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Cosmological tests of modified gravity models with non-zero gravitational slip

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Introduction

The universe is undergoing a phase of accelerated expansion and this is a fact supported by a vast amount of observational data collected over the years. The first probe of such an acceleration came from Supernova Type Ia (SNIa) measurements in 1998 [1]- [2] and it was later confirmed by independent cosmological observational data from baryon acoustic oscillations (BAO), cosmic microwave background (CMB) and large scale structures (LSS) [3]. However, a satisfactory explanation of the cosmological phenomena, dubbed "dark energy" (DE), remains until today completely elusive.
In the years that followed the discovery of the expansion of the universe a landscape of theories were proposed, the simplest of them is the one that modifies the r.h.s of the Einstein's field equations introducing the so called cosmological constant Λ. It works as a negative pressure against gravity in such a way to produce an accelerated expansion. This theory corresponds to the cosmological concordance model ΛCDM (Lambda Cold Dark Matter) that consists in a flat universe in which the gravity is described by general relativity (GR) and the evolution history of the universe is the one predicted by the standard cosmological Hot Big Bang model in which the energy content of the universe is roughly subdivided in this way: 25% of cold dark matter, 5% baryons and 70% dark energy. This simple scenario is in impressive agreement with observations: GR has been shown to be consistent with all current observations, including tests in the solar system and binary pulsars; moreover data analysis of Planck 2018 combined with SN Ia and BAO data constrain the value of the equation of state (EOS) ($w_{DE} = p_{DE}/\rho_{DE}$) to be close to "$-1$". Thus, up to now the standard six-parameters ΛCDM scenario represents the current cosmological concordance model that is in general agreement with observed phenomena. However, this model suffers some theoretical issues, among them there is the difficulty to conciliate the vacuum energy density theoretically predicted by particle physics ($\rho_{\rm vac} \simeq 10^{74}\rm GeV^4$) with the observed value of dark energy ($\rho_\Lambda \simeq 10^{-47}\rm GeV^4$) [4]- [106]. Furthermore, beside these theoretical problems, it was recently disclosed that there is a substantial discordance inferred by ΛCDM between observations of large-scale structures (e.g. weak lensing measurements from KiDS (Kilo Degree Survey)) and CMB measurements made by Planck .
These theoretical and observational problems have motivated the searching for alternative solutions. In particular, they can be divided in two categories: dark energy (DE) and modified gravity (MG) models. The former are models in which the r.h.s of the Einstein's field equations is modified, i.e. the energy-momentum tensor $T^{\mu\nu}$; the latest are models in which the geometrical part of the Einstein's field equations is modified (l.h.s).
In order to be in agreement with observational data a theory of gravity should have an expansion history that is indistinguishable from that of ΛCDM and thus it should give rise to a sufficiently long radiation and matter eras, as well as a transition to a stable the Sitter era [58]; it should also satisfy local gravity constraints. However, such a theory can still differs, with respect to ΛCDM, in the evolution of the cosmological perturbations and thus predicts different growth of structures and also a non-standard relation between the scalar potentials that characterize the scalar-mode perturbations in the Newtonian gauge, i.e. $\Psi \neq \Phi$
In this dissertation we consider MG models that predict such a non-standard relation between the scalar potentials, i.e theories in which there is GRAVITATIONAL SLIP.
The thesis is organized as follows:

Chapter 1 We write down the closed system of equations which describe the evolution of cosmological perturbations at linear order in GR. Then, in order to measure the deviation of the phenomenology of LSS from GR in a model-independent way, we introduce the functions $\mu(a,k)$, $\eta(a,k)$ and $\Sigma(a,k)$. Departures from GR can be fully characterized by using only two functions of scale and time. In the last part of this chapter we briefly list other popular parametrizations commonly used in the literature to measure deviations from GR.
Chapter 2 We focus on gravitational slip making a distinction between the anisotropic stress sourced by matter and the gravitational slip itself. Then we quickly mention how LSS can be used to probe gravitational slip and how, in a certain class of theories, it is related with the non-standard propagation of the tensor modes.
Chapter 3 We introduce the effective field theory (EFT) formalism by considering two different parametrizations: the first is the most general EFT approach which consists in an unified description of DE and MG models; the second, the α-parametrization, is specifically designed to study the phenomenological aspects of Horndeski theories and, by adding new parameters, of its extensions. Using these formalisms we write down the modified Poisson equation, the modified anisotropy constraint and, consequently, we obtain the analytic expressions for the phenomenological functions μ, η and Σ in the quasi-static approximation. Then we focus on Horndeski gravity and beyond Horndeski models specifying the above-mentioned functions for these two classes of theories.
Chapter 4 We account for the recent binary neutron-star merger that put a very stringent bound on the tensor speed at low redshifts. In particular, we classify the survived Horndeski theories in terms of the α-parameters finding for which values of these property functions the theories exhibit gravitational slip. We also consider the expressions of μ, η and Σ, for both Horndeski and beyond Horndeski theories, in two limiting cases (sub-Compton and super-Compton regimes) and we discuss how, using these expressions, cosmological observations can be used to rule out subclasses of these theories.
Chapter 5 This chapter is devoted to Horndeski gravity. In particular, we summarize what we find in the previous chapter and following [13] we discuss the conjecture $(\mu-1)(\Sigma-1)\geq 0$ which characterizes this class of models. We finally reports the results of two recent analysis intended to constrain these theories using observational data and the publicly available Boltzmann codes EFTCMB and EFTCosmoMC.
Chapter 6 We shortly list possible sources of degeneracies that one has to take into account when testing for departures from GR and that can significantly limit our ability in constraining cosmological parameters (and in particular MG parameters).
Chapter 7 In this final chapter we report the results of some forecasting analysis in order to have an overview of how future surveys will improve our ability in constraining MG parameters and thus, testing departures from GR.

Units and conventions

In this dissertation we adopt the metric signature $(-,+,+,+)$ and we use the natural unit such that the speed of light is $c=1$, the reduced Planck's constant is $\hbar=1$ and the Boltzmann's constant is also $k_B=1$.

