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Open mapping theorems of higher order

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Introduction

Open mapping theorems are a useful tool to derive necessary conditions for minimizers. In the general case, we have a Banach space X, a function $\phi : X \to \mathbb{R}$ to minimize and some constraint of the form $f(x) = q$ where $f : X \to \mathbb{R}^m$ and $q \in \mathbb{R}^m$ is fixed. We are interested in solving the problem

$$\min\{\phi(x) \ : \ x \in X, f(x) = q\}. \tag{1}$$

Let $M := \{x \in X : f(x) = q\}$. If $f \in C^1(X; \mathbb{R}^m)$ and $x_0 \in M$ is regular for f, i.e., $d_{x_0} f$ is surjective, the set M coincides with a C^1 graph around x_0. We can remove the constraint in (1) by consider the function ϕ along the graph if we are interested in local minimizers. Then, we look for necessary conditions of minimizers for the unconstrained problem.

The problem rises when $x_0 \in M$ is singular for F. We need a different approach to find necessary optimality conditions. Thus, we consider the extended map

$$F : X \to \mathbb{R}^{m+1}, \quad F(x) = (\phi(x), f(x)). \tag{2}$$

If a point $x_0 \in X$ is a local minimizer for (1), then the map F cannot be open at x_0. If we can prove sufficient differential conditions at a point so that F is open, we gain new necessary conditions on the possible minimizers of (1).

An important example of (1) is given by the extended end-point map in the context of sub-Riemmanian geometry. Let $M \subset \mathbb{R}^m$ be a smooth manifold and $\Delta \subset TM$ a distribution of rank $2 \leq k \leq \dim(M)$. Then for every point $q_0 \in M$ there exists a neighbourhood $U \subset M$ of q_0 and k linearly independent smooth vector fields $f_1, \ldots, f_k \in \mathrm{Vec}(U)$ so that $\Delta = \mathrm{span}\{f_1, \ldots, f_k\}$ on U. If we work locally around a given point, we may assume $U = M$.

We fix $q_0 \in M$ and $X = L^2([0, 1]; \mathbb{R}^k)$ the set of controls. The end-point map is the map

$$\mathcal{E} = \mathcal{E}_{q_0} : X \to M \quad \text{defined by } \mathcal{E}(u) = \gamma(1)$$

where $\gamma \in AC([0, 1], M), \gamma(0) = q_0$, satisfies $\dot{\gamma} \in \Delta_\gamma$ a.e. on $[0, 1]$. Then, there exists a unique control $u \in X$ such that

$$\dot{\gamma} = \sum_{j=1}^{k} u_j f_j(\gamma) \quad \text{a.e. on } [0, 1]. \tag{3}$$

We identify $\gamma = \gamma_u$ through the corresponding control u. We fix on Δ the metric that makes $f_1, \ldots, f_k$ orthonormal and thus

$$\mathrm{length}(\gamma_u) = ||u||_{L^1([0,1];\mathbb{R}^k)} \leq ||u||_{L^2([0,1];\mathbb{R}^k)} \quad \text{by Hölder inequality.}$$

Moreover, if γ_u has constant speed, then it is length-minimizing if and only if it is a minimizer for the energy functional

$$\mathcal{L} : X \to \mathbb{R}, \quad \mathcal{L}(u) = \frac{1}{2}||u||^2_{L^2([0,1];\mathbb{R}^k)}.$$

Finally, the extended end-point map is the map $F : X \to \mathbb{R}^m \times \mathbb{R}$ defined by

$$F(u) = (\mathcal{E}(u), \mathcal{L}(u)).$$

The extended end-point map is smooth. A proof of the C^∞-regularity of $\mathcal{E} : X \to M$ can be found in [4, Appendix D]. With simple computations, it can be proved that $\mathcal{L} : X \to \mathbb{R}$ is Fréchet-differentiable at every $u \in X$ with $D\mathcal{L}(u) = u$; namely,

$$D\mathcal{L}(u) : L^2([0,1];\mathbb{R}^k) \to \mathbb{R}, \quad D\mathcal{L}(u)[v] = \int_0^1 \langle u(x), v(x)\rangle \, dx.$$

Since the map $u \mapsto D\mathcal{L}(u)$ is just the identity, it is also C^∞ and thus $\mathcal{L} \in C^\infty(X;\mathbb{R})$.

To any curve γ_u there corresponds a unique control u, which is either regular or singular for the end-point map. In the first case, it can be proved that the curve γ_u is actually smooth. However, if u is singular for $\mathcal{E}$, the best possible regularity for γ_u is still an open problem, see [4, Section 10.1]. An ongoing research topic is the study of the regularity of singular length-minimizing curves and the openness argument is one of the tools that are used.

Motivated by this example, the topic of our thesis is the study of open mapping theorems of higher order for smooth maps. To simplify, we consider $x_0 = 0 \in X$ and $F \in C^\infty(X, \mathbb{R}^m), m \in \mathbb{N}, F(0) = 0$. Since we assume d_0F not surjective, the idea is to look at higher order differentials to recover the vectors of $\mathbb{R}^m \backslash \mathrm{Im}(d_0F)$. For this reason, it is useful to define

$$\mathrm{corank}(d_0F) := \dim(\mathrm{coker}(d_0F)), \quad \mathrm{coker}(d_0F) := \mathbb{R}^m/\mathrm{Im}(d_0F), \tag{4}$$

and $\mathrm{proj} : \mathbb{R}^m \to \mathrm{coker}(d_0F)$ as the standard projection.

In Chapter 1, we present the theory of regular differentials, a key notion in our thesis. For $n \in \mathbb{N}, n \geq 1$, we define a n-th differential $D^n_0F : X^n \to \mathbb{R}^m$ by

$$D^n_0F(v_1, \dots v_n) := \frac{\partial^n}{\partial s^n} F\left(\sum_{h=1}^n \frac{s^h v_h}{h!}\right)\Bigg|_{s=0}, \quad v_1, \dots, v_n \in X. \tag{5}$$

Then, for $n \in \mathbb{N}, n \geq 2$, we define the so-called intrinsic n-differential $\mathcal{D}^n_0F : \mathrm{dom}(\mathcal{D}^n_0F) \to \mathrm{coker}(d_0F)$ in the following way:

$$\mathrm{dom}(\mathcal{D}^2_0F) = \ker(d_0F), \quad \mathrm{dom}(\mathcal{D}^n_0F) := \{v \in \mathrm{dom}(\mathcal{D}^{n-1}_0F) \times X \mid D^{n-1}_0F(v) = 0\} \quad \text{for } n > 3,$$

$$\mathcal{D}^n_0F(v) := \mathrm{proj}(D^n_0F(v, *)), \quad v \in \mathrm{dom}(\mathcal{D}^n_0F), * \in X \quad \text{for all } n \geq 2.$$

We discuss properly these definitions later in Section 1.1.

In our opinion, it is worth of interest to recall some results for second and third order open mapping theorems. We denote the standard Hessian of F at 0 as

$$H_0F : X \to \mathbb{R}^m, \quad H_0F(v) := \frac{\partial^2}{\partial t^2} F(tv)\bigg|_{t=0}.$$

In [5], Agrachev and Sachkov proved sufficient conditions on H_0F for openness through Morse's index theory. For $\lambda \in \mathrm{coker}(d_0F), \lambda \neq 0$, they define a λ-scalarization of the Hessian as

$$\lambda H_0F : \ker(d_0F) \to \mathbb{R}, \quad \lambda H_0F(v) := \langle \lambda, H_0F(v) \rangle.$$

They define the negative index of λH_0F as the number

$$\mathrm{ind}_-\lambda H_0F := \max\left\{ \dim(L) \ : \ L \text{ subspace of } \ker(d_0F), \ \lambda H_0F\big|_{L\setminus\{0\}} < 0 \right\} \in \mathbb{N} \cup \{\infty\}.$$

Theorem 20.3 of [5] states that if we have

$$\mathrm{ind}_-\lambda H_0F \geq l \quad \forall \lambda \in \mathrm{coker}(d_0F), \lambda \neq 0, \quad \text{where } l = \mathrm{corank}(d_0F),$$

then the map F is open at 0.

We refer the reader to [6] for a third order open mapping theorem. This case is harder than the previous one: the domain of $\mathcal{D}_0^3F$ is no longer a linear space and we need additional assumptions so that it contains non-trivial elements. The strategy is to compose F with a suitable function ϕ and look at the Taylor expansion of $F \circ \phi$ at 0, which is also our approach.

Here, we prove sufficient conditions for openness involving intrinsic n-differential of arbitrary order. Roughly speaking, an intrinsic n-differential $\mathcal{D}_0^nF : \mathrm{dom}(\mathcal{D}_0^nF) \to \mathrm{coker}(d_0F)$ is regular if there exists a continuous polynomial function $w : \mathbb{R}^l \to \mathrm{dom}(\mathcal{D}_0^nF)$ such that the map

$$f : \mathbb{R}^l \to \mathrm{coker}(d_0F), \quad f(t) := \mathcal{D}_0^nF(w(t)),$$

is a homeomorphism, where $l = \mathrm{corank}(d_0F)$.

The main result of Chapter 1 is the following theorem.

Theorem 1. *Let $F \in C^\infty(X;\mathbb{R}^m)$ be such that $F(0) = 0$ and $\mathrm{corank}(d_0F) = l \in \{1,\dots,m\}$. If there exists $n \in \mathbb{N}, n \geq 2$, such that $\mathcal{D}_0^nF$ is regular, then F is open at 0.*

In general, this result provides only sufficient conditions for openness, as we shall see in Chapter 3. The theory of regular differentials and the proof of Theorem 1 were developed by Alessandro Socionovo in his PhD thesis [1] and [7]. We made some improvements to Socionovo's work, which we are going to point out and comment later in the thesis.

In Chapter 2 we apply the theory to functions $F \in C^\infty(X;\mathbb{R}^m)$ with $\mathrm{corank}(d_0F) = 1$. In this case we have equivalent conditions for the existence of a regular differential, see Proposition 2.1. The first new result we prove in this chapter is the following:

Theorem 2. *Let $F \in C^\infty(\mathbb{R}^m;\mathbb{R}^m), m \geq 2$, be such that $F(0) = 0$ and $\mathrm{corank}(d_0F) = 1$. Then F is open at 0 if and only if there exists $n \in \mathbb{N}$ such that $\mathcal{D}_0^nF$ is regular.*

One implication is a direct consequence of Theorem 1, while we use an implicit function argument to prove the other direction. The idea is the following: up to re-ordering rows and columns, we can assume

$$d_0F = \begin{pmatrix} * & M \\ * & * \end{pmatrix}, \quad M \in GL_{m-1}(\mathbb{R}). \tag{6}$$

If $F = (F_j)_j$ is open at 0, then for all $\varepsilon > 0$ there exists $\delta > 0$ so that the set of equations

$$F_j(z) = 0 \quad \forall 1 \leq j \leq m-1, \quad F_m(z) = \nu, \tag{7}$$

has a solution $z_\nu \in B(0,\varepsilon)$ for all $|\nu| < \delta$. However, for $\varepsilon > 0$ small enough the set of solutions for the first $m-1$ equations in $B(0,\varepsilon)$ coincides with the graph of a C^∞ function φ by the implicit function theorem. Thus, (7) reads

$$F_m(\text{graph}(\varphi)(x)) = F_m(x,\varphi(x)) = \nu, \quad x \text{ in a open interval } U \text{ centred at } 0, \quad |\nu| < \delta. \tag{8}$$

In (8), we basically require the map $F_m \circ \text{graph}(\varphi) \in C^\infty(U;\mathbb{R})$ to be open at 0.

In Subsection 2.2.1 we state and prove a stronger version of Theorem 2. Through direct computations and a simple counting argument, we show that, in the hypothesis of Theorem 2, if F is open at 0 and we call

$$\bar{n} := \min\{n \ : \ \mathcal{D}_0^n F \text{ is regular}\} \quad (\bar{n} \text{ exists finite by Theorem 2}),$$

then $\bar{n}$ is odd and $\mathcal{D}_0^n F$ is regular if and only if n is a non-zero multiple of $\bar{n}$.

In Section 2.3 we consider functions $F \in C^\infty(X;\mathbb{R}^m)$ with $F(0) = 0, \text{corank}(d_0F) = 1$, and prove equivalent conditions for openness at 0. We consider two cases:

- $d_0F \neq 0$. We adapt the implicit function argument and derive necessary conditions satisfied by open maps, see Proposition 2.3. We prove that they are also sufficient for openness in Theorem 2.3 and thus we have equivalent conditions, described in Definition 2.2.
- $d_0F = 0$. By the definition of corank in (4) we have $F \in C^\infty(X;\mathbb{R})$. Using this fact, we easily adapt the proofs of the $d_0F \neq 0$ case and obtain the same equivalent conditions for openness of Definition 2.2.

In the end, we can summarize the entire second chapter into the following statement:

Theorem 3. *Let $F \in C^\infty(X;\mathbb{R}^m)$ be such that $F(0) = 0$ and $\text{corank}(d_0F) = 1$. Then F is open at 0 if and only if F satisfies the regularity condition of Definition 2.2.*

In the process, we address and partially solve the first Open Problem in [1], concerning a new definition of regular differential independent of the function $w(t)$. As far as we know, the second chapter contains new results. We believe that Section 2.3 is the most interesting.

In the third chapter we address the $\text{corank}(d_0F) = 2$ case, which is harder. In particular, we investigate whether openness at 0 is equivalent to the existence of a regular differential for maps F satisfying

$$F \in C^\infty(\mathbb{R}^2;\mathbb{R}^2), \quad F(0) = 0, \quad d_0F = 0.$$

In Section 3.1 we prove that the map

$$F : \mathbb{R}^2 \to \mathbb{R}^2, \quad F(x,y) = \begin{pmatrix} x^2 - y^2 \\ xy \end{pmatrix},$$

is open at 0, but admits no regular differentials. This counterexample shows also that, in general, Theorem 1 provides sufficient, but not necessary, conditions for openness.

Chapter 1

Regular differentials

1.1 Definitions and preliminary results

Consider a Banach space $(X, |\cdot|_X)$ and $F \in C^\infty(X;\mathbb{R}^m), m \in \mathbb{N}$. For any $n \in \mathbb{N}$ we define the n-th differential of F at $0 \in X$ as the map $d_0^n F : X \to \mathbb{R}^m$

$$d_0^n F(v) := \frac{\partial^n}{\partial s^n} F(sv)\big|_{s=0}, \quad v \in X. \tag{1.1}$$

With the same notation, we also indicate the n-multilinear differential $d_0^n F : X^n \to \mathbb{R}^m$:

$$d_0^n F(v_1, \dots, v_n) := \frac{\partial^n}{\partial s_1 \dots \partial s_n} F\left(\sum_{h=1}^n s_h v_h\right)\Big|_{s_1=\dots=s_n=0}.$$

Actually, we could have defined the first differential just as the particular case of the second one when $v_1 = \dots = v_n$. This new differential is symmetric, i.e., for every $\sigma \in S_n$ we have

$$d_0^n F(v_1, \dots, v_n) = d_0^n F(v_{\sigma(1)}, \dots, v_{\sigma(n)}).$$

Another differential is the map $D_0^n F : X^n \to \mathbb{R}^m$

$$D_0^n F(v_1, \dots v_n) := \frac{\partial^n}{\partial s^n} F\left(\sum_{h=1}^n \frac{s^h v_h}{h!}\right)\Big|_{s=0}, \quad v_1, \dots, v_n \in X. \tag{1.2}$$

We define a set of multi-indices: given $h \in \mathbb{N}$, we set

$$\mathcal{F}_h := \{\alpha \in \mathbb{N}^h \ : \ \alpha_1, \dots, \alpha_h \geq 1\}.$$

Finally, we let $\mathcal{F}_h^0$ be the set of multi-indices $\alpha \in \mathbb{N}^h$ with $\alpha_1, \dots, \alpha_h \geq 0$.

