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The Yang-Mills Plateau Problem in dimension four

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Chapter 1

Introduction

Yang-Mills theories underlie contemporary particle physics and in particular they are significant for the description of fundamental forces of nature, excluding gravitation. The key idea behind them is that each interaction force can be interpreted as the curvature of a connection form over some principal bundle. The choice of the structure group for the bundle is dictated by the internal symmetries of the specific fundamental interaction we are considering, while the base manifold usually coincides with the Minkowsky space of special relativity. In the standard model of particle physics the gauge group is $U(1)\times SU(2)\times SU(3)$, where $U(1)\times SU(2)$ expresses the electroweak interaction, while $SU(3)$ models the strong force. Chapter 2 will be devoted to the study of the geometrical framework of these physical theories.

In the first section we define the concept of Lie group G and the associated Lie Algebra $\mathfrak{g}$. In particular, we also analyse the action of Lie groups on manifolds which underlies the theory of connections. In the second section instead we develop the theory of bundles, focusing on vector bundles and principal fibre bundles over some manifold M. We characterize principal bundles proving that given a Lie group G, a covering $\mathcal{U} = \{U_i\}_{i\in I}$ of the base manifold M, and a family of maps $g_{ij} \in C^\infty(U_i \cap U_j, G)$ satisfying the cocycle conditions, then there exists a unique G-principal bundle (up to isomorphism) $\pi : P \to M$, whose transition functions are exactly the family $\{g_{ij}\}$. This characterization for bundles will be useful in the attempt of defining Sobolev bundles.

We define connections both on vector bundles and principal fibre bundles. The first one are not essential to this work, but they are studied here because in Chapter 5 the Levi-Civita connection, which is a special connection on Riemannian manifolds (completely determined by the metric itself), will emerge naturally in the study of the Euler-Lagrange equations of weakly harmonic maps. For the same reason, we will also recall the concept of Shape operator.

Given a principal fibre bundle $\pi : P \to M$ with structure group G, we present two equivalent definitions of connection. Indeed, one may see a connection as a right equivariant horizontal distribution Γ. To such a distribution Γ we can uniquely associate a $\mathfrak{g}$-valued differential 1-form ω on P, called connection form, that is a pseudotensorial form of type Ad, and whose values on a fundamental vector field $A^\sharp$ associated to an $A \in \mathfrak{g}$ are identically A. We show that also the inverse is true, namely that given a $\mathfrak{g}$-valued differential 1-form ω on P satisfying the two aforementioned properties, we can build a unique equivariant horizontal distribution Γ associated to ω. Therefore, the two definitions are actually equivalent.

The physical fields Yang-Mills theories deal with, that in physics literature are called gauge potentials, are representations on the base manifold of these connection forms. Namely, given a G-principal bundle $\pi : P \to M$ and $\mathcal{A} = \{(U_i, \chi_i)\}_{i\in I}$ an atlas for it, then the gauge potentials of a connection ω are the local $\mathfrak{g}$-valued differential 1-forms $A_i := s_i^*(\omega)$, where $s_i : U_i \to \pi^{-1}(U_i)$ are the local trivial cross sections associated to the atlas $\mathcal{A}$. In Proposition 2.3.25, we show that the family $\{A_i\}_{i\in I}$ satisfies the fundamental relation

$$A_j = g_{ij}^{-1} dg_{ij} + g_{ij}^{-1} A_i g_{ij} \ \text{ in } \ U_i \cap U_j \tag{1.1}$$

called compatibility condition, where G is a matrix Lie group, and g_{ij} are the transition functions associated to $\mathcal{A}$. It is also shown that conversely if a family $\{A_i\}_{i\in I}$ of $\mathfrak{g}$-valued 1-forms on the covering $\mathcal{U}$ satisfies (1.1), then there exists a unique connection ω on P such that $s_i^*(\omega) = A_i$ for each $i \in I$. We will use this characterization to define Sobolev connections.

Once the theory of connections is established we define the most important geometrical object, the one that from a physical point of view should express the forces determined by the interacting particles: the curvature form Ω. It is defined as the differential of a connection form ω computed along the horizontal directions.
Just as a connection admits a local representation so does its curvature Ω , whose pull-back $s^*\Omega$ via some cross section s is called local field strength, and it is denoted as F_A, where $A := s^*\omega$. Under a local change of gauge g, as the local gauge potential trans-

forms from A to A^g via the compatibility condition (1.1), the field strength F_A transforms to F_{A^g}, which turns out to be the adjoint through g^{-1} of F_A. Using Cartan structure equation, one proves that $F_A = dA + [A, A]$, where the $\mathfrak{g}$-valued two form $[A, A]$ is defined by $[A, A](X, Y) = A(X)A(Y) - A(Y)A(X)$ for each vector field X, Y. Once we endow the Lie algebra $\mathfrak{g}$ with an Ad-invariant scalar product, and the base manifold M with a metric, we can norm pointwise the local field strength in M. The square of such a norm will be our Lagrangian, and the part of Analysis will be entirely devoted to the study of the properties of the corresponding functional, which will be then defined on the space of local gauge potentials.

In the first chapter devoted to the analytic part, called Hodge theory, we start by defining differential forms over some domain in $\mathbb{R}^n$. In particular we generalize these definitions in order to get Sobolev differential forms, and we state some technical results that will be useful in order to develop properly the concepts of Sobolev bundles and Sobolev connections.

Indeed, the study of the minimization problem requires more generic structures, and we need to relax the regularity of both the bundle and the connection. We consider for an open bounded smooth domain $\Omega \subset \mathbb{R}^n$ a covering $\mathcal{U} = \{U_i\}_{i \in I}$, and define a $W^{2,p}$ G-principal Sobolev bundle $\mathcal{P}$, for $1 < p < \infty$, as a family of maps $g_{ij} \in W^{2,p}(U_i \cap U_j, G)$ satisfying the cocycle conditions. As already observed by T.Isobe in [19], $W^{2,p}(U_{ij}, G)$ is a topological group, and so the cocycle conditions on the family $\{g_{ij}\}$ still make sense. We conclude the introduction of Sobolev bundles by generalizing the geometrical concept of equivalent bundles, to the case of Sobolev bundles.
Once a $W^{2,p}$-principal Sobolev bundle $\mathcal{P} = \{(U_{ij}, g_{ij})\}$ is given, we define a Sobolev connection on it as a family of maps
$A_i \in (W^{1,p} \cap L^{2p})(U_i, T^*U_i \otimes \mathfrak{g})$ such that the compatibility condition holds in the overlaps U_{ij}. The request for the connection to be also in L^{2p} is fundamental for the compatibility condition to hold. However, when $2p = n$, the Sobolev embedding $W^{2,p}(U_i, T^*U_i \otimes \mathfrak{g}) \hookrightarrow L^{2p}(U_i, T^*U_i \otimes \mathfrak{g})$ holds, and therefore it is sufficient to ask $W^{1,p}$-regularity for the family $\{A_i\}_{i \in I}$. For this reason we call the dimension $n = 2p$ critical dimension. In this work we focus on the case $n = 4$, $p = 2$ and we will therefore work in the critical setting.

In critical dimension, we consider a Sobolev bundle $\mathcal{P} = \{(U_{ij}, g_{ij})\}$ in which is defined a Coulomb connection $\{A_i\}_{i \in I}$, namely a connection satisfying $d^\star A_i = 0$, for each $i \in I$. These bundles are of particular interest because the Coulomb condition on the connection, leads to a PDE solved by the transition functions g_{ij}, namely

$$\Delta g_{ij} = dg_{ij} \cdot A_j - A_i \cdot dg_{ij} \text{ in } U_{ij} \tag{1.2}$$

which increases the regularity of g_{ij}. In particular we will show that each $W^{2,2}$-Coulomb bundle is locally a $W^{2,(2,1)}$-bundle (see [33]) and by the Sobolev embedding $W^{2,(2,1)} \hookrightarrow C^0$ in dimension four, it holds that $g_{ij} \in C_{\mathrm{loc}}(U_{ij}, G)$. Moreover, if we assume that the L^4-norm of the Coulomb connection is under a certain threshold, we can even say that the bundle is $W^{2,p}$ locally, with $2 < p < 4$. Since such a threshold is scale invariant, from this last result we can infer that there exists a refinement of the covering, with respect to which the bundle is $W^{2,p}$. The Hölder's regularity of Coulomb bundles in critical dimension was proved in [40], and here it will be crucial in Theorem 4.3.10. Finally, in the last subsection we state and prove a remarkable result of K.Uhlenbeck [44] which tells us that if two $W^{2,p}$-bundles, with $2 < p < 4$, are L^∞-close enough (depending on the cardinality of the covering of Ω) then they are $W^{2,p}$-equivalent.

Once all the geometric and analytic tools are defined, we are ready to present the Plateau Problem for the Yang-Mills functional. We will mainly work with the trivial bundle $P = B^4 \times G$, where B^4 is endowed with the Euclidean metric, and consider over it the space of connections $W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g})$.
Choosing a $\mathfrak{g}$-valued 1-form η out of the space $H^{\frac{1}{2}}(\partial B^4, T^*\partial B^4 \otimes \mathfrak{g})$, the Plateau problem consists in proving whether the infimum

$$\inf \left\{ YM(A) := \int_{B^4} |F_A|^2 dx : \; A \in W^{1,2}_\eta(B^4, T^*B^4 \otimes \mathfrak{g}) \right\} \tag{1.3}$$

is attained by some $A_0 \in W^{1,2}_\eta(B^4, T^*B^4 \otimes \mathfrak{g})$, where $W^{1,2}_\eta(B^4, T^*B^4 \otimes \mathfrak{g})$ is the space of connections over the trivial bundle whose tangential component is equal to η. Since the pointwise norm of the field strength is Ad-invariant, the Lagrangians of two gauge equivalent potentials always coincide, and this realizes a huge invariance group for the Yang-Mills functional. Classical variational methods here fail, since such invariance group makes the functional non coercive, and therefore minimizing sequences are not necessarily weakly compact. We will study the problem both in the Abelian and in the non Abelian cases.

In the first case the non linear term $[A, A]$ inside the field strength

vanishes identically, and thus we bound the Yang-Mills functional from above by the more regular Dirichlet energy functional E, which agrees with the first if and only if the connection is Coulomb.
This bears similarities with the approach to the classical Plateau problem, where instead of studying the surface area functional $A(u)$ one works with the Dirichlet integral $D(u)$, which agrees with the first one when u is weakly conformal. In particular, we see that Coulomb connections in the Yang-Mills Plateau problem are the equivalent of weakly conformal maps in the classical Plateau problem. This parallelism is explored in Subsection 4.2.1.
The functional E turns out to be coercive in $W^{1,2}_\eta(B^4, T^*B^4 \otimes \mathfrak{g})$ and together with its convexity this leads to the existence of a unique minimizer for E. In particular the Euler-Lagrange equations point out that it has harmonic components, and is a Coulomb connection. From this, thanks to the existence of a Coulomb gauge for each connection proved in Proposition 4.2.1, and the invariance of the Lagrangian of the Yang-Mills functional with respect to a change of gauge, one proves that the minimum for E is a minimum also for the Yang-Mills functional, and the Plateau problem when $G = U(1)$ is solved.

When G is a compact and connected non Abelian matrix Lie group, the above technique does not work anymore due to the presence of the nonlinear term $[A, A]$ in the field strength, and therefore we need to proceed differently. In order to find a bounded minimizing sequence we present a result by K.Uhlenbeck [44], which assures that if the value of the Yang-Mills functional is under a certain threshold then one can find a change of gauge with respect to which the $W^{1,2}$-norm of the gauge equivalent connection is bounded from above by YM. This result is actually valid for each domain diffeomorphic to B^4, and the threshold, which depends on G and on the domain, is scale invariant. By exhibiting a bound from above of the Yang-Mills functional in terms of the $H^{\frac{1}{2}}$-norm of the fixed boundary potential η, we identify a subdivision in the study of the minimization problem.

Indeed, when the norm of the prescribed boundary connection is small enough, by virtue of K.Uhlenbeck Small Energy Theorem, each element of a minimizing sequence in $W^{1,2}_\eta$ admits a gauge change such that the resulting new sequence is bounded, and therefore weakly converging. The tangential component of the minimum turns out to be gauge equivalent to the fixed boundary potential η for a gauge on the boundary $g \in H^{\frac{3}{2}}(\partial B^4, G)$. Here the Yang-Mills

Plateau problem meets the extension problem for Sobolev maps between manifolds. Indeed, if a $W^{2,2}$-extension of g exists then this provides us the existence of a solution to the minimization problem. Unluckily this is not always possible, and in Chapter 5 we will give some counterexamples. However, a good bound on the $H^{\frac{1}{2}}$-norm of dg allows us to extend g to the whole B^4, as it will be proved in Chapter 5. In this case, the condition on the $H^{\frac{1}{2}}$-norm of η translates in a condition on the $H^{\frac{1}{2}}$-norm of dg, and therefore assuming $\|\eta\|_{H^{\frac{1}{2}}}$ small enough we can extend g to the whole B^4.

The problem becomes more subtle when we relax the hypothesis on the norm $\|\eta\|_{H^{\frac{1}{2}}}$, since in this case we cannot apply K.Uhlenbeck Theorem globally on B^4. However, we prove that up to a finite set of points $\{P_1, ..., P_N\} \subset \overline{B^4}$, called singularities, one can locally apply the Small Energy Theorem to a minimizing sequence of connections $\{A_k\}_k \subset W^{1,2}_{\eta}$, producing therefore a sequence of Coulomb Sobolev bundles $\mathcal{P}_k := \{(g_k^{ij}, U_{ij})\}$ with base manifold $C_\delta \subset B^4$, that are $W^{2,p}$-equivalent to the trivial bundle, for a fixed $2 < p < 4$. The submanifold C_δ of B^4 is given by the complement in B^4 of neighbourhoods of radii δ of the singularities, and $\mathcal{U} = \{U_i\}_{i\in I}$ is a finite covering for C_δ, that in the proof of the theorem will be made of balls. The diameter of each element of the covering depends on δ. These results are obtained in Theorem 4.3.10, and here the aforementioned observation on the scale invariance of the threshold both of the Small Energy Theorem and of Lemma 3.2.15 on Hölder's regularity of Coulomb bundles, is crucial.
The sequence $\{\mathcal{P}_k\}_k$ converges weakly in $W^{2,p}$ to a limit bundle $\mathcal{P}_\infty$ in which is defined the limit connection A_∞. We can apply Lemma 3.2.18 on the $W^{2,p}$-equivalence of L^∞-near Sobolev bundles getting therefore that also the limit bundle $\mathcal{P}_\infty$ is trivial.
We repeat the above argument taking always smaller neighbourhoods of the singularities points $P_1, ..., P_N$, letting therefore $\delta \to 0$, in order to obtain a minimizer in $W^{1,2}_{\mathrm{loc}}(B^4 \setminus \{P_1, ..., P_N\})$. At this point we apply an improved version of the celebrated result by K.Uhlenbeck on the removability of singularities, see [45], that indeed guarantees the existence of a local gauge change that transforms the minimizer in a connection with finite $W^{1,2}(B^4)$-norm. While the Removable singularities Theorem due to K.Uhlenbeck is proved only for Yang-Mills fields, namely solutions to the Euler-Lagrange of YM, the one we prove works for every $W^{1,2}$ connection with singularities, under the only condition of finite Yang-Mills energy. It was first observed by T.Rivière and M.Petrache in [33], and

the local $W^{2,(2,1)}$-regularity of $W^{2,2}$-Coulomb bundles underlie its proof.
Once we have removed the singularities of the minimizer, we show that its tangential component is gauge equivalent to the prescribed boundary connection for a gauge $g \in H^{\frac{3}{2}}(\partial B^4, G)$. In contrast with the previous case, now that we have dropped the condition on the $H^{\frac{1}{2}}$-norm of η, we are no longer able to bound properly the $H^{\frac{1}{2}}$-norm of dg and therefore, we cannot assure the existence of a $W^{2,2}$ extension of it. The most we can say therefore, is that there exists a minimizer of the Yang-Mills functional in the space of connections with tangential component gauge equivalent to η, for some gauge of the boundary.

In the last chapter of this thesis we study the extension problem. In particular, relating it to the problem of weakly harmonic maps, we exhibit some counterexample of functions, with values in some manifold, that do not admits extensions with the required regularity. In the last subsection, instead, we give a proof of the extendibility of a $g \in H^{\frac{3}{2}}(\partial B^4, G)$ which has $H^{\frac{1}{2}}$-norm of the differential dg under a certain threshold, depending on the target group G, and the domain. The proof is obtained following a reasoning of F.Bethuel in [5], which allows to extend the map to some $U \in W^{2,2}(B^4 \setminus B_\rho(0), G)$, with $0 < \rho < 1$. One can prove that U is continuous in $\partial B_\rho(0)$, and using the condition on the $H^{\frac{1}{2}}$-norm of dg, we extend U to the whole B^4 thanks to the exponential map.

Chapter 2

Differential geometry

2.1 Lie groups

Definition 2.1.1. A **lie group** is a group G endowed with a differentiable structure relative to which the product $G \times G \to G$ $((g_1, g_2) \mapsto g_1 \cdot g_2)$, and the inverse $G \to G$ $(g \mapsto g^{-1})$ are differentiable maps. We will denote the identity with $e \in G$.

If G is a Lie group and $h \in G$, we define the **left** and **right** multiplication for h as follows:

$$L_h : G \longrightarrow G \qquad R_h : G \to G$$
$$g \mapsto h \cdot g \qquad\qquad g \mapsto g \cdot h$$

and these maps are clearly diffeomorphisms.

Example 2.1.2. The following are some examples of Lie Groups

1) $(\mathbb{R} \setminus \{0\}, \cdot)$ is a Lie group.
2) If we call $M_n(\mathbb{R})$ the set of real $n \times n$ matrices, we can identify it with $\mathbb{R}^{n^2}$. Then we have that $GL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det(A) \neq 0\}$ is a open subset of $\mathbb{R}^{n^2}$, and thus it is a differentiable manifold. Moreover, the composition of matrices is smooth with respect to this differentiable structure, and therefore $GL(n, \mathbb{R})$ is a Lie Group.

Definition 2.1.3. Let G be a Lie group and H a subgroup of G. We say that H is a **regular Lie subroup** of G if it is a submanifold of G.
If H is an immersed submanifold and at the same time a Lie group with respect to this differential structure, we call H a **Lie subgroup** of G.

Clearly a regular Lie subgroup is a Lie group. Indeed $\cdot : H \times H \to H$ can be decomposed as $i : H \times H \hookrightarrow G \times G \to G$ where the last row is the product. By definition of submanifold we have that the embedding of H into G is differentiable, and so is also the composition of the embedding of $H \times H$ into $G \times G$ with the algebraic product.

Example 2.1.4. The subgroup $O(n) := \left\{ A \in GL(n, \mathbb{R}) : AA^T = \mathbb{I}_n \right\}$ of $GL(n, \mathbb{R})$ is a regular Lie subgroup. To see this observe that the map $S : GL(n, \mathbb{R}) \to M_n(\mathbb{R})$ defined as $S(A) = AA^T$, is differentiable and has no critical points[1]. Thus, the preimage of the identity $\mathbb{I}_n$, which clearly coincides with $O(n)$, is a submanifold of $GL(n, \mathbb{R})$. Moreover, $SO(n)$ is the connected component of $O(n)$ containing the identity, and thus it is also a regular Lie subgroup.

Example 2.1.5. With a similar argument one can prove that also $GL(n, \mathbb{C})$ is a Lie group, and that both $U(n) = \left\{ U \in GL(n, \mathbb{C}) : U\overline{U}^T = \mathbb{I}_n \right\}$ and $SU(n)$ are regular Lie subgroups.

2.1.1 Lie Algebra

Definition 2.1.6. Consider on a vector space V a bilinear map $[\cdot, \cdot] : V \times V \to V$ such that :

1) $[v, w] = -[w, v]$ for every $v, w \in V$
2) $[v, [w, u]] + [w, [u, v]] + [u, [v, w]] = 0$ for every $v, w, u \in V$

Then we call V a **Lie algebra**.

Example 2.1.7. The following are examples of Lie Algebras

1) Let M be a manifold, and $\tau(M)$ the space of vector fields over M, then $\tau(M)$ with the usual bracket Lie operation is a Lie Algebra.
2) $(\mathbb{R}^3, \times)$, where $\times$ is the classical external product, is a Lie Algebra.

Definition 2.1.8. If V is a Lie algebra, and W is a subspace of V such that $[v, w] \in W$ $\forall w, v \in W$, then we say that W is a **subalgebra**.
Furthermore, if V_1 and V_2 are Lie algebras, we say that a linear map

[1]If M, N are manifolds and $\phi : M \to N$ is differentiable, then $p \in M$ is a critical point if $d\phi_p : T_pM \to T_{\phi(p)}N$ is not surjective

$f : V_1 \to V_2$ is a **lie algebra morphism** if $f([v, w]) = [f(v), f(w)]$, for each $v, w \in V_1$.

Example 2.1.9. A remarkable example of Lie Algebra morphism is the following. Let M, N be two n-dimensional manifolds and $\phi : M \to N$ a diffeomorphism. Furthermore, let $\phi_* : \tau(M) \to \tau(N)$, the map that to each $X \in \tau(M)$ associates the vector field over N, $\phi_*(X)_y = (d\phi)_{\phi^{-1}(y)}(X)$ for each $y \in N$. Then we have that $[\phi_*(X), \phi_*(Y)] = \phi_*([X, Y])$ for each $X, Y \in \tau(M)$.

Definition 2.1.10. Let G be a Lie group. We define the following subalgebra of $\tau(G)$. If $X \in \tau(G)$ we say that X is **left invariant** if for every $h \in G$, $(dL_h)_g(X_g) = X_{hg}$.

The following proposition establishes that the set of left invariant vector fields is actually a vector space, and furthermore it is a sublagebra of $\tau(G)$.

Proposition 2.1.11. *The set* $\mathfrak{g}$ *of all left invariant vector fields is a subalgebra of* $\tau(G)$*. Moreover, there is an isomorphism between* $\mathfrak{g}$ *and* T_eG

Proof. If $X, Y \in \mathfrak{g}$, $h \in G$ and $a, b \in \mathbb{R}$ then $(dL_h)_g(aX + bY)_g = a(dL_h)_g(X_g)+b(dL_h)_g(Y_g) = (aX+bY)_{hg}$. Besides, $(dL_h)_g[X, Y]_g = [dL_h(X), dL_h(Y)]_{hg}$ since L_h is a diffeomorphism, and the last term is equal to $[X, Y]_{hg}$ because X and Y are both left invariant.
Finally observe that we can define the following map

$$f : T_eG \longrightarrow \mathfrak{g}$$

$$v \mapsto X, \quad X_g := (dL_g)_e v$$

which is clearly linear and injective. f is also onto, indeed if $X \in \mathfrak{g}$, then $f(X_e)$ is exactly X. Observe that this in particular means that the values of a left invariant vector field are completely determined by its value at the identity. □

Definition 2.1.12. We call the **Lie Algebra** of G the subalegbra of $\tau(G)$ made of left invariant vector fields. If $v, w \in T_eG$ we define $[v, w] := [f(v), f(w)]_e$, and T_eG with this operation is a Lie Algebra too. Thanks to the proposition above we see that f is a Lie algebra isomorphism. From now on we will identify T_eG with $\mathfrak{g}$.

It is well known that a vector field over a manifold produces locally a flow. What is new now, is that if G is a Lie group, and the vector field is in the Lie algebra of G, then the flow is a local group homomorphism between a neighbourhood of zero in $\mathbb{R}$ and G itself. More precisely we have the below relevant proposition. Before stating it, we will need the following definition.

Definition 2.1.13. Let G be a connected Lie group. A (global) **1-parameter subgroup** of G is a C^∞ map $\sigma : \mathbb{R} \to G$ which is also a group homomorphism.

Proposition 2.1.14. *Let G be a Lie Group and $X \in \mathfrak{g}$. Then the followings hold:*

1) *The integral curve of X starting at e is a 1-parameter subgroup of G.*
2) *If $\sigma : \mathbb{R} \to G$ is a 1-parameter subgroup of G, and $\sigma(0)' = X_e$ then σ is the integral line of X at e.*

Proof. 1)Let $\sigma : (-\varepsilon, \varepsilon) \to G$ be the maximal integral curve of X starting from e. Then if we call $\gamma : (-\varepsilon, \varepsilon) \to G$, the map defined as $\gamma(t) = \sigma(t_0)\sigma(t)$, where $t_0 \in (-\varepsilon, \varepsilon)$, we see that

$$\frac{d}{dt}\gamma(t) = dL_{\sigma(t_0)}(\sigma(t)^{'}) = dL_{\sigma(t_0)}(X_{\sigma(t)}) = X_{\sigma(t_0)\sigma(t)} = X_{\gamma(t)}$$

This equation implies that $\gamma(t)$ is the integral curve of X starting from t_0, but for uniqueness we get that

$$\gamma(t) = \sigma(t_0)\sigma(t) = \sigma(t_0 + t)$$

Moreover, we clearly see that the maximal interval of integration is actually $(-\infty, \infty)$.
2)Since σ is a 1-parameter subgroup, we have that $\sigma(t_0 + t) = L_{\sigma(t_0)}\sigma(t)$, and from this we deduce

$$\sigma(t_0)^{'} = \frac{d}{dt}(\sigma(t_0 + t))|_{t=0} = dL_{\sigma(t_0)}(X_e) = X_{\sigma(t_0)}$$

and therefore we have proved also the second statement. □

Thus, for each left invariant vector field X on G, there exists a unique 1-parameter subgroup σ of G, such that $\sigma(0)' = X_e$, and it coincides with the integral curve of X at e. This motivates the following definition.

Definition 2.1.15. Let $X \in \mathfrak{g}$ and $\sigma_X(t)$ be the 1-parameter subgroup associated to it. We define the **exponential map** as $\exp : \mathfrak{g} \to G$ where $\exp(X) := \sigma_X(1)$.

Example 2.1.16. Let X be a left invariant vector field of $GL(n, \mathbb{R})$, and $\sigma(t)$ the 1-parameter subgroup of $GL(n, \mathbb{R})$ such that $\sigma(0)^{'} = X_e$. Since it coincides with the integral curve associated to X, then $\sigma(s)' = X_{\sigma(s)} = dL_{\sigma(s)}(X_e)$. We are dealing with product of matrices and then for each $g \in GL(n, \mathbb{R})$ and $A \in \tau(G)$, $dL_g(A) = g \cdot A$, which implies that $\sigma(s)' = \sigma(s) \cdot X_e$. The solution to this ODE is trivial and we get $\exp(X) = \sum_{k=0}^{\infty} \frac{(X_e)^k}{k!}$.

A significant feature of the exponential map is that in a suitable neighbourhood of $0 \in \mathfrak{g}$ it is a diffeomorphism with image in a neighbourhood of $e \in G$. In the following Theorem we gather some of the main properties of the exponential map.

Theorem 2.1.17. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. The followings hold:*

1) $\exp : \mathfrak{g} \to G$ *is* $C^\infty(\mathfrak{g}, G)$

2) $(d\exp)_0 : \mathfrak{g} \to T_eG$ *is the identity, i.e.* $(d\exp)_0(X) = X_e$. *Furthermore there exists a neighbourhood* V_0 *of* $0 \in \mathfrak{g}$ *such that* $\exp|_{V_0} : V_0 \to \exp(V_0)$ *is a diffeomorphism and* $\exp(V_0) = V_e$ *is an open neighbourhood of* $e \in G$.

Proof. 1)Let $X \in \mathfrak{g}$ and Θ_X the flow associated to X. Now in the manifold $G \times \mathfrak{g}$ we build the following vector field

$$Y_{g,Z} := (X_g, 0) \in T_g(G) \times T_Z\mathfrak{g} \cong T_{g,Z}(G \times \mathfrak{g})$$

If σ_Y is the flow of Y then: $\sigma_Y(1,(g,Z)) = (\Theta_X(1,g), Z) = (\exp(X), Z)$. But then if you call $\pi_1 : G \times \mathfrak{g} \to G$ the canonical projection on the first component, we get $\exp(X) = \pi_1 \circ \sigma_Y(1,(g,Z))$ and so the map is a composition of C^∞ function.
2)Now fix X in the lie algebra of G, and consider the map $(-\varepsilon, \varepsilon) \ni t \to \exp(tX)$. We know that it is differentiable, and
$X = \frac{d}{dt}\exp(tX)|_{t=0} = (d\exp)_0 X$.
Now since the differential at the point $0 \in \mathfrak{g}$ is an isomorphism, then by the Inverse mapping Theorem we conclude the proof of the second point. □

Remark 2.1.18. By the second point of the above theorem we deduce in particular that every $g \in G$ admits a neighbourhood V_g diffeomorphic to V_0. Indeed, we just have to compose the left action L_g with the exponential map:

$$\exp|_{V_0} : V_0 \to \exp(V_0) \subset G \to L_g \exp(V_0) := V_g$$
$$h \mapsto \exp(h) \mapsto g\exp(h) \tag{2.1}$$

If ω is a differential 1-form over a Lie group G, we say that it is **left invariant** if $(L_g)^*\omega = \omega$[2], namely $\omega_{gh}((dL_g)_h X_h) = \omega_h(X_h)$, for each $h, g \in G$ and $X \in \tau(G)$. It is easy to see that the subspace of $\wedge^1(G)$ made of left invariant 1-forms on G is the dual of $\mathfrak{g}$, and thus it has the same dimension.

[2]If M, N are manifolds, and $\phi \in C^\infty(M,N)$ then we will denote with ϕ^* the pull-back. See [1] for further details.

Definition 2.1.19. We define the **canonical $\mathfrak{g}$-valued 1-form** Θ on G by

$$\Theta_g : T_gG \to T_eG \cong \mathfrak{g}$$
$$v \mapsto \Theta_g(v) := (dL_{g^{-1}})_g v$$

which is clearly a left invariant differential form. Equivalently we can define it as the left invariant $\mathfrak{g}$-valued 1-form such that $\Theta(A) = A$ for each $A \in \mathfrak{g}$.

2.1.2 Action of Lie groups

Le M be a manifold and G a Lie group.

Definition 2.1.20. We say that G **acts** differentially on M on the left if

1) $G \times M \to M$ such that $(g, m) \mapsto g \cdot m$ is differentiable.

2) The map $m \mapsto g \cdot m$ for a fixed $g \in G$ is an automorphism of M.

3) $(gh) \cdot m = g \cdot (h \cdot m)$ for each $g, h \in G$ and $m \in M$.

We will also say that G acts **freely** (resp. **effectively**) if $g \cdot m = m$ for some $m \in M$ implies $g = e$ (resp. for all $m \in M$ implies $g = e$).

From 3) and 2) we deduce that the map $m \to e \cdot m$ is the identity of M.
If $A \in \mathfrak{g}$ we can associate to it a vectorfield $A^\sharp \in \tau(M)$. It is given by the action of the one parameter subgroup $\exp(tA)$ of G, in the following way

$$(A^\sharp)_m = \frac{d}{dt}(\exp(tA) \cdot m)|_{t=0} \quad \forall m \in M \tag{2.2}$$

If we define $L_m : G \to M$ by $L_m(g) = g \cdot m$ then $(A^\sharp)_m = (dL_m)_e A_e$ for each $m \in M$. This construction will turn out to be useful later, for connections on principal bundles. We have the following proposition (see [22], Proposition 4.1).

Proposition 2.1.21. *Let G and M as above. The mapping $\sigma : \mathfrak{g} \to \tau(M)$ such that $\sigma(A) = A^\sharp$ is a Lie algebra homomorphism. If G acts effectively then σ is an isomorphism of $\mathfrak{g}$ into $\sigma(\mathfrak{g})$, and if furthermore G acts freely then (for $A \neq 0$) $A^\sharp$ never vanishes.*

Definition 2.1.22. We define the **conjugation action** $I : G \times G \to G$, by $I_g(x) = I(g, x) := gxg^{-1}$. Taking its push forward we get an action on $\mathfrak{g}$, $Ad : G \times \mathfrak{g} \to \mathfrak{g}$, such that $Ad_g(X) := dI_g(X)$, or more explicitly $Ad_g(X)_x = (dI_g)_{I_g^{-1}(x)}(X_{I_g^{-1}(x)})$ for every $x \in G$, which is called the **adjoint action**.

If we apply Proposition 2.1.21 to the adjoint action then we get the Lie Algebra morphism $\sigma : \mathfrak{g} \to \Gamma(\mathfrak{g})$, and to denote this particular morphism, instead of σ, we will use $ad : \mathfrak{g} \to \Gamma(\mathfrak{g})$ and for every $X \in \mathfrak{g}$ we write the image as ad_X, which is a vector field over $\mathfrak{g}$. It is easy to verify that $ad_X|_Y = [X, Y]$ for every $Y \in \mathfrak{g}$. Observe that another way to see ad is as a map that to each element in $\mathfrak{g}$ associates a linear map on $\mathfrak{g}$ as follows:

$$
\begin{aligned}
ad : \mathfrak{g} &\to \mathcal{L}(\mathfrak{g}, \mathfrak{g}) \\
X &\mapsto ad_X : \mathfrak{g} \to \mathfrak{g} \\
&\qquad\qquad Y \mapsto [X, Y]
\end{aligned}
$$

where $\mathcal{L}(\mathfrak{g}, \mathfrak{g})$ is the monoid of endomorphisms of $\mathfrak{g}$.

Definition 2.1.23. Let G be a Lie group with Lie algebra $\mathfrak{g}$. We construct a symmetric bilinear form called the **Killing form**

$$
\begin{aligned}
K : \mathfrak{g} \times \mathfrak{g} &\to \mathbb{R} \\
(A, B) &\mapsto K(A, B) := -\operatorname{Tr}(ad_X \circ ad_Y)
\end{aligned}
\tag{2.3}
$$

where Tr is the trace of an endomorphism.

Proposition 2.1.24. *Let G be a Lie group, with Lie Algebra $\mathfrak{g}$. Then the Killing form K is Ad invariant, namely*

$$K(Ad_g(X), Ad_g(Y)) = K(X, Y) \quad \forall g \in G, \ \forall X, Y \in \mathfrak{g}$$

Proof. Since $Ad_g : \mathfrak{g} \to \mathfrak{g}$ is an automorphism of $\mathfrak{g}$ then, for every $X, Y \in \mathfrak{g}$

$$ad_{Ad_g(X)}|_Y = [Ad_g(X), Y] = Ad_g \circ [X, Ad_{g^{-1}} Y] = Ad_g \circ ad_X \circ Ad_{g^{-1}}(Y)$$

therefore $ad_{Ad_g(X)} = Ad_g \circ ad_X \circ Ad_{g^{-1}}$. So we get

$$K(Ad_g(X), Ad_g(Y)) = -\operatorname{Tr}(ad_{Ad_g(X)} \circ ad_{Ad_g(Y)}) =$$

$$= -\operatorname{Tr}(Ad_g \circ ad_X \circ Ad_{g^{-1}} \circ Ad_g \circ ad_Y \circ Ad_{g^{-1}}) = -\operatorname{Tr}(ad_X \circ ad_Y) = K(X, Y)$$

□

Remark 2.1.25. In particular we can prove that if the Lie Algebra is semisimple[3] then the bilinear form is non degenerate, and if G is compact and connected it is also positive definite (see [18]). Since we are interested only in such groups then (2.3) induces a scalar product on $\mathfrak{g}$, and then also a norm on it.
Actually we can do even more, by using (2.3) to define a metric $\mathbf{g}$ on the whole G, such to promote G to Riemannian manifold $(G, \mathbf{g})$ This can be done as follows. For every $a \in G$ and every $v, w \in T_a(G)$ we set

$$\mathbf{g}_a(v, w) := K(dL_{a^{-1}}(v), dL_{a^{-1}}(w)) = K(\Theta_a(v), \Theta_a(w)) \tag{2.4}$$

Proposition 2.1.26. *The metric* $\mathbf{g}$ *is bi-invariant in* G*. This means that for each* $v, w \in T_g G$ *and for each* $\tilde{g} \in G$ *the following equations hold*

$$\begin{aligned} \mathbf{g}(dR_{\tilde{g}}(v), dR_{\tilde{g}}(w)) &= \mathbf{g}(v, w) \\ \mathbf{g}(dL_{\tilde{g}}(v), dL_{\tilde{g}}(w)) &= \mathbf{g}(v, w) \end{aligned} \tag{2.5}$$

Proof. The second equation is clearly true by definition. Indeed,

$$\mathbf{g}(dL_{\tilde{g}}(v), dL_{\tilde{g}}(w)) = K(dL_{g^{-1}\tilde{g}^{-1}}(dL_{\tilde{g}}(v)), dL_{g^{-1}\tilde{g}^{-1}}(dL_{\tilde{g}}(w))) =$$
$$= K(dL_{g^{-1}}(v), dL_{g^{-1}}(w)) = \mathbf{g}(v, w)$$

We also see that the property of K of being Ad-invariant is automatically transferred to the metric $\mathbf{g}$. Indeed, we see that

$$\mathbf{g}(Ad_{\tilde{g}}(v), Ad_{\tilde{g}}(w)) = \mathbf{g}(dL_{\tilde{g}}dR_{\tilde{g}^{-1}}(v), dL_{\tilde{g}}dR_{\tilde{g}^{-1}}(w)) =$$
$$= \mathbf{g}(dR_{\tilde{g}^{-1}}(v), dR_{\tilde{g}^{-1}}(w)) = K(dL_{\tilde{g}}dL_{g^{-1}}dR_{\tilde{g}^{-1}}(v), dL_{\tilde{g}}dL_{g^{-1}}dR_{\tilde{g}^{-1}}(w)) =$$
$$= K(Ad_{\tilde{g}}dL_{g^{-1}}(v), Ad_{\tilde{g}}dL_{g^{-1}}(w)) = K(dL_{g^{-1}}(v), dL_{g^{-1}}(w)) = \mathbf{g}(v, w)$$

which also proves that $\mathbf{g}$ is right invariant. □

2.2 Bundles

This section is devoted to the development of the theory of bundles, which are key geometrical structures in the study of gauge theories. If M and S are two manifolds, then a bundle over M with fibre S is, roughly speaking, a manifold that locally in $U \subset M$ looks like $U \times S$. In what follows we are only concerned with vector bundles and principal fibre bundles. For the first subsection we will refer mainly to [1], while the second subsection is based both on [1] and [22].

[3] A Lie Algebra is semisimple if it is a direct sum of simple subalgebras, i.e. non abelian algebras with trivial ideals.

2.2.1 Vector Bundles

Let M be a manifold.

Definition 2.2.1. A manifold E is called a **vector bundle** of rank r over M if the followings are true.

1) There exists a surjective differential map $\pi : E \to M$, called the **projection**, such that $\pi^{-1}(x) = E_x$ is a real vectorspace of dimension r for each $x \in M$

2) For each $x \in M$, there exists a neighbourhood $U \subset M$ of x and a diffeomorphism $\chi : \phi^{-1}(U) \to U \times \mathbb{R}^r$ called **local trivialization**, such that $\pi_1 \circ \chi = \pi$, namely the following diagram commutes

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\chi"] \arrow[d, "\pi"] & U \times \mathbb{R}^r \arrow[dl, "\pi_1"] \\ U & \end{tikzcd} \tag{2.6}$$

where $\pi_1 : U \times \mathbb{R}^r \to U$ is the projection on the first component. We also require that χ restricted to E_x is an isomorphism between E_x and $\{x\} \times \mathbb{R}^r$ for each $x \in M$.

By point 2) of Definition 2.2.1 we see that if $\pi : E \to M$ is a vector bundle then there exists a covering $\{U_i\}_{i \in I}$ of M and local trivializations $\{\chi_i\}_{i \in I}$ defined on the covering. We call the family $\mathcal{A} = \{(U_i, \chi_i)\}_{i \in I}$ an **atlas** of the vector bundle E. Note that in $U_i \cap U_j \neq \emptyset$ we have that

$$\chi_i \circ \chi_j^{-1} : (U_i \cap U_j) \times \mathbb{R}^r \to (U_i \cap U_j) \times \mathbb{R}^r$$

$$(x, v) \longmapsto (x, \phi_{ij}(x)(v)) \tag{2.7}$$

where the family of maps $\phi_{ij} : U_i \cap U_j \to GL(r, \mathbb{R})$ satisfies the following conditions

$$\begin{gathered} \phi_{ij}\phi_{jl} = \phi_{il} \ \text{ in } U_i \cap U_j \cap U_l \neq \emptyset \\ \phi_{ij}\phi_{ji} = e \ \text{ in } \ U_i \cap U_j \neq \emptyset \\ \phi_{ii} = e \ \text{ in } \ U_i \end{gathered} \tag{2.8}$$

which are called the **cocycle conditions**.

Definition 2.2.2. Let $\pi : E \to M$ be a vector bundle endowed with an atlas $\mathcal{A} = \{(U_i, \chi_I)\}_{i \in I}$. The family of smooth maps $\phi_{ij} : U_i \cap U_j \to GL(r, \mathbb{R})$ defined by equation (2.7) are called **transition functions** of E with respect to the atlas $\mathcal{A}$.

We now give some well known examples of vector bundles. After that we will define morphisms between vector bundles, and spot a characterization for bundles that are isomorphic. In particular this last characterization is significant, and it is strictly related to the idea of Čech Cohomology, see Appendix B.

Example 2.2.3. The manifold given by $M \times \mathbb{R}^r$, and endowed with the projection on the first component $\pi : M \times \mathbb{R}^r \to M$ is an example of vector bundle of rank r over M, and it is called **trivial bundle**.

Example 2.2.4. The most important example of vector bundle over a manifold M is the **tangent bundle** $\pi : TM \to M$, where $TM = \bigcup_{x \in M} T_x M$. An atlas $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$ for M induces the local trivializations

$$\chi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^n$$
$$(v_j \partial_{x_i^j}|_x) \mapsto (x, v)$$

where $v = (v_1, ..., v_n)$ and $\varphi_i = (x_i^1, ..., x_i^n)$. An easy computation also shows that the transition functions are $\phi_{ij} = \frac{\partial x_i}{\partial x_j}$, where $\frac{\partial x_i}{\partial x_j}$ is the Jacobian matrix of the change of coordinates $\varphi_i \circ \varphi_j^{-1}$. Therefore, we have obtained from $\mathcal{A}$, an atlas $\tilde{\mathcal{A}} := \{(U_i, \chi_i)\}_{i \in I}$ for TM.
Similarly one sees that also $T^*M = \bigcup_{x \in M} T_x^* M$, called **cotangent bundle**, is a vector bundle over M.

Definition 2.2.5. Let $\pi_1 : E_1 \to M_1$ and $\pi_2 : E_2 \to M_2$ be two vector bundles. Then a **morphism** between them is a couple of differentiable maps $f : E_1 \to E_2$ and $F : M_1 \to M_2$, such that

1) $\pi_2 \circ f = F \circ \pi_1$, namely the following diagram commutes

$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[d, "\pi_1"] & E_2 \arrow[d, "\pi_2"] \\ M_1 \arrow[r, "F"] & M_2 \end{tikzcd} \tag{2.9}$$

2) $f|_{(E_1)_x} : (E_1)_x \to (E_2)_{F(x)}$ is linear for each $x \in M_1$.

If both f and F are diffeomorphism, then the morphism is called **isomorphism**. If $M_1 = M_2$ and $F = id$, and f is an isomorphism, then we will say that $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ are **equivalent**.

Remark 2.2.6 (**Characterization for equivalent vector bundles**). Let $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ be two equivalent vector bundles of rank r over M. Let $\mathcal{A}_1 = \{(U_i, \chi_i)\}_{i\in I}$ and $\mathcal{A}_2 = \{(U_i, \tilde{\chi}_i)\}_{i\in I}$ be two atlases respectively for E_1 and E_2, and denote with ϕ_{ij} and $\tilde{\phi}_{ij}$ the corresponding families of transition functions. Note that it is not restrictive to assume the trivializations of E_1 and E_2 over the same covering of M. This is because one can always find a common refinement of two different coverings.
Let $f : E_1 \to E_2$ be the equivalence, then we have

$$\begin{array}{ccc} \pi_1^{-1}(U_i) & \xrightarrow{f} & \pi_2^{-1}(U_i) \\ \Big\downarrow{\scriptstyle \chi_i} & & \Big\downarrow{\scriptstyle \tilde{\chi}_i} \\ U_i \times \mathbb{R}^r & \xrightarrow{\hat{\sigma}_i} & U_i \times \mathbb{R}^r \end{array} \tag{2.10}$$

where $\hat{\sigma}_i := \tilde{\chi}_i \circ f \circ \chi_i^{-1}$, and since $f|_{(E_1)_x}$ is an isomorphism for each $x \in M$, we get that

$$\hat{\sigma}_i : U_i \times \mathbb{R}^r \to U_i \times \mathbb{R}^r$$

$$(x, v) \longmapsto (x, \sigma_i(x)(v))$$

where $\sigma_i \in C^\infty(U_i, GL(r, \mathbb{R}))$. In particular we have that

$$\tilde{\chi}_i^{-1} \circ \hat{\sigma}_i \circ \chi_i = \tilde{\chi}_j^{-1} \circ \hat{\sigma}_j \circ \chi_j \quad \text{in} \qquad \pi_1^{-1}(U_i \cap U_j)$$

$$\Downarrow$$

$$\tilde{\chi}_i \circ \tilde{\chi}_j^{-1} = \hat{\sigma}_i \circ \chi_i \circ \chi_j^{-1} \hat{\sigma}_j^{-1} \qquad \text{in} \;\; (U_i \cap U_j) \times \mathbb{R}^r$$

$$\Downarrow$$

$$\tilde{\phi}_{ij} = \sigma_i \phi_{ij} \sigma_j^{-1} \qquad\quad \text{in} \;\; U_i \cap U_j \tag{2.11}$$

Conversely, let $\pi : E_1 \to M$ and $\pi_2 : E_2 \to M$ be two vector bundles over M of rank r, with atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ as above, and transition functions ϕ_{ij} and $\tilde{\phi}_{ij}$. If there exists a family of smooth maps $\sigma_i \in C^\infty(U_i, GL(r, \mathbb{R}))$ satisfying

$$\tilde{\phi}_{ij} = \sigma_i \phi_{ij} \sigma_j^{-1}$$

then the two bundles are equivalent, namely there exists a diffeomorphism $f : E_1 \to E_2$ such that $\pi_2 \circ f = \pi_1$ and $f|_{(E_1)_x}$ is a vector space isomorphism for each $x \in M$. Indeed, following the reasoning above, if we call $\hat{\sigma}_i$, the maps

$$\hat{\sigma}_i : U_i \times \mathbb{R}^r \to U_i \times \mathbb{R}^r$$
$$(x, v) \longmapsto (x, \sigma_i(x)(v))$$

then defining $f_i := \tilde{\chi}_i^{-1} \circ \hat{\sigma}_i \circ \chi_i$, we have that $f_i = f_j$ in $\pi^{-1}(U_i \cap U_j)$, and therefore we have a well defined map $f : E_1 \to E_2$. It is immediate to see that f satisfies all the requirements in order to be an equivalence.

We have seen that a vector bundle endowed with an atlas leads to the existence of a family of transition functions which satisfy the cocycle conditions. A remarkable property is that also the inverse is true.

Proposition 2.2.7. *Let M be a manifold endowed with an atlas $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$. We are also given a family of maps $\phi_{ij} : U_i \cap U_j \to GL(r, \mathbb{R})$ satisfying the cocycle conditions. Then there exists a unique vector bundle E (up to isomorphism) over M with transition functions ϕ_{ij}.*

Proof. Consider $\tilde{E} = \bigcup_{i \in I}(U_i \times \mathbb{R}^r)$. We denote with $E = \tilde{E}/\sim$ the quotient of $\tilde{E}$ with respect to the equivalence relation $\sim$ defined as follows. Take $(x, v) \in U_i \times \mathbb{R}^r$ and $(y, w) \in U_j \times \mathbb{R}^r$, then $(x, v) \sim (y, w)$ if and only if $x = y$ and $\phi_{ij}(x)(w) = v$. The cocycle conditions guarantee that $\sim$ is actually an equivalence relation.
It is clear that the projection on the first component $\pi : E \to M$ is surjective and that $\pi^{-1}(U_i) = (U_i \times \mathbb{R}^r)/\sim$. Since we have that different elements in $U_i \times \mathbb{R}^r$ are not equivalent, then we can define a bijection $\chi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^r$. If $x \in U_i \cap U_j$ and $v \in \mathbb{R}^r$, then the unique element in $U_j \times \mathbb{R}^r$ that is equivalent to $(x, v) \in U_i \times \mathbb{R}^r$ is $(x, \phi_{ji}(x)(v))$, which implies that $\chi_j \circ \chi_i^{-1}(x, v) = (x, \phi_{ji}(x)(v))$. In particular it holds that for each $i \in I$, $\pi_1 \circ \chi_i = \pi|_{\pi^{-1}(U_i)}$. This last two results are enough to say that E is a vector bundle with transition functions ϕ_{ij} with respect to the atlas $\{(U_i, \chi_i)\}_{i \in I}$. Indeed, if $x \in U_i$, then $\chi_i|_{E_x}$ is a bijection by construction. Therefore we can endow E_x with a structure of vector space as follows. For each $u_1, u_2 \in E_x$ we define their sum as

$$u_1 + u_2 := \chi_i^{-1}(x, v_1 + v_2)$$

where $u_l = \chi_i^{-1}(x, v_l)$ for $l = 1, 2$, and if $\lambda \in \mathbb{R}$, the product λu_1 is defined as

$$\lambda u_1 := \chi_i^{-1}(x, \lambda v_1)$$

Now we need to show that these two operations do not depend on the choice of the local trivialization. Then let $j \in I$ such that $U_j \cap U_i \neq \emptyset$, and $\chi_j(u_l) = (x, w_l)$. We have that $(x, v_l) = \chi_i \circ \chi_j^{-1}(x, w_l) = (x, \phi_{ij}(x)(w_l))$, and thus

$$\chi_i^{-1}(x, v_1 + v_2) = \chi_i^{-1}(x, \phi_{ij}(x)w_1 + \phi_{ij}(x)w_2)) =$$

$$\chi_i^{-1}(x, \phi_{ij}(v_1 + v_2)) = \chi_i^{-1} \circ (\chi_i \circ \chi_j^{-1}(x, w_1 + w_2)) = \chi_j^{-1}(x, w_1 + w_2)$$

which proves that the sum does not depend on the choice of the local trivialization. Similarly one proves the same for the scalar product. Consider $\tilde{U}_i := \pi^{-1}(U_i)$, and the maps on them $\tilde{\varphi}_i := (\varphi_i, id) \circ \chi_i$. Then we have that

$$\tilde{\varphi}_i \circ \tilde{\varphi}_j^{-1} = (\varphi_i \circ \varphi_j^{-1}, \phi_{ij})$$

are C^∞-maps, and therefore we can endow E with a manifold structure, with atlas $\tilde{\mathcal{A}} := \{(\tilde{U}_i, \tilde{\varphi}_i)\}$. It is easy to check that $\pi : E \to M$ satisfies all the properties in order to be a vector bundle of rank r. It is only left to prove that the bundle is unique up to an isomorphism. Let $\hat{\pi} : \hat{E} \to M$ be a vector bundle over M, with transition functions ϕ_{ij}, and trivializations $\hat{\chi}_i : \hat{\pi}^{-1}(U_i) \to U_i \times \mathbb{R}^r$. We define $f_i : \pi^{-1}(U_i) \to \hat{\pi}^{-1}(U_i)$ as $f_i := \hat{\chi}_i^{-1} \circ \chi_i$. We see that if $U_i \cap U_j \neq \emptyset$, then in $\pi^{-1}(U_i) \cap \pi^{-1}(U_j)$ it holds that $f_i = f_j$, indeed this is true if and only if

$$\hat{\chi}_i^{-1} \circ \chi_i = \hat{\chi}_j^{-1} \circ \chi_j \Leftrightarrow \chi_i \circ \chi_j^{-1} = \hat{\chi}_i \circ \hat{\chi}_j^{-1}$$

and this last relation is true since the two vector bundles have the same transition functions. By the fact that for each $i \in I$, the maps f_i are diffeomorphisms which are linear on the fibres and that $\hat{\pi} \circ f_i = \pi$, we conclude that $f : E \to \hat{E}$ is an isomorphism (actually an equivalence) of vector bundles. This concludes the proof. $\square$

Definition 2.2.8. Let $\pi : E \to M$ be a vector bundle over M, and let $U \subset M$ be open. Then a differentiable map $s : U \to E$ is called a **local section** if $\pi \circ s = id_U$. A **global section** is a differentiable map $s : M \to E$ such that $\pi \circ s = id_M$. We denote the vector space of global sections as $E(M)$.

Each vector bundle admits always global sections, as one can easily prove, see for instance [1]. However, it is not always true that it is possible to build a global section that never vanishes. We highlight some important sections using the above examples of vector bundles.

Example 2.2.9. If M is a manifold, then the vector space of **vector fields** $\tau(M)$, is the space of sections of the tangent bundle $\pi : TM \to M$. Similarly the space of **differential** **1-forms** over M, denoted with $\bigwedge^1(M)$, coincides with the sections of the cotangent bundle $\pi : T^*M \to M$.

Before concluding we define a richer structure of vector bundle, which will arise in the study of principal bundles. In what follows let M be the base manifold, and G a Lie group.

Definition 2.2.10. Let $\pi : E \to M$ be a vector bundle of rank r. Then we say that it is a G**-vector bundle** if there exists

1) an action $\theta : G \times \mathbb{R}^r \to \mathbb{R}^r$ of G on $\mathbb{R}^r$

2) an atlas $\mathcal{A} = \{(U_i, \chi_i)\}_{i \in I}$ and a family of maps $g_{ij} : U_i \cap U_j \to G$ such that if ψ_{ij} are the transition functions of $\mathcal{A}$, then

$$\psi_{ij}(x, v) = \theta(g_{ij}(x), v) \tag{2.12}$$

for each $x \in U_i \cap U_j$ and $v \in \mathbb{R}^r$.

Example 2.2.11. We observe that every vector bundle $\pi : E \to M$ of rank r is a $GL(r, \mathbb{R})$-vector bundle with fibre $\mathbb{R}^r$. Other examples of G-vector bundles will be shown in the subsection devoted to Principal Fibre bundles.

2.2.2 Principal Fibre Bundles

Definition 2.2.12. Let M be a manifold and G a Lie group. We call **principal fibre bundle** over M with structure group G, a manifold P with a right action of G, such that

1) G acts freely on the right $(p, g) \in P \times G \to \sigma_g(p) = p \cdot g \in P$

2) M is the quotient space of the equivalence relation[4] induced by G, and the canonical projection $\pi : P \to M = P/G$ is differentiable.

3) For each $x \in M$ there exists a neighbourhood U of x in M and a diffeomorphism $\chi : \pi^{-1}(U) \to U \times G$ such that $\chi(p) = (\pi(p), \phi(p))$, namely the following diagram commutes

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\chi"] \arrow[d, "\pi"] & U \times G \arrow[dl, "\pi_1"] \\ U & \end{tikzcd} \tag{2.13}$$

[4]Two point p_1, p_2 of P are equivalent if there exists $g \in G$ such that $\sigma_g(p_1) = p_2$.

and $\phi : \pi^{-1}(U) \to G$ satisfies $\phi(p \cdot g) = \phi(p)g$ for each $g \in G$. The map π_1 is the projection on the first component.

We call P the **total space**, M the **base space**, and G the **structure group**. Since the canonical projection is a submersion, $\pi^{-1}(x)$ is a closed submanifold of P for each $x \in M$, and it is diffeomorphic to G. We call $\pi^{-1}(x)$ the **fibre** in x, and by definition if $u \in \pi^{-1}(x)$ then $\pi^{-1}(x) = \{u \cdot g : g \in G\}$.
If P is a fibre bundle, then by 3) there exists a covering $\{U_i\}_{i \in I}$ of M and a local trivialization $\{\chi_i\}_{i \in I}$ associated to it. We call the family $\mathcal{A} = \{(U_i, \chi_i)\}_{i \in I}$ an **atlas** of the bundle.
One may observe that for $x \in U_i \cap U_j$ with $i \neq j$ and $p \in \pi^{-1}(x)$ the product $\phi_i(p \cdot g)(\phi_j(p \cdot g))^{-1} = \phi_i(p)\phi_j(p)^{-1}$ for each $g \in G$ does not depend on $p \in \pi^{-1}(x)$ but just on x. Therefore, we can define for $U_i \cap U_j \neq \emptyset$ the maps

$$\begin{aligned} g_{ij} : U_i \cap U_j &\to G \\ x &\longmapsto g_{ij}(x) := \phi_i(u)\phi_j(u)^{-1} \end{aligned} \tag{2.14}$$

where $u \in \pi^{-1}(x)$. If $(x, g) \in (U_i \cap U_j) \times G$ then we see that

$$\chi_i \circ \chi_j^{-1}(x, g) = (x, g_{ij}(x)g)$$

and this family of maps g_{ij} satisfy the relations

$$\begin{aligned} g_{il}g_{lj} &= g_{ij} \quad \text{in } \; U_i \cap U_j \cap U_l \neq \emptyset \\ g_{ij}g_{ji} &= e \quad \text{in } \; U_i \cap U_j \neq \emptyset \\ g_{ii} &= e \quad \text{in } \; U_i \end{aligned} \tag{2.15}$$

which are called the **cocycle conditions**. We clearly see the analogy with the vector bundle transition functions, and also in this case we have the following definition.

Definition 2.2.13. Let $\pi : P \to M$ be a principal fibre bundle, with structure group G over M, endowed with an atlas $\mathcal{A} = \{(U_i, \chi_I)\}_{i \in I}$. The family of smooth maps $g_{ij} : U_i \cap U_j \to G$ defined by equation (2.14) are called **transition functions** of P with respect to the atlas $\mathcal{A}$.

Example 2.2.14. If M is a manifold and G a Lie group, then the product manifold $P = M \times G$, endowed with the projection on the first component $\pi : P \to M$, is a principal fibre bundle over M. Here the action of G on P is $\sigma_{g_1}(x, g) = (x, g \cdot g_1)$ for each $x \in M$ and $g, g_1 \in G$.

Example 2.2.15 (**The Frame Bundle**). Starting from a vector bundle $\pi : E \to M$ of rank r we can build a principal bundle as follows. For each fibre E_x we can consider the set $F(E)_x$ of all basis of E_x. We denote with $F(E) = \bigcup_{x\in M} F(E)_x$ and with $\tilde{\pi} : F(E) \to M$ the canonical projection on M. Local trivializations of $\pi : E \to M$ induce local trivializations of $F(E)$. Indeed, if $\{\sigma_1, ..., \sigma_r\}$ is a local frame for E in $U \subset M$, then we can build

$$\tilde{\chi} : \tilde{\pi}^{-1}(U) \to U \times GL(r, \mathbb{R})$$

that associates to each basis $\{e_1, ..., e_r\}$ in $F(E)_x = \tilde{\pi}^{-1}(x)$ the couple (x, A), where $A = (a_h^k) \in GL(r, \mathbb{R})$ is the unique matrix such that

$$e_h = \sum_{k=1}^{r} a_h^k \sigma_k(x)$$

for $x \in U$. Moreover, $GL(r, \mathbb{R})$ acts on each fibre. Therefore, $F(E)$ is a principal fibre bundle with structure group $G = GL(r, \mathbb{R})$.

Definition 2.2.16. Let $\pi_1 : P_1 \to M_1$ and $\pi_2 : P_2 \to M_2$ be two principal fibre bundles, both with structure group G. A **principal bundle map** from P_1 to P_2 is defined as a differentiable map $f : P_1 \to P_2$ such that

$$f(p \cdot g) = f(p) \cdot g \quad \forall p \in P_1, \ \ \forall g \in G \tag{2.16}$$

Notice that this means that f maps the fibre of p to the fibre of $f(p)$. Condition (2.16) tells us more, each fibre in P_1 is carried diffeomorphically onto a fibre of P_2. Therefore, we can define a map $\tilde{f} : M_1 \to M_2$ such that $\pi_2 \circ f = \tilde{f} \circ \pi_1$, namely the following diagram commutes

$$\begin{array}{ccc} P_1 & \xrightarrow{f} & P_2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow{\tilde{f}} & M_2 \end{array} \tag{2.17}$$

If P_1 and P_2 are principal bundles over the same base space M, we say that a principal bundle map f from P_1 to P_2 is an **equivalence** if the induced map $\tilde{f} : M \to M$ is the identity. In this case it is easy to verify that f is a diffeomorphism and the inverse is also an equivalence. If moreover $P_1 = P_2$ and f is an equivalence, the we call it an **automorphism**.

As we already did for equivalent vector bundles, we now introduce a characterization for equivalent principal fibre bundles. We will see that it is really similar to the characterization for vector bundles, and it is strictly related to the concept of Čech cohomology, which is defined in Appendix B. In particular we will show that there is a one to one correspondence between the set of all G-principal fibre bundles (up to equivalence) over a manifold M and the classes of the Čech cohomology with coefficients in the sheaf of smooth G-valued functions on M.

Remark 2.2.17 (**Characterization for equivalent principal bundles**). Let $\pi : P \to M$ and $\tilde{\pi} : \tilde{P} \to M$ be two principal fibre bundles with structure group G, and base manifold M. If $f : P \to \tilde{P}$ is an isomorphism of principal bundles, and $\mathcal{A} = \{(U_i, \chi_i)\}_{i\in I}$, $\tilde{\mathcal{A}} = \{(U_i, \tilde{\chi}_i)\}_{i\in I}$ are two atlases for P and $\tilde{P}$ respectively, then we can define the maps $\hat{h}_i := \tilde{\chi}_i \circ f \circ \chi_i^{-1}$, and the following diagram commutes

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{f} & \tilde{\pi}^{-1}(U_i) \\ \downarrow{\scriptstyle \chi_i} & & \downarrow{\scriptstyle \tilde{\chi}_i} \\ U_i \times G & \xrightarrow{\hat{h}_i} & U_i \times G \end{array} \tag{2.18}$$

It is easy to verify that there exists a family of maps $h_i \in C^\infty(U_i, G)$ such that

$$\hat{h}_i(x, g) = (x, h_i(x)g) \;\; \forall (x, g) \in U_i \times G$$

Since f is well defined, we have the relation

$$\tilde{\chi}_i^{-1} \circ \hat{h}_i \circ \chi_i = \tilde{\chi}_j^{-1} \circ \hat{h}_j \circ \chi_j \quad \text{in} \;\; \pi^{-1}(U_i \cap U_j)$$

which easily implies that

$$\tilde{g}_{ij} = h_i g_{ij} h_j^{-1} \quad \text{in} \;\; U_i \cap U_j \neq \emptyset \tag{2.19}$$

where $\{g_{ij}\}$ and $\{\tilde{g}_{ij}\}$ are the transition functions of $\mathcal{A}$ and $\tilde{\mathcal{A}}$.
Conversely if $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are as above, and their transition functions $\{g_{ij}\}$ and $\{\tilde{g}_{ij}\}$ satisfy the equation (2.19) for a family $h_i \in C^\infty(U_i, G)$, then with a construction similar to the one of Remark 2.2.6 we can build an equivalence between P and $\tilde{P}$.

We have seen how a principal fibre bundle leads to a family of transition functions satisfying the cocycle conditions. The following proposition, which is the equivalent of Proposition 2.2.7 for principal bundles, guarantees that given a manifold M, a covering $\mathcal{U} = \{U_i\}_{i\in I}$ and a family of maps $g_{ij} \in C^\infty(U_i \cap U_j, G)$ satisfying the cocycle conditions, we can build a principal fibre bundle over M with structure group G.

Proposition 2.2.18. *Let M be a manifold and $\{U_i\}_{i\in I}$ a covering for M. If $g_{ij} : U_i \cap U_j \to G$ is a family of differentiable functions satisfying the cocycle conditions, then there exists a principal fibre bundle P with structure group G and base manifold M with transition functions g_{ij}.*

The proof is essentially the same of Proposition 2.2.7. For further details see [22], Proposition 5.2.

Definition 2.2.19. Let $\pi : P \to M$ be a principal bundle with structure group G. A **local cross section** $s : V \subset M \to \pi^{-1}(V)$ is a smooth map from an open subset V of M into P such that $\pi \circ s = id_V$.

Definition 2.2.20. Let $\pi : P \to M$ be a principal fibre bundle over M, with structure group G. If $s : V \to \pi^{-1}(V)$ is a local cross section, and $g \in C^\infty(V, G)$, then we define the **local gauge transformation** $s^g(x) := s(x)\cdot g(x)$ for each $x \in V$, which is clearly again a local cross section.

The following Proposition highlights that there always exists a gauge transformation between two different local cross sections defined in the same open subset of the base manifold.

Proposition 2.2.21. *Let $\pi : P \to M$ be a principal fibre bundle over M, and structure group G. Let $V \subset M$ be open, then:*

1) If $h, s : V \to \pi^{-1}(V)$ are two local cross sections, then they are gauge equivalent.

2) Every change of cross section in V, induces a local automorphism $f : \pi^{-1}(V) \to \pi^{-1}(V)$, and also the inverse is true.

Proof. 1) If $h, s : V \to \pi^{-1}(V)$ are both cross sections then there exists a gauge $g : V \to G$ such that $s^g = h$, and this is due to the fact that for every $x \in V$ the elements $h(x)$ and $s(x)$ are in the same fibre, and so there exists a unique $g(x) \in G$ such that $s(x) \cdot g(x) = h(x)$.

2) Indeed, if $s : V \to \pi^{-1}(V)$ is a local cross section, then each fibre in $\pi^{-1}(V)$ can be written as $\pi^{-1}(x) = \{s(x)h| \quad h \in G\}$ for $x \in V$. Then, if $g \in C^\infty(V, G)$ is a local gauge transformation, we can define the automorphism

$$\pi^{-1}(V) \ni s(x)h \mapsto s^g(x)h \in \pi^{-1}(V)$$

Conversely if $f : \pi^{-1}(V) \to \pi^{-1}(V)$ is a local automorphism and $s : V \to \pi^{-1}(V)$ is a cross section, then $V \ni x \to f^{-1}(s(x)) \in \pi^{-1}(V)$ is clearly a cross section, and by 1) we obtain a local gauge transformation. $\square$

In Example 2.2.15 we saw that starting from a vector bundle we can build a principal fibre bundle, called the frame bundle. The following Theorem tells us that also the inverse is true. Namely, if $\pi : P \to M$ is a principal fibre bundle, with structure group G, and such that G acts on the left on some vector space $\mathcal{V}$, then there is a rule that let us build a vector bundle with fibre $\mathcal{V}$ and structure group G, associated to P in some sense. Actually the below result is more general, and it holds for principal bundles with structure group acting on the left on manifolds (and not only vector spaces). For further details see for instance [1].

Theorem 2.2.22. *Let $\pi : P \to M$ be a principal fibre bundle with structure group G, $\mathcal{V}$ a vector space and $\rho : G \to GL(\mathcal{V})$ a group representation*[5]*. Then it holds*

1) The map

$$
\begin{aligned}
R : (P \times \mathcal{V}) \times G &\to P \times \mathcal{V} \\
((p, v), g) &\mapsto (p \cdot g, \rho(g^{-1})(v))
\end{aligned}
\tag{2.20}
$$

is a free right action of G on $P \times \mathcal{V}$.

2) The quotient space $P \times_G \mathcal{V} = (P \times \mathcal{V})/G$ has a unique structure of manifold, with respect to which the quotient map $\psi : P \times \mathcal{V} \to P \times_G \mathcal{V}$ is a submersion.

3) If $\pi_1 : P \times \mathcal{V} \to P$ is the projection on the first component, then the following diagram

$$
\begin{array}{ccc}
P \times \mathcal{V} & \xrightarrow{\psi} & P \times_G \mathcal{V} \\
\Big\downarrow{\scriptstyle \pi_1} & & \Big\downarrow{\scriptstyle \overline{\pi}} \\
P & \xrightarrow{\pi} & M
\end{array}
\tag{2.21}
$$

defines a map $\overline{\pi} : P \times_G \mathcal{V} \to M$ with respect to which $P \times_G \mathcal{V}$ is a G-vector bundle with fibre $\mathcal{V}$, with left action given by the group representation ρ.

[5]A group homomorphism $\rho : G \to GL(\mathcal{V})$, where $GL(\mathcal{V})$ denote the group of automorphisms of a vector space $\mathcal{V}$, is called group representation

Proof. 1) This first step is an easy exercise.
2) First of all we need to show that the map $\overline{\pi}$ is well defined. This request is equivalent to prove that $\pi \circ \pi_1$ is constant on the orbits of R, which is true. Indeed, if $g \in G$ and $(p, v) \in P \times \mathcal{V}$, then $\pi \circ \pi_1(p, s) = \pi(p) = \pi(p \cdot g) = \pi \circ \pi_1(p \cdot g, \rho(g^{-1})(v))$.
Let $\mathcal{A} = \{(U_i, \chi_i)\}_{i \in I}$ be an atlas for $\pi : P \to M$, with transition functions $g_{ij} : U_i \cap U_j \to G$. We define the following family of maps

$$\begin{aligned} \overline{\xi}_i : U_i \times \mathcal{V} &\to \overline{\pi}^{-1}(U_i) \\ (x, v) &\longmapsto \psi(\chi_i^{-1}(x, e), v) \end{aligned} \tag{2.22}$$

We show that for each $x \in U_i$ the map $\overline{\xi}_i(x, \cdot) : \mathcal{V} \to \overline{\pi}^{-1}(x)$ is bijective. This is true if and only if for each $\psi(p, v') \in \overline{\pi}^{-1}(x)$ there exists and is unique $v \in \mathcal{V}$ such that $\psi(p, v') = \psi(\chi_i^{-1}(x, e), v)$. This last request is satisfied if and only if there exists and is unique $g \in G$ such that $(p \cdot g, \rho(g^{-1})(v')) = (\chi_i^{-1}(x, e), v)$. The solution $g \in G$ exists and is unique, and one fixes $v = \rho(g^{-1})(v')$.
Therefore, all the $\overline{\xi}_i$ are bijective, and we denote $\overline{\chi}_i := \overline{\xi}_i^{-1} : \overline{\pi}^{-1}(U_i) \to U_i \times \mathcal{V}$. We have that

$$\begin{aligned} \overline{\chi}_j^{-1}(x, v) &= \psi(\chi_j^{-1}(x, e), v) = \psi(\chi_i^{-1}(x, g_{ij}e), v) \\ &= \psi(\chi_i^{-1}(x, e), \rho(g_{ij})(v)) = \overline{\chi}_i^{-1}(x, \rho(g_{ij})(v)) \end{aligned}$$

In particular $\overline{\mathcal{A}} := \{(U_i, \overline{\chi}_i)\}_{i \in I}$ is an atlas for $P \times_G \mathcal{V}$, and it induces a differentiable structure on it. Moreover, since $\overline{\xi}_i$ are all differentiable, we get that also ψ is differentiable, and it is actually a submersion. It is easy to prove that also $\overline{\pi}$ is differentiable and surjective, a precise proof of this last fact can be found in Exercise 2.83 of [1]. □

Remark 2.2.23. If $\pi : P \to M$ is a principal fibre bundle with structure group G, we know that $Ad : G \times \mathfrak{g} \to \mathfrak{g}$ is a left action on the Lie Algebra $\mathfrak{g}$. Therefore thanks to Theorem 2.2.22, we can build a G-vector bundle $\overline{\pi} : P \times_G \mathfrak{g} \to M$ over M, with fibre $\mathfrak{g}$. This is called the **Adjoint bundle** associated to the principal fibre bundle $\pi : P \to M$.

2.3 Connections

Connections on vector bundles are geometrical tools, born in order to define the concept of derivation of a section along a curve on the base manifold. The classical idea used in $T(\mathbb{R}^n) \cong \mathbb{R}^n \times \mathbb{R}^n$ of defining the derivative as limit of the difference quotient, do not

work for general vector bundles, since a section usually gives us vectors in different vector spaces (although with the same dimension) for different points on the base manifold. For this first subsection we refer to [1], while for the third subsection, where we will define connections also for principal fibre bundles, we will mainly use as reference [22].
In the second subsection, instead, we will define the Second fundamental form, and the Shape operator, for a Riemannian manifold. These last interesting geometrical constructions, are presented here just for an Analytic purpose, since the Shape operator naturally arises in the Euler-Lagrange equations of weakly harmonic maps, that will be studied in Chapter 5. The relation between the Analytic problem of weakly Harmonic maps and the Shape operator is explored for instance in [42].

2.3.1 Connections on a Vector Bundle

Let M be a manifold, and $\pi : E \to M$ a vector bundle on it.

Definition 2.3.1. A **connection** over the vector bundle E is a function

$$\begin{aligned} \nabla : \tau(M) \times E(M) &\to E(M) \\ (X, s) &\longmapsto \nabla_X s \end{aligned} \tag{2.23}$$

such that the following are satisfied

1) $\forall X, Y \in \tau(M)$ and $\forall f_1, f_2 \in C^\infty(M, \mathbb{R})$

$$\nabla_{f_1 X + f_2 Y} s = f_1 \nabla_X s + f_2 \nabla_Y s \tag{2.24}$$

for each $s \in E(M)$.

2) $\forall X \in \tau(M)$ and $\forall s_1, s_2 \in E(M)$ it holds

$$\nabla_X a_1 s_1 + a_2 s_2 = a_1 \nabla_X s_1 + a_2 \nabla_X s_2 \tag{2.25}$$

for each $a_1, a_2 \in \mathbb{R}$.

3) $\forall X \in \tau(M)$ and $\forall s \in E(M)$ it holds

$$\nabla_X (fs) = X(f)s + f \nabla_X s \qquad \textbf{(Liebniz's rule)} \tag{2.26}$$

for each $f \in C^\infty(M, \mathbb{R})$.

The section $\nabla_X s \in E(M)$ is called the **covariant derivative** of $s \in E(M)$ along $X \in \tau(M)$. If $E = TM$ then we will call ∇ a **linear connection**, and later we will see a fundamental example of it.

If ∇ is a connection over the vector bundle $\pi : E \to M$, then the value of $(\nabla_X s)(x)$ depends just on the value of X_x and on the behaviour of s in a neighbourhood of $x \in M$. The following Lemma makes precise this assertion

Lemma 2.3.2. *Let ∇ be a connection over the vector bundle $\pi : E \to M$. Then*

1) *If $X, \tilde{X} \in \tau(M)$ and $s, \tilde{s} \in E(M)$ are such that $X_x = \tilde{X}_x$ and $s = \tilde{s}$ in a neighbourhood of $x \in M$, then $(\nabla_X s)(x) = (\nabla_{\tilde{X}} \tilde{s})(x)$.*

In particular we can improve this first point as follows

2) *If $X \in \tau(M)$ and $s, \tilde{s} \in E(M)$, and there exists a curve $\gamma : (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = x$ and $\gamma'(0) = X_x$ and moreover $s \circ \gamma = \tilde{s} \circ \gamma$, then $(\nabla_X s)(x) = (\nabla_X \tilde{s})(x)$.*

A proof for this technical result can be found in [1]. In particular using Lemma 2.3.2, one can prove the following result, which we will just state for the sake of completeness and whose proof can be found for instance in [1].

Theorem 2.3.3. *Each vector bundle $\pi : E \to M$ admits a connection.*

One may want to express a connection locally. Let then (U, ϕ) be a local chart for M that also trivializes locally the vector bundle $\pi : E \to M$ of rank r, namely such that there exists a local trivialization for E

$$\chi : \pi^{-1}(U) \to U \times \mathbb{R}^r$$

Then the canonical basis $e_1, ..., e_r$ of $\mathbb{R}^r$ determines a local frame for $\pi^{-1}(U)$, i.e. $\overline{e}_1, ...\overline{e}_r \in E(U)$ as follows

$$\begin{aligned} \overline{e}_j : U &\to \pi^{-1}(U) \\ x &\longmapsto \chi^{-1}(x, e_j) \end{aligned}$$

While the chart $\phi : U \to \phi(U) \subset \mathbb{R}^n$ determines a local frame $\{\partial_{x_1}, ..., \partial_{x_n}\}$ for the tangent bundle TM. Then we obtain the existence of $\Gamma^k_{jh} \in C^\infty(U, \mathbb{R})$, called **coefficients of the connection**[6], such that

$$\nabla_{\partial_{x_j}} \overline{e}_h = \sum_{k=1}^{r} \Gamma^k_{jh} \overline{e}_k \tag{2.27}$$

where $j = 1, ..., n$ and $k, h = 1, ..., r$. If $X \in \tau(U)$ and $s \in E(U)$, using equation (2.27) and properties 1),2) and 3) of Definition 2.3.1 we get

$$\nabla_X s = (X(s^k) + \Gamma^k_{jh} X^j s_h) \overline{e}_k \tag{2.28}$$

[6]If $E = TM$, then the coefficients of the connection are called Christoffel symbols

Connection as a differential form Let $\pi : E \to M$ be a vector bundle over the manifold M. We know that k-differential forms over M are sections of the bundle $\wedge^k T^*M$. Similarly k-differential forms with values in E are sections of the bundle $\wedge^k T^*M \otimes E$, and we will denote them with $A^k(E)$. We now give an alternative definition of connection over E.

Definition 2.3.4. A connection over the vector bundle $\pi : E \to M$ is a $\mathbb{R}$-linear operator

$$\nabla : E(M) \to A^1(E) \tag{2.29}$$

such that $\nabla(fs) = f\nabla s + df \otimes s$, for each $s \in E(M)$ and $f \in C^\infty(M, \mathbb{R})$.

The relation with Definition 2.3.1 is the following. If ∇ is a connection over $\pi : E \to M$, then for each $X \in \tau(M)$, we have

$$\nabla_X s = \langle \nabla s, X \rangle \tag{2.30}$$

where in this context $\langle \cdot, \cdot \rangle$ is defined as follows. If $\omega \in A^1(E)$ it can be expressed as $\omega = \sum_i \omega_i \otimes s_i$ where s_i are sections of E, and $\omega_i \in \wedge^1(M)$. Then we have $\langle \omega, X \rangle := \sum_i \omega_i(X) s_i$.

Remark 2.3.5. Let $\pi : E \to M$ be a vector bundle over M, and ∇ a connection over it. Moreover assume that (U, ϕ) is a local chart for M, such that there exists a local trivialization

$$\chi : \pi^{-1}(U) \to U \times \mathbb{R}^r$$

Then if $\{\overline{e}_1, ..., \overline{e}_r\}$ is the associated frame for E in U, we have that

$$\nabla \overline{e}_j = \sum_{k=1}^{r} \omega_j^k \otimes \overline{e}_k \tag{2.31}$$

The local chart ϕ let us express the one forms ω_j^k locally as follows

$$\omega_j^k = \sum_{i=1}^{n} \Gamma_{ij}^k dx^i$$

for proper $\Gamma_{ij}^k \in C^\infty(U, \mathbb{R})$. The smooth functions Γ_{ij}^k are exactly the coefficients of the connection, indeed

$$\nabla_{\partial_{x_i}} \overline{e}_j = \langle \nabla \overline{e}_j, \partial_{x_i} \rangle = \sum_{k=1}^{r} \omega_j^k(\partial_{x_i}) \overline{e}_k = \sum_{k=1}^{r} \Gamma_{ij}^k \overline{e}_k$$

Definition 2.3.6. The matrix ω whose entries are the differential 1-forms ω_j^k of equation (2.31) is called the 1-**form of the connection**, with respect to the fixed frame.

Changing the local frame we get of course a different 1-form of the connection, which is related to the first one by a significant equation. We develop the calculations here because the aforementioned equation is relevant to us, as it will be extensively explained in Example 2.3.30.
Let $\{\tilde{e}_1, ...\tilde{e}_r\}$ be another frame for E in U. Then it exists and is unique $A \in C^\infty(U, GL(r, \mathbb{R}))$ such that

$$\tilde{e}_j = \sum_{k=1}^{r} a_h^k \overline{e}_k$$

where $A = (a_h^k)_{h,k=1,...,r}$. If $\tilde{\omega} = (\tilde{\omega}_i^h)$ is the 1-form of the connection with respect to the frame $\tilde{e}_1, ..., \tilde{e}_r$, then we have

$$\nabla \tilde{e}_i = \tilde{\omega}_i^h \otimes \tilde{e}_h = \tilde{\omega}_i^h \otimes a_j^k \overline{e}_k = a_h^k \tilde{\omega}_i^h \otimes \overline{e}_k$$

but also, thanks to Definition 2.3.4, we get

$$\begin{aligned} \nabla \tilde{e}_i = \nabla(a_i^k \overline{e}_k) = a_i^k \nabla \overline{e}_k + da_i^k \otimes \overline{e}_k = a_i^k \omega_k^l \otimes \overline{e}_l + da_i^k \otimes \overline{e}_k \\ = (a_i^j \omega_j^k + da_i^k) \otimes \overline{e}_k \end{aligned}$$

where $\omega = (\omega_i^k)$ is the 1-form of the connection with respect to the frame $\overline{e}_1, ..., \overline{e}_r$, and in the last identity we have just renamed the indexes. These last two equations lead to

$$a_h^k \tilde{\omega}_i^h = a_i^j \omega_j^k + da_i^k$$

which in matrix representation reads as

$$A\tilde{\omega} = \omega A + dA \Rightarrow \tilde{\omega} = A^{-1} dA + A^{-1} \omega A \tag{2.32}$$

Levi-Civita Connection

In this subsection we introduce a fundamental connection, called **Levi-Civita connection**, on the tangent bundle TM of a Riemannian manifold $(M, \mathbf{g})$. It has a deep geometrical meaning, and as we will see it is completely determined by the metric we are considering on M.

Definition 2.3.7. Let $(M, \mathbf{g})$ be a Riemannian manifold, and ∇ a connection on the tangent bundle TM. Then we say that ∇ is **compatible** with the metric $\mathbf{g}$ if for each $X, Y, Z \in \tau(M)$, it holds

$$Z(\mathbf{g}(X, Y)) = \mathbf{g}(\nabla_Z X, Y) + \mathbf{g}(X, \nabla_Z Y) \tag{2.33}$$

There are several equivalent definitions of connection compatible with the metric of a Riemannian manifold, many of them regarding the parallel transport, see [1]. However, since this tool is not fundamental for this work, we will just state one characterization for compatible connections, which is actually the compatibility condition expressed in a generic local chart for M.

Proposition 2.3.8. *Let $(M, \mathbf{g})$ be a Riemannian manifold and ∇ a connection on the tangent bundle. The followings are equivalent*

1) ∇ is compatible with the metric $\mathbf{g}$

2) In each coordinate system (U, φ) for M it holds

$$\partial_{x_k}\mathbf{g}_{ij} = \mathbf{g}_{lj}\Gamma^l_{ki} + \mathbf{g}_{il}\Gamma^l_{kj} \tag{2.34}$$

where Γ^k_{ij} are the Christoffel symbols.

Proof. 1)$\Rightarrow$ 2) Let (U, φ) be a local chart for M, and $\partial_{x_k}, \partial_{x_i}, \partial_{x_j} \in \tau(U)$, then we have

$$\partial_{x_k}\mathbf{g}_{ij} = \partial_{x_k}\mathbf{g}(\partial_{x_i}, \partial_{x_j}) = \mathbf{g}(\nabla_{\partial_{x_k}}\partial_{x_i}, \partial_{x_j}) + \mathbf{g}(\partial_{x_i}, \nabla_{\partial_{x_k}}\partial_{x_j})$$

where the second identity is of course due to 1). Now we have that $\nabla_{\partial_{x_k}}\partial_{x_i} = \Gamma^l_{ki}\partial_{x_l}$, and therefore

$$\partial_{x_k}\mathbf{g}_{ij} = \mathbf{g}(\Gamma^l_{ki}\partial_{x_l}, \partial_{x_j}) + \mathbf{g}(\partial_{x_i}, \Gamma^l_{kj}\partial_{x_l}) = \mathbf{g}_{lj}\Gamma^l_{ki} + \mathbf{g}_{il}\Gamma^l_{kj}$$

2)$\Rightarrow$ 1) Since the compatibility condition holds for every local chart, then of course it holds for every vector field over M. $\square$

Example 2.3.9. If $M = \mathbb{R}^n$ and we endow it with the Euclidean metric, then it is clear that for each $X, Y, Z \in \tau(\mathbb{R}^n)$, we have

$$Z(X \cdot Y) = Z(X) \cdot Y + X \cdot Z(Y)$$

Therefore the flat connection is compatible with the Euclidean metric.

Proposition 2.3.10. *Let ∇ be a connection on the tangent bundle TM of a manifold M. Then we have that the operator*

$$\begin{aligned} \kappa : \tau(M) \times \tau(M) &\to \tau(M) \\ (X, Y) &\longmapsto \nabla_X Y - \nabla_Y X - [X, Y] \end{aligned} \tag{2.35}$$

is $C^\infty(M, \mathbb{R})$-linear, and antisymmetric.

Proof. The antisymmetry is obvious. Take $f \in C^\infty(M,\mathbb{R})$ and $X, Y \in \tau(M)$, then

$$\kappa(fX,Y) = \nabla_{fX}Y - \nabla_Y(fX) - [fX,Y] = f\nabla_X Y - f\nabla_Y X - Y(f)X +$$
$$-f[X,Y] + Y(f)X = f\kappa(X,Y)$$

and this proves the assertion. □

Definition 2.3.11. Let M be a manifold, and ∇ a connection on the tangent bundle TM. Then we say that ∇ is **symmetric** if for each $X, Y \in \tau(M)$ the following holds

$$\kappa(X,Y) = 0 \tag{2.36}$$

Remark 2.3.12. Note that this definition is the generalization to an arbitrary manifold, of the obvious property of the flat connection on $\mathbb{R}^n$

$$X(Y) - Y(X) - [X,Y] = 0$$

where $X, Y \in \mathbb{R}^n$.

Also in this case we have equivalent definitions of symmmetric connection over the tangent bundle TM of some manifold M.

Proposition 2.3.13. *Let ∇ be a connection on the tangent bundle TM of a manifold M. Then the followings are equivalent*

1) ∇ is symmetric

2) In each coordinate system one has that the Christoffel symbols are symmetric, namely

$$\Gamma^j_{hi} = \Gamma^j_{ih} \tag{2.37}$$

Proof. We prove 1)⇒ 2), which is actually just a computation. Indeed, if (U,φ) is a local chart for M, then we have that for $\partial_{x_i}, \partial_{x_j} \in \tau(U)$

$$\kappa(\partial_{x_i}, \partial_{x_j}) = \nabla_{\partial_{x_i}}\partial_{x_j} - \nabla_{\partial_{x_j}}\partial_{x_i} - \underbrace{[\partial_{x_i}, \partial_{x_j}]}_{=0} = (\Gamma^k_{ij} - \Gamma^k_{ji})\partial_{x_k} = 0$$

and this leads us to the wanted symmetry identity.
2)⇒ 1) is obtained by expressing locally two generic vector fields $X, Y \in \tau(M)$, and then using the symmetry of the Christoffel symbols. □

One can prove that there are infinite connections over the tangent bundle TM of a Riemannian manifold $(M, \mathbf{g})$ that are compatible with the metric. However, the following Theorem shows us that there exists and is unique a connection on TM which is both compatible with the metric and symmetric. This connection is the candidate to be the Levi-Civita connection.

Theorem 2.3.14 (Levi-Civita). *Let $(M, \mathbf{g})$ be a Riemannian manifold. Then there exists and is unique a connection ∇ on the bundle TM which is both symmetric and compatible with the metric $\mathbf{g}$. Furthermore it holds*

$$\mathbf{g}(\nabla_X Y, Z) = \frac{1}{2}\Big(X\mathbf{g}(Y,Z) + Y\mathbf{g}(Z,X) - Z\mathbf{g}(X,Y) + \\ \mathbf{g}([X,Y],Z) - \mathbf{g}([Y,Z],X) + \mathbf{g}([Z,X],Y)\Big) \tag{2.38}$$

which expressed in local coordinates gives us the following equation for the Christoffel symbols[7]

$$\Gamma^k_{ij} = \frac{1}{2}\mathbf{g}^{kl}\left(\frac{\partial \mathbf{g}_{lj}}{\partial x_i} + \frac{\partial \mathbf{g}_{il}}{\partial x_j} - \frac{\partial \mathbf{g}_{ij}}{\partial x_l}\right) \tag{2.39}$$

A proof for this important result can be found in [1].

Example 2.3.15 (**The Levi-Civita connection of a Lie Group endowed with a bi-invariant metric**). Let G be a compact and connected Lie Group endowed with a bi-invariant metric $\mathbf{g}$, and ∇ the Levi-Civita connection on it. Then if X, Y are left-invariant vector fields it holds

$$\nabla_X Y = \frac{1}{2}[X,Y] \tag{2.40}$$

To prove this, first we observe that if X, Y, Z are left-invariant vector fields, then

$$\mathbf{g}([X,Y],Z) = -\mathbf{g}(Y,[X,Z]) \tag{2.41}$$

Indeed, we already observe that if $g \in G$ then $[X,Y]_g = \frac{d}{dt}Ad_{\gamma(t)}(Y)|_{t=0}$, where γ is the integral curve of X such that $\gamma(0) = g$, and since $\mathbf{g}$ is Ad-invariant we get

$$0 = \frac{d}{dt}\mathbf{g}(Y,Z)|_{t=0} = \frac{d}{dt}\mathbf{g}(Ad_{\gamma(t)}Y, Ad_{\gamma(t)}Z)|_{t=0} = \\ = \mathbf{g}([X,Y],Z) + \mathbf{g}(Y,[X,Z])$$

[7] If (U, ϕ) is a local chart for M, then if $x \in U$ we denote with $(\mathbf{g}(x)^{kl})_{k,l}$ the inverse matrix of $(\mathbf{g}(x)_{ij})_{ij}$ in the chosen local system of coordinates

Since $\mathbf{g}$ is bi-invariant, it is left invariant, which means that $X(\mathbf{g}(Y,Z)) = Y(\mathbf{g}(Z,X)) = Z(\mathbf{g}(X,Y)) = 0$, for each X,Y,Z left-invariant vector fields. Therefore under these assumptions equation (2.38) becomes

$$\mathbf{g}(\nabla_X Y, Z) = \frac{1}{2}\left(\mathbf{g}([X,Y],Z) - \mathbf{g}(X,[Y,Z]) + \mathbf{g}([Z,X],Y)\right)$$

which thanks to equation (2.41) can be rewritten as

$$\mathbf{g}(\nabla_X Y, Z) = \frac{1}{2}\mathbf{g}([X,Y],Z)$$

and proves the wanted equation (2.40).

We now exhibit a last example before moving on.

Example 2.3.16. Let $(\tilde{M},\tilde{\mathbf{g}})$ be a Riemannian manifold, and let M be a submanifold of $\tilde{M}$. We endow $\tilde{M}$ with the Levi-Civita connection $\tilde{\nabla}$, and in M we consider the induced metric[8]. In this example we show that the Levi-Civita connection ∇ for $(M, i^*\tilde{\mathbf{g}})$ is given by

$$\nabla_X Y = \top(\tilde{\nabla}_X Y) \ \ \forall X,Y \in \tau(M) \tag{2.42}$$

where $\top : T\tilde{M} \to TM$ is the orthogonal projection. First we show that it is compatible with the induced metric $i^*\tilde{\mathbf{g}}$. Indeed, we consider T_xM as a subspace of $T_x\tilde{M}$ for each $x \in M$, and if $X,Y,Z \in \tau(M)$, then we have

$$Zi^*\tilde{\mathbf{g}}(X,Y) = Z\tilde{\mathbf{g}}(X,Y) = \tilde{\mathbf{g}}(\tilde{\nabla}_Z X, Y) + \tilde{\mathbf{g}}(X, \tilde{\nabla}_Z Y) = (*)$$

and since it is clear that $\tilde{\mathbf{g}}(\tilde{\nabla}_Z X, W) = \tilde{\mathbf{g}}(\top(\tilde{\nabla}_Z X), W)$ for each $W \in \tau(M)$, then we have that

$$(*) = \tilde{\mathbf{g}}(\top(\tilde{\nabla}_Z X), Y) + \tilde{\mathbf{g}}(X, \top(\tilde{\nabla}_Z Y))$$

which proves that ∇ is compatible with the induced metric. Finally we see that

$$\nabla_X Y - \nabla_Y X = \top(\tilde{\nabla}_X Y - \tilde{\nabla}_Y X) = \top([X,Y])$$

but of course $[X,Y] \in \tau(M)$, and therefore $\top([X,Y]) = [X,Y]$, and we conclude.

2.3.2 Second Fundamental Form & The Shape Operator

In this subsection we define a tool which is useful in order to study the relations between the geometry of a Riemannian manifold, and a

[8] If $i : M \hookrightarrow \tilde{M}$ is a submanifold of the Riemannian manifold $(\tilde{M},\tilde{\mathbf{g}})$, then the induced metric on M is given by the pull-back $i^*\tilde{\mathbf{g}}$

submanifold endowed with the induced metric, see [1]. However, we introduce it just for an Analytic purpose regarding weakly harmonic maps, see Chapter 5.
Let M be a submanifold of a Riemannian manifold $(\tilde{M}, \tilde{\mathbf{g}})$, and we endow M with the induced metric.

Definition 2.3.17. We denote with $\top : T\tilde{M} \to TM$ and with $\perp : T\tilde{M} \to (TM)^{\perp}$ the orthogonal projections. We call $\tau(M, \tilde{M})$ the space of sections over M of $T\tilde{M}|_M$, and with $\mathcal{N}(M)$ the subspace of sections of $T\tilde{M}|_M$ orthogonal to TM.

Definition 2.3.18. Let M be a submanifold of a Riemannian manifold $(\tilde{M}, \tilde{\mathbf{g}})$. The **second fundamental form** is the linear form

$$II : \mathcal{N}(M) \times \tau(M) \times \tau(M) \to C^{\infty}(M, \mathbb{R}) \tag{2.43}$$

given by $II(N, X, Y)_x = \tilde{\mathbf{g}}_x(\tilde{\nabla}_X N, Y)$ where $\tilde{\nabla}$ is the Levi-Civita connection of $\tilde{M}$, and $x \in M$.

Proposition 2.3.19. *Let M, $\tilde{M}$ and II be as in Definition 2.3.18. Then for each $N \in \mathcal{N}(M)$ and $X, Y \in \tau(M)$ it holds*

$$II(N, X, Y) = -\tilde{\mathbf{g}}(N, \tilde{\nabla}_X Y) \quad \textbf{\textit{Weingarten's Formula}} \tag{2.44}$$

Moreover II is $C^{\infty}(M, \mathbb{R})$-linear, and symmetric with respect to the last two entries.

Proof. The $C^{\infty}(M, \mathbb{R})$-linearity with respect to the last entry is obvious. As far as the second is concerned, let $f \in C^{\infty}(M, \mathbb{R})$, then we have that

$$\begin{aligned} II(N, fX, Y) &= \tilde{\mathbf{g}}(\tilde{\nabla}_{fX} N, Y) = \tilde{\mathbf{g}}(f\tilde{\nabla}_X N, Y) = f\tilde{\mathbf{g}}(\tilde{\nabla}_X N, Y) = \\ &= fII(N, X, Y) \end{aligned}$$

While for the first component we observe that

$$\begin{aligned} II(fN, X, Y) &= \tilde{\mathbf{g}}(\tilde{\nabla}_X fN, Y) = \tilde{\mathbf{g}}(f\tilde{\nabla}_X N + X(f)N, Y) = \\ &= \tilde{\mathbf{g}}(f\tilde{\nabla}_X N, Y) = f\tilde{\mathbf{g}}(\tilde{\nabla}_X N, Y) = fII(N, X, Y) \end{aligned}$$

where the third identity is due to the fact that $X(f)N \in \mathcal{N}(M)$, and therefore $\tilde{\mathbf{g}}(X(f)N, Y) = 0$. In particular since $\tilde{\nabla}$ is the Levi-Civita connection, then we have

$$II(N, X, Y) = \tilde{\mathbf{g}}(\tilde{\nabla}_X N, Y) = X(\tilde{\mathbf{g}}(N, Y)) - \tilde{\mathbf{g}}(N, \tilde{\nabla}_X Y) = -\tilde{\mathbf{g}}(N, \tilde{\nabla}_X Y)$$

which proves the Weingarten's Formula. Finally, using always the properties of the Levi-Civita connection, we obtain

$$II(N, X, Y) - II(N, Y, X) = \tilde{\mathbf{g}}(N, -\tilde{\nabla}_X Y + \tilde{\nabla}_Y X) = \tilde{\mathbf{g}}(N, [Y, X]) = 0$$

which gives us the symmetry with respect to the last two components of the second fundamental form. $\square$

We define now another fundamental operator, strictly related to the second fundamental form, which will be used in Chapter 5 when we will introduce weakly harmonic maps.

Definition 2.3.20. Let M be a submanifold of a Riemannian manifold $(\tilde{M}, \tilde{\mathbf{g}})$. The **shape operator** of M, is the operator

$$\begin{aligned} S : \tau(M) \times \tau(M) &\to \mathcal{N}(M) \\ (X, Y) &\longmapsto -\perp(\tilde{\nabla}_X Y) \end{aligned} \tag{2.45}$$

Remark 2.3.21. The second fundamental form and the shape operator are related by the following formula. As always let $N \in \mathcal{N}(M)$ and $X, Y \in \tau(M)$, then

$$II(N, X, Y) = \tilde{\mathbf{g}}(N, S(X, Y)) \tag{2.46}$$

This identity is obtained thanks to the Weingarten's formula. As a consequence we get that the Shape operator is $C^\infty(M, \mathbb{R})$-linear and symmetric. This means that for each $x \in M$ the shape operator defines a symmetric bilinear operator

$$S_x : T_xM \times T_xM \to (T_xM)^\perp$$

Moreover, from Example 2.3.16, we deduce the following equation which relates the Shape operator with the Levi-Civita connection in $\tilde{M}$, and the induced Levi-Civita connection on M

$$\tilde{\nabla}_X Y = \nabla_X Y - S(X, Y) \tag{2.47}$$

for each $X, Y \in \tau(M)$.

2.3.3 Connections on Principal Fibre Bundles

Let $\pi : P \to M$ be a principal fibre bundle over M and with structure group G. For each $p \in P$ we call G_p the subspace of T_pP made of vectors tangent to the fibre $\{p \cdot g \mid \ g \in G\}$ in p.

Definition 2.3.22. A **connection** Γ in $\pi : P \to M$ is a differentiable distribution H_p over P such that

1) $T_pP = H_p \oplus G_p$ for each $p \in P$
2) The distribution is **right invariant** under the action of the structure Lie group G, i.e. $H_{p \cdot g} = (d\sigma_g)_p H_p$ for each $p \in P$ and $g \in G$.

We call G_p the **vertical subspace** and H_p the **horizontal subspace** of T_pP.

By 1) we deduce that every vector $X \in T_pP$ can be uniquely decompose in the sum of its horizontal component X^H and vertical component X^V, i.e $X = X^H + X^V$ where $X^V \in G_p$ and $X^H \in H_p$.
We can associate to each connection Γ in P a $\mathfrak{g}$-valued 1-form ω on P. The construction for ω is the following.
The action of G on P induces, as stated in Proposition 2.1.21, a Lie algebra morphism $\sigma : \mathfrak{g} \to \tau(P)$, where $\sigma(A) = A^\sharp$, and since the action is free then σ is an isomorphism into its image. It is called the **fundamental vectorfield corresponding to** A.
Moreover, $A^\sharp$ was defined as $A^\sharp_p := \frac{d}{dt}(p \cdot \exp(tA))|_{t=0}$ for each $p \in P$ therefore $A^\sharp_p \in G_p \quad \forall p \in P$. But then this means that for every $p \in P$ the map $\mathfrak{g} \ni A \mapsto A^\sharp_p \in G_p$ is an isomorphim, and therefore for every $X \in T_pP$ there exists a unique $A \in \mathfrak{g}$ such that $X = X^H + A^\sharp_p$.
We define $\omega_p(X) := A$.
Obviously $\omega(X) = 0$ if and only if X is horizontal.

Definition 2.3.23. Let $\pi : P \to M$ be a principal fibre bundle with structure group G. Then if Γ is a connection on P, we call the $\mathfrak{g}$-valued 1-form ω associated to Γ, defined in the above construction, **connection form**.

The following proposition is the characterization of connection forms on P. Namely we pinpoint the necessary and sufficient conditions for a $\mathfrak{g}$-valued 1-form on P in order to be a connection form.

Proposition 2.3.24. *Let $\pi : P \to M$ be a principal fibre bundle with structure group G. A connection form ω on P satisfies the following conditions:*

a) $\omega(A^\sharp) = A$ for every $A \in \mathfrak{g}$

b) $(\sigma_g)^*\omega = Ad_{g^{-1}} \circ \omega$, that is to say $\omega_{p \cdot g}((d\sigma_g)_p X_p) = Ad_{g^{-1}}(\omega_p(X_p))$ for every vector field X on P and $g \in G$.

Conversely given a $\mathfrak{g}$-valued 1-form ω on P satisfying condition a) and b) there is a unique connection in P whose connection form is ω.

Proof. Let ω be a connection form over the principal fibre bundle $\pi : P \to M$. Point a) is a straightforward consequence of the definition. For point b) assume that $p \in P$ and $X \in T_pP$. Then we can decompose $X = X^H + X^V = X^H + A^\sharp_p$, where $A \in \mathfrak{g}$ is the unique element of the Lie Algebra $\mathfrak{g}$, such that $A_p^\sharp = X^V$. Then we have that

$$(\sigma_g^*\omega)_p(X) = \omega_{\sigma_g(p)}(d\sigma_g(X)) = \omega_{\sigma_g(p)}(d\sigma_g(X^H) + d\sigma_g(A_p^\sharp)) =$$
$$= \omega_{\sigma_g(p)}(d\sigma_g(A_p^\sharp))$$

where the last identity is due to the fact that $d\sigma_g(X^H) \in H_{\sigma_g(p)}$ by definition of horizontal subspace. Now, as already observed in the first section of this chapter, if we call $L_p : G \to P$ the map defined by $L_p(h) := \sigma_h(p)$, then

$$A_p^\sharp = dL_p(A_e)$$

where $A_e \in T_eG \cong \mathfrak{g}$. Therefore, we get

$$d\sigma_g(A_p^\sharp) = (d\sigma_g \circ dL_p)(A_e) = d(\sigma_g \circ L_p)(A_e) = \frac{d}{dt}\left(p \cdot \exp(tA_e)g\right)|_{t=0} =$$
$$= \frac{d}{dt}\left(\sigma_g(p) \cdot g^{-1}\exp(tA_e)g\right)|_{t=0} = \frac{d}{dt}\left(\sigma_g(p) \cdot \exp(Ad_{g^{-1}}(tA_e)\right)|_{t=0} =$$
$$=: (Ad_{g^{-1}}(A))^\sharp_{\sigma_g(p)}$$

which means that $(\sigma_g^*\omega)_p(X) = Ad_{g^{-1}}(A) = Ad_{g^{-1}}(\omega_p(X))$, proving the first part of the proposition.
Conversely let ω be $\mathfrak{g}$-valued 1-form on P that satisfies both a) and b). For each $p \in P$ we define the following linear subspace of T_pP,

$$H_p := \{X \in T_pP : \ \omega_p(X) = 0\} \tag{2.48}$$

If $X \in T_pP$ is tangent to the fibre, then $X = A_p^\sharp$ and thanks to point a) $\omega_p(A_p^\sharp) = A \in \mathfrak{g}$. This proves that for each $p \in P$ the subspace H_p has always the same dimension, and moreover $T_pP = G_p \oplus H_p$. Furthermore, $p \to H_p$ is differentiable, since ω is a differentiable form, thus H_p is a smooth distribution. Using point b) we easily obtain also that if $X \in H_p$, for $p \in P$, then $d\sigma_g(X) \in H_{\sigma_g(p)}$ for each $g \in G$. This concludes the proof. $\square$

Suppose now that $\mathcal{A} = \{(U_i, \chi_i)\}_{i \in I}$ is an atlas for the principal fibre bundle $\pi : P \to M$, and ω is a connection form in P. As always we will denote the family of transition functions with $\{g_{ij}\}$. If we consider the family of **trivial** local **cross sections**

$$s_i : U_i \to \pi^{-1}(U_i)$$
$$x \longmapsto s_i(x) := \chi_i^{-1}(x, e) \tag{2.49}$$

then we can build locally a $\mathfrak{g}$-valued one form on U_i using the pull-back

$$A_i := s_i^* \omega \quad \text{in} \quad U_i$$

A_i is called **local gauge potential** on U_i in the cross section s_i. Suppose now that $U_i \cap U_j \neq \emptyset$ for some $i, j \in I$. Then in this intersection A_i and A_j satisfy the below **compatibility condition**.

Proposition 2.3.25. *Let Θ be the left invariant $\mathfrak{g}$-valued canonical 1-form. For each non empty $U_i \cap U_j$ let $\Theta_{ij} := g_{ij}^* \Theta$. Then in the above hypothesis*

$$A_j = Ad_{g_{ij}^{-1}} \circ A_i + \Theta_{ij} \quad \textit{in} \;\; U_i \cap U_j \neq \emptyset \tag{2.50}$$

Conversely if a family $\{A_i\}_{i \in I}$ of $\mathfrak{g}$-valued 1 forms, defined on the covering $\mathcal{U} = \{U_i\}_{i \in I}$, satisfies condition (2.50), there is a unique connection form ω on P such that $A_i = s_i^* \omega$.

Proof. We first prove that if ω is a connection form and $A_i := s_i^* \omega$ is the above family of $\mathfrak{g}$-valued one forms, then they satisfy the compatibility condition. It is easy to verify that $s_j(x) = s_i(x) \cdot g_{ij}(x)$ for every $x \in U_i \cap U_j$. This relation show that s_j in $U_i \cap U_j$ can be seen as the following composition of maps

$$\begin{aligned} U_i \cap U_j &\longrightarrow P \times G \longrightarrow P \\ x &\mapsto (s_i(x), g_{ij}(x)) \mapsto s_i(x) \cdot g_{ij}(x) \end{aligned} \tag{2.51}$$

and so by the Leibniz's rule (see [22], Proposition 1.4) we have that for $x \in U_i \cap U_j$ and $v \in T_x M$

$$(ds_j)_x v = (d\sigma_{g_{ij}(x)})_{s_i(x)} (ds_i)_x v + (dL_{s_i(x)})_{g_{ij}(x)} (dg_{ij})_x v$$

where by an abuse of notation we have called $L_{s_i(x)} : G \to P$ the map defined as $L_{s_i(x)} g = s_i(x) \cdot g$ for every $g \in G$. Now observe that $(dg_{ij})_x v = A_{g_{ij}(x)}$ for some $A \in \mathfrak{g}$. This means that:

$$(g_{ij}^* \Theta) v = \Theta(dg_{ij}(v)) = \Theta(A_{g_{ij}(x)}) = A$$

and since $(dL_{s_i(x)})_{g_{ij}(x)} (dg_{ij})_x v = (dL_{s_i(x)})_{g_{ij}(x)} (A_{g_{ij}(x)}) = A^{\sharp}_{s_i(x) g_{ij}(x)}$ then we can write

$$(ds_j)_x v = (d\sigma_{g_{ij}(x)})_{s_i(x)} (ds_i)_x v + [(g_{ij}^* \Theta)(v)]^{\sharp}_{s_j(x)}$$

If we compute $\omega_{s_j(x)}$ both on the right and left of the preceding equation we find

$$(A_j)_x v := \omega_{s_j(x)}((ds_j)_x v) = \omega_{s_j(x)}((d\sigma_{g_{ij}(x)})_{s_i(x)} (ds_i)_x v) + (g_{ij}^* \Theta) v$$

and the first term on the right hand side, thanks to Proposition 2.3.24, is equal to $Ad_{g_{ij}^{-1}(x)}(A_i)_x v$. Therefore we get the wanted compatibility condition.
We have to prove now that given a family of $\mathfrak{g}$-valued 1-forms $\{A_i\}_{i\in I}$ such that (2.50) holds, we can get a unique connection form ω on P such that $s_i^*(\omega) = A_i$ for every $i \in I$, where $s_i : U_i \to \pi^{-1}(U_i)$ are the trivial cross sections, defined in (2.49).
We start by building a connection form on $\pi^{-1}(U_i)$ for every $i \in I$. If $p = \chi_i^{-1}(x, g)$ then for every $w \in T_pP$ exists and is unique a couple $(v_1, v_2) \in T_{(x,g)}(U_i \times G)$ such that $w = (d\chi_i^{-1})_{(x,g)}(v_1, v_2)$. Since the s_i are local trivial sections, then $\chi_i^{-1}(x, g) = s_i(x)g$. Differentiating we obtain

$$(d\chi_i^{-1})_{(x,g)}(v_1, v_2) = (d\sigma_g)_{s_i(x)}(ds_i)_x(v_1) + (dL_{s_i(x)})_g(v_2)$$

where we have used the Leibniz's rule. Now, $v_2 = (dL_g)(A_e)$ for some $A \in \mathfrak{g}$, and therefore the last addendum in the previous equation is $(dL_{s_i(x)})_g((dL_g)(A_e)) = dL_p(A_e) = A_p^\sharp$. So we get

$$(d\chi_i^{-1})_{(x,g)}(v_1, v_2) = (d\sigma_g)_{s_i(x)}(ds_i)_x(v_1) + A_p^\sharp$$

If $p = s_i(x)$ then for every $w \in T_pP$ we define

$$\omega_i(w) = (\omega_i)_p((ds_i)_x(v_1) + A_p^\sharp) := (A_i)_x(v_1) + A \tag{2.52}$$

since we have just proved that $w = (ds_i)_x(v) + A_p^\sharp$ for $v \in T_xM$ and $A \in \mathfrak{g}$.

If instead $p = s_i(x)g$, where $e \neq g \in G$, then for $w \in T_pP$ we generalize the above equation

$$(\omega_i)_p(w) := Ad_{g^{-1}} \circ (\omega_i)_{s_i(x)}((d\sigma_{g^{-1}})_p(w)) \tag{2.53}$$

We have to prove now that this $\mathfrak{g}$-valued 1-form ω_i in $\pi^{-1}(U_i)$ is a connection form. Thanks to Proposition 2.3.24 it is sufficient to check that:

1) $\omega_i(A^\sharp) = A$ for every $A \in \mathfrak{g}$

2) $\sigma_g^*\omega_i = Ad_{g^{-1}} \circ \omega$ for every $g \in G$.

We start with the first point.
1)Let $A \in \mathfrak{g}$ and $p = s_i(x)$ for some $x \in U_i$. By equation (2.52) we trivially obtain the wanted relation. Suppose now that $p = s_i(x)g$ for $g \in G$ and $x \in U_i$. Equation (2.53) tells us that

$$(\omega_i)_p(A_p^\sharp) = Ad_{g^{-1}} \circ (\omega_i)_{s_i(x)}((d\sigma_{g^{-1}})_{s_i(x)g}A^\sharp) =$$

$$= Ad_{g^{-1}} \circ (\omega_i)_{s_i(x)}((Ad_g \circ A)^\sharp_{s_i(x)}) = A$$

2)As far as the second point is concerned let $w \in T_p(P)$ where $p = s_i(x)h$ with $h \in G$, then

$$(\sigma_g^*\omega_i)_p(w) = (\omega_i)_{pg}((d\sigma_g)_p(w)) = (\omega_i)_{s_i(x)hg}((d\sigma_g)_{s_i(x)h}(w)) =$$

$$= Ad_{g^{-1}h^{-1}} \circ (\omega_i)_{s_i(x)}((d\sigma_{g^{-1}h^{-1}})_{s_i(x)hg}(d\sigma_g)_{s_i(x)h}w) =$$

$$= Ad_{g^{-1}} \circ Ad_{h^{-1}} \circ (\omega_i)_{s_i(x)}((d\sigma_{h^{-1}})_{s_i(x)h}w) = Ad_{g^{-1}} \circ (\omega_i)_p(w)$$

where the thirds equivalence is given by (2.53).

Finally we have only to prove that in $\pi^{-1}(U_i \cap U_j)$ the connections ω_i and ω_j coincide. We will focus only on the subset $s_j(U_i \cap U_j) \subset \pi^{-1}(U_i \cap U_j)$, the generalization is easy. Let $p = s_j(x)$ with $x \in U_j \cap U_i$. We pick $w \in T_{s_j(x)}P$ and we know that $w = (ds_j)_x(v) + A^\sharp_p$ for some $v \in T_xM$ and $A \in \mathfrak{g}$. Then

$$(\omega_j)_p(A^\sharp_p) = A = (\omega_i)_p(A^\sharp_p)$$

While

$$(\omega_j)_p((ds_j)_x v) = (A_j)_x v = Ad_{g_{ij}^{-1}} \circ (A_i)_x v + (g_{ij}^*\Theta)v$$

where the last equivalence is by hypothesis. We have already proved that for every $v \in T_xM$, $(ds_i)_x(v) = (d\sigma_{g_{ji}})_{s_j(x)}(ds_j)_x v + [g_{ji}^*\Theta(v)]^\sharp_{s_i(x)}$, and then we can rewrite the last equation as:

$$Ad_{g_{ij}^{-1}} \circ (\omega_i)_{s_i(x)}((d\sigma_{g_{ij}^{-1}})_{s_j(x)}(ds_j)_x v) - (g_{ij}^*\Theta)(v) + (g_{ij}^*\Theta)v =$$

$$= (\omega_i)_{s_j(x)}((ds_j)_x(v))$$

where an easy calculation shows that $Ad_{g_{ij}^{-1}}((g_{ji}^*\Theta)v) = -(g_{ij}^*\Theta)v$. This concludes the proof of the theorem. $\square$

Remark 2.3.26. Let M be a manifold, and G a Lie group. As a consequence of the previous Proposition every $\mathfrak{g}$-valued 1-form defined globally on M is the pull-back via some global section of a connection form defined on the trivial bundle $P := M \times G$. To clearly see it, just take the trivial atlas $\mathcal{A} = \{(M, \chi)\}$ for P where

$$\chi : M \times G \to M \times G$$

$$(x, g) \longmapsto (x, \psi(x)g)$$

for some $\psi \in C^\infty(M, G)$, and the compatibility condition is trivially verified.

Remark 2.3.27. Suppose that in Proposition 2.3.25 G is a matrix Lie group, with composition of matrices as group operation. If $w \in T_xM$ and $\beta(t)$ is its integral curve with starting point x, then

$$\Theta_{ij}(w) = \Theta_{g_{ij}(x)}((dg_{ij})_x w) = (dL_{g_{ij}(x)^{-1}})_{g_{ij}(x)}((dg_{ij})_x w) =$$
$$= \frac{d}{dt}(g_{ij}(x)^{-1} g_{ij}(\beta(t)))|_{t=0} = g_{ij}(x)^{-1}\frac{d}{dt}(g_{ij}(\beta(t)))|_{t=0} =$$
$$= g_{ij}(x)^{-1} dg_{ij}(x) w$$

This means that if G is a matrix Lie group then, in the hypothesis of Proposition 2.3.25, we can rewrite (2.50) as

$$A_j = g_{ij}^{-1} dg_{ij} + g_{ij}^{-1} A_i g_{ij} \tag{2.54}$$

Remark 2.3.28. In the previous proposition we have assumed that the sections are canonical. We want to see how compatibility condition changes if we consider a generic cross section.
Let ω be a connection form and $\mathcal{A} = \{(U_i, \chi_i)\}_{i\in I}$ an atlas for the principal fibre bundle $\pi : P \to M$, with transition functions $\{g_{ij}\}$. Let $\tilde{s}_i : U_i \to \pi^{-1}(U_i)$ and $\tilde{s}_j : U_j \to \pi^{-1}(U_j)$ be two generic cross sections, and $U_i \cap U_j \neq \emptyset$. Then as we have already argued there exist smooth $h_i \in C^\infty(U_i, G)$ and $h_j \in C^\infty(U_j, G)$ such that

$$\chi_i^{-1}(x, h_i(x)) = \tilde{s}_i(x) \quad \text{and} \quad \chi_j^{-1}(x, h_j(x)) = \tilde{s}_j(x)$$

Then we create a new atlas $\tilde{\mathcal{A}} := \{(U_i, \tilde{\chi}_i)\}_{i\in I}$, such that

$$\tilde{\chi}_i(p) := (\pi(p), h_i^{-1}(\pi(p))\cdot\phi_i(p)) \qquad \tilde{\chi}_j(p) := (\pi(p), h_j^{-1}(\pi(p))\cdot\phi_j(p))$$

and in this way $\tilde{s}_i(x) = \tilde{\chi}_i^{-1}(x, e)$ and the same holds for j, which means that $\tilde{s}_i$ are now trivial cross sections with respect to the new atlas $\tilde{\mathcal{A}}$. So if we call $\tilde{A}^i := \tilde{s}_i^*(\omega)$ and $\tilde{A}_j = \tilde{s}_j^*(\omega)$, then by Proposition 2.3.25 we get the identity

$$\tilde{A}_j = \tilde{g}_{ij}^{-1} d\tilde{g}_{ij} + \tilde{g}_{ij}^{-1} \tilde{A}_i \tilde{g}_{ij}$$

where $\tilde{g}_{ij}(x) := h_i^{-1}(x) g_{ij}(x) h_j(x)$ are the new transition functions.

Definition 2.3.29. Let ω_1 and ω_2 be two connection forms over the principal fibre bundle $\pi : P \to M$. We say that they are **gauge equivalent** if there exists an automorphism of the bundle $f : P \to P$ such that

$$f^*\omega_1 = \omega_2$$

Example 2.3.30. In Example 2.2.15 we showed how from a vector bundle $\pi : E \to M$ of rank r, we can build a principal fibre bundle $\tilde{\pi} : F(E) \to M$ with structure group $G(r, \mathbb{R})$ over M, called the frame bundle of E.
In this example we show that given a connection ∇ on the vector bundle $\pi : E \to M$, we can endow also $F(E)$ with a connection, completely determined by ∇. If $\mathcal{A} = \{(\chi_i, U_i)\}_{i\in I}$ is an atlas for $\pi : E \to M$, then we already saw how to obtain an atlas $\tilde{\mathcal{A}} = \{(\tilde{\chi}_i, U_i)\}_{i\in I}$ for $\tilde{\pi} : F(E) \to M$.
Once we are given the atlas $\mathcal{A}$, we can associate to the connection ∇ the family of matrices of the connection $\{\omega_i\}_{i\in I}$, where each ω_i is considered with respect to the frame $\{\overline{e}_1, ..., \overline{e}_r\}$ in U_i associated to χ_i, namely for $j = 1, ..., r$ we have

$$\begin{aligned} \overline{e}_j : U_i &\to \pi^{-1}(U_i) \\ x &\longmapsto \overline{e}_j(x) := \chi_i^{-1}(x, e_j) \end{aligned}$$

where $\{e_1, ..., e_r\}$ is the canonical basis of $\mathbb{R}^r$. These matrices of differential forms are related by the equation

$$\omega_j = g_{ij}^{-1} dg_{ij} + g_{ij}^{-1}\omega_i g_{ij} \tag{2.55}$$

where $g_{ij} \in C^\infty(U_i \cap U_j, GL(r, \mathbb{R}))$ are such that
$\chi_j^{-1}(x, e_h) = \chi_i^{-1}(x, g_{ij}e_h) = g_{ij}\chi_i^{-1}(x, e_h)$. It is immediate to see that g_{ij} are also the transition functions of $F(E)$, namely

$$\tilde{\chi}_i \circ \tilde{\chi}_j^{-1}(x, A) = (x, g_{ij}A) \ \ \forall (x, A) \in (U_i \cap U_j) \times GL(r, \mathbb{R})$$

Therefore, thanks Proposition 2.3.25 we conclude. Indeed, observe that each $\omega_i \in M_r(\mathbb{R}) \cong \mathfrak{g}$, where $\mathfrak{g}$ is the Lie Algebra of the structure group $G = GL(r, \mathbb{R})$.

One sees that once a connection is defined over a principal fibre bundle then for every $p \in P$ the differential of the canonical projection $d\pi_p : T_pP \to T_{\pi(p)}M$ maps isomorphically H_p into $T_{\pi(p)}M$.

Definition 2.3.31. Let $X \in \tau(M)$, we define its **horizontal lift** X^*, as the unique horizontal vector field in P such that for every $p \in P$, $d\pi_p(X^*) = X_{\pi(p)}$.

Proposition 2.3.32. *Given a connection over P and $X \in \tau(M)$, there exists and is unique the horizontal lift X^*, and furthermore it is right invariant. Conversely if $Y \in \tau(P)$ and it is horizontal and right invariant, then there exists a unique vector field over M whose horizontal lift coincides with Y.*

Remark 2.3.33. Let $\pi : P \to M$ a principal fibre bundle with a connection, and $(U, \phi = (x_1, ..., x_n))$ a local chart for M. Let ∂_i^* be the horizontal lift in $\pi^{-1}(U)$ of the local vector field ∂_{x_i}. Then $\partial_1^*, ..., \partial_n^*$ is a local frame for the horizontal distribution $p \to H_p$ in $\pi^{-1}(U)$.

Before concluding we make the following observation. Later it will be useful in order to achieve a deeper geometric interpretation on the Yang-Mills functional. If ω is some connection over a principal bundle $\pi : P \to M$ and $s : U \to \pi^{-1}(U)$ a cross section, then by definition $\pi \circ s = id_U$. Differentiating we get that $d\pi \circ ds(X) = X$ for every $X \in \tau(U)$. In particular $X = d\pi(ds(X)^H + ds(X)^V) = d\pi(ds(X)^H)$ and therefore $ds(X)^H = X^*$.

2.4 Curvature form

Let $\pi : P \to M$ be a principal fibre bundle. We define an important class of tensors over P. Let $\mathcal{V}$ be a vector space and $\rho : G \to GL(\mathcal{V})$ be a group representation.

Definition 2.4.1. We say that a $\mathcal{V}$-valued k-form α over P is a **pseudotensorial of type** ρ, if $\sigma_g^* \alpha = \rho(g^{-1}) \circ \alpha$ for every $g \in G$, where $\sigma_g : P \to P$ is the right action of the group G on P, i.e. $\sigma_g(p) = p \cdot g$.
A pseudotensorial of type ρ which vanishes on vertical tangent vectors, namely $\forall p \in P$ if at least one of the tangent vectors $v_1, ..., v_k \in T_pP$ is vertical then $\alpha_p(v_1, ..., v_k) = 0$, is called **tensorial of type** ρ.

We already encountered a pseudotensorial. Indeed if we consider as ρ the Adjoint representation $Ad : G \to GL(\mathfrak{g})$, then every connection form ω over P is actually a 1-pseudotensorial of type Ad.
It is easy to verify that if ϕ is some pseudotensorial form then, also $d\phi$ is pseudotensorial, but if ϕ is tensorial then its differential is not necessarily tensorial too. With the following definition we introduce the covariant exterior derivative, and from each pseudotensorial form we will be able to get a tensorial form.

Definition 2.4.2. Let ω be a connection form over a principal fibre bundle $\pi : P \to M$. If ϕ is any $\mathcal{V}$-valued pseudotensorial form of type ρ where $\rho : G \to GL(\mathcal{V})$ is a group representation, then we define the **exterior covariant derivative** $d^\omega \phi$ as

$$(d^\omega \phi)_p(v_1, ..., v_{k+1}) := d\phi_p(v_1^H, ..., v_{k+1}^H) \quad v_1, ..., v_{k+1} \in T_pP \quad (2.56)$$

Proposition 2.4.3. *In the same hypothesis of the above definition, we have that $d^\omega\phi$ is a $(k+1)$-tensorial form of type ρ.*

Proof. The $(k+1)$-form is clearly vanishing on vertical vectors. We just have to prove that it is pseudotensorial. Let $v_1, ..., v_{k+1} \in T_pP$ and $\sigma_g : P \to P$ defined as always. Then

$$\sigma_g^*(d^\omega\phi)(v_1, ..., v_{k+1}) = d^\omega\phi_{\sigma_g(p)}(d\sigma_g v_1, ..., d\sigma_g v_{k+1}) =$$

$$= d\phi_{\sigma_g(p)}((d\sigma_g v_1)^H, ..., (d\sigma_g v_{k+1})^H) =$$

but since $(d\sigma_g(v_i))^H = d\sigma_g(v_i^H)$, then the last equation is equal to

$$= d\phi_{\sigma_g(p)}(d\sigma_g(v_1^H), ..., d\sigma_g(v_{k+1}^H)) = (\sigma_g^* d\phi)(v_1^H, ..., x_{k+1}^H) =$$

$$= (d\sigma_g^*\phi)(v_1^H, ..., x_{k+1}^H) = (d(\rho(g^{-1}) \circ \phi))(v_1^H, ..., v_{k+1}^H)$$

and since $\rho(g^{-1})$ is linear we finally find:

$$\rho(g^{-1}) \circ d^\omega\phi(v_1, ..., v_{k+1})$$

□

We introduce now a fundamental example of a $\mathfrak{g}$-valued tensorial 2-form of type Ad, and we get it trough the covariant exterior derivative of a connection form.

Definition 2.4.4. Let $\pi : P \to M$ be a principal fibre bundle and ω a connection form on P. We define the **curvature form** Ω of ω, as the exterior covariant derivative of ω:

$$\Omega := d^\omega\omega \tag{2.57}$$

Since it is not always convenient to compute the horizontal component of a vector field over the bundle, we state the following important formula for the curvature form, called the **Cartan structure equation**.

Theorem 2.4.5. *Let $\pi : P \to M$ be a principal fibre bundle and ω a connection on it. Then we can rewrite its curvature form as*

$$\Omega = d\omega + [\omega, \omega] \tag{2.58}$$

where we define the $\mathfrak{g}$-valued 2-form $[\omega, \omega](v, w) := [\omega(v), \omega(w)]$ for each $v, w \in T_pP$ and $p \in P$. Furthermore, if $X, Y \in \tau(P)$ then

$$\Omega(X, Y) = -\omega([X^H, Y^H]) \tag{2.59}$$

Proof. Fix $v, w \in T_pP$, we want to prove that

$$\Omega_p(v,w) = d\omega_p(v,w) + [\omega(v), \omega(w)] \tag{2.60}$$

Since both sides are bilinear, then it is sufficient to check the equation for the following three cases:

1) v, w are both vertical
2) v, w are both horizontal
3) v is horizontal and w is vertical.

We start with 1). Since v, w are vertical then the left hand side of (2.60) is vanishing, and furthermore there exist $A, B \in \mathfrak{g}$ such that $A^\sharp_p = v$ and $B^\sharp_p = w$. This means that thanks to the Cartan formula for the exterior derivative:

$$d\omega_p(v,w) = (d\omega(A^\sharp, B^\sharp))_p = B^\sharp(\omega(A^\sharp))_p - A^\sharp(\omega(B^\sharp))_p - \omega([A^\sharp, B^\sharp])_p =$$
$$= -\omega([A,B]^\sharp) = -[A,B]$$

where the equivalence in the last equation is due to the fact that the map $\mathfrak{g} \ni A \mapsto A^\sharp \in \tau(P)$ is a Lie Algebra morphism. Since $[\omega(A^\sharp), \omega(B^\sharp)] = [A,B]$, then the first point is proved.
2) We suppose now that v, w are both horizontal. Then

$$\Omega_p(v,w) = d\omega_p(v^H, w^H) = d\omega_p(v,w)$$

and $[\omega(v), \omega(w)] = [0,0] = 0$ and so also this case is proved.
3)We extend v to a horizontal vectorfield V in P, and there exists $B \in \mathfrak{g}$ such that $B^\sharp_p = w$. Then $\Omega(v,w) = 0$, while the right hand side of (2.60) is

$$d\omega(V, B^\sharp)_p + [0,B] = B^\sharp(\omega(V))_p - V(\omega(B^\sharp))_p - \omega([V, B^\sharp])_p = -\omega([V, B^\sharp])_p$$

So we have to prove that $\omega([V, B^\sharp]) = 0$, namely $[V, B^\sharp]$ is horizontal. Indeed, observe that the flow of $B^\sharp$ around $p \in P$ is $\sigma_{b(t)}$, where $b(t)$ is the integral curve of B in G. Then

$$[V, B^\sharp] = -\lim_{t \to 0} \frac{d\sigma_{b(t)}(V) - V}{t}$$

and since the horizontal distribution is right invariant then $[V, B^\sharp]$ is horizontal.

Now let X, Y be two vector fields over P, then by definition one easily obtain that for every $p \in P$

$$\Omega_p(X,Y) = \Omega_p(X^H, Y^H) = (d\omega(X^H, Y^H))_p$$

and by the Cartan formula for the exterior differential

$$d\omega(X^H, Y^H) = Y^H(\omega(X^H)) - X^H(\omega(Y^H)) - \omega([X^H, Y^H])$$

Since ω is vanishing on horizontal vector fields, then

$$\Omega(X, Y) = -\omega([X^H, Y^H]).$$

$\square$

As we already did for the connection form, we can build a local representation of the curvature form by considering some local section over the principal bundle $\pi : P \to M$.

Definition 2.4.6. Let ω be a connection form over a principal fibre bundle $\pi : P \to M$, and let Ω be its curvature form. If $s : V \to \pi^{-1}(V)$ is some local cross section, where V is an open set in M, we define the **local field strength** in gauge s as:

$$F := s^*\Omega \tag{2.61}$$

which is a $\mathfrak{g}$-valued 2-form over $V \subset M$.

Observe that if in V we have $A = s^*\omega$, then we can write F in terms of the local gauge potential A. Indeed,

$$F = s^*\Omega = s^*d\omega + s^*([\omega, \omega]) = d(s^*\omega) + [s^*\omega, s^*\omega] = dA + [A, A]$$

For this reason every time we fix a connection form ω and a section s such that $s^*\omega = A$ we will write F_A instead of F to specify the local gauge potential, and therefore the local cross section.

Theorem 2.4.7. *Let $\pi : P \to M$ be a principal fibre bundle and ω be a connection form over it. Let V_1 and V_2 be two open sets in M such that $V_1 \cap V_2 \neq \emptyset$, and $s_1 : V_1 \to \pi^{-1}(V_1)$ and $s_2 : V_2 \to \pi^{-1}(V_2)$ be two local cross sections. Then if Ω is the curvature form:*

$$s_2^*\Omega = Ad_{g_{12}^{-1}} \circ s_1^*\Omega \quad \textit{in } V_1 \cap V_2$$

where $s_2 = s_1 \cdot g_{12}$ and $g_{12} : V_1 \cap V_2 \to G$.

Proof. Let $x_0 \in V_1 \cap V_2$ and $v, w \in T_{x_0}M$, then

$$(s_2^*\Omega)(v, w) = \Omega_{s_2(x_0)}(ds_2(v), ds_2(w)) = \Omega_{s_2(x_0)}(d(s_1 g_{12})(v), d(s_1 g_{12})(w))$$

and thanks to the Leibniz's rule, as we already saw, it holds that

$$d(s_1 g_{12})(v) = d\sigma_{g_{12}}(ds_1(v)) + dL_{s_1}(dg_{12}(v))$$

where the map $L_{s_1} : G \to P$ is defined as $L_{s_1}(g) = s_1 \cdot g$. Now if we call A the element of the Lie algebra of G such that $A_{g_{12}(x_0)} = dg_{12}(v)$ in $g_{12}(x_0)$, then $dL_{s_1}(dg_{12}(v)) = A^*_{s_1 \cdot g_{12}}$, which means that it is vertical, and its image through Ω is therefore zero. So we have proved:

$$(s_2^*\Omega)(v,w) = \Omega_{s_2(x_0)}(d\sigma_{g_{12}}(ds_1(v)), d\sigma_{g_{12}}(ds_1(w)) =$$
$$= \sigma^*_{g_{12}} s_1^* \Omega_{x_0}(v,w) = Ad_{g_{12}^{-1}} \circ (s_1^*\Omega)(v,w)$$

where the last equivalence is given by Proposition 2.4.3. This concludes the proof of the theorem. □

In particular if ω is some connection form over the principal fibre bundle $\pi : P \to M$ and s_1,s_2 are defined as in the theorem, then if we call $A_1 = s_1^*\omega$ and $A_2 = s_2^*\omega$ the equation in Theorem 2.4.7 becomes

$$F_{A_2} = Ad_{g_{12}^{-1}} \circ F_{A_1}$$

Unlike the classical exterior derivative, usually $d^\omega \circ d^\omega \neq 0$. However if we consider as pseudotensorial form of type Ad the connection form inducing the exterior covariant derivative itself then the vanishing relation holds.

Theorem 2.4.8. *Let $\pi : P \to M$ be a principal bundle and ω a connection form. Then*

$$d^\omega(d^\omega\omega) = d^\omega\Omega = 0 \qquad \textbf{\textit{(Bianchi Identity)}} \qquad (2.62)$$

Proof. Let $v, w, z \in T_pP$. If any one of them is vertical then $d^\omega\Omega_p(v,w,z) = 0$. So we restrict to the case where v, w, z are horizontal. We extend them to horizontal vector fields V, W, Z and we find thanks to the Cartan structure equation:

$$d^\omega\Omega_p(v,w,z) = (d\Omega(V,W,Z))_p = (\underbrace{dd\omega}_{=0}(V,W,Z))_p + d([\omega,\omega])(V,W,Z)_p$$

The last summand in the above equation is equal to

$$V([\omega,\omega](W,Z))_p - W([\omega,\omega](V,Z))_p + Z([\omega,\omega](V,W))_p$$
$$-([\omega,\omega]([V,W],Z))_p - ([\omega,\omega](V,[W,Z]))_p = 0$$

since V, W, Z are horizontal vector fields, and ω is vanishing on horizontal vectors. □

Chapter 3

Hodge Theory

3.1 Differential Forms & Hodge operator

In this section we define $\mathbb{R}^m$-valued differential forms, extending them from smooth objects to less regular one, a necessary requirement when studying variational problems that a priori cannot be solved working with smooth differential forms.
If M is a manifold we know that a k-form is defined as a section of the vector bundle $\wedge^k M$[1]. When M is a open domain in some $\mathbb{R}^n$, then any k-form admits a global representation, using the Euclidean coordinates of $\mathbb{R}^n$.

Definition 3.1.1. Let $\Omega \subseteq \mathbb{R}^n$ be open. We define a smooth $\mathbb{R}^m$-valued differential k-form as a section of the bundle $\wedge^k\Omega \otimes \mathbb{R}^m$. In coordinates any k-form can be written as

$$\omega(x) = \sum_{i_1<\ldots<i_k} \omega_{i_1,\ldots,i_k}(x) dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

where $\omega_{i_1,\ldots,i_k} \in C^\infty(\Omega, \mathbb{R}^m)$. We denote as $C^\infty(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m)$ the space of $\mathbb{R}^m$-valued k-forms over Ω.

When working with $\mathbb{R}^m$-valued differential k-forms, one can extend the idea of exterior product as in the following definition. From now on we assume that $\mathbb{R}^m$ is endowed with a scalar product $\cdot : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$.

Definition 3.1.2. We define the following operator

$$\begin{aligned} \wedge : C^\infty(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m) \times C^\infty(\Omega, \wedge^l T^*\Omega \otimes \mathbb{R}^m) &\to \wedge^{l+k}(\Omega) \\ (\alpha, \beta) &\longmapsto \alpha \wedge \beta \end{aligned} \tag{3.1}$$

[1] $\wedge^k M$ is the vector bundle defined as the disjointed union of $\wedge^k(T_p^* M)$ over $p \in M$. For further details see for instance [1]

and $\alpha \wedge \beta$ is the $(l+k)$-differential form defined as[2]

$$\alpha\wedge\beta(v_1, ...v_{k+l}) = \sum_{\sigma\in S(k,k+l)} \operatorname{sgn}(\sigma)\alpha(v_{\sigma(1)}, ..., v_{\sigma(k)})\cdot\beta(v_{\sigma(k+1)}, ..., v_{\sigma(k+l)})$$

where $v_1, ..., v_{l+k}$ are arbitrary vectorfields in Ω.

As one expects we can also define the standard exterior derivative d of a smooth $\mathbb{R}^m$-valued k-form.

Definition 3.1.3. Let $\Omega \subset \mathbb{R}^n$, and $\alpha \in C^\infty(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m)$. Then we set

$$d\alpha = \sum_{i_1<...<i_k} \left(\sum_j \frac{\partial \alpha_{i_1,...,i_k}}{\partial x_j} dx^j \right) \wedge dx^{i_1} \wedge ... \wedge dx^{i_k} \tag{3.2}$$

Now we define a fundamental operator in Hodge theory. We will define it for any open domain $\Omega \subset \mathbb{R}^n$ with a generic Riemannian metric $\mathbf{g}$ on it. As we will see, this new operator let us construct a formal definition of pointwise norm of differential forms. The following definitions can easily be generalized to any Riemannian manifold.

Definition 3.1.4. Let $\Omega \subseteq \mathbb{R}^n$ open, and let $\mathbf{g}$ be a Riemannian metric on it.

1) Let $X_1, ..., X_n$ be a $\mathbf{g}$-orthonormal frame in Ω. This defines a pointwise product in $C^\infty(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m)$, given by

$$\langle \cdot, \cdot \rangle_{\mathbf{g}} : C^\infty(\Omega, \wedge^k T^*\Omega\otimes\mathbb{R}^m)\times C^\infty(\Omega, \wedge^k T^*\Omega\otimes\mathbb{R}^m) \to C^\infty(\Omega, \mathbb{R})$$
$$(\alpha, \beta) \longmapsto \sum_{i_1<...<i_k} \alpha(X_{i_1}, ..., X_{i_k})\cdot\beta(X_{i_1}, ..., X_{i_k}) \tag{3.3}$$

We denote with $|\alpha|^2 := \langle \alpha, \alpha \rangle$ the pointwise square norm of α.

2) The **Hodge operator** $\star_{\mathbf{g}} : C^\infty(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m) \to C^\infty(\Omega, \wedge^{n-k} T^*\Omega \otimes \mathbb{R}^m)$ is determined by

$$\beta \wedge \star_{\mathbf{g}}\alpha = \langle \beta, \alpha \rangle_{\mathbf{g}} \eta \qquad \forall \beta \in C^\infty(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m) \tag{3.4}$$

where η is the volume form on Ω induced by the Riemannian metric $\mathbf{g}$. In coordinates we have $\eta := \sqrt{|\det(\mathbf{g})|}dx^1\wedge...\wedge dx^n$.[3] It is clear that $\star_{\mathbf{g}}$ depends explicitly on the chosen metric.

[2] We denote with $S(k,n)$ for $k \leq n$, the set of all permutations of $\{1, ..., n\}$, such that $\sigma(1) < ... < \sigma(k)$ and $\sigma(k+1) < ... < \sigma(k+l)$

[3] A volume form over an n-dimensional manifold M is any differential n-form, and we have that it is nowhere vanishing if and only if the manifold is orientable. Each open subset of $\mathbb{R}^n$, is of course orientable, and therefore it admits a volume form.

Note that Definition 3.1.4 does not depend on the particular choice of the $\mathbf{g}$-orthonormal frame.

Remark 3.1.5. We will denote the Hodge star operator associated to the Euclidean metric with $\star$, without any subscript.
With this metric we have that $\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\}$ is an orthonormal frame, and therefore for any $\alpha, \beta \in C^\infty(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m)$ we can rewrite equation (3.3) as

$$\langle \alpha, \beta \rangle(x) = \sum_{i_1 < ... < i_k} \alpha_{i_1,...,i_k}(x) \cdot \beta_{i_1,...,i_k}(x) \tag{3.5}$$

with $x \in \Omega$. The volume form associated to the Euclidean metric is $dx^1 \wedge ... \wedge dx^n$, and if in the above definition we choose $\beta := e_j dx^{i_{n-k+1}} \wedge ... \wedge dx^{i_n}$, where $\{e_i\}_{i=1,...,m}$ is the canonical basis of $\mathbb{R}^m$, then

$$\alpha^j_{i_{n-k+1},...,i_n} = \langle \beta, \alpha \rangle = (\beta \wedge \star\alpha)_{1,...,n} = \operatorname{sgn}(\sigma)(\star\alpha)^j_{i_1,...,i_{n-k}}(x)$$

where $\sigma(\{1, ..., n\}) = \{i_{n-k+1}, ..., i_n\} \cup \{i_1, ..., i_{n-k}\}$, for some permutation σ.

Remark 3.1.6. In Definition 3.1.4 we have defined the pointwise scalar product between two $\mathbb{R}^m$-valued k-forms α and β. If now we assume that α is still $\mathbb{R}^m$-valued but β has values in $\mathbb{R}$, then with abuse of notation we define $\langle \alpha, \beta \rangle(x) := \sum_{i_1 < ... < i_k} \alpha_{i_1...i_k}(x)\beta_{i_1...i_k}(x)$, which is not a scalar product anymore.

The following proposition establishes the most important properties of the Hodge star operator.

Proposition 3.1.7. *Let $\Omega \subset \mathbb{R}^n$ be open and endowed with a Riemannian metric $\mathbf{g}$.*

1) *$\star_{\mathbf{g}} : C^\infty(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m) \to C^\infty(\Omega, \wedge^{n-k} T^*\Omega \otimes \mathbb{R}^m)$, is a linear operator. Furthermore it holds $\alpha \wedge \star_{\mathbf{g}}\beta = \beta \wedge \star_{\mathbf{g}}\alpha$.*

2) *$\star_{\mathbf{g}}(\eta) = 1$ and $\star_{\mathbf{g}}1 = \eta$, where η is the volume form associated to $\mathbf{g}$.*

3) *If β is a $\mathbb{R}^m$-valued k-form, then $\star_{\mathbf{g}}(\star_{\mathbf{g}}\beta) = (-1)^{k(n-k)}\beta$*

4) *If $\phi : \Omega_1 \to \Omega$ is a diffeomorphism, then*

$$\phi^* \circ \star_{\mathbf{g}} = \star_{\phi^*(\mathbf{g})} \circ \phi^* \tag{3.6}$$

where ϕ^* is the pull-back of ϕ.[4]

[4]If $\phi : \Omega_1 \to \Omega$ is a diffeomorphism and Ω is endowed with a Riemannian metric g, then we can "transport" this metric on Ω_1 through the pull-back as follows. For each $x \in \Omega_1$ and $v, w \in T_x\Omega_1$, we have $\phi^*(g)_x(v, w) = g_{\phi(x)}(d\phi(v), d\phi(w))$.

Proof. Points 1),2) and 3) are simply direct consequences of the definition, and therefore a proof is not necessary. For details see [39].
4)Fix $\alpha \in C^\infty(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m)$, and for every smooth $\mathbb{R}^m$-valued k-form in Ω_1, we have by definition of Hodge star operator that

$$\beta \wedge \star_{\phi^*(\mathbf{g})} \phi^*(\alpha) = \langle \beta, \phi^*(\alpha) \rangle_{\phi^*(\mathbf{g})} \eta$$

where η is the volume form associated to $\phi^*(\mathbf{g})$. We fix a $\phi^*(\mathbf{g})$-orthonormal frame in Ω_1, $\{X_1, ..., X_n\}$. If we compute the right hand side of the above equation in $y \in \Omega_1$, we obtain

$$\sum_{i_1 < ... < i_k} \beta_y(X_{i_1}, ..., X_{i_k}) \cdot \alpha_{\phi(y)}(d\phi(X_{i_1}), ..., d\phi(X_{i_k}))\eta_y = (*)$$

and since $\phi : \Omega_1 \to \Omega$ is a diffeomorphism, then if $x = \phi(y)$

$$(*) = \sum_{i_1 < ... < i_k} \beta_{\phi^{-1}(x)}(d\phi^{-1}(Y_{i_1}), ..., d\phi^{-1}(Y_{i_k})) \cdot \alpha_x(Y_{i_1}, ..., Y_{i_k})\eta_{\phi^{-1}(x)}$$

where $d\phi(X_i) = Y_i$ for each $i = 1, ..., n$. In particular by definition of $\phi^*(\mathbf{g})$, we have that $\{Y_1, ..., Y_n\}$ is a $\mathbf{g}$-orthonormal frame in Ω, and also that $(\phi^*)^{-1}(\eta) = \omega$ is the volume form associated to $\mathbf{g}$ in Ω. Therefore we get for each $y \in \Omega_1$

$$(*) = \phi^*(\langle (\phi^*)^{-1}(\beta), \alpha \rangle_{\mathbf{g}} \omega)_y$$

So finally we have that

$$\beta \wedge \star_{\phi^*(\mathbf{g})} \phi^*(\alpha) = \langle \beta, \phi^*(\alpha) \rangle_{\phi^*(\mathbf{g})} \eta = \phi^*(\langle (\phi^*)^{-1}(\beta), \alpha \rangle_{\mathbf{g}} \omega) =$$
$$= \phi^*((\phi^*)^{-1}(\beta) \wedge \star_{\mathbf{g}} \alpha) = \beta \wedge \phi^*(\star_{\mathbf{g}} \alpha)$$

and this concludes the proof.

□

From now on we will work exclusively with the Euclidean metric and the Hodge star operator associated to it, if not differently specified.
Now that we have introduced the concept of pointwise scalar product for differential forms, we are ready to define the Sobolev spaces of differential forms.

Definition 3.1.8. For $1 \leq p < \infty$ we consider the subspace of $C^\infty(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m)$ made of $\mathbb{R}^m$-valued k-forms α such that

$$\|\alpha\|_{L^p} := \left(\int_\Omega |\alpha|^p dx \right)^{\frac{1}{p}} < \infty \tag{3.7}$$

Smooth $\mathbb{R}^m$-valued k-forms are a subspace of measurable $\mathbb{R}^m$-valued k-forms, and therefore we define
$L^p(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m) = \overline{C^\infty(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m)}^{\|\cdot\|_{L^p}}$. This space coincides with the space of $\mathbb{R}^m$-valued k-forms α, whose components $\alpha_{i_1,\dots,i_k}$ are chosen in $L^p(\Omega, \mathbb{R}^m)$, and the following is clearly an equivalent norm:

$$\|\alpha\|_{L^p(\Omega)} := \sum_{i_1<\dots<i_k} \|\alpha_{i_1,\dots,i_k}\|_{L^p(\Omega,\mathbb{R}^m)} \tag{3.8}$$

Definition 3.1.9. For $r \in \mathbb{N}$ and $1 < p < \infty$ we also define the space of Sobolev $\mathbb{R}^m$-valued k-forms

$$\begin{aligned} &W^{r,p}(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m) = \\ &\quad = \left\{\omega \in L^p(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m) : \ \omega_{i_1,\dots,i_k} \in W^{r,p}(\Omega, \mathbb{R}^m)\right\} \end{aligned} \tag{3.9}$$

with the norm

$$\|\omega\|_{W^{r,p}(\Omega)} := \sum_{i_1<\dots<i_k} \|\omega_{i_1,\dots,i_k}\|_{W^{r,p}(\Omega,\mathbb{R}^m)} \tag{3.10}$$

and in this fashion we can define the usual function spaces but for $\mathbb{R}^m$-valued k-differential forms, just working componentwise. Observe that if $m = 1$ we have the usual k-forms.

We introduce now the notion of inner product, that soon will let us define the concept of tangential and normal component of a differential form.

Definition 3.1.10. Let $0 \le k, l \le n$ and α a $\mathbb{R}^m$-valued k-differential form, and β a l-form. We define the **inner product** of α with β as:

$$\beta \lrcorner \alpha := (-1)^{n(k-l)} \star (\beta \wedge (\star\alpha)) \tag{3.11}$$

If $k = l$ one see that $\beta \lrcorner \alpha = \langle \alpha, \beta \rangle$, while if $l > k$ then $\beta \lrcorner \alpha = 0$.

Remark 3.1.11. The results of Proposition 3.1.7 can be easily generalized to the Sobolev space $W^{r,p}(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m)$ using density of smooth $\mathbb{R}^m$-valued k-differential forms in this space.

3.1.1 Tangential and Normal component

In what follows we consider Ω an open bounded an sufficiently smooth subset of $\mathbb{R}^n$, and if $x \in \partial\Omega$ we call $\nu(x)$ the outer unit normal at x.
The unit normal is a vectorfield restricted to $\partial\Omega$, in the sense that

$\nu \in \tau(\Omega|_{\partial\Omega})$, but anyway we will often identify it with a 1-form, as follows. If $\nu(x) = (\nu_1(x), ..., \nu_n(x))$ then we associate to it the 1-form $\nu(x) = \nu_1(x)dx^1 + ... + \nu_n(x)dx^n$. Observe that we can always identify a vectorfield with a 1-form if we define on Ω a Riemannian metric, which in this case is the norm induced by the standard scalar product on $\mathbb{R}^n$.
We give two definitions of tangential component, and normal component. The Proposition below would underline in what sense they are equivalent.

Definition 3.1.12. Let $\alpha \in W^{1,p}(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m)$. We define the **restriction** of α to $\partial\Omega$ as

$$\alpha|_{\partial\Omega} = \sum_{i_1<...<i_k} \alpha_{i_1,...,i_k}|_{\partial\Omega} dx^{i_1} \wedge ... \wedge dx^{i_k}$$

where thanks to the Trace Theorem for Sobolev spaces one has that $\alpha_{i_1,...,i_k}|_{\partial\Omega}$ is well defined in $W^{1-\frac{1}{p},p}(\partial\Omega, \mathbb{R}^m)$. By definition this means that $\alpha|_{\partial\Omega} \in W^{1-\frac{1}{p},p}(\partial\Omega, \wedge^k T^*\Omega|_{\partial\Omega} \otimes \mathbb{R}^m)$.

Definition 3.1.13. The **tangential component** of α on $\partial\Omega$ is the $(k+1)$-form $\nu \wedge \alpha \in W^{1-\frac{1}{p},p}(\partial\Omega, \wedge^{k+1} T^*\Omega|_{\partial\Omega} \otimes \mathbb{R}^m)$. The **normal component** of α on $\partial\Omega$ is the $(k-1)$-form
$\nu \lrcorner \alpha \in W^{1-\frac{1}{p},p}(\partial\Omega, \wedge^{k-1} T^*\Omega|_{\partial\Omega} \otimes \mathbb{R}^m)$.

Let α be as above. We introduce the other two equivalent definitions for tangential and normal component.

Definition 3.1.14. Let $\{X_1, ..., X_n\}$ be a basis of $\mathbb{R}^n$ at $x \in \partial\Omega$ with $\{X_1, ..., X_{n-1}\}$ spanning $T_x(\partial\Omega)$ and X_n orthogonal to $T_x(\partial\Omega)$, then we call **tangential component** $\alpha_T \in W^{1-\frac{1}{p},p}(\partial\Omega, \wedge^k T^*\Omega|_{\partial\Omega} \otimes \mathbb{R}^m)$ the k-form defined as:

$$\alpha_T(x)(X_{i_1}, ..., X_{i_k}) = \alpha(x)(X_{i_1}, ..., X_{i_k}) \qquad 1 \leq i_1 < ... < i_k < n$$

and

$$\alpha_T(x)(X_{i_1}, ..., X_{i_k}) = 0 \qquad 1 \leq i_1 < ... < i_k = n$$

While the **normal component** is simply $\alpha_N(x) := \alpha(x) - \alpha_T(x)$ for each $x \in \partial\Omega$. Obviously α_N is in $W^{1-\frac{1}{p},p}(\partial\Omega, \bigwedge^k T^*\Omega|_{\partial\Omega} \otimes \mathbb{R}^m)$ too.

Remark 3.1.15. In Definition 3.1.14 we have defined α_T as a differential form which is identically zero outside the tangent bundle of $\partial\Omega$. Thus we can identify α_T as an object of $W^{1-\frac{1}{p},p}(\partial\Omega, \bigwedge^k T^*\partial\Omega \otimes \mathbb{R}^m)$ without losing any information.

In the following proposition we higlight the relation between the two different definitions of tangential and normal component. For a proof in the case $m = 1$ see [11].

Proposition 3.1.16. *Let* $\alpha \in W^{1,p}(\Omega, \bigwedge^k T^*\Omega \otimes \mathbb{R}^m)$*, then*

$$\alpha_T = \nu \lrcorner (\nu \wedge \alpha) \quad \text{and} \quad \alpha_N = \nu \wedge (\nu \lrcorner \alpha) \tag{3.12}$$

Corollary 3.1.17. *The map*

$$T : W^{1,p}(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m) \to W^{1-\frac{1}{p},p}(\partial\Omega, \wedge^k T^*\partial\Omega \otimes \mathbb{R}^m)$$

$$\alpha \mapsto \alpha_T$$

is linear and continuous.

Proof. The linearity is clear by Definition 3.1.14. Using the previous proposition and the standard Trace Theorem for Sobolev spaces

$$\|\alpha_T\|_{W^{1-\frac{1}{p},p}} = \|\nu \lrcorner (\nu \wedge \alpha)\|_{W^{1-\frac{1}{p},p}} \leq C\|\alpha\|_{W^{1-\frac{1}{p},p}} \leq C\|\alpha\|_{W^{1,p}}$$

and this inequality establishes continuity. □

We conclude this subsection with a result on the properties of the Hodge star operator.

Proposition 3.1.18. *Let* $\alpha \in W^{1,p}(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m)$*, then we have the following equivalences*

$$(\star\alpha)_T = \star(\alpha_N) \quad \text{and} \quad (\star\alpha)_N = \star(\alpha)_T$$

Proof. Let $x \in \partial\Omega$ and $\{X_1, ..., X_n\}$ a basis for $T_x\Omega$ with $X_1, ..., X_{n-1}$ spanning $T_x\partial\Omega$ and X_n orthogonal to $T_x\partial\Omega$. Then using the definition of $\star\alpha$, and fixing a permutation $\sigma(1, ..., n) = (i_1, ..., i_n)$

$$(\star\alpha)_T(X_{i_{k+1}}...X_{i_n}) = \star\alpha(X_{i_{k+1}}...X_{i_n}) \text{ if } X_n \notin \{X_{i_{k+1}}, ..., X_{i_n}\}$$

$$(\star\alpha)_T(X_{i_{k+1}}...X_{i_n}) = 0 \text{ if } X_n \in \{X_{i_{k+1}}, ..., X_{i_n}\}$$

Since $\star\alpha(X_{i_{k+1}}...X_{i_n}) = \text{sgn}(\sigma)\alpha(X_{i_1}...X_{i_k})$ then the last two equations imply

$$(\star\alpha)_T(X_{i_{k+1}}...X_{i_n}) = \text{sgn}(\sigma)\alpha(X_{i_1}...X_{i_k}) \text{ if } X_n \in \{X_{i_1}, ..., X_k\}$$

$$(\star\alpha)_T(X_{i_{k+1}}...X_{i_n}) = 0 \text{ if } X_n \notin \{X_{i_1}, ..., X_k\}$$

and this means that $(\star\alpha)_T = \star(\alpha_N)$. The other equations can be proved similarly. □

3.1.2 Gaffney's Inequality

We say that α has **vanishing tangential component** if $\alpha_T = 0$, and thanks to the Proposition above this condition is equivalent to $\nu \wedge \alpha = 0$ on $\partial\Omega$.
Similarly we say that α has **vanishing normal component** if $\alpha_N = 0$, or $\nu \lrcorner \alpha = 0$ on $\partial\Omega$. We call

$$W^{r,p}_T(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m) := \{\alpha \in W^{r,p} : \alpha_T = 0\}$$

$$W^{r,p}_N(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m) := \{\alpha \in W^{r,p} : \alpha_N = 0\}$$

The theorem below is the integration by parts formula for differential forms, and it will give us a characterization for differential forms with vanishing tangential (or normal) component. First we need to define the following operator, called the codifferential. We also define the Laplacian for differential forms. This last definition is fundamental and heavily used throughout this thesis, since it allows us to describe the Laplacian of a differential form as the composition of the differential and codifferential, as follows.

Definition 3.1.19. If α is a k-form in $W^{r,p}(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m)$ (for some $p > 1$ and $r \geq 1$) then we define the **codifferential** of α as

$$d^\star\alpha = (-1)^{n(k-1)} \star (d(\star\alpha))$$

Definition 3.1.20. Let $\alpha \in W^{r,p}(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m)$, then we define the **Laplacian** of α, as

$$\Delta\alpha := (dd^\star + d^\star d)\alpha \tag{3.13}$$

Theorem 3.1.21 ([11], Theorem 3.28)**.** *Let Ω be an open bounded smooth domain, then $\forall \alpha \in W^{1,p}(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m)$ and $\beta \in W^{1,p'}(\Omega, \wedge^{k+1} T^* \otimes \mathbb{R}^m)$ we have the following identity:*

$$\int_\Omega \langle d\alpha, \beta\rangle + \int_\Omega \langle \alpha, d^\star\beta\rangle = \int_{\partial\Omega} \langle \nu \wedge \alpha, \beta\rangle = \int_{\partial\Omega} \langle \alpha, \nu \lrcorner \beta\rangle \tag{3.14}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

So we can also define $W^{1,p}_T(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m)$ as the set of $\alpha \in W^{1,p}$ such that

$$\int_\Omega \langle d\alpha, \beta\rangle + \int_\Omega \langle \alpha, d^\star\beta\rangle = 0 \qquad \forall \beta \in C^\infty(\overline{\Omega}, \wedge^{k+1} T^*\Omega \otimes \mathbb{R}^m) \tag{3.15}$$

Now we state a first version of the Gaffney's inequality, that later will be generalized. A proof of the following theorem can be found in [11].

Theorem 3.1.22. *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, and $\omega \in W_T^{1,2}(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m) \cup W_N^{1,2}(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m)$. Then the following inequality holds*

$$\|\omega\|_{W^{1,2}(\Omega)} \leq C(\|d\omega\|_{L^2(\Omega)} + \|d^\star\omega\|_{L^2(\Omega)} + \|\omega\|_{L^2(\Omega)}) \tag{3.16}$$

We see that the above Theorem allows us to bound from above the $W^{1,2}$-norm of a differential form $\omega \in W_T^{1,2}(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m) \cup W_N^{1,2}(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m)$ by means of the L^2-norm of $d\omega$ and $d^\star\omega$ and the L^2-norm of the differential form itself. This is not a trivial result, indeed on the right hand side of (3.16) we are not considering the L^2 norm of all the partial derivatives of ω, unlike in the classical definition of $W^{1,2}$-norm of ω.
Now we introduce the space of harmonic fields, which will be used in the generalization of the Gaffney's inequality.

Definition 3.1.23. The set of $\mathbb{R}^m$-valued **harmonic fields** is defined as

$$H(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m) = \left\{\omega \in W^{1,2}(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m) : d\omega = 0,\ d^\star\omega = 0\right\} \tag{3.17}$$

Furthermore we define

$$H_T(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m) := H(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m) \cap W_T^{1,2}(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m)$$

$$H_N(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m) := H(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m) \cap W_N^{1,2}(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m) \tag{3.18}$$

We list now some properties of harmonic fields, that will be useful in stating the main result of this section, the generalized Gaffney's inequality.

Theorem 3.1.24. *Let Ω be an open bounded smooth domain in $\mathbb{R}^n$, then the followings hold*

1) $H(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m) \subset C^\infty(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m)$
2) $H_T(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m)$ *and* $H_N(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m)$ *are both finite dimensional*

If Ω is also contractible[5] *then*

[5] We say that a set Ω is contractible if there exists $x_0 \in \Omega$ and $F \in C^\infty([0,1] \times \Omega, \Omega)$ such that $F(0,x) = x_0$ and $F(1,x) = x$ for each $x \in \Omega$.

3) For $0 \leq k \leq n-1$ the space $H_T(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m) = \{0\}$, and for $1 \leq k \leq n$ the space $H_N(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m) = \{0\}$

Proof. 1) The first property is an easy consequence of the Weyl's Lemma (see [47]). Indeed, if $\omega \in H(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m)$ then for every $\phi \in C_c^\infty(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m)$ thanks to the integration by parts formula in Theorem 3.1.21 for differential forms

$$\int_\Omega \langle \omega, \Delta\phi \rangle dx = \int_\Omega \langle \omega, d^*d\phi + dd^*\phi \rangle dx =$$
$$= \int_\Omega \langle d\omega, d\phi \rangle dx + \int_\Omega \langle d^*\omega, d^*\phi \rangle dx = 0$$

and therefore each component of ω is smooth.
2) We will prove this point using Riesz Theorem (see for instance [8]). Consider a $W^{1,2}$-bounded sequence $\{\omega_n\}_n \subset H_T(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m)$, then by reflexivity it admits a subsequence weakly converging to some $\tilde{\omega}$ in $W^{1,2}$. Corollary 3.1.17 tells us that the map

$$T : W^{1,2}(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m) \to W^{\frac{1}{2},2}(\partial\Omega, \wedge^k T^*\partial\Omega \otimes \mathbb{R}^m)$$

is linear and continuous for the strong topology of $W^{1,2}$, and therefore also for the weak one, which implies that $\tilde{\omega} \in W_T^{1,2}(\Omega, T^*\Omega \otimes \mathbb{R}^m)$. The lower semicontinuity of the norm with respect to the weak convergence implies that $d\tilde{\omega} = 0$ and $d^*\tilde{\omega} = 0$. Then $\tilde{\omega} \in H_T(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m)$. Finally by Rellich-Kondrakov we have that $W^{1,2}(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m) \hookrightarrow L^2(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m)$ compactly, and thus there is a strongly L^2 converging subsequence $\{\omega_{n_k}\}_k$ of $\{\omega_n\}_n$. This last observation and Theorem 3.1.22 imply that

$$\|\omega_{n_k} - \tilde{\omega}\|_{W^{1,2}(\Omega)} \leq C\Bigg(\| \underbrace{d(\omega_{n_k} - \tilde{\omega})}_{=0} \|_{L^2(\Omega)} + \| \underbrace{d^\star(\omega_{n_k} - \tilde{\omega})}_{=0} \|_{L^2(\Omega)} +$$
$$+ \|\omega_{n_k} - \tilde{\omega}\|_{L^2(\Omega)} \Bigg)$$

and therefore ω_{n_k} converges strongly to $\tilde{\omega} \in H_T(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m)$ in $W^{1,2}$. By Riesz theorem this implies that $H_T(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m)$ is finite dimensional.
3)Let Ω be contractible, and $1 \leq k \leq n$. If $\omega \in H_N(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m)$ then $d\omega = 0$ and by Poincaré Lemma there exists
$\beta \in W^{1,2}(\Omega, \wedge^{k-1} T^*\Omega \otimes \mathbb{R}^m)$ such that $d\beta = \omega$. We apply the integration by parts formula (3.14) and get

$$\|\omega\|_{L^2}^2 = \int_\Omega \langle \omega, \omega \rangle dx = \int_\Omega \langle d\beta, \omega \rangle dx =$$

$$= -\int_\Omega \langle \beta, d^\star \omega\rangle dx + \int_{\partial\Omega} \langle \beta, \nu \lrcorner \omega\rangle d\sigma = 0$$

Similarly we obtain the same result for $H_T(\Omega, \wedge^k\Omega \otimes \mathbb{R}^m)$ for $0 \leq k \leq n-1$

□

The following theorem from [6], is a generalization of Gaffney's inequality and it will be really useful since it allows us to endow the $W^{r,p}(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m)$ spaces with an equivalent but more versatile norm.

Theorem 3.1.25. *Let $n > 2$, $r \geq 1$ and $1 < p < \infty$. Let Ω be a bounded open and smooth domain in $\mathbb{R}^n$, with exterior normal ν. Then there exists a constant $C_1 > 0$ such that for every $\omega \in W^{r,p}(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m)$ with $1 \leq k \leq n-1$*

$$\|\omega\|_{W^{r,p}(\Omega)} \leq C_1 \left(\|d\omega\|_{W^{r-1,p}(\Omega)} + \|d^\star\omega\|_{W^{r-1,p}(\Omega)}\right) + $$
$$+C_1 \left(\|\nu \wedge \omega\|_{W^{r-\frac{1}{p},p}(\partial\Omega)} + \sum_{i=1}^{B_{n-k}} \left| \int_{\partial\Omega} \langle \omega, \nu \wedge z_i\rangle \right| \right) \tag{3.19}$$

where $\{z_i\}_{i=1,\dots,B_{n-k}}$ is a basis of $H_N(\Omega^c, \wedge^{k-1}T^*\Omega \otimes \mathbb{R}^m)$[6]*. It holds also the following inequality, where instead we consider the normal component,*

$$\|\omega\|_{W^{r,p}(\Omega)} \leq C_2(\|d\omega\|_{W^{r-1,p}(\Omega)} + \|d^\star\omega\|_{W^{r-1,p}(\Omega)}) + $$
$$+C_2 \left(\|\nu \lrcorner \omega\|_{W^{r-\frac{1}{p},p}(\partial\Omega)} + \sum_{i=1}^{B_k} \left| \int_{\partial\Omega} \langle \omega, \nu \lrcorner y_i\rangle \right| \right) \tag{3.20}$$

where $\{y_i\}_{i=1,\dots,B_k}$ is a basis of $H_T(\Omega^c, \wedge^{k+1})$

Remark 3.1.26. Observe that if Ω is a contractible domain and $1 \leq k \leq n-1$, then we have $H_T(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m) = H_N(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m) = \{0\}$, and since the following equations hold

$$\dim(H_T(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m)) = \dim(H_N(\Omega^c, \wedge^{k-1} T^*\Omega \otimes \mathbb{R}^m))$$
$$\dim(H_N(\Omega, \wedge^k T^*\Omega \otimes \mathbb{R}^m)) = \dim(H_T(\Omega^c, \wedge^{k+1} T^*\Omega \otimes \mathbb{R}^m)) \tag{3.21}$$

as proved in [23], we can rewrite (3.19) and (3.20) as

$$\|\omega\|_{W^{r,p}(\Omega)} \leq C_1 \left(\|d\omega\|_{W^{r-1,p}(\Omega)} + \|d^\star\omega\|_{W^{r-1,p}(\Omega)} + \|\nu \wedge \omega\|_{W^{r-\frac{1}{p},p}(\partial\Omega)}\right) \tag{3.22}$$

[6]The space $H_N(\Omega^c, \wedge^{k-1}T^*\Omega \otimes \mathbb{R}^m))$ is defined analogously to $H_N(\Omega, \wedge^{k-1}T^*\Omega \otimes \mathbb{R}^m))$ with the further requirement that if $|x| \to \infty$ then $\omega(x) \to 0$ uniformly. We call $B_{n-k} = \dim(H_N(\Omega^c, \wedge^{k-1}T^*\Omega \otimes \mathbb{R}^m))$ **Betti's number**. Similarly one define also $H_T(\Omega^c, \wedge^{k+1}T^*\Omega \otimes \mathbb{R}^m)$, and $\dim(H_T(\Omega^c, \wedge^{k+1}T^*\Omega \otimes \mathbb{R}^m) = B_k$

$$\|\omega\|_{W^{r,p}(\Omega)} \leq C_2 \left(\|d\omega\|_{W^{r-1,p}(\Omega)} + \|d^\star\omega\|_{W^{r-1,p}(\Omega)} + \|\nu \lrcorner \omega\|_{W^{r-\frac{1}{p},p}(\partial\Omega)} \right) \tag{3.23}$$

If furthermore we assume also that
$\omega \in W^{r,p}_T(\Omega, T^*\Omega \otimes \mathbb{R}^m) \cup W^{r,p}_N(\Omega, T^*\Omega \otimes \mathbb{R}^m)$ then

$$\|\omega\|_{W^{r,p}(\Omega)} \leq C_1 \left(\|d\omega\|_{W^{r-1,p}(\Omega)} + \|d^\star\omega\|_{W^{r-1,p}(\Omega)} \right) \tag{3.24}$$

3.2 Sobolev Bundles & Sobolev Connections

In section 2.2 we have given a definition of principal fibre bundle through a family of transition functions satisfying the cocycle conditions, as follows.
Let M be a m-dimensional manifold and G a matrix Lie Group, that for our purposes will be compact and connected, and naturally embedded in some $\mathbb{R}^{n^2}$, for $n \geq 1$. Then a principal bundle with base manifold M and structure Lie group G is the data of an open covering $\{U_i\}_{i\in I}$ of M and a family of smooth maps $g_{ij} : U_i \cap U_j \to G$ satisfying the cocycle conditions.
In the Analytic part of this thesis we mainly deal with bundles that have as base manifolds smooth bounded and connected open subsets Ω of $\mathbb{R}^m$. So from now on we will always identify $M = \Omega$.
When dealing with a variational problem, one need to consider less restrictive structures, since the search for a minimum often requires variational methods based on reflexivity of function spaces. For this reason we introduce the concept of Sobolev bundles.
To achieve our purpose we let the transition maps to be in a proper Sobolev space. Let $k \geq 1$ and $1 \leq p \leq \infty$, then for every open $U \subset \Omega$ we define the following space

$$W^{k,p}(U, G) = \left\{ g \in W^{k,p}(U, \mathbb{R}^{n^2}) | g(x) \in G \text{ for } a.e.\ x \in U \right\}$$

Definition 3.2.1. We define a **Sobolev principal bundle** of class $W^{k,p}$ with base manifold Ω and structure group G as the data of an open covering $\{U_i\}_{i\in I}$ of Ω, and a family of functions
$g_{ij} \in W^{k,p}(U_i \cap U_j, G)$, for $U_i \cap U_j \neq \emptyset$, such that the cocycle conditions hold

$$g_{ij} g_{jl} = g_{il} \quad \text{in } U_i \cap U_j \cap U_l \neq \emptyset$$
$$g_{ij} g_{ji} = e \quad \text{in } U_i \cap U_j \neq \emptyset$$

where in the above two equations we have considered the pointwise product. We will denote such a bundle with $\mathcal{P} = \{(g_{ij}, U_{ij})\}$, where for convenience we called $U_{ij} = U_i \cap U_j$.

Anyway, it is not clear if this pointwise operation is defined in $W^{k,p}$, since usually the product of two Sobolev functions loses regularity. The following theorem guarantees us that it is well defined, making $W^{k,p}(U,G)$ a topological group[7] for every open subset $U \subset \Omega$ smooth enough.

Theorem 3.2.2. *The space $W^{k,p}(U,G)$ defined as in Definition 3.2.1 is a topological group with respect to pointwise multiplication $W^{k,p}(U,G) \times W^{k,p}(U,G) \ni (f,g) \mapsto f \cdot g \in W^{k,p}(U,G)$ and pointwise inversion $W^{k,p}(U,G) \ni f \mapsto f^{-1} \in W^{k,p}(U,G)$.*

Proof. Before checking the continuity of the operations one should clearly prove that the product and inversion are actually defined in $W^{k,p}(U,G)$. Since the approach is similar to the one we will perform in order to get continuity, we skip this first part of the proof, assuming that the operations are well defined.
The Sobolev norm is defined as always $\|u\|_{k,p} = \sum_{|\alpha|\le k} \|D^\alpha u\|_{L^p}$, where we adopt the classical multi index notation. Let $\{f_n\}_n$ and $\{g_n\}_n$ be two sequences in $W^{k,p}(U,G)$ converging in this space to f and g respectively.

$$\|f_n g_n - fg\|_{k,p} = \|f_n g_n - fg\|_{L^p} + \sum_{0<|\alpha|\le k} \|D^\alpha(f_n g_n) - D^\alpha(fg)\|_{L^p} \le$$

$$\le \|f_n g_n - fg\|_{L^p} + C(\alpha,\beta) \sum_{0<|\alpha|\le k} \sum_{\beta\le\alpha} \underbrace{\|D^{\alpha-\beta} f_n D^\beta g_n - D^{\alpha-\beta} f D^\beta g\|_{L^p}}_{(I)}$$

Clearly $\|f_n g_n - fg\|_{L^p}$ is converging, indeed

$$\begin{aligned} \|f_n g_n - fg\|_{L^p} &\le \|f_n(g_n - g)\|_{L^p} + \|(f_n - f)g\|_{L^p} \\ &\le \|f_n\|_{L^\infty} \|g_n - g\|_{L^p} + \|f_n - f\|_{L^p} \|g\|_{L^\infty} \end{aligned}$$

and the convergence then follows by the fact that $|f_n| \le M$ a.e., and the same obviously holds also for g, since they all have image in the compact G, which implies that $W^{k,p}(U,G) \subset L^\infty(U,\mathbb{R}^{n^2})$. (I) is bounded from above by

$$(I) \le \|(D^{\alpha-\beta} f_n - D^{\alpha-\beta} f) D^\beta g_n\|_{L^p} + \|D^{\alpha-\beta} f (D^\beta g_n - D^\beta g)\|_{L^p}$$

We focus on the convergence of the first term on the right hand side (the second can be studied similarly) when $\beta \ne 0, \alpha$. Applying Hölder's inequality we get

$$\|(D^{\alpha-\beta} f_n - D^{\alpha-\beta} f) D^\beta g_n\|_{L^p} \le \|D^{\alpha-\beta} f_n - D^{\alpha-\beta} f\|_{L^{\frac{pk}{k-|\beta|}}} \|D^\beta g_n\|_{L^{\frac{pk}{|\beta|}}} \tag{3.25}$$

[7] A topological group is a group equipped with a Hausdorff topology with respect to which the group operations are continuous

Since $W^{k,p}(U,G) \subset L^\infty(U,\mathbb{R}^{n^2})$, Gagliardo Nirenberg inequality (see [29]) implies that

$$\max_{|\gamma|=|\beta|} \|D^\gamma g_n\|_{L^q} \leq C_1 \|g_n\|_{L^\infty}^{(1-\frac{|\beta|}{k})} \max_{|\delta|=k} \|D^\delta g_n\|_{L^p}^{\frac{|\beta|}{k}} + C_2 \|g_n\|_{L^p}$$

where $q = \frac{kp}{|\beta|}$, and therefore the last term in (3.16) is bounded. Now we apply the same inequality to $D^{\alpha-\beta} f_n - D^{\alpha-\beta} f$:

$$\max_{|\lambda|=|\alpha-\beta|} \|D^\lambda f_n - D^\lambda f\|_{L^s} \leq C_1 \|f_n - f\|_{L^\infty}^{(1-\frac{|\alpha-\beta|}{k})} \max_{|\mu|=k} \|D^\mu f_n - D^k f\|_{L^p}^{\frac{|\alpha-\beta|}{k}} +$$
$$+ C_2 \|f_n - f\|_{L^p}$$

where $s = \frac{kp}{|\alpha-\beta|}$. The right hand side of the above inequality goes to zero by hypothesis, and so does also the left hand side. Notice that in general $f_n \nrightarrow f$ with respect the L^∞-norm, the term $\|f_n - f\|_{L^\infty}$ is simply bounded. Since $U \subset \Omega$ which is bounded, then $L^{\frac{pk}{k-|\beta|}} \subset L^{\frac{pk}{|\alpha-\beta|}}$, and therefore the first term on the right hand side of (3.25) is converging. If $\beta = 0$, or $\beta = \alpha$ then one can get the convergence with a reasoning similar to the one we have exhibited for the L^p convergence. Finally the continuity of the pointwise inversion can be proved with similar arguments. □

The above theorem holds also when M is any compact Riemannian m-dimensional manifold. The generalization, though not difficult, requires some geometrical tools that go beyond the purpose of this work. A more complete proof can be found in [19]. If in the above theorem we assume $kp > m$, then the Sobolev embedding $W^{k,p}(U,G) \hookrightarrow C^0(\overline{U},G)$ implies also that the bundle is actually continuous.

In the section devoted to the development of the theory of bundles, we have also seen a characterization for isomorphic principal fibre bundles. Namely, if $\pi : P \to M$ and $\tilde{\pi} : \tilde{P} \to M$ are two principal fibre bundles and $\mathcal{A} := \{(U_i, \chi_i)\}_{i \in I}$ $\tilde{\mathcal{A}} := \{(U_i, \tilde{\chi}_i)\}_{i \in I}$ are their atlases, then they are equivalent (or isomorphic), if and only if there exists a family of maps $h_i \in C^\infty(U_i, G)$ such that

$$\tilde{g}_{ij} = h_i g_{ij} h_j^{-1} \quad \text{in } U_i \cap U_j \neq \emptyset$$

where $\{g_{ij}\}$ and $\{\tilde{g}_{ij}\}$ are the transition functions corresponding respectively to the atlases $\mathcal{A}$ and $\tilde{\mathcal{A}}$, see Remark 2.2.17.
We want to do the same thing for principal Sobolev bundles, and this

motivates the following definition, which, as always in the analytic part of this work, is stated for $M = \Omega$.

Definition 3.2.3. Let $\Omega \subset \mathbb{R}^m$ be a bounded and smooth domain, and $\mathcal{U} = \{U_i\}_{i \in I}$ a covering of Ω. If $\mathcal{P} = \{(U_{ij}, g_{ij})\}$ and $\tilde{\mathcal{P}} = \{(U_{ij}, \tilde{g}_{ij})\}$ are two $W^{k,p}$-Sobolev bundles, then we say they are $W^{k,p}$**-equivalent**, if there exists a refinement[8] $\mathcal{V} = \{V_j\}_{j \in J}$ of $\mathcal{U} = \{U_i\}_{i \in I}$ and a family of maps $h_j \in W^{k,p}(V_j, G)$ satisfying

$$\tilde{g}_{\phi(i)\phi(j)} = h_i g_{\phi(i)\phi(j)} h_j^{-1} \quad \text{in} \quad V_i \cap V_j =: V_{ij} \tag{3.26}$$

where ϕ is the refinement map. We will say that a $W^{k,p}$ bundle over Ω is trivial if it is $W^{k,p}$-equivalent to the trivial bundle $M \times G$.

Remark 3.2.4. Definition 3.2.3 has been stated assuming that $\mathcal{P}$ and $\tilde{\mathcal{P}}$ are defined over the same covering $\mathcal{U} = \{U_i\}_{i \in I}$. This is not restrictive, since one can always choose a common refinement of two different coverings of the same manifold M.

Now that we have defined a Sobolev principal bundle, we would like to define also the concept of Sobolev connection. In section 2.3 we have seen how to define a connection on a bundle, once it was given an open covering for the base manifold, and the family of transition functions, via what we have called the compatibility condition. We restrict ourselves to the aforementioned case when $M = \Omega$, and consider an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ for it. We recall that if $\{g_{ij}\}$ is a family of smooth transition functions, then we can define a connection on the bundle as a family $\{A_i\}_{i \in I}$ of $\mathfrak{g}$-valued 1-forms on $\mathcal{U}$ such that in the overlapping $U_i \cap U_j \neq \emptyset$ the compatibility condition holds:

$$A_j = g_{ij}^{-1} dg_{ij} + g_{ij}^{-1} A_i g_{ij} \tag{3.27}$$

Let $\mathcal{P} = \{(U_{ij}, g_{ij})\}$ be a $W^{k,p}$-Sobolev principal bundle over $\Omega \subset \mathbb{R}^m$ bounded and smooth, where $\mathcal{U} = \{U_i\}_{i \in I}$ is the chosen covering for Ω, and $k \geq 1$ and $1 \leq p < \infty$. We consider only the case of $k = 1$, $k = 2$ since they will be the most used throughout this work.

Definition 3.2.5. Let $\mathcal{P} = \{(U_{ij}, g_{ij})\}$ be the $W^{k,p}$-Sobolev bundle on Ω defined above. Then we have that

- If $k = 1$, then a **Sobolev connection** on $\mathcal{P}$ is given by a family $\{A_i\}_{i \in I}$ in $L^p(U_i, T^*U_i \otimes \mathfrak{g})$ such that the condition (3.27) is satisfied a.e. $\forall i, j$ such that $U_i \cap U_j \neq \emptyset$.

[8] If $\mathcal{U} = \{U_i\}_{i \in I}$ is a covering of a manifold M, then we say that $\mathcal{V} = \{V_j\}_{j \in J}$ is a refinement, if it is still a covering of M, and there exists $\phi : J \to I$ such that $V_j \subset\subset U_{\phi(j)}$ for each $j \in J$. We call ϕ refinement map.

- If $k = 2$, then a **Sobolev connection** on $\mathcal{P}$ is given by a family $\{A_i\}_{i\in I}$ in $L^{2p} \cap W^{1,p}(U_i, T^*U_i \otimes \mathfrak{g})$ such that the condition (3.27) is satisfied a.e. $\forall i, j$ such that $U_i \cap U_j \neq \emptyset$.

Remark 3.2.6. If $k = 2$ we have asked the family $\{A_i\}_{i\in I}$ to be in $(W^{1,p} \cap L^{2p})(U_i, T^*U_i \otimes \mathfrak{g})$, and not just in $W^{1,p}$. This is because if $U_i \cap U_j \neq \emptyset$ and $g_{ij} \in W^{2,p}(U_i \cap U_j, G)$ then the condition $A_j = g_{ij}^{-1} dg_{ij} + g_{ij}^{-1} A_i g_{ij}$ is required, but though $g_{ij}^{-1} dg_{ij} \in W^{1,p}(U_i \cap U_j, T^*(U_i \cap U_j) \otimes \mathfrak{g})$ thanks to Theorem 3.2.2, the same generally does not hold for the term $g_{ij}^{-1} A_i g_{ij}$ unless we ask A_i to be also in $L^{2p}(U_i, T^*U_i \otimes \mathfrak{g})$. Indeed, if we assume $A_i \in (L^{2p} \cap W^{1,p})(U_i, T^*U_i \otimes \mathfrak{g})$ for each $i \in I$, then we get that $g_{ij}^{-1} A_i g_{ij} \in L^{2p}(U_i \cap U_j, T^*(U_i \cap U_j) \otimes \mathfrak{g})$ since $g_{ij} \in G$ a.e. and G is compact. While we see that for each $k = 1, ..., m$ we have that $\partial_{x_k}(g_{ij}^{-1} A_i g_{ij}) = \partial_{x_k} g_{ij}^{-1} A_i g_{ij} + g_{ij}^{-1} \partial_{x_k} A_i g_{ij} + g_{ij}^{-1} A_i \partial_{x_k} g_{ij}$, and the second term is clearly in L^p. For the first term instead we can perform the bound

$$\|\partial_{x_k} g_{ij}^{-1} A_i g_{ij}\|_{L^p(U_i \cap U_j)} \leq C \|\partial_{x_k} g_{ij}^{-1} A_i\|_{L^p(U_i \cap U_j)} \leq$$
$$\leq C \|\partial_{x_k} g_{ij}^{-1}\|_{L^{2p}(U_i \cap U_j)} \|A_i\|_{L^{2p}(U_i \cap U_j)}$$

where in the last inequality we have used Hölder's inequality, and $\partial_{x_k} g_{ij}^{-1} \in L^{2p}(U_i \cap U_j)$ thanks to Gagliardo-Nirenberg inequalilty.

If $A \in W^{1,p}(\Omega, T^*\Omega \otimes \mathfrak{g})$ then it is a Sobolev connection in the trivial bundle $P := \Omega \times G$ by Definition 3.2.5, and if $g \in W^{2,p}(\Omega, G)$ then the $\mathfrak{g}$-valued 1-form $A^g := g^{-1}dg + g^{-1}Ag$ is still a connection on P. Note that the definition we have given for Sobolev connections tells us that the family $\{A, A^g\}$ is a connection in the bundle given by the trivial covering $\{\Omega\}$ with transition function g in Ω.

If $m = 4$ and $p = 2$, then we see that if $A \in W^{1,2}(\Omega, T^*\Omega \otimes \mathfrak{g})$ is a connection on the trivial bundle $\Omega \times G$, then it is automatically in the space $L^4(\Omega, T^*\Omega \otimes \mathfrak{g})$, thanks to the Sobolev embedding $W^{1,2}(\Omega, T^*\Omega \otimes \mathfrak{g}) \hookrightarrow L^4(\Omega, T^*\Omega \otimes \mathfrak{g})$. This embedding, if $p = 2$, does not hold in higher dimensions than four, which means that if $m > 4$, then we will have to reintroduce the request $A \in (W^{1,2} \cap L^4)(\Omega, T^*\Omega \otimes \mathfrak{g})$. For this reason we will refer to the dimension 4 as **critical dimension**. We have the following technical results, in critical setting, that will be used several times throughout this thesis.

Proposition 3.2.7. *Let $\Omega \subset \mathbb{R}^4$ be an open bounded smooth domain and $A_n \in W^{1,2}(\Omega, T^*\Omega \otimes \mathfrak{g})$ such that $A_n \to A$ in $W^{1,2}(\Omega, T^*\Omega \otimes \mathfrak{g})$, and the sequence of gauges $g_n \in W^{2,2}(\Omega, G)$ converging to $g \in W^{2,2}(\Omega, G)$. Then $A_n^{g_n} \to A^g$ in $W^{1,2}(\Omega, T^*\Omega \otimes \mathfrak{g})$.*

Proof. We have the following estimates

$$\|A_n^{g_n} - A^g\|_{W^{1,2}} \leq \|g_n^{-1}dg_n - g^{-1}dg\|_{W^{1,2}} + \|g_n^{-1}A_n g_n - g^{-1}Ag\|_{W^{1,2}} \leq$$
$$\leq \|g_n^{-1}dg_n - g^{-1}dg\|_{W^{1,p}} + \|(g_n^{-1} - g^{-1})A_n g_n\|_{W^{1,2}} +$$
$$+ \|g^{-1}(A_n - A)g_n\|_{W^{1,2}} + \|g^{-1}A(g_n - g)\|_{W^{1,2}}$$

The first term in the second inequality goes to zero by Theorem 3.2.2. Using the fact that G is a compact group and therefore $g_n, g \in L^\infty \cap W^{2,2}(\Omega, G)$ and Hölder's inequality we get the convergence also of the other terms. □

Proposition 3.2.8. *Let $\Omega \subset \mathbb{R}^4$ be a bounded and smooth domain, and $A \in W^{1,2}(\Omega, T^*\Omega \otimes \mathfrak{g})$ with $\|A\|_{W^{1,2}(\Omega)} \leq \varepsilon$ and let $g \in W^{2,2}(\Omega, G)$ be a gauge such that also $\|g\|_{W^{2,2}(\Omega)} \leq \varepsilon$. Then there exists a constant $C(G)$ depending on the gauge group G such that*

$$\|A^g\|_{W^{1,2}} \leq C(G)\varepsilon \tag{3.28}$$

Proof. We have that $\|A^g\|_{W^{1,2}} \leq \|A^g\|_{L^2} + \sum_{i=1}^4 \|\partial_{x_i} A^g\|_{L^2}$ The L^2-norm of A^g is bounded by

$$\|A^g\|_{L^2} \leq \|g^{-1}dg\|_{L^2} + \|g^{-1}Ag\|_{L^2} \leq \|g^{-1}dg\|_{L^2} + \|A\|_{L^2}$$
$$C\|dg\|_{L^2} + \varepsilon \leq \varepsilon(C+1)$$

The L^2-norm of the first derivatives of A^g has similar estimates, for every $i = 1, ..., 4$

$$\|\partial_{x_i} A^g\|_{L^2} \leq \|\partial_{x_i} g^{-1}dg\|_{L^2} + \|g^{-1}d(\partial_{x_i} g)\|_{L^2} + 2C\|\partial_{x_i} g\|_{L^4}\|A\|_{L^4} + \|\partial_{x_i} A\|_{L^2}$$
$$\leq \|\partial_{x_i} g^{-1}\|_{L^4}\|dg\|_{L^4} + C\|d(\partial_{x_i} g)\|_{L^2} + 2C\varepsilon^2 + \varepsilon \leq$$
$$(2C+1)\varepsilon^2 + (C+1)\varepsilon$$

and this concludes the proof. □

3.2.1 Hölder's regularity of Coulomb Bundles in critical dimension

In this subsection we study Sobolev bundles in critical dimension, endowed with a particular type of connection, called Coulomb connection. As we will soon see, these bundles are particularly interesting because from the compatibility condition we can extract a PDE satisfied by the transition functions. This PDE will lead us to an higher regularity of the transition functions.
We start with the definition of Coulomb bundle.

Definition 3.2.9. Let $\Omega \subset \mathbb{R}^4$ be bounded and smooth, and $\mathcal{P} = \{(g_{ij}, U_{ij})\}$ be a $W^{2,2}$-Sobolev principal bundle on Ω. If in $\mathcal{P}$ we are given a $W^{1,2}$-connection $\{A_i\}_{i\in I}$ such that $\forall i \in I$

$$d^\star A_i = 0 \text{ in } U_i \tag{3.29}$$

we call $\mathcal{P}$ a **Coulomb bundle**, and $\{A_i\}_{i\in I}$ a **Coulomb connection**.

Remark 3.2.10. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be a covering of $\Omega \subset \mathbb{R}^4$ bounded and smooth, and $\mathcal{P} = \{(U_{ij}, g_{ij})\}$ be a $W^{2,2}$-Coulomb bundle over Ω. Then if $\{A_i\}_{i\in I}$ is the Coulomb connection defined in $\mathcal{P}$, the compatibility condition

$$A_j = g_{ij}^{-1} dg_{ij} + g_{ij}^{-1} A_i g_{ij} \text{ in } U_i \cap U_j$$

leads to the equation

$$dg_{ij} = g_{ij} A_j - A_i g_{ij} \quad \text{in } U_i \cap U_j \tag{3.30}$$

We take the codifferential on both sides of equation (3.30), and using the hypothesis $d^\star A_i = 0$ for each $i \in I$, then we obtain[9]

$$\Delta g_{ij} = d^\star dg_{ij} = dg_{ij} \cdot A_j - A_i \cdot dg_{ij} \text{ in } U_i \cap U_j \tag{3.31}$$

Equation (3.31) was the PDE, solved by the transition functions of a Coulomb bundle, we were referring to at the beginning of this subsection. From this PDE we can deduce some regularity results on the family $\{g_{ij}\}$. In particular first we show that the transition functions g_{ij} are in $C^0_{\mathrm{loc}}(U_{ij}, G)$, and then we will show that one can choose a suitable refinement $\{V_j\}_{j\in J}$ of the covering $\{U_i\}_{i\in I}$ such that the transition functions in this new covering are $C^{0,\alpha}$-continuous for each $0 \le \alpha < 1$.

Lemma 3.2.11 ([33])**.** *Let $\mathcal{P} = \{(g_{ij}, U_{ij})\}$ be a $W^{2,2}$-Coulomb bundle over $\Omega \subset \mathbb{R}^4$ bounded and smooth. Then $g_{ij} \in W^{2,(2,1)}_{loc}(U_{ij}, G)$. Furthermore, for each compact $K_{ij} \subset\subset U_{ij}$ there exists $\overline{g}_{ij} \in G$, and a constant $C_{ij} := C(K_{ij}, U_{ij}, G) > 0$ such that*

$$\|g_{ij} - \overline{g}_{ij}\|_{W^{2,(2,1)}(K_{ij})} \le C_{ij} \Big(\|A_i\|_{W^{1,2}(U_{ij})} + \|A_j\|_{W^{1,2}(U_{ij})} + \\ + \left(\|A_i\|_{W^{1,2}(U_{ij})} + \|A_j\|_{W^{1,2}(U_{ij})}\right)^2 \Big) \tag{3.32}$$

where $\{A_i\}_{i\in I}$ is the Coulomb connection defined on $\mathcal{P}$.

[9]If α and β are two $\mathfrak{g}$-valued 1-forms, then we denote $\alpha \cdot \beta = \sum_i \alpha_i \beta_i$, where $\alpha_i \beta_i$ is the composition of the matrix α_i with the matrix β_i

Proof. As already discussed, the compatibility condition for the Coulomb connection $\{A_i\}_{i\in I}$

$$A_j = g_{ij}^{-1} dg_{ij} + g_{ij}^{-1} A_i g_{ij} \ \text{ in } \ U_{ij} = U_i \cap U_j$$

leads to equation (3.30), that gives us the bound

$$\|dg_{ij}\|_{L^{(4,2)}(U_{ij})} \leq C(\|A_i\|_{L^{(4,2)}(U_{ij})} + \|A_j\|_{L^{(4,2)}(U_{ij})}). \tag{3.33}$$

since $g_{ij} \in G$ a.e. in U_{ij} and G is compact. The embedding $L^{(2,1)}(U_{ij}) \hookrightarrow L^4(U_{ij})$ and Poincaré inequality imply that

$$\|g_{ij} - \tilde{g}_{ij}\|_{L^{(2,1)}(U_{ij})} \leq C\|dg_{ij}\|_{L^4(U_{ij})} \tag{3.34}$$

where $\tilde{g}_{ij} := \frac{1}{|U_{ij}|}\int_{U_{ij}} g(x)dx$, and the embedding $L^{(4,2)}(U_{ij}) \hookrightarrow L^4(U_{ij})$ gives us the estimate

$$\|g_{ij} - \tilde{g}_{ij}\|_{L^{(2,1)}(U_{ij})} \leq C\|dg_{ij}\|_{L^{(4,2)}(U_{ij})} \tag{3.35}$$

By hypothesis we know that $d^\star A_i = d^\star A_j = 0$, and therefore g_{ij} satisfies the PDE (3.31). We want now to bound the $L^{2,1}$-norm of the Laplacian of g_{ij}, and applying the Lorentz embedding $L^{4,2}(U_{ij}) \times L^{4,2}(U_{ij}) \hookrightarrow L^{2,1}(U_{ij})$ to (3.31) we find:

$$\|\Delta g_{ij}\|_{L^{2,1}(U_{ij})} \leq C\|dg_{ij}\|_{L^{4,2}(U_{ij})}\left(\|A_j\|_{L^{4,2}(U_{ij})} + \|A_i\|_{L^{4,2}(U_{ij})}\right) \tag{3.36}$$

The Sobolev embedding $W^{1,2}(U_{ij}, T^*U_{ij} \otimes \mathfrak{g}) \hookrightarrow L^{4,2}(U_{ij}, T^*U_{ij} \otimes \mathfrak{g})$, see Theorem A.3.8, implies that inside the parentheses in the right hand side of (3.36) one can replace the $L^{4,2}$-norm with the $W^{1,2}$ one, and so we get

$$\|\Delta g_{ij}\|_{L^{(2,1)}(U_{ij})} \leq C\left(\|A_i\|_{W^{1,2}(U_{ij})} + \|A_j\|_{W^{1,2}(U_{ij})}\right)^2$$

If we now fix a compact subset $K_{ij} \subset\subset U_{ij}$, following the arguments of Example A.2.9, we find that

$$\begin{aligned}
\|D^2 g_{ij}\|_{L^{(2,1)}(K_{ij})} \leq \hat{C}_{ij}\Bigg(& \|g_{ij} - \tilde{g}_{ij}\|_{L^{(2,1)}(U_{ij})} + \\
& + \|dg_{ij}\|_{L^{(2,1)}(U_{ij})} + \|\Delta g_{ij}\|_{L^{(2,1)}(U_{ij})}\Bigg) \leq \\
\leq C\left(\|A_i\|_{W^{1,2}(U_{ij})} + \|A_j\|_{W^{1,2}(U_{ij})} + \left(\|A_i\|_{W^{1,2}(U_{ij})} + \|A_j\|_{W^{1,2}(U_{ij})}\right)^2\right)
\end{aligned} \tag{3.37}$$

where $\hat{C}_{ij}$ depends on the group G, on K_{ij} and U_{ij}. Therefore we have obtained that

$$\|g_{ij} - \tilde{g}_{ij}\|_{W^{2,(2,1)}(K_{ij})} \leq \tilde{C}_{ij}\Bigg(\|A_i\|_{W^{1,2}(U_{ij})} + \|A_j\|_{W^{1,2}(U_{ij})} +$$

$$+ \left(\|A_i\|_{W^{1,2}(U_{ij})} + \|A_j\|_{W^{1,2}(U_{ij})}\right)^2 \Big) \tag{3.38}$$

and clearly also here the constant $\tilde{C}_{ij}$ depends on the group G, on the compact K_{ij} and on U_{ij}. The element $\tilde{g}_{ij}$ is not necessarily in G, but anyway we can substitute it. Indeed, G is compact and $g_{ij}(x) \in G$ for a.e. $x \in U_{ij}$, therefore there exists $\overline{g}_{ij} \in G$, such that $|\tilde{g}_{ij} - \overline{g}_{ij}| = \inf\{|y - \tilde{g}_{ij}| : \, y \in \overline{Im(g_{ij}) \cap G}\}$, which implies that for a.e. $x \in U_{ij}$ it holds

$$|g_{ij}(x) - \overline{g}_{ij}| \leq |g_{ij}(x) - \tilde{g}_{ij}| + |\tilde{g}_{ij} - \overline{g}_{ij}| \leq 2|g_{ij}(x) - \tilde{g}_{ij}|$$

This last inequality in addition to the embedding $W^{2,(2,1)}(K_{ij}) \hookrightarrow L^\infty(K_{ij})$, see Theorem A.3.10 in appendix A, leads to

$$\|g_{ij} - \overline{g}_{ij}\|_{L^\infty(K_{ij})} \leq C\|g_{ij} - \overline{g}_{ij}\|_{W^{2,(2,1)}(K_{ij})} \leq C\|g_{ij} - \tilde{g}_{ij}\|_{W^{2,(2,1)}(K_{ij})} \leq$$

$$\leq C_{ij}\left(\|A_i\|_{W^{1,2}(U_{ij})} + \|A_j\|_{W^{1,2}(U_{ij})} + \left(\|A_i\|_{W^{1,2}(U_{ij})} + \|B_{ij}\|_{W^{1,2}(U_{ij})}\right)^2\right) \tag{3.39}$$

□

Let $\mathcal{P} = \{(U_{ij}, g_{ij})\}$ and $\{A_i\}_{i\in I}$ be as in Remark 3.2.10. We know that the family of transition functions $\{g_{ij}\}$ satisfies the PDE

$$\Delta g_{ij} = dg_{ij} \cdot A_j - A_i \cdot dg_{ij} \text{ in } U_i \cap U_j = U_{ij}$$

The following two Lemmas give us some important elliptic estimates of solutions of the above equation, under suitable conditions on the L^4-norm of $\{A_i\}_{i\in I}$, improving significantly the regularity of the transition functions.
In what follows remember that we assumed $G \hookrightarrow \mathbb{R}^{n^2}$ for some $n \in \mathbb{N}$.

Lemma 3.2.12. *Let $\Omega \subset \mathbb{R}^4$ be bounded and smooth, and $A \in L^4(\Omega, T^*\Omega \otimes \mathbb{R}^{n^2})$. If $\alpha \in W_0^{2,2}(\Omega, \mathbb{R}^{n^2})$ satisfies*

$$\Delta\alpha = A \cdot d\alpha + F \text{ in } \Omega \tag{3.40}$$

with $F \in L^p(\Omega, \mathbb{R}^{n^2})$ with $2 < p < 4$, then $\exists \varepsilon_1 = \varepsilon_1(n^2, p, \Omega) > 0$ such that if $\|A\|_{L^4(\Omega)} < \varepsilon_1$, then $\alpha \in W_0^{2,p}(\Omega, \mathbb{R}^{n^2})$ and the following estimate is true

$$\|\alpha\|_{W^{2,p}(\Omega)} \leq C_1\|F\|_{L^p(\Omega)} \tag{3.41}$$

for $C_1 = C_1(n^2, p, \Omega) > 1$. The constant ε_1 is scale invariant[10]*, and invariant with respect to translations of the domain.*

[10] We say that ε_1 is scale invariant if $\varepsilon_1(n^2, p, \Omega_r) = \varepsilon_1(n^2, p, \Omega)$, where for $r > 0$ we have called $\Omega_r = \{xr : \, x \in \Omega\}$.

Proof. This result is obtained through a fixed point argument. For each $v \in W_0^{2,p}(\Omega, \mathbb{R}^{n^2})$ let $T(v)$ be the solution to

$$\begin{cases} \Delta T(v) = A \cdot dv + F \text{ in } \Omega \\ T(v) = 0 \text{ in } \partial\Omega \end{cases}$$

Now since $A \in L^4(\Omega, \mathbb{R}^{n^2})$ and $dv \in L^{\frac{4p}{4-p}}(\Omega)$, by Hölder's inequality, we have that $\Delta T(v) \in L^p(\Omega)$. Therefore, from Calderón-Zygmund inequality, see Appendix A, we obtain that $D^2 T(v) \in L^p(\Omega)$, which means that $T(v) \in W_0^{2,p}(\Omega, \mathbb{R}^{n^2})$. From classical elliptic estimates we have that for each $v, w \in W_0^{2,p}(\Omega, \mathbb{R}^{n^2})$ it holds

$$\|T(v) - T(w)\|_{W_0^{2,p}(\Omega)} \leq C\|\Delta T(v) - \Delta T(w)\|_{L^p(\Omega)}$$

and therefore

$$\|T(v) - T(w)\|_{W_0^{2,p}(\Omega)} \leq C\|A\|_{L^4(\Omega)}\|dv - dw\|_{L^{\frac{4p}{4-p}}(\Omega)} \leq$$

$$\leq C\|A\|_{L^4(\Omega)}\|v - w\|_{W_0^{2,p}(\Omega)}$$

This last estimate implies that if the L^4-norm of A is small enough, then we can apply the Shrinking Lemma (Lemma 1.1 in [24]). Therefore, there exists and is also unique a solution to equation (3.40) in $W_0^{2,p}(\Omega, \mathbb{R}^{n^2})$. Since $\alpha \in W_0^{2,2}(\Omega, \mathbb{R}^{n^2})$ is a solution to the same equation we get the estimates

$$\|\alpha - v\|_{W^{2,2}(\Omega)} \leq C\|d\alpha - dv\|_{W^{1,2}(\Omega)} \leq C\|d^\star d\alpha - d^\star dv\|_{L^2(\Omega)} \leq$$

$$\leq \|A\|_{L^4(\Omega)}\|d\alpha - dv\|_{L^4(\Omega)} \leq C\|A\|_{L^4(\Omega)}\|\alpha - v\|_{W^{2,2}(\Omega)}$$

where the second inequality is a consequence of Gaffney's inequality (3.19). Therefore, if $\|A\|_{L^4}$ is small enough, we must have $\alpha = v$, which implies finally that $\alpha \in W_0^{2,p}(\Omega, \mathbb{R}^{n^2})$. The estimate (3.41) is obtained by observing that

$$\|\alpha\|_{W^{2,p}(\Omega)} \leq C\left(\|A\|_{L^4(\Omega)}\|d\alpha\|_{L^{\frac{4p}{4-p}}(\Omega)} + \|F\|_{L^p(\Omega)}\right)$$

and therefore for $\|A\|_{L^4(\Omega)}$ small enough, we get that

$$\|\alpha\|_{W^{2,p}(\Omega)} \leq C_1\|F\|_{L^p(\Omega)}$$

where C_1 depends on ε_1. The scale invariance is easy to obtain. Indeed, if α, A, F satisfy (3.40) in Ω_r, then for $x \in \Omega$ we define the new functions, $\tilde{\alpha}(x) = \alpha(rx)$, $\tilde{A}(x) = rA(rx)$ and $\tilde{F}(x) = r^2 F(rx)$, and an easy computation shows that

$$\Delta\tilde{\alpha} = \tilde{A} \cdot d\tilde{\alpha} + \tilde{F} \text{ in } \Omega$$

and since $\|\tilde{A}\|_{L^4(\Omega)} = \|A\|_{L^4(\Omega_r)}$, we can conclude. The invariance with respect to translations is trivial. $\square$

Lemma 3.2.13. *Let $\Omega \subset \mathbb{R}^4$ be bounded and smooth, and $A \in L^4(\Omega, T^*\Omega \otimes \mathbb{R}^{n^2})$. If $\|A\|_{L^4(\Omega)} \leq \varepsilon_1$, where ε_1 has been defined in Lemma 3.2.12 and $2 < p < 4$ is fixed, then for every $\alpha \in W^{2,2}(\Omega, \mathbb{R}^{n^2})$ solution of*

$$\Delta\alpha = A \cdot d\alpha \;\; in \;\; \Omega \tag{3.42}$$

it holds $\alpha \in W^{2,p}_{loc}(\Omega, \mathbb{R}^{n^2})$. Furthermore, for every $K \subset\subset \Omega$ there exists a constant $C_2 = C_2(n^2, p, \Omega, K)$ such that

$$\|\alpha\|_{W^{2,p}(K)} \leq C_2 \|\alpha\|_{W^{2,2}(\Omega)} \tag{3.43}$$

The constant ε_1 is still scale invariant with respect to translations of the domain.

Proof. Let us fix $K \subset\subset \Omega_1 \subset\subset \Omega$. Then we choose $\phi \in C_c^\infty(\Omega_1)$, such that $\phi = 1$ in K. We have that $\phi\alpha \in W_0^{2,2}(\Omega_1, \mathbb{R}^{n^2})$. We can easily extend $\phi\alpha$ in the following way

$$v := \begin{cases} \phi\alpha & \text{in } \;\Omega_1 \\ 0 & \text{in } \;\Omega \setminus \Omega_1 \end{cases} \tag{3.44}$$

and we get that $v \in W_0^{2,2}(\Omega, \mathbb{R}^{n^2})$. Since α is a solution to (3.42) we have that

$$\Delta(\phi\alpha) = A \cdot d(\phi\alpha) + \alpha(\Delta\phi - A \cdot d\phi) + 2d\phi \cdot d\alpha \quad \text{in } \;\Omega_1 \tag{3.45}$$

and moreover we see that

$$\Delta v = A \cdot dv \;\; \text{in } \;\Omega \setminus \Omega_1 \tag{3.46}$$

since $v = 0$ in $\Omega \setminus \Omega_1$. Finally we set

$$\tilde{F} = \begin{cases} \alpha(\Delta\phi - A \cdot d\phi) + 2d\phi \cdot d\alpha \;\; \text{in } \;\Omega_1 \\ 0 \;\; \text{in } \;\Omega \setminus \Omega_1 \end{cases} \tag{3.47}$$

and a simple computation shows that $\tilde{F} \in L^p(\Omega, \mathbb{R}^{n^2})$, for each $2 < p < 4$. Therefore, we have found that

$$\Delta v = A \cdot dv + \tilde{F} \quad \text{in } \;\Omega \tag{3.48}$$

and if $\|A\|_{L^4(\Omega)} < \varepsilon_1(n^2, p, \Omega)$, we can apply Lemma 3.2.12 to v in Ω, since equation (3.48) is in the same form of (3.40).
Therefore, we have obtained that $\phi\alpha \in W_0^{2,p}(\Omega_1, \mathbb{R}^{n^2})$, and since

$\phi = 1$ in K, then $\alpha \in W^{2,p}(K, \mathbb{R}^{n^2})$. It is left to prove the inequality, which anyway follows easily from (3.41). Indeed,

$$\|\alpha\|_{W^{2,p}(K)} \leq \|\phi\alpha\|_{W^{2,p}(\Omega_1)} \leq C\|\alpha(\Delta\phi - A \cdot d\phi) + 2d\phi \cdot d\alpha\|_{L^p(\Omega_1)} \leq$$
$$\leq C_2\|\alpha\|_{W^{2,2}(\Omega)}$$

where of course the last constant $C_2 > 0$ depends also on K.

□

Remark 3.2.14. From the proof of Lemma 3.2.13, we deduce that if $\|A\|_{L^4(\Omega)} \leq \varepsilon_1(p, n^2, \Omega)$, then for each $V \subset \Omega$ and for each $\alpha \in W^{2,2}(V, \mathbb{R}^{n^2})$ solution of

$$\Delta\alpha = A \cdot d\alpha \quad \text{in } V$$

it holds that $\alpha \in W^{2,p}_{\text{loc}}(V, \mathbb{R}^{n^2})$, even though we did not explicitly require that $\|A\|_{L^4(V)} \leq \varepsilon_1(n^2, p, V)$. To clearly see it, fix $K \subset\subset \Omega_1 \subset\subset V \subset \Omega$, and $\phi \in C^\infty_c(\Omega_1)$ such that $\phi = 1$ in K. Then we can build, as we did in Lemma 3.2.13, the function

$$v := \begin{cases} \phi\alpha & \text{in } \Omega_1 \\ 0 & \text{in } \Omega \setminus \Omega_1 \end{cases}$$

and thanks to the same reasoning we get

$$\Delta v = A \cdot dv + \tilde{F} \ \text{ in } \ \Omega$$

where

$$\tilde{F} := \begin{cases} \alpha(\Delta\phi - A \cdot d\phi) + 2d\phi \cdot d\alpha & \text{in } \Omega_1 \\ 0 & \text{in } \Omega \setminus \Omega_1 \end{cases}$$

is clearly L^p. Then, all the hypothesis in order to apply Lemma 3.2.12 are satisfied, and $v \in W^{2,p}_0(\Omega, \mathbb{R}^{n^2})$, which means that $\alpha \in W^{2,p}(K, \mathbb{R}^{n^2})$. Moreover, the inequality
$\|\alpha\|_{W^{2,p}(K)} \leq C(K, n^2, p, V)\|\alpha\|_{W^{2,2}(V)}$ still clearly holds.

Let again $\mathcal{P} = \{(g_{ij}, U_{ij})\}$ and $\{A_i\}_{i\in I}$ be as in Remark 3.2.10. We use the elliptic estimates obtained in the above two Lemmas to prove that the transition functions $g_{ij} \in W^{2,p}_{\text{loc}}(U_{ij}, G)$ for any $2 < p < 4$, if $\|A_i\|_{L^4(U_i)}$ is small enough for each $i \in I$.

Lemma 3.2.15. *Let $\mathcal{P} = \{(g_{ij}, U_{ij})\}$ be a $W^{2,2}$-Coulomb bundle over $\Omega \subset \mathbb{R}^4$ bounded and smooth, and $\mathcal{U} = \{U_i\}_{i\in I}$ the covering for Ω. There exists $\varepsilon_2 := \varepsilon_2(n^2, p, \mathcal{U}) > 0$ for each fixed $2 < p < 4$, such that if the Coulomb connection $\{A_i\}_{i\in I}$ defined on $\mathcal{P}$ satisfies*

$$\|A_i\|_{L^4(U_i)} \leq \varepsilon_2 \ \ \forall i \in I \tag{3.49}$$

then $g_{ij} \in W^{2,p}_{loc}(U_{ij}, G)$. *Moreover, for each* $K_{ij} \subset\subset U_{ij}$ *there exists a constant* $M_{ij} = M_{ij}(K_{ij}, U_{ij}, n^2, p)$ *such that*

$$\|g_{ij}\|_{W^{2,p}(K_{ij})} \leq M_{ij} \|g_{ij}\|_{W^{2,2}(U_{ij})} \tag{3.50}$$

Proof. From the compatibility condition

$$A_j = g_{ij}^{-1} dg_{ij} + g_{ij}^{-1} A_i g_{ij} \ \text{ in } \ U_i \cap U_j = U_{ij}$$

we have already deduced the partial differential equation for the transition functions $g_{ij} \in W^{2,2}(U_{ij}, G)$

$$\Delta g_{ij} = d^\star dg_{ij} = dg_{ij} \cdot A_j - A_i \cdot dg_{ij} \quad \text{in} \ \ U_i \cap U_j = U_{ij}$$

in Remark 3.2.10, and it is an equation of the same form of (3.42). This means that if

$$\|A_i\|_{L^4(U_i)} \leq \varepsilon_2 \quad \text{with} \quad \varepsilon_2 := \min_{i \in I} \left(\frac{\varepsilon_1(n^2, p, U_i)}{4} \right) \tag{3.51}$$

thanks to Remark 3.2.14, we can apply Lemma 3.2.13 to the family $\{g_{ij}\}$ and thus get $g_{ij} \in W^{2,p}_{\text{loc}}(U_{ij}, G)$ for each $2 < p < 4$. The estimate (3.50) follows trivially from (3.43). $\square$

Remark 3.2.16 ([41],Theorem 16). Let $\Omega \subset \mathbb{R}^4$ be bounded and smooth, $\mathcal{U} = \{U_i\}_{i \in I}$ be a covering. Moreover, let $\mathcal{P} = \{(g_{ij}, U_{ij})\}$ be a $W^{2,2}$-Coulomb bundle over Ω, where the Coulomb connection is $\{A_i\}_{i \in I}$.
One may observe that even though we do not have a proper bound on the L^4-norm of $\{A_i\}_{i \in I}$, we can take a refinement $\mathcal{V} = \{V_j\}_{j \in J}$ of $\mathcal{U} = \{U_i\}_{i \in I}$ such that there is an enlarged cover $\mathcal{V}' := \{V_j'\}_{j \in J}$, which is also a refinement of $\mathcal{U}$, with the same refinement map $\phi : J \to I$ of $\mathcal{V}$ and we have

$$\|A_{\phi(j)}\|_{L^4(V_j')} \leq \frac{\varepsilon_1(n^2, p, V_j)}{4} \qquad \forall j \in J$$

with $V_j \subset\subset V_j'$. In this way we have that $\tilde{\mathcal{P}} = \{(V_{ij}, g_{\phi(i),\phi(j)})\}$ is an $W^{2,p}$-Sobolev bundle. This is possible thanks to the fact that ε_1 is scale invariant.

3.2.2 $W^{2,p}$-equivalence of L^∞-near $W^{2,p}$-Sobolev Bundles

This subsection is devoted to an important result, due to K.Uhlenbeck, on equivalence of $W^{2,p}$ bundles (with $p > 2$) in the critical dimension four. We will see in Lemma 3.2.18 that if $\mathcal{P} = \{(g_{ij}, U_{ij})\}$ and

$\mathcal{P}' = \{(h_{ij}, U_{ij})\}$ are two $W^{2,p}$-Sobolev bundles over $\Omega \subset \mathbb{R}^4$, that are L^∞-near, namely

$$\|g_{ij} - h_{ij}\|_{L^\infty(U_{ij})} < \delta$$

with δ small enough, then they are $W^{2,p}$ equivalent. The constant δ will be shown to be dependent on the number of elements of the covering.
In what follows we denote with V_e a neighbourhood of $e \in G$, where the map $\exp^{-1}$ is well defined and differentiable.

Lemma 3.2.17 ([44], Lemma 3.1.)**.** *Let G be a compact connected matrix Lie Group, endowed with a bi-invariant metric. There exists a constant $\delta_0 > 0$ such that if $h, g, \rho \in G$, and $|\exp^{-1}(hg)| \le \delta_0$ and also $|\exp^{-1}(\rho)| \le \delta_0$, then $h\rho g \in V_e$ and*

$$|\exp^{-1}(h\rho g)| \le 2(|\exp^{-1}(hg)| + |\exp^{-1}(\rho)|) \tag{3.52}$$

Proof. There exists a neighbourhood V_0 of $0 \in \mathfrak{g}$, such that the map

$$\begin{aligned} Q : V_0 \times V_0 &\to \mathfrak{g} \\ (k, u) &\mapsto Q(k, u) := \exp^{-1}(\exp(k)\exp(u)) \end{aligned} \tag{3.53}$$

is well defined and differentiable. It is clear that $Q(0, 0) = 0$ and also that $|dQ(0, 0)| = 1$. We fix the neighbourhood $V_0 \supseteq \mathcal{O} := \{x \in \mathfrak{g} : |x| \le \delta_0\}$ of $0 \in \mathfrak{g}$, such that $|dQ(k, u)| \le 2$ for each $(k, u) \in \mathcal{O} \times \mathcal{O}$. Thanks to the convexity of $\mathcal{O}$ and the differentiability of Q, we can apply the mean value theorem, which tells us that

$$|Q(k, u)| \le 2(|k| + |u|) \qquad \forall (k, u) \in \mathcal{O} \times \mathcal{O}$$

We have proved the result. Indeed, if we set $k = \exp^{-1}(hg)$ and $u = Ad_{g^{-1}} \exp^{-1}(\rho)$, we get

$$Q(k, u) = \exp^{-1}(hg \exp(Ad_{g^{-1}} \exp^{-1}(\rho))) = \exp^{-1}(h\rho g)$$

and the following estimate holds

$$\begin{aligned} |Q(k, u)| \le 2(|\exp^{-1}(hg)| + |Ad_{g^{-1}}(\exp^{-1}(\rho))|) &= \\ = 2(|\exp^{-1}(hg)| + |\exp^{-1}\rho|) \end{aligned}$$

where the last identity is due to the hypothesis that the metric is bi-invariant. $\square$

Lemma 3.2.18 ([44], Proposition 3.2 & Corollary 3.3)**.** *Let $\Omega \subset \mathbb{R}^4$ be bounded, smooth and connected, and $\mathcal{U} = \{U_i\}_{i \in I}$ a finite covering of Ω. Let $\mathcal{P} := \{(g_{ij}, U_{ij})\}$ and $\mathcal{P}' := \{(h_{ij}, U_{ij})\}$ be two*

$W^{2,p}$-Sobolev bundles over Ω. Then there exists a constant $\delta_I > 0$, depending on the cardinality of I and the group G, such that if for each $i, j \in I$ satisfying $U_i \cap U_j \neq \emptyset$ it holds

$$m = \| \exp^{-1}(h_{ji}g_{ij})\|_{L^\infty(U_{ij})} \leq \delta_I \tag{3.54}$$

then $\mathcal{P}$ is $W^{2,p}$-equivalent to $\mathcal{P}'$. In particular for each refinement $\mathcal{V} = \{V\}_{i\in I}$ of $\mathcal{U}$ satisfying $\overline{V}_i \subset U_i$ and $\cup_i V_i \supset \Omega$, there exists a family $\sigma_i \in W^{2,p}(V_i, G)$, such that

$$h_{ij} = \sigma_i g_{ij} \sigma_j^{-1} \quad \textit{in} \;\; V_i \cap V_j \neq \emptyset \tag{3.55}$$

Proof of Lemma 3.2.18. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be as in the statement. The result is proved by induction on the number of elements of the cover $\mathcal{U} = \{U_i\}_{i\in I}$. If $|I| = 1$, then $g_{11} = e = h_{11}$ and therefore $\sigma_1 = e \in G$.
Now suppose we have constructed for $k \in I$, ($|I| > 1$) and for $1 \leq i, j \leq k$,

$1)_k$ $\sigma_i \in W^{2,p}(W_{i,k}, G)$ satisfying

$$h_{ij} = \sigma_i g_{ij} \sigma_j^{-1} \;\text{ on }\; W_{i,k} \cap W_{j,k}$$

where $\overline{V}_i \subset W_{i,k} \subset U_i$.

Moreover we also assume that

$2)_k$ $\| \exp^{-1} \sigma_i\|_{L^\infty(W_{i,k})} \leq c_k m$, where $c_k > 1$ is a constant depending on k.

We claim that if m is small enough, then we can continue our construction from k to $k+1$, namely that $1)_k, 2)_k$ imply $1)_{k+1}$,$2)_{k+1}$, proving therefore the Lemma by induction. Let $l = k + 1$, if we assume $m \leq \frac{\delta_0}{c_k}$, then we have

$$\| \exp^{-1}(h_{li}g_{il})\|_{L^\infty(U_i\cap U_j)} \leq m \leq \frac{\delta_0}{c_k} \qquad \text{and also}$$

$$\| \exp^{-1} \sigma_i\|_{L^\infty(W_{i,k})} \leq c_k m \leq \delta_0$$

and thanks to Lemma 3.2.17, we have that for $i \leq k = l - 1$ the map

$$f_l : W_{i,k} \cap U_l \to \mathfrak{g}$$

$$x \mapsto f_l(x) := \exp^{-1}(h_{li}(x)\sigma_i(x)g_{il}(x)) \tag{3.56}$$

is well defined and actually $W^{2,p}$. Furthermore, we also have that

$$\|f_l\|_{L^\infty(W_{i,k}\cap U_l)} \leq 2(1 + c_k)m = c_l m$$

and since the families $\{g_{ij}\}$ and $\{h_{ij}\}$ satisfy the cocycle conditions, then an easy computation shows that f_l is well defined in the whole $U_l \cap \left(\bigcup_{i\leq k} W_{i,k}\right)$. We now choose a cutoff function $\phi_l \in C^\infty(\mathbb{R}^4)$ such that

$$\begin{cases} \phi_l = 0 & \text{in } \ U_l \setminus \left(\bigcup_{i\leq k} W_{i,k}\right) \\ \phi_l = 1 & \text{in } \ \bigcup_{i\leq k} \tilde{V}_i \end{cases}$$

where $\overline{V}_i \subset \tilde{V}_i \subset W_{i,k}$ is arbitrary. We also set

$$W_{i,l} := W_{i,k} \cap \operatorname{int}\{x \in \mathbb{R}^4 : \ \phi_l(x) = 1\} \ \ \forall i \leq k \tag{3.57}$$

Then we define $\sigma_l : U_l \to G$ as follows

$$\sigma_l := \begin{cases} \exp(\phi_l f_l) \ \text{ in } \ U_l \cap \left(\bigcup_{i\leq k} W_{i,k}\right) \\ \sigma_l = 1 \ \text{ in } \ U_l \setminus \left(\bigcup_{i\leq k} W_{i,k}\right) \end{cases} \tag{3.58}$$

It is clear that $\| \exp^{-1} \sigma_l\|_{L^\infty(U_l)} \leq \|\phi_l f_l\|_{L^\infty(U_l)} \leq 2(1+c_k)m = c_l m$. Finally we choose $W_{l,l}$ any open subset, satisfying $\overline{V}_l \subset W_{l,l} \subset U_l$. The map $\sigma_l \in W^{2,p}(W_{ll}, G)$ by construction, and properties $1)_{k+1}$ and $2)_{k+1}$ are true for the new family of sets $W_{i,l}$ and maps σ_i.

$\square$

Chapter 4

The Plateau Problem for the YM functional

4.1 YM functional and its properties

We will consider connections over the trivial fibre bundle $P = B^4 \times G$. If $A \in C^\infty(B^4, T^*B^4 \otimes \mathfrak{g})$ then its field strength is defined by $F_A := dA + [A, A]$. Let us endow the Lie Algebra $\mathfrak{g}$ with the norm induced by the Killing form. Then the **Yang-Mills energy** of A is the L^2-norm square of F_A

$$YM(A) := \int_{B^4} |F_A|^2 dx = \sum_{i<j} \int_{B^4} |F_A^{ij}|^2 dx \tag{4.1}$$

The functional admits a geometrical interpretation that concerns the integrability of the horizontal distribution associated to A. Indeed we remember that from A we can build a unique connection form ω on the bundle $B^4 \times G$, such that if $s : B^4 \to P$ is the trivial section, then $s^*\omega = A$. The distribution on P given by $P \ni p \to H_p := \ker(\omega_p) \subset T_pP$ is horizontal and right equivariant. From the vector fields $\partial_{x_1}, ..., \partial_{x_4}$ we can find a global frame for the distribution H, using their **horizontal lifts** $\partial^*_{x_1}, ..., \partial^*_{x_4}$. By Frobenius theorem (see for instance [1]) the distribution H on P is integrable if and only if it is involutive, namely $[\partial^*_{x_i}, \partial^*_{x_j}]_p \in H_p$ for every $p \in P$ and $i, j = 1, ..., 4$. This is true if and only if $[\partial^*_{x_i}, \partial^*_{x_j}]^V = 0$, or equivalently $\omega([\partial^*_{x_i}, \partial^*_{x_j}]) = 0$ for any i, j. Theorem 2.4.5 tells us that $\Omega(\partial^*_{x_i}, \partial^*_{x_j}) = -\omega([\partial^*_{x_i}, \partial^*_{x_j}])$ and therefore the horizontal distribution is integrable if and only if $\Omega(\partial^*_{x_i}, \partial^*_{x_j}) = 0$ for every i, j. Now since $F_A = F_A^{ij} dx^i \wedge dx^j$, we have that

$$F_A^{ij} = F_A(\partial_{x_i}, \partial_{x_j}) = (s^*\Omega)(\partial_{x_i}, \partial_{x_j}) = \Omega(ds(\partial_{x_i}), ds(\partial_{x_j})) =$$

$$= -\omega([ds(\partial_{x_i})^H, ds(\partial_{x_j})^H]) = -\omega([\partial^*_{x_i}, \partial^*_{x_j}])$$

Therefore, one can interpret the Yang-Mills energy of A as the L^2 measure of lack of integrability of the horizontal distribution associated to A.

We now consider a more generic framework, passing from smooth connections to Sobolev one. In particular we can extend the field strength to connections $A \in W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g})$, and the Yang-Mills energy of A will be still well defined, as the following proposition shows.

Proposition 4.1.1. *For each $A \in W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g})$, the $\mathfrak{g}$-valued 2-form $F_A \in L^2(B^4, \wedge^2 T^*B^4 \otimes \mathfrak{g})$.*

Proof. By hypothesis $dA \in L^2(B^4, \wedge^2 T^*B^4 \otimes \mathfrak{g})$. The 2-form $[A, A]$ defined as $[A, A](\partial_{x_i}, \partial_{x_j}) := [A(\partial_{x_i}), A(\partial_{x_j})] = [A_i, A_j]$, where $A = A_i dx^i$ and the Lie bracket $[\cdot, \cdot]$ is the commutator of matrices, is in L^2 too. Indeed

$$\|[A, A]\|_{L^2} = \sum_{i<j} \|A_i A_j - A_j A_i\|_{L^2} \leq 2 \sum_{i<j} \|A_i\|_{L^4} \|A_j\|_{L^4}$$

where we have used Hölder's inequality. The last term is finite thanks to the Sobolev embedding
$W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g}) \hookrightarrow L^4(B^4, T^*B^4 \otimes \mathfrak{g})$ and so we conclude. □

We define the **Yang-Mills functional**:

$$YM : W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g}) \to \mathbb{R}$$

$$A \mapsto YM(A) := \int_{B^4} |dA + [A, A]|^2 dx,$$

which is continuous, as the following proposition shows.

Proposition 4.1.2. *Let us consider $\{A_k\} \subset W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g})$ such that $A_k \to A \in W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g})$. Then*

$$F_{A^k} \to F_A \tag{4.2}$$

strongly in $L^2(B^4, \bigwedge^2 T^*B^4 \otimes \mathfrak{g})$.

Proof. We have the following Sobolev embedding:

$$W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g}) \hookrightarrow L^4(B^4, T^*B^4 \otimes \mathfrak{g}) \tag{4.3}$$

and thus $A_k \to A$ in $W^{1,2}$ implies the convergence also in L^4. By Hölder's inequality we get that $[A_k, A_k] \to [A, A]$ in L^2. Finally

$$\|F_{A_k} - F_A\|_{L^2} = \|dA_k + [A_k, A_k] - dA - [A, A]\|_{L^2} \leq$$

$$\leq \|dA_k - dA\|_{L^2} + \|[A_k, A_k] - [A, A]\|_{L^2} \tag{4.4}$$

therefore $F_{A_k} \to F_A$ strongly in L^2 $\square$

Remark 4.1.3. The above result can be extended in the following sense. If instead of $W^{1,2}$ connections we consider $W^{1,p}(B^4, T^*B^4 \otimes \mathfrak{g})$ as domain for YM, one can find that F_{A_k} converges strongly in L^p, when $2 < p < 4$, and A_k is a converging sequence in $W^{1,p}$. This is obtained thanks to the Sobolev embedding

$$W^{1,p}(B^4, T^*B^4 \otimes \mathfrak{g}) \hookrightarrow L^{\frac{4p}{4-p}}(B^4, T^*B^4 \otimes \mathfrak{g}) \tag{4.5}$$

and Hölder's inequality.

The key property of the Yang-Mills functional is the **gauge invariance**, namely if A, B are two gauge potentials and there exists $g \in W^{2,2}(B^4, G)$ such that $A = B^g$, then they have the same Yang-Mills energy.

Proposition 4.1.4. *Let $A \in W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g})$ and $g \in W^{2,2}(B^4, G)$. Then*

$$YM(A^g) = YM(A) \tag{4.6}$$

Proof. Observe that if A is a smooth connection and g is a smooth gauge then by Theorem 2.4.7, we have that $F_{A^g} = g^{-1}F_A g$. Since the norm induced by the killing form in $\mathfrak{g}$ is Ad-invariant, then

$$|F^{ij}_{A^g}| = |F^{ij}_A| \text{ in } B^4$$

which clearly implies the identity $YM(A^g) = YM(A)$. For Sobolev connections we use a density argument. We consider a sequence of smooth connections $A_n \in C^\infty(B^4, T^*B^4 \otimes \mathfrak{g})$ converging in $W^{1,2}$ to A, and a sequence of smooth gauges $g_n \in C^\infty(B^4, G)$ converging in $W^{2,2}$ to g[1]. As we have already argue

$$|F_{A_n^{g_n}}| = |F_{A_n}| \ \ \forall x \in B^4 \tag{4.7}$$

and by Proposition 3.2.7 $A_n^{g_n} \to A^g$ in $W^{1,2}$, which implies thanks to Proposition 4.1.2 that

$$YM(A^g) = YM(A).$$

$\square$

This huge group of invariances makes the functional non coercive[2], as the following easy example shows.

[1]Note that generally if we have two compact manifolds M and N then $C^\infty(M, N)$ is not necessarily dense in $W^{k,p}(M, N)$ for any k, p. It has been proved that this is true for $kp = dim(M)$. See [37],[38]

[2]Let V be a normed space, we say that $F : V \to \mathbb{R}$ is coercive if for every $\{x_n\}_n \subset V$ such that $\|x_n\| \to \infty$ then $|F(x_n)| \to \infty$ too.

Example 4.1.5. Fix any $A \in W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g})$ and a sequence $g_n \in W^{2,2}(B^4, G)$ such that $\|dg_n\|_{L^2} \to \infty$. Then clearly $YM(A^{g_n}) = YM(A)$ is constant and therefore finite, but $\|A_n^{g_n}\|_{W^{1,2}} \geq \|A^{g_n}\|_{L^2} \geq \|g_n^{-1}dg_n\|_{L^2} - \|g_n^{-1}Ag_n\|_{L^2} \geq m\|dg_n\|_{L^2} - \|A\|_{L^2} \to \infty$, where $m = \min_{g\in G}|g|$.

In particular this means that minimizing sequences are not automatically bounded, and therefore not necessarily weakly converging. The first result of K.Uhlenbeck we will present deals with this problem when the connections have small enough Yang-Mills energy.
Similarly to the classical Plateau problem, in which we fix some curve Γ and try to find a map $u : D^2 \to \mathbb{R}$ such that $Im(u|_{\partial D^2}) = \Gamma$ and whose graph has minimum surface measure, we build the **Yang-Mills plateau problem**. Let $\eta \in H^{\frac{1}{2}}(\partial B^4, T^*\partial B^4 \otimes \mathfrak{g})$ be fixed, then we want to find some
$A \in W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g})$ solution of the following minimization problem:

$$\inf \left\{ YM(A) : A \in W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g}) \text{ and } A_T = \eta \right\} \tag{4.8}$$

If $\eta \in H^{\frac{1}{2}}(\partial B^4, T^*\partial B^4 \otimes \mathfrak{g})$, then we will denote
$W^{1,2}_\eta(B^4, T^*B^4 \otimes \mathfrak{g}) := \{A \in W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g}) : \ A_T = \eta\}$.
Remark 4.1.6. At the beginning of this section we have given an interpretation of the Yang-Mills energy of a connection as the L^2 measure of lack of integrability of the associated distribution. Similarly if we fix the boundary form η, which can be seen as a Sobolev connection on the boundary, one can consider the minimization problem as the attempt to find a gauge potential in B^4 that extends the boundary potential, and at the same time has minimum lack of integrability[3]

4.2 Abelian gauge group: $U(1)$

We aim to solve the minimization problem in the case $G = U(1)$. The Yang-Mills functional reads as

$$YM(A) = \int_{B^4} |dA|^2 dx \tag{4.9}$$

since $U(1)$ is an abelian group and then $[A, A] = 0$ for each
$A \in W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g})$. The request thus is to see if there exists a

[3]As a smooth connection defines a horizontal distribution, so does a Sobolev connection, with the difference that the distribution is not more smooth but Sobolev in some sense. One can find a detalied definition of Sobolev distributions in [39]

minimum of

$$\left\{\int_{B^4} |dA|^2 dx : A \in W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g}) \quad \text{and} \quad A_T = \eta\right\} \tag{4.10}$$

where η is some given 1-form in $H^{\frac{1}{2}}(\partial B^4, T^*\partial B^4 \otimes \mathfrak{g})$.

Proposition 4.2.1. *For each $B \in W^{1,2}_\eta(B^4, T^*B^4 \otimes \mathfrak{g})$ there exists a gauge $g \in W^{2,2}(B^4, G)$ such that $d^\star B^g = 0$ and at the boundary $B^g_T = \eta$.*

Proof. Indeed, let $\varphi \in W^{1,2}_0(B^4, \mathbb{R})$ be the unique solution of

$$\begin{cases} -i\Delta\varphi = d^\star B & \text{in } D'(B^4) \\ \varphi|_{\partial B^4} = 0 \end{cases} \tag{4.11}$$

which is in $W^{2,2}(B^4, \mathbb{R})$, for the L^2-regularity theory for elliptic systems (see Theorem 4.14 [14]). Note that we have multiplied $\Delta\varphi$ by i because if $G = U(1)$, then $\mathfrak{g} = i\mathbb{R}$. Hence we have that

$$\begin{cases} d(B + id\varphi) = dB & \text{in } D'(B^4) \\ d^\star(B + id\varphi) = 0 & \text{in } D'(B^4) \\ (B + id\varphi)_T = \eta \end{cases} \tag{4.12}$$

Taking $g := \exp(i\varphi)$ we have that $B^g = B + id\varphi$, and thanks to (4.12), (4.11) we get $d^\star B^g = 0$. $\square$

In order to find a minimizer of (4.10) we will need to introduce a backup functional, reminiscent in some way to the Classical Plateau Problem (see the subsection below). Let us define the **energy functional**

$$E : W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g}) \to \mathbb{R}$$

$$A \longmapsto E(A) := \int_{B^4} \left(|dA|^2 + |d^\star A|^2\right) dx$$

Remark 4.2.2. If $\eta \in H^{\frac{1}{2}}(\partial B^4, T^*\partial B^4 \otimes \mathfrak{g})$, then the energy functional E is strictly convex in $W^{1,2}_\eta(B^4, T^*B^4 \otimes \mathfrak{g})$. Indeed, if $A, B \in W^{1,2}_\eta(B^4, T^*B^4 \otimes \mathfrak{g})$ then for each $h \in (0,1)$

$$E(hA+(1-h)B) = \int_{B^4} \left(|hdA + (1-h)dB|^2 + |hd^*A + (1-h)d^*B|^2\right) dx \tag{4.13}$$

the square norm $|\cdot|^2$ is strictly convex therefore the last term is bounded by

$$hE(A) + (1-h)E(B)$$

and the equivalence holds true if and only if $dA = dB$ and $d^*A = d^*B$ almost everywhere in B^4. Therefore if we call $\tilde{B} := A - B$ we have that $\tilde{B}$ is a harmonic field with $\tilde{B}_T = 0$. In Theorem 3.1.24 we have proved that the space of harmonic fields with vanishing tangential component over a contractible domain is $\{0\}$. The identity in (4.13) then holds if and only if $A = B$, and we conclude.

Lemma 4.2.3. *Fix $\eta \in H^{\frac{1}{2}}(\partial B^4, T^*\partial B^4 \otimes \mathfrak{g})$. Then E admits a unique minimizer $\tilde{A}$ in $W^{1,2}_\eta(B^4, T^*B^4 \otimes \mathfrak{g})$ which satisfies*

$$\begin{cases} d^\star d\tilde{A} = 0 & in \;\; D'(B^4) \\ d^\star \tilde{A} = 0 & in \;\; D'(B^4) \\ \tilde{A}_T = \eta \end{cases} \tag{4.14}$$

where the first two equations are the Euler-Lagrange of E.

Proof. Let $A \in W^{1,2}_\eta(B^4, T^*B^4 \otimes \mathfrak{g})$, we want to establish

$$\|A\|^2_{W^{1,2}} \leq C\left[E(A) + \|\eta\|^2_{H^{\frac{1}{2}}(\partial B^4)}\right] \tag{4.15}$$

To this purpose we apply Gaffney's inequality (3.22) which implies that

$$\|A\|_{W^{1,2}(B^4)} \leq C\left(\|dA\|_{L^2(B^4)} + \|d^\star A\|_{L^2(B^4)} + \|\nu \wedge A\|_{H^{\frac{1}{2}}(\partial B^4)}\right), \tag{4.16}$$

where ν is the outer unit normal of ∂B^4. Since on the boundary $A = A_N + A_T$, by Proposition 3.1.16 we get

$$\nu \wedge A = \nu \wedge A_T \tag{4.17}$$

and this gives

$$\|\nu \wedge A\|_{H^{\frac{1}{2}}} = \|\nu \wedge A_T\|_{H^{\frac{1}{2}}} \leq C\|A_T\|_{H^{\frac{1}{2}}} = C\|\eta\|_{H^{\frac{1}{2}}(\partial B^4)}$$

Then (4.16) implies that

$$\|A\|_{W^{1,2}(B^4)} \leq C\left(\|dA\|^2_{L^2} + \|d^\star A\|^2_{L^2} + \|\eta\|^2_{H^{\frac{1}{2}}(\partial B^4)}\right)^{\frac{1}{2}}$$

and squaring we obtain (4.15). Therefore if A_n is a minimizing sequence in $W^{1,2}_\eta(B^4, T^*B^4 \otimes \mathfrak{g})$, we have that it is bounded. By reflexivity the sequence admits a subsequence A_{n_k} weakly converging to some $\tilde{A} \in W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g})$. By Corollary 3.1.17 we have that the trace operator for the tangential component $T : W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g}) \to H^{\frac{1}{2}}(\partial B^4, T^*\partial B^4 \otimes \mathfrak{g})$ such that $T(A) = A_T$ is continuous, and since it is linear it is also weakly continuous. Therefore $\tilde{A}_T = \eta$. It is easy to see that E is continuous with respect to the strong topology, and thus it is also lower semicontinuous. But a convex functional which is lower semicontinuous for the strong topology is automatically lower semicontinuous for the weak one, as a consequence of Mazur's Corollary [8], and therefore

$$E(\tilde{A}) \leq \liminf_k E(A_{n_k})$$

which means that $\tilde{A}$ is a minimum. By Remark 4.2.2 we have also that $\tilde{A}$ is the unique minimum. Let us compute the Euler-Lagrange equations of E. Let $\phi \in C_c^\infty(B^4, \mathfrak{g})$ then since $\tilde{A}$ is a minimum

$$0 = \frac{d}{dt} E(\tilde{A} + td\phi)\big|_{t=0} = \int_{B^4} \langle d(d\phi), d\tilde{A}\rangle + \langle d^\star d\phi, d^\star \tilde{A}\rangle dx \quad (4.18)$$

and the first term inside the integral is trivially equal to zero. As far as the second is concerned observe that since ϕ is a zero-form, then $\Delta\phi = d^\star d\phi$. But since for every $f \in C_c^\infty(B^4, \mathfrak{g})$ there exists ϕ such that $\Delta\phi = f$, equation (4.18) reads as

$$\int_{B^4} \langle f, d^\star \tilde{A}\rangle dx = 0 \quad \forall f \in C_c^\infty(B^4, \mathfrak{g})$$

which implies $d^\star \tilde{A} = 0$. If now instead of $d\phi$ in (4.18) we choose any $\psi \in C_c^\infty(B^4, T^*B^4 \otimes \mathfrak{g})$ then we obtain

$$0 = \int_{B^4} \langle d\psi, d\tilde{A}\rangle dx = -\int_{B^4} \langle \psi, d^\star d\tilde{A}\rangle dx$$

where we have used Theorem 3.1.21 in addition to the fact that ψ has compact support in B^4, and so we get $d^\star d\tilde{A} = 0$. $\square$

Theorem 4.2.4. *For any $\eta \in H^{\frac{1}{2}}(\partial B^4, T^*\partial B^4 \otimes \mathfrak{g})$ there exists a minimizer $\tilde{A} \in C^\infty(B^4, T^*B^4 \otimes \mathfrak{g})$ of (4.10).*

Proof. Since YM is not coercive then it could be that a minimizing sequence does not admit a subsequence weakly converging. Anyway, we notice that

$$YM(A) \leq E(A) \quad \forall A \in W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g})$$

with equality if and only if $d^\star A = 0$.
Thanks to Lemma 4.2.3 we know that E admits a unique minimizer $\tilde{A} \in W_\eta^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g})$ which is a solution of the Euler-Lagrange equations (4.14). By the first two equations we deduce

that $\tilde{A}$ has harmonic components, and therefore it is smooth. Moreover $YM(\tilde{A}) = E(\tilde{A})$. We claim that $\tilde{A}$ is a minimizer also for YM. Indeed by contradiction suppose
$B \in W^{1,2}_{\eta}(B^4, T^*B^4 \otimes \mathfrak{g})$ and $YM(B) < YM(\tilde{A})$. By Proposition 4.2.1 we can find a gauge $g \in W^{2,2}(B^4, G)$ such that $YM(B) = YM(B^g) = E(B^g)$ and therefore this leads to $E(B^g) = YM(B^g) = YM(B) < YM(\tilde{A}) \leq E(\tilde{A})$ which is a contradiction, since $\tilde{A}$ was the unique minimizer for E. $\square$

4.2.1 The $U(1)$ Yang-Mills Plateau Problem & The Classical Plateau Problem

The Plateau problem for the Yang-Mills functional when $G = U(1)$ is very similar to the classical Plateau problem, and the idea used to solve it displays some analogy.
In the classical Plateau problem we fix a Jordan oriented curve Γ in $\mathbb{R}^n$ and search for a map $u : D^2 \to \mathbb{R}^n$ whose trace is an oriented parametrization of Γ and whose surface area measure is minimum. Therefore the functional we need to study is

$$\mathcal{A}(u) = \int_{D^2} \left| \frac{\partial u}{\partial x} \wedge \frac{\partial u}{\partial y} \right| dxdy \tag{4.19}$$

and we take u in the following set

$$C(\Gamma) = \{u \in W^{1,2}(D^2, \mathbb{R}^n) : u|_{\partial D^2} \in C^0(\mathbb{S}^1, \mathbb{R}^n) \text{ is an oriented parametrization of } \Gamma\} \tag{4.20}$$

It is easy to see that the group of invariances for $\mathcal{A}$ is the group of diffeomorphisms of the disc, and for this reason the functional is easily showed to be non coercive. The problem is treated introducing the Dirichlet functional

$$\mathcal{A}(u) \leq \mathcal{D}(u) := \frac{1}{2} \int_{D^2} |\nabla u|^2 dx \tag{4.21}$$

which is invariant for a smaller group, but anyway non compact, namely the conformal diffeomorphisms of D^2. The two functionals above coincide if and only if u is weakly conformal.[4] By Morrey-Lichtenstein Theorem, we have that

$$\inf_{u \in C(\Gamma)} \mathcal{A}(u) = \inf_{u \in C(\Gamma)} \mathcal{D}(u) \tag{4.22}$$

[4] a map u is weakly conformal if $\left|\frac{\partial u}{\partial x}\right| = \left|\frac{\partial u}{\partial y}\right|$ and $\frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} = 0$

This result would have been trivial if for every u, a diffeomorphism η of D^2 had existed such that $u \circ \eta$ is weakly conformal. Therefore now the problem is to minimize $\mathcal{D}$. Note that the study of $\mathcal{D}$ would have been easier if we had prescribed the trace of u. Indeed, in this last case the existence of the minimum is a well known result, see for instance [14]. However by slightly restricting $C(\Gamma)$ by the so called *three points condition*, we can achieve the existence of a minimum for $\mathcal{D}$ and thus for $\mathcal{A}$.

Now that we have briefly recalled the Classical Plateau Problem, let us spot the similarities and differences with the Abelian Yang-Mills one.
As we have already seen the Yang-Mills functional has a big group of invariances too that prevents YM from being coercive. The strategy we applied to overcome this lack of regularity was based on the introduction of the backup functional E, which bounds from above YM, similarly to what is done in (4.21) for $\mathcal{A}$ with $\mathcal{D}$ in the Classical Plateau Problem. We showed that

$$\inf_{A \in W^{1,2}_\eta} YM(A) = \inf_{A \in W^{1,2}_\eta} E(A) \tag{4.23}$$

and we actually got this result thanks to Proposition 4.2.1, that assures the existence of a Coulomb gauge for every $A \in W^{1,2}_\eta(B^4, T^*B^4 \otimes \mathfrak{g})$, that preserves the tangential component. Note that here coulomb gauges play the same role of weakly conformal parametrizations, that make the functionals $\mathcal{A}$ and $\mathcal{D}$ coincide. However remember that the existence of a weakly conformal parametrization for every u is not true, and we got (4.22) thanks to Morrey-Lichtenstein Theorem.

4.3 Non Abelian Gauge Group

We now consider the minimization problem when G is a non abelian Lie Group. In this case due to the non vanishing term $[A, A]$ in F_A we cannot be sure anymore that $YM(A) \leq E(A)$ for every $A \in W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g})$. Also in this case one can prove the existence of a Coulomb gauge, but this does not mean necessarily that for such a gauge the two functionals coincide. So we cannot discuss the existence of the minimizer with the same method of $G = U(1)$. Since, as we already discussed, YM is non coercive, classical methods for the minimization do not work. In 1982 K.Uhlenbeck proved in [44] that every $W^{1,2}$ connection A with a good enough bound on the L^2 norm of its curvature F_A admits a Coulomb gauge $g \in W^{2,2}$, such that A^g has $W^{1,2}$-norm controlled by its Yang-Mills energy . This fundamental result tells us that from a sequence of connections with a suitable L^2 bound on their curvature, we can build a sequence of gauge equivalent connections that is weakly compact in $W^{1,2}$.

Theorem 4.3.1 (Small Energy Theorem, [44])**.** *Let G be a compact and connected matrix Lie group. There exists two positive constants ε_G and C_G, both of them depending on G, such that for each $A \in W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g})$ satisfying*

$$\int_{B^4} |F_A|^2 dx < \varepsilon_G \tag{4.24}$$

there exists $g \in W^{2,2}(B^4, G)$ such that:

$$\begin{cases} \|A^g\|_{L^4(B^4)} + \|DA^g\|_{L^2(B^4)} \leq C_G \|F_A\|_{L^2(B^4)} \\ d^\star A^g = 0 \quad in \;\; B^4 \\ (A^g)_N = 0 \end{cases} \tag{4.25}$$

Proof. We will first prove the result for $2 < p < 4$, and then extend it to the case $p = 2$ with a density argument. In the third step of the theorem it will be clear why we need to do so. Fix $2 < p < 4$, and for $\varepsilon > 0$ we introduce the following two sets:

$$U^\varepsilon := \left\{ A \in W^{1,p}(B^4, T^*B^4 \otimes \mathfrak{g}) \mid YM(A) < \varepsilon \right\}$$

$$V^\varepsilon_{\tilde{C}} := \Bigg\{ A \in U^\varepsilon \Big| \;\; \exists g \in W^{2,p}(B^4, G) \text{ s.t. } \|dA^g\|^q_{L^q(B^4)} \leq \tilde{C} \|F_A\|^q_{L^q(B^4)}$$

$$q = 2, p \;\text{ and }\; d^\star A^g = 0, \quad (A^g)_N = 0 \Bigg\}$$

We are going to prove that for a suitable ε and $\tilde{C}$ the previous two sets coincide, and this will prove the theorem for $2 < p < 4$. In order to accomplish this, we will prove the following three steps:

1) U^ε is path connected (and therefore topologically connected)

2) $V^\varepsilon_{\tilde{C}}$ is closed in U^ε with respect to the $W^{1,p}$ topology

3) $V^\varepsilon_{\tilde{C}}$ is open in U^ε with respect to the $W^{1,p}$ topology.

First step.

Since the zero connection is in U^ε we can establish path connectedness by proving that for every $A \in U^\varepsilon$ there exists a continuous map $[0, 1] \ni t \mapsto A_t \in U^\varepsilon$ such that $A_0 = 0$ and $A_1 = A$. So let us define $A_t \in W^{1,p}(B^4, T^*B^4 \otimes \mathfrak{g})$ as follows

$$A_t(x) := tA(tx) \tag{4.26}$$

Let us prove that $A_t \in U^\varepsilon$ for every $t \in [0,1]$.
Observe that $F_{A_t}(x) = t^2 F_A(tx)$, as the following calculation shows

$$F_{A_t}(x) = dA_t(x) + [A_t(x), A_t(x)] = t^2 dA(tx) +$$
$$+ t^2[A(tx), A(tx)] = t^2 F_A(tx)$$

We set $B_t^4 = \{tx : \ x \in B^4\}$ and we see that $YM(A_t) < \varepsilon$

$$\int_{B^4} |F_{A_t}(x)|^2 dx = \int_{B^4} t^4 |F_A(tx)|^2 dx = \int_{B_t^4} |F_A(x)|^2 dx < \varepsilon$$

therefore $A_t \in U_\varepsilon$ for each $t \in [0,1]$. Now it is left to prove that the path is continuous. By the embedding $L^{p^*}(B^4, T^*B^4 \otimes \mathfrak{g}) \hookrightarrow L^p(B^4, T^*B^4 \otimes \mathfrak{g})$ we get

$$\|A_t\|_{L^p(B^4)} \leq C\|A_t\|_{L^{p^*}(B^4)} = Ct^{1-\frac{4}{p^*}}\|A\|_{L^{p^*}(B_t^4)} \leq Ct^{1-\frac{4}{p^*}}\|A\|_{L^{p^*}(B^4)} \tag{4.27}$$

where p^* is the Sobolev conjugate of p; and similarly we also have

$$\|DA_t\|_{L^p(B^4)} = t^{2-\frac{4}{p}}\|DA\|_{L^p(B_t^4)} \leq t^{2-\frac{4}{p}}\|DA\|_{L^p(B^4)} \tag{4.28}$$

By the last two inequalities we deduce that $A_t \to 0$ in $W^{1,p}(B^4, T^*B^4 \otimes \mathfrak{g})$ for $t \to 0$. As a straightforward consequence of (4.27) and (4.28) we also get the path continuity. Thus we have proved that U^ε is path connected.

Second step
Now we prove that $V^\varepsilon_{\tilde{C}}$ is closed in U^ε for the $W^{1,p}$ topology. Consider a sequence $\{A_k\}_k$ in $V^\varepsilon_{\tilde{C}}$ and assume that A_k converges strongly in $W^{1,p}(B^4, T^*B^4 \otimes \mathfrak{g})$ to some limit $A_\infty \in U^\varepsilon$. We want to prove that $A_\infty \in V^\varepsilon_{\tilde{C}}$. By definition of $V^\varepsilon_{\tilde{C}}$ there exists a sequence $\{g_k\}_k \subset W^{2,p}(B^4, G)$ such that the followings hold

$$\int_{B^4} |dA_k^{g_k}|^q dx \leq \tilde{C} \int_{B^4} |F_{A_k}|^q dx \quad \text{ for } \ q = 2, p \tag{4.29}$$

$$d^\star A_k^{g_k} = 0 \qquad (A_k^{g_k})_N = 0 \tag{4.30}$$

As a consequence of Proposition 4.1.2 we have that the right hand side of (4.29) is uniformly bounded and then $\|dA_k^{g_k}\|_{L^p(B^4)}$ is uniformly bounded too. Gaffney's inequality (3.23) applied to $A_k^{g_k}$ gives

$$\|A_k^{g_k}\|_{W^{1,p}(B^4)} \leq C\left(\|dA_k^{g_k}\|_{L^p(B^4)} + \|d^\star A_k^{g_k}\|_{L^p(B^4)} + \|\nu \lrcorner A_k^{g_k}\|_{W^{1-\frac{1}{p},p}(\partial B^4)}\right) =$$
$$= C\|dA_k^{g_k}\|_{L^p(B^4)}$$

where the last equality is given by (4.30) together with the fact that $0 = (A_k^{g_k})_N = \nu(\nu \lrcorner A_k^{g_k})$ by equation (3.12).
From this last result we get also the boundedness of dg_k with respect to the $W^{1,p}$-norm. Indeed $A_k^{g_k} = g_k^{-1} dg_k + g_k^{-1} A_k g_k$, and this relation can be rewritten as

$$dg_k = g_k A_k^{g_k} - A_k g_k \tag{4.31}$$

By definition of norm we have that

$$\|dg_k\|_{W^{1,p}(B^4)} = \|dg_k\|_{L^p(B^4)} + \|D^2 g_k\|_{L^p(B^4)} \tag{4.32}$$

and using (4.31) we see that the first term in the right hand side of (4.32) is bounded

$$\|dg_k\|_{L^p(B^4)} \leq C \left(\|A_k^{g_k}\|_{L^p(B^4)} + \|A_k\|_{L^p(B^4)} \right) < C \tag{4.33}$$

since both A_k and $A_k^{g_k}$ are bounded in $W^{1,p}(B^4, T^*B^4 \otimes \mathfrak{g})$ by hypothesis. The other term instead can be estimated observing that for $i, j = 1, ..., 4$

$$\frac{\partial^2 g_k}{\partial_{x_i} \partial_{x_j}} = \partial_{x_i} g_k (A_k^{g_k})_j + g_k \partial_{x_i} (A_k^{g_k})_j - \partial_{x_i} (A_k)_j g_k - (A_k)_j \partial_{x_i} g_k \tag{4.34}$$

The estimates (4.33) together with Hölder's inequality and the embedding (4.5) lead to the boundedness of $D^2 g_k$ in $L^p(B^4)$. Now since g_k is bounded in $W^{2,p}(B^4, G)$ and this last space embeds compactly in $C^0(\overline{B^4}, G)$, then there exists a subsequence which converges strongly to some $g_\infty \in C^0(\overline{B^4}, G)$. Furthermore $W^{2,p}(B^4, \mathbb{R}^{m^2})$ is a reflexive space and so $g_{k'} \rightharpoonup g_\infty$ weakly in $W^{2,p}(B^4, \mathbb{R}^{m^2})$. The weak limit coincides with the limit in $C^0(\overline{B^4}, G)$ because of the compact embedding of $W^{2,p}(B^4, G) \hookrightarrow C^0(\overline{B^4}, G)$. Thus we deduce the following weak convergence in $W^{1,p}(B^4, T^*B^4 \otimes \mathfrak{g})$

$$g_{k'}^{-1} dg_{k'} + g_{k'}^{-1} A_{k'} g_{k'} \rightharpoonup g_\infty^{-1} dg_\infty + g_\infty^{-1} A_\infty g_\infty \tag{4.35}$$

So we have found that $A_{k'}^{g_{k'}} \rightharpoonup A_\infty^{g_\infty}$ weakly in $W^{1,p}(B^4, T^*B^4 \otimes \mathfrak{g})$. The $W^{1,p}$-norm is weakly lower semicontiuous, therefore

$$\begin{aligned} \|A_\infty^{g_\infty}\|_{W^{1,p}(B^4)} &\leq \liminf_{k'} \|A_{k'}^{g_{k'}}\|_{W^{1,p}(B^4)} \leq \\ &\leq \tilde{C} \liminf_{k'} \|F_{A_{k'}}\|_{L^p(B^4)} = \tilde{C} \|F_{A_\infty}\|_{L^p(B^4)} \end{aligned} \tag{4.36}$$

where the second inequality is given by (4.29), while the equality holds since $F_{A_{k'}} \to F_A$ in $L^p(B^4, \wedge^2 T^*B^4 \otimes \mathfrak{g})$. Notice that the

above inequality holds true also for $p = 2$.
The codifferential $d^\star : W^{1,p}(B^4, T^*B^4 \otimes \mathfrak{g}) \to L^p(B^4, \mathfrak{g})$ is linear and continuous with respect to the strong topology, and therefore it is linear and continuous also if in both spaces we consider the weak topology. This leads us to $0 = d^\star A_{k'}^{g_{k'}} \rightharpoonup d^\star A_\infty^{g_\infty}$ weakly in $L^p(B^4, \mathfrak{g})$. Corollary 3.1.17 implies also that $0 = (A_{k'}^{g_{k'}})_N \rightharpoonup (A_\infty^{g_\infty})_N$ weakly in $W^{1-\frac{1}{p},p}(\partial B^4, T^*B^4|_{\partial B^4} \otimes \mathfrak{g})$ which finally means $A_\infty \in V^\varepsilon_{\tilde{C}}$.
Third step
$V^\varepsilon_{\tilde{C}}$ is open in U_ε with respect to the $W^{1,p}$-topology if for every $A \in V^\varepsilon_{\tilde{C}}$ we can find an open neighbourhood V of A in $W^{1,p}$ such that $V \subset V^\varepsilon_{\tilde{C}}$. Fix then $A \in V^\varepsilon_{\tilde{C}}$, and consider the gauge $g \in W^{2,p}(B^4, G)$, associated to A, given in the definition of $V^\varepsilon_{\tilde{C}}$. Then we have that the following map is continuous

$$g : W^{1,p}(B^4, T^*B^4 \otimes \mathfrak{g}) \to W^{1,p}(B^4, T^*B^4 \otimes \mathfrak{g})$$
$$B \mapsto B^g := g^{-1}dg + g^{-1}Bg$$

and clearly $g(V_{\tilde{C}^\varepsilon}) = V_{\tilde{C}^\varepsilon}$[5]. Then if for $\delta > 0$ small enough the neighbourhood $V_\delta := \{A^g + \omega : \ \|\omega\|_{W^{1,p}(B^4)} < \delta\}$ of A^g is such that $V_\delta \subset V^\varepsilon_{\tilde{C}}$, we have that $g^{-1}(V_\delta)$ is a neighbourhood of A and $g^{-1}(V_\delta) \subset V^\varepsilon_{\tilde{C}}$.
So we can assume from the beginning that $d^\star A = 0$ and $A_N = 0$, and furthermore that for $q = 2, p$ it holds $\|dA\|^q_{L^q(B^4)} \leq \tilde{C}\|F_A\|^q_{L^q(B^4)}$, where now we have fixed $\tilde{C} > 1$. We are looking for the existence of $\delta > 0$ sufficiently small such that $(A + \omega) \in V^\varepsilon_{\tilde{C}}$ for each $\omega \in W^{1,p}(B^4, T^*B^4 \otimes \mathfrak{g})$ with $\|\omega\|_{W^{1,p}(B^4)} < \delta$. The requirement for $A + \omega$ to be in $V^\varepsilon_{\tilde{C}}$ reads as: there exists $g \in W^{2,p}(B^4, G)$ such that

$$\begin{cases} d^\star(A+\omega)^g = 0 \\ (A+\omega)^g_N = 0 \\ \|d(A+\omega)^g\|^q_{L^q(B^4)} \leq \tilde{C}\|F_{A+\omega}\|^q_{L^q(B^4)} \qquad q = 2, p \end{cases} \tag{4.37}$$

In order to prove the first two equations of (4.37) we introduce the following C^1 map

$$\mathcal{N} : W^{1,p}(B^4, T^*B^4 \otimes \mathfrak{g}) \times W^{2,p}(B^4, \mathfrak{g}) \to L^p(B^4, \mathfrak{g}) \times W^{1-\frac{1}{p},p}(\partial B^4, T^*B^4|_{\partial B^4} \otimes \mathfrak{g})$$
$$(\omega, U) \mapsto (d^\star(A+\omega)^{g_U}, (A+\omega)^{g_U}_N) \tag{4.38}$$

where $g_U := \exp(U) \in W^{2,p}(B^4, G)$, see Lemma 4.3.3. If we prove that the Fréchet derivative of the map in $(0, 0)$ with respect to the second variable is an isomorphism, then we can apply the implicit

[5] If $E \subset W^{1,p}(B^4, T^*B^4 \otimes \mathfrak{g})$ then $g(E) := \{B^g : \ B \in E\}$

function theorem (see [24]), and since by hypothesis $\mathcal{N}(0,0) = (0,0)$, we will find that for each ω, in a proper neighbourhood of $0 \in W^{1,p}(B^4, T^*B^4 \otimes \mathfrak{g})$, there exists a gauge g such that the first two equations of (4.37) hold.
By Lemma 4.3.3 the derivative of $\mathcal{N}$ along the U direction at $(0,0)$ is

$$\begin{aligned} \partial_U \mathcal{N}(0,0) : W^{2,p}(B^4, \mathfrak{g}) &\longrightarrow H \subset L^p(B^4, \mathfrak{g}) \times W^{1-\frac{1}{p},p}(\partial B^4, T^*B^4|_{\partial B^4} \otimes \mathfrak{g}) \\ V &\longmapsto \partial_U \mathcal{N}(0,0) \cdot V = (\Delta V + (A, dV), (dV)_N) \end{aligned} \tag{4.39}$$

where H is the hyperplane of $L^p(B^4, \mathfrak{g}) \times W^{1-\frac{1}{p},p}(\partial B^4, T^*B^4|_{\partial B^4} \otimes \mathfrak{g})$ made of couples (f, g) such that

$$\int_{B^4} f(x) dx = \int_{\partial B^4} \nu \lrcorner g d\sigma^3 \tag{4.40}$$

We can establish (4.40) applying Green's identity, since (A, dV)[6] has null integral over B^4, as a straightforward consequence of the integration by parts formula for 1-forms in (3.1.21), remembering that 0-forms have zero normal component.
Applying Gaffney's inequality (3.23) to dV we have the following a priori estimate for any $V \in W^{2,p}(B^4, \mathfrak{g})$:

$$\|dV\|_{W^{1,p}(B^4)} \leq C \left(\|d^\star dV\|_{L^p(B^4)} + \|\langle \nu, dV \rangle\|_{W^{1-\frac{1}{p},p}(\partial B^4)} \right) \tag{4.41}$$

where observe that for a zero form as V holds $\Delta V = d^\star dV$, and moreover the product $\langle \nu, dV \rangle$ coincides with $\partial_r V$. Furthermore if we ask that

$$V \in \left\{ U \in W^{2,p}(B^4, \mathfrak{g}) : \int_{B^4} U = 0 \right\} := E$$

one can apply Poincaré inequality to V, $\|V\|_{L^p(B^4)} \leq C \|dV\|_{L^p(B^4)}$. This last inequality and (4.41) leads to:

$$\begin{aligned} \|V\|_{W^{2,p}(B^4)} &\leq C \left(\|\Delta V\|_{L^p(B^4)} + \|\partial_r V\|_{W^{1-\frac{1}{p},p}(\partial B^4)} \right) \leq \\ &\leq C \left(\|\partial_U \mathcal{N}(0,0) \cdot V\|_H + \|(A, dV)\|_{L^p(B^4)} \right) \leq \\ &\leq C \left(\|\partial_U \mathcal{N}(0,0) \cdot V\|_H + C \|A\|_{L^4(B^4)} \|dV\|_{L^{p^*}(B^4)} \right) \end{aligned} \tag{4.42}$$

Using Gaffney's inequality and the given hypothesis on $A \in V_{\tilde{C}}^{\varepsilon}$ we have

$$\|A\|_{W^{1,2}(B^4)} \leq C \|dA\|_{L^2(B^4)} \leq C \tilde{C}^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} \tag{4.43}$$

[6] If α and β are two $\mathfrak{g}$-valued 1-forms, then we denote $(\alpha, \beta) = \alpha \cdot \beta - \beta \cdot \alpha$ where $\alpha \cdot \beta = \sum_i \alpha_i \beta_i$ and $\alpha_i \beta_i$ is the composition of the matrix α_i with the matrix β_i.

and the embedding $W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g}) \hookrightarrow L^4(B^4, T^*B^4 \otimes \mathfrak{g})$ clearly implies $\|A\|_{L^4(B^4)} \leq C\tilde{C}^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}$. Substituting this inequality in (4.42) we find

$$\|V\|_{W^{2,p}(B^4)} \leq C\left(\|\partial_U \mathcal{N}(0,0)\cdot V\|_H + C\tilde{C}^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}\|dV\|_{L^{p^*}}\right)$$

and by the Sobolev embedding $W^{1,p}(B^4, T^*B^4 \otimes \mathfrak{g}) \hookrightarrow L^{p^*}(B^4, T^*B^4 \otimes \mathfrak{g})$ we get

$$\left(1 - C\tilde{C}^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}\right)\|V\|_{W^{2,p}(B^4)} \leq C\|\partial_U \mathcal{N}(0,0)\cdot V\|_H \tag{4.44}$$

and for ε sufficiently small therefore $\partial_U \mathcal{N}(0,0)$ is injective in E. We know by elliptic theory that the following linear map is invertible

$$\begin{aligned} \mathcal{L}_0 : E &\to H \\ V &\mapsto (\Delta V, (dV)_N) \end{aligned} \tag{4.45}$$

and applying the method of continuity (see [15] Theorem 5.2) to the following family of linear and continuous maps

$$\begin{aligned} \mathcal{L}_t : E &\to H \\ V &\mapsto (\Delta V + t(A, dV), (dV)_N) \end{aligned}$$

with $t \in [0,1]$, we have that $\mathcal{L}_1 = \partial_U \mathcal{N}(0,0)$ is invertible too in E. Finally we can apply the implicit function theorem. So there exists a neighbourhood U_0 of $0 \in E$ and a $\delta > 0$ such that $\forall \omega \in W^{1,p}(B^4, T^*B^4 \otimes \mathfrak{g})$ satisfying $\|\omega\|_{W^{1,p}(B^4)} < \delta$ there exists and is unique $V_\omega \in U_0$ such that $\mathcal{N}(\omega, V_\omega) = 0$.
Now it is left only to check the third equation in (4.37), namely

$$\|d(A+\omega)^{g_\omega}\|^q_{L^q(B^4)} \leq \tilde{C}\|F_{A+\omega}\|^q_{L^q(B^4)} \tag{4.46}$$

for $q = 2, p$. To prove this last inequality, which concludes the proof of the third step, observe that

$$\|d(A+\omega)^{g_\omega}\|_{L^q(B^4)} \leq \|F_{A+\omega}\|_{L^q(B^4)} + \|[(A+\omega)^{g_\omega}, (A+\omega)^{g_\omega}]\|_{L^q(B^4)} \tag{4.47}$$

The last addendum on the right hand side of (4.47) can be bounded, thanks to Hölder's inequality, by

$$\underbrace{\|(A+\omega)^{g_\omega}\|_{L^4(B^4)}}_{(I)}\|(A+\omega)^{g_\omega}\|_{L^{q^*}(B^4)} \tag{4.48}$$

and the first multiplicand is such that

$$(I) \leq C(\|A\|_{L^4(B^4)} + \|\omega\|_{L^4(B^4)} + \|dg_\omega\|_{L^4(B^4)}) \leq$$

$$\leq C(\|dA\|_{L^2(B^4)} + \delta + \|dg_\omega\|_{L^4(B^4)}) \leq C(\varepsilon + \delta + \|dg_\omega\|_{L^4(B^4)})$$

where $\|dg_\omega\|_{L^4(B^4)} < C\delta$, since $g_\omega := \exp(V_\omega)$ with $V_\omega \in U_0$ and U_0 is small with δ. We already proved that $\|A\|_{L^4(B^4)} \leq C\|dA\|_{L^2(B^4)}$. Finally, since we have just showed that $d^\star(A+\omega)^{g_\omega} = 0$ and also $(A+\omega)_N^{g_\omega} = 0$, thanks to Gaffney's inequality (3.23) and the classical Sobolev embedding $W^{1,q}(B^4, T^*B^4 \otimes \mathfrak{g}) \hookrightarrow L^{q^*}(B^4, T^*B^4 \otimes \mathfrak{g})$

$$\|(A+\omega)^{g_\omega}\|_{L^{q^*}(B^4)} \leq C\|d(A+\omega)^{g_\omega}\|_{L^q(B^4)} \tag{4.49}$$

Therefore, these last estimates and equations (4.47) and (4.48) imply that

$$\|d(A+\omega)^{g_\omega}\|_{L^q(B^4)} \leq \frac{1}{1 - C(\varepsilon+\delta)}\|F_{A+\omega}\|_{L^q(B^4)}$$

and since for ε and δ small enough, $\frac{1}{1-C(\varepsilon+\delta)} < \tilde{C}^{\frac{1}{q}}$, we conclude.

End of the Proof Now that we have proved $V_{\tilde{C}}^\varepsilon = U^\varepsilon$ for ε and $\tilde{C}$ as above, we need to substitute $W^{1,p}$ with $W^{1,2}$.
So let the 1-form $A \in W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g})$ such that condition (4.24) holds. By density there exists $A_n \in C^\infty(\overline{B^4}, T^*B^4 \otimes \mathfrak{g})$ converging strongly to A in $W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g})$. Thus one obtain that $F_{A_n} \to F_A$ strongly in $L^2(B^4, \wedge^2 T^*B^4 \otimes \mathfrak{g})$ and therefore

$$\|F_{A_n}\|^2_{L^2(B^4)} < \varepsilon \qquad n \geq N \tag{4.50}$$

where N is taken large enough. Then $A_n \in V_{\tilde{C}}^\varepsilon$ and so there exists $g_n \in W^{2,p}(B^4, G)$ such that

$$\|dA_n^{g_n}\|^2_{L^2(B^4)} \leq \tilde{C}\|F_{A_n}\|^2_{L^2} \tag{4.51}$$

holds and furthermore $d^\star A_n^{g_n} = 0$ and $(A_n^{g_n})_N = 0$. As we have already seen, by Gaffney's inequality we find that $A_n^{g_n}$ is uniformly bounded in $W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g})$.
Repeating a reasoning not different from the one of equations (4.31)-(4.34) we find out that the sequence of gauges $\{g_n\}$ is bounded in $W^{2,2}(B^4, G)$, and therefore it weakly converges to a $g_0 \in W^{2,2}(B^4, \mathbb{R}^{m^2})$. By Rellich-Kondrakov theorem it also converges strongly to g_0 in $L^p(B^4)$ for each $1 \leq p < \infty$, and thus $g_n \to g_0$ a.e. which means $g_0 \in W^{2,2}(B^4, G)$. We easily find that

$$A_n^{g_n} \rightharpoonup A^{g_0} \ \text{ weakly in } \ W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g}) \tag{4.52}$$

As we have already observed, the map $d^\star : W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g}) \to L^2(B^4, \mathfrak{g})$ is a linear and continuous map for the norm topology, and so it is also for the weak one, i.e. if we substitute in both spaces the

strong topology with the weak one then the map $d^\star$ is still continuous. Thus by $d^\star A_n^{g_n} = 0$ we easily obtain $d^\star A^{g_0} = 0$.
As far the normal component is concerned, we know by Corollary 3.1.17 that $N : W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g}) \to H^{\frac{1}{2}}(\partial B^4, T^*B^4|_{\partial B^4} \otimes \mathfrak{g})$ is linear and continuous and then $0 = (A_n^{g_n})_N \rightharpoonup A_N^{g_0}$ weakly in $H^{\frac{1}{2}}(\partial B^4, T^*B^4|_{\partial B^4} \otimes \mathfrak{g})$.
Finally the lower semicontinuity of the L^2-norm together with this weak convergence leads us to:

$$\|dA^{g_0}\|^2_{L^2(B^4)} \leq \tilde{C} \|F_A\|^2_{L^2(B^4)}$$

That was the last step of the proof. $\square$

Remark 4.3.2. We have constructed the proof of Theorem 4.3.1, by first proving the result for $A \in W^{1,p}(B^4, T^*B^4 \otimes \mathfrak{g})$ with $2 < p < 4$, and then generalizing it when $A \in W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g})$ thanks to a density argument.
This choice was necessary since the map $\mathcal{N}$, introduced in the third step, is not C^1 when the gauges are not at least continuous. Thanks to the embedding $W^{2,p}(B^4, G) \hookrightarrow C^0(\overline{B}^4, G)$, valid for $2 < p < 4$, this was the case.

Lemma 4.3.3. *Let $B^4 \subset \mathbb{R}^4$ be the unit ball, and $2 < p < 4$. Then:*

1) *The map $W^{2,p}(B^4, \mathfrak{g}) \ni V \mapsto \exp(V) := \sum_{k=0}^{\infty} \frac{V^k}{k!}$ has image in $W^{2,p}(B^4, G)$. Moreover it is Fréchet differentiable.*

2) *The map $\mathcal{N}$ in (4.38) is C^1, and the derivative in $(0,0)$ with respect to the second direction is $\partial_U \mathcal{N}(0,0) \cdot V = (-\Delta V + (A, dV), (dV)_N)$ for $V \in W^{2,p}(B^4, \mathfrak{g})$.*

Proof. 1) $\exp(V)$ is the composition between $V : B^4 \to \mathfrak{g}$ and $\exp : \mathfrak{g} \to G$. By hypothesis $V \in W^{2,p}(B^4, \mathfrak{g}) \hookrightarrow C^0(\overline{B}^4, \mathfrak{g})$ and therefore it has bounded image. On the other hand $\exp \in C^\infty(\mathfrak{g}, G)$, and thus $\exp(V) \in W^{2,p}(B^4, G)$.
Now we focus on the Fréchet differentiability of exp. We fix $U \in W^{2,p}(B^4, \mathfrak{g})$ and we claim that the directional derivative of $\exp(U)$ in the V direction is

$$D_V \exp(U) = V + \sum_{k=2}^{\infty} \frac{U^{k-1}}{k!} \cdot V + V \cdot \sum_{k=2}^{\infty} \frac{U^{k-1}}{k!}$$

for any $V \in W^{2,p}(B^4, \mathfrak{g})$. We have to check that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|\exp(U+V) - \exp(U) - D_V \exp(U)\|_{W^{2,p}(B^4)} < \varepsilon \|V\|_{W^{2,p}(B^4)} \quad (4.53)$$

for any V with $\|V\|_{W^{2,p}(B^4)} < \delta$. We can rewrite the left hand side of (4.53) as:

$$\Big\| \sum_{k=0}^{\infty} \frac{(U+V)^k}{k!} - \sum_{k=0}^{\infty} \frac{U^k}{k!} - V - \sum_{k=2}^{\infty} \frac{U^{k-1}}{k!} \cdot V - V \cdot \sum_{k=2}^{\infty} \frac{U^{k-1}}{k!} \Big\|_{W^{2,p}(B^4)} =$$

$$= \Big\| \sum_{k=2}^{\infty} \frac{(U+V)^k}{k!} - \sum_{k=2}^{\infty} \frac{U^k}{k!} - \sum_{k=2}^{\infty} \frac{U^{k-1}}{k!} \cdot V - V \cdot \sum_{k=2}^{\infty} \frac{U^{k-1}}{k!} \Big\|_{W^{2,p}(B^4)}$$

The last equation is bounded from above by:

$$\sum_{k=2}^{\infty} \frac{1}{k!} \big\| (U+V)^k - U^k - U^{k-1}V - VU^{k-1} \big\|_{W^{2,p}(B^4)} = (I)$$

and if we expand $(U+V)^k$ we see that for any V such that $\|V\|_{W^{2,p}(B^4)} <$ δ:

$$(I) \leq CM\delta^p \|V\|_{W^{2,p}(B^4)}$$

where M is a constant depending on the $W^{2,p}$-norm of U and on δ. If we choose δ small enough then we obtain (4.53).
2)Now we prove the differentiability of

$$\mathcal{N} : W^{1,p}(B^4, T^*B^4 \otimes \mathfrak{g}) \times W^{2,p}(B^4, \mathfrak{g}) \to L^p(B^4, \mathfrak{g}) \times W^{1-\frac{1}{p},p}(\partial B^4, T^*B^4|_{\partial B^4} \otimes \mathfrak{g})$$

$$(\omega, U) \mapsto (d^\star (A+\omega)^{g_U}, (A+\omega)^{g_U}_N)$$

which is clearly a composition of maps, and $g_U := \exp(U)$. In particular the first component, call it $\mathcal{N}^{(1)}$, can be written as

$$\mathcal{N}^{(1)}(\omega, U) = d^\star \circ F^{(1)}(\omega, U) + d^\star \circ T^{(1)}(\omega, U)$$

with $d^\star : W^{1,p}(B^4, T^*B^4 \otimes \mathfrak{g}) \to L^p(B^4, \mathfrak{g})$, where we have defined

$$F^{(1)}(\omega, U) = g_U^{-1} dg_U \ \text{ and } \ T^{(1)}(\omega, U) := g_U^{-1}(A+\omega) g_U$$

Since $d^\star$ is linear and continuous, it is trivially Fréchet differentiable. Thus, in order to prove the differentiability of $\mathcal{N}^{(1)}$, we just have to check that

$$(\omega, U) \mapsto (A+\omega)^{g_U} = F^{(1)}(\omega, U) + T^{(1)}(\omega, U)$$

is smooth. Indeed composition of Fréchet differentiable functions is still Fréchet differentiable, see for instance [24]. We see that $F^{(1)}(\omega, U) = b(g_U^{-1}, dg_U) = b \circ (g_U^{-1}, d \circ g_U)$ where b is the bilinear form

$$b : W^{2,p}(B^4, \mathbb{R}^{m^2}) \times W^{1,p}(B^4, T^*B^4 \otimes \mathbb{R}^{m^2}) \to W^{1,p}(B^4, T^*B^4 \otimes \mathbb{R}^{m^2})$$

$$(A,B) \longmapsto b(A,B) := AB_i dx^i$$

with $B = \sum_{i=1}^4 B_i dx^i$. We have already proved in 1) that $W^{2,p}(B^4, \mathfrak{g}) \ni U \mapsto g_U \in W^{2,p}(B^4, G)$ is Fréchet differentiable, and therefore so is $d \circ g_U : W^{2,p}(B^4, \mathfrak{g}) \mapsto W^{1,p}(B^4, T^*B^4 \otimes TG)$ since, as already discussed, composition of Fréchet differentiable maps is Fréchet differentiable. Similarly also the map $W^{2,p}(B^4, \mathfrak{g}) \ni U \mapsto \exp(-U) = g_U^{-1} \in W^{2,p}(B^4, G)$ is Fréchet differentiable too. The bilinear map b is bounded

$$\|b(A,B)\|_{W^{1,p}(B^4)} \leq C\|A\|_{W^{2,p}(B^4)}\|B\|_{W^{1,p}(B^4)}$$

indeed

$$\|b(A,B)\|_{W^{1,p}(B^4)} := \sum_{i=1}^4 \|AB_i\|_{L^p(B^4)} + \sum_{i,j=1}^4 \|(\partial_{x_j}A)B_i + A\partial_{x_j}B_i\|_{L^p(B^4)} \leq$$

$$\leq \|A\|_{C^0}\|B\|_{L^p(B^4)} + \sum_{j=1}^4 \|\partial_{x_j}A\|_{L^{p^*}(B^4)}\|B\|_{L^4(B^4)} +$$

$$+ \sum_{i,j=1}^4 \|A\|_{C^0}\|\partial_{x_j}B_i\|_{L^p(B^4)} \leq C\|A\|_{W^{2,p}}\|B\|_{W^{1,p}}$$

The boundedness of the bilinear form let us apply Proposition 3.3 of [3] and so establish finally the differentiability of $F^{(1)}$. Arguing in a similar way, we can prove that also $T^{(1)}$ is differentiable.
This concludes the proof of the Fréchet differentiability, because, also the second component of $\mathcal{N}$, call it $\mathcal{N}^{(2)}$, is the composition of $F^{(1)} + T^{(1)}$ with the linear and continuous map that to each $B \in W^{1,p}(B^4, T^*B^4 \otimes \mathfrak{g})$ gives its normal component $B_N \in W^{1-\frac{1}{p},p}(\partial B^4, T^*B^4|_{\partial B^4} \otimes \mathfrak{g})$.

We need now to compute the Fréchet derivative of the map $\mathcal{N}$ at the point $(0,0)$ in the second direction. We will start by determining the value of $\partial_V(\mathcal{N}^{(1)})(0,0)$ with $V \in W^{2,p}(B^4, \mathfrak{g})$. We have that $\mathcal{N}^{(1)} = d^\star \circ F^{(1)} + d^\star \circ T^{(1)}$ and by linearity of the Fréchet derivative, we have

$$\partial_V(\mathcal{N}^{(1)})(0,0) = \underbrace{\partial_V(d^\star \circ F^{(1)})(0,0)}_{(I)} + \underbrace{\partial_V(d^\star \circ T^{(1)})(0,0)}_{(II)}$$

Since $d^\star : W^{1,p}(B^4, T^*B^4 \otimes \mathfrak{g}) \to L^p(B^4, \mathfrak{g})$ is linear and continuous we have that $D_f(d^\star)(g) = d^\star(f)$, for each $f, g \in W^{1,p}(B^4, T^*B^4 \otimes \mathfrak{g})$.

Therefore, using the chain rule for maps between Banach spaces, see for instance [3], we get

$$\partial_V(d^\star \circ F^{(1)})(0,0) = D_{\partial_V F^{(1)}(0,0)}(d^\star)(F^{(1)}(0,0)) = d^\star(\partial_V F^{(1)}(0,0))$$

and so (I) is obtained by computing $D_V(F^{(1)})(0,0)$. When a map is Frechét differentiable then it is also Gateaux differentiable, and these two derivatives agree. The Gateaux differential in the V direction of $F^{(1)}$ in $(0,0)$ is

$$\lim_{h\to 0} \frac{g_{hV}^{-1} dg_{hV}}{h} \tag{4.54}$$

where the limit is considered with respect the $W^{1,p}(B^4, T^*B^4 \otimes \mathfrak{g})$ topology. Since we already know that a limit in $W^{1,p}(B^4, T^*B^4 \otimes \mathfrak{g})$ exists, we can try to compute it pointwise. For $x \in \Omega$ we have

$$\frac{\exp(-hV(x))d\exp(hV(x))}{h} = \frac{\exp(-hV(x))D\exp(hV(x))hdV(x)}{h} =$$
$$= \exp(-hV(x))D\exp(hV(x))dV(x)$$

where $D\exp(hV(x))$ is the differential of $\exp : \mathfrak{g} \to G$ computed in the point $hV(x) \in \mathfrak{g}$. Taking the pointwise limit, $\exp(-hV) \to e$ and $D(\exp(hV))$ turns to be the identity map between $T_0\mathfrak{g} \cong \mathfrak{g}$ and $T_e(G) \cong \mathfrak{g}$. Therefore, we have that

$$\partial_V(d^\star \circ F^{(1)})(0,0) = d^\star(\partial_V F^{(1)}(0,0)) = d^\star dV = \Delta V$$

As far as the other term is concerned we will apply a slightly different strategy. We compute the Gateaux derivative of $d^\star \circ T^{(1)}$ at $(0,0)$ in the V direction. Therefore, we need to study the limit

$$\lim_{h\to 0} \frac{d^\star(g_{hV}^{-1} A g_{hV})}{h} = (*)$$

since by hypothesis $0 = d^\star A = d^\star(T^{(1)}(0,0))$. Then we have

$$d^\star(g_{hV}^{-1} A g_{hV}) = dg_{hV}^{-1} \cdot A g_{hV} + g_{hV}^{-1} A \cdot dg_{hV}$$

Using the previous computations of dg_{hV}, we get the pointwise limit

$$(*) = \lim_{h\to 0} \frac{d^\star(g_{hV}^{-1} A g_{hV})}{h} = -dV \cdot A + A \cdot dV = (A, dV)$$

and so we have obtained $\partial_V(\mathcal{N}^{(1)})(0,0) = \Delta V + (A, dV)$ as wanted. We now need to compute $\partial_V(\mathcal{N}^{(2)})(0,0)$. Similarly to what we have done before we note that

$$D_V(\mathcal{N}^{(2)})(0,0) = D_V(N \circ F^{(1)})(0,0) + D_V(N \circ T^{(1)})(0,0)$$

since $\mathcal{N}^{(2)} = N \circ F^{(1)} + N \circ T^{(1)}$, where the normal component map $N : W^{1,p}(B^4, T^*B^4 \otimes \mathfrak{g}) \to W^{1-\frac{1}{p},p}(\partial B^4, T^*B^4|_{\partial B^4} \otimes \mathfrak{g})$ is linear and continuous. Reasoning as we did for the map $d^\star$, we get that

$$\partial_V(N \circ F^{(1)})(0,0) = N(\partial_V(F^{(1)})(0,0)) = N \circ dV = (dV)_N = \langle \nu, dV \rangle$$

where ν is the external normal to B^4. If we compute now $\partial_V(N \circ T^{(1)})(0,0)$ using the Gateaux derivative, we see that

$$\lim_{h \to 0} \frac{N \circ T^{(1)}(0,hV) - N \circ T^{(1)}(0,0)}{h} = 0$$

since $T^{(1)}(0,0) = A$ and $A_N = 0$ by hypothesis. While $N \circ T^{(1)}(0,hV) = (g_{hV}^{-1} A g_{hV})_N = g_{hV}^{-1} A_N g_{hV} = 0$. $\square$

It is clear that Theorem 4.3.1 holds for every ball in $\mathbb{R}^4$, and not only for B^4. In the following proposition we show that the constants ε_G and C_G are the same, independently from the ball considered.

Proposition 4.3.4. *The constants ε_G and C_G appearing in Theorem 4.3.1, are invariant under translations and also scale invariant.*

Proof. The invariance for translations is trivial. We will prove only the invariance for dilations of the domain. Suppose that $A \in W^{1,2}(B^4_r, T^*B^4_r \otimes \mathfrak{g})$ satisfies equations (4.24) in B^4_r. Then, we define the following connection

$$\tilde{A}(x) := rA(rx) \quad x \in B^4 \tag{4.55}$$

and clearly $\tilde{A} \in W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g})$. As proved in Part One of Theorem 4.3.1, we have that for $x \in B^4$

$$F_{\tilde{A}}(x) = r^2 F_A(rx) \tag{4.56}$$

and integrating its square norm we obtain

$$\int_{B^4} |F_{\tilde{A}}(x)|^2 dx = \int_{B^4} |r^2 F_A(rx)|^2 dx = \int_{B^4_r} |F_A|^2 dy$$

where the second inequality is due to the change of variable $rx = y$. Therefore if $\|F_A\|^2_{L^2(B^4_r)} \leq \varepsilon_G$, we can apply Theorem 4.3.1 to $\tilde{A}$, getting a gauge $g \in W^{2,2}(B^4, G)$ such that equations (3.47) hold. Then we have that $\tilde{A}^{g(x)}(x) = rA^{g(x)}(rx)$, and therefore it can be easily proved that

$$d^\star A^{\overline{g}} = 0, \quad A^{\overline{g}}_N = 0 \text{ where } \overline{g}(y) := g\left(\frac{y}{r}\right) \text{ for } y \in B^4_r \tag{4.57}$$

and clearly $\overline{g} \in W^{2,2}(B^4_r, G)$. Since that $A^{\overline{g}}(y) = \frac{1}{r}\tilde{A}^g(x)$, where $y = rx$, then

$$\|A^{\overline{g}}\|_{L^4(B^4_r)} = \|\tilde{A}^g\|_{L^4(B^4)}$$

and a similar computation shows also that $\|DA^{\overline{g}}\|_{L^2(B^4_r)} = \|D\tilde{A}^g\|_{L^2(B^4)}$. Therefore we have seen that both ε_G and C_G do not change by dilations of B^4. $\square$

The Small Energy Theorem is actually valid for each domain Ω which is diffeomorphic to B^4. Indeed, the second and third steps of the proof of Theorem 4.3.1 are clearly valid also when working on a generic contractible smooth bounded four dimensional domain. The only step which is not immediate is the first. In the following proposition we show that it holds true also for domains that are diffeomorphic to the unit ball.

Proposition 4.3.5. *Let $\Omega \subset \mathbb{R}^4$, and $\phi : B^4 \to \Omega$ be a diffeomorphism. Then there exists $\varepsilon(\Omega, G)$, depending also on Ω, such that for each $A \in W^{1,2}(\Omega, T^*\Omega \otimes \mathfrak{g})$ satisfying*

$$\int_\Omega |F_A|^2 dx < \varepsilon(\Omega, G) \tag{4.58}$$

there exists $g \in W^{2,2}(\Omega, G)$ and a constant $C(\Omega, G)$ such that

$$\begin{cases} \|A^g\|_{L^4(\Omega)} + \|DA^g\|_{L^2(\Omega)} \le C(\Omega, G)\|F_A\|_{L^2(\Omega)} \\ d^\star A^g = 0 \quad in \ \ \Omega \\ (A^g)_N = 0 \end{cases} \tag{4.59}$$

Proof. The idea is to show that for every $\varepsilon > 0$, the set

$$U^\varepsilon_\Omega := \left\{ A \in W^{1,2}(\Omega, T^*\Omega \otimes \mathfrak{g}) : \ \int_\Omega |F_A|^2 dx < \varepsilon \right\} \tag{4.60}$$

is path connected. The only obstruction is that Ω is not necessarily a star domain, and therefore performing a path between two connections in U^ε_Ω is not straightforward. The diffeomorphism $\phi : B^4 \to \Omega$, induces the following isomorphism:

$$\begin{aligned} \phi^* : C^\infty(\Omega, T^*\Omega \otimes \mathfrak{g}) &\to C^\infty(B^4, T^*B^4 \otimes \mathfrak{g}) \\ A &\longmapsto \phi^*(A) \end{aligned} \tag{4.61}$$

through the pull-back. Of course by density we have that it naturally extends to an isomorphism

$$\phi^* : W^{1,2}(\Omega, T^*\Omega \otimes \mathfrak{g}) \to W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g})$$

Therefore, if we prove that $\phi^*(U^\varepsilon_\Omega)$ is path connected, then the same property holds also for U^ε_Ω. We see that for each $A \in U^\varepsilon_\Omega$, it holds

$$\int_\Omega |F_A|^2 dx = \int_{B^4} \phi^*(F_A \wedge \star F_A) dx = \int_{B^4} \phi^*(F_A) \wedge \phi^*(\star F_A) dx = (I) \tag{4.62}$$

where the Hodge star operator is taken with respect to the Euclidean metric. The Hodge star operator generally does not commute with the pull-back, but anyway, as proved in Proposition 3.1.7, the following significant equation holds

$$\phi^* \circ \star = \star_{\phi^*(E)} \circ \phi^* \tag{4.63}$$

where $\phi^*(E)$ is the pull-back of the Euclidean metric. Note that by linearity of the pull-back we have that $\phi^*(dA + [A, A]) = \phi^*(dA) + \phi^*([A, A])$, and since it is well known that the pull-back commutes with the differential, and that $\phi^*([A, A]) = [\phi^*(A), \phi^*(A)]$, we get $F_{\phi^*(A)} = \phi^*(F_A)$. So we can write

$$(I) = \int_{B^4} \phi^*(F_A) \wedge \star_{\phi^*(E)} \phi^*(F_A) dx = \int_{B^4} F_{\phi^*(A)} \wedge \star_{\phi^*(E)} F_{\phi^*(A)} dx = \\ = \int_{B^4} |F_{\phi^*(A)}|^2 dx$$

Therefore, if we consider the Riemannian manifold $(B^4, \phi^*(E))$, we have that

$$\phi^*(U^\varepsilon_\Omega) = U^\varepsilon_{B^4} \tag{4.64}$$

and the set on the right hand side is easily proved to be path connected. $\square$

Remark 4.3.6. The constants $\varepsilon(\Omega, G)$ and $C(\Omega, G)$ are still scale invariant, and also invariant by translations. The proof of this fact is the same of Proposition 4.3.4.

4.3.1 Small boundary connection norm

In what follows we prove the existence of the minimizer for the minimization problem (4.8) assuming that the prescribed boundary connection η has a small enough trace norm. This is fundamental, indeed we will be able to bound the value of $YM(B)$ for some $B \in W^{1,2}_\eta$ from above with the norm of η, and therefore apply the Small Energy Theorem to minimizing sequences.

Theorem 4.3.7. *There exists $\delta > 0$ such that for each $\eta \in H^{\frac{1}{2}}(\partial B^4, T^*\partial B^4 \otimes \mathfrak{g})$ satisfying $\|\eta\|_{H^{\frac{1}{2}}} < \delta$ the minimization problem (4.8) is solved by a connection $A^0 \in W^{1,2}_\eta(B^4, T^*B^4 \otimes \mathfrak{g})$.*

Proof. First of all we will prove that we can apply the Small Energy Theorem to a minimizing sequence $\{A_n\} \subset W^{1,2}_\eta(B^4, T^*B^4 \otimes \mathfrak{g})$, under a suitable condition on the norm of $\eta \in H^{\frac{1}{2}}(\partial B^4, T^*\partial B^4 \otimes \mathfrak{g})$. So let $B \in W^{1,2}_\eta(B^4, T^*B^4 \otimes \mathfrak{g})$ then it holds that

$$YM(B) \leq 2\|dB\|^2_{L^2(B^4)} + 2\int_{B^4} \big|[B,B]\big|^2 dx$$

and applying Hölder's inequality

$$YM(B) \leq C\left(\|B\|^4_{L^4(B^4)} + \|dB\|^2_{L^2(B^4)}\right)$$

By the embedding $W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g}) \hookrightarrow L^4(B^4, T^*B^4 \otimes \mathfrak{g})$ and the obvious inequality $\|dB\|^2_{L^2(B^4)} \leq \|B\|^2_{W^{1,2}(B^4)}$ we get

$$YM(B) \leq C\left(\|B\|^4_{W^{1,2}(B^4)} + \|B\|^2_{W^{1,2}(B^4)}\right) \tag{4.65}$$

By Lemma 4.2.3 we know that the functional E, defined in Section 4.2, admits a unique minimizer $\tilde{A}$ in $W^{1,2}_\eta(B^4, T^*B^4 \otimes \mathfrak{g})$ and if we choose $B = \tilde{A}$, then B coincides with the Harmonic extension of η, and from classical elliptic estimates (see Lemma 7.1 in [11]) we get

$$\|B\|_{W^{1,2}(B^4)} \leq C\|\eta\|_{H^{\frac{1}{2}}(\partial B^4)}$$

which leads to the following energy bound

$$YM(B) \leq C\left(\delta^2 + \delta^4\right) \tag{4.66}$$

We choose δ such that $C(\delta^2+\delta^4) < \varepsilon_G$. If $A_k \in W^{1,2}_\eta(B^4, T^*B^4 \otimes \mathfrak{g})$ is a minimizing sequence then we can assume without loss of generality that for each k, $YM(A_k) \leq YM(B) < \varepsilon_G$. Then by Theorem 4.3.1 we have that there exists $g_k \in W^{2,2}(B^4, G)$ such that

$$\begin{cases} \|A_k^{g_k}\|_{L^4(B^4)} + \|DA_k^{g_k}\|_{L^2(B^4)} \leq C_G\|F_{A_k}\|_{L^2(B^4)} \\ d^\star A_k^{g_k} = 0 \ \text{ in } \ B^4 \end{cases} \tag{4.67}$$

So the new sequence $A_k^{g_k}$ is bounded in $W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g})$, and up to a subsequence then $A_k^{g_k} \rightharpoonup A_\infty$ weakly in $W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g})$, for some $A_\infty \in W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g})$.
We claim that there exists $g_\infty \in W^{2,2}(B^4, G)$ such that $(A_\infty^{g_\infty})_T = \eta$. First observe that

$$(A_k^{g_k})_T := g_k^{-1}(dg_k)_T + g_k^{-1}\eta g_k \rightharpoonup A_\infty$$

weakly in $H^{\frac{1}{2}}(\partial B^4, T^*\partial B^4 \otimes \mathfrak{g})$, by Corollary 3.1.17. By the Sobolev embedding

$$H^{\frac{1}{2}}(\partial B^4, T^*\partial B^4 \otimes \mathfrak{g}) \hookrightarrow L^3(\partial B^4, T^*\partial B^4 \otimes \mathfrak{g})$$

$(A_k^{g_k})_T$ is converging weakly in $L^3(\partial B^4, T^*\partial B^4 \otimes \mathfrak{g})$ too. Thus also $g_k\big|_{\partial B^4}$ converges weakly to some $g_\infty \in W^{1,3}(\partial B^4, \mathbb{R}^{m^2})$. This is true because $d(g_k|_{\partial B^4}) = (dg_k)_T$. The space $W^{1,3}(\partial B^4, \mathbb{R}^{m^2})$ embeds compactly in $L^q(\partial B^4, \mathbb{R}^{m^2})$ for each $q > 1$, which means that g_k converges strongly in $L^q(\partial B^4, \mathbb{R}^{m^2})$ for each $q > 1$. This last note implies that $g_k \to g_\infty$ a.e. in ∂B^4, and therefore g_∞ has values a.e. in G (for the sphere measure) which means that $g_\infty \in W^{1,3}(\partial B^4, G)$. Another consequence of this strong convergence is

$$g_\infty^{-1} dg_\infty + g_\infty^{-1} \eta g_\infty = (A_\infty)_T \tag{4.68}$$

Now we need to extend g_∞ to a $\tilde{g} \in W^{2,2}(B^4, G)$. If we can do this then we have proved the claim. To this purpose we state the following theorem, which will be proved in Chapter 5.

Theorem 4.3.8. *There exists a constant ε_3 such that for any $g \in H^{\frac{3}{2}}(\partial B^4, G)$ satisfying*

$$\|dg\|_{H^{\frac{1}{2}}(\partial B^4)} < \varepsilon_3$$

there exists an extension $\tilde{g} \in W^{2,2}(B^4, G)$ of g.

Thus, in order to apply the previous theorem we need to bound properly the $H^{\frac{1}{2}}$-norm of dg_∞. By the relation (4.68) we have that

$$\|dg_\infty\|_{H^{\frac{1}{2}}} \leq C \left(\|g_\infty A_\infty\|_{H^{\frac{1}{2}}} + \|\eta g_\infty\|_{H^{\frac{1}{2}}} \right)$$

Thanks to the following embedding,

$$L^\infty \cap W^{1,3}(\partial B^4) \cdot H^{\frac{1}{2}}(\partial B^4) \hookrightarrow H^{\frac{1}{2}}$$

we find that the first addendum in the last inequality, is well defined, and in particular:

$$\|g_\infty A_\infty\|_{H^{\frac{1}{2}}} \leq C \left(\|g_\infty\|_{L^\infty} + \|g_\infty\|_{W^{1,3}} \right) \|A_\infty\|_{H^{\frac{1}{2}}}$$

and a similar bound holds also for the second addendum. Thus, since by the lower semincontinuity of the norm with respect to the weak convergence we have that $\|A_\infty\|_{H^{\frac{1}{2}}} \leq \liminf_k \|A_k^{g_k}\|_{H^{\frac{1}{2}}} < C(\delta^2 + \delta^4)$ we have that:

$$\|dg_\infty\|_{H^{\frac{1}{2}}} \leq C \left(\|g_\infty\|_{L^\infty} + \|g_\infty\|_{W^{1,3}} \right) (\delta^2 + \delta^4) \tag{4.69}$$

In particular if δ is small enough then we can apply Theorem 4.3.8, and obtain then the extension $\tilde{g} \in W^{2,2}(B^4, G)$. We define $A^0 =$

$A_\infty^{\tilde{g}^{-1}}$, which is a minimum for our problem and its tangential component on the boundary coincides with η. Indeed, since YM is a lower semicontinuous functional[7] for the $W^{1,2}$ weak topology, then:

$$YM(A_\infty^{\tilde{g}^{-1}}) = YM(A_\infty) \leq \liminf_k YM(A_k^{g_k}) = \liminf_k YM(A_k)$$

and since A_k is a minimizing sequence then A^0 is a minimum. As far as the boundary condition is concerned, one can observe that if $g \in W^{2,2}(B^4, G)$, then by the relation $g \cdot g^{-1} = id$ one gets

$$\partial_{x_i} g \cdot g^{-1} = -g \cdot \partial_{x_i} g^{-1}$$

for each $i = 1, ..., 4$. Then

$$A_T^0 = (A_\infty^{\tilde{g}^{-1}})_T = \tilde{g} d(\tilde{g}^{-1})_T + \tilde{g}(A_\infty)_T \tilde{g}^{-1} =$$

$$g_\infty d((g_\infty)^{-1}) + g_\infty \left((g_\infty)^{-1} dg_\infty + (g_\infty)^{-1} \eta g_\infty\right)(g_\infty)^{-1} = \eta$$

and this concludes the proof. □

4.3.2 Arbitrary boundary connection norm

We have proved that for a small enough $H^{\frac{1}{2}}$-norm of the given boundary connection η, we can establish the existence of a minimum for the minimization problem. This result was achieved using K.Uhlenbeck's Small Energy Theorem on the whole B^4, since thanks to an estimate of YM in terms of $\|\eta\|_{H^{\frac{1}{2}}}$ we could make YM smaller than the constant ε_G.
If we relax the hypothesis on the norm of the boundary connection, we cannot apply anymore the Small Energy Theorem to a minimizing sequence of connections $\{A_k\}$ on the whole B^4, but anyway we can use it locally in the sense of the following proposition.

Proposition 4.3.9. *Let $\{A_k\}$ be a sequence in $W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g})$ such that $YM(A_k)$ is bounded, and ε a positive constant. Then there exists a subsequence of $\{A_k\}$ and N points $P_1, ..., P_N \in \overline{B^4}$ such that* $\forall \delta > 0 \quad \exists \rho > 0$

$$\sup_n \int_{B_\rho(y) \cap B^4} |F_{A_{k_n}}|^2 dx \leq \varepsilon \qquad \forall y \in B^4 \setminus \cup_{i=1}^N B_\delta(P_i)$$

Proof. We define the map $\rho_k : \overline{B^4} \to \mathbb{R}$ where

$$\rho_k(x) := \sup \left\{ 0 < \rho < 1 : \int_{B_\rho(x) \cap B^4} |F_{A_k}|^2 dx < \varepsilon \right\}$$

[7] We can get this property using for example Tonelli's Theorem

Claim 4.3.9.1. There is a finite set of points $\{P_1, ..., P_N\} \subset \overline{B^4}$ such that $\inf_k \rho_k(P_i) = 0$.

If the claim is true then $\rho(x_0) := \inf_k \rho_k(x_0) > 0$ for each $x_0 \in \overline{B^4} \setminus \cup_{i=1}^N B_\delta(P_i) =: C_\delta$, where $\delta > 0$ is a constant we fix.
Now observe that each ρ_k is continuous. Indeed, take $\{x_n\}_{n\in\mathbb{N}} \subset C_\delta$ converging to x_0, and fix $\epsilon > 0$. Then there exists $\tilde{\rho}$ such that

$$\tilde{\rho} < \rho_k(x_0) < \tilde{\rho} + \epsilon \quad \text{and} \quad \int_{B_{\tilde{\rho}}(x_0)\cap B^4} |F_{A_k}|^2 dx < \varepsilon$$

There exists $N(\epsilon) \in \mathbb{N}$ large enough such that $\forall n \geq N(\epsilon)$ it holds $B_{\tilde{\rho}-\epsilon}(x_n) \subset B_{\tilde{\rho}}(x_0)$ and so $\rho_k(x_n) > \tilde{\rho} - \varepsilon$. It is also clear by construction that $\rho_k(x_n) < \rho_k(x_0) + |x_n - x_0|$. Putting together these inequalities we get:

$$-|x_n - x_0| < \rho_k(x_0) - \rho_k(x_n) < \tilde{\rho} + \epsilon - \tilde{\rho} + \epsilon = 2\epsilon \quad \forall n \geq N(\epsilon)$$

Since ρ_k is continuous, then $\inf_k \rho_k = \rho$ is continuous too in C_δ, and by the claim it is also positive. Thus, it admits a minimum, $\overline{\rho} := \min_{x\in C} \rho(x) > 0$. This last thing means that

$$\sup_k \int_{B_{\overline{\rho}}(y)\cap B^4} |F_{A_k}|^2 dx \leq \varepsilon$$

for every $y \in C_\delta$. So we need to prove the claim now.

Proof of claim 1. Suppose by contradiction that

$$S := \left\{ P \in \overline{B^4} : \inf_k \rho_k(P) = 0 \right\}$$

is at least countable. Choose a couple $(P_1, \rho_1) \in S \times \mathbb{R}^+$ such that $S \setminus B_{\rho_1}(P_1)$ is still infinite, and a subsequence A_{k_1} of A_k with $\int_{B_{\rho_1}(P_1)} |F_{A_{k_1}}|^2 dx > \varepsilon$. Out of this subsequence we choose a generic $\tilde{A}_1$.
Now let $(P_2, \rho_2) \in S \times \mathbb{R}^+$ such that $B_{\rho_1}(P_1) \cap B_{\rho_2}(P_2) = \emptyset$ and $S \setminus (B_{\rho_1}(P_1) \cup B_{\rho_2}(P_2))$ is still infinite. Then there is a subsequence A_{k_2} of A_{k_1}, such that $\int_{B_{\rho_2}(P_2)} |F_{A_{k_2}}|^2 dx > \varepsilon$, and we select a connection $\tilde{A}_2$.
Going on in this fashion, we find $(P_n, \rho_n) \in S \times \mathbb{R}^+$ such that $\cap_{i=1}^n B_{\rho_i}(P_i) = \emptyset$ and $S \setminus \cup_{i=1}^n B_{\rho_i}(P_i)$ is still infinite, and there is a subsequence A_{k_n} of $A_{k_{n-1}}$ with $\int_{B_{\rho_n}(P_n)} |F_{A_{k_n}}|^2 dx > \varepsilon$, and we pick

$\tilde{A}_n$ from A_{k_n}. We have thus constructed a subsequence $\tilde{A}_k$ such that

$$\int_{B^4} |F_{\tilde{A}_k}|^2 dx \geq \int_{\cup_{i=1}^k B_{\rho_i}(P_i)} |F_{\tilde{A}_k}|^2 dx > \sum_{i=1}^k \varepsilon = k\varepsilon,$$

and clearly $YM(\tilde{A}_k)$ is not bounded, contradiction.

$\square$

At this point we know that we can give a cover $\mathcal{U} = \{U_i\}_{i\in I}$ of the submanifold C_δ defined as in Proposition 4.3.9, such that in each U_i (choosing $\varepsilon < \varepsilon_G$) we can apply the Small Energy Theorem producing a sequence of families of gauges $g_k^{ij} \in W^{2,2}(U_i \cap U_j, G)$, that defines a sequence of bundles. We are ready to state and prove the Theorem for the existence of the minimizer, up to a gauge of the fixed boundary connection.

Theorem 4.3.10. *Let G be a compact and connected matrix Lie group. For any 1-form on the boundary $\eta \in H^{\frac{1}{2}}(\partial B^4, T^*\partial B^4 \otimes \mathfrak{g})$ there exists $A^0 \in W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g})$ such that*

$$YM(A_0) = \inf_{A\in W_\eta^{1,2}(B^4,T^*B^4\otimes\mathfrak{g})} YM(A)$$

and $(A_0)_T = \eta^g$ for some $g \in H^{\frac{3}{2}}(\partial B^4, G)$.

Proof. Consider a minimizing sequence A_k in $W_\eta^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g})$, and let us fix any $2 < p < 4$. Then if we choose

$$\varepsilon \leq \inf_{\substack{y\in\overline{B^4} \\ \rho\in(0,1)}} \left\{ \varepsilon(B^4 \cap B_\rho(y), G), \frac{\varepsilon_1(n^2, p, B_\rho(y) \cap B^4)^2}{16C(B_\rho(y) \cap B^4, G)^2} \right\} \tag{4.70}$$

in the previous proposition, where $\varepsilon(B^4\cap B_\rho(y), G)$,$C(B^4\cap B_\rho(y), G)$ and
$\varepsilon_1(n^2, p, B_\rho(y) \cap B^4)$ have been defined respectively in Proposition 4.3.4 and in Lemma 3.2.13, we know that there are at most N points $P_1, ..., P_N$ in $\overline{B^4}$ such that for any $\delta > 0$ there exists a $\rho > 0$

$$\sup_{y\in C_\delta} \sup_k \left\{ \int_{B_\rho(y)\cap B^4} |F_{A_k}|^2 dx < \varepsilon \right\} \tag{4.71}$$

where $C_\delta := B^4 \backslash \cup_{i=1}^N B_\delta(P_i)$. The constant ε is non vanishing thanks to the fact that the three constants involved in its definition are all scale invariant and also invariant by translations of the domain. The

proof of the theorem is divided in two main steps.
1)We start assuming that $\{P_1, ..., P_N\} = \emptyset$. Then we know that there exists $\rho > 0$ such that

$$\sup_k \sup_{y\in B^4} \int_{B^4\cap B_\rho(y)} |F_{A_k}|^2 dx < \varepsilon \tag{4.72}$$

and we choose a good finite covering[8] by balls $\mathcal{B}_{\frac{\rho}{2}} := \left\{B_{\frac{\rho}{2}}(x_i)\right\}_{i\in I}$ of $\overline{B^4}$ with $x_i \in \overline{B^4}$ for each $i \in I$. For convenience we will denote

$$B_{\frac{\rho}{2}}(x_i) =: B^i_{\frac{\rho}{2}} \quad \forall i \in I \ \text{ and } \quad B^i_{\frac{\rho}{2}} \cap B^j_{\frac{\rho}{2}} := B^{ij}_{\frac{\rho}{2}} \quad \forall i, j \in I$$

Therefore, by the Small Energy Theorem 4.3.1, we know that there exists for each $i \in I$ a gauge $g^i_k \in W^{2,2}(B^i_\rho \cap B^4, G)$, such that

$$\|A_k^{g^i_k}\|_{L^4(B^i_\rho\cap B^4)} + \|DA_k^{g^i_k}\|_{L^2(B^i_\rho\cap B^4)} \leq C(B^i_\rho\cap B^4, G)\left(\int_{B^i_\rho\cap B^4} |F_{A_k}|^2 dx\right)^{\frac{1}{2}}$$

$$d^\star A_k^{g^i_k} = 0 \ \text{ in } \ B^i_\rho \cap B^4 \tag{4.73}$$

From now on we will not indicate anymore the intersections of the balls with B^4, just meaning it whenever it is necessary.
The sequence of families $g^{ij}_k := (g^i_k)^{-1} g^j_k \in W^{2,2}(B^{ij}_\rho, G)$ defines a sequence of $W^{2,2}$-principal Sobolev G-bundles $\mathcal{P}_k = \{(B^{ij}_\rho, g^{ij}_k)\}$, and in each one of them we have the Sobolev connection $\{A_k^{g^i_k}\}_{i\in I}$ that satisfies the classical compatibility condition

$$A_k^{g^j_k} = (g^{ij}_k)^{-1} dg^{ij}_k + (g^{ij}_k)^{-1} A_k^{g^i_k} g^{ij}_k \ \text{ in } \ B^{ij}_\rho \neq \emptyset \tag{4.74}$$

We want to prove that the sequence of bundles converges to a trivial Sobolev bundle. Since $\mathcal{P}_k$ is a Coulomb bundle for each $k \in \mathbb{N}$, and by equation (4.73) we have that

$$\|A_k^{g^i_k}\|_{L^4(B^i_\rho)} \leq \frac{\varepsilon_1(n^2, p, B^i_\rho)}{4}$$

then by Lemma 3.2.13 we have that for the fixed $2 < p < 4$ it holds

$$g^{ij}_k \in W^{2,p}\left(B^{ij}_{\frac{3}{4}\rho}, G\right) \hookrightarrow C^{0,\alpha}\left(B^{ij}_{\frac{3}{4}\rho}, G\right)$$

where the Sobolev embedding is compact. This tells us that if we consider the refinement $\mathcal{B}_{\frac{3}{4}\rho} := \left\{B^i_{\frac{3}{4}\rho}\right\}_{i\in I}$, we have that $\mathcal{P}'_k :=$

[8] We say that a finite open covering is good if the intersections of the sets are all contractible

$\left\{\left(B^{ij}_{\frac{3}{4}\rho}, g^{ij}_k\right)\right\}$ is a sequence of Hölder continuous bundles. Furthermore, Lemma 3.2.13 also gives us the bound

$$\|g^{ij}_k\|_{W^{2,p}\left(B^{ij}_{\frac{3}{4}\rho}\right)} \leq C\|g^{ij}_k\|_{W^{2,2}\left(B^{ij}_{\frac{3}{4}\rho}\right)} \tag{4.75}$$

Using the compatibility condition (4.74) we get the equation

$$dg^{ij}_k = g^{ij}_k A^{g^j_k}_k - A^{g^i_k}_k g^{ij}_k$$

that we use in order to obtain the following two bounds

$$\|dg^{ij}_k\|_{L^4\left(B^{ij}_{\frac{3}{4}\rho}\right)} \leq C\left(\|A^{g^i_k}_k\|_{L^4(B^i_\rho)} + \|A^{g^j_k}_k\|_{L^4(B^j_\rho)}\right)$$

$$\|D^2 g^{ij}_k\|_{L^2\left(B^{ij}_{\frac{3}{4}\rho}\right)} \leq C\left(\|A^{g^i_k}_k\|_{L^4(B^i_\rho)} + \|A^{g^j_k}_k\|_{L^4(B^j_\rho)}\right)^2 +$$
$$+C\left(\|DA^{g^i_k}_k\|_{L^2(B^i_\rho)} + \|DA^{g^j_k}_k\|_{L^2(B^j_\rho)}\right) \tag{4.76}$$

Therefore, using these estimates and equation (4.75) we have found that there exists a constant $M > 0$, depending also on ε_1 and G, such that

$$\|g^{ij}_k\|_{W^{2,p}\left(B^{ij}_{\frac{3}{4}\rho}\right)} \leq M$$

Thus, the sequence of transitions functions g^{ij}_k is uniformly bounded in $W^{2,p}$, and since $W^{2,p}$ is a reflexive space then there exists a subsequence of g^{ij}_k such that

$$g^{ij}_k \rightharpoonup g^{ij}_\infty \quad \text{weakly in } W^{2,p}\left(B^{ij}_{\frac{3}{4}\rho}\right) \tag{4.77}$$

and moreover from the inequality in (4.73) we deduce that

$$A^{g^i_k}_k \rightharpoonup A^i_\infty \quad \text{wealky in } W^{1,2}(B^i_\rho, T^*B^i_\rho \otimes \mathfrak{g})$$

for each $i \in I$. The weak convergence in $W^{2,p}$ and the compact embedding $W^{2,p} \hookrightarrow C^0$ imply that

$$\|g^{ij}_k - g^{ij}_\infty\|_{L^\infty\left(B^{ij}_{\frac{3}{4}\rho}\right)} \to 0 \text{ for } k \to \infty \tag{4.78}$$

This also means that $g^{ij}_k \to g^{ij}_\infty$ a.e. in $B^{ij}_{\frac{3}{4}\rho}$, hence $g^{ij}_\infty \in W^{2,p}\left(B^{ij}_{\frac{3}{4}\rho}, G\right)$. It is straightforward to see that in each $B^{ij}_{\frac{3}{4}\rho} \neq \emptyset$ it holds

$$A^j_\infty = (g^{ij}_\infty)^{-1} dg^{ij}_\infty + (g^{ij}_\infty)^{-1} A^i_\infty g^{ij}_\infty \tag{4.79}$$

and furthermore the family of limits $\{g^{ij}_\infty\}$ still respects the cocycle conditions

$$g^{ij}_\infty g^{jl}_\infty = g^{il}_\infty \text{ in } B^{ijl}_{\frac{3}{4}\rho}$$
$$g^{ij}_\infty g^{ji}_\infty = e \text{ in } B^{ij}_{\frac{3}{4}\rho} \tag{4.80}$$

This means that the family of maps g^{ij}_∞ defines a $W^{2,p}$-bundle $\mathcal{P}_\infty := \left\{\left(B^{ij}_{\frac{3}{4}\rho}, g^{ij}_\infty\right)\right\}$, which is actually a Coulomb bundle, since clearly $d^\star A^i_\infty = 0$ in $B^i_{\frac{3}{4}\rho}$ $\forall i \in I$.

The next step is to show that the limit bundle $\mathcal{P}_\infty$ is trivial. This fundamental result is achieved thanks to Lemma 3.2.18, that can be easily applied to $g^{ij} = g^{ij}_{\overline{k}}$ and $h^{ij} = g^{ij}_\infty$ for a $\overline{k} \in \mathbb{N}$ large enough, thanks to equation (4.78). The refinement we choose is $\mathcal{B}_{\frac{\varrho}{2}}$, and we get the existence of a family $\sigma^i_{\overline{k}} \in W^{2,p}\left(B^i_{\frac{\varrho}{2}}, G\right)$, such that

$$g^{ij}_\infty = (\sigma^i_{\overline{k}})^{-1} g^{ij}_{\overline{k}} \sigma^j_{\overline{k}} \quad \text{in } B^{ij}_{\frac{\varrho}{2}} \tag{4.81}$$

Now equation (4.81) together with the fact that for each $k \in \mathbb{N}$ it holds $g^{ij}_k = (g^i_k)^{-1} g^j_k$ in $B^{ij}_{\frac{\varrho}{2}}$ leads to the triviality also of the cocycle g^{ij}_∞

$$g^{ij}_\infty = (g^i_{\overline{k}}\sigma^i_{\overline{k}})^{-1} g^j_{\overline{k}}\sigma^j_{\overline{k}} \text{ in } B^{ij}_{\frac{\varrho}{2}} \tag{4.82}$$

Then, the connection $\{A^i_\infty\}_{i\in I}$ in the trivial bundle reads as

$$A^0 := f_i^{-1} df_i + f_i^{-1} A^i_\infty f_i = (A^i_\infty)^{f_i} \quad \text{in } B^i_{\frac{\varrho}{2}} \tag{4.83}$$

where $f_i = (g^i_{\overline{k}}\sigma^i_{\overline{k}})^{-1} \in W^{2,p}\left(B^i_{\frac{\varrho}{2}}, G\right)$, and $A^0 \in W^{1,2}(B^4, T^*B^4, \otimes\mathfrak{g})$ minimizes the YM functional. Indeed we can extract from the family $\mathcal{B}_{\frac{\varrho}{2}}$ a covering which is disjointed, defining $U_1 := B^1_{\frac{\varrho}{2}}$, $U_2 := B^2_{\frac{\varrho}{2}} \setminus B^1_{\frac{\varrho}{2}}$, $U_3 := B^3_{\frac{\varrho}{2}} \setminus (B^1_{\frac{\varrho}{2}} \cup B^2_{\frac{\varrho}{2}})$,..., $U_t := B^t_{\frac{\varrho}{2}} \setminus (\cup_{i=1}^{t-1} B^i_{\frac{\varrho}{2}})$, and so we get

$$\int_{B^4} |F_{A^0}|^2 dx = \int_{\cup_{i=1}^t U_i} |F_{A^i_\infty}|^2 dx \leq$$
$$\leq \sum_{i=1}^t \liminf_k \int_{U_i} |F_{A^i_k}|^2 dx = \liminf_k \int_{B^4} |F_{A_k}|^2 dx \tag{4.84}$$

The last step before concluding the proof of part 1) of the Theorem, is to show that there exists a gauge $g \in H^{\frac{3}{2}}(\partial B^4, G)$ such that $A^0_T = \eta^g$.

Consider any $i \in I$ such that $B^i_{\frac{\varrho}{2}} \cap \mathbb{S}^3 \neq \emptyset$. We know that in $B^i_{\frac{\varrho}{2}} \cap B^4$ the sequence $A_k^{g_k^i} \rightharpoonup A^i_\infty$ weakly in $W^{1,2}$. As we have already proved in Theorem 4.3.7, by this last consideration we deduce that

$$(A_k^{g_k^i})_T := (g_k^i)^{-1} d(g_k^i\big|_{V_i}) + (g_k^i)^{-1}((A_k)_T) g_k^i \rightharpoonup (A^i_\infty)_T \tag{4.85}$$

weakly in $H^{\frac{1}{2}}(V_i, T^*V_i \otimes \mathfrak{g})$, where $V_i := \partial(B^i_{\frac{\varrho}{2}} \cap B^4)$. Furthermore if we call $S_i := V_i \cap \mathbb{S}^3$, then the last convergence is true also in $H^{\frac{1}{2}}(S_i, T^*S_i \otimes \mathfrak{g})$ and the sequence $g_k^i\big|_{S_i} \rightharpoonup \tilde{g}^i$ weakly in $W^{1,3}(S_i, G)$, and one proves, always with the same arguments of Theorem 4.3.7, that $\tilde{g}^i \in H^{\frac{3}{2}}(S_i, G)$. Passing to the limit in (4.85) we get that

$$(\tilde{g}^i)^{-1} d\tilde{g}^i + (\tilde{g}^i)^{-1}(\eta)\tilde{g}^i = (A^i_\infty)_T \quad \text{in } \; S_i \tag{4.86}$$

Therefore, we have obtained that in each S_i the following identity is true

$$A^0_T = ((A^i_\infty)^{f_i})_T = ((A^i_\infty)_T)^{f_i} = (\eta)^{\tilde{g}^i \cdot f_i}$$

The second identity is justified by the fact that $(df_i)_T = d(f_i|_{S_i})$. Finally note that $\cup_i S_i = \mathbb{S}^3$, and that in particular $\tilde{g}^i f_i = \tilde{g}^j f_j$ in each intersection $S_i \cap S_j \neq \emptyset$. Indeed, the weak convergence $g_k^i \rightharpoonup \tilde{g}^i$ in $W^{1,3}(S_i, G)$ implies by Rellich-Kondrakov that the convergence is actually strong in L^q and therefore it also converges pointwise a.e. in S_i. In $S_i \cap S_j$ we have that $f_i = g^{ij}_\infty f_j$. If we call $f_k^i := g_k^{ij} f_j$ then such a sequence by (4.77) converges weakly to f_i in $W^{2,2}\left(B^{ij}_{\frac{\varrho}{2}}, G\right)$ and therefore its trace converges too weakly in $H^{\frac{3}{2}}\left(\partial\left(B^{ij}_{\frac{\varrho}{2}}\right), G\right)$, and we clearly understand that

$$f_k^i \to f_i \; \text{ a.e. in } \; S_i \cap S_j$$

Finally we have that

$$g_k^i f_k^i = g_k^i g_k^{ij} f_k^j = g_k^j f_k^j \; \text{ in } \; S_i \cap S_j$$

and the pointwise convergence gives us $\tilde{g}^i f_i = \tilde{g}^j f_j$ a.e. in $S_i \cap S_j$.

2) Suppose now that $\{P_1, ..., P_N\} \neq \emptyset$. Repeating the same arguments of part 1) we find a finite good open covering $\mathcal{B}_{\frac{\varrho}{2}} := \left\{B^i_{\frac{\varrho}{2}}\right\}$ of $C_\delta := B^4 \setminus \cup_{l=1}^N B_\delta(P_l)$ and a family of gauge changes $g_k^i \in W^{2,2}(B^i_\rho, G)$ such that

$$\begin{cases} A_k^{g_k^i} \rightharpoonup A^i_\infty \; \text{ weakly in } \; W^{1,2}(B^i_\rho, T^*B^i_\rho \otimes \mathfrak{g}) \\ g_k^{ij} \rightharpoonup g^{ij}_\infty \; \text{ weakly in } \; W^{2,p}\left(B^{ij}_{\frac{3\rho}{4}}\right) \end{cases} \tag{4.87}$$

Each g_∞^{ij} is a $W^{2,p}$-cocycle, and applying the same arguments adopted in the previous step of the proof we get that g_∞^{ij} are still trivial for the Čech Cohomology $\check{H}^1(C_\delta, C^0(G))$, see Appendix B. This is due to the fact that Lemma 3.2.18 can be applied to any four dimensional bounded and smooth domain, and not only to the ball.
Thus we can build a connection A_δ^0 defined on the whole C_δ, such that in $\partial B^4 \setminus \cup_{l=1}^N B_\delta(P_l)$ it is gauge equivalent to η and

$$\int_{C_\delta} |F_{A_\delta^0}|^2 dx \leq \liminf_k C \int_{B^4} |F_{A_k}|^2 dx \tag{4.88}$$

Suppose now that $\delta_1 < \delta$, and A_δ^0 and $\left\{ B_{\frac{\rho}{2}}^i \right\}_{i\in I}$ are as above. Then we consider the following open covering for C_{δ_1}

$$\mathcal{B} = \left\{ B_{\frac{\rho}{2}}^i \right\}_{i\in I} \cup \left\{ B_{\frac{\rho_1}{2}}^j \right\}_{j\in J} \tag{4.89}$$

where $\cup_{j\in J} B_{\frac{\rho_1}{2}}^j \supseteq C_{\delta_1}$, and ρ_1 is defined as always. Using $\mathcal{B}$ we can then extend A_δ^0 to a $\mathfrak{g}$-valued 1-form $A_{\delta_1}^0 \in W^{1,2}(C_{\delta_1}, T^*C_{\delta_1} \otimes \mathfrak{g})$.
Taking $\delta \to 0$, we find an $A_0 \in W_{\mathrm{loc}}^{1,2}(B^4 \setminus \{P_1, ..., P_N\})$ which is a minimizer for YM and at the boundary is gauge equivalent to η for some $g \in H^{\frac{3}{2}}(\partial B^4, G)$. Thanks to the Removable Singularities Theorem 4.3.14, we can find a local gauge $g \in W_{\mathrm{loc}}^{2,2}(U_{P_l}, G)$ on a neighbourhood U_{P_l} of each point P_l such that $(A^0|_{U_{P_l}})^g \in W^{1,2}(U_{P_l}, T^*U_{P_l} \otimes \mathfrak{g})$. Therefore, we have found finally a gloabal connection $\tilde{A}^0 \in W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g})$ which is a minimizer, gauge equivalent to η at the boundary ∂B^4, namely there exists $g \in H^{\frac{3}{2}}(\partial B^4, G)$ such that $(A_0)_T = \eta^g$.

□

Remark 4.3.11. We note that, similarly to Theorem 3.5.4, we arrived in this proof to a minimizing connection whose tangential component is gauge equivalent, for a $g \in H^{\frac{3}{2}}(\partial B^4, G)$, to the prescribed η. However, unlike the previous theorem, since we dropped the condition on the norm of η, now we do not have anymore a suitable small bound on the $H^{\frac{1}{2}}$-norm of dg, and so generally g does not admit an extension on the whole B^4, see Chapter 5 for further details. Therefore, all we can say, in this case, is that if A is our minimum, then $A_T = \eta^g$ for some gauge of the boundary $g \in H^{\frac{3}{2}}(\partial B^4, G)$.

4.3.3 The Removable singularities Theorem

The Removable singularities Theorem, first proved by K.Uhlenbeck in [45], for Yang-Mills fields[9] with a singularity, asserts that it is possible to find a local gauge, such that the field in this new gauge has lost its singularity.
The following result, established by T.Rivière and M.Petrache in [33], is an improved version of this Theorem, indeed here we do not assume that the connection is a minimizer but only that it has finite Yang-Mills energy. The key point, that allows such a generalization, is based on Lemma 3.2.11, also proved in [33], and the embedding $W^{2,(2,1)} \hookrightarrow L^\infty$ in dimension four.
We will prove the theorem for the punctured ball $B^4 \setminus \{0\}$, namely for connections whose blow up point for the $W^{1,2}$-norm coincides with the origin. Before stating the main Theorem of this section we will need the following preliminary Lemma. We fix the notation for $k \in \mathbb{N}$

$$T_k = B_{2^{-k+4}} \setminus B_{2^{-k-4}}, \qquad S_k = B_{2^{-k+3}} \setminus B_{2^{-k-3}}$$

Lemma 4.3.12. *There exists $\tilde{\varepsilon}$, not depending on k, such that if $A \in W^{1,2}(T_k, T^*T_k \otimes \mathfrak{g})$ satisfies*

$$\int_{T_k} |F_A|^2 dx < \tilde{\varepsilon}$$

then there exists $g \in W^{2,2}(S_k, G)$ and a constant $\tilde{C}$ such that

$$\begin{cases} d^\star A^g = 0 \ \ in \ \ S_k \\ \|A^g\|_{L^4(S_k)} + \|DA^g\|_{L^2(S_k)} \leq \tilde{C}\|F_A\|_{L^2(T_k)} \end{cases} \tag{4.90}$$

Remark 4.3.13. We see that $T_k = 2^{-k}T_0$ and also $S_k = 2^{-k}S_0$. Therefore performing the same computations of Proposition 4.3.4 we get that the constants $\tilde{\varepsilon}$ and $\tilde{C}$ are independent from k.

Proof. We start by assuming that $\|F_A\|^2_{L^2(T_0)} \leq \varepsilon$, where the constant ε will be specified later, in order to satisfy some requests.
Thanks to Remark 4.3.13 it is sufficient to prove the theorem for the case $k = 0$. We can take a covering $\mathcal{U} = \{U_1, U_2\}$ for T_0 made of two open sets diffeomorphic to the ball B^4, and with smooth intersection $U_1 \cap U_2 =: U_{12}$.
By choosing

$$\varepsilon \leq \min_{i=1,2} \left\{ \varepsilon(U_i, G), \frac{\varepsilon_1(n^2, p, U_i)^2}{16C(U_i, G)^2} \right\}$$

[9] A connection A is a Yang-Mills field if it is a minimizer of the Yang-Mills functional

for a fixed $2 < p < 4$, we can apply the small energy theorem on both U_1 and U_2, as a consequence of Proposition 4.3.5, which gives the existence of $g_i \in W^{2,2}(U_i, G)$ such that

$$\begin{cases} d^\star A^{g_i} = 0 \text{ in } U_i \\ \|A^{g_i}\|_{L^4(U_i)} + \|DA^{g_i}\|_{L^2(U_i)} \leq C(U_i, G)\|F_A\|_{L^2(U_i)} \end{cases} \tag{4.91}$$

This leads to the $W^{2,2}$-Coulomb bundle $\mathcal{P} = \{(g_{12}, U_{12})\}$ on T_0, with Coulomb connection $\{A^{g_i}\}_{i=1,2}$. As already discussed in Lemma 3.2.15 we have that $g_{21} := g_2^{-1}g_1 \in W^{2,p}_{\text{loc}}(U_{12}, G)$. In particular, as already proved before, we have the existence of a $\overline{g} \in G$, such that in $\tilde{U}_{12} := S_0 \cap U_{12}$, we have the estimates

$$\|g_{21} - \overline{g}\|_{W^{2,p}(\tilde{U}_{1,2})} \leq C\Bigg(\|A^{g_1}\|_{W^{1,2}(U_{12})} + \|A^{g_2}\|_{W^{1,2}(U_{12})} +$$

$$+ \left(\|A^{g_1}\|_{W^{1,2}(U_{12})} + \|A^{g_2}\|_{W^{1,2}(U_{12})}\right)^2 \Bigg) \leq C\left(\|F_A\|_{L^2(T_0)} + \|F_A\|^2_{L^2(T_0)}\right) \tag{4.92}$$

and the last inequality is given by equations (4.91). We rescale one of the two gauges as follows,

$$h_2 := g_2\overline{g}, \qquad h_1 := g_1 \tag{4.93}$$

so that we have that both A^{h_1} and A^{h_2} still satisfy equations

$$\begin{cases} \|A^{h_i}\|_{L^4(U_i)} + \|DA^{h_i}\|_{L^2(U_i)} \leq C(U_i, G)\|F_A\|_{L^2(U_i)} \\ d^\star A^{h_i} = 0 \text{ in } U_i \end{cases} \tag{4.94}$$

and the inequality (4.92) together with the embedding $W^{2,p}(\tilde{U}_{12}, G) \hookrightarrow C^0(\tilde{U}_{12}, G)$ implies that

$$\|h_{21} - e\|_{L^\infty(\tilde{U}_{12})} \leq C\|h_{21} - e\|_{W^{2,p}(\tilde{U}_{12})} \leq C\left(\|F_A\|_{L^2(T_0)} + \|F_A\|^2_{L^2(T_0)}\right) \tag{4.95}$$

Therefore for ε small enough there exists $V_{21} \in W^{2,p}(\tilde{U}_{12}, \mathfrak{g})$, such that

$$h_{21} = \exp(V_{21}) \text{ in } \tilde{U}_{12}$$

and it also holds that

$$\|V_{21}\|_{W^{2,p}(\tilde{U}_{12})} \leq C\left(\|F_A\|_{L^2(T_0)} + \|F_A\|^2_{L^2(T_0)}\right)$$

Now we can extend the map V_{21} to the whole $\tilde{U}_2 := S_0 \cap U_2$, by the classical extension theorem for Sobolev functions. So we get

$\tilde{V} \in W^{2,p}(\tilde{U}_2, \mathfrak{g})$, such that $\tilde{V} = V_{21}$ in $\tilde{U}_{12}$, and $\|\tilde{V}\|_{W^{2,p}(\tilde{U}_2)} \leq C\|V_{21}\|_{W^{2,p}(\tilde{U}_{12})}$.

We **claim** that the extension $\tilde{V}$ can be chosen such that if we define the gauge

$$h(x) := \begin{cases} h_1(x) \text{ if } x \in \tilde{U}_1 \\ h_2(x)\tilde{h}(x) \text{ if } x \in \tilde{U}_2 \end{cases} \tag{4.96}$$

where $\tilde{h} := \exp(\tilde{V}) \in W^{2,p}(\tilde{U}_2, G)$, then it satisfies the wanted equations

$$\begin{cases} d^\star A^h = 0 \text{ in } S_0 \\ \|A^h\|_{L^4(S_0)} + \|DA^h\|_{L^2(S_0)} \leq C\|F_A\|_{L^2(T_0)} \end{cases} \tag{4.97}$$

To prove the claim we consider the following map between Banach spaces

$$\begin{aligned} \tilde{\mathcal{N}} : W^{2,p}(\tilde{U}_2, \mathfrak{g}) \times W^{2,p}_0(\overline{U}_2, \mathfrak{g}) &\to L^p(\overline{U}_2, \mathfrak{g}) \\ (V_1, V_2) &\longmapsto d^\star A^{h_2 \exp(V_1+V_2)} \end{aligned} \tag{4.98}$$

where we have define $\overline{U}_2 := \tilde{U}_2 \setminus \tilde{U}_{12}$. We see that this map, which is similar to $\mathcal{N}$ defined in Theorem 4.3.1, is also a C^1 map, as one can easily prove following the same arguments of Lemma 4.3.3. We see that $\tilde{\mathcal{N}}(0,0) = 0$ by hypothesis, and the idea is to apply the implicit function theorem to $\tilde{\mathcal{N}}$, just as we did in Theorem 4.3.1. To do so we have to prove that the Fréchet derivative of $\tilde{\mathcal{N}}$ with respect to the second component $\partial_U \tilde{\mathcal{N}}(0,0) : W^{2,p}_0(\overline{U}_2, \mathfrak{g}) \to L^p(\overline{U}_2, \mathfrak{g})$ is an isomorphism. Computing it we find

$$\partial_U \tilde{\mathcal{N}}(0,0) \cdot V_2 = \Delta V_2 + (A^{h_2}, dV_2) \tag{4.99}$$

for each $V_2 \in W^{2,p}_0(\overline{U}_2, \mathfrak{g})$. Using Gaffney's inequality (3.24) in $\overline{U}_2$ we get the following bound for the $W^{1,p}$-norm of dV_2

$$\|dV_2\|_{W^{1,p}(\overline{U}_2)} \leq C\|\Delta V_2\|_{L^p(\overline{U}_2)}$$

Then Poincaré inequality holds and therefore

$$\begin{aligned} \|V_2\|_{W^{2,p}_0(\overline{U}_2)} &\leq C\|\Delta V_2\|_{L^p(\overline{U}_2)} \leq \\ &\leq C\left(\|\partial_{V_2}\tilde{\mathcal{N}}(0,0)\|_{L^p(\overline{U}_2)} + \|(A^{h_2}, dV_2)\|_{L^p(\overline{U}_2)}\right) \end{aligned} \tag{4.100}$$

Using Hölder's inequality and the Sobolev embedding $W^{1,p}(\overline{U}_2, \mathfrak{g}) \hookrightarrow L^{\frac{4p}{4-p}}(\overline{U}_2, \mathfrak{g})$ we get

$$\|V_2\|_{W^{2,p}_0(\overline{U}_2)} \leq C\left(\|\partial_{V_2}\tilde{\mathcal{N}}(0,0)\|_{L^p(\overline{U}_2)} + \underbrace{\|A^{h_2}\|_{L^4(\overline{U}_2)}}_{\leq C\varepsilon}\|dV_2\|_{L^{\frac{4p}{4-p}}(\overline{U}_2)}\right)$$

and the last equation leads to

$$(1 - C\varepsilon^{\frac{1}{2}})\|V_2\|_{W_0^{2,p}(\overline{U}_2)} \le C\|\partial_{V_2}\tilde{\mathcal{N}}(0,0)\|_{L^p(\overline{U}_2)}$$

and this finally means that for ε small enough $\partial\tilde{\mathcal{N}}(0,0)$ is injective. It remains only to show that it is surjective, and after this we will be finally able to apply the Implicit Function Theorem.
As we already did in Theorem 4.3.1, we apply the method of continuity to the family of linear maps

$$\begin{aligned} \mathcal{L}_t : W_0^{2,p}(\overline{U}_2, \mathfrak{g}) &\to L^p(\overline{U}_2, \mathfrak{g}) \\ V_2 &\longmapsto \Delta V_2 + t(A^{h_2}, dV_2) \end{aligned} \tag{4.101}$$

The surjectivity of $\mathcal{L}_0$ is guaranteed by the existence of the solution for the classical Dirichlet problem for the Laplace equation. Therefore, by the method of continuity $\mathcal{L}_1 = \partial_U\tilde{\mathcal{N}}(0,0)$ is surjective too, and thus an isomorphism.
We can finally apply the Implicit Function Theorem, and infer that there exists a $\delta > 0$ such that if we call $V_\delta := \{V \in W^{2,p}(\tilde{U}_2, \mathfrak{g}) : \|V\|_{W^{2,p}} < \delta\}$, then

$$\forall V_1 \in V_\delta \;\; \exists! V_2 \in W_0^{2,p}(\overline{U}_2, \mathfrak{g}) \;\text{ such that }\; \tilde{\mathcal{N}}(V_1, V_2) = 0$$

and furthermore we have the bound

$$\|V_2\|_{W_0^{2,p}(\overline{U}_2)} \le C\delta$$

So if we choose ε small enough then $\tilde{V} \in V_\delta$, and then there exists $\tilde{V}_2$ such that $\tilde{\mathcal{N}}(\tilde{V}, \tilde{V}_2) = 0$. We define $\tilde{h} := \exp(\tilde{V} + \tilde{V}_2)$, and set the gauge $h \in W^{2,2}(S_0, G)$ as in equation (4.96). By construction we have that $d^\star A^h = 0$ in S_0. The inequality in equations (4.97) also follows easily by construction. □

Theorem 4.3.14. *Let $A \in W^{1,2}_{loc}(B^4 \setminus \{0\}, T^*(B^4 \setminus \{0\}) \otimes \mathfrak{g})$ such that*

$$\int_{B^4} |F_A|^2 dx < \infty \tag{4.102}$$

Then there exists $g \in W^{2,2}_{loc}(B^4 \setminus \{0\}, G)$ such that $A^g \in W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g})$.

Proof. We start by assuming, without loss of generality that $YM(A) < \delta$, with δ smaller then the constant $\tilde{\varepsilon}$. We divide the ball in concentric annuli T_k around the origin. Then in each one of them we can apply Lemma 4.3.12, and therefore we find a sequence $g_k \in W^{2,2}(S_k, G)$

such that

$$\begin{cases} \|A^{g_k}\|_{L^4(S_k)} + \|DA^{g_k}\|_{L^2(S_k)} \leq \tilde{C}\|F_A\|_{L^2(T_k)} \\ d^\star A^{g_k} = 0 \text{ in } S_k \end{cases} \tag{4.103}$$

In the intersections $S_k \cap S_{k+1}$ we have the compatibility condition

$$A^{g_{k+1}} = g_{k,k+1}^{-1} dg_{k,k+1} + g_{k,k+1}^{-1} A^{g_k} g_{k,k+1} \tag{4.104}$$

where $g_{k,k+1} := (g_k)^{-1} g_{k+1}$. Since $d^\star A^{g_k} = 0$ for each k then by Lemma 3.2.11, we find that $g_{k,k+1} \in W^{2,(2,1)}_{\text{loc}}(S_k \cap S_{k+1}, G)$, and also that there exists $\overline{g}_{k,k+1}$ such that

$$\|g_{k,k+1} - \overline{g}_{k,k+1}\|_{C^0(\tilde{S}_k \cap \tilde{S}_{k+1})} \leq C\|g_{k,k+1} - \overline{g}_{k,k+1}\|_{W^{2,(2,1)}(\tilde{S}_k \cap \tilde{S}_{k+1})} \leq$$

$$C \leq \left(\|F_A\|_{L^2(T_k \cup T_{k+1})} + \|F_A\|^2_{L^2(T_k \cup T_{k+1})} \right) \tag{4.105}$$

where we have denoted

$$\tilde{S}_k := B_{2^{-k+2}} \setminus B_{2^{-k-2}}$$

We now build a new family of gauges $\{h_k\}_k$ that are all $W^{2,(2,1)}$-near to the same element of the gauge group G. To do so consider

$$\sigma_k := \prod_{l=1}^{k-1} \overline{g}_{l,l+1} \in G \tag{4.106}$$

Our new family of gauges will be $h_k := g_k \sigma_k^{-1} \in W^{2,2}(\tilde{S}_k, G)$. First observe that for each $k \in \mathbb{N}$ the connection A^{h_k} clearly satisfies

$$\begin{cases} \|A^{h_k}\|_{L^4(\tilde{S}_k)} + \|DA^{h_k}\|_{L^2(\tilde{S}_k)} \leq \tilde{C}\|F_A\|_{L^2(T_k)} \\ d^\star A^{h_k} = 0 \text{ in } \tilde{S}_k \end{cases} \tag{4.107}$$

Furthermore, we have that in $\tilde{S}_k \cap \tilde{S}_{k+1}$ the usual compatibility condition

$$A^{h_{k+1}} = (h_{k,k+1})^{-1} dh_{k,k+1} + h_{k,k+1}^{-1} A^{h_k} h_{k,k+1},$$

with $h_{k,k+1} = \sigma_k g_{k,k+1} \sigma_{k+1}^{-1}$. Observing that $\overline{g}_{k,k+1} = \sigma_k^{-1} \sigma_{k+1}$, and that the σ_k are all constant we get the wanted estimate

$$\|h_{k,k+1} - e\|_{L^\infty(\tilde{S}_k \cap \tilde{S}k+1)} \leq C\|g_{k,k+1} - \overline{g}_{k,k+1}\|_{L^\infty(\tilde{S}_k \cap \tilde{S}_{k+1})}$$

$$\leq C \left(\|F_A\|_{L^2(T_k \cup T_{k+1})} + \|F_A\|^2_{L^2(T_k \cup T_{k+1})} \right) \tag{4.108}$$

Therefore by Theorem 2.1.17, choosing δ small enough we have the existence of a family $U_{k,k+1} \in W^{2,(2,1)}(\tilde{S}_k \cap \tilde{S}_{k+1}, \mathfrak{g})$ such that

$$h_{k,k+1} = \exp(U_{k,k+1}) \text{ in } \tilde{S}_k \cap \tilde{S}_{k+1} \tag{4.109}$$

and we have a small $W^{2,(2,1)}$-bound also for $U_{k,k+1}$, specifically

$$\|U_{k,k+1}\|_{W^{2,(2,1)}(\tilde{S}_k \cap \tilde{S}_{k+1})} \leq C\left(\|F_A\|_{L^2(T_k \cup T_{k+1})} + \|F_A\|^2_{L^2(T_k \cup T_{k+1})}\right)$$

We now glue together all the gauges h_k, in order to get a global gauge. We consider a new family of annuli given by

$$Q_k := B_{2^{-k+\frac{3}{2}}} \setminus B_{2^{-k}}$$

and in each one of them we give new gauges as follows. We take a smooth map $\rho \in C_c^\infty([0,2])$ such that $\rho \equiv 1$ in $[1, \sqrt{2}]$. We build the family of cutoff functions

$$\rho_k(x) := \rho(|x|2^k)$$

and the new gauges are $h_k \tau_k \in W^{2,2}(Q_k, G)$ with

$$\tau_k = \exp(\rho_k U_{k,k+1}) \text{ in } Q_k \tag{4.110}$$

Then we note that in $Q_k \cap Q_{k+1}$ we have $\tau_k = h_{k,k+1}$ and $\tau_{k+1} = e$, which means that

$$h_k(x)\tau_k(x) = h_{k+1}(x)\tau_{k+1}(x) \quad \text{if} \quad x \in Q_k \cap Q_{k+1}$$

and we define therefore the gauge

$$g(x) = h_k(x)\tau_k(x) \quad \text{if } x \in Q_k$$

We prove that the connection $A|_{\cup_k Q_k}$ in this new gauge g

$$\tilde{A} := (A|_{\cup_k Q_k})^g \tag{4.111}$$

has $W^{1,2}$-norm bounded in $\cup_k Q_k$. Since we already know that

$$\|A^{h_k}\|_{L^4(Q_k)} + \|DA^{h_k}\|_{L^2(Q_k)} \leq \tilde{C}\|F_A\|_{L^2(T_k)}$$

then by Proposition 3.2.8 we just need to prove that τ_k has a good enough $W^{2,2}$-bound in Q_k. Note that

$$\|\tau_k\|_{L^2(Q_k)} \leq C\left(\|F_A\|_{L^2(T_k \cup T_{k+1})} + \|F_A\|^2_{L^2(T_k \cup T_{k+1})}\right)$$

by construction. Computing its first derivatives we get $\partial_{x_i}\tau_k = \partial_{x_j}\exp(\rho_k U_{k,k+1})(\partial_{x_i}\rho_k U^j_{k,k+1} + \rho_k \partial_{x_i} U^j_{k,k+1})$, and we have therefore the following L^2-bound for each $i = 1, ..., 4$

$$\|\partial_{x_i}\tau_k\|_{L^2(Q_k)} \leq C\left(\|F_A\|_{L^2(T_k\cup T_{k+1})} + \|F_A\|^2_{L^2(T_k\cup T_{k+1})}\right)(\|\partial_{x_i}\rho_k\|_{L^2(Q_k)}+1) \tag{4.112}$$

and since $\rho_k = \rho(|x|2^k)$, there exists $M > 0$ such that for each k the pointwise estimate $|\partial_{x_i}\rho_k| \leq M2^k$ holds. Then, we have the measure $|Q_k| = \frac{\pi^2}{2}2^{-4k}(2^6-1)$, which gives

$$\|\partial_{x_i}\rho_k\|_{L^2(Q_k)} \leq C2^{-k}$$

In particular $C\left(\|F_A\|_{L^2(T_k\cup T_{k+1})} + \|F_A\|^2_{L^2(T_k\cup T_{k+1})}\right)$ controls the $W^{1,2}$-norm of τ_k. Similar estimates show that the same bound holds for the $W^{2,2}$-norm of τ_k. Therefore, applying Proposition 3.2.8, we get

$$\|A^{h_k\tau_k}\|_{W^{1,2}(Q_k)} \leq C\int_{T_k\cup T_{k+1}}|F_A|^2dx + C\left(\int_{T_k\cup T_{k+1}}|F_A|^2dx\right)^{\frac{1}{2}} \tag{4.113}$$

Then summing over $k\in\mathbb{N}$ we have

$$\|\tilde{A}\|_{W^{1,2}(\cup_k Q_k)} \leq \sum_k \|A^{h_k\tau_k}\|_{W^{1,2}(Q_k)} \leq$$

$$\leq 14C\int_{T_k\cup T_{k+1}}|F_A|^2dx + 14C\left(\int_{T_k\cup T_{k+1}}|F_A|^2dx\right)^{\frac{1}{2}} \tag{4.114}$$

Therefore, $\tilde{A} \in W^{1,2}(\cup_k Q_k, T^*(\cup_k Q_k)\otimes\mathfrak{g})$, and extending g from $\cup_k Q_k$ to the whole B^4 is just a technical exercise.

□

Chapter 5

The Extension Problem

In Chapter 4 we found that given $\eta \in H^{\frac{1}{2}}(\partial B^4, T^*\partial B^4 \otimes \mathfrak{g})$ then

1) If we assume $\|\eta\|_{H^{\frac{1}{2}}(\partial B^4)} < \delta$, for δ specified in Theorem 4.3.7, then there exists a solution to the minimization problem (4.8), i.e. there exists $A_0 \in W^{1,2}_\eta(B^4, T^*B^4 \otimes \mathfrak{g})$ which minimizes the Yang-Mills functional in $W^{1,2}_\eta(B^4, T^*B^4 \otimes \mathfrak{g})$

2) If instead we do not make any assumption on the boundary connection $\eta \in H^{\frac{1}{2}}(\partial B^4, T^*\partial B^4 \otimes \mathfrak{g})$, we can only say that there exists $A_0 \in W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g})$ such that
$$YM(A_0) = \inf_{A \in W^{1,2}_\eta} YM(A) \tag{5.1}$$
and $(A_0)_T = \eta^g$, where $g \in H^{\frac{3}{2}}(\partial B^4, G)$. This was Theorem 4.3.10.

In the proof of Theorem 4.3.7 we also arrived to the existence of $A_0 \in W^{1,2}(B^4, T^*B^4 \otimes \mathfrak{g})$ such that (5.1) holds and $(A_0)_T = \eta^g$, for some $g \in H^{\frac{3}{2}}(\partial B^4, G)$. We were able to say that the minimization problem (4.8) was actually solved, because the condition on the $H^{\frac{1}{2}}$-norm of η translated to a condition on the $H^{\frac{1}{2}}$- norm of dg, and this let us extend g to the whole B^4 to a $\tilde{g} \in W^{2,2}(B^4, G)$ thanks to Theorem 4.3.8, and therefore $A_0^{\tilde{g}^{-1}}$ was a solution.
One may try to see if it is possible to extend always a Sobolev map $g \in H^{\frac{3}{2}}(\partial B^4, G)$ to a $\tilde{g} \in W^{2,2}(B^4, G)$, also without any condition on the norm of dg, so that the Plateau Problem for the Yang-Mills functional always has a solution. However, this is not generally possible, and in the following section we will be able to give some counterexamples when the boundary of the domain is $\mathbb{S}^2$ and $\mathbb{S}^1$.
Note that we have found that the Yang-Mills Plateau problem is

strictly related to the problem of extension of Sobolev maps between manifolds, where the target manifold is a compact connected matrix Lie Group G.

5.1 The Extension Problem & Weakly Harmonic maps

In this section we briefly define the concept of weakly harmonic map and local minimizer of the Dirichlet energy functional. We will refer mainly to [14] and [17]. Let (N, γ) be a Riemannian manifold, and by Nash's Theorem, we know that there exists $p \in \mathbb{N}$ such that $N \hookrightarrow \mathbb{R}^p$ isometrically, where in $\mathbb{R}^p$ we are considering the Euclidean metric. As we already did for maps with values in the group G, we now define

$$W^{1,2}(B^m, N) := \{u \in W^{1,2}(B^m, \mathbb{R}^p) : \ u(x) \in N, \text{ for a.e. } x \in B^m\} \tag{5.2}$$

and similarly one defines also the space of traces with values in N. Let us fix $u \in H^{\frac{1}{2}}(\mathbb{S}^{m-1}, N)$, and we define the (eventually empty) set

$$\mathcal{A}_u := \{\tilde{u} \in W^{1,2}(B^m, N) : \ \tilde{u}|_{\partial B^4} = u\} \tag{5.3}$$

The extension problem consists in proving whether for a $u \in H^{\frac{1}{2}}(\mathbb{S}^{m-1}, N)$ the set $\mathcal{A}_u$ is non empty. For each one of these traces we build the minimization problem

$$\inf_{\tilde{u} \in \mathcal{A}_u} \int_{B^m} |d\tilde{u}|^2 dx \tag{5.4}$$

where $\mathcal{E}(u) := \int_{B^m} |du|^2 dx$ is called the **Dirichlet Energy functional**. The following Proposition shows that the extension problem and the minimization problem (5.4) are actually related.

Proposition 5.1.1. *Let $u \in H^{\frac{1}{2}}(\mathbb{S}^{m-1}, N)$ then we have*

$$\mathcal{A}_u \neq \emptyset \Leftrightarrow \ \textit{the minimization problem } (5.4) \ \textit{admits a solution}$$

Proof. ($\Leftarrow$) This implication is obvious.
($\Rightarrow$) If $\tilde{u} \in \mathcal{A}_g$, then we have the existence of a minimizing sequence $\{u_n\}_n \subset \mathcal{A}_u$ for the minimization problem (5.4). We define the sequence $\{v_n\}_n \subset W_0^{1,2}(B^m, \mathbb{R}^p)$ where $v_n := u_n - \tilde{u}$ for each $n \in \mathbb{N}$, and by Poincaré inequality we have that $\|v_n\|_{L^2(B^m)} \leq C\|dv_n\|_{L^2(B^m)}$, which leads to

$$\|v_n\|^2_{W^{1,2}(B^m)} \leq C \int_{B^m} |dv_n|^2 dx$$

Since the right hand side of the last inequality is bounded by hypothesis, then $\{u_n\}_n$ is bounded in $W^{1,2}$ and therefore $u_n \rightharpoonup \hat{u} \in W^{1,2}$. This weak convergence, thanks to Rellich-Kondrakov Theorem, gives us that $\hat{u} \in N$ almost everywhere. Furthermore we have the weak convergence in $H^{\frac{1}{2}}(\mathbb{S}^{m-1}, \mathbb{R}^p)$ also of the traces $u_n|_{\mathbb{S}^{m-1}} = u$, and therefore $\hat{u}|_{\mathbb{S}^{m-1}} = u$. □

Definition 5.1.2. Let $\tilde{u} \in W^{1,2}_{\text{loc}}(B^m, N)$. Then, we say that $\tilde{u}$ is a **local minimizer** of the Dirichlet energy functional if for every $B_\rho(x_0) \subset\subset B^m$ and for every $v \in W^{1,2}(B_\rho(x_0), N)$ with $v = u$ in $\partial B_\rho(x_0)$, we have

$$\int_{B_\rho(x_0)} |d\tilde{u}|^2 dx \leq \int_{B_\rho(x_0)} |dv|^2 dx \tag{5.5}$$

It can be proved that if $\tilde{u} \in W^{1,2}_{\text{loc}}(B^m, N)$ is a local minimizer of the Dirichlet Energy functional, then it satisfies the Euler Lagrange

$$\Delta\tilde{u} + \sum_{i=1}^{4} S_{\tilde{u}}(\partial_{x_i}\tilde{u}, \partial_{x_i}\tilde{u}_i) = 0 \ \text{ in } \ D'(B_\rho(x_0)) \tag{5.6}$$

for each $B_\rho(x_0) \subset\subset B^m$. This result is obtained using inner variations of $\tilde{u}$, and the fact that it is a local critical point for the function $\mathcal{E}$, see for instance [17]. The operator S has already been defined, and it is the shape operator of N.

Definition 5.1.3. Let $\tilde{u} \in W^{1,2}_{\text{loc}}(B^m, N)$. Then we say that $\tilde{u}$ is a **weakly harmonic map** if it satisfies equation (5.6) in each $B_\rho(x_0) \subset\subset B^m$.

Remark 5.1.4. Note that each local minimizer is a weakly harmonic map, but the inverse is generally false.

Let $u \in H^{\frac{1}{2}}(\mathbb{S}^{m-1}, N)$. It is clear that if a solution $\tilde{u} \in W^{1,2}(B^m, N)$ to (5.4) exists then it is also a global minimizer for the Dirichlet Energy functional in $\mathcal{A}_u$ and thus it satisfies the set of equations

$$\begin{cases} \Delta\tilde{u} + \sum_{i=1}^{4} S_{\tilde{u}}(\partial_{x_i}\tilde{u}, \partial_{x_i}\tilde{u}_i) = 0 \ \text{ in } \ D'(B^m) \\ \tilde{u}|_{\mathbb{S}^{m-1}} = u \end{cases} \tag{5.7}$$

Therefore, if for some $u \in H^{\frac{1}{2}}(\mathbb{S}^{m-1}, N)$ the set $\mathcal{A}_u$ is non empty, then there exists $\hat{u} \in \mathcal{A}_u$ satisfying (5.7).
Now we use the theory on the regularity of local minimizers of $\mathcal{E}$ in order to show some counterexamples on the extension problem. Namely we show that it is not true that for each $u \in H^{\frac{1}{2}}(\mathbb{S}^{m-1}, N)$ the set $\mathcal{A}_u$ is non empty.

Example 5.1.5 (**Counterexamples on the extension problem**). In what follows we produce two examples of maps from $\mathbb{S}^{m-1}$ to some manifold N, for $m = 2, 3$, such that they do not admit extensions with enough regularity. In particular:

- Let $N = \mathbb{S}^2$ and consider the map $u \in H^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{S}^2)$ defined by $u(x) = x$. By contradiction let us assume that $\mathcal{A}_u \neq \emptyset$. We apply Proposition 5.1.1 getting the existence of a solution $\hat{u} \in \mathcal{A}_u$ of (5.4).
  Then it is clear that $\hat{u}$ is a weakly harmonic map, and by the regularity theory for 2-dimensional weakly harmonic maps, see [20], we have that $\hat{u} \in C^\infty(B^2, \mathbb{S}^1)$. This is clearly a contradiction since $\hat{u}$ is a continuous retraction of B^2 to its boundary.

- This counterexample is taken from [17], and the idea used to build it is essentially the same of the above example. Let $N = SO(2) \cong \mathbb{S}^1$, and consider the map $g : \mathbb{S}^2 \subset \mathbb{R}^3 \to \mathbb{S}^1$ defined as
  $$g(x_1, x_2, x_3) := \frac{(x_1, x_2)}{|(x_1, x_2)|} \quad \text{where}$$
  $$(x_1, x_2, x_3) \in \mathbb{S}^2 \setminus \{(0, 0, 1), (0, 0, -1)\}$$
  and we have identified $\mathbb{S}^2 := \{x \in \mathbb{R}^3 : \ |x| = 1\}$ and $\mathbb{S}^1 = \{x \in \mathbb{R}^3 : \ x = (x_1, x_2, 0) \ \text{ and } \ |x| = 1\}$. One can easily prove that $g \in H^{\frac{1}{2}}(\mathbb{S}^2, \mathbb{S}^1)$. If there exists an extension $\tilde{g} \in W^{1,2}(B^3, \mathbb{S}^1)$ of g, then we are able to find also a map $\hat{g} \in W^{1,2}(B^3, \mathbb{S}^1)$ solution of the minimization problem (5.4).
  Using the regularity theory for local minimizers of the Dirichlet energy functional (see [14], Theorem 10.11) we have that $\hat{g}$ is Hölder's continuous in $B^3 \setminus \Sigma$, where Σ is a subset of B^3 made of isolated points. Therefore, there exists an hyperplane π of $\mathbb{R}^3$, such that $\hat{g}|_{B^3 \cap \pi} : B^3 \cap \pi \to \mathbb{S}^1$ is continuous. This is not possible since $\hat{g}|_{\partial(B^3 \cap \pi)}$ is an homeomorphism, and therefore we would be able to build a continuous retraction from the disc $B^3 \cap \pi$ to its boundary.

5.2 The Extension Theorem for gauges with proper bound on the norm

We now move to the case $N = G$, for G a compact connected matrix Lie group, and $m = 4$. In G, as always, we consider the metric $\mathbf{g}$ induced by the killing form. In the previous section we have established that if $g \in H^{\frac{1}{2}}(\mathbb{S}^3, G)$ then $\mathcal{A}_g \neq \emptyset$ if and only if the

minimization problem (5.4) admits a solution.
For our purposes we will ask $g \in H^{\frac{3}{2}}(\mathbb{S}^3, G) \hookrightarrow H^{\frac{1}{2}}(\mathbb{S}^3, G)$, and we will show that if we have a proper bound on the $H^{\frac{1}{2}}$-norm of dg, then $\mathcal{A}_g \neq \emptyset$, which means that the extension can be chosen as a solution of (5.7). Furthermore, we will show that, under these assumptions on dg, we can find an extension $\tilde{g} \in W^{2,2}(B^4, G)$, which is the result we wanted to prove in this chapter.

We give a proof of Theorem 5.2.1, and we follow the same idea of F.Bethuel in [5], Theorem 2. There, it is proved that each map $W^{1-\frac{1}{4},4}(\mathbb{S}^3, G)$ admits an extension in $W^{1,4}(B^4, G)$ if and only if each $u \in C^0(\mathbb{S}^3, G)$ admits a continuous extension. Since this last property does not hold when the domain is B^4 and the target manifold is any compact and connected matrix Lie group G, we need some further requirement. In particular, we will ask to the $H^{\frac{1}{2}}$-norm of the differential of the boundary gauge to be under a suitable threshold.

Theorem 5.2.1. *There exists a constant $\varepsilon_3 > 0$ such that for each $g \in H^{\frac{3}{2}}(\partial B^4, G)$ satisfying*

$$\|dg\|_{H^{\frac{1}{2}}(\partial B^4)} < \varepsilon_3 \tag{5.8}$$

there exists an extension $\tilde{g} \in W^{2,2}(B^4, G)$ of g.

Proof. We start by extending $g \in H^{\frac{3}{2}}(\mathbb{S}^3, G)$ in a suitable submanifold of B^4. In particular for some $0 < \delta < 1$ we identify,

$$\begin{aligned} \phi : (\overline{B^4} \setminus B_{1-\rho}(0)) &\to \mathbb{S}^3 \times [0, \delta] \\ x &\longmapsto (P_{\mathbb{S}^3}(x), d(x, \mathbb{S}^3)) \end{aligned} \tag{5.9}$$

where $P_{\mathbb{S}^3}$ is the projection on $\mathbb{S}^3$, while $d(x, \mathbb{S}^3)$ is the distance of $x \in \mathbb{R}^4$ from the boundary $\mathbb{S}^3$. Note that the projection map $P_{\mathbb{S}^3} : \overline{B^4} \setminus B_{1-\rho}(0) \to \mathbb{S}^3$ is well defined for each $0 < \delta < 1$. Now, for $x' \in \mathbb{S}^3$ and $h \in [0, \delta]$ we define the following map

$$v(x', h) = \frac{1}{|B_h(x')|} \int_{B_h(x')} g(y) d\sigma^3(y) \tag{5.10}$$

where $B_h(x')$ is the geodesic ball in $\mathbb{S}^3$ centred in x' and with radius h. It can be proved that $v \in W^{2,2}(\mathbb{S}^3 \times [0, \delta], \mathbb{R}^p)$ (see for instance [25]) where $G \hookrightarrow \mathbb{R}^p$ isometrically. Actually, one also has that v is C^1 in $]0, \delta] \times \mathbb{S}^3$.
We show that for each $(x', h) \in \mathbb{S}^3 \times [0, \delta]$, there exists $\overline{z} \in B_h(x')$ and a constant $C > 0$ such that

$$|v(x', h) - g(\overline{z})| \leq C\varepsilon_3 \tag{5.11}$$

Indeed, note that for each $z \in B_h(x')$

$$v(x',h) - g(z) = \frac{1}{|B_h(x')|}\int_{B_h(x')} g(y) - g(z) d\sigma^3(y)$$

We integrate both sides with respect to $z \in B_h(x')$, and following the computations in [5], we get

$$\frac{1}{|B_h(x')|}\int_{B_h(x')} |v(x',h) - g(z)|^4 d\sigma^3(z) \leq$$
$$\leq C \int_{B_h(x')}\int_{B_h(x')} \frac{|g(y) - g(z)|^4}{|y-z|^6} d\sigma^3(y) d\sigma^3(z) \tag{5.12}$$

Using Poincaré inequality on the sphere $\mathbb{S}^3$, see for instance [36], we get that

$$\|g - \overline{g}\|_{H^{\frac{3}{2}}(\mathbb{S}^3)} \leq C\varepsilon_3 \quad \text{where } \overline{g} = \frac{1}{|\mathbb{S}^3|}\int_{\mathbb{S}^3} g(x) d\sigma^3(x) \tag{5.13}$$

and the Sobolev embedding $H^{\frac{3}{2}}(\mathbb{S}^3) \hookrightarrow W^{1-\frac{1}{4},4}(\mathbb{S}^3)$ guarantee therefore that the left hand side of equation (5.12) is bounded from above by $C\varepsilon_3$. Therefore, we have obtained that inequality (5.11) holds. This means that for ε_3 small enough the image of v is all contained in a tubular neighbourhood of G, where the projection map Π on G is well-defined. Therefore, we can define the map $U_\delta \in W^{2,2}(B^4 \setminus B_{1-\rho}(0), G)$ as,

$$U_\delta := \Pi \circ v \circ \phi \tag{5.14}$$

The next step is to extend to the whole B^4 the map U_δ. As we already did several times in this thesis, we can substitute in equation (5.13) the mean value $\overline{g}$ with an element of the group $\hat{g} \in G$, and obtain

$$\|g - \hat{g}\|_{H^{\frac{3}{2}}(\mathbb{S}^3)} \leq C\varepsilon_3 \tag{5.15}$$

We see that if we fix $\delta < 1$, then we get

$$|v(x',\delta) - \hat{g}| \leq \frac{1}{|B_\delta(x')|}\int_{B_\delta(x')} |g(y) - \hat{g}| d\sigma^3(y) \leq \frac{1}{|B_\delta(x')|} C\varepsilon_3$$

where in the last inequality we have used the Sobolev embedding $H^{\frac{3}{2}}(\mathbb{S}^3) \hookrightarrow L^1(\mathbb{S}^3)$ together with the estimate (5.15). This implies that there exists $\hat{U} \in G$, such that

$$\|U_\delta - \hat{U}\|_{L^\infty(\partial B_{1-\rho}(0))} \leq C_\delta \varepsilon_3$$

and therefore for ε_3 small enough, there exists $V \in C^1 \cap H^{\frac{3}{2}}(\partial B_{1-\delta}(0), \mathfrak{g})$ such that

$$U_\delta|_{B_{1-\delta}(0)} = \hat{U} \exp(V)$$

and we extend V harmonically to a $\tilde{V} \in C^1 \cap W^{2,2}(B_{1-\delta}(0), \mathfrak{g})$. Therefore, the function defined by

$$\tilde{g} := \begin{cases} U_\delta & \text{in } \; B^4 \setminus B_{1-\delta}(0) \\ \hat{U} \exp(\tilde{V}) & \text{in } \; B_{1-\delta}(0) \end{cases}$$

is a $W^{2,2}$-extension for $g \in H^{\frac{3}{2}}(\mathbb{S}^3, G)$. $\square$

Appendices

Appendix A

Sobolev and Lorentz-Sobolev spaces

In this Appendix we introduce some important function spaces that we have used throughout this work, and we will give formal definitions of them. The first spaces we define are Lorentz spaces, which can be considered as intermediate spaces of the classical L^p, and for them we will present two equivalent formulations, the first based on the concept of decreasing rearrangement and the second one will require interpolation theory. For the Fractional Sobolev spaces and Besov spaces we will instead workout just their interpretation as interpolation spaces since with this method we can easily obtain some important embeddings, that can be considered as a generalization of the classical Sobolev embeddings.
The first section of this Appendix is therefore devoted to the development of the interpolation theory, that we will heavily use in the other two sections. We will follow essentially [4].

A.1 Interpolation Theory

Let $\mathcal{N}$ be the category[1] of normed vector spaces, and we will consider as morphisms between to objects A and B, the set of all linear and continuous maps $T : A \to B$.

Definition A.1.1. More generally let A_0 and A_1 be two topological vector spaces. Then we shall say that they are **compatible** if there

[1] A category C is the data of a class of objects $Ob(C)$ of C and morphisms $Hom_C(A, B)$ between $A, B \in Ob(C)$ such that if $f \in Hom_C(A, B)$ and $g \in Hom_C(B, D)$ then $g \circ f \in Hom_C(A, D)$ and associativity holds. Furthermore for every object A in $Ob(C)$ there exists a morphism $1_A \in Hom_C(A, A)$ called identity such that for every $f \in Hom_C(A, D)$ and $g \in Hom_C(B, A)$ one has $1_A \circ f = f$ and $g \circ 1_A = g$ for B, D objects of C.

is an Hausdorff topological vector space U, such that A_0 and A_1 are both subspaces of U.

If A_0 and A_1 are two compatible normed vector spaces then it can be easily proved that $A_0 \cap A_1$ is a normed vector space with the norm

$$\|a\|_{A_0 \cap A_1} = \max\left(\|a\|_{A_0}, \|a\|_{A_1}\right) \quad a \in A_0 \cap A_1$$

and also $A_0 + A_1$ is a normed vector space with the norm

$$\|a\|_{A_0+A_1} = \inf_{a=a_0+a_1} \left(\|a_0|_{A_0} + \|a_1\|_{A_1}\right) \quad a \in A_0 + A_1$$

Proposition A.1.2. *Let A_0 and A_1 be compatible Banach vector spaces. Then $A_0 \cap A_1$ and $A_0 + A_1$ are Banach too.*

Proof. We prove the statement for $A_0 \cap A_1$, using the characterization of Banach spaces. Consider a sequence $\{a_n\}_n \subset A_0 \cap A_1$, such that

$$\sum_n \|a_n\|_{A_0 \cap A_1} < \infty$$

This means that both $\sum_n \|a_n\|_{A_0}$ and $\sum_n \|a_n\|_{A_1}$ are finite. Then they both converge, since A_0 and A_1 are Banach. Moreover these two spaces are compatible, therefore the limits of the series (in the two different norms) are coinciding and then it is in $A_0 \cap A_1$. Thus the series $\sum_n a_n$ converges also in $A_0 \cap A_1$. □

Let now $\mathcal{C}$ denote any subcategory[2] of $\mathcal{N}$, such that the morphisms between two objects A, B in $\mathcal{C}$ are still all the linear and continuous operators from A to B. Then with $\mathcal{C}_1$ we indicate a new category made of couples $\overline{A} = (A_0, A_1)$ of compatible vector spaces, such that $A_0 + A_1$ and $A_0 \cap A_1$ are objects in $\mathcal{C}$. The morphisms $T : (A_0, A_1) \to (B_0, B_1)$ in $\mathcal{C}_1$ are all the bounded and linear maps from $A_0 + A_1$ to $B_0 + B_1$ such that their restrictions $T_{A_0} : A_0 \to A_1$ and $T_{B_0} : B_0 \to B_1$ are morphisms is $\mathcal{C}$. Now we consider $\mathcal{C}$ such that it is also closed under the operations of sum and intersection.

Definition A.1.3. Let $\overline{A} = (A_0, A_1)$ be a given couple in $\mathcal{C}_1$. Then a space A in $\mathcal{C}$ is an **intermediate space** between A_0 and A_1 if $A_0 \cap A_1 \subset A \subset A_0 + A_1$, with continuous inclusions. The space A is called an **interpolation space** between A_0 and A_1 if in addition $T : \overline{A} \to \overline{A}$ implies $T : A \to A$.
More generally let $\overline{A}$ and $\overline{B}$ be two couples in $\mathcal{C}_1$. Then we say that two spaces A and B in $\mathcal{C}$ are **interpolation spaces** with respect to $\overline{A}$ and $\overline{B}$ if A and B are intermediate spaces with respect to $\overline{A}$ and $\overline{B}$ and if $T : \overline{A} \to \overline{B}$ implies $T : A \to B$.

[2]A subcategory S of a category C is a category whose objects are objects of C and whose morphisms are morphisms in C with the same identities and composition of morphisms.

Example A.1.4. Let $\overline{A}$ and $\overline{B}$ be two couples in $\mathcal{C}_1$. Then clearly $\Delta(\overline{A}) := A_0 \cap A_1$ and $\Delta(\overline{B})$ are interpolation spaces with respect to $\overline{A}$ and $\overline{B}$. The same holds true for $\sum(\overline{A}) := A_0 + A_1$ and $\sum(\overline{B})$. In particular it is easy to see that if $A = \Delta(\overline{A})$ and $B = \Delta(\overline{B})$ then for any $T : \overline{A} \to \overline{B}$ we have[3]

$$\|T\|_{A,B} \leq \max\left(\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1}\right) \tag{A.1}$$

Interpolations spaces are classified as follows

Definition A.1.5. If $\overline{A}$ and $\overline{B}$ are as above, and A, B are interpolation spaces with respect to $\overline{A}$ and $\overline{B}$, then we say that they are **exact interpolation spaces** if (A.1) holds for every $T : \overline{A} \to \overline{B}$. If equation (A.1) holds but with a multiplicative constant $C \neq 1$ on the right hand side, then we say that A and B are **uniform interpolation spaces**.
Finally the interpolation spaces A and B are said to be of **exponent** θ if

$$\|T\|_{A,B} \leq C\|T\|^{1-\theta}_{A_0,B_0}\|T\|^{\theta}_{A_1,B_1} \tag{A.2}$$

for every $T : \overline{A} \to \overline{B}$. If $C = 1$ then we say that A and B are **exact of exponent** θ, where $0 \leq \theta \leq 1$.

We now define what is an interpolation functor, namely a method of constructing interpolation spaces. In the following subsection we will present the two main real interpolation functors, the K-method and the J-method.

Definition A.1.6. By an **interpolation functor** on $\mathcal{C}$ we mean a functor[4] F from $\mathcal{C}_1$ to $\mathcal{C}$ such that if $\overline{A}$ and $\overline{B}$ are couples in $\mathcal{C}_1$ then $F(\overline{A})$ and $F(\overline{B})$ are interpolation spaces with respect to $\overline{A}$ and $\overline{B}$. Moreover we shall write $F(T) = T$ for every $T : \overline{A} \to \overline{B}$. We will say that F is a **uniform** (or exact) **interpolation functor** if $F(\overline{A})$ and $F(\overline{B})$ are uniform (or exact) interpolation spaces with respect to $\overline{A}$ and $\overline{B}$. Similarly we deduce the definition of **functor of exponent** θ.

A.1.1 The K-Method & The J-Method

We now introduce two families of interpolation functors on $\mathcal{N}$. The theory we develop follows essentially the work of Jaak Peetre [31].

[3] If $T : A \to B$ is linear and continuous, with A and B normed vector spaces, then we define $\|T\|_{A,B} := \sup_{\|a\|_A=1} \|Ta\|_B$

[4] A (covariant) functor F from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a mapping such that $F(A) \in Ob(\mathcal{D})$ for every $A \in Ob(\mathcal{C})$, and $F(f) \in Hom_{\mathcal{D}}(F(A), F(B))$ for every $f \in Hom_{\mathcal{C}}(A, B)$. Furthermore if $f \in Hom_{\mathcal{C}}(A, B)$ and $g \in Hom_{\mathcal{C}}(B, E)$ then $F(g \circ f) = F(f) \circ F(g)$ and $F(1_A) = 1_{F(A)}$ for every $A \in Ob(\mathcal{C})$.

The K-Method We start by defining the following function for every fixed couple $\overline{A} = (A_0, A_1)$ in the category $\mathcal{N}_1$,

$$K : \mathbb{R}^+ \times (A_0 + A_1) \to \mathbb{R}^+$$

$$(t, a) \longrightarrow K(t, a; \overline{A}) := \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}) \tag{A.3}$$

and for every fixed $t > 0$, it is clearly an equivalent norm on $A_0 + A_1$. In particular we have the following proposition.

Proposition A.1.7. *For every fixed $a \in A_0 + A_1$, $K(t, a)$ is a positive, increasing and concave function of t. In particular we have the following inequality*

$$K(t, a) \leq \max\left(1, \frac{t}{s}\right) K(s, a) \tag{A.4}$$

Proof. The proof of $K(t, a)$ being positive and increasing for every fixed $a \in A_0 + A_1$ is straightforward. While for the concavity we see that if $t, h \in \mathbb{R}^+$ and $a \in A_0 + A_1$ is fixed, then

$$K\left(\frac{t+h}{2}, a\right) = \inf_{a=a_0+a_1} \left(\frac{1}{2}\|a_0\|_{A_0} + \frac{t}{2}\|a_1\|_{A_1} + \frac{1}{2}\|a_0\|_{A_0} + \frac{h}{2}\|a_1\|_{A_1}\right) \geq$$

$$\geq \frac{1}{2} \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}) + \frac{1}{2} \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + h\|a_1\|_{A_1}) =$$

$$= \frac{1}{2}K(t, a) + \frac{1}{2}K(h, a)$$

Instead, for inequality (A.4) we choose $t, s \in \mathbb{R}^+$, and see that for each fixed $a \in A_0 + A_1$, we have

$$K(t, a) = \inf_{a=a_0+a_1} \left(\|a_0\|_{A_0} + \frac{t}{s} s\|a_1\|_{A_1}\right) \leq \max\left(1, \frac{t}{s}\right) K(s, a)$$

□

We define the functional $\Phi_{\theta,q}$ over measurable functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, for $0 \leq \theta \leq 1$ and $1 \leq q \leq \infty$:

$$\Phi_{\theta,q}(\phi) := \left(\int_0^\infty \left(t^{-\theta}\phi(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \quad \text{if } 1 \leq q < \infty$$

$$\Phi_{\theta,q}(\phi) := \sup_t t^{-\theta}\phi(t) \quad \text{if } q = \infty \tag{A.5}$$

Definition A.1.8. Let $0 < \theta < 1$ and $1 \leq q \leq \infty$, either $0 \leq \theta \leq 1$ and $q = \infty$. For these values we let $\overline{A}_{\theta,q;K} = K_{\theta,q}(\overline{A})$ denote the set of all elements $a \in A_0 + A_1$ such that

$$\Phi_{\theta,q}(K(\cdot, a)) < \infty \tag{A.6}$$

This is a normed subspace of $A_0 + A_1$, endowed with the norm $\|a\|_{\theta,q} = \Phi_{\theta,q}(K(\cdot, a))$.

Theorem A.1.9. *$K_{\theta,q}$ is an exact interpolation functor of exponent θ on the category $\mathcal{N}$. Moreover we have the inequality*

$$K(s, a; \overline{A}) \leq C_{\theta,q} s^{\theta} \|a\|_{\theta,q} \tag{A.7}$$

Proof. Inequality (A.4) can be rewritten as

$$\min\left(1, \frac{t}{s}\right) K(s, a) \leq K(t, a)$$

for each fixed $a \in A_0 + A_1$. Applying then the functional $\Phi_{\theta,q}$ on both sides we get

$$\Phi_{\theta,q}\left(\min\left(\frac{t}{s}, 1\right)\right) K(s, a) \leq \|a\|_{\theta,q}$$

and a computation, see [4], shows that $\Phi_{\theta,q}\left(\min\left(\frac{t}{s}, 1\right)\right) = s^{-\theta} C_{\theta,q}^{-1}$, where $C_{\theta,q}^{-1}$ is a constant depending on θ and q.
Inequality (A.7), with $s = 1$, tells us that $\overline{A}_{\theta,q} \hookrightarrow A_0 + A_1$. For the remaining inclusion we note that if $a \in A_0 \cap A_1$, then

$$K(t, a) \leq \min(1, t)\|a\|_{A_0 \cap A_1} \Rightarrow \|A\|_{\theta,q} \leq \Phi(\min(1, t))\|a\|_{A_0 \cap A_1}$$

and therefore $A_0 \cap A_1 \hookrightarrow \overline{A}_{\theta,q}$. So far, we have proved that $\overline{A}_{\theta,q}$ is an intermediate space. Now we show that $K_{\theta,q}$ is an exact interpolation functor. Let $\overline{A} = (A_0, A_1)$, and $\overline{B} = (B_0, B_1)$, and consider the linear and continuous operator $T : \overline{A} \to \overline{B}$. Then if we call $M_i = \|T\|_{A_i,B_i}$ for $i = 0, 1$, we have that

$$K(t, Ta; \overline{B}) \leq \inf_{a=a_0+a_1} (\|Ta_0\|_{A_0} + t\|Ta_1\|_{A_1}) \leq M_0 K\left(t\frac{M_1}{M_0}, a; \overline{A}\right)$$

Applying on both sides the functional $\Phi_{\theta,a}$, after a computation we get

$$\|Ta\|_{\overline{B}_{\theta,q}} \leq M_0^{1-\theta} M_1^{\theta} \|a\|_{\overline{A}_{\theta,q}}$$

□

The J-Method We define the following function for every fixed couple $\overline{A} = (A_0, A_1)$ in $\mathcal{N}_1$

$$\begin{aligned} J : \mathbb{R}^+ \times A_0 \cap A_1 &\to \mathbb{R}^+ \\ (t, a) &\mapsto J(t, a; \overline{A}) := \max\left(\|a\|_{A_0}, t\|a\|_{A_1}\right) \end{aligned} \tag{A.8}$$

and for every $t \in \mathbb{R}^+$ the function $J(t, \cdot)$ is an equivalent norm in $A_0 \cap A_1$. We collect some useful properties of J in the following proposition.

Proposition A.1.10. *For every fixed $a \in A_0 \cap A_1$ the function $J(\cdot, a)$ is a positive, increasing and convex function of t. Furthermore the following inequalities hold:*

$$J(t, a) \leq \max\left(1, \frac{t}{s}\right) J(s, a)$$

$$K(t, a) \leq \min\left(1, \frac{t}{s}\right) J(s, a) \tag{A.9}$$

We are now ready to define the interpolation spaces obtained through this interpolation method.

Definition A.1.11. For $0 < \theta < 1$ and $1 \leq q \leq \infty$, either $0 \leq \theta \leq 1$ and $q = 1$, we define $\overline{A}_{\theta,q;J} = J_{\theta,q}(\overline{A})$ as the normed subspace of $A_0 + A_1$ made of $a \in A_0 + A_1$ such that they can be represented by $a = \int_0^\infty u(t) \frac{dt}{t}$ (the convergence is in $A_0 + A_1$)[5] where $u(t)$ is measurable with values in $A_0 \cap A_1$ and $\Phi_{\theta,q}(J(t, u(t)) < \infty$. The norm we consider for $\overline{A}_{\theta,q;J}$ is $\|a\|_{\theta,q;J} := \inf_u \Phi_{\theta,q}(J(t, u(t))$.

Theorem A.1.12. *Let $J_{\theta,q}$ be defined as above. Then $J_{\theta,q}$ is an exact interpolation functor of exponent θ on the category $\mathcal{N}$. Moreover we have the following inequality*

$$\|a\|_{\theta,q;J} \leq C s^{-\theta} J(s, a; \overline{A}) \tag{A.10}$$

for every $a \in A_0 \cap A_1$, where C is a constant independent of θ and q.

The J-method admits a discrete formulation which turns out to be useful in some situations. Here we state a Lemma, that we will apply later when studying Lorentz-Sobolev spaces.

Lemma A.1.13. *Let $a \in A_0 + A_1$, then $a \in J_{\theta,q}(\overline{A})$ if and only if there exists a sequence $a_n \in A_0 \cap A_1$ (with $-\infty < n < \infty$) such that*

$$a = \sum_{-\infty}^{\infty} a_n \tag{A.11}$$

---

[5]Here the integral is the Bochner integral, see for instance [26]

where the convergence is in A_0+A_1, and $2^{-n\theta}J(2^n,a_n)^q \in l^1$.[6] *Moreover*

$$\|a\|_{\theta,q;J} \sim \inf_{a_n}\left(\sum_{-\infty}^{\infty} 2^{-n\theta}J(2^n,a_n)^q\right)^{\frac{1}{q}} \tag{A.12}$$

where the infimum is taken over all sequences $\{a_n\}$ satisfying (A.11).

It can be proved that the K-method and the J-method are equivalent as far as $\theta \neq 0, 1$. This result is called the **equivalence theorem**.

Theorem A.1.14. *If $0 < \theta < 1$ and $1 \leq q \leq \infty$ then $J_{\theta,q}(\overline{A}) = K_{\theta,q}(\overline{A})$ with equivalent norms.*

We now state some few simple properties of interpolation spaces. These results are stated for $\theta \in (0,1)$ which implies that the K and J methods produce the same spaces, therefore we will drop the index K and J. In this case we will denote the space $\overline{A}_{\theta,q}$ also with $(A_0,A_1)_{\theta,q}$, always assuming that $0 < \theta < 1$. When $\theta = 0, 1$ we will specify what method we are using.

Proposition A.1.15. *Let $\overline{A} = (A_0, A_1)$ be a couple in $\mathcal{N}_1$, and $0 < \theta < 1$, $1 \leq q \leq \infty$. Then we have the followings properties:*

a) $(A_0,A_1)_{\theta,q} = (A_1,A_0)_{1-\theta,q}$ with equal norms

b) $\overline{A}_{\theta,q} \subset \overline{A}_{\theta,r}$ if $q \leq r$

c) If furthermore A_0 and A_1 are complete then so is $\overline{A}_{\theta,q}$

d) if $q < \infty$ then $A_0 \cap A_1$ is dense in $\overline{A}_{\theta,q}$.

e) If A_0 and A_1 are both reflexive Banach spaces and $q < \infty$, then $\overline{A}_{\theta,q}$ is reflexive too.

Proof. b)If $r = \infty$ then by equation (A.7) we get that $t^{-\theta}K(t,a) \leq C\|a\|_{\theta,q}$, and therefore $a \in \overline{A}_{\theta,\infty}$. Now let $1 \leq q \leq r < \infty$ and $a \in \overline{A}_{\theta,q}$, then

$$\|a\|_{\theta,r} = \left(\int_0^\infty (t^{-\theta}K(t,a))^q \frac{1}{t}((t^{-\theta}K(t,a))^{r-q}dt\right)^{\frac{1}{r}}$$

Using again inequality (A.7) in the third factor inside the integral we easily get that $\|a\|_{\theta,r} \leq C\|a\|_{\theta,q}$.
c)To prove this we use a characterization of Banach spaces.[7] Take

[6] The space l^1 is the space of all real sequences a_n such that $\sum_n |a_n| < \infty$.

[7] A normed space $(N, \|\cdot\|)$ is a Banach space if and only if for every sequence $\{a_n\}_{n\in\mathbb{N}}$ such that $\sum \|a_n\| < \infty \Rightarrow \sum a_n$ converges in N.

therefore a sequence $\{a_n\}$ in $\overline{A}_{\theta,q}$ such that $\sum \|a_n\|_{\theta,q}$ is finite. Then this sum is clearly finite also if we substitute the norm of $\overline{A}_{\theta,q}$ with the one of $A_0 + A_1$. But $A_0 + A_1$ is a Banach space, and thus there exists $\tilde{a} \in A_0 + A_1$, such that $\sum_n a_n = \tilde{a}$. Now observe that $K(t, \tilde{a}) \leq \sum_n K(t, a_n)$, therefore applying the functional $\Phi_{\theta,q}$ to both sides we get $\|\tilde{a}\|_{\theta,q} \leq \sum_n \|a_n\|_{\theta,q}$, and this concludes the proof.
e)A Banach space $(E, \|\cdot\|_E)$ is reflexive, if the canonical injection[8] $J : E \to E^{**}$ is also surjective. Then by hypothesis we have that $J : A_i \to A_i^{**}$ is surjective for $i = 0, 1$. Using Theorem 3.7.1 in [4], we get that $(A_0, A_1)_{\theta,q}^{**} = (A_0^{**}, A_1^{**})_{\theta,q}$, and therefore the surjectivity of $J : \overline{A}_{\theta,q} \to \overline{A}_{\theta,q}^{**}$ easily follows.

□

Remark A.1.16. Point d) of Proposition A.1.15 has a consequence that can be useful. If C is a normed vector space that is dense in $A_0 \cap A_1$ then it is also dense in $\overline{A}_{\theta,q}$, as far we are assuming $q < \infty$. Indeed, if $f \in \overline{A}_{\theta,q}$ and $\{\phi_n\}_n \subset A_0 \cap A_1$ is a converging sequence to f, we can consider for each fixed n a sequence $\{\phi_m^n\}_m$ in C that converges to ϕ_n in $A_0 \cap A_1$, by density of C in $A_0 \cap A_1$. Then for every $\varepsilon > 0$, we have that

$$\begin{aligned} \|f - \phi_n^m\|_{\overline{A}_{\theta,q}} &\leq \|f - \phi_n\|_{\overline{A}_{\theta,q}} + \|\phi_n - \phi_n^m\|_{\overline{A}_{\theta,q}} \leq \\ &\leq \|f - \phi_n\|_{\overline{A}_{\theta,q}} + C\|\phi_n - \phi_n^m\|_{A_0 \cap A_1} < \varepsilon \end{aligned}$$

for a proper choice of n and m.

A.1.2 The Reiteration Theorem

This section is devoted to a fundamental result of the real interpolation method. If two spaces X_0 and X_1 are obtained from a given couple $\overline{A} = (A_0, A_1)$ in $\mathcal{N}_1$ by means of the real interpolation method, and if X is constructed from $\overline{X} = (X_0, X_1)$ by means of the real method too then X can be directly built from $\overline{A}$ always by the real interpolation method. In what follows we give some definitions that will allow us to specify the proper conditions under which we can apply such a theorem.

Definition A.1.17. Let $\overline{A} = (A_0, A_1)$ be a given couple of normed vector spaces. Suppose that X is an intermediate space with respect to $\overline{A}$. For $0 \leq \theta \leq 1$ we say that

- X is of class $C_K(\theta, \overline{A})$ if $K(t, a; \overline{A}) \leq Ct^\theta \|a\|_X$, with $a \in X$

[8] If $(E, \|\cdot\|_E)$ is a Banach spaces, then the canonical injection $J : E \to E^{**}$ is defined as $J(f) = \langle \cdot, f \rangle : E^* \to \mathbb{R}$ for each$f \in E$, where in this context we denoted with $\langle \cdot, \cdot \rangle$ the dual coupling.

- X is of class $C_J(\theta, \overline{A})$ if $\|a\|_X \le Ct^{-\theta} J(t, a; \overline{A})$, with $a \in A_0 \cap A_1$
- We say that X is of class $C(\theta, \overline{A})$ if X is of class $C_K(\theta, \overline{A})$ and of class $C_J(\theta, \overline{A})$.

Sometimes it is convenient to use the following characterization of the above definition. In particular we have this result.

Proposition A.1.18. *Let $\overline{A} = (A_0, A_1)$ be a given couple of normed vector spaces, and let X be an intermediate space. Then*

a) X is of class $C_K(\theta, \overline{A}) \Leftrightarrow A_0 \cap A_1 \subset X \subset \overline{A}_{\theta,\infty;K}$

b) X is of class $C_J(\theta, \overline{A}) \Leftrightarrow \overline{A}_{\theta,1;J} \subset X \subset A_0 + A_1 \Leftrightarrow$ we have $\|a\|_X \le C\|a\|_{A_0}^{1-\theta}\|a\|_{A_1}^{\theta}$ for each $a \in A_0 \cap A_1$.

Note that we already know that $\overline{A}_{\theta,q}$ is of class $C(\theta, \overline{A})$ if $0 < \theta < 1$. We are now ready to state the reiteration theorem.

Theorem A.1.19. *Let $\overline{A} = (A_0, A_1)$ and $\overline{X} = (X_0, X_1)$ be two couples of spaces in $\mathcal{N}_1$, and assume that X_i are complete and of class $C(\theta_i, \overline{A})$, where $0 \le \theta_i \le 1$ and $\theta_0 \ne \theta_1$. Then for $1 \le q \le \infty$ $\overline{X}_{\eta,q} = \overline{A}_{\theta,q}$ where $\theta = (1-\eta)\theta_0 + \eta\theta_1$ with $\eta \in (0, 1)$. As a consequence if $0 < \theta_i < 1$ and $\overline{A}_{\theta_i,q}$ are complete then $(\overline{A}_{\theta_0,q_0}, \overline{A}_{\theta_1,q_1})_{\eta,q} = \overline{A}_{\theta,q}$ and $1 < q_0, q_1 \le \infty$*

A.2 Lorentz spaces

In this section we will introduce Lorentz spaces, and we will define them in two different but equivalent ways. First we will get these spaces using the decreasing rearrangement function. Our second definition instead is based on the interpolation by means of the real method of the most known L^p spaces.

A first definition of Lorentz spaces Let Ω be an open subset of $\mathbb{R}^n$, and let f be a scalar measurable function which is finite a.e. (for the Lebesgue measure). We introduce the function[9]

$$m_f(\sigma) = |\{x \in \Omega : \ |f(x)| > \sigma\}| \tag{A.13}$$

The function $m_f : \mathbb{R}^+ \to \mathbb{R}^+$ is non increasing and continuous on the right.

Definition A.2.1. Let $f : \Omega \to \mathbb{R}^+$ be measurable. Then its **decreasing rearrangement** f^* is the following function on $\mathbb{R}^+$

$$f^*(t) := \inf\{\sigma \in \mathbb{R}^+ : \ m_f(\sigma) \le t\} \tag{A.14}$$

[9] If A is any Lebesgue measurable subset of $\mathbb{R}^n$, then by $|A|$ we denote its Lebesgue measure.

The decreasing rearrangement is clearly a non negative, non increasing function on $\mathbb{R}^+$ and it satisfies $m_f(\sigma) = m_{f^*}(\sigma)$ for every $\sigma \geq 0$. In the following proposition we will state some of its properties that will turn out to be useful later. Further significant features of this function can be found in [16].

Proposition A.2.2. *Let $f, g : \Omega \to \mathbb{R}$ be two measurable functions. Then :*

1) $(f+g)^*(t_1+t_2) \leq f^*(t_1) + g^*(t_2)$

2) $(fg)^*(t_1+t_2) \leq f^*(t_1) g^*(t_2)$

3) $(|f|^p)^* = (f^*)^p$ *for* $0 < p < \infty$

4) $\int_\Omega |f|^p = \int_0^{|\Omega|} (f^*)^p$ *for* $0 < p < \infty$

5) If $f \in L^\infty(\Omega)$*, then* $\|f\|_{L^\infty} = f^*(0)$*.*

Proof. 1)& 2) We define the following two sets $A := \{s_1 | \; m_f(s_1) \leq t_1\}$ and $B := \{s_2 | \; m_g(s_2) \leq t_2\}$. Then if we call $C := \{s| \; m_{fg}(s) \leq t_1 + t_2\}$ and $S = \{s| \; mf+g \leq t_1+t_2\}$ we clearly have $A \cdot B \subset C$, and $A+B \subset S$[10]. In particular we have that $(fg)^*(t_1+t_2) = \inf C \leq s_1 \cdot s_2$ and $(f+g)^*(t_1+t_2) = \inf S \leq s_1 + s_2$ for each $s_1 \in A$ and $s_2 \in B$. Then taking the infimum over all elements in A and B we finally get the wanted inequalities.
3)Note that the following sets coincide
$\{s| \; m(s, |f|^p) \leq t\} = \{\sigma^p | \; m(\sigma, f) \leq t\}$. Therefore, taking the infimum in both sides we deduce that $(|f|^p)^* = (f^*)^p$.
5)By the fact that $m_f(\|f\|_{L^\infty}) = 0$ we get that $f^*(0) \leq \|f\|_{L^\infty}$. Conversely one sees that if there exists $s < \|f\|_{L^\infty}$ such that $m_f(s) = 0$, then s must be larger or equal to $\|f\|_{L^\infty}$, which is a contradiction. $\square$

Definition A.2.3. For $1 \leq p \leq \infty$ we define the **Lorentz space** $L^{p,q}(\Omega)$, as the space of all measurable functions $f : \Omega \to \mathbb{R}$, such that $\|f\|_{L^{p,q}} < \infty$ where

$$\|f\|_{L^{p,q}} := \left(\int_0^\infty \left(t^{\frac{1}{p}} f^*(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \qquad \text{if} \quad q < \infty$$

$$\|f\|_{L^{p,q}} := \sup_{t>0} t^{\frac{1}{p}} f^*(t) \qquad \text{if} \quad q = \infty \tag{A.15}$$

[10]If A and B are two subsets of $\mathbb{R}$, then we define $A \cdot B = \{s_1 \cdot s_2 | \; s_1 \in A \text{ and } s_2 \in B\}$

Remark A.2.4. Note that $L^{p,p}(\Omega) = L^p(\Omega)$, for every $1 \le p < \infty$. Indeed by the above definition we have that

$$\|f\|_{L^{p,p}} = \left(\int_0^\infty f^*(t)^p dt\right)^{\frac{1}{p}} = \left(\int_\Omega |f|^p dx\right)^{\frac{1}{p}} = \|f\|_{L^p}$$

where the penultimate equality is due to the fourth point of Proposition A.2.1. Nevertheless it is important to highlight the fact that if $p \neq q$ then $\|\cdot\|_{L^{(p,q)}}$, defined in equation (A.15), is not a norm but a seminorm.

We now give a second definition of Lorentz spaces based on interpolation theory, and after that we will state the most important properties of these spaces.

An interpolation Formulation We start with the following theorem, that allows us to define the Lorentz spaces as interpolation spaces of L^p spaces. As above let Ω be an open subset of $\mathbb{R}^n$.

Theorem A.2.5. *Suppose that $f \in L^p(\Omega) + L^\infty(\Omega)$ for $1 \le p < \infty$. Then $K(t, f; L^p(\Omega), L^\infty(\Omega)) \sim \left(\int_0^{t^p} (f^*(s))^p ds\right)^{\frac{1}{p}}$. Moreover with $1 \le p_0 < p_1 \le \infty$ we have*

$$(L^{p_0}(\Omega), L^{p_1}(\Omega))_{\theta,q} = L^{p,q}(\Omega) \tag{A.16}$$

if $p_0 < q \le \infty$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Proof. Let $1 \le p < \infty$, and consider the decomposition $f = f_0 + f_1$, where

$$f_0(x) := \begin{cases} f(x) - f^*(t^p) & \text{if } |f(x)| > f^*(t^p) \\ 0 & otherwise \end{cases}$$

and clearly $f_1 := f - f_0$. The function f_1 by definition is $L^\infty(\Omega)$, while it is easy to show that $f_0 \in L^p(\Omega)$. If we call $E = \{x \in \Omega |\ f_0(x) \neq 0\}$, we have by definition of decreasing rearrangement of f that $|E| \le t^p$, and since f^* is constant in the interval $[|E|, t^p]$ we get

$$K(t, f; L^p(\Omega), L^\infty(\Omega)) \le \|f_0\|_{L^p} + t\|f_1\|_{L^\infty} =$$

$$= \left(\int_E |f(x) - f^*(t^p)|^p dx\right)^{\frac{1}{p}} + tf^*(t^p) = (*)$$

By the fourth point of Proposition A.2.1 we get

$$(*) = \left(\int_0^{t^p} (f^*(s) - f^*(t^p))^p ds\right)^{\frac{1}{p}} + \left(\int_0^{t^p} (f^*(t^p))^p\right)^{\frac{1}{p}} \le$$

$$\leq C \left(\int_0^{t^p} f^*(s) ds \right)^{\frac{1}{p}}$$

Consider now any decomposition $f = f_0 + f_1$ with $f_0 \in L^p(\Omega)$ and $f_1 \in L^\infty(\Omega)$. Then by the first point of Proposition A.2.2 we have that for $1 > \varepsilon > 0$, $f^*(s) = (f_0 + f_1)^*(s) \leq f_0^*((1-\varepsilon)s) + f_1^*(\varepsilon s)$, and this implies that

$$\left(\int_0^{t^p} (f^*(s))^p ds \right)^{\frac{1}{p}} \leq \left(\int_0^{t^p} (f_0^*((1-\varepsilon)s)^p ds \right)^{\frac{1}{p}} + \left(\int_0^{t^p} (f_1^*(\varepsilon s))^p \right)^{\frac{1}{p}}$$

By point 5) of Proposition A.2.2, and by the fact that the decreasing rearrangement is decreasing, we get

$$\left(\int_0^{t^p} (f^*(s))^p ds \right)^{\frac{1}{p}} \leq (1-\varepsilon)^{-\frac{1}{p}} \|f_0\|_{L^p} + t \|f_1\|_{L^\infty}$$

Taking the infimum over the decompositions $f = f_0 + f_1$, and letting $\varepsilon \to 0$ we get the wanted inequality.
We will prove (A.16) first for $p_1 = \infty$ and $p_0 < p$ fixed. Let $f \in L^{p,q}(\Omega)$, with $\theta = 1 - \frac{p_0}{p}$ then we have

$$\|f\|_{\theta,q} = \left(\int_0^\infty (t^{-\theta} K(t,f))^q \frac{dt}{t} \right)^{\frac{1}{q}} = (*)$$

using the proportionality we have just proved, we get

$$(*) \sim \left(\int_0^\infty t^{-\theta q} \left(\int_0^{t^{p_0}} (f^*(s))^{p_0} ds \right)^{\frac{q}{p_0}} \frac{dt}{t} \right)^{\frac{1}{q}} =$$

$$= \left(\int_0^\infty \left(\int_0^1 t^{\frac{1-\theta}{p_0}} f^*(st^{p_0})^{p_0} \frac{ds}{t^{\frac{p_0}{q}}} \right)^{\frac{q}{p_0}} dt \right)^{\frac{1}{q}} \leq$$

$$\leq C \left(\int_0^1 \left(\int_0^\infty s^{(\theta-1)\frac{q}{p_0}+1} \left(t^{\frac{1-\theta}{p_0}} f^*(t) \right)^q \frac{dt}{t} \right)^{\frac{p_0}{q}} ds \right)^{\frac{1}{p_0}}$$

where between the second and third line we have used the Minkowski's integral inequality. The last term is easily seen to be controlled by $C\|f\|_{L^{p,q}}$. If now $f \in (L^{p_0}, L^\infty)_{\theta,q}$, and $\frac{1}{p} := \frac{1-\theta}{p}$, then for the converse inequality we have

$$\|f\|_{L^{p,q}} = \left(\int_0^\infty \left(t^{\frac{1}{p}} f^*(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} = p_0^{\frac{1}{q}} \left(\int_0^\infty \left(t^{\frac{p_0}{p}} f^*(t^{p_0}) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \leq$$

$$\leq C\left(\int_0^\infty\left(t^{\frac{p_0}{p}-1}\left(\int_0^{t^{p_0}} f^*(s)ds\right)^{\frac{1}{p_0}}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \sim$$

$$\sim\left(\int_0^\infty \left(t^{-\theta}K(t,f)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}$$

and the last term is equal to $\|f\|_{\theta,q}$, and therefore we have concluded. It is left only to drop the condition $p_1=\infty$, and this can be easily obtained by the Reiteration Theorem. Indeed for $1\leq p_0<p_1<\infty$, we have that

$$(L^{p_0}(\Omega),L^{p_1}(\Omega))_{\theta,q}=((L^r(\Omega),L^\infty(\Omega))_{\theta_0,p_0},(L^r(\Omega),L^\infty(\Omega))_{\theta_1,q_1})_{\theta,q}=$$

$$=(L^r(\Omega),L^\infty(\Omega))_{\eta,q}=L^{p,q}(\Omega)$$

where the second equality is due to the Reiteration theorem and $r=\frac{1-\theta_0}{p_0}$. □

Theorem A.2.6. *Let $1\leq p<\infty$ and $q\leq r\leq\infty$. Then*

1) $L^{p,q}(\Omega)$ are all Banach spaces.

2) $L^{p,q}(\Omega)\subset L^{p,r}(\Omega)$

3) $C^\infty(\Omega)\cap L^{p,q}(\Omega)$ is dense in $L^{p,q}(\Omega)$ for $1\leq q<\infty$.

4) If Ω has finite Lebesgue measure, then for $r>p$ and $1\leq q\leq\infty$ we have $L^r(\Omega)\hookrightarrow L^{p,q}(\Omega)$.

Proof. Points 1) and 2) are straightforward consequences of points c) and b) of Proposition A.1.15 respectively.
The third point is a direct consequence of Remark A.1.16 together with the fact that $C^\infty(\Omega)\cap L^p(\Omega)$ is dense in $L^p(\Omega)$ for each $1\leq p<\infty$.
4)It is enough to prove the result for $q=1$ since $L^{(p,1)}(\Omega)\subset L^{(p,q)}(\Omega)$ if $q>1$. Let then $f\in L^{(p,1)}(\Omega)=(L^1(\Omega),L^\infty(\Omega))_{1-\frac{1}{p},1}$. Then it holds

$$\|f\|_{L^{(p,1)}(\Omega)}=\int_0^\infty t^{\frac{1}{p}-1}K(t,f)\frac{dt}{t}=\int_0^\infty t^{\frac{1}{p}-1}\int_0^t f^*(s)ds\frac{dt}{t}=$$

$$=\int_0^{|\Omega|} t^{\frac{1}{p}-1}\int_0^t f^*(s)ds\frac{dt}{t}+\int_{|\Omega|}^\infty t^{\frac{1}{p}-1}\int_0^t f^*(s)ds\frac{dt}{t}=(*)$$

Applying Hölder's inequality in the first term above we find that

$$(*)\leq\int_0^{|\Omega|} t^{\frac{1}{p}-\frac{1}{r}}\|f\|_{L^r(\Omega)}\frac{dt}{t}+\int_{|\Omega|}^\infty t^{\frac{1}{p}-1}\|f\|_{L^1(\Omega)}\frac{dt}{t}$$

Since $\frac{1}{p} - \frac{1}{r} > 0$ by hypothesis, then the first integral converges. Thanks to the embedding $L^r(\Omega) \hookrightarrow L^1(\Omega)$ we conclude.

$\square$

A.2.1 The Calderón-Zygmund inequality

We now define the concept of kernel, which will be useful in studying the regularity of the Laplace equation. For this subsection we will refer to [14].

Definition A.2.7. We say that a function $k : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is a **Calderón-Zygmund kernel** if

1) $k(x) = \frac{\omega(x)}{|x|^n}$ for each $x \in \mathbb{R}^n$, where ω is a zero-homogeneous function[11]

2) $\omega|_{\partial B(0)} \in L^\infty$

3) $\int_{\partial B(0)} k d\sigma = 0$.

Let k be a Calderón-Zygmund kernel, and $k_\varepsilon := k\chi_{\mathbb{R}^n \setminus B_\varepsilon(0)}$. Then we define the convolution

$$T_\varepsilon(f) := k_\varepsilon * f(x) := \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} k(x-y)f(y)dy \tag{A.17}$$

where $f \in L^p(\mathbb{R}^n)$. A classical Theorem of Calderón-Zygmund states that if $f \in L^p(\mathbb{R}^n)$ with $1 < p < \infty$ then the limit $T(f)$ of $T_\varepsilon(f)$ for $\varepsilon \to 0$ exists in L^p and furthermore

$$\|T(f)\|_{L^p(\mathbb{R}^n)} \leq C(p)\|f\|_{L^p(\mathbb{R}^n)} \tag{A.18}$$

where $C(p)$ is constant depending on p. See for instance Theorem 7.22 [14]. Using the interpolation theory we have developed, we will extend this result to Lorentz spaces.

Corollary A.2.8. *Let $k : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be a Calderón-Zygmund kernel, with Lipshitz continuous restriction on $\partial B(0)$. Then we have that if $f \in L^{p,r}(\mathbb{R}^n)$, $1 < p < \infty$*

$$\|T(f)\|_{L^{p,r}(\mathbb{R}^n)} \leq A\|f\|_{L^{p,r}(\mathbb{R}^n)} \tag{A.19}$$

where $0 < r \leq \infty$.

[11] A function $f : \mathbb{R}^n \to \mathbb{R}$ is called homogeneous of degree n if $f(kx) = k^n f(x)$ for each $k \in \mathbb{R}$ and $x \in \mathbb{R}^n$

Proof. The proof is straightforward. Indeed, we have already seen that $L^{p,q}(\mathbb{R}^n) = (L^{p_0}(\mathbb{R}^n), L^{p_1}(\mathbb{R}^n))_{\theta,q}$ for $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then this means that for $p > 1$ we can choose both $p_0, p_1 > 1$. Then the map T is continuous from $L^{p_i}(\mathbb{R}^n)$ to $L^{p_i}(\mathbb{R}^n)$ with $i = 0, 1$ by Calderón-Zygmund theorem. Since $L^{p,q}(\mathbb{R}^n)$ is an interpolation space, then $T : L^{p,q}(\mathbb{R}^n) \to L^{p,q}(\mathbb{R}^n)$ is continuous too. □

The following example is significant. Indeed, it uses the above Corollary, in the framework of the Laplace equation, to get an higher regularity for the solution. It is fundamental also because, as already discussed, the improved version of the Removable singularities theorem of T.Riviére partially relies in some consequences of it.

Example A.2.9. Let Ω be a bounded domain in $\mathbb{R}^n$, and $f \in W^{2,2}(\Omega, \mathbb{R})$ such that $\|\Delta f\|_{L^{2,1}(\Omega)} \leq M$ and also $\|f\|_{L^{2,1}(\Omega)}, \|\nabla f\|_{L^{2,1}(\Omega)} \leq M$. We show that this is enough to prove that $f \in W^{2,(2,1)}_{\text{loc}}(\Omega, \mathbb{R}^m)$. Take $\eta \in C^\infty_c(\Omega, \mathbb{R})$ a cut-off function satisfying $\eta \equiv 1$ in K, where $K \subset \Omega$ is compact. We build the following Dirichlet problem

$$\begin{cases} \Delta V = \Delta(\eta f) \quad \text{in } \Omega \\ V = 0 \quad \text{in } \partial\Omega \end{cases} \tag{A.20}$$

Clearly the solution exists and is unique $V = \eta f \in W^{2,2}_0(\Omega, \mathbb{R}^m)$. Moreover, we know that we can also rewrite it as

$$V(x) = \int_{\mathbb{R}^n} \Gamma(x-y)\Delta(\eta(y)f(y))dy = \int_\Omega \Gamma(x-y)\Delta(\eta(y)f(y))dy \tag{A.21}$$

where Γ is the Newtonian potential, and $\frac{\partial^2\Gamma}{\partial x_i \partial x_j}$ can be proved to be a Calderón-Zygmund Kernel for each $i, j = 1, ..., n$ (see for instance [14]). So in particular thanks to Calderón-Zygmund inequality and Corollary A.2.8 we have that

$$\|D^2 V\|_{L^{2,1}(\Omega)} \leq C\|\Delta(\eta f)\|_{L^{2,1}(\Omega)}$$

as far as the $L^{2,1}$-norm of the right hand side is finite, which by hypothesis is the case. Since $\eta = 1$ in K and $\Delta(\eta f) = \Delta\eta f + 2\nabla f \cdot \nabla\eta + \eta\Delta f$, then the above inequality leads to

$$\begin{aligned} \|D^2 f\|_{L^{2,1}(K)} &\leq C\|\Delta(\eta f)\|_{L^{2,1}(\Omega)} \leq \\ &\leq \tilde{C}\left(\|f\|_{L^{2,1}(\Omega)} + \|\nabla f\|_{L^{2,1}(\Omega)} + \|\Delta f\|_{L^{2,1}(\Omega)}\right) \leq \tilde{C}M \end{aligned} \tag{A.22}$$

where $\tilde{C}$ is a constant depending on K and Ω.

A.3 Sobolev and Lorentz-Sobolev spaces

We will now introduce Fractional Sobolev spaces, Lorentz-Sobolev spaces and Besov spaces. We will give just an interpolation formulation of them, and it is important to know that this choice does not allow us to describe them all, but only those spaces that are needed in our treatment. A more complete construction, which uses the Bessel and Riesz potentials, can be found for example in [4]. In this section we will refer for the most to [43].

Definition A.3.1. For $1 \le p \le \infty$ and $0 < s < 1$, the **Fractional Sobolev space** $W^{s,p}(\mathbb{R}^n)$ is defined as

$$W^{s,p}(\mathbb{R}^n) := (W^{1,p}(\mathbb{R}^n), L^p(\mathbb{R}^n))_{1-s,p} \tag{A.23}$$

For $1 \le p, q \le \infty$ and $0 < s < 1$ one defines also the **Besov space** $B^s_{p,q}(\mathbb{R}^n)$, as

$$B^s_{p,q}(\mathbb{R}^n) := (W^{1,p}(\mathbb{R}^n), L^p(\mathbb{R}^n))_{1-s,q} \tag{A.24}$$

We extend this definition to non integers $s > 1$ as follows.

Definition A.3.2. Let $k \in \mathbb{N}$ and $k < s < k+1$. Then if $m \in \mathbb{N}$ such that $m \ge k+1$, for $(1-\theta)m = s$, we define

$$W^{s,p}(\mathbb{R}^n) := (W^{m,p}(\mathbb{R}^n), L^p(\mathbb{R}^n))_{\theta,p} \tag{A.25}$$

and for $1 \le q \le \infty$

$$B^s_{p,q}(\mathbb{R}^n) := (W^{m,p}(\mathbb{R}^n), L^p(\mathbb{R}^n))_{\theta,q} \tag{A.26}$$

The last definition seems to depend on the choice of the integer m, but as we will see it is well defined. We will get this using the Reiteration Theorem.

Proposition A.3.3. *Let $1 \le p < \infty$, and $k \in \mathbb{N}$. If m is any integer such that $m \ge k+1$, then we have that*

$$W^{k,p}(\mathbb{R}^n) \text{ is of class } C\left(\frac{m-k}{m}; \overline{A}\right) \tag{A.27}$$

where $\overline{A} = (W^{m,p}(\mathbb{R}^n), L^p(\mathbb{R}^n))$. In particular Definition A.3.2 is well defined.

Proof. We will skip the proof of $W^{k,p}(\mathbb{R}^n)$ being of class $C\left(\frac{m-k}{m}, \overline{A}\right)$ since it is technical, anyway it can be found in [43]. The last statement is proved thanks to the Reiteration Theorem. Let $m_1, m_2 \in \mathbb{N}$, such that $m_1, m_2 \ge k+1$, and $k < s < k+1$. Furthermore, suppose that $m_1 < m_2$. Then we have $W^{m_1,p}(\mathbb{R}^n)$ is of class $C\left(\frac{m_2-m_1}{m_2}, \overline{A}\right)$, and L^p is of class $C\left(1, \overline{A}\right)$, where $\overline{A} = (W^{m_2,p}(\mathbb{R}^n), L^p(\mathbb{R}^n))$. Therefore, applying the reiteration theorem we get that

$$(W^{m_1,p}(\mathbb{R}^n), L^p(\mathbb{R}^n))_{\frac{m_1-s}{m_1},q} = (W^{m_2,p}(\mathbb{R}^n), L^p(\mathbb{R}^n))_{\theta,q}$$

where $\theta = \left(1 - \frac{m_1-s}{m_1}\right)\frac{m_2-m_1}{m_2} + \frac{m_1-s}{m_1} = \frac{m_2-s}{m_2}$. This concludes the proof of our statement. □

Remark A.3.4. Applying again the Reiteration Theorem we can rewrite $B^s_{p,q}(\mathbb{R}^n)$ and $W^{s,p}(\mathbb{R}^n)$, when s is a positive non integer in the following way:

$$B^s_{p,q}(\mathbb{R}^n) = (W^{s_1,p}(\mathbb{R}^n), W^{s_2,p}(\mathbb{R}^n))_{\theta,q} \text{ for } s = (1-\theta)s_1 + \theta s_2 \tag{A.28}$$

and $0 < s_1 < s < s_2$.

Remark A.3.5. When s is a positive non integer then by the above definitions we have that $B^s_{p,p}(\mathbb{R}^m) = W^{s,p}(\mathbb{R}^n)$. This is not anymore true when s is an integer. Anyway, we can obtain $W^{m,p}(\mathbb{R}^n)$ by interpolating on the exponent of integrability, rather than the order of (weak) differentiability. Before stating this result we introduce another Sobolev space.

Definition A.3.6. For $m \in \mathbb{N}$, and $1 \le p < \infty$ and $1 \le q \le \infty$, we define the **Sobolev-Lorentz space**

$$W^{m,(p,q)}(\mathbb{R}^n) = \{f \in L^{p,q}(\mathbb{R}^n) | \ D^\alpha f \in L^{p,q}(\mathbb{R}^n) \ \forall \alpha \text{ such that } |\alpha| \le m\} \tag{A.29}$$

which is clearly a generalization of the classical Sobolev spaces. Here we assume that the weak derivatives are in Lorentz spaces, rather then just in L^p.

Theorem A.3.7. *Let $k \in \mathbb{N}$ and $1 < p < \infty$. Then we can rewrite $W^{k,p}(\mathbb{R}^n)$ as an interpolation space:*

1) $W^{k,p}(\mathbb{R}^n) = (W^{k,p_0}(\mathbb{R}^n), W^{k,p_1}(\mathbb{R}^n))_{\theta,p}$, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

2) $W^{k,(p,q)}(\mathbb{R}^n) = (W^{k,p_0}(\mathbb{R}^n), W^{k,p_1}(\mathbb{R}^n))_{\theta,q}$ where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

These two last results, even though they seem intuitive, are not easy to prove. A proof of them can be found for instance in [12] or [10].

In particular writing these last spaces as interpolation spaces will let us prove easily some improved Sobolev embeddings, that otherwise would require a greater effort. The following results are mainly due to Jaak Peetre [30] who first got some of these new embeddings thanks to interpolation techniques.

Theorem A.3.8. *We have the following embeddings.*

1) Let $0 < s < \frac{n}{p}$ for $1 < p < \infty$, then

$$B^s_{p,q}(\mathbb{R}^n) \hookrightarrow L^{p(s),q}(\mathbb{R}^n) \quad \text{for} \quad \frac{1}{p(s)} := \frac{1}{p} - \frac{s}{n} \tag{A.30}$$

and similarly if $k < \frac{n}{p}$ we have

$$W^{k,(p,q)}(\mathbb{R}^n) \hookrightarrow L^{p(k),q}(\mathbb{R}^n) \quad \text{for} \quad \frac{1}{p(k)} := \frac{1}{p} - \frac{k}{n} \tag{A.31}$$

where $1 \leq q \leq \infty$.

2) For every $s > 0$

$$B^s_{p,q_1}(\mathbb{R}^n) \subset B^s_{p,q_2}(\mathbb{R}^n) \ \text{for} \ q_1 < q_2 \tag{A.32}$$

and furthermore when $k < s_1 < s_2 < k+1$ for $k \in \mathbb{N}$ we have that

$$W^{k+1,p}(\mathbb{R}^n) \subset W^{s_2,p}(\mathbb{R}^n) \subset W^{s_1,p}(\mathbb{R}^n) \subset W^{k,p}(\mathbb{R}^n) \tag{A.33}$$

Proof. 1)By definition $B^s_{p,q}(\mathbb{R}^n) = (W^{m,p}(\mathbb{R}^n), L^p(\mathbb{R}^n))_{\theta,q}$ with $s = (1-\theta)m$ and $s < m \in \mathbb{N}$. Now we use the Reiteration theorem as follows. We can choose m such that $mp > n$, then using the Gagliardo-Nirenberg inequality we have that for every $f \in W^{m,p}(\mathbb{R}^n)$, the following holds

$$\|f\|_{L^\infty(\mathbb{R}^n)} \leq C \left(\sum_{|\alpha|=m} \|D^\alpha f\|_{L^p} \right)^{\frac{n}{mp}} \|f\|_{L^p}^{1-\frac{n}{mp}} \tag{A.34}$$

which by Proposition A.1.15 implies that $(W^{m,p}(\mathbb{R}^n), L^p(\mathbb{R}^n))_{\theta_1,1} \subset L^\infty(\mathbb{R}^n)$ with $\theta_1 = 1 - \frac{n}{mp}$. Now clearly we have that $(W^{m.p}(\mathbb{R}^n), L^p(\mathbb{R}^n)_{1,1} \subset L^p(\mathbb{R}^n)$. These two embeddings together with the Reiteration Theorem let us conclude that

$$(W^{m,p}(\mathbb{R}^n), L^p(\mathbb{R}^n))_{\eta,q} \subset (L^p(\mathbb{R}^n), L^\infty(\mathbb{R}^n))_{\lambda,q} \tag{A.35}$$

where $\eta = (1-\lambda) + \lambda\theta_1$. Using therefore $\eta = \theta$ we find that $B^s_{p,q}(\mathbb{R}^n) \subset (L^p(\mathbb{R}), L^\infty(\mathbb{R}^n)))_{\frac{ps}{n},q}$, and this last space is $L^{p(s),q}(\mathbb{R}^n)$, with $\frac{1}{p(s)} = \frac{1}{p} - \frac{s}{n}$.
Let us consider the space $W^{k,(p,q)}(\mathbb{R}^n)$ with $k < \frac{n}{p}$ an integer. By

Theorem A.3.7 we have that
$W^{k,(p,q)}(\mathbb{R}^n) = (W^{k,p_1}(\mathbb{R}^n), W^{k,p_2}(\mathbb{R}^n))_{\theta,q}$ where $p_1 < p < p_2$ and $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$. We choose both p_1 and p_2 such that $p_1 k < n$ and $p_2 k < n$, and applying the classical Sobolev embeddings we get that $W^{k,p_i}(\mathbb{R}^n) \hookrightarrow L^{p_i^*}(\mathbb{R}^n)$ where $p_i^* = \frac{np_i}{n-kp_i}$ for each $i = 1, 2$. Then by interpolating we get that

$$W^{k,p}(\mathbb{R}^n) = (W^{k,p_1}(\mathbb{R}^n), W^{k,p_2}(\mathbb{R}^n))_{\theta,q} \hookrightarrow (L^{p_1^*}(\mathbb{R}^n), L^{p_2^*}(\mathbb{R}^n))_{\theta,q} \tag{A.36}$$

and the last space is $L^{p^*,q}(\mathbb{R}^n)$ and this concludes the first point.
2)The embedding (A.32) is a direct consequence of point b) of Proposition A.1.15. If s_2 is a positive non integer such that $k + 1 > s_2$, then we can find two positive integers m_1, m_2, such that $m_1 \leq s_2 \leq m_2$ and $m_1, m_2 \leq k + 1$. Therefore, we can write $W^{s_2,p}(\mathbb{R}^n) = (W^{m_1,p}(\mathbb{R}^n), W^{m_2,p}(\mathbb{R}^n))$ where $W^{k+1,p}(\mathbb{R}^n)$ embeds in $W^{m_i,p}(\mathbb{R}^n)$ for each $i = 1, 2$. This last observation leads to the first inclusion in (A.33). The others can be proved with an analogous reasoning. □

So far we have defined these function spaces when the domain is $\mathbb{R}^n$. We now generalize these definitions when the domain is some open subset Ω of $\mathbb{R}^n$ regular enough. In particular the following result is valid for every open domain of $\mathbb{R}^n$ with the extension property, but we will state it only for bounded Lipschitz domains.

Proposition A.3.9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then if we call*

$$X^s_{p,q}(\Omega) = \{f|_\Omega \ | \ f \in B^s_{p,q}(\mathbb{R}^n)\}$$

with the norm $\|f\|_{X^s_{p,q}(\Omega)} := \inf_{F|_\Omega = f} \|F\|_{B^s_{p,q}(\mathbb{R}^n)}$, and

$$Y^s_{p,q}(\Omega) = (W^{m_1,p}(\Omega), W^{m_2,p}(\Omega))_{\theta,q}$$

for s a positive non integer, and $1 \leq p < \infty$, $1 \leq q \leq \infty$, then $X^s_{p,q} = Y^s_{p,q}$.

Proof. Note that the restriction map $R_\Omega : W^{m_i,p}(\mathbb{R}^n) \to W^{m_i,p}(\Omega)$ is linear and continuous for $i = 1, 2$. Then it is also linear and continuous from $B^s_{p,q}(\mathbb{R}^n)$ to $Y^s_{p,q}(\Omega)$, which means that $X^s_{p,q}(\Omega) \subset Y^s_{p,q}(\Omega)$.
Conversely, since Ω is bounded and Lipschitz it has the extension property, namely we have the existence of the extension map $E : W^{m_i,p}(\Omega) \to W^{m_i,p}(\mathbb{R}^n)$ for $i = 1, 2$. Then by interpolation we get that $E : Y^s_{p,q}(\Omega) \to B^s_{p,q}(\mathbb{R}^n)$, and so every $f \in Y^s_{p,q}(\Omega)$ is the restriction of some map $E(f)$ in $B^s_{p,q}(\mathbb{R}^n)$, which implies $Y^s_{p,q}(\Omega) \subset X^s_{p,q}(\Omega)$. □

When $kp = n$ we know that for Ω bounded and Lipschitz it holds the embedding $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q \in [1, \infty[$. One could try to see what happens when instead of $W^{k,p}(\Omega)$ we consider the slightly smaller space $W^{k,(p,1)}(\Omega)$. In particular we have the following result that we state for a bounded Lipschitz domain. The proof is based on the discrete version of the J-method, see Lemma A.1.13, and on the interpolation formulation of the Lorentz-Sobolev spaces as in Theorem A.3.7.

Theorem A.3.10. *Let $\Omega \subset \mathbb{R}^n$ be a bounded and Lipschitz domain. Then if $pk = n$ we have the following embeddings*

$$W^{k,(p,1)}(\Omega) \hookrightarrow C^0(\overline{\Omega}) \hookrightarrow L^\infty(\Omega) \tag{A.37}$$

Proof. By Theorem A.3.7 we have the identity $W^{k,(p,1)}(\Omega) = (W^{k,p_0}(\Omega), W^{k,p_1}(\Omega))_{\theta,1}$, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. We choose $p_0 = Mp$ with $1 < M < 2$, which of course implies that $kp_0 > n$ and leads to the embedding $W^{k,p_0}(\Omega) \hookrightarrow C^0(\overline{\Omega})$. Furthermore, we select $p_1 = mp$ with $m := \frac{M}{1+M^2-M} < 1$. In this way we have that $\theta = \frac{m(M-1)}{M-m} = \frac{n}{kp_0}$. This particular choice for p_0 and p_1 will be clear soon.
By the discrete formulation of the real J-method, see Lemma A.1.13, we have that for each $f \in W^{k,(p,1)}(\Omega)$, there exists a sequence $f_n \in W^{2,p_0}(\Omega)$ such that $f = \sum_n f_n$, where the convergence is in $W^{2,p_1}(\Omega)$ and furthermore it holds that $2^{-n\theta} J(2^n, f_n) \in l^1$, and its norm is controlled by $\|f\|_{W^{k,(p,1)}(\Omega)}$. So we have that

$$\|f\|_{L^\infty(\Omega)} \leq \sum_n \|f_n\|_{L^\infty(\Omega)} = (*)$$

Now we apply Theorem 5.8 in [2] to each f_n, which gives us the following bound

$$\|f_n\|_{L^\infty(\Omega)} \leq C \|f_n\|^{\theta}_{W^{k,p_0}(\Omega)} \|f_n\|^{1-\theta}_{L^{p_0}(\Omega)} \tag{A.38}$$

and therefore we have that

$$(*) \leq C_1 \sum_n \|f_n\|^{\theta}_{W^{k,p_0}(\Omega)} \|f_n\|^{1-\theta}_{L^{p_0}(\Omega)} = (*_1)$$

where the constant C_1 depends on p_0, k, n and the domain Ω. The previous choice for $M < 2$, implies that $p_1^* > p_0$ so that now $W^{k,p_1}(\Omega) \hookrightarrow L^{p_1^*}(\Omega) \hookrightarrow L^{p_0}(\Omega)$, and thus inequality (A.38) is bounded from above by

$$\leq C_2 \|f_n\|^{\theta}_{W^{k,p_0}(\Omega)} \|f_n\|^{1-\theta}_{W^{k,p_1}(\Omega)}$$

Using Proposition A.1.10, we finally get

$$(*_1) \leq C_2 \sum_n 2^{-n\theta} J(2^n, f_n) \leq C\|f\|_{W^{k,(p,1)}(\Omega)}$$

and the last inequality is given always by Lemma A.1.13.
By the well known fact that $C^1(\Omega) \cap W^{k,p_0}(\Omega)$ is dense in $W^{k,p_0}(\Omega)$ and thanks to Remark A.1.16, we have that for each $f \in W^{k,(p,1)}(\Omega)$, there exists $\{f_n\}_n \subset W^{k,(p,1)}(\Omega) \cap C^1(\Omega)$, such that $f_n \to f$. This observation and the inequality we have proved imply

$$\|f - f_n\|_{L^\infty(\Omega)} \leq C\|f - f_n\|_{W^{k,(p,1)}(\Omega)}$$

and therefore there exists $\tilde{f}$ continuous such that $\tilde{f} = f$ almost everywhere. $\square$

Appendix B

Čech Cohomology

In what follows we briefly workout a definition for Čech cohomology with coefficients in the sheaf of smooth G-valued functions on a manifold M, and prove that there is a one to one correspondence between the possible different principal bundles over M (up to isomorphism) and the classes in its Čech cohomology $\check{H}^1(M, C^\infty(G))$. We refer mainly to [46], where the theory requires an accurate definition of sheaves, presheaves and some category tools. We will skip these definitions and present a more straightforward construction. Let M be a manifold, $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering for M, and G a Lie group. We choose a family of C^∞ functions

$$\varphi := \{\phi_{i_0 \dots i_n} : U_{i_0 \dots i_n} \to G\} \tag{B.1}$$

where we have adopted the notation $U_{i_0\dots i_n} := U_{i_0} \cap \dots \cap U_{i_n}$. We call this collection a differentiable **Čech n-cochain**, and we indicate with $\check{C}^n(\mathcal{U}, C^\infty(G))$ the set of all n-cochains.

Definition B.0.1. We define the **coboundary operator** for 1-cochains as the map $\delta : \check{C}^1(\mathcal{U}, C^\infty(G)) \to \check{C}^2(\mathcal{U}, C^\infty(G))$ such that

$$\begin{aligned} (\delta\phi)_{ijl} : U_i \cap U_j \cap U_l \to G \\ x \longmapsto (\delta\phi)_{ijl}(x) := \phi_{ij}(x)\phi_{jl}(x)\phi_{li}(x) \end{aligned} \tag{B.2}$$

while for 0-cochains we have $\delta : \check{C}^0(\mathcal{U}, C^\infty(G)) \to \check{C}^1(\mathcal{U}, C^\infty(G))$ defined by

$$\begin{aligned} (\delta\phi)_{ij} : U_i \cap U_j \to G \\ x \longmapsto (\delta\phi)_{ij}(x) := \phi_i(x)\phi_j(x)^{-1} \end{aligned} \tag{B.3}$$

If a 1-cochain ϕ is such that $\delta\phi = e$, where e is the constant function equal to the indentity of G, then we say that ϕ is a **Čech 1-cocycle**.

We denote the set of 1-cocycles with $\check{Z}^1(\mathcal{U}, C^\infty(G))$.
If a 1-cochain ϕ is such that $\phi = \delta\alpha$ for some 0-cochain α then ϕ is called a **Čech coboundary**. Note that a Čech coboundary is automatically 1-Čech cocycle.

Definition B.0.2. We say that two 1-cocycles $\phi, \psi \in \check{Z}^1(\mathcal{U}, C^\infty(G))$ are **cohomologus**, and we write $\phi \sim \psi$, if there exists a 0-cochains β such that for each i, j

$$\phi_{ij}(x) = \beta_i(x)\psi_{ij}(x)\beta_j^{-1}(x) \tag{B.4}$$

It is easy to check that $\sim$ is an equivalence relation in $\check{Z}^1(\mathcal{U}, C^\infty(G))$.

Definition B.0.3. We define **Čech cohomology** with coefficients in the sheaf of smooth G-valued functions of the covering $\mathcal{U} = \{U_i\}_{i\in I}$ as the quotient space

$$H^1(\mathcal{U}, C^\infty(G)) := \frac{\check{Z}^1(\mathcal{U}, C^\infty(G))}{\sim}$$

of the 1-cocycles via the equivalence relation $\sim$.

We now introduce two fundamental categories in the study of Čech cohomology $\check{H}^1(M, C^\infty(G))$, which allow a more precise exposition of the basic concepts of this theory.

Definition B.0.4. We call $\mathrm{Cov}(M)$ the category whose objects $\mathrm{Ob}(\mathrm{Cov}(M))$ are open coverings of M, and if $\mathcal{V}, \mathcal{U} \in \mathrm{Ob}(\mathrm{Cov}(M))$ are two objects then the set of all morphisms $\mathrm{Hom}_{\mathrm{Cov}(M)}(\mathcal{V}, \mathcal{U})$ coincides with the set of all maps

$$\tau : J \to I \;\text{ such that }\; V_j \subset U_{\tau(j)} \;\; \forall j \in J$$

where $\mathcal{U} = \{U_I\}_{i\in I}$ and $\mathcal{V} = \{V_j\}_{j\in J}$.

We define also the category Set_* whose objects $\mathrm{Ob}(\mathrm{Set}_*)$ are pointed sets, namely couples (X, x) where X is a set and $x \in X$ is fixed. If (X, x), $(Y, y) \in \mathrm{Ob}(\mathrm{Set}_*)$, then we choose

$$\mathrm{Hom}_{\mathrm{Set}_*}((X, x), (Y, y)) := \{f : X \to Y : \; f(x) = y\}$$

If $\mathcal{V}$, $\mathcal{U} \in \mathrm{Ob}(\mathrm{Cov}(M))$ are two objects, then each morphism $\tau \in \mathrm{Hom}_{\mathrm{Cov}(M)}(\mathcal{V}, \mathcal{U})$ induces the following map

$$\tau^* : \check{Z}^1(\mathcal{U}, C^\infty(G)) \to \check{Z}^1(\mathcal{V}, C^\infty(G))$$

$$\phi \longmapsto \tau^*(\phi) := \left\{ \phi_{\tau(j)\tau(j')}\big|_{V_j \cap V_{j'}} \to G \right\} \tag{B.5}$$

If two 1-cocycles ϕ, $\psi \in \check{Z}^1(\mathcal{U}, C^\infty(G))$, are such that $\phi \sim \psi$ then also $\tau^*(\phi) \sim \tau^*(\psi)$, therefore the map τ^* defined in equation (B.5) is well defined if we pass to the quotient spaces

$$\begin{aligned} \tau^* : \check{H}^1(\mathcal{U}, C^\infty(G)) &\to \check{H}^1(\mathcal{V}, C^\infty(G)) \\ [\phi]_\sim &\mapsto [\tau^*(\phi)]_\sim \end{aligned} \tag{B.6}$$

and it maps the equivalence class of the identity (the map costantly equal to the identity of G) to itself. Thus, if we consider $(\check{H}^1(\mathcal{U}, C^\infty(G)), [1]_\sim)$ as objects in Set_*, (B.6) becomes a morphism in the category of pointed sets Set_*. In particular we have the following proposition. The proof is just an exercise.

Proposition B.0.5. *The map between the categories $Cov(M)^{opp}$ and Set_**

$$\begin{aligned} F : Cov(M)^{opp} &\to Set_* \\ \mathcal{U} &\mapsto F(\mathcal{U}) := \check{H}^1(\mathcal{U}, C^\infty(G)) \end{aligned} \tag{B.7}$$

and such that

$$Hom(\mathcal{U}, \mathcal{V}) \ni \tau \mapsto F(\tau) := \tau^* \in Hom(F(\mathcal{U}), F(\mathcal{V}))$$

is a (covariant) functor.

Now that we have gathered all the necessary tools, we are finally ready to define $\check{H}^1(M, C^\infty(G))$.

Definition B.0.6. We define the **Čech cohomology** with coefficients in the sheaf of smooth G-valued functions on M as

$$\check{H}^1(M, C^\infty(G)) = \left(\coprod_{\mathcal{U}} \check{H}^1(\mathcal{U}, C^\infty(G)) \right) / \sim \tag{B.8}$$

where the disjointed union is taken over the open coverings of M. If $x \in \check{H}^1(\mathcal{U}, C^\infty(G))$ and $y \in \check{H}^1(\mathcal{V}, C^\infty(G))$ we say that $x \sim y$ if and only if there exists a refinement $\mathcal{W} = \{W_k\}_{k \in K}$ of both $\mathcal{U} = \{U_i\}_{i \in I}$ and $\mathcal{V} = \{V_j\}_{j \in J}$, with inclusion maps $\tau_1 : K \to I$ and $\tau_2 : K \to J$, such that $\tau_1^*(x) = \tau_2^*(y)$.

Observe that the equivalence relation above, roughly speaking, is saying that two cocycles (defined in different coverings $\mathcal{U}$ and $\mathcal{V}$) are in the same class in $\check{H}^1(M, C^\infty(G))$ if there exists a refinement $\mathcal{W}$ of both $\mathcal{U}$ and $\mathcal{V}$ such that the restrictions of this two cocycles in $\mathcal{W}$ are cohomologus in $H^1(\mathcal{W}, C^\infty(G))$.

The following theorem highlight the connection between principal fibre bundles and Čech cohomology.

Theorem B.0.7. *Let G be a Lie group and M a manifold. Then we have the following isomorphism in the category of Set$_*$*

$$\gamma : \left\{ \begin{array}{c} \textit{principal bundles with structure group } G \\ \textit{and base manifold } M \textit{ (up to isomorphism)} \end{array} \right\} \to \check{H}^1(M, C^\infty(G)) \tag{B.9}$$

Proof. Let $\pi : P \to M$ be a principle bundle with structure group G and base manifold M. We consider an atlas $\mathcal{A} = \{(U_\alpha, \chi_\alpha)\}$ for P. Then we know that the local trivializations lead to a 1-cocycle $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to G\}$ (transition maps), which then corresponds to a class $[g_{\alpha\beta}]$ in $\check{H}^1(\mathcal{U}, C^\infty(G))$. We define $\gamma(P)$ as the class of $[g_{\alpha\beta}]$ in $\check{H}^1(M, C^\infty(G))$. Now we prove that the map is well defined, that is to say it does not depend on the choices of different, but isomorphic, bundles.

Let $\pi : P \to M$ and $\tilde{\pi} : \tilde{P} \to M$ be two principal fibre bundles with structure group G, and base manifold M. Moreover, let $\mathcal{A} = \{(U_\alpha, \chi_\alpha)\}$ and $\tilde{\mathcal{A}} = \{(U_\alpha, \tilde{\chi}_\alpha)\}$ be two atlases for P and $\tilde{P}$ respectively, with transition functions $\{g_{\alpha\beta}\}$ and $\{\tilde{g}_{\alpha\beta}\}$. If $f : P \to \tilde{P}$ is an isomorphism of principal bundles, we have seen in Remark 2.2.17 that there exists a family of smooth maps $h_\alpha \in C^\infty(U_\alpha, G)$, such that

$$\tilde{g}_{\alpha\beta} = h_\alpha g_{\alpha\beta} h_\beta^{-1} \quad \text{in } U_\alpha \cap U_\beta \neq \emptyset$$

This last equation shows that $\{g_{\alpha\beta}\}$ and $\{\tilde{g}_{\alpha\beta}\}$ are cohomologus in $\check{H}^1(\mathcal{U}, C^\infty(G))$, and then they represent the same class in $\check{H}^1(M, C^\infty(G))$. Observe that the invariance of $\gamma(P)$ by isomorphism, implies also invariance by different choices of local parametrizations. We have proved the map γ is well defined.

Now we prove that it is an isomorphism in the category of pointed sets. Note that if two bundles P and $\tilde{P}$ are such that $\gamma(P) = \gamma(\tilde{P})$, then by definition of class in $\check{H}^1(M, C^\infty(G))$, there exist two atlases $\mathcal{A} = \{(U_\alpha, \chi_\alpha)\}$ and $\tilde{\mathcal{A}} = \{(U_\alpha, \tilde{\chi}_\alpha)\}$, such that the transition functions are cohomologus in
$\check{H}^1(\mathcal{U}, C^\infty(G))$. Then we can easily build a isomorphism between P and $\tilde{P}$. This proves that γ is injective. Finally if $g \in \check{H}^1(M, C^\infty(G))$, then we choose a representative $\{g_{\alpha\beta}\}$ of g in some $\check{H}^1(\mathcal{U}, C^\infty(G))$, and thanks to Proposition 2.2.18 there exists a bundle whose transition functions are exactly $g_{\alpha\beta}$ and so the map is also surjective. To conclude, the image of the trivial bundle is the identity class, and so γ is an isomorphism in the category of Set$_*$.

□

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