = 2$*, then we have* $N^1(X) \cong N_1(X)$.

Definition/Proposition 5.2.9 (Neron-Severi bilattice). *Let* V *be a gDP-variety of type* (n, m) *and*

$$V = V_m \xrightarrow{\sigma_m} V_{m-1} \xrightarrow{\sigma_{m-1}} V_{m-2} \dots \xrightarrow{\sigma_2} V_1 \xrightarrow{\sigma_1} V_0 = \mathbb{P}^n$$

be a blowing-down structure. The Neron-Severi bilattice is a pair:

$$N(V) = (N^1(V), N_1(V))$$

with the pairing

$$\begin{aligned} i\colon N^1(V) \times N_1(V) &\to \mathbb{Z} \\ (D, \gamma) &\mapsto i(D, \gamma) \end{aligned}$$

$$N^1(V) = \mathbb{Z}h_0 + \mathbb{Z}h_1 + \dots + \mathbb{Z}h_m, \quad N_1(V) = \mathbb{Z}l_0 + \mathbb{Z}l_1 + \dots + \mathbb{Z}l_m$$

where

$$h_0 = (\sigma_1 \circ .. \circ \sigma_m)^{-1}(H), \quad H \textit{ is a hyperplane in } \mathbb{P}^n$$

$$h_i = (\sigma_i \circ .. \circ \sigma_m)^{-1}(x^i), \quad i = 1, \dots, m,$$

$$l_0 = (\sigma_1 \circ .. \circ \sigma_m)^{-1}(l), \quad l \textit{ is a line in } \mathbb{P}^n,$$

$$h_0 = (\sigma_{i+1} \circ .. \circ \sigma_m)^{-1}(l_i), \quad l_i \textit{ is a line in } \sigma_i^{-1} \cong \mathbb{P}^{n-1}, i = 1, \dots, m.$$

such that

$$i(h_0, l_0) = 1, \quad i(h_i, l_i) = -1, \quad i \neq 0, \quad i(h_i, h_j) = 0, \quad i \neq j.$$

The gDP variety of type $(2, m)$ plays an important role in the study of Del Pezzo surfaces as will see in the following proposition.

Definition 5.2.10. *A point set* $\hat{x} \in \hat{\mathbb{P}_n^m}$ *is in general position if*

(i) $\hat{x}$ *does not contain infinitely near points,*

(ii) *no* 3 *points from* $\hat{x}$ *are collinear,*

(iii) *no* 6 *points from* $\hat{x}$ *lie on a conic,*

(iv) *if* $m = 8$*,* $\hat{x}$ *does not lie on a cubic with a singular point at one points from* $\hat{x}$*.*

Proposition 5.2.11. *Any Del Pezzo surface is isomorphic to a gDP variety of type* $(2, m)$ *with* $m \leq 8$ *obtained by blowing up a point set* $\hat{x} \in \hat{\mathbb{P}}_n^m$ *in general position.*

Proof. The proof follow immediately from the definition of Del Pezzo surfaces and the fact that these are obtained by blowing point $\mathbb{P}^2$ in general position and that $\hat{\mathbb{P}}_2^m$ and $\mathbb{P}^2$ are birationally equivalent. □

5.2.12 GEOMETRIC MARKING OF gDP-VARIETIES

Definition 5.2.13 (Lattice)**.** *A lattice L is a free abelian group of finite rank equipped with symmetric bilinear form*

$$\begin{aligned} m\colon L \times L &\to \mathbb{Z} \\ (v, v') &\mapsto v \circ v'. \end{aligned}$$

Remark 5.2.14. *Tensoring L by* $\mathbb{R}$ *defines a quadratic form on the real vector space* $L_{\mathbb{R}}$ *and therefore we can speak about signature, rank etc. However we need a more general concept namely bilattice.*

Definition 5.2.15. *A bilattice is a pair* (L_1, L_2) *of lattices equipped with a bilinear form*

$$\begin{aligned} L_1 \times L_2\colon &\to \mathbb{Z} \\ (v_1, v_2) &\mapsto v_1 \circ v_2 \end{aligned}$$

Remark 5.2.16. *A lattice L is considered as a bilattice* (L, L)*.*

Definition 5.2.17. *A morphism of bilattices is*

$$\begin{aligned} \varphi := (\varphi_1, \varphi_2) : (L_1, L_2)\colon &\to (L_1', L_2') \\ (v_1, v_2) &\mapsto (\varphi_1(v_1), \varphi_2(v_2)) \end{aligned}$$

with

$$\varphi_i : L_i \to L_i' \quad \textit{a homomorphism of lattices}$$

satisfying

$$\varphi_1(v_1) \circ \varphi_2(v_2) = v_1 \circ v_2, \quad \textit{for any} \quad v_1 \in L_1, \quad v_2 \in L_2.$$

Remark 5.2.18. *Every bilattice* (L_1, L_2) *admits natural morphisms to the bilattice* (L_i, L_i^*)*, for* $i = 1, 2$ *where*

$$L_i^* = Hom_{\mathbb{Z}}(L_i, \mathbb{Z}),$$

is the dual abelian group, and

$$x.x^* = x^*(x), \quad \textit{for every } x \in L_i \textit{ and } x^* \in L_i^* \textit{ for } i = 1, 2$$

. Morever a bilattice is said to be unimodular if these morphisms are isomorphisms.

