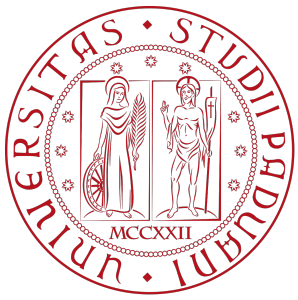

UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Fisica e Astronomia "Galileo Galilei"

Corso di Laurea in Fisica

Tesi di Laurea

Gruppi finiti e mescolamento dei neutrini

Finite groups and neutrino mixing

Relatore

Dr. Arsenii Titov

Laureando

Daniel Tassetto

Anno Accademico 2024/2025

Abstract

I gruppi sono un concetto matematico di fondamentale importanza, la loro applicazione spazia in tutti gli ambiti della fisica classica e moderna. I gruppi descrivono e classificano le simmetrie di un sistema fisico, in accordo con il teorema di Noether: "Ad ogni simmetria corrisponde una legge di conservazione" quindi identificare quale gruppo rappresenta una determinata simmetria ci consente di comprendere più a fondo il sistema fisico in esame. L'obiettivo di questa tesi è incentrato sullo studio e sull'applicazione di una ristretta tipologia di gruppi: i gruppi finiti.

Inizialmente verranno trattati i concetti principali della teoria dei gruppi finiti e della teoria delle rappresentazioni, questi concetti sono illustrati con esempi concreti, prendendo in considerazione, i gruppi finiti non abeliani S_3 e A_4. Infine, uno di questi gruppi verrà applicato alla descrizione del mescolamento dei neutrini.

Contents

Introduction **2**

1 Finite groups **3**
1.1 Basic concepts 3
1.2 Permutations 8

2 Representations **11**
2.1 Basics 11
2.2 Character tables 14
2.3 Kronecker products and C-G coefficients 17

3 Applications **19**
3.1 Neutrino oscillations 19
3.2 Tri-bimaximal mixing 21

Conclusions **23**

Bibliography **25**

Introduction

Symmetry is a property of an object in which its shape exhibits regular repetitions.

Human beings, by nature, seek harmony, beauty, and order in things, that is why it comes naturally to us to look for symmetries in nature.

In history, symmetries have always been studied by mathematicians and philosophers to understand nature's patterns. However the mathematical structures that generate symmetrical patterns were not systematically explored and categorized until the nineteenth century by Évariste Galois. We now call these mathematical structures groups.

In light of this, from a modern perspective, the initial definition of symmetry can be extended to: Symmetry is a property of an object that remains invariant under a specified group of transformations. Groups split up in two main categories: Lie groups and finite groups. Our attention is focused on the latter.

Finite groups describe structures with a finite number of elements such as the symmetry of polygons or permutations. For our purposes it is important to study these groups and their proprieties. In particular we are interested in two groups: S_3 and A_4, respectively the symmetry group of permutations on three objects and the symmetry group of even permutations on four objects.

In the first two chapters of this thesis we will introduce some theory about finite groups to be further applied in a physical context.

The physical context in question is neutrino mixing. The phenomenon of neutrino mixing occurs when neutrinos produced in a weak interaction with a specific flavor such as ν_e, ν_μ, ν_τ, do not coincide with the mass eigenstates ν_1, ν_2, ν_3,. As a result, a neutrino created as ν_e is actually a superposition of mass states, which evolve differently as it travels. This effect leads to neutrino oscillations, namely the neutrino can assume different flavor during its propagation, these oscillations are a quantum mechanical interference effect on macroscopic distances. The effect has been confirmed with solar, atmospheric and reactor neutrinos.

The relationship between flavor and mass states is described by the Pontecorvo–Maki–Nakagawa–Sakata matrix, simply known as PMNS matrix. This matrix encodes the mixing angles and possible CP-violating phases. One simple approximation of this matrix is the tribimaximal mixing. We will show how it can arise assigning to the three generations of leptons A_4 irriducible triplet

The purpose of this thesis is to demonstrate how symmetric forms of the PMNS matrix can originate from finite groups.

Chapter 1

Finite groups

In this chapter, we will introduce the concept of finite groups and their general axioms. We will discuss in detail some important groups up to the order twelve and their multiplication tables, as well as permutations and basic concepts such as conjugation, conjugacy classes, and simple groups. These concepts will be illustrated using as examples S_3 and A_4. This chapter is based on [1], [2].

1.1 Basic concepts

Def. 1.1.1. *A **finite group** G is a collection of a finite number of elements:*

$$G : \{a_1, a_2, ...a_n\} \tag{1.1}$$

with a "$\cdot$" operation with the following properties:

- *Closure*

  For every ordered pair of elements, a_i and a_j, there exists a unique element

  $$a_i \cdot a_j = a_k \tag{1.2}$$

- *Associativity*

  The operation "$\cdot$" is associative:

  $$(a_i \cdot a_j) \cdot a_k = a_i \cdot (a_j \cdot a_k) \tag{1.3}$$

- *Unit element*

  The set G contains a unique element e such that

  $$e \cdot a_i = a_i \cdot e = a_i \tag{1.4}$$

- *Inverse element*

  Corresponding to every element a_i , there exists a unique element of G, the inverse a_i^{-1} such that

  $$a_i \cdot a_i^{-1} = a_i^{-1} \cdot a_i = e \tag{1.5}$$

Therefore G is called finite group of order n.

Def. 1.1.2. *A group is said to be Abelian when for all $a_i, a_j \in G$*

$$a_i \cdot a_j = a_j \cdot a_i = a_k \tag{1.6}$$

As the order of the group increases, it is more convenient to introduce a new way to describe the groups, namely, a group presentation:

Def. 1.1.3. *A **group presentation** $\langle S|R\rangle$ contains a set of generators S and a set of relations R between elements constructed in terms of the generators in S.*

We now turn to the study of the characteristics of some groups up to the twelfth order, followed by a focused analysis on S_3 and A_4.

Let's start by introducing the two groups of order two and three respectively: $\mathcal{Z}_2$ and $\mathcal{Z}_3$. These simple groups belong to the $\mathcal{Z}_n$ family of groups called "Cyclic groups",these groups represent $\frac{2\pi}{n}$ rotations in then plain. So, this group is defined as:

$$\mathcal{Z}_n = \langle \rho|\rho^n = e\rangle \tag{1.7}$$

There are only two groups of the fourth order: $\mathcal{Z}_4$ (from now on, cyclic groups will not be mentioned as they are assumed to be familiar) and the simplest group D_2, part of a family of groups known as Dihedral groups D_n of order $2n$. This family of groups represent the symmetries of a regular n-sided polygon. D_2 is the only abelian group among all the others. In general, D_n is defined as

$$D_n = \langle a, b|a^n = b^2 = e, bab^{-1} = a^{-1}\rangle. \tag{1.8}$$

We now turn our attention to the sixth order, a fundamental group for our purpose is S_3, this group represent the symmetry of an equilateral triangle. It is the set of all reflections, rotations, and combinations of these two, that leaves invariant the triangle. Also, it is the first non-Abelian group that we encounter .

$$S_3 = \langle a, b|a^3 = b^2 = e, bab^{-1} = a^{-1}\rangle \tag{1.9}$$

The multiplication table is:

S_3	e	a	a^2	b	ab	a^2b
e	e	a	a^2	b	ab	a^2b
a	a	a^2	e	ab	a^2b	b
a^2	a^2	e	a	a^2b	b	ab
b	b	a^2b	ab	e	a^2	a
ab	ab	b	a^2b	a	e	a^2
a^2b	a^2b	ab	b	a^2	a	e

With two generators, S_3 is said to have rank two. Its elements of order three describe the 120° rotation about O, the center of the triangle. Its three elements of order two are the reflections about the three axis O A, O B, and OC, from each vertex to its center, see figure 1.1.

Another way to represent this group consists in the permutations on the three letters A, B, C which label the vertices of the triangle.

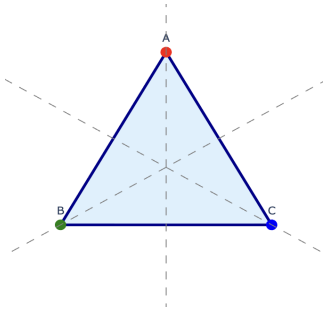

Figure 1.1: Geometrical representation of S_3 action on a equilateral triangle

The **reflection** about the vertical axis which interchanges the vertices B and C is the permutation on two letters, called transposition, and denoted by the symbol (BC). The other two reflections are simply (AB) and (AC).

The **120° rotations** about the center are permutations on three letters, denoted by the symbol (ABC), meaning the transformations $A \to B \to C \to A$. We call **transpositions** two-cycles, and we call **permutations** like (ABC), (ACB) three-cycles. Thus the six elements of S_3 break up into the unit element, three transpositions, and two three-cycles.

