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A modern approach to De Giorgi's Elliptic regularity theory and its application to Navier-Stokes system and Stochastic PDE's

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Introduction

One widely renowned technique in Mathematical Analysis for enhancing regularity of solutions is ε-regularity. Its applications encompass various fields and problems but can be divided into two main groups: Partial Differential Equations and Minimal Surface Theory. An ε-regularity argument involves proving the existence of a global constant $\varepsilon > 0$ such that, if the energy of the selected solution of our PDE is smaller than this constant then the solution is bounded. In the field of Minimal Surfaces, this technique translates into Allard's type regularity argument. In this case, this energy is also known as "excess" and measures the difference in norm between the approximate tangent plane of the rectifiable set and a fixed plane (see [ACM14], [De 17] for the theory of Minimal Surfaces).

In this thesis, we focus on the application of ε-regularity results in the context of Partial Differential Equations. One of the significant applications of this technique, marking its very birth, is undoubtedly the solution to Hilbert's XIX problem by Ennio De Giorgi. In his article "Sulla differenziabilità e analiticità delle estremali degli integrali multipli regolari" [De 57], he showed that, if u is a Sobolev function belonging in H^1 that solves a classical elliptic equation of type

$$\operatorname{div}(A\nabla u) = 0, \tag{1}$$

(with suitable ellipticity assumptions, see Theorem 1.1) then there exists a constant ε such that, if the energy $U = \|u\|_{L^2(B(1))}$ is smaller than ε, then $\|u\|_{L^\infty(B(1/2))}$ is finite. In other words, he was able to reverse the classical Hölder's inequality, obtaining

$$\|u\|_{L^\infty(B(1/2))} \lesssim \|u\|_{L^2(B(1))}.$$

In the first chapter, we show how to obtain such an inequality. To do this, we follow a more modern approach than the original one by De Giorgi, using instead some ideas from the works of L. Caffarelli and A. Vasseur, published between 2005 and 2015 (in particular [CV10], [Vas14]). For the proof a specific setting is necessary and we postpone the details to the body of this thesis. For the sake of this introduction, we just sketch the main idea: a decay estimate which is also a recurring theme throughout this thesis. Let us introduce the solution truncations and the related truncated energies:

$$u_k = [u - (1 - 2^{-k})]_+ \quad \text{and} \quad U_k = \int_{B_k} u_k^2.$$

where with B_k we indicate a ball centred at the origin with radius $1/2(1+2^{-k})$.
The core of the proof is to show the decay of the sequence of the truncated energies. In particular we show that there exists β strictly greater than 1 such that:

$$0 \leq U_{k+1} \leq C U_k^{\beta} \quad \forall k \in \mathbb{N}. \tag{2}$$

To prove such a decay we follow the original idea of Ennio De Giorgi: we employ a combination of Sobolev embeddings, Chebyshev's inequality, and energy inequality to derive the energy decay described in Equation (2).

Once it is shown that the truncated energy decays geometrically with an exponent greater than one, we can conclude that if the initial energy term is sufficiently small, i.e. $U_0 \leq \varepsilon$, then the sequence will tend to zero, i.e. $U_\infty = 0$. In this way, using the definition of U_k, we obtain that

$$U_0 = \|u_+\|_{L^2(B(1))} \leq \varepsilon \quad \Rightarrow \quad U_\infty = 0,\ i.e.\ u \leq 1 \text{ a.e in } B(1/2).$$

If Chebyshev's inequality and Sobolev's embeddings are valid for general functions the energy inequality is strongly related to our PDE and is the point where the main difficulties arise. In the literature it takes different names depending on the context in which it is used: in the case of the elliptic equation, for example, it is called Caccioppoli-Leray while, for Minimal Surfaces, it is called Tilt-Excess inequality. The common feature of the various energy inequalities we will see is the idea of how to obtain them: we will test the weak formulation of our PDE with a function that itself depends on the solution.

We briefly sketch here the plan of this thesis.

- In Chapter I, we present a modern proof of XIX Hilbert's problem's solution using an ε-regularity argument. We show that if $u \in H^1_{\text{loc}}$ solves (1), then u is locally α-Hölder continuous. In particular, we use the Caccioppoli-Leray inequality to prove the decay
$$U_{k+1} \leq C 2^{4k} U_k^{1+\frac{2}{n}}.$$
Then, since $1+\frac{2}{n}$ is larger than the critical value 1, we can reverse Hölder's inequality. To conclude the proof, we establish an oscillation decay estimate of the form
$$\operatorname{osc}_{B_{1/2}} u \leq \lambda \operatorname{osc}_{B_1} u,$$
with $\lambda \ll 1$. Once we have this oscillation estimate, we can easily conclude that u is Hölder continuous. This last part is ultimately based on the isoperimetric inequality.

- In Chapter II, we show the solution a counterpart of XIX Hilbert's problem in the parabolic setting. If $u \in L^\infty((0,T); L^2(\Omega)) \cap L^2((0,T); H^1(\Omega))$ is a weak solution to
$$\partial_t u - \operatorname{div}\big(A(t,x)\nabla_x u\big) = 0 \quad \forall (t,x) \in (0,T) \times \Omega \tag{3}$$

then it is Hölder continuous. We proceed in a similar way as we did in Chapter I. The main difference is the Energy inequality: the different geometry of the problem and the different initial regularity of the solution suggest us the new choice of energy and allow us to prove a Parabolic energy inequality which works in the same way as the Caccioppoli-Leray's inequality. Indeed it reverses the Sobolev embedding and controls the gradient of the solution with the solution itself.

- In Chapters III and IV, we deal with the Navier-Stokes equations:

$$\begin{aligned} &\partial_t u + (u \cdot \nabla) u + \nabla p = \Delta u \quad (t,x) \in (0,T) \times \mathbb{R}^3 \\ &\operatorname{div} u = 0 \end{aligned} \tag{4}$$

  where $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ is the velocity field of an incompressible fluid and $p : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ is its pressure. One among the Millennium problems of the Clay Institute ([Fef00]) asks for a proof of the existence of a smooth solution (u,p) to the system (4).

  Up to now, the most important result in this direction is the Caffarelli-Kohn-Nirenberg theorem [CKN82], which states that the singularities of a particular class of solutions, called suitable solutions, of the Navier-Stokes equations cannot persist along a space-time curve. To prove it, we follow the work of Vasseur, who proposed a new proof based on an ε-regularity argument [Vas07].

  Even though the problem is highly nonlinear, we can adapt the previous scheme to our new setting. With a particular choice of energy and a clever use of the suitability condition, we are able to prove the following energy decay:

$$U_k \leq C^k (1 + \|p\|_{L^{p,1}([-1,1] \ \times B(1))}) U_{k-1}^{\beta},$$

  which guarantees our thesis. For the sake of readability, we do not report here the choice of Energy and the energy inequality, but there are many similarities with the parabolic case. Besides the energy inequality, a new problem arises in the Navier-Stokes system: the non-locality of the pressure conflicts with our choice of shrinking cylinders (the parabolic counterparts of the previous shrinking balls B_k). However, an intelligent pressure decomposition between a local part and a harmonic one guarantees the applicability of the Calderón-Zygmund theory to obtain the necessary estimates.

- In Chapter V, we add a generalized Wiener process to the equation from Chapter II and demonstrate how, under reasonable assumptions, Hölder continuity can be achieved (main reference [HWW20]). The stochastic partial differential equation we consider is given by:

$$\partial_t u = \operatorname{div}(A \nabla u) + f(t,x,u) + \sum_{i \geq 1} g_i(t,x,u) \dot{w}_t^i.$$

  In this case the energy inequality is derived using the Itô formula for stochastic processes in Hilbert spaces. and energy decay takes the form:

$$U_{k,a} \leq \frac{C^k}{a^{2/(n+1)}}(U_{k-1,a} + X^*_{k-1,a})U^{1/(n+1)}_{k-1,a},$$

where the term $X^*_{k-1,a}$ collects the non-deterministic part of the problem. By employing martingale tail control of the stochastic integral, we establish a relationship between the energy $U_{k-a,a}$ and the stochastic term $X^*_{k-1,a}$. Once we establish the connection between these two terms, we can demonstrate the stochastic counterpart of the reverse Hölder inequality:

$$\mathbb{P}\left\{\|u^+\|_{L^\infty([T,2T]\times\mathbb{R}^n)} > a, M\|u^+\|_{L^{4,2}([0,2T]\times\mathbb{R}^n)} \leq a\right\} \leq e^{-M^{1/(n+1)}},$$

which allows us to conclude the Hölder continuity of the solution.

In conclusion, studying the detailed proof of Hilbert's XIX problem allows us to understand that the depth of ideas required for its resolution goes beyond the usual techniques used for elliptic regularity. In fact, ε-regularity, although originally used in the context of linear PDEs, can be adapted to heavily nonlinear contexts such as the Navier-Stokes equations or even in the presence of non-deterministic components as in the case of stochastic PDEs. Therefore, we can confidently state that De Giorgi's ideas are still fundamental today in various problems in analysis.

Chapter 1

De Giorgi's scheme for XIX Hilbert's problem

De Giorgi's solution to Hilbert's XIX problem is a significant achievement in the theory of regularity for partial differential equations. In this chapter, we delve into his solution, highlighting the key steps. While the original article [De 57] is highly precise, grasping the brilliant ideas behind the solution can be challenging.

In more recent years (around 2000), a modern approach to De Giorgi's theory was developed by L. Caffarelli and A. Vasseur. They approached the problem from a unique perspective, enabling them to identify precisely where the ellipticity of the problem was utilized. This allowed them to isolate the underlying framework of the original solution and extend it to a more general context. The main references for this modern approach are [Vas14] and [CV10]. The solution we present follows their works, and we also outline the general ideas behind De Giorgi's scheme.

Theorem 1.1 (De Giorgi)**.** *Fix $A : \Omega \subset \mathbb{R} \to \mathbb{R}^{n\times n}$ a Borel function. Suppose $\lambda I \leq A(x) \leq \Lambda I$ (I is the identity matrix) almost everywhere on Ω for $\lambda, \Lambda \in \mathbb{R}$. If $u \in H^1_{loc}(\Omega; \mathbb{R})$ solves in the sense of distribution*

$$Lu = -\operatorname{div}(A\nabla u) = 0 \tag{1.1}$$

then there exists a constant $\alpha = \alpha(\lambda, \Lambda, n)$ such that for every $\Omega' \Subset \Omega$, u is α-Hölder continuous in Ω'.

The work is organized in two steps:

1. The Height estimate (also called $L^\infty - L^2$ estimate). Thanks to the study of the behaviour of the energy between the super level sets of the solution u we prove the estimate
$$\|u\|_{L^\infty(B(1))} \lesssim_A \|u\|_{L^2(B(2))}.$$
2. Oscillation decay. With a more geometric approach we obtain
$$\|u\|_{C^\alpha(B(1/2))} \lesssim_A \|u\|_{L^\infty(B(1))}$$
where the balls are centred in the origin.

Before going into the details we spend some words about the procedure of zooming/scaling. This is a very general approach when dealing with PDE's and it will be also useful in the next chapters.

Zooming

We start with some considerations about the matrix A and the choice of the domains Ω, Ω'. When we speak about the ellipticity condition of the matrix A we refer to the bounds
$$\lambda I \leq A(x) \leq \Lambda I$$
which hold x-almost everywhere.

Consider L a constant $n \times n$ matrix with real coefficients and $c \in \mathbb{R}^n$. Then the function $v(x) = u(L(x) + c)$ solves
$$-\operatorname{div}(A'\nabla v) = 0$$
with $A' = A(L(x) + c)$. The interesting fact is that A' and A satisfy the same ellipticity condition (since the condition is uniform). This implies that both, u and v, have the same Holder constant α.

With this observation, we can notice that is not restrictive to take $\Omega = B(2)$ and $\Omega' = B(1/2)$. Indeed suppose we have proved the theorem for such domains and consider now general Ω and Ω'. Set $d = \operatorname{dist}(\Omega', \Omega^c)$ and for every $x_0 \in \Omega'$ set
$$u'(x) = u\left(x_0 + \frac{d}{2}x\right), \quad x \in B(2).$$

By previous considerations, note that u' solves (1.1) with matrix $A'(x) = A\,(x + dy)$ which verifies the same uniform elliptic estimates. By hypothesis u' is C^α in $B(1/2)$ and u is

C^α in Ω'. To be precise this conclusion is correct because α does not depend on d since, in the theorem, α does not depend on the choice of Ω' or Ω.

Remark. *An important remark, applicable to the entire thesis, is that we are considering uniform conditions, such as the ellipticity condition. This ensures that our problem behaves consistently under translations, as suggested in the Zooming section. Thanks to this assumption, going forward, we will assume that all the balls are centered at the origin unless explicitly stated otherwise.*

1.1 The Height estimate

The goal of the first part of the proof is to verify the following ε-regularity statement:

Proposition 1.1. *If u solves (1.1) there exists a constant $\varepsilon = \varepsilon(n, A)$ such that if $||u_+||^2_{L^2(B_1)} < \varepsilon$, then*

$$\|u_+\|_{L^\infty(B(1/2))} := (\text{ess})\sup_{B(1/2)} u \leq 1 \tag{1.2}$$

where $u_+ = \max(u, 0)$.

The Height estimate is a consequence of this proposition and it provides a relationship between the L^∞ norm and the L^2 norm of a function. Specifically, it states that if a function satisfies certain conditions and its L^2 norm is bounded on a certain set, then its L^∞ norm is also bounded on a smaller set:

Corollary 1.1 (Height Estimate)**.** *If u solves (1.1) there exists a constant $\varepsilon = \varepsilon(n, A)$ such that*

$$\|u\|_{L^\infty(B(1))} \leq \frac{1}{\varepsilon} \|u\|_{L^2(B(2))}.$$

Proof. For any $x \in B(1/2)$, consider

$$\tilde{u}(y) = \frac{\varepsilon}{\|u\|_{L^2(B(2))}} u(x+y),$$

which is still a solution for the considerations of the Zooming section. Notice also that $\|\tilde{u}\|_{L^2(B(1))} < \varepsilon$ and so, by Proposition 1.1,

$$\tilde{u}(y) \leq 1 \quad \text{for } y \in B(1/2).$$

We get the bound from below by applying the same result to $-\tilde{u}$. Collecting the two estimates we write

$$\|\tilde{u}\|_{L^\infty(B(1/2))} \leq 1.$$

Now using the arbitrariness of x in $B(1/2)$ we have the thesis. □

Notice that the thesis of Corollary 1.1 holds immediately if u is a harmonic function: it is trivially implied by the mean value property. Even if we are starting from the weaker hypothesis of u just solving the elliptic equation (1.1) instead of being harmonic, we are going to prove that we have the same property. To put it differently, even without the general mean value property, if $Lu = 0$ (for an elliptic operator L), we can still establish the reverse Hölder's inequality stated in Corollary 1.1. This highlights the power of the constraints in compensating for the absence of the mean value property and enabling the derivation of such inequalities..

We underline that we are using the constraints $Lu = 0$ to find an inequality that competes with a universally applicable inequality (such as Hölder's inequality). This parallel brings to mind the Caccioppoli-Leray where the ellipticity assumptions allow us to find an inequality which rivals with Sobolev embeddings (that holds for every general function). In both cases, the prize we must pay to get the reverse inequalities is the restriction of the domain of the left-hand side.

The settings

To prove Proposition 1.1 the idea is to study the decay of the energy between super-level sets of u. For this purpose, we introduce the truncation u_k

$$u_k = [u - (1 - 2^{-k})]_+ \tag{1.3}$$

and a sequence of decreasing balls defined as

$$B_k = B\left(\frac{1}{2}(1 + 2^{-k})\right).$$

The importance of this choice is that $B_0 = B(1)$ and the sequence of radii is decreasing up to $B_\infty = B(1/2)$. With these two, we can define the energy U_k as

$$U_k = \int_{B_k} u_k^2 = \|u_k\|^2_{L^2(B_k)}.$$

Truncation, shrinking domains, and the choice of energy are the key components in setting up De Giorgi's scheme. As we delve into the subsequent chapters, these quantities will undergo transformations influenced by the geometry of the problem, while preserving their original purpose.

The strategy now is to apply the following crucial Lemma to the energy U_k:

Lemma 1.1 (Geometric decay). *Suppose that a sequence satisfies*

$$0 \leq U_{k+1} \leq \alpha\, C^k U_k^\beta \tag{1.4}$$

for $C > 1$*,* $\alpha > 0$ *and* $\beta > 1$ *then there exists a constant* ε *such that if* $0 < U_0 < \varepsilon$ *then*

$$\lim_{k \to +\infty} U_k = 0.$$

Supposing that we are able to prove that U_k has such a decay, then if $U_0 = \|u_+\|_{L^2(B(1)} < \varepsilon$ we have $U_k \to 0$, which implies

$$\int_{B(1/2)} [u-1]_+^2 = 0 \quad \Longrightarrow \quad \text{ess sup}_{B(1/2)}\, u \leq 1.$$

Hence, to prove the Proposition 1.1, we have to verify that U_k satisfies (1.4). Before proving this fact we spend some comments on the Lemma 1.1 since it will be useful also in other applications.

Geometric decay

The Lemma 1.1 says that if our sequence decays faster than a geometric sequence we can choose a first term U_0 small enough in such a way that the sequence U_k goes to 0 at infinity. It is a consequence of the trivial fact that $x^n \to 0$ if $x < 1$.
Proving that β is strictly greater than 1 is what guarantees us the convergence of the sequence U_k. Now we give an elementary proof of the Geometry decay Lemma:

Proof. Let us denote

$$V_k = C^{\frac{k}{\beta-1}} C^{\frac{1}{(\beta-1)^2}} U_k.$$

By hypothesis we have

$$0 \leq V_{k+1} \leq C^{\frac{k+1}{\beta-1}} C^{\frac{1}{(\beta-1)^2}} \alpha C^k U_k^\beta = \alpha\, C^{\frac{k\beta}{\beta-1}} C^{\frac{\beta}{(\beta-1)^2}} U_k^\beta = \alpha\, V_k^\beta$$

and, in particular

$$V_k \leq \alpha^t (V_0)^{\beta k}.$$

Developing the computation in U instead of V we get

$$C^{\frac{k}{\beta-1}} C^{\frac{1}{(\beta-1)^2}} U_k \leq \alpha^t (C^{\frac{1}{(\beta-1)^2}} U_0)^{\beta k} \quad \Longrightarrow \quad U_k \lesssim \left(\alpha\, C^{\frac{1}{\beta-1}} C^{\frac{\beta}{(\beta-1)^2}} U_0^\beta\right)^k.$$

Hence, for a small enough U_0, we have that $\left(\alpha\, C^{\frac{1}{\beta-1}} C^{\frac{\beta}{(\beta-1)^2}} U_0^\beta\right) < 1$ and so $U_k \to 0$. □

Energy decay estimate

We are left to prove that our U_k decays faster than a geometric sequence. Overcoming the gap between 1 and $\beta > 1$ is where difficulties arise (notice that the inequality $U_{k+1} \leq U_k$ is trivial). To this aim, truncation plays the leading role: we are going to see that taking only the positive part allows us to use a combination of Hölder and Chebychev's inequality to put our exponent β over the value 1.

In particular, we are going to verify the decay for these exponents

$$U_{k+1} \leq C 2^{4k} U_k^{1+\frac{2}{n}}. \tag{1.5}$$

To obtain this result, we need to combine three theorems: Sobolev embeddings, Energy inequality, and Chebyshev's inequality. We would like to note in advance that when we refer to Sobolev embeddings, we also tacitly include the use of the Poincaré inequality when the function is compactly supported. We present here the Energy inequality that we will use, but its proof is postponed.

Theorem. *(Elliptic Energy inequality) Suppose $u \geq 0$ is a weak subsolution to (1.1) in $\Omega \subset \mathbb{R}^n$ ($Lu \leq 0$). Then for every $\varphi \in C_c^\infty(\Omega)$ we have*

$$\int_\Omega \left(\nabla\left(\varphi u_+\right)\right)^2 dx \leq C \int_{\Omega \,\cap\, spt\, \varphi} u_+^2 |\nabla\varphi|^2 dx$$

with C depending only on the matrix A.

We begin considering the following family of test functions:

$$\varphi_k \equiv 1 \quad \text{in } B_k,$$
$$\varphi_k \equiv 0 \quad \text{in } B_{k-1}^c,$$
$$|\nabla\varphi_k| \leq C2^k.$$

Noticing that $\varphi_k \equiv 1$ in B_k and that $\{\varphi_k u_k > 0\} = \{\varphi_k u_k \neq 0\}$ (the functions φ_k and u_k are positive) we have

$$U_{k+1} \leq \int \left(\varphi_{k+1} u_{k+1}\right)^2 = \int \left(\varphi_{k+1} u_{k+1}\right)^2 \mathbf{1}_{\{\varphi_{k+1} u_{k+1} > 0\}}.$$

Since

$$\frac{1}{p^*/2} + \frac{1}{n/2} = \frac{2(n-2)}{2n} + \frac{2}{n} = 1$$

we can use Hölder's inequality on the right-hand side with exponents $p^*/2$ and $n/2$:

$$U_{k+1} \leq \left[\int \left(\varphi_{k+1} u_{k+1}\right)^{p^*}\right]^{2/p^*} \cdot \left|\{\varphi_{k+1} u_{k+1} > 0\}\right|^{\frac{2}{n}}. \tag{1.6}$$

This inequality is the starting point to prove that our energy decays faster than a sequence with exponent 1. We stress that without taking the positive part in the truncation we would not have the second term on the right-hand side of (1.6). Now we work separately with the two terms on the right-hand side. The reader is encouraged to pay close attention to both of them.

To get the estimate on the first term we use the Sobolev embeddings (for a proof of $u_k \in H^1$ see [ACM10]):

$$\left[\int (\varphi_{k+1} u_{k+1})^{p^*}\right]^{2/p^*} \leq C \int (\nabla(\varphi_{k+1} u_{k+1}))^2 .$$

The truncation u_{k+1} is positive and $Lu_{k+1} = 0$, hence we can use the Energy inequality to get

$$C \int (\nabla(\varphi_{k+1} u_{k+1}))^2 \leq C \int_{\operatorname{spt} \varphi_{k+1}} u_{k+1}^2 |\nabla \varphi_{k+1}|^2 \leq C 2^{2k} \int_{\operatorname{spt} \varphi_{k+1}} u_{k+1}^2 .$$

Since $u_{k+1} \leq u_k$ and $\operatorname{spt}(\varphi_{k+1}) \subset B_k$ (by definition of φ_{k+1}) we obtain

$$\left[\int (\varphi_{k+1} u_{k+1})^{p^*}\right]^{2/p^*} \leq C 2^{2k} \int_{B_k} u_k^2 \leq C 2^{2k} U_k. \tag{1.7}$$

The opposite action of the Sobolev embeddings and the Energy inequality manages to get the desired estimate. As we can see, we have not increased the exponent of U_k yet, but the second term of (1.6) plays an important role to this aim. Notice that

$$\varphi_{k+1}(x) > 0 \Rightarrow x \in B_k \Rightarrow \varphi_k(x) = 1,$$

$$u_{k+1} > 0 \Rightarrow u > 1 - 2^{-(k+1)} \Rightarrow u > 1 - 2^{-k} + 2^{-(k+1)} \Rightarrow u_k > 2^{-(k+1)}$$

and so we have

$$\{\varphi_{k+1} v_{k+1} > 0\} \subset \left\{\varphi_k v_k > 2^{-(k+1)}\right\} .$$

Hence, by Chebychev, we conclude

$$|\{\varphi_{k+1} u_{k+1} > 0\}|^{\frac{2}{n}} \leq \left|\{\varphi_k u_k > 2^{-(k+1)}\}\right|^{\frac{2}{n}} \leq C 2^{\frac{4k}{n}} \left(\int (\varphi_k u_k)^2\right)^{\frac{2}{n}} = C 2^{\frac{4k}{n}} U_k^{\frac{2}{n}}. \tag{1.8}$$

By combining the two estimates (1.7) and (1.8) with (1.6), we obtain the thesis:

$$U_{k+1} \leq C 2^{4k} U_k^{1+\frac{2}{n}} .$$

Energy inequality

As we have seen Energy inequality competes with the Sobolev Embeddings, namely it has to control the norm of the gradient with the norm of the solution. We have already explained how Corollary 1.1 is the reverse of Hölder inequality. In other words, to find the reverse Hölder we need to reverse first the Sobolev embeddings and the information of this inversion is contained in the Energy inequality.

Notice also that we have not used any hypothesis about u being a solution to the elliptic problem. The fact that u solves (1.1) is used only to obtain the Energy inequality. The proof we present uses the same arguments of the Caccioppoli-Leray inequality, in particular, we test the weak solution to $Lu = 0$ with a function depending on the solution itself.

Theorem 1.2. *(Elliptic Energy inequality) Suppose $u \geq 0$ is a weak subsolution to (1.1) in $\Omega \subset \mathbb{R}^n$ ($Lu \leq 0$). Then for every $\varphi \in C_0^\infty(\Omega)$ we have*

$$\int_\Omega (\nabla(\varphi u))^2\, dx \leq C \int_{\Omega\, \cap\; spt\; \varphi} u^2 |\nabla \varphi|^2 dx$$

with C depending only on the matrix A.

Proof. We multiply $Lu = -\operatorname{div}(A\nabla u)$ by $\varphi^2 u > 0$ and we integarte by parts:

$$\int_\Omega A\nabla u \cdot \nabla(\varphi^2 u) \leq 0.$$

Now we use $\nabla(\varphi^2 u) = \varphi\nabla(\varphi u) + \varphi u \nabla\varphi$ and we re-write the left hand side as

$$\int_\Omega A\nabla u \cdot \nabla(\varphi^2 u) = \int_\Omega A(\varphi\nabla u)\cdot\nabla(\varphi u) + \int_\Omega \varphi u A\nabla u \cdot \nabla\varphi$$

and developing the first term of the right-hand side we get

$$\int_\Omega A\nabla u \cdot \nabla(\varphi^2 u) = \int_\Omega A\nabla(\varphi u)\cdot\nabla(\varphi u) - \int_\Omega uA\nabla\varphi\cdot\nabla(\varphi u) + \int_\Omega \varphi u A\nabla u\cdot\nabla\varphi \leq 0$$

or, equivalently,

$$\int_\Omega A\nabla(\varphi u)\cdot\nabla(\varphi u) \leq \int_\Omega uA\nabla\varphi\cdot\nabla(\varphi u) - \int_\Omega \varphi u A\nabla u\cdot\nabla\varphi. \tag{1.9}$$

The right-hand side is equal to

$$\int_\Omega u\varphi A\nabla\varphi\cdot\nabla u + \int_\Omega u^2 A\nabla\varphi\cdot\nabla\varphi - \int_\Omega \varphi u A\nabla u\cdot\nabla\varphi.$$

The second term can be bound using the ellipticity condition. For the other two, we notice

$$\int_\Omega u\varphi A\nabla\varphi\cdot\nabla u-\int_\Omega \varphi u A\nabla u\cdot\nabla\varphi=\int_\Omega u\nabla(\varphi u)\cdot A\nabla\varphi-\int_\Omega uA\nabla(\varphi u)\cdot\nabla\varphi$$

and using, again the ellipticity of A:

$$\left|\int_\Omega u\nabla(\varphi u)\cdot A\nabla\varphi-uA\nabla(\varphi u)\cdot\nabla\varphi\right|\leq 2\Lambda\left\|\nabla\left(\varphi u\right)\right\|_{L^2}\left\|u\nabla\varphi\right\|_{L^2}.$$

Since $\lambda I\leq A(x)$ we can estimate with

$$\frac{2\Lambda}{\lambda}\left(\int\nabla\left(\varphi u\right)\cdot A\nabla\left(\varphi u\right)\right)^{1/2}\left\|u\nabla\varphi\right\|_{L^2}. \tag{1.10}$$

The key idea is to use the Young's inequality (which holds for every ε):

$$ab\leq\frac{\varepsilon a^p}{p}+\frac{b^q}{\varepsilon q}\quad(\text{with }1/p+1/q=1).$$

Then (1.10) is less or equal then

$$\frac{1}{2\varepsilon^2}\int\nabla\left(\varphi u\right)\cdot A\nabla\left(\varphi u\right)+2\frac{\Lambda^2\varepsilon^2}{\lambda^2}\int_\Omega u^2(\nabla\varphi)^2.$$

Summarizing, we can re-write (1.9) in the form:

$$\int_\Omega A\nabla(\varphi u)\cdot\nabla(\varphi u)\leq\int_\Omega u^2A\nabla\varphi\cdot\nabla\varphi+\frac{1}{2\varepsilon^2}\int\nabla\left(\varphi u\right)\cdot A\nabla\left(\varphi u\right)+2\frac{\Lambda^2\varepsilon^2}{\delta^2}\int_\Omega u^2(\nabla\varphi)^2. \tag{1.11}$$

With an appropriate choice of ε ($\varepsilon=\sqrt{2}$), we can absorb the second term on the left-hand side and we conclude

$$\frac{\lambda}{4}\int_\Omega A\nabla(\varphi u)\cdot\nabla(\varphi u)\leq\left(\Lambda+2\frac{\Lambda^2\varepsilon^2}{\lambda^2}\right)\int_\Omega u^2(\nabla\varphi)^2.$$ □

We conclude the section with a consequence of the previous result.

Corollary 1.2 (Caccioppoli-Leray). *If we consider $\Omega''\Subset\Omega'\Subset\Omega$ then there exists a constant $C=C(A,\Omega'',\Omega')$ such that*

$$\int_{\Omega''}\left(\nabla u\right)^2dx\leq C\int_{\Omega'}u^2dx$$

Proof. Consider $0\leq\varphi\leq 1$ in Ω such that $\varphi\in C_c^\infty(\Omega')$ and $\varphi\equiv 1$ in Ω''. Since $\|\nabla\varphi\|_{\Omega'}$ is bounded we conclude

$$\int_{\Omega''}(\nabla u)^2\leq\int_\Omega(\nabla(\varphi u))^2\leq C\int_{\Omega'}u^2|\nabla\varphi|^2\leq C\int_{\Omega'}u^2$$ □

1.2 Oscillation decay

Once we have obtained the estimate of Corollary 1.1 we can study the behaviour of the oscillations of the solution. As a definition of oscillation of u in a set B, we take

$$\operatorname{osc}_B u = (\text{ess}) \sup_B u - (\text{ess}) \inf_B u.$$

In particular, we are going to prove the following estimate:

Lemma 1.2. *If u solves (1.1) in $B(2)$ then there exists $\delta = \delta(n, A) < 1$ such that*

$$\operatorname{osc}_{B_{1/2}} u \leq \lambda \operatorname{osc}_{B_1} u.$$

We start showing that this lemma implies the α-Hölder continuity of u. Consider $x_0 \in B_{1/2}$ and set the following sequence of functions:

$$u_1(x) = u(x_0 + x/4) \quad u_n(x) = u_{n-1}(x/4).$$

Since δ depends only on (A, n) the functions u_n satisfy the same decay estimate of u (see Zooming).

Now we proceed using recursively the oscillation decay:

$$\begin{aligned}\sup_{|x-x_0|\leq 4^{-n}} |u(x_0) - u(x)| &= \sup_{x\in B_{1/2}} |u(x_0) - u(x_0 + x/4^{n-1})| = \sup_{x\in B_{1/2}} |u(x_0) - u_{n-1}(x)| \leq \\ &\leq \operatorname{osc}_{B_{1/2}} u_{n-1} \leq \delta \operatorname{osc}_{B_2} u_{n-1} \leq \cdots \leq \delta^{n-1} \operatorname{osc}_{B_2} u_1 = \\ &= \delta^{n-1} \operatorname{osc}_{x\in B_2} u(x_0 + x/4) \leq 2\delta^{n-1} \|u\|_{L^\infty(B_1)}.\end{aligned}$$

The right-hand side does not depend on the choice of x_0, hence we have

$$\sup_{|x-y|\leq 4^{-n}} |u(y) - u(x)| \leq 2\delta^{n-1} \|u\|_{L^\infty(B_1)}$$

Now take $x, y \in B_{1/2}$ then $4^{-n} \leq |x - y| \leq 4^{-(n-1)}$ for a particular n. Combining the previous estimates we obtain

$$\frac{|u(y) - u(x)|}{|x - y|^\alpha} \leq 4^{n\alpha} \delta^{n-2} \|u\|_{L^\infty(B_1)}.$$

Choosing $\alpha = -\ln_4 \delta$ we have that $u \in C^\alpha(B_{1/2})$.

Proof of Lemma 1.2

To prove Lemma (1.2) we need an intermediate step. To do this we are going to use that that an H^1 function can not have a jump discontinuity:

Proposition 1.2 (Isoperimetric Inequality for H^1). *Consider u such that $\int_{B_1} |\nabla u_+|^2 \, dx \leq C$ and set*

$$|N| = |\{u \leq 0\} \cap B_1|,$$

$$|B| = |\{u \geq 1/2\} \cap B_1|,$$

$$|M| = |\{0 < u < 1/2\} \cap B_1|.$$

Then we have

$$C|M| \geq \frac{1}{4}\left(|B||N|^{1-\frac{1}{n}}\right)^2.$$

Proof and remarks of this Proposition, which relies on the isoperimetric inequality, are postponed to the last section of this Chapter.

What we are going to prove now, indeed, is that if v is a solution of (1.1), smaller than one in B_1, and is "far from 1 " in a set of non-trivial measure, it cannot get too close to 1 in $B_{1/2}$.

Proposition. *Let v a solution to (1.1) in $B(2)$. Assume $v \leq 1$ and $|B_1 \cap \{v \leq 0\}| \geq \mu > 0$. Then $\sup_{B_{1/2}} v \leq 1 - \lambda$, where λ depends only on μ, A, and n.*

Proof. The strategy now is to utilize the Height Estimate. We aim to derive a contradiction by constructing a sequence of solutions in which the L^2 norm decreases while the supremum does not. This contradicts the essence of the Height Estimate, which states that if the L^2 norm is controlled within a certain region, then the function cannot exhibit excessive growth, and its supremum value remains bounded.

Consider the map $S(x) = 2x - 1$ and set the following sequence of functions:

$$v_0 = v \quad v_{k+1} = S(v_k) \quad \text{or, alternatively} \quad v_k = 2^k v - 2^k + 1.$$

By construction we have

1. $v(x) = 1$ if and only if $v_k(x) = 1$ for at least one k;
2. At every step we decrease the L^2 norm of $(v_k)_+$.

Since v_k remains a solution and the L^2 norm of its positive part is decreasing, according to Corollary 1.1, we expect the supremum of v_k over $B(1/2)$ to decrease as well. Consequently, we can ensure that $v_k < 1$. As 1 is a fixed point of S (refer to Figure), we

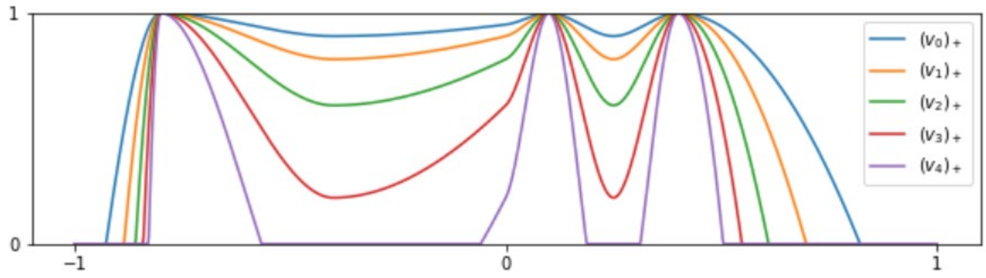

Figure 1.1: *The first iterations of our construction (for a continuous map) when the function v reaches the value* 1.

can conclude that $v < 1$ as well.