Example 5.2.19. *Our main example of lattice is the standard hyperbolic lattice of rank* $m + 1$

$$H_m = \mathbb{Z}e_0 + \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_m,$$

Where

$$e_0.e_0 = 1, \quad e_i.e_i = -1, \quad i \neq 0, \quad e_i.e_j = 0, \quad i \neq j.$$

Remark 5.2.20. *Another example of bilattice is the Neron-Severi bilattice* $N(V)$ *of a smooth complete variety.*

In particular the Neron-Severi bilatitice is intrinsically related to the standard hyperbolic lattice when V is a gDP-variety.

Proposition 5.2.21. *Let* V *be gDP variety. Then following the notation of definition (5.2.9) we have that the following group morphism :*

$$\begin{aligned} \varphi^1 : H_m &\to N^1(V), \\ e_i &\mapsto h_i, \end{aligned}$$

$$\begin{aligned} \varphi_1 : H_m &\to N_1(V), \\ e_i &\mapsto l_i, \end{aligned}$$

define an isomorphism of bilattices.

$$\varphi = (\varphi^1, \varphi_1) : H_m \to N(V).$$

As a consequence, $N(V)$ *is unimodular.*

Definition 5.2.22. *Let* $L = (L_1, L_2)$ *be a bilattice. An L-marking of smooth complete variety* V *is an isomorphism of bilattices:*

$$\varphi : L \to N(V).$$

And L-marked variety V *is a pair* (V, φ) *where* φ *is a L-marking.*

Lemma 5.2.23. *Let $f : X \dashrightarrow X'$ be a pseudo-isomorphism of smooth complete varieties. Assume that the Neron-Severi bilattices of X and X' are unimodular. Then there exists a natural isomorphism of these bilattices:*

$$f^* : N(X') \to N(X).$$

Proof. See ([3]) page 69. □

Here we state our main definition

Definition 5.2.24 (A strict geometric marking.). *A strict geometric marking of gDP- variety V of type (n, m) is an H_m-marking*

$$\begin{aligned} \varphi := (\varphi^1, \varphi_1) : (H_m, H_m) : \ &\to (N^1(V), N_1(V)) \\ (e_i, e_i) &\mapsto (\varphi^1(e_i), \varphi_1(e_i)) \end{aligned}$$

where $\varphi^1(e_i) = h_i, \quad \varphi_1(e_i) = l_i, \quad i = 0, 1, ..., m.$

5.2.25 The Weyl groups

In this section we are looking for a group that acts on the G-orbits in $\hat{\mathbb{P}}_n^m$ by acting on the geometric markings. The natural candidate is the group of "isometries" of the lattice H_m that we denote $O(H_m)$. It is clear that it acts on H_m-markings

$$\varphi : H_m \to N(V)$$

by composing them with isometries

$$\sigma : H_m \to H_m.$$

However since this action is not stable under any subset of geometric markings, we have to look for a "nice" subgroup of $O(H_m)$ which consists of isometries of H_m preserving the set of geometric markings.It turns out that the right candidate is the Weyl group of a certain natural root basis in H_m.

Definition 5.2.26. *Let L be a lattice (5.2.13). An isometry morphism of L is a bijection*

$$\sigma : L \to L,$$

such that

$$m(v, v') = m(\sigma(v), \sigma(v')), \quad \textit{for all } v, v' \in L.$$

Remark 5.2.27. *We denote by $O(L)$ the set of all isometries of L. Moreover, it is obviously a group.*

Definition 5.2.28 (Root Basis.). *A root basis in a bilattice $L = (L_1, L_2)$ is a pair $(B, \hat{B})$ of subsets of L_1 and L_2, respectively, together with a bijection*

$$\begin{aligned} T\colon B &\to \hat{B} \\ \alpha &\mapsto \hat{\alpha} \end{aligned}$$

satisfying :

(i) $\alpha.\hat{\alpha} = -2$ *;*

(ii) $\alpha.\hat{\beta} \geq 0$ *for any* $\alpha, \beta \in B, \quad \alpha \neq \beta$.

Remark 5.2.29. *A root basis is symmetric if the following additional property holds:*

(iii) $\alpha.\hat{\beta} = \beta.\hat{\alpha}$ *for any* $\alpha, \beta \in B$.

Definition 5.2.30 (Simple reflections)**.** *Let* $L = (L_1, L_2)$ *be a bilattice and* $(B, \hat{B})$ *be a root basis. Let* $\alpha \in B$ *and* $\hat{\alpha} \in \hat{B}$*. Then the a simple reflections*

$$\begin{aligned} S_\alpha\colon L_1 &\to L_1 \\ x_1 &\mapsto x_1 + (x_1.\hat{\alpha})\alpha, \end{aligned}$$

and

$$\begin{aligned} \hat{S}_\alpha\colon L_2 &\to L_2 \\ x_2 &\mapsto x_2 + (x_2.\alpha)\hat{\alpha} \end{aligned}$$

are linear involutions of L_1 *and* L_2.

Definition 5.2.31. *The subgroup of* $GL(L_1)$ *(resp. of* $GL(L_2)$ *) generated by* S_α *(resp.* $\hat{S}_\alpha$ *) are called Weyl group of the root basis* $(B, \hat{B})$ *and are denoted* W_B *(resp.* $W_{\hat{B}}$*).*

Proposition 5.2.32. *The following map :*

$$\begin{aligned} \psi\colon W_B &\to W_{\hat{B}} \\ w &\mapsto \hat{w} \end{aligned}$$

is an isomorphism.