As the order increases, from order eight, we encounter groups that can be expressed as the product of two smaller groups, yet are not isomorphic to any of the previously mentioned. For example $\mathcal{D}_2 \times \mathcal{Z}_2$ and $\mathcal{Z}_4 \times \mathcal{Z}_2$ are order eight groups not isomorphic to $\mathcal{Z}_8$ neither to each other. Vice versa, if we return to consider groups of the fourth order we realize that we haven't mentioned $\mathcal{Z}_2 \times \mathcal{Z}_2$, but this group is isomorphic to D_2 so they are the same group. Other groups of the eighth order are D_4 and Q, this new group is called the quaternion group. The elements of this group are the unit quaternions: $Q = \{-1, 1, i, -i, j, -j, k, -k\}$. Quaternions are a type of numbers that extend complex numbers, for example $q = a+bi+cj+dk$ is a quaternion, where $a, b, c, d \in \mathbb{R}$ are coefficients and i, j, k are imaginary units. The definition of Q by its presentation is: :

$$Q = \langle i, j | i^4 = e, i^2 = j^2, jij^{-1} = i^{-1} \rangle \tag{1.10}$$

where $jij^{-1} = k$ At the twelfth order there are two Abelian groups and three non-Abelian groups. We are interested only in the non-Abelian A_4. The group A_4 is part of a family of groups known as Alternating groups A_n. The presentation of the group is:

$$A_4 = \langle a, b | a^2 = b^3 = (ab)^3 = e \rangle \tag{1.11}$$

Using this definition we can find all the twelve elements: $\{e, a, b, b^2, ab, ab^2, ba, b^2a, aba, ab^2a, bab, b^2ab\}$

From a geometric perspective, the group A_4 can be visualized as the group of rotational symmetries of a regular tetrahedron, see fig 1.2. Each element of A_4 written in terms of its two generators corresponds to a even permutation of the four vertices.

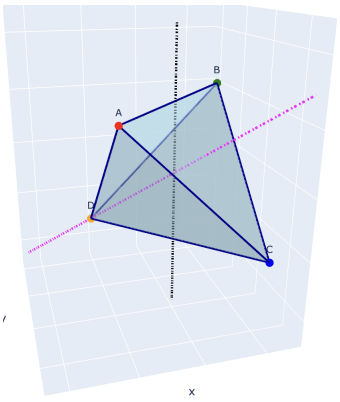

Figure 1.2: Geometrical representation of A_4 action on a regular tetrahedron

The twelve elements are divided in three categories, using the permutation notation:

- identity: (trivial)
- three double transpositions: It is a rotation of 180° around an axis passing through the midpoints of two opposite edges.
- eight 3-cycles: This operation corresponds to a rotation of either 120° or 240° around an axis that passes through a vertex of the tetrahedron and the center of the opposite face.

We have seen some groups of order up to 12. In general, the order of a group can be arbitrarily large. At this point, it becomes important to organize and classify groups in order to distinguish which groups can be viewed as fundamental and which cannot. To do so, we need to introduce the following concepts.

Def. 1.1.4. *Let $a, b, \tilde{b} \in G$ it is said that they are **conjugate** when $\tilde{b} = aba^{-1}$*

We can then use conjugacy to organize group elements:

Def. 1.1.5. *A **class** is a subset $C \subset G$ defined by conjugacy: $C_b = \{\tilde{b} \in G : \tilde{b} = gbg^{-1}, g, b \in G\}$*

Example 1.1.1. S_3 has three conjugacy classes:

$$1C^1(e) = \{e\}, \quad 2C^3(a) = \{a, a^2\}, \quad 3C^2(b) = \{b, ab, a^2b\} \tag{1.12}$$

Let's verify each class:
Verifying the identity class is trivial.
Elements in $2C^3(a)$:

$$ea(e)^{-1} = a, \quad a^2a(a^2)^{-1} = a, \quad ba(b)^{-1} = a^2, \quad aba(ab)^{-1} = a^2, \quad a^2ba(a^2b)^{-1} = a^2 \tag{1.13}$$

Elements in $3C^2(b)$:

$$eb(e)^{-1} = b, \quad ab(a)^{-1} = a^2b, \quad a^2b(a^2)^{-1} = ab, \quad abb(ab)^{-1} = a^2b, \quad a^2bb(a^2b)^{-1} = ab \tag{1.14}$$

Same process is applied to find A_4 classes but we will limit ourselves just to report them:

$$1C^1(e) = \{e\}, \quad 3C^2(a) = \{a, bab^2, b^2ab\}, \quad 4C^3(b) = \{b, ab, ba, aba\}, \quad 4C^3(b^2) = \{b^2, ab^2, b^2a, bab\} \tag{1.15}$$

Def. 1.1.6. *Let G be a group with an internal operation $\cdot$. A subset H of G is called a **subgroup** of G if H also forms a group under the operation $\cdot$. A subgroup H of a group G gives rise to two partitions of G:*

Right cosets: sets of the form $Hg = \{hg : h \in H \ g \in G\}$

Left cosets: sets of the form $gH = \{gh : h \in H \ g \in G\}$

Now, we can state an important result of finite group theory:

Theorem 1.1.1. *Lagrange Theorem: If a group G of order N has a subgroup H of order n, then N is necessarily an integer multiple of n.*

This theorem links the size of a group to the size of its subgroups, providing constraints on the type of subgroup that can exist.

Def. 1.1.7. *A subgroup $N \subset G$ is called a **normal subgroup** (and we write $N \triangleleft G$) if it is self-conjugate:*

$$\forall g \in G, \ b, \tilde{b} \in N : \tilde{b} = gbg^{-1} \tag{1.16}$$

Example 1.1.2. Let's analyze how many normal subgroups do S_3 and A_4 have (except the trivial ones as identity and the group itself):
Normal subgroups in S_3:

$$N = \mathcal{Z}_3 = \{e, a, a^2\} \tag{1.17}$$

Normal subgroups in A_4:

$$N = \{e, a, bab^2, b^2ab\} \tag{1.18}$$

Not all classes form normal subgroups: S_3 and A_4 have another non-trivial classe that do not form normal subgroups because they fail to satisfy the closure axiom.

Def. 1.1.8. *If H is a normal subgroup of G, we denote the set of left or right cosets by G/H and we call it **quotient group**. We define an operation on G/H by the rule*

$$(Ha)(Hb) = Hab \ \forall a, b \in G. \tag{1.19}$$

Example 1.1.3. Since S_3 and A_4 have a normal subgroup, $S_3/\mathcal{Z}_3,\ A_4/N$ are quotient groups.
$\forall g \in S_3$ we can compute $g\mathcal{Z}_3$ obtaining the following two cosets:

$$S_3/\mathcal{Z}_3 = \{\mathcal{Z}_3 = \{e, a, a^2\},\ b\mathcal{Z}_3 = \{b, ab, a^2b\}\} \tag{1.20}$$

The quotient group must have $6 : 2 = 3$ elements, which are exactly the two cosets that we have just found.
Let's verify group axioms:

- unit element:$\mathcal{Z}_3 \times \mathcal{Z}_3 = \mathcal{Z}_3$
- inverse: $b\mathcal{Z}_3 \times (b\mathcal{Z}_3)^{-1} = b\mathcal{Z}_3 \times b\mathcal{Z}_3 = \mathcal{Z}_3$
- closure: $\mathcal{Z}_3 \times b\mathcal{Z}_3 = b\mathcal{Z}_3,\ b\mathcal{Z}_3 \times b\mathcal{Z}_3 = \mathcal{Z}_3$
- associativity: $(b\mathcal{Z}_3 \times \mathcal{Z}_3) \times b\mathcal{Z}_3 = b\mathcal{Z}_3 \times (\mathcal{Z}_3 \times b\mathcal{Z}_3) = \mathcal{Z}_3$

The conclusion is:

$$S_3/\mathcal{Z}_3 \simeq \mathcal{Z}_2 \tag{1.21}$$

The same procedure is applied to A_4/N and the conclusion is:

$$A_4/N \simeq \mathcal{Z}_3 \tag{1.22}$$

Def. 1.1.9. *A **simple group** is a group which does not have any normal subgroup.*

Example 1.1.4. As first examples of simple groups there are $A_n,\ n \geq 5$ and the entire family $\mathcal{Z}_n$.

All finite groups can be systematically constructed from simple groups in a procedure called composition series: We can decompose any group in terms of its normal subgroups by seeking its largest normal subgroup H_1. If it has a normal subgroup, the group elements split into H_1 and the quotient group G/H_1. Then we can reapply the procedure to H_1 and so on until reaching the following composition:

$$G \triangleright H_1 \triangleright H_2 \triangleright ... \triangleright H_k \triangleright e \tag{1.23}$$

Each of them generates the quotient group

$$G/H_1, H_1/H_2, ..H_k \tag{1.24}$$

The process of seeking the largest subgroup with the purpose of building a quotient group is also known as "abelianization".

