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Non-linear cosmic structure formation in generalized theories of gravity

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Abstract

The cosmological standard model is built upon the theory of General Relativity and the two symmetry assumptions that the Universe appears on average as homogeneous and isotropic to freely falling observers. This model is remarkably successful, but some of its open questions suggest that General Relativity might be replaced by a more general theory. The success of the standard model, however, should imply that such a theory should have cosmological consequences not far from those of General Relativity.

With the mean-field approach to the gravitational interaction between dark-matter particles in the kinetic field theory (KFT) of cosmic structure formation, it has become possible to study the statistical properties of non-linear cosmic structures analytically. Specifically, the non-linear density-fluctuaton power spectrum predicted by KFT depends on the cosmological background model only via the cosmic expansion function, and via a possible time dependence of the gravitational coupling. The analytic form of the KFT approach allows taking functional derivatives of the non-linear power spectrum with respect to the background functions, and thus to functionally Taylor-expand the non-linear density-fluctuation power spectrum around the expectation from General Relativity. In the proposed Master's thesis, this approach will be taken to study the generic, first-order response of the non-linear power spectrum not to specific, but to all generalisations of General Relativity whose cosmological models deviate little from the standard model.

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Introduction

This master thesis work will be organised as follows. The first three chapters make up an introduction into the topic, which is aimed to provide the mathematical and conceptual fundamentals for the later discussion and to motivate the importance to carry out a research in this field.
In particular, the first chapter will be a standard introduction to cosmic structure formation and to the standard model of cosmology. The second chapter, on the other hand, will provide an overview of possible ways to overcome such a model. As we will see, the fundamental ingredient of a cosmological model is the underlying theory of gravity and for this reason, several alternatives to General Relativity have been developed in recent research works. These theories need of course to be tested and the aim of this thesis is to present a new method for this purpose.
The mathematical framework in which this method will be developed is the Kinetic Field Theory (KFT), a statistical field theory for deterministic particle ensembles, which can successfully be applied to cosmic structure formation. The main concepts of this formalism will be illustrated in chapter three. We will see that a very important result of KFT is the possibility of computing the non-linear evolution of the matter power spectrum analytically.
Chapter four and five will include the topics on which I actually worked. The purpose is to develop a general method for including in the power spectrum the alternative gravity theories and this is covered in the fourth chapter. The fundamental hypothesis is that possible observable deviations of an alternative theory of gravity from General Relativity must be small (as suggested by observations). This allows a first order Taylor expansion of the non-linear power spectrum around the GR value of two functions: the cosmic expansion function and the gravitational coupling.
Finally, in the fifth chapter some results from specific examples of alternative gravity are presented and discussed.

Chapter 1

Cosmic structure formation

1.1 Our universe

Modern cosmology is founded on the so called *Cosmological Principle*, which states that, on large enough scales, our universe is spatially homogeneous and isotropic.
The claim of isotropy stems from astronomical observations of large scale structures and, primarily, of the Cosmic Microwave Background (CMB).
Isotropy, combined with the copernican principle, which expects no point of space to be special or preferred, suggest then homogeneity.
Indeed, modern observations hint that the scale at which the universe starts to look homogeneous and isotropic is approximately $\sim 100\,Mpc$.
As it is clear from our own experience, as we follow a single region of universe to smaller scales we observe matter organizing in complex structures, from the cosmic web, down to galaxy clusters, galaxies, stars.

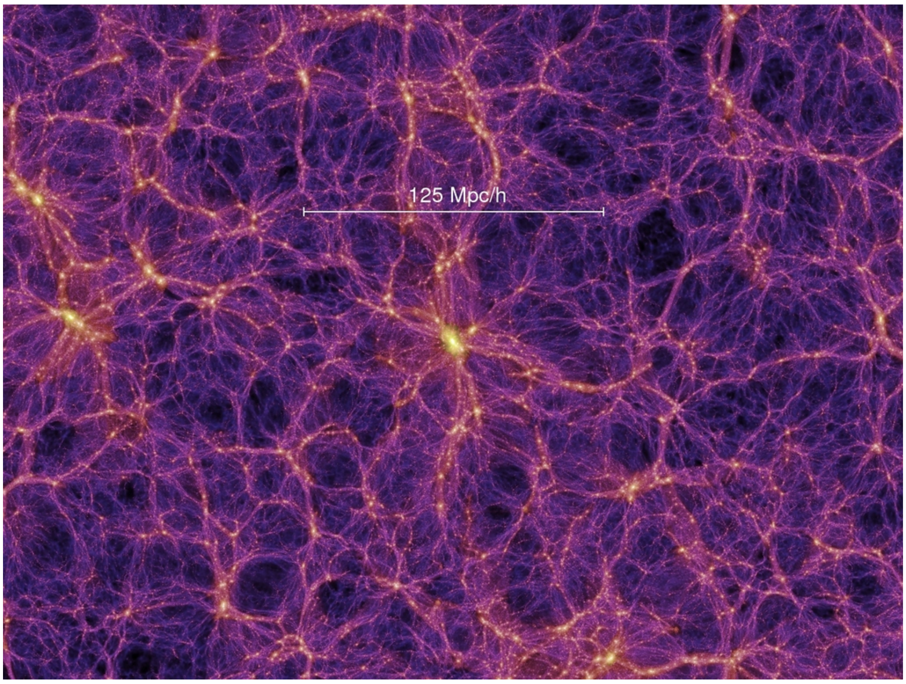

Figure 1.1: The cosmic web. Credits: Millenium simulation project

Hence, the cosmological principle can be exploited to describe the universe as a whole. Within the framework of General Relativity, homogeneity and isotropy allow an elegant description of spacetime structure, through the so called Friedman-Lemaitre-Robertson-Walker metric [28]:

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu = -c^2dt^2 + R^2(t)\gamma_{ij}(x^i)dx^i dx^j \tag{1.1}$$

with the coordinate set $\{t, \chi, \theta, \phi\}$.
The spatial part reads:

$$\gamma_{ij}(x^i)dx^i dx^j = \frac{d\chi^2}{1 - K\chi^2} + \chi^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.2}$$

The first coordinate is the cosmic time t, the function $R(t)$ the cosmic world radius, while χ, θ and ϕ are the spherical comoving spatial coordinates.
With a suitable normalization, the curvature parameter K can be always reduced to one of the values $(-1, 0, +1)$. The case $K = -1$ and $K = +1$ represent respectively a negatively curved (hyperbolic) spacetime and a positively curved (spherical) one. The case $K = 0$, on the other hand, represents a flat spacetime.
The FLRW metric can be rearranged in the following way, collecting an overall adimensional scale factor $a(t) = R(t)/R_0$:

$$ds^2 = a^2(t)\big[-d\tau^2 + dr^2 + R_0^2 f_K^2(r/R_0)(d\theta^2 + \sin^2\theta d\phi^2)\big] \tag{1.3}$$

with the conformal time

$$d\tau = \frac{c}{a(t)}dt \tag{1.4}$$

and the curvature function

$$f_K(r/R_0) = \begin{cases} \sin(r/R_0) & \text{if} \quad K = +1 \\ r/R_0 & \text{if} \quad K = 0 \\ \sinh(r/R_0) & \text{if} \quad K = -1 \end{cases} \tag{1.5}$$

As we implicitly assumed so far, and as it is evident from observations, the universe is not static, but it actually expands. The equations which govern the evolution of the universe, can be coherently derived in the framework of General Relativity, namely from the Einstein equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}. \tag{1.6}$$

We defined the Ricci tensor $R_{\mu\nu}$ as the contraction of the Riemann tensor $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$, which reads

$$R^\rho_{\mu\sigma\nu} = \partial_\sigma\Gamma^\rho_{\mu\nu} - \partial_\nu\Gamma^\rho_{\mu\rho} + \Gamma^\rho_{\alpha\sigma}\Gamma^\alpha_{\mu\nu} - \Gamma^\rho_{\alpha\nu}\Gamma^\alpha_{\mu\sigma} \tag{1.7}$$

The object Γ is usually called Christoffel symbol or Levi-Civita connection. It is a combination of first derivatives of the metric and cannot be considered a tensor due to its non-covariant transformation rule under a diffeomorphism acting on coordinates. The complete expression is

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\big(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}\big) \tag{1.8}$$

The scalar R is usually called Ricci scalar and can be computed as the trace of the Ricci tensor, namely $R = g^{\mu\nu}R_{\mu\nu}$.
The so called Dark Energy, which produces the expansion of the universe, is effectively included in equation (1.6) through the cosmological constant Λ.
The energy-momentum tensor $T_{\mu\nu}$ on the right-hand side represents the matter-energy content of the universe and we can model it with the ideal fluid form

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right)u_\mu u_\nu + pg_{\mu\nu} \tag{1.9}$$

where u_μ is the fluid four-velocity, while ρ and p are, respectively, energy density and pressure. Furthermore, the energy momentum-tensor must obey the conservation law

$$\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\alpha} T^{\alpha\nu} + \Gamma^\nu_{\mu\alpha} T^{\mu\alpha} = 0 \tag{1.10}$$

where ∇_μ is the covariant derivative.
Equation (1.6) and (1.10), applied to the FLRW metric, yield respectively, the Friedmann equations

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} + \frac{\Lambda c^2}{3} \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}\left(\rho + 3\frac{p}{c^2}\right) + \frac{\Lambda c^2}{3} \end{aligned} \tag{1.11}$$

and the energy density evolution equation

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) \tag{1.12}$$

Note that equation (1.12) holds for every species individually. Therefore, it is important to specify the different types of matter-energy that populate the universe. In general it is possible to express the equation of state in the simple barotropic form

$$p = wc^2\rho. \tag{1.13}$$

The equation of state parameter w is ~ 0 for non-relativistic matter (usually addressed in cosmology as dust), while it is $\sim 1/3$ for extremely relativistic particles like photons. Note that also the cosmological constant produces energy density and negative pressure, with an equation of state given by $w = -1$. For a species i with a generic equation of state of the form (1.13), equation (1.12) is solved by

$$\rho_i(t) \propto a^{-3(1+w_i)}, \tag{1.14}$$

while the Friedman equation gives

$$a(t) \propto t^{\frac{2}{3(1+w)}} \tag{1.15}$$

and the Hubble parameter

$$H(t) = \frac{\dot{a}}{a} = \frac{2}{3(1+w)t}. \tag{1.16}$$

Considering a flat universe in which only Dark Energy is present, one obtains the constant $H \propto \sqrt{\Lambda}$, and the exponential expansion

$$a(t) \propto e^{Ht}. \tag{1.17}$$

This case is usually addressed as De Sitter model and it is very important in the study of inflation.
In order to characterise cosmological models, it is convenient to define for the species i the density parameter

$$\Omega_i = \frac{\rho_i}{\rho_c} \tag{1.18}$$

with $\rho_c = 3H^2/8\pi G$ critical density.
In terms of density parameters, the Friedman equation can be rewritten as

$$\sum_i \Omega_i - \frac{Kc^2}{a^2} = \sum_i \Omega_i + \Omega_K = 1 \tag{1.19}$$

with the definition of the curvature parameter Ω_K, which behaves like an energy density contribution with equation of state parameter $w = -1/3$.
It is worth to stress that the sign of the curvature parameter cannot change during the history of the universe, hence, the corresponding parameter will remain as well positive, negative or null, according with the initial conditions.

The standard model of cosmology is usually called ΛCDM since it includes the cosmological constant Λ as source of Dark Energy and the so called Cold Dark Matter. This latter constituent is represented by an unknown form of non-baryonic matter which does not feel the electromagnetic interaction and which is non-relativistic (cold) during the most of universe thermal history.
During the evolution of the universe different epochs can be distinguished in which the contribution from a certain species dominates among all, as each one evolves with a different scaling according with its equation of state.
The density parameters corresponding to different constituents today in the ΛCDM model can be measured using data from CMB, supernovae Ia and baryonic acoustic oscillations (BAO).
The values are approximatively (see [2]):

$$\begin{aligned} &\text{Non-relat. matter:} && \Omega_{m,0} = \Omega_{\text{baryon},0} + \Omega_{dm,0} \approx 0.3 \\ &\text{Dark energy:} && \Omega_{\Lambda,0} \approx 0.7 \\ &\text{Relativistic species:} && \Omega_{r,0} \approx 10^{-5} \end{aligned} \tag{1.20}$$

while the curvature parameter has been constrained to be $|\Omega_K| < 10^{-2}$. It is clear that today we observe a universe which is strongly Dark Energy-dominated.

1.2 Newtonian perturbation theory

As already mentioned, on scales $< 100\,Mpc$ the matter in the universe appears finely organized in gravitational structures. However, we know that in the early stages, our universe was homogeneous even on small scales, up to very tiny fluctuations. The structures that we observe today, must be, therefore, the effect of a process of evolution led by gravity, of these primordial perturbations. This evolution initially proceeds in a linear regime, in which perturbations can be described by small variations of some physical quantities on top of a stationary background.
In such regime, newtonian hydrodynamics provides a pretty good description, even though a more rigorous treatment would of course require General Relativity and the employment of Boltzmann equation.
This regime breaks down when perturbations start to grow non-linearly and effects like shell-crossing start to be effective.
Effects of relativistic free streaming of particles, which can also play a role in structure formation, are not included in this classical treatment.
The system of equations is constituted by continuity, Euler and Poisson equation:

$$\begin{aligned} &\partial_t \rho + \nabla_r \cdot (\rho \vec{v}) = 0 \\ &\partial_t \vec{v} + (\vec{v} \cdot \nabla_r)\vec{v} + \frac{\nabla_r p}{\rho} + \nabla_r \Phi = 0 \\ &\nabla_r^2 \Phi = 4\pi G \rho \end{aligned} \tag{1.21}$$

Let's now introduce the perturbations:

$$\rho = \rho_0 + \delta\rho\,, \quad p = p_0 + \delta p\,, \quad \vec{v} = \vec{v}_0 + \delta\vec{v}\,, \quad \Phi = \Phi_0 + \phi \tag{1.22}$$

Observe that, assuming adiabaticity, $\delta p = c_S^2 \delta\rho$, where c_S is the sound speed, which depends on the equation of state.
Let's now pass to the comoving coordinate system:

$$\vec{r}(t) = a(t)\vec{x}\,, \quad \nabla_r = \frac{1}{a(t)}\nabla_x\,, \quad \partial_t|_r = \partial_t|_x - \frac{1}{a(t)}\vec{v_0} \cdot \nabla_x \tag{1.23}$$

Furthermore, it is convenient to pass to Fourier space. Consider as an example the density perturbation expansion in Fourier modes:

$$\delta\rho(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{x}\cdot\vec{k}} \delta\rho(\vec{k}, t) \tag{1.24}$$

We will indicate the Fourier transform with the same notation, just as a function of wave vector instead of coordinates.
The other relevant quantities can be expanded analogously.
The system now reads:

$$\begin{aligned} &\partial_t \delta\rho + 3H\delta\rho + \frac{i\rho_0}{a}\vec{k}\cdot\delta\vec{v} = 0 \\ &\partial_t \delta\vec{v} + H\delta\vec{v} + \frac{i\vec{k}}{a\rho_0}c_S^2\delta\rho + \frac{i\vec{k}}{a}\phi = 0 \\ &k^2\phi + 4\pi G a^2 \delta\rho = 0 \end{aligned} \tag{1.25}$$

Consider first the so called vortical modes, for which $\vec{k}\cdot\delta\vec{v} = 0$.
Assuming $\delta\rho = \phi = 0$, Euler equation reduces to

$$\partial_t \delta\vec{v} + H\delta\vec{v} = 0 \tag{1.26}$$

which is solved by $v \propto a^{-1}$. These modes are rapidly suppressed as the universe expands and the cosmic fluid becomes quickly nearly irrotational.
Consider now the adiabatic modes. From now on we will use the notation $\bar{\rho} = \rho_0$. It is convenient to define the so called density contrast

$$\delta(\vec{k}, t) = \frac{\rho(\vec{k}, t) - \bar{\rho}(t)}{\bar{\rho}(t)} \tag{1.27}$$

The continuity equation can be rewritten as

$$\partial_t \delta + \frac{i}{a}\vec{k}\cdot\delta\vec{v} = 0 \tag{1.28}$$

Differentiating it with respect to time and substituting $\partial_t \delta v$ from Euler and δv from (1.28) we obtain

$$\ddot{\delta} + 2H\dot{\delta} + \left(\frac{c_S^2 k^2}{a^2} - 4\pi G\bar{\rho}\right)\delta = 0 \tag{1.29}$$

According to Jeans criterion, only perturbations with larger wavelenght compared to the Jeans lenght can grow. Mathematically:

$$\lambda = \frac{2\pi a}{k} > \lambda_J = c_S\sqrt{\frac{\pi}{G\bar{\rho}}} \tag{1.30}$$

Since the Jeans wavelenght is directly proportional to the speed of sound, the above condition can be fulfilled more easily if the pressure of the fluid is tiny with respect to the energy density.
Considering only modes with wavelenght which are much larger compared to λ_J, we can simplify equation (1.29) to

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\bar{\rho}\delta = 0 \tag{1.31}$$

that is generally solved by:

$$\delta(\vec{k}, t) = \delta_+(\vec{k})D_+(t) + \delta_-(\vec{k})D_-(t). \tag{1.32}$$

The function $D_+(t)$ is called cosmic growth factor, and will be fundamental for our later discussion.
It is important to stress that structure formation is mainly efficient in the matter-dominated epoch, namely in the stage of universe history in which the larger fraction of energy density is represented by the contribution of non-relativistic matter. For this reason, from now on we will assume $\Omega_m = 1$. This model, namely a flat universe in which only matter is present, is addressed as Einstein-de Sitter universe (mind the difference from de Sitter model, in which Dark Energy is the only content).
In such model the cosmic growth factor scales like

$$D_+(t) \propto a(t) \propto t^{2/3}. \tag{1.33}$$

Let's now rearrange a bit equation (1.31), passing from time variable to scale factor:

$$\delta'' + \left(\frac{3}{a} + \frac{E'(a)}{E(a)}\right)\delta' - \frac{3\Omega_m}{2a^2}\delta = 0 \tag{1.34}$$

Note that we indicated with $'$ the derivative with respect to a.
We also introduced the cosmic expansion function $E(a) = H(t)/H_0$.
It can be expressed from the Friedmann equation in terms of the density parameters observed today, with their proper scaling:

$$\begin{aligned} E(a) &= \sqrt{\sum_i \Omega_{i,0} a^{-3(1+w_i)} + \Omega_K a^{-2}} = \\ &= \sqrt{\Omega_{m,0} a^{-3} + \Omega_{r,0} a^{-4} + \Omega_{\Lambda,0} + \Omega_K a^{-2}}. \end{aligned} \tag{1.35}$$

Using the Ansatz from the first term of (1.32), we have, for a given wave number, that δ_+ is constant with time, and therefore with the scale factor. Thus, we can plug the Ansatz inside (1.34) and obtain a second order differential equation for $D_+(a)$:

$$D_+''(a) + \left(\frac{3}{a} + \frac{E'(a)}{E(a)}\right)D_+'(a) - \frac{3\Omega_m}{2a^2}D_+(a) = 0. \tag{1.36}$$

This equation will be fundamental later, when we will deal with functional differentiation of the cosmic growth factor in order to study its variation with respect to some parameters.