The differentials $d_0^n F$ and $D_0^n F$ are related by Faà di Bruno formula:

Proposition 1.1 (Faà di Bruno)**.**

$$D_0^n F(v_1, \dots, v_n) = \sum_{h=1}^n \sum_{\alpha \in \mathcal{F}_h, |\alpha|=n} \frac{n!}{h! \alpha!} d_0^h F(v_\alpha), \quad v_\alpha = (v_{\alpha_1}, \dots, v_{\alpha_h}). \tag{1.3}$$

Proof. A proof of (1.3) can be found in [3]. □

Finally, we define $\mathrm{proj} : \mathbb{R}^m \to \mathrm{coker}(d_0F)$ as the standard projection on

$$\mathrm{coker}(d_0F) := \mathbb{R}^m/\mathrm{Im}(d_0F).$$

Definition 1.1. *Let $F \in C^\infty(X;\mathbb{R}^m)$. We say that $0 \in X$ is a critical point of F with corank $l \in \{1,\dots,m\}$ if $\dim(\mathrm{coker}(d_0F)) = l$.*

Now we give the first important definition:

Definition 1.2 (Intrinsic n-differential)**.** *Let $F \in C^\infty(X;\mathbb{R}^m)$. For $n \geq 2$, we define a map $\mathcal{D}_0^n F : \mathrm{dom}(\mathcal{D}_0^n F) \to \mathrm{coker}(d_0F)$, called the intrinsic n-differential of F at 0, and its domain $\mathrm{dom}(\mathcal{D}_0^n F) \subseteq X^{n-1}$ as follows:*

- *When $n = 2$, we define*

$$\mathrm{dom}(\mathcal{D}_0^2 F) := \{v \in X \mid d_0F(v) = 0\} = \ker(d_0F),$$

$$\mathcal{D}_0^2 F(v) := \mathrm{proj}(D_0^2 F(v,*)), \quad v \in \mathrm{dom}(\mathcal{D}_0^2 F), * \in X.$$

- *By induction, for $n > 2$ we set*

$$\mathrm{dom}(\mathcal{D}_0^n F) := \{v \in \mathrm{dom}(\mathcal{D}_0^{n-1} F) \times X \mid D_0^{n-1} F(v) = 0\},$$

$$\mathcal{D}_0^n F(v) := \mathrm{proj}(D_0^n F(v,*)), \quad v \in \mathrm{dom}(\mathcal{D}_0^n F), * \in X.$$

The functions in Definition 1.2 are well defined. Indeed $(v,*) \mapsto \mathrm{proj}(D_0^n F(v,*))$ is independent of the last variable:

$$D_0^n F(v,*) = d_0F(*) + \sum_{h=2}^{n} \sum_{\alpha \in \mathcal{F}_h, |\alpha|=n} \frac{n!}{h!\alpha!} d_0^h F(v_\alpha), \quad \text{by (1.3)}.$$

When we project on $\mathrm{coker}(d_0F)$, the term $d_0F(*)$ disappears.

In the general case, it is difficult to describe $\mathrm{dom}(\mathcal{D}_0^n F)$ for $n \geq 3$. However, under mild assumptions, we have a clear description of its possible elements via an implicit function argument.

We briefly recall the statement of the implicit function theorem:

Theorem 1.1 (Implicit function theorem)**.** *Consider $F \in C^k(U \times W; Y), k \in \mathbb{N}$, where Y is a Banach space and $U \subset X_1, W \subset X_2$ are open subsets of Banach spaces. Suppose that $F(u_0, w_0) = 0$ and $d_{(u_0,w_0)}F\big|_{X_2} \in \mathrm{Inv}(X_2, Y)$ for some $u_0 \in U$ and $w_0 \in X$. Then there exist open neighbourhoods U_0,W_0 of u_0, w_0, respectively, and a function $\varphi \in C^k(U_0, W_0)$ such that*

1. *$F(u,\varphi(u)) \equiv 0$ for all $u \in U_0$;*
2. *$F(u,w) = 0$ for $(u,w) \in U_0 \times W_0$ implies that $w = \varphi(u)$.*

Proof. A complete proof can be found in [2]. □

It is useful to fix coordinates on X and $\mathbb{R}^m$ so that

$$X = \ker(d_0F) \oplus \mathbb{R}^{m-l}, \quad \mathbb{R}^m = \mathrm{Im}(d_0F) \oplus \mathbb{R}^l \quad \text{and} \quad d_0F(x) = d_0F(u,w) = \begin{pmatrix} w \\ 0 \end{pmatrix}. \tag{1.4}$$

The element $x \in X$ in (1.4) can be written as $x = (u,w)$ where $u \in \ker(d_0F)$ and $w \in \mathbb{R}^{m-l}$.

Using (1.4) with $l \neq m$, we apply Theorem 1.1 to the equation

$$\tilde{F}(x) = (F_1(x), \dots, F_{m-l}(x)) = 0, \quad x \in X. \tag{1.5}$$

If we assume $F(0) = 0$, then $\tilde{F}(0) = 0$. By (1.4), $d_0\tilde{F}\big|_{\mathbb{R}^{m-l}} = \mathbb{I}_{\mathbb{R}^{m-l}} \in \mathrm{Inv}(\mathbb{R}^{m-l}, \mathbb{R}^{m-l})$. We denote by $\varphi \in C^\infty$ the function given by the implicit function theorem applied to (1.5) at $0 \in X$.

Proposition 1.2. *Let $F = (F_1, \dots, F_m) \in C^\infty(X; \mathbb{R}^m)$ be such that $F(0) = 0$ and $d_0F \neq 0$. Assume that* $\mathrm{corank}(d_0F) = l \in \{1, \dots, m-1\}$ *and fix coordinates like in* (1.4)*. Then for every $n \geq 2$*

$$\mathrm{dom}(\mathcal{D}_0^n F) \subseteq \{\big(u_1, \varphi_u'(0)), \dots, (u_{n-1}, \varphi_u^{(n-1)}(0)\big) : u \in \ker(d_0F)^{n-1}\}, \tag{1.6}$$

where $u = (u_1, \dots, u_{n-1}) \in \ker(d_0F)^{n-1}$ and $\varphi_u(t) = \varphi\big(\sum_{h=1}^{n-1} \frac{u_h}{h!} t^h\big)$.

Proof. We prove (1.6) by showing that for every $n \geq 2$

$$\mathrm{dom}(\mathcal{D}_0^n \tilde{F}) = \{\big(u_1, \varphi_u'(0)), \dots, (u_{n-1}, \varphi_u^{(n-1)}(0)\big) : u \in \ker(d_0F)^{n-1}\}. \tag{1.7}$$

The thesis follows from the fact that $\mathrm{dom}(\mathcal{D}_0^n F) \subseteq \mathrm{dom}(\mathcal{D}_0^n \tilde{F})$.

We prove (1.7) by induction on n. For $n = 2$, recall that $\mathrm{dom}(\mathcal{D}_0^2 \tilde{F}) := \ker(d_0\tilde{F})$. By hypothesis, the map φ satisfies

$$\tilde{F}(u, \varphi(u)) = 0 \tag{1.8}$$

for every u in an open neighbourhood U of $0 \in \ker(d_0F)$. Given $u_1 \in \ker(d_0F)$, there exists $t(u_1) > 0$ so that $tu_1 \in U$ for every $|t| < t(u_1)$. Thus

$$\tilde{F}(tu_1, \varphi(tu_1)) = \tilde{F}(tu_1, \varphi_{u_1}(t)) = 0 \quad \forall |t| < t(u_1). \tag{1.9}$$

By differentiating once (1.9) and evaluating at $t = 0$ we obtain

$$\frac{\partial}{\partial t} \tilde{F}(tu_1, \varphi(tu_1))\bigg|_{t=0} = d_0\tilde{F}(u_1, \varphi_{u_1}'(0)) = 0 \Longrightarrow (u_1, \varphi_{u_1}'(0)) \in \ker(d_0\tilde{F}). \tag{1.10}$$

When we compute the n-th derivative of the composition at 0, it is not restrictive to replace the inner function with its n-th Taylor polynomial at 0. Thus, we can compute the derivative in (1.10) using (1.1) and the fact that $\varphi_{u_1}(t) = \varphi_{u_1}(0) + \varphi_{u_1}'(0)t + \dots = \varphi_{u_1}'(0)t + \dots$ since $\varphi_{u_1}(0) = \varphi(0) = 0$.

Using the set of coordinates (1.4) we have

$$d_0F(x) = d_0F(u,w) = \begin{pmatrix} d_0\tilde{F}(u,w) \\ 0 \end{pmatrix} = \begin{pmatrix} w \\ 0 \end{pmatrix} \quad \forall x = (u,w) \in \ker(d_0F) \times \mathbb{R}^{m-l} = X,$$

and $\ker(d_0F) = \ker(d_0\tilde{F})$. This implies $\varphi_{u_1}'(0) = 0$ for every $u_1 \in \ker(d_0F)$. Thus

$$\ker(d_0\tilde{F}) = \{(u_1, 0) \ : \ u_1 \in \ker(d_0F)\} = \{(u_1, \varphi_{u_1}'(0)) \ : \ u_1 \in \ker(d_0F)\}. \tag{1.11}$$

We assume (1.7) to be true for n and we prove it for $n+1$. By definition

$$\mathrm{dom}(\mathcal{D}_0^{n+1}\tilde{F}) = \{v \in \mathrm{dom}(\mathcal{D}_0^n \tilde{F}) \times X \;:\; D_0^n \tilde{F}(v) = 0\}.$$

An element $v \in \mathrm{dom}(\mathcal{D}_0^{n+1}\tilde{F})$ has n components each in X and the first $n-1$ are of the form

$$\big(u_1, \varphi_u'(0)\big), \dots, \big(u_{n-1}, \varphi_u^{(n-1)}(0)\big),$$

for a suitable $u \in \ker(d_0\tilde{F})^{n-1}$, by induction hypothesis. We fix u and call v_n the last component of v.

For $u_n \in \ker(d_0F)$, we define $\bar{u} = (u, u_n) \in \ker(d_0F)^{n-1} \times \ker(d_0F)$ and observe that

$$D_0^n \tilde{F}\big((u_1, \varphi_{\bar{u}}'(0)), \dots, (u_n, \varphi_{\bar{u}}^{(n)}(0))\big) = 0. \tag{1.12}$$

We use (1.8) with $\sum\limits_{h=1}^{n} \frac{u_h}{h!} t^h$ in place of u, differentiate n times and evaluate at $t=0$.

On the other hand, v_n must satisfy

$$D_0^n \tilde{F}\big(u_1, \varphi_u'(0)), \dots, (u_{n-1}, \varphi_u^{(n-1)}(0)), v_n\big) = 0. \tag{1.13}$$

We subtract (1.12) and (1.13), and by (1.3) we get

$$d_0\tilde{F}(v_n - (u_n, \varphi_{\bar{u}}^{(n)}(0))) = 0 \Longleftrightarrow v_n - (u_n, \varphi_{\bar{u}}^{(n)}(0)) \in \ker(d_0\tilde{F}).$$

By (1.11) this is equivalent to

$$u_n = (u_n + \lambda, \varphi_{\bar{u}}^{(n)}(0) + \varphi_\lambda'(0)), \quad \lambda \in \ker(d_0F). \tag{1.14}$$

Define $\hat{u} = (u, u_n + \lambda)$. We claim that

$$(u_n + \lambda, \varphi_{\hat{u}}^{(n)}(0)) = (u_n + \lambda, \varphi_{\bar{u}}^{(n)}(0) + \varphi_\lambda'(0)). \tag{1.15}$$

The proof of (1.15) is simple: we need to verify the equality only for the component in $\mathbb{R}^{m-l}$.

$$\begin{aligned}
\varphi_{\hat{u}}^{(n)}(0) &= \frac{\partial^n}{\partial t^n} \varphi\Bigg(\sum_{h=1}^{n} \frac{u_h}{h!} t^h + \frac{\lambda}{n!} t^n \Bigg)\Bigg|_{t=0} = D_0^n \varphi\big(u_1, \dots, u_n + \lambda\big) \\
&= D_0^n \varphi(u_1, \dots, u_n) + d_0\varphi(\lambda) \quad \text{by linearity in the last component,} \\
&= \varphi_{\bar{u}}^{(n)}(0) + \varphi_\lambda'(0).
\end{aligned}$$

We can conclude because our element $v \in \mathrm{dom}(\mathcal{D}_0^{n+1}\tilde{F})$ is necessarily of the form

$$v = \big(u_1, \varphi_{\hat{u}}'(0)\big), \dots, \big(u_n + \lambda, \varphi_{\hat{u}}^{(n)}(0)\big), \quad \hat{u} = (u, u_n + \lambda) \in \ker(d_0F)^n.$$

□

Although Proposition 1.2 does not guarantee that $\mathrm{dom}(\mathcal{D}_0^n F)$ contains non-trivial elements, it will be crucial in Section 2.3 to prove equivalent conditions for openness when the corank is one.

In [1, Proposition 2.8], Alessandro Socionovo proved that, under suitable assumptions, $\mathrm{dom}(\mathcal{D}_0^n F)$ is diffeomorphic to $\ker(d_0F)^{n-1}$. In Proposition 1.2 we showed that if $d_0F \neq 0$, then $\mathrm{dom}(\mathcal{D}_0^n F) \simeq \ker(d_0F)^{n-1}$ is actually the best possible case and we have an explicit form for all its possible elements.

Now we prove the existence of a polynomial function whose image belongs to $\mathrm{dom}(\mathcal{D}_0^n F)$.

Proposition 1.3. *Let $F \in C^\infty(X;\mathbb{R}^m)$ such that $\mathcal{D}_0^h F = 0$ for all $2 \le h < n$ for some $n \ge 2$. Then for any choice of $v_1^1,\dots,v_1^l \in \ker(d_0F)$, $l \in \mathbb{N}$, there exist elements $v_j^\beta \in X$ such that the function $w \in C^\infty(\mathbb{R}^l; X^{n-1})$ with components*

$$w_j(t) := \sum_{\beta\in\mathcal{F}_l^0, |\beta|=j} t^\beta v_j^\beta, \quad j = 1,\dots,n-1, \tag{1.16}$$

satisfies $w(t) \in \operatorname{dom}(\mathcal{D}_0^n F)$ for every $t \in \mathbb{R}^l$. In particular, $v_1^{e_i} = v_1^i$ for every element of the canonical base of $\mathbb{R}^l$.

To prove Proposition 1.3, we need an auxiliary result:

Lemma 1.1. *Consider $F \in C^\infty(X;\mathbb{R}^m)$. For $l, n \in \mathbb{N}$ define the function*

$$g(t) := \sum_{\beta\in\mathcal{F}_l^0, |\beta|=n} t^\beta g(v_n^\beta), \quad t \in \mathbb{R}^l, \quad g(v_n^\beta) \in \mathbb{R}^m, \tag{1.17}$$

where $g(v_n^\beta) = d_0F(v_n^\beta) + h^\beta$ with $h^\beta \in \mathbb{R}^m$ fixed for every β and $v_n^\beta \in X$. Assume $g(t) \in \operatorname{Im}(d_0F)$ for every $t \in \mathbb{R}^l$. Then there exist elements $v_n^\beta \in X$ so that $g(t) \equiv 0$.

Proof. We work by induction on l.

For $l = 1$ and n arbitrary, the function in (1.17) is $g(t) = t^n g(v_n^{ne_1})$ for all $t \in \mathbb{R}$. By hypothesis, the vector $g(v_n^{ne_1})$ belongs to $\operatorname{Im}(d_0F)$. Thus

$$\operatorname{Im}(d_0F) \ni g(v_n^{ne_1}) = d_0F(v_n^{ne_1}) + h^{ne_1} \Longrightarrow \operatorname{Im}(d_0F) \ni h^{ne_1} = d_0F(v) \quad \exists v \in X.$$

If we choose $v_n^{ne_1} = -v$, we are done.

Assume now the thesis to be true for $1, 2, \dots, l-1$ (and all n) and we prove it for l (and all n). If we restrict $g(t)$, as in (1.17), to the hyperplane $t_1 = 0$, we obtain a function in $l-1$ variables:

$$g(0, t_2, \dots, t_l) = \sum_{\beta\in\mathcal{F}_l^0, |\beta|=n, \beta_1=0} t^\beta g(v_n^\beta) \in \operatorname{Im}(d_0F) \quad \forall t_2,\dots,t_l.$$

We can re-index the sum over the multi-indices $\tilde\beta \in \mathcal{F}_{l-1}^0, |\tilde\beta| = n$ since β_1 is fixed. We can use the induction hypothesis and deduce that $g(v_n^\beta) = 0$ for all $\beta \in \mathcal{F}_l^0, |\beta| = n, \beta_1 = 0$ for suitable v_n^β. If we repeat the argument for all subspaces of $\mathbb{R}^l$ of the form

$$\{t_h = 0\ \forall h \in H\}, \quad \emptyset \neq H \subseteq \{1,\dots,l\},$$

we can set $g(v_n^\beta) = 0$ for all β with at least one component equal to zero.

If $n - 1 < l$, then all the multi-indices β in (1.17) have at least one component equal to zero; so $g(t) \equiv 0$. Otherwise, the remaining multi-indices have all components ≥ 1. Thus,

$$g(t) = \sum_{\beta\in\mathcal{F}_l^0, |\beta|=n, \beta_h\ge 1 \forall h} t^\beta g(v_n^\beta) = (t_1, \cdots t_l) \sum_{\beta\in\mathcal{F}_l^0, |\beta|=n, \beta_h\ge 1 \forall h} \frac{t^\beta}{t_1\cdots t_l} g(v_n^\beta) = (t_1\cdots t_l)\tilde g(t).$$

The assumption $g(t) \in \operatorname{Im}(d_0F)$ for all $t \in \mathbb{R}^l$ implies $\tilde g(t) \in \operatorname{Im}(d_0F)$ for all $t \in \mathbb{R}^l$ with $t_i \neq 0$. By the continuity of $\tilde g$ and the fact that $\operatorname{Im}(d_0F)$ is closed, latter property extends to the whole space. At this point, we repeat the procedure on $\tilde g(t)$. After a finite number of steps, we are able to set all the coefficients $g(v_n^\beta)$ in (1.17) to zero for suitable v_n^β. □

Now we prove Proposition 1.3.

Proof. We prove the existence of $w(t)$ as in (1.16) by induction on n.

