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Photon perturbations induced by a Gravitational Wave

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Abstract

We have studied both the linear approximation and the exact form of a plane gravitational wave, which interacts with an electromagnetic wave. Using the linearized Einstein equations we calculated the changes induced on the four-potential field A^{μ} of the electromagnetic wave by describing the null geodesics followed by photons and we have identified the associated physical effects, namely phase shift, change of polarization vector, angular deflection and delay. Moreover, we have calculated the null geodesics of a space-time describing a sandwich pp-wave which is an exact solution of vacuum Einstein equations, and representing an exact plane gravitational wave. In this way we were able to identify the same physical effects calculated in the linearized theory. Finally, we have calculated the response of a Michelson laser interferometer (e.g. the LIGO and VIRGO detectors) to a linearized and exact gravitational wave.

Introduction

Gravitational waves are the radiative component of the gravitational field, propagating at the same speed of light in vacuum and originated by the rearrangements of energy/matter distributions. (Misner, Thorne and Wheeler, 1973) Their existence is not prescribed only by Einstein's theory of Gravitation, but also by any theory based on a metric tensor. On September 14th 2015 gravitational waves have finally been detected by the LIGO detectors (Abbott et al., 2016). An entire century has gone by since Einstein formulated his theory of gravitation and such a peculiar aspect of his theory has been tested with a positive result.

The purpose of this thesis is to study the linearized theory of gravitational waves, and a particular exact solution of the vacuum Einstein equations, which corresponds to the propagation of a strong gravitational wave. In particular, we study the interaction of linearized and exact gravitational waves with an electromagnetic wave with the aim of describing the response of Michelson interferometers, such as LIGO and VIRGO, to a gravitational wave. We eventually compare the two calculations, and we will show that physical effects of the linearized and exact gravitational wave on an electromagnetic wave (or equivalently the effects on null geodesics followed by photons) are the same.

The plan of this thesis is as follows.

In chapter 1 we will introduce the linearized theory of Einstein equations. We then discuss the use of a particular coordinate system, the TT-gauge to set the right framework to proceed with calculations. We must note that the TT-gauge doesn't represent a physical system that can be defined using clocks and measuring rods, but since we are interested in measurable quantities that are scalars we can calculate them in this particular gauge, where it's easier to perform calculations and then we know that the scalars obtained are the same in any coordinate system. In section 1.4 we write Maxwell equations in curved space-times. We then study how the electromagnetic wave is affected by the passage of a plane gravitational wave.

In section 1.6 we will calculate the effects that the gravitational wave causes when it encounters the electromagnetic wave. We calculate both the general situation in which the electromagnetic wave vector, polarization vector, gravitational wave vector and principal direction of the polarization tensor have arbitrary directions and a particular situation in which all expressions take a particular simple form. We identify an angle θ between the two wave vectors and other two angles ϕ and ψ which altogether give us the three Euler angles. Once the general calculation is over we study the case where $\theta = \frac{\pi}{2}$, which means that the two waves are propagating along mutually orthogonal directions. We will further assume that the electromagnetic polarization vector has only one component.

In chapter 2 we will analyze a particular exact solution of Einstein equations. This solution belongs to a particular class of plane waves, defined to be pp-waves in which the field components are the same at every point of the wave surfaces.

In section 2.1 we will mathematically describe the situation in which we are in. The space-time throughout which the strong wave runs can be divided into three regions. Region I and region III represent flat regions of space-time and region II represent the region of the wave. It will be using this fact that we will manage to describe this physical problem. Then we start calculating the geodesics that describe the path followed by photons in these three regions. We proceed writing the matching conditions between the three regions.

Section 2.3 allows us to "neglect" the wave region and connect the in-photons with the out-photons. In section 2.4 we will perform a coordinate transformation in region III since it's a property of gravitational wave to distort the metric at its passage (Landau and Lifschitz, 1976).

In section 2.6, we will study the interactions between the two waves as we did in the linearized regime and we conclude by showing that the results obtained in the exact theory reminds those of the linearized theory.

Chapter 1

Linearized Theory

1.1 Einstein Equations

Einstein equations read

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \frac{8\pi G}{c^4} T^{\mu\nu}. \tag{1.1.1}$$

It is a set of ten-second order partial derivative-equations (Landau and Lifschitz, 1976); here G is the gravitational constant and c is the speed of light; $g_{\mu\nu}$ is the metric tensor which defines the infinitesimal distance between two events occurring in the space time: $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. With the metric tensor we can construct the Christoffel symbols

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}\, g^{\alpha\delta}\left(\partial_\nu g_{\delta\mu} + \partial_\mu g_{\delta\nu} - \partial_\delta g_{\mu\nu}\right) \tag{1.1.2}$$

and the Riemann Tensor

$$R^\alpha_{\ \mu\beta\nu} = \partial_\beta \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\beta} + \Gamma^\eta_{\mu\nu}\Gamma^\alpha_{\eta\beta} - \Gamma^\alpha_{\eta\nu}\Gamma^\eta_{\mu\beta}\,, \tag{1.1.3}$$

which represents the curvature of space-time. The Ricci tensor $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$ is then obtained contracting the Riemann tensor and $R = g^{\mu\nu} R_{\mu\nu}$ is the curvature scalar. In the end we have the stress-energy tensor $T^{\mu\nu}$ that satisfies equations

$$T^{\mu\nu}{}_{;\nu} = 0, \tag{1.1.4}$$

which represent the equation of motion of the matter under the influence of the gravitational field (Hobson and others, 2006). The symbol ";" represents the covariant derivative

$$A^\mu_{;\nu} = \partial_\nu A^\mu + \Gamma^\mu_{\nu\delta} A^\delta. \tag{1.1.5}$$

It is important to show that because of Bianchi's identity we have

$$(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu})_{;\nu} = 0. \tag{1.1.6}$$

We can therefore state that the conservation of energy and momentum is implemented in Einstein equations. For this reason, it's not possible to assign a priori the sources distribution: the energy-stress tensor is one of the variables of our problem. The ten equations are sufficient to find the energy density and three of the four components of the matter's velocity (since the fourth is given by the relation $u_\mu u^\mu = 1$), in addition to six of the ten components of the metric tensor $g_{\mu\nu}$. The missing components give rise to the freedom of choice with which we can execute four coordinate transformation $x'^\mu = x'^\mu(x^\nu)$ (Landau and Lifschitz, 1976). As in electromagnetism we choose a particular gauge in general relativity we chose a particular coordinate system.

Using this gauge freedom we can impose the Hilbert-De Donder condition on $g_{\mu\nu}$ which reminds us the Lorentz condition in electromagnetism on the four-potential A^{μ}:

$$\partial_{\nu}\left(\sqrt{-g}g^{\mu\nu}\right) = 0$$

with $g = \det\left(g_{\mu\nu}\right)$. We know that the Lorentz condition does not univocally fix the potentials which are defined unless a restricted gauge transformation, so the De Donder condition fixes the coordinate system unless transformations $x'^{\mu} = x'^{\mu}(x^{\nu})$ such as

$$\Box x'^{\mu} = 0. \tag{1.1.7}$$

By taking advantage of this gauge freedom we will successfully linearize Einstein equations.

1.2 Linearized equations

We know that in the case of a null gravitational field the space-time is flat. Then, in the presence of a weak gravitational field, the space-time must remain almost flat. In this situation the metric tensor can be written as (Hobson and others, 2006; Landau and Lifschits, 1976)

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \tag{1.2.1}$$

where $\eta_{\mu\nu}$ represents a flat space-time and $h_{\mu\nu}$, with $\left|h_{\mu\nu}\right| \ll 1$, represents the perturbation on the space-time induced by a weak gravitational wave. We are using the convention $(+---)$ for $\eta^{\mu\nu}$. By using the linear approximation, neglecting all the terms $O(h^2)$, we will obtain a linearized form of Einstein equations. The radiative solutions of these equations will represent the gravitational waves. In this approximation we have:

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}\eta^{\alpha\beta}\left(\partial_{\mu}h_{\beta\nu} + \partial_{\nu}h_{\beta\mu} - \partial_{\beta}h_{\mu\nu}\right), \tag{1.2.2}$$

and similar equations hold for the Riemann tensor, the Ricci tensor and the curvature scalar. We can now write the linearized Einstein equations (here we are using the SI convention):

$$\Box h_{\mu\nu} - \partial_{\mu}\partial_{\nu}h + \partial_{\mu}\partial_{\alpha}h^{\alpha}_{\nu} + \partial_{\nu}\partial_{\alpha}h^{\alpha}_{\mu} - \eta_{\mu\nu}\Box \mathrm{h} - \eta_{\mu\nu}\partial_{\alpha}\partial_{\beta}h^{\alpha\beta} = \frac{16\pi G}{c^4}T_{\mu\nu}. \tag{1.2.3}$$

We change variables using

$$\Psi_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\,, \tag{1.2.4}$$

where $h = \eta^{\mu\nu}h_{\mu\nu}$. We obtain

$$\psi = -h\,, \tag{1.2.5}$$

$$h_{\mu\nu} = \psi_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\psi. \tag{1.2.6}$$

If we now express equation (1.2.3) in terms of $\psi_{\mu\nu}$ and use the De Donder condition, where we have $-g = -\det(g_{\mu\nu}) = 1 + h$ hence $\sqrt{-g} \cong 1 + \frac{1}{2}h$ we have

$$\partial_\nu \Psi^{\mu\nu} = 0 \tag{1.2.7}$$

and so Einstein equations become

$$\Box\Psi^{\mu\nu} = \frac{16\pi G}{c^4} T^{\mu\nu}. \tag{1.2.8}$$

The solution to the homogenous associated equation gives as a solution a linearized gravitational wave.

1.3 TT Gauge

We'll now study a special coordinate system to study in an easier way the gravitational waves. In this coordinate system the tensor $h_{\mu\nu}$ has two only independent components. This system is called TT gauge (Transverse-Traceless) and will be obtained by completely fixing the gauge by imposing a condition on the residual gauge as well (Landau and Lifschitz, 1976).
We start writing

$$\Box\Psi^{\mu\nu} = 0 \tag{1.3.1}$$

and

$$\partial_\nu \Psi^{\mu\nu} = 0. \tag{1.3.2}$$

We consider the solution for monochromatic plane waves

$$\psi_{\mu\nu} = A_{\mu\nu} e^{i\chi_\alpha x^\alpha}. \tag{1.3.3}$$

$A_{\mu\nu}$ for now, are ten arbitrary constants. If we want equation (1.3.3) to be a solution of equations (1.3.1), the following equation must be verified

$$\chi^\alpha \chi_\alpha = \chi_0^2 - |\vec{\chi}|^2 = 0. \tag{1.3.4}$$

If we apply the De Donder condition to equation (1.3.3) we have

$$A_{\mu\nu}\chi^\nu = 0, \tag{1.3.5}$$

which are four conditions to impose on the ten constants. We now have six independent constants. We can still execute a coordinate transformation $x'^\mu = x'^\mu(x^\nu)$ of residual gauge type, that satisfy the De Donder condition. We must have then

$$\Box x'^\mu = 0. \tag{1.3.6}$$

We are in the linearized theory regime, so the transformation executed must be infinitesimal, of the kind

$$x'^{\mu} = x^{\mu} + \xi^{\mu}, \tag{1.3.7}$$

where ξ^{μ}, which is of the same order of h (Hobson and others, 2006) are four functions and for them we can neglect the powers higher than the second. Following this idea the functions $\psi_{\mu\nu}$ transform in the following way:

$$\psi'_{\mu\nu} = \psi_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\xi^{\alpha}{}_{,\alpha} \tag{1.3.8}$$

and it follows

$$\partial_{\mu}\psi'^{\mu\nu} = \partial_{\mu}\psi^{\mu\nu} + \Box\xi^{\mu}. \tag{1.3.9}$$

In this way we see that with the condition

$$\Box\xi^{\mu} = 0, \tag{1.3.10}$$

the De Donder condition is satisfied by ξ^{μ}. A solution of the previous equation is

$$\xi^{\mu} = -iC^{\mu}e^{i\chi_{\alpha}x^{\alpha}}, \tag{1.3.11}$$

where C^{μ} are four arbitrary constants. Substituting equations (1.3.3) in equations (1.3.8) we obtain how the $A^{\mu\nu}$ transform using equations (1.3.7):

$$A'_{\mu\nu} = A_{\mu\nu} - C_{\mu}\chi_{\nu} - C_{\nu}\chi_{\mu} + \eta_{\mu\nu}(C^{\alpha}\chi_{\alpha}). \tag{1.3.12}$$

From equation (1.3.4) we can easily see that $\chi_0 \neq 0$ since χ_1, χ_2, χ_3 can't be all zero at the same time; thanks to this fact we can always divide for χ_0. From the transformation rule (1.3.12) of $A_{\mu\nu}$, by imposing that

$$A'_{i0} = 0, \tag{1.3.13}$$

we can fix the arbitrary constants C_i

$$C_i = \frac{1}{\chi_0}(A_{i0} - C_0\chi_i). \tag{1.3.14}$$

The De Donder condition tells us then that $A'_{00} = 0$. We managed to fix three of the four arbitrary constants C_{μ}, we still have to put the condition on C_0. We can manage to do this by saying

$$A'^{\mu}{}_{\mu} = tr(A') = 0. \tag{1.3.15}$$

From equation (1.3.12) we also have

$$C_0 = \frac{1}{4\chi_0}(-A^{\alpha}{}_{\alpha} + 2A_{00}) \tag{1.3.16}$$

We finally managed to impose 8 conditions on the arbitrary constants $A_{\mu\nu}$ so to have only two independent components. If we choose the z-axis as the propagation direction of the wave, we'll have

$$A_{13} = A_{23} = A_{33} = 0 \tag{1.3.17}$$

and the only non-zero components are $A_{12} = A_{21}$ and $A_{11} = -A_{22}$. We note that in this gauge

$$\psi = \psi^{\alpha}_{\ \alpha} = A^{\alpha}_{\ \alpha} e^{i\chi_\beta x^\beta} \tag{1.3.18}$$

and from equation (1.2.6) we have

$$h_{\mu\nu} = \psi_{\mu\nu}. \tag{1.3.19}$$

We can therefore rewrite the form that takes the perturbation of the metric in this particular coordinate system in the hypothesis of propagation along the z-axis

$$h^{TT}_{\mu\nu} = h_+(\epsilon_+)_{\mu\nu} e^{i\chi_\alpha x^\alpha} + h_\times(\epsilon_\times)_{\mu\nu} e^{i\chi_\alpha x^\alpha}, \tag{1.3.20}$$

where h_+ and $h_\times$ are the amplitudes of the two polarization states of the wave and ϵ_+ and $\epsilon_\times$ are the two tensors that represent the two states of polarization:

$$\epsilon_+ \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.3.21}$$

$$\epsilon_\times \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.3.22}$$

In the case in which the wave is not propagating along one of the TT-axis, only the $\epsilon_{\mu 0}$ components will remain zero because of equation (1.3.13) while it will remain true the following

$$^{TT}h^{\mu\nu}{}_{,\nu} = 0\ (transversality) \tag{1.3.23}$$

$$^{TT}h\ = 0\ (traceless) \tag{1.3.24}$$

$$^{TT}h_{\mu 0} = 0\ (synchronous\ gauge). \tag{1.3.25}$$

1.4 Maxwell Equations in General Relativity

We now want to rewrite Maxwell equations in a form suitable to General Relativity. To this end we will substitute the usual derivative with the covariant derivative with the form

$$A^{\mu}_{;\nu} = \partial_\nu A^\mu + \Gamma^{\mu}_{\nu\delta} A^\delta. \tag{1.4.1}$$

We remember that in special relativity Maxwell equations take the form (Landau L. and Lifschitz E., 1976)

$$F^{\mu\nu}{}_{,\nu} = -\frac{4\pi}{c} j^{\mu} \qquad (1.4.2)$$

and

$$F_{[\mu\nu,\alpha]} = 0. \qquad (1.4.3)$$

Here, the symbol "," represents the ordinary derivative:

$$F^{\mu\nu}{}_{,\nu} = \partial_{\nu} F^{\mu\nu}, \qquad (1.4.4)$$

while

$$F_{[\mu\nu,\alpha]} = \partial_{\alpha} F_{\mu\nu} + \partial_{\nu} F_{\alpha\nu} + \partial_{\mu} F_{\nu\alpha}\,, \qquad (1.4.5)$$

where $F^{\mu\nu}$ is the electromagnetic tensor

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -H_z & H_y \\ -E_y & H_z & 0 & -H_x \\ -E_z & -H_y & H_x & 0 \end{pmatrix} \qquad (1.4.6)$$

Since equations (1.4.2,3) are of the first order can be easily generalized to a curved space-time:

$$F^{\mu\nu}{}_{;\nu} = -\frac{4\pi}{c} j^{\mu} \qquad (1.4.7)$$

$$F_{[\mu\nu;\alpha]} = 0, \qquad (1.4.8)$$

and we know that from equation (1.4.4) we can deduce the existence of a four-vector A_{μ} such that

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}. \qquad (1.4.9)$$

Similarly, in a curved space-time we have

$$F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu}, \qquad (1.4.10)$$

but because of the symmetry of Christoffel symbols we have

$$A_{\mu;\nu} - A_{\nu;\mu} = A_{\mu,\nu} - A_{\nu,\mu} - \Gamma^{\delta}_{\nu\mu} A_{\delta} + \Gamma^{\delta}_{\mu\nu} A_{\delta} = A_{\mu,\nu} - A_{\nu,\mu}. \qquad (1.4.11)$$

We now want to see what condition has to obey A_μ such that equations (1.4.9) satisfies equation (1.4.7). We obtain

$$A^{\nu;\mu}{}_{\nu} - A^{\mu;\nu}{}_{\nu} = -\frac{4\pi}{c} j^\mu. \qquad (1.4.12)$$

In the flat space-time we have

$$A^{\nu,\mu}{}_{\nu} - A^{\mu,\nu}{}_{\nu} = -\frac{4\pi}{c} j^\mu, \qquad (1.4.13)$$

and at this point we use the Lorentz condition $A^\nu{}_{,\nu} = 0$. Now we know that the usual derivative commutes, and for this reason the first term of equation (1.4.13) is equal to zero and therefor we obtain

$$\Box A^\mu = -\frac{4\pi}{c} j^\mu. \qquad (1.4.14)$$

Differently from what we have in a flat space-time, in a curved space-time we have to apply to the Lorentz condition $A^\nu{}_{;\nu} = 0$ the commutative rule of covariant derivative

$$A^{\nu;\mu}{}_{\nu} = A^{\nu}{}_{;\nu}{}^{\mu} + A^\delta R^{\mu}_{\delta} = A^\delta R^{\mu}_{\delta}. \qquad (1.4.15)$$

Using this last condition Maxwell equations become

$$-A^{\mu;\nu}{}_{\nu} + A^\delta R_\delta{}^\mu = -\frac{4\pi}{c} j^\mu, \qquad (1.4.16)$$

which goes under the name of De Rahm equations.

1.5 Electromagnetic wave in the field of plane gravitational wave

We have just seen how the space-time curvature modifies Maxwell equations. We now want to study the solution of these equations having in the background a monochromatic gravitational wave. By doing so we can use the results obtained in the linearized theory. To simplify our study, we will consider a gravitational wave propagating along the z-axis polarized $+$ and an electromagnetic wave propagating along the x axis with a general polarization. We will se later that even though here we are thinking of a general polarization for the electromagnetic wave, we will find a polarization vector with only one component.

Let's first introduce the principal direction of the polarization tensor. The angle ψ can be identified with the third Euler angle that is needed to represent the gravitational wave in the reference system of the electromagnetic wave defined by the electromagnetic wave vector $\vec{k}$, the electromagnetic wave polarization vector $\vec{e}$ and $\vec{k}\times\vec{e}$. In general, we can describe a linearized gravitational wave as

$$h_{\mu\nu} = h(\epsilon)_{\mu\nu} e^{i\phi_g}, \qquad (1.5.1)$$

where

$$(\epsilon)_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos 2\psi & \sin 2\psi & 0 \\ 0 & \sin 2\psi & -\cos 2\psi & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{1.5.2}$$

$\phi_g = \chi_\mu x^\mu$ is the phase and $\rho = 2\psi$. In fact, the physical angle is ψ and 2ψ is the angle between the principal direction of polarization and the arm of the Michelson interferometer.

We get the relations

$$\chi_\mu = \partial_\mu \phi_g \qquad \epsilon_{\mu\nu}\chi^\nu = 0 \qquad \chi_\mu \chi^\mu = 0 \qquad \epsilon_{\mu\mu} = 0, \tag{1.5.3}$$

where $\chi_\mu = (\chi_0, \vec{\chi})$ is the gravitational wave vector. The second represents the transverse nature of the wave, the third tells us that the wave is propagating at the speed of light and the last one is the traceless condition.

We now remember that Einstein equations in the vacuum are

$$R_{\mu\nu} = 0, \tag{1.5.4}$$

and the equation (1.4.16) for an electromagnetic wave ($j^\mu = 0$) becomes

$$A^{\mu;\nu}{}_{\nu} = 0 \tag{1.5.5}$$

or

$$g^{\nu\lambda} A^{\mu}{}_{;\nu} = 0. \tag{1.5.6}$$

The linearized metric tensor is given by

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{1.5.7}$$

with the condition

$$g^{\nu\lambda} g_{\lambda\mu} = \delta^\nu_\mu \tag{1.5.8}$$

so that

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}. \tag{1.5.9}$$

We now write D'Alembert operator $\Box$ and we think it in the flat space-time:

$$\Box = \Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = -\eta^{\mu\nu}\partial_\mu \partial_\nu. \tag{1.5.10}$$

Using now the notion of covariant derivative and the (1.5.9) the (1.5.6) becomes

$$(\eta^{\nu\lambda} - h^{\nu\lambda})\left(\partial_\nu A^\mu + \Gamma^\mu_{\nu\alpha} A^\alpha\right)_{;\lambda} = 0, \tag{1.5.11}$$

and if we forget about the second derivative by h we get

$$(\eta^{\nu\lambda} - h^{\nu\lambda})\left(\partial_\lambda\partial_\nu A^\mu + \Gamma^\mu_{\lambda\delta}\partial_\nu A_\delta - \Gamma^\delta_{\lambda\nu}\partial_\delta A^\mu + \partial_\lambda\Gamma^\mu_{\nu\delta}A^\delta + \Gamma^\mu_{\nu\delta}\partial_\lambda A^\delta\right) = 0. \qquad (1.5.12)$$

If we use the linearized approximation in Christoffel symbols and the traceless transverse condition, we obtain

$$-\Box A^\mu - \delta^\mu_\delta h^{\nu\lambda}\partial_\nu\partial_\lambda A^\delta + \left(h^{\mu\lambda}{}_{,\delta} + h^\mu{}_{\delta,}{}^\lambda - h^\lambda{}_{\delta,}{}^\mu\right)\partial_\lambda A^\delta = 0 \qquad (1.5.13)$$

or

$$\Box A^\mu + L^\mu{}_\delta A^\delta = 0, \qquad (1.5.14)$$

where has been introduced the linear operator (Braginsky and others, 1990)

$$L^\mu{}_\delta = \delta^\mu_\delta h^{\nu\lambda}\partial_\nu\partial_\lambda + \left(h^{\mu\lambda}{}_{,\delta} + h^\mu{}_{\delta,}{}^\lambda - h^\lambda{}_{\delta,}{}^\mu\right)\partial_\lambda. \qquad (1.5.15)$$

The 4-potential vector that represents the unperturbed electromagnetic field reads

$$A^\mu_{(0)} = A_0 e^\mu e^{i\phi_e}, \qquad (1.5.16)$$

where A_0 represents the wave amplitude, e^μ is the space-like polarization vector and $\phi_e = k_\alpha x^\alpha$ is the electromagnetic wave phase with $k_\alpha = (k_0, \vec{k})$ wave vector.
In what follows these relations will be helpful:

$$k_\mu = \partial_\mu\phi_e \qquad e^\mu k_\mu = 0 \qquad e^0 = 0 \qquad e^\mu e_\mu = -1, \qquad (1.5.17)$$

where the second relation represents the transverse nature of the electromagnetic wave and the third is a convenient choice for the gauge and the last one represents the normalization of the polarization vector.
Now we know that the equation (1.5.16) is the unperturbed solution of the usual Maxwell equations with the Lorentz gauge in flat space-time

$$\Box A^\mu_{(0)} = 0 \qquad\qquad \partial_\mu A^\mu_{(0)} = 0. \qquad (1.5.18)$$

We can calculate the perturbation induce on the second term of (1.5.14) using $A^\mu_{(0)}$ instead of A^μ. We have (Braginsky and others, 1990)

$$\begin{aligned} L^\mu{}_\delta A^\delta_{(0)} = A_0 h\big[&\left(-e^\mu\epsilon^{\nu\lambda}k_\nu k_\lambda\right) + \left(\epsilon^{\mu\lambda}k_\lambda\chi_\delta e^\delta\right) + \left(\epsilon^\mu{}_\delta e^\delta\chi^\lambda k_\lambda\right) \\ &- \left(\epsilon^\lambda{}_\delta k_\lambda e^\delta\chi^\mu\right)\big]e^{i(\phi_e+\phi_g)} \end{aligned} \qquad (1.5.19)$$

or

$$L^\mu{}_\delta A^\delta_{(0)} = A_0 h b^\mu e^{i(\phi_e+\phi_g)}, \qquad (1.5.20)$$

where we see that

$$b^\mu = -(\epsilon^{\nu\lambda}k_\nu k_\lambda)e^\mu + (\epsilon^{\mu\lambda}k_\lambda)\chi_\delta e^\delta + (\epsilon^\mu{}_\delta e^\delta)\chi^\lambda k_\lambda \\ - (\epsilon^\lambda{}_\delta k_\lambda e^\delta)\chi^\mu. \tag{1.5.21}$$

Maxwell equations for the vector potential take the form

$$\Box A^\mu = -A_0 h b^\mu e^{i(\phi_e+\phi_g)}. \tag{1.5.22}$$

The solution to this equation is given by the sum of the solution of the associated homogeneous equation and a particular solution to the equation. The first one is given by the (1.5.16) while the second one has the form

$$A^\mu_{(part)} = A^\mu_{(1)} = A_0 b^\mu F e^{i\phi_e}. \tag{1.5.23}$$

The unknown function F (x^μ) will be proportional to h and will oscillate with the gravitational wave.
If we now use this hypothetical solution in equation (1.5.22) we have

$$-\partial_\nu\partial^\nu\left(A_0 b^\mu F e^{i\phi_e}\right) = A_0 b^\mu e^{i\phi_e}(-2i\partial^\nu\phi_e\partial_\nu F + \Box\mathrm{F}) \\ = -A_0 h b^\mu e^{i(\phi_e+\phi_g)}, \tag{1.5.24}$$

so that F must satisfy

$$\Box\mathrm{F} - 2ik^\nu\partial_\nu F = -he^{i\phi_g}. \tag{1.5.25}$$

The Lorentz gauge in TT is given by

$$A^\mu{}_{;\mu} = \partial_\mu A^\mu + \Gamma^\mu_{\mu\nu}A^\nu = \partial_\mu A^\mu, \tag{1.5.26}$$

since the Γ term reduces to zero for the traceless condition. If we apply the last equation to the 4-potential $A^\mu = A^\mu_{(0)} + A^\mu_{(1)}$ we obtain

$$\partial_\mu\left[A_0(e^\mu + Fb^\mu)e^{i\phi_e}\right] = A_0\left(ik_\mu e^\mu + ik_\mu b^\mu F + \partial_\mu F b^\mu\right)e^{i\phi_e} \\ = A_0\left(ik_\mu b^\mu F + \partial_\mu F b^\mu\right)e^{i\phi_e} = 0 \tag{1.5.27}$$

or

$$b^\mu\left(ik_\mu F + \partial_\mu F\right) = 0. \tag{1.5.28}$$

It is possible to demonstrate that $b^\mu k_\mu = -b^\mu\chi_\mu$ so that the (1.5.28) becomes

$$b^\mu\left(-i\chi_\mu F + \partial_\mu F\right) = 0 \tag{1.5.29}$$

and for the first of the (1.5.3) we have

$$b^\mu\left(Fe^{-i\phi_g}\right)_{,\mu} = 0. \tag{1.5.30}$$

We assume now that the region of space we are working in is much smaller than the typical wavelength of a gravitational wave. Therefore we can give less importance in equation

(1.5.28) to the terms proportional to $k\chi$ in respect to the terms k^2, where $k = |\vec{k}|$ and $\chi = |\vec{\chi}|$ By doing so the last equation becomes (Braginsky and others, 1990)

$$e^{\mu}\left(Fe^{-i\phi_g}\right)_{,\mu} = 0. \tag{1.5.31}$$

Let's remember that in our coordinate system the electromagnetic wave is propagating along the x-axis so that we have $k^0 = k^1 = k$ and we can think that the function F takes the form:

$$F = g(x)he^{i\phi_g}. \tag{1.5.32}$$

In this coordinates we have $e^3 = 0$ and so the equation (1.5.31) is automatically satisfied. If we put the equation (1.5.32) inside (1.5.25) we have an equation with the term $g(x)$ that resembles the harmonic oscillator equation. Hence it is useful to write

$$g(x) = f(x)e^{i(k+\chi\cos\theta)x} \tag{1.5.33}$$

to obtain the actual harmonic oscillator equation. By doing so the function F becomes

$$F = f(x)he^{i[\phi_g+(k+\chi\cos\theta)x]}. \tag{1.5.34}$$

Let's explicit how reads the gravitational wave in this coordinate system. We have

$$\phi_g = \chi_\mu x^\mu = \chi[ct - \sin\theta(\cos\phi x + \sin\phi y) - \cos\theta z], \tag{1.5.35}$$

where θ is the angle between $\vec{k}$ and $\vec{\chi}$ and ϕ is the angle that completes the three Euler angles altogether with ψ and θ. Let's put this expression inside the expression (1.5.34) and it reads

$$F = f(x)he^{i[\chi[ct-\sin\theta(\cos\phi x+\sin\phi y)]+(kz)]} \tag{1.5.36}$$

We remember now that in this coordinate system $k^0 = k^1 = k$ the equation (1.5.25) becomes

$$\frac{d^2 f}{dx^2} + [(k+\chi)^2 - \chi^2(\sin\theta)^{2]} f = -e^{-i(k+\chi\cos\theta)x} \tag{1.5.37}$$

and if we define

$$\Omega = k + \chi\cos\theta \qquad \Omega_0 = \sqrt{(k+\chi)^2 - \chi^2(\sin\theta)^2} \tag{1.5.38}$$

the last equation becomes

$$f'' + {\Omega_0}^2 f = -e^{-i\Omega x} \tag{1.5.39}$$

which is the harmonic oscillator equation. If we impose the initial conditions

$$f(0) = 0 \; f'(0) = 0 \tag{1.5.40}$$

the solution of (1.5.39) takes the form

$$f(x) = \frac{e^{-i\Omega x}}{2\Omega_0}\left[\frac{e^{i(\Omega-\Omega_0)x} - 1}{\Omega_0 - \Omega} + \frac{e^{i(\Omega+\Omega_0)x} - 1}{\Omega_0 + \Omega}\right] \tag{1.5.41}$$

and so we can write

$$F = f(x)he^{i(\phi_g+\Omega x)} \tag{1.5.42}$$

and using (1.5.41) we obtain

$$F = h\frac{e^{i\phi_g}}{2\Omega_0}\left[\frac{e^{i(\Omega-\Omega_0)x} - 1}{\Omega_0 - \Omega} + \frac{e^{i(\Omega+\Omega_0)x} - 1}{\Omega_0 + \Omega}\right] \tag{1.5.43}$$

1.6 Perturbations on the electromagnetic wave

The aim of this section is to calculate the effects that the gravitational wave induces on the electromagnetic wave such as: variation in amplitude, variation in phase and rotation of the polarization angle. We can therefore compare the wave as we defined it earlier, which takes the form

$$A^\mu = A_0(e^\mu + Fb^\mu)e^{i\phi_e} \tag{1.6.1}$$

with the formal expression of a perturbed wave (Braginsky and others, 1990):

$$\begin{aligned} A^\mu = A_0 &\left(1 + \frac{\delta A}{A}\right)(e^\mu + \delta e^\mu)e^{i(\phi_e+\delta\phi_e)} \\ &= \left[\left(1 + \frac{\delta A}{A} + i\delta\phi_e\right)e^\mu + \delta e^\mu\right], \end{aligned} \tag{1.6.2}$$

where the last equality has been given expanding to the first order the exponential. In this situation $\frac{\delta A}{A}$ indicates the perturbation in respect to the unperturbed potential. From this comparison we deduce

$$b^\mu(\Re(F) + i\Im(F)) = \left(\frac{\delta A}{A} + i\delta\phi_e\right)e^\mu + \delta e^\mu \tag{1.6.3}$$

and if we multiply times e^μ we obtain

$$\frac{\delta A}{A} = -(b^\mu e_\mu)\Re(F) + \Re(\delta e^\mu e_\mu) \tag{1.6.4}$$

$$\delta\phi_e = -(b^\mu e_\mu)\Im(F) + \Im(\delta e^\mu e_\mu) \tag{1.6.5}$$

Let's remember now that in this coordinate system the electromagnetic wave is propagating along the x-axis. Let's put $\rho = 0$ and a^μ is the versor that completes the spatial tern in this system. Moreover we assume that e^μ has only the z-component. We also choose $\theta = \frac{\pi}{2}$ and ϕ is kept generic. We have

$$e^{\mu} = (0,0,0,1) \; a^{\mu} = (0,0,1,0) \; k^{\mu} = (k,k,0,0) \; \delta e^{\mu} = (\delta e^{0}, \delta e^{1}, \delta e^{2}, \delta e^{3}) \tag{1.6.6}$$

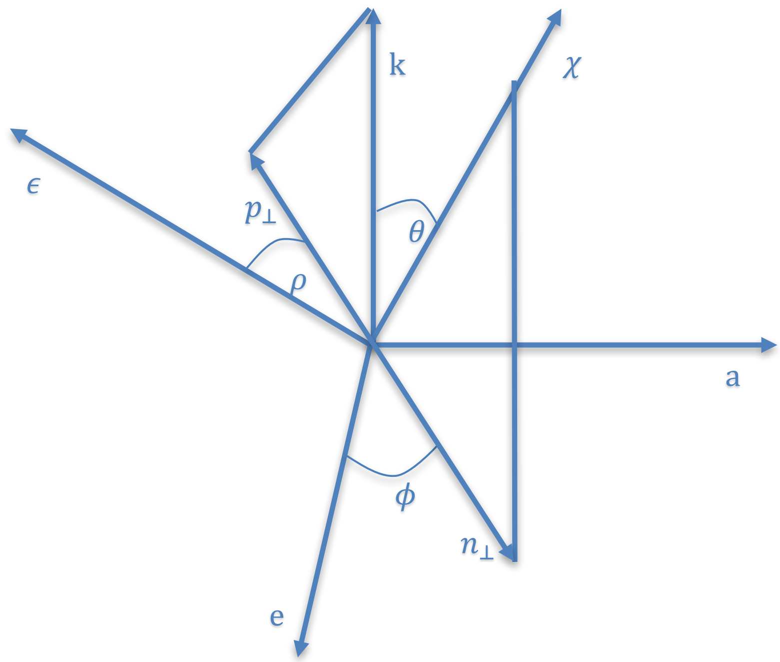


Figure 1.6.1: $\vec{k}$ and $\vec{\chi}$ are respectively the electromagnetic and gravitational wave vectors, $\vec{e}$ is the polarization vector, $\vec{a}$ is the vector orthogonal to $\vec{k}$ and $\vec{e}$, $\vec{n}_\perp$ is the unitary projection of $\vec{\chi}$ in the plane orthogonal to $\vec{k}$, $\vec{p}_\perp$ is the unitary projection of $\vec{k}$ in the plane orthogonal to $\vec{\chi}$ and $\vec{\epsilon}$ is the principal direction of the polarization tensor.

If we multiply the equation (1.6.3) times a_μ we have

$$b^{\mu} a_{\mu} (\Re(F) + i\Im(F)) = \Re(\delta e^{\mu} a_{\mu}) + i\Im(\delta e^{\mu} a_{\mu}) \tag{1.6.7}$$

and so we have

$$\delta\epsilon = -(b^{\mu} a_{\mu}) \Re(F) \tag{1.6.8}$$

Phase Shift

In the last paragraph we saw that a gravitational wave induces a phase shift on the electromagnetic wave that reads

$$\delta\phi_e = -(b^{\mu} e_{\mu}) \Im(F) + \Im(\delta e^{\mu} e_{\mu}) \tag{1.6.9}$$

This equation quantifies an electromagnetic wave phase shift that can be observed by an observer if the electromagnetic wave has propagated having in the background a gravitational wave, differently than what would have happened if we didn't have a gravitational wave at all. Staying in the TT Gauge we can give a physical explanation to this phase shift. If we write $\phi_e = k_0 x^0 + k_i x^i$ we see that $\delta\phi_e = k_0 c \Delta t$. We can conclude that in TT the phase shift is related to the delayed propagation of the electromagnetic wave caused by the gravitational perturbation on the flowing of time.

Let's calculate equation (1.6.9). Using equation (1.5.21) we obtain

$$\left(b^{\mu}e_{\mu}\right) = -\left(\epsilon^{\nu\lambda}k_{\nu}k_{\lambda}\right)e^{\mu}e_{\mu} + \left(\epsilon_{\mu\delta}e^{\mu}e^{\delta}\right)\chi^{\lambda}k_{\lambda} \tag{1.6.10}$$

and we remember that in the limit $\chi \ll k$ we neglect the second term in respect to the first one obtaining

$$\left(b^{\mu}e_{\mu}\right) = \left(\epsilon^{\nu\lambda}k_{\nu}k_{\lambda}\right) \tag{1.6.11}$$

In the hypothesis explained earlier, where we have the electromagnetic wave propagating along the x-axis and putting the $\epsilon_{0\mu} = 0$ as we are in TT gauge the last equation becomes

$$\left(b^{\mu}e_{\mu}\right) = k^2\epsilon_{33} = k^2\,(\sin\theta)^2(\cos\varrho\cos 2\psi - \sin\varrho\sin 2\psi) \tag{1.6.12}$$

Remembering the definition of F we have

$$\Im F = \frac{h}{2\Omega_0}\left[\frac{\sin\left[\phi_g + (\Omega - \Omega_0)x\right] - \sin\phi_g}{\Omega_0 - \Omega} + \frac{\sin\left[\phi_g + (\Omega + \Omega_0)x\right] - \sin\phi_g}{\Omega_0 + \Omega}\right] \tag{1.6.12}$$

In the limit $\chi \ll k$ in equations (1.5.38) we obtain:

$$\begin{gathered}\Omega_0 = \sqrt{k^2 + 2k\chi} = k + \chi\\ \Omega_0 - \Omega = \chi(1 - \cos\theta)\\ \Omega_0 + \Omega = 2k + \chi(1 + \cos\theta)\end{gathered} \tag{1.6.13}$$

In this limit we have

$$\begin{aligned}\Im F = {} & \frac{h}{2(k + \chi)}\left[\frac{\sin[\phi_g + \chi(\cos\theta - 1)x] - \sin\phi_g}{\chi(1 - \cos\theta)}\right.\\ & \left. + \frac{\sin[\phi_g + (2k + \chi(1 + \cos\theta))x] - \sin\phi_g}{2k + \chi(1 + \cos\theta)}\right]\\ = {} & \frac{h}{2k}\frac{\sin[\phi_g + \chi(\cos\theta - 1)x] - \sin\phi_g}{\chi(1 - \cos\theta)},\end{aligned} \tag{1.6.14}$$

where we neglected the terms of order $\frac{1}{k+\chi}$ with respect to the terms $\frac{1}{\chi}$. So now we are able to calculate the first of equation (1.6.9):

$$\begin{aligned}&\left(b^{\mu}e_{\mu}\right)\Im(F)\\ &= (h_{+}\cos 2\psi)\frac{k\,(\sin\theta)^2}{2}\frac{\sin[\phi_g + \chi(\cos\theta - 1)x] - \sin\phi_g}{\chi(1 - \cos\theta)}\end{aligned} \tag{1.6.15}$$

Let's evaluate the scalar product $\delta e^{\mu} e_{\mu}$ that appears in equation (1.6.9), we use the relations expressed in equation (1.6.6) and the following:

$$e^{\mu} + \delta e^{\mu} = (\delta e^0, \delta e^1, \delta e^2, 1 + \delta e^3) \tag{1.6.16}$$

Using the fact that the polarization vector has to be unitary we have

$$\begin{aligned} -1 &= (e^{\mu} + \delta e^{\mu})(e^{\nu} + \delta e^{\nu})\left(\eta_{\mu\nu} + h_{\mu\nu}\right) \\ &= -1 + 2\eta_{\mu\nu} e^{\mu} \delta e^{\nu} + h_{\mu\nu} e^{\mu} e^{\nu} \end{aligned}$$

hence

$$\delta e^1 = \frac{1}{2}^{TT} h_{11}$$

$$\delta e^{\mu} e_{\mu} = -\frac{1}{2}^{TT} h_{11}. \tag{1.6.17}$$

In the limit $\frac{k}{\chi} \gg 1$ the term in (1.6.17) goes with h, and we can neglect it with respect to the term (1.6.15) that goes with h$\frac{k}{\chi}$. In the end we have

$$\delta \phi_e = -(h_+ \cos 2\psi) \frac{k\,(\sin\theta)^2}{2} \frac{\sin[\phi_g + \chi(\cos\theta - 1)L] - \sin\phi_g}{\chi(1 - \cos\theta)}, \tag{1.6.18}$$

where we supposed that the electromagnetic source is placed in the origin of the coordinate system and the observer is put at a distance x=L.
We can write

$$\begin{aligned} \delta \phi_e &= -k\,(\sin\theta)^2 (h_+ \cos 2\psi) \frac{\sin\left[\frac{\chi(\cos\theta - 1)L}{2}\right]}{\chi(1 - \cos\theta)} \cos\Bigg[\phi_g \\ &\quad + \frac{\chi(\cos\theta - 1)L}{2}\Bigg] \\ &= -\frac{Lk}{2} (\sin\theta)^2 (h_+ \cos 2\psi) \frac{\sin[\eta(\cos\theta - 1)]}{\eta(1 - \cos\theta)} \cos\Bigg[\phi_g \\ &\quad + \frac{\chi(\cos\theta - 1)L}{2}\Bigg], \end{aligned} \tag{1.6.19}$$

where we introduced the parameter

$$\eta = \chi \frac{L}{2} \tag{1.6.20}$$

that represents the relation between the dimensions of a interferometer and the length of the gravitational wave.

Rotation of the polarization vector

We saw that the gravitational wave applies a rotation of the polarization vector in the plane orthogonal to the direction of the propagation of the wave.

We will calculate the effect denoted by $\delta\epsilon$ with $\psi = 0$.

We remember that

$$\delta\epsilon = -\left(b^\mu a_\mu\right)\Re F \tag{1.6.21}$$

Remembering the expression of b^μ we obtain

$$\left(b^\mu a_\mu\right) = \left(\epsilon_{\mu\lambda} a^\mu k^\lambda\right)\chi_\delta e^\delta + \left(\epsilon_{\mu\delta} a^\mu e^\delta\right)\chi^\lambda k_\lambda - \left(\epsilon_{\lambda\delta} k^\lambda e^\delta\right)\chi^\mu a_\mu, \tag{1.6.22}$$

where we used the second of expressions (1.5.17). We want to write the vector k^i in the two components parallel and orthogonal to the vector χ^i; obviously $p_\parallel^i = \frac{\vec{\chi}}{\chi}$

$$k^i = k(p_\parallel^i \cos\theta + p_\perp^i \sin\theta) \tag{1.6.23}$$

We can execute a similar decomposition to the spatial part of χ^i in two vectors $n_\parallel$ and $n_\perp$ respectively parallel and orthogonal to the vector k^i:

$$\chi^i = \chi(n_\parallel^i \cos\theta + n_\perp^i \sin\theta) \tag{1.6.24}$$

So we can finally calculate the terms in equation (1.6.22)

$$\left(\epsilon_{\mu\lambda} a^\mu k^\lambda\right) = k\epsilon_{ij} a^i (p_\parallel^j \cos\theta + p_\perp^j \sin\theta) = k \sin\theta\, \epsilon_{ij} a^i p_\perp^j, \tag{1.6.25}$$

where we used the fact that $\epsilon_{\mu 0}$ valid in TT. We note the fact that the components $p_\parallel^i$ don't contribute because of the transverse nature of the gravitational wave.

In fact from the second of (1.5.3) we have

$$\chi\left(\epsilon_{ij} p_\parallel^i\right) = \epsilon_{ij}\chi^i = 0 \tag{1.6.26}$$

Similarly

$$\chi_\delta e^\delta = e_i \chi(n_\parallel^i \cos\theta + n_\perp^i \sin\theta) = \chi \sin\theta\, e_i n_\perp^i \tag{1.6.27}$$

because the second and the third of equations (1.5.17) give

$$k\left(e_i p_\parallel^i\right) = e_i k^i = 0 \tag{1.6.28}$$

Now we calculate the second term of equation (1.6.22)

$$\begin{gathered}\left(\epsilon_{\mu\delta} a^\mu e^\delta\right) k^\lambda \chi_\lambda = \\ \left(\epsilon_{ij} a^i e^j\right)[k^0 \chi_0 + k(p_\parallel^i \cos\theta + p_\perp^i \sin\theta)\chi_i] = \left(\epsilon_{ij} a^i e^j\right) k\chi(1 - \cos\theta)\end{gathered} \tag{1.6.29}$$

From equation (1.6.22) we also have

$$\left(\epsilon_{\lambda\delta}k^{\lambda}e^{\delta}\right) = \left(\epsilon_{ij}k^{i}e^{j}\right) = k\sin\theta\,\epsilon_{ij}e^{i}p_{\perp}^{j} \tag{1.6.30}$$

At last we have

$$\chi^{\mu}a_{\mu} = \chi(n_{\parallel}^{i}\cos\theta + n_{\perp}^{i}\sin\theta)a_{i} = \chi\sin\theta n_{\perp}^{i}a_{i}, \tag{1.6.31}$$

where in the last equality we used the fact that a^i is orthogonal to k^i

$$k\left(a_{i}n_{\parallel}^{i}\right)a_{i}k^{i} = 0 \tag{1.6.32}$$

It is possible to see that (Figure 1.6.1)

$$\left(e_{i}n_{\parallel}^{i}\right) = \cos\phi\ \left(a_{i}n_{\perp}^{i}\right) = \sin\phi \tag{1.6.33}$$

From the previous equations we deduce that

$$\begin{aligned}\left(b^{\mu}a_{\mu}\right) = k\chi\,(\sin\theta)^{2}\cos\phi\left(\epsilon_{ij}a^{i}p_{\perp}^{j}\right) + k\chi(1-\cos\theta)\left(\epsilon_{ij}a^{i}e^{j}\right)\\ - k\chi\,(\sin\theta)^{2}\sin\phi\left(\epsilon_{ij}e^{i}p_{\perp}^{j}\right)\end{aligned} \tag{1.6.34}$$

It is possible to deduce from figure (1.6.1) the following components

$$p_{\perp}^{1} = -\cos\theta\cos\phi \qquad p_{\perp}^{2} = -\cos\theta\sin\phi \qquad p_{\perp}^{3} = \sin\theta \tag{1.6.35}$$

and we obtain the scalar products in equation (1.6.34)

$$\begin{gathered}\left(\epsilon_{ij}a^{i}p_{\perp}^{j}\right) = -\cos\theta\sin\phi\\ \left(\epsilon_{ij}a^{i}e^{j}\right) = \cos\phi\sin\phi\,[1+(\cos\theta)^{2}]\\ \left(\epsilon_{ij}e^{i}p_{\perp}^{j}\right) = -\cos\theta\cos\phi.\end{gathered} \tag{1.6.36}$$

And so equation (1.6.34) becomes

$$\left(b^{\mu}a_{\mu}\right) = k\chi(1-\cos\theta)\{\frac{1}{2}\ [1+(\cos\theta)^{2}]\sin 2\phi\}. \tag{1.6.37}$$

From the definition of $\delta\epsilon$ from equation (1.6.21), we have that the rotation of the polarization vector is given by

$$\delta\epsilon = \frac{1}{2}\left[\frac{h_{+}}{2}\ [1+(\cos\theta)^{2}\right]\sin 2\phi\left\{\cos[\phi_{g} + \chi(\cos\theta - 1)L\right] - \cos\phi_{g}\}, \tag{1.6.38}$$

where $\Re F$ has been deduced from equation (1.6.12) substituting to the imaginary part of the complex exponents, the real part. If we now use the parameter η defined earlier in this last equation we have

$$\delta\epsilon = \frac{\chi L}{2}\left[\frac{h_+}{2}\ [1+(\cos\theta)^2\right]\sin 2\phi\,\frac{\sin[\eta\,(1-\cos\theta)]}{\eta}\sin[\phi_g + \frac{\chi(\cos\theta - 1)L}{2}]$$
$$\approx \frac{\chi L}{2}(1-\cos\theta)\left[\frac{h_+}{2}\ [1+(\cos\theta)^2\right]\sin 2\phi\sin\phi_g, \tag{1.6.39}$$

where the last equation has been obtained by expanding considering $\eta \ll 1$.

Delay and Deflection

So far we have seen that the interaction between the electromagnetic and the gravitational wave gives two important effects i.e. phase shift and rotation of the polarization vector. Now we will discuss briefly other two effects, namely delay and deflection. In fact, the electromagnetic wave is subject to a delay in time and a change in direction because of the warping of space-time due to the presence of the gravitational perturbation.
To calculate these two terms, we remember that:

$$k_\mu = \partial_\mu \phi_e. \tag{1.6.40}$$

Earlier we calculated the phase shift $\delta\phi_e$and we can therefore calculate the delay and the deflection remembering that $k_\mu = (k_0, \vec{k})$. We will calculate this two terms in a particular situation, we are considering $\theta = \frac{\pi}{2}$ and $\phi = \frac{\pi}{2}$.
We have

$$\partial_t \delta\phi_e = -\frac{Lk}{2}h_+\sin\left[\chi(t-y) - \frac{\chi L}{2}\right]\chi, \tag{1.6.41}$$

$$\partial_y\ \delta\phi_e = \frac{Lk}{2}h_+\sin\left[\chi(t-y) - \frac{\chi L}{2}\right]\chi. \tag{1.6.42}$$

In this section we also give the expression for the phase shift using the same conventions as above, namely

$$\delta\phi_e = \frac{Lk}{2}h_+\cos[\phi_g - \frac{\chi L}{2}], \tag{1.6.43}$$

We must note now that what has been obtained is not taking in consideration the round trip of the photons. In fact, L is the length of the Michelson interferometer, therefore the round trip of the photons consists of $2L$. If we take into account that there are two photon beams, one for each arm of the interferometer, we have that the total phase shift is:

$$\delta{\phi_e}^{(tot)} = Lkh_+\cos[\phi_g - \chi L]. \tag{1.6.44}$$

Therefore the delay and the deflection of the photon beams take the form

$$\partial_t \delta{\phi_e}^{(tot)} = -Lkh_+\sin[\chi(t-y) - \chi L]\chi, \tag{1.6.45}$$

$$\partial_y\ \delta{\phi_e}^{(tot)} = Lkh_+\sin[\chi(t-y) - \chi L]\chi. \tag{1.6.46}$$

Because of how the interferometer is built the phase shift is the only measurable effect. As far as it regards the rotation of the polarization vector we keep its expression with a generic ϕ angle. We must note that this effect exists but it's not measured by the interferometers.

What has been done here describes the interaction between gravitational and electromagnetic waves in general. The results obtained can be particularized to recall some well known phenomena i.e. gravitational lensing, redshift, etc.

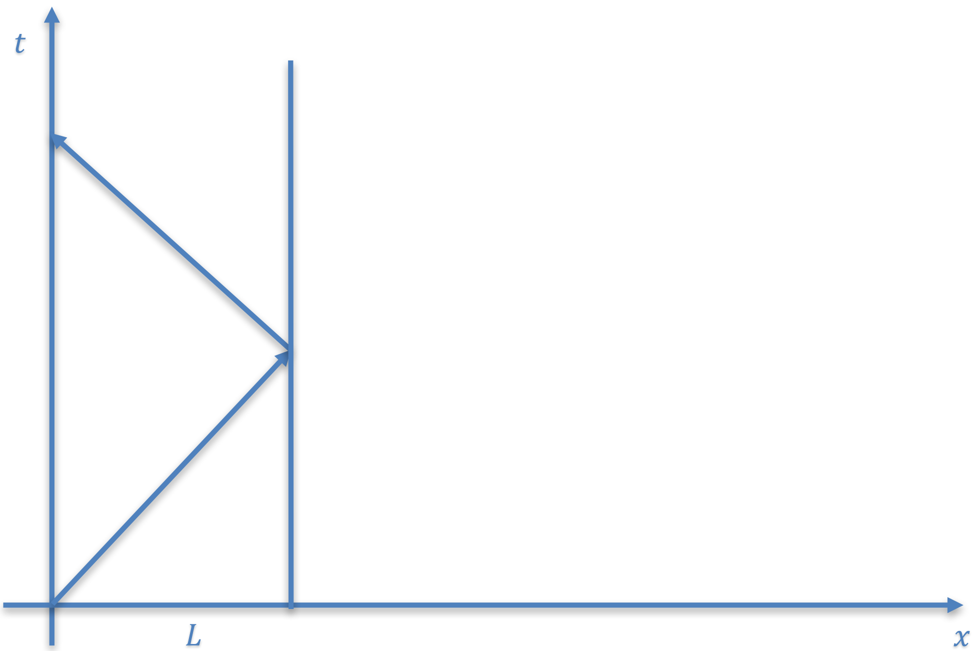


Figure 1.6.2: Round trip of the photon beams. The vertical line represents the geodetic of one mirror.

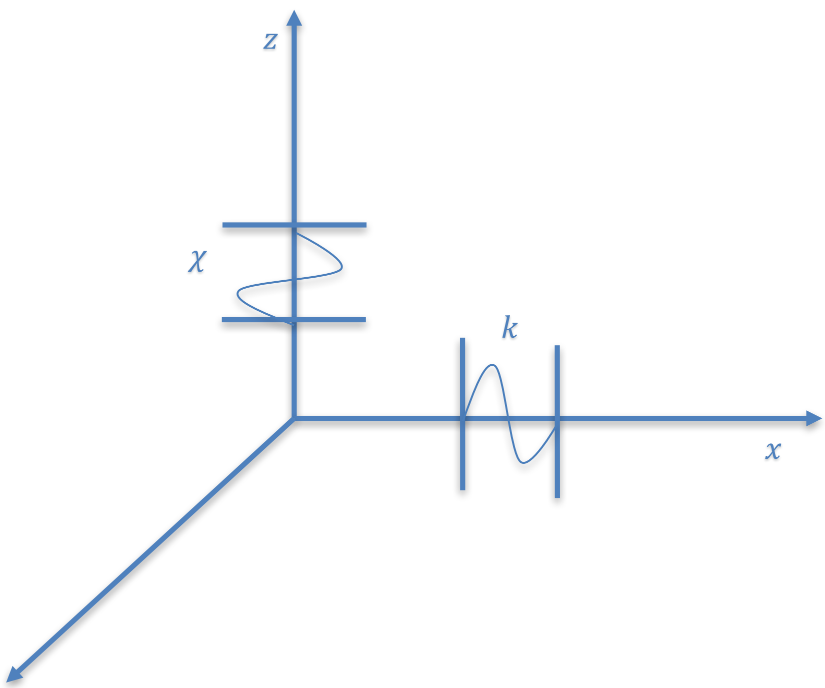


Figure 1.6.3: Directions of propagation of the electromagnetic and gravitational waves.

Chapter 2

Exact Theory

2.1 Metric

We now want to examine an exact solution of Einstein equations in the vacuum. In General Relativity, an important class of exact solutions of Einstein field equations are known as pp-waves. These are plane-fronted gravitational waves with parallel rays. They are defined by the property that they admit a covariantly constant null vector field. It is possible to interpret such a field as the rays of gravitational waves. These solutions model radiation that travels at the speed of light. In Einstein–Maxwell theory, the particular class of plane waves are defined to be pp-waves in which the field components are the same at every point of the wave surfaces. In this sense they are said to have 'plane symmetry'.

The space-time metric of an exact gravitational plane wave with a single + state polarization can be written as (Bini, Fortini, Haney and Ortolan, 2011):

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ &= -dt^2 + dz^2 + F^2(t-z)dx^2 + G^2(t-z)dy^2 \end{aligned} \tag{2.1.1}$$

It is now convenient to introduce two null coordinates u and v that are related to a standard temporal coordinate t and a spatial coordinate z (the direction of propagation of the wave) by the transformation

$$u = t - z,\; v = t + z \tag{2.1.2}$$

which implies

$$du = \frac{1}{2}\,(dt - dz), dv = \frac{1}{2}\,(dt + dz). \tag{2.1.3}$$

The inverse of this transformation is then

$$t = \frac{(u+v)}{2}, z = \frac{(v-u)}{2} \tag{2.1.4}$$

and therefore we have

$$dt = du - dv, dz = dv + du \tag{2.1.5}$$

The vacuum Einstein field equations associated with the metric described above reduce to the single equation $R_{uu} = 0$, i.e.

$$\frac{F''(u)}{F(u)} + \frac{G''(u)}{G(u)} = 0, \tag{2.1.6}$$

where a prime denotes differentiation with respect to u. The wave is then propagating along the positive z-axis with axes of polarization aligned with the coordinate axes x and y. In the following we will consider a sandwich-wave solution, i.e. a curved space-time region in the interval $u \in \left[0, \frac{a^2}{\tau}\right]$ between the two Minkowskian regions $u \in (-\infty, 0) \cup \left(\frac{a^2}{\tau}, \infty\right)$, where the constant parameters a and τ have been introduced, with τ representing the duration of the interaction of particles or fields with the wave and $\frac{1}{a}$ the overall curvature of the wave region.

Figure 2.1.1: It shows the gravitational wave propagating along the u-axes and the three regions that characterize the problem.

A possible choice of metric functions is:

$$F(u) = \begin{cases} 1 & u \leq 0 & \text{(I)} \\ \cos\frac{u}{a} & 0 \leq u \leq \frac{a^2}{\tau} & \text{(II)} \\ \alpha + \beta u & \frac{a^2}{\tau} \leq u & \text{(III)} \end{cases}$$

$$G(u) = \begin{cases} 1 & u \leq 0 & \text{(I)} \\ \cosh\frac{u}{a} & 0 \leq u \leq \frac{a^2}{\tau} & \text{(II)} \\ \gamma + \delta u & \frac{a^2}{\tau} \leq u & \text{(III)} \end{cases} \tag{2.1.7}$$

where labels I, II and III refer to in-zone, wave-zone and out-zone, respectively. The constants α, β, γ and δ can be found y requiring C^1 regularity conditions at the boundary of the sandwich, $u = 0$ and $u = \frac{a^2}{\tau}$, that is

$$\alpha = \cos\frac{a}{\tau} + \frac{a}{\tau}\sin\frac{a}{\tau}, \qquad \beta = -\frac{1}{a}\sin\frac{a}{\tau},$$

$$\gamma = \cosh\frac{a}{\tau} - \frac{a}{\tau}\sinh\frac{a}{\tau}, \qquad \delta = \frac{1}{a}\sinh\frac{a}{\tau}. \tag{2.1.8}$$

We point out that for $u < 0$ with the choice $F(u) = 1 = G(u)$ the space-time is flat (Minkowski) and the metric reduces to

$$g_{\mu\nu} = \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \sqrt{-\det(g)} = \frac{1}{2} \tag{2.1.9}$$

The coordinates (t, x, y, z) associated with (u, v, x, y) by means of equation (2.1.2) are then standard Cartesian coordinates. Finally, in order to compare the results of the present analysis with the existing literature we recall that the metric functions F and G in the linearized approximation and in the transverse-traceless gauge are usually such that

$$F(u)^2 = 1 + h_+(u) \quad G(u)^2 = 1 - h_+(u), \tag{2.1.10}$$

where h_+ is a first-order perturbation to the flat background.

2.2 Geodesics

Let us now consider null geodesics with respect to the coordinates (u, v, x, y) in all space-time regions. Denoting by p^α the vector tangent to these world lines and by λ an affine parameter (associated with proper time), we have $p^\alpha = \frac{dx^\alpha}{d\lambda}$, with

$$p^\alpha \nabla_\alpha p^\beta = \rho, \quad p^\alpha \mathrm{p}_\alpha = \rho, \tag{2.2.1}$$

where the parameter ρ discriminates among the classes of geodesics we are considering, i.e. $\rho = -1,0,1$ corresponding to time-like, null and space-like geodesics, respectively (Bini, Fortini, Haney and Ortolan, 2011). The solution for the geodesics, using u as a convenient parameter, can be written as follows

$$u(\lambda) = -2p_v\lambda + u_c, \quad v(u) = \frac{1}{4p_v^2}\int^u \left(-\rho + \frac{p_x^2}{F(u)^2} + \frac{p_y^2}{G(u)^2}\right) du + v_{c,}$$

$$x(u) = -\frac{p_x}{2p_v}\int^u \frac{du}{F(u)^2} + x_c, \quad y(u) = \frac{p_y}{2p_v}\int^u \frac{du}{G(u)^2} + y_{c,} \tag{2.2.2}$$

where the quantities p_v, p_x and p_y (covariant components of the momentum) are Killing constants, while (u_c, v_c, x_c, y_c) mark coordinates of a generic point.
From now on we will only consider null geodesics since we are working with massless particles, i.e. $\rho = 0$.
The associated momentum is

$$p = -2p_v\partial_u - \frac{1}{2p_v}\left(\frac{p_x^2}{F(u)^2} + \frac{p_y^2}{G(u)^2}\right)\partial_v + \frac{p_x^2}{F(u)^2}\partial_x + \frac{p_y^2}{G(u)^2}\partial_y \tag{2.2.3}$$

or equivalently

$$p^b = \frac{1}{4p_v}\left(\frac{p_x^2}{F(u)^2} + \frac{p_y^2}{G(u)^2}\right)du + p_v dv + p_x dx + p_y dy. \tag{2.2.4}$$

Note that within the linear approximation (small overall curvature of the wave region, $a \to \infty$) we recover the results in the wave region:

$$p^b{}_{(lin)} = \frac{1}{4p_v}\left[p_x^2(1-h_+) + p_y^2(1+h_+)\right]du + p_v dv + p_x dx + p_y dy. \tag{2.2.5}$$

With the choice of F and G functions given in equation (2.1.7) we can also identify

$$h_+(u) \cong -\frac{u^2}{a^2} + O\left(\frac{u^3}{a^3}\right). \tag{2.2.6}$$

By specifying the metric in the various regions of space-time we find C^1 solutions of the geodesic equations before, during and after the passage of the wave.

- *Region* I. The geodesics are straight lines from (u_s, v_s, x_s, y_s) to $(0, v_0, x_0, y_0)$:

$$u(\lambda) = -2p_v\lambda + u_s, \quad v(u) = \frac{1}{4p_v^2}\left(p_x^2 + p_y^2\right)u + \widetilde{v_s}, \tag{2.2.7}$$

$$x(u) = -\frac{p_x}{2p_v}u + \widetilde{x_s}, \quad y(u) = -\frac{p_y}{2p_v}u + \widetilde{y_s},$$

with

$$x_s = \frac{p_x}{2p_v}u_s + x_s, \quad \widetilde{y_s} = \frac{p_y}{2p_v}u_s + y_s, \quad \widetilde{v_s} = v_s - \frac{1}{4p_v^2}\left(p_x^2 + p_y^2\right)u_s. \tag{2.2.8}$$

The associated momentum is

$$p_{\mathrm{I}} = -2p_v\partial_u - \frac{1}{2p_v}(p_x^2 + p_y^2)\partial_v + p_x\partial_x + p_y\partial_y. \tag{2.2.9}$$

- *Region* II. The geodesics connect the space-time points from $(0, v_0, x_0, y_0)$ to $(\frac{a^2}{\tau}, v_1, x_1, y_1)$:

$$u(\lambda) = -2p_v\lambda + u_s, \quad v(u) = \frac{1}{4p_v^2}\left(p_x^2 a \tan\frac{u}{a} + p_y^2 a \tanh\frac{u}{a}\right) + v_0, \tag{2.2.10}$$

$$x(u) = -\frac{p_x}{2p_v}a\tan\frac{u}{a} + x_0, \quad y(u) = -\frac{p_y}{2p_v}\tanh\frac{u}{a} + y_0.$$

The associated momentum is

$$p_{\mathrm{II}} = -2p_v\partial_u - \frac{1}{2p_v}\left(\frac{p_x^2}{(\cos\frac{u}{a})^2} + \frac{p_y^2}{(\cosh\frac{u}{a})^2}\right)\partial_v + \frac{p_x}{(\cos\frac{u}{a})^2}\partial_x + \frac{p_y}{(\cosh\frac{u}{a})^2}\partial_y. \tag{2.2.11}$$

In the linearized regime the above relations have the following limit:

$$u(\lambda) = -2p_v\lambda + u_s, \quad v(u) = \frac{u}{4p_v^2}\left[(p_x^2 + p_y^2) + \frac{(p_x^2 - p_y^2)}{3a^2}u^2\right] + v_0 = v_s - \frac{\lambda}{2p_v}(p_x^2 + p_y^2) - \frac{\lambda}{2p_v}(p_x^2 - p_y^2)f_+(\lambda)$$

$$x(u) = -\frac{p_x}{2p_v}\left(1 + \frac{u^2}{3a^2}\right)u + x_0, \quad y(u) = -\frac{p_y}{2p_v}\left(1 - \frac{u^2}{3a^2}\right)u + y_0, \tag{2.2.12}$$

where, (M. Rakhmanov,2009)

$$f_+(\lambda) = \frac{1}{\lambda}\int_0^\lambda h_+\big(u(\lambda)\big)d\lambda \cong -\frac{1}{a^2}\lambda\int_0^\lambda u^2(\lambda)d\lambda = \frac{1}{6p_v a^2\lambda}\left[(u_s - 2p_v\lambda)^3 - u_s^3\right] \tag{2.2.13}$$

denotes the average amplitude of the gravitational wave and hence

$$\begin{aligned} p_{\mathrm{II}} &= -2p_v\partial_u - \frac{1}{2p_v}\Bigg[(p_x^2 + p_y^2) + \frac{u^2}{a^2}(p_x^2 - p_y^2)]\partial_v \\ &\quad + p_x\left(1 + \frac{u^2}{a^2}\right)\partial_x + p_y\left(1 - \frac{u^2}{a^2}\right)\partial_y \\ &= p_{\mathrm{I}} + \frac{u^2}{a^2}\left[(p_x^2 - p_y^2)\partial_v + p_x\partial_x - p_y\partial_y\right]. \end{aligned} \tag{2.2.14}$$

- *Region* III . The geodesics connect the space-time points $(\frac{a^2}{\tau}, v_1, x_1, y_1)$ and (u_e, v_e, x_e, y_e), where P_e denoted an arbitrary point in the out-zone:

$$u(\lambda) = -2p_v\lambda + u_s, \quad v(u) = -\frac{1}{4p_v^2}\left(\frac{p_x^2}{\beta(\alpha + \beta u)} + \frac{p_y^2}{\delta(\gamma + \delta u)}\right) + \widetilde{v_1}$$

$$x(u) = \frac{p_x}{2p_v}\frac{1}{\beta(\alpha + \beta u)} + \widetilde{x_1} \quad y(u) = \frac{p_y}{2p_v}\frac{1}{\delta(\gamma + \delta u)} + \widetilde{y_1}, \tag{2.2.15}$$

with

$$\widetilde{x_1} = x_1 + \frac{p_x}{2p_v}\frac{a}{\sin\frac{a}{\tau}\cos\frac{a}{\tau}}, \quad \widetilde{y_1} = y_1 - \frac{p_y}{2p_v}\frac{a}{\sinh\frac{a}{\tau}\cosh\frac{a}{\tau}},$$

$$\widetilde{v_1} = v_1 + \frac{a}{4p_v^2}\left(\frac{p_x^2}{\sin\frac{a}{\tau}\cos\frac{a}{\tau}} + \frac{p_y^2}{\sinh\frac{a}{\tau}\cosh\frac{a}{\tau}}\right). \tag{2.2.16}$$

The associated momentum is

$$p_{\mathrm{III}} = -2p_v\partial_u - \frac{1}{2p_v}\left(\frac{p_x^2}{(\alpha+\beta u)^2} + \frac{p_y^2}{(\gamma+\delta u)^2}\right)\partial_v + \frac{p_x}{(\alpha+\beta u)^2}\partial_x + \frac{p_y}{(\gamma+\delta u)^2}\partial_y. \tag{2.2.17}$$

2.3 Matching Conditions

By imposing a matching condition at the boundaries $\mathrm{I}-\mathrm{II}$ and $\mathrm{II}-\mathrm{III}$, we can relate the solutions in all three regions to the initial space-time points (u_s, v_s, x_s, y_s) (Bini, Fortini, Haney and Ortolan, 2011).

i) Matching conditions of the boundary $\mathrm{I}-\mathrm{II}$. We find

$$\lambda_0 = \frac{u_s}{2p_v}, \quad v_0 = v_s - \frac{1}{4p_v^2}(p_x^2+p_y^2)u_s, \quad x_0 = \frac{p_x}{2p_v}u_s + x_s,$$

$$y_0 = \frac{p_y}{2p_v}u_s + y_s. \tag{2.3.1}$$

ii) Marching of the boundary $\mathrm{II}-\mathrm{III}$. We find

$$\lambda_1 = \frac{u_s - \frac{a^2}{\tau}}{2p_v}, \quad x_1 = -\frac{p_x}{2p_v}a\tan\frac{a}{\tau} + x_0 = x_s + \frac{p_x}{2p_v}\left[u_s - a\tan\frac{a}{\tau}\right],$$

$$y_1 = -\frac{p_y}{2p_v}a\tanh\frac{a}{\tau} + y_0 = y_s + \frac{p_y}{2p_v}\left[u_s - a\tan\frac{a}{\tau}\right],$$

$$v_1 = \frac{a}{4p_v^2}\left(p_x^2\tan\frac{a}{\tau} + p_y^2\tanh\frac{a}{\tau}\right) + v_0 = \frac{1}{4p_v^2}\left[p_x^2\left(a\tan\frac{a}{\tau} - u_s\right) + p_y^2\left(a\tanh\frac{a}{\tau} - u_s\right)\right] + v_s. \tag{2.3.2}$$

The associated momenta at the boundaries are

$$p_{\mathrm{I-II}} = -2p_v\partial_u - \frac{1}{2p_v}(p_x^2+p_y^2)\partial_v + p_x\partial_x + p_y\partial_y,$$

and

$$p_{\mathrm{II-III}} = -2p_v\partial_u - \frac{1}{2p_v}\left(\frac{p_x^2}{(\cos\frac{u}{a})^2} + \frac{p_y^2}{(\cosh\frac{u}{a})^2}\right)\partial_v + \frac{p_x}{(\cos\frac{u}{a})^2}\partial_x + \frac{p_y}{(\cosh\frac{u}{a})^2}\partial_y. \tag{2.3.3}$$

2.4 Coordinate transformation in region III

In the flat region III it is convenient to restore Cartesian coordinates. This is practically achieved by the double mapping $(u, v, x, y,) \to (U, V, X, Y) \to (T, Z, X, Y)$, as specified above, namely

$(u, v, x, y,) \to (U, V, X, Y)$:

$$U = u, \quad X = F(u)x, \quad Y = G(u)y, \quad V = v + F(u)F'(u)x^2 + G(u)G'(u)y^2, \tag{2.4.1}$$

and

$(U, V, X, Y) \to (T, Z, X, Y)$:

$$T = \frac{U+V}{2}, \quad Z = \frac{V-U}{2}, \quad X = X, \quad Y = Y. \tag{2.4.2}$$

Thus, in region III we obtain

$$U(\lambda) = -2p_v\lambda + u_s,$$

$$X(U) = \frac{p_x}{2p_v\beta} + \widetilde{x_1}(\alpha + \beta U), \quad Y(U) = \frac{p_y}{2p_v\delta} + \widetilde{y_1}(\gamma + \delta U),$$

$$V(U) = \widetilde{x_1}^2\beta(\alpha + \beta U) + \widetilde{y_1}^2\delta(\gamma + \delta U) + \frac{p_x}{p_v}\widetilde{x_1} + \frac{p_y}{p_v}\widetilde{y_1} + \widetilde{v_1}, \tag{2.4.3}$$

which can be represented in the same way as the geodesics in region I, i.e.

$$X(U) = -\frac{Q_x}{2Q_v}U + \widetilde{X_s}, \quad Y(U) = -\frac{Q_y}{2Q_v}U + \widetilde{Y_s},$$

$$V(U) = \frac{1}{4Q_v^2}\left(Q_x^2 + Q_y^2\right)U + \widetilde{V_s}, \quad U(\lambda) = -2Q_v\lambda + U_s \tag{2.4.4}$$

with

$$U_s = u_s, \quad Q_v = p_v, \quad Q_x = -2p_v\beta\widetilde{x_1}, \quad Q_y = -2p_v\delta\widetilde{y_1}, \quad \widetilde{X_s} = \frac{p_x}{2p_v\beta} + \alpha\widetilde{x_1},$$

$$\widetilde{Y_s} = \frac{p_y}{2p_v\delta} + \gamma\widetilde{y_1}, \quad \widetilde{V_s} = \widetilde{v_1} + \widetilde{x_1}\left(\frac{p_x}{p_v} + \alpha\beta\widetilde{x_1}\right) + \widetilde{y_1}\left(\frac{p_y}{p_v} + \gamma\delta\widetilde{y_1}\right). \tag{2.4.5}$$

The associated transformed momentum in region III is then

$$p_{\mathrm{III}} = -2p_v\{\partial_U + [(\beta\widetilde{x_1})^2 + (\delta\widetilde{y_1})^2]\partial_V + (\beta\widetilde{x_1})\partial_X + (\delta\widetilde{y_1})\partial_Y\}, \tag{2.4.6}$$

or equivalently, by using the quantities Q_v, Q_x, Q_y defined above:

$$p_{\mathrm{III}} = -2Q_v\partial_U - \frac{1}{2Q_v}\left(Q_x^2 + Q_y^2\right)\partial_V + Q_X\partial_X + Q_Y\partial_Y. \tag{2.4.7}$$

By passing to the standard Cartesian temporal and spatial coordinates (T, Z, X, Y) we rewrite the momentum as

$$\begin{aligned} p_{\mathrm{III}} = -Q_v\left[\frac{1}{4Q_v^2}\left(Q_x^2 + Q_y^2\right) + 1\right]\partial_T - Q_v\left[\frac{1}{4Q_v^2}\left(Q_x^2 + Q_y^2\right) - 1\right]\partial_Z \\ + Q_X\partial_X + Q_Y\partial_Y. \end{aligned} \tag{2.4.8}$$

2.5 Scattering of electromagnetic wave by the gravitational wave

Maxwell equations in the Lorentz gauge

$$\Box A_\alpha \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu A_\alpha = 0, \quad \nabla_\mu A^\mu = 0, \tag{2.5.1}$$

once solved for the vector potential A, leads to

$$A^b = \frac{A^0}{\sqrt{FG}}e^{i\phi}e^b. \tag{2.5.2}$$

This solution represents a field, which is not a wave in general, propagating in a direction associated with positive v, x, y coordinates. Since it is a wave in region I, we will refer to ϕ as the phase and e_μ as the polarization vector of the field also in the other regions.
In general, the phase ϕ is given by

$$\begin{aligned} \phi = (\int^u p_u du) + p_v v + p_x x + p_y y, \quad p_u = p_u(u) \\ = \frac{1}{4p_v}\left(\frac{p_x^2}{F^2} + \frac{p_y^2}{G^2}\right), \end{aligned} \tag{2.5.3}$$

and the polarization vector by

$$e^b = e_A^b \sin\theta + e_B^b \cos\theta\,, \tag{2.5.4}$$

which is orthogonal to p and a linear combination of two independent vectors e_A and e_B:

$$e_A^b = \frac{p_x}{2p_vF}du + Fdx, \quad e_B^b = \frac{p_y}{2p_vG}du + Gdy, \tag{2.5.5}$$

both also orthogonal to p.

The phase ϕ is constant along the integral curves of p, namely

$$\nabla_p \phi = p^\alpha \partial_\alpha \phi = p^\alpha p_\alpha = 0. \tag{2.5.6}$$

The polarization vector is parallely transported along the integral curves of p:

$$\nabla_p e^b = 0, \tag{2.5.7}$$

and its contravariant expression is represented by

$$e = -\frac{1}{p_v}\left(\frac{p_x}{F}\cos\theta + \frac{p_y}{G}\sin\theta\right)\partial_v + \frac{\cos\theta}{F}\partial_x + \frac{\sin\vartheta}{G}\partial_y. \tag{2.5.8}$$

It should be stressed that the solution found above has two associated invariants; therefore, in general, this field is non singular, even if in the flat space-time before the passage of the gravitational wave it represents an electromagnetic wave. After the passage of the wave we have again a wave-like behavior. For our analysis of the response of the interferometer we have not considered the electromagnetic field inside the wave region since we are only interested in the emerging field.

Electromagnetic field before the passage of the wave: region I

The space-time before the passage of the wave, corresponding to $F_{\mathrm{I}} = 1 = G_{\mathrm{I}}$, is flat. The metric is

$$ds^2 = -du dv + dx^2 + dy^2, \tag{2.5.9}$$

and it can be reduced to its standard form by using the transformation in equation (2.1.2). The solution for the vector potential A in this region is

$$A_{\mathrm{I}}^b = A_0 e^{i\phi_{\mathrm{I}}} e_{\mathrm{I}}^b, \tag{2.5.10}$$

where the phase ϕ_{I} is given by

$$\phi_{\mathrm{I}} = \frac{1}{4p_v}(p_x^2 + p_y^2)u + p_v v + p_x x + p_y y + C_{\mathrm{I}} = p_\alpha x^\alpha + C_{\mathrm{I}},$$

$$p_\alpha = const., \tag{2.5.11}$$

with $p^b = p_\alpha dx^\alpha$ the null vector seen above and a constant

$$C_{\mathrm{I}} = -\frac{1}{4p_v}(p_x^2 + p_y^2)u_s. \tag{2.5.12}$$

The polarization vector reads

$$e_{\mathrm{I}} = -\frac{1}{p_v}\left[p_x\cos\theta + p_y\sin\theta\right]\partial_v + \cos\theta\,\partial_x + \sin\theta\,\partial_y. \tag{2.5.13}$$

Electromagnetic field after the passage of the wave: region III

After the passage of the wave, i.e. when $u > \frac{a^2}{\tau}$, the metric functions are $F_{\mathrm{III}} = \alpha + \beta u$ and $G_{\mathrm{III}} = \gamma + \delta u$, but the space-time is still flat. Maxwell equations are solved by

$$A^b_{\mathrm{III}} = \frac{A_0}{\sqrt{F_{\mathrm{III}} G_{\mathrm{III}}}} e^{i\phi_{\mathrm{III}}} e^b_{\mathrm{III}} = \frac{A_0}{\sqrt{(\alpha + \beta u)(\gamma + \delta u)}} e^{i\phi_{\mathrm{III}}} e^b_{\mathrm{III}}. \tag{2.5.14}$$

This solution of Maxwell equations have a "phase"

$$\begin{aligned}\phi_{\mathrm{III}} = -\frac{1}{4p_v}\left(\frac{p_x^2}{\beta(\alpha + \beta u)} - \frac{p_y^2}{\delta(\gamma + \delta u)}\right) + p_v v + p_x x + p_y y \\ + C_{\mathrm{III}},\end{aligned} \tag{2.5.15}$$

or equivalently, with the coordinate transformation induced in equation (2.4.1):

$$\phi_{\mathrm{III}} = C_{\mathrm{III}} + p_v V - \frac{p_v \beta}{\alpha + \beta u}\left(\frac{p_x}{2p_v\beta} - X\right)^2 - \frac{p_v \delta}{\alpha + \beta u}\left(\frac{p_y}{2p_v\delta} - Y\right)^2 \tag{2.5.16}$$

with a constant

$$C_{\mathrm{III}} = \frac{a}{4p_v}\left(-\frac{p_x^2}{\sin\frac{a}{\tau}\cos\frac{a}{\tau}} + \frac{p_y^2}{\sinh\frac{a}{\tau}\cosh\frac{a}{\tau}}\right). \tag{2.5.17}$$

The phase in region III, from equation (2.5.15), is generically a function of the coordinates (u, v, x, y) (or equivalently, from equation (2.5.16), of the coordinates (U, V, X, Y)). It is dominated by its value along the null geodesics, namely

$$\begin{aligned}\phi_{\mathrm{III\,(d)}} = \frac{1}{4Q_v}\left(Q_x^2 + Q_y^2\right)U + Q_v V + Q_x X + Q_y Y + \widetilde{C_{\mathrm{III}}} \\ = Q_\alpha X^\alpha + \widetilde{C_{\mathrm{III}}},\end{aligned} \tag{2.5.18}$$

with

$$\widetilde{C_{\mathrm{III}}} = C_{\mathrm{III}} + Q_v\left(\widetilde{V_s} - \widetilde{v_1}\right). \tag{2.5.19}$$

In fact, let us consider the "phase" given by equation (2.5.16) along the generic curve $X^\alpha = X^\alpha(\lambda)$ as a function of the parameter λ along the curve, and require its variation to be vanishing:

$$\frac{d}{d\lambda}\phi_{\mathrm{III}} = 0, \tag{2.5.20}$$

in order to determine the dominant part. We find that this extremal condition is satisfied exactly by the null geodesics given by equation (2.4.8). In addition, we can say that even if our general solution for the electromagnetic field after the passage of the gravitational wave is not exactly a plane wave, it is dominated by a plane wave with the wave vector aligned with that of a null geodesic of the background, with the phase given by equation (2.5.18).

The polarization vector is given by

$$e_{\mathrm{III}} = -\frac{1}{p_v}\left[\frac{p_x}{\alpha+\beta u}\cos\theta + \frac{p_y}{\gamma+\delta u}\sin\theta\right]\partial_v + \frac{\cos\theta}{\alpha+\beta u}\partial_x + \frac{\sin\theta}{\gamma+\delta u}\partial_y, \tag{2.5.21}$$

or, transformed, by

$$e_{\mathrm{III}} = \left[2\beta\frac{\cos\theta}{\alpha+\beta U}\left(X-\frac{p_x}{2p_v\beta}\right) + 2\delta\frac{\sin\theta}{\gamma+\delta U}\left(Y-\frac{p_y}{2p_v\delta}\right)\right]\partial_V + \cos\theta\,\partial_X + \sin\theta\,\partial_Y. \tag{2.5.22}$$

Similar to what happens for the phase, the polarization vector is also dominated by the corresponding value along the null geodesics as in equation (2.4.8), in the sense that the e_{III}^X and e_{III}^Y components do not depend on the curve, while the e_{III}^V component reaches its extremal value on the null geodesics, namely

$$e_{\mathrm{III}(d)} = -\frac{1}{Q_v}\left[Q_x\cos\theta + Q_y\sin\theta\right]\partial_V + \cos\theta\,\partial_X + \sin\theta\,\partial_Y, \tag{2.5.23}$$

with $\widetilde{V_s}$ and $Q_\alpha = const.$ given by equation (2.4.5) where the Q_α are the components of the dominant wave vector as emerging after the scattering by the gravitational wave. Summarizing, the dominant part of the electromagnetic field can be written as

$$A_{\mathrm{III}(d)} = A_0 e^{i\phi_{\mathrm{III}(d)}} e_{\mathrm{III}(d)} \tag{2.5.24}$$

and represents the electromagnetic wave emerging after the interaction.

Variation in the wave and polarization vector, phase shift

We will now consider the variation in the properties of the electromagnetic wave by comparing the dominant parts of the solutions before and after the passage of the gravitational wave. Concerning the covariant components of the wave vector we find $Q_\alpha = p_\alpha + \Delta p_\alpha$ with

$$\begin{aligned}
\Delta p_u &= \left[p_v\frac{x_0^2}{a^2} - \frac{p_x^2}{4p_v}\right]\left(\sin\frac{a}{\tau}\right)^2 + \left[p_v\frac{y_0^2}{a^2} + \frac{p_y^2}{4p_v}\right]\left(\sinh\frac{a}{\tau}\right)^2 + p_x\left(\sin\frac{a}{\tau}\cos\frac{a}{\tau}\right)\left(\frac{x_0}{a}\right) \\
&\quad - p_y\left(\sinh\frac{a}{\tau}\cosh\frac{a}{\tau}\right)\left(\frac{y_0}{a}\right), \\
\Delta p_x &= 2p_v\sin\frac{a}{\tau}\left(\frac{x_0}{a}\right) - p_x\left[1-\cos\frac{a}{\tau}\right], \\
\Delta p_y &= -2p_v\sinh\frac{a}{\tau}\left(\frac{y_0}{a}\right) - p_y\left[1-\cosh\frac{a}{\tau}\right], \\
\Delta p_v &= 0,
\end{aligned} \tag{2.5.25}$$

where equation (2.2.17) has been used. For the contravariant components of the wave vector we find $Q^\alpha = p^\alpha + \Delta p^\alpha$ with

$$\Delta p^u = 0, \quad \Delta p^x = \Delta p_x, \quad \Delta p^y = \Delta p_y, \quad \Delta p^v = -2\Delta p_u. \tag{2.5.26}$$

The contravariant polarization vector has a variation only in the v-component, namely

$$\begin{aligned} \Delta e_v = &\left[\left(1-\cos\frac{a}{\tau}\right)\frac{p_x}{p_v} - 2\sin\frac{a}{\tau}\left(\frac{x_0}{a}\right)\right]\cos\theta \\ &+ \left[\left(1-\cosh\frac{a}{\tau}\right)\frac{p_y}{p_v} - 2\sinh\frac{a}{\tau}\left(\frac{y_0}{a}\right)\right]\sin\theta. \end{aligned} \tag{2.5.27}$$

After the passage of the gravitational wave and in terms of the dominant mode analysis discussed above, the phase of the electromagnetic wave is shifted by

$$\begin{aligned} \Delta\phi = \phi_{\mathrm{III}} - \phi_{\mathrm{I}} \\ &= Q_x\widetilde{X_s} + Q_y\widetilde{Y_s} + Q_v\widetilde{V_s} - p_x\widetilde{x_s} - p_y\widetilde{y_s} - p_v\widetilde{v_s} + \widetilde{C_{\mathrm{III}}} \\ &- C_{\mathrm{I}} = -\frac{a}{4p_v}\left(p_x\tan\frac{a}{\tau} + p_y\tanh\frac{a}{\tau}\right) + p_v(v_s - v_0). \end{aligned} \tag{2.5.28}$$

Note that the transformed coordinates (U, V, X, Y) are Cartesian, so that the new metric functions are such that $F_{\mathrm{III}} = G_{\mathrm{III}} = 1$. As a consequence, the amplitude of the dominant part of the electromagnetic field is unaffected by the passage of the gravitational wave.

2.6 Photon moving along one axis

Let us now consider the motion of photons along x- or y-axes which represent the direction of the arm of a Michelson interferometer with the beam splitter in the origin (Fortini and Ortolan, 1991). The photons start at the beam splitter (denoted by $*$) and are reflected once by an end mirror at a distance L from the origin, denoted by small s (we regard the mirrors as fixed and therefore do not consider the time-like geodesics associated with them). At the start of the proper time, $\lambda = 0$, the photons are assumed at the generic point $P_{s,x}$ or $P_{s,y}$ on the mirror (where $x_s = L$ and $y_s = 0$ or $x_s = 0$ and $y_s = L$), where the momentum is p_x or p_y in the negative x- or y-direction (towards the origin). In the points $P_{s,x}$ and $P_{s,y}$ we have imposed $v_s = u_s$, thereby ensuring $z_s = 0$ at the start. The momenta p_x and p_y are constrained by demanding $z_* = 0$ in the origin, namely (Bini, Fortini, Haney and Ortolan, 2011)

$$p_x = 2p_v, \quad p_y = 2p_v, \quad p_v < 0. \tag{2.6.1}$$

The choice of negative momentum p_v ensures that u increases with λ.
In this case the parametric equations for the unperturbed photon are

$$u = -2p_v\lambda + u_s,$$

$$x = k_x[(u - u_s) - L], \quad y = k_y[(u - u_s) - L], \quad z = 0, \tag{2.6.2}$$

where the choice of k_x and k_y distinguishes the motion of photons. A factor $k_x = 1$ signifies the motion of a photon in positive x-direction (towards the beam splitter). Positive and negative y-directions are distinguished along the same lines.

For photons after the interaction we use the parametric equations (2.4.4) and (2.4.5) with p_x, x_s, p_y and y_s defined as above, and we obtain

$$U = -2p_v\lambda + u_s,$$

$$X = k_x\left[\frac{Q_x}{2p_v}U - \widetilde{X_s}\right],$$

$$Y = k_y\left[\frac{Q_y}{2p_v}U - \widetilde{Y_s}\right],$$

$$Z = \frac{1}{2}\left[\left(k_x^2\frac{Q_x^2}{4p_v^2} + k_y^2\frac{Q_y^2}{4p_v^2} - 1\right)U + \widetilde{V_s}\right], \tag{2.6.3}$$

where

$$\widetilde{V_s} = \widetilde{v_1} + k_x^2\widetilde{x_1}(2 + \alpha\beta\widetilde{x_1}) + k_y^2\widetilde{y_1}(2 + \gamma\delta\widetilde{y_1}),$$

$$\widetilde{v_1} = u_s - k_x^2\left[u_s + a\cot\frac{a}{\tau}\right] - k_y^2\left[u_s - a\coth\frac{\alpha}{\tau}\right]. \tag{2.6.4}$$

The directions of motion (towards the mirror or the origin, along the x-or the y-axis) are again specified by the choice of factors k_x and k_y.
The motion of the photons emerging from the gravitational wave is dependent on the coordinate time u_s at the start of the proper time, the initial momentum p_v of the photon, the interferometer arm length L, the curvature $\frac{1}{a}$ and duration τ of the gravitational wave; explicitly

$$\widetilde{x_1} = L + u_s + a\cot\frac{a}{\tau},$$

$$\widetilde{y_1} = L + u_s - a\coth\frac{\alpha}{\tau},$$

$$Q_x = 2p_v\left[\frac{L+u_s}{a}\sin\frac{a}{\tau} + \cos\frac{a}{\tau}\right], \qquad Q_y = -2p_v\left[\frac{L+u_s}{a}\sinh\frac{a}{\tau} - \cosh\frac{a}{\tau}\right],$$

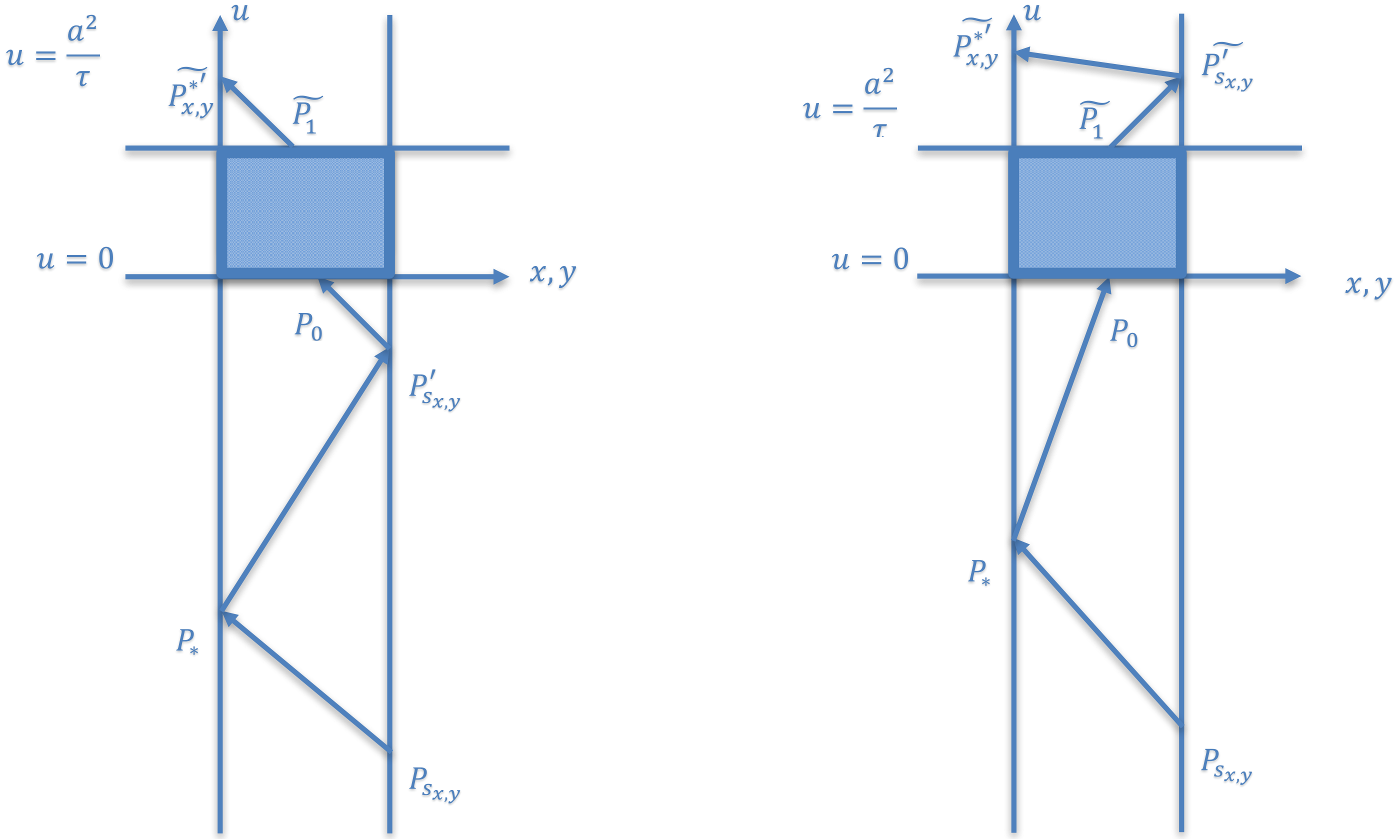
$$\widetilde{X_s} = \left[L + u_s + \frac{a^2}{\tau}\right]\cos\frac{a}{\tau} + \left[\frac{a}{\tau}(L+u_s) - a\right]\sin\frac{a}{\tau},$$

$$\widetilde{Y_s} = \left[L + u_s + \frac{a^2}{\tau}\right]\cosh\frac{a}{\tau} - \left[\frac{a}{\tau}(L+u_s) + a\right]\sinh\frac{a}{\tau}. \tag{2.6.5}$$

We consider two photon beams, making the round trip trough the interferometer along the x- and y- axes, respectively, and arriving again at the beam splitter afterwards (see figure 2.6.1).

The photons start from the origin at $u^* = u_s + L$, and we can foresee three possible scenarios.

Scenario I. The photons travel from the origin to the mirror unperturbed and are reflected by the mirror (in $P'_{s,x}$ or $P'_{s,y}$) at $u'_s = u_s + 2L$. On the return trip they encounter the gravitational wave, and return to the origin at $\widetilde{U}_x^{*'}$ or $\widetilde{U}_y^{*'}$ respectively.



(a) *(b)*

Figure 2.6.1: The two possible scenarios for the interaction of a photon with a gravitational wave during an interferometer round trip: (a) scenario I, and (b) scenario II. (The slopes of the geodesics have been exaggerated for graphic depiction)

- *Scenario II.* The photons encounter the gravitational wave on the way to the mirror, where they are reflected at $\widetilde{U}'_{s,x}$ or $\widetilde{U}'_{s,y}$ (depending on the interferometer arm we consider). They return to the origin in the post-wave region and arrive there again at $\widetilde{U}_x^{*'}$ or $\widetilde{U}_y^{*'}$, respectively.
- *Scenario III.* The photons pass the interferometer round trip without encountering the gravitational wave. They are reflected by the mirror (in $P'_{s,x}$ or $P'_{s,y}$) at $u'_s = u_s + 2L$, and arrive in the origin again at $u^{*'} = u_s + 3L$.

The two photon beams emerging from the gravitational wave experience a deflection in Z-direction of

$$\tilde{Z}_x^{*'} = \frac{1}{2}\left[\left(\frac{Q_x}{p_v} + 1\right)L + u_s - \widetilde{U}_x^{*'}\right], \qquad \tilde{Z}_y^{*'} = \frac{1}{2}\left[\left(\frac{Q_y}{p_v} + 1\right)L + u_s - \widetilde{U}_y^{*'}\right], \tag{2.6.6}$$

respectively, and arrive at the origin at a coordinate time

$$\widetilde{U}_x^{*\prime} = \frac{a^2}{\tau} + a\frac{2L + (L + u_s)\cos\frac{a}{\tau} - a\sin\frac{a}{\tau}}{(L + u_s)\sin\frac{a}{\tau} + a\cos\frac{a}{\tau}},$$

$$\widetilde{U}_x^{*\prime} = \frac{a^2}{\tau} + a\frac{2L + (L + u_s)\cos\frac{a}{\tau} - a\sin\frac{a}{\tau}}{(L + u_s)\sin\frac{a}{\tau} + a\cos\frac{a}{\tau}}. \tag{2.6.7}$$

We can regard the path of a single photon travelling through the interferometer as the centre of a photon beam. The deflection decreases the intensity of the interference pattern of the two photon beams, but the magnitude of the deflection compared to the cross section of the photon beam is very small, whereas the change in the interference pattern of the two photon beams due to their shifted phase is a much greater effect.

It should also be noted that the expressions for the deflection and delay of photons arriving at the origin after the passage of the wave are the same for scenarios I and II. They are distinguished by the relation between u_s and L. In the scenario I, the photons have to leave the origin at $L < |u^*| < 2L$ to meet the wave on the return trip towards the origin, while in scenario II the photons leave from the origin $0 < |u^*| < L$ in order to encounter the wave on the way to the mirror.

When the two photon beams arrive at the beam splitter again after the round trip, they have a relative phase shift

$$\Delta\phi = \Delta\phi_x - \Delta\phi_y = p_v\left[\tanh\frac{a}{\tau} - \tan\frac{a}{\tau}\right] \tag{2.6.8}$$

And a relative change in polarization

$$\begin{aligned}\Delta e_v = 2&\left[1 - \left(\cos\frac{a}{\tau}\right) - \frac{u_s + L}{a}\sin\frac{a}{\tau}\right]\cos\theta \\ &- 2\left[1 - \left(\cosh\frac{a}{\tau}\right) + \frac{u_s + L}{a}\sinh\frac{a}{\tau}\right]\sin\theta\end{aligned} \tag{2.6.9}$$

The scenario considered (I or II) is distinguished by the choice of u_s in terms of L as above, yielding different expressions for the relative change in polarization. The relative phase shift depends only on the dimension of the gravitational wave, not on the construction of the interferometer.

Conclusions

We have considered the propagation of a test electromagnetic field on the background of an exact gravitational plane wave with a linear polarization.

Furthermore, we determined the phase shift between the ingoing electromagnetic wave and the dominant part of the outgoing field as the significant response of a Michelson interferometer to the presence of an exact gravitational wave. In addition, we have calculated the change of the polarization vector, the angular deflection and the delay of photon beams making the round trip of photons in the interferometer. No matter how small these effects are, they could potentially be measured by way of different detection methods.

It is now possible to show that the results obtained through the exact theory give the same results obtained in the linearized regime using an opportune transformation. In fact, both, the linearized gravitational waves and the strong gravitational waves, have the same symmetry group. Moreover, we found that the effects of both waves on the propagation of an electromagnetic wave are qualitatively the same. Although, as said earlier, all of these effects are present, the LIGO/VIRGO interferometers now in operation are able to measure only the phase shift of the electromagnetic wave.

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