1.3 Statistical tools

The density contrast δ can be described as a random field. It is reasonable to assume, that it is actually a gaussian random field with zero mean. However, some deviation from gaussianity could be present in the distribution of δ due to primordial mechanisms like inflation or topological defects and they could have, for instance, some observable effects on CMB and clusters mass function (see [39]). Nevertheless, observations constrain these non-gaussianities to be small enough to be safely neglected for our purposes.
The simplest statistical feature that characterises a random field is the two-point correlation function

$$\xi_\delta(\vec{x}, \vec{x}') = \langle \delta(\vec{x})\delta(\vec{x}')\rangle \tag{1.37}$$

where we indicated with $\langle\cdot\rangle$ the expectation value.
Thanks to homogeneity and isotropy, we claim that ξ only depends on the modulus $r = |\vec{x} - \vec{x}'|$.
The two-point function can be expressed through its spectral decomposition in Fourier space, by

$$\langle \delta(\vec{k})\delta^*(\vec{k}')\rangle = (2\pi)^3 \delta_D(\vec{k} - \vec{k}') P_\delta(\vec{k}) \tag{1.38}$$

where the sign * indicates the complex conujugate.
The above relation defines the density fluctuation power spectrum P_δ which is an extremely efficient tool in theoretical cosmology.
Due to the fact that ξ is real, we have $P_\delta(\vec{k}) = P_\delta^*(-\vec{k})$.
Furthermore, thanks to isotropy the power spectrum only depends on the wave vector modulus, namely $P_\delta(\vec{k}) = P_\delta(k)$.
The two-point correlation function can be expressed in terms of P_δ as

$$\begin{aligned} \xi_\delta(r) &= \frac{1}{(2\pi)^3}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \,\sin\theta \int_0^{\infty} dk\, k^2 P_\delta(k) e^{ikr\cos\theta} \\ &= \frac{1}{2\pi^2}\int_0^{\infty} dk\, k^2 P_\delta(k) j_0(kr) \end{aligned} \tag{1.39}$$

where $j_0(y) = \sin y/y$ is a spherical Bessel function.
Using again the Ansatz from (1.32) it is easy to show that in linear regime, the power spectrum evolves according with

$$P_\delta^{lin}(k,t) = \left(\frac{D_+(t)}{D_+^{(i)}}\right)^2 P_\delta^{(i)}(k). \tag{1.40}$$

This regime breaks down when $\delta \sim 1$, i.e. when the amplitude of density perturbation starts to be comparable to the background values. At that point, the so called non-linear evolution sets in and perturbation theory cannot be employed anymore.
Non-linear structure formation is usually investigated by means of numerical N-body simulations, which yield excellent result, but do not provide an actual theoretical picture of the physics behind them.
However, as we will see in chapter 3, the recently developed formalism of Kinetic Field Theory can describe theoretically the non-linearly evolved density power spectrum, within some approximations.
This quick discussion concludes the first chapter, in which a brief overview of structure formation in cosmology was given. In the next chapters we will widely employ the tools developed so far.

Chapter 2

Gravity theories

2.1 General Relativity and beyond

In the previous chapter we introduced the standard model of cosmology, usually addressed as ΛCDM. The two core pillars of this model are the cosmological principle and General Relativity. Indeed, gravity is the one among the four fundamental interactions which most affects phenomena at cosmological scales and therefore, the theory that we employ to describe it is an important discriminant between different models.
General Relativity is nowadays universally accepted for the description of gravity at *intermediate* scales, as it is remarkably constrained by the most of the experimental tests (even though some small discrepancies have been found also in cosmological observations, see [23]). However, there are two main theoretical puzzles that could conceal some general gaps within the theory.
The first problem is the quantization of gravity, which becomes necessary at Planck scale and above and which is probably the most involved dilemma in theoretical physics. The issue arises from the failure of the canonical quantization procedure for fields, as the underlying theory of gravity is not renormalizable and can only hold effectively up to a certain energy scale.
The second one regards the cosmological constant. In fact, all the theoretical attempts of renormalization led to a zero-point energy which does not agree with observations [31].
For these reasons, several alternative theories of gravity have been proposed in literature and our aim is now to provide a brief overview.
In the previous chapter we introduced some mathematical objects from General Relativity and the Einstein Equation, without any motivation or derivation. It is worth, then, to go a bit further in the key concepts of this theory in order to identify the points in which generalisations can be grafted.
From a physical point of view, General Relativity is built upon the equivalence principle. For a detailed discussion of the different statements see Carrol's book [12] or [40] (the latter reference is more focused on generalizations). The stricter formulation, however, is known as Strong Equivalence Principle (SEP). This statement claims that the result of a local test on gravity is independent of the frame and uniquely leads to General Relativity. In particular, to build an alternative theory of gravity, a relaxation of SEP is, at least, necessary.
The (burdensome) mathematical implementation of the physical insight of Equivalence Principle leads eventually to all the astonishing quantitative predictions of General Relativity. Let's go more in depth in the fundamental concepts.
Gravity can be viewed through different perspectives. One possibility is to proceed with a geometric interpretation. This is the way which Einstein chose, namely to represent space time as a (Pseudo-Riemannian) differentiable manifold. In particular, General Relativity describes gravity as the curvature on this manifold, through the Riemann tensor, that we have already introduced. Actually, curvature is not the only geometrical object which can be employed to model gravity, but it can be shown that torsion and non-metricity (scheme in figure (2.1)) lead to equivalent equation of motion (Einstein Equation) up to boundary terms (see [23] for a thorough discussion about this topic).
However, in modern theoretical physics it is customary to prefer the interpretation of field theory. In

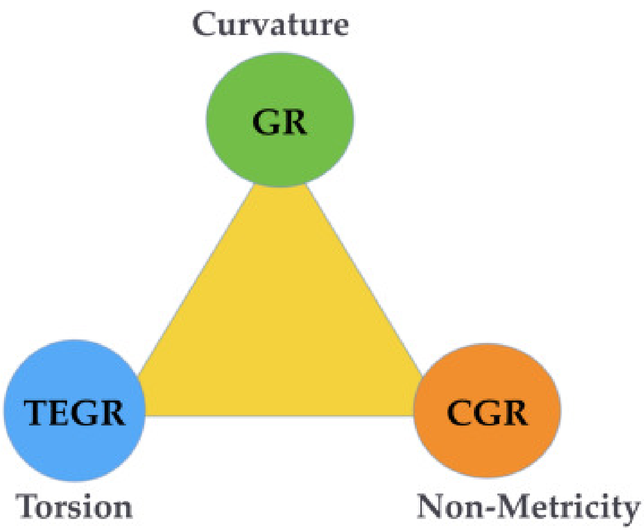

Figure 2.1: Equivalent geometric interpretations of gravity in terms of different mathematical objects. Standard GR based on curvature, Teleparallel Equivalent of GR based on torsion and Coincident GR based on non-metricity. Image from [23]

this picture, gravity, analogously to other fundamental interactions, is conveyed by a mediator, which is associated to a field with certain transformation properties.
Indeed, the requirement of a consistent theory for a Lorentz-invariant spin-2 mediator, which correspond to the metric tensor $g_{\mu\nu}$, unambiguously leads to General Relativity, while the introduction of additional fields set up an alternative theory of gravity.
This unicity is stated by Lovelock's theorem. It claims that GR is the only possible theory arising from an action which just contains the metric tensor and its first and second derivatives in four dimensions. This action, with the cosmological constant, reads:

$$S = \frac{M_p^2}{2} \int d^4x \sqrt{-g}\big(R - 2\Lambda\big) + S_m = S_{EH} + S_m \tag{2.1}$$

We introduced the reduced Planck mass $M_p^2 = \hbar c/8\pi G$, the determinant of the metric g and the Ricci scalar R.
The first term (without Λ) is called Einstein-Hilbert action and only regards gravity, while S_m is the action associated to the matter content. The extremization of the action varying $g_{\mu\nu}$ leads to the Einstein Equation. In particular, the variation of the Einstein Hilbert action leads to the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \tag{2.2}$$

while the variation of S_m produces the energy-momentum tensor

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}\,. \tag{2.3}$$

In the realm of geometric interpretation, one can try to extend General Relativity substituting to the Ricci scalar some function $f(R)$ in the Einstein-Hilbert action. The same procedure holds for equivalent geometrical formulations where other scalars constructed from torsion or non-metricity replace R.

On the other hand, in the frame of field theory, assuming locality, unitarity and local Lorentz invariance, one can add new degrees of freedom to the Einstein-Hilbert action. These new degrees of freedom necessary appear through additional fields (scalars, vectors, tensors) which interact with $g_{\mu\nu}$ and represent new propagating mediators.
Note that the term Λ in equation (2.1) can be included in the action as it does not spoil its mathematical properties and it is motivated by the need of dark energy source. However, in modified theories of gravity dark energy is usually attributed to one or more dynamic fields and the cosmological constant

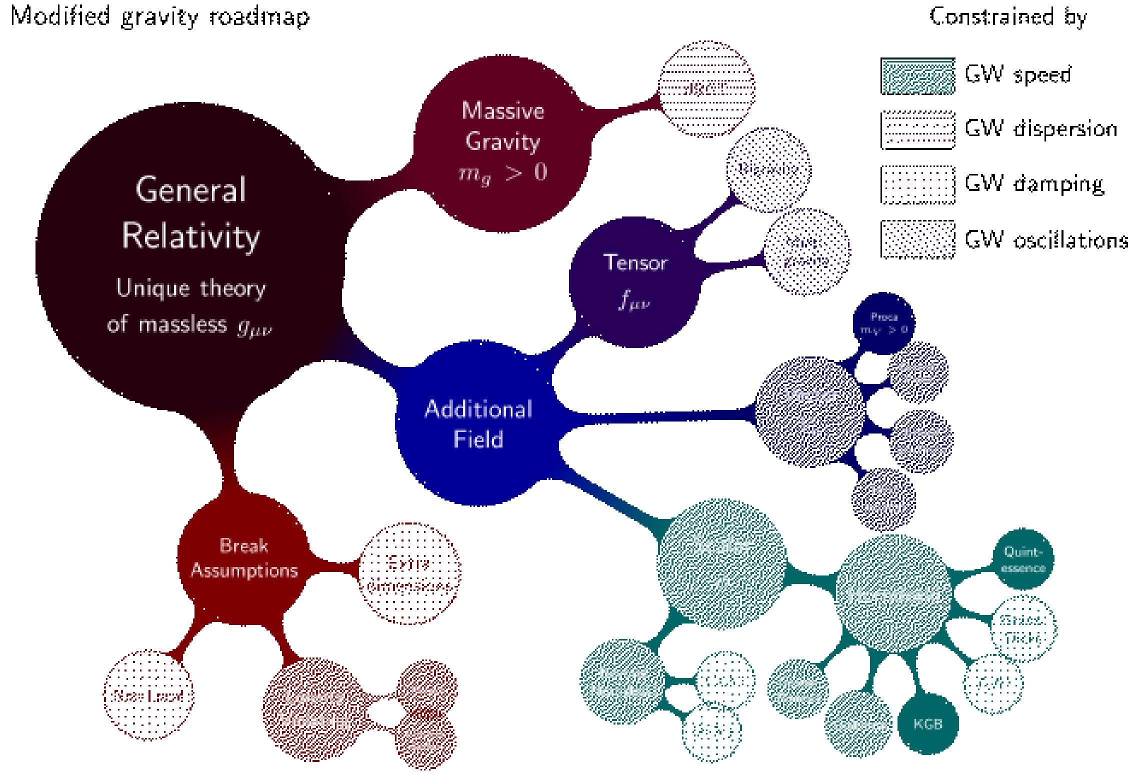

Figure 2.2: Schematic classification of alternative theories of gravity. Image from [17]

is not needed anymore. This, as anticipated, allows to overcome the cosmological constant issue.
The classification of the different alternative gravity theories relies on the types of fields involved and on the way they interact among each other and with the gravity sector.
Some theories can present particular problems related to the mathematical nature of specific terms. These problems are called instabilities. Some of them arise at a certain energy scale, therefore a model with such a pathology can anyways be stable as an effective field theory within a safe energy domain. It is worth to mention the main examples of instability (see [23, 25] for more details).
A ghost instability appears when a field has the wrong sign on its kinetic term (depending on signature choice). This produce a so-called ghost particle with negative energy in the quantization procedure. This can be very problematic since the vacuum state would then be unstable to a decay process with zero energy cost and, therefore, infinite rate.
On the other hand, a laplacian instability occurs when a spatial derivative term with the wrong sign appears. The effect of such kind of term, is an unbound exponential growing solution, which clearly makes the theory unstable.
Another interesting example is the tachyonic instability, arising in presence of a field with negative squared mass.
Let's now take a brief overview of the different theories, according with the nature of the new degrees of freedom they introduce. A schematic representation of the general classification is showed in figure (2.2).
The first class of alternative gravity is represented by theories in which a scalar field interacts with the tensor $g_{\mu\nu}$. Theories of this family are therefore named Scalar-Tensor Theories and they come from a relaxation of SEP.
The introduction of a simple scalar field in the general action is the simplest way to model cosmic inflation. This kind of scalar field is usually called quintessence, as it stands as a fifth force besides the fundamental four.
This approach can be opened up with the so-called K-essence model, in which a ϕ-dependent kinetic term appears, and then further with covariant Galileon and Horndesky theories, which are the most general theories with second order equations of motion. Dropping this requirement, we find other consistent examples like beyond-Horndesky theories and Degenerate Higher-Order Scalar-Tensor

theories (DHOST) with more general interactions.
However, a theory with equations of motion with order higher than two runs into the so-called Ostrogadsky instability, i.e. the appearence of a ghost term. This problem can be worked around in an effective field theory context.
An interaction could also exist between scalar fields and matter fields. A particularly relevant example is a theory in which a quintessence scalar field interacts with dark matter. In this case we would have a quintessence-dependent mass of dark matter particles which would modify its non-relativistic scaling through the history of the universe. This effect also provides an interesting description of dark energy.
Another important class is represented by the so called vector-tensor theories. The allowed interactions and symmetries are of course depending on whether the vector field is massive (Proca field) or not (Maxwell field). The most general theory of this kind is the so called Generalised Proca.
Moreover, the most general scalar-tensor and vector-tensor theories can be unified in consistent scalar-vector-tensor theories.
Finally, another class is represented by tensor-tensor theories, like massive gravity.
Some of these models will be described in more details in chapter 5.
The predictions of all these theoretical proposals, must of course face up to experimental tests. The hypothesis of a fifth force generated, for instance, from a scalar field should be detectable where we have the possibility to put tighter bounds.
However, the gravitational coupling predicted by General Relativity has been constrained with a high degree of accuracy (see [25]) both through solar system physics and stellar evolution.
For this reason, many screening mechanisms have been proposed in literature, which would inhibit the fifth force in the environments where we can perform tests. An example is the chameleon mechanism, mainly related to scalar fields. Thanks to this effect, the field would acquire an effective mass which increases with the density of the environment, and the interaction range of the fifth force would be consequently suppressed outside the surface of astronomical bodies like the Sun or the Earth.
Another important example is the Vainshtein mechanism which arises from non-linear self-interaction terms (usually kinetic term for scalars). This mechanism suppresses the fifth force in the region around a gravitational source within the so-called Vainshtein radius (dependent on the specific theory). The contributions of the hidden degrees of freedom would however play a role at larger scales.
Besides solar system physics and stellar evolution and dynamics, another fundamental way to test the nature of gravity comes from gravitational waves. In particular, the detection of a binary neutron star merger marked the beginning of multimessenger astrophysics era. The combined analysis of the gravitational wave event GW170817 and the electromagnetic counterpart GRB170817A (a gamma-ray burst) allowed to constrain the propagation speed for gravitational waves to the value of c with $\Delta c/c \sim 10^{-15}$ [1].
The strict requirement $c_t = c$ (without assuming fine tuning among functions in the lagrangian) had a tremendous impact on the alternative gravity theories landscape as many models predicted a deviation from this value coming from high order couplings [16].
Finally, cosmological observations provide nowadays an important feedback for alternative theories of gravity. The aim of this work is to estimate the predictions of some models in the non-linearly evolved matter power spectrum, in comparison with General Relativity. It would be interesting in the future to confront this analytical method with results from numerical simulations and observations.

2.2 General-relativistic perturbation theory

We have already described the newtonian perturbation theory in cosmology and we want now to extend the same approach in a covariant manner.
One first has to express the metric and the energy-momentum tensor respectively as:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \,, \qquad T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu} \,. \tag{2.4}$$

The linearised Einstein equations can be arranged as:

$$\hat{\mathcal{L}}[\,\bar{g}_{\mu\nu}]\delta g_{\mu\nu} = \delta T_{\mu\nu} \,, \tag{2.5}$$

where $\hat{\mathcal{L}}[\,\cdot\,]$ is a suitable second-order differential operator which acts on the background metric.
The metric perturbation can be decomposed in scalar, vector and tensor modes, according with the irreducible representation of $SO(3)$ group. The components of the metric perturbation therewith read:

$$\begin{aligned} \delta g_{00} &= -2a^2\phi \\ \delta g_{0j} &= a^2(\omega_j + \partial_j\omega) \\ \delta g_{ij} &= a^2\Big[2\psi\delta_{ij} + \Big(\partial_i\partial_j - \frac{\delta_{ij}}{3}\partial^k\partial_k\Big)\chi + \partial_i\chi_j + \partial_j\chi_i + \chi_{ij}\Big]. \end{aligned} \tag{2.6}$$

The functions ϕ, ω, ψ, χ are scalar perturbations, while ω_j, χ_j are vectors and χ_{ij} pure tensors, i.e. gravitational waves.
The matter fields enclosed in $T_{\mu\nu}$ can be organised in a similar manner. The density perturbations $\delta\rho$, which are the base of structure formation analysis, are for instance scalar perturbations.
Thanks to the diffeomorphism symmetry of General Relativity, one enjoys the freedom to choose a certain gauge to carry out the calculation (see [23] for an overview of the main gauge choices in cosmological context).
One can instead decide to proceed with gauge-invariant quantities without choosing any specific gauge. This latter approach has the advantage that final equations are free from unphysical terms, at the price, however, of a huger calculation burden.
From the perturbed Einstein equation one can obtain a generalised Poisson equation in terms of the gauge invariant Bardeen's potential Ψ which reads:

$$k^2\,\Psi + 4\pi G\bar{\rho}a^2\delta = 0\,. \tag{2.7}$$

This is clearly an extension of the newtonian Poisson equation (2.5).
This kind of analysis allows, of course, to include possible additional fields from alternative gravity theories. In this case the concerned fields must be properly perturbed and the system (2.5) must be expanded to include the equations of motions for the additional degrees of freedom.
A common approach is to view the modification of gravity as variation of the gravitational coupling, which can be in general promoted to an effective function of time and scale:

$$G \longmapsto G_{\text{eff}}(t,\,k) \tag{2.8}$$

where the explicit expression for G_{eff} depends on the details of the theory.
Introducing a gauge-invariant density contrast δ, in quasi-static approximation, it is also possible to recover the density perturbation evolution equation as

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G_{\text{eff}}\bar{\rho}\delta \simeq 0\,. \tag{2.9}$$

At this point we have an idea of how the fundamental equations of cosmic structure formation can be extended to alternative theories of gravity.
In the next chapters we will develop a method for the computation of the non-linear matter power spectrum which can smoothly include different theories simply knowing G_{eff} and the background expansion function.

Chapter 3

Kinetic Field Theory

3.1 Motivation

As emerged from many theoretical works in the last decade, there are several examples of systems in physics, which could be well described with a statistical field theory adapted to classical degrees of freedom [18, 32].
The idea of Kinetic Field Theory formalism comes from this necessity and can be successfully applied to different problems in physics. In this work we will focus on its application to cosmic structures formation.
As we will see, the advantages of this theoretical approach are several. First of all, the deterministic evolution of particles in phase space follow hamiltonian trajectories, which allow to get rid of the shell-crossing problem arising in hydrodynamics [11].
Furthermore, as we will see more in details in the next section, KFT allows the analytical computation of the non-linear power spectrum through a mean-field approximation. The closed analytical form of the spectrum is extremely useful for studying the role of single physical parameters. This approach is very promising for testing the effects of different theories of gravity on structures formation.