For $n=2$ we just take

$$w(t)=w_1(t)=\sum_{h=1}^{l} t_h v_1^{e_h}, \quad \text{because } \ker(d_0F) \text{ is a linear subspace.}$$

We assume now the statement to be true for $n-1$ and we prove it for n. By induction hypothesis, $w_1(t),\dots,w_{n-2}(t)$ are fixed and they satisfy

$$D_0^{n-2}F(w_1(t),\dots,w_{n-2}(t))=0 \quad \forall t\in\mathbb{R}^l.$$

We are left to find elements $v_{n-1}^{\beta}\in X$ for $\beta\in\mathcal{F}_{n-1}^0, |\beta|=n-1$ so that

$$w_{n-1}(t)=\sum_{\beta\in\mathcal{F}_l^0,|\beta|=n-1} t^\beta v_{n-1}^\beta \tag{1.18}$$

satisfies $D_0^{n-1}F(w_1(t),\dots,w_{n-1}(t))\equiv 0$. By Faà Di Bruno formula in Proposition 1.1,

$$D_0^{n-1}F(w_1(t),\dots,w_{n-1}(t))=d_0F(w_{n-1}(t))+\sum_{h=1}^{n-1}\sum_{\alpha\in\mathcal{F}_h,|\alpha|=n-1}\frac{(n-1)!}{h!\alpha!}d_0^hF(w_\alpha(t)). \tag{1.19}$$

We call $g(t)$ the right hand side of (1.19). The hypothesis $\mathcal{D}_0^{n-1}F\equiv 0$ implies that $g(t)\in \mathrm{Im}(d_0F)$ for all $t\in\mathbb{R}^l$. Besides this,

$$d_0F(w_{n-1}(t))=\sum_{\beta\in\mathcal{F}_l^0,|\beta|=n-1} t^\beta d_0F(v_{n-1}^\beta),$$

by linearity of d_0F, and

$$\begin{aligned} d_0^hF(w_\alpha(t)) &= d_0^hF(w_{\alpha_1}(t),\dots,w_{\alpha_h}(t)) \\ &= \sum_{\beta^1\in\mathcal{F}_l^0,|\beta^1|=\alpha_1}\cdots\sum_{\beta^h\in\mathcal{F}_l^0,|\beta^h|=\alpha_h} t^\beta d_0^hF(v_{\alpha_1}^{\beta^1},\dots,v_{\alpha_h}^{\beta^h}), \end{aligned}$$

by multilinearity of d_0^hF. In the end, we can rewrite $g(t)$ as

$$g(t)=\sum_{\beta\in\mathcal{F}_l^0,|\beta|=n-1} t^\beta g(v_{n-1}^\beta),$$

where

$$g(v_{n-1}^\beta)=d_0F(v_{n-1}^\beta)+\sum_{h=1}^{n-1}\sum_{\alpha\in\mathcal{F}_h,|\alpha|=n-1}\sum_{\beta^1+\cdots\beta^h=\beta}\frac{(n-1)!}{h!\alpha!}d_0^hF(v_{\alpha_1}^{\beta^1},\dots,v_{\alpha_h}^{\beta^h}).$$

All the $v_{\alpha_h}^{\beta^h}$ are fixed by induction hypothesis. We have to prove that $g(t)\equiv 0$ for suitable v_{n-1}^β, but this is true by Lemma 1.1. □

For every $t\in\mathbb{R}^l$ with $t_i\neq 0$ we define $\mathrm{sgn}(t):=(\mathrm{sgn}(t_1),\dots,\mathrm{sgn}(t_l))$ and an orthant as any subset of $\mathbb{R}^l$ where $\mathrm{sgn}(t)$ is constant. Given $2l$ elements $v_1^{1,\pm},\dots,v_1^{l,\pm}\in\ker(d_0F)$, Proposition 1.2 gives us an extension $w^{\mathrm{sgn}(t)}(t)\in\mathrm{dom}(\mathcal{D}_0^nF)$ of $v_1^{1,\mathrm{sgn}(t_1)},\dots,v_1^{l,\mathrm{sgn}(t_l)}$ that in each orthant has coordinates

$$w_j^{\mathrm{sgn}(t)}(t)=\sum_{\beta\in\mathcal{F}_l^0,|\beta|=j} t^\beta v_j^{\beta,\mathrm{sgn}(t)} \quad 2\le j\le n-1. \tag{1.20}$$

Proposition 1.4. *Let $F \in C^\infty(X;\mathbb{R}^m)$ such that $\mathcal{D}_0^h F \equiv 0$ for all $2 \le h < n$. For any $2l$ elements $v_1^{1,\pm},\dots,v_1^{l,\pm} \in \ker(d_0F)$, the function $w^{\mathrm{sgn}(t)}(t)$, as in (1.20) for $t_i \neq 0$, admits a continuous extension $w \in C^0(\mathbb{R}^l;\mathrm{dom}(\mathcal{D}_0^n F))$.*

Proof. We work by induction on $j = 1,\dots,n-1$.

For $j=1$ we have

$$w_1^{\mathrm{sgn}(t)}(t) = \sum_{h=1}^{l} t_h v_1^{h,\mathrm{sgn}(t_h)}.$$

Given $\emptyset \neq H \subseteq \{1,\dots,l\}$, let $\bar t \in \mathbb{R}^l$ such that

$$\bar t_h = 0 \quad \forall h \in H \quad \text{and} \quad \bar t_h \neq 0 \quad \forall h \notin H. \tag{1.21}$$

We set

$$w_1(\bar t) := \lim_{t\to\bar t} w_1^{\mathrm{sgn}(t)}(t) = \sum_{h\neq H} \bar t_h v_1^{h,\mathrm{sgn}(\bar t_h)}. \tag{1.22}$$

Notice that for $h \notin H$, the vectors $v_1^{h,\mathrm{sgn}(\bar t_h)}$ are well defined because $\bar t_h \neq 0$.

To be precise, we should write $t \to \bar t$ with $t_j \neq 0$ since the function we want to extend is defined only inside the orthants. If $H = \{1,\dots,l\}$, we set $w_1(0) = 0 \in X$.

It is easy to verify that the right hand side in (1.21) is the right value for the limit:

$$\begin{aligned}
\Big| \sum_{h\neq H} \bar t_h v_1^{h,\mathrm{sgn}(\bar t_h)} - \sum_{h=1}^{l} t_h v_1^{h,\mathrm{sgn}(t_h)} \Big|_X &= \Big| \sum_{h\neq H} \Big(\bar t_h v_1^{h,\mathrm{sgn}(\bar t_h)} - t_h v_1^{h,\mathrm{sgn}(t_h)} \Big) - \sum_{h\in H} t_h v_1^{h,\mathrm{sgn}(t_h)} \Big|_X \\
&\le \Big| \sum_{h\neq H} \Big(\bar t_h v_1^{h,\mathrm{sgn}(\bar t_h)} - t_h v_1^{h,\mathrm{sgn}(t_h)} \Big) \Big|_X + \Big| \sum_{h\in H} t_h v_1^{h,\mathrm{sgn}(t_h)} \Big|_X \\
&\le \Big| \sum_{h\neq H} v_1^{h,\mathrm{sgn}(\bar t_h)} (\bar t_h - t_h) \Big|_X + \sum_{h\in H} \Big| t_h v_1^{h,\mathrm{sgn}(t_h)} \Big|_X \quad \text{for } t \text{ close to } \bar t \\
&\le \max_h \big| v_1^{h,\pm} \big|_X \cdot \Big(\sum_{h\neq H} \big| \bar t_h - t_h \big| + \sum_{h\in H} |t_h| \Big) \longrightarrow 0.
\end{aligned}$$

Since $\mathrm{dom}(\mathcal{D}_0^2 F) = \ker(d_0F)$ is closed, $w_1(\bar t) \in \ker(d_0F)$ as limit of a sequence inside $\ker(d_0F)$.

We assume the extension to exist for the first $j \le n-2$ components and we prove that it can be defined also for the $(j+1)$-th one. Given $\emptyset \neq H \subseteq \{1,\dots,l\}$, we consider $\bar t \in \mathbb{R}^l$ as in (1.21) and define the following set

$$\mathcal{F}_{l,H}^0 := \{\beta \in \mathcal{F}_l^0 \ : \ \beta_h = 0 \ \forall h \in H\}.$$

Like in (1.22), the value of the limit would be

$$w_{j+1}(\bar t) := \lim_{t\to\bar t} w_{j+1}^{\mathrm{sgn}(t)}(t) = \sum_{\beta\in\mathcal{F}_{l,H}^0} \bar t^\beta v_{j+1}^{\beta,\mathrm{sgn}(\bar t)}, \tag{1.23}$$

but it is not clear if the vectors $v_{j+1}^{\beta,\mathrm{sgn}(\bar t)}$ do exist for $\beta \in \mathcal{F}_{l,H}^0$.

The vectors $v_{j+1}^{\beta,\mathrm{sgn}(t)}$ are solutions of

$$d_0F(v_{j+1}^{\beta,\mathrm{sgn}(t)}) + \sum_{h=2}^{n-1} \sum_{\alpha\in\mathcal{F}_h, |\alpha|=n-1} \sum_{\beta^1+\dots+\beta^h=\beta} \frac{(n-1)!}{h!\alpha!} d_0^h F(v_{\alpha_1}^{\beta_1,\mathrm{sgn}(t)},\dots,v_{\alpha_h}^{\beta_h,\mathrm{sgn}(t)}) = 0 \tag{1.24}$$

inside each orthant. By induction hypothesis, the limits

$$w_k(\bar t) := \lim_{t\to\bar t} w_k^{\mathrm{sgn}(t)}(t) = \sum_{\beta\in\mathcal{F}^0_{l,H}} \bar t^\beta v_k^{\beta,\mathrm{sgn}(\bar t)} \quad \text{exist for all } 1\le k\le j. \tag{1.25}$$

In particular, the vectors $v_k^{\beta,\mathrm{sgn}(\bar t)}$, $\beta\in\mathcal{F}^0_{l,H}$, in (1.25) are well defined. Therefore, when we took the limit as $t\to\bar t$ in (1.24), the $v_{\alpha_h}^{\beta_h,\mathrm{sgn}(\bar t)}$ exist since $\beta\in\mathcal{F}^0_{l,H}$ implies $\beta_1,\dots,\beta_h\in\mathcal{F}^0_{l,H}$. By (1.24) with $t=\bar t$ we can define $v_{j+1}^{\beta,\mathrm{sgn}(\bar t)}$.

In the end, the limit in (1.23) do exist and

$$D_0^{j+2}F(w_1(\bar t),\dots,w_{j+1}(\bar t)) = 0.$$

□

Definition 1.3 (Regular extension)**.** *We call $w(t)\in C^0(\mathbb{R}^l;\mathrm{dom}(\mathcal{D}_0^nF))$, as in Proposition 1.4, the regular extension of $v^{1,\pm},\dots,v^{l,\pm}\in\ker(d_0F)$ to $\mathrm{dom}(\mathcal{D}_0^nF)$.*

To conclude this introductory paragraph, we give the definition of regular n-differential.

Definition 1.4 (Regular n-differential)**.** *Let $F\in C^\infty(X;\mathbb{R}^m)$ be such that $0\in X$ is a critical point of corank $l\in\{1,\dots,m\}$ for F. We say that $\mathcal{D}_0^nF$ is regular if*

- *n is even and there exist $2l$ elements $v^{1,\pm},\dots,v^{l,\pm}\in\ker(d_0F)$ such that there exists $w(t)$, as in Definition 1.2, so that the map*

$$f:\mathbb{R}^l\to\mathrm{coker}(d_0F) \quad f(t) := \mathcal{D}_0^nF(w(\varrho((t))) \tag{1.26}$$

  is a homeomorphism, where

$$\varrho(t) := (\mathrm{sgn}(t_1)|t_1|^{\frac{1}{n}},\dots,\mathrm{sgn}(t_l)|t_l|^{\frac{1}{n}}). \tag{1.27}$$

- *n is odd and there exist l elements $v^1,\dots,v^l\in\ker(d_0F)$ such that there exists $w(t)$, as in Proposition 1.3, so that the map $f(t)$ defined in (1.26) is a homeomorphism.*

Remark 1.1. *At the end of the proof of Theorem 1.2, we will implicitly use the fact that*

$$\exists L>0 \;:\; |f^{-1}(\tau)|\le L|\tau| \quad \forall\tau\in\mathrm{coker}(d_0F),$$

or equivalently

$$\exists L>0 \;:\; |t|\le L|f(t)| \quad \forall t\in\mathbb{R}^l.$$

The function $f(t)$ defined in (1.26) is 1-homogeneous. Thus, for $t\ne 0$,

$$|t|\le L|f(t)| \iff |t|\le L|t|\left|f\left(\frac{t}{|t|}\right)\right| \iff \frac{1}{L}\le\min_{\mathbb{S}^{l-1}}|f|.$$

The minimum exists by continuity of f and it is not zero since f is bijective and $f(0)=0$. So we are able to choose the constant $L>0$ as we need. In [1, Definition 2.13], the same map $f(t)$ is required to have bounded inverse at 0, namely, there exists $0<L<+\infty$ such that

$$|f^{-1}(\tau)|\le L|\tau|, \quad \forall\tau\in\mathrm{coker}(d_0F).$$

However, we just showed that such constant L always exists as $f(t)$ is a 1-homogeneous continuous bijection.

1.2 Sufficient conditions for openness

Now we are ready to prove sufficient conditions for a smooth map to be open at 0:

Theorem 1.2. *Let $F \in C^\infty(X;\mathbb{R}^m)$ be such that $F(0)=0$. Assume there exists $n \in \mathbb{N}, n \geq 2$, so that $\mathcal{D}_0^n F$ is regular with $0 \in X$ a critical point of corank $l \in \{1,\dots,m\}$. Then F is open at 0.*

Proof. Assume n to be even, the proof for the other case is identical.
By hypothesis there exist $2l$ elements $v^{1,\pm},\dots,v^{l,\pm} \in \ker(d_0F)$ that admit a regular extension, according to Definition 1.2,

$$w(t) = (w_1(t),\dots,w_{n-1}(t)), \quad t \in \mathbb{R}^l,$$

so that the function $f(t)$, see Definition 1.4, is a homeomorphism.

We define the map $\Phi : \mathbb{R}^m \to X$ by

$$\Phi(r,t) = r + \sum_{j=1}^{n-1} \frac{w_j(t)}{j!}, \quad (r,t) \in \mathbb{R}^{m-l} \times \mathbb{R}^l.$$

We first prove the following Taylor expansion at $0 \in \mathbb{R}^m$:

$$F(\Phi(r,t)) = d_0F(r) + D_0^n F(w(t),0) + R(r,t), \tag{1.28}$$

where the remainder satisfies

$$\lim_{(r,t)\to 0} \frac{R(r,t)}{|r| + |t|^n} = 0. \tag{1.29}$$

We consider the Taylor expansion of $F \circ \Phi$ of order n.

The function $w_j(t)$ is j-homogeneous by construction, so for any $s \geq 0$

$$\Phi(r,st) = r + \sum_{j=1}^{n-1} \frac{w_j(t)}{j!} s^j.$$

For fixed $t \in \mathbb{R}^l$, we set $\phi(s) := F(\Phi(0,st))$, $s \geq 0$. This function has the following Taylor expansion at 0 of order n

$$\phi(s) = \sum_{j=1}^{n} \frac{\phi^{(j)}(0)}{j!} s^j + \frac{\phi^{(n+1)}(\bar{s})}{(n+1)!} \bar{s}^{n+1} \quad \exists \bar{s} \in [0,s]. \tag{1.30}$$

By hypothesis $\phi(0) = F(0) = 0$. By construction we have

$$\phi^{(j)}(0) = D_0^j F(w_1(t),\dots,w_j(t)) \quad \forall 1 \leq j \leq n. \tag{1.31}$$

Since $w(t) \in \mathrm{dom}(\mathcal{D}_0^n F)$ for all t, we have $\phi^{(j)}(0) = 0$ for $j \in \{1,\dots,n-1\}$, while

$$\phi^{(n)}(0) = D_0^n F(w(t),0).$$

Therefore (1.30) reads for $s=1$

$$\phi(1) = F(\Phi(0,t)) = D_0^n F(w(t),0) + E_t(t) \tag{1.32}$$

with

$$|E_t(t)| \leq \left| \frac{\phi^{(n+1)}(\bar{s})}{(n+1)!} \bar{s}^{n+1} \right| \leq C|t|^{n+1} \quad \exists C > 0. \tag{1.33}$$

When we develop $F \circ \Phi$ in the variable r we have

$$F(\Phi(r,0)) = d_0F(r) + E_r(r), \quad E_r(r) = o(|r|^2). \tag{1.34}$$

The error appearing in (1.28) can be estimated by

$$|R(r,t)| \leq \sum_{0\leq i\leq 2, 0\leq j\leq n+1} c_{ij}|r|^i|t|^j, \quad c_{ij} \geq 0, \tag{1.35}$$

where $c_{0j} = 0$ for $0 \leq j \leq n$ by (1.32), (1.33) and $c_{10} = 0$ by (1.34).