Proof. For all $\alpha \in B$, the morphism which associates S_α to $\hat{S}_\alpha$ extends to an isomorphism

$$\psi \colon W_B \to W_{\hat{B}}$$
$$w \mapsto \hat{w}.$$

□

Proposition 5.2.33. *The Weyl group W_B is isomorphic to a subgroup of the isometry group $O(L)$.*

Proof. Let us prove that W_B is a subgroup of the isometry group $O(L)$. Let x_1, x_2 be respectively in L_1 and L_2 then we have

$$m(w(x_1), \hat{w}(x_2)) = m(x_1, x_2).$$

□

Definition 5.2.34. *An element of a W_B-orbit of B in L_1 (resp. of $\hat{B}$ in L_2) is called a B-root (resp. $\hat{B}$-root). The set of such elements is denoted by R_B (resp. $\hat{R_B}$).*

In fact

$$R_B = \{w(\alpha), \quad \text{with} \quad \alpha \in B, \quad w \in W_B\},$$

and

$$R_{\hat{B}} = \{\hat{w}(\hat{\alpha}), \quad \text{with} \quad \hat{\alpha} \in \hat{B}.\}$$

Remark 5.2.35. *An element of B (resp. of $\hat{B}$) is called a simple B- root (resp. $\hat{B}$- root.)*

Proposition 5.2.36. *There is a natural bijection between the set of B- roots R_B and the set of $\hat{B}$- roots $R_{\hat{B}}$ i.e.*

$$R_B \cong R_{\hat{B}}.$$

Proof. The bijection between simple B-roots and simple $\hat{B}$-roots

$$T \colon B \cong \hat{B}$$
$$\alpha \mapsto \hat{\alpha}$$

can be extended naturally to a bijection between R_B and $R_{\hat{B}}$ i.e

$$Q \colon R_B \to R_{\hat{B}}$$
$$w(\alpha) \mapsto \hat{w}(\hat{\alpha}).$$

□

Definition 5.2.37. *A B-root $w(\alpha)$ with $\alpha \in B$ is called positive (resp. negative) if it can be written as a linear combination of simple B-roots with integral non-negative (resp.non positive) coefficients, i.e*

$$w(\alpha) = \sum_{i \in \mathbb{Z}} n_i \beta_i, \quad \textit{with} \quad n_i \in \mathbb{Z}_{\geq 0} \quad \textit{and } \beta_i \textit{ a simple B-root.}$$

And α be a negative B-root of course if all integers n_i are negative.

Remark 5.2.38. *Let*

$$R_B^+ = \{w(\alpha) = \sum_{i \in \mathbb{Z}} n_i \beta_i, \quad \textit{with} \quad n_i \in \mathbb{Z}_{\geq 0} \quad \textit{and } \beta_i \textit{ a simple B-root. }\}$$

be the set of positive B-roots and

$$R_B^- = -R_B^+$$

the set of negative B-roots.

Example 5.2.39 (The Weyl group of Hyperbolic lattice H_m). *Coming back to our situation when $L = H_m$ and $m \geq n+1 \geq 3$, we set the canonical root basis of type $n > 1$ in H_m by posing:*

$$B_n = \{\alpha_0, ..., \alpha_{m-1}\}$$
$$\hat{B_m} = \{\hat{\alpha_0}, ..., \hat{\alpha_{m-1}}, \}$$

where

$$\alpha_0 = e_0 - e_1 - ... - e_{n+1}, \quad \alpha_i = e_i - e_{i+1}, \quad i = 1, ..., m-1,$$

$$\hat{\alpha_0} = (n-1)e_0 - e_1 - ... - e_{n+1}, \quad \hat{\alpha_i} = e_i - e_{i+1}, \quad i = 1, ..., m-1.$$

It is a symmetric root basis. Moreover we denote by $W_{n,m}$ its corresponding Weyl group.

5.2.40 Discriminant conditions

Definition 5.2.41. *Let $\varphi : H_m \to N(V)$ be a geometric marking of a gDP-variety V of type (n, m) and $(B.\hat{B})$ be a canonical root basis of type n in H_m. A B-roots α (resp. $\hat{B}$-roots $\hat{\alpha}$) is effective or nodal with respect to the geometric marking φ if $\varphi^1(\alpha)$ is effective (resp. $\varphi_1(\hat{\alpha})$ is effective).*

Remark 5.2.42. *We set :*

$$R_B(\varphi)^+ = \{\alpha \in R_B : \quad \varphi^1(\alpha) \quad \textit{is effective}\}$$

the set of all effective B-roots with respect to the geometric marking φ. And

$$R_{\hat{B}}(\varphi)^+ = \{\hat{\alpha} \in R_{\hat{B}} : \quad \varphi_1(\hat{\alpha}) \quad \textit{is effective}\}$$

the set of all effective $\hat{B}$-roots with respect to the geometric marking φ

Definition 5.2.43. *Let $x \in \hat{\mathbb{P}_n^m}$ and $\varphi_x : H_m \to N(V(x))$ the corresponding strict geometric marking we set*

$$R_B(x)^+ = R_B(\varphi_x)^+$$

$$R_{\hat{B}}(x)^+ = R_{\hat{B}}(\varphi_x)^+$$

the set of discriminant conditions on the point set x.

Definition 5.2.44. *A point set x (resp. a geometric marking φ) is said to be unnodal if*

$$R_B(x)^+ = \varnothing \quad (\textit{resp.} \quad R_B(\varphi) = \varnothing).$$

Moreover a gDP-variety V is unnodal if all of its geometric markings are unnodal.