3.2 Generating functional and equation of motion

Let's start with the introduction of a generic state variable, mathematically described by the field φ and governed by the probability distribution $P(\varphi)$.
In full analogy to quantum statistical field theory, let's introduce also a generator field J and the following scalar product, defined by means of an integral over time:

$$\langle J, \varphi \rangle = \int_0^t dt' \, J(t')\varphi(t') \tag{3.1}$$

The collective dynamics of the system is described by a generating functional, which has the role of the grand canonical partition function in equilibrium
thermodynamics:

$$Z[\,J\,] = \int \mathcal{D}\varphi \, P(\varphi) e^{i\langle J,\varphi\rangle} \tag{3.2}$$

with the functional integration measure $\mathcal{D}\varphi$ expressed with the standard path integral notation.
A generic moment of the field φ can be easily computed as an appropriate functional derivative of the generating functional:

$$\langle \varphi^n \rangle = \left(-\, i\frac{\delta}{\delta J}\right)^n Z[\,J\,]\bigg|_{J=0} \tag{3.3}$$

Setting a time $t = 0$, let's assume we have the initial configuration $\varphi^{(i)}$, $P(\varphi^{(i)})$.
As the system evolves out of equilibrium, for a generic time $t > 0$, we will have the final state φ

with the corresponding probability distribution. The initial configuration is mapped in the final one through the transition probability $P(\varphi^{(i)}|\varphi)$. Then:

$$P(\varphi) = \int \mathcal{D}\varphi^{(i)}\, P(\varphi^{(i)}) P(\varphi^{(i)}|\varphi). \tag{3.4}$$

Defining the initial integration measure $\mathcal{D}\Gamma^{(i)} = \mathcal{D}\varphi^{(i)} P(\varphi^{(i)})$, one can express the generating functional as:

$$Z[\,J\,] = \int \mathcal{D}\Gamma^{(i)} \int \mathcal{D}\varphi\, P(\varphi^{(i)}|\varphi) e^{i\langle J,\varphi\rangle}. \tag{3.5}$$

So far we provided an abstract description which holds in principles for every statistical ensemble. Let's now restrict to the case of N classical particles of equal mass m.
A point in phase space is defined by a vector which includes spatial coordinates q_i and momenta p_i.

$$x_i = (q_i, p_i), \qquad \text{with} \qquad i \in [1, N] \tag{3.6}$$

The trajectories of all particles can be combined in a single object:

$$\mathbf{x}(t) = x_i(t) \otimes e_i \tag{3.7}$$

with the orthonormal basis $\{e_i\}$.
The tensor $\mathbf{x}(t)$ will replace the abstract state variable φ in this specific case. The generator field and the scalar product can be defined accordingly:

$$\mathbf{J}(t) = J_i(t) \otimes e_i = \begin{pmatrix} J_{q_i}(t) \\ J_{p_i}(t) \end{pmatrix} \otimes e_i \tag{3.8}$$

$$\langle \mathbf{J}, \mathbf{x}\rangle = \langle J_i(t) \otimes e_i,\, x_j(t) \otimes e_j\rangle = \langle J_i, x_j\rangle \delta_{ij} = \int_0^t dt'\, J_i \cdot x_i \tag{3.9}$$

Here comes the main conceptual difference between a quantum and a classical ensemble. While in the first case the transition probability has an amplitude in phase space due to uncertainty principle, in the classical case the evolution is fully deterministic.
Therefore, given some equations of motion $E(x) = 0$ the transition probability will be non vanishing only for trajectories which satisfy them. Mathematically, this translates into a Dirac delta distribution, with its functional representation:

$$P(\mathbf{x}|\mathbf{x}^{(i)}) = \delta_D[\,E(\mathbf{x})\,] = \int \mathcal{D}\boldsymbol{\chi}\, e^{i\langle \boldsymbol{\chi}, E(\mathbf{x})\rangle} \tag{3.10}$$

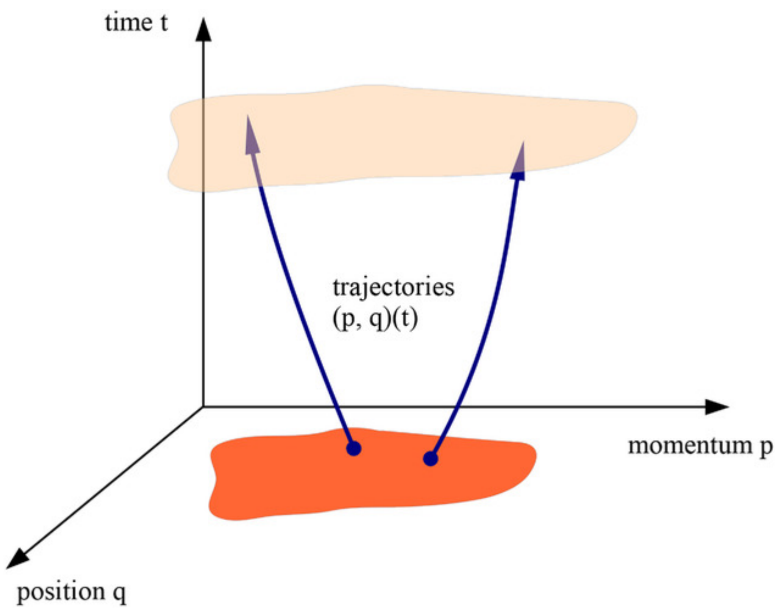

Figure 3.1: Representation of trajectories in phase space. Image from [11]

Let's now introduce a field K conjugate to χ and rewrite the generating functional:

$$Z[\mathbf{J},\mathbf{K}] = \int \mathcal{D}\Gamma^{(i)} \int \mathcal{D}\mathbf{x} \int \mathcal{D}\boldsymbol{\chi}\, e^{i\langle\boldsymbol{\chi},E(\mathbf{x})\rangle + i\langle\mathbf{J},\mathbf{x}\rangle + i\langle\mathbf{K},\boldsymbol{\chi}\rangle}. \tag{3.11}$$

In analogy to the quantum field theory path integral, we can view the argument of the exponential as the action associated to a certain field, defined as integral over time of the corresponding lagrangian:

$$\langle\boldsymbol{\chi}, E(\mathbf{x})\rangle = \int_0^t dt'\, \boldsymbol{\chi}\cdot\mathbf{E} = \int_0^t dt'\, \mathcal{L} = S[\boldsymbol{\chi}] \tag{3.12}$$

At this point it is convenient to express the equation of motion operator as a sum of a free and an interaction contribution, namely:

$$E(\mathbf{x}) = \dot{\mathbf{x}} + E_0(\mathbf{x}) + E_I(\mathbf{x}). \tag{3.13}$$

Ignoring the interaction term, one obtains the free generating functional:

$$\begin{aligned} Z_0[\mathbf{J},\mathbf{K}] &= \int \mathcal{D}\Gamma^{(i)} \int \mathcal{D}\mathbf{x} \int \mathcal{D}\boldsymbol{\chi}\, e^{i\langle\boldsymbol{\chi},E(\mathbf{x})\rangle + i\langle\mathbf{J},\mathbf{x}\rangle + i\langle\boldsymbol{\chi},\dot{\mathbf{x}}+E_0(\mathbf{x})+\mathbf{K}\rangle} = \\ &= \int \mathcal{D}\Gamma^{(i)} \int \mathcal{D}\mathbf{x}\, \delta_D[\dot{\mathbf{x}} + E_0(\mathbf{x}) + \mathbf{K}] e^{i\langle\mathbf{J},\mathbf{x}\rangle} = \\ &= \int \mathcal{D}\Gamma^{(i)}\, e^{i\langle\mathbf{J},\bar{\mathbf{x}}(\mathbf{K})\rangle}, \end{aligned} \tag{3.14}$$

where $\mathbf{K}$ has replaced the interaction term in the equation of motion and the corresponding solution $\bar{\mathbf{x}}(\mathbf{K})$.
Let's now go more specifically into the equation of motion. An ensamble of N classical particle, such as the one we introduced, is well described by the Hamilton equations, which can be written in a compact manner for each particle as:

$$\dot{x}_i = \mathcal{J}\partial_{x_i} H \tag{3.15}$$

with the hamiltonian function:

$$\mathcal{H}(q_i, p_i) = \frac{|p_i|^2}{2m} + V(q_i) \tag{3.16}$$

and the symplectic matrix:

$$\mathcal{J} = \begin{pmatrix} 0 & \mathbb{I}_3 \\ -\mathbb{I}_3 & 0 \end{pmatrix}. \tag{3.17}$$

Therefore, equation (3.15) reads explicitelly:

$$\begin{pmatrix} \dot{q}_i \\ \dot{p}_i \end{pmatrix} = \begin{pmatrix} \partial_{p_i} \\ -\partial_{q_i} \end{pmatrix} \left(\frac{|p_i|^2}{2m} + V(q_i) \right) = \begin{pmatrix} p_i/m \\ -\partial_{q_i} V \end{pmatrix} \tag{3.18}$$

Let's study the mathematical form of the solutions, starting from the free case, in which the potential energy term vanishes ($V = 0$). In this rather simple case, the system reads:

$$\begin{pmatrix} \dot{q}_i \\ \dot{p}_i \end{pmatrix} = \begin{pmatrix} 0 & (1/m)\mathbb{I}_3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} q_i \\ p_i \end{pmatrix} = M \begin{pmatrix} q_i \\ p_i \end{pmatrix} \tag{3.19}$$

and it is solved by direct exponentiation of the matrix M:

$$\bar{x}_i(t) = exp\Big(\int_{t_0}^t dt'\, M(t') \Big) x_i(t_0) = \begin{pmatrix} \mathbb{I}_3 & ((t-t_0)/m)\mathbb{I}_3 \\ 0 & \mathbb{I}_3 \end{pmatrix} x_i(t_0) \tag{3.20}$$

The matrix in the last passage is the so called Green's function:

$$G(t, t_0) = \begin{pmatrix} \mathbb{I}_3 & ((t-t_0)/m)\mathbb{I}_3 \\ 0 & \mathbb{I}_3 \end{pmatrix} = \begin{pmatrix} g_{qq}(t,t_0) & g_{qp}(t,t_0) \\ 0 & g_{pp}(t,t_0) \end{pmatrix}. \tag{3.21}$$

The Green's function formalism is very convenient to move to the interacting case in which $V \neq 0$ in which the solution reads:

$$\bar{x}_i(t) = G(t.t_0)x_i(t_0) - \int_{t_0}^{t} dt'\, G(t.t') \begin{pmatrix} 0 \\ -\partial_{q_i} V(t') \end{pmatrix} \tag{3.22}$$

All these results in the hamiltonian framework can be rewritten in a tensorial form, according with (3.7), in order to plug the equation of motion into the generating functional of KFT.
Define then the tensors

$$\mathfrak{I} = \mathcal{I} \otimes \mathbb{I}_N\,, \qquad \boldsymbol{\nabla} = \partial_{x_i} \otimes e_i\,, \qquad \mathbf{G}(t,t_0) = G(t,t_0) \otimes \mathbb{I}_N \tag{3.23}$$

with which the final form of the solution reads:

$$\bar{\mathbf{x}}(t) = \mathbf{G}(t,0)x^{(i)} - \int_0^t dt'\, \mathbf{G}(t,t') \begin{pmatrix} 0 \\ -\partial_q V(t') \end{pmatrix}. \tag{3.24}$$

3.3 Density operator

Since we are dealing with classical point-particles, the numerical density of the system is a combination of Dirac deltas:

$$\rho(\vec{q},t) = \sum_{i=1}^{N} \rho_i(\vec{q},t) = \sum_{i=1}^{N} \delta_D[\vec{q} - \vec{q}_i(t)]. \tag{3.25}$$

The Fourier representation of a single particle density reads:

$$\rho_j(1) = exp(-i\vec{k}_1 \cdot \vec{q}_j(t_1)) \tag{3.26}$$

where we defined $1 = (t_1, \vec{k}_1)$. It describes the contribution of particle j at time $t = t_1$ and wave number $\vec{k} = \vec{k}_1$.
As it is usual in field theories, ordinary functions can be promoted to operators through the action of an appropriate derivative. As phase space coordinates are the state variables in KFT, we can use the following definition using generator fields:

$$q_i(t_1) \longmapsto -i\frac{\delta}{\delta J_{q_i}(t_1)}. \tag{3.27}$$

The density is promoted accordingly to the operator

$$\hat{\rho}_i(1) = \sum_{i=1}^{N} exp\Big(-\vec{k}_1 \cdot \frac{\delta}{\delta J_{q_i}(t_1)}\Big) \tag{3.28}$$

It can be demonstrated that the exponential of a differential operator produces a finite translation. In fact, the infinitesimal translations are generated in group theory by first order differential operators, and this can be viewed as the first term of a Taylor expansion. Introducing the infinite sum over higher order terms one recovers the definition of exponential. Hence, applying the density operator to the the generating functional we shift the value of the field J at which the functional itself is evaluated.

$$\hat{\rho}_i(1)\; Z[\mathbf{J},\mathbf{K}]|_{\mathbf{J}=0} = exp\Big(-\vec{k}_1 \cdot \frac{\delta}{\delta J_{q_i}(t_1)}\Big)\; Z[\mathbf{J},\mathbf{K}]|_{\mathbf{J}=0} = Z[\mathbf{L},\mathbf{K}] \tag{3.29}$$

The amount of shift depends on the wave vector of the mode associated to $\hat{\rho}$:

$$\begin{aligned} \langle \mathbf{L}, \bar{\mathbf{x}}(\mathbf{K})\rangle &= -\vec{k}_1 \cdot \frac{\delta}{\delta J_{q_i}(t_1)}\, \langle \mathbf{J}, \bar{\mathbf{x}}(\mathbf{K})\rangle|_{\mathbf{J}=0} = \\ &= -\vec{k}_1 \langle \delta_D(t-t_1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes e_i,\, \bar{\mathbf{x}}(\mathbf{K})\rangle = \\ &= -\vec{k}_1[\bar{q}_i(t_1) - \bar{K}_i(t_1)]. \end{aligned} \tag{3.30}$$

where $\bar{q}_i(t_1) = q_i + t_1 p_i$ and $\bar{K}_i(t_1) = \int_0^{t_1} dt' \, [\bar{K}_{q_i}(t') + (t-t')\bar{K}_{p_i}(t')]$ from the general solution of Hamilton equation.
Applying the density operator r times one obtains:

$$\hat{\rho}_{i_1}(1) \cdots \hat{\rho}_{i_r}(r) \; Z[\mathbf{J}, \mathbf{K}]|_{\mathbf{J}=0} = Z[\mathbf{L}, \mathbf{K}] \tag{3.31}$$

with

$$\langle \mathbf{L}, \bar{\mathbf{x}}(\mathbf{K}) \rangle = - \sum_{s=1}^{r} \vec{k}_s [\bar{q}_{i_s}(t_s) - \bar{K}_{i_s}(t_s)]. \tag{3.32}$$

This Ansatz for density can be exploited for expressing the interaction potential, namely:

$$V(q) = \sum_{i=1}^{N} v(q - q_i) = \sum_{i=1}^{N} \int_{q'} v(q - q')\delta_D(q' - q_i) = \int_{q'} v(q - q')\rho(q') \tag{3.33}$$

where we introduced the following shorthand notation for integrals over positions:

$$\int_{q'} = \int d^3 q'. \tag{3.34}$$

Therefore, the gradient of the potential evaluated at position $q = q_i$ reads:

$$\begin{aligned} \partial_q V(q)|_{q=q_i} &= \int_q \delta_D(q - q_i)\partial_q V(q) = \int_q \int_{q'} \rho_i(q)\partial_q v(q - q')\rho(q') = \\ &= - \int_q \int_{q'} \partial_q \rho_i(q) v(q - q')\rho(q'). \end{aligned} \tag{3.35}$$

where in the last passage we performed the integration by parts.
With these last expressions we are finally able to derive the interaction contribution to the equation of motion. In fact, we can define the interaction lagrangian for hamiltonian equation of motion, like we did in equation (3.12) :

$$\mathcal{L}_I = \chi \cdot \mathbf{E_I} = \chi_{p_i} \partial_{q_i} V = - \int_q B(q) V(q). \tag{3.36}$$

In the last passage the response field $B(q) = \chi_{p_i} \partial_q \rho_i(q)$ has been defined. Moving to Fourier representation, one obtains the elegant expression:

$$B(1) = i k_1 \chi_{p_i} \rho_i(1), \qquad\qquad \chi \cdot E_I = - \int_k B(-k) v(k) \rho(k) \tag{3.37}$$

with the compact notation for integrals over wave numbers:

$$\int_k = \int \frac{d^3 k}{(2\pi)^3}. \tag{3.38}$$

Finally, following the same procedure that we employed for positions, we can promote B to an operator as:

$$\hat{B}(1) = k_1 \frac{\delta}{\delta K_{p_i}(t_1)} \hat{\rho}_i(1). \tag{3.39}$$

Then we can write the full interaction action in operator form, which will be suitably applied to the free generating functional:

$$\hat{S}_I = \int dt \, \hat{\chi} \cdot \hat{E}_I = - \int d1 \, \hat{B}(-1) v(1) \hat{\rho}(1). \tag{3.40}$$

This interaction can be cast in a more elegant form ([8]) through the introduction of the doublet field

$$\hat{\Phi}(\vec{k}, t) = \begin{pmatrix} \hat{\rho}(\vec{k}, t) \\ \hat{B}(\vec{k}, t) \end{pmatrix} \tag{3.41}$$

and the suitable metric:

$$\sigma(1,2) = -\nu(1)\delta_D(1-2)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.42}$$

In this way the interaction term becomes:

$$\hat{S}_I = -\int d1 \int d2\, \hat{\Phi}^T(-1)\sigma(1,2)\hat{\Phi}(2). \tag{3.43}$$

Finally, thanks to these last passages, the interacting generating functional can be expressed as the action of the interaction operator on the free one in a very compact form, by:

$$Z[\mathbf{J},\mathbf{K}] = e^{i\hat{S}_I} Z_0[\mathbf{J},\mathbf{K}]. \tag{3.44}$$

This conclude the general overview of KFT formalism. It is evident how KFT can provide macroscopic and statistical correlations in phase space starting from microphysical assumptions about fundamental constituent of the system in hamiltonian framework.
At this point we are ready to apply this general expressions to the context of cosmic structure formation.

3.4 Application to cosmology

The main step to apply the KFT general framework to the context of cosmology is the choice of proper coordinates and variables. As it is intuitive, the best choice is a system of comoving coordinates $\{q\}$, which are related to physical ones through the scale factor by $q = r/a(t)$ ([6]).
Regarding the cosmic structures, we will introduce them like perturbations in the linear regime on top of a globally homogeneous expanding universe.
The equation of motion of a test particle in the background universe can be derived from the classical lagrangiang:

$$\mathcal{L}(r,\dot{r},t) = \frac{m}{2}\dot{r}^2 - m\,\Phi(r). \tag{3.45}$$

The gravitational potential is the solution to the modified Poisson's equation:

$$\nabla_r^2\Phi = 4\pi G\rho - \Lambda \tag{3.46}$$

where the expansion of the universe is introduce through the gravitational constant Λ.
The passage to comoving coordinate is straightforward and yields:

$$\mathcal{L}(q,\dot{q},t) = L\frac{m}{2}a^2\dot{q}^2 - m\,\phi\ , \qquad \nabla_q^2\phi = 4\pi G a^2(\rho - \bar{\rho}) \tag{3.47}$$

where ϕ is the so called peculiar gravitational potential. We stress that we are operating in the linear regime in which the condition $\delta \ll 1$ on the density contrast holds.
It is convenient to rescale the time coordinate using the cosmic growth factor:

$$t \longmapsto t = D_+(a) - D_+^{(i)} \tag{3.48}$$

normalized such that $a(t_i) = 1$, $t_i = 0$, i.e. with $D_+^{(i)} = 1$.
Transforming the time derivative accordingly and imposing that the action is left unchanged by this transformation, one obtains a suitable lagrangian, which can be written with the standard form, introducing a time dependent mass and the potential $\varphi = (a^2/mH_0^2)\phi$:

$$\mathcal{L}(q,\dot{q},t) = \frac{m(t)}{2}\dot{q}^2 - m(t)\,\varphi \tag{3.49}$$

The time-dependent mass reads:

$$m(t) = a^2 D_+ f E \tag{3.50}$$

with the function f defined by

$$f = \frac{dlnD_+}{dlna}, \tag{3.51}$$

$E(t)$ is the cosmic expansion function introduced in chapter 1 and we will employ it widely when we deal with the effect of the gravity theory on KFT predictions.
Note that the potential φ fulfills the Poisson's equation:

$$\nabla^2\varphi = \frac{3a}{2m^2}\delta. \tag{3.52}$$

The particle trajectories are given by the Euler-Langrange equation:

$$\ddot{q} + \frac{\dot{m}}{m}\dot{q} + \nabla\varphi = 0. \tag{3.53}$$

With this choice of time coordinate, particle trajectories can be described as inertial, introducing the effective force

$$f = -\frac{\dot{m}}{m}\dot{q} - \nabla\varphi. \tag{3.54}$$

Impose now that the solution of (3.53) has the hamiltonian form, i.e. the one constructed in terms of Green functions:

$$q(t) = q^{(i)} + g_{qp}(t, 0) + \int_0^t dt\, g_{qp}(t, t')f(t'). \tag{3.55}$$

The term g_{qp} has the so called Zel'dovich propagator form:

$$g_{qp}(t, t') = t - t' \tag{3.56}$$

while the effective force reads:

$$f = -\nabla\varphi - \frac{\dot{m}}{m^2}\left[p^{(i)} - \int_0^t dt'\, m\nabla\varphi\right] \tag{3.57}$$

Now that this formalism has been developed, it is important to characterize properly the initial state, i.e. to build a suitable initial probability distribution function, relying on some fundamental statistical assumptions.
In the context of structure formation, it is convenient to sample phase space coordinates from the pair $(\delta^{(i)}, p^{(i)})$, which we can constrain using hydrodynamics equations.
Cosmological observations suggest that we can assume a gaussian power spectrum for the initial density fluctuation distribution. From this homogeneous configuration in the early universe, the formation of cosmic structures will be described in KFT evolving phase space configurations through the hamiltonian deterministic flux. Thanks to the scaling of vortical perturbation modes, it is also safe to assume that, particle momenta have vanishing vorticity, i.e. vanishing curl. This translates in the mathematical statement that the corresponding vector field can be expressed as the gradient of a scalar potential ψ, namely:

$$p^{(i)} = \nabla\psi. \tag{3.58}$$

The advantage of this approach is that ψ is a statistically homogeneous, isotropic, gaussian random field, and it is therefore, fully characterized by its power spectrum, in analogy to $\delta^{(i)}$.
Furthermore, these two quantities can be related using continuity equation:

$$\dot{\delta} + \nabla \cdot p = \frac{\partial}{\partial t}(\delta^{(i)}t) + \nabla \cdot \nabla\psi = 0 \tag{3.59}$$

with the linear Ansatz from the first term of (1.32) for δ. This leads to

$$\delta^{(i)} = -\nabla^2\psi. \tag{3.60}$$

Using the definition of power spectrum, we obtain, as a consequence, the important statistical relation between our two fundamental random fields in Fourier space:

$$k^4 P_\psi(k) = P_\delta(k). \tag{3.61}$$

From now on, we will drop the superscript (i) for the initial state since no ambiguity can arise anyway. At this point we are ready to derive the expression from the probability distribution in phase space, applying the definition (3.4) and using the transition probabilities to sample (q, p) from $(\delta, \nabla\psi)$:

$$P(\mathbf{q}, \mathbf{p}) = \int \mathcal{D}\delta \int \mathcal{D}(\nabla\psi)\, P(q|\delta)P(p|\nabla\psi)P(\delta, \nabla\psi). \tag{3.62}$$

The transition probability from $\nabla\psi \longmapsto p$ is simply the Dirac delta $\delta_D(p - \nabla\psi)$, while the one from $\delta \longmapsto q$ can be expressed as

$$P(q|\delta) = \frac{\rho}{N} = \frac{\bar{\rho}(1+\delta)}{N}. \tag{3.63}$$

The joint probability then reads:

$$P(\mathbf{q}, \mathbf{p}) = \frac{\bar{\rho}}{N} \int \mathcal{D}\delta\, (1+\delta)P(\delta, p). \tag{3.64}$$

After some mathematical manipulations, it can be demonstrated (see [10] and [19]) that this intial joint probability can be expressed in a nearly gaussian form, after the introduction of a suitable kernel matrix $\bar{C}_{pp}$. Namely:

$$P(\mathbf{q}, \mathbf{p}) \approx \frac{V^{-N}}{\sqrt{(2\pi)^{3N} det\bar{C}_{pp}}} \exp\Big(-\frac{1}{2}\mathbf{p}^T \bar{C}_{pp}^{-1}\mathbf{p}\Big) = \mathcal{N} \exp\Big(-\frac{1}{2}\mathbf{p}^T \bar{C}_{pp}^{-1}\mathbf{p}\Big). \tag{3.65}$$

Let's describe more in depth the momentum correlation matrix.
Using the properties of tensor product:

$$\bar{C}_{pp} = \langle \mathbf{p} \otimes \mathbf{p} \rangle = \langle p_j \otimes p_k \rangle \otimes (e_j \otimes e_k) = \frac{\sigma_p^2}{3} \mathbb{1}_3 \otimes \mathbb{1}_N + C_{p_j p_k} \otimes E_{jk} \tag{3.66}$$

where in the last passage we separated the diagonal part ($j = k$) from the off-diagonal terms ($j \neq k$) and the matrix $E_{jk} = e_J \otimes e_k$ was defined.
The term σ_p^2 is the momentum dispersion. The generic form of a moment of order n is:

$$\sigma_n^2 = \int_k k^{2(n-2)} P_\delta(k) = \frac{1}{2\pi^2} \int_0^\infty dk\, k^{2n-2} P_\delta(k) \tag{3.67}$$

with the initial density fluctuation power spectrum $P_\delta(k)$.
Given the momenta p_j and p_k separated by a distance q, their correlation matrix is represented by the term

$$C_{p_j p_k} = \langle \nabla\psi \otimes \nabla\psi \rangle = -(\nabla \otimes \nabla)\xi_\psi(q) \tag{3.68}$$

where the function $\xi_\psi(q)$ is the one point correlation function of the potential ψ and reads:

$$\xi_\psi(q) = \frac{1}{2\pi^2} \int_0^\infty \frac{dk}{k^2} P_\delta(k) j_0(kq) \tag{3.69}$$

where $j_0(kq)$ is the spherical Bessel function of zeroth order.
It is convenient to introduce the projectors on the parallel and perpendicular direction to the unitary separation vector $\hat{q}$:

$$\pi^{\parallel} = \hat{q} \otimes \hat{q}\ , \qquad \pi^{\perp} = \mathbb{1}_3 - \pi^{\parallel}. \tag{3.70}$$

Therefore, we will have:

$$C_{p_j p_k} = -\pi^{\parallel} \xi_\psi''(q) - \pi^{\perp} \frac{\xi_\psi'(q)}{q}. \tag{3.71}$$

Note that for $q = 0$ the momentum dispersion reads simply:

$$\sigma_p^2 = \int_k k^2 P_\psi(k). \tag{3.72}$$

With these ingredients we are able to see how power spectra are described in KFT.

3.5 Power spectrum and mean-field approach

In this section we will start to tackle the problem of describing structure formation in different regimes (linear and non-linear) through the power spectrum formalism within the KFT framework.
Let's start from the simple case of synchronous (i.e. $t_1 = t_2 = t$) two-point free density correlation function, using the density operators which we previously introduced.

$$\begin{aligned} G_{\rho\rho} &= \langle \rho_2(-1)\rho_1(1)\rangle = \hat{\rho}_2(-1)\hat{\rho}_1(1)Z_0[\,\mathbf{J},\mathbf{K}\,]|_{\mathbf{J}=0} = Z_0[\,\mathbf{J},\mathbf{K}\,] = \\ &= e^{ik_1}(\bar{K}_2(t) - \bar{K}_1(t)) \int \mathcal{D}\Gamma^{(i)}\, e^{-ik_1(\bar{q}_2(t)-\bar{q}_1(t))} \end{aligned} \tag{3.73}$$

The particle trajectories in phase space are the ones which solve the hamiltonian equation, namely the inertial trajectories under the effective force that we defined previously.

$$\bar{q}_{1,2}(t) = q_{1,2} + tp_{1,2}. \tag{3.74}$$

Let's define the spatial separation between positions 1 and 2 and the corresponding momentum shift:

$$q = q_2 - q_1 \ , \qquad \mathbf{L}_p = -k_1 t \otimes (e_2 - e_1) \tag{3.75}$$

which lead to:

$$\begin{aligned} Z_0[\,\mathbf{L},0\,] &= \int \mathcal{D}\Gamma^{(i)}\, e^{-ik_1q} e^{i\mathbf{L}_p\cdot\mathbf{p}} = \\ &= \int d\mathbf{q}\, d\mathbf{p}\, \mathcal{N}\, exp\Big(-\frac{1}{2}\mathbf{p}^T \bar{C}_{pp}^{-1}\mathbf{p}\Big) e^{-ik_1q} e^{i\mathbf{L}_p\cdot\mathbf{p}} = \\ &= \int dq\, exp\Big(-\frac{1}{2}\mathbf{L}_p^T \bar{C}_{pp}^{-1}\mathbf{L}_p\Big) e^{-ik_1q} = \\ &= \int dq\, exp\Big(-t^2\Big(\frac{\sigma_p^2 k_1^2}{2} + \mathbf{k}_1^T C_{p1p2}\mathbf{k}_1\Big)\Big). \end{aligned} \tag{3.76}$$

In the third passage we integrated out the momenta degrees of freedom using the gaussian integral rule for multivariate gaussians.
Introduce now the director cosine between the unitary wave vector and the spatial separation versor $\mu = \hat{k}_1 \cdot \hat{q}$ which allows to express in a more operative form the projection operation from (3.70). Accordingly, let's define the two functions

$$a_{\parallel}(q) = \mu^2 \xi_\psi''(q) + (1-\mu^2)\frac{\xi_\psi'(q)}{q} \ , \qquad Q_D = \frac{\sigma_p^2 t^2 k_1^2}{3} \tag{3.77}$$

Equipped with these definitions, we can rewrite the free generating functional as

$$Z_0[\,\mathbf{L},0\,] = e^{-Q_D}\int dq\, e^{t^2k_1^2 a_{\parallel}(q)} e^{-ik_1q}. \tag{3.78}$$

Observe that through this expression the physical meaning of the quadratic form Q_D is clear. It represents a damping term in time evolution, coming from momentum dispersion which tend to hamper the matter concentration and the consequent formation of structures.
Since this last integral yields a term which is proportional to a Dirac delta, it is convenient to define the power spectrum for this initial case as

$$\mathscr{P}(1) = Z_0[\,\mathbf{L},0\,] - \delta_D(k_1) = e^{-Q_D}\int_q \Big[e^{t^2k_1^2a_{\parallel}(q)} - 1\Big] e^{-ik_1q}. \tag{3.79}$$

Observe that, neglecting for a second the damping term, one can linearize the time evolution of $\mathscr{P}$ for small enough values of the exponent, through the Taylor expansion

$$e^{t^2k_1^2a_{\parallel}(q)} = 1 + t^2k_1^2a_{\parallel}(q) + ... \tag{3.80}$$

which produces the time scaling

$$\mathscr{P}(k_1,t) \approx t^2 P_\delta^{(i)}(k_1) = P_\delta^{lin}(k_1), \tag{3.81}$$

that we already found in equation (1.40). The great advantage of KFT is the possibility to study analytically the non linear evolution of density power spectrum through a mean-field approach for the interaction term (see [7, 9]), extending the mathematical description beyond the linear regime. We will conclude this chapter describing how this approach works.
Qualitatively, we expect the gravitational interaction among particles to counteract the damping from particles free streaming represented by Q_D.
Quantitatively, we will use a suitable average of the interaction term $\langle S_I[\mathbf{L}]\rangle$ and employ the Burger's approximation to express the non-linear power spectrum as

$$P_\delta^{nl}(k) \approx e^{i\langle S_I(k)\rangle} P_\delta^{lin}(k). \tag{3.82}$$

Consider the interaction lagrangian (3.12) from two particles suffering the mutual influence through the effective force f. It can be averaged as:

$$\begin{aligned}\langle \mathcal{L}_{I,21}\rangle = \chi_{p_2}\cdot \int_{q_2}\int_{q_1} \langle \rho_2(q_2) f(q_1-q_2)\rho_1(q_1)\rangle = \\ = \chi_{p_2}\cdot \int_{q_1}\int_{q} f(q)\xi(q) = V_{\chi_{p_2}}\cdot \int_q f(q)\xi(q)\end{aligned} \tag{3.83}$$

where we reintroduced the spatial separation $q = q_2 - q_1$ and the two point function $\xi(q)$. The volume $V_{\chi_{p_2}}$ comes from the first integral.
It is well known that an ordinary product is mapped to a convolution in Fourier space. This implies that the integrand function in (3.83) can be Fourier-transformed as:

$$f(k) * Z_0[\mathbf{L},\mathbf{K}]. \tag{3.84}$$

Furthermore, remember that any field can be promoted to an operator through a suitable functional derivative with respect to the conjugate:

$$\chi_{p_2} \longmapsto -i\frac{\delta}{\delta K_{p_2}(t_1)} \tag{3.85}$$

Therefore, applying it to the generating functional, we get

$$-i\frac{\delta}{\delta K_{p_2}(t_1)} Z_0[\mathbf{L},\mathbf{K}]\Big|_{K=0} = ik(t-t_1)Z_0[\mathbf{L},0] = ik(t-t_1)\mathscr{P}(1) \tag{3.86}$$

which leads in Fourier space to the mean interaction action

$$i\langle S_I\rangle(k,t) = -k\int_0^t dt_1\,(t-t_1)f(1) * \mathscr{P}(1). \tag{3.87}$$

In linear approximation, as anticipated, we have

$$\mathscr{P}(1) \approx e^{-Q_D}k^2t^2\int_q a_\parallel(q)e^{-ikq} = e^{-Q_D}P_\delta(1) = \bar{P}_\delta(1) \tag{3.88}$$

Where we defined the damped density fluctuation power spectrum linearly evolved to t_1 $\bar{P}_\delta(1)$. Hence:

$$i\langle S_I\rangle(k,t) \approx -k\cdot\int_0^t dt_1\,(t-t_1)f(1) * \bar{P}_\delta(1) \tag{3.89}$$

The general expression for the non linear power spectrum in mean-field approximation is finally:

$$\mathscr{P}(k,t) = e^{i\langle S_I\rangle - Q_D}\int_q \left[e^{t^2k^2a_\parallel(q)} - 1\right]e^{-ikq}. \tag{3.90}$$

The last passage is to find a closed expression for the mean interaction term, starting from a model of gravitational interaction potential.
A physically motivated shape in Fourier space for the gravitational potential (see [9]) is:

$$\tilde{v}(k) = -\frac{A_\varphi}{\bar{\rho}(k_0^2 + k^2)} \tag{3.91}$$

The term k_0, that gives the shape of a Yukawa propagator, represents the scale value at which non-linear evolution sets on.
The amplitude A_φ comes from the definition of the potential φ, namely:

$$\varphi = \phi - A_\varphi D_+ \psi \tag{3.92}$$

Then, since the spatial gradient is mapped to Fourier space as $\nabla \longmapsto i\vec{k}$, we can express the convolution in equation (3.87) as

$$\vec{k} \cdot (\nabla \tilde{v} * \bar{P})(k) = -\frac{iA_\varphi}{\bar{\rho}} \int_{k'} \frac{\vec{k} \cdot (\vec{k} - \vec{k}')}{k_0^2 + (\vec{k} - \vec{k}')^2} \bar{P}(k') \tag{3.93}$$

The integral can be simplified introducing spherical polar coordinates, with the polar axis chosen as the direction of $\vec{k}$ and the calling μ the cosine of the angle defined by $\vec{k}$ and $\vec{k}'$ and introducing the new variables $y = k'/k$ and $y_0 = k_0/k$. The result is

$$\vec{k} \cdot (\nabla \tilde{v} * \bar{P})(k) = -\frac{iA_\varphi}{\bar{\rho}} \frac{k^3}{(2\pi)^2} \int_0^\infty dy\, y^2 \bar{P}(ky, t') J(y, y0) \tag{3.94}$$

with

$$\begin{aligned} J(y, y_0) &= \int_{-1}^{+1} d\mu \, \frac{1 - y\mu}{1 + y_0^2 + y^2 - 2ky\mu} = \\ &= 1 + \frac{1 - y^2 - y_0^2}{4y} \ln \frac{y_0^2 + (1+y)^2}{y_0^2 + (1-y)^2} \end{aligned} \tag{3.95}$$

Given the damped initial power spectrum $\bar{P}^{(i)}$, it is convenient to define the moment

$$\sigma_J^2 = \frac{k^3}{(2\pi)^2} \int_0^\infty dy\, y^2 \bar{P}^{(i)}(ky) J(y, y_0) \tag{3.96}$$

which allows to write

$$\vec{k} \cdot (\nabla \tilde{v} * \bar{P})(k) = -\frac{iA_\varphi}{\bar{\rho}} D_+^2 \sigma_J^2 = -\frac{i\dot{m}}{\bar{\rho} m} D_+^2 \sigma_J^2. \tag{3.97}$$

In the second passage we plug directly the expression of A_φ in terms of the functions m and $\dot{m} = 3aD_+/2m$.
At the end we get for the average interaction term the closed expression

$$i\langle S_I \rangle(k, t) = 2 \int_0^t dt'(t - t') \frac{\dot{m}}{m} \Big[D_+ \sigma_J^2 - \frac{1}{m} \int_0^{t'} d\bar{t} \dot{m} D_+ \sigma_J^2 \Big]. \tag{3.98}$$

This formula will be the starting point for the discussion of the next chapter, in which the functional dependence of this interaction term with respect to some important gravity-related parameters will be investigated.

Chapter 4

Functional variation of non-linear power spectrum

4.1 Gravitational parameters

In the previous chapter we discussed the KFT formalism and its application to cosmic structure formations. In particular, we concluded with the derivation of the mean-field interaction term:

$$\begin{aligned} i\langle S_I\rangle(k,t) &= 2\int_0^t dt'(t-t')\frac{\dot{m}}{m}\left[D_+\sigma_J^2 - \frac{1}{m}\int_0^{t'} d\bar{t}\dot{m}D_+\sigma_J^2\right] = \\ &= 2\int_0^t dt'(t-t')f(t'). \end{aligned} \tag{4.1}$$

Let's point out some mathematical aspects of this object. As it is clear, it is, first of all, an ordinary function of time and scale. However, we are now interested in the hidden functional dependence, which is related to the choice of a gravity theory.
In particular, it turns out that this choice only affects (4.1) through the cosmic expansion function $E(a)$ and the effective gravitational coupling $G(a)$.
Since cosmological observations constrain eventual deviation from General Relativity to be very small, we expect these two functions to depart just slightly from the ΛCDM background values.
This point is fundamental, as it allows to provide a mathematical description of such deviations which is only partially dependent on the gravity theory itself. In other words, we are able to perform a functional Taylor expansion of the non-linear density power spectrum around these two parameters, where the functional derivatives are evaluated in the GR values and are, then, independent of the theory itself.
Note that we have chosen the time coordinate as $t = D_+ - 1$, so, given the variations $E \to E + \delta E$, $G \to G + \delta G$, it will experience a variation δt which can be expressed through the functional derivatives of $D_+(t)$ with respect to $E(t)$ and $G(t)$. The integrated time variables t' and $\bar{t}$ are however unaffected. This happens because the physical effect of a small change in the gravitational theory is just a shift δt in the evolution of the interaction operator.