Moreover, we have $|R(r,t)| \leq C(|r|^2 + |r||t| + |t|^{n+1})$ for a suitable constant $C > 0$. By Young's inequality,

$$|r||t| \leq \frac{n}{n+1}|r|^{\frac{n+1}{n}} + \frac{1}{n+1}|t|^{n+1}.$$

Thus

$$|R(r,t)| \leq C\bigg(|r|^2 + \frac{n}{n+1}|r|^{\frac{n+1}{n}} + \frac{n+2}{n+1}|t|^{n+1}\bigg). \tag{1.36}$$

By (1.36) we obtain (1.29).

The second step of the proof is the following

Lemma 1.2. *If $F \circ \Phi$ is open at $0 \in \mathbb{R}^m$ then F is open at $0 \in X$.*

Proof. For every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$B(0, \delta(\varepsilon)) \subseteq (F \circ \Phi)(B(0, \varepsilon)). \tag{1.37}$$

In particular,

$$B(0, \delta(\varepsilon)) \subseteq F(\Phi(B(0, \varepsilon))). \tag{1.38}$$

Given $(r,t) \in B(0, \varepsilon)$,

$$\begin{aligned}
|\Phi(r,t)|_X \leq |r|_X + \bigg|\sum_{j=1}^{n-1} \frac{w_j(t)}{j!}\bigg|_X &\leq \varepsilon + \sum_{j=1}^{n-1} \frac{|w_j(t)|_X}{j!} \\
&\leq \varepsilon + A\sum_{j=1}^{n-1} \frac{\varepsilon^j}{j!} \quad \text{there exists } A > 0 \text{ because } w_j(t) \text{ are polynomial,} \\
&= \bigg(1 + A\sum_{j=0}^{n-2} \frac{\varepsilon^j}{j!}\bigg)\varepsilon \leq \frac{3}{2}\varepsilon \quad \text{for } 0 < \varepsilon < \varepsilon_0.
\end{aligned}$$

Therefore for all $0 < \varepsilon < \varepsilon_0$

$$B(0, \delta(\varepsilon)) \subseteq F(\Phi(B(0, \varepsilon))) \subseteq F(B(0, \frac{3}{2}\varepsilon)),$$

i.e., F is open at $0 \in X$. □

Recall the map $\varrho(t)$ defined in (1.27):

$$\varrho(t) = (\text{sgn}(t_1)|t_1|^{\frac{1}{n}}, \ldots, \text{sgn}(t_l)|t_l|^{\frac{1}{n}}), \quad t \in \mathbb{R}^l.$$

This map is a homeomorphism whose inverse is

$$\varrho^{-1}(t) = (\mathrm{sgn}(t_1)|t_1|^n, \dots, \mathrm{sgn}(t_l)|t_l|^n), \quad t \in \mathbb{R}^l.$$

Thus, $F(\Phi(r,t))$ is open at $0 \in \mathbb{R}^l$ if and only if $\Psi(r,t) = F(\Phi(r,\varrho(t)))$ is open at $0 \in \mathbb{R}^l$.

We prove that Ψ is open at 0 via a fixed point theorem. From (1.28) we have

$$\Psi(r,t) = d_0F(r) + D_0^nF(w(\varrho(t)),0) + R(r,\varrho(t)). \tag{1.39}$$

Moreover,

$$D_0^nF(w(\varrho(t)),0) = (\mathcal{D}_0^nF(w(\varrho(t))), g(t)) = (f(t), g(t)). \tag{1.40}$$

We just stressed out the $\mathrm{coker}(d_0F)$ and $\mathrm{Im}(d_0F)$ components of $D_0^nF(w(\varrho(t)))$.

By construction the function $g(t)$ in (1.40) is continuous and 1-homogeneous: hence there exists $C_1 > 0$ such that

$$|g(t)| \leq C_1|t|, \quad \forall t \in \mathbb{R}^l. \tag{1.41}$$

With respect to the factorization $(r,t) \in \mathbb{R}^{m-l} \times \mathbb{R}^l$, we introduce a family of norms that depend on a parameter $\lambda_0 > 0$:

$$||(r,t)||_{\lambda_0} := \max\{|r|, \lambda_0|t|\}. \tag{1.42}$$

We will fix the value of λ_0 later. We denote by B_δ the closed ball, with respect to the norm $||\cdot||_{\lambda_0}$, centred at 0 and with radius $\delta > 0$. These sets are both convex and compact.

Since the Euclidean norm and $||\cdot||_{\lambda_0}$ are equivalent, the map Ψ is open at 0 if and only if for all $\varepsilon > 0$ small enough there exists $\delta > 0$ such that

$$B_\delta \subseteq \Psi(B_\varepsilon). \tag{1.43}$$

We fix $\varepsilon > 0$. For any $(\xi,\tau) \in B_\delta$ we look for $(r,t) \in B_\varepsilon$ such that $\Psi(r,t) = (\xi,\tau)$.

Using the splitting $\mathbb{R}^m = \mathrm{Im}(d_0F) \oplus \mathbb{R}^l$, the equation $\Psi(r,t) = (\xi,\tau)$ is equivalent to the system

$$\begin{cases} d_0F(r) + g(t) + R_1(r,\varrho(t)) = \xi \\ f(t) + R_2(r,\varrho(t)) = \tau. \end{cases} \tag{1.44}$$

The remainders $R_1(r,\varrho(t))$, $R_2(r,\varrho(t))$ are the $\mathrm{Im}(d_0F)$ and $\mathrm{coker}(d_0F)$ components of $R(r,\varrho(t))$, respectively. By (1.29), for any $0 < \sigma < 1$ there exists $\varepsilon > 0$ such that

$$|R(r,\varrho(t))| \leq \sigma(|r| + |\varrho(t)|^n) \quad \forall ||(r,t)||_{\lambda_0} < \varepsilon. \tag{1.45}$$

The limit $(r,t) \to 0$ in (1.29) is meant with respect to the norm $||\cdot||_{\lambda_0}$. We will fix a suitable value for $\sigma > 0$ later in the proof.

By definition of $\varrho(t)$ in (1.27), for $t = (t_1, \dots, t_l)$

$$|\varrho(t)|^n = \left(\sum_{k=1}^{l} |t_k|^{\frac{2}{n}}\right)^{\frac{n}{2}} \quad \text{is both continuous and 1-homogeneous.}$$

Thus we are able to find a constant $C_2 > 0$ so that $|\varrho(t)|^n \leq C_2|t|$ for every t. So (1.45) reads

$$|R_1(r,\varrho(t))|, |R_2(r,\varrho(t))| \leq |R(r,\varrho(t))| \leq \sigma(|r| + C_2|t|) \quad \forall ||(r,t)||_{\lambda_0} < \varepsilon. \tag{1.46}$$

We are ready to apply the fixed point theorem argument: the system (1.44) is equivalent to

$$\begin{cases} r = d_0F^{-1}(\xi - g(t) - R_1(r, \varrho(t))) = h_1(r,t) \\ t = f^{-1}(\tau - R_2(r, \varrho(t))) = h_2(r,t). \end{cases} \tag{1.47}$$

Since $\mathcal{D}_0^n F$ is regular, the map f is invertible with f^{-1} continuous.

The final step of the proof is to show that

$$h : B_\varepsilon \subseteq \mathbb{R}^m \to \mathbb{R}^m, \quad h(r,t) = (h_1(r,t), h_2(r,t)), \quad h \in C^0(\mathbb{R}^m, \mathbb{R}^m),$$

maps B_ε into itself for suitable $\delta > 0, \sigma > 0, \lambda_0 > 0$. Then we conclude with Brouwer fixed point theorem.

1. We estimate $|h_1(r,t)|$:

$$\begin{aligned} |h_1(r,t)| &\leq ||d_0F^{-1}||(|\xi| + |g(t)| + |R_1(r,t)|) \\ &\leq ||d_0F^{-1}||(|\xi| + C_1|t| + \sigma(|r| + C_2|t|)) \quad \text{by } (1.41) \text{ and } (1.46) \\ &\leq C_3\left(\delta + \frac{\varepsilon}{\lambda_0} + \sigma\frac{\lambda_0 + C_2}{\lambda_0}\varepsilon\right) \quad \text{there exists } C_3 > 0. \end{aligned} \tag{1.48}$$

   By (1.42), if $(r,t) \in B_\varepsilon$ then

$$|r| \leq \varepsilon \quad \text{and} \quad |t| \leq \frac{\varepsilon}{\lambda_0} \Longrightarrow |r| + C_2|t| \leq \frac{\lambda_0 + C_2}{\lambda_0}\varepsilon.$$

2. We estimate $|h_2(r,t)|$:

$$\begin{aligned} |h_2(r,t)| &\leq L(|\tau| + |R_2(r,t)|) \quad \text{see Remark 1.1} \\ &\leq \left(\frac{\delta}{\lambda_0} + \sigma\frac{\lambda_0 + C_2}{\lambda_0}\varepsilon\right). \end{aligned} \tag{1.49}$$

By $(1.48), (1.49)$ the condition $h(r,t) \in B_\varepsilon$ reads

$$\begin{cases} C_3\big(\delta + \frac{\varepsilon}{\lambda_0} + \sigma\frac{\lambda_0+C_2}{\lambda_0}\varepsilon\big) \leq \varepsilon \\ \big(\frac{\delta}{\lambda_0} + \sigma\frac{\lambda_0+C_2}{\lambda_0}\varepsilon\big) \leq \frac{\varepsilon}{\lambda_0}. \end{cases} \tag{1.50}$$

It is not restrictive to assume $\delta = A\varepsilon, A > 0$, so (1.50) becomes

$$\begin{cases} C_3\big(A + \frac{1}{\lambda_0} + \sigma\frac{\lambda_0+C_2}{\lambda_0}\big) \leq 1 \\ A + \sigma(\lambda_0 + C_2) \leq 1. \end{cases} \tag{1.51}$$

We look for $A > 0, \sigma > 0, \lambda_0 > 0$ so that (1.51) holds.

From the second equation, $\sigma(\lambda_0 + C_2) \leq 1 - A$. Hence

$$C_3\left(\delta + \frac{\varepsilon}{\lambda_0} + \sigma\frac{\lambda_0 + C_2}{\lambda_0}\varepsilon\right) \leq C_3\left(A + \frac{1}{\lambda_0} + \sigma\frac{1-A}{\lambda_0}\right) = C_3\left(\frac{1+\sigma}{\lambda_0} + \frac{\lambda_0 - \sigma}{\lambda_0}A\right) \tag{1.52}$$

We pick $\lambda_0 = 2C_3$: this choice is independent of ε, A, σ. So (1.52) reads

$$\frac{1+\sigma}{2} + \frac{2C_3 - \sigma}{2}A \leq \frac{1+\sigma}{2} + C_3A \leq \frac{3}{4} + C_3A \leq 1 \tag{1.53}$$

if we choose $0 < \sigma \leq \frac{1}{2}$ and $0 < A \leq \frac{1}{4C_3}$.

To conclude, we have to check whether $\sigma(\lambda_0 + C_2) = \sigma(2C_3 + C_2) \leq 1 - A$ holds for some σ, A. The inequality can be rewritten as

$$\sigma \leq \frac{1 - A}{2C_3 + C_2}. \tag{1.54}$$

It is not restrictive to assume $0 < A < 1$. Thus, the right hand side in (1.54) is strictly positive. Summarizing

$$\lambda = 2C_3, \quad 0 < A < \min\left\{1, \frac{1}{4C_3}\right\}, \quad 0 < \sigma < \min\left\{\frac{1}{2}, \frac{1 - A}{2C_3 + C_2}\right\}.$$

The proof of Theorem 1.2 is finished. □

Chapter 2

Corank $l = 1$ case

So far, we proved sufficient conditions for a smooth map to be open at 0. However, it is complicated to prove that a differential is regular according to Definition 1.4. Luckily, when the corank is equal to one, see Definition 1.1, there are equivalent and simpler conditions for a differential to be regular.

Proposition 2.1. *Let X be a Banach space and $F \in C^\infty(X; \mathbb{R}^m)$ be such that $F(0) = 0$ and $0 \in X$ is a critical point of corank $l = 1$; let $n \geq 2$:*

- *If n is even then $\mathcal{D}_0^n F$ is regular if and only if there exist two elements $v^\pm \in \mathrm{dom}(\mathcal{D}_0^n F)$ such that*

$$\mathcal{D}_0^n F(v^+) > 0 \quad \textit{and} \quad \mathcal{D}_0^n F(v^-) < 0. \tag{2.1}$$

- *If n is odd then $\mathcal{D}_0^n F$ is regular if and only if there exists $v \in \mathrm{dom}(\mathcal{D}_0^n F)$ such that*

$$\mathcal{D}_0^n F(v) \neq 0. \tag{2.2}$$

Proof. One implication is simple: since $l = 1$, the function $f(t)$ in Definition 1.4 maps the real line into itself. If $\mathcal{D}_0^n F$ is regular, then f has to be surjective. Thus, if $\mathcal{D}_0^n F$ was either ≥ 0 (≤ 0) or identically 0, it would not be surjective.

Now we prove the converse implication. We assume n even and that there exist $v^\pm \in \mathrm{dom}(\mathcal{D}_0^n F)$ as in (2.1). Let

$$v^+ = (v_1^+, \ldots, v_{n-1}^+) \quad \text{and} \quad v^- = (v_1^-, \ldots, v_{n-1}^-).$$

We claim that the function

$$w(t) = \begin{cases} (tv_1^+, t^2 v_2^+, \ldots, t^{n-1} v_{n-1}^+), & t \geq 0, \\ (-tv_1^-, t^2 v_2^-, \ldots, -t^{n-1} v_{n-1}^-), & t < 0, \end{cases}$$

is a regular extension that makes $\mathcal{D}_0^n F$ regular. Notice that $w(1) = v^+$ and $w(-1) = v^-$. We need to check that

$$w(t) \in \mathrm{dom}(\mathcal{D}_0^n F) \quad \forall t \in \mathbb{R}.$$

For $t=0$, this is obviously true. Take $t<0$: using Faà di Bruno formula in Proposition 1.1

$$\begin{aligned}
D_0^{n-1}F(w(t)) &= d_0F(w_{n-1}(t)) + \sum_{h=2}^{n-1}\sum_{\alpha\in\mathcal{I}_h,|\alpha|=n-1}\frac{(n-1)!}{h!\alpha!}d_0^hF(w_\alpha(t))\\
&= -t^{n-1}d_0F(v_{n-1}^-) + \sum_{h=2}^{n-1}\sum_{\alpha\in\mathcal{I}_h,|\alpha|=n-1}\frac{(n-1)!}{h!\alpha!}\prod_{j=1}^h(-1)^{\alpha_j}t^{\alpha_j}d_0^hF(v_\alpha^-)\\
&= t^{n-1}\Big(-d_0F(v_{n-1}^-) + \sum_{h=2}^{n-1}\sum_{\alpha\in\mathcal{I}_h,|\alpha|=n-1}\frac{(n-1)!}{h!\alpha!}\prod_{j=1}^h(-1)^{\alpha_j}d_0^hF(v_\alpha^-)\Big)\\
&= -t^{n-1}D_0^{n-1}F(v^-) = 0.
\end{aligned}$$

We just use the fact that, fixed $\alpha\in\mathcal{I}_h$ with $|\alpha|=n-1$, then

$$\prod_{j=1}^h(-1)^{\alpha_j} = -1,$$

because $n-1$ is odd, hence $|\{j : \alpha_j \text{ odd}\}|$ is an odd number.

Similarly, it can be proved that

$$D_0^jF(w_1^-(t),\dots,w_j^-(t)) = (-t)^jD_0^jF(v_1^-,\dots,v_j^-) = 0 \quad \forall 1\le j<n-1$$

and, when $t>0$,

$$D_0^jF(w_1^+(t),\dots,w_j^+(t)) = t^jD_0^jF(v_1^+,\dots,v_j^+) = 0 \quad \forall 1\le j\le n-1.$$

In the end, we have

$$f(t) = \begin{cases}\mathcal{D}_0^nF(v^+)|t|, & t\ge 0,\\ \mathcal{D}_0^nF(v^-)|t|, & t<0.\end{cases}$$

The function $f:\mathbb{R}\to\mathbb{R}$ is bijective due to (2.1). The case n odd is pretty much the same:

$$w(t) := (tv_1, t^2v_2,\dots,t^{n-1}v_{n-1}) \quad \text{and} \quad f(t) = \mathcal{D}_0^nF(v)t.$$

By (2.2), the function $f(t)$ is clearly a bijection. □

A weaker version of Proposition 2.1 for the corank one case can be found in [1, Proposition 2.14]. Socionovo assumed $\mathcal{D}_0^hF=0$ for all $2\le h\le n-1$ so that for any choice of v or $v^\pm$ inside $\ker(d_0F)$ it was possible to define the polynomial extension $w:\mathbb{R}^l\to\mathrm{dom}(\mathcal{D}_0^nF)$ and so that $\mathrm{dom}(\mathcal{D}_0^nF)\simeq\ker(d_0F)^{n-1}$.

In Proposition 2.1 we characterized the notion of regular differential when $\mathrm{corank}(d_0F)=1$ in terms of suitable elements of $\mathrm{dom}(\mathcal{D}_0^nF)$. Therefore, it is independent of the function $w(t)$ and so the hypothesis $\mathcal{D}_0^hF=0$ for all $2\le h\le n-1$ is not needed. Moreover, in Section 2.3 we are going to prove that if a smooth map of corank one is open at 0, then some of its domains contain non-trivial elements.

2.1 Implicit function argument

Up to this point, we considered $F \in C^\infty(X;\mathbb{R}^m)$ where X was a generic Banach space. In Section 2.1 and 2.2, we will work with $X = \mathbb{R}^m$ for $m \geq 2$.

Let $F \in C^\infty(\mathbb{R}^m;\mathbb{R}^m)$ be such that $F(0) = 0$ and $\operatorname{corank}(d_0F) = 1$, i.e., $\operatorname{rk}(d_0F) = m-1$. This means that d_0F has a minor $M \in GL_{m-1}(\mathbb{R})$. We may assume that the Jacobian of F at 0 has the following form:

$$d_0F = \begin{pmatrix} * & M \\ * & * \end{pmatrix}.$$

If F were open at 0, then for all $\varepsilon > 0$ the system

$$\begin{cases} F_1(z) = 0 \\ \dots \\ F_{m-1}(z) = 0 \\ F_m(z) = \nu \end{cases} \tag{2.3}$$

would have a solution $z_\nu \in B(0,\varepsilon) \subset \mathbb{R}^m$ for every $\nu \in (-\delta(\varepsilon), \delta(\varepsilon))$, where $\delta(\varepsilon) > 0$ has to be found. To start, we focus on the first $m-1$ equations. We know that

1. $z \mapsto \tilde{F}(z) = (F_1(z), \dots, F_{m-1}(z))$ is a $C^\infty(\mathbb{R}^m;\mathbb{R}^{m-1})$ function;
2. $\tilde{F}(0) = 0$ and $d_0\tilde{F} = \begin{pmatrix} * & M \end{pmatrix}$.

We are in the hypothesis of Dini theorem: there exist real numbers $\lambda, \mu > 0$ and a function $\varphi \in C^\infty(B(0,\lambda); B(0,\mu))$ such that

$$\{(x,y) \in B(0,\lambda) \times B(0,\mu) \ : \ \tilde{F}(x,y) = 0\} = \{(x,\varphi(x)) \ : \ x \in B(0,\lambda)\}.$$

The last equation of (2.3) reads

$$F_m(x,\varphi(x)) = \nu, \quad x \in B(0,\lambda).$$

So, we are left to check whether the map

$$B(0,\lambda) \ni x \mapsto F_\varphi(x) := F_m(x,\varphi(x)) \in \mathbb{R}$$

is open at 0. Since it is a C^∞ function from the real line into itself, F_φ is open at 0 if and only if the order of its first non-zero derivative at $x = 0$ is odd. We will prove this fact in the next paragraph.

Before moving on, we point out that Dini theorem can be applied to functions $F \in C^\infty(\mathbb{R}^d;\mathbb{R}^m)$ with $d \geq m$ as long as $F(0) = 0$ and $d_0F \neq 0$. We consider the particular case $d = m$ because we can prove a sharp result connecting openness and regular differentials.

2.2 Existence of a regular differential in a particular case

In this paragraph, we are going to prove the following

Theorem 2.1. *Let $F \in C^\infty(\mathbb{R}^m;\mathbb{R}^m)$, $m \geq 2$, be such that $F(0) = 0$ and $\operatorname{corank}(d_0F) = 1$. Then F is open at 0 if and only if there exists $n \in \mathbb{N}$ such that $\mathcal{D}_0^n F$ is regular.*

We start to prove the theorem with the easy implication. If $\mathcal{D}_0^n F$ is regular, for some $n \in \mathbb{N}$, then F is open at 0 thanks to Theorem 1.2.

Now we prove the converse implication. Suppose that F is open at 0. We can assume

$$d_0F = \begin{pmatrix} * & M \\ * & * \end{pmatrix}, \quad M \in GL_{m-1}(\mathbb{R}),$$

up to switching the order of columns and rows. We fix some notation:

- we let $\varphi \in C^\infty(B(0,\lambda); B(0,\mu))$, for suitable $\lambda, \mu > 0$, be the implicit function of Dini theorem applied to $(F_1, \dots, F_{m-1})$, where $F = (F_1, \dots, F_m)$;
- we define $\tilde{F}(z) := (F_1(z), \dots, F_{m-1}(z))$ for $z \in \mathbb{R}^m$;
- for $t = (t_1, \dots, t_n) \in \mathbb{R}^n$, we define $f_t(x) := \sum\limits_{h=1}^{n} \frac{t_h}{h!} x^h$ and $\varphi_t(x) := (\varphi \circ f_t)(x)$ for $x \in \mathbb{R}$;
- finally, we let $F_\varphi(x) := F_m(x, \varphi(x))$.

Now we prove the fact anticipated at the end of the previous paragraph:

Lemma 2.1. *Consider $\lambda > 0$ and $g \in C^\infty((-\lambda, \lambda); \mathbb{R})$ with $g(0) = 0$. Then g is open at 0 if and only if the order of the first non-zero derivative of $g(x)$ at $x = 0$ is odd.*

Proof. Consider the Taylor expansion at 0 of $g(x)$:

$$g(x) = \frac{g^{(n)}(0)}{n!} x^n + o(x^n) = \left(\frac{g^{(n)}(0)}{n!} + \sigma(x) \right) x^n, \quad n \geq 1, \tag{2.4}$$

where $g^{(n)}(0)$ is the first derivative different from 0 and $\sigma(x)$ continuous at $x = 0$ with $\sigma(0) = 0$. For $|x|$ small enough, we have

$$\frac{g^{(n)}(0)}{n!} + \sigma(x) \neq 0.$$

So, $g(x) \approx x^n$ in a small neighbourhood of 0. If n is even, then $g(x)$ will have constant sign around the origin. Thus, n has to be odd if $g(x)$ is open at 0.

Conversely, assume $g(x)$ to have the Taylor expansion at 0 given by (2.4) with n odd. For all $\varepsilon > 0$ small enough, we look for $\delta(\varepsilon) > 0$ so that $(-\delta(\varepsilon), \delta(\varepsilon)) \subseteq g((-\varepsilon, \varepsilon))$. It is not restrictive to assume that $\frac{g^{(n)}(0)}{n!} + \sigma(x) > 0$ for all $|x| < \varepsilon$; the other case is identical. Then

$$g(x) > 0 \quad \forall x \in (0, \varepsilon) \quad \text{and} \quad g(x) < 0 \quad \forall x \in (-\varepsilon, 0).$$

This implies that $g(-\varepsilon, \varepsilon)$ contains an open neighbourhood of 0. □

Since $d_0F \neq 0$ because $m \geq 2$ and $\operatorname{corank}(d_0F) = 1$, we can use the implicit function argument. Therefore, $F_\varphi \in C^\infty((-\lambda, \lambda); \mathbb{R})$, $F_\varphi(0) = 0$ is open at 0 as F is open at 0. We apply Lemma 2.1 to $g(x) = F_\varphi(x)$ and define

$$\bar{n} := \min\{n \in \mathbb{N} \ : \ F_\varphi^{(n)}(0) \neq 0\}, \ \bar{n} \text{ is odd.} \tag{2.5}$$

Remark 2.1. *It is important to notice that* $\bar{n} \neq 1$. *Since* $\mathrm{corank}(d_0 F) = 1$, $d_0 F_m$ *is a linear combination of the* $d_0 F_j$ *for* $j \in \{1, \ldots, m-1\}$ *and by construction*

$$\tilde{F}(x, \varphi(x)) = 0 \quad \forall |x| < \lambda \Longrightarrow \frac{\partial}{\partial x}\big(\tilde{F}(x, \varphi(x))\big)\Big|_{x=0} = d_0 \tilde{F}(1, \varphi'(0)) = 0.$$

Thus, $d_0 F_m(1, \varphi'(0)) = 0$. *On the other hand, if* $\bar{n} = 1$ *the natural notion of regular differential of the first order would imply the surjectivity of* $d_0 F$.

The next step is to prove the following

Claim 2.1. *For* $\bar{n}$ *as in* (2.5), *the differential* $\mathcal{D}_0^{\bar{n}} F$ *is regular.*

We need another lemma.

Lemma 2.2. *For any* $n \in \{1, \ldots, \bar{n}\}$ *and* $t \in \mathbb{R}^n$ *we have*

$$D_0^n F_m\big((t_1, \varphi_t'(0)), \ldots, (t_n, \varphi_t^{(n)}(0))\big) = \begin{cases} 0, & \text{for } 1 \le n \le \bar{n} - 1, \\ t_1^{\bar{n}} F_\varphi^{(\bar{n})}(0), & n = \bar{n}. \end{cases}$$

Proof. Recall the definition of $D_0^n F_m$ given in (1.2):

$$D_0^n F_m\big((t_1, \varphi_t'(0)), \ldots, (t_n, \varphi_t^{(n)}(0))\big) := \frac{\partial^n}{\partial s^n} F_m\bigg(\sum_{h=1}^n \frac{s^h}{h!}(t_h, \varphi_t^{(h)}(0))\bigg)\bigg|_{s=0}.$$

This derivative can be rewritten as follows:

$$\frac{\partial^n}{\partial s^n} F_m\bigg(\sum_{h=1}^n \frac{s^h}{h!}(t_h, \varphi_t^{(h)}(0))\bigg)\bigg|_{s=0} = \frac{\partial^n}{\partial s^n} F_\varphi\bigg(\sum_{h=1}^n \frac{s^h}{h!} t_h\bigg)\bigg|_{s=0} = (*). \tag{2.6}$$

We are differentiating n times two different functions:

$$F_m\bigg(\sum_{h=1}^n \frac{s^h}{h!}(t_h, \varphi_t^{(h)}(0))\bigg) \quad \text{and} \quad F_m\bigg(\sum_{h=1}^n \frac{s^h}{h!} t_h, \varphi\bigg(\sum_{h=1}^n \frac{s^h}{h!} t_h\bigg)\bigg).$$

However, we notice that the inner function on the left is just the n-th Taylor polynomial at 0 of the one on the right. Thus, identity (2.6) holds.

We apply Faà di Bruno formula, Proposition 1.1, to the right hand of side of (2.6) and we obtain

$$(*) = D_0^n F_\varphi(t_1, \ldots, t_n) = \sum_{h=1}^n \sum_{\alpha \in \mathcal{F}_h, |\alpha| = n} \frac{n!}{h! \alpha!} F_\varphi^{(h)}(0)\big(t_{\alpha_1} \cdots t_{\alpha_h}\big).$$

We can easily conclude since $F_\varphi^{(h)}(0) = 0$ for all $1 \le h \le \bar{n} - 1$ by the definition of $\bar{n}$. For $n < \bar{n}$, all the terms are equal to 0. When $n = \bar{n}$, the non-zero terms are obtained for $h = n$ and the only multi-index $\alpha \in \mathcal{F}_{\bar{n}}$ with $|\alpha| = \bar{n}$ is $(1, \ldots, 1) \in \mathbb{R}^n$. □

At this point, we can prove Claim 2.1 namely that F being open at 0 implies the existence of $n \in \mathbb{N}$ so that $\mathcal{D}_0^n F$ is regular:

1. Since $f_t \in C^0$ and $f_t(0) = 0$, there exists $\lambda(t) > 0$ such that $|f_t(x)| \leq \lambda$ for all $|x| < \lambda(t)$:
$$\tilde{F}(x, \varphi(x)) \equiv 0 \; \forall |x| < \lambda \Longrightarrow \tilde{F}(f_t(x), \varphi(f_t(x))) \equiv 0 \; \forall |x| < \lambda(t). \tag{2.7}$$
We deduce that
$$0 = \frac{\partial^n}{\partial x^n} \tilde{F}(f_t(x), \varphi_t(x))\bigg|_{x=0} = D_0^n\big(\tilde{F}(t_1, \varphi_t'(0)), \ldots, (t_n, \varphi_t^{(n)}(0))\big) \quad \forall n \in \mathbb{N}, \tag{2.8}$$
by Faà di Bruno formula for the n-th derivative of the composition in Proposition 1.1. If we look back at Definition 1.2, using induction and (2.8) we obtain
$$\big((t_1, \varphi_t'(0)), \ldots, (t_n, \varphi_t^{(n)}(0))\big) \in \operatorname{dom}(\mathcal{D}_0^{n+1}\tilde{F}) \quad \forall t \in \mathbb{R}^n. \tag{2.9}$$
2. Once more by induction, we deduce that for every $t \in \mathbb{R}^{\bar{n}-1}$
$$\big((t_1, \varphi_t'(0)), \ldots, (t_{\bar{n}-1}, \varphi_t^{(\bar{n}-1)}(0))\big) \in \operatorname{dom}(\mathcal{D}_0^{\bar{n}}F),$$
by combining (2.9) for $n = \bar{n} - 1$, Lemma 2.2 and the fact that
$$\operatorname{dom}(\mathcal{D}_0^{\bar{n}}F) = \operatorname{dom}(\mathcal{D}_0^{\bar{n}}\tilde{F}) \cap \operatorname{dom}(\mathcal{D}_0^{\bar{n}}F_m).$$
3. Since $\operatorname{corank}(d_0F) = 1$, there exists $0 \neq w \in \mathbb{R}^m$ such that
$$\operatorname{Im}(d_0F) = \{z \in \mathbb{R}^m \;:\; \langle w, z\rangle = 0\}.$$
So, $\operatorname{coker}(d_0F) = \langle w\rangle$ and $\operatorname{proj}(z) = \langle w, z\rangle w$. It is easy to see that $\operatorname{Im}(d_0F)$ can be generated by $m-1$ vectors of the form
$$\begin{pmatrix} 1 \\ 0 \\ \cdots \\ \cdots \\ a_1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ \cdots \\ \cdots \\ a_2 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 1 \\ a_{m-1} \end{pmatrix} \in \mathbb{R}^m, \quad a_1, \ldots, a_{m-1} \in \mathbb{R},$$
obtained by suitable linear combinations of the last $m-1$ columns of d_0F. Taking this basis, it is clear that $w_m \neq 0$. The vector w is the solution of the system:
$$\begin{cases} w_1 + a_1 w_m = 0 \\ w_2 + a_2 w_m = 0 \\ \cdots \\ w_{n-1} + a_{m-1} w_m = 0. \end{cases}.$$
If $w_m = 0$, then all the other coordinates of w would be zero. Impossible.
4. Let $v_t := \big((t_1, \varphi_t'(0)), \ldots, (t_{\bar{n}-1}, \varphi_t^{(\bar{n}-1)}(0))\big)$ for an arbitrary $t \in \mathbb{R}^{\bar{n}-1}$ with $t_1 \neq 0$. Then
$$\begin{aligned} \mathcal{D}_0^{\bar{n}}F(v_t) &= \operatorname{proj}(D_0^{\bar{n}}F(v_t, *)), \quad \forall * \in \mathbb{R}^m, \\ &= \operatorname{proj}(D_0^{\bar{n}}F(v_t, (\bar{t}_{\bar{n}}, \varphi_{\bar{t}}^{(\bar{n})}(0)))), \quad \bar{t} = (t, \bar{t}_{\bar{n}}) \in \mathbb{R}^{\bar{n}-1} \times \mathbb{R}, \\ &= w_m D_0^{\bar{n}}F_m\big((\bar{t}_1, \varphi_{\bar{t}}'(0)), \ldots, (\bar{t}_{\bar{n}}, \varphi_{\bar{t}}^{(\bar{n})}(0))\big) \\ &= w_m t_1^{\bar{n}} F_\varphi^{(\bar{n})}(0) \neq 0 \quad \text{by construction.} \end{aligned}$$
5. We use Proposition 2.1 to conclude that $\mathcal{D}_0^{\bar{n}}F$ is regular in the odd case. Remember that $\bar{n}$ is odd due to Lemma 2.1.

2.2.1 Uniqueness

Theorem 2.1 ensures the existence of a regular differential, according to Definition 1.4, under suitable assumptions. We wonder whether, in the same setup, there exist $n \neq \bar{n}$, see (2.5), such that $\mathcal{D}_0^n F$ is regular. In this paragraph, we are going to prove a more precise version of Theorem 2.1.

Theorem 2.2. *Let $F \in C^\infty(\mathbb{R}^m;\mathbb{R}^m), m \geq 2$, be such that $F(0) = 0$ and corank$(d_0F) = 1$. Then the following statements are true:*

1. *F is open at 0 if and only if there exists $n \in \mathbb{N}$ such that $\mathcal{D}_0^n F$ is regular.*
2. *If F is open at 0 and we define $\bar{n}$ as in (2.5), then*

$$\bar{n} = \min\{n \in \mathbb{N} \ : \ \mathcal{D}_0^n F \text{ is regular}\}$$

and $\mathcal{D}_0^n F$ is regular if and only if n is a multiple of $\bar{n}$ different from zero.

As in Theorem 2.1, it is not restrictive to assume

$$d_0F = \begin{pmatrix} * & M \\ * & * \end{pmatrix}, \quad M \in GL_{m-1}(\mathbb{R}).$$

We recall the notation used in the proof of Theorem 2.1:

- we let $\varphi \in C^\infty(B(0,\lambda);B(0,\mu))$, for suitable $\lambda,\mu > 0$, be the implicit function of Dini theorem applied to $(F_1,\ldots,F_{m-1})$, where $F = (F_1,\ldots,F_m)$;
- we define $\tilde{F}(z) := (F_1(z),\ldots,F_{m-1}(z))$ for $z \in \mathbb{R}^m$;
- for $t = (t_1,\ldots,t_n) \in \mathbb{R}^n$, we define $f_t(x) := \sum\limits_{h=1}^{n} \frac{t_h}{h!}x^h$ and $\varphi_t(x) := (\varphi \circ f_t)(x)$ for $x \in \mathbb{R}$;
- finally, we let $F_\varphi(x) := F_m(x,\varphi(x))$.

The first statement of Theorem 2.2 is just Theorem 2.1. We begin the proof of the second part with the following result:

Proposition 2.2. *For $n \in \{1,\ldots,\bar{n}\}$ we have*

$$\operatorname{dom}(\mathcal{D}_0^n F) = \{\big((t_1,\varphi_t'(0)),\ldots,(t_{n-1},\varphi_t^{(n-1)}(0))\big) \ : \ t \in \mathbb{R}^{n-1}\}. \tag{2.10}$$

Proof. We prove (2.10) by induction on n.
If $n = 2$, then $\operatorname{dom}(\mathcal{D}_0^2 F) := \ker(d_0F)$. By differentiating once the identity in (2.7) and evaluating it at $x = 0$, we have

$$d_0\tilde{F}(t_1,\varphi_t'(0)) = 0 \Longrightarrow (t_1,\varphi_t'(0)) = t_1(1,\varphi'(0)) \in \ker(d_0\tilde{F}) \quad \forall t_1 \in \mathbb{R},$$

but $\ker(d_0\tilde{F})$ coincides with $\ker(d_0F)$ because the $\operatorname{corank}(d_0F) = 1$. Since $\dim(\ker(d_0F)) = 1$, it is actually

$$\ker(d_0F) = \langle(1,\varphi'(0))\rangle. \tag{2.11}$$

We assume the statement to be true for $2 \leq n \leq \bar{n}-1$ and we prove it for $n+1$. By definition,

$$\operatorname{dom}(\mathcal{D}_0^{n+1}F) = \{v \in \operatorname{dom}(\mathcal{D}_0^n F) \times \mathbb{R}^m \ : \ D_0^n F(v) = 0\}.$$

An element $v \in \operatorname{dom}(\mathcal{D}_0^{n+1}F)$ has n components each in $\mathbb{R}^m$ and the first $n-1$ are of the form

$$\big((t_1, \varphi_t'(0)), \dots, (t_{n-1}, \varphi_t^{(n-1)}(0))\big),$$

for a suitable $t \in \mathbb{R}^{n-1}$, by induction. We fix the vector t.

We are left to find the last component of v, let it be v_n. The condition $D_0^n F(v) = 0$ is equivalent to the following system:

$$\begin{cases} D_0^n \tilde{F}\big((t_1, \varphi_t'(0)), \dots, (t_{n-1}, \varphi_t^{(n-1)}(0)), v_n\big) = 0 \\ D_0^n F_m\big((t_1, \varphi_t'(0)), \dots, (t_{n-1}, \varphi_t^{(n-1)}(0)), v_n\big) = 0. \end{cases} \tag{2.12}$$

By (2.8) we know that

$$D_0^n \tilde{F}\big((t_1, \varphi_{\bar{t}}'(0)), \dots, (t_n, \varphi_{\bar{t}}^{(n)}(0))\big) = 0, \tag{2.13}$$

where $\bar{t} = (t, t_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ for any t_n. We choose an arbitrary t_n.

Recall that

$$D_0^n \tilde{F}(w) = d_0\tilde{F}(w_n) + \sum_{h=1}^{n} \sum_{\alpha \in \mathcal{F}_h, |\alpha| = n} \frac{n!}{h!\alpha!} d_0^h \tilde{F}(w_\alpha), \quad w = (w_1, \dots, w_n).$$

So by subtracting the first equation of (2.12) with (2.13) we obtain

$$d_0\tilde{F}\big(v_n - (t_n, \varphi_{\bar{t}}^{(n)}(0))\big) = 0, \quad \text{i.e,} \quad v_n - (t_n, \varphi_{\bar{t}}^{(n)}(0)) \in \ker(d_0\tilde{F}).$$

By (2.11), it is equivalent to

$$v_n = (t_n + \lambda, \varphi_{\bar{t}}^{(n)}(0) + \lambda\varphi'(0)), \quad \lambda \in \mathbb{R}.$$

Define $\hat{t} = (t, t_n + \lambda)$. We claim that

$$\varphi_{\bar{t}}^{(n)}(0) + \lambda\varphi'(0) = \varphi_{\hat{t}}^{(n)}(0). \tag{2.14}$$

First, notice that $f_{\hat{t}}(x) = f_{\bar{t}}(x) + \frac{\lambda}{n!}x^n$. Besides this, we have

$$\varphi_{\hat{t}}^{(n)}(0) = \frac{\partial^n}{\partial s^n}\varphi\bigg(\sum_{h=1}^{n} \frac{t_h}{h!}s^h + \frac{\lambda}{n!}s^n\bigg)\bigg|_{s=0} = D_0^n\varphi(t_1, \dots, t_n + \lambda).$$

By Faà di Bruno formula, Proposition 1.1,

$$D_0^n\varphi(t_1, \dots, t_n + \lambda) = d_0\varphi(t_n + \lambda) + \sum_{h=1}^{n} \sum_{\alpha \in \mathcal{F}_h, |\alpha| = n} d_0^h\varphi(t_\alpha) = d_0\varphi(\lambda) + D_0^n\varphi(t_1, \dots, t_n).$$

The identity (2.14) is proved since $d_0\varphi = \varphi'(0)$.

At this point, if our element v belongs to $\operatorname{dom}(\mathcal{D}_0^{n+1}F)$, it is of the form

$$v = \big((t_1, \varphi_{\hat{t}}'(0)), \dots, (t_n + \lambda, \varphi_{\hat{t}}^{(n-1)}(0))\big), \quad \hat{t} = (t_1, \dots, t_n + \lambda) \in \mathbb{R}^n;$$

but then it satisfies the second equation of system (2.12) by Lemma 2.2 for $1 \leq n \leq \bar{n}-1$. Thus, identity (2.10) is proved. □

A simple consequence of Proposition 2.2 is the following:

Corollary 2.1. *$\mathcal{D}_0^{\bar n}F$ is the regular differential of smallest order. Equivalently,*

$$\bar n := \min\{n \in \mathbb{N} \ : \ F_\varphi^{(n)}(0) \neq 0\} = \min\{n \in \mathbb{N} \ : \ \mathcal{D}_0^n F \textit{ is regular}\}.$$

Proof. We have already proved that $\mathcal{D}_0^{\bar n}F$ is regular in Theorem 2.1.

By Proposition 2.2, an element $v \in \operatorname{dom}(\mathcal{D}_0^n F)$, for $2 \leq n \leq \bar n - 1$, is of the form

$$v = \big((t_1, \varphi_t'(0)), \dots, (t_{n-1}, \varphi_t^{(n-1)}(0))\big), \quad t \in \mathbb{R}^{n-1}.$$

As a consequence,

$$\mathcal{D}_0^n F(v) = w_m D_0^n F_m((t_1, \varphi_{\bar t}'(0)), \dots, (t_n, \varphi_{\bar t}^{(n)}(0)) \quad \text{for } \bar t = (t, t_n) \in \mathbb{R}^{n-1} \times \mathbb{R}.$$

However, the right hand side is always equal to zero by Lemma 2.2 for $2 \leq n \leq \bar n - 1$. □

Now we look at $\mathcal{D}_0^n F$ for $n > \bar n$. Let's consider $\mathcal{D}_0^{\bar n+1}F$. By Definition 1.2,

$$\operatorname{dom}(\mathcal{D}_0^{\bar n+1}F) = \{v \in \operatorname{dom}(\mathcal{D}_0^{\bar n}F) \times \mathbb{R}^m \ : \ D_0^{\bar n}F(v) = 0\}.$$

Given any $v \in \operatorname{dom}(\mathcal{D}_0^{\bar n+1}F)$, then

$$v = \big((t_1, \varphi_t'(0)), \dots, (t_{\bar n-1}, \varphi_t^{(\bar n-1)}(0)), v_{\bar n}\big)$$

for some $t \in \mathbb{R}^{\bar n-1}$ and suitable $v_{\bar n} \in \mathbb{R}^m$. As we did in the proof of Proposition 2.2, the last component is of the form

$$v_{\bar n} = (t_{\bar n} + \lambda, \varphi_t^{(\bar n)}(0) + \lambda\varphi'(0)), \quad t_{\bar n}, \lambda \in \mathbb{R}.$$

Therefore,

$$v = \big((t_1, \varphi_{\hat t}'(0)), \dots, (t_{\bar n} + \lambda, \varphi_{\hat t}^{(\bar n)}(0))\big), \quad \hat t = (t_1, \dots, t_n + \lambda).$$

We use Lemma 2.2 for $n = \bar n$ and obtain

$$D_0^{\bar n}F_m(v) = t_1^{\bar n} \cdot F_\varphi^{(\bar n)}(0) = 0 \Longleftrightarrow t_1 = 0 \quad \text{by definition of } \bar n.$$

$$\Longrightarrow \operatorname{dom}(\mathcal{D}_0^{\bar n+1}F) = \{\big((0, \varphi_t'(0)), \dots, (t_{\bar n}, \varphi_t^{(\bar n)}(0))\big) \ : \ t = (0, t_2, \dots, t_{\bar n}) \in \mathbb{R}^{\bar n}\}. \tag{2.15}$$

Remark 2.2. *Since $\varphi_t'(0) = t_1\varphi'(0)$, all the elements of $\operatorname{dom}(\mathcal{D}_0^{\bar n+1}F)$ have the origin as their first component. The same holds for all the successive domains.*

For $v \in \operatorname{dom}(\mathcal{D}_0^{\bar n+1}F)$ we have

$$\mathcal{D}_0^{\bar n+1}F(v) = w_m D_0^{\bar n+1}F_m(v) = w_m\Bigg(\sum_{h=1}^{\bar n+1} \sum_{\alpha \in \mathcal{F}_h, |\alpha| = \bar n+1} \frac{(\bar n+1)!}{h!\alpha!} F_\varphi^{(h)}(0)\big(t_{\alpha_1} \cdots t_{\alpha_h}\big)\Bigg), \tag{2.16}$$

for some $t = (t_1, \dots, t_n) \in \mathbb{R}^{\bar n+1}$. The second equality, up to the non-zero constant w_m, has been already proved in Lemma 2.2.

We want to compute the last term in (2.16). Recall that $F_\varphi^{(h)}(0) = 0$ for all $1 \le h \le \bar{n}-1$ by definition of $\bar{n}$, see (2.5). So (2.16) can be simplified:

$$\mathcal{D}_0^{\bar{n}+1}F(v) = w_m\Bigg(F_\varphi^{(\bar{n}+1)}(0)t_1^{\bar{n}+1} + \sum_{\alpha\in\mathcal{F}_{\bar{n}},|\alpha|=\bar{n}+1} \frac{\bar{n}+1}{\alpha!}F_\varphi^{(\bar{n})}(0)\big(t_{\alpha_1}\cdots t_{\alpha_h}\big)\Bigg), \tag{2.17}$$

the only relevant terms are for $h = \bar{n}, \bar{n}+1$.

The multi-indices α in the sum have length $\bar{n}$ and weight $\bar{n}+1$; so at least one of the components is equal to one. Besides this, (2.15) implies that $t_1 = 0$ and so the right hand side of (2.17) is actually equal to zero. In the end, $\mathcal{D}_0^{\bar{n}+1}F$ cannot be regular because it is identically zero on its domain.

Now we study the regularity of $\mathcal{D}_0^n F$ for $n > \bar{n}+1$:

1. First, we compute $\mathrm{dom}(\mathcal{D}_0^n F)$. By definition,

$$\mathrm{dom}(\mathcal{D}_0^n F) = \{v \in \mathrm{dom}(\mathcal{D}_0^{n-1}F)\times\mathbb{R}^m \ : \ D_0^{n-1}F(v) = 0\}.$$

Repeating the proof of Proposition 2.2 and using Remark 2.2, if $v \in \mathrm{dom}(\mathcal{D}_0^n F)$, then it is of the form

$$v = \big(0, (t_2, \varphi_t^{(2)}(0)), \ldots, (t_{n-1}, \varphi_t^{(n-1)}(0))\big), \quad t \in \{0\}\times\mathbb{R}^{n-2}.$$

However, we still have to verify if

$$D_0^{n-1}F_m(v) = 0. \tag{2.18}$$

By explicit computations, the left hand side of (2.18) is

$$\begin{aligned} D_0^{n-1}F_m(v) &= \sum_{h=1}^{n-1}\sum_{\alpha\in\mathcal{F}_h,|\alpha|=n-1}\frac{(n-1)!}{h!\alpha!}F_\varphi^{(h)}(0)\big(t_{\alpha_1}\cdots t_{\alpha_h}\big) \\ &= \sum_{h=\bar{n}}^{n-1}\sum_{\alpha\in\mathcal{F}_h,|\alpha|=n-1}\frac{(n-1)!}{h!\alpha!}F_\varphi^{(h)}(0)\big(t_{\alpha_1}\cdots t_{\alpha_h}\big). \end{aligned} \tag{2.19}$$

The non-zero terms are given by the multi-indices α with all components ≥ 2 because $t_1 = 0$. So, in (2.19) we only look at the $\alpha \in \mathcal{F}_h, h \ge \bar{n}$, such that

$$\alpha_j \ge 2 \quad \forall j = 1,\ldots,h \quad \text{and} \quad |\alpha| = n-1. \tag{2.20}$$

The fact that α has $h \ge \bar{n}$ components ≥ 2 implies that $|\alpha| \ge 2h \ge 2\bar{n}$. This condition and (2.20) are compatible only if $n-1 \ge 2\bar{n}$, i.e, $n \ge 2\bar{n}+1$. Thus, for all $n \in \{\bar{n}+2,\ldots,2\bar{n}\}$,

$$\mathrm{dom}(\mathcal{D}_0^n F) = \{\big(0, (t_2, \varphi_t^{(2)}(0)), \ldots, (t_{n-1}, \varphi_t^{(n-1)}(0))\big) \ : \ t \in \{0\}\times\mathbb{R}^{n-2}\}, \tag{2.21}$$

because all the multi-indices α in (2.19) have at least one component equal to one.

2. We study the regularity of $\mathcal{D}_0^n F$ for $n \in \{\bar{n}+2,\ldots,2\bar{n}\}$. Given $v \in \mathrm{dom}(\mathcal{D}_0^n F)$, it is of the form

$$v = \big(0, (t_2, \varphi_t^{(2)}(0)), \ldots, (t_{n-1}, \varphi_t^{(n-1)}(0))\big), \quad t \in \{0\}\times\mathbb{R}^{n-2} \quad \text{by (2.21)}.$$

Therefore

$$\begin{aligned} \mathcal{D}_0^n F(v) &= w_m D_0^n F_m\big(0, (t_2, \varphi_t^{(2)}(0)), \ldots, (t_{n-1}, \varphi_t^{(n-1)}(0))\big) \\ &= w_m\Bigg(\sum_{h=\bar{n}}^{n}\sum_{\alpha\in\mathcal{F}_h,|\alpha|=n}\frac{n!}{h!\alpha!}F_\varphi^{(h)}(0)\big(t_{\alpha_1}\cdots t_{\alpha_h}\big)\Bigg). \end{aligned} \tag{2.22}$$

Using (2.20) with n in place of $n-1$, we realize that the non-zero terms in (2.22) occur only if $n \geq 2\bar{n}$. Thus, for $n \in \{\bar{n}+2, \dots, 2\bar{n}-1\}$ the differential $\mathcal{D}_0^n F$ is identically zero on its domain. This is not the case when $n = 2\bar{n}$. Indeed, the only "relevant" multi-index α has length $\bar{n}$ and weight $2\bar{n}$: it is $(2, \dots, 2)$. So

$$\mathcal{D}_0^{2\bar{n}} F(v) = w_m D_0^{2\bar{n}} F_m(v) = w_m t_2^{\bar{n}} F_\varphi^{(\bar{n})}(0) \neq 0 \text{ as long as } 0 \neq t_2 \in \mathbb{R}. \tag{2.23}$$

Although $2\bar{n}$ is now even, t_2 is elevated to the power $\bar{n}$ that is odd by Lemma 2.1, and so we have regularity by Proposition 2.1 in the even case.