In particular, if we can write $v_{k_0} \leq \eta$ (for some k_0) in $B_{1/2}$, for $0 < \eta < 1$, then

$$v < 1 - 2^{-k_0}(1 - \eta)$$

which implies the thesis. To this aim consider a small $\eta > 0$, if we can prove that there exists k_0 such that

$$\int_{B_1} (v_{k_0})_+^2 \, dx \leq \eta \, \varepsilon.$$

then, by Corollary 1.1, $v_k \leq \eta$.

To achieve this, it is important to note that for any k, we have $v_k \leq 1$. Thus, by applying the Corollary 1.2 of with domains $B(1)$ and $B(2)$, we obtain the following inequality:

$$\int_{B_1} \left|\nabla (v_k)_+\right|^2 dx \leq C$$

By hypothesis we have also $|\{v_k \leq 0\} \cap B_1| \geq |\{v \leq 0\} \cap B_1| \geq \mu$. As long as $2v_k$ satisfies

$$\int_{B_1} (v_{k+1})_+^2 \, dx \geq \eta\varepsilon$$

we get, by definition of v_k,

$$|\{2v_k \geq 1\} \cap B_1| = |\{v_{k+1} \geq 0\} \cap B_1| \geq \int_{B_1} (v_{k+1})_+^2 \, dx \geq \eta\varepsilon.$$

Summarizing we have

1. $|\{v_k \leq 0\} \cap B_1| \geq \mu > 0$
2. $|\{v_k \geq 1/2\} \cap B_1| \geq \eta\varepsilon > 0$

So, from Proposition 1.2, there exists a positive constant α, which does not depend on k, such that

$$|\{0 < v_k < 1/2\} \cap B_1| \geq \alpha.$$

Then

$$|\{v_k \leq 0\} \cap B_1| \geq |\{v_{k-1} \leq 0\} \cap B_1| + \alpha \geq \mu + k\alpha.$$

This fails after a finite number of k. At this k_0 we have for sure that

$$\int_{B_1} (v_{k_0+1})_+^2 \, dx \leq \eta\varepsilon.$$

□

We can prove that the previous proposition implies the oscillation decay (1.2), which concludes Hilbert's 19th problem. Take u to be a solution of (1.1) and consider the function:

$$v(x) = \frac{2}{\operatorname{osc}_{B_2} u}\left(u(x) - \frac{\sup_{B_2} u + \inf_{B_2} u}{2}\right).$$

Notice that $-1 \leq v \leq 1$ and

$$\operatorname{osc}_{B_{1/2}} v = \frac{2 \operatorname{osc}_{B_{1/2}} u}{\operatorname{osc}_{B_2} u}.$$

Assume that $|B_1 \cap \{v \leq 0\}| \leq \mu$ (or use $-v$ instead of v). Then we can apply the proposition on v which gives that $\sup_{B_{1/2}} \leq 1 - \lambda$ and so $\operatorname{osc}_{B_{1/2}} v \leq 2 - \lambda$. Hence $\operatorname{osc}_{B_{1/2}} u \leq (1 - \lambda/2) \operatorname{osc} u_{B_2}$.

Conclusion by means of the isoperimetric inequality

In this last part, we prove the Proposition 1.2 that we report:

Proposition. *Consider u such that $\int_{B_1} |\nabla u_+|^2 \, dx \leq C$ and set*

$$|N| = |\{u \leq 0\} \cap B_1|,$$

$$|B| = |\{u \geq 1/2\} \cap B_1|,$$

$$|M| = |\{0 < u < 1/2\} \cap B_1|.$$

Then we have

$$C|M| \geq \frac{1}{4}\left(|B||N|^{1-\frac{1}{n}}\right)^2.$$

As reported in [ACM10] this result is the reason why De Giorgi's regularity argument, even if so analytic, is deeply geometric in spirit. Indeed we are going to see how it is ultimately based on the isoperimetric inequality. The following proof, which is particularly elegant, is taken from [Vas14].

Proof. We set $\bar{u} = \sup(0, \inf(u, 1/2))$ which implies $\nabla\bar{u} = (\nabla u_+)\,\mathbf{1}_{\{0\leq u\leq 1/2\}}$. For x in N and y in B, we have

$$1/2 = \bar{u}(y) - \bar{u}(x) = \int_0^1 (y-x)\cdot\nabla\bar{u}(x+t(y-x))dt \leq \int_0^{|y-x|} |\nabla\bar{u}|\left(x + s\frac{y-x}{|y-x|}\right) ds.$$

Assigning the value 0 to $\nabla\bar{u}$ outside of B_1, we get

$$1/2 \leq \int_0^\infty |\nabla\bar{u}|\left(x + s\frac{y-x}{|y-x|}\right) ds$$

integrating this inequality for all $y \in B$, we get

$$\frac{|B|}{2} \leq \int_B \left(\int_0^\infty |\nabla\bar{u}|\left(x + s\frac{y-x}{|y-x|}\right) ds\right) dy \leq \int_{B_1} \left(\int_0^\infty |\nabla\bar{u}|\left(x + s\frac{y-x}{|y-x|}\right) ds\right) dy.$$

Writing the first integral in polar coordinates for $y-x = r\sigma$, and noticing that the function does not depend on r we get

$$\begin{aligned}\frac{|B|}{2} &\leq \int_0^2 r^{n-1}\int_{\mathbb{S}^1}\left(\int_0^\infty |\nabla\bar{u}|\,(x+s\sigma)\,ds\right)d\sigma dr \leq \int_{\mathbb{S}^1}\int_0^\infty |\nabla\bar{u}|\,(x+s\sigma)\,dsd\sigma \\ &= \int_{\mathbb{S}^1}\int_0^\infty s^{n-1}\frac{|\nabla\bar{u}|\,(x+s\sigma)}{s^{n-1}}dsd\sigma = \int_{B(1)}\frac{|\nabla\bar{u}|(y)}{|x-y|^{n-1}}dy.\end{aligned}$$

Now integrating the variable $x \in N$ we get

$$\frac{|N||B|}{2} \leq \int_{B(1)} |\nabla\bar{u}|(y)\left(\int_N \frac{1}{|x-y|^{n-1}}dx\right)dy$$

By the isoperimetric inequality, the integral in dx is maximized by the ball of radius $|N|^{1/n}$ centred in y. In particular

$$\frac{|N||B|}{2} \leq |N|^{1/n}\int_{B(1)} |\nabla\bar{u}|(y)dy.$$

Now we use Hölder's inequality to obtain

$$\int_{B(1)} |\nabla\bar{u}|(y)dy \leq \left(\int_M |\nabla u_+|^2\right)^{1/2} |M|^{1/2}.$$ □

Chapter 2

Parabolic case

We will now show how to establish the Hölder regularity of the solution in the parabolic setting. This analysis builds upon the same ideas presented in the previous chapter. The objective is to become acquainted with the methodology before delving into more intricate problems such as the Navier-Stokes equations or the stochastic heat equation.

The equation we are going to consider is the following

$$Lu = \partial_t u - \operatorname{div}\big(A(t,x)\nabla_x u\big) = 0 \quad \forall (t,x) \in (0,T)\times\Omega \tag{2.1}$$

where $A : (0,T)\times\Omega \to \mathbb{R}^{n\times n}$ is a Borel function that satisfies

$$\lambda I \leq A(t,x) \leq \Lambda I \quad \forall (t,x) \in (0,T)\times\Omega. \tag{2.2}$$

A weak solution of (2.1) is a function $u \in L^\infty((0,T);L^2(\Omega)) \cap L^2((0,T);H^1(\Omega))$ satisfing (2.1) in a distributional sense. For simplicity, when we speak about weak solutions we implicitly assume that A satisfies (2.2).

The regularity result we are going to prove is:

Theorem 2.1. *If u is a weak solution of (2.1) in $(0,T)\times\Omega$ then there exists a constant $0 < \alpha < 1$ such that, for every $0 < s < T$ and $\Omega' \Subset \Omega$, u is α-Holder continous in $(s,T)\times\Omega'$.*

According to Remark 1 and using the Zooming technique of the previous Chapter one can check that the thesis is equivalent to proving that if u is a solution of (2.1) in $(-2,1)\times B(2)$ then $u \in C^\alpha((-1/2,1)\times B(1/2))$[1].

[1] Dealing with negative times is just a mathematical choice to simplify the readability of the proof

2.1 De Giorgi's scheme for parabolic equation

The proof follows the pattern introduced for the elliptic case. The main goal of this section is the parabolic counterpart of Proposition 1.1:

Theorem 2.2. *If u is a weak solution of (2.1) and A satisfies (2.2) then there exists $\varepsilon > 0$ such that if $\|u_+\|^2_{L^2(Q_1)} \leq \varepsilon$ then*

$$u_+ \leq 1/2 \quad \text{in } Q'_{1/2}$$

The setting

Since we are in a parabolic context it is more natural to consider cylinders instead of balls:

$$Q_k = (T_k, 1) \times B_k$$

where $B_k = B(\frac{1}{2}(1 + 2^{-k}))$ and

$$T_k = -\frac{1}{2}(1 + 2^{-k}).$$

This time we have $Q_0 = (-1, 1) \times B(1)$ and $Q_\infty = (-1/2, 1) \times B(1/2)$.
As in the elliptic case we set the energy

$$U_k = \int_{Q_k} |u_k|^2_+ dxdt$$

with u_k the truncation

$$u_k = [u - (1 - 2^{-k})]_+.$$

Parabolic Energy inequality

We remind the importance of this equation. As we have already said we need to control the norm of the gradient with the norm of the solution. In the parabolic case, we have the following result.

Theorem 2.3. *(Parabolic Energy inequality) Suppose $u \geq 0$ is a weak subsolution to (2.1) in $\Omega \subset \mathbb{R}^n$ ($Lu \leq 0$). Then for every $\varphi \in C_0^\infty(\Omega)$ and every $s \leq t$ we have*

$$\left(\int_\Omega \varphi^2 u^2\right)(t) + \int_s^t \int_\Omega (\nabla(\varphi u))^2 \leq \left(\int_\Omega \varphi^2 u^2\right)(s) + C \int_s^t \int_\Omega u^2 |\nabla \varphi|^2 \tag{2.3}$$

with C depending only on the matrix A.

Proof. We multiply (2.1) by $\varphi^2 u$ and we use the divergence theorem in space. In this way we get

$$\left(\int_\Omega \varphi^2 u^2\right)(t) + \int_s^t \int_\Omega \nabla\left(\varphi^2 u\right) \cdot A\nabla_x u \leq \left(\int_\Omega \varphi^2 u^2\right)(s).$$

For the second term of the left inside, we proceed (fixing the time) in the same way as the Elliptic Energy Inequality. □

With some extra work, we can prove this Corollary, whose importance will be clear in the next part. A very similar collection of ideas will be used also for the Navier-Stokes equations.

Corollary 2.1. *With the same hypothesis of the previous theorem we have*

$$\sup_{T_{k+1}\leq t\leq 1}\left(\int_\Omega \varphi^2 u^2\right)(t) + \int_{T_{k+1}}^1 \int_\Omega \left(\nabla\left(\varphi u\right)\right)^2 \leq 2^{k+2}\int_{T_k}^{T_{k+1}}\int_\Omega \varphi^2 u^2 + C\int_{T_k}^1\int_\Omega u^2|\nabla\varphi|^2$$

where T_k is defined as before and $\varphi \in C_0^\infty(\Omega)$.

Proof. The idea is to restrict to the case $T_k \leq s \leq T_{k+1} \leq t \leq 1$. Using this range we can write (2.3) in the form

$$\left(\int_\Omega \varphi^2 u^2\right)(t) + \int_{T_{k+1}}^t \int_\Omega \left(\nabla\left(\varphi u\right)\right)^2 \leq \left(\int_\Omega \varphi^2 u^2\right)(s) + C\int_{T_k}^1\int_\Omega u^2|\nabla\varphi|^2.$$

Now we average between T_k and T_{k+1} in the variable s and we obtain

$$\left(\int_\Omega \varphi^2 u^2\right)(t) + \int_{T_{k+1}}^t \int_\Omega \left(\nabla\left(\varphi u\right)\right)^2 \leq 2^{k+2}\int_{T_k}^{T_{k+1}}\int_\Omega \varphi^2 u^2 + C\int_{T_k}^1\int_\Omega u^2|\nabla\varphi|^2.$$

Since the estimate is true for every $t \in [T_{k+1}, 1]$ we can find the maximum for the left-hand side and conclude

$$\sup_{T_{k+1}\leq t\leq 1}\left(\int_\Omega \varphi^2 u^2\right)(t) + \int_{T_{k+1}}^1 \int_\Omega \left(\nabla\left(\varphi u\right)\right)^2 \leq 2^{k+2}\int_{T_k}^{T_{k+1}}\int_\Omega \varphi^2 u^2 + C\int_{T_k}^1\int_\Omega u^2|\nabla\varphi|^2. \ \square$$

Notice that the left-hand side is simply

$$W_{k+1}(\varphi, u) := \|\varphi u\|_{L^{\infty,2}(Q_{k+1})} + \|\nabla(u\varphi)\|_{L^{2,2}(Q_{k+1})}.$$

It is immediate to see that this quantity is a localisation of the assumption on u belonging to $L^\infty((0,T);L^2(\Omega)) \cap L^2((0,T);H^1(\Omega))$.

Energy decay

To conclude De Giorgi's scheme we have to prove that

$$U_{k+1} \leq C^k U_k^{1+\frac{2}{2+n}} \tag{2.4}$$

which, as in the elliptic case, implies (2.2).

We define the test functions $0 \leq \varphi_k \leq 1$ as

$$\begin{aligned} &\varphi_k \equiv 1 \quad \text{in } B_k \\ &\varphi_k \equiv 0 \quad \text{in } B_{k-1}^c \\ &|\nabla \varphi_k| \leq C2^k, \end{aligned}$$

and we apply Corollary 2.1 with φ_{k+1} and u_{k+1} as subsolution. In this way we have

$$W_{k+1} := W_{k+1}(u_{k+1}, \varphi_{k+1}) \leq 2^{k+2} \int_{T_k}^{T_{k+1}} \int_\Omega \varphi_{k+1}^2 u_{k+1}^2 + C \int_{T_k}^1 \int_\Omega u_{k+1}^2 |\nabla \varphi_{k+1}|^2. \tag{2.5}$$

Since φ_{k+1} is compactly supported in B_k and $u_{k+1} \leq u_k$, the right hand side is bounded by

$$2^{k+1} \int_{T_k}^1 \int_{B_k} u_k^2 + C2^{4k} \int_{T_k}^1 \int_{B_k} u_k^2 \leq C^k U_k \quad \text{with } \; C > 1.$$

For the left-hand side of (2.5), we introduce a new concept. Typically, when working with functions that depend on both time and space, it is beneficial to obtain a uniform bound for both variables. This estimate is obtained through an interpolation argument and is closely tied to the initial regularity assumptions of the weak solution of the partial differential equation. We state it now and we will continue to use it in the next chapters.

Lemma 2.1 (Time-Space Interpolation). *Let $f \in L^{p,q} \cap L^{p',q'}$ with $1 \leq p, q, p', q' \leq +\infty$. For any $\lambda \in [0,1]$ let p_λ, q_λ be so that*

$$\frac{1}{p_\lambda} = \frac{\lambda}{p} + \frac{1-\lambda}{p'} \quad \textit{and} \quad \frac{1}{q_\lambda} = \frac{\lambda}{q} + \frac{1-\lambda}{q'}$$

Then

$$\|f\|_{p_\lambda, q_\lambda} \leq \|f\|_{p,q}^{\lambda} \|f\|_{p',q'}^{1-\lambda}$$

In our scenario, we have the following result:

Lemma 2.2. *There exists a constant $C > 0$ such that*

$$\|\varphi_{k+1} u_{k+1}\|_{L^{\frac{2(2+n)}{n}}(Q_{k+1})} \leq C W_{k+1}$$

Proof. The bound
$$\|\varphi_{k+1}u_{k+1}\|_{L^{\infty,2}(Q_{k+1})} \leq W_{k+1}$$
is trivial and, by Sobolev Embeddings, we have also
$$\|\varphi_{k+1}u_{k+1}\|_{L^{2,\frac{2n}{n-2}}(Q_{k+1})} \leq W_{k+1}.$$
Now we interpolate between the two estimates and we get
$$\|\varphi_{k+1}u_{k+1}\|_{L^{p_\lambda,q_\lambda}(Q_{k+1})} \leq W_{k+1}$$
for any p_λ, q_λ satisfing (for $\lambda \in [0,1]$)
$$\frac{1}{p_\lambda} = \frac{1-\lambda}{2} \quad \text{and} \quad \frac{1}{q_\lambda} = \frac{\lambda}{2} + \frac{1-\lambda}{2n/(n-2)}.$$
Imposing $p_\lambda = q_\lambda$ we find $p_\lambda = q_\lambda = 2(2+n)/n$ which implies the thesis. □

Collecting the results we have proved, we obtain
$$\|\varphi_{k+1}u_{k+1}\|_{L^{\frac{2(2+n)}{n}}(Q_{k+1})} \leq W_{k+1} \leq C^k U_k \tag{2.6}$$
and therefore,
$$U_{k+1} = \int_{Q_{k+1}} |u_{k+1}|^2 = \int_{Q_{k+1}} |u_{k+1}|^2 \mathbf{1}_{\{u_{k+1}>0\}} \leq \int_{Q_{k+1}} \varphi_{k+1}^2 |u_{k+1}|^2 \mathbf{1}_{\{u_{k+1}>0\}}$$
where the first inequality is given by the definition of φ_{k+1}. Using Hölder and Chebyshev inequalities, we can derive the following conclusion:
$$\begin{aligned} U_{k+1} &\leq \|\varphi_{k+1}u_{k+1}\|_{L^{\frac{2(2+n)}{n}}(Q_{k+1})} \cdot |\{x \in Q_{k+1} : u_{k+1} > 0\}|^{\frac{2}{2+n}} \\ &\leq C^k U_k |\{x \in Q_{k+1} : u_k > 2^{-(k+1)}\}|^{\frac{2}{2+n}} \\ &\leq C^k U_k \left(2^{2(k+1)} \int_{Q_{k+1}} |u_k|^2\right)^{\frac{2}{2+n}} \\ &\leq C^k U_k^{1+\frac{2}{2+n}}. \end{aligned}$$

2.2 The Hölder regularity of the heat equation

We follow De Giorgi's idea from the previous chapter. We need to introduce
$$\tilde{Q} = [-3/2, 0] \times B(1).$$
The key point is the counterpart of the elliptic case

Proposition 2.1. *Let v be a solution of (2.1) in Q'_2. Assume $v \leq 1$ and*

$$|\{u \leq 0\} \cap \tilde{Q}| \geq \frac{|\tilde{Q}|}{2}.$$

Then $\sup_{Q'_{1/2}} u \leq 1 - \lambda$, where $0 < \lambda \ll 1$ depends only on μ, A and n.

Once we have this result we apply it to the following function

$$v(x) = \frac{2}{\operatorname{osc}_{Q'_2} u} \left(u(x) - \frac{\sup_{Q'_2} u + \inf_{Q'_2} u}{2} \right),$$

which implies the oscillation decay estimate

$$\operatorname{osc}_{Q'(1/2)} u \leq \lambda \operatorname{osc}_{Q'(2)} u.$$

Proceeding (exactly) as we did in the previous Chapter we get the Hölder regularity.

Proof of Proposition (2.1)

In De Giorgi's proof we worked with the domains $B(1/2) \subset B(1) \subset B(2)$ and so we are expected to use $Q'(1/2) \subset Q'(1) \subset Q'(2)$. The introduction of $\tilde{Q}$ does not modify deeply the proof of the Proposition 2.1 but we need the following inequality (whose proof is postponed).

Proposition 2.2. *There exists $\alpha > 0$ such that: if u is a solution to (2.1) in Q'_2 the following holds. If we denote,*

$$\begin{aligned} |B| &= |\{u \geq 1/2\} \cap Q_1| \\ |N| &= |\{u \leq 0\} \cap \tilde{Q}| \\ |M| &= \left|\{0 < u < 1/2\} \cap \left(Q_1 \cup \tilde{Q}\right)\right| \end{aligned}$$

and $|B| \geq \delta, |N| \geq |\tilde{Q}|/2$ then

$$|M| \geq \alpha$$

We sketch the proof of Proposition 2.1

Proof. We set

$$v_0 = v \quad v_{k+1} = 2v_k - 1.$$

Since we have (2.2), as in the elliptic case, we have just to prove that, up to choosing a big k, we can make $\|v_k\|_{L^2(Q_1)}$ small as we want.
By contradiction, suppose that for every k

$$\int_{Q'_1} (v_k)_+^2 \geq \delta$$

for some $\delta > 0$. Then we have

$$|\{v_k \geq 1/2\} \cap Q_1'| = |\{v_{k+1} \geq 0\} \cap Q_1'| \geq \int_{Q_1'} (v_{k+1})^2 \geq \delta.$$

Moreover, by definition of v_k,

$$|\{v_k \leq 0\} \cap \tilde{Q}| \geq |\{v \leq 0\} \cap \tilde{Q}| \geq \frac{|\tilde{Q}|}{2}.$$

Then, by the Proposition 2.2, we get

$$|\{0 < v_k < 1/2\} \cap (Q_1' \cup \tilde{Q})| \geq \alpha.$$

We conclude

$$|\{v_k \leq 0\} \cap (Q_1' \cup \tilde{Q})| \geq |\{v_{k-1} \leq 0\} \cap (Q_1' \cup \tilde{Q})| + |\{0 \leq v_{k-1} \leq 1/2\} \cap (Q_1' \cup \tilde{Q})| \geq \frac{|\tilde{Q}|}{2} + k\alpha$$

and passing to the limit $k \to +\infty$ we have the contradiction. □

2.3 Isoperimetric inequality for parabolic equation

The proof of Proposition 2.2 is technical but deeply related to the geometry of the parabolic equation.
Suppose, by contradiction, that we can find a sequence u_k of solutions to (2.1) in Q_2' such that

$$|\{u_k \geq 1/2\} \cap Q_1'| \geq \delta$$
$$|u_k \leq 0| \cap \tilde{Q}| \geq |\tilde{Q}|/2$$
$$\left|\{0 < u_k < 1/2\} \cap (Q_1' \cup \tilde{Q})\right| \leq 1/k.$$

Step 1

In this step we prove that $v_k = (u_k)_+$ is uniformly bounded in $L^2([-2, 1]; H^1(B(1)))$ and also in $L^2([-2, 1]; L^2(B(1)))$. To help the readability we use the notation $L^2 + L^2$ to indicate the space $L^2([-2, 1]; L^2(B(1)))$ and similarly for other functional spaces.
Since $v_k \geq 0$ is still a solution to (2.1) we can use the Energy inequality (2.3) that we report:

$$\left(\int_\Omega \varphi^2 u^2\right)(t) + \int_s^t \int_\Omega |\nabla(\varphi u)|^2 \leq \left(\int_\Omega \varphi^2 u^2\right)(s) + C\int_s^t \int_\Omega u^2 |\nabla \varphi|^2$$

Setting $\varphi \equiv 1$ in $B(1)$ and compactly supported in $B(2)$ the inequality becomes

$$\left(\int_{B(1)} v_k^2\right)(t) + \int_s^t \int_{B(1)} (\nabla v_k)^2 \leq \left(\int_{B(2)} v_k^2\right)(s) + C\int_s^t \int_{B(2)} v_k^2$$

where we have used that also $\nabla\varphi$ is bounded. Since $v_k \leq 1$ we conclude

$$\left(\int_{B(1)} v_k^2\right)(t) + \int_s^t \int_{B(1)} (\nabla v_k)^2 \leq |B(2)| + C(t-s).$$

Choosing $s=-2$ and $t=1$ we have that v_k is uniformly bounded in L^2+L^2 while if we integrate both sides in t between $[-2,1]$ we get the uniform bound for v_k in L^2+L^2. We conclude that v_k is uniformly bounded in L^2+H^1.
We claim that $\partial_t v_k$ is uniformly bounded in L^2+H^{-1}. Indeed

$$\|\partial_t v_k\|_{L^2+H^{-1}} = \int_{-2}^1 \|\partial v_k(t,\cdot)\|^2_{H^{-1}(B(1))}\,dt = \int_{-2}^1 \left[\sup_{\substack{\psi\in H_0^1(B(1))\\ \|\psi\|_{H^1}=1}} \int_{B(1)} \partial_t v_k\ \psi\right]^2 dt.$$

Since v_k is a solution, the last term is equal to

$$\int_{-2}^1 \left[\sup_{\substack{\psi\in H_0^1(B(1))\\ \|\psi\|_{H^1}=1}} \int_{B(1)} \operatorname{div}(A\nabla v_k)\ \psi\right]^2 dt \leq \Lambda \int_{-2}^1 \left[\sup_{\substack{\psi\in H_0^1(B(1))\\ \|\psi\|_{H^1}=1}} \int_{B(1)} \operatorname{div}(\nabla v_k)\ \psi\right]^2 dt$$

where we have used the ellipticity of A.
We integrate by parts (remember that $\psi\equiv 0$ on $\partial\Omega$ and we do not care about signs)

$$\Lambda\int_{-2}^1 \left[\sup_{\substack{\psi\in H_0^1(B(1))\\ \|\psi\|_{H^1}=1}} \int_{B(1)} \nabla v_k\cdot\nabla\psi\right]^2 dt \lesssim \int_{-2}^1 \left[\sup_{\substack{\psi\in H_0^1(B(1))\\ \|\psi\|_{H^1}=1}} \|\nabla v_k\|_{L^2(B(1))}\|\nabla\psi\|_{L^2(B(1))}\right]^2 dt.$$

Since the right-hand side is controlled by $\|\nabla v_k\|_{L^2+L^2}$ we have proved the claim.

Step 2

We want to prove that, up to a subsequence, v_k is converging in $L^2([-2,1];L^2(B(1)))$. To this aim, we need Aubin Lion Lemma (see [BPB13] Theorem II.5.16):

Lemma (Aubin Lion Lemma)**.** *Let X_0, X and X_1 be three Banach spaces with $X_0\subset X\subset X_1$. Suppose that X_0 is compactly embedded in X and that X is continuously embedded in X_1. Suppose that $1<p,q<\infty$ and*

$$W=\{u\in L^p([0,T];X_0) : \partial_t u\in L^q([0,T];X_1)\}$$

Then W is compactly embedded into $L^p([0,T];X)$.

Our sequence

$$H^1 \subset L^2 \subset H^{-1}$$

satisfies the hypothesis and so we have that

$$\{u \in L^2 + H^1 : \partial_t u \in L^2 + H^{-1}\}$$

is compactly embedded into $L^2 + L^2$. This implies that v_k, up to extracting a subsequence, is converging to some v in $L^2 + L^2$.
Since the convergence in L^2 implies the convergence in probability (using Chebychev) we have, for any $\varepsilon > 0$,

$$\lim_{k \to +\infty} |\{|v_k - v| \geq \varepsilon\} \cap [-2, 1] \times B(1)| = 0. \tag{2.7}$$

Step 3

Notice that all the properties of u_k are inherited by v_k, moreover we have that $|\{v_k = 0\} \cap \tilde{Q}| \geq |\tilde{Q}|/2$ because v_k is positive. Also, v behaves well, indeed we have this lemma:

Lemma 2.3. *In the previous assumptions and setting the followings facts hold for v*

1. $|\{v \geq 1/2\} \cap Q_1'| \geq \delta$,
2. $|\{v = 0\} \cap \tilde{Q}| \geq |\tilde{Q}|/2$,
3. $|\{0 < v < 1/2\} \cap (Q_1 \cup \tilde{Q})| = 0$.

Proof. We prove separately the three points

1. If $v_k \geq 1/2$, then either $|v - v_k| \geq \varepsilon$ or $v \geq 1/2 - \varepsilon$. Hence
$$\delta \leq |\{v_k \geq 1/2\} \cap Q_1'| \leq |\{|v - v_k| \geq \varepsilon\} \cap Q_1'| + |\{v \geq 1/2 - \varepsilon\} \cap Q_1'|.$$
Using (2.7) and the arbitrariness of ε we conclude the point 1.
2. In the same way, if $v_k \leq 0$, then either $|v - v_k| \geq \varepsilon$ or $v \leq \varepsilon$. So
$$\frac{|\tilde{Q}|}{2} \leq |\{v_k \leq 0\} \cap \tilde{Q}| \leq |\{|v - v_k| \geq \varepsilon\} \cap \tilde{Q}| + |\{v \leq \varepsilon\} \cap \tilde{Q}|$$
and we conclude again by (2.7).
3. For the third time we have that if $\varepsilon \leq v \leq 1/2 - \varepsilon$ then either $|v - v_k| \geq \varepsilon$ or $0 < v_k < 1/2$. So
$$|\{\varepsilon \leq v \leq 1/2 - \varepsilon\} \cap (Q_1' \cup \tilde{Q})| \leq |\{|v - v_k| \geq \varepsilon\} \cap (Q_1' \cup \tilde{Q})| + |\{0 < v_k < 1/2\} \cap (Q_1' \cup \tilde{Q})|.$$
Using (2.7) we have that the right-hand side goes to zero. Since it is true for every ε we get the thesis.

□

Step 4

The previous Lemma leads to a contradiction with the isoperimetric inequality for the elliptic case. Indeed, the function v jumps between sets where it is bigger than 1/2 and sets smaller than 0. If we can formalize this concept we have the contradiction we need. The third point of the previous Lemma implies that for almost every $t \in [-3/2, 1]$ we have

$$|\{0 < v(t, \cdot) < 1/2\} \cap B(1))| = 0.$$

For almost every $t \in [-2, 1]$, $\nabla v(t, \cdot) \in L^2(B(1))$ we can apply De Giorgi's isoperimetric inequality (Proposition 1.2), to get that, for almost every $t \in [-3/2, 1]$, we have either

$$v(t, \cdot) = 0 \quad \text{in} \quad B(1) \qquad \text{or} \qquad v(t, \cdot) \geq 1/2 \quad \text{in} \quad B(1). \tag{2.8}$$

Now, by point 2 of Lemma 2.3, there are some times $-3/2 < s < -1$ such that $v(t, \cdot) \leq 0$ in $B(1)$. Choose one of such s and set a cutoff function compactly supported in $B(1)$. By the energy inequality we obtain, for $t > s$:

$$\left(\int_\Omega \varphi^2 v_k^2 \right)(t) \leq \left(\int_\Omega \varphi^2 v_k^2 \right)(s) + C(t - s) \tag{2.9}$$

and letting $k \to +\infty$ (with $v(\cdot, s) = 0$ in $B(1)$) it implies

$$\left(\int_{B(1)} \varphi^2 v^2 \right)(t) \leq C(t - s). \tag{2.10}$$

Consider the quantity $M = \|\varphi\|_2^2/4$ and notice that by (2.10) for every $t \in (s, s + M/C)$ we have

$$\left(\int_{B(1)} \varphi^2 v^2 \right)(t) < M = \frac{1}{4} \int_{B(1)} \varphi^2$$

If $v(t, \cdot) \geq 1/2$ we have a contradiction then, by (2.8), $v(\cdot, t) \equiv 0$ in $B(1)$ for $t \in (s, s+M/C)$. Bootstrapping the argument we conclude

$$v = 0 \quad \text{in} \quad Q_1'$$

which is a contradiction with point 1 of Lemma 2.3.

Remark. *The limit $k \to +\infty$ is well defined. Indeed $\|\varphi v_k\|_{L^\infty + L^2}$ is uniformly bounded by the estimate (2.9) and by the fact that $\varphi v_k \leq 1$ with φ compactly supported in $B(1)$.*

Remark. *Each step of the lemma relies on the equation (2.7), which holds in the interval $[-2, 1] \times B(1)$. This interval contains both Q_1' and $\tilde{Q}$. The reason we can obtain such a precise estimate is closely connected to the Energy Inequality in the parabolic case. This inequality allows us to have significant flexibility in the time variable, as the test function is defined only in the spatial variable. Consequently, we can exert control over future events based on our knowledge of the past. The inequality (2.9) provides us with information about Q_1' by utilizing the information we have about $\tilde{Q}$.*

In a sense, the parabolic isoperimetric inequality is more powerful than its elliptic counterpart. While the assumptions control the values of u *separately in* Q_1' *and* $\tilde{Q}$*, the conclusion still holds for the union of the two regions. This is due to the fact that* $\tilde{Q} = [-3/2, -1] \times B(1)$ *represents the past of* $\tilde{Q} = [-1, 1] \times B(1)$*, and we can exert control over the latter using the former.*

Chapter 3

Regularity for Navier-Stokes equations

The objective of this transitional chapter is to motivate the reader regarding the significance of the Caffarelli-Kohn-Nirenberg theorem. To accomplish this, we will provide a brief overview of the Navier-Stokes equations problem, aiming to give an intuitive understanding of the regularity assumptions that we will be imposing.

The issue of existence and smoothness of the Navier-Stokes equations is regarded as one of the most challenging problems in contemporary mathematics. It is one of the "Millennium Prize Problems." The current formulation of the problem was given by Charles Fefferman in 2000 [Fef00] and can be accessed on the official website.

The incompressible Navier-Stokes equations in dimension three are[1]

$$\begin{aligned} &\partial_t u + (u \cdot \nabla)u + \nabla p - \nu \Delta u = f \quad (t, x) \in (0, \infty) \times \mathbb{R}^3 \\ &\text{div} = 0 \end{aligned} \tag{3.1}$$

where $u : (0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ is the **velocity vector field** of the fluid and and $p : (0, +\infty) \times \mathbb{R} \to \mathbb{R}$ is the **pressure**. The term f is a given, applied external force and $\nu > 0$ is the viscosity constant. The problem is endowed with the initial condition:

$$u(0, \cdot) = u_0 \tag{3.2}$$

where u_0 is a divergence-free vector field.

Since we are working with an unbounded domain we have to control the behaviour of

[1] The gradient and the Laplacian are taken on on the spacial variables

the solution when $|x| \to \infty$. Hence we will require

$$\begin{aligned} &|\partial_x^\alpha u_0(x)| \leq C_{(\alpha,K)}(1+|x|)^{-K} \quad \text{for any } \alpha \text{ and } K \\ &|\partial_x^\alpha \partial_t^m f(x)| \leq C_{(\alpha,m,K)}(1+|x|+t)^{-K} \quad \text{for any } \alpha, m, K. \end{aligned} \tag{3.3}$$

For physical reasons, we consider accetable only smooth solutions with bounded energy (at every time), namely

$$\begin{aligned} &p, u \in C^\infty\left((0,\infty)\times\mathbb{R}^3\right) \\ &\int_{\mathbb{R}^3} |u(t,x)|^2 dx \leq C \quad \text{for any } t. \end{aligned} \tag{3.4}$$

The Clay Mathematics Institute of Cambridge ask for a proof of

First formulation of the Millenium Problem

Take $\nu > 0$ and u_0 any smooth divergence-free vector field verifying (3.3) (only the first equation). Take f identically zero. Then there exist u, p functions on $(0,\infty)\times\mathbb{R}^3$ solving (3.1), (3.2) and (3.4).