4.2 Functional derivatives

Given a generic functional $F[\phi]$ and the perturbation $\phi \longmapsto \phi + \delta\phi$, the functional variation reads:

$$\delta F = F[\phi + \delta\phi] - F[\phi] = \int dy \left(\frac{\delta F}{\delta\phi(y)}\right)\delta\phi(y) + O(\delta\phi^2). \tag{4.2}$$

This formula defines the functional derivative of F with respect to ϕ, $\delta F/\delta\phi$. It has all the standard properties of derivatives, like Leibnitz and chain rule, properly generalized.

Also, defining the identity functional $I_d : \phi(y) \longmapsto \phi(x)$ as

$$I_d[\phi] = \phi(x) = \int dy\, \delta_D(y-x)\phi(y) \tag{4.3}$$

one notices that the following fundamental identity holds:

$$\frac{\delta\phi(x)}{\delta\phi(y)} = \delta_D(x-y). \tag{4.4}$$

Equation (4.2) can be applied, as anticipated, to the non-linear power spectrum of a generic alternative theory of gravity, in order to Taylor expand it around the General Relativity values of $E(a)$ and $G(a)$.

$$P^{nl}_{\delta,\,\mathrm{Alt}}(t,k) \approx P^{nl}_{\delta}\Big|_{GR}(t,k) + \int dx\, \frac{\delta P^{nl}_{\delta}}{\delta E(x)}\Big|_{GR} \delta E(x) + \int dx\, \frac{\delta P^{nl}_{\delta}}{\delta G(x)}\Big|_{GR} \delta G(x) \tag{4.5}$$

where δE and δG are the only ingredients which depend specifically on the theory of gravity we are considering, since all functional derivatives are evaluated in the GR-predicted values.

Let's start with the variation of the cosmic growth factor $D_+(a)$ with respect to the gravitational parameters $E(a)$ and $G(a)$. In both cases, we are not able to derive the explicit functional dependence, but we can exploit the linear density perturbation evolution equation for directly computing the functional derivative (see [36]). We will explain the procedure for $E(a)$ and the one for $G(a)$ will be fully analogous.

Since $\delta/\delta E$ commutes with the derivative with respect to the scale factor a, we can apply it to equation (1.34) and solve it for the functional derivative:

$$\frac{\delta}{\delta E(x)}\left[D''_+(a) + \left(\frac{3}{a} + \frac{E'(a)}{E(a)}\right)D'_+(a) - \frac{3\Omega_m(a)}{2a^2}D_+(a)\right] = 0 \tag{4.6}$$

Performing the functional derivative, one gets:

$$\begin{aligned}\frac{d^2}{da^2}\frac{\delta D_+(a)}{\delta E(x)} + \left(\frac{3}{a} + \frac{d\ln E(a)}{da}\right)\frac{d}{da}\frac{\delta D_+(a)}{\delta E(x)} - \frac{3\Omega_m(a)}{2a^2}\frac{\delta D_+(a)}{\delta E(x)} &= \\ = -\frac{d}{da}\left(\frac{\delta_D(a-x)}{E(a)}\right)D'_+(a) - \frac{3D_+(a)\Omega_m(a)}{a^2E(a)}\delta_D(a-x).&\end{aligned} \tag{4.7}$$

It is reasonable to introduce the ansatz

$$\frac{\delta D_+(a)}{\delta E(x)} = C(a,x)D_+(a) \tag{4.8}$$

and the causality condition

$$\frac{\delta D_+(a)}{\delta E(x)} \propto \Theta_H(a-x) \tag{4.9}$$

where Θ_H is the Heaviside step function.

Given these assumptions, equation (4.7) is solved by

$$\frac{\delta D_+(a)}{\delta E(x)} = D_+(a)g(x)\Gamma(x,a) \tag{4.10}$$

with

$$g(x) = xD_+^2(x)\Omega_m(x)\left[\Omega_m^{2\gamma-1}(x) - \frac{3}{2}\right] \tag{4.11}$$

and

$$\Gamma(x,a) = \Theta_H(a-x)\int_x^a \frac{dy}{y^3D_+^2(y)E(y)}. \tag{4.12}$$

The exponent γ in equation (4.15) is implicitly given by the logarithmic derivative

$$\frac{d\ln D_+(a)}{d\ln a} = \Omega_m^\gamma(a). \tag{4.13}$$

A similar procedure can be applied for computing the functional derivative of D_+ with respect to the gravitational coupling [33].
The final result is:

$$\frac{\delta D_+(a)}{\delta G(x)} = D_+(a) f(x) \Gamma(x,a) \tag{4.14}$$

with

$$f(x) = \frac{3x D_+(x)^2 E(x) \Omega_m(x)}{2G(x)}. \tag{4.15}$$

Equipped with (4.10) and (4.14) we are able to compute all the variations we need with respect to these two functions.

4.3 The non-linear power spectrum

Consider now the variation $E \to E + \delta E$, $G \to G + \delta G$, which produces $t[E,G] \to t[E,G] + \delta t[E,G]$ and assume they are all of order $\epsilon \ll 1$.
Using equation (3.81) and (3.82) we can compute the corresponding variation of the non-linear power spectrum:

$$P_\delta^{nl}[E + \delta E, G + \delta G] = P_\delta^{nl}[E,G] + \delta P_\delta^{nl} \tag{4.16}$$

with

$$\begin{aligned} \delta P_\delta^{nl} &= e^{i\langle S_I\rangle} \delta P_\delta^{lin} + \delta\big(e^{i\langle S_I\rangle}\big) P_\delta^{lin} = \\ &= e^{i\langle S_I\rangle} P_\delta^{(i)} D_+ \big(2\delta D_+ + D_+ \delta(i\langle S_I\rangle)\big) + O(\epsilon^2). \end{aligned} \tag{4.17}$$

where the first term δD_+ can be obtained integrating equation (4.10) and (4.14) according with the definition of functional derivative (4.2).
The second term needs, on the other hand, some passages.
The functions $m(t')$, $\dot m(t')$ change accordingly as an effect of the variation of E and G.

$$\begin{aligned} m[E + \delta E, G + \delta G] &= (a + \delta a)^3 (D'_+ + \delta D'_+)(E + \delta E) = \\ &= a^3 D'_+ E + 3a^2 D'_+ E\,\delta a + a^3 E\,\delta\left(\frac{dD_+}{da}\right) + a^3 D'_+ \delta E + O(\epsilon^2) = \\ &= m[E,G] + \delta m\,. \end{aligned} \tag{4.18}$$

One may immediately notice that there are two terms involving respectively the variation of the scale factor and the variation of the derivative of the cosmic growth factor w.r.t. the scale factor. We have assumed that the time t' which appears as integration variable is not directly varied as a consequence of the perturbation of E and G. What varies is, however, the way the scale factor depends on t', since the ordinary dependence which maps $t' \longmapsto a(t')$ is functionally mediated by E and G. The complete expression for δm can be therefore expressed as:

$$\begin{aligned} \delta m = \Bigg(&3a^2\,E \frac{dD_+}{da}\frac{da}{dt'}\frac{\delta D_+}{\delta E} + a^3\,E \frac{d}{da}\frac{\delta D_+}{\delta E} + \\ &+ a^3 D'_+\Bigg)\delta E + \left(3a^2\,E \frac{dD_+}{da}\frac{da}{dt'}\frac{\delta D_+}{\delta G} + a^3\,E\frac{d}{da}\frac{\delta D_+}{\delta G}\right)\delta G = \\ = &\left(3a^2\,E\frac{\delta D_+}{\delta E} + a^3\,E\frac{d}{da}\frac{\delta D_+}{\delta E} + a^3 D'_+\right)\delta E + \\ &+ \left(3a^2\,E\frac{\delta D_+}{\delta G} + a^3\,E\frac{d}{da}\frac{\delta D_+}{\delta G}\right)\delta G\,. \end{aligned} \tag{4.19}$$

The functional derivatives of D_+ derived w.r.t. a can be further processed using equations (4.10) and (4.14):

$$\begin{aligned}\frac{d}{da}\left(\frac{\delta D_+}{\delta E}\right) &= \frac{d}{da}\left(\Theta_H(a-x)\,D_+(a)g(x)\int_x^a \frac{dy}{y^3D_+^2(y)E(y)}\right) = \\ &= \delta_D(a-x)\,D_+(a)g(x)\int_x^a \frac{dy}{y^3D_+^2(y)E(y)} + \\ &+ \Theta_H(a-x)\,\frac{dD_+(a)}{da}g(x)\int_x^a \frac{dy}{y^3D_+^2(y)E(y)} + \\ &+ \Theta(a-x)\,D_+(a)g(x)\frac{1}{a^3D_+^2(a)E(a)}\,.\end{aligned} \tag{4.20}$$

Observe that the first term vanishes since the Dirac Delta is $\neq 0$ only for $x = a$ and this force the two extrema of the integral to coincide. Then:

$$\begin{aligned}\frac{d}{da}\left(\frac{\delta D_+}{\delta E}\right) = \Theta_H(a-x)g(x)\Bigg[&\frac{dD_+(a)}{da}\int_x^a \frac{dy}{y^3D_+^2(y)E(y)} + \\ &+ D_+(a)\frac{1}{a^3D_+^2(a)E(a)}\Bigg]\,.\end{aligned} \tag{4.21}$$

At this point we have the full variation δm and we can express $\delta\dot{m}$ in terms of it:

$$\begin{aligned}\dot{m}[E+\delta E, G+\delta G] &= \frac{3aD_+}{2(m+\delta m)} = \frac{3aD_+}{2m}\left(1-\frac{\delta m}{m}+O(\epsilon^2)\right) = \\ &= \frac{3aD_+}{2m} - \frac{3aD_+}{2m^2}\delta m + O(\epsilon^2) = \dot{m}[E,G] + \delta\dot{m} + O(\epsilon^2)\end{aligned} \tag{4.22}$$

Using the definition of functional derivative we have:

$$\delta m(t') = \int d\tau\left(\delta E(\tau)\frac{\delta m(t')}{\delta E(\tau)} + \delta G(\tau)\frac{\delta m(t')}{\delta G(\tau)}\right) + O(\epsilon^2) \tag{4.23}$$

which, compared with equation (4.18) and with all the substitutions yields:

$$\begin{aligned}\frac{\delta m(t')}{\delta E(\tau)} &= 3a^2(t')\,E(t')\frac{\delta D_+(t')}{\delta E(\tau)} + a^3(t')\,E(t')\frac{d}{da}\frac{\delta D_+(t')}{\delta E(\tau)} + \\ &\quad + \frac{m(t')}{E(t')}\delta_D(t'-\tau) \\ \frac{\delta m(t')}{\delta G(\tau)} &= 3a^2(t')\,E(t')\frac{\delta D_+(t')}{\delta G(\tau)} + a^3(t')\,E(t')\frac{d}{da}\frac{\delta D_+(t')}{\delta G(\tau)}\,.\end{aligned} \tag{4.24}$$

Using the same procedure for $\dot{m}$ one gets:

$$\begin{aligned}\frac{\delta\dot{m}(t')}{\delta E(\tau)} &= -\frac{3a(t')D_+(t')}{2m^2(t')}\frac{\delta m(t')}{\delta E(\tau)} = \\ &= -\dot{m}(t')\left[2\frac{a^2(t')\,E(t')}{m(t')}\frac{\delta D_+(t')}{\delta E(\tau)} + \frac{a^3(t')\,E(t')}{m(t')}\frac{d}{da}\left(\frac{\delta D_+(t')}{\delta E(\tau)}\right) + \right. \\ &\qquad \left. + \frac{\delta_D(t'-\tau)}{E(t')}\right] \\ \frac{\delta\dot{m}(t')}{\delta G(\tau)} &= -\frac{3a(t')D_+(t')}{2m^2(t')}\frac{\delta m(t')}{\delta G(\tau)} = \\ &= -\dot{m}(t')\left[2\frac{a^2(t')\,E(t')}{m(t')}\frac{\delta D_+(t')}{\delta E(\tau)} + \frac{a^3(t')\,E(t')}{m(t')}\frac{d}{da}\left(\frac{\delta D_+(t')}{\delta E(\tau)}\right)\right]\end{aligned} \tag{4.25}$$

Then, coming back to our mean interaction term, we have:

$$\begin{aligned}\langle S_I\rangle[E+\delta E, G+\delta G] = 2\int_0^{t+\delta t} dt'(t+\delta t-t')\frac{\dot m+\delta\dot m}{m+\delta m}\times \\ \times\left[D_+\sigma_J^2 - \frac{1}{m+\delta m}\int_0^{t'} d\bar t(\dot m+\delta\dot m)D_+\sigma_J^2\right] + O(\epsilon^2) = \\ = 2\int_0^{t+\delta t} dt'(t+\delta t-t')\left(\frac{\dot m}{m}+\frac{\delta\dot m}{m}-\frac{\dot m}{m^2}\delta m\right)\times \\ \times\left[D_+\sigma_J^2 - \left(\frac{1}{m}-\frac{\delta m}{m^2}\right)\int_0^{t'} d\bar t(\dot m+\delta\dot m)D_+\sigma_J^2\right] + O(\epsilon^2).\end{aligned} \tag{4.26}$$

Proceeding further with simplifications and neglecting second order terms, one obtains:

$$\begin{aligned}2\int_0^{t+\delta t} dt'(t+\delta t-t')\left[\frac{\dot m D_+\sigma_J^2}{m}+\frac{D_+\sigma_J^2}{m}\delta\dot m - \frac{\dot m D_+\sigma_J^2}{m^2}\delta m + \right.\\ -\frac{\dot m}{m^2}\int_0^{t'} d\bar t\,\dot m D_+\sigma_J^2 - \left(\frac{\delta\dot m}{m^2}-2\frac{\dot m}{m^3}\delta m\right)\int_0^{t'} d\bar t \dot m D_+\sigma_J^2 + \\ \left. -\frac{\dot m}{m^2}\int_0^{t'} d\bar t\, D_+\delta\dot m\sigma_J^2\right] = \\ = 2\int_0^t dt'\,(t-t')f(t') + 2\int_t^{t+\delta t} dt'\,(t-t')f(t') + 2\delta t\int_0^{t+\delta t} dt' f(t') + \\ + 2\int_0^{t+\delta t} dt'(t-t')\delta f = \\ = \langle S_I\rangle[E,G] + 2\delta t\int_0^t dt' f(t') + 2\int_0^t dt'(t-t')\delta f.\end{aligned} \tag{4.27}$$

Note that, in the last passage, some term has been neglected since they yields second order contributions. Namely:

$$\begin{aligned}2\int_t^{t+\delta t} dt'\,(t-t')f(t') = 2\delta t\; f(t')(t-t')\big|_{t'=t} + O(\epsilon^2) = O(\epsilon^2) \\ 2\int_t^{t+\delta t} dt'\,\delta f(t') = \delta t^2 f(t) + O(\epsilon^2) = O(\epsilon^2).\end{aligned} \tag{4.28}$$

It was also introduced the functional variation of $f(t')$, which reads:

$$\begin{aligned}\delta f(t') = \frac{D_+\sigma_J^2}{m}\delta\dot m - \frac{\dot m D_+\sigma_J^2}{m^2}\delta m - \left(\frac{\delta\dot m}{m^2}-\frac{2\dot m}{m^3}\delta m\right)\int_0^{t'} d\bar t\dot m D_+\sigma_J^2 + \\ -\frac{\dot m}{m^2}\int_0^{t'} d\bar t\, D_+\sigma_J^2\delta\dot m.\end{aligned} \tag{4.29}$$

The variations δm and $\delta\dot m$ can then be expressed in terms of δE through equation (4.18) and (4.22) and using explicit expressions (4.24) and (4.25). The full expression would be really cumbersome, so let's do some further considerations first. The functional derivatives of $m(t')$ and $\dot m(t')$ w.r.t. $E(\tau)$, compared to the ones w.r.t. $G(\tau)$ possess an extra term which is proportional to the Dirac Delta and comes from the direct dependence on the expansion function. Since we need to integrate those functions, it is worth to separate the terms containing Dirac Deltas as they are going to yield analytic expressions after the integration. On the other hand, all the remaining integrals will be computed numerically and we will refer to them as $\mathcal{N}_E(t') + \mathcal{N}_G(t')$ as they results from a contribution from the variation of E and G separetely.

The contributions from these analytical terms to δf read as follows:

$$\begin{aligned}\delta f(t') = \int_0^t d\tau\, \delta E(\tau) \Bigg\{ \delta_D(t'-\tau) \Bigg[-\frac{2\dot{m} D_+ \sigma_J^2}{mE} + \frac{3\dot{m}}{m^2 E} \int_0^{t'} d\bar{t}\, \dot{m} D_+ \sigma_J^2 \Bigg] + \\ + \frac{\dot{m}}{m^2} \int_0^{t'} d\bar{t}\, \delta_D(\bar{t}-\tau) \frac{\dot{m} D_+ \sigma_J^2}{E} \Bigg\} + \mathcal{N}(t') = \\ = \int_0^t d\tau\, \delta E(\tau) \Bigg\{ \delta_D(t'-\tau) \Bigg[-\frac{2\dot{m} D_+ \sigma_J^2}{mE} + \frac{3\dot{m}}{m^2 E} \int_0^{t'} d\bar{t}\, \dot{m} D_+ \sigma_J^2 \Bigg] + \\ + \frac{\dot{m}}{m^2}(t') \times \frac{\dot{m} D_+ \sigma_J^2}{E}(\tau) \Theta_H(t'-\tau) \Bigg\} + \mathcal{N}_E(t') + \mathcal{N}_G(t').\end{aligned} \tag{4.30}$$

The Heaviside step function has been introduced in order to extend the integration extremum to ∞ and apply properly the Dirac delta.
This result can be plugged in the last term of equation (4.27):

$$\begin{aligned}2\int_0^t dt'\,(t-t')\delta f = \int_0^t d\tau\, \delta E(\tau) \Bigg\{ (t-\tau)\Theta_H(t-\tau) \Bigg[-\frac{2\dot{m} D_+ \sigma_J^2}{mE} + \\ + \frac{3\dot{m}}{m^2 E} \int_0^{\tau} d\bar{t}\, \dot{m} D_+ \sigma_J^2 \Bigg] + \frac{\dot{m}}{m^2} \int_{\tau}^{t} dt'\,(t-t') \frac{\dot{m}}{m^2} \Bigg\} + \\ + \int_0^t dt'\,(t-t')\big(\mathcal{N}_E(t') + \mathcal{N}_G(t')\big)\,.\end{aligned} \tag{4.31}$$

Note that this last expression only depends on the time variables t and τ.
At this point we only need to expand the variation δt in equation (4.27). The second term as whole, then, reads:

$$2\delta t \int_0^t dt'\, f(t') = 2\int_0^t d\tau \left(\frac{\delta D_+(t)}{\delta E(\tau)} \delta E(\tau) + \frac{\delta D_+(t)}{\delta G(\tau)} \delta G(\tau) \right) \int_0^t dt'\, f(t'). \tag{4.32}$$

The first order variation of the average interaction term is:

$$\begin{aligned}\delta\langle S_I\rangle = \langle S_I\rangle[E+\delta E, G+\delta G] - \langle S_I\rangle[E,G] = \\ = \int d\tau \left[\frac{\delta\langle S_I\rangle}{\delta E(\tau)} \delta E(\tau) + \frac{\delta\langle S_I\rangle}{\delta G(\tau)} \delta G(\tau) \right]\end{aligned} \tag{4.33}$$

Comparing this expression with the terms computed previously, one obtains the full expressions for the functional derivatives of the average interaction operator with respect to $E(\tau)$ and $G(\tau)$:

$$\begin{aligned}\frac{\delta\langle S_I\rangle}{\delta E(\tau)} = 2\frac{\delta D_+(t)}{\delta E(\tau)} \int_0^t dt'\, f(t') + (t-\tau)\Theta_H(t-\tau) \Bigg[-\frac{4\dot{m} D_+ \sigma_J^2}{mE} + \\ + \frac{6\dot{m}}{m^2 E} \int_0^{\tau} dt'\, \dot{m} D_+ \sigma_J^2 \Bigg] + \frac{2\dot{m} D_+ \sigma_J^2}{E} \int_{\tau}^{t} dt'\,(t-t') \frac{\dot{m}}{m^2} + \\ + \int_0^t dt'\,(t-t')\mathcal{N}_E(t') \\ \frac{\delta\langle S_I\rangle}{\delta G(\tau)} = 2\frac{\delta D_+(t)}{\delta G(\tau)} \int_0^t dt'\, f(t') + \int_0^t dt'\,(t-t')\mathcal{N}_G(t').\end{aligned} \tag{4.34}$$

At this point, we have all the explicit mathematical expressions that we need for studying the variation of the non-linear power spectrum through equation (4.5) and we can move to the numerical evaluation of remaining integral functions.

4.4 Numerical evaluation

For the numerical evaluation of integrals we used the GNU Scientific Library (GSL), see [21] as a reference.
We need to perform numerically some integrals over time, which we derived in the previous section, but it is more convenient to move to the scale factor as integration variable. Since we chose time as the cosmic growth factor, which is related to scale factor through a one-to-one correspondence, we can smoothly change variable introducing the Jacobian:

$$\int dD_+(\cdots) = \int da\, \mathcal{J}(a)(\cdots) = \int da\, \frac{dD_+}{da}(\cdots) \tag{4.35}$$

It is possible to test the accuracy of the integration routine comparing the numerical results with an analytic solution.
For a generic cosmological model we can express the matter density parameter as

$$\Omega_m(a) = \frac{\Omega_{m,i}}{a^3 E^2(a)} \approx \frac{1}{a^3 E^2(a)} \tag{4.36}$$

Note that, as we are considering the matter-dominated epoch, we can safely assume the initial matter density parameter to be unitary, i.e. $\Omega_{m,i} \approx 1$.
Furthermore, in numerical evaluation, we will normalize the scale factor in units today's value. This will, of course, hold for both a and x.
In the Einstein-De Sitter model it is possible to get closed expressions for the functional derivatives of the cosmic growth factor from equations (4.10) and (4.14). In such model we have:

$$\Omega_m(a) = 1\,, \qquad E(a) = a^{-3/2}\,, \qquad D_+(a) = a \tag{4.37}$$

and, therefore:

$$\frac{\delta D_+(a)}{\delta E(x)} = \frac{ax^3}{5}\Theta_H(a-x)\left(a^{-5/2} - x^{-5/2}\right); \tag{4.38}$$

$$\frac{\delta D_+(a)}{\delta G(x)} = \frac{3ax^{3/2}}{5}\Theta_H(a-x)\left(x^{-5/2} - a^{-5/2}\right). \tag{4.39}$$

In figure (4.1) we showed the absolute relative difference between the numerical and analytical predictions for these two quantities at $x = 0.001$ as a function of a, namely:

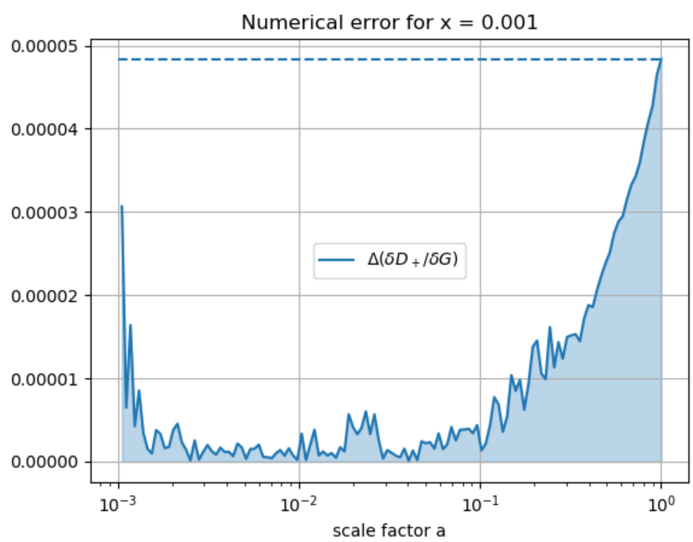

Figure 4.1: Absolute relative difference between numerical and analytic computation of functional derivatives of cosmic growth factor with respect to G for x = 0.001.

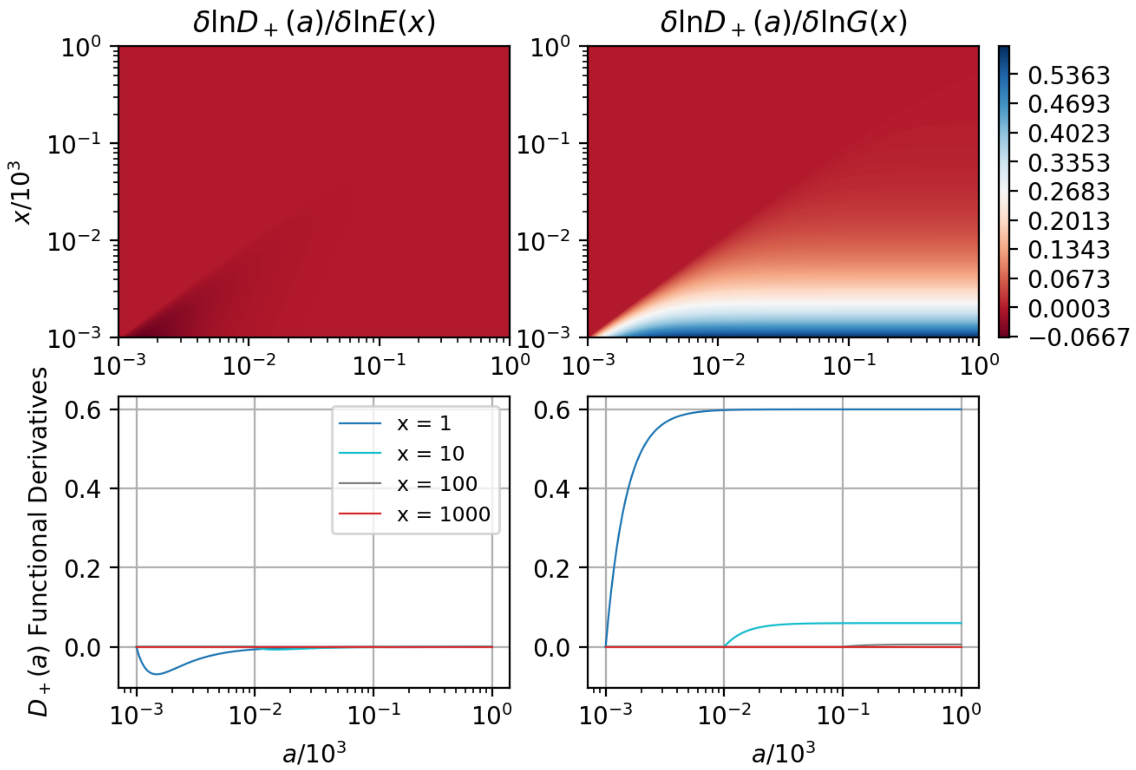

Figure 4.2: Logarithmic functional derivatives of D_+ with respect to E and G.

$$\Delta\left(\frac{\delta D_+}{\delta E, G}\right) = \left(\left.\frac{\delta D_+}{\delta E, G}\right|_{\text{analytic}} - \left.\frac{\delta D_+}{\delta E, G}\right|_{\text{numerical}}\right) \Bigg/ \left.\frac{\delta D_+}{\delta E, G}\right|_{\text{analytic}}. \tag{4.40}$$

It is evident that the error is small enough to trust the numerical integration procedure.
After this dutiful estimate of the magnitude of numerical error, we can proceed to evaluate relevant functions from previous section for the ΛCDM model.
The logarithmic functional derivatives of the cosmic growth factor with respect to the cosmic expansion function and the effective gravitational constant are represented in figure (4.2) as a function of both the ordinary scale factor a and the perturbation scale factor x. The effect of the Heaviside function, whose causal interpretation was already discussed, is here clearly evident. In fact, below the diagonal, namely for $a < x$, the functional derivatives are identically zero as a perturbation at a certain x cannot affect structure formation at earlier times.
The functional derivatives of the interaction term, are represented in figure (4.3) evaluated at the today value of scale factor $a = a_0$.
The conclusion of this chapter will be a brief note about the normalization of the power spectrum that will be adopted for the following graphic representations.
As it is evident from equation (4.17) the variation of the power spectrum produced by a deviation of E and G can be viewed as the sum of two terms. The first one is purely time dependent and its effect is a "rigid" amplitude change. The second one, which comes from the mean interaction term, is, on the other hand, scale-dependent and will the shape of the power spectrum at fixed time.
Since we are mostly interested in the shape distortion due to gravity modification, a convenient choice is to normalize the power spectrum with the amplitude of ΛCDM power spectrum at wavenumber $k = 0.01\, h\, Mpc^{-1}$ and at fixed time (or scale factor).

$$P^{nl}_{\text{alt}}(k,t) \longrightarrow \frac{P^{nl}_{\text{GR}}(0.01,t)}{P^{nl}_{\text{alt}}(0.01,t)} P^{nl}_{\text{alt}}(k,t)\,. \tag{4.41}$$

In the next chapter we will observe how this Taylor expansion method behaves with some specific alternative gravity theories.

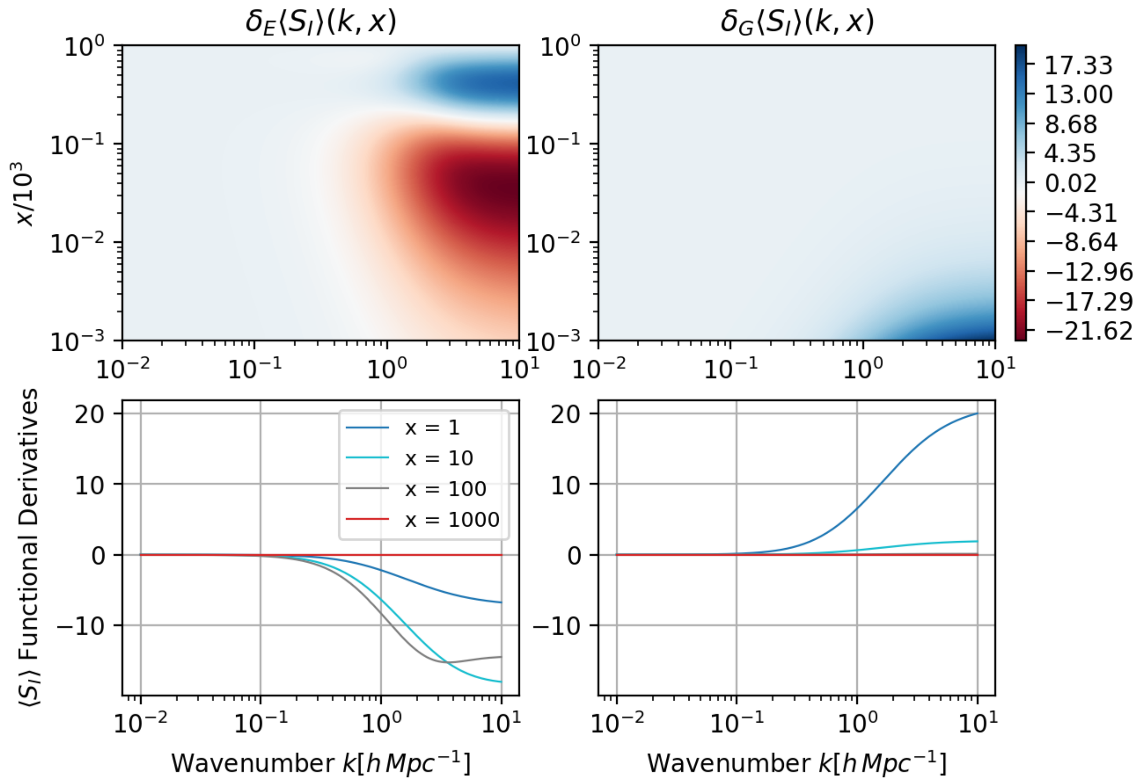

Figure 4.3: Functional derivatives of S_I evaluated at a_0 with respect to x and k.

Chapter 5

Application to specif models

5.1 Proca theory

The first order Taylor expansion method, which we discussed so far, should be now applied to a concrete alternative gravity theory.
In the second chapter a schematic overview of the different ways to extend GR was provided and we want now to focus on some of them.
The first theory that we are going to discuss is the generalized Proca theory, i.e. the most general example of vector-tensor gravity.
The choice of starting with such a complex model and then moving to simpler ones could sound unusual. Nevertheless, the non-linear KFT power spectrum has already been applied to it in an exact manner in [24] and therefore it is worth to start herewith the application of the Taylor expansion method.
After that, we will focus on scalar-tensor theories.
The Generalized Proca lagrangian can be expressed as a sum of some terms [22], namely $\mathcal{L}_P = -F^{\mu\nu}F_{\mu\nu}/4 + \sum_{i=2}^{6} \mathcal{L}_i$.
Consider a massive vector field A_μ and define the strength $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ and its dual $\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}/2$. For convenience, define also the tensor $K_{\mu\nu} = \nabla_\mu A_\nu$ (∇_μ be the usual covariant derivative).
Define also the scalars $X = -A^\mu A_\mu/2$, $F = -F^{\alpha\beta}F_{\alpha\beta}/4$ and $Y = A^\mu A^\nu F^\alpha_\mu F_{\alpha\nu}$.
The terms $\mathcal{L}_i$ read:

$$\begin{aligned}
\mathcal{L}_2 &= G_2(X, F, Y), \\
\mathcal{L}_3 &= G_3(X)[K], \\
\mathcal{L}_4 &= G_4(X)R + G_4, X[[K]^2 - [K^2]], \\
\mathcal{L}_5 &= G_5(X)G_{\mu\nu}K^{\mu\nu} - \frac{1}{6}G_{5,X}[[K]^2 - 3[K][K^2] + 2[K^3]] - g_5(X)\tilde{F}^{\alpha\mu}\tilde{F}^\beta_\mu K_{\alpha\beta}, \\
\mathcal{L}_6 &= G_6(X)L^{\mu\nu\alpha\beta}K_{\mu\nu}K_{\alpha\beta} + \frac{1}{2}G_{6,X}\tilde{F}^{\alpha\beta}\tilde{F}^{\mu\nu}K_{\alpha\beta}K_{\mu\nu}.
\end{aligned} \tag{5.1}$$

The tensor $L^{\mu\nu\alpha\beta} = \epsilon^{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\delta}R_{\rho\sigma\gamma\delta}$ is the so called double-dual Riemann tensor, while G_i and g_5 are generic functions. Note that we used the compact notation $,X$ for derivative with respect to X.
The following step will be to apply these equations to a specific cosmological model. First of all we can express the background FLRW line element in the form

$$ds^2 = -N(t)^2dt^2 + a(t)^2\vec{dx} \cdot \vec{dx}, \tag{5.2}$$

with the lapse function $N(t)$.
Due to the symmetries of this background, the vector field is forced into the configuration $A^\mu = (A^0(t), \vec{0})^T$. In fact, a non-vanishing spatial part or a spatial dependence in A^0 would shatter the homogeneity and isotropy requirement.

The time component can be chosen as $A^0(t) = \phi(t)/N(t)$. In this way we have a set of scalar degrees of freedom represented by $(N(t), a(t), \phi(t))$ whose evolution can be studied through the variational principle.
In particular, we have the usual density-evolution equation for matter content:

$$\dot{\rho}_m + 3H\rho_m = 0\,. \tag{5.3}$$

Besides this, one gets two equations from the variation of the action with respect to $g_{\mu\nu}$ and one more varying it with respect to A^μ [20]:

$$\begin{aligned} &G_2 G_{2,X}\phi^2 3G_{3,X}H\phi^3 + 6G_4 H^2 6(2G_{4,X} + G_{4,XX}\phi^2)H^2\phi^2 + \\ &\qquad + G_{5,XX}H^3\phi^5 + 5G_{5,X}H^3\phi^3 = \rho_m\,, \\ &G_2 - \dot{\phi}\phi^2 G_{3,X} + 2G_4(3H^2 + 2\dot{H}) - 2G_{4,X}\phi(3H^2\phi + 2H\dot{\phi} + 2\dot{H}\phi) + \\ &\qquad - 4G_{4,XX}H\dot{\phi}\phi^3 + G_{5,XX}H^2\dot{\phi}\phi^4 + G_{5,X}H\phi^2(2\dot{H}\phi + \\ &\qquad + 2H^2\phi + 3H\dot{\phi}) = 0 \\ &\phi(G_{2,X} + 3G_{3,X}H\phi + 6G_{4,x}H^2 + 6G_{4,x,X}H^2\phi^2 + \\ &\qquad - 3G_{5,X}H^3\phi - G_{5,X,X}H^3\phi^3) = 0\,. \end{aligned} \tag{5.4}$$

In total the system consists of four equation of motions, but only three of them are independent since we are dealing with a massive vector field which is endowed with three physical degrees of freedom. In fact, one can immediately notice that last equation is algebraic.
One trivial solution for the time component is $\phi = 0$. Let's consider however the other solution, given implicetely by:

$$\begin{aligned} G_{2,X} + 3G_{3,X}H\phi + 6G_{4,x}H^2 + &6G_{4,x,X}H^2\phi^2 \\ &- 3G_{5,X}H^3\phi - G_{5,X,X}H^3\phi^3 = 0\,. \end{aligned} \tag{5.5}$$

Once we know the time evolution of the background functions, we can introduce perturbations $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$ and $A_\mu = \bar{A}_\mu + \delta A_\mu$.
Without going into details, we directly show the effective gravitational coupling $G_{\rm eff}$ for Generalised Proca theory:

$$G_{\rm eff} = \frac{H}{4\pi\phi}\left(\frac{\mu_2\mu_3 - \mu_1\mu_4}{\mu_5}\right). \tag{5.6}$$

See [15] for the explicit expression of functions μ_i.
For the specific form of functions G_i we employed the model described in [15]. The functions are set as:

$$\begin{aligned} G_2 &= b_2 X^{p_2} + F \\ G_3 &= b_3 X^{p_3} \\ G_4 &= \frac{1}{16\pi G} + b_4 X^{p_4} \\ G_5 &= b_5 X^{p_5} \end{aligned} \tag{5.7}$$

with the exponents:

$$\begin{aligned} p_3 &= \frac{1}{2}(p + 2p_2 - 1) \\ p_4 &= p + p_2 \\ p_5 &= \frac{1}{2}(3p + 2p_2 - 1) \end{aligned} \tag{5.8}$$

and allow the solution $\phi^p \propto H^{-1}$.
It is convenient to introduce the usual density parameters and the adimentional functions:

$$y \equiv \frac{8\pi G b_2 \phi^{2p_2}}{3H^2 2^{p_2}}\,, \qquad \beta_i \equiv \frac{p_i b_i}{2^{2p_i - p2} p_2 b_2}(\phi^p H)^{i-2}\,. \tag{5.9}$$

Thanks to equation (5.5) β_3 can be expressed in terms of β_4 and β_5 through:

$$1 + 3\beta_3 + 6(2p + 2p_2 - 1)\beta_4 - (3p + 2p_2)\beta_5 = 0. \tag{5.10}$$

The dark energy density parameter reads:

$$\Omega_{DE} = 1 - \Omega_m = \frac{6p_2^2(2p + 2p_2 - 1)\beta_4 - p_2(p + p_2)(1 + 4p_2\beta_5)}{p_2(p + p_2)} y. \tag{5.11}$$

The equation of state parameter for dark energy is itself dependent on Ω_{DE} as:

$$w_{DE} = -\frac{1 + s}{1 + s\Omega_{DE}}, \tag{5.12}$$

with $s = p_2/p$.
One has then all the ingredients to compute the expansion function $E(a)$ predicted by this theory. Finally, in the examined model, the effective gravitational constant reads:

$$\frac{G_{\text{eff}}}{G} = \frac{(p + p_2)\{q_V u^2 - 2p_2 y[1 - 6\beta_4(2p + 2p_2 - 3) + 2\beta_5(3p + 2p_2 - 3)]]\}}{\mathcal{F}_G} \tag{5.13}$$

where q_V is

$$q_V = G_{2,F} + 2G_{2,Y}\phi^2 - 4g_5 H\phi + 2G_6 H^2 + 2G_{6,X} H^2\phi^2 \tag{5.14}$$

and $\mathcal{F}_G$ is a function of the parameters p, p_2, β_4, β_5 which reads

$$\begin{aligned}\mathcal{F}_G =& q_V\, u^2\{p + p_2 + 6\beta_4\, p_2\, y + p_2(p + p_2)[1 - 6\beta_4(1 + 2p + 2p_2) + \\ &+ 2\beta_5(3 + 3p + 2p_2)]\, y\} + 2p_2\, y\{(p + p_2)[-1 + 6\beta_4(2p + 2p_2 - 3) + \\ &+ \beta_5(6 - 6p - 4p_2)] + 6p_2[18\beta^4(2p + 2p_2 - 1) + \\ &- \beta_4(1 + \beta_5(30p + 28p_2 - 6)) + 6\beta_5^2(p + p_2)]y\}\,.\end{aligned} \tag{5.15}$$

Now, equipped with the cosmic expansion function and the effective gravitational coupling for this model, we are able to compute the variations δE and δG in order to apply our Taylor expansion model. In figure (5.1) the relative variation of the non-linear power spectrum for Generalised Proca Theory with respect to the ΛCDM one is shown for two models (see the parameters in the caption) and for different values of q_V.

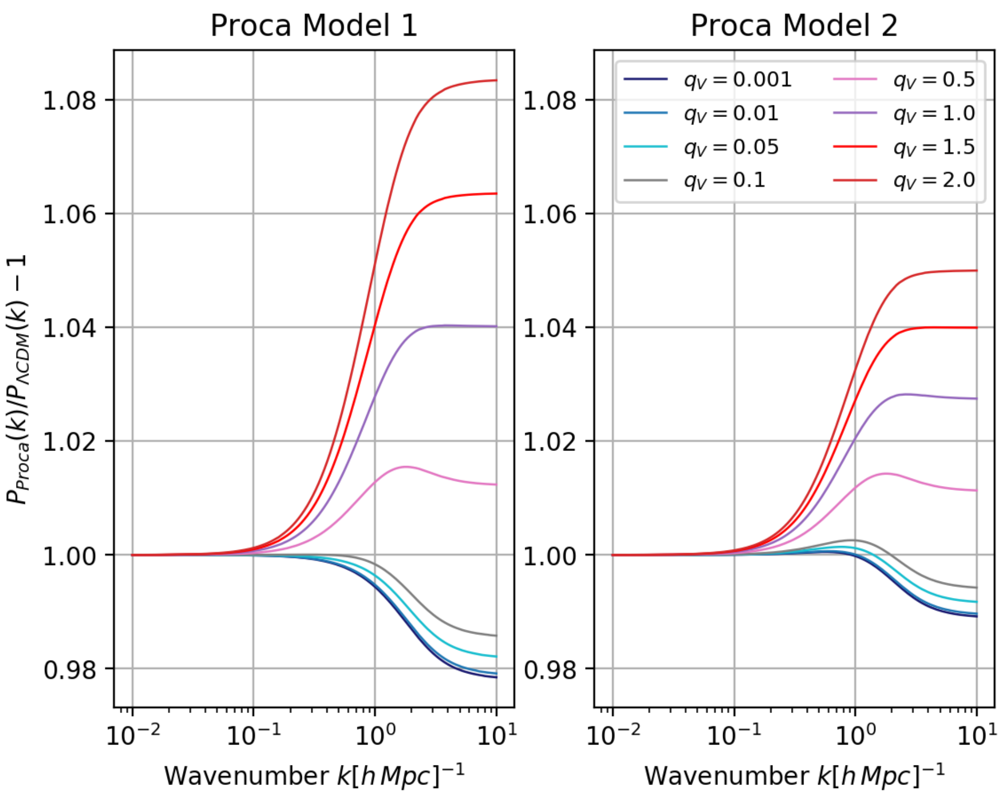

Figure 5.1: Relative non-linear power spectrum deviation for two models within Proca theory. In both model we have $p = 2.5$. $p_2 = 0.5$ and $\lambda = 0.86$, model 1 then has $\beta_4 = 10^{-4}$, $\beta_5 = 0.052$ while model 2 has $\beta_4 = \beta_5 = 0$.

5.2 Scalars in gravity: Jordan VS Einstein frame

Let's take a few steps back. Consider the following action [29] describing a generic theory of gravity which include a scalar field ϕ interacting with the metric tensor and its derivatives.

$$S = \int d^4x \sqrt{-g}\left[\frac{A(\phi)\,R}{16\pi G} - \frac{B(\phi)}{2} g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi) + \mathcal{L}_m(e^{2\alpha(\phi)}g,\,\phi)\right] \tag{5.16}$$

This can be obtained relaxing the Strong Equivalence Principle (SEP), i.e. the last and stricter statement.

However, one enjoys the freedom to simplify a bit this expression, thanks to two important symmetries: invariance of the action under scalar field redefinition and conformal transformation of the metric.

There are two main possible convenient choices. The first is to take $\alpha = 0$ and $A = \phi$. This brings to:

$$S = \int d^4x \sqrt{-g}\left[\frac{\phi\,R}{16\pi G} - \frac{B(\phi)}{2} g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi) + \mathcal{L}_m(g,\,\phi)\right]. \tag{5.17}$$

It is immediate to observe that the modification, namely the presence of the scalar field, is gathered in the purely gravitational part of the action. The matter part, on the other hand, does not contain ϕ and the matter fields follow the geodesics of Jordan metric, in absence of external forces. In this frame scalar-tensor theories can in general be reduced to $f(R)$ theories with an appropriate choice of scalar potential $V(\phi)$. Furthermore, Jordan frame is particularly suitable to test the stability of different theories and to check if they have ghosts.

The latter choice, namely the Einstein frame, consists of imposing $A = B = 1$. Then:

$$S = \int d^4x \sqrt{-\tilde{g}}\left[\frac{\tilde{R}}{16\pi G} - \frac{1}{2}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi - V(\phi) + \mathcal{L}_m(e^{2\alpha(\phi)}\tilde{g},\,\phi)\right]. \tag{5.18}$$

Unlike Jordan frame, with this choice one obtains a kinetic term for gravity which is independent on ϕ, but the gravitational interaction suffered by matter particles is mediated by the scalar field.

5.3 Scalar quintessence

In this paragraph we will apply our method to some examples of scalar-tensor theories. We will start with the simplest models in which the scalar field only interact with gravity through the scalar product in the kinetic term of the lagrangian. After that, we will move to models in which the scalar field is coupled to dark matter and finally to the most general Horndesky theories.
As anticipated, the simpler scalar field which we can introduce in a cosmological model, as we already anticipated, is the so called quintessence $\phi(x)$ and it is described by the lagrangian

$$\mathcal{L}(\phi, X) = X - V(\phi) \tag{5.19}$$

with $X = \frac{1}{2}\partial^\mu\phi\, g_{\mu\nu}\,\partial^\nu\phi$.
The equation of motion is the Klein-Gordon equation:

$$\frac{1}{\sqrt{-g}}\partial_\nu\big(\sqrt{-g}\,g^{\mu\nu}\partial_\mu\phi\big) + \frac{\partial V}{\partial\phi} = \ddot{\phi} + 3H\dot{\phi} - \frac{\nabla^2\phi}{a^2} + \frac{\partial V}{\partial\phi} = 0\,. \tag{5.20}$$

The energy-momentum tensor reads:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}(X + V) \tag{5.21}$$

Since we are now interested in the background evolution, we can safely assume our scalar field to be homogeneous, according with cosmological principle. This implies $\phi = \phi(t)$, so all the spatial derivatives of ϕ vanish.
Energy density and pressure can be computed from the components of the energy-momentum tensor as

$$\rho_\phi = -T^0_0 = X + V = \frac{\dot{\phi}^2}{2} + V\,, \qquad p_\phi = T^i_i = X - V = \frac{\dot{\phi}^2}{2} - V\,. \tag{5.22}$$

Introducing the equation of state parameter $w = p_\phi/\rho_\phi$, it is easy to observe that it is ≤ -1 for every value value of ϕ, $\dot{\phi}$.
Observe that we can use w to express $\dot{\phi}$ and $V(\phi)$ as

$$\dot{\phi}^2 = (1+w)\rho_\phi\,, \qquad 2V(\phi) = (1-w)\rho_\phi\,. \tag{5.23}$$

On the other hand, the density evolves according to the well-known continuity equation

$$\dot{\rho}_\phi = -3H(1+w)\rho_\phi \tag{5.24}$$

We we can choose a simple exponential form for the potential:

$$V(\phi) = V_0 e^{-\lambda\phi/M_p}. \tag{5.25}$$

With this choice we can write

$$\frac{\dot{V}}{V} = -\frac{\lambda\dot{\phi}}{M_p} = -\frac{\lambda}{M_p}\sqrt{\rho_\phi(1+w)} \tag{5.26}$$

but also:

$$2\dot{V} = (1-w)\dot{\rho}_\phi - \rho_\phi\dot{w} \tag{5.27}$$

Putting together these expressions, substituting $\dot{\rho}_\phi$ from the continuity equation and isolating the time derivative, we get an evolution equation for w, namely:

$$\dot{w} = -3H(1-w^2) + (1-w)\frac{\lambda}{M_p}\sqrt{3(1+w)\rho_\phi} \tag{5.28}$$

Consider now a cosmological model which only contains non-relativistic matter (made up of baryons and dark matter) and dark energy coming from the scalar field. Their evolution with the scale factor can be described as:

$$\rho_m = \rho_{m,0} a^{-3}\,, \qquad \rho_\phi = \rho_{\phi,0} f_\phi(a) \tag{5.29}$$

All the information we need can be enclosed in a system of three equations, which reads:

$$\begin{aligned} \frac{dw}{d\ln a} &= -3(1-w^2) + \frac{\lambda}{E}(1-w)\sqrt{3(1+w)\Omega_{DE0,0} f_\phi} \\ E &= \sqrt{\Omega_{m,0} a^{-3} + \Omega_{DE,0} f_\phi} \\ \frac{d\ln f_\phi}{d\ln a} &= -3(1+w) \end{aligned} \tag{5.30}$$

It is usually employed in cosmology the Chevallier-Polarski-Linder (CPL) parametrization (see for instance [30, 35]), which describes the dependence of w on the scale factor through the simple Ansatz

$$w(a) = w_0 + w_a(1-a). \tag{5.31}$$

With this choice, the scalar model is defined by the two parameters (w_0, w_a). The constant λ can be expressed then as

$$\lambda = \frac{w_a - 3(1-w_0^2)}{(1-w_0)\sqrt{3(1+w_0)\Omega_{DE,0}}} \tag{5.32}$$

The third equation of the system (5.30) is hereby directly solved, and yields

$$f_\phi(a) = a^{-3(1+w_0+w_a)} e^{3w_a(a-1)}. \tag{5.33}$$

In this simple model, the scalar field represents a form of dark energy which affects the expansion function making it deviate form the one predicted in the ΛCDM framework. No modification of the gravitational attraction among particles is however introduced, therefore the gravitational constant remains the usual one.
The application to our Taylor expansion method is rather simple. In figure (5.2) the relative deviation of non-linear power spectrum for different choices of λ is shown. The value of w_0 has been fixed to $w_0 = -0.95$.
Increasing λ above the values represented in the plot would produce huge relative deviations.

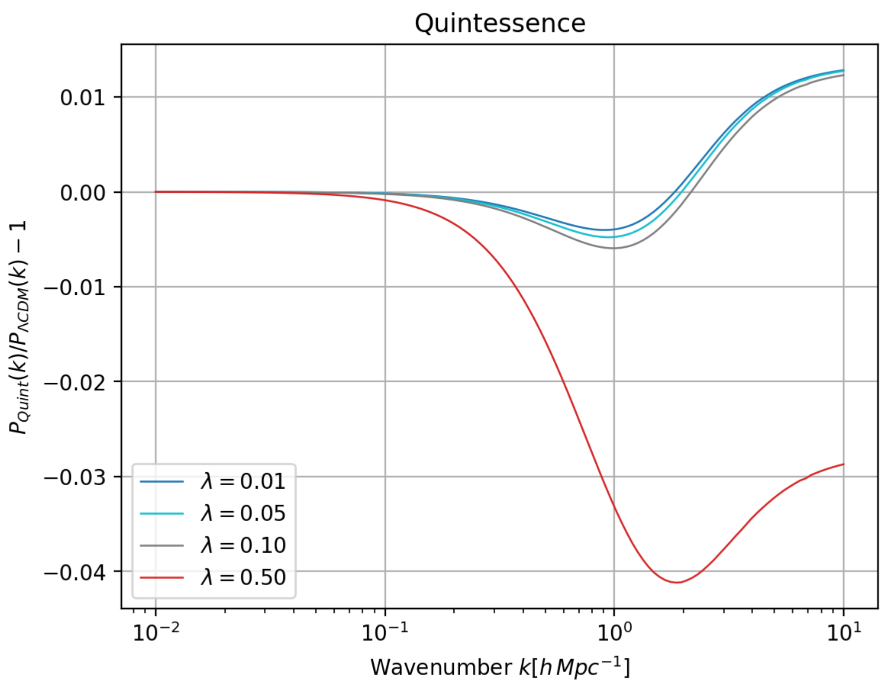

Figure 5.2: Relative non-linear power spectrum deviation for a simple scalar quintessence model with CPL parametrization for w.

5.4 Scalar interacting with dark matter

In some models the scalar quintessence can be coupled to dark matter [3, 14, 34, 37, 38] or to matter in general in the so-called coupled quintessence (CQ) [4, 5].
An example is a Yukawa-type coupling [14]

$$\mathcal{L}_{I,\mathrm{Yukawa}} \propto f(\phi/M_p)\bar{\psi}\psi \tag{5.34}$$

where ψ, $\bar{\psi}$ are, respectively a Dirac spinor and its conjugate and they describe fermionic dark matter particles.
In Einstein frame, this is realized introducing a modified metric tensor in the matter lagrangian corresponding to dark matter [38]. The new metric is related to the standard one through a transformation which depends on ϕ. The general transformation is the following:

$$\tilde{g}_{\mu\nu} = C(\phi)\, g_{\mu\nu} + D(\phi)\, \partial_\mu\phi\partial_\nu\phi\,, \tag{5.35}$$

where $C(\phi)$ is the conformal part, while $D(\phi)$ is the disformal one. Let's consider only the former. This translates into a ϕ-dependent mass of dark-matter particles. The example we will focus on is:

$$m(\phi) = m_0 e^{-\tilde{c}(\phi/M_p)}\,. \tag{5.36}$$

Thus, the density of non-relativistic dark matter can be expressed as the product of a scale factor-dependent number density and a ϕ-dependent mass, namely:

$$\rho_{DM} = n_{DM}(a)\, m(\phi) = \rho_{DM,0}\, a^{-3}\, e^{-\tilde{c}(\phi-\phi_0)/M_p}\,. \tag{5.37}$$

All the quantities with the subscript 0 are evaluated today.
Expression (5.37) can be plugged in Friedmann equation (neglect relativistic contributions):

$$3H^2M_p^2 = \rho_{b,0}\, a^{-3} + \rho_{DM,0}\, a^{-3}\, e^{-\tilde{c}(\phi-\phi_0)/M_p} + \rho_\phi\,. \tag{5.38}$$

However, the Klein-Gordon equation which rules the evolution of the scalar field must contain an additional forcing term related to this new coupling, namely

$$\begin{aligned}\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial\phi} &= -\rho_{DM,0}\, a^{-3}\, \frac{\partial}{\partial\phi}\, e^{-\tilde{c}(\phi-\phi_0)/M_p} = \\ &= \tilde{c}\, \rho_{DM,0}\, a^{-3}\, e^{-\tilde{c}(\phi-\phi_0)/M_p}\end{aligned} \tag{5.39}$$

Multiplying both sides by $\dot{phi}$ one can recognize the time derivative of the scalar energy density $\dot{\rho}_\phi = \dot{\phi}(\ddot{\phi} + \partial_\phi V)$:

$$\dot{\rho}_\phi + 3H\dot{\phi}^2 = \tilde{c}\, \rho_{DM,0}\, a^{-3}\, e^{-\tilde{c}(\phi-\phi_0)/M_p}\, \dot{\phi}\,. \tag{5.40}$$

Using now the first equation in (5.23) one can get rid of the $\dot{\phi}$ term:

$$\dot{\rho}_\phi + 3H\, \rho_\phi(1+w_\phi) = \tilde{c}\, \rho_{DM,0}\, a^{-3}\, e^{-\tilde{c}(\phi-\phi_0)/M_p} \sqrt{\rho_\phi(1+w_\phi)}\,. \tag{5.41}$$

From this last result it is clear that the usual continuity equation for the scalar energy density is not valid anymore. The physical reason is that the energy-momentum tensor for the scalar field and the one for dark matter are not conserved separately.
The fact that these two energy contributions are tangled together throughout the evolution of the universe require a definition of effective dark energy which results from scalar field and interacting dark matter:

$$\rho_{DE}^{eff} = \rho_\phi + \rho_{DM,0}\, a^{-3}\big(e^{-\tilde{c}(\phi-\phi_0)/M_p} - 1\big) \tag{5.42}$$

This picture allows to disentangle the different energy contributions, therefore dark matter will scale in the usual way, without ϕ-dependent factor:

$$\rho_{DM} = \rho_{DM,0}\, a^{-3} \tag{5.43}$$

On the other hand the effective dark energy will satisfy the homogeneous continuity equation:

$$\dot{\rho}_{DE}^{eff} + 3H(1 + w_{eff})\rho_{DE}^{eff} = 0\,. \tag{5.44}$$

Deriving (5.42) with respect to time one gets:

$$\dot{\rho}_{DE}^{eff} = -3H\left[\rho_{DM,0}\, a^{-3}\big(e^{-\tilde{c}(\phi-\phi_0)/M_p} - 1\big) + (1 + w_\phi)\rho_\phi\right], \tag{5.45}$$

which compared to (5.44), yields:

$$w_{eff} = -1 + \frac{1}{\rho_{DE}^{eff}}\left[\rho_{DM,0}\, a^{-3}\big(e^{-\tilde{c}(\phi-\phi_0)/M_p} - 1\big) + (1 + w_\phi)\rho_\phi\right]. \tag{5.46}$$

The first thing to notice, is that this w_{eff} can be < -1, unlike w_ϕ.
Let's use again for the scalar field the simple exponential potential (5.25) so that the evolution equation for w_ϕ is the same.
Passing from time to scale factor, our system reads:

$$\begin{aligned}
\frac{dw}{d\ln a} &= (1 - w_\phi)\left[-3(1 + w_\phi) + \frac{\lambda}{H}\sqrt{3(1 + w_\phi)\rho_\phi}\right] \\
\frac{d\rho_{DE}^{eff}}{d\ln a} &= -3\left[\rho_{DM,0}\, a^{-3}\big(e^{-\tilde{c}(\phi-\phi_0)/M_p} - 1\big) + (1 + w_\phi)\rho_\phi\right] \\
\frac{d\phi}{d\ln a} &= \frac{\sqrt{\rho_\phi(1 + w_\phi)}}{H} \\
\frac{d\rho_\phi}{d\ln a} &= -3(1 + w_\phi)\rho_\phi + \tilde{c}\,\frac{\rho_{DM,0}\, a^{-3}}{H}\, e^{-\tilde{c}\,(\phi-\phi_0)/M_p}\sqrt{\rho_\phi(1 + w_\phi)} \\
E(a) &= \sqrt{(\Omega_{b,0} + \Omega_{DM,0})a^{-3} + \rho_{DE}^{eff}(a)/(3H_0^2 M_p^2)}
\end{aligned} \tag{5.47}$$

We can require that today $w_{eff} = w_\phi$. This yields immediately the initial condition for the first equation. Furthermore, this condition implies $\rho_{\phi,0} = \rho_{DE,0}^{eff} = \rho_{DE,0}$. The parameters left to be set are the constant λ, the coupling $\tilde{c}$ and the value of the scalar field today ϕ_0.
From the solution of the system we obtain the alternative expansion function, i.e. the first term of our Taylor-expanded power spectrum. However, the fifth force suffered by dark matter particles appears as an effective change in the gravitational coupling. In the simple model we are considering, this simply reads:

$$G_{\text{eff}} = G(1 + 2\tilde{c}^2)\,. \tag{5.48}$$

In figure (5.3) the results for different values of the coupling constant $\tilde{c}$ are shown.
It is important to stress that in both the simple quintessence and in the coupled-quintessence model we employed for w the CPL parameterization, which is a fit holding for low redshifts. This choice is aimed to produce an alternative expansion function which does not deviate too much from the ΛCDM one, in order to safely assume $\delta E \ll 1$. The direct solution of the evolution equation would provide a trend for w which is rather far from the linearity of CPL parameterization at early times. The effect on the non-linear power spectrum would be in this case dramatic, with deviations of order ~ 10 or more from ΛCDM, regardless of values of the characteristic parameters. Such an outcome could however been interpreted as a strong constraint on this kind of theories.
As we will see in the next paragraph, a more complex model of scalar-tensor gravity could predict convincing non-linear power spectra even with the direct solution of the equations of motion and without strong assumptions on the trend of w.