Example 1.1.5. The largest A_4 non trivial normal subgroup is $N = \{e, a, bab^2, b^2ab\}$. This group is isomorphic to the V_4 group, also know as Klein-four group.

$$V_4 = \langle a, b | a^2 = b^2 = (ab)^2 = e \rangle \tag{1.25}$$

. Let's seek its normal subgroups (Except trivial ones):
In V_4 there exist three normal subgroups each of them of the second order, isomorphic between them to $\mathcal{Z}_2$. In general, normal subgroups are not simple, but these are constructed each time out of the largest one, so they must be simple.
The composition series of A_4 is:

$$A_4/V_4,\ V_4/\mathcal{Z}_2,\ \mathcal{Z}_2 \tag{1.26}$$

Now, we can enunciate a serie of theorems that go under the name of **Sylow's criteria** which state:

Theorem 1.1.2. *Let G be a group of order n. Decompose its order in powers of primes: $n = p^m r$ where p is a prime, and r is integer not multiple of p. Then:*

- *G contains n_p Sylow p-subgroups of order p^m*
- *All G_p^i are isomorphic to one another, related by $G_p^j = gG_p^k g^{-1}, g \in G$*

- n_p *is a divisor of* r
- $np = 1 \ \ mod(p)$

Sylow's criteria provide a partial converse to Lagrange's theorem since it's only valid for p-groups. It's very useful for small groups because it tells us what subgroup orders must actually exist.

Def. 1.1.10. *Let G and K be two groups with elements $\{g_a\}, a = 1...n_g$ and $\{k_i\}, i = 1...n_k$ respectively. We assemble new elements (g_a, k_i), with multiplication rule*

$$(g_a, k_i)(g_b, k_j) = (g_a \cdot k_i, g_b \cdot k_j) \tag{1.27}$$

They clearly satisfy the group axioms, forming a group of order $n_g n_k$ called the ***direct product*** *group $G \times K$.*

There is another way to construct a group out of two groups, it requires two groups G and K, and the action of one on the other.

Def. 1.1.11. *Given a group G, a subgroup H, and a normal subgroup $N \triangleleft G$, we define $G = N \rtimes H$ as* ***semi-direct product****. G is the product of subgroups, $G = NH$, and these subgroups have trivial intersection: $N \cap H = \{e\}$*

This definition of product induces an homomorphism $\phi : H \to Aut(N)$, where $Aut(N)$ denotes the group of all automorphisms of N, which is a group under composition. ϕ is defined with conjugation:

$$\phi_h(n) = hnh^{-1} \forall h \in H, n \in N \tag{1.28}$$

The subgroups N and H determine G up to isomorphisms.

Now that we have introduced direct and semi-direct product, we can see that simple groups constitute the "fundamental building blocks" of group theory, as all other groups can be obtained from them through these compositions.

1.2 Permutations

Permutations play a central role in the study of finite groups because it can be shown that any finite group of order n can be viewed as permutations on n letters. Let's start by introducing what a permutation is:

Def. 1.2.1. *A* ***permutation*** *on n object is the act of shuffling them into a different order.*

We can represent permutation in two ways: Let's say that initially the n objects are $\{a_1, a_2, a_3, ..a_n \in X\}$ then we can write permutation in this notation:

$$\begin{pmatrix} a_1 & a_2 & a_3 & ... & a_n \\ \phi(a_1) & \phi(a_2) & \phi(a_3) & ... & \phi(a_n) \end{pmatrix} \tag{1.29}$$

The other way to represent a permutation is with cycles.

This is actually more efficient for large n. A permutation that shuffles $k < n$ objects into themselves, leaving the others untouched is called a k-cycle. For example the permutation d is a 2-cycle:

$$d = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \to (1)(23) \tag{1.30}$$

since it leaves untouched a_1 and swaps a_2, a_3. Vice versa the permutation p is a 3-cycle:

$$p = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \to (123) \tag{1.31}$$

Matematically, a permutation can be viewed as a map $\phi : X \to X$. It preserves all the group axioms, for instance the composition of two permutations it is also a permutation.

Example 1.2.1. Given the presentation of $A_4 = \langle a, b | a^2 = b^3 = (ab)^3 = e \rangle$, we want to write some elements as permutations:
To do so, we need first to find the expression of the two generators a, b and then use the composition law to find all the other elements:
We have 4 objects. From a geometrical perspective a must be a double transposition since $a^2 = e$ then $a \to (12)(34)$ while b is a three-cycle, because $b^3 = e$, then $b \to (123)$ It is easy now to find all the other elements:

$$b^2 \to (123)(123) \to (132) \quad ab \to (12)(34)(123) \to (243) \tag{1.32}$$

$$ba \to (123)(12)(34) \to (134), \quad ab^2 \to (124)(234) \to (143) \tag{1.33}$$

Def. 1.2.2. ***The Permutation group (Symmetric Group)*** *of n elements S_n is the group of automorphisms from the set $X = \{a_1, a_2, a_3, .., a_n\}$ to itself.*

As we said when introducing permutations, there is an important result in finite group theory known as **Cayley's theorem**:

Theorem 1.2.1. *Every group of finite order n is isomorphic to a subgroup of the permutation group S_n .*

Thanks to this theorem, it does not matter how strange a group looks, we can always view it as a subgroup of S_n.

Example 1.2.2. The elements of A_4 are the even permutations on four objects while S_4 contains $4! = 24$ objects, that correspond to both even and odd permutations on four objects. Thus $A_4 \subset S_4$.

Chapter 2

Representations

In the previous chapter, we examined finite groups in their abstract form. We now turn to describing a frame work for representing these groups concretely, so that we can apply them effectively in physical contexts.
First, we will begin by introducing the concepts of reducible and irreducible representations[1], followed by character tables, and then we will see how representations combine through the Kronecker product. This chapter is based on [1], [2].

2.1 Basics

Def. 2.1.1. *A **linear representation** of a group G is a map $\rho : G \to GL(N, \mathbb{C})$ that preserves the composition law of the group G: Let $g \in G$, $D(g)$ is the linear representation of g.*

- $\rho(g_1 \cdot g_2) = \rho(g_1) \cdot \rho(g_2)$
- $\rho(e) = \mathbb{I}_{N\times N}$
- $\rho(g^{-1}) = \rho(g)^{-1}$

We can also think of a linear representation as a map $\rho : GV \to V$ where V is a vector space, usually $\mathbb{C}^N$ where N is the dimension of the representation.

Since we are representing abstract groups using the GL group (GL stands for general linear group, it is the set of all $N \times N$ invertible matrices), it might be that the map does not preserve all the information of the group. For example, we will see that S_3 permits three irreducible representations: the one dimensional irreducible representation discriminate only between permutations and transpositions, assigning them respectively a value of "$+1$" and "-1". We need to distinguish faithful from unfaithful representations: a faithful rep is induced by isomorphism and an unfaithful rep by homomorphism.

The formalism of quantum mechanics is built on Hilbert spaces, so it is important to study representations that act in these spaces. From now on, the vector space V in which we define representations of the groups will be assumed to be complete and equipped with a positive-defined norm. Let's introduce unitary representations.

Def. 2.1.2. *A **unitary representation** of a group G is a map D such that $\forall g \in G$ and $|x\rangle, |y\rangle \in V$*

$$\langle x|y\rangle = \langle D(g)x|D(g)y\rangle \iff D(g)^\dagger D(g) = \mathbb{I} \tag{2.1}$$

hence

$$D(g^{-1}) = D(g)^{-1} = D(g)^\dagger \tag{2.2}$$

Theorem 2.1.1. *Any representation of a finite group is equivalent to a unitary representation.*

[1]For notational simplicity, we will often refer to irreducible representation as irreps. and representations as reps.

Thanks to this theorem, we will be able to apply the results we develop here in the next chapter.

Among all the representations of a group, physicists prefer those that are irreducible, as these form the "fundamental building blocks" of representation theory. Now we will explain the reason.

Def. 2.1.3. *Two representations D_1 and D_2 are **equivalent** if there exists a matrix S called similarity matrix such that:*

$$D_1(g) = S^{-1} D_2(g) S, \quad \forall g \in G \tag{2.3}$$

Def. 2.1.4. *A representation is said **fully reducible** if it can be written as a direct sum of other representations up to a change of basis.*

$$D(g) = D_1 \oplus D_2 \oplus ... \oplus D_k = \begin{pmatrix} D_1(g) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & D_k(g) \end{pmatrix} \tag{2.4}$$

*Otherwise a representation is said **reducible** if it can be put in the block triangular form:*

$$D(g) = \begin{pmatrix} D_1(g) & ... & B_k(g) \\ 0 & \ddots & \vdots \\ 0 & 0 & D_k(g) \end{pmatrix} \tag{2.5}$$

Def. 2.1.5. *A representation is said **irreducible** if it is not reducible.*

The intuition behind these two definitions is that a representation is irreducible if there are no invariant subspaces under its action. Otherwise, if the action of the rep can be split up in two or more different spaces it is a reducible rep.

Example 2.1.1. A particular simple representation of groups is the regular representation. This rep comes from permutations: Let's see an example of this rep for S_3 and A_4 and if this rep is reducible.
S_3:
We need the representation of just three elements: e, a, b which correspond to identity and its two generators, then we can find all the other matrices using the composition rules of the group.

$$\rho_{\mathcal{R}}(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho_{\mathcal{R}}(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho_{\mathcal{R}}(b) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.6}$$

In order to see that the regular rep is reducible or rather: $\rho(g) = \mathbf{1} \oplus \mathbf{2}$, we need to perform a change of basis:

$$\rho(g)^{'} = \mathcal{U}^{\dagger} \rho(g)_{\mathcal{R}} \mathcal{U} \tag{2.7}$$

Where the unitary matrix $\mathcal{U}$ is build up using three linear independent vectors: The normalized vector $|e_1\rangle = \frac{1}{\sqrt{3}}(1,1,1)$ which is an invariant vector in S_3 since any vector with all coordinates equal is fixed. The other two vectors $|e_2\rangle, |e_3\rangle$, which provide a basis for the two dimensional irrep, must both satisfy: $\langle e_2|e_1\rangle = 0, \langle e_3|e_1\rangle = 0$ this lead us to the condition: $(x, y, -x-y)$.
Choosing $x = 1$, $y = \pm 1 \rightarrow z = -2, 0$ the normalized vectors are:

$$|e_2\rangle = \frac{1}{\sqrt{6}}(1,1,-2), \quad |e_3\rangle = \frac{1}{\sqrt{2}}(1,-1,0) \tag{2.8}$$

Now we can build up the matrix $\mathcal{U}$:

$$\mathcal{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 1 \\ \sqrt{2/3} & 1/\sqrt{3} & -1 \\ \sqrt{2/3} & -2/\sqrt{3} & 0 \end{pmatrix} \tag{2.9}$$

Finally, we can compute the elements of the group in a block diagonal form (we provide only the block diagonal form for the generators):

$$\rho(a)^{'} = \mathcal{U}^{\dagger}\rho(a)_{\mathcal{R}}\mathcal{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & -1/2 \end{pmatrix} \quad \rho(b)^{'} = \mathcal{U}^{\dagger}\rho(b)_{\mathcal{R}}\mathcal{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.10}$$

We recognize that the 2×2 blocks correspond to the 2D irrep, let's check composition law of Eq.(1.8):

$$\rho(a)^3_{\mathbf{2}} = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho(b)^2_{\mathbf{2}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.11}$$

$$\rho(b)_{\mathbf{2}}\rho(a)_{\mathbf{2}}\rho(b)^{-1}_{\mathbf{2}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} = \rho(a)^{-1}_{\mathbf{2}} \tag{2.12}$$

A_4:
While S_3 has 3 irreps, A_4 has four of them, three are 1D and one is 3D. Now, our purpose is to seek a block-diagonal form for the reducible rep induced by four dimensional regular rep such that: $\rho(g) = \mathbf{1} \oplus \mathbf{3}$.
The regular rep of A_4's generators:

$$\rho(a)_{\mathcal{R}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \rho(b)_{\mathcal{R}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.13}$$

As we just did for S_3 we notice that the vector $|e_0\rangle = \frac{1}{2}(1,1,1,1)$ is invariant in A_4. Now, imposing $\langle w|e_1\rangle = 0$ we get a condition on the coordinates: $x + y + w + z = 0$
By choosing $x = 1,0,0 \;\; y = 0,1,0 \;\; w = 0,0,1$ we get the three vectors: $|w_1\rangle = (1,0,0,-1), \quad |w_2\rangle = (0,1,0,-1), \quad |w_3\rangle = (0,0,1,-1)$. These vectors do not form an orthonormal basis. In order to build it, we use Gram-Schmidt process on them and the result is:

$$|e_1\rangle = \frac{1}{\sqrt{2}}(1,0,0,-1), \quad |e_2\rangle = \frac{1}{\sqrt{6}}(-1,2,0,-1), \quad |w_3\rangle = \frac{1}{2\sqrt{3}}(-1,-1,3,-1) \tag{2.14}$$

Therefore, the matrix $\mathcal{U}$ is:

$$\mathcal{U} = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{2\sqrt{3}} \\ \frac{1}{2} & 0 & \frac{2}{\sqrt{6}} & -\frac{1}{2\sqrt{3}} \\ \frac{1}{2} & 0 & 0 & \frac{3}{2\sqrt{3}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{2\sqrt{3}} \end{pmatrix} \tag{2.15}$$

Now it is easy to conclude that:

$$\rho(a)^{'} = \mathcal{U}^{\dagger}\rho(a)_{\mathcal{R}}\mathcal{U} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{3} & -\frac{\sqrt{6}}{3} \\ 0 & \frac{\sqrt{3}}{3} & -\frac{2}{3} & -\frac{\sqrt{2}}{3} \\ 0 & -\frac{\sqrt{6}}{3} & -\frac{\sqrt{2}}{3} & -\frac{1}{3} \end{pmatrix}, \quad \rho(b)^{'} = \mathcal{U}^{\dagger}\rho(b)_{\mathcal{R}}\mathcal{U} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{6} & \frac{\sqrt{6}}{3} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{6} & -\frac{\sqrt{2}}{3} \\ 0 & 0 & \frac{2\sqrt{2}}{3} & -\frac{1}{3} \end{pmatrix} \tag{2.16}$$

The element (11) of the matrices is the trivial 1D irrep, while the 3×3 blocks correspond to the 3D irrep. Let's verify that these matrices respect the composition rule of the group:

$$\rho(a)^{'\,2}_{\mathbf{3}} = \begin{pmatrix} 0 & \frac{\sqrt{3}}{3} & -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{3}}{3} & -\frac{2}{3} & -\frac{\sqrt{2}}{3} \\ -\frac{\sqrt{6}}{3} & -\frac{\sqrt{2}}{3} & -\frac{1}{3} \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; \rho(b)^{'\,3}_{\mathbf{3}} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{6} & \frac{\sqrt{6}}{3} \\ \frac{\sqrt{3}}{2} & -\frac{1}{6} & -\frac{\sqrt{2}}{3} \\ 0 & \frac{2\sqrt{2}}{3} & -\frac{1}{3} \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.17}$$

$$(\rho(a)^{'}_{\mathbf{3}}\rho(b)^{'}_{\mathbf{3}})^3 = (\begin{pmatrix} 0 & \frac{\sqrt{3}}{3} & -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{3}}{3} & -\frac{2}{3} & -\frac{\sqrt{2}}{3} \\ -\frac{\sqrt{6}}{3} & -\frac{\sqrt{2}}{3} & -\frac{1}{3} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{6} & \frac{\sqrt{6}}{3} \\ \frac{\sqrt{3}}{2} & -\frac{1}{6} & -\frac{\sqrt{2}}{3} \\ 0 & \frac{2\sqrt{2}}{3} & -\frac{1}{3} \end{pmatrix})^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.18}$$

Now, we introduce a theorem that will become important later when discussing 1D irreps of A_4.

Theorem 2.1.2. *Irreducible representations on $\mathbb{C}$ for an Abelian group are 1-dimensional.*

2.2 Character tables

At this point, the questions are: How can we tell when two irreps are equivalent? How many irreps does a group have? Is there a limit to the number of dimension of each irrep?

We can answer these questions by introducing the character table. This table is organized with the conjugacy classes of the group as its columns, and the irreps as its rows. Each entry in the table corresponds to the value of the character of a given irrep.

First, we need to discuss an important theorem and what a character is.

In order to introduce the Great Orthogonality Theorem (GOT) and the concept of orthogonality itself, we need to define an inner product:

$$(\phi\,,\psi) = \frac{1}{\#G}\sum_{g}^{\#G}\phi(g)\psi(g),\ \forall g\in G,\ \phi,\psi\in\mathbb{C}^{\#G} \tag{2.19}$$

Notice that $\rho_{ij} : G \to \mathbb{C}$, in this way matrix element can be viewed as complex functions defined over G. The scalar product on this space acts by summing the values of the functions ϕ, $\psi \in \mathbb{C}^{\#G}$ on their components.

Def. 2.2.1. *The **character** of a representation is a function $\chi_D : G \to \mathbb{C}$ defined as:*

$$\chi_D(g) = TrD(g),\ \forall g\in G \tag{2.20}$$

We can immediately see that characters possess important proprieties:

$$\chi_D(\tilde{g}) = Tr(D(h)D(g)D(h^{-1})) = Tr(D(h^{-1})D(h))TrD(g) = Tr(D(g)) = \chi_D(g) \tag{2.21}$$

(In this derivation it has been used cyclic invariant propriety of the traces). It's clear that characters are constant in each conjugacy class. Another fundamental propriety of characters is that since they also are functions: characters associated to unequivalent irreps are orthogonal:

$$\langle\chi_a|\chi_b\rangle = \delta_{ab} \tag{2.22}$$

Theorem 2.2.1. *The **Great Orthogonality Theorem**:*

Given two different irreducible representations of G: $\rho_1(g)$, $\rho_2(g)$, therefore:

$$(\rho^a_{ij}(g)\,,\rho^b_{lm}(g)) = \frac{\delta^{ab}}{\#G}\delta_{il}\delta_{jm} \tag{2.23}$$

where a, b denote the dimension of the irreps, while $\rho^a_{ij}(g)$, $\rho^b_{lm}(g)$ are the vectors containing the $\#G$ matrix elements, one for each element of the group.