We will work in this direction. We suppose $\nu = 1$ and instead of $\mathbb{R}^3$ we consider a bounded and regular region $\Omega \subset \mathbb{R}^3$. Then the condition (3.3) is substituted by

$$u(t,x) = 0 \quad \text{in } (0,\infty)\times\partial\Omega.$$

From now on when we speak about (u,p) solving the Navier-Stokes equations in $(0,T)\times\Omega$ with $T > 0$ we mean:

$$\begin{aligned} &\partial_t u + (u\cdot\nabla)u + \nabla p = \Delta u \quad (t,x) \in (0,T)\times\mathbb{R}^3 \\ &\operatorname{div} u = 0 \end{aligned} \tag{3.5}$$

with boundary conditions:

$$\begin{aligned} &u(0,\cdot) = u_0 \quad \text{in } \Omega \\ &u(t,x) = 0 \quad \text{in } (0,T)\times\partial\Omega \end{aligned} \tag{3.6}$$

with u_0 divergence-free.

3.1 Weak solutions

In this section, we present a constructive definition of suitable weak solutions. We aim to provide an intuitive understanding of the hypotheses we impose.

When analyzing solutions of the Navier-Stokes equations, it is advantageous to consider spaces of divergence-free functions. We begin by defining the space of the solution as follows:

$$\mathcal{V}(\Omega) := \{u \in C_c^\infty(\Omega) : \operatorname{div}(u) = 0\},$$

Next, we consider the closures of this space in L^2 and H^1, respectively:

$$H(\Omega) := \overline{\mathcal{V}(\Omega)}^{\|\cdot\|_2} \quad \text{and} \quad V(\Omega) := \overline{\mathcal{V}(\Omega)}^{\|\cdot\|_{1,2}}.$$

These subspaces are closed and thus remain Hilbert spaces when equipped with the L^2 and H^1 norms, respectively.

Now, we present the most general definition of a weak solution:

Definition 3.1. *Given $u_0 \in \mathcal{V}(\Omega)$ we say that (u, p) is a global weak solution of (3.5) in $(0, T) \times \Omega$ if:*

1. $u \in L^2\left((0,T); V(\Omega)\right) \cap L^\infty\left((0,T); H(\Omega)\right);$
2. (u, p) *solves the first equation of* (3.5) *in a distributional sense*;
3. $u(0, \cdot) = u_0$ *a.e. in* Ω.

In the literature, it is common for simplicity to only require the following condition instead of point 1 of Definition 3.1.

$$u \in L^2((0,T); H_0^1(\Omega)) \cap L^\infty((0,T); L^2(\Omega)).$$

Moreover, the condition div $= 0$ is imposed in a weak sense, meaning that (u, p) satisfies the two points of equation (3.5) in a distributional sense.
Let's delve further into the two key points:

1. Up until this point, we have not imposed any restrictions on the regularity of the pressure term in the Navier-Stokes equations;
2. We have not discussed the uniqueness of the solution:sS far, we have not addressed the issue of the uniqueness of the solution to the Navier-Stokes equations.

Let us begin with the first of these two points, which concerns the pressure. We want to emphasize that the issues regarding its regularity are not trivial but rather necessary to address in order to tackle the Caffarelli-Kohn-Nirenberg theorem. Before doing so, we state an intermediate but crucial result:

Proposition 3.1. *Under the previous assumptions on the regularity of u, we have*

$$u \in L^2\left((0,T); L^6(\Omega)\right) \cap L^{10/3}\left((0,T)\times\Omega\right). \tag{3.7}$$

Moreover, $(u\cdot\nabla)u \in L^2((0,T); L^{3/2})$.

Proof. By Sobolev Embeddings, since in dimension 3 we have $2^* = 6$, we get $u \in L^2\left((0,T); L^6(\Omega)\right)$. By space-time interpolation between $L^{2,6}$ and $L^{\infty,2}$ we get also $u \in L^{10/3,10/3}$.
For the last implication, notice that $[(u\cdot\nabla)u]_i = \sum_j u_j\partial_j u_i$. Then, using Holder with exponents 4 and 4/3 (fixing t) we get

$$\|u_j\partial_j u_i\|_{L^{3/2}(\Omega)} \le \|u_j\|^{\alpha}_{L^6(\Omega)}\|\partial_j u_i\|^{\beta}_{L^2(\Omega)} < +\infty$$

for some positive α, β. □

A strong result of pressure regularity is the following:

Proposition 3.2. *If (u,p) is a weak solution of the Navier-Stokes then*

$$p \in L^2_{loc}((0,T)\times\Omega) \quad \textit{and} \quad p \in L^{5/3}((0,T)\times\Omega).$$

Proof. Using a regularity result concerning the Stokes equation one can prove that, providing u_0 is regular enough, we get

$$\|\nabla p\|_{L^t((0,T);L^s(\Omega))} \le C(\|(u\cdot\nabla)u\|)_{L^t((0,T);L^s(\Omega))}$$

with $\frac{2}{t}+\frac{3}{s} \le 4$ and $1 < s < 3/2$ [2].

By previous proposition we have that the norm $\|\nabla p\|_{L^t_{loc}((0,T);L^s(\Omega))}$ is bounded if

$$\frac{2}{t}+\frac{3}{s} \le 4 \quad \text{with} \quad 1 < s < 3/2 \quad \text{and} \quad t \le 2.$$

Using the Sobolev embeddings [3] ($s^* = 3s/(3-s)$) we get that $\|p\|_{L^t_{loc}((0,T);L^{s^*}(\Omega))}$ is bounded if

$$\frac{2}{t}+\frac{3}{s^*} \le 3 \quad \text{with} \quad 3/2 < s^* < 3 \quad \text{and} \quad t \le 2.$$

By choosing $T = s^* = 2$ we have the best (space-time homogeneous) bound possible.

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[2] A discussion of this kind of regularity can be found in the lecture notes of Milan Pokorny, starting from [Pok].

[3] We are using, implicitly the Poincare inequality. The compactness of the support of p is given by (3.8) in combination with (A.3)

For global regularity, we use weak divergence on the first equation of (3.5) and we get

$$-\Delta p = \operatorname{div}\left((u \cdot \nabla)u\right). \tag{3.8}$$

Fixing t and computing the right-hand side we have that, for almost every t, the function $p(t)$ solves

$$-\Delta p(t) = \sum_{ij} \partial^2_{ij} u_j(t) u_i(t)$$

where we have used $\operatorname{div}(u) = 0$. We can apply Calderón-Zygmund theory (see (A.3)) to get

$$\|p(t)\|^{5/3}_{L^{5/3}(\Omega)} \leq C \|u_j(t) u_i(t)\|^{5/3}_{L^{5/3}(\Omega)}.$$

Integrating in the time variable we finally get

$$\|p\|^{5/3}_{L^{5/3}((0,T)\times\Omega)} \leq C \|u_j u_i\|^{5/3}_{L^{5/3}((0,T)\times\Omega)}.$$

Since $u \in L^{10/3}((0,T) \times \Omega)$, we have reached the desired conclusion. □

We conclude by mentioning two well-known regularity results regarding u, which will be useful in the next subsection. The proofs of these results are omitted here but can be found in the references [Cau01] and [Gal00].

Proposition 3.3. *If (u, p) is a weak solution of the Navier-Stokes equation we have*

1. $u \in C^0_w\left([0,T); H^1(\Omega)\right)$;
2. $\partial_t u \in L^{4/3}\left((0,T); H^{-1}(\Omega)\right)$.

Having $u \in C^0_w([0,T]; H^1(\Omega))$ is a weak way to satisfy the boundary condition. In fact, for every $f \in H^1(\Omega)$, the map

$$t \mapsto \int_\Omega u(x,t) f(x) dx$$

is continuous.

The fact that $\partial_t u \in L^{4/3}((0,T); H^{-1}(\Omega))$ is non-trivial but guarantees us that ∂_t is a distribution representable by an L^1_{loc} function.

3.1.1 Suitable weak solution

In the Caffarelli-Kohn-Nirenberg theorem, a particular class of weak solutions is used: suitable weak solutions or simply suitable solutions.

Definition 3.2. *We say that (u,p) is a suitable weak solution in $(0,T)\times\Omega$ if it satisfies points 1 and 2 of Definition 3.1 and, in a distributional sense, we have the following energy inequality*

$$\partial_t \frac{|u|^2}{2} + \operatorname{div}\left(u\left(\frac{|u|^2}{2}+p\right)\right) + |Du|^2 - \Delta\frac{|u|^2}{2} \le 0 \quad \forall t\in(0,T),\ x\in\Omega. \tag{3.9}$$

Theorem 3.1. *Suppose $u_0 \in \mathcal{V}(\Omega)$. Then there exists at least one suitable weak solution of Navier-Stokes equations in $(0,T)\times\Omega$ satisfying*

1. *$u(t)\to u_0$ weakly in H^1 as $t\to 0$.*
2. *For all functions $\varphi\in C_c^\infty([0,T]\times\Omega)$ with $\varphi\ge 0$ and $\varphi = 0$ near $(0,T)\times\partial\Omega$, we have*

$$\begin{aligned}\int_\Omega \frac{|u|^2}{2}(x,t)\varphi(x,t)dx + \int_0^t\int_\Omega |Du|^2\varphi \le \int_\Omega \frac{|u_0|^2}{2}(x)\varphi(x,0)dx \\ + \int_0^t\int_\Omega \frac{|u|^2}{2}(\varphi_t+\Delta\varphi) + \left(\frac{|u|^2}{2}+p\right)u\cdot\nabla\varphi\end{aligned} \tag{3.10}$$

For the proof see [FM00]. Notice also that the hypothesis on the regularity of the initial data u_0 can be weakned.
Similarly one can prove that, under the same hypothesis, the following holds

Corollary 3.1 (see [Gal00])**.** *For every $\varphi\in C_c^\infty(\Omega)$ with $\varphi\ge 0$ and every $0<s<t$ we have:*

$$\begin{aligned}&\int_\Omega \frac{|u|^2}{2}(x,t)\varphi(x)dx + \int_s^t\int_\Omega |Du|^2\varphi \\ &\qquad \le \int_\Omega \frac{|u|^2}{2}(x,s)\varphi(x)dx + \int_s^t\int_\Omega \frac{|u|^2}{2}\Delta\varphi + \left(\frac{|u|^2}{2}+p\right)u\cdot\nabla\varphi\end{aligned} \tag{3.11}$$

Now, let us discuss the regularity problem. To address this, we introduce the following definition:

Definition 3.3. *Let (u,p) be a weak solution (or suitable weak solution) of the Navier-Stokes equations. Let $(t_0,x_0)\in((0,+\infty)\times\Omega)$. It will be said that (t_0,x_0) is a singular point if u is not L^∞ in any neighborhood of (t_0,x_0). The remaining points, those where u is locally bounded, will be called regular points. The standard notation is:*

$$\begin{aligned}Reg(u) &:= \{(x,t) : u\in L^\infty(U) \text{ for some neighborhood } (x,t)\in U\}\\ Sing(u) &:= (Reg(u))^c\end{aligned}$$

To gain insight into the significance of obtaining a description of the set $Sing(u)$, we turn to the work of Serrin (see [Ser62]). He proved that in order for u to belong to the class C^∞ in a neighbourhood of (t_0, x_0), it suffices to have certain estimates, specifically L^r in time and L^s in space, with sufficiently large values of r and s. Subsequently, there have been several improvements in this line of research, and we present here the theorem from [Gal00] (in particular Theorem 5.2):

Theorem 3.2. *Let u a weak solution (or suitable) to the Navier-Stokes equation in $(0,T)\times\Omega$. Suppose Ω is regular and*

$$u \in L^r((0,T); L^s(\Omega)), \text{ for some } r, s \text{ such that } \frac{3}{s} + \frac{2}{t} = 1,\ s \in (3, +\infty],$$

then

$$u \in C^\infty((0,T] \times \overline{\Omega}).$$

The Theorem 3.2 guarantees us that if we prove the solution's boundness (in space-time), we also have smoothness. In particular, the Caffarelli-Kohn-Nirenberg theorem estimates how big the set of singularities of the suitable weak solution u is.

3.2 Caffarelli-Kohn-Nirenberg

Theorem 3.3 (Caffarelli-Kohn-Nirenberg). *If u is a suitable weak solution of the Navier-Stokes equations, then $\mathcal{P}^1(Sing(u)) = 0$. With $\mathcal{P}$ we mean the Parabolic Hasudorff measure.*

As we have mentioned earlier, studying the set where u is unbounded provides crucial information about the regularity of the Navier-Stokes equations. With this theorem, we establish that this set cannot be contained within a curve, specifically, the set of singularities of u cannot include a space-time curve of the form $\{(t,x) : x = \phi(t)\}$. Currently, this result is undoubtedly considered the most significant achievement in the study of the Navier-Stokes equations. The proof relies completely on the following ε-regularity statement:

Theorem 3.4. *The exists a positive constant ε such that if u is a weak solution of the Navier-Stokes equations in $(0,\infty) \times \Omega$ and for some pair $(t,x) \in (0,\infty) \times \Omega$ we have*

$$\limsup_{r \to 0} \frac{1}{r} \int_{C_r(x,t)} |Du|^2 < \varepsilon, \tag{3.12}$$

then $(x,t) \in Reg(u)$. With $C_r(x,t) \subset \mathbb{R} \times \mathbb{R}^3$ we have denoted the cylinder $(t - r^2, t + r^2) \times B_r(x)$.

Before showing the details we anticipate that we are going to deal with negative times which is just a mathematical choice: shifting from $(0,T)$ to $(-1,T)$ does not change any result that we have already obtained.

We begin by demonstrating how Theorem 3.4 implies Theorem 3.3, and then we prove Theorem 3.4 itself. The proof of the latter theorem is divided into two major steps: we first establish an intermediate result (Theorem 3.5), and then we conclude the proof. Both the intermediate result and the conclusion rely on two propositions: Proposition 3.4 and Proposition 3.5. These propositions will be proven in the subsequent chapter.

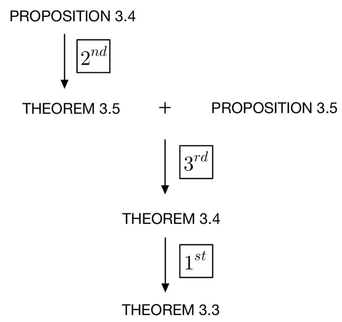

Figure 3.1: Scheme of the proof: the numbers indicate the order of the proofs.

From Theorem 3.4 to Theorem 3.3

To state the theorem we need to introduce the Parabolic Hausdorff measure.

Definition 3.4. *Given $E \in \mathbb{R}^3 \times \mathbb{R}$ and two parameters $\alpha \geq 0$ and $\delta > 0$, we define*

$$\mathcal{P}^\alpha_\delta(E) := \inf \left\{ \sum_i \omega_\alpha r_i^\alpha : E \subset \bigcup_i B_{r_i}(x_i) \times (t_i - r_i^2, t_i + r_i^2) \text{ and } 2r_i\sqrt{1 + r_i^2} < \delta \ \forall i \right\}$$

where $\omega_\alpha = \pi^{\alpha/2}/\Gamma(1+\frac{\alpha}{2})$ (with Γ we mean the Gamma function). The parabolic Hausdorff measure of the set E is then

$$\mathcal{P}^\alpha(E) := \lim_{\delta \to 0} \mathcal{P}^\alpha_\delta(E) = \sup_{\delta > 0} \mathcal{P}^\alpha_\delta(E).$$

Notice that it is very similar to the classical Hausdorff measure but we consider cylinders instead of balls. Moreover, $2r_i\sqrt{1 + r_i^2}$ is the diameter of the cylinder $B_{r_i}(x_i) \times (t_i - r_i^2, t_i + r_i^2)$.
Going from the Theorem 3.4 to the Theorem 3.3 is a classical covering argument. To do so we have to remind the Besicovitch property of the cylinders:

Lemma. *There exists a number N with the following property: if $\mathcal{F} = \{\bar{C}_r(x,t)\}$ is a family of cylinders then there exists N subfamilies of $\mathcal{F}$, each consisting of pairwise disjoint cylinders, which cover the set $\{(x,t) : \exists \bar{C}_r(x,t) \in \mathcal{F}\}$.*

Similarly to the Balls case, N depends only on the dimension of the ambient space, in our setting $n = 3$.

Proof of Theorem 3.3

Fix $\delta > 0$ and set

$$\mathcal{F}_\delta := \left\{ \bar{C}_r(x,t) \in (0,\infty) \times \Omega : 2r\sqrt{1 + r^2} < \delta \text{ and } \int_{C_r(x,t)} |Du|^2 \geq \varepsilon r \right\}.$$

By Theorem 3.4 the set of centers of $\mathcal{F}_\delta$ contains $Sing(u)$. Now we use the Besicovitch property and we consider the N subfamilies $\mathcal{F}_\delta^1, \dots, \mathcal{F}_\delta^N$ of the previous Lemma. Then, by definition of the Parabolic Hausdorff measure, we have

$$\mathcal{P}_\delta^1(Sing(u)) \leq \sum_{\bar{C}_r(x,t) \in \bigcup_{i=n}^N \mathcal{F}_\delta^i} \omega_1 r \leq \sum_{i=1}^N \sum_{\bar{C}_r(x,t) \in \mathcal{F}_\delta^i} \frac{\omega_1}{\varepsilon} \int_{\bar{C}_r(x,t)} |Du|^2.$$

Defining $R_\delta^i := \bigcup_{\bar{C}_r(x,t) \in \mathcal{F}_\delta^i} C_r(x,t)$, we rewrite

$$\mathcal{P}_\delta^1(Sing(u)) \leq \sum_{i=1}^N \frac{\omega_1}{\varepsilon} \int_{R_\delta^i} |Du|^2.$$

If we manage to prove that $\lim_{\delta \to 0} |R_\delta^i| = 0$ then we have the thesis. Notice that

$$\mathcal{L}^4(C_r) \approx r^5.$$

Now, using the definition of $\mathcal{F}_\delta$, we get that that $r^4 \leq \delta^2$ and $\varepsilon r \leq \| |Du| \|^2_{L^2(C_r)}$, we get

$$|R_\delta^i| \leq C \sum_{\bar{C}_r(x,t) \in \mathcal{F}_\delta^i} r^4 \cdot r \leq CN \frac{\delta^2}{\varepsilon} \int_{(0,\infty) \times \Omega} |Du|^2.$$

Since $Du \in L^2$ (by assumption on weak solutions) we conclude that the limit for $\delta \to 0$ of the Parabolic Hausdorff measure is zero.

From Proposition 3.4 to Theorem 3.5

The intermediate result is:

Theorem 3.5. *For any $p > 1$, there exists a constant C, such that any suitable weak solution in $Q_0 = [-1, 1] \times B(1)$ verifying*

$$\|u\|^2_{L^{\infty,2}(Q_0)} + \|Du\|^2_{L^{2,2}(Q_0)} + \|p\|^2_{L^{p,1}(Q_0)} \leq C$$

is bounded by 1 almost everywhere in $[-1/2, 1] \times B(1/2)$.

The main difference between the work of Vasseur and the original one by Caffarelli-Kohn-Nirenberg relies on this part. The first great idea is to put ourselves in De Giorgi's setting: we introduce a sequence of decreasing balls shrinking at $B(1/2)$ (always centred at the origin as anticipated in Remark 1)

$$B_k = B(1/2(1 + 2^{-3k}).$$

The choice of the exponent $-3k$ (instead of just $-k$) will be clear much later (see pressure decomposition), for the moment just notice that $B_0 = B(1)$ and $B_\infty = B(1/2)$. As in the parabolic case, we also introduce a sequence of time

$$T_k = \frac{1}{2}(-1 - 2^{-k})$$

and a sequence of specific cylinders

$$Q_k = [T_k, 1] \times B_k,$$

where $Q_0 = [-1, 1] \times B(1)$ and $Q_\infty = [-1/2, 1] \times B(1/2)$. The truncation is defined as

$$v_k = [|u| - (1 - 2^{-k})]_+,$$

and the energy is given by

$$U_k = \|v_k\|^2_{L^{\infty,2}(Q_k)} + \|d_k\|^2_{L^2(Q_k)},$$

where

$$d_k^2 = \frac{(1 - 2^{-k})\mathbf{1}_{|u|\geq(1-2^{-k})}}{|u|}|\nabla|u||^2 + \frac{v_k}{|u|}|Du|^2 = \frac{(1 - 2^{-k})}{|u|}|\nabla v_k|^2 + \frac{v_k}{|u|}|Du|^2.$$

The introduction of the term d_k is the first significant difference compared to the parabolic settings. Without delving into technical details, it should be noted that the term d_k depends on the derivatives of v_k and, consequently, on the derivatives of u. This is in line with the formulation of Theorem 3.5. In this case, the universal constant C needs to control not only u but also Du, unlike in the elliptic-parabolic context. One might wonder why we do not simply take $d_k = |\nabla v_k|$. The reason is that when we substitute v_k into the first set of Navier-Stokes equations, the nonlinearity causes the term d_k to emerge. However, all the computations will be presented in the next chapter.

The strategy is to prove that $U_k \to 0$, in that case we would have $\|v_\infty\|_{L^{\infty,2}(Q_\infty)} = 0$ which is equivalent to

$$\sup_{t\in[-1/2,1]} \left(\int_{B(1/2)} [|u| - 1]_+^2 dx \right) = 0 \quad \Longrightarrow \quad |u| \leq 1 \text{ a.e } \in [-1/2, 1] \times B(1/2). \tag{3.13}$$

The key for proving such a result is the following fact:

Proposition 3.4. *Let $p > 1$. Then there exist constants $C, \beta > 1$ depending only on p such that for any suitable weak solution in $[-1, 1] \times B(1)$, if $U_0 \leq 1$, then we have for every $k > 0$*

$$U_k \leq C^k (1 + \|p\|_{L^{p,1}([-1,1],B(1))}) U_{k-1}^\beta.$$

For this decay estimate we use a combination of De Giorgi's scheme and a pressure decomposition suggested by Vasseur. The proof is postponed to the next chapter.

To prove how Proposition 3.4 implies Theorem 3.5, we begin by noting that

$$U_0 = \|u\|^2_{L^{\infty,2}(Q_0)} + \|Du\|^2_{L^2(Q_0)}.$$

Assuming the hypotheses of Theorem 3.5, we have the existence of a constant C that bounds both U_0 and $\|p\|_{L^{p,1}(Q_0)}$. Let us set a new constant, also denoted as C:

$$C = \inf\{C, 1\}.$$

Then we have $U_0 \leq 1$ and $\|p\|_{L^{p,1}(Q_0)} \leq 1$. By applying Proposition 3.4 (Note that the constant C^k is different from the previous one), we obtain:

$$U_k \leq 2C^k U^\beta_{k-1}.$$

Using Lemma 1.1, we can find a constant $\varepsilon_0 > 0$ such that, if $U_0 \leq \varepsilon_0$, we have

$$U_k \longrightarrow 0.$$

By redefining the constant as $C = \inf\{C, \varepsilon_0\}$ (refer to (3.13)), we have completed the proof.

Conclusion

Consider a pair (t_0, x_0) in $(0, \infty) \times \Omega$ and define the rescaled solutions for a fixed $\lambda < 1$

$$u_k(t, x) = \lambda^k u(\lambda^{2k} t + t_0, \lambda^k x + x_0),$$
$$p_k(t, x) = \lambda^{2k} p(\lambda^{2k} t + t_0, \lambda^k x + x_0).$$

It is easy to verify that if (u, p) is a suitable weak solution to the Navier-Stokes equations, then (u_k, p_k) is also a suitable weak solution. The advantage of this construction is that for every (t_0, x_0) in the interior of $(0, \infty) \times \Omega$, there exists a sufficiently large k such that (u_k, p_k) is a suitable weak solution in Q_0. This property allows us to focus on the behaviour of the solutions in the reference cylinder Q_0 and analyze their regularity and properties in a controlled setting.

Now, for $1 < p \leq 4/3$, we define a sequence

$$V_k = \|u_k\|^2_{L^{\infty,2}(Q_0)} + \frac{1}{\lambda^8}\|p_k - \bar{p}_k\|^2_{L^{p,2}(Q_0)} \quad \text{where} \quad \bar{p}_k(t) = \fint_{B_0} p_k(t, x)dx.$$

Since $\nabla p_k = \nabla(p_k - \bar{p}_k)$ the pair $(u_k, p_k - \bar{p}_k)$ is still a suitable weak solution.

Proposition 3.5. *For $1 < p < 2$ there exists $\lambda < 1$ and $\varepsilon_0 \leq C/2$ (where C is the constant of Theorem 3.5) such that for every suitable weak solution and every $(t_0, x_0) \in (0, \infty) \times \Omega$ the following holds. If*

$$\limsup_{r \to 0} \frac{1}{r} \int_{C_r(x_0,t_0)} |Du|^2 < \varepsilon_0$$

there exists $k_0 > 0$ such that, for any $k \geq k_0$,

$$V_{k+1} \leq \frac{V_k}{4} + \frac{C}{4}. \tag{3.14}$$

We remind that we have to prove that Theorem 3.5 and Proposition 3.5 imply Theorem 3.4. To this aim, assume that Proposition 3.5 holds. Since in Proposition 3.2 we have proved that $p \in L^2_{loc}((0, \infty) \times \Omega)$ we have that V_1 is finite. Setting $M = \max\{C, V_1\}$, by (3.14), $V_k \leq M$ for every k, it follows that $\limsup_k V_k$ is finite. In particular, again by (3.14),

$$\limsup_{k \to +\infty} V_k \leq C/3.$$

Moreover

$$\|Du_k\|^2_{L^2(Q_0)} = \frac{1}{\lambda^k} \int_{t_0 - \lambda^{2k}}^{t_0 + \lambda^{2k}} \int_{x_0 + B(\lambda^k)} |Du|^2. \tag{3.15}$$

Since $\lambda < 1$ we have that $\lambda^k \to 0$ and so

$$\limsup_{k \to +\infty} \|Du_k\|^2_{L^2(Q_0)} \leq \limsup_{r \to 0} \frac{1}{r} \int_{C_r(x_0,t_0)} |Du|^2 < \varepsilon_0 \leq \frac{C}{2}.$$

Then, for a big enough k_1, we conclude

$$V_{k_1} + \|Du_{k_1}\|^2_{L^2(Q_0)} \leq \frac{C}{2} + \frac{C}{3} < C,$$

which implies, using $\|p_k - \bar{p}_{k_1}\|^2_{L^{p,1}(Q_0)} \leq \lambda^{-8} \|p_k - \bar{p}_{k_1}\|^2_{L^{p,2}(Q_0)}$, that

$$\|u_{k_1}\|^2_{L^2(Q_0)} + \|p_k - \bar{p}_{k_1}\|^2_{L^{p,1}(Q_0)} + \|Du_{k_1}\|^2_{L^2(Q_0)} < 1$$

and by Theorem 3.5 u_{k_1} is bounded almost everywhere in $[-1/2, 1] \times B(1/2)$. This implies that there exists a neighbourhood of (t_0, x_0) where u is essentially bounded.

Chapter 4

Proof of Caffarelli-Kohn-Nirenberg in De Giorgi's style

As we have already discussed the Caffarelli-Kohn-Nirenberg theorem can be reduced to the Propositions 3.4 and 3.5. The heart of the work is the first one, which we remind:

Proposition. *Let $p > 1$. Then there exist constants $C, \beta > 1$ depending only on p such that for any suitable weak solution in $[-1,1] \times B(1)$, if $U_0 \leq 1$, then we have for every $k > 0$*

$$U_k \leq C^k(1 + \|p\|_{L^{p,1}([-1,1],B(1))})U_{k-1}^{\beta}.$$

The proof is organised in two steps: the first is an application of De Giorgi's scheme in the Navier-Stokes setting and the second is Vasseur's idea to decompose the pressure. We conclude the chapter by proving also the second Proposition. Of course, the main reference is [Vas07].

4.1 De Giorgi's scheme for Navier Stokes

The first part of the proof will follow the De Giorgi's scheme. The objective is to prove that the energy decreases with an exponent greater than 1. In this first part, we will not yet address the decrease of the pressure, for which an additional important observation will be necessary (the technique will be developed in the next section).

The key elements of the method include the Chebyshev inequality, Sobolev Embedding, and Energy Inequality. However, in our specific context, we must adapt these principles and make technical observations regarding our truncations beforehand.

The truncation

We start fixing some notation

1. If A, B are $n \times n$ matrix we denote $A : B$ the scalar
$$A : B = \sum_{i,j} A_{ij} B_{ij}$$

2. Consider $u, \bar{u} : \mathbb{R}^3 \to \mathbb{R}^3$ then $u \otimes \bar{u}$ is a $n \times n$ matrix such that
$$[u \otimes \bar{u}]_{ij} = u_i \bar{u}_j$$

3. With the previous u we denote $D(u)$ the matrix with components
$$[D(u)]_{ij} = \partial_j u_i$$

Now we compute some useful identities.

Proposition 4.1. *Consider $u, \bar{u} : \mathbb{R}^3 \to \mathbb{R}^3$ and $v : \mathbb{R}^3 \to \mathbb{R}$. Suppose $u, \bar{u}, v$ are regular enough. Then*

1. $D(uv) = u \otimes \nabla v + v D(u)$
2. $u \cdot (u^T D(u)) = u \cdot [(u \cdot \nabla) u]$
3. $\operatorname{div}\left(\bar{u}^T D(u)\right) = \bar{u} \cdot \Delta(u) + D(\bar{u}) : D(u)$
4. $(u \otimes \nabla v) : D(u) = (u^T D(u)) \cdot \nabla v$
5. $D(uv) : D(u) = (u^T D(u)) \cdot \nabla v + v |D(u)|^2$
6. $u^T D(u) = |u| \nabla |u|$

Proof. We proceed proving each point.

1. $$[D(uv)]_{ij} = \partial_j (u_i v) = u_i \partial_j v + v \partial_j u_i = [u \otimes \nabla v]_{ij} + v [D(u)]_{ij}.$$

2. We compute the left and the right-hand side:
$$u \cdot (u^T D(u)) = \textstyle\sum_i u_i (u^T D(u))_i = \sum_i u_i \sum_j u_j \partial_i u_j = \sum_{ij} u_i u_j \partial_i u_j,$$
$$u \cdot [(u \cdot \nabla) u] = \textstyle\sum_i u_i [(u \cdot \nabla) u_i] = \sum_i u_i \sum_j u_j \partial_j u_i = \sum_{ij} u_i u_j \partial_j u_i.$$

3.

$$\begin{aligned}\operatorname{div}\big(\bar{u}^T D(u)\big) &= \sum_i \partial_i [\bar{u}^T D(u)]_i = \sum_i \partial_i \bigg(\sum_j \bar{u}_j \partial_i u_j \bigg) = \sum_{ij} \partial_i (\bar{u}_j \partial_i u_j) = \\ &= \sum_{ij} \bar{u}_j \partial^2_{i,i} u_j + \partial_i \bar{u}_j \partial_i u_j = \bar{u} \cdot \Delta(u) + D(\bar{u}) : D(u).\end{aligned}$$

4.

$$\begin{aligned}(u \otimes \nabla v) : D(u) =& (u^T D(u)) = \sum_{ij} u_i \partial_j v \partial_j u_i = \\ &= \sum_j \bigg(\sum_i u_i \partial_j u_i \bigg) \partial_j v = (u^T D(u)) \cdot \nabla v\end{aligned}$$

5. Combining point 1 and 4 we get

$$D(uv) : D(u) = (u \otimes \nabla v + vD(u)) : D(u) = (u^T D(u)) \cdot \nabla v + v|D(u)|^2.$$

6. We compute separately the two quantities

$$[u^T D(u)]_i = \textstyle\sum_j u_j \partial_i u_j,$$

$$[\nabla |u|]_i = \partial_i \bigg(\textstyle\sum_j u_j^2 \bigg)^{1/2} = \frac{2 \sum_j u_j \partial_i u_j}{2|u|}.$$

□

We remind the choice of truncation

$$v_k = [|u| - (1 - 2^{-k})]_+$$

and

$$d_k^2 = \frac{(1 - 2^{-k}) \mathbf{1}_{|u| \geq \{1 - 2^{-k}\}}}{|u|} |\nabla |u||^2 + \frac{v_k}{|u|} |Du|^2.$$

Lemma 4.1. *The function u can be split in the following way:*

$$u = u \frac{v_k}{|u|} + u \left(1 - \frac{v_k}{|u|} \right)$$

where

$$\left| u \left(1 - \frac{v_k}{|u|} \right) \right| \leq 1 - 2^{-k}$$

Moreover, we have the following bounds:

$$\frac{v_k}{|u|} |Du| \leq d_k, \qquad \mathbf{1}_{\{|u| \geq 1 - 2^{-k}\}} |\nabla |u|| \leq d_k, \quad |\nabla v_k| \leq d_k, \qquad \left| D \frac{u v_k}{|u|} \right| \leq 3 d_k.$$

Proof. Notice that

$$v_k = 0 \qquad \text{if } |u| \le 1 - 2^{-k}$$
$$v_k = |u| - (1 - 2^{-k}) \quad \text{if } |u| \ge 1 - 2^{-k}$$

which implies

$$\left| u\left(1 - \frac{v_k}{|u|}\right) \right| \le 1 - 2^{-k}.$$

Notice also that $d_k = 0$ when $|u| < 1 - 2^{-k}$, so the only interesting case is when $|u| \ge 1 - 2^{-k}$, as otherwise all terms vanish. Let us suppose $|u| \ge 1-2^{-k}$ and so $v_k = |u| - (1-2^{-k})$.

1. Since $v_k \le |u|$ we have

$$d_k^2 \ge \frac{v_k}{|u|}|Du|^2 \ge \left(\frac{v_k}{|u|}|Du|\right)^2.$$

2. We use $|\nabla|u|| \le |Du|$ to get

$$d_k^2 \ge \frac{(1 - 2^{-k}) + v_k}{|u|}|\nabla|u||^2 = |\nabla|u||^2.$$

3. Notice that $\nabla v_k = \nabla|u|$ and we conclude as before.

4. Compute the left-hand side using twice Proposition 4.1 and

$$\nabla(|u|^{-1}) = -|u|^{-2}\nabla|u|.$$

   Thus

$$D\frac{uv_k}{|u|} = \frac{u}{|u|} \otimes \nabla v_k + v_k D\left(\frac{u}{|u|}\right) = \frac{u}{|u|} \otimes \nabla v_k + \frac{v_k}{|u|}Du - \frac{v_k u}{|u|^2} \otimes \nabla|u|$$

   and every summand can be bound by d_k.

□

Sobolev Emeddings

Now we want to obtain a counterpart of the estimate (2.6) (obtained in the parabolic case) but adapted to the Navier-Stokes context. Proceeding with the same technique, we utilize a combination of Sobolev Embeddings and space-time interpolation. Although we exploit the initial regularity of the weak solution, it is important to highlight that no properties related to the geometry of the Navier-Stokes equations or the suitable weak solution condition are involved.

The result is contained in the following lemma, where, unlike in the parabolic case, we do not yet use a test function. Therefore, we need to pay attention because our function does not have compact support, and thus we cannot use the Poincaré inequality (which we implicitly used in previous chapters).