Chapter 1

Parametrizing deviations from GR in the evolution of cosmological perturbations

In this chapter we see how deviations from GR can be characterized using a model-independent approach. In order to do so we introduce some of the possible parametrizations used in the literature and we analyse the behaviour of these deviations from GR in the sub-horizon regime and in the super-horizon regime. We thus introduce the gravitational slip and its most popular parametrizations.

1.1 Evolution of linear cosmological perturbations in general relativity

To derive the evolution equations for the linear cosmological perturbations one needs to perturb the FLRW metric and the energy-momentum tensor $T_{\mu\nu}$

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu}, \qquad T_{\mu\nu} = T^{(0)}_{\mu\nu} + \delta T_{\mu\nu} \tag{1.1}$$

where $g^{(0)}_{\mu\nu}$ and $T^{(0)}_{\mu\nu}$ are the background terms and $\delta g_{\mu\nu}$, $\delta T_{\mu\nu}$ are their small perturbations.
In particular, we specialize to the conformal Newtonian (or longitudinal) gauge because it is the most convenient to our purpose, so that the perturbed energy-momentum tensor is

$$T^0_0 = -(\rho + \delta\rho) \tag{1.2}$$

$$T^0_i = -T^i_0 = (\rho + p)v_i \tag{1.3}$$

$$T^i_j = (p + \delta p)\delta^i_j + \Pi^i_j \tag{1.4}$$

where ρ and p are the density and the pressure of the unperturbed perfect fluid, v_i the velocity field and $\Pi^i_j \equiv T^i_j - T^k_k \delta^i_j/3$ denotes the traceless component of the energy-momentum tensor.
The line element of the perturbed spatially-flat FRW metric, in the Newtonian gauge, is

$$ds^2 = a(\tau)^2[-(1+\Psi)d\tau^2 + (1-\Phi)\delta_{ij}dx^i dx^j] \tag{1.5}$$

where τ is the conformal time, $a(\tau)$ the scale factor and Ψ and Φ are respectively the Newtonian potential and the curvature potential. In the Newtonian limit the potential Ψ is that which enters in the matter equation of motion $\ddot{\vec{x}} = -\vec{\nabla}\Psi$, while Φ enters in the Poisson equation $\nabla^2\Phi = -4\pi G a^2 \delta\rho$; from this limit one can understand that the first determines the geodesic motion

of matter, the second is the potential generated by matter sources and, in general, they do not coincide.
The evolution of the perturbations can be obtained by linearising the Einstein's field equations and subtracting the background evolution. The equations that characterize the evolution of linear perturbations are the Poisson equation and the anisotropy constraint which are obtained respectively by combining the time-time and time-space component and by the traceless part of the space-space component. The former relates the over-density to the curvature potential[1], while, the second, describes the relationship among the two scalar potentials and the shear stress.
In GR they can be written, in the Fourier space, as

$$k^2\Phi = -4\pi G a^2 \sum_i \bar{\rho}_i \Delta_i \tag{1.6}$$

$$k^2(\Psi - \Phi) = -12\pi G a^2 \sum_i \bar{\rho}_i (1 + w_i)\sigma_i \tag{1.7}$$

where the index i denotes the sum over different matter species. Then ρ_i is the density of the i-species, $w_i(\rho, S) = P_i(\rho, S)/\rho_i$, Δ_i and σ_i are respectively the comoving density perturbation and the dimensionless shear stress perturbation[2] which are defined by:

$$\Delta = \delta + 3\frac{\mathcal{H}}{k^2}(1+w)\theta, \qquad (\rho + p)\sigma \equiv -(\hat{k}_i\hat{k}^j - \delta_i^j/3)\Pi_j^i \tag{1.8}$$

here $\mathcal{H} = (da/d\tau)a^{-1} = aH$, $\delta = (\rho - \bar{\rho})/\bar{\rho}$ is the density contrast, $ik^j\delta T_j^0 = (\rho + p)\theta$ [3].
Then one needs the energy-momentum conservation equation to close the system[4] : $T^{\mu\nu}_{;\mu} = 0$

$$\delta' = -(1+w)(\theta - 3\Phi') + 3\mathcal{H}\delta\left(w - \frac{\delta p}{\delta\rho}\right) \tag{1.9}$$

$$\theta' = -\left[(1-3w)\mathcal{H} + \frac{w'}{1+w}\right]\theta + k^2\left[\frac{\delta p/\delta\rho}{1+w}\delta + \Psi - \sigma\right] \tag{1.10}$$

where we considered the averaged quantities on the i-species and the prime denotes the derivation with respect to the conformal time. Using these two equations one finds the evolution equation for Δ

$$\Delta' = 3(1+w)(\Phi' + \mathcal{H}\Psi - \mathcal{H}\sigma) + 3\mathcal{H}w\Delta - [k^2 + 3(\mathcal{H}^2 - \mathcal{H}')]\frac{\theta}{k^2}(1+w) \tag{1.11}$$

The motion of relativistic particles is determined by the equation for the Weyl potential $\Phi_W = (\Phi + \Psi)/2$ which is obtained by combining (1.6) and (1.7)

$$k^2(\Psi + \Phi) = -8\pi G a^2 \left[\sum_i \bar{\rho}_i \Delta_i + 3\bar{\rho}_i(1+w_i)\sigma_i/2\right] \tag{1.12}$$

1.2 Parametrized evolution of linear cosmological perturbations in a general theory of gravity

Departures from GR can be characterized using two approaches: a) one writes down the Lagrangian and derives a set of predictions which can be tested to constrain the parameters of the

[1] there is an analogous equation for the Newtonian potential.

[2] This quantity is related to the anisotropic stress perturbation Π defined by Kodama & Sasaki (1984) through $\frac{3}{2}(\rho_m + p_m)\sigma = p_m\Pi$.