3. At this point, we consider $\mathcal{D}_0^{2\bar{n}+1} F$. Using the same argument of point 1, we study the domain. An arbitrary element $v \in \operatorname{dom}(\mathcal{D}_0^{2\bar{n}+1} F)$ will be of the form

$$\big(0, (t_2, \varphi_t^{(2)}(0)), \dots, (t_{2\bar{n}}, \varphi_t^{(2\bar{n})}(0))\big), \quad t \in \{0\} \times \mathbb{R}^{2\bar{n}-1},$$

and it will solve

$$D_0^{2\bar{n}} F_m(v) = t_2^{\bar{n}} F_\varphi^{(\bar{n})}(0) = 0. \tag{2.24}$$

Since $F_\varphi^{(\bar{n})}(0) \neq 0$ by the definition of $\bar{n}$, (2.24) holds if and only if $t_2 = 0$. Therefore

$$\operatorname{dom}(\mathcal{D}_0^{2\bar{n}+1} F) = \{\big(0, 0, (t_3, \varphi_t^{(3)}(0)), \dots, (t_{2\bar{n}}, \varphi_t^{(2\bar{n})}(0))\big) \;:\; t \in \{0\}^2 \times \mathbb{R}^{2\bar{n}-2}\}. \tag{2.25}$$

4. We repeat the procedure of point 2, but now we are interested only in the multi-indices α with all components ≥ 3. We deduce that

$$\operatorname{dom}(\mathcal{D}_0^n F) = \{\big(0, 0, (t_3, \varphi_t^{(3)}(0)), \dots, (t_n, \varphi_t^{(n)}(0))\big) \;:\; t \in \{0\}^2 \times \mathbb{R}^{n-3}\}$$

for all $n \in \{2\bar{n}+2, \dots, 3\bar{n}\}$, and also that

$$\mathcal{D}_0^n F(v) = \begin{cases} 0, & \text{for } n \in \{2\bar{n}+1, \dots, 3\bar{n}-1\}, \\ w_m t_3^{\bar{n}} F_\varphi^{(\bar{n})}(0), & n = 3\bar{n}, \end{cases}$$

for $v \in \operatorname{dom}(\mathcal{D}_0^n F)$. Thus, only $\mathcal{D}_0^{3\bar{n}} F$ is regular by Proposition 2.1.

5. Reiterating this algorithm, it is not difficult to realize that $\mathcal{D}_0^n F$ is regular if and only if n is a non-zero multiple of $\bar{n}$. In the end, we have uniqueness modulus $\bar{n}$.

2.3 New definition of regularity

We would like to give a definition of regular differential that does not depend on the existence of the polynomial extension, see Proposition 1.3, and that provides equivalent conditions for openness, at least for the corank one case.

Proposition 2.1 suggests the following new definition of regular differential:

Definition 2.1 (Regular n-differential)**.** *Let $F \in C^\infty(X; \mathbb{R}^m)$ be such that $0 \in X$ is a critical point of corank one for F. We say that $\mathcal{D}_0^n F, n \geq 2$, is regular if*

- *n is even and there exist 2 elements $v^\pm \in \mathrm{dom}(\mathcal{D}_0^n F)$ such that*

$$\mathcal{D}_0^n F(v^+) > 0 \quad \textit{and} \quad \mathcal{D}_0^n F(v^-) < 0.$$

- *n is odd and there exist an element $v \in \mathrm{dom}(\mathcal{D}_0^n F)$ such that*

$$\mathcal{D}_0^n F(v) \neq 0.$$

In the particular case of Theorem 2.1, we proved that the openness at 0 of the map was equivalent to the existence of a regular n-differential according to Definition 2.1.

We wonder whether the statement of Theorem 2.1 can be generalized, keeping the corank one: namely, if a map $F \in C^\infty(X;\mathbb{R}^m)$ with $F(0) = 0, d_0F \neq 0$ and corank one is open at $0 \in X$ if and only if there exists a regular n-differential as in Definition 2.1.

We start from the following result. As for Proposition 1.2, we fix coordinates on both X and $\mathbb{R}^m$ so that

$$X = \ker(d_0F) \oplus \mathbb{R}^{m-1}, \quad \mathbb{R}^m = \mathrm{Im}(d_0F) \oplus \mathbb{R} \quad \text{and} \quad d_0F(x) = d_0F(u,w) = \begin{pmatrix} w \\ 0 \end{pmatrix} \tag{2.26}$$

for every $X \ni x = (u,w) \in \ker(d_0F) \times \mathbb{R}^{m-1}$.

Proposition 2.3. *Let X be a Banach space and $F \in C^\infty(X;\mathbb{R}^m)$ be such that $F(0) = 0$, $d_0F \neq 0$ and* $\mathrm{corank}(d_0F) = 1$. *Fix coordinates like in (2.26) and assume F to be open at $0 \in X$. Then at least one of the following situations must occur:*

1. *There exist $n \geq 2$ odd and $v \in \mathrm{dom}(\mathcal{D}_0^n F)$ such that*

$$\mathcal{D}_0^n F(v) \neq 0. \tag{2.27}$$

2. *There exist $n^\pm \geq 2$ even and elements $v^\pm \in \mathrm{dom}(\mathcal{D}_0^{n^\pm} F)$ such that*

$$\mathcal{D}_0^{n^+} F(v^+) > 0 \quad \textit{and} \quad \mathcal{D}_0^{n^-} F(v^-) < 0. \tag{2.28}$$

Proof. Using the notation of Theorem 2.1, F being open at 0 implies that

$$F_\varphi \in C^\infty(U;\mathbb{R}), \quad F_\varphi(u) := F_m(u,\varphi(u)) \in \mathbb{R}$$

is also open at $0 \in \ker(d_0F)$, where U is an open neighbourhood of $0 \in \ker(d_0F)$. The map φ is given by the implicit function theorem applied to $\tilde{F} = (F_1,\dots,F_{m-1}) = 0$.

For fixed $\sigma > 0$, consider the curves satisfying

$$\gamma \in C^\infty((-\sigma,\sigma);U) \quad \text{with} \quad \gamma(0) = 0 \tag{2.29}$$

and define

$$F_{\varphi,\gamma}(s) := F_\varphi(\gamma(s)) = F_m(\gamma(s),\varphi(\gamma(s))), \quad s \in (-\sigma,\sigma). \tag{2.30}$$

For γ like in (2.29), we let

$$n_\gamma := \min\left\{ n \ : \ \frac{\partial^n}{\partial s^n} F_{\varphi,\gamma}(s)\Big|_{s=0} \neq 0 \right\} \geq 2, \quad n_\gamma = +\infty \quad \text{if } F_{\varphi,\gamma} \equiv 0. \tag{2.31}$$

The coordinates in (2.26) imply $d_0F_m = 0$ and so $d_0F_\varphi = 0$ by the chain rule.

Since F_φ is open at 0, then at least one of the following situations must occur:

1. There exists γ like in (2.29) such that n_γ is odd.

2. There exist curves $\gamma^\pm$ such that $n_{\gamma^\pm}$ are both even,

$$\frac{\partial^{n_{\gamma^+}}}{\partial s^{n_{\gamma^+}}} F_{\varphi,\gamma^+}(s)\bigg|_{s=0} > 0 \quad \text{and} \quad \frac{\partial^{n_{\gamma^-}}}{\partial s^{n_{\gamma^-}}} F_{\varphi,\gamma^-}(s)\bigg|_{s=0} < 0.$$

If neither of the previous conditions occurs, for every γ we have n_γ even and

$$\frac{\partial^{n_\gamma}}{\partial s^{n_\gamma}} F_{\varphi,\gamma}(s)\bigg|_{s=0} \geq 0. \tag{2.32}$$

It is not restrictive to assume the derivative in (2.32) to be non-zero. Otherwise, $F_{\varphi,\gamma} \equiv 0$ by the definition of n_γ in (2.31). The condition (2.32) holds for every γ: F_φ is locally concave around 0 so it cannot be open at 0. The case ≤ 0 is identical.

Now, assume there exists γ like in (2.29) such that n_γ is odd. By the definition of n_γ we have:

$$\frac{\partial^h}{\partial s^h} F_m(\gamma(s), \varphi(\gamma(s)))\bigg|_{s=0} = 0 \quad \forall 2 \leq h \leq n_\gamma - 1, \tag{2.33}$$

$$\frac{\partial^{n_\gamma}}{\partial s^{n_\gamma}} F_m(\gamma(s), \varphi(\gamma(s)))\bigg|_{s=0} \neq 0. \tag{2.34}$$

We can easily reformulate (2.33) and (2.34) in terms of the differentials of F_m:

$$D_0^h F_m\big((\gamma'(0), \varphi_\gamma'(0)), \ldots, (\gamma^{(h)}(0), \varphi_\gamma^{(h)}(0))\big) = 0 \quad \forall 2 \leq h \leq n_\gamma - 1, \tag{2.35}$$

$$D_0^{n_\gamma} F_m\big((\gamma'(0), \varphi_\gamma'(0)), \ldots, (\gamma^{(n_\gamma)}(0), \varphi_\gamma^{(n_\gamma)}(0))\big) \neq 0, \tag{2.36}$$

where $\varphi_\gamma(s) = \varphi\big(\sum_{h=1}^{n_\gamma} \frac{\gamma^{(h)}(0)}{h!} s^h\big)$. In (2.33), (2.34), it is not restrictive to replace the inner function with its Taylor polynomial at 0 when computing the derivative of the composition.

We claim that the conditions (2.35) imply

$$v = \big((\gamma'(0), \varphi_\gamma'(0)), \ldots, (\gamma^{(n_\gamma-1)}(0), \varphi_\gamma^{(n_\gamma-1)}(0))\big) \in \operatorname{dom}(\mathcal{D}_0^{n_\gamma} F). \tag{2.37}$$

In the proof of Proposition 1.2, we showed that

$$\operatorname{dom}(\mathcal{D}_0^{n_\gamma} \tilde{F}) = \{\big(u_1, \varphi_u'(0)), \ldots, (u_{n_\gamma-1}, \varphi_u^{(n_\gamma-1)}(0)\big) : u \in \ker(d_0 F)^{n_\gamma-1}\}. \tag{2.38}$$

The element v in (2.37) is like the ones in (2.38) for $u = (\gamma'(0), \ldots, \gamma^{(n_\gamma-1)}(0)) \in \ker(d_0 F)^{n_\gamma-1}$. The conditions (2.35) imply that $v \in \operatorname{dom}(\mathcal{D}_0^{n_\gamma} F_m)$ so (2.37) holds since $F = (\tilde{F}, F_m)$.

Moreover, (2.36) implies (2.27) since, by (2.26), the map proj is just the projection on the last component of $\mathbb{R}^m$ and

$$\mathcal{D}_0^{n_\gamma} F(v) := \operatorname{proj}(D_0^{n_\gamma} F(v, *)) \quad \text{for any choice of } * \in X\text{, we choose } * = (\gamma^{(n_\gamma)}(0), \varphi_\gamma^{(n_\gamma)}(0)).$$

We obtain (2.27) by taking $n = n_\gamma$ and v like in (2.37).

Using a similar argument for the even case, we obtain (2.28). □

If (2.27) holds, then there exists a regular differential of odd order as in Definition 2.1. The problem rises when $\mathcal{D}_0^n F \equiv 0$ for all $n \geq 2$ odd. In (2.28) it is not guaranteed that $n^+ = n^-$ and so there may not exist a regular differential of even order.

By Proposition 2.3, we conclude that Definition 2.1 has to be weakened in order to obtain candidate equivalent conditions for openness. Thus, a possible definition of regularity for the corank one case may be following:

Definition 2.2. *Let $F \in C^\infty(X;\mathbb{R}^m)$ be such that $F(0) = 0$, $d_0F \neq 0$ and $\operatorname{corank}(d_0F) = 1$. Fix coordinates like in (2.26). We say that F is regular at 0 if at least one of the following conditions is satisfied:*

1. *There exist $n \geq 2$ odd and $v \in \operatorname{dom}(\mathcal{D}_0^n F)$ such that*

$$\mathcal{D}_0^n F(v) \neq 0. \tag{2.39}$$

2. *There exist $n^\pm \geq 2$ even and elements $v^\pm \in \operatorname{dom}(\mathcal{D}_0^{n^\pm} F)$ such that*

$$\mathcal{D}_0^{n^+} F(v^+) > 0 \quad \textit{and} \quad \mathcal{D}_0^{n^-} F(v^-) < 0. \tag{2.40}$$

2.3.1 Equivalent conditions for openness

Now we show that Definition 2.2 actually gives equivalent conditions for openness. We fix coordinates on both X and $\mathbb{R}^m$ so that

$$X = \ker(d_0F) \oplus \mathbb{R}^{m-1}, \quad \mathbb{R}^m = \operatorname{Im}(d_0F) \oplus \mathbb{R} \quad \text{and} \quad d_0F(x) = d_0F(u,w) = \begin{pmatrix} w \\ 0 \end{pmatrix} \tag{2.41}$$

for all $X \ni x = (u,w) \in \ker(d_0F) \times \mathbb{R}^{m-1}$.

Theorem 2.3. *Let $F \in C^\infty(X;\mathbb{R}^m)$ be such that $F(0) = 0$. Assume that $d_0F \neq 0$ and $0 \in X$ is a critical point of corank $l = 1$. Fix coordinates as in (2.41). Then F is open at 0 if and only if F is regular at 0 according to Definition 2.2.*

Proof. If F is open at 0, we just use Proposition 2.3. We prove the other implication.

Assume F to be regular at 0. If (2.39) occurs, we just repeat the proof of Theorem 1.2 for the n odd case. So without loss of generality we can suppose (2.40) to hold. We can also assume $n^+ > n^-$. We basically repeat the argument of Theorem 1.2 with minor adjustments.

Let

$$v^+ = (v_1^+, \ldots, v_{n^+-1}^+) \quad \text{and} \quad v^- = (v_1^-, \ldots, v_{n^--1}^-). \tag{2.42}$$

Define

$$w : \mathbb{R} \to X^{n^+-1} \quad w(t) = \begin{cases} (tv_1^+, t^2v_2^+, \ldots, t^{n^+-1}v_{n^+-1}^+) & t \geq 0, \\ (-tv_1^-, t^2v_2^-, \ldots, -t^{n^--1}v_{n^--1}, 0, \ldots, 0) & t < 0. \end{cases} \tag{2.43}$$

It is important to notice that $w \in C^0(\mathbb{R}; X^{n^+-1})$, like the regular extension of Definition 1.3.

Define

$$\Phi : \mathbb{R}^{m-1} \times \mathbb{R} \to X \quad \Phi(r,t) = r + \sum_{j=1}^{n^+-1} \frac{w_j(t)}{j!}, \quad w(t) = (w_1(t), \ldots, w_{n^+-1}(t)). \tag{2.44}$$

At this point, we compute the Taylor expansion at 0 of $F \circ \Phi$.

We fix $t \in \mathbb{R}$ and define $\phi(s) := F(\Phi(0, st))$ for $s \geq 0$. For all $1 \leq j \leq n^+ - 1$, $w_j(t)$ is homogeneous of degree j. The function $\phi(s)$ has a Taylor expansion at 0 of the form

$$\phi(s) = \sum_{j=1}^{n^+} \frac{\phi^{(j)}(0)}{j!} s^j + \frac{\phi^{(n^+ + 1)}(\bar{s})}{j!} \bar{s}^{n^+ + 1}, \quad \exists \bar{s} \in [0, s]. \tag{2.45}$$

By construction, $\phi^{(j)}(0) = D_0^j F(w_1(t), \ldots, w_j(t))$ for all j. The value of $\phi^{(j)}(0)$ depends on both j and t:

1. For $1 \leq j \leq n^- - 1$, $\phi^{(j)}(0) = 0$ for every t.
2. For $j = n^-$,
$$\phi^{(n^-)}(0) = \begin{cases} 0 & t \geq 0, \\ D_0^{n^-} F(v^-, 0)(-t)^{n^-} & t < 0. \end{cases}$$
3. For $n^- + 1 \leq j \leq n^+ - 1$,
$$\phi^{(j)}(0) = \begin{cases} 0 & t \geq 0, \\ D_0^j F(v^-, 0, \ldots, 0)(-t)^j & t < 0. \end{cases}$$
4. Finally, for $j = n^+$
$$\phi^{(n^+)}(0) = \begin{cases} D_0^{n^+} F(v^+, 0) t^{n^+} & t \geq 0, \\ D_0^{n^+} F(v^-, 0, \ldots, 0)(-t)^{n^+} & t < 0. \end{cases}$$

Thus

$$F(\Phi(0, t)) = \begin{cases} D_0^{n^+} F(v^+, 0) \frac{t^{n^+}}{(n^+)!} + E^+(t) & t \geq 0, \\ D_0^{n^-} F(v^-, 0) \frac{(-t)^{n^-}}{(n^-)!} + \sum_{j=n^-+1}^{n^+} D_0^j F(v^-, 0 \ldots, 0) \frac{(-t)^j}{j!} + E^-(t) & t < 0, \end{cases} \tag{2.46}$$

where

$$|E^+(t)| \leq C^+ |t|^{n^+ + 1} \quad \text{and} \quad |E^-(t)| \leq C^- |t|^{n^+ + 1} \quad \exists C^+, C^- > 0. \tag{2.47}$$

We can rewrite (2.46), (2.47) as

$$F(\Phi(0, t)) = \begin{cases} D_0^{n^+} F(v^+, 0) \frac{t^{n^+}}{(n^+)!} + E^+(t) & t \geq 0, \\ D_0^{n^-} F(v^-, 0) \frac{(-t)^{n^-}}{(n^-)!} + E^-(t) & t < 0, \end{cases} \tag{2.48}$$

where

$$|E^+(t)| \leq C^+ |t|^{n^+ + 1} \quad \text{and} \quad |E^-(t)| \leq C^- |t|^{n^- + 1} \quad \exists C^+, C^- > 0. \tag{2.49}$$

At this point, we obtain the following expansion:

$$F(\Phi(r, t)) = \begin{cases} d_0 F(r) + D_0^{n^+} F(v^+, 0) \frac{t^{n^+}}{(n^+)!} + R^+(r, t) & t \geq 0, \\ d_0 F(r) + D_0^{n^-} F(v^-, 0) \frac{t^{n^-}}{(n^-)!} + R^-(r, t) & t < 0 \end{cases} \tag{2.50}$$

where

$$\lim_{(r,t)\to 0} \frac{R^+(r,t)}{|r|+|t|^{n^+}} = 0 \quad \text{and} \quad \lim_{(r,t)\to 0} \frac{R^-(r,t)}{|r|+|t|^{n^-}} = 0. \tag{2.51}$$

We "normalize" using the following function, which is a generalization of $\varrho(t)$ in Definition 1.4:

$$\bar{\varrho}(t) = \begin{cases} \operatorname{sgn}(t) \sqrt[n^+]{|t|} & t \geq 0, \\ \operatorname{sgn}(t) \sqrt[n^-]{|t|} & t < 0. \end{cases} \tag{2.52}$$

Therefore

$$F(\Phi(r, \bar{\varrho}(t))) = \begin{cases} d_0F(r) + D_0^{n^+} F(v^+, 0) \frac{|t|}{(n^+)!} + R^+(r, \bar{\varrho}(t)) & t \geq 0, \\ d_0F(r) + D_0^{n^-} F(v^-, 0) \frac{|t|}{(n^-)!} + R^-(r, \bar{\varrho}(t)) & t < 0. \end{cases} \tag{2.53}$$

Now we stress out the $\operatorname{Im}(d_0F)$ and $\operatorname{coker}(d_0F)$ components of $F(\Phi(r, \bar{\varrho}(t)))$. Using the set of coordinates (2.41)

$$F(\Phi(r, \bar{\varrho}(t))) = \begin{cases} \begin{pmatrix} d_0F(r) + g^+(t) + R_1^+(r, \bar{\varrho}(t)) \\ L^+|t| + R_2^+(r, \bar{\varrho}(t)) \end{pmatrix} & t \geq 0, \\ \begin{pmatrix} d_0F(r) + g^-(t) + R_1^-(r, \bar{\varrho}(t)) \\ L^-|t| + R_2^-(r, \bar{\varrho}(t)) \end{pmatrix} & t < 0. \end{cases} \tag{2.54}$$

We simplified the expression by calling $L^+ = \frac{\mathcal{D}_0^{n^+} F(v^+)}{(n^+)!} > 0$ and $L^- = \frac{\mathcal{D}_0^{n^-} F(v^-)}{(n^-)!} < 0$.

We repeat the fixed point argument on (2.54) and the proof is finished. The function $g^{\pm}(t)$ is the $\operatorname{Im}(d_0F)$ component of $D_0^{n^{\pm}} F(v^{\pm}, 0) \frac{|t|}{(n^{\pm})!}$ and it is both continuous and 1-homogeneous.

In our case, the function $f(t)$ that appears in the proof of Theorem 1.2 is

$$f(t) = \begin{cases} L^+|t| & t \geq 0, \\ L^-|t| & t < 0. \end{cases} \tag{2.55}$$

Since $L^+L^- < 0$, $f(t)$ in (2.55) is a homeomorphism and we are able to find the constant $L > 0$ as in Remark 1.1. □

In conclusion, we provided a complete description of maps $F \in C^\infty(X, \mathbb{R}^m)$, $m \in \mathbb{N}$, $F(0) = 0$, with corank one that are open at 0:

1. If $\operatorname{corank}(d_0F) = 1$ and $d_0F \neq 0$, Theorem 2.3 gives us equivalent condition for openness.
2. Otherwise, $\operatorname{corank}(d_0F) = 1$ and $d_0F = 0$: $m = 1$ by Definition 1.1. Despite $d_0F = 0$, we can repeat the argument of Proposition 2.3 with the curves γ: the map F has only one component. Thus, if F is open at 0, then it is regular at 0 as in Definition 2.2. To prove the opposite implication, we just repeat the proof of Theorem 2.3: the only difference is that $\Phi(r, t) = \Phi(t)$ as $m = 1$, see (2.44).

Chapter 3

Corank $l = 2$ case

The next step is to study the case of corank $l = 2$. We consider functions satisfying

$$F \in C^\infty(\mathbb{R}^2; \mathbb{R}^2), \quad F(0) = 0, \quad d_0 F = 0. \tag{3.1}$$

We wonder whether openness and regular differentials, as in Definition 1.4, are equivalent for functions as in (3.1). By Theorem 1.2, we know that the existence of a regular differential always implies openness. However, in the general case the converse implication does not hold. In Section 3.1, we will provide a map satisfying (3.1) that is open at 0, but has no regular differentials.

3.1 A counterexample in the plane

In this section, we prove the following result.

Theorem 3.1. *The map $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $F(x, y) = (x^2 - y^2, xy)$ is open at 0, but there exists no $n \in \mathbb{N}$ such that $\mathcal{D}_0^n F : \mathbb{R}^2 \to \operatorname{coker}(d_0 F) = \mathbb{R}^2$ is regular according to Definition 1.4.*

We need a different criterion to identify open maps, in particular the homogeneous ones.

Proposition 3.1. *Let $F : \mathbb{R}^d \to \mathbb{R}^m$ be homogeneous of degree $k \in \mathbb{N}, k \neq 0$, and continuous. Assume that $F(z) = 0$ if and only if $z = 0$. Then F is open at the origin if and only if F is surjective.*

Proof. F is open at 0 if and only if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$B(0, \delta) \subset F(B(0, \varepsilon)). \tag{3.2}$$

Suppose F to be open at 0. Since $F(B(0, \varepsilon))$ contains a ball and thus all directions, F is surjective. By (3.2), for any $\nu \in B(0, \delta/2)$ there is $z \in B(0, \varepsilon)$ such that $F(z) = \nu$. Then

$$F\left(\frac{z}{\sqrt[k]{|\nu|}}\right) = \frac{\nu}{|\nu|} \in \mathbb{S}^{m-1} \Longrightarrow \{\lambda \nu \ : \ \lambda \geq 0\} \subseteq F(\langle z \rangle) \quad \text{by } F \text{ being homogeneous.}$$

The image of F contains every half line starting from the origin and so the whole $\mathbb{R}^m$.

Now, assume F to be surjective and fix $\varepsilon > 0$. We need to find $\delta > 0$ as in (3.2). By surjectivity, for every $\nu \in \mathbb{S}^{m-1}$ there exists $0 \neq z_\nu \in \mathbb{R}^d$ such that $F(z_\nu) = \nu$. Thus

$$F\left(\frac{\varepsilon}{|z_\nu|} z_\nu\right) = \left(\frac{\varepsilon}{|z_\nu|}\right)^k \nu \Longrightarrow \left|F\left(\frac{\varepsilon}{|z_\nu|} z_\nu\right)\right| = \left(\frac{\varepsilon}{|z_\nu|}\right)^k, \quad \frac{\varepsilon}{|z_\nu|} z_\nu \in \partial B(0, \varepsilon). \tag{3.3}$$

The map F is homogeneous so

$$\left\{\lambda\nu \ : \ 0 \leq \lambda < \left(\frac{\varepsilon}{|z_\nu|}\right)^k\right\} \subseteq F(B(0,\varepsilon)).$$

Since $F(z) = 0$ if and only if $z = 0$, for every $\varepsilon > 0$

$$\min_{\partial B(0,\varepsilon)} |F(z)| > 0. \tag{3.4}$$

The minimum exists and it is finite because F is continuous. Therefore

$$\inf\left\{\left(\frac{\varepsilon}{|z_\nu|}\right)^k \ : \ \nu \in \mathbb{S}^{m-1}\right\} \geq \min_{\partial B(0,\varepsilon)} |F(z)| > 0.$$

We choose $\delta = \min_{\partial B(0,\varepsilon)} |F(z)|$. □

We begin the proof of Theorem 3.2. We notice that:

- F is polynomial and 2-homogeneous;
- $F(x, y) = 0$ if and only if $x = y = 0$;
- F is surjective.

By Proposition 3.1, F is open at 0.

Remark 3.1. *In our setup, we have*

$$F(v) = \frac{1}{2} d_0^2 F(v), \quad v \in \mathbb{R}^2,$$

and the map $\mathrm{proj} : \mathbb{R}^2 \to \mathrm{coker}(d_0 F) = \mathbb{R}^2$ *is just the identity map on* $\mathbb{R}^2$.

Now we prove the second statement of Theorem 3.1. For $n \geq 2$ and $v \in \mathbb{R}^{2n}$, we apply Faà di Bruno formula (Proposition 1.1):

$$\begin{aligned} D_0^n F(v) &= \sum_{\alpha \in \mathcal{F}_2, |\alpha| = n} \frac{n!}{2 \cdot \alpha!} d_0^2 F(v_\alpha) \quad \text{because only } d_0^2 F \text{ is not trivial} \\ &= \begin{cases} \sum_{k=1}^{[\frac{n}{2}]} \binom{n}{k} d_0^2 F(v_k, v_{n-k}), & n \text{ odd}, \\ \sum_{k=1}^{\frac{n}{2}-1} \binom{n}{k} d_0^2 F(v_k, v_{n-k}) + \frac{1}{2} \binom{n}{n/2} d_0^2 F(v_{n/2}, v_{n/2}), & n \text{ even}. \end{cases} \end{aligned} \tag{3.5}$$

As a consequence, we are able to compute explicitly the domains of all the differentials:

Proposition 3.2. *Let* $n \in \mathbb{N}, n \geq 1$. *Then*

$$\mathrm{dom}(\mathcal{D}_0^{2n+1} F) = \{(0, \ldots, 0, v_{n+1}, \ldots, v_{2n}) \in \mathbb{R}^{4n} \ : \ v_{n+1}, \ldots, v_{2n} \in \mathbb{R}^2\}, \tag{3.6}$$

$$\mathrm{dom}(\mathcal{D}_0^{2n} F) = \{(0, \ldots, 0, v_n, \ldots, v_{2n-1}) \in \mathbb{R}^{2(2n-1)} \ : \ v_n, \ldots, v_{2n-1} \in \mathbb{R}^2\}. \tag{3.7}$$

Proof. The proof consists of three steps:

1. The base case: $\mathrm{dom}(\mathcal{D}_0^2 F) = \ker(d_0 F) = \mathbb{R}^2$ satisfies (3.7) for $n = 1$;

2. Assume, for some $n \geq 1$, that

$$\mathrm{dom}(\mathcal{D}_0^{2n} F) = \{(0, \dots, 0, v_n, \dots, v_{2n-1}) \in \mathbb{R}^{2(2n-1)} \ : \ v_n, \dots, v_{2n-1} \in \mathbb{R}^2\}.$$

   An element $v \in \mathbb{R}^{4n}$ belongs to $\mathrm{dom}(\mathcal{D}_0^{2n+1} F)$ if

$$v \in \mathrm{dom}(\mathcal{D}_0^{2n} F) \times \mathbb{R}^2 \Longrightarrow v = (0, \dots, 0, v_n, \dots, v_{2n})$$

   and

$$\begin{aligned} D_0^{2n} F(v) &= \sum_{k=1}^{n-1} \binom{2n}{k} d_0^2 F(v_k, v_{2n-k}) + \frac{1}{2} \binom{2n}{n} d_0^2 F(v_n, v_n) \\ &= \frac{1}{2} \binom{2n}{n} d_0^2 F(v_n, v_n) = 0 \Longleftrightarrow v_n = 0 \quad \text{by Remark 3.1.} \end{aligned}$$

   Thus, $\mathrm{dom}(\mathcal{D}_0^{2n+1} F)$ satisfies (3.6).

3. To conclude, it is enough to prove that (3.6) implies (3.7). Assume now that

$$\mathrm{dom}(\mathcal{D}_0^{2n+1} F) = \{(0, \dots, 0, v_{n+1}, \dots, v_{2n}) \in \mathbb{R}^{4n} \ : \ v_{n+1}, \dots, v_{2n} \in \mathbb{R}^2\}$$

   for some $n \in \mathbb{N}$. An element $v \in \mathbb{R}^{2(2n+1)}$ belongs to $\mathrm{dom}(\mathcal{D}_0^{2(n+1)} F)$ if

$$v \in \mathrm{dom}(\mathcal{D}_0^{2n+1} F) \times \mathbb{R}^2 \Longrightarrow v = (0, \dots, 0, v_{n+1}, \dots, v_{2n+1})$$

   and

$$D_0^{2n+1} F(v) = \sum_{k=1}^{n} \binom{2n+1}{k} d_0^2 F(v_k, v_{(2n+1)-k}) = 0,$$

   but the first n components of v are zero and so does the sum. There are no additional constraints for v so we proved (3.7).

□

Now, we evaluate each differential at a generic element of its domain. By (3.5) and Remark 3.1, we can explicitly compute any differential.

Corollary 3.1. *For every* $n \geq 1$ *,* $\mathcal{D}_0^{2n+1} F = 0$.

Proof. Every $v \in \mathrm{dom}(\mathcal{D}_0^{2n+1} F)$ is of the form

$$v = (0, \dots, 0, v_{n+1}, \dots, v_{2n}), \quad v_k \in \mathbb{R}^2 \quad \forall n+1 \leq k \leq 2n \quad \text{by (3.6).}$$

Then for any $* \in \mathbb{R}^2$ we have

$$\mathcal{D}_0^{2n+1} F(v) = D_0^{2n+1} F(v, *) = \sum_{k=1}^{n} \binom{2n+1}{k} d_0^2 F(v_k, v_{(2n+1)-k}) \equiv 0$$

since $v_k = 0$ for every $1 \leq k \leq n$. □

Corollary 3.2. *For every $n \geq 1$ and $v \in \operatorname{dom}(\mathcal{D}_0^{2n}F)$,*

$$\mathcal{D}_0^{2n}F(v) = \frac{1}{2}\binom{2n}{n}\, d_0^2F(v_n, v_n),$$

where v_n is the n-th component of v.

Proof. An element $v \in \operatorname{dom}(\mathcal{D}_0^{2n}F)$ is of the form

$$v = (0, \ldots, 0, v_n, \ldots, v_{2n-1}), \quad v_k \in \mathbb{R}^2 \ \forall n \leq k \leq 2n-1 \quad \text{by (3.7).}$$

So for any $* \in \mathbb{R}^2$ we have

$$\mathcal{D}_0^{2n}F(v) = D_0^{2n}F(v,*) = \sum_{k=1}^{n-1}\binom{2n}{k}\, d_0^2F(v_k, v_{2n-k}) + \frac{1}{2}\binom{2n}{n}\, d_0^2F(v_n, v_n)$$

with $v_k = 0$ for every $1 \leq k \leq n-1$. The sum is equal to zero and only the last term remains. □

Therefore, if there existed a regular differential, its order would be even. So assume that there exists a regular extension $w(t) : \mathbb{R}^2 \to \operatorname{dom}(\mathcal{D}_0^{2n}F)$, see Definition 1.3, such that $\mathcal{D}_0^{2n}F : \mathbb{R}^2 \to \mathbb{R}^2$ is regular, for some n. We require the function

$$\mathbb{R}^2 \ni t \mapsto f(t) = \mathcal{D}_0^{2n}F(w(\varrho(t))) = \binom{2n}{n} F(w_n(\varrho(t))) \tag{3.8}$$

to be a homeomorphism.

The map $t \mapsto w_n(\varrho(t))$ has to be a bijection from $\mathbb{R}^2$ onto a subset of the plane where F is both injective and surjective. For instance, we can take

$$A = \{(0, y) \ : \ y \geq 0\} \cup \{(x, y) \ : \ x > 0, y \in \mathbb{R}\}.$$

The map $w_n(\varrho(t))$ is a bijective if and only if $w_n(t)$ is. Indeed,

$$t \mapsto \varrho(t) = \left(\operatorname{sgn}(t_1)\sqrt[2n]{|t_1|}, \operatorname{sgn}(t_2)\sqrt[2n]{|t_2|}\right)$$

is a bijection from $\mathbb{R}^2$ onto itself.

However, $w_n(t)$ is a n-homogeneous polynomial:

- If n is even, then $w_n(t)$ is an even function. So w_n cannot be injective.
- If n is odd, then $w_n(t)$ is an odd function. Since $A = \operatorname{Im}(w_n)$ contains the half line

  $$\{(0, y) \ : \ y \geq 0\},$$

  $\operatorname{Im}(w_n)$ will actually contain the entire y-axis. We lose the injectivity of $F \circ w_n$:

  $$\{(x, 0) \ : \ x \leq 0\} = F(\{(0, y) \ : \ y \leq 0\}) = F(\{(0, y) \ : \ y \geq 0\}).$$

In the end, $f(t)$ in (3.8) cannot be a bijection: there is no regular differential of even order. The proof of Theorem 3.1 is finished.

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