Proposition 5.2.45. *Assume that gDP-surface V admits an unnodal geometric marking. Then V is unnodal.*

Proof. As we will see in the next section, for every geometric marking

$$\varphi : H_m \to N(V)$$

$$\psi : H_m \to N(V)$$

there exists $w \in W_{2,m}$ such that

$$\psi = \varphi \circ w$$

Thus :

$$\alpha \in R_B(\psi)^+ \Leftrightarrow \quad w(\alpha) \in R_B(\varphi)^+$$

□

Corollary 5.2.46. *Let V be a gDP-surface. Assume $m \leq 8$. The following properties are equivalent:*

(i) V is unnodal for some geometric marking $\varphi : H_m \to N(V)$,

(ii) V is a Del Pezzo surface.

Proof. See ([3]) page 80. □

5.3 Cremona Action

Let Σ_m be the permutation group on m letters. It naturally acts on the varieties P_n^m via its natural action on $\mathbb{P}_n^m$. In this section we will see that this action can be extended to a birational action of the Weyl group $W_{n,m}$. In fact, this action occurs by applying to the point sets certains types of Cremona transformations of $\mathbb{P}_n$.

Definition 5.3.1. *The standard Cremona tranformation T_0 in $\mathbb{P}_n$ is the birational transformation of $\mathbb{P}_n$ defined by the formula:*

$$\begin{aligned} T_0 : \mathbb{P}_n &\to \mathbb{P}_n \\ (t_0, \dots, t_n) &\mapsto (t_1 \dots t_n, \dots, t_0 \dots \hat{t_i} \dots t_n, \dots, t_0 \dots t_{n-1}) \end{aligned}$$

where $\hat{t_i}$ means we omit t_i in the product.

Remark 5.3.2. *The linear system of hypersurfaces defining T_0 consists of hypersurfaces of degree n that pass through the points $x^i = (0, ..., 1, ..., 0)$ with multiplicity $n-1$. The choice of the basis in this linear system is determined by the fact that:*

$$T_0^2 = Id_{\mathbb{P}_n}.$$

Moreover, T_0 is defined everywhere except at the points x^i which are transformed to the hypersurfaces:

$$H_i = \{(t_0, ..., t_n) \in \mathbb{P}_n \quad : \quad t_i = 0.\}$$

Lemma 5.3.3. *There exists a commutative diagram of birational maps:*

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "f"] \\ \mathbb{P}_n \arrow[r, "T_0"] & \mathbb{P}_n \end{tikzcd}$$

where g is an isomorphism, and f is the composition

$$Y = Y_{m-1} \xrightarrow{f_{m-1}} Y_{n-2} \xrightarrow{f_{m-2}} Y_{m-3} \dots \xrightarrow{f_2} V_1 \xrightarrow{f_1} Y_0 = \mathbb{P}^n,$$

where

$$f_k : Y_k \to Y_{k-1} \textit{is a blowing up for the proper transforms of the subspaces}$$

$$H_{i_1} \cap \dots \cap H_{i_{n+1-k}}, \quad 0 \le i_1 < \dots < i_{n+1-k} \le n+1, \quad \textit{under} \quad f_{k-1} \quad (f_0 = id),$$

and

$$(f \circ g)(f^{-1}(H_{i_1} \cap \dots \cap H_{i_{n+1-k}})) = H_{i_1} \cap \dots \cap H_{j_{n+1-k}},$$

where $\{i_1, ..., i_k\}$ and $\{j_1, ..., j_{n+1-k}\}$ are complementary subsets of $\{1, ..., n+1\}$.
Moreover, under some identification of $f^{-1}(H_{i_1} \cap \dots \cap H_{i_n})$ and H_{j_1} with $\mathbb{P}_{n-1}$, the rational map

$$f \circ g \circ f_{n-1}^{-1} : \mathbb{P}_{n-1} \dashrightarrow \mathbb{P}_{n-1}$$

is a standard Cremona transformation.

Corollary 5.3.4. *Let $\sigma : V(x) \to \mathbb{P}_n$ be the blowing-up of the point set $x = (x^1, \ldots, x^{n+1})$, $\varphi_x : H_{n+1} \to N(V(x))$ be the corresponding strict geometric marking. Then there exist a pseudo-isomorphism*

$$h : V(x) \dashrightarrow V(x)$$

and commutative diagrams:

$$\begin{tikzcd} V(x) \arrow[r, dotted, "h"] \arrow[d, "\sigma"] & V(x) \arrow[d, "\sigma"] \\ \mathbb{P}_n \arrow[r, dotted, "T_0"] & \mathbb{P}_n \end{tikzcd}$$

$$\begin{tikzcd} H_{n+1} \arrow[r, "\varphi_x"] \arrow[d, "S_{\alpha_0}"] & N(V(x)) \arrow[d, "h^*"] \\ H_{n+1} \arrow[r, "\varphi_x"] & N(V(x)) \end{tikzcd}$$

with φ_x an isomorphism.