Lemma 4.2. *There exists a constant C such that for every k, and every function $f \in L^\infty(Q_k)$ with $\nabla f \in L^2(Q_k)$*

$$\|f\|_{L^{10/3}(Q_k)} \leq C\left(\|f\|_{L^{\infty,2}(Q_k)} + \|f\|^{2/5}_{L^{\infty,2}(Q_k)}\|\nabla f\|^{3/5}_{L^2(Q_k)}\right)$$

Proof. We use Sobolev embedding with $p = 2$ and $p^* = 6$ on the space variables to get

$$\|f\|_{L^{2,6}(Q_k)} \leq C\|f\|_{H^1(Q_k)} \leq C\left(\|f\|_{L^{\infty,2}(Q_k)} + \|\nabla f\|_{L^2(Q_k)}\right).$$

We underline that the constant does not depend on the domain (and so on k) because the domains are controlled: $[-1/2, 1] \times B(1/2) \subset Q_k \subset [-1, 1] \times B(1)$. Now we use space-time interpolation (2.1) with coefficients with $p = \infty$, $q = 2$, $p' = 2$, $q' = 6$ and $\lambda = 2/5$ to obtain the estimate

$$\|f\|_{L^{10/3}} \leq \|f\|^{2/5}_{L^{\infty,2}} + \|f\|^{3/5}_{L^{2,6}},$$

which implies the thesis. □

With a combination of Lemma 4.1 (point 3) and Lemma 4.2, we obtain the following result (where every norm is considered in Q_k):

$$\begin{aligned}\|v_k\|_{L^{10/3}} \leq& C\left(\|v_k\|_{L^{\infty,2}} + \|v_k\|^{2/5}_{L^{\infty,2}}\ \|\nabla v_k\|^{3/5}_{L^2}\right)\\ \leq& C\left(\|v_k\|_{L^{\infty,2}} + \|v_k\|^{2/5}_{L^{\infty,2}}\ \|d_k\|^{3/5}_{L^2}\right).\end{aligned} \tag{4.1}$$

Since $U_k = \|v_k\|^2_{L^{\infty,2}} + \|d_k\|^2_{L^2}$ we have $\|v_k\|_{L^{\infty,2}(Q_k)} \leq U_k^{1/2}$ $\|d_k\|_{L^2(Q_k)} \leq U_k^{1/2}$. Hence, using (4.1), we get the counterpart of (2.6):

$$\boxed{\|v_k\|_{L^{10/3}(Q_k)} \leq CU_k^{1/2}} \tag{4.2}$$

where the constant C does not depend on k.

Chebyshev's inequality

Continuing to work without directly using the Navier-Stokes equations but only the regularity of its solution, we can prove "our Chebyshev". In particular, there exists a universal constant C (independent from k) such that for all $k \geq 1$ and $q > 1$ we have:

$$
\begin{aligned}
&\|\mathbf{1}_{\{v_k>0\}}\|_{L^q(Q_{k-1})} \le C2^{\frac{10k}{3q}} U_{k-1}^{\frac{5}{3q}} \\
&\|\mathbf{1}_{\{v_k>0\}}\|_{L^{\infty,q}(Q_{k-1})} \le C2^{\frac{2k}{q}} U_{k-1}^{\frac{1}{q}}
\end{aligned}
\tag{4.3}
$$

Proof. By definition if $v_k > 0$ then $v_{k-1} > 2^{-k}$. Using Chebyshev and the estimate (4.2) we find

$$
\begin{aligned}
\|\mathbf{1}_{\{v_k>0\}}\|^q_{L^q(Q_{k-1})} &= \int_{Q_{k-1}} \mathbf{1}_{\{v_k>0\}} \le |\{v_{k-1} > 2^{-k}\} \cap Q_{k-1}| \\
&\le 2^{10k/3} \int_{Q_{k-1}} |v_{k-1}|^{10/3} \le C2^{10k/3} U_{k-1}^{5/3}.
\end{aligned}
$$

Indeed for every $t \in [T_{k-1}; 1]$:

$$
\begin{aligned}
\|\mathbf{1}_{\{v_k(t,\cdot)>0\}}\|^q_{L^q(B_{k-1})} &\le |\{v_{k-1}(t,\cdot) > 2^{-k}\} \cap B_{k-1}| \le 2^{2k} \int_{B_{k-1}} |v_{k-1}(t,x)|^2 dx \\
&\le 2^{2k} \sup_{s\in[T_{k-1},1]} \int_{B_{k-1}} |v_{k-1}(t,x)|^2 dx \le 2^{2k} U_{k-1}
\end{aligned}
$$

and taking the q-th root we obtained the desired conclusion. □

Energy inequality

The third ingredient of De Giorgi's scheme is an Energy inequality. In this case, the geometry of the problem comes into play and guarantees the existence of an inequality to counterbalance Sobolev Embeddings. As expected, we will proceed with the Caccioppoli-Leray technique.

Before attempting this, we need to make use of a technical result whose difficulty is due to the strong nonlinearity of the Navier-Stokes equations. Special attention should be paid to the need for the assumption of a suitable weak solution in order to handle the truncation term v_k, which, unlike in previous chapters, is no longer a solution itself. It is observed how, in order to handle the nonlinearity, the term d_k emerges, demonstrating its usefulness. This is an example of why the De Giorgi (and Nash) solutions, despite being used to solve a linear problem, can be adapted to other non linear contexts.

Lemma 4.3. *Let (u, p) be a suitable weak solution in $Q = (0, \infty) \times \Omega$. Then v_k verifies in the sense of the distribution*

$$
\partial_t \frac{v_k^2}{2} + \operatorname{div}\left(u\frac{v_k^2}{2}\right) + d_k^2 - \Delta\frac{v_k^2}{2} + \operatorname{div}(up) + \left(\frac{v_k}{|u|} - 1\right) u \cdot \nabla p \le 0 \tag{4.4}
$$

Proof. Recall that the pair (u,p), assuming suitability, satisfies:

$$\partial_t \frac{|u|^2}{2} + \operatorname{div}\left(u\frac{|u|^2}{2}\right) - \Delta\frac{|u|^2}{2} + \operatorname{div}(up) + |Du|^2 \leq 0. \tag{4.5}$$

At first, we write v_k^2 as

$$\frac{v_k^2}{2} = \frac{|u|^2}{2} + \frac{v_k^2 - |u|^2}{2}.$$

We can use (4.5) for the first term while, for the second, notice that

$$\begin{aligned}\partial_\alpha\left(\frac{v_k^2 - |u|^2}{2}\right) &= v_k\partial_\alpha v_k - u\cdot\partial_\alpha u\\ &= v_k\partial_\alpha|u| - u\cdot\partial_\alpha u\\ &= v_k\frac{u}{|u|}\cdot\partial_\alpha u - u\cdot\partial_\alpha u\\ &= u\left(\frac{v_k}{|u|} - 1\right)\cdot\partial_\alpha u.\end{aligned}$$

Since $\left|u\left(\frac{v_k}{|u|}-1\right)\right|$ is bounded by 1 we have a uniform bound on the space-time derivatives of $\left(\frac{v_k^2-|u|^2}{2}\right)$ and we are allowed to compute what follows.

We have

$$\begin{aligned}\operatorname{div}\left(u\frac{v_k^2 - |u|^2}{2}\right) &= u\cdot\nabla\left(\frac{v_k^2 - |u|^2}{2}\right) = u\cdot u^T\left(\frac{v_k}{|u|}-1\right)D(u)\\ &= [(u\cdot\nabla)u]\cdot\left[u\left(\frac{v_k}{|u|}-1\right)\right]\end{aligned} \tag{4.6}$$

where for the first equality we have used that $\operatorname{div}(u) = 0$, for the second the previous computation and for the third Proposition 4.1 point 2. Trivially we get also

$$\partial_t\frac{v_k^2 - |u|^2}{2} = \partial_t u\cdot\left[u\left(\frac{v_k}{|u|}-1\right)\right]. \tag{4.7}$$

Then consider the Navier-Stokes equation

$$\partial_t u + (u\cdot\nabla)u + \nabla p - \Delta u = 0$$

and multiply it by $u\left(\frac{v_k}{|u|}-1\right)$. Using (4.6) and (4.7) we obtain

$$\partial_t\frac{v_k^2 - |u|^2}{2} + \operatorname{div}(u\frac{v_k^2 - |u|^2}{2}) + \left[u\left(\frac{v_k}{|u|}-1\right)\right]\cdot\nabla p - \left[u\left(\frac{v_k}{|u|}-1\right)\right]\cdot\Delta u = 0 \tag{4.8}$$

Adding up (4.5) and (4.8) we get

$$\partial_t\frac{v_k^2}{2} + \operatorname{div}\left(u\frac{v_k^2}{2}\right) + \left[u\left(\frac{v_k}{|u|}-1\right)\right]\cdot\nabla p + \operatorname{div}(up) - \left[u\left(\frac{v_k}{|u|}-1\right)\right]\cdot\Delta u - \Delta\frac{|u|^2}{2} + |\nabla u|^2 \leq 0.$$

We are left to prove that

$$-\left[u\left(\frac{v_k}{|u|}-1\right)\right]\cdot\Delta u-\Delta\frac{|u|^2}{2}+|\nabla u|^2=d_k^2-\Delta\frac{v_k^2}{2}. \tag{4.9}$$

We use Proposition 4.1 point 3 with $\bar{u}=u\big(\frac{v_k}{|u|}-1\big)$ and we write

$$\operatorname{div}\left(u^T\left(\frac{v_k}{|u|}-1\right)D(u)\right)=\left[u\left(\frac{v_k}{|u|}-1\right)\right]\cdot\Delta u+D\left(u\left(\frac{v_k}{|u|}-1\right)\right):D(u).$$

For the second term of the right-hand side, we use point 5 of Proposition 4.1:

$$D\left(u\left(\frac{v_k}{|u|}-1\right)\right):D(u)=(u^TD(u))\cdot\nabla\left(\frac{v_k}{|u|}\right)+\left(\frac{v_k}{|u|}-1\right)|D(u)|^2$$

while for the left-hand side, we get

$$\operatorname{div}\left(u^T\left(\frac{v_k}{|u|}-1\right)D(u)\right)=\operatorname{div}\left(\nabla\left(\frac{v_k^2-|u|^2}{2}\right)\right)=\Delta\frac{v_k^2-|u|^2}{2}.$$

Combining the last three equations we have that

$$-[u\left(\frac{v_k}{|u|}-1\right)]\cdot\Delta u=-\Delta\frac{v_k^2-|u|^2}{2}+(u^TD(u))\cdot\nabla\left(\frac{v_k}{|u|}\right)+\left(\frac{v_k}{|u|}-1\right)|D(u)|^2. \tag{4.10}$$

Putting (4.10) into (4.9) we have that:

$$(u^TD(u))\cdot\nabla\left(\frac{v_k}{|u|}\right)+\left(\frac{v_k}{|u|}-1\right)|D(u)|^2=d_k^2-|D(u)|^2.$$

We claim that

$$(u^TD(u))\cdot\nabla\left(\frac{v_k}{|u|}\right)=\frac{(1-2^{-k})\mathbf{1}_{\{|u|\geq\{1-2^{-k})\}}}{|u|}|\nabla|u||^2. \tag{4.11}$$

As always, if $|u|\leq 1-2^{-k}$ everything is trivial, so suppose $|u|\geq 1-2^{-k}$ and the equation (4.11) becomes

$$(u^TD(u))\cdot\nabla\left(\frac{|u|-(1-2^{-k})}{|u|}\right)=\frac{(1-2^{-k})}{|u|}|\nabla|u||^2.$$

The left-hand side is equal to

$$\begin{aligned}(u^TD(u))\cdot\nabla\left(1-\frac{(1-2^{-k})}{|u|}\right)&=-(1-2^{-k})(u^TD(u))\cdot\nabla\left(\frac{1}{|u|}\right)\\&=\frac{(1-2^{-k})}{|u|^2}(u^TD(u))\cdot\nabla|u|\\&=\frac{(1-2^{-k})}{|u|^2}(|u|\nabla|u|)\cdot\nabla|u|\\&=\frac{(1-2^{-k})}{|u|}|\nabla|u||^2\end{aligned}$$

where the third equality is given by Proposition 4.1. □

We are ready to formulate the Energy Inequality for the Navier-Stokes equation. We multiply Equation (4.4) by a positive test function φ defined in x, and integrate it over the entire space and time interval $s < t$.

$$\begin{aligned}
&\left(\int \varphi \frac{|v_k|^2}{2} dx\right)(t) + \int_s^t \int \varphi \, d_k^2 \, dx d\tau \leq \left(\int \varphi \frac{|v_k|^2}{2} dx\right)(s) \\
&+ \int_s^t \int \nabla\varphi \cdot u \frac{|v_k|^2}{2} \, dx d\tau + \int_s^t \int \Delta\varphi \frac{|v_k|^2}{2} \, dx d\tau - \\
&- \int_s^t \int \varphi \left[\operatorname{div}(up) + \left(\frac{v_k}{|u|} - 1\right) u \cdot \nabla p \right] dx d\tau.
\end{aligned} \tag{4.12}$$

To be truly precise, we cannot, a priori, work in this manner, as the test function must have compact support in the space-time domain. To formalize everything, we proceed as in Corollary 3.1, following the idea of [Gal00].

As in the parabolic case, we need to carefully choose the test function in order to refine the previous result.

Corollary 4.1. *We have the following estimate*

$$\begin{aligned}
&\sup_{t\in[T_k,1]} \left(\int_{B_k} \frac{|v_k|^2}{2} dx\right)(t) + \int_{T_k}^t \int_{B_k} d_k^2 \, dx d\tau \right) \leq 2^{k+1} \int_{Q_{k-1}} \frac{|v_k|^2}{2} dx d\tau \\
&+ C2^{3k} \int_{Q_{k-1}} u \frac{|v_k|^2}{2} \, dx d\tau + C2^{6k} \int_{Q_{k-1}} \frac{|v_k|^2}{2} \, dx d\tau + \\
&+ \int_{T_{k-1}}^1 \left| \int \eta_k \operatorname{div}(up) + \left(\frac{v_k}{|u|} - 1\right) u \cdot \nabla p dx \right| d\tau.
\end{aligned} \tag{4.13}$$

where η_k is defined above.

Proof. In (4.12) we choose $\varphi = \eta_k \in C^\infty(\mathbb{R}^3; \mathbb{R})$ such that

$$\eta_k(x) = 1 \quad \text{in } B_k$$

$$\eta_k(x) = 0 \quad \text{in } B_{k-1}^C$$

$$0 \leq \eta(x) \leq 1$$

$$|\nabla \eta_k| \leq C2^{3k}$$

$$|\Delta \eta_k| \leq C2^{6k}$$

A very important observation for the upcoming sections (in particular pressure decomposition) is that the result continues to hold even if η_k is compactly supported in sets smaller than B_{k-1}. The only hypothesis we need is that such sets are between B_k and B_{k-1}. After using the estimates on $\nabla\eta_k$ and $\Delta\eta_k$ we get:

$$\begin{aligned}&\left(\int \eta_k \frac{|v_k|^2}{2}dx\right)(t) + \int_s^t \int \eta_k\, d_k^2\, dx d\tau \leq \left(\int \eta_k \frac{|v_k|^2}{2}dx\right)(s)\\ &+ C2^{3k}\int_s^t \int_{\mathrm{spt}(\eta_k)} u\frac{|v_k|^2}{2}\, dxd\tau + C2^{6k}\int_s^t \int_{\mathrm{spt}(\eta_k)} \frac{|v_k|^2}{2}\, dxd\tau\\ &- \int_s^t \int \eta_k \left[\mathrm{div}(up) + \left(\frac{v_k}{|u|} - 1\right)u\cdot\nabla p\right] dxd\tau.\end{aligned}$$

Playing with the range of s and t, we obtain the following inequality for $T_{k-1} \leq s \leq T_k \leq 1$:

$$\begin{aligned}&\left(\int \eta_k \frac{|v_k|^2}{2}dx\right)(t) + \int_{T_k}^t \int \eta_k\, d_k^2\, dx d\tau \leq \left(\int \eta_k \frac{|v_k|^2}{2}dx\right)(s)\\ &+ C2^{3k}\int_{T_{k-1}}^1 \int_{\mathrm{spt}(\eta_k)} u\frac{|v_k|^2}{2}\, dxd\tau + C2^{6k}\int_{T_{k-1}}^1 \int_{\mathrm{spt}(\eta_k)} \frac{|v_k|^2}{2}\, dxd\tau\\ &+ \int_{T_{k-1}}^1 \left|\int \eta_k \left| \mathrm{div}(up) + \left(\frac{v_k}{|u|} - 1\right)u\cdot\nabla p dx\right| d\tau.\end{aligned}$$

Using the properties of the test function and the fact that t is only on the left-hand side we get that, for $T_{k-1} \leq s \leq T_k$,

$$\begin{aligned}&\sup_{t\in[T_k,1]}\left(\int_{B_k} \frac{|v_k|^2}{2}dx\right)(t) + \int_{T_k}^t \int_{B_k} d_k^2\, dx d\tau\Bigg) \leq \left(\int_{B_{k-1}} \frac{|v_k|^2}{2}dx\right)(s)\\ &+ C2^{3k}\int_{Q_{k-1}} u\frac{|v_k|^2}{2}\, dxd\tau + C2^{6k}\int_{Q_{k-1}} \frac{|v_k|^2}{2}\, dxd\tau +\\ &+ \int_{T_{k-1}}^1 \left|\int \eta_k\, \mathrm{div}(up) + \left(\frac{v_k}{|u|} - 1\right)u\cdot\nabla p dx\right| d\tau.\end{aligned}$$

We want to eliminate the dependence from s of the right-hand side. The idea is to apply the averaging operator

$$\fint_{T_{k-1}}^{T_k} ds$$

to each term of the previous inequality. This operator does not act as the identity only on the first term of the right-hand side, resulting in the following expression:

$$\begin{aligned}&\sup_{t\in[T_k,1]}\left(\int_{B_k} \frac{|v_k|^2}{2}dx\right)(t) + \int_{T_k}^t \int_{B_k} d_k^2\, dx d\tau\Bigg) \leq 2^{k+1}\int_{Q_{k-1}} \frac{|v_k|^2}{2}dxds\\ &+ C2^{3k}\int_{Q_{k-1}} u\frac{|v_k|^2}{2}\, dxd\tau + C2^{6k}\int_{Q_{k-1}} \frac{|v_k|^2}{2}\, dxd\tau\\ &+ \int_{T_{k-1}}^1 \left|\int \eta_k \left[\mathrm{div}(up) + \left(\frac{v_k}{|u|} - 1\right)u\cdot\nabla p\right] dx\right| d\tau.\end{aligned}$$

□

The energy decay

We are now ready to merge the results above and derive (3.4). Once again, the Energy inequality competes with Sobolev embeddings and the Chebyshev's inequality enables us to surpass the critical exponent of 1.

In particular, the left-hand side of Corollary 4.1 satisfies

$$U_k \leq 4\,\text{sup}_{t\in[T_k;1]}\left(\int_{B_k}\frac{|v_k(t,x)|^2}{2}dx+\int_{T_k}^t\int_{B_k}d_k^2(\tau,x)dxd\tau\right),$$

and so, up to a multiplicative constant,

$$\begin{aligned}U_k \leq& 2^{k+1}\int_{Q_{k-1}}|v_k(\tau,x)|^2dxd\tau+C2^{3k}\int_{Q_{k-1}}u|v_k(\tau,x)|^2dxd\tau\\&+C2^{6k}\int_{Q_{k-1}}|v_k(\tau,x)|^2dxd\tau\\&+2\int_{T_{k-1}}^1\left|\int_{\mathbb{R}^3}\eta_k(x)\left[\text{div}(up)+\left(\frac{v_k}{|u|}-1\right)u\cdot\nabla p\right]dx\right|d\tau.\end{aligned}\tag{4.14}$$

Now we focus on the second term of the right-hand side. We begin by employing the following decomposition:

$$uv_k^2=\left[u\left(1-\frac{v_k}{|u|}\right)+\frac{uv_k}{|u|}\right]v_k^2$$

and using Lemma 4.1 we get

$$\left|u\left(1-\frac{v_k}{|u|}\right)v_k^2\right|\leq v_k^2\qquad\left|\frac{u}{|u|}v_kv_k^2\right|=v_k^3.$$

This allows us to bound $uv_k^2\leq v_k^2+v_k^3$. By substituting this inequality into (4.14) and rearranging the terms (while also replacing τ with s for simplicity), we obtain:

$$\begin{aligned}U_k \leq& C2^{6k}\int_{Q_{k-1}}|v_k(s,x)|^2dxds+C2^{3k}\int_{Q_{k-1}}|v_k(s,x)|^3dxds\\&+2\int_{T_{k-1}}^1\left|\int_{\mathbb{R}^3}\eta_k(x)\left[\text{div}(up)+\left(\frac{v_k}{|u|}-1\right)u\cdot\nabla p\right]dx\right|ds\end{aligned}\tag{4.15}$$

Now let us consider the first two terms on the right-hand side, and applying Hölder's inequality, we can write:

$$\begin{aligned}&C2^{6k}\int_{Q_{k-1}}|v_k(s,x)|^2+C2^{3k}\int_{Q_{k-1}}|v_k(s,x)|^3\\&\leq C^{6k}\|v_k^2\|_{L^{5/3}(Q_{k-1})}\|\mathbf{1}_{\{v_k>0\}}\|_{L^{5/2}(Q_{k-1})}\\&+C^{3k}\||v_k|^3\|_{L^{10/9}(Q_{k-1})}\|\mathbf{1}_{\{v_k>0\}}\|_{L^{10}(Q_{k-1})}\end{aligned}$$

Then, we can continue bounding the right-hand side. In fact,

$$\|v_k^2\|_{L^{5/3}(Q_{k-1})} = \|v_k\|^2_{L^{10/3}(Q_{k-1})} \leq \|v_{k-1}\|^2_{L^{10/3}(Q_{k-1})} \leq CU_{k-1}$$
$$\|v_k^3\|_{L^{10/9}(Q_{k-1})} = \|v_k\|^3_{L^{10/3}(Q_{k-1})} \leq CU_{k-1}^{3/2}.$$

Finally, thanks to the estimates (4.3) in combination with Hölder's inequality, we can continue the calculation as follows:

$$\begin{aligned} C2^{6k} \int_{Q_{k-1}} |v_k(s,x)|^2 + C2^{3k} \int_{Q_{k-1}} |v_k(s,x)|^3 \\ \leq & C2^{6k} U_{k-1} 2^{4k/3} U_{k-1}^{2/3} + C2^{3k} U_{k-1}^{3/2} 2^{k/3} U_{k-1}^{1/6} \\ = & C2^{6k+4k/3} U_{k-1}^{5/3}. \end{aligned}$$

Summarizing, we have proved:

$$U_k \leq C\ 2^{6k+4k/3}\ U_{k-1}^{5/3} + 2\int_{T_{k-1}}^1 \left| \int \eta_k(x) \left[\operatorname{div}(up) + \left(\frac{v_k}{|u|} - 1 \right) u \cdot \nabla p \right] dx \right| ds. \quad (4.16)$$

We are left with the pressure term.

4.2 Pressure Estimate

We need an estimate for

$$\int_{T_{k-1}}^1 \left| \int \eta_k \left[\operatorname{div}(up) + \left(\frac{v_k}{|u|} - 1 \right) u \cdot \nabla p \right] dx \right| ds.$$

We have previously observed (see (3.8)) that our pressure p satisfies

$$-\Delta p = \sum_{ij} \partial^2_{ij}(u_i u_j). \quad (4.17)$$

A promising approach to gain regularity for p is to utilize the Calderón–Zygmund theory applied to (4.17). However, there is a crucial issue: even if we are interested in the values of p only in a smaller set than Ω (such as the support of η_k), the Calderón–Zygmund theory only guarantees an estimate of the form

$$\|p\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)},$$

without providing a good bound for $\|p\|_{L^p(Q_{k-1})}$. The brilliant idea of Vasseur is to decompose the pressure into two components, denoted as p_1 and p_2 with $p = p_1 + p_2$. One of them, referred to as the local component, solves

$$-\Delta p_2 = \sum_{ij} \varphi \partial^2_{ij}(u_i u_j),$$

where φ is a test function compactly supported in Q_{k-1}. This decomposition allows us to obtain the useful bound:

$$\|p_2\|_{L^p(Q_{k-1})} \leq \|u\|_{L^p(Q_{k-1})}.$$

as seen in (A.3) for a compactly supported right-hand side.

Regarding the other component, p_1, it is harmonic when $\varphi \equiv 1$. The harmonicity provides us with additional regularity for this term.

To be precise we have to fix the set where φ assumes the value 1. To this aim, we need to introduce some intermediate balls:

$$B_{k-1/3} := B(1/2(1 + 2 \cdot 2^{-3k}))$$
$$B_{k-2/3} := B(1/2(1 + 4 \cdot 2^{-3k}))$$

One can check that we have this inclusion

$$B_k \subset B_{k-1/3} \subset B_{k-2/3} \subset B_{k-1}.$$

For the sake of readability, we use the following notation:

$$Q_{k-1/3} = [T_{k-1}, 1] \times B_{k-1/3}$$
$$Q_{k-2/3} = [T_{k-1}, 1] \times B_{k-2/3}.$$

From now on η_k is compactly supported in $B_{k-1/3}$ (instead of B_{k-1}). Notice that we are allowed to proceed in this way thanks to the remark contained in Corollary 4.1.

Pressure decomposition

For the sake of completeness, we present the following proposition that captures the technical details of the pressure decomposition, which we are going to prove.

Prior to stating it, we underline the structure of our localization test function $\varphi_k \in C_c^\infty(B_{k-1})$, which will be equal to 1 in $B_{k-2/3}$. In this way, our non-local part (which will be called p_{k1}) will be harmonic in $B_{k-2/3}$. To obtain our estimate, we also impose the conditions $|\nabla \varphi_k| \leq C2^{3k}$ and $|\Delta \varphi_k| \leq C2^{6k}$.

Proposition 4.2. *Suppose $p > 1$, $1 \leq i, j \leq 3$ and let p a solution in Q_{k-1} to*

$$-\Delta p = \sum_{ij} \partial^2_{ij} G_{ij}.$$

Then we can decompose $p|_{B_{k-2/3}}$ into two parts $p|_{B_{k-2/3}} = p_{k1}|_{B_{k-2/3}} + p_{k2}|_{B_{k-2/3}}$, where:

- *p_{k2} solves*
$$-\Delta p_{k2} = \sum_{ij} \partial^2_{ij}(\varphi_k G_{ij}) \quad \text{in } [T_{k-1}, 1] \times \mathbb{R}^3$$
with φ_k defined as above.

- *For every $(t,x) \in Q_{k-1/3}$ we have the following estimate:*
$$|p_{k1}(t,x)| + |\nabla p_{k1}(t,x)| \leq C 2^{12k} \left(\int_{B_{k-1}} |p(t,y)| dy + \int_{B_{k-1}} \sum_{ij} |G_{ij}(t,y)| dy \right) \quad (4.18)$$

Proof. We start using the definition of p and p_{k2} and do the following computation (Einstein's notation):

$$\begin{aligned} &-\Delta(\varphi_k p) = \varphi_k \Delta p + \Delta\varphi_k p + \nabla\varphi_k \cdot \nabla p. \\ &-\Delta p_{k_2} = \partial_i \varphi_k \partial_j G_{ij} + \partial_j \varphi_k \partial_i G_{ij} + \partial^2_{ij}\varphi_k G_{ij} + \varphi_k \partial^2_{ij} G_{ij}. \end{aligned}$$

Combining the two equations we obtain:

$$-\Delta(p_{k2} - \varphi_k p) = \partial_i \varphi_k \partial_j G_{ij} + \partial_j \varphi_k \partial_i G_{ij} + \partial^2_{ij}\varphi_k G_{ij} - \Delta\varphi_k p - \nabla\varphi_k \cdot \nabla p.$$

Remembering that $\frac{1}{4\pi|x|}$ is the fundamental solution of the Laplacian in $\mathbb{R}^3$ for a Dirac delta in the origin we set

$$\begin{aligned} p_0^1 &= \frac{1}{4\pi|x|} * (\partial_i \varphi_k \partial_j G_{ij}), \\ p_0^2 &= \frac{1}{4\pi|x|} * (\partial_j \varphi_k \partial_i G_{ij}), \\ p_0^3 &= \frac{1}{4\pi|x|} * (\partial^2_{ij} \varphi_k G_{ij}), \\ p_0^4 &= -\frac{1}{4\pi|x|} * (\Delta\varphi_k \, p), \\ p_0^5 &= -\frac{1}{4\pi|x|} * (\nabla\varphi_k \cdot \nabla p). \end{aligned}$$

Finally, we set $p_0 = p_0^1 + p_0^2 + p_0^3 + p_0^4 + p_0^5$. With this notation we have

$$p_{k2} = \varphi_k p - p_0;$$

$$p_{k1} = (1 - \varphi_k)p + p_0.$$

The first three terms of p_0 can be bounded in the same way. We show only the first one:

$$p_0^1 = \partial_j \frac{1}{4\pi|x|} * (\partial_i \varphi_k G_{ij}) - \frac{1}{4\pi|x|} * (\partial^2_{ij} \varphi_k G_{ij}).$$

Hence we have (we are not writing the sum and the constants):

$$p_0^1(t,x) = \int_{B_{k-1}\setminus B_{k-2/3}} \frac{-1}{|x-y|^2}\partial_i\varphi_k(y)G_{ij}(t,y)dy - \int_{B_{k-1}\setminus B_{k-2/3}} \frac{1}{|x-y|}\partial^2_{ij}\varphi_k(y)G_{ij}(t,y)dy$$

where we have used that $\nabla\varphi_k \equiv 0$ in $B_{k-2/3} \cup B^c_{k-1}$. If $x \in B_{k-1/3}$ we have that $|x-y| \geq 2^{-3k}$ and so:

$$\begin{aligned}|p_0^1(t,x)| &\leq 2^{6k}\|\varphi_k\|_{C^1(B_{k-1})}\int_{B_{k-1}}|G_{ij}(t,y)|dy + 2^{3k}\|\varphi_k\|_{C^2(B_{k-1})}\int_{B_{k-1}}|G_{ij}(t,y)|dy \\ &\leq 2^{9k}\int_{B_{k-1}}|G_{ij}(t,y)|dy.\end{aligned}$$

Similarly, for the last two terms (again $x \in B_{k-1/3}$), we get

$$|p_0^4(t,x)| \leq 2^{9k}\int_{B_{k-1}}|p(t,y)|dy$$

and so we conclude

$$|p_0(t,x)| \leq 2^{9k}\left(\int_{B_{k-1}}|p(t,y)|dy + \int_{B_{k-1}}|G_{ij}(t,y)|dy\right).$$

Since $1-\varphi_k$ is null in $Q_{k-2/3}$ we have the same estimate for p_{k1}.

For ∇p_{k1}, we can utilize the fact that the gradient of a harmonic function stays harmonic. We can follow a similar procedure as before. By distributing the new derivative to the term $\frac{1}{|x-y|}$, we increase its exponent by one, resulting in the estimate being multiplied by a factor of 2^{3k}. Indeed we get

$$|\nabla p_{k1}(t,x)| \leq 2^{12k}\left(\int_{B_{k-1}}|p(t,y)|dy + \int_{B_{k-1}}|G_{ij}(t,y)|dy\right). \qquad \square$$

The pressure decomposition is essentially an adaptation of the previous proposition to our specific context. We have just to choose our G_{ij} to be u_iu_j as suggested by (3.8).

Theorem (Pressure Decomposition)**.** *If (u,p) is a suitable weak solution in Q_{k-1} then we can decompose $p|_{B_{k-2/3}}$ into two parts*

$$p|_{B_{k-2/3}} = p_{k1}|_{B_{k-2/3}} + p_{k2}|_{B_{k-2/3}}$$

where:

- p_{k2} *solves*

$$-\Delta p_{k2} = \sum_{i,j}\partial^2_{ij}(\varphi_k u_j u_i) \quad \textit{in } [T_{k-1},1]\times\mathbb{R}^3$$

with φ_k as before.

- *Moreover, for every $p > 1$, we have the following bound*

$$\|\nabla p_{k1}\|_{L^{p,\infty}(Q_{k-1/3})} + \|p_{k1}\|_{L^{p,1}(Q_{k-1/3})} \leq C2^{12k}\big(\|p\|_{L^{p,1}(Q_{k-1})} + 1\big) \tag{4.19}$$

Proof. We fix the quantity $G_{ij} = u_i u_j$ such that

$$-\Delta p = \sum_{i,j} \partial_i \partial_j G_{ij}.$$

Integrating (4.18) for $p > 1$, we obtain

$$\|\nabla p_{k1}\|_{L^{p,\infty}(Q_{k-1/3})} + \|p_{k1}\|_{L^{p,\infty}(Q_{k-1/3})} \leq C2^{12k}\left(\|p\|_{L^{p,1}(Q_{k-1})} + \sum_{i,j} \|G_{ij}\|_{L^{p,1}(Q_{k-1})}\right).$$

By applying Holder's inequality, we have

$$\sum_{i,j} \|G_{ij}\|_{L^{p,1}(Q_{k-1})} \lesssim \sum_{i,j} \|G_{ij}\|_{L^{\infty,1}(Q_{k-1})} \lesssim \|u\|^2_{L^{\infty,2}(Q_{k-1})} \leq 1,$$

where the last inequality follows from $\|u\|^2_{L^{\infty,2}(Q_{k-1})} \leq \|u\|^2_{L^{\infty,2}([-1,1]\times B(1))} \leq U_0 \leq 1$, which is our starting assumption in Proposition 3.4. □

We conclude the section on Pressure Decomposition with a straightforward Corollary that will be used at the end of the chapter to prove Proposition 3.5.

Corollary 4.2. *Consider (u, p) a a suitable weak solution in Q_0 and define for every t*

$$\overline{u}(t) = \fint_{B(1)} u(t,x)dx \quad \text{and} \quad \overline{p}(t) = \fint_{B(1)} p(t,x)dx$$

Then we can decompose $(p - \overline{p})_{[-1/2,1/2]\times B(1/2)}$ into two parts: p_1 and p_2, where

1. *p_2 is solution to*

$$-\Delta p_2 = \sum_{ij} \partial_i \partial_j [\varphi_1 (u_j - \overline{u}_j)(u_i - \overline{u}_i)]$$

   with φ_1 a test function compactly supported in B_0 and equal to 1 in $B_{1-2/3}$. We require also $|\nabla \varphi_1| \leq C2^3$ and $|\nabla^2 \varphi_1| \leq C2^6$.