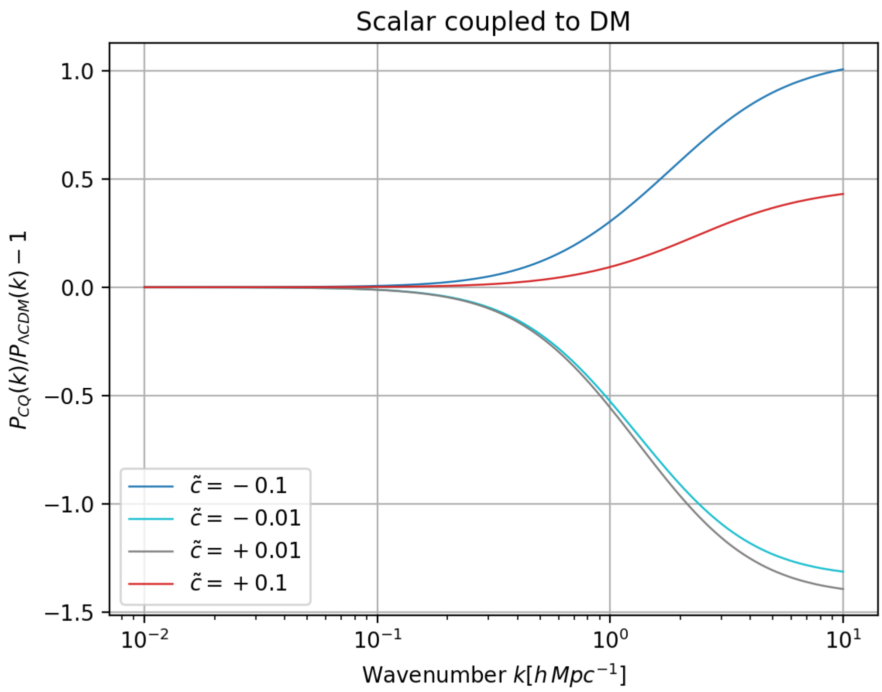

Figure 5.3: Relative non-linear power spectrum deviation for the coupled quintessence model. A value of $\lambda = 0.5$ has been set in order to see the role of scalar field-DM coupling.

5.5 Horndeski theory

As already mentioned, the Horndeski theory is the most general scalar-tensor theory with second order equation of motion. Similarly to the Proca theory, its lagrangian can be expressed as the sum of the following terms [16, 26, 29]

$$\begin{aligned}
\mathcal{L}_2 =& G_2(\phi, X) \\
\mathcal{L}_3 =& -G_3(\phi, X)\Box\phi \\
\mathcal{L}_4 =& G_4(\phi, X)\, R + G_{4,X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)\right] \\
\mathcal{L}_5 =& G_5(\phi, X)\, G_{\mu\nu}(\nabla^\mu\nabla^\nu\phi) - \frac{1}{6}G_{5,X}\left[(\Box\phi)^3 + \right. \\
&\left. - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi) + 2(\nabla^\mu\nabla_\alpha\phi)(\nabla^\alpha\nabla_\beta\phi)(\nabla^\beta\nabla_\mu\phi)\right].
\end{aligned} \tag{5.49}$$

Observe that it is easy to recover from this general action the simpler scalar-tensor theories. For instance, the choice:

$$G_2 = G_2(\phi, X), \qquad G_3 = 0, \qquad G_4 = \frac{M_p^2}{2}, \qquad G_5 = 0 \tag{5.50}$$

leads to quintessence and K-essence, where the former comes from the particular case $G_2(\phi, X) = X - V(\phi)$ [13].
Considering again the general lagrangian and expressing again the line element as in (5.2) one can take the variation w.r.t. to the lapse function $N(t)$ and the scale factor $a(t)$ [16], obtaining respectively:

$$\begin{aligned}
\sum_{i=2}^{5} \mathcal{E}_i = -\rho_m \\
\sum_{i=2}^{5} \mathcal{P}_i = 0
\end{aligned} \tag{5.51}$$

with the functions

$$
\begin{aligned}
\mathcal{E}_2 &= 2X\,G_{2,X} - G_2\,, \\
\mathcal{E}_3 &= 6X\dot{\phi}\,H\,G_{3,X} - 2X\,G_{3,\phi}\,, \\
\mathcal{E}_4 &= -6H^2\,G_4 + 24H^2\dot{X}(G_{4,X} + X\,G_{4,XX}) - 12H\,X\dot{\phi}\,G_{4,\phi X} - 6H\,\dot{\phi}G_{4,\phi}\,, \\
\mathcal{E}_5 &= 2H^3\,X\dot{\phi}(5G_{5,X} + 2X\,G_{5,XX}) - 6h^2\,X(3G_{5,\phi} + 2X\,G_{5,\phi X})\,;
\end{aligned}
\tag{5.52}
$$

and

$$
\begin{aligned}
\mathcal{P}_2 &= G_2\,, \\
\mathcal{P}_3 &= -2X(G_{3,\phi} + \ddot{\phi}\,G_{3,X})\,, \\
\mathcal{P}_4 &= 2(3H^2 + 2\dot{H})G_4 - 12H^2\,X\,G_{4,X} - 4H\,\dot{X}\,G_{4,X} - 8\dot{H}\,X\,G_{4,X} + \\
&\quad - 8H\,X\,\dot{X}\,G_{4,XX} + 2(\ddot{\phi} + 2H\dot{\phi})G_{4,\phi} + 4X\,G_{4,\phi\phi} \\
&\qquad + 4X(\ddot{\phi} - 2H\dot{\phi})G_{4,\phi X}\,, \\
\mathcal{P}_5 &= -2X\,(2H^3\dot{\phi} + 2H\,\dot{H}\,\dot{\phi} + 3H^2\ddot{\phi})\,G_{5,X} - 4H^2\,X^2\ddot{\phi}\,G_{5,XX} + \\
&\quad + 4H\,X\,(\dot{X} - H\,X)\,G_{5,\phi X} + 2[2(\dot{H}\,X + H\,\dot{X}) + 3H^2\,X]\,G_{5,\phi} + \\
&\qquad + 4H\,X\,\dot{\phi}\,G_{5,\phi\phi}\,.
\end{aligned}
\tag{5.53}
$$

The equation of motion for ϕ can be written in the elegant form

$$
a^{-3}\frac{d}{dt}(a^3\,J) = P_\phi \tag{5.54}
$$

with

$$
\begin{aligned}
J &= \dot{\phi}\,G_{2,X} + 6H\,X\,G_{3,X} - 2\dot{\phi}\,G_{3,\phi} + 6H^2\dot{\phi}(G_{4,X} + 2X\,G4, XX) + \\
&\quad - 12H\,X\,G_{4,\phi X} + 2H^3\,X\,(3G_{5,X} + 2X\,G_{5,XX}) - 6H^2\,\dot{\phi}(G_{5,\phi} + X\,G_{5,\phi X})\,, \\
P_\phi &= G_{2,\phi} - 2X(G_{3,\phi\phi} + \ddot{\phi}\,G_{3,\phi X}) + 6(2H^2 + \dot{H})\,G_{4,\phi} + \\
&\quad + 6H(\dot{X} + 2H\,X)\,G_{4,\phi X} - 6H^2\,X\,G_{5,\phi\phi} + 2H^3\,X\dot{\phi}\,G_{5,\phi X}\,.
\end{aligned}
\tag{5.55}
$$

Finally, the matter density evolution is ruled by the usual continuity equation (5.3).
We focus now on a concrete model (model D2 in [26]), given by the lagrangian:

$$
\mathcal{L} = (1 - 6Q^2)F(\phi)\,X - m^3\phi + \beta_3\,X\,\Box\phi + \frac{M_p^2}{2}F(\phi)\,R\,. \tag{5.56}
$$

with $F(\phi) = e^{-2Q(\phi-\phi_0)}$. One needs to set β_3, Q, and ϕ_0. The cosmological evolution of this model is studied extensively and even in a Beyond-Horndeski framework, in [27].
The term $\sim X\,\Box\phi$ characterises the so-called cubic Galileon, as in the limit of minkoski spacetime the equation of motion of such a model satisfies the galilean symmetry

$$
\partial_\mu\phi \longmapsto \partial_\mu\phi + b_\mu\,. \tag{5.57}
$$

For a constant four-vector b_μ.
The scalar field is non-minimally coupled to gravity through the function $F(\phi)$, which produce a fifth force, which is efficiently screened by a mechanism of chameleon type (high density suppression).
In order to solve the equations of motion for this model, it is convenient to introduce the following dimensionless variables:

$$
\begin{aligned}
x_1 &= \frac{\dot{\phi}}{\sqrt{6}M_p\,H}\,, & \Omega_{\phi 2} &= \frac{m^3\phi}{3M_p^2H^2\,F}\,, & \Omega_{\phi 3} &= -\frac{\beta_3\dot{\phi}}{M_p^2H^2\,F}\,, \\
\Omega_r &= \frac{\rho_r}{3M_p^2H^2\,F}\,, & \Omega_m &= \frac{\rho_m}{3M_p^2H^2\,F}\,, & \lambda &= -\frac{M_p}{\phi}\,.
\end{aligned}
\tag{5.58}
$$

The density parameter for matter can be however expressed in terms of the other variables thanks to the Friedmann equation:

$$\Omega_m = 1 - (1-6Q^2)x_1^2 - 2\sqrt{6}Q\,x_1 - \Omega_{\phi 2} - \Omega_{\phi 3} - \Omega_r\,. \tag{5.59}$$

The differential system, using the notation $' = d/d\ln a$, reads:

$$\begin{aligned}
x_1' &= x_1(\epsilon_\phi - h)\,,\\
\Omega_{\phi 2}' &= \Omega_{\phi 2}[\sqrt{6}(2Q-\lambda)x_1 - 2h]\,,\\
\Omega_r' &= 2\Omega_r(\sqrt{6}Qx_1 - 2 - h)\,,\\
\Omega_{\phi 3}' &= \Omega_{\phi 3}[2\sqrt{6}Qx_1 + 3\epsilon_\phi - h]\,,\\
\lambda' &= \sqrt{6}\lambda^2 x_1\,.
\end{aligned} \tag{5.60}$$

where the functions $\epsilon_\phi = \ddot{\phi}/(H\dot{\phi})$ and $h = \dot{H}/H^2$ have been defined. Their full expressions are:

$$\begin{aligned}
h = &-\{\Omega_{\phi 3}(6 + 2\Omega_r - 6\Omega_{\phi 2} + 3\Omega_{\phi 3}) + \sqrt{6}\Omega_{\phi 3}(2Q - \lambda\Omega_{\phi 2})x_1 +\\
&+ 2[3 + \Omega_r - 3\Omega_{\phi 2} + 6\Omega_{\phi 3} - 6\lambda Q\,\Omega_{\phi 2} + 6Q^2(1 - \Omega_r +\\
&+ 3\Omega_{\phi 2} - 2\Omega_{\phi 3})]x_1^2 + 2\sqrt{6}Q(6Q^2-1)(\Omega_{\phi 3} - 2)x_1^3 +\\
&+ 6(12Q^4 - 8Q^2 + 1)x_1^4\}/D\,,\\
\epsilon_\phi = &\{\sqrt{6}\Omega_{\phi 3}(\Omega_r - 3\Omega_{\phi 2} - 3) + 12[Q(\Omega_r - 1 - 3\Omega_{\phi 2} - 2\Omega_{\phi 3}) +\\
&+ \lambda\Omega_{\phi 2}]x_1 + 3\sqrt{6}[\Omega_{\phi 3} - 4 + 2Q^2(4 + \Omega_{\phi 3})]x_1^2 +\\
&+ 12Q(5 - 6Q^2)x_1^3\}/(\sqrt{6}D)
\end{aligned} \tag{5.61}$$

with

$$D = 4x_1^2 + 4\Omega_{\phi 3} + 4\sqrt{6}Qx_1\Omega_{\phi 3} + \Omega_{\phi 3}^2\,. \tag{5.62}$$

Finally, the equation of state parameter for dark energy reads:

$$\begin{aligned}
w_{DE} = &-\{3 + 2h - [3 + 2h + 3(1+2Q^2)x_1^2 - 3\Omega_{\phi 2} - \epsilon_\phi\,\Omega_{\phi 3}\\
&- 2\sqrt{6}Q\,x_1(2+\epsilon_\phi)]\}\,F\,\times\{3 - 3[1 - \Omega_{\phi 2} - \Omega_{\phi 3} +\\
&+ (6Q^2-1)x_1^2 - 2\sqrt{6}Q\,x_1]\,F\,\}^{-1}\,.
\end{aligned} \tag{5.63}$$

At this point one is able to obtain the expansion function integrating these equations.
The effective gravitational coupling reads:

$$G_{\text{eff}} = (1+2Q^2)G\,. \tag{5.64}$$

In figure (5.4) the relative variation of the non-linear power spectrum is presented. Note that more negative values of the coupling Q tend to lower the power spectrum at small scales with respect to the standard one.

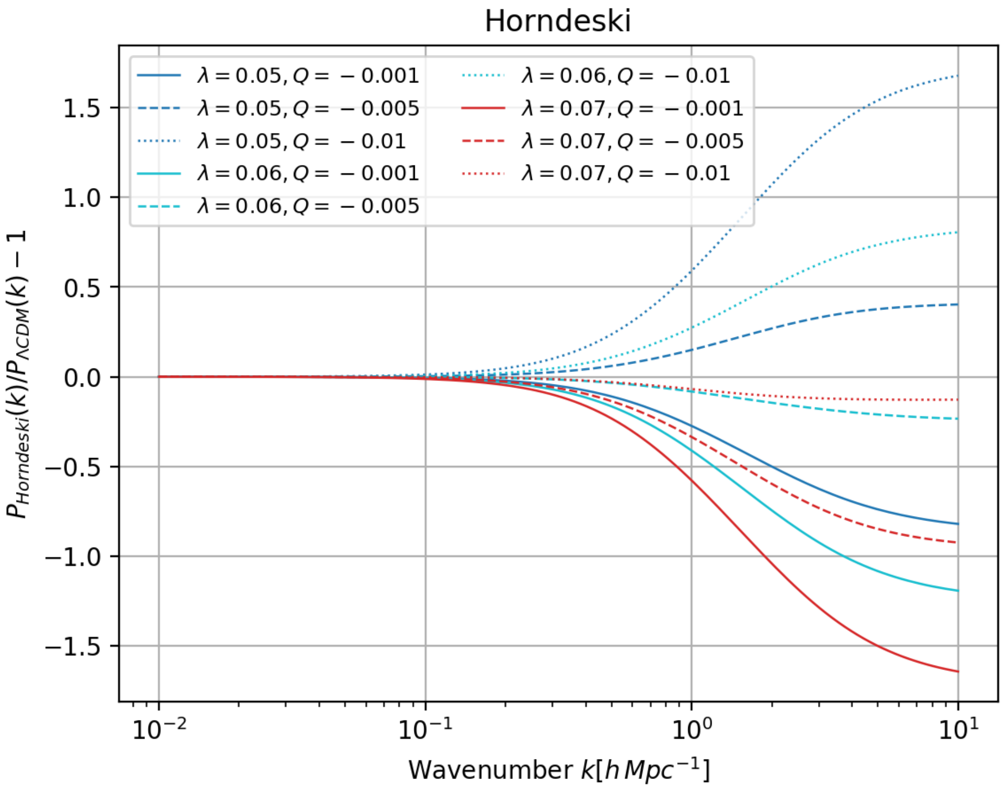

Figure 5.4: Relative non-linear power spectrum deviation for the Horndeski model discussed in this chapter. Each color is associated to a value of the initial condition λ and different traits represent different values of Q.

5.6 Conclusions and future prospects

In this last chapter some concrete applications of the KFT Taylor expansion method were presented. Even though different models were compared, some common features can be observed.
First of all, the departure of non-linear power spectra predicted by alternative gravity theories with respect to standard model always occur at $k \gtrsim 0.1 h/Mpc$, i.e. at the scale of galaxy clusters or beneath. Indeed, it is at small scales that one expects to see the most evident non-linear gravitational effects and, consequently, the possible signatures of additional degrees of freedom from alternative gravity.
Some interesting physical intuitions about the role of different parameters within a certain theory can also be inferred from the plots of this last chapter. In the Proca theory for instance, it is clear how increasing parameter q_V, which measures the magnitude of the contribution from vector modes, results in an enhanced two-point correlation. This is in agreement with [24], even though a different normalization choice was used in that work.
A peculiar behavior can be seen in the simple quintessence model, as increasing in λ seems to inhibit the correlation. This suppression is effective up to $k \sim 1.0$, while the trend is inverted at very small scales.
Something similar, but more regular is observable in Horndeski. In particular, the initial condition λ, which is inversely proportional to the scalar field, tends to suppress the correlation. On the other hand, the coupling Q enhances it.
It is finally important to stress that the first order truncation of the functional Taylor expansion would require δE, $\delta G \ll 1$. This could be not always the case, especially for E, which in a dynamic dark energy model at very early times can deviate considerably from the expansion function induced by ΛCDM. Nevertheless, the wide applicability of this method could provide interesting constraints on many alternative gravity theories and on a wide scale range.
In fact, it would be interesting to extend this method to many other models which were recently proposed in literature and try to put constraints on specific parameters.

It would be also important in the future to compare these results with the ones from numerical simulations. The application of the KFT mean interaction operator to ΛCDM non-linear power spectrum yielded a good agreement with numerical results even at small scales (see [7,9]), even though a possible underestimation of the correlation at $k \gtrsim 5h/Mpc$ has been pointed out in [7]. We have to bear in mind that, as already discussed, the KFT mean-field approach stands upon some approximations and could in principle be extended to consider for instance three-particle correlations or a different interaction potential. In general, while at large scales the matter is characterised by a rather smooth collective behavior, as we move to smaller scales, an appropriate modeling of fundamental physics becomes more and more involved.
Anyways, as we leave ΛCDM to explore alternative models, a comparison with numerical simulations could also be crucial for having a clearer idea of possible limits of the Taylor expansion approach developed so far.

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