Proof. From the lemma above, T_0 induces a rational map

$$h = (f_{n-1} \circ \ldots \circ f_2) \circ g \circ (f_{n-1} \circ \ldots \circ f_2)^{-1} : Y_1 = V(x) \dashrightarrow Y_1$$

which is a pseudo-isomorphism that sends the strict geometric marking

$$\varphi_x : H_{n+1} \to N(Y_1)$$

of Y_1 to the geometric marking $\psi = h^* \circ \varphi_x : H_{n+1} \to N(Y_1)$ defined by

$$\psi^1(e_0) = n\varphi^1(e_0) - (n-1)(\varphi^1(e_1) + \cdots + \varphi^1(e_{n+1})),$$
$$\psi^1(e_i) = \varphi^1(e_0) - \varphi^1(e_1) + \ldots + \varphi^1(e_{n+1}) + \varphi^1(e_i), \quad i = 1, \ldots, n+1,$$

$$\psi_1(e_0) = n\varphi_1(e_0) - (\varphi_1(e_1) + \cdots + \varphi_1(e_{n+1})),$$

$$\psi_1(e_i) = (n-1)\varphi_1(e_0) - (\varphi_1(e_1) + \cdots + \varphi_1(e_{n+1})) + \varphi_1(e_i), \quad i = 1, \ldots, n+1.$$

From the action of the simple reflection S_{α_0}, we have :

$$\psi = \varphi_x \circ S_{\alpha_0}.$$

Hence in its natural action on the set of markings $\varphi : H_{n+1} \to N(V)$, the reflection S_{α_0} transforms a strict geometric marking $\varphi = \varphi_x$ of V defined by a point set $x = (x^1, \ldots, x^{n+1})$ into a geometric marking $\psi = h^* \circ \varphi$. Similarly, a simple reflection S_{α_i} transforms φ_x into φ_y, where

$$y = (x^1, ..., x^{i-1}, x^{i+1}, x^i, x^{i+2}, ..., x^{n+1}).$$

□

Remark 5.3.5. *The proof of the corollary above suggests that the whole group $W^{n,m}$ acts on the set of pseudo-isomorphism classes of geometric markings of any gDP-variety V of type (n, m).*

Proposition 5.3.6. *Let $x = (x^1, ..., x^m) \in \mathbb{P}_n^m$ and assume that all point x^i are distinct and the first $n+1$ points span $\mathbb{P}_n$. Then for every $i = 0, ..., m-1$, there exist a point set $y = (y^1, ..., y^m) \in \mathbb{P}_n^m$, a pseudo-isomorphism $f: V(x) \dashrightarrow V(Y)$, a birational transformation*

$$T_i : \mathbb{P}_n \dashrightarrow \mathbb{P}_n,$$

and commutative diagrams:

$$\begin{tikzcd} V(x) \arrow[r, dotted, "f"] \arrow[d, "\sigma"] & V(y) \arrow[d, "\sigma'"] \\ \mathbb{P}_n \arrow[r, dotted, "T_i"] & \mathbb{P}_n \end{tikzcd}$$

$$\begin{tikzcd} H_{n+1} \arrow[r, "\varphi_y"] \arrow[d, "S_{\alpha_i}"] & N(V(y)) \arrow[d, "f^*"] \\ H_{n+1} \arrow[r, "\varphi_x"] & N(V(x)) \end{tikzcd}$$

with φ_x and φ_y two isomorphism.

Proof. Let

$$\sigma : V(x) = V_m \to V_{m-1} \to ... \to V_1 \to \mathbb{P}_n$$

be the corresponding blowing-down structure on $V(x)$. Assume first that $i \neq 0$. We choose a projective transformation T_i of $\mathbb{P}_n$ which permutes the points x^i and x^{i+1} and sends the remaining points x^j to some points y^j. The define the point set y by

$$y = (y^1, ..., y^{i-1}, x^i, x^{i+1}, y^{i+2}, ..., y^m).$$

The composition $T_i \circ \sigma$ maps the exceptional divisors $E_1, ..., E_m$ of σ to the points $y^1, ..., y^m$ of y respectively. Let $(v(y), \sigma')$ be the blowing-down structure corresponding to y. The composition

$$T_i \circ \sigma : V(x) \to \mathbb{P}_n$$

blows up the same set y. By the uniqueness of the blowing-up, there exists an isomorphism $f : V(x) \to V(y)$ defining the first diagram in the statement. It is clear that

$$f^*([\sigma'^{-1}(y^j)]) = [\sigma^{-1}(x^j)] \quad j \neq i, i+1,$$

$$f^*([\sigma'^{-1}(y^i)]) = [\sigma^{-1}(x^{i+1})],$$
$$f^*([\sigma'^{-1}(y^i)]) = [\sigma^{-1}(x^i)].$$

This implies that

$$\varphi_x \circ S_{\alpha_i} = f^* \circ \varphi_y,$$

which proves our statement. Moreover, let $i = 0$, since $x^1, ..., x^{n+1}$ span $\mathbb{P}_n$, we may substitute x by a projectively equivalent set to assume that $x^i = (0, ..., 1, ..., 0)$, $i = 1, ..., n+1$. Let

$$f' : V_{n+1} \dashrightarrow V_{n+1}$$

be the pseudo-isomorphism defined in the previous corollary. It extends to a pseudo-isomorphism

$$f : V(x) \dashrightarrow V(y),$$

where

$$y = T_0(x) = (x^1, ..., x^{n+1}, T_0(x^{n+2}), ..., T_0(x^m)).$$

It is immediately verified, using the corollary, that $\varphi_x \circ S_i = f^* \circ \varphi_y$. □

Remark 5.3.7. *From the previous proposition we can apply any product of simple reflections to a geometric marking to obtain another geometric marking, provided that at every step the resulting point set $x = (x^1, ..., x^m)$ satisfies:*

(i) x does not contain infinitely near points;

(ii) $< x^1, ..., x^{n+1} >= \mathbb{P}_n$

This can be stated in terms of the strict geometric marking

$$\varphi_x : H_m \to N(V(x)),$$

by saying that all simple B-roots α_i are not effective.

Observe that

$$w(R_B(\varphi)^+) = R_B(\varphi \circ w^{-1})^+, \quad \textit{for any} \quad w \in W_B.$$

This shows that we can apply every $w \in W_B = W_{n,m}$ to x if

$$R_B(x)^+ = \varnothing,$$

meaning that if x is unnodal in the sense of the previous section.

Therefore we have led to study the orbits in the set

$$(\hat{\mathbb{P}_n^m})^{un} := \hat{\mathbb{P}_n^m} - Z$$

where

$$Z = \cup_{\alpha \in R_B} Z(\alpha),$$

$$Z(\alpha) = \{x \in \hat{\mathbb{P}_n^m} : \quad \alpha \in R_B(x)^+\}$$

Moreover, for $i > j > 0$, $\alpha = e_j - e_i = \alpha_i + \ldots + \alpha_{j-1} \in R_B$, *and*

$$Z(\alpha) = \hat{\Delta}_{ij}(m).$$

This shows that

$$(\hat{\mathbb{P}_n^m})^{un} \subset \hat{\mathbb{P}_n^m} - \hat{\Delta}(m) \cong \mathbb{P}_n^m - \Delta(m),$$

Thus it allows us to use

$$(\hat{\mathbb{P}_n^m})^{un}, (\mathbb{P}_n^m)^{un}$$

To denote the same set. Taking $\alpha = e_0 - e_{i_1} - \ldots - e_{i_{n+1}}$, *we obtain that*

$$Z(\alpha) = \{x \in \hat{\mathbb{P}_n^m} : \quad \textit{no } n+1 \textit{ points lie in a hyperplane}\}$$

It follows from the criterion of stability of point sets that

$$(\hat{\mathbb{P}_n^m})^{un} \subset (\mathbb{P}_n^m - \Delta(m))^s.$$

Set

$$(P_n^m)^{un} = \phi((\hat{\mathbb{P}_n^m})^{un}) \subset P_n^m - \hat{\mathscr{D}}.$$

Let us see first that $(\mathbb{P}_n^m)^{un}$ is not empty.

Theorem 5.3.8. *For every B-root* α *the subset*

$$Z(\alpha) = \{x \in \hat{\mathbb{P}_n^m} : \quad \alpha \in R_B(x)^+\}$$

is a closed subset of $\hat{\mathbb{P}_n^m}$. *Furthermore, its restriction to* $(\hat{\mathbb{P}_n^m} - \hat{\Delta}(m))^s$ *is an irreducible hypersurface.*

Proof. Let

$$\alpha = a_0 e_0 - a_1 e_1 - \ldots - a_m e_m \in R_B.$$

Assume $\varphi_x(\alpha) \geq 0$. Since $h_0 = \varphi_x(e_0)$ is numerically effective,

$$\varphi_x(\alpha) h_0 = (a_0 h_0 - a_1 h_1 - \ldots - a_m h_m) h_0 = a_0 \geq 0,$$

If $a_0 = 0, \alpha = e_i - e_j$ for some $i, j > 0$ (Proposition 4 [3] of chapter 5). Hence

$$\varphi_x(\alpha) = h_i - h_j \geq 0 \text{iff } x^j \text{ is infinitely near to } x^i \;,$$

and $Z(\alpha) = \hat{\Delta}_{ij}(m)$ in this case. By proposition(5.3.8), it is a hypersurface.

Assume $a_0 > 0$. By proposition 4 from [3] page 74 we have :

$$a_i \geq 0, i = 1, ..., m.$$

Assume that x does not contain infinitely near points. Then

$$\varphi_x(\alpha) = a_0 h_0 - a_1 h_1 - ... - a_m h_m$$

is the class of effective divisor D if and only if there exists a hypersurface in $\mathbb{P}_n$ of degree a_0 that passes through the point x^i with multiplicity $\geq a_i$. In this case

$$D = D' + K_1 E_1 + ... + K_m E_m,$$

where D' is the proper inverse transform of the hypersurface. The existence of a hypersurface is expressed by algebraic equations in the coordinates of the points x^i. This proves that $Z(\alpha) \cap (\hat{\mathbb{P}_n^m} - \hat{\Delta}(m))$ is a closed subset of $(\hat{\mathbb{P}_n^m} - \hat{\Delta}(m))$.

Now assume $x \in \hat{\Delta}(m)$. For simplicity we also assume that $x \notin \hat{\Delta}_l(m)$ with $|l| > 2$. Without loss of generality we may take x in $\hat{\Delta}_{12}(m)$. If $K = \sup\{-a_1 + a_2, 0\}$, any effective divisor with class $\varphi_x(\alpha)$ contains $K(E_1 - E_2)$, where $\varphi_x(e_i) = E_i$. Thus $\varphi_x(\alpha) \geq 0$ if and only if there exists a hypersurface in $\mathbb{P}_n$ of degree a_0 passing through x^1 with multiplicity $\geq a_1 + K$, passing through the infinitely near point $x^2 \to x^1$ with multiplicity $\geq a_2$, and passing through the remaining points with multiplicity $\geq a_i, i > 2$. This expressed by algebraic condition on the coordinates of the x^i's. And this proves that $Z(\alpha)$ is a closed subset $\hat{\mathbb{P}_n^m}$.

□

Corollary 5.3.9. *Assume that the canonical root system of type n in H_m is of finite type. Then $(\mathbb{P}_n^m)^{un}$ (resp. $(P_n^m)^{un}$) is an open zariski subset of $\mathbb{P}_n^m$) (resp. P_n^m).*

Remark 5.3.10. *If $(B, \hat{B})$ is of finite type, R_B and $W_{n,m}$ are finite and $W_{n,m}$ acts biregularly on the open set $(P_n^m)^{un}$. In general, $W_{n,m}$ does not act regularly on any open subset of P_n^m.*

However, the next theorem shows, that at least in the csae $n = 2$, that the Weyl group acts transitively on the set of unnodal geometric markings of the same gDP-variety.

Theorem 5.3.11. *Let $\varphi : H_m \to N(V)$ and $\psi : H_m \to N(V)$ be two geometric markings of a gDP-surface. Then there exists $w \in W_{2,m}$ that*

$$\psi = \varphi \circ w.$$

Proof. Let

$$\varphi(e_i) = h_i, \quad i = 0, ..., m,$$

$$\psi(e_0) = a_0 h_0 - a_1 h_1 - ... - a_m h_m = \varphi(a_0 e_0 - a_1 e_1 - ... - a_m e_m).$$

Since $\psi(e_0)$ is numerically effective,

$$a_0 = \psi(e_0).h_0 > 0, \quad a_i = \psi(e_0).h_i \geq 0, \quad i > 0.$$

Set

$$v = a_0 e_0 - a_1 e_1 - \dots - a_m e_m,$$

so that

$$\varphi(v) = \psi(e_0).$$

Suppose we show that there exists an element $w \in W_{2,m}$ such that

$$w(v) = e_0.$$

Then

$$w^{-1} \circ \varphi^{-1} \circ \psi(e_0) = e_0$$

thus

$$w^{-1} \circ \varphi^{-1} \circ \psi(e_i) = e_{\sigma(i)}, \quad i = 1, \dots, m,$$

for some permutation σ of $\{1, \dots, m\}$. Replacing w by $w \circ \sigma$, we may assume that

$$w^{-1} \circ \varphi^{-1} \circ \psi(e_i) = e_i, \quad i = 1, \dots, m.$$

This certainly implies that

$$\psi = \varphi \circ w.$$

To show that such a w exists we assume first that φ is unnodal. By assumption,

$$R_B(\varphi)^+ = \varnothing.$$

Then

$$R_B(\varphi \circ w)^+ = \varnothing, \quad \text{for any } w \in W_{2,n}.$$

thus for every $w \in W_{2,n}$ the composition $\varphi \circ w$ is unnodal geometric marking. Obviously, $\varphi(v) = \psi(e_0)$ is represented by an irreducible curve. Thus there exixts an irreducible plane curve of degree a_0 with a_i-multiple points at the x^i's. Applying an element of Σ_m we may assume that

$$a_1 \geq a_2 \geq \dots \geq a_m \geq 0.$$

This implies that $\psi(e_0)$ satisfies the assumptions of Noether's inequality ([3]), and

$$a = a_0 - a_1 - a_2 - a_3 < 0$$

unless $v = e_0$, in which case we are done. If $v \neq e_0$ we apply S_{α_0} to v to obtain

$$w(v) = v' = (a_0 + a)e_0 - (a_1 + a)e_1 - (a_2 + a)e_2 - (a_3 + a)e_3 - a_4e_4 - \ldots - a_me_m.$$

Since $\varphi \circ S_{\alpha_0}$ is a geometric marking, $\varphi(S_{\alpha_0}(v))$ is the class of a numerically effective divisor. Thus

$$0 < a_0 + a < a_0, \quad a_i + a \geq 0, \quad i = 1, 2, 3.$$

Proceeding in this way, we decrease the coefficient at e_0 until we reach the case

$$w(v) = e_0$$

for some $w \in W_{2,m}$.

Now assuming that φ is any geometric marking and let x be a generic point set, that is the generic point of $\mathbb{P}_2^m$. Let

$$D \in \mathrm{Pic}(V(x))$$

represent the class $\varphi_x(v)$. We know that

$$D^2 = 1, \quad D.K_{V(x)} = -3.$$

Since

$$(K_{V(x)} - D).\varphi_x(e_0) = -3 - a_0 < 0,$$

It follows that $h^0(K_{V(x)} - D) = 0$. By Rieman-Roch

$$h^0(D) \geq 3$$

and we may assume that $D \geq 0$. Specializing x to the point set $\bar{x}$ representing the geometric marking φ we obtain that D specializes to an element of the irreducible linear system $|\varphi(v)| = |\psi(e_0)|$ on $V(\bar{x})$. Thus we can choose D to be irreducible. This easily implies that the linear system $|D|$ is of dimension 2 and defines a birational morphism

$$V(x) \to \mathbb{P}_2.$$

Thus there exists a geometric marking ψ' of $V(x)$ such that

$$D = \psi'(e_0).$$

By theorem (5.3.8), x is unnodal. hence we are in the previous situation and can find $w \in W_{2,m}$ for which $w(v) = f_0$.

□

Theorem 5.3.12. *Assume $m \leq 8$ and $char(k) = 0$. Then the quotient space*

$$(P_2^m)^{un}/W_{2,m} \cong \mathcal{M}_{DP}(m)$$

where $\mathcal{M}_{DP}(m)$ is the coarse moduli space of Del Pezzo surface of degree $9 - m$

Proof. First let us recall a construction of the latter space. If $m = 4$, $\mathcal{M}_{DP}(4)$ is a one-point set. Since any set points of four points in general position is projectively equivalent to the set of reference points $[1,0,0]$, $[0,1,0]$, $[0,0,1]$, $[1,1,1]$, we obtain that all nonsingular Del Pezzo surfaces of degree 5 are isomorphic.

If $m = 5$ the anticanonical linear system $|-K_V|$ maps the surface V isomorphically onto the intersection of two quadrics in $\mathbb{P}_4$. In this case $\mathcal{M}_{DP}(5)$ can be realized as an appropriate quotient of an open subset in the Grassmann variety of pencils of quadrics in $\mathbb{P}_4$. If $m = 6$, $|-K_V|$ maps V isomorphically onto a nonsingular cubic surface in $\mathbb{P}_3$. In this case $\mathcal{M}_{DP}(6)$ is constructed by standard methods of geometric invariant theory. If $m = 7$, $-K_V$ defines a double cover of degree 2 onto $\mathbb{P}_2$ branched along a nonsingular quadric curve. Thus $\mathcal{M}_{DP}(7)$ is isomorphic to a certain quotient of an open subset of space quadric curves. Finally, if $m = 8$, $|-2K_V|$ defines a double cover onto a singular quadratic cone Q in $\mathbb{P}_3$ and ramifies along a curve of degree 6 cut out on Q by a cubic. the construction of $\mathcal{M}_{DP}(8)$ in this case is similar to the previous case.

Let

$$(P_2^m)^{un} \to \mathcal{M}_{DP}(m)$$

be the map defined by forgetting the blowing-down structure. It follows from theorem (5.3.11) Proposition (5.2.45) that this map factors through the quotient by the finite group $W_{2,m}$ and defines a bijective map

$$(P_2^m)^{un}/W_{2,m} \to \mathcal{M}_{DP}(m).$$

Since both spaces are normal algebraic varieties, the assertion follows from Zariski's main theorem ([6], page 410).

□

Remark 5.3.13 ([3], page 94). *It is believed that the birational action of the finite Weyl groups $W_{n,m}$ on P_n^m can be extended by a biregular action on $\hat{P}_n^m$. The quotient $\hat{P}_2^m/W_{2,m}$ ($m \leq 8$) would be a certain compactification of the moduli space $\mathcal{M}_{DP}(m)$.*

6

Appendix

Let k be an algebraically closed field. This section treats of the notion of nonsingular projective curve C_K with function field equal to K, where K is a finitely generated extension field of k with transcendence degree 1 (function field of dimension 1).

Remark 6.0.1. *Before we define what is a nonsingular projective curve C_K, it is important to recall some definitions and properties that will enable us to have a better understanding of what the latter are. One knows, that for any given non singular curve Y and a point $P \in Y$, the local ring $\mathcal{O}_P$ of Y at P will be a regular local ring of dimension one ([6], chapter 1). Thus by ([4], chapter 12) $\mathcal{O}_P$ is a discrete valuation ring whose quotient field is the function field K of Y, and since $k \subseteq \mathcal{O}_P$, hence it is a valuation ring of K/k.*

Definition 6.0.2. *Let Y be a nonsingular curve Y and $P \in Y$, we define the set C_K as the set of all discrete valuation rings of K/k.*

Remark 6.0.3. *Thus set of the local rings of Y is a subset of C_K. the set C_K is a topological space by taking its closed sets the finite subsets and the whole space itself.*

Definition 6.0.4. *An abstract nonsingular curve is an open subset $U \subseteq C_K$, where K is a function field of dimension over k.*

Theorem 6.0.5 ([6], page 44)**.** *Let K be a function field of dimension 1 over k. Then the abstract nonsingular curve C_K defined above is isomorphic to a nonsingular projective curve.*

Proof. See [[6], page 42] □

References

[1] O. Bergvall, *Commutative algebra and algebraic geometry*, 2005. [Online]. Available: https://docs.google.com/document/d/1QPGg_Pd3oVZ8U1nGX3d8W_MSyKOSKyNhfTOpiM-mHjg/edit

[2] C. De Concini and C. Procesi, "A characteristic free approach to invariant theory," in *Young Tableaux in Combinatorics, Invariant Theory, and Algebra*. Elsevier, 1982, pp. 169–193.

[3] I. V. Dolgachev and D. Ortland, *Point sets in projective spaces and theta functions*. Société mathématique de France, 1988, vol. 165.

[4] A. Gathmann, *commutative algebra*, 2002. [Online]. Available: https://www.mathematik.uni-kl.de/~gathmann/en/commalg.php

[5] A. Grothendieck, "Éléments de géométrie algébrique (rédigés avec la collaboration de jean dieudonné): Ii. étude globale élémentaire de quelques classes de morphismes," *Publications Mathématiques de l'IHÉS*, vol. 8, pp. 5–222, 1961.

[6] R. Hartshorne, *Algebraic geometry*. Springer Science & Business Media, 2013, vol. 52.

[7] H. Matsumura, *Commutative ring theory*. Cambridge university press, 1989, no. 8.

[8] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory.*, 3rd ed., ser. Ergeb. Math. Grenzgeb. Berlin: Springer-Verlag, 1994, vol. 34. [Online]. Available: hdl.handle.net/2433/102881

[9] P. Newstead, "Introduction to moduli problems and orbit spaces. tifr lect," *Notes*, vol. 51, 1978.

[10] Z. Reichstein, "Stability and equivariant maps," *Invent. Math.*, vol. 96, no. 2, pp. 349–383, 1989.

[11] I. R. Šafarevič, *Basic algebraic geometry. 2,[Schemes and complex manifolds]*. Springer, 1996.

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