The orthogonality relation provided by the GOT for characters results:

$$(\chi^{[\alpha]},\,\chi^{[\beta]}) = \frac{1}{\#G}\sum_{a} n_a\overline{\chi^{[\alpha]}_a}\chi^{[\beta]}_a \tag{2.24}$$

Where the sum on a is intended on the number of classes, while n_a is the number of elements in each class.

This theorem answers to our first question: It enables us to understand when two irreps are not equivalent. In case the two irreps have different dimensions, it is obvious that they correspond to two

different irreps and effectively G.O.T tells us that the inner product is zero and the vectors containing the matrix elements must be orthogonal.

As a consequence of this theorem we come up to following result:

$$\sum_a N_a^2 \leq \#G \tag{2.25}$$

This result can be deduced by observing that for a given N_a dimensional irrep, there exist N_a^2 independent and mutually orthogonal matrix elements. Since the space in which these functions are defined is $\mathbb{C}^{\#G}$, summing up on all irreps we obtain the previous inequality.

Example 2.2.1. Verify the GOT for the 2D irrep of S_3 given in example 2.1.1:
First, let's start by writing all the six matrices of the 2D S_3's irrep (We will not write the trivial irrep):

$$\rho(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \rho(a) = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix},\ \rho(b) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.26}$$

$$\rho(a^2) = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix},\ \rho(ab) = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix},\ \rho(a^2b) = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, \tag{2.27}$$

1 and **2** are orthogonal due to the GOT:

$$\langle \rho_i(g) | \rho_{11}(g) \rangle = \frac{1}{6}(1\cdot 1 + 1\cdot(-1) + 1\cdot(-\frac{1}{2}) + 1\cdot\frac{1}{2} + 1\cdot(-\frac{1}{2}) + 1\cdot(\frac{1}{2})) = 0 \tag{2.28}$$

In **2**, let's verify the second part of the GOT:

$$\langle \rho_{12}(g) | \rho_{12}(g) \rangle = \frac{1}{6}(0\cdot 0 + 0\cdot 0 + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} + (-\frac{\sqrt{3}}{2})\cdot(-\frac{\sqrt{3}}{2}) + (-\frac{\sqrt{3}}{2})\cdot(-\frac{\sqrt{3}}{2}) + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}) = \frac{1}{2} \tag{2.29}$$

$$\langle \rho_{22}(g) | \rho_{12}(g) \rangle = \frac{1}{6}(1\cdot 0 + 1\cdot 0 + (-\frac{1}{2})\cdot\frac{\sqrt{3}}{2} + (-\frac{1}{2})\cdot(-\frac{\sqrt{3}}{2}) + (-\frac{1}{2})\cdot(-\frac{\sqrt{3}}{2}) + (-\frac{1}{2})\cdot\frac{\sqrt{3}}{2}) = 0 \tag{2.30}$$

Now we can answer the last questions posed before with the two following theorems.

Theorem 2.2.2. *The number of inequivalent irreducible representations of a finite group is equal to the number of conjugacy classes.*

The next theorem will just extend a consequence of the GOT that we have already seen:

Theorem 2.2.3. *Given a group G and the full set of its irreducible representation, of dimension N_a, we have:*

$$\sum_a N_a^2 = \#G \tag{2.31}$$

This is a fundamental result in representation theory. This theorem links the order of a finite groups to the dimensions of its inequivalent irreducible representations.

Example 2.2.2. Character table of S_3:
From the last theorems we get: $6 = n_1^2 + n_2^2 + n_3^2$ since S_3 has only three conjugacy classes. This equation is solved only for $n_1 = 1$, $n_2 = 1$, $n_3 = 2$. This equation tells us that S_3 has 2 1D irreps and one 2D irrep.

S_3	$1C_e$	$2C_a$	$3C_b$
1	1	1	1
1′	1	x	y
2	2	s	t

As first step, let's fill the cells of trivial characters such as the first column. Each character corresponds to the sum of the diagonal elements of identity in each irrep, while in the first row we just fill with "1" since those characters correspond to the trivial irrep.
For all the other elements let's use orthogonality relation between characters:

$$(\chi^{[\mathbf{1}]}, \chi^{[\mathbf{1'}]}) = 1(1)(1) + 2(1)(x) + 3(1)(y) = 0, \quad \langle\chi^{[\mathbf{1'}]}|\chi^{[\mathbf{1'}]}\rangle = 1(1)(1) + 2(x)(x) + 3(y)(y) = 6 \tag{2.32}$$

$$(\chi^{[\mathbf{1}]}, \chi^{[\mathbf{2}]}) = 1(1)(2) + 2(1)(s) + 3(1)(t) = 0, \quad (\chi^{[\mathbf{2}]}, \chi^{[\mathbf{2}]}) = 1(2)(2) + 2(s)(s) + 3(t)(t) = 6 \tag{2.33}$$

The solutions of the two previous systems are the two couples $(x, y) = (1, -1)$, $(s, t) = (-1, 0)$, so the complete character table looks:

S_3	$1C_e$	$2C_a$	$3C_b$
$\mathbf{1}$	1	1	1
$\mathbf{1'}$	1	1	-1
$\mathbf{2}$	2	-1	0

From this character table emerges an irrep that we have not seen yet: the $\mathbf{1'}$ irrep is called "alternating" because it maps permutations and transpositions respectively in "+1" and "−1".
Character table of A_4:
A_4 has four conjugacy classes so by setting $12 = n_1^2 + n_2^2 + n_3^2 + n_4^2$, we see that the unique solution is: $n_1 = 1$, $n_2 = 1$, $n_3 = 1$, $n_4 = 3$. First, we observe that C_b and C_{b^2} are the inverse of each other. This fact translates in $\chi^{[\alpha]}(g) = \bar{\chi}^{[\alpha]}(g^{-1})$, while C_a is its own inverse, so its characters are real. As we did before, let's set up the character table by filling in the trivial characters and introducing variables for the rest cells.

A_4	$1C_e$	$3C_a$	$4C_b$	$4C_{b^2}$
$\mathbf{1}$	1	1	1	1
$\mathbf{1'}$	1	s	η	$\bar{\eta}$
$\mathbf{1''}$	1	t	ω	$\bar{\omega}$
$\mathbf{3}$	3	u	σ	$\bar{\sigma}$

For the 1D irreps the third order elements must satisfy $g^3 = e \to \chi^{[\alpha]}(g^3) = {\chi^{[\alpha]}}^3(g) = \chi^{[\alpha]}(e) = 1$. This put a constrain on the complex value η, ω, in fact they must be third roots of the unity. Solving a simple complex equation, we are lead to the two value: $\eta = \bar{\omega} = e^{\frac{2\pi i}{3}}$, $\bar{\eta} = \omega = e^{\frac{4\pi i}{3}}$, where the equality between $\eta, \bar{\omega}$ and $\omega, \bar{\eta}$ are estabilished by the fact that if two irreps have the same set of characters they are the same irrep. All the other characters follow from orthogonality relations:

$$(\chi^{[\mathbf{1}]}, \chi^{[\mathbf{1'}]}) = 1 + 3s + 4(\eta + \bar{\eta}) = 0, \quad (\chi^{[\mathbf{1'}]}, \chi^{[\mathbf{1'}]}) = 1 + 3s^2 + 8|\eta|^2 = 12 \tag{2.34}$$

Using: $\eta + \bar{\eta} = 2Re(\eta)$, $|\eta| = 1$ and $Re(\eta) = Re(e^{\frac{2\pi i}{3}}) = -\frac{1}{2}$ we obtain $s = 1$. Same method is applied to find $t = 1$. The equation is the same up to $\eta, \to \bar{\omega}$.
Let's impose orthogonality relations for the 3D irrep:

$$(\chi^{[\mathbf{1}]}, \chi^{[\mathbf{3}]}) = 3 + 3u + 4(\sigma + \bar{\sigma}) = 0, \quad (\chi^{[\mathbf{2}]}, \chi^{[\mathbf{2}]}) = 9 + 3u^2 + 8|\sigma|^2 = 12 \tag{2.35}$$

From these two equations we get: $Im(\sigma) = 0 \to \sigma \in \mathbb{R}$ in particular $\sigma = 0$ and $u = -1$.
The complete character table takes the following form:

A_4	$1C_e$	$3C_a$	$4C_b$	$4C_{b^2}$
$\mathbf{1}$	1	1	1	1
$\mathbf{1'}$	1	1	$e^{\frac{2\pi i}{3}}$	$e^{\frac{4\pi i}{3}}$
$\mathbf{1''}$	1	1	$e^{\frac{4\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$
$\mathbf{3}$	3	-1	0	0

At this point, we recognize the 3D irrep seen in example 2.1.1. We can check the characters and indeed they are correct. On the other hand this table shows us the existence of other two 1D irreps: $\mathbf{1}$ and $\mathbf{1'}$ are complex irreps. This happens because as we have seen in the first chapter in example 1.1.3 the abelianization of A_4 is: $A_4/V_4 \simeq \mathcal{Z}_3$ and due to theorem 2.1.2, A_4 must have three 1D irreps.

2.3 Kronecker products and C-G coefficients

The last topic we will face is decomposition of irreducible representations, known as Kronecker products. To do so we need to introduce the Kronecker product.

Def. 2.3.1. *Let G be a group and $\rho_\alpha(g)$, $\rho_\beta(g)$ two of its irreducible representations defined on $\mathbb{C}^\alpha$ and $\mathbb{C}^\beta$. The **Kronecker product** of two representations is defined as [3]:*

$$\rho_\alpha(g) \otimes \rho_\beta(g) = \sum_\gamma d(\alpha, \beta|\gamma)\rho_\gamma(g)$$

where $\rho_\gamma(g) \in \mathbb{C}^\gamma$ the coefficients $d(\alpha, \beta|\gamma)$ are given from orthogonality relations as:

$$d(\alpha, \beta|\gamma) = \frac{1}{\#G} \sum_{n_a} n_a \overline{\chi^{[\alpha]}\chi^{[\beta]}}\chi^{[\gamma]}$$

with n_a being the number of elements in the a conjugacy classes and $\#G$ the cardinality of the group.

It is important to observe that this product is commutative. In general, a product of irreps is not an irrep but most likely a reducible representation, which is decomposed in a direct sum of irreps. Let's see some examples of how to decompose a product of representations:

Example 2.3.1. Kronecker product of S_3 irreps:
In S_3 we have just $\mathbf{1}$, $\mathbf{1'}$, $\mathbf{2}$, it is easy to show that $\mathbf{1} \otimes \mathbf{R} = \mathbf{R}$ where $\mathbf{R}$ is any non trivial irrep. Non trivial products are:

- $\mathbf{1'} \otimes \mathbf{1'}$
$$d(\mathbf{1'}, \mathbf{1'}|\mathbf{1}) = 1,\ d(\mathbf{1'}, \mathbf{1'}|\mathbf{1'}) = 0,\ d(\mathbf{1'}, \mathbf{1'}|\mathbf{2}) = 0$$
- $\mathbf{2} \otimes \mathbf{1'}$
$$d(\mathbf{2}, \mathbf{1'}|\mathbf{1}) = 0,\ d(\mathbf{2}, \mathbf{1'}|\mathbf{1'}) = 0,\ d(\mathbf{2}, \mathbf{1'}|\mathbf{2}) = 1$$
- $\mathbf{2} \otimes \mathbf{2}$
$$d(\mathbf{2}, \mathbf{2}|\mathbf{1}) = 1,\ d(\mathbf{2}, \mathbf{2}|\mathbf{1'}) = 1,\ d(\mathbf{2}, \mathbf{2}|\mathbf{2}) = 1$$

In summary, we can decompose the three products as:

$$\mathbf{1'} \otimes \mathbf{1'} = \mathbf{1}, \quad \mathbf{2} \otimes \mathbf{1'} = \mathbf{2}, \quad \mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{1'} \oplus \mathbf{2}$$

For the last one, we say that $\mathbf{2} \otimes \mathbf{2}$ representation splits up in two singlets and a doublet.
Kronecker products of A_4 irreps:
Given the four irreps $\mathbf{1}$, $\mathbf{1'}$, $\mathbf{1''}$, $\mathbf{3}$, the non trivial possibilities of products are:

- $\mathbf{1'} \otimes \mathbf{1'}$:
$$d(\mathbf{1'}, \mathbf{1'}|\mathbf{1}) = 0, \quad d(\mathbf{1'}, \mathbf{1'}|\mathbf{1'}) = 0, \quad d(\mathbf{1'}, \mathbf{1'}|\mathbf{1''}) = 1, \quad d(\mathbf{1'}, \mathbf{1'}|\mathbf{3}) = 0$$
- $\mathbf{1''} \otimes \mathbf{1''}$:
$$d(\mathbf{1''}, \mathbf{1''}|\mathbf{1}) = 0, \quad d(\mathbf{1''}, \mathbf{1''}|\mathbf{1'}) = 1, \quad d(\mathbf{1''}, \mathbf{1''}|\mathbf{1''}) = 0, \quad d(\mathbf{1''}, \mathbf{1''}|\mathbf{3}) = 0$$
- $\mathbf{1'} \otimes \mathbf{1''}$:
$$d(\mathbf{1'}, \mathbf{1''}|\mathbf{1}) = 1, \quad d(\mathbf{1'}, \mathbf{1''}|\mathbf{1'}) = 0, \quad d(\mathbf{1'}, \mathbf{1''}|\mathbf{1''}) = 0, \quad d(\mathbf{1'}, \mathbf{1''}|\mathbf{3}) = 0$$
- $\mathbf{3} \otimes \mathbf{1'}$:
$$d(\mathbf{3}, \mathbf{1'}|\mathbf{1}) = 0, \quad d(\mathbf{3}, \mathbf{1'}|\mathbf{1'}) = 0, \quad d(\mathbf{3}, \mathbf{1'}|\mathbf{1''}) = 0, \quad d(\mathbf{3}, \mathbf{1'}|\mathbf{3}) = 1$$
- $\mathbf{3} \otimes \mathbf{1''}$:
$$d(\mathbf{3}, \mathbf{1''}|\mathbf{1}) = 0, \quad d(\mathbf{3}, \mathbf{1''}|\mathbf{1'}) = 0, \quad d(\mathbf{3}, \mathbf{1''}|\mathbf{1''}) = 0, \quad d(\mathbf{3}, \mathbf{1''}|\mathbf{3}) = 1$$

- $\mathbf{3} \otimes \mathbf{3}$:

$$d(\mathbf{3},\mathbf{3}|\mathbf{1}) = 1, \quad d(\mathbf{3},\mathbf{3}|\mathbf{1}') = 1, \quad d(\mathbf{3},\mathbf{3}|\mathbf{1}'') = 1, \quad d(\mathbf{3},\mathbf{3}|\mathbf{3}) = 2$$

Thus, the decompositions of each Kronecker product in A_4 reads:

$$\mathbf{1}' \otimes \mathbf{1}' = \mathbf{1}'', \quad \mathbf{1}'' \otimes \mathbf{1}'' = \mathbf{1}', \quad \mathbf{1}' \otimes \mathbf{1}'' = \mathbf{1}, \quad \mathbf{3} \otimes \mathbf{1}' = \mathbf{3} \otimes \mathbf{1}'' = \mathbf{3}, \quad \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}'' \oplus \mathbf{3} \oplus \mathbf{3}$$

In this case, we say that $\mathbf{3} \otimes \mathbf{3}$ splits up in three singlets and two triplets.

Before leaving the discussion about representations of finite groups, we would like to point out that the terminology of "singlets", "doublets", "triplets" and so on, is not casual at all: It's borrowed from physics.

From a physical prospective the dimension of an irreducible representation corresponds to the number of independent states it can contain. In the next chapter we will refer to irreps as "multiplets".

Kronecker products must be independent of the bases chosen for the irreps. By fixing the bases it is possible to derive the Clebsh-Gordan coefficients that are basis dependent. Denoting the bases of two multiplets as $(|\alpha_1\rangle\ |\alpha_2\rangle\ ...\ |\alpha_n\rangle)$ and $(|\beta_1\rangle\ |\beta_2\rangle\ ...\ |\beta_n\rangle)$, the resulting representation with components $(|\gamma_1\rangle\ |\gamma_2\rangle\ ...\ |\gamma_n\rangle)$ is obtained from:

$$|\gamma_k\rangle = \sum_{i,j} C_{ij}^k\, |\alpha_i\rangle \otimes |\beta_j\rangle \tag{2.36}$$

Example 2.3.2. Clebsh-Gordan coefficients for Kronecker products of S_3 irreps [4]:

- $\mathbf{1}' \otimes \mathbf{1}' = \mathbf{1}$:
  Let $|\alpha\rangle$, $|\beta\rangle$ be two singlets:
  Let's see how the two generators $\rho_{\mathbf{1}'}(a) = 1,\ \rho_{\mathbf{1}'}(b) = -1$ act on these bases:

$$\rho_{\mathbf{1}'}(a)\,|\alpha\rangle = |\alpha\rangle\,, \quad \rho_{\mathbf{1}'}(a)\,|\beta\rangle = |\beta\rangle \tag{2.37}$$

$$\rho_{\mathbf{1}'}(b)\,|\alpha\rangle = -\,|\alpha\rangle\,, \quad \rho_{\mathbf{1}'}(b)\,|\beta\rangle = -\,|\beta\rangle \tag{2.38}$$

$$\rho_{\mathbf{1}'}(a)\,|\alpha\rangle\,\rho_{\mathbf{1}'}(a)\,|\beta\rangle = |\alpha\rangle\,|\beta\rangle\,, \quad \rho_{\mathbf{1}'}(b)\,|\alpha\rangle\,\rho_{\mathbf{1}'}(b)\,|\beta\rangle = (-1)\,|\alpha\rangle\,(-1)\,|\beta\rangle = |\alpha\rangle\,|\beta\rangle \tag{2.39}$$

  Thus, a basis for $\mathbf{1}$ is $|\alpha\rangle\,|\beta\rangle$ For the rest of the Kronecker products we just provide the results:

- $\mathbf{2} \otimes \mathbf{1}'$:

$$|\alpha\rangle \begin{pmatrix} -\,|\beta_2\rangle \\ |\beta_1\rangle \end{pmatrix} \tag{2.40}$$

- $\mathbf{2} \otimes \mathbf{2} = \mathbf{1} + \mathbf{1}' + \mathbf{2}$:
  $\mathbf{1}$:

$$|\alpha_1\rangle\,|\beta_1\rangle + |\alpha_2\rangle\,|\beta_2\rangle\,, \tag{2.41}$$

  $\mathbf{1}'$:

$$|\alpha_1\rangle\,|\beta_2\rangle - |\alpha_2\rangle\,|\beta_1\rangle\,, \tag{2.42}$$

  $\mathbf{2}$:

$$\begin{pmatrix} |\alpha_2\rangle\,|\beta_2\rangle - |\alpha_1\rangle\,|\beta_1\rangle \\ |\alpha_1\rangle\,|\beta_2\rangle + |\alpha_2\rangle\,|\beta_1\rangle \end{pmatrix} \tag{2.43}$$

Chapter 3

Applications

In this chapter we will apply what we learned about finite groups to a physical model.

We will briefly introduce neutrino oscillations and mixing from a theoretical perspective. Therefore, once the problem has been framed we will consider the exaple of tri-bimaximal neutrino mixing.

3.1 Neutrino oscillations

The hypothesis of neutrino was first proposed by Wolfgang Pauli in 1930 to explain the continuum spectrum in beta decays. Since then, we now know that there exist three generations of flavors of leptons organized in three $SU(2)_L$ doublets [5]:

$$\begin{pmatrix} e^- \\ \nu_e \end{pmatrix}, \begin{pmatrix} \mu^- \\ \nu_\mu \end{pmatrix}, \begin{pmatrix} \tau^- \\ \nu_\tau \end{pmatrix} \tag{3.1}$$

The standard model predicts that neutrinos are massless, but in 1998 Super Kamiokande experiment demonstated that this prediction is incorrect.

The indirect evidence that neutrinos have mass, comes from the phenomenon of neutrino oscillations. When a neutrino of a given flavor is produced, for example an electron neutrino, it is actually a superposition of mass eigenstates. Each mass component evolves in time with a different phase, so after traveling a sufficient distance there is a non-zero probability of detecting the neutrino as a different flavor for example muon neutrino [5].

The matrix responsible of the relation between flavor states and mass eigenstates is called U_{PMNS}, after Pontecorvo, Maki, Nakagawa and Sakata.

U_{PMNS} is a unitary matrix, frequently parametrized by a product of three rotation matrices. These matrices respectively contain: atmospheric, reactor and solar mixing angles [5]:

$$U_{PMNS} = U_{atm} U_{reac} U_{sol} = \tag{3.2}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_{23}) & \sin(\theta_{23}) \\ 0 & -\sin(\theta_{23}) & \cos(\theta_{23}) \end{pmatrix} \begin{pmatrix} \cos(\theta_{13}) & 0 & \sin(\theta_{13})e^{-i\delta} \\ 0 & 1 & 0 \\ -\sin(\theta_{13})e^{i\delta} & 0 & \cos(\theta_{13}) \end{pmatrix} \begin{pmatrix} \cos(\theta_{12}) & \sin(\theta_{12}) & 0 \\ -\sin(\theta_{12}) & \cos(\theta_{12}) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \tag{3.3}$$

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \tag{3.4}$$

where c, s stand for cosine and sine. θ_{12}, θ_{23}, θ_{13} are called mixing angles and δ is the CP-violating phase. As we said, using this matrix we can relate flavor states to mass eigenstates:

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \\ |\nu_\tau\rangle \end{pmatrix} = U_{PMNS} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \\ |\nu_3\rangle \end{pmatrix} \tag{3.5}$$

Now, we may consider a simple case of two neutrino oscillations to prove that this phenomenon implies they are having masses.

Example 3.1.1. Two neutrino oscillations: $|\nu_e\rangle$, $|\nu_\mu\rangle$. Since the oscillation happens between only two flavors, U_{PMNS} is reduced to [5]:

$$U_{PMNS} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \tag{3.6}$$

Consider a $|\nu_e\rangle$ produced at $t = 0$ then $|\nu(0)\rangle = |\nu_e\rangle$. Evolving the state in time, for a given t we have:

$$|\nu_e(t)\rangle = e^{\frac{-iHt}{\hbar}} |\nu(0)\rangle = \cos(\theta) e^{\frac{-iE_1 t}{\hbar}} |\nu_1\rangle + \sin(\theta) e^{\frac{-iE_2 t}{\hbar}} |\nu_2\rangle \tag{3.7}$$

where $H |\nu_k\rangle = E_k |\nu_k\rangle$, $k = 1, 2$, neutrino are ultra-relativistic:

$$E_k = \sqrt{p^2c^2 + m_k^2 c^4} \approx pc(1 + \frac{m_k^2 c^4}{2p^2c^2}) \tag{3.8}$$

The probability to find $|\nu_e\rangle$ after an interval of time t is:

$$P_{\nu_e \to \nu_e} = |\langle \nu_e(t) | \nu_e \rangle|^2 = \cos^4(\theta) + \sin^4(\theta) + 2\cos^2(\theta)\sin^2(\theta)\cos(\frac{1}{2}\frac{\Delta m_{12}^2 c^4}{\hbar c}\frac{L}{pc}) = \tag{3.9}$$

$$= 1 - \sin^2(2\theta)\sin^2(\frac{1}{4}\frac{\Delta m_{12}^2 c^4}{\hbar c}\frac{L}{pc}) \tag{3.10}$$

while the probability to observe $|\nu_\mu\rangle$ is simply:

$$P_{\nu_e \to \nu_\mu} = 1 - P_{\nu_e \to \nu_e} = \sin^2(2\theta) sin^2\left(\frac{1}{4}\frac{\Delta m_{12}^2 c^4}{\hbar c}\frac{L}{pc}\right) \tag{3.11}$$

where we set $m_{12}^2 = m_{\nu_1}^2 - m_{\nu_2}^2$, $L = ct$. We can see that the probability of transition depends on the difference of the two neutrinos masses, if neutrinos were massless, we would not observe this phenomenon.

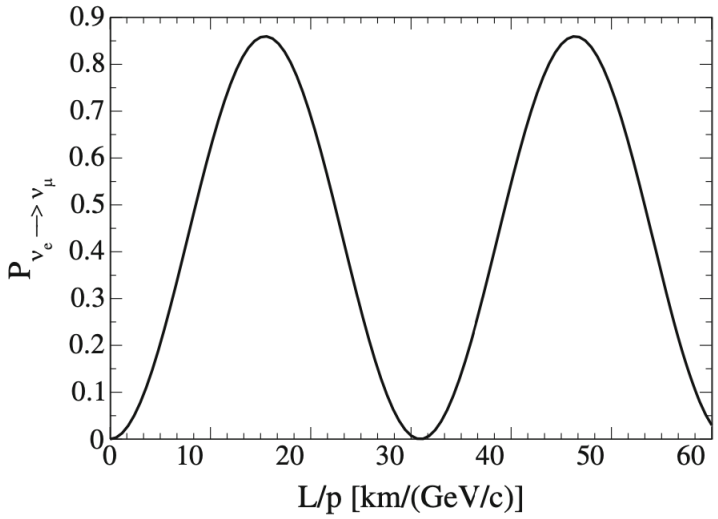

Figure 3.1: Probability of transition from electron neutrino to muon neutrino, from [5]

The graph in fig 3.1 displays the probability of transition from $|\nu_e\rangle \to |\nu_\mu\rangle$ where $\theta = 34°$, $\Delta m_{12}^2 = 8 \cdot 10^{-5} eV/c^4$ then the probability depends only on the distance L over momentum p. We can see that when $\frac{L}{p} \approx 15 Km/(GeV/c)$ the probability of transition is 85%.

3.2 Tri-bimaximal mixing

The tri-bimaximal (TBM) mixing is a proposed form of the U_{PMNS} matrix, which assumes large θ_{12}, maximal θ_{23} and zero θ_{13} [6].

First, let's clarify the name. It is called tri-bimaximal because $|\nu_2\rangle$ is maximal mixed on the three flavors, while $|\nu_1\rangle$, $|\nu_3\rangle$ are respectively maximal mixed on two flavors. To be clear, maximal mixed means that the eigenstate of mass is equiprobable on two (bi) or three (tri) flavors.

Initially, this approximation provided a good fit to neutrino oscillation data of CHOOZ and KamLAND experiments [4]. However, in 2012, experiments demonstrated that $\theta_{13} \neq 0°$ [5]. Despite this, the TBM form remains of theoretical and historical interest, as applying finite group symmetries to this framework continues to yield accurate predictions.

In the standard model extended with massive neutrinos, U_{PMNS} can be written as:

$$U_{PMNS} = U_l^\dagger U_\nu \tag{3.12}$$

where U_l, U_ν are the matrices that diagonalize $M_l^\dagger M_l$ and $M_\nu^\dagger M_\nu$. The two matrices M_l, M_ν tell us the masses of the leptons and how their flavor states are related to mass eigenstates. We will not go further in this discussion, since it is beyond the scope of this thesis.

As first step, let's observe that the generators of the 3D irrep of A_4 in Eq. (2.17) are related by a similarity matrix to the following generators better know as Altarelli-Feruglio basis [7], this is a canonical choice in flavor symmetries.

$$\sigma(a) = W\rho(a)W^\dagger = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad \sigma(b) = W\rho(b)W^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \tag{3.13}$$

We assume that the neutrino mass matrix is invariant under the action of $\mathcal{Z}_2^a$ symmetry, implying:

$$\sigma(a)^\dagger M_\nu M_\nu^\dagger \sigma(a) = M_\nu M_\nu^\dagger \implies [M_\nu M_\nu^\dagger, \sigma(a)] = 0 \tag{3.14}$$

When two matrices commute, they can be diagonalized simultaneously by the same matrix, since:

$$U_\nu^\dagger M_\nu M_\nu^\dagger U_\nu = diag(M_\nu M_\nu^\dagger) = diag(m_1^2, m_2^2, m_3^2) \tag{3.15}$$

we are lead to:

$$U_\nu^\dagger \sigma(a) U_\nu = \sigma_{diag}(a) \tag{3.16}$$

Thus, to find the matrix U_ν, we need to diagonalize the generator $\sigma(a)$:

$$\sigma(a) - \lambda\mathbb{I} = \frac{1}{3}\begin{pmatrix} -1-\lambda & 2 & 2 \\ 2 & -1-\lambda & 2 \\ 2 & 2 & -1-\lambda \end{pmatrix} \to p(\lambda) = (\lambda-1)(\lambda+1)^2 \tag{3.17}$$

Therefore $\sigma(a)$ diagonalized result:

$$\sigma_{diag(a)} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{3.18}$$

By plugging each eigenvalue in Eq.(3.8) we find the following constrains for the eigenstates:

$$\lambda = 1 : x = y = z \implies v_1 = \frac{1}{\sqrt{3}}(1,1,1) \tag{3.19}$$

$$\lambda = -1 : x + y + z = 0, \text{ with: } x = 2, 0\, y = -1, 1 \implies z = -1, -1 \tag{3.20}$$

$$\implies v_2 = \frac{1}{\sqrt{6}}(2,-1,-1),\ v_3 = \frac{1}{\sqrt{2}}(0,1,-1) \tag{3.21}$$

We can apply then the same steps to find U_l but it is not necessary because $\sigma(b)$ is already diagonal. This implies that $U_l = \mathbb{I}$, therefore U_{PMNS} reduces to:

$$U_{PMNS} = U_\nu = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix} R_{13}(\theta) \tag{3.22}$$

where θ is a free angle parameter. In this form, we rename $U_{PMNS} = U_{TBM}R(\theta)$. For $\theta = 0°$, Eq. (3.2) becomes:

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \\ |\nu_\tau\rangle \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \\ |\nu_3\rangle \end{pmatrix} \tag{3.23}$$

By comparing the tri-bimaximal mixing matrix U_{TBM} with the parametrization of U_{PMNS} given in Eq.(3.3), we can derive the values of the mixing angles. from element (1,3):

$$\sin(\theta_{13}) = 0 \implies \theta_{13} = \sin^{-1}(0) = 0° \tag{3.24}$$

from element (1,1):

$$\cos(\theta_{12}) \cdot 1 = \sqrt{\frac{2}{3}} \implies \theta_{12=} \cos^{-1}(\sqrt{\frac{2}{3}}) = 35,2° \tag{3.25}$$

from element (2,3):

$$\sin(\theta_{23}) \cdot 1 = \frac{1}{\sqrt{2}} \implies \theta_{23=} \sin^{-1}(\frac{1}{\sqrt{2}}) = 45° \tag{3.26}$$

This example shows how discrete flavor symmetries and, in particular, residual symmetries of mass matrices, can shape the form of U_{PMNS}, predicting specific values of the neutrino mixing angles.

We can immediately see that the theoretical results of Eq.(3.25),(3.26),(3.27) match experimental data in first approximation, except for θ_{13}.

For completeness, we report the results of the global analysis of neutrino oscillation data performed by the NuFit collaboration [8] in the following table:

	θ_{12}°	θ_{23}°	θ_{13}°
TBM	35	45	0
Nu-FIT 6.0 (2024)	$33.68^{+0.73}_{-0.70}$	$48.5^{+0.7}_{-0.9}$	$8.52^{+0.11}_{-0.1}$

We conclude that discrete A_4 symmetry provides an elegant and predictive model of neutrino mixing. However, in its minimal version it is unable to explain why $\theta_{13} \neq 0°$. Finally we note that non zero value of θ_{13}, opens the door to leptonic CP violation, parametrized by the δ phase.

Conclusions

In the first chapter of the thesis, we focused on the properties of finite groups, examining their definition, composition laws, and their realization through permutations. As examples, we derived the multiplication table of the symmetric group S_3. We have seen the subgroups of A_4 and S_3, the quotient groups generated from A_4 and S_3. Furthermore we explored their geometrical action, respectively as symmetries of a regular tetrahedron and of an equilateral triangle.

In the second chapter, we introduced the theory of group representations, with focus on irreducible representations, character tables, and Kronecker products. We exemplified these concepts using the groups A_4 and S_3. In particular, we constructed the character tables of both groups, and we showed how to derive all of their irreducible representations. We then discussed Kronecker products, providing a definition and showing how to decompose them into irreducible components. Finally, we derived the Clebsch–Gordan coefficients and observed that while the multiplicities of irreducible representations are basis independent, the Clebsch–Gordan coefficients depend on the chosen basis.

In the final chapter of the thesis, we introduced the historical hypothesis behind the neutrino and discussed by deriving the probability of transition of two neutrinos, how the experimentally observed phenomenon of neutrino oscillations provides direct evidence that neutrinos have mass. Then, we discussed U_{PMNS} matrix, its parametrization and its form in the Standard Model extended with massive neutrinos. By assuming that the neutrino mass matrix is invariant under a subgroup of A_4, we derived a specific form of U_{PMNS} known as the tri-bimaximal mixing matrix U_{TBM}. From this matrix, one can extract mixing angles that are consistent with earlier experimental data. However, it fails to explain more recent experimental data that yielded a $\theta_{13} \neq 0°$. This measurement opened up a path to establishing the status of CP violation in the lepton sector.

Bibliography

[1] Gianguido Dall'Agata. *Group theory and physics*. Padova, 2017. pp.23-80.

[2] Pierre Ramond. *Group theory A physicist's survey*. Cambridge university press, New York, 2010. pp.4-48.

[3] C. Luhn S. F. King. Neutrino mass and mixing with discrete symmetry. `https://arxiv.org/abs/1301.1340v3`, 2016. pp.14-25, 36-37.

[4] H. Ohki H. Ishimori, T. Kobayashi2, H. Okada, Y. Shimizu, and M. Tanimoto. Non-abelian discrete symmetries in particle physics. `https://arxiv.org/abs/1003.3552v2`, 2010. pp.7-11, 125-126.

[5] C. Scholz B. Povh, K. Rith, F. Zetsche, and W. Rodejohann. *Particles and nuclei*. Springer Berlin, Heidelberg, 2015. pp.167-183.

[6] P. F. Harrison, D. H. Perkins, and W. G. Scott. Tri-bimaximal mixing and the neutrino oscillation data. *Phys. Lett. B*, 530:167, 2002.

[7] Guido Altarelli and Ferruccio Feruglio. Tri-bimaximal neutrino mixing, A(4) and the modular symmetry. *Nucl. Phys. B*, 741:215–235, 2006.

[8] NuFIT Collaboration. Nufit 6.0 (2024). `https://www.nu-fit.org/`.