[3] θ is the divergence of the peculiar velocity v_i and their relation in the Fourier space is $ik^i v_i = \theta$.

[4] The semicolon stands for the covariant derivative, thus the energy-momentum tensor is covariantly conserved.

theory; b) one parametrizes the deviations from GR using model-independent functions which can be reconstructed by observations, then one can obtain their specific form for each model and can use them to rule out the theories which are not consistent with observations.
Modified gravity models, in general, affect both the background and the perturbation evolution. However, many of these theories of gravity, can mimic the same background evolution history but they can still differ in the evolution of perturbations and thus predict measurable differences in the growth rate of large-scale structures. In particular, the observational constraints on H(t), reduce the number of viable models to those in which the background evolution is close to that predicts by the cosmological concordance model, ΛCDM, which fit very well the observational data.
Using the approach b) the deviations from GR can be completely characterized by two phenomenological functions which depend on scale and time. A possible choice is the following

$$k^2\Psi = -4\pi G\mu(a,k)a^2\sum_i \bar{\rho}_i\Delta_i \tag{1.13}$$

$$\frac{\Phi}{\Psi} = \eta(a,k) \tag{1.14}$$

Here $G\mu \equiv G_{matter}$ is the effective gravitational coupling felt by non relativistic particles; η is the gravitational slip parameter and measure the mismatch between the two scalar potentials; in the following we refer to gravitational slip as that determined only by modifications of the l.h.s. of Einstein's equations[5].
Sometimes, however, can be useful to introduce the function

$$\Sigma(a,k) = \mu\frac{(1+\eta)}{2} \tag{1.15}$$

It enters in the equation for the Weyl potential and characterizes the effective gravitational coupling felt by relativistic particles ($G_{light} \equiv G\Sigma$)

$$k^2(\Phi+\Psi) = -8\pi G\Sigma(a,k)a^2\sum_i \bar{\rho}_i\Delta_i \tag{1.16}$$

It is directly sensitive to weak lensing measurements as well as to the Integrated Sachs-Wolfe (ISW) effect, thus it is more constrained than η and μ by this kind of observations. Moreover, since η is not directly determined by observations, in the literature the deviations from GR are often described in terms of Σ and μ.
Another set of popular phenomenological functions are $Q(a,k)$ and $R(a,k)$. The first appears in the Poisson equation for Φ and in combination with the second in the anisotropy constraint

$$k^2\Phi = -4\pi GQ(a,k)a^2\sum_i \bar{\rho}_i\Delta_i \tag{1.17}$$

$$k^2(\Psi - R(a,k)\Phi) = -12\pi Ga^2\sum_i \bar{\rho}_i(1+w_i)\sigma_i Q(a,k) \tag{1.18}$$

they are linked with the other phenomenological functions by the following relations:

$$Q = \mu\eta, \qquad R = \frac{1}{\eta}, \qquad \frac{Q(1+R)}{2} \tag{1.19}$$

All these functions, thus, are not independent, indeed the deviations from GR can by fully characterized by choosing two of them. The couples $\{\mu,\eta\}$ and $\{\mu,\Sigma\}$ have been widely adopted

[5]The free-streaming of relativistic particles also determines a difference in these two potentials, but this is a non-linear effect at late time.

in the literature.
In [16] is pointed out that in a general theory of gravity the dynamics of linear perturbations may not match that one obtains by averaging over small scale fluctuations. In particular, N-body simulations, used to study the dynamics in the non-linear regime, in scalar-tensor theories of chameleon type reduced to the linear one on large-scale. But this fact is not general and thus the relation between Ψ and Δ could be non-linear leading to non-linear phenomenological functions.
In what follows we consider, for simplicity, matter of type "dust", i.e $w = 0$, in particular we only consider CDM, for which the sound velocity is negligible at least at linear order[6]. The equations (1.10), (1.9) and (1.11) become

$$\delta' = -\theta + 3\Phi' \tag{1.20}$$

$$\theta' = -\mathcal{H}\theta + k^2\Psi \tag{1.21}$$

$$\Delta' = 3(\Phi' + \mathcal{H}\Psi) - [k^2 + 3(\mathcal{H}^2 - \mathcal{H}')]\frac{\theta}{k^2} \tag{1.22}$$

One can recast (1.13) in the following way

$$\Psi = -\frac{a^2\rho}{2M_p^2k^2}\mu\Delta \tag{1.23}$$

where $M_p \equiv (8\pi G)^{-1}$ is the bare Planck mass.
It is straightforward to compute the derivative of the Newtonian potential with respect to conformal time and then, using (1.14), the derivative of the curvature potential

$$\Psi' = -\frac{a^2\rho}{2M_p^2k^2}[(\mu' - \mu\mathcal{H})\Delta + \mu\Delta'], \qquad \Phi' = \eta'\Psi + \eta\Psi' \tag{1.24}$$

Substituting these expressions in (1.20)-(1.22) we get

$$\Delta' = \left\{-\frac{3a^2\rho}{2M_p^2}\mu\eta\left[\mathcal{H}\frac{(1-\eta)}{\eta} + \frac{(\mu\eta)'}{\mu\eta}\right]\Delta - \left[k^2 + 3(\mathcal{H}^2 - \mathcal{H}')\right]\theta\right\}\left[k^2 + \frac{3a^2\rho}{2M_p^2}\mu\eta\right]^{-1} \tag{1.25}$$

$$\theta' = -\mathcal{H}\theta - \frac{a^2\rho}{2M_p^2}\mu\Delta \tag{1.26}$$

By solving these equations one finds Δ and θ, then the solutions for the potentials follow from (1.13) and (1.14).