2. *p_1 is harmonic in $[-1,1] \times B_{1-2/3}$ (and in particular in $[-1/2,1/2] \times B(1/2)$). Moreover, we have the estimate for $p > 1$:*

$$\|p_1\|_{L^{p,\infty}((-1/2,1/2]\times B(1/2))} \leq C\left(\|p - \overline{p}\|_{L^{p,1}(Q_0)} + \|u - \overline{u}\|^2_{L^{p,2}(Q_0)}\right)$$

Proof. We have

$$-\Delta(p-\overline{p}) = \sum_{i,j} \partial_i \partial_j [(u_j - \overline{u}_j)(u_i - \overline{u}_i)].$$

Setting $G_{ij} = (u_j - \overline{u}_j)(u_i - \overline{u}_i)$ and integrating in time (4.18) we get, for $p > 1$,

$$\begin{aligned} \|p_{k1}\|_{L^{p,\infty}(B_{1-1/3})} &\leq C\big(\|p-\overline{p}\|_{L^{p,1}(Q_0)} + \sum_{i,j} \|G_{ij}\|_{L^{p,1}(Q_0)}\big) \\ &\leq C\big(\|p-\overline{p}\|_{L^{p,1}(Q_0)} + \|u-\overline{u}\|_{L^{p,2}(Q_0)}\big) \end{aligned}$$

where we have used $k = 1$ and $\|u_i - \overline{u}_i\| \leq \|u - \overline{u}\|$. Noticing that

$$B(1/2) \subset B_{1-1/3}$$

we have done. □

Non-local term

Once we have the pressure decomposition we can estimate the pressure with the sum of two components:

$$\begin{aligned} &\int_{T_{k-1}}^{1} \left| \int_{B_{k-1/3}} \eta_k \left[\operatorname{div}(u p_{k1}) + \left(\frac{v_k}{|u|} - 1 \right) u \cdot \nabla p_{k1} \right] dx \right| dt + \\ &\int_{T_{k-1}}^{1} \left| \int_{B_{k-1/3}} \eta_k \left[\operatorname{div}(u p_{k2}) + \left(\frac{v_k}{|u|} - 1 \right) u \cdot \nabla p_{k2} \right] dx \right| dt. \end{aligned}$$

By (4.18) we have that $\nabla p_{k1} \in L^{p,\infty}(Q_{k-1/3})$ and, since $u \in L^{\infty,2}$, we have

$$\operatorname{div}(u p_{k1}) = u \cdot \nabla p_{k1} \quad \in L^{p,2}(Q_{k-1/3}).$$

Then we can expand the non-local term and obtain:

$$\int_{T_{k-1}}^{1} \left| \int_{B_{k-1/3}} \eta_k \frac{v_k u}{|u|} \nabla p_{k1} dx \right| dt \leq \int_{Q_{k-1/3}} |v_k \nabla p_{k1}| \,. \tag{4.20}$$

To obtain an estimate for (4.20), we need to consider three different cases based on the values of p. In all cases, we utilize a combination of Hölder's inequality, the estimate for v_k (4.2), and Chebyshev's estimate (4.3), which we include here for the sake of readability:

$$\begin{aligned} &\|v_k\|_{L^{10/3}(Q_k)} \leq C U_k^{1/2} \\ &\|\mathbf{1}_{\{v_k>0\}}\|_{L^q(Q_{k-1})} \leq C 2^{\frac{10k}{3q}} U_{k-1}^{\frac{5}{3q}} \\ &\|\mathbf{1}_{\{v_k>0\}}\|_{L^{\infty,q}(Q_{k-1})} \leq C 2^{\frac{2k}{q}} U_{k-1}^{\frac{1}{q}}. \end{aligned} \tag{4.21}$$

Case $p > 10$

In this case, we estimate (4.20) using the following bound:

$$C\|v_k\|_{L^{10/3}(Q_{k-1})}\|\nabla p_{k1}\|_{L^{p,\infty}(Q_{k-1/3})}\|\mathbf{1}_{\{v_k>0\}}\|_{L^{q,10/7}(Q_{k-1/3})}, \tag{4.22}$$

where q satisfies ($p > 1$)

$$\frac{3}{10} + \frac{1}{q} + \frac{1}{p} = 1 \quad \Longrightarrow \frac{1}{q} = \frac{7}{10} - \frac{1}{p} \quad \Longrightarrow q > \frac{10}{7}.$$

Since $q > 10/7$, we can bound (4.22) with

$$C\|v_k\|_{L^{10/3}(Q_{k-1})}\|\nabla p_{k1}\|_{L^{p,\infty}(Q_{k-1/3})}\|\mathbf{1}_{\{v_k>0\}}\|_{L^{q}(Q_{k-1})}.$$

Using the estimates from (4.21), we have

$$\|v_k\|_{L^{10/3}(Q_{k-1})}\|\mathbf{1}_{\{v_k>0\}}\|_{L^{q}(Q_{k-1})} \leq CU_{k-1}^{1/2}2^{10k/(3q)}U_{k-1}^{5/(3q)} = C2^{(7k/3-10k/3p)}U_{k-1}^{5/3(1-1/p)}.$$

Then, using the estimate from Pressure Decomposition, in particular, (4.19) we conclude that for $p > 10$ we can control the non-local part with:

$$\int_{Q_{k-1/3}} |v_k \nabla p_{k1}| \leq \; C(1 + \|p\|_{L^{p,1}(Q_{k-1})})2^{\frac{7k}{3}-\frac{10k}{3p}}U_{k-1}^{\frac{5}{3}(1-\frac{1}{p})}$$

which is compatible with the thesis of Proposition 3.4.

Case $2 \leq p \leq 10$

We bound (4.20) by

$$C\|v_k\|_{L^{\infty,2}(Q_{k-1/3})}\|\nabla p_{k1}\|_{L^{p,\infty}(Q_{k-1/3})}\|\mathbf{1}_{\{v_k>0\}}\|_{L^{p',2}(Q_{k-1/3})}. \tag{4.23}$$

Since $p' \leq 2$, we can control (4.23) with

$$C\|v_k\|_{L^{\infty,2}(Q_{k-1/3})}\|\nabla p_{k1}\|_{L^{p,\infty}(Q_{k-1/3})}\|\mathbf{1}_{\{v_k>0\}}\|_{L^{2}(Q_{k-1})}.$$

By (4.21), we have

$$\|v_k\|_{L^{\infty,2}(Q_{k-1/3})}\|\mathbf{1}_{\{v_k>0\}}\|_{L^{2}(Q_{k-1})} \leq C2^{5k/3}U_{k-1}^{4/3},$$

and we conclude as before.

Case $p \leq 2$

Again, we bound (4.20) by

$$C\|v_k\|_{L^{\infty,2}(Q_{k-1/3})}\|\nabla p_{k1}\|_{L^{p,\infty}(Q_{k-1/3})}\|\mathbf{1}_{\{v_k>0\}}\|_{L^{p',2}(Q_{k-1/3})}$$

and this time with

$$C\|v_k\|_{L^{\infty,2}(Q_{k-1/3})}\|\nabla p_{k1}\|_{L^{p,\infty}(Q_{k-1/3})}\|\mathbf{1}_{\{v_k>0\}}\|_{L^{p'}(Q_{k-1})}\|\mathbf{1}_{\{v_k>0\}}\|_{L^{\infty,\frac{2p}{2-p}}(Q_{k-1})}$$

where we have used the fact that

$$1 = \frac{1}{2} + \frac{1}{p'} + \frac{2-p}{2p}.$$

By applying all three points of (4.21), we obtain

$$\|v_k\|_{L^{\infty,2}(Q_{k-1/3})}\|\mathbf{1}_{\{v_k>0\}}\|_{L^{p'}(Q_{k-1})}\|\mathbf{1}_{\{v_k>0\}}\|_{L^{\infty,\frac{2p}{2-p}}(Q_{k-1})} \leq C2^{\frac{7k}{3}-\frac{4k}{3p}}U_{k-1}^{\frac{5}{3}-\frac{2}{3p}}.$$

Summarizing we have that the non-local pressure term is bounded by

$$C2^{k\alpha_p}U_{k-1}^{\beta_p}\left(\|p\|_{L^{p,1}(Q_{k-1})}+1\right) \tag{4.24}$$

where:

1. For $p > 10$ we have $\alpha_p = 12 + \frac{7}{3} - \frac{10}{3p}$ and $\beta_p = \frac{5}{3}\left(1 - \frac{1}{p}\right)$.
2. For $2 \leq p \leq 10$ we have $\alpha_p = 12 + \frac{5}{3}$ and $\beta_p = \frac{4}{3}$.
3. For $p < 2$ we have $\alpha_p = 12 + \frac{7}{3} - \frac{4}{3p}$ and $\beta_p = \frac{5}{3} - \frac{2}{3p}$.

Notice that β_p is greater than 1 for very $p > 1$. Moreover if $p > 10$ the exponent β_p is greater then $3/2$.

Local term

We have to find an estimate for

$$\int_{T_{k-1}}^{1}\left|\int_{B_{k-1/3}}\eta_k\left[\operatorname{div}(up_{k2}) + \left(\frac{v_k}{|u|}-1\right)u\cdot\nabla p_{k2}\right]dx\right|ds. \tag{4.25}$$

As we anticipated, dealing with a local term allows us to apply Calderón-Zygmund theory to gain regularity of p_{k2}. We start splitting once again p_{k2} into three components: p_{k21}, p_{k22}, and p_{k23}, where for $1 \leq h \leq 3$,

$$-\Delta p_{k2h} = \sum_{ij} \partial^2_{ij} f^h_{ij}.$$

and

$$\begin{aligned} f^1_{ij} &= \varphi_k u_j \left(1 - \frac{v_k}{|u|}\right) u_i \left(1 - \frac{v_k}{|u|}\right), \\ f^2_{ij} &= 2\varphi_k u_j \left(1 - \frac{v_k}{|u|}\right) u_i \frac{v_k}{|u|}, \\ f^3_{ij} &= \varphi_k u_j \frac{v_k}{|u|} u_i \frac{v_k}{|u|}. \end{aligned}$$

Then we split the local term (4.25) in three parts

$$\begin{aligned} PT1 &= \int_{T_{k-1}}^{1} \left| \int_{B_{k-1/3}} \eta_k(x) \left[\operatorname{div}(u p_{k21}) + \left(\frac{v_k}{|u|} - 1\right) u \cdot \nabla p_{k21} \right] dx \right| ds, \\ PT2 &= \int_{T_{k-1}}^{1} \left| \int_{B_{k-1/3}} \eta_k(x) \left[\operatorname{div}(u p_{k22}) + \left(\frac{v_k}{|u|} - 1\right) u \cdot \nabla p_{k22} \right] dx \right| ds, \\ PT3 &= \int_{T_{k-1}}^{1} \left| \int_{B_{k-1/3}} \eta_k(x) \left[\operatorname{div}(u p_{k23}) + \left(\frac{v_k}{|u|} - 1\right) u \cdot \nabla p_{k23} \right] dx \right| ds. \end{aligned}$$

The idea is that f^1_{ij} is bounded by Lemma 4.1 and, using Calderón–Zygmund theory, this gives us enough regularity on p_{k21} to find the estimate for PT1. For f^2_{ij} and f^3_{ij} we do not have the L^∞ property but we can restrict the study to $\{u > 1 - 2^{-k}\}$. Otherwise, if $v_k = 0$ we have that p_{k22} and p_{k23} are harmonics and so smooth. This allows us to expand the divergence term and trivially we get $PT2 = PT3 = 0$.
Before going into the details, let us introduce some basic facts about the Calderón–Zygmund theory.

PT1 Estimate

By Lemma 4.1, we have $\left| u \left(1 - \frac{v_k}{|u|}\right) \right| \leq 1$, which implies that $\varphi_k u_j \left(1 - \frac{v_k}{|u|}\right) \in L^q$ for every $1 < q < \infty$.

Applying Calderón-Zygmund theory (specifically, the Calderón-Zygmund estimate (A.3)), we obtain $\|p_{k21}\|_{L^q(Q_{k-1})} \leq C_q$ for every $1 < q < \infty$.

Furthermore, by Calderón-Zygmund theory, we know that ∇p_{k21} has the same summability as

$$\nabla\left(\varphi_k u_j\left(1-\frac{v_k}{|u|}\right)u_i\left(1-\frac{v_k}{|u|}\right)\right) = \nabla\varphi_k u_j\left(1-\frac{v_k}{|u|}\right)u_i\left(1-\frac{v_k}{|u|}\right) + \varphi_k\nabla\left(u_j\left(1-\frac{v_k}{|u|}\right)\right)u_i\left(1-\frac{v_k}{|u|}\right) + \varphi_k u_j\left(1-\frac{v_k}{|u|}\right)\nabla\left(u_i\left(1-\frac{v_k}{|u|}\right)\right).$$

The first term of the right-hand side belongs to L^∞. The second and third terms are in L^2 because, thanks to Lemma 4.1

$$D\left(u\left(1-\frac{v_k}{|u|}\right)\right) = Du - D\left(u\frac{v_k}{|u|}\right) \quad \in L^2.$$

Then we conclude ∇p_{k21} belongs to L^2 and so we can compute:

$$\operatorname{div}(up_{k21}) + \left(\frac{v_k}{|u|}-1\right)u\cdot\nabla p_{k21} = u\frac{v_k}{|u|}\cdot\nabla p_{k21} = \operatorname{div}\left(\frac{v_k u}{|u|}p_{k21}\right) - p_{k21}\operatorname{div}\left(\frac{uv_k}{|u|}\right)$$

and estimate

$$\begin{aligned} PT1 \leq & \int_{T_{k-1}}^1 \left|\int p_{k21}v_k\frac{u}{|u|}\cdot\nabla\eta_k dx\right|ds + \int_{T_{k-1}}^1\left|\int p_{k21}\operatorname{div}\left(\frac{uv_k}{|u|}\right)\eta_k dx\right|ds \\ \leq & C2^{3k}\int_{Q_{k-1}} v_k|p_{k21}|dxds + \int_{Q_{k-1}}|p_{k21}|\left|D\frac{uv_k}{|u|}\right|dxds \\ \leq & C2^{3k}\int_{Q_{k-1}} v_k|p_{k21}|dxds + C\int_{Q_{k-1}}|p_{k21}|d_k\ dxds. \end{aligned}$$

It is easy to check that $d_k \leq \sqrt{2}d_{k-1}$ and so $\|d_k\|_{L^2(Q_{k-1})} \leq \sqrt{2}U_{k-1}^{1/2}$. In particular, for $q \geq 10/3$, we get

$$\begin{aligned} PT1 \leq & C2^{3k}\|v_k\|_{L^{10/3}}\|p_{k21}\|_{L^q}\|\mathbf{1}_{\{v_k>0\}}\|_{L^{\frac{10q}{7q-10}}} + C\|d_k\|_{L^2}\|p_{k21}\|_{L^q}\|\mathbf{1}_{\{v_k>0\}}\|_{L^{\frac{2q}{q-2}}} \\ \leq & C\left(U_{k-1}^{\frac{5}{3}(1-\frac{1}{q})} + U_{k-1}^{\frac{4}{3}+\frac{5}{3q}}\right) \end{aligned}$$

where for the first bound we have used Hölder with exponents

$$\frac{3}{10}+\frac{1}{q}+\frac{7q-10}{10q} = 1 \qquad \frac{1}{2}+\frac{1}{q}+\frac{q-2}{2q} = 1.$$

Hence, for q large enough, we have

$$PT1 \leq C^k U_{k-1}^{\frac{5}{4}}. \tag{4.26}$$

Estimate PT2 and PT3

By Calderón–Zygmund we have

$$\begin{aligned}\|p_{k22}\|_{L^{10/3}} &\leq C\sum_{i,j}\|u_j(1-v_k/|u|)\|_{L^\infty}\|v_k u_i/|u|\|_{L^{10/3}}\\ &\leq C\|v_k\|_{L^10/3} \leq U_{k-1}^{1/2} \quad \text{and}\\ \|p_{k23}\|_{L^{5/3}} &\leq C\sum_{i,j}\|u_j v_k/|u|)\|_{L^{10/3}}\|v_k u_i/|u|\|_{L^{10/3}}\\ &\leq C\|v_k\|^2_{L^{10/3}} \leq U_{k-1}.\end{aligned} \tag{4.27}$$

This time we need to control their gradients too.

Lemma 4.4. *Writing*

$$\begin{aligned}\nabla p_{k22} =& G_{221} + G_{222} + G_{223}\\ \nabla p_{k23} =& G_{231} + G_{232}\end{aligned}$$

we have

1. $|G_{221}\|_{L^{\frac{10}{3}}(Q_{k-1/3})} \leq C2^{3k}\|v_k\|_{L^{10/3}(Q_{k-1})} \leq C2^{3k}U^{1/2}$.
2. $\|G_{222}\|_{L^2(Q_{k-1/3})} \leq C\|d_k\|_{L^2(Q_{k-1})} \leq CU^{1/2}$.
3. $\|G_{223}\|_{L^{\frac{5}{4}}(Q_{k-1/3})} \leq C\|v_k\|_{L^{10/3}(Q_{k-1})}\|d_k\|_{L^2(Q_{k-1})} \leq CU$.
4. $\|G_{231}\|_{L^{\frac{5}{3}}(Q_{k-1/3})} \leq C2^{3k}\|v_k\|^2_{L^{10/3}(Q_{k-1})} \leq C2^{3k}U$.
5. $\|G_{232}\|_{L^{\frac{5}{4}}(Q_{k-1/3})} \leq C\|v_k\|_{L^{10/3}(Q_{k-1})}\|d_k\|_{L^2(Q_{k-1})} \leq CU$.

Before proving the lemma let us see how we can conclude. As we have already discussed we can consider the integrals only where $\{|u| \geq 1-2^{-k}\}$:

$$\begin{aligned}PT2+PT3 =& \int_{T_{k-1}}^1 \left|\int_{B_{k-1/3}} \eta_k(x)\left[\operatorname{div}(u(p_{k22}+p_{k23}) + \left(\frac{v_k}{|u|}-1\right)u\cdot\nabla(p_{k22}+p_{k23}\right]dx\right|ds\\ \leq& \int_{T_{k-1}}^1\int_{B_{k-1/3}} |\nabla\eta_k||u||p_{k22}+p_{k23}|dxdt + \int_{T_{k-1}}^1\int_{B_{k-1/3}}(|\nabla p_{k22}|+|\nabla p_{k23}|)dxdt\\ \leq& C2^{3k}\int_{T_{k-1}}^1\int_{B_{k-1/3}}(1+v_k)|p_{k22}+p_{k23}|dxdt + \int_{T_{k-1}}^1\int_{B_{k-1/3}}(|\nabla p_{k22}|+|\nabla p_{k23}|)dxdt\end{aligned}$$

For the first term on the right-hand side, we use Hölder's inequality to get, up to a multiplicative constant,

$$2^{3k}\left(\|\mathbf{1}_{\{|u|\geq 1-2^{-3k}\}}\|_{L^{10/7}(Q_{k-1})}\|p_{k22}\|_{L^{10/3}(Q_{k-1/3})} + \|\mathbf{1}_{\{|u|\geq 1-2^{-3k}\}}\|_{L^{5/2}(Q_{k-1})}\|p_{k23}\|_{L^{5/3}(Q_{k-1/3})}\right) +$$
$$2^{3k}\left(\|v_k\|_{L^{10/3}(Q_{k-1})}\|\mathbf{1}_{\{|u|\geq 1-2^{-3k}\}}\|_{L^{5/2}(Q_{k-1})}\|p_{k22}\|_{L^{10/3}(Q_{k-1/3})}\right) +$$
$$2^{3k}\left(\|v_k\|_{L^{10/3}(Q_{k-1})}\|\mathbf{1}_{\{|u|\geq 1-2^{-3k}\}}\|_{L^{10}(Q_{k-1})}\|p_{k23}\|_{L^{5/3}(Q_{k-1/3})}\right)$$

and using (4.27), (4.3) and (4.2) we obtain the estimate

$$C2^{\alpha k}\left(U_{k-1}^{7/6}U_{k-1}^{1/2}+U_{k-1}^{2/3}U_{k-1}+U_{k-1}^{1/2}U_{k-1}^{2/3}U_{k-1}^{1/2}+U_{k-1}^{2/3}U_{k-1}+U_{k-1}^{1/2}U_{k-1}^{1/6}U_{k-1}\right)=C2^{\alpha k}U_{k-1}^{5/3}. \tag{4.28}$$

For the second term, we decompose the gradients of the pressure and, by Hölder's inequality, we get the estimate, up to a multiplicative constant,:

$$\left(\|\mathbf{1}_{\{|u|\geq 1-2^{-3k}\}}\|_{L^{10/7}(Q_{k-1})}\|G_{221}\|_{L^{10/3}(Q_{k-1/3})}+\|\mathbf{1}_{\{|u|\geq 1-2^{-3k}\}}\|_{L^{5/2}(Q_{k-1})}\|G_{231}\|_{L^{5/3}(Q_{k-1/3})}\right)+$$
$$\|\mathbf{1}_{\{|u|\geq 1-2^{-3k}\}}\|_{L^{5}(Q_{k-1})}\left(\|G_{223}\|_{L^{5/4}(Q_{k-1/3})}+\|G_{232}\|_{L^{5/3}(Q_{k-1/3})}\right)+$$
$$\left(\|\mathbf{1}_{\{|u|\geq 1-2^{-3k}\}}\|_{L^{2}(Q_{k-1})}\|G_{222}\|_{L^{2}(Q_{k-1/3})}\right).$$

Using Lemma 4.4 with (4.3) we can keep bound with:

$$C2^{\alpha k}\left(U_{k-1}^{7/6}U_{k-1}^{1/2}+U_{k-1}^{2/3}U_{k-1}\right)+C\left(U_{k-1}^{1/3}U_{k-1}+U_{k-1}^{1/3}U_{k-1}+U_{k-1}^{5/6}U_{k-1}^{1/2}\right)=C2^{\alpha k}U_{k-1}^{5/3}+CU_{k-1}^{4/3}. \tag{4.29}$$

Collecting (4.28) and (4.29) we conclude

$$PT2+PT3\leq C2^{\alpha k}U_{k-1}^{5/3}+CU_{k-1}^{4/3}$$

We are left to prove Lemma 4.4:

Proof. Recall that p_{k23} is a solution of

$$-\Delta p_{k23}=\sum_{i,j}\partial_{ij}\varphi_k u_j\frac{v_k}{|u|}u_i\frac{v_k}{|u|}=\sum_{i,j}\partial_{ij}f_{ij}^3.$$

We decompose $\nabla f_{ij}^3 =: g_{ij}^{31}+g_{ij}^{32}$ where

$$\begin{aligned} g_{ij}^{31} &= \nabla\varphi_k\frac{u_jv_k}{|u|}\frac{u_iv_k}{|u|}\\ g_{ij}^{32} &= \varphi_k\nabla\left(\frac{u_jv_k}{|u|}\right)\frac{u_iv_k}{|u|}+\varphi_k\nabla\left(\frac{u_iv_k}{|u|}\right)\frac{u_jv_k}{|u|}\end{aligned}$$

and so ∇p_{k23} in G_{231} and G_{232} with

$$\begin{aligned} -\Delta G_{231} &= \sum_{i,j}\partial_i\partial_j g_{ij}^{31}\\ -\Delta G_{232} &= \sum_{i,j}\partial_i\partial_j g_{ij}^{32}.\end{aligned}$$

Using, as always, Lemma 4.1 we obtain

$$\begin{aligned} |g_{ij}^{31}| &\leq C2^{3k}v_k^2\\ |g_{ij}^{32}| &\leq Cd_kv_k\end{aligned}$$

and, by Calderón-Zygmund theory and Hölder's inequality, we conclude

$$\|G_{231}\|_{L^{5/3}(Q_{k-1/3})} \le C2^{3k}\|v_k\|^2_{L^{10/3}(Q_{k-1})}$$
$$\|G_{232}\|_{L^{5/4}(Q_{k-1/3})} \le C\|v_k\|_{L^{10/3}(Q_{k-1})}\|d_k\|_{L^2(Q_{k-1})}.$$

For p_{k22} we proceed in the same way: since

$$-\Delta p_{k22} = \sum_{i,j} \partial_{ij} 2\varphi_k u_j \left(1 - \frac{v_k}{|u|}\right) u_i \frac{v_k}{|u|} = \sum_{i,j} \partial_{ij} f^2_{ij}.$$

we decompose $\nabla f^2_{ij} =: g^{21}_{ij} + g^{22}_{ij} + g^{23}_{ij}$ (up to a multiplicative factor 2) where

$$g^{21}_{ij} = \nabla\varphi_k u_j \left(1 - \frac{v_k}{|u|}\right) \frac{u_i v_k}{|u|}$$
$$g^{22}_{ij} = \varphi_k u_j \left(1 - \frac{v_k}{|u|}\right) \nabla\left(\frac{u_i v_k}{|u|}\right) + \varphi_k u_i \left(1 - \frac{v_k}{|u|}\right) \nabla u_j \frac{v_k}{|u|}$$
$$g^{23}_{ij} = -\varphi_k u_j \nabla\left(\frac{v_k}{|u|}\right) \frac{u_i v_k}{|u|}.$$

By means of Lemma 4.1 and by noticing that

$$u_j \nabla\left(\frac{v_k}{|u|}\right) = \frac{u_j}{|u|}\nabla v_k - \frac{v_j u_j}{|u|^2}\nabla|u|$$

we can prove

$$|g_{ij}|^{21} \le C2^{3k} v_k,$$
$$|g_{ij}|^{22} \le C3d_k + Cd_k = Cd_k,$$
$$|g_{ij}|^{23} \le Cd_k v_k$$

and we conclude as before.

□

4.3 Proof of Proposition 3.5

We conclude the Caffarelli-Kohn-Nirenberg's Theorem with the proof of Proposition 3.5. As we have mentioned earlier, there is nothing new in this proof, but we include it here for completeness. Proposition 3.5 states:

Proposition. *For $1 < p \le 4/3$ there exists $\lambda < 1$ and $\varepsilon_0 \le C/2$ (where C is the constant of Theorem 3.5) such that for every suitable weak solution and every $(t_0, x_0) \in (0,\infty)\times\Omega$ the following holds. If*

$$\limsup_{r\to 0} \frac{1}{r}\int_{C_r(x_0,t_0)} |Du|^2 < \varepsilon_0$$

there exists $k_0 > 0$ such that, for any $k \ge k_0$,

$$V_{k+1} \le \frac{V_k}{4} + \frac{C}{4}.$$

We remind that

$$u_k(t,x) = \lambda^k u(\lambda^{2k}t + t_0, \lambda^k x + x_0),$$
$$p_k(t,x) = \lambda^{2k} p(\lambda^{2k}t + t_0, \lambda^k x + x_0),$$

and

$$V_k = \|u_k\|^2_{L^{\infty,2}(Q_0)} + \frac{1}{\lambda^8}\|p_k - \bar{p}_k\|^2_{L^{p,2}(Q_0)} \quad \text{where} \quad \bar{p}_k(t) = \fint_{B_0} p_k(t,x)dx.$$

For the moment we choose $\lambda < 2^{-3}$. Notice that for $(t,x) \in Q_0$:

$$u_{k+1}(t,x) = \lambda u_k(\lambda^2 t, \lambda x).$$

For $-1 \leq t \leq \lambda^2$ and $x \in B_0$ we introduce the backward heat kernel

$$\psi_\lambda(t,x) = \frac{1}{(2\lambda^2 - t)^{3/2}} e^{-\frac{|x|^2}{4(2\lambda^2 - t)}},$$

which verifies

$$\begin{aligned}
&|\psi_\lambda| \leq 1 \quad \text{for } t = -1\\
&|\psi_\lambda| \geq \frac{C}{\lambda^3} \quad \text{for } |x| \leq \lambda,\ -\lambda^2 \leq t \leq \lambda^2\\
&|\Delta\psi_\lambda + |\nabla\psi_\lambda| \leq C \quad \text{for } x \in B^c_{\frac{1}{8}},\ -1 \leq t \leq \lambda^2\\
&|\nabla\psi_\lambda| \leq \frac{C}{\lambda^4} \quad \text{for } x \in \mathbb{R}^3,\ -1 \leq t \leq \lambda^2.
\end{aligned} \tag{4.30}$$

We also define $\eta_1 \in C_0^\infty(B_1)$ to be a standard cutoff function with $\eta_1 \equiv 1$ in $B(1/2)$ (and so $\eta_1 \equiv 1$ in $B(\lambda)$). We set also

$$\overline{u}_k(t) := \fint_{B(1)} u_k(x,t)dx,$$
$$\overline{|u_k|^2}(t) := \fint_{B(1)} |u_k(x,t)|^2 dx.$$

Since (u_k, p_k) is a suitable weak solution, following Corollary 3.1, we multiply the suitable condition (3.9) by $\psi_\lambda \eta_1$ and we integrate in $[-1,s] \times \mathbb{R}^3$ for $-1 \leq s \leq \lambda^2$. Using that space derivatives of $\overline{|u_k|^2}$ and $\overline{p}_k$ are null we obtain

$$\begin{aligned}
\int_{-1}^s \int \partial_t \frac{|u_k|^2}{2}\psi_\lambda \eta_1 \leq &\int_{-1}^s \int \left(\frac{|u_k|^2 - \overline{|u_k|^2}}{2}\right) u_k \cdot \nabla(\eta_1 \psi_\lambda)\\
&+ \int_{-1}^s \int (\psi_\lambda \Delta\eta_1 + 2\nabla\eta_1 \cdot \nabla\psi_\lambda + \eta_1\ \Delta\psi_k)\frac{|u_k|^2}{2}\\
&+ \int_{-1}^s \int \nabla(\eta_1\psi_\lambda) \cdot u_k(p_k - \overline{p}_k).
\end{aligned}$$

Notice that the right-hand side is equal to

$$\int \frac{|u_k(x,s)|^2}{2}\psi_\lambda(x,s)\eta_1(x) - \int \frac{|u_k(x,-1)|^2}{2}\psi_\lambda(x,-1)\eta_1(x) - \int_{-1}^{s}\int \frac{|u_k|^2}{2}\partial_t(\psi_\lambda)\eta_1$$

and, since ψ_λ is the backward heat kernel,

$$\int_{-1}^{s}\int \eta_1\Delta\psi_k\frac{|u_k|^2}{2} = -\int_{-1}^{s}\int \eta_1\partial_t\psi_k\frac{|u_k|^2}{2}.$$

Hence we can delete the common terms and we get

$$\begin{aligned}\int \frac{|u_k(x,s)^2|}{2}\psi_\lambda(x,s)\eta_1(x)\,dx \le &\int \frac{|u_k(x,-1)|^2}{2}\psi_\lambda(x,-1)\eta_1(x)\,dx\\ &+\int_{-1}^{\lambda^2}\int\left|\left(\frac{|u_k|^2-\overline{|u_k|^2}}{2}\right)u_k\cdot\nabla(\eta_1\psi_\lambda)dx\right|dt\\ &+\int_{-1}^{\lambda^2}\left|\int(\psi_\lambda\Delta\eta_1+2\nabla\eta_1\cdot\nabla\psi_\lambda)\frac{|u_k|^2}{2}dx\right|dt\\ &+\int_{-1}^{\lambda^2}\left|\int\nabla(\eta_1\psi_\lambda)\cdot u_k(p_k-\overline{p}_k)dx\right|dt.\end{aligned} \tag{4.31}$$

The left-hand-side is controlled from below by

$$\frac{C}{\lambda^3}\int_{B(\lambda)}\frac{|u_k(s,x)|^2}{2}dx \ge \frac{C}{\lambda^2}\int_{B(1)}|u_{k+1}|^2\left(\frac{s}{\lambda^2},x\right)dx \ge \frac{C}{\lambda^2}V_{k+1},$$

The first summand of the right-hand side is bounded by:

$$\int \frac{|u_k(x,-1)|^2}{2}\psi_\lambda(x,-1)\eta_1(x)\,dx \le V_k.$$

Collecting the results:

$$\begin{aligned}V_{k+1} \le& C\lambda^2V_k + \frac{1}{\lambda^8}\|p_{k+1}-\overline{p}_{k+1}\|^2_{L^{2,p}(Q)} + C\lambda^2\left(\int_{-1}^{\lambda^2}\int\left|\left(\frac{|u_k|^2-\overline{|u_k|^2}}{2}\right)u_k\cdot\nabla(\eta_1\psi_\lambda)dx\right|dt\right)\\ &+C\lambda^2\left(\int_{-1}^{\lambda^2}\left|\int(\psi_\lambda\Delta\eta_1+2\nabla\eta_1\cdot\nabla\psi_\lambda)\frac{|u_k|^2}{2}dx\right|dt+\int_{-1}^{\lambda^2}\left|\int\nabla(\eta_1\psi_\lambda)\cdot u_k(p_k-\overline{p}_k)dx\right|dt\right).\end{aligned}$$

For the second term notice that

$$\left\|u_k\left(\frac{|u_k|^2}{2}-\frac{\overline{|u_k|^2}}{2}\right)\right\|_{L^1(Q_0)} \le \|u_k\|_{L^{\infty,2}(Q_0)}\left\|\frac{|u_k|^2}{2}-\frac{\overline{|u_k|^2}}{2}\right\|_{L^{1,2}(Q_0)}$$

and

$$\begin{aligned}\left\|\frac{|u_k|^2}{2}-\frac{\overline{|u_k|^2}}{2}\right\|_{L^{1,2}(Q_0)} \leq & \left\|\frac{|u_k|^2}{2}-\frac{\overline{|u_k|^2}}{2}\right\|_{L^{1,3}(Q_0)}\\ \leq & C\left\|D\frac{|u_k|^2}{2}\right\|_{L^{1,3/2}(Q_0)}\\ \leq & C\|u_k\|_{L^{2,6}(Q_0)}\|\nabla u_k\|_{L^{2,2}(Q_0)}\\ \leq & C\|Du_k\|_{L^{2,2}(Q_0)}\left(\|u_k\|_{L^{\infty,2}(Q_0)}+\|\nabla u_k\|_{L^{2,2}(Q_0)}\right)\end{aligned}$$

where we have used Sobolev embeddings for the second inequality, Hölder's inequality with exponents 4 and 4/3 for the third inequality, and interpolation along with Sobolev embeddings for the last inequality.
Now using (4.30) we conclude

$$\begin{aligned}\int_{-1}^{\lambda^2}\int\left|\left(\frac{|u_k|^2-\overline{|u_k|^2}}{2}\right)u_k\cdot\nabla(\eta_1\psi_\lambda)\right| \leq & \|\nabla\psi\|_{L^\infty(Q_0)}\|u_k\|_{L^{\infty,2}(Q_0)}\left\|\frac{|u_k|^2}{2}-\frac{\overline{|u_k|^2}}{2}\right\|_{L^{1,2}(Q_0)}\\ \leq & \frac{C}{\lambda^4}\|Du_k\|_{L^{2,2}(Q_0)}V_k+\frac{C}{\lambda^4}\|Du_k\|^2_{L^{2,2}(Q_0)}\sqrt{V_k}\end{aligned}$$

For the third term notice that $\nabla u = 0$ in $B_{1/8}$ and so we can restrict the domain to $B(1)\cap B^c_{1/8}$ where both ψ_λ and $\nabla\psi_\lambda$ are bounded (see (4.30)). Hence we have

$$\int_{-1}^{\lambda^2}\left|\int(\psi_\lambda\Delta\eta_1+2\nabla\eta_1\cdot\nabla\psi_\lambda)\frac{|u_k|^2}{2}dx\right|dt\leq C\|u_k\|_{L^{\infty,2}(Q_0)}\leq CV_k.$$

The last term is controlled by

$$\frac{C}{\lambda^4}\|u_k\|_{L^{\infty,2}(Q_0)}\|p_k-\overline{p}_k\|_{L^{p,2}(Q_0)}\leq C\left(\|u_k\|^2_{L^{\infty,2}(Q_0)}+\frac{\|p_k-\overline{p}_k\|^2_{L^{p,2}(Q_0)}}{\lambda^8}\right)\leq CV_k$$

where we have used Young's inequality.
Summarizing we have proved:

$$V_{k+1}\leq C\lambda^2V_k+\frac{1}{\lambda^8}\|p_{k+1}-\overline{p}_{k+1}\|^2_{L^{2,p}(Q)}+\frac{C}{\lambda^2}\|Du_k\|_{L^{2,2}}V_k+\frac{C}{\lambda^2}\|Du_k\|^2_{L^{2,2}}\sqrt{V_k} \tag{4.32}$$

and we are left with the pressure term.

Using the Corollary 4.2 we decompose $p_k-\overline{p}_k = p_{1k}+p_{2k}$. Since $[-\lambda^2,\lambda^2]\times B(\lambda)$ is contained in $[-1/2,1/2]\times B(1/2)$ the part p_{1k} is harmonic in this latter set and so smooth. Hence, for every $t\in[-\lambda^2,\lambda^2]$, we use Poincaré–Wirtinger inequality to get

$$\frac{1}{|B_\lambda|}\int_{B_\lambda}\left|p_{1k}(x,t)-\fint p_{1k}(y,t)dy\right|^2dx\leq C\frac{\lambda^2}{\lambda^3}\|\nabla p_{1k}(t,\cdot)\|^2_{L^2(B_\lambda)}\leq C\frac{\lambda^2\lambda^3}{\lambda^3}\|\nabla p_{1k}(t,\cdot)\|^2_{L^\infty(B_\lambda)}$$

and, in particular,

$$\frac{1}{|B_\lambda|}\int_{B_\lambda}\left|p_{1k}(x,t)-\fint p_{1k}(y,t)dy\right|^2 dx \le C\lambda^2\|\nabla p_{1k}(t,\cdot)\|^2_{L^\infty(B(1/8)}.$$

Now, we use Theorem 2.10 of [GT83]:

Theorem. *Let u harmonic in Ω and Ω' any compact subset in Ω. Then, for any multi-index α, we have*

$$\sup_{\Omega'}|D^\alpha u| \le \left(\frac{n|\alpha|}{d}\right)^{|\alpha|}\sup_{\Omega}|u|$$

where $d = dist(\Omega', \partial\Omega)$.

In our case we get

$$\|\nabla p_{k1}(t,\cdot)\|^2_{L^\infty(B(1/8))} \le C\|p_{k1}(t,\cdot)\|^2_{L^\infty(B(1/4)} \le C\|p_{k1}(t,\cdot)\|^2_{L^2(B(1/2)}$$

where the last inequality is a consequence of the mean value property for the harmonic function. Combining the previous results we conclude

$$\frac{1}{|B_\lambda|}\int_{B_\lambda}\left|p_{1k}(x,t)-\fint p_{1k}(y,t)dy\right|^2 dx \le C\lambda^2\int_{B(1/2)}|p_{k1}(t,x)|^2dx. \tag{4.33}$$

For p_{2k} notice that, since $p \le 4/3$, then $4/(2-p) \le 6$ and so, by Sobolev embeddings,

$$\|u_k-\overline{u}_k\|^{2/p}_{L^{2,\frac{4}{2-p}}} \le \|u_k\|^{2/p}_{L^{2,2}}.$$

Then Calderon-Zygmund theory, together with Hölder's inequality, gives for $p < 4/3$ that

$$\begin{aligned}
\|p_{2k}\|_{L^{p,2}} &\le C\||u_k-\overline{u}_k|^{2(1-1/p)}\|_{L^{\infty,\frac{p}{p-1}}}\||u_k-\overline{u}_k|^{2/p}\|_{L^{p,\frac{2p}{2-p}}}\\
&\le C|u_k-\overline{u}_k\|^{2(1-1/p)}_{L^{\infty,2}}\|u_k-\overline{u}_k\|^{2/p}_{L^{p,\frac{4}{2-p}}}\\
&\le C|u_k-\overline{u}_k\|^{2(1-1/p)}_{L^{\infty,2}}\|Du_k\|^{2/p}_{L^{2,2}}\\
&\le CV_k^{1-1/p}\|Du_k\|^{2/p}_{L^{2,2}}.
\end{aligned}$$

Using $p_{k+1}(t,x) = \lambda^2 p_k(\lambda^2 t, \lambda x) = \lambda^2\left(p_{1k}(\lambda^2 t,\lambda x)+p_{2k}(\lambda^2 t,\lambda x)\right)$ and a change of variables we get for very $t \in [-1,1]$:

$$\begin{aligned}
&\|p_{k+1}-\overline{p}_{k+1}\|^2_{L^2(B(1))}\\
&\le 2\lambda^4\frac{1}{|B(\lambda)|}\int_{B(\lambda)}\left|p_{1k}(\lambda^2t,x)-\frac{1}{|B(\lambda)|}\int_{B(\lambda)}p_{1k}(\lambda^2t,y)dy\right|^2dx+\\
&+2\lambda^4\frac{1}{|B(\lambda)|}\int_{B(\lambda)}\left|p_{2k}(\lambda^2t,x)-\frac{1}{|B(\lambda)|}\int_{B(\lambda)}p_{2k}(\lambda^2t,y)dy\right|^2dx\\
&\le C\lambda^6\int_{B(1/2)}|p_{k1}(\lambda^2t,x)|^2dx+4\lambda^4\frac{1}{|B(\lambda)|}\int_{B(\lambda)}|p_{2k}(\lambda^2t,x)|^2dx\\
&\le C\lambda^6\int_{B(1/2)}|p_{k1}(\lambda^2t,x)|^2dx+C\lambda\int_{B(\lambda)}|p_{2k}(\lambda^2t,x)|^2dx.
\end{aligned}$$

where we have used (4.33). Therefore

$$\|p_{k+1}-\overline{p}_{k+1}\|^2_{L^{p,2}(Q_0)} \leq C\lambda^{6-\frac{4}{p}}\|p_{1k}\|^2_{L^{p,2}([-\lambda^2,\lambda^2]\times B(1/2))} + C\lambda^{1-\frac{4}{p}}\|p_{2k}\|^2_{L^{p,2}([-\lambda^2,\lambda^2]\times B(\lambda))}$$

and, in particular,

$$\|p_{k+1}-\overline{p}_{k+1}\|^2_{L^{p,2}(Q_0)} \leq C\lambda^{6-\frac{4}{p}}\|p_{1k}\|^2_{L^{p,2}([-1/2,1/2]\times B(1/2))} + C\lambda^{1-\frac{4}{p}}\|p_{2k}\|^2_{L^{p,2}([-1/2,1/2]\times B(1/2))}.$$

The Corollary 4.2 gives the bound:

$$\begin{aligned}\|p_{k1}\|_{L^{p,\infty}([-1/2,1/2]\times B(1/2)} &\leq C\left(\|p_k-\overline{p}_k\|_{L^{p,1}(Q_0)} + \|u_k-\overline{u}_k\|^2_{L^{p,2}(Q_0)}\right)\\ &\leq C\left(\|p_k-\overline{p}_k\|_{L^{p,1}(Q_0)} + \|Du_k\|^2_{L^2(Q_0)}\right)\end{aligned}$$

where in the last inequality we have used $p<2$ and Poincarè inequality.
Finally, collecting the previous estimates we obtain

$$\begin{aligned}\|p_{k+1}-\overline{p}_{k+1}\|^2_{L^{p,2}(Q_0)} &\leq C\lambda^{6-\frac{4}{p}}\|p_{1k}\|^2_{L^{p,2}([-1/2,1/2]\times B(1/2))} + C\lambda^{1-\frac{4}{p}}\|p_{2k}\|^2_{L^{p,2}([-1/2,1/2]\times B(1/2))}\\ &\leq C\lambda^{6-\frac{4}{p}}\left(\|p_k-\overline{p}_k\|^2_{L^{p,1}(Q_0)} + \|Du_k\|^4_{L^2(Q_0)}\right)\\ &\quad + C\lambda^{1-\frac{4}{p}}V_k^{2-2/p}\|Du_k\|^{4/p}_{L^{2,2}(Q_0)}.\end{aligned}$$

Finally, using (4.32), we have

$$\begin{aligned}V_{k+1} \leq& C\lambda^2V_k + \frac{1}{\lambda^8}C\lambda^{6-\frac{4}{p}}\left(\|p_k-\overline{p}_k\|^2_{L^{p,1}} + \|Du_k\|^4_{L^2}\right)\\ &+C\lambda^{1-\frac{4}{p}}V_k^{2-2/p}\|Du_k\|^{4/p}_{L^{2,2}} + \frac{C}{\lambda^2}\|Du_k\|_{L^{2,2}}V_k + \frac{C}{\lambda^2}\|Du_k\|^2_{L^{2,2}}\sqrt{V_k}\\ =&\frac{C}{\lambda^{2+\frac{4}{p}}}\|Du_k\|^4_{L^2} + C(\lambda^2+\lambda^{6-\frac{4}{p}})V_k + \frac{C}{\lambda^2}\|Du_k\|_{L^{2,2}}V_k\\ &+\frac{C}{\lambda^2}\|Du_k\|^2_{L^{2,2}}\sqrt{V_k} + \frac{C}{\lambda^{7+\frac{4}{p}}}\|Du_k\|^{4/p}_{L^2}V_k^{2-2/p}.\end{aligned}$$

Now we use that $V_k^q \leq 1+V_k$ for $0\leq q\leq 1$. Moreover we fix λ in such a way that $C(\lambda^2+\lambda^{6-\frac{4}{p}}) < 1/8$. In this way, we get

$$\begin{aligned}V_{k+1} \leq& C_\lambda\|Du_k\|^4_{L^2} + \frac{1}{8}V_k + C_\lambda\|Du_k\|_{L^{2,2}}V_k\\ &+C_\lambda\|Du_k\|^2_{L^{2,2}}(V_k+1) + C_\lambda\|Du_k\|^{4/p}_{L^2}(V_k+1)\\ =&\frac{1}{8}V_k + A + BV_k,\end{aligned}$$

where C_λ depends on λ and

$$\begin{aligned}A &= \left(C_\lambda\|Du_k\|^4_{L^{2,2}} + C_\lambda\|Du_k\|^2_{L^2} + C_\lambda\|Du_k\|^{4/p}_{L^{2,2}}\right)\\ B &= \left(C_\lambda\|Du_k\|_{L^{2,2}} + C_\lambda\|Du_k\|^2_{L^2} + C_\lambda\|Du_k\|^{4/p}_{L^{2,2}}\right).\end{aligned}$$

Now we choose ε_0 in such a way that if $\|Du_k\|_{L^2} \leq 2\varepsilon_0$ we have

$$\max\{A, B\} \leq \min\left\{\frac{1}{8}, \frac{C}{4}\right\}$$

where C is the constant of the Theorem 3.5. Thanks to (3.15) we have reached the desired conclusion.

Chapter 5

Hölder regularity for Stochastic Heat equation

The regularity theory of De Giorgi, with the assumption of measurability and ellipticity of the diffusion matrix, can be extended successfully to the case of random matrices. The introduction of White Noise does not pose a problem either, as long as its behaviour is assumed to be sufficiently regular. In this chapter, we will follow the article [HWW20] to prove the stochastic counterpart of what we did in Chapter II. For readers who are not familiar with stochastic analysis in infinite dimensions, a brief introduction to the subject can be found in the appendix.

The equation we are dealing with is the stochastic heat equation. Fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ and consider the following stochastic partial differential equation

$$du_t = \tilde{A}(t, \omega, u_t)dt + \tilde{B}(t, u_t)dW_t \tag{5.1}$$

where $t \in [0, \infty)$. Notice that $\tilde{A}$ and $\tilde{B}$ depend also on the time variable. Furthermore $\tilde{A}$ has also a random part described by $\omega \in \Omega$.

The generalised Wiener process we are going to consider is the so-called white noise or cylindrical Wiener process:

$$W = \sum_{i \geq 1} w^i z_i \tag{5.2}$$

where $(z_i)_i$ is an orthonormal basis of $\ell^2(\mathbb{R})$ and $(w^i)_i$ is a family of i.i.d., standard, real Brownian motions. This process is actually an $\ell^2(\mathbb{R})$-valued generalised Wiener process with covariance operator $Q = Id_{\ell^2}$.

The Gelfand Triple we are considering is the classical

$$H_0^1(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \subset H^{-1}(\mathbb{R}^n).$$

Nothing change if we consider, for example, a bounded domain instead of all $\mathbb{R}^n$.

Since the covariance operator is the identity map on $\ell^2(\mathbb{R})$, the operator $\tilde{B}$ is defined in the following spaces (we are implicitly using the fact that we do not need a Hilbert-Schmidt extension of l^2 since our covariance matrix is an identity):

$$\tilde{B} : [0,\infty) \times H_0^1(\mathbb{R}^n) \longrightarrow L_2\left(l^2(\mathbb{R}), L^2(\mathbb{R}^n)\right).$$

Its action on $z \in l^2(\mathbb{R})$ is defined as follows

$$\tilde{B}(t,v)(z) = \sum_i g_i(t,\cdot,v(\,\cdot\,))z_i \;\in L^2(\mathbb{R}^n)$$

where $g = (g_i)_i$ is defined in the spaces $g : [0,\infty) \times \mathbb{R}^n \times \mathbb{R} \to l^2(\mathbb{R})$. Regularity assumptions on g are postponed.

The interesting part of the problem relies on the non-deterministic part of the operator $\tilde{A}$ which is defined in

$$\tilde{A} : [0,\infty) \times H_0^1(\mathbb{R}^n) \longrightarrow H^{-1}(\mathbb{R}^n)$$

where we have omitted the dependence of Ω. Its action on an element of $w \in H_0^1(\mathbb{R}^n)$ is defined as

$$\begin{aligned}\tilde{A}(t,v)(w) &= \int_{\mathbb{R}^n} \operatorname{div}\left(A(t,x)\nabla v(x)\right) w(x) + f(t,x,v(x))w(x)dx \\ &= -\int_{\mathbb{R}^n} \left(A(t,x)\nabla v(x)\right)\cdot \nabla w(x) + f(t,x,v(x))w(x)dx\end{aligned} \tag{5.3}$$

with $A = A(t,x,\omega)$ is an $n \times n$ matrix depending also on $\omega \in \Omega$ and f is a real function defined on $[0,\infty) \times \mathbb{R}^n \times \mathbb{R}$.

An alternative and more intuitive way to see our SPDE is:

$$\partial_t u = \operatorname{div}(A\nabla u) + f(t,x,u) + \sum_{i\geq 1} g_i(t,x,u)\dot{w}_t^i$$

where the diffusion coefficients of A are random. The symbol $\dot{w}_t^i$ is very common in the literature and can be seen as a sort of derivative of the Brownian motion (even though, formally, it is not differentiable).

The hypotheses we require on A, f, g are of two types: one is about measurability and the other one is about regularity. Since we will primarily focus on the regularity aspects, the reader is encouraged to pay closer attention to them. As for the measurability, we assume standard hypotheses:

M1 the diffusion coefficients of A are $(\mathcal{F}_t)_t$ progressive measurable (for fixed x and t);

M2 for fixed x and progressively measurable process h, the process $(g(t,x,h_t))_t$ is also progressively measurable with respect to $(\mathcal{F}_t)_t$.

For regularity, we ask:

R1 Uniform ellipticity: there is a positive constant λ such that

$$\lambda I \leq A(t,x,\omega) \leq \lambda^{-1} I \quad \text{for all } (t,x,\omega) \in [0,\infty) \times \mathbb{R}^n \times \Omega;$$

R2 Linear growth: there exists a non-negative function $K \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and a positive constant Λ such that

$$|f(t,x,v)| + |g(t,x,v)|_{l^2} \leq K(x) + \Lambda |v| \quad \text{for all } (t,x,v) \in [0,\infty) \times \mathbb{R}^n \times \mathbb{R}.$$

We will now proceed to demonstrate that our initial hypotheses are sufficiently strong to ensure the well-definition of our stochastic partial differential equation.

Recalling property 3 of the definition of stochastic PDE (B.7), we observe that equation (5.1) is well-defined if the process $\tilde{B}(t,u_t)$ is stochastically integrable. However, it is important to note that our definition of stochastic integrability only applies to processes defined on intervals of the form $[0,T]$. When the time domain is $[0,\infty)$, we say that a process is stochastically integrable in $[0,\infty)$ if it is stochastically integrable in $[0,T]$ for every $T > 0$. It is important to be careful in this case since we are not allowed to integrate over the entire interval $[0,\infty)$.

Proposition. *Under the previous hypotheses we have that if $[0,\infty) \ni t \mapsto u_t \in H_0^1(\mathbb{R}^n)$ is a square-integrable and progressive measurable process then $(\tilde{B}(t,u_t))_t$ is stochastically integrable in $[0,\infty)$.*

Proof. For simplicity set $\varphi_t = \tilde{B}(t,u_t)$ and consider an arbitrary $T > 0$. We have to prove that

1. φ is predictable with respect to $(\mathcal{F}_t)_t$;
2. $\|\varphi\|_{\mathcal{H}_T^2} < \infty$.

The first thesis is given by M2 (progressive measurable implies predictable) while for the second we have to verify that

$$\mathbb{E}\left[\int_0^T \mathrm{Tr}[(\varphi_t \mathrm{Q}^{1/2})((\varphi_t \mathrm{Q}^{1/2})^*]\mathrm{dt}\right] < \infty.$$

Since in our case $Q = Id_{l^2}$ we have to check that

$$\mathbb{E}\left[\int_0^T \sum_{k\geq 1} \langle \varphi_t \varphi_t^* e_k, e_k \rangle_{L^2(\mathbb{R}^n)}\, dt\right] = \mathbb{E}\left[\int_0^T \sum_{k\geq 1} \|\varphi_t^* e_k\|^2_{l^2(\mathbb{R})}\, dt\right] < \infty. \tag{5.4}$$

We start looking for the adjoint of $\varphi_t \in L_2(\ell^2(\mathbb{R}); L^2(\mathbb{R}^n))$. To this aim consider $z \in \ell^2$ and $f \in L^2$ (the following computations are allowed by R2):

$$\begin{aligned}\langle \varphi_t z, f\rangle_{L^2} = \langle B(t,u_t)z, f\rangle_{L^2} &= \int_{\mathbb{R}^n} \sum_i g_i(t,x,u_t(x)) z_i f(x)\, dx \\ &= \langle z, \int_{\mathbb{R}^n} g(t,x,u_t(x)) f(x)\, dx\rangle_{l^2}.\end{aligned}$$

Hence the adjoint of φ_t is, for $f \in L^2(\mathbb{R}^n)$,

$$\varphi_t^*(f) = \left(\int_{\mathbb{R}^n} g_i(t,x,u_t(x)) f(x)\, dx\right)_i \in l^2(\mathbb{R}).$$

Hence we have

$$\begin{aligned}\sum_{k\geq 1} \|\varphi_t^* e_k\|^2_{l^2(\mathbb{R})} &= \sum_{k\geq 1} \left|\int_{\mathbb{R}^n} g(t,x,u_t(x)) e_k(x)\, dx\right|^2_{l^2} \leq \sum_{k\geq 1} \left(\int_{\mathbb{R}^n} |g(t,x,u_t(x))|_{l^2} e_k(x)\, dx\right)^2 \\ &\leq \left(\sum_{k\geq 1} \int_{\mathbb{R}^n} |g(t,x,u_t(x))|_{l^2} e_k(x)\, dx\right)^2 \leq \left(\sum_{k\geq 1} \int_{\mathbb{R}^n} \left(K(x) + \Lambda |u_t(x)|\right) e_k(x)\, dx\right)^2 \\ &= \left(\sum_{k\geq 1} \langle K, e_k\rangle_{L^2} + \Lambda \langle u_t, e_k\rangle_{L^2}\right)^2 = \left(\|K\|_{L^2} + \Lambda \|u_t\|_{L^2}\right)^2\end{aligned}$$

Finally the term (5.4) is controlled by

$$\mathbb{E}\left[\int_0^T 2\|K\|^2_{L^2} + 2\Lambda \|u_t\|^2_{L^2} dt\right] = 2T\, \|K\|^2_{L^2} + 2\Lambda \mathbb{E}\left[\int_0^T \|u_t\|^2_{L^2} dt\right]$$

where the last term is finite since $(u_t)_t$ is square-integrable. □

Finally, we have proved that equation (5.1) makes sense and we can define, according to (B.8), the notion of weak solution.

Definition 5.1. *A weak solution to* (5.1) *is an* $(\mathcal{F}_t)_{t\in[0,\infty)}$*-adapted process* $[0,\infty) \times \Omega \ni (t,\omega) \mapsto u_t(\omega) \in H_0^1(\mathbb{R}^n)$ *if (up to a modification* [1]*)*

$$u \in L^2(\Omega \times [0,\infty); H_0^1(\mathbb{R}^n)),$$

[1] The process v is a modification of u if $\forall t\ \mathbb{P}(v_t = u_t) = 1$.

and for every $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $t \in [0,\infty)$, $\mathbb{P}$-a.s.

$$\langle u_t(\cdot), \varphi(\cdot)\rangle = \langle u_0, \varphi\rangle - \int_0^t \langle A(s,\cdot)\nabla u_s(\cdot), \nabla\varphi(\cdot)\rangle ds + \int_0^t \langle f(s,\cdot,u_s(\cdot)), \varphi(\cdot)\rangle ds$$
$$+ \sum_{i\geq 1} \int_0^t \langle g_i(s,\cdot,u_s(\cdot)), \varphi(\cdot)\rangle dw_s^i.$$

where every inner product is in $L^2(\mathbb{R}^n)$ and we have explicated the variable with respect to which we are doing such product (with the notation $(\cdot)$). The dependence of u_t and A from $\omega \in \Omega$ are omitted.

Remark. *One may ask why the last term of the previous definition has that form. We do not show a formal proof but we give a precise idea of why we have the following equality:*

$$\int_0^t \langle \tilde{B}(s,u_s(\cdot))dW_s, \varphi(\cdot)\rangle = \sum_{i\geq 1} \int_0^t \langle g_i(s,\cdot,u_s(\cdot)), \varphi(\cdot)\rangle dw_s^i.$$

We start reminding our white noise (5.2)

$$W = \sum_{i\geq 1} w^i z_i$$

where w^i are Brownian motions and $(z_i)_i$ is a basis for l^2. Since W does not depend on the space variable and formally we have $dW = \sum_i dw^i z_i$ we write

$$\int_0^t \langle \tilde{B}(s,u_s(\cdot))dW_s, \varphi(\cdot)\rangle = \sum_{i\geq 1} \int_0^t \langle \tilde{B}(s,u_s(\cdot))z_i, \varphi(\cdot)\rangle dw_s^i.$$

Using the definition of $\tilde{B}$ the last term is equal to

$$\sum_{i\geq 1} \int_0^t \langle \sum_j g_j(t,\cdot,v(\,\cdot\,))(z_i)_j, \varphi(\cdot)\rangle dw_s^i = \sum_{i\geq 1} \int_0^t \langle \sum_j g_j(t,\cdot,v(\,\cdot\,))\delta_{ij}, \varphi(\cdot)\rangle dw_s^i$$

where the last equality holds because $(z_i)_i$ is a basis.

We conclude by emphasizing that we do not have sufficient hypotheses to establish the existence of such a weak solution. However, we will show that if a solution exists, then it possesses α-Hölder continuity.

5.1 De Giorgi's scheme for a Stochastic PDE

The main result of the work is the following estimate.

Theorem 5.1. *Let u be a weak solution (according to definition (5.1)) solution of the SPDE (5.1) with deterministic initial data $u_0 \in C^\infty(\mathbb{R}^n)$. Then there exists a constant $C = C(n, \lambda, \Lambda, T, p)$ such that*

$$\mathbb{E}\left[\|u\|^p_{L^p([0,2T];L^2(\mathbb{R}^n))}\right] + \mathbb{E}\left[\|u\|^p_{L^\infty([T,2T]\times\mathbb{R}^n)}\right] \leq C\left(\|u_0\|_{L^2(\mathbb{R}^n)} + \|K\|_{L^2(\mathbb{R}^n)} + \|K\|_{L^\infty(\mathbb{R}^n)}\right)^p$$

This result will imply the α-Hölder continuity of the solution.

Before going into the proof we spend some words on the theorem.

Remark. *As always the choice of the time interval is arbitrary: during the previous chapters, we have taken $[-1, 1]$ shrinking to $[-1/2, 1]$ according to Vasseur's work. In the stochastic case, it is convenient to use $[0, 2T]$ and $[T, 2T]$. The reason is contained in the classical approach to Martingales and Brownian motions: since the starting points of these particular stochastic processes are $t = 0$ we work with $[0, 2T]$ and $[T, 2T]$. It should be clear that this change is completely irrelevant (since we can scale).*

We conclude by noticing that since we are dealing with an initial data value we have to require also that our weak solution u is a $\mathbb{P}$-almost surely continuous process with respect to the time variable.

The setting

We are going to deal with cylinders because the equation is parabolic. Since the spatial domain is not bounded we set

$$Q_k = [T_k, 2T] \times \mathbb{R}^n.$$

with $T_k = (1 - 2^{-k})T$. In this way, we have $Q_0 = [0, 2T] \times \mathbb{R}^n$ and $Q_\infty = [T, 2T] \times \mathbb{R}^n$. As a truncation we consider, for $a > 1$,

$$u_{k,a} = [u - a(1 - 2^{-k})]_+$$

and the energy is:

$$U_{k,a} := \|u_{k,a}\|^2_{L^{4,2}(Q_k)} = \sqrt{\int_{T_k}^{2T}\left(\int_{\mathbb{R}^n} |u_{k,a}(x,t)|^2 dx\right)^2 dt}.$$

Another crucial quantity that will control the stochastic contribution is

$$X^*_{k-1,a} := \sup_{T_k \leq s \leq t \leq 2T} \int_s^t \sum_i \langle g(\tau, \cdot, u_\tau(\cdot)), u_{k,a}(\tau)(\cdot)\rangle_{L^2} dw^i_\tau$$

Energy Inequality

We need to use the Itô formula in infinite dimensions to obtain the desired inequality. For simplicity, we will state only the formula in the case of a function that does not depend on the time variable. However, a slightly more general version and its proof can be found in [DZ14] (section 4.4).

Given a Hilbert space H and a functional $F \in L(H;\mathbb{R})$ we write $D^1_x F \in L(H; L(H;\mathbb{R})$, $D^2_{xx} F \in L(H \otimes H; L(H;\mathbb{R})$ for the first and second Fréchet derivatives respectively. $C^2(H;\mathbb{R})$ is the space of the functional twice continuous, Fréchet differentiable in H.

Theorem (Itô formula)**.** *Let $T > 0$, U, H be Hilbert spaces, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space, carrying a generalised Wiener process $(W_t)_{t\geq 0}$ defined on U with covariance operator Q and assume*

1. *$\psi \in L^2(\Omega \times [0,\infty); H)$;*
2. *φ is a $L_2(U_0; H)$ valued stochastically integrable process;*

and $X_0 \in H$. Then given $F \in C^2(H;\mathbb{R})$ and a stochastic process, $(X_t)_{t\geq 0}$, satisfying,

$$X_t = X_0 + \int_0^t \psi_s ds + \int_0^t \varphi_s dW_s$$

it holds that,

$$F(X_t) = F(X_0) + \int_0^t D^1_x F(X_s)(\psi_s) ds + \int_0^t D^1_x F(X_s)(\varphi_s dW_s) \\ + \frac{1}{2}\int_0^t Tr[D^2_{xx} F(X_s)(\varphi_s Q^{1/2})(\varphi_s Q^{1/2})^*] ds.$$

Proof. The proof can be found in [DZ14] □

Now we show how to use the formula in our context.

Step 1

The idea is to apply the formula on u_t with decomposition

$$u_t = u_0 + \int_0^t \tilde{A}(s, u_s) ds + \int_0^t \tilde{B}(s, u_s) dW_s \tag{5.5}$$

where $U = l^2(\mathbb{R})$, $H = L^2(\mathbb{R}^n)$ and $Q = Id_{l^2}$. As ψ_s, φ_s we are considering $\tilde{A}(s, u_s)$ and $\tilde{B}(s, u_s)$, respectively.

The assumptions on $\tilde{B}$ guarantees us that φ is a $L_2(l_0^2; L^2(\mathbb{R}^n))$ valued stochastically integrable process. The regularity on ψ is given by $R1$ and $R2$. The only problem is that, a priori, $\tilde{A}(s, u_s) = \psi_s$ is $H^{-1}(\mathbb{R}^n)$-valued and not $L^2(\mathbb{R}^n)$-valued. To solve this we remind that we are assuming the existence of a weak solution in

$$u \in L^2(\Omega \times [0, \infty); H_0^1(\mathbb{R}^n)),$$

which implies that u_s is $L^2(\mathbb{R}^n)$-valued process. Since both u_t and $\tilde{B}(s, u_s)(W_s)$ are $L^2(\mathbb{R}^n)$-valued process, by (5.5), we conclude that $\tilde{A}(s, u_s)$ is an $L^2(\mathbb{R}^n)$-valued process.

Since we have checked the hypotheses we can consider as function the following $F : L^2(\mathbb{R}^n) \to \mathbb{R}$ defined as

$$F(f) := \|f_{k,a}\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |f(x) - a(1 - 2^{-k})|_+^2 dx.$$

Formally the Fréchet derivatives are

$$D_x^1 F(f)(g) = 2 \int_{\mathbb{R}^n} f_{k,a}(x) g(x) dx,$$
$$D_{xx}^2 F(f)(g)(h) = 2 \int_{\mathbb{R}^n} \mathbf{1}_{\{f_{k,a} > 0\}} g(x) h(x) dx.$$

Notice that F does not belong to $C^2(L^2(\mathbb{R}^n); \mathbb{R})$ and thus we are not allowed to use the Itô formula. To fully justify this, refer to [Kry10] or [HWW20].

Step 2

We analyse separately every term of our Itô formula.

1. $F(X_t) = \|u_{k,a}(t)\|^2_{L^2(\mathbb{R}^n)}$ and $F(X_0) = \|u_{k,a}(0)\|^2_{L^2(\mathbb{R}^n)}$

2. Using (5.3)

$$\int_0^t D_x^1 F(X_s)(\psi_s) ds = 2 \int_0^t \langle u_{k,a}(s), \tilde{A}(s, u_s) \rangle_{L^2(\mathbb{R}^n)} ds$$
$$= -2 \int_0^t \int_{\mathbb{R}^n} (A(s,x) \nabla u_s(x)) \cdot \nabla u_{k,a}(s)(x) dx ds + 2 \int_0^t \int_{\mathbb{R}^n} f(t, x, u_s(x)) u_{k,a}(s)(x) dx ds$$
$$= -2 \int_0^t \langle A(s, \cdot) \nabla u_{k,a}(s)(\cdot), \nabla u_{k,a}(s)(\cdot) \rangle_{L^2} ds + 2 \int_0^t \int_{\mathbb{R}^n} f(t, x, u_s(x)) u_{k,a}(s)(x) dx ds.$$

3. By the definition of $\tilde{B}$ we have

$$\begin{aligned}\int_0^t D_x^1 F(s,X_s)(\varphi_s dW_s) &= 2\int_0^t \langle u_{k,a}(s)(\cdot), \tilde{B}(s,u_s(\cdot))dW_s\rangle \\ &= 2\int_0^t \int_{\mathbb{R}^n} \sum_i g_i(t,x,u_s(x))u_{k,a}(s)(x)dxdw_s^i\end{aligned}$$

4. For the last term we have to spend some words on the notation. The expression

$$D_{xx}^2 F(X_s)(\varphi_s Q^{1/2})(\varphi_s Q^{1/2})^* = D_{xx}^2 F(u_s)(\tilde{B}(s,u_s))(\tilde{B}(s,u_s))^*$$

should be seen as an operator from $l^2(\mathbb{R})$ to $\mathbb{R}$. It follows that the trace acts on a basis $(z_i)_i$ of $l^2(\mathbb{R})$ as follows

$$\begin{aligned}\sum_i [D_{xx}^2 F(u_s)(\tilde{B}(s,u_s))(\tilde{B}(s,u_s))^*]z_i &= 2\int_{\mathbb{R}^n}\sum_i \left[\tilde{B}(s,u_s)(z_i)(x)\right]^2 \mathbf{1}_{\{u_{k,a}(s)>0\}}dx \\ &= 2\int_{\mathbb{R}^n}\sum_i [g_i(s,x,u_s)]^2\,\mathbf{1}_{\{u_{k,a}(s)>0\}}dx \\ &= 2\int_{\mathbb{R}^n} |g(s,x,u_s)|_{l^2}^2 \mathbf{1}_{\{u_{k,a}(s)>0\}}dx.\end{aligned}$$

By collecting all the results and writing everything in stochastic notation we get

$$\begin{aligned}d\|u_{k,a}(t)\|_{L^2}^2 = &-2\langle A(t,\cdot)\nabla u_{k,a}(t)(\cdot), \nabla u_{k,a}(t)(\cdot)\rangle_{L^2}dt + 2\sum_i \langle g_i(t,\cdot,u_t(\cdot)), u_{k,a}(t)(\cdot)\rangle_{L^2}dw_s^i \\ &+ \left[\int_{\mathbb{R}^n}\left\{|g(t,x,u_t)|_{l^2}^2 + 2u_{k,a}(t)(x)f(t,x,u_t(x))\right\}\mathbf{1}_{\{u_{k,a}(t)>0\}}dx\right]dt.\end{aligned} \tag{5.6}$$

Step 3

We need some extra work on the equation (5.6) to get our inequality. The reader is invited to note the similarity of the techniques with the Navier-Stokes case.

First, we use the elliptic assumptions on A and integrate between t_0 and t, where $T_k \le t_0 \le T_{k-1} \le t \le 2T$:

$$\begin{aligned}\|u_{k,a}(t)\|_{L^2}^2 + 2\lambda\int_{t_0}^t \|\nabla u_{k,a}(s)\|_{L^2}ds \le \|u_{k,a}(t_0)\|_{L^2}^2 + \int_{t_0}^t 2\sum_i \langle g_i(u_s), u_{k,a}(s)\rangle_{L^2}dw_s^i \\ + \int_{t_0}^t\left[\int_{\mathbb{R}^n}\left\{|g(u_s)|_{l^2}^2 + 2u_{k,a}(s)f(u_s\right\}\mathbf{1}_{\{u_{k,a}(s)>0\}}dx\right]ds,\end{aligned} \tag{5.7}$$

where we have denoted $f(u_s) = f((s, x, u_s(x))$ and $g(u_s) = g((s, x, u_s(x))$. Additionally, we have omitted the spatial dependence, which is evident from the preceding computations.

Now, we work on the last term of the previous inequality. If $u_{k,a} > 0$ then

$$0 < u \leq (1 + 2^k)u_{k-1,a}. \tag{5.8}$$

By Chebychev we have also

$$|\{u_{k,a}(t) > 0\}| = |\{u_{k-1,a}(t) > 2^{-k}a\}| \leq \left(\frac{2^k}{a}\right)^2 \|u_{k-1,a}(t)\|_2^2.$$

These facts, together with the regularity assumptions on g, give us the following estimate

$$\begin{aligned}
&\int_{t_0}^t \left[\int_{\mathbb{R}^n} |g(u_s)|_{l^2}^2 \mathbf{1}_{\{u_{k,a}(s)>0\}} dx\right] ds \leq 2 \int_{t_0}^t \left[\int_{\mathbb{R}^n} (K(x)^2 + \Lambda^2 |u_s(x)|^2) \mathbf{1}_{\{u_{k,a}(s)>0\}} dx\right] ds \\
&\quad \leq 2\|K\|_{L^\infty}^2 \int_{t_0}^t |\{x : u_{k,a}(s)(x) > 0\}| \, ds + 2\Lambda^2 \int_{t_0}^t \int_{\mathbb{R}^n} |u_s(x)|^2 dx ds \\
&\quad \leq \|K\|_\infty^2 C^k \int_{t_0}^t \|u_{k-1,a}(s)\|_2^2 ds + C^k \int_{t_0}^t \int_{\mathbb{R}^n} |u_{k-1,a}(s)(x)|^2 dx ds \\
&\quad = C^k (1 + \|K\|_\infty^2) \int_{t_0}^t \|u_{k-1,a}(s)\|_2^2 ds \\
&\quad \leq C^k (1 + \|K\|_\infty^2) U_{k-1,a}
\end{aligned}$$

where the last inequality is given by Jensen's inequality and the constant C depends on Λ. For the term involving f, we proceed in the same way (using also Hölder's inequality) and we get

$$\int_{t_0}^t \int_{\mathbb{R}^n} 2u_{k,a}(s) f(u_s) \mathbf{1}_{\{u_{k,a}(s)>0\}} dx ds \leq C^k (1 + \|K\|_\infty^2) U_{k-1,a}.$$

Collecting all the results we transform (5.7) in

$$\begin{aligned}
&\|u_{k,a}(t)\|_{L^2}^2 + 2\lambda \int_{t_0}^t \|\nabla u_{k,a}(s)\|_{L^2}^2 ds \\
&\qquad \leq \|u_{k,a}(t_0)\|_{L^2}^2 + \int_{t_0}^t 2 \sum_i \langle g_i(u_s), u_{k,a}(s)\rangle_{L^2} dw_s^i + C^k (1 + \|K\|_\infty^2) U_{k-1,a}.
\end{aligned}$$

Taking the supremum over t in $[T_k, 2T]$, we have that

$$\sup_{t\in[T_k,2T]} \|u_{k,a}(t)\|_{L^2}^2 + \int_{t_0}^{2T} \|\nabla u_{k,a}(s)\|_{L^2}^2 ds \\ \leq C\|u_{k,a}(t_0)\|_{L^2}^2 + CX_{k-1,a}^* + C^k(1+\|K\|_\infty^2)U_{k-1,a}. \tag{5.9}$$

for some constant C depending only on n, λ and Λ.

Energy decay estimate

Since we have our Energy Inequality, we can proceed to prove the geometric decay of the Energy. Our objective is to be able in a position to use Lemma 1.1.

Following the previous chapters, we compute our space-time interpolation. The choice of the exponents will become clear later on.

Lemma 5.1. *We have the following estimate for general functions:*

$$\|f\|_{\frac{4(n+1)}{n},\frac{2(n+1)}{n}} \leq \|f\|_{\infty,2}^2 + \|f\|_{2,\frac{2n}{n-2}}^2. \tag{5.10}$$

Proof. We start using use time-space interpolation (see Lemma (2.1)) with $p_\lambda = \frac{4(n+1)}{n}$, $p=\infty$, $p'=2$ and $q_\lambda = \frac{2(n+1)}{n}$, $q=2$, $q' = \frac{2n}{n-2}$. This gives us

$$\|f\|_{\frac{4(n+1)}{n},\frac{2(n+1)}{n}} \leq \|f\|_{\infty,2}^{\frac{n+2}{2(n+1)}} \|f\|_{2,\frac{2n}{n-2}}^{\frac{n}{2(n+1)}}.$$

Now, using Young's inequality

$$ab \leq a^p + b^q$$

with $p = 2(n+1)/(n+2)$ and $q = 2(n+l)/n$ we conclude. □

We are now prepared to prove that the Energy decay is geometric. This result will conclude the first and main part of our proof. Notice that it is precisely the Chebyshev inequality that generates the exponent $-\frac{1}{2}(n+1)$, thus pushing the decay index beyond the critical value of 1.

Theorem 5.2. *Assume that the function K satisfies $\|K\|_{L^\infty} \leq 1$. Then for a constant $C = C(n,\lambda,\Lambda,T)$ we have*

$$U_{k,a} \leq \frac{C^k}{a^{2/n+1}}(U_{k-1,a} + X_{k-1,a}^*)U_{k-1,a}^{1/(n+1)}.$$

Proof. Using Hölder's inequality with exponents $(n+1)/n$ and $n+1$ we obtain

$$\|u_{k,a}(t)\|_2^2 \leq \|u_{k,a}(t)\|_{2(n+1)/n}^2 \cdot |\{u_{k,a}(t) > 0\}|^{1/(n+1)} \tag{5.11}$$

where we have denoted by $\|\cdot\|_2$ the norm $\|\cdot\|_{L^2(\mathbb{R}^n)}$. By Chebychev's ineuqality we have:

$$|\{u_{k,a}(t) > 0\}| = |\{u_{k-1,a}(t) > 2^{-k}a\}| \leq \left(\frac{2^k}{a}\right)^2 \|u_{k-1,a}(t)\|_2^2.$$

Squaring (5.11) and integrating on $[T_k, 2T]$ we have

$$U_{k,a}^2 \leq \left(\frac{2^k}{a}\right)^{4/(n+1)} \int_{T_k}^{2T} \|u_{k,a}(t)\|_{2(n+1)/n}^4 \|u_{k-1,a}(t)\|_2^{4/(n+1)} dt.$$

Again, for Hölder exponents with the same values:

$$\begin{aligned} U_{k,a} &\leq \left(\frac{2^k}{a}\right)^{2/(n+1)} \left(\int_{T_k}^{2T} \|u_{k,a}(t)\|_{2(n+1)/n}^{4(n+1)/n} dt\right)^{n/2(n+1)} \left(\int_{T_k}^{2T} \|u_{k-1,a}(t)\|_2^4 dt\right)^{1/2(n+1)} \\ &= \left(\frac{2^k}{a}\right)^{2/(n+1)} \|u_{k,a}\|_{L^{4(n+1)/n, 2(n+1)/n}(Q_k)}^2 U_{k-1,a}^{1/(n+1)}. \end{aligned} \tag{5.12}$$

Applying (5.10) on $u_{k,a}$ we obtain

$$\|u_{k,a}\|_{L^{4(n+1)/n, 2(n+1)/n}(Q_k)}^2 \leq \sup_{t \in [T_k, 2T]} \|u_{k,a}(t)\|_2^2 + \int_{T_k}^1 \|u_{k,a}(t)\|_{2n/(n-2)}^2 dt.$$

Applying Sobolev's Embeddings to the last term we obtain

$$\|u_{k,a}\|_{L^{4(n+1)/n, 2(n+1)/n}(Q_k)}^2 \leq \sup_{t \in [T_k, 2T]} \|u_{k,a}(t)\|_2^2 + \int_{T_k}^{2T} \|\nabla u_{k,a}(t)\|_2^2 dt.$$

which is controlled, using Energy inequality, by

$$C\|u_{k,a}(t_0)\|_{L^2}^2 + C^k X_{k-1,a}^* + C^k(1 + \|K\|_\infty^2) U_{k-1,a},$$

The inequality (5.12) becomes

$$U_{k,a} \leq \frac{C^k}{a^{2/(n+1)}} \left(\|u_{k,a}(t_0)\|_{L^2}^2 + X_{k-1,a}^* + U_{k-1,a}\right) U_{k-1,a}^{1/(n+1)}$$

where we have used $\|K\|_{L^\infty} \leq 1$. Since t_0 is free to move in the interval $[T_{k-1}, T_k]$ we can use the mean value theorem to select a value of t_0 in such interval such that

$$\|u_{k,a}(t_0)\|_{L^2}^2 = \fint_{T_{k-1}}^{T_k} \|u_{k,a}(t)\|_{L^2}^2 dt \leq 2^k T^{-1} U_{k-1,a}.$$

□

5.2 Height Estimate

The objective of this section is to derive an estimate of the $L^\infty - L^2$ type, commonly referred to as the Height estimate. The techniques we will employ in this section differ from those used in the previous chapters and they ultimately rely on the tail property of martingales.

For now, we have proved that, if $\|K\|_{L^\infty} \le 1$, then there exists $C = C(n, \lambda, \Lambda, T)$ such that

$$U_{k,a} \le \frac{C^k}{a^{2/n+1}}(U_{k-1,a} + X^*_{k-1,a})U^{1/(n+1)}_{k-1,a}$$

where

$$X^*_{k-1,a} = \sup_{T_k \le s \le t \le 2T} \int_s^t \sum_i \langle g_i(\tau, \cdot, u_\tau(\cdot)), u_{k,a}(\tau)(\cdot)\rangle_{L^2} dw^i_\tau.$$

As one might expect, the presence of the stochastic term $X^*_{k-1,a}$ requires some additional considerations. The final part of the proof contains elements from classical literature on stochastic analysis. We closely follow the approach outlined in [HWW20].

To prove the α-Hölder regularity of the solution, we begin with the following basic fact:

Lemma. *Suppose $(M_t)_t$ is a continuous local martingale. Then we have*

$$\mathbb{P}\left\{\sup_{l \le s \le t \le S} (M_t - M_s) \ge a, \langle M\rangle_S - \langle M\rangle_l \le b\right\} \le 2e^{-a^2/4b}$$

Proof. By the Theorem of Dambis, Dubins-Schwarz (see [RY99] Chapter V, Section 1, Theorem 1.6) there exists a Brownian motion B such that

$$M_t - M_0 = B_{\langle M\rangle_t}.$$

Hence we have the following sequence of contained events:

$$\begin{aligned}
\mathbb{P}\Big\{\sup_{l \le s \le t \le S} (M_t - M_s) \ge a, &\langle M\rangle_S - \langle M\rangle_l \le b\Big\} \\
&= \mathbb{P}\left\{\sup_{l \le s \le t \le S} (B_{\langle M\rangle_t} - B_{\langle M\rangle_s}) \ge a, \langle M\rangle_S - \langle M\rangle_l \le b\right\} \\
&\le \mathbb{P}\left\{\sup_{0 \le s \le t \le b} (B_t - B_s) \ge a\right\} \\
&\le \mathbb{P}\left\{2 \sup_{0 \le t \le b} B_t \ge a\right\}
\end{aligned}$$

where we have used that the quadratic variation is an increasing process.
By the symmetries of the Brownian motion we obtain

$$\mathbb{P}\left\{\sup_{0\le t\le b} 2B_t \ge a\right\} \le 2\mathbb{P}\left\{B_b \ge a/2\right\} \le 2e^{-a^2/4b}$$

where the last inequality holds for the Gaussian variable $B_b \sim \mathcal{N}(0, \sqrt{b})$. □

This basic lemma contains the idea to find an estimate for our $X^*_{k,a}$. Indeed consider

$$X_t := \int_0^t \sum_i \langle g_i(\tau, \cdot, u_\tau(\cdot)), u_{k+1,a}(\tau)(\cdot)\rangle_{L^2} dw^i_\tau$$

which implies $X^*_{k,a} = \sup_{T_{k+1}\le s\le t\le 2T}(X_t - X_s)$. By the standard theory of stochastic integration, X_t is a continuous martingale (see the appendix) and so we can use the previous lemma.

Proposition. *Assume $\|K\|_\infty \le 1$. Then there exists a constant $C = C(n, \lambda, \Lambda)$ such that for all positive α, β it holds*

$$\mathbb{P}\left\{X^*_{k,a} \ge \alpha\beta,\ U_{k,a} \le \beta\right\} \le Ce^{-\alpha^2/C^k}.$$

Proof. If we can show

$$\langle X\rangle_{2T} - \langle X\rangle_{T_{k+1}} \le C^k U^2_{k,a} \tag{5.13}$$

then we have

$$\left\{X^*_{k,a} \ge \alpha\beta,\ U_{k,a} \le \beta\right\} \subset \left\{\sup_{T_{k+1}\le s\le t\le 2T}(X_t - X_s) \ge \alpha\beta,\ \langle X\rangle_{2T} - \langle X\rangle_{T_{k+1}} \le C^k\beta^2\right\}$$

and by previous Lemma, we have done. To prove (5.13) we start with

$$\langle X\rangle_{2T} - \langle X\rangle_{T_{k+1}} = \int_{T_{k+1}}^{2T} \sum_i \langle g_i(\tau, \cdot, u_\tau(\cdot)), u_{k+1,a}(\tau)(\cdot)\rangle^2_{L^2} d\tau.$$

Now we proceed similarly as we have already done in the energy inequality. We start using Minkovsky inequality

$$\sum_i \left(\int_{\mathbb{R}^n} g_i(\tau, x, u_\tau(x)) u_{k+1}(\tau)(x)dx\right)^2 \le \left(\int_{\mathbb{R}^n} |g(\tau, x, u_\tau(x))|_{l^2}\ u_{k+1}(\tau)(x)dx\right)^2.$$

Using R2 we can control the right-hand side with

$$\begin{aligned}\left(\int_{\mathbb{R}^n} \left[K(x) + \Lambda u_\tau(x)\right] u_{k+1,a}(\tau)(x)dx\right)^2 &\le \left(\int_{\mathbb{R}^n} K(x)u_{k+1,a}(\tau)(x) + \Lambda u_\tau(x)u_{k+1,a}(\tau)(x)dx\right)^2 \\ &\le \left(\int_{\mathbb{R}^n} u_{k+1,a}(\tau)(x)dx + C^k\int_{\mathbb{R}^n} u^2_{k,a}(\tau)(x)dx\right)^2\end{aligned}$$

where we have used (5.8). For the first term we use Hölder's inequality with exponent 2 and Chebychev we conclude

$$\sum_i \left(\int_{\mathbb{R}^n} g_i(\tau, x, u_\tau(x)) u_{k+1}(\tau)(x) dx \right)^2 \le C^k \left(\int_{\mathbb{R}^n} u_{k,a}^2(\tau)(x) dx \right)^2 .$$

Finally, integrating in $[T_{k+1}, 2T]$ we have reached our conclusion. □

We are ready to present the stochastic counterpart of the Height estimate. Notice that we have not used the classical Lemma 1.1, but we have relied on the tail property of the martingales, instead. Therefore, we introduce the so-called tail estimate for $T = 1$.

Proposition 5.1. *Assume $K_\infty \le 1$. Then there exists a constant $M_0 = M_0(n, \lambda, \Lambda)$ such that for all $a \ge 1$ and $M > M_0$,*

$$\mathbb{P}\left\{ \|u^+\|_{L^\infty(Q_\infty)} > a, M\|u^+\|_{L^{4,2}(Q_0)} \le a \right\} \le e^{-M^{\frac{1}{n+1}}}$$

Proof. The first trivial observation is that $\{\|u^+\|_{L^\infty(Q_\infty)} > a\} \subset G_a^c$ where $G_a = \{\lim_{k\to\infty} U_{k,a} = 0\}$. Now consider the family of events

$$\mathcal{E}_k = \left\{ U_{k,a} \le \left(\frac{a}{M}\right)^2 \gamma^k \right\}$$

where M and $\gamma < 1$ will be determined later. These events satisfy:

1. By definition of $U_{k,a}$ we have $\{M\|u^+\|_{L^{4,2}(Q_0)} \le a\} = \mathcal{E}_0$;
2. $G_a^c \subset \bigcup_{k\ge 0} \mathcal{E}_k^c$ because $\gamma^\infty = 0$. Decomposing

$$\bigcup_{k\ge 0} \mathcal{E}_k^c = \mathcal{E}_0^c \cup \left[\bigcup_{k\ge 1} (\mathcal{E}_k^c \cap \mathcal{E}_{k-1}) \right]$$

We have that

$$\mathbb{P}\left\{ \|u^+\|_{L^\infty(Q_\infty)} > a, M\|u^+\|_{L^{4,2}(Q_0)} \le a \right\} \le \mathbb{P}\{G_a^c \cap \mathcal{E}_0\} \le \sum_{k\ge 1} \mathbb{P}\{\mathcal{E}_k^c \cap \mathcal{E}_{k-1}\}.$$

Now we want to estimate $\mathbb{P}\{\mathcal{E}_k^c \cap \mathcal{E}_{k-1}\}$. To this aim, we use a combination of Theorem (5.2) and the previous estimate. The goal is to prove that

$$\{\mathcal{E}_k^c \cap \mathcal{E}_{k-1}\} \subset \{X_{k-1,a}^* > \alpha\beta, U_{k-1,a} \le \beta\} \tag{5.14}$$

for $\alpha = (2C)^{k/2} M^{\frac{1}{n+1}}$ with C the constant of the previous proposition and $\beta = \frac{a^2\gamma^{k-1}}{M^2}$. Notice immediately that $\{U_{k-1,a} \le \beta\} = \mathcal{E}_{k-1}$ from the choice of β. Now (5.14) is equivalent to assuming that the event $\mathcal{E}_{k-1}$ is true (has probability one) and proving that

$$\{X_{k-1,a}^* > \alpha\beta\}^c \subset \mathcal{E}_k \quad \text{or, equivalently} \quad X_{k-1,a}^* \le \alpha\beta \Rightarrow U_{k,a} \le \left(\frac{a}{M}\right)^2 \gamma^k = \gamma\beta.$$

By Theorem 5.2 we have

$$U_{k,a} \leq \frac{C_1^k}{a^{1/2\delta}}(\beta + \alpha\beta)\beta^\delta = \frac{(C_1\gamma^\delta)^k(1+(2C)^{k/2}M^\delta)}{\gamma^{1+\delta}M^{2\delta}}\gamma\beta$$

where C_1 is a constant different from C and $\delta = 1/(n+1)$. Now if γ small enough and M is big enough we have

$$U_{k,a} \leq \gamma\beta.$$

Now using the previous Proposition on (5.14) we find

$$\mathbb{P}\{\mathcal{E}_k^c \cap \mathcal{E}_{k-1}\} \leq Ce^{-\alpha^2/C^k} = Ce^{-2^k M^{2\delta}}.$$

Hence, for a large M,

$$\mathbb{P}\left\{\|u^+\|_{L^\infty(Q_\infty)} > a, M\|u^+\|_{L^{4,2}(Q_0)} \leq a\right\} \leq \mathbb{P}\{G_a^c \cap \mathcal{E}_0\} \leq C\sum_{k\geq 1} e^{-2^k M^{2\delta}} \leq e^{-M\delta}. \quad \square$$

5.3 Holder continuity

We are ready to prove Theorem 5.1. By scaling we can assume $T = 1$, $\|K\|_2 + \|K\|_\infty \leq 1$ and $\|u_0\|_2 \leq 1$. For readability, we report the scaled version of Theorem 5.1:

Theorem. *Let u be a weak solution (according to definition (5.1)) of the SPDE (5.1) with deterministic initial data $u_0 \in C^\infty(\mathbb{R}^n)$. Assume $\|K\|_2 + \|K\|_\infty \leq 1$ and $\|u_0\|_2 \leq 1$. Then there exists a constant $C = C(n, \lambda, \Lambda, p)$ such that*

$$\mathbb{E}\left[\|u\|^p_{L^p([0,2];L^2(\mathbb{R}^n))}\right] + \mathbb{E}\left[\|u\|^p_{L^\infty([1,2]\times\mathbb{R}^n)}\right] \leq C.$$

Proof. We show that there exists a constant $C = C(n, \lambda, \Lambda, p)$ such that

$$\mathbb{E}\left[\int_0^2 \|u(t)\|^p_{L^2} dt\right] \leq C \quad \text{and} \quad \mathbb{E}\left[\|u\|^p_{L^\infty([1,2]\times\mathbb{R}^n)}\right] \leq C. \tag{5.15}$$

We can assume that $p \geq 4$. Indeed suppose that we have proved the theorem for $p \geq 4$ and consider the case $p < 4$:

$$\mathbb{E}\left[\int_0^2 \|u(t)\|^p_{L^2} dt\right] = \mathbb{E}\left[\int_0^2 \left(\int_{\mathbb{R}^n} |u_t(x)|^2 dx\right)^{p/2} dt\right] \leq \mathbb{E}\left[\int_0^2 \left(1 + \int_{\mathbb{R}^n} |u_t(x)|^2 dx\right)^{p/2} dt\right]$$
$$\leq \mathbb{E}\left[\int_0^2 \left(1 + \int_{\mathbb{R}^n} |u_t(x)|^2 dx\right)^{\frac{p}{2}\frac{4}{p}} dt\right] \leq \mathbb{E}\left[\int_0^2 2 + 2\left(\int_{\mathbb{R}^n} |u_t(x)|^2 dx\right)^2 dt\right]$$
$$\leq 4 + 2\mathbb{E}\left[\int_0^2 \|u_t\|^4_{L^2} dt\right] \leq C.$$

Step 1

For the first bound the idea is to use the Itô formula with the function $F(f) = \|f\|_{L^2} + 1$. With the same work that we have already done for the Energy inequality, we can check that

$$d\varphi(t) = \varphi(t)\left(H(t)dt + dG_t\right) \tag{5.16}$$

where $\varphi(t) := \|u_t(\cdot)\|^2_{L^2} + 1$ and

$$H(t) := \frac{-\langle A(t,\cdot)\nabla u_t(\cdot), \nabla u_t(\cdot)\rangle_{L^2} + \langle f(t,\cdot,u_t(\cdot)), u_t(\cdot)\rangle_{L^2} + \||g(t,\cdot,u_t(\cdot))|_{l^2}\|^2_{L^2}}{\|u_t(\cdot)\|^2_{L^2} + 1}$$

$$G(t) := \int_0^t \sum_i \frac{\langle g_i(s,\cdot,u_s(\cdot), u_s(\cdot)\rangle_{L^2}}{\|u_s(\cdot)\|^2_{L^2} + 1} dw^i_s.$$

The solution of (5.16) is well known and given by

$$\varphi(t) = \varphi(0)\exp\left[\int_0^t H(s)ds + G_t - \frac{1}{2}\langle G\rangle_t\right].$$

Since our stochastic integral is a square-integrable martingale we have that, for $0 \le t \le 2$,

$$\begin{aligned}\langle G\rangle_t &\le \int_0^2 \left(\frac{\int_{\mathbb{R}^n} |g(s,x,u_s(x)|_{l^2} u_s(x)dx}{\|u_s(\cdot)\|^2_{L^2}+1}\right)^2 ds \le \int_0^2 \left(\frac{\int_{\mathbb{R}^n} (K(x) + \Lambda|u_s(x)|)\, u_s(x)dx}{\|u_s(\cdot)\|^2_{L^2}+1}\right)^2 ds \\ &\le \int_0^2 \left(\frac{\int_{\mathbb{R}^n} K(x)u_s(x) + \Lambda u_s^2(x)dx}{\|u_s(\cdot)\|^2_{L^2}+1}\right)^2 ds \le \int_0^2 \left(\frac{\|K\|_2\|u_s\|_2 + \Lambda\|u_s\|^2_2}{\|u_s\|^2_2+1}\right)^2 ds\end{aligned}$$

where the last inequality is given by Hölder's inequality.
Using the assumption on K we get

$$\langle G\rangle_t \le \int_0^2 \left(\frac{\|u_s\|_2 + \Lambda\|u_s\|^2_2}{\|u_s\|^2_2+1}\right)^2 ds \le 2(\Lambda+1)^2, \tag{5.17}$$

where we have used

$$\|u_s\|_2 + \Lambda\|u_s\|^2_2 \le (\|u_s\|^2_2 + 1)(\Lambda + 1)$$

(which can be proved by considering separately the cases $\|u_s\|_2 < 1$ and $\|u_s\|_2 \ge 1$). By Novikov's condition (see [Nov72]) the estimate (5.17) guarantees that

$$\exp\left[G_t - \frac{1}{2}\langle G\rangle_t\right]$$

is a martingale for $0 \leq t \leq 2$. In a similar way, one can check that $H(t) \leq 4(\Lambda+1)^2$ (starting by noticing that $-\langle A\nabla u, \nabla u\rangle$ is always negative by the ellipticity assumptions on A). Since everything is uniformly bounded in time we obtain

$$\mathbb{E}[\varphi(t)^p] = \varphi(0)^p \mathbb{E}\left\{\exp\left[p\left(\int_0^t F(s)ds + G_t - \frac{1}{2}\langle G\rangle_t\right)\right]\right\} \leq C\varphi(0)^p.$$

In particular,

$$\mathbb{E}\left[\int_0^2 \|u_t\|_{L^2}^2 dt\right] \leq 2C\varphi(0)^p = 2C(\|u_0\|_{L^2}^2 + 1)^p \leq C.$$

Step 2

For the second bound, we consider the quantities

$$X = \|u\|_{L^\infty([1,2]\times\mathbb{R}^n)} \quad \text{and} \quad Y = \left(\int_0^2 \|u_t(\cdot)\|_{L^2}^4 dt\right)^{\frac{1}{4}}.$$

By considering u and $-u$ we have, using Proposition 5.1, that

$$\mathbb{P}\left\{X > a, Y \leq \frac{a}{M}\right\} \leq 2e^{-M^{1/(n+1)}}$$

for every $a \geq 1$ and $M > M_0$. Set $I = \max\{1, M_0\}$ and use the previous inequality with $a > I^2$ and fix $M = \sqrt{a} > I \geq M_0$:

$$\mathbb{P}\left\{X > a, Y \leq \sqrt{a}\right\} \leq 2e^{-a^{1/2(n+1)}}.$$

By the first inequality in (5.15) and Jensen's inequality we have

$$\mathbb{E}[Y^{2p}] \leq C.$$

Hence,

$$\begin{aligned}
\mathbb{E}\left[\|u\|_{L^\infty([1,2]\times\mathbb{R}^n)}^p\right] &= \|X\|_{L^p(\Omega,\mathbb{P})}^p = p\int_0^\infty \mathbb{P}(X > a)a^{p-1}da \\
&\leq M_0^{2p} + p\int_{M_0^2}^\infty \mathbb{P}(Y > \sqrt{a})a^{p-1}da + p\int_{M_0^2}^\infty \mathbb{P}(X > a, Y \leq \sqrt{a})a^{p-1}da \\
&\leq M_0^{2p} + p\int_0^\infty \mathbb{P}(Y^2 > a)a^{p-1}da + p\int_{M_0^2}^\infty \mathbb{P}(X > a, Y \leq \sqrt{a})a^{p-1}da \\
&\leq M_0^{2p} + \mathbb{E}[Y^{2p}] + 2p\int_{M_0^2}^\infty e^{-a^{\frac{1}{2(n+1)}}} a^{p-1}da \leq C. \square
\end{aligned}$$

Now that we have proved Theorem 5.1 we have all the elements to conclude the work and prove the α-Hölder continuity of the solution.

Theorem 5.3. *Let u be a solution of (5.1). Then there exists a positive $\alpha = \alpha(n, \lambda, \Lambda)$ such that $u \in C^\alpha([T, 2T] \times \mathbb{R}^n)$ for every $T > 0$. Furthermore, we have a quantitative estimate: indeed there exists a constant $C = C(n, \lambda, \Lambda, p)$ such that*

$$\mathbb{E}\left[\|u\|^p_{C^\alpha([T,2T]\times\mathbb{R}^n)}\right] \leq C\left(\|u_0\|_{L^2(\mathbb{R}^n)} + \|K\|_{L^2(\mathbb{R}^n)} + +\|K\|_{L^\infty(\mathbb{R}^n)}\right)^p$$

Remark. *By assuming $T = 1$ and scaling up to $\|u_0\|_{L^2(\mathbb{R}^n)} \leq 1$ and $\|K\|_{L^2(\mathbb{R}^n)} + \|K\|_{L^\infty(\mathbb{R}^n)} \leq 1$ we re-write the thesis as*

$$\mathbb{E}\left[\|u\|^p_{C^\alpha([1,2]\times\mathbb{R}^n)}\right] \leq C. \tag{5.18}$$

We provide a brief overview of the proof since it involves elements of stochastic analysis that go beyond the scope of this thesis. However, all the details can be found in [HWW20].

Proof. The crucial idea to conclude is contained in [DDH14] and consists in considering the solution v to the same SPDE with constant coefficients.

$$dv = \Delta v dt + \sum_i g_i(u) dw_t^i \quad v(1/2) = 0.$$

For this simpler equation, Kyrlov's theory applies and $v \in C^{\alpha_1}((1/2, 2] \times \mathbb{R}^n)$.
The function $\varphi = u - v$ solves,

$$\partial_t \varphi = \operatorname{div}(A\nabla u) + \operatorname{div}(A\nabla v) - \Delta v + f(\varphi + v) \quad \text{in } [1/2, 2] \times \mathbb{R}^n. \tag{5.19}$$

Since (5.19) does not have a stochastic perturbation, the usual regularity theory of Chapter II (with small modifications) applies and we have $\phi \in C^{\alpha_2}([1, 2] \times \mathbb{R}^n)$ for some exponent $\alpha_2 \in (0, 1)$.

Now that we have established the Hölder continuity of both φ and v, we can proceed to prove the Hölder continuity of u for $\alpha = \min(\alpha_1, \alpha_2)$. However, to prove (5.18), we need to exert additional effort to control the p-power of the solution. For this purpose, we rely on quantitative estimates provided in works such as [Kry96]. □

Appendix A

Calderon-Zygmund theory

Let us denote Λ as the fundamental solution of Laplace's equation for the Dirac delta in the origin ($\Delta\Lambda = \delta_0$). Then, for an integrable function f on a general bounded domain Ω, we define the Newtonian potential of f as the function $\mathcal{N}(f)$ defined on $\mathbb{R}^n$:

$$\mathcal{N}(f)(x) = (\Lambda \star f)(x) = \int_\Omega \Lambda(x-y)f(y)\, dy.$$

The main regularity theorem (for the proof, see [GT83] Theorem 9.9) of the Newtonian potential states:

Theorem A.1. *Let $f \in L^p(\Omega)$, $1 < p < \infty$. Then, $\mathcal{N}(f) \in W^{2,p}(\Omega)$, $\Delta\mathcal{N}(f) = f$ a.e. in Ω, and*

$$\|D^2\mathcal{N}(f)\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}, \tag{A.1}$$

where $C = C(n,p)$.

In other words, the theorem states that the L^p norm of the second derivatives of the solution to the Poisson equation can be bounded by the L^p norm of the Laplacian of the solution. It is important to note that even if f has compact support in Ω, the Newtonian potential $\mathcal{N}(f)$ is not necessarily compactly supported in Ω.

Since we would like $\mathcal{N}(f)$ to have compact support, we will proceed with a local construction of the Newton potential, denoted as $\overline{\mathcal{N}}(f)$. To this aim, consider f with compact support in Ω and denote $d = \operatorname{dist}(\operatorname{spt} f, \partial\Omega)$.

- Consider the following covering of Ω:

  $$\mathcal{F} = \{B_x(d/2)\}_{x\in\Omega}.$$

  By Besicovitch's covering theorem, there exists a number $N = N(n)$, depending only on the dimension n, and subfamilies $\mathcal{F}_1, \dots, \mathcal{F}_N$ of $\mathcal{F}$ such that:

1. $\Omega \subset \bigcup_{i=1}^{N} \mathcal{F}_i$;
2. for every i the elements of $\mathcal{F}_i$ are pairwise disjoint.

Since the balls are disjoint, there is a finite number of them and we indicate with $y_1, \ldots, y_M$ their centres.

- We construct our operator in the following recursive way:

If $x \in B_{y_1}(d/2)$, then

$$\overline{\mathcal{N}}(f)(x) := \int_{B_{y_1}(d/2)\cap\Omega} \Lambda(x-y)f(y)\,dy.$$

If $x \in B_{y_i}(d/2)$ and $x \notin \bigcup_{j=1}^{i-1} B_{y_j}(d/2)$, then

$$\overline{\mathcal{N}}(f)(x) := \int_{B_{y_i}(d/2)\cap\Omega} \Lambda(x-y)f(y)\,dy.$$

It is easy to see that, this time, $\overline{\mathcal{N}}(f)$ is compactly supported in Ω. A reasonable question is if there is an analogous of Theorem (A.1) for $\overline{\mathcal{N}}(f)$. Luckily, we have

Theorem. *Let f compactly supported in Ω, $1 < p < \infty$, then $\overline{\mathcal{N}}(f) \in W^{2,p}(\Omega)$, $\Delta\overline{\mathcal{N}}(f) = f$ in Ω and*

$$\|D^2\overline{\mathcal{N}}(f)\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}. \tag{A.2}$$

where $C = C(n,p)$

Proof. By Theorem (A.1), we know that $\Delta\overline{\mathcal{N}}(f)(x) = f(x)$ holds, up to a set of null measure S_1 in B_{y_1}. Similarly, this equation holds in $B_{y_2} \setminus B_{y_1}$, up to a set of null measure S_2. Continuing this process, we find that $\Delta\overline{\mathcal{N}}(f)(x) = f(x)$ holds almost everywhere in Ω, up to the set $S_1 \cup \cdots \cup S_M$. Since M is finite, we conclude:

$$\Delta\overline{\mathcal{N}}(f) = f \quad \text{a.e. in } \Omega.$$

Similarly, we can prove that $\overline{\mathcal{N}}(f) \in W^{2,p}(\Omega)$. However, in order to establish the bound, we rely on the Besicovitch property of our covering. [1]

$$\|D^2\overline{\mathcal{N}}(f)\|^p_{L^p(\Omega)} = \int_{\bigcup_{i=1}^{N}\mathcal{F}_i} |D^2\overline{\mathcal{N}}(f)|^p \leq \sum_{i=1}^{N} \sum_{B_y(d/2)\in\mathcal{F}_i} \int_{B_y(d/2)} |D^2\overline{\mathcal{N}}(f)|^p.$$

Without loss of generality assume that for every i:

$$\sum_{B_y(d/2)\in\mathcal{F}_1} \int_{B_y(d/2)} |D^2\overline{\mathcal{N}}(f)|^p \geq \sum_{B_y(d/2)\in\mathcal{F}_i} \int_{B_y(d/2)} |D^2\overline{\mathcal{N}}(f)|^p.$$

[1] Without utilizing this property, we could encounter overlapping of the balls, and the number of overlapping would depend on the number of balls, which in turn depends on the domain Ω. Consequently, the constant C in the bound would also depend on the domain.

Now we use (A.1) together with the fact that the constant C in Theorem A.1 does not depend on the choice of the domain:

$$\begin{aligned}\|D^2\overline{\mathcal{N}}(f)\|^p_{L^p(\Omega)} &\le N \sum_{B_y(d/2)\in\mathcal{F}_1} \int_{B_y(d/2)} |D^2\overline{\mathcal{N}}(f)|^p \le C^p N \sum_{B_y(d/2)\in\mathcal{F}_1} \int_{B_y(d/2)} |f|^p \\ &= C \int_{\bigcup_{\mathcal{F}_1} B_y(d/2)} |f|^p \le C\|f\|^p_{L^p(\Omega)}\end{aligned}$$

where in the last inequality if $\bigcup_{\mathcal{F}_1} B_y(d/2) \supset \Omega$ we use that $f \equiv 0$ outside Ω. □

We can now present a crucial result concerning regularity in elliptic equations.

Lemma. *Suppose that u solves in the distributional sense in Ω*

$$-\Delta u = \sum_{i,j} \partial^2_{ij} f$$

Then, for every $1 < p < \infty$, there exists a constant $C = C(n,p)$ such that

$$\|u\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)}.$$

Moreover, if f is compactly supported in Ω also u can be assumed to be compactly supported in Ω.

Proof. We use the dual definition of the L^p norm and the regularity of the Newtonian potential:

$$\|u\|_{L^p(\Omega)} = \sup_{\|w\|_{L^{p'}}=1} \int_\Omega uw = \sup_{\|w\|_{L^{p'}}=1} \int_\Omega u\Delta\overline{\mathcal{N}}(w) = \sup_{\|w\|_{L^{p'}}=1} \int_\Omega \nabla u \nabla\overline{\mathcal{N}}(w),$$

where for the third equality we have used that, by density, we can assume $w \in C^\infty_c(\Omega)$ and in this case $\overline{\mathcal{N}}(w)$ is compactly supported. Using the definition of distributional derivatives (again by the compactness of the support of $\overline{\mathcal{N}}(w)$) and the estimate (A.2) we get

$$\begin{aligned}\|u\|_{L^p(\Omega)} &= \sup_{\|w\|_{L^{p'}}=1} \int_\Omega \sum_{i,j} \partial^2_{ij} f\, \overline{\mathcal{N}}(w) = \sup_{\|w\|_{L^{p'}}=1} \int_\Omega \sum_{i,j} f\, \partial^2_{ij}\overline{\mathcal{N}}(w) \\ &\le C \sup_{\|w\|_{L^{p'}}=1} \sum_{i,j} \|f\|_{L^p(\Omega)}\|w\|_{L^{p'}(\Omega)} = C\|f\|_{L^p(\Omega)}.\end{aligned}$$

To prove the last point notice that if f is compactly supported in Ω also $\sum_{i,j} \partial^2_{ij} f$ is compactly supported. Then, using the operator $\overline{\mathcal{N}}$ we can conclude. □

With a slight modification to the previous proof, we can obtain the following result, to which we refer when we talk about the Calderón-Zygmund theory.

If u is a solution, in Ω, to

$$-\Delta u = \sum_{i,j} \partial^2_{ij} f_{ij}$$

then

$$\|u\|_{L^p(\Omega)} \leq C \sum_{i,j} \|f_{ij}\|_{L^p(\Omega)} \tag{A.3}$$

where $\{f_{ij}\}_{ij}$ is a finite family of functions.
Moreover, if every f_{ij} is compactly supported in Ω also u is compactly supported in Ω.

Appendix B

Stochastic Analysis in Infinite dimension

We present a self-contained introduction to Stochastic Analysis in Hilbert spaces, with a focus on Stochastic Partial Differential Equations (SPDEs). We underline that our aim in this introduction is not on the existence and uniqueness of solutions to SPDEs. Instead, we aim to provide a basic overview of the ingredients of a Stochastic Partial Differential Equation. Our main reference for this topic is the book by Da Prato and Zabczyk [DZ14].

B.1 Introduction to probability theory in Infinite dimension

Operator in Infinite Dimensional Spaces

Given Banach spaces E, F we denote $L(E; F)$ the set of bounded, linear operators between E and F. This space is also a Banach Space (with operator norm) and we denote by $L(E)$ the space $L(E; E)$.

Consider two Hilbert spaces U and H and consider a bounded, linear operator between them $O \in L(U; H)$.

- We denote with $O^* \in L(H; U)$ its adjoint operator which satisfies

$$\langle Ox, h\rangle_H = \langle x, O^*h\rangle_U \quad \text{for all } x \in U,\ h \in H.$$

- We say that an operator $O \in L(H)$ is positive if

$$\langle Oh, h\rangle_H \geq 0 \quad \text{for all } h \in H.$$

Now we proceed by introducing two important functional operator spaces that play a crucial role in the study of Stochastic Analysis in infinite dimensions: the Trace Class operators (or simply Trace operators) and the Hilbert-Schmidt operators.

Definition B.1. *Let U be an Hilbert space. We say that a symmetric* [1] *and non-negative operator $T \in L(U)$ is a trace class if there exists a basis (when we say basis we mean orthonormal basis) $(e_k)_k$ for U such that*

$$\mathrm{TrT} := \sum_{\mathrm{k}=1}^{\infty} \langle \mathrm{Te_k}, \mathrm{e_k} \rangle_{\mathrm{U}} < \infty.$$

[2] *In such case, we say $T \in L_1(U)$.*

One can prove that the space $L_1(U)$ equipped with the natural norm $\|T\|_{L_1} = |\mathrm{TrT}|$ is a Banach space.

Definition B.2. *Let U, H be Hilbert spaces with basis $(e_k)_k$, $(f_k)_k$ respectively. Then $T \in L(U; H)$ is said to be a Hilbert-Schmidt operator if*

$$\|T\|^2_{L_2} := \sum_{k,l=1}^{\infty} \langle Te_k, f_l \rangle^2_H < \infty.$$

In such case, we say $T \in L_2(U)$.

The space $L_2(U)$ of these operators is a separable Hilbert space with the natural scalar product

$$\langle T, S \rangle = \sum_{k=1}^{\infty} \langle Te_k, Se_k \rangle_H \quad \text{for } T, S \in L_2(U; H).$$

The standard regularity theory for Hilbert spaces guarantees us that if $Q \in L_1(U)$ is symmetric and non-negative then $Q^{1/2} \in L_2(U)$ and, in particular, there exists a basis $(e_k)_k \subset U$ and a summable real sequence $(\lambda_k)_k$ such that we have the following representation

$$Q^{1/2}x = \sum_{k \geq 1} \sqrt{\lambda_k} \langle x, e_k \rangle_U \, e_k. \tag{B.1}$$

Random variables

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(E, \mathcal{B}(E))$ a measurable, separable, Banach space, with $\mathcal{B}(E)$ the Borel sigma algebra induced by the norm $\| \cdot \|_E$.

[1] A symmetric operator $T \in L(U)$ is an operator T such that $\langle Tx, y \rangle = \langle x, Ty \rangle$ for every $x, y \in U$

[2] It can be proved that if the series converges for a basis then it converges for any basis

A measurable map $X : \Omega \to E$ is called a random variable and $\mathcal{L}(X) = X_*\mathbb{P}$ is its law (which is a probability measure on E). A well-known fact is that X is an E-valued random variable if and only if $\varphi(X)$ is a real random variable for every $\varphi \in E^*$. This implies that if X is an E valued random variable, then $\Omega \ni \omega \mapsto \|X(\omega)\|_E$ is a real random variable (see [DZ14] Proposition 1.2).

Now, we aim to integrate a random variable with respect to the measure $\mathbb{P}$. We follow a similar approach as in the real case: we begin by integrating only simple random variables, and then expand the construction using a density argument.

A simple E-valued random variable is a random variable which can be written as:

$$X(\omega) = \sum_{k=1}^{N} x_k \mathbf{1}_{A_k}(\omega)$$

where $N > 0$ is an integer, $x_k \in E$ and $A_k \in \mathcal{F}$ for very k. The natural integration is given by

$$\mathbb{E}[X] := \int_\Omega X(\omega) d\mathbb{P}(\omega) = \sum_{k=1}^{N} x_k \mathbb{P}(A_k) \ \in E$$

and we say that X is integrable if $\|X(\cdot)\|_E$ is integrable.

The density argument is the following (for proof see [May21]):

Proposition. *Let X be an E-valued random variable with $(E, \|\cdot\|_E)$ a Banach space. Then there exists a sequence of simple random variables $(X_n)_{n\geq 1}$ such that $\|X(\omega) - X_n(\omega)\|_E \to 0$ almost everywhere.*

The integral of X is defined as the limit of the integral along an approximating sequence of simple random variables. For $p \in [1, \infty]$ we define naturally the spaces $L^p(\Omega; E)$ as the completion of the simple random variables under the norms

$$\|X\|_p := \begin{cases} \mathbb{E}\left[\|X\|_E^p\right]^{\frac{1}{p}}, & \text{if } p \in [1, \infty), \\ \operatorname*{ess\,sup}_{\omega\in\Omega} \|X(\omega)\|_E, & \text{if } p = \infty. \end{cases}$$

Gaussian variables

Just as in the real case, Gaussian variables play a fundamental role in the study of non-deterministic phenomena in the context of Hilbert spaces. In what follows, we identify the Hilbert space U with its dual space U^*, and we denote the action of g on h as $\langle h, g\rangle$. Before proceeding, let us recall the definition of the characteristic function.

For a measure μ defined on a Hilbert space U, the characteristic function of μ is defined as follows:

$$\hat{\mu}(h) = \int_U e^{i\langle h,g\rangle} d\mu(g) \quad \text{for all } h \in U.$$

Definition B.3. *Let X be a U-valued random variable. Then we say that X is Gaussian if and only if $\langle X, g\rangle$ is a real Gaussian random variable, for every $g \in U$. We say that a measure $\mu \in \mathcal{P}(U)$ is Gaussian if $\mu = \mathcal{L}(X)$ for a Gaussian random variable X.*

The following theorem presents a strong characterization of Gaussian random variables in the context of Hilbert spaces. It bears similarities with the characterization in the real case.

Theorem. *Let U be a Hilbert space. A measure $\mu \in \mathcal{P}(U)$ is Gaussian if and only if there exist $m \in U$ and $Q \in L_1(U)$ symmetric and non-negative such that*

$$\hat{\mu}(h) = e^{i\langle m,h\rangle - \frac{1}{2}\langle Qh,h\rangle}, \qquad \text{for all } h \in U.$$

Furthermore one has the identity,

$$\int_U \|h\|_U^2 \, \mathrm{d}\mu(h) = \operatorname{Tr} Q.$$

Finally, if $\mu = \mathcal{L}(X)$ then we say that $X \sim \mathcal{N}(m, Q)$.

Proof. Suppose μ is Gaussian then, by definition, can be written as $\mu = X_*\mathbb{P}$ for a Gaussian random variable X. By Riesz representation theorem (every continuous linear functional on a Hilbert space can be uniquely represented as the inner product between a fixed vector in the Hilbert space and a variable vector) there exist $m \in U$ and $Q \in L(U)$ symmetric and non-negative, such that for all $h, g \in U$,

$$\mathbb{E}[\langle X, h\rangle] = \int_U \langle x, h\rangle \mathrm{d}\mu(x) = \langle m, h\rangle$$
$$\mathbb{E}[\langle X, h\rangle\langle X, g\rangle] = \int_U \langle x, h\rangle\langle x, g\rangle \mathrm{d}\mu(x) = \langle Qh, g\rangle.$$

Assume without loss of generality that $m = 0$ (centred Gaussian variable). By definition of push forward, we have

$$\hat{\mu}(h) = \int_H e^{i\langle h,g\rangle} d\mu(g) = \int_\Omega e^{i\langle h,X(\omega)\rangle} d\mathbb{P}(\omega) = \mathbb{E}[e^{iY}]$$

where $Y = \langle X, h\rangle$ is a real Gaussian variable. Then, by the characterization of the characteristic function of real centred Gaussian variables, we have

$$\mathbb{E}[e^{iY}] = e^{-\frac{1}{2}\sigma^2}$$

where $\sigma^2 = \mathbb{E}[Y^2] = \mathbb{E}[\langle X, h\rangle\langle X, h\rangle] = \langle Qh, h\rangle$. We are left to prove that Q is Trace class. By Fernique's Theorem (see [May21] Theorem 1.4.11), a Gaussian measure has a finite moment of all orders, in particular

$$\int_U \|h\|_U^2 \, \mathrm{d}\mu(h) < \infty.$$

Now let $(e_n)_{n\geq 1} \subset U$ be a basis, so that we are allowed to apply Lebesgue's dominated convergence in the first step, we have,

$$\int_U \|h\|_U^2 \, \mathrm{d}\mu(h) = \sum_{k=1}^{\infty} \int_U \langle h, e_k\rangle^2 \, \mathrm{d}\mu(h) = \sum_{k=1}^{\infty} \mathbb{E}[\langle X, e_k\rangle\langle X, e_k\rangle] = \sum_{k=1}^{\infty} \langle Qe_k, e_k\rangle = \operatorname{Tr} Q.$$

For the converse implication, we consider $Q \in L_1(U)$ symmetric and non-negative. Then there exist a summable sequence $(\lambda_k)_k \subset \mathbb{R}$ and a basis $(e_k)_k$ for U such that

$$Qe_k = \lambda_k e_k.$$

Now take a sequence of i.i.d real standard Gaussian variables $(\beta_k)_k$ and set

$$X := \sum_{k=1}^{\infty} \sqrt{\lambda_k} e_k \beta_k. \tag{B.2}$$

We claim that X defines an U-valued Gaussian variable. To prove that B.2 is well defined consider the partial sums

$$X_m = \sum_{k=1}^{m} \sqrt{\lambda_k} e_k \beta_k$$

and notice that $(X_m)_m$ is a Cauchy sequence with respect to the $\|\cdot\|_2$ norm defined above. Indeed for $n > m$ we have

$$\|X_n - X_m\|_2^2 = \mathbb{E}[\|X_n - X_m\|_U^2] = \mathbb{E}\left[\left(\sum_{k=m}^{n} \sqrt{\lambda_k}\beta_k\right)^2\right] \leq \sum_{k=m}^{n} \lambda_k$$

and since λ_k is summable it is also a Cauchy sequence (which implies $(X_m)_m$ is a Cauchy sequence as well). Since the space $L^2(\Omega, U)$ is complete we have that $(X_m)_m$ is converging in $\|\cdot\|_2$, then there exists a subsequence of partial sums converging $\mathbb{P}$-almost everywhere in U. □

From now on Q will be called the covariance operator of X.

Generalised Gaussian variables

Following the idea presented in (B.2) we have that X is a Gaussian variable if and only if it can be written as

$$X = \sum_{k=1}^{\infty} \beta_k Q^{1/2} e_k = \sum_{k=1}^{\infty} \beta_k \sqrt{\lambda_k} e_k \tag{B.3}$$

where $(\lambda_k)_k$ is a summable sequence and $Q \in L_1(U)$ is the covariance operator of X. We stress that the last identity is possible because Q is a Trace class operator. The leading idea behind the definition of the Generalised Gaussian variable is to consider $Q \in L(U)$ symmetric and non-negative only. In this way the formula

$$X = \sum_{k=1}^{\infty} \beta_k Q^{1/2} e_k \tag{B.4}$$

is not well defined but the strategy is to extend the space U to a bigger space U_{ex} in a way that

$$\sum_{k=1}^{\infty} \mathbb{E}[\beta_k^2] \langle Q^{1/2} e_k, Q^{1/2} e_k \rangle_{U_{ex}} < \infty, \tag{B.5}$$

which implies that the partial sums of (B.4) are converging almost everywhere. Once we have the convergence we can define the action of X on $h \in U$ as

$$h \mapsto X_h := \sum_{k=1} \beta_k \left\langle Q^{1/2} e_k, e_k \right\rangle_{U_{ex}} \langle h, e_k \rangle_{U_{ex}}$$

which is also linear. We start with a very important example.

Example. Consider the so-called White Noise

$$W := \sum_{k=1}^{\infty} \beta_k e_k$$

which corresponds to $Q = Id_U$, which is not a Trace class operator. Notice that if $h \in U$ the real random variable $\langle W, h \rangle_U$ is not well defined (the variable W does not converge to any element in U). Now we consider the extension U_{ex} defined as the completion of U under the norm

$$\|h\|_{U_{ex}}^2 = \sum_{k=1}^{\infty} \frac{1}{k^2} \langle h, e_k \rangle^2$$

where $(e_k)_k$ is an orthonormal basis for U. Notice that the map

$$h \mapsto W_h = \langle W, h \rangle_{U_{ex}} := \sum_{k=1}^{\infty} \beta_k \langle h, e_k \rangle_{U_{ex}}$$

is well-defined. Indeed we have

$$\sum_{k=1}^{\infty} \langle h, e_k \rangle_{U_{ex}} \leq \sum_{k=1}^{\infty} \|h\|_{U_{ex}}^2 \|e_k\|_{U_{ex}}^2 = \|h\|_{U_{ex}}^2 \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

Moreover we have also, for $h, g \in U$,

$$\mathbb{E}[\langle W, h \rangle_{U_{ex}}] = 0, \quad \mathbb{E}[\langle W, h \rangle_{U_{ex}} \langle W, g \rangle_{U_{ex}}] = \langle h, g \rangle_U.$$

In the example it was easy to find the suitable extension of the space U under which W converges. For a more general X we have to understand which properties we need on these extensions to make everything work. For the moment we present the definition of the generalised Gaussian variable and after we show how it is related to the previous construction.

Definition B.4. *Given a Hilbert space U, we say that a linear map $h \mapsto X_h$ defines a generalised Gaussian random variabl, if X_h is a real Gaussian random variable for every $h \in U$ and if*

$$\mathbb{E}\left[|X_h - X_{h_n}|^2\right] \to 0 \quad \text{as } h_n \to h \in U$$

Notice that the previous map

$$h \mapsto W_h := \langle W, h \rangle_{U_{ex}}$$

defines a generalised Gaussian variable. The extension idea we have presented is closely connected to generalized Gaussian variables through this powerful result (for a complete proof see [May21]):

Theorem. *Let U, U_{ex} be Hilbert spaces and $Q \in L(U)$ (symmetric and non-negative) be such that $Q^{1/2}(U) =: U_0 \subseteq U_{ex}$ with Hilbert-Schmidt embedding $iQ^{1/2} : U \to U_{ex}$. Then* (B.4) *defines a generalised Gaussian random variable on U. Moreover it can be proved that the definition of X as a generalised Gaussian random variable on U is independent of the choice of U_{ex} and i.*

Hence the property that we require on our extension is that the map

$$iQ^{1/2} : U \to U_{ex}$$

is a Hilbert-Schmidt operator. Indeed in the example with $Q = Id_U$ one can easily prove that $i : U \to U_{ex}$ is Hilbert-Schmidt.

Proof. Our aim is to prove that X as in (B.4) is converging in U_{ex}. This is equivalent to check (B.5):

$$\sum_{k=1}^{\infty} \mathbb{E}[\beta_k^2] \langle Q^{1/2} e_k, Q^{1/2} e_k \rangle_{U_{ex}} = \sum_{k=1}^{\infty} \langle iQ^{1/2} e_k, iQ^{1/2} e_k \rangle_{U_{ex}} = \sum_{k=1}^{\infty} \langle \left(iQ^{1/2}\right)^* iQ^{1/2} e_k, e_k \rangle_U$$
$$= Tr\left(\left(iQ^{1/2}\right)^* iQ^{1/2}\right) < \infty$$

where we have used that if $iQ^{1/2}$ is Hilbert-Schmidt then $\left(iQ^{1/2}\right)^* iQ^{1/2}$ is Trace class. □

Martingales

The results concerning the Martingales in an infinite dimensional space are very similar to the real case. Every proof of this subsection can be found in [DZ14].

Theorem. *Let X be a E valued random variable which is Bochner integrable and $\mathcal{G} \subseteq \mathcal{F}$ be a sub-sigma-algebra of $\mathcal{F}$. Then there exists a $\mathbb{P}$-unique random variable Z such that*

1. *Z is integrable;*
2. *Z is $\mathcal{G}$-measurable;*
3. $$\int_\Omega X(\Omega)\mathbf{1}_A d\mathbb{P}(\omega) = \int_\Omega Z(\omega)\mathbf{1}_A d\mathbb{P}(\omega), \qquad \textit{for all } A \in \mathcal{G}$$

In this case we write $Z = \mathbb{E}[X \mid \mathcal{G}]$.

Once we have this construction we can define the Martingales.

Definition B.5. *Let $(M_t)_{t\geq 0}$ be an E-valued stochastic process and $(\mathcal{F})_{t\geq 0}$ a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$. Then we say that $(M_t)_{t\geq 0}$ is an $(\mathcal{F})_t$-martingale if*

1. *for every $t \geq 0$ the random variable M_t is integrable;*
2. *M_t is $\mathcal{F}_t$ measurable;*
3. *$\mathbb{E}[M_t \mid \mathcal{F}_s] = M_s$ $\mathbb{P}$-a.s. for all $0 \leq s \leq t < \infty$.*

We are particularly interested in continuous, square integrable martingales, i.e., E-valued, continuous, $(\mathcal{F}_t)_t$-martingales such that

$$\|M\|_{\mathcal{M}_T^2} := \sup_{t\in[0,T]} \mathbb{E}[\|M_t\|_E^2] < \infty.$$

It is possible to prove that $\mathcal{M}_T^2(E)$ is a closed subspace of $L^2(\Omega; C([0,T]; E))$ and so it is itself a Banach space.

Wiener Processes and Generalised Wiener Processes

Definition B.6. *Given a Hilbert space U and a trace class operator $Q \in L_1(U)$, we say that $(W_t)_{t\geq 0}$ is a standard Q-Wiener process with respect to $(\mathcal{F}_t)_{t\geq 0}$ if,*

1. *W_0=0;*

2. *$W_{t\geq 0}$ is $\mathcal{F}_{t\geq 0}$ measurable for every $t \geq 0$;*

3. *the map $t \mapsto W_t \in U$ is $\mathbb{P}$-a.s. continuous;*

4. *$W_t - W_s \sim \mathcal{N}(0, (t-s)Q)$ for any $0 \leq s < t < \infty$;*

5. *$W_t - W_s$ is indipendent from $\mathcal{F}_s$ for any $0 \leq s < t < \infty$.*

If we do not explicitly write the filtration we are assuming that $(\mathcal{F}_t)_t$ is the normal filtration induced by the process $(W_t)_t$.
Let $Q \in L_1(U)$ and suppose that $Qe_k = \lambda_k e_k$ for an orthonormal basis $(e_k)_k$ for U with $(\lambda_k)_k \subset \mathbb{R}$. One can prove that $(W_t)_{t\geq 0}$ is a Q-Wiener process if and only if

$$W_t = \sum_k \sqrt{\lambda_k}\beta_t^k e_k$$

where $(\beta_k)_k$ are independent, real, Brownian motions.
Suppose $(W_t)_{t\geq 0}$ and consider again the orthonormal basis generated by Q. Then we have

$$\sup_{t\in[0,T]} \mathbb{E}[\|W_t\|_U^2] = \sup_{t\in[0,T]} \mathbb{E}\left[\sum_{k=1}^{\infty} |\langle W_t, e_k\rangle|^2\right] \leq \sup_{t\in[0,T]} \sum_{k=1}^{\infty} \langle tQe_k, e_k\rangle = \sup_{t\in[0,T]} t\, Tr(Q) < \infty$$

which implies that the Wiener process belongs to $\mathcal{M}_T^2(E)$.
We define the generalised Wiener process using the same extension argument that brings from a Gaussian variable to the Generalised Gaussian variable.

Definition B.7. *Given a Hilbert space U, we say that a family of linear maps $[0,T]\times U \ni (t,h) \mapsto W_{h;t} \in \mathbb{R}$ defines a generalised Wiener process if for each $h \in U$ the map $t \mapsto W_{h,t}$ defines a real Wiener process and for any subsequence $(h_n)_n$ converging to $h \in U$ we have*

$$\mathbb{E}[|W_{h,t} - W_{h_n,t}|^2] \to 0 \quad \textit{for all } t \geq 0.$$

Employing the same ideas used for the Gaussian variables we can understand a generalised Wiener process as a Wiener process taking values in a space slightly larger than U. In particular, we can write W_t as a formal sum

$$W_t = \sum_{k\geq 1} \beta_t^k Q^{1/2} e_k$$

where $(\beta_k)_k$ is a family of i.i.d, standard, real Brownian motions. We may consider the Wiener process $(W_{h,t})_{t,h}$ as a U_{ex} valued Q-Wiener process on U provided that

$$iQ^{1/2} : U \to U_{ex}$$

is Hilbert-Schmidt. This brings to the fact that the generalised Wiener process is an element of $\mathcal{M}_T^2(U_{ex})$.

B.2 Stochastic integration

We conclude by giving meaning to integrals of the form

$$[0,T] \ni t \mapsto \int_0^t \varphi_s dW_S$$

where $(\varphi_t)_t$ is a suitable operator valued process. The strategy is similar to the one for the finite-dimensional case, the work is organised as follows

1. Define the stochastic integration for a class of simple $L(U;H)$-valued processes called $\mathcal{E}_T(U;H)$;
2. We notice that the integration is an isometry between $(\mathcal{E}_T(U;H), \|\cdot\|_{\mathcal{H}^2_T})$ and the space of square-integrable martingales $(\mathcal{M}^2_T(H), \|\cdot\|_{\mathcal{M}^2_T})$, where the norm $\mathcal{H}^2_T$ has to be carefully defined;
3. With a density argument we extend the definition of stochastic integration to every predictable process $(\varphi_t)_t$ such that $\|\varphi\|_{\mathcal{H}^2_T}$ is finite.

Simple processes

We say that a process $[0,T] \ni t \mapsto \varphi_t \in L(U;H)$ is simple if there exists a sequence of times $0 = t_0 < t_1 < \cdots < t_{n-1} = T$, and a set of $L(U;H)$ valued, $(\mathcal{F}_{t_m})_{m=0}^{n-1}$ measurable, random variables $(\varphi_m)_{m=1}^{n-1}$, taking only **finitely many values**, such that

$$\varphi_t = \varphi_0 \mathbf{1}_0(t) + \sum_{m=0}^{n-1} \varphi_m \mathbf{1}_{(t_m,t_{m+1}]}(t).$$

We write $\mathcal{E}_t(U;H)$ for the set of $L(U;H)$ valued simple processes on $[0,T]$.
For a simple process φ one defines the stochastic integral by the formula

$$\int_0^t \varphi_s dW_s = \sum_{m=0}^{n-1} \varphi_m (W_{t_{m+1}\wedge t} - W_{t_m \wedge t})$$

and denoted by $\varphi \cdot W(t)$.

The norm $\mathcal{H}^2_T$

We start introducing the subspace $U_0 = Q^{1/2}(U)$ of U which is a Hilbert space with the inner product

$$\langle u, v\rangle_{U_0} = \langle Q^{-1/2}u, Q^{-1/2}v\rangle_U$$

for $u, v \in U_0$ and $Q^{-1/2} = \left(Q^{1/2}\right)^*$.

Proposition. *We have the inclusion $L(U;H) \subset L_2^0(U;H)$ also denoted as $L_2(U_0;H)$.*

Proof. Let $\varphi \in L(U;H)$ and we prove that $\varphi \in L_2(U_0;H)$ or, equivalently, $\varphi^*\varphi \in L_1(U_0;H)$. Indeed consider $(u_k)_k$ a basis for U_0

$$\sum_k \langle \varphi^*\varphi u_k, u_k\rangle_{U_0} = \sum_k \langle Q^{-1/2}\varphi^*\varphi u_k, Q^{-1/2}u_k\rangle_U = \sum_k \langle Q^{1/2}Q^{-1/2}\varphi^*\varphi u_k, u_k\rangle_{U_0} < \infty.$$

The last series is finite because $Q^{1/2}Q^{-1/2}\varphi^*\varphi$ is trace class. Indeed $Q^{1/2}Q^{-1/2}$ is trace class and $\varphi^*\varphi \in L(U)$. Since the composition of a trace class operator with a bounded linear operator is still a trace class we have done. □

This inclusion guarantees that simple processes are measurable, $L_2(U_0;H)$-valued processes. The advantage is that $L_2(U_0;H)$ is also a separable Hilbert space, equipped with the norm

$$\|\varphi\|^2_{L_2^0} := Tr\left[(Q^{1/2}\varphi)(Q^{1/2}\varphi)^*\right].$$

This allows us to define the set of square integrable processes with respect to a Q-Wiener process. For a measurable process, $[0,T] \ni t \mapsto \varphi_t \in L_2^0(U_0;H)$, we set

$$\|\varphi\|_{\mathcal{H}_T^2} := \mathbb{E}\left[\int_0^T \|\varphi_s\|^2_{L_2^0}\right].$$

With respect to this norm, we have the isometry we need. Indeed the map

$$\begin{aligned} \mathcal{E}_T(U;H) &\longrightarrow \mathcal{M}_T^2(H) \\ \varphi &\longrightarrow \varphi \cdot W \end{aligned}$$

is an isometry between $(\mathcal{E}_T(U;H), \|\cdot\|_{\mathcal{H}_T^2})$ and the space of square-integrable martingales $(\mathcal{M}_T^2(H), \|\cdot\|_{\mathcal{M}_T^2})$. These results are collected in the following proposition.

Proposition. *Let $\varphi \in \mathcal{E}_T(U;H)$. Then $(\varphi \cdot W_t)_t$, defined as before, is a continuous, square-integrable, H-valued martingale, adapted to the natural filtration of $(W_t)_t$. Furthermore, one has*

$$\|\varphi \cdot W\|_{\mathcal{M}_T^2} = \sup_{t\in[0,T]} \mathbb{E}[\|\varphi \cdot W_t\|_H^2] = \|\varphi\|_{\mathcal{H}_T^2}$$

For the proof see [May21].

Density extension

We wish to extend the definition of integration to every $L_2^0(U;H)$-valued process such that $\|\varphi\|_{\mathcal{H}_T^2} < \infty$. Since we want our stochastic integration to preserve the martingale property we have to restrict to predictable processes.

Definition B.8. *Given $(\mathcal{F}_t)_{t\in[0,T]}$ a filtration, then we define $\mathcal{P}_T$ as*

$$\mathcal{P}_T := \sigma\left(\{(s,t] \times F : 0 \leq s < t \leq T, F \in \mathcal{F}_s\} \cup \{\{0\} \times F : F \in \mathcal{F}_0\}\right).$$

Given a Hilbert space H, we say that a process $\varphi : [0,T] \times \Omega \to H$ is predictable if it is $\mathcal{P}_T$-measurable.

We say that a process $[0,T] \times \Omega \ni (t,\omega) \mapsto L_2^0(U;H)$ is **stochastically integrable** with respect to a Wiener process W if

1. it is predictable with respect to the natural filtration of W;
2. $\|\varphi\|_{\mathcal{H}_T^2} < \infty$.

This space is denoted by $\mathcal{H}_T^2(W)$ and observe that it is a Hilbert space with inner product $\langle \varphi, \psi \rangle_{\mathcal{H}_T^2} = \int_0^T \langle \varphi_t, \psi_t \rangle_{L_2^0} dt$.

Theorem. *If $\varphi \in \mathcal{H}_T^2(W)$ then there exists a sequence of simple processes $(\varphi^n)_n \subset \mathcal{E}_T(U;H)$ such that $\|\varphi - \varphi^n\|_{\mathcal{H}_T^2} = 0$. This allows us to define, by density, the stochastic integral indicated as*

$$[0,T] \ni t \mapsto \int_0^t \varphi_s dW_s := \varphi \cdot W_t$$

which is a continuous, square-integrable, H-valued martingale.

With some extra work but with the same ingredients one can define the integration with respect to a generalised Q-Wiener processes $(W_t)_t$. Since the embedding $i : U_0 \mapsto U_{ex}$ is Hilbert-Schmidt by definition one can see W as a Wiener process with values in U_{ex}. Hence we can define the integrals for predictable processes taking values in $L_2(Q^{1/2}(U_{ex});H)$, where we have replaced U with U_{ex}. However, this is unsatisfactory, since we know that definition of a generalised Wiener process does not depend on the choice of the extension U_{ex} and so it would be natural for the integration to retain this property. It turns out that we can retain the same space of integrands as in the previous theorem.

Theorem. *Let $(W_t)_{t\in[0,T]}$ be a generalsied Wiener process on $\varphi \in \mathcal{H}_T^2(W)$. Then, letting $i : U_0 \to U_{ex}$ be an Hilbert-Schmidt extension then the integral*

$$[0,T] \in t \mapsto \varphi \cdot W_t := \int_0^t (\varphi_s \circ i^{-1}) dW_s$$

is a continuous, H-valued square-integrable, martingale.

B.3 Stochastic PDE's

Before introducing Stochastic Partial Differential Equations (SPDEs), it is helpful to review some basic concepts from Functional Analysis that are commonly used in the study of time-dependent partial differential equations (PDEs) in infinite-dimensional spaces. These concepts provide a foundation for understanding the behaviour and properties of solutions to such equations. We begin by recalling the notion of unbounded operators on Banach spaces.

Definition B.9. *Given Banach spaces E, F, an unbounded linear operator A from E to F is a pair $(A, D(A))$ where $D(A) \subset E$ is a linear dense subspace and $A : D(A) \to F$ is a linear map.*

If A is densely defined from E to F and continuous on its domain then there exists a unique, continuous extension of A defined on all E.
Another important concept in the study of PDEs is the Gelfand triple. We recall the definition:

Definition B.10. *A Gelfand triple consists of three spaces: a separable, reflexive Banach space denoted by V, a separable Hilbert space denoted by H, and the dual space of V denoted by V^*. The triple is defined as follows:*

$$V \subset H \cong H^* \subset V^*.$$

Here, H^* represents the dual space of H, which is isomorphic to H itself due to the Hilbert space duality.

Let $A : D(A) \subset V \to V^*$ be an unbounded linear operator, $T > 0$ and $B \in L^2([0,T]; H)$ be a square-integrable map. Then we consider the deterministic, linear PDE

$$\begin{cases} \partial_t u - A(u) = B_t \\ u|_{t=0} = u_0. \end{cases} \tag{B.6}$$

We leave any specification of spatial domain and any boundary conditions to the definition of the spaces V and H.
For example, consider the homogeneous heat equation, which corresponds to $B = 0$ and $A = \Delta$. The Gelfand triple we have to consider is

$$H_0^1(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \subset \left(H_0^1(\mathbb{R}^n)\right)^* = H^{-1}(\mathbb{R}^n).$$

The unbounded operator $\Delta : C_c^2(\mathbb{R}^3) \subset H_0^1(\mathbb{R}^n) \to H^{-1}(\mathbb{R}^n)$ is a continous map from $C_c^2(\mathbb{R}^3)$ to $H^{-1}(\mathbb{R}^n)$. Since $C_c^2(\mathbb{R}^3)$ is dense in $H_0^1(\mathbb{R}^n)$ the operator Δ can be uniquely extended to $H_0^1(\mathbb{R}^3)$.

We are ready to add stochastic rumour to the equation (B.6). Fix $T > 0$ and a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$. Consider also a Gelfand triple (V, H, V^*) of Hilbert spaces and the stochastic partial differential equation

$$\begin{cases} du_t = A(u_t)dt + B(u_t)dW_t, \\ u|_{t=0} = u_0, \end{cases} \tag{B.7}$$

where

1. $(W_t)_{t\in[0,T]}$ is a U-valued generalised Wiener process with respect to the filtration $(\mathcal{F}_t)_t$, with covariant operator Q and U is a fourth Hilbert space.
2. $A : D(A) \subset V \to V^*$ is an unbounded operator;
3. $B : V \to L_2(U_0, H)$ and for every square integrable process $[0, T] \ni t \mapsto u_t \in V$ we have $(B(u_t))_{t\in[0,T]} \in \mathcal{H}_t^2(W)$.

A weak solution to (B.7) is an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted process $[0, T] \times \Omega \ni (t, \omega) \mapsto u_t(\omega) \in V$ if

$$u \in L^2(\Omega \times [0, T]; V), \tag{B.8}$$

and for every $v \in V$ and $t \in [0, T]$, $\mathbb{P}$-a.s.

$$\langle u_t, v\rangle = \langle u_0, v\rangle + \int_0^t \langle A(u_s), v\rangle ds + \int_0^t \langle v, B(u_s)dW_s\rangle.$$

Notice that we have given the definition only in the case when the operators A and B do not depend on (t, ω). As we are going to see this does not modify the definition of a weak solution.
The hypotheses are very similar to the one we ask for a deterministic PDE, the only difference is represented by the ones on B. To understand them we remind that $\mathcal{H}_t^2(W)$ represent the space of stochastically integrable processes, namely $\varphi \in \mathcal{H}_t^2(W)$ if it is a process $[0, T] \times \Omega \ni (t, \omega) \mapsto L_2^0(U; H)$ such that

1. it is predictable with respect to the natural filtration of W;
2. $\|\varphi\|_{\mathcal{H}_T^2} < \infty$.

Hence the hypothesis of $(B(u_t))_{t\in[0,T]} \in \mathcal{H}_t^2(W)$ for every square-integrable process $[0, T] \ni t \mapsto u_t \in V$ is exactly what we need for a good definition of (B.7).

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