1.2.1 Super-horizon evolution

In [38] it is shown that in any theory of gravity in which the energy-momentum tensor is covariantly conserved, if the entropy perturbations can be neglected and the long wavelength curvature perturbations have a well defined behaviour in the infrared limit[7], the following consistency relation must hold [63]:

$$\zeta\left[\frac{1}{a}\left(\frac{a}{\mathcal{H}}\right)' + O(k^2)\right] = \frac{1}{a^2}\left(\frac{a^2\Phi}{\mathcal{H}}\right)' + \Psi - \Phi \tag{1.27}$$

where ζ is the curvature perturbation on hypersurfaces of uniform density [79] and k (in this section) denotes the comoving wave number. On super-horizon scales ($k << aH$) it is independent by time if one neglects terms of order $O(k^2\zeta)$, thus one can use this expression to define ζ

[6]It is also negligible for baryons at late time.
[7]We are assuming also flat background

in this way[8]

$$\zeta = \Phi + \frac{\Phi' + \mathcal{H}\Psi}{\mathcal{H}(1 - \mathcal{H}'/\mathcal{H}^2)}, \qquad (k=0) \tag{1.28}$$

The conservation of this quantity on super-horizon scales yields a second order differential equation for Φ

$$\dot{\zeta} = \ddot{\Phi} - \frac{\ddot{H}}{\dot{H}}\dot{\Phi} + \dot{\Psi} + \left(\frac{\dot{H}}{H} - \frac{\ddot{H}}{\dot{H}}\right)\Psi = 0, \qquad (k=0) \tag{1.29}$$

where here the overdot denotes derivative with respect to lna [9]. Thus, once the relation between the two gravitational potentials is specified, i.e. once the gravitational slip is known, this equation solves the evolution of linear perturbations on super-horizon scales. Then, if all the assumptions we made hold, the time and scale dependence in this regime must factorize and thus one obtains

$$\Psi(k,t) = F(a)\zeta(k) + O(k^2\zeta) \tag{1.30}$$

$$\Phi(k,t) = \eta(a)\Psi(k,t) + O(k^2\zeta) \tag{1.31}$$

Therefore, substituting $\Phi = \eta\Psi$ in (1.29) one gets [16]

$$\ddot{\Psi} + \left[2\frac{\dot{\eta}}{\eta} + \frac{1}{\eta} - \frac{\ddot{H}}{\dot{H}}\right]\dot{\Psi} + \left[\frac{\ddot{\eta}}{\eta} - \frac{\ddot{H}}{\dot{H}}\frac{\dot{\eta}}{\eta} + \left(\frac{\dot{H}}{H} - \frac{\ddot{H}}{\dot{H}}\right)\frac{1}{\eta}\right]\Psi = 0, \qquad (k=0) \tag{1.32}$$

One can show that this equation can be equivalently obtained by combining (1.10), (1.11), (1.13) and (1.14) in the limit $k^2/(a^2H^2\mu\eta) \to 0$ and assuming that η and μ are close to their GR values[10]. The equation (1.32) shows that in this regime the scalar potential evolution is independent from μ, this means that on super-horizon scales the only degree of freedom, for a generic theory of gravity, is η.

1.2.2 Sub-horizon evolution

This regime corresponds to $k >> aH$ and in this limit the equation for θ and Δ are the following:

$$\Delta' = -\theta \tag{1.33}$$

$$\theta' = -\mathcal{H}\theta - \frac{a^2\rho}{2M_p^2}\mu\Delta \tag{1.34}$$

$$\Longrightarrow \qquad \Delta'' + \mathcal{H}\Delta' - \frac{a^2\rho}{2M_p^2}\mu\Delta = 0 \tag{1.35}$$

This is a second order differential equation in Δ where the only phenomenological function that appears is μ; it means that on sub-horizon scales the growth of matter perturbations is affected only by μ.

[8]This approximation is valid if the Jeans length is smaller than the cosmological scale we are considering; this is true for $z < 30$ [38].

[9]In terms of derivatives of "lna" and $H = da/dt$ (with t the standard cosmological time) the curvature perturbation is $\zeta = \Phi - (\dot{\Phi} + \Psi)H/\dot{H}$, where $\dot{} \equiv d/d\text{ln}a$.

[10]Such an assumption is indeed in well accordance with observations.

1.2.3 The ΛCDM limit

In this limit μ and η are equal to one so that the equations (1.25) and (1.26) become

$$\Delta' = -\theta\left[k^2 + 3(\mathcal{H}^2 - \mathcal{H}')\right]\left[k^2 + \frac{3a^2\rho}{2M_p^2}\right]^{-1} \tag{1.36}$$

$$\theta' = -\mathcal{H}\theta - \frac{a^2\rho}{2M_p^2}\Delta \tag{1.37}$$

if the radiation component can be neglected then we have $3(\mathcal{H}^2 - \mathcal{H}') = 3a^2\rho/2M_p^2$ and thus the equation (1.36) reduces to

$$\Delta' = -\theta \tag{1.38}$$

this last equation combined with (1.37) gives the second order differential equation for Δ

$$\Delta'' + \mathcal{H}\Delta' - \frac{a^2\rho}{2M_p^2}\Delta = 0 \tag{1.39}$$

from the above expression we can see that it looks like the equation obtained in the sub-horizon regime except that, in the first, there is a factor μ in the expression. This shows that in this case the time evolution of Δ is scale-independent.

1.3 Other parametrizations

1.3.1 Parametrized Post-Friedmann (PPF) formalism

The parametrized post-Friedmann (PPF) formalism has been introduced with the purpose of parametrizing in a phenomenological and model-independent way the new degrees of freedom arising from modification of general relativity at cosmological scales; it was the analogous of the parametrized post-Newtonian (PPN) formalism which was introduced in the 1970s to test alternative gravity theories in the solar system and also in binary systems[11]. Indeed, PPN and PPF, can be thought of as complementary parametrizations that cover different gravitational regimes.

There are different formulations of the PPF parametrization, among them we find the one proposed by Hu and Sawicki in [5]. In this paper they consider three different regimes of modified gravity: a) the super-horizon scales in which the evolution of cosmological perturbations has to be compatible with the background expansion; b) the intermediate scales, i.e. quasi-static regime, in which the Poisson equation is modified; c)the non-linear regime in which the additional degrees of freedom must be suppressed in order to satisfy local gravity constraints. Thus, they develop a PPF framework that includes all these three regimes and which gives rise to accelerated expansion without dark energy. Their parametrization consists in three functions which characterize deviations from GR and one parameter that controls the transition between the first two regimes. In particular, in the super-horizon regime and in the quasi-static regime, they define the metric ratio function that in the Newtonian gauge is

$$g(a,k) = \frac{\Phi - \Psi}{\Phi + \Psi} \tag{1.40}$$

this function specifies the relation between the two scalar potential and determines the evolution of the metric fluctuations. In particular, in [5], is pointed out that even if the definition

[11] The PPN formalism consist in a parametrization of the metric tensor that satisfies a set of reasonable assumptions in the slow-motion, weak-gravitational field limit.

of the function holds in the two regimes, the metric evolution is not the same, indeed in the quasi-static regime one can have variations of the effective Newton's constant. In terms of the gravitational slip parameter η this function is

$$g = \frac{\eta - 1}{\eta + 1} \tag{1.41}$$

so that a non zero value implies deviation from GR. Then using the Weyl potential $\Phi_W = (\Phi + \Psi)/2$ one has that

$$\Phi = (1+g)\Phi_W, \qquad \Psi = (1-g)\Phi_W \tag{1.42}$$

This means that under the assumptions made in section 1.2.1, on super-horizon scales, the scalar metric perturbations can be completely specified by the metric ratio "g" and "H". In GR this function is completely determined by the ratio of anisotropic stress to energy density; but since the matter anisotropic stress is negligible at linear order at late time, this quantity is determined by the anisotropic stress of dark energy; however, in dark energy models based on scalar fields it is also negligible at linear order giving $g = 0$.
The other two functions used by Hu and Sawicki are $f_\zeta(a)$ and $f_G(a)$. The former expresses the super-horizon relationship between the metric and density, in particular, the choice of this function is not important from an observational point of view. The latest is introduced for the quasi-static regime to allow modification of the effective gravitational coupling in the following way

$$k^2\Phi_W = \frac{4\pi G}{1+f_G} a^2 \rho\Delta \tag{1.43}$$

The parameter introduced in order to make a bridge between the two linear regimes is c_Γ (for details see [5].
Finally for the non-linear regime, in order to satisfy local gravity constraints and thus to suppress additional degrees of freedom, they develop a non-linear ansatz based on the halo model of non-linear clustering that gives a qualitative description of the main features of this regime in modified gravity. In particular, they decompose the non-linear matter power spectrum in the sum of two pieces, one accounts for the correlations between DM halos and the other for the correlations within DM halos. Thus it can be written as

$$P(k) = I_1(k) + I_2^2(k) P_L(k) \tag{1.44}$$

where P_L is the linear power spectrum of the density fluctuations and the expressions for $I_i(k)$ $(i = 1,2)$ can be found in [5]).
An other formulation of the PPF formalism has been introduced in [6]. The idea is to construct a parametrization that allows for modifications of the Einstein's field equations without specifying any precise form. In particular, these modifications can introduce new scalar degrees of freedom and also they can modify the degrees of freedom already present in GR.
The starting point is to add to the linearised field equations a piece arising from these modifications

$$\begin{aligned} \delta G_{\mu\nu} &= 8\pi G \delta T_{\mu\nu} + \delta U_{\mu\nu} \\ \delta U_{\mu\nu} &= \delta U_{\mu\nu}^{metric} + \delta U_{\mu\nu}^{d.o.f.} + \delta U_{\mu\nu}^{matter} \end{aligned} \tag{1.45}$$

Thus they decompose the additional term $\delta U_{\mu\nu}$ in three parts among which the third can be eliminated in favour of the first two. Therefore the additional terms due to modified gravity are those arising from scalar perturbation of the metric and of the new degrees of freedom. Then, they choose, as derivative order of the parametrization, the second order so that the Ostrogradski's instability is avoided. Moreover they require gauge-invariance of the parametrization and that the coefficients of the parametrization must be functions of the zero order background quantities, for FRW universe this means that they must be functions of scale and time.

They expand $\delta U^{metric}_{\mu\nu}$ and $\delta U^{d.o.f.}_{\mu\nu}$ in terms of gauge-invariant variables, in particular, for the first, they choose to use the Bardeen potential $\hat{\Phi}$ and a linear combination of the two Bardeen potentials $\hat{\Gamma} = (\dot{\hat{\Phi}} + H\hat{\Psi})/k$; for the latest they introduce the gauge-invariant variable $\hat{\chi}$, that represents a gauge-invariant parametrization of the additional scalar degree of freedom. The coefficients of this expansion, which are function of time and scale, are twenty-two and are not all independent; moreover their number can be reduced by imposing restrictions on the types of theories of gravity one is considering. This parametrization covers scales at which the linear perturbation theory holds, a detailed analysis can be found in [5].

1.3.2 Caldwell's gravitational slip

The Caldwell's gravitational slip is a cosmological parameter introduced in [62] in order to quantify the departure from GR at cosmological scales and it is defined implicitly by

$$\Psi(a,k) = [1 + \varpi(a,k)]\Phi(a,k) \tag{1.46}$$

Thus this parameter measures the difference between the two scalar potentials or equivalently, the scalar shear fluctuations in a dark energy component. It can be thought as the similar of the PPN parameter γ[12] in a cosmological context. Its relation with this parameter is $\varpi \simeq 1 - \gamma$ in the limit of weak departure from GR. On scales of tens kiloparsec the constraint on the PPN parameter translates in a constraint on the Caldwell's parameter so that $\varpi = 0.02 \pm 0.07$ (68% CL); while on few hundred kiloparsec scales comparison between weak lensing and X-ray based masses gives the value $\varpi = 0.03 \pm 0.10$ (68% CL) [62].
A non-zero value of the Caldwell's gravitational slip affect the anisotropic constraint (eq. (1.7)) in the following way

$$k^2(\Psi - \Phi) = -12\pi G a^2 \bar{\rho}\sigma|_{\gamma,\nu} + \varpi k^2 \Phi \tag{1.47}$$

where the first contribution is that due to the free-streaming of the relativistic particles and it is negligible at late time. Thus, a non vanishing value of this parameter, can be interpreted as a non-standard relation between the two scalar potentials (i.e. $\Phi \neq \Psi$) due to modifications of the gravity; the specific form of ϖ can be obtained for a given theory of gravity.
In [62] the authors start from a universe described by the ΛCDM model and introduce the Caldwell's gravitational slip as an extension of this simple model that accounts for the accelerated expansion of the late time universe when the dark energy component becomes important; the resulting model was denoted as $\varpi\Lambda$CDM in [39]. Since in this model the gravitational slip appears only at late time they use the following phenomenological parametrization

$$\varpi(a,\bar{k}) = \varpi_0(a,k)\frac{\rho_{DE}(a,k)}{\rho_m(a,k)} \tag{1.48}$$

where the parameter ϖ_0 has to be $|\varpi_0| \lesssim 10$ to satisfy local gravity constraints and $|\varpi_0| \lesssim 100$ on the scale of a galaxy halo.
Then one can expand to first order in perturbations around the FRW background obtaining

$$\varpi(a,k) \simeq \varpi_0 \frac{\bar{\rho}_{DE}(a)}{\bar{\rho}_m(a)} = \varpi_0 \frac{\Omega_{DE}}{\Omega_m}(1+z)^{-3} = \varpi(a) \tag{1.49}$$

where $\bar{\rho}_{DE}$ and $\bar{\rho}_m$ are the background values of respectively the DE density and the matter density.

[12] It is the PPN parameter which measures the amount of spacetime curvature per unit mass; it was strongly constrained to be close to one (that is the GR value of this parameter) in the solar system by the Cassini mission [70] which measured $\gamma - 1 = (2.1 \pm 2.3)\cdot 10^{-5}$. On galactic size scales (kiloparsec scales) its value has been measured to be 0.98 ± 0.07 (68% CL) by comparing, in two different ways, the mass of 15 elliptical lensing galaxies from the Sloan Lens ACS Survey [71].

From (1.49) one can see that, at the lowest order in perturbations, the scale dependence disappears, so the Caldwell's gravitational slip depends only by time (or redshift), while the gravitational slip parameter η introduced in section (1.2) is function of time and scale. If the scale dependence can be neglected, the relation between the two gravitational slip parameters is the following

$$\varpi(a) = \eta(a)^{-1} - 1 \tag{1.50}$$

1.3.3 The E_G-parameter test

The E_G parameter is a useful quantity because its determination combines three different probes of LSS [7]- [8]: galaxy-galaxy lensing, galaxy clustering and galaxy redshift distortions. This combination ensures that this parameter is insensitive to the galaxy bias but very sensitive to modification of gravity that generates gravitational slip or a difference in the rate of growth of structures with respect to GR. Indeed, the galaxy-galaxy lensing, is sensitive to the Weyl potential $\Phi_W = (\Psi + \Phi)/2$, while the galaxy clustering depends on the Newtonian potential Ψ and, therefore, they can be used to probe any non-standard relation between the two scalar potentials. On the other hand, the galaxy redshift distortions is sensitive to the growth of the structures and thus can be used to discriminate among different theories of gravity that predict different growth rates.
Following Zhang et al. (2007) [7], we define E_G as the ratio of the Laplacian of the Weyl potential over the peculiar velocity divergence

$$E_G(k) = \left[\frac{\nabla^2(\Psi + \Phi)}{3H_0^2(1+z)\theta(k)} \right]_{\bar{z}, k=\ell/\bar{\chi}} \tag{1.51}$$

where $\bar{\chi}$ is the mean angular comoving distance at the redshift $\bar{z}$ and $\theta = \nabla \vec{v}/H(z)$ is the velocity field divergence. This latter can be written in term of the matter perturbation and the growth factor ($f(z)$) using the equation (1.20) in the sub-horizon limit as $\theta = f(z)\delta_m$. In the ΛCDM model E_G is scale-independent and is given by

$$E_G^{GR} = \frac{\Omega_{0m}}{f(z)} \tag{1.52}$$

where Ω_{0m} is the relative matter density today. The E_G parameter is scale-independent also in models with large sound speed and negligible anisotropic stress, but potentially, in a general theory of gravity, this parameter can exhibit scale-dependence.
The quantity defined above corresponds to the expectation value of the estimator $\hat{E}_G$, i.e. $\langle \hat{E}_G \rangle = E_G$. The estimator $\hat{E}_G$ is obtained by the ratio of the crossing galaxy-lensing power spectrum over the velocity galaxy cross power spectrum

$$\hat{E}_G(\ell, \Delta\ell) = \frac{C_{kg}(\ell, \Delta\ell)}{3H_0^2(1+z)\sum_\alpha f_\alpha(\ell, \Delta\ell)P^\alpha} \tag{1.53}$$

here $C_{kg}(\ell, \Delta\ell)$ is the band power of the crossing galaxy-lensing power spectrum centred in ℓ with band width $\Delta\ell$; P^α is the band power of the galaxy velocity cross power spectrum $P_{g\theta}$ between k_α and $k_{\alpha+1}$ and f_α is the corresponding weighting function.
In order to obtain this estimator they use the relation between the matter overdensity and the peculiar velocity field (equation 1.20 on sub-horizon scale); in this way, by measuring the velocity field at a given redshift, one is able to extract the corresponding matter overdensity. The velocity field can be obtained by the redshift space distortions (RSD) of the galaxy power spectrum; while the lensing signal at the corresponding redshift is measured using the cross correlation of the galaxies and the lensing maps reconstructed from background galaxies.

Chapter 2

Gravitational slip

The gravitational slip is the mismatch between the two scalar potentials appearing in the perturbed FRW metric, in the Poisson gauge (or in the Newtonian gauge), due to modifications of the geometrical part of the Einstein's field equations. It is a general signature of non-minimal gravitational coupling and thus, it is a key quantity in order to characterize the nature of the dark energy.

The traceless part of the (ij)-component of the linearized Einstein's equations, i.e. the anisotropy constraint (or shear equation), in a general theory of gravity can be recast in the following form [32]- [33]

$$\Psi(a,k) - \Phi(a,k) = \sigma(a)\Pi + \pi_m \tag{2.1}$$

where the quantity Π is a function of the parameters which characterize the theory, while $\sigma(a)$ is a background function and it depends only by the parameters of the theory. Finally π_m is the anisotropic stress sourced by the matter sector and it is non vanishing for imperfect fluids. In GR we have seen that the anisotropy constraint equation is given by equation (1.7) and specifying the anisotropic stress contribution one gets

$$k^2(\Psi - \Phi) = -32\pi G a^2[\rho_\gamma \Theta_2 + \rho_\nu \mathcal{N}_2] \tag{2.2}$$

one can see that the only source of anisotropic stress comes from the quadrupole moments of relativistic particles, i.e. photons and neutrinos, that are generated by the free-streaming of these particles. However, the contribution of photons is very small and that of neutrinos becomes negligible at late time[1], thus, at linear order in perturbations, it disappears and the relation between the two gravitational potentials becomes trivial $\Phi = \Psi$, this means that $\eta = 1$. At second order in perturbations, the contribution of the anisotropic stress is not vanishing but it is constrained in the late universe by the bound $|\eta - 1| \lesssim 10^{-3}$.

However, if an imperfect fluid drove the cosmic acceleration, its contribution to the anisotropic stress might be not negligible at linear order.

In [33]- [32] two fundamental point are highlighted:

- There is a subtle distinction between anisotropic stress sourced by imperfect-fluid and gravitational slip, the former is a property of matter, the latest is a purely geometrical effect.They both give rise to a difference between the two scalar potentials but, as we will see later, only the second leads to a non-standard propagation of the tensor modes (GWs) in many theories of modified gravity.
- The gravitational slip parameter $\eta(a,k) = \Phi/\Psi$ can depend in principle by the reference frame; however, one can remove this ambiguity by adopting an operational definition

[1] The contribution of neutrinos to the anisotropic stress in the ΛCDM scenario is significant only after decoupling during the radiation-dominated era when $\eta - 1 = \frac{2}{5}\left(\frac{\rho_\nu + p_\nu}{\bar{\rho} + \bar{p}}\right)$; while during the matter-dominated era this contribution becomes negligible since it scales as $\eta - 1 \propto a^{-1}$.

which is model independent. This consist in reconstructing this parameter by comparing the evolution of redshift space distortions of the galaxy power spectrum with weak-lensing tomography, thus, η is a "bona fide" observable and not only a phenomenological function [10].

The presence of such a difference between the scalar potentials affects the LSS in several ways; in particular, a non-vanishing gravitational slip will affect the growth of structures, the CMB temperature anisotropies, the CMB polarization maps and it also might affect the sign of CMB and LSS cross-correlations [63]- [39].

2.1 Probing gravitational slip with large-scale structures

In section 1.2.1 we have seen that on large scale the only degree of freedom necessary to fully specify a general theory of gravity is η. In [63] Bertschinger and Zukin distinguish between two classes of theories of gravity:

- *Scale-independent modified gravity models*: are theories in which the terms of order $O(\zeta k^2)$ can be neglected also at sub-horizon scales which are higher than Jeans length scale. Then in this case the gravitational slip completely characterizes deviations from GR and following [63], it can be parametrized as follows

$$\eta(a) = 1 + \beta a^s \tag{2.3}$$

  where β and $s > 0$ are constant parameters.
  Thus, all the observable quantities of a scale-independent theory of gravity, can be specified in terms of (2.3).

- *Scale-dependent modified gravity models*: are more general theories in which the functions that describe departures from GR depend on time and scale and can be parametrized, at linear order in perturbations, as follows

$$\mu(a,k) = \frac{1+\alpha_1 k^2 a^s}{1+\alpha_2 k^2 a^s} \qquad \eta(a,k) = \frac{1+\beta_1 k^2 a^s}{1+\beta_2 k^2 a^s} \tag{2.4}$$

  where $s > 0$ is a constant and α_i, β_i $(i = 1,2)$ are arbitrary constants with the units of length squared. Moreover, to avoid divergence in the above expressions, one must require $\alpha_2, \beta_2 > 0$. Finally, in order to have a theory of gravity which is attractive, one needs $\alpha_1 > 0$. In this parametrization there are at least three physical scales; in particular, considering theories in which the α_i and β_i are all comparable, we have: the Hubble scale $(1/H)$, the transition scale $\sqrt{\alpha_1} = a^{1+s/2}$ and the non -linear length scale for LSS formation $(\sim 10$ today).

2.1.1 Growth of structures

At sub-horizon scales and at late time, the evolution of growth of density perturbation[2] is given by

$$\delta'' + \mathcal{H}\delta' = -k^2\Psi = \begin{cases} 4\pi G a^2 \rho\delta & \text{scale-independent case} \\ 4\pi G\mu(k,a) a^2 \rho\delta & \text{scale-dependent case} \end{cases} \tag{2.5}$$

where we have used (1.20)-(1.22). This equation has two solutions: a decaying mode and a growing mode. We are interesting in the latter which is the solution that leads to the structures

[2] We are considering non relativistic pressureless perturbations or CDM.

formation. In GR the linear growth factor of perturbations, denoted as $D_+(a)$, is the quantity that relates the overdensity $\delta(a)$ at a given "a" to that at some initial "a_i"

$$\delta(a) = \frac{D_+(a)}{D_+(a_i)}\delta(a_i) \tag{2.6}$$

where $D_+(a_i)$ and $\delta(a_i)$ are fixed by the initial conditions. From this quantity one can define an other quantity relevant from an observational point of view, this is the growth rate and is defined as follows

$$f(a) = \frac{\mathrm{d\ lnD}}{\mathrm{d\ lna}} \tag{2.7}$$

Then one can define the theoretical matter power spectrum as

$$\langle\delta(z,\boldsymbol{k})\delta(z,\boldsymbol{k}')\rangle = (2\pi)^3 P(z,k)\delta_D^{(3)}(\boldsymbol{k}-\boldsymbol{k}') \tag{2.8}$$

$$P(z,k) = A_s k^{n_s} T^2(k) G^2(z) \tag{2.9}$$

where $\delta_D^{(3)}$ is the delta of Dirac, $A_s k^{n_s}$ is the primordial fluctuations power spectrum (with $n_s = 0.9652 \pm 0.0062$ from observation); $T(k)$ is the transfer function which describes the evolution of perturbations through the horizon crossing and the radiation-matter transition. Finally $G(a) \equiv D_+(a)/a$ is the scale-independent growth at late time.
Then, to connect this theoretical prediction with observations coming from galaxy surveys, one has to account for the galaxy bias $b(z,k) = \delta_g(z,k)/\delta_m(z,k)$ and the redshift space distortions (RSD) that introduces a factor $f(z)\mu_{RSD}^2$ (μ_{RSD} is the cosine of the angle to the line of sight. Thus the galaxy power spectrum is

$$P_{gg}^s(z,k,\mu_{RSD}) = A_s k^{n_s} T^2(k) G^2(z)\left[b(z,k) + f(z)\mu_{RSD}^2\right]^2 \tag{2.10}$$

On small scales, the power spectrum must be modified to include non-linear effects.
Departures from GR can affect $T(k)$, $G(z)$ and also the growth rate $f(z)$. As a consequence, the galaxy power spectrum can be modified in shape and amplitude.
In the scale-independent class of models, as we just said above, the deviations from GR are characterized by $\eta(a)$, therefore, in these theories, the matter power spectrum can be affected only through an enhancement or diminution of the Newtonian potential (with respect to GR case) that correspond respectively to $\eta < 1$ and $\eta > 1$.
Instead, in the scale-dependent case, a new degree of freedom is introduced in order to characterize modifications of gravity (in the above parametrization $\mu(a,k)$) and, as we have seen in section 1.2.3, in this case the modification of the growth rate depends only by $\mu(a,k)$ which is independent by $\eta(a,k)$.

2.1.2 CMB temperature anisotropies and CMB polarizations

The observations of CMB (Cosmic Microwave Background) are fundamental in order to understand the early universe and its evolution. In particular, its temperature anisotropies and its polarization maps provide a powerful tool in testing modified gravity.
The CMB temperature anisotropies were measured for the first time by the COBE satellite in 1992 [77], then other experiments gave more precise measures of these temperature fluctuations. The CMB polarization maps can be decomposed in curl-free E-modes and gradient-free B-modes; these latter are generated by gravitational lensing of CMB by LSS. B-modes have been observed by two independent experiments [21]- [22] and are very interesting because they give us information about primordial gravitational waves [3] as well as about the lensing potential.

[3] By measuring primordial tensor perturbations, i.e GWs, one obtains information about the energy scale of inflation.

The CMB power spectrum is affected by modifications of gravity in several ways, as an example it is modified at low multipoles (large angular scales) by the late ISW (Integrated-Sachs-Wolfe) effect caused by the time variation of the gravitational potentials. Moreover, modified gravity theories that predict gravitational slip, affect the B-mode spectra in two different ways: 1) the first effect is the modification of the lensing potential as well as of the TT, EE and BB spectra; 2) the second effect is a possible shift of the position of the primordial B-mode peak which is determined by the tensor speed c_T, thus a theory which predicts $c_T \neq 1$ predicts also a change in the position of this peak.

Integrated Sachs-Wolfe effect

The ISW effect is a secondary anisotropy of the CMB temperature fluctuations due to the fact that photons from the last scattering surface reach us crossing potential wells and voids. At late time ($z < 2$), this effect can be induced only by departures from GR: the difference in the energy that photons gain by falling down to the potential wells and that they loose by climbing out of these wells can differ from zero because of modification of gravity or the presence of a dark energy component; while the photons which cross voids reduce their energy and thus their temperature.
The ISW effect is given by the following integral

$$\frac{\Delta T}{T}(\hat{\boldsymbol{n}}) = -\int_{\tau_0}^{\tau_*} d\tau \frac{\partial(\Psi + \Phi)}{\partial \tau} \tag{2.11}$$

here T is the CMB temperature, τ_* the conformal time at the last scattering surface, τ_0 is the conformal time of the observer and $\hat{\boldsymbol{n}}$ the photon's direction.
This effect modifies the CMB temperature power spectrum at larger angular scales ($\ell < 10$). The presence of gravitational slip, as well as a different evolution in the scalar potentials, modifies the CMB temperature power spectrum. In particular, in [63], they show that for scale-independent models with $0.2 < \eta < 1$ this effect reduces the low multipoles part of this spectrum because of the destructive interference with the primary anisotropy contribution.

CMB weak lensing

As photons travel from the last scattering surface to us, their paths are gravitationally deflected by the LSS. These deviations induce very small distortions in the CMB temperature and polarization maps that, in turn, lead to the generation of 3 and 4 point correlation functions and to the conversion of the E-mode polarization into B-modes. These non-Gaussian B-mode signals can be used to constrain different cosmological parameters, among these the sum of neutrino's masses.
In order to calculate the weak lensing effect on the CMB anisotropies, we define the lensing potential as [78]

$$\psi(\hat{\boldsymbol{n}}) = -2\int_0^{\chi_*} d\chi \frac{f_K(\chi_* - \chi)}{f_K(\chi_*) f_K(\chi)} \Phi_W(\chi\hat{\boldsymbol{n}}; \tau_0 - \chi) \tag{2.12}$$

where $\Phi_W = (\Psi + \Phi)/2$ is the Weyl gravitational potential; χ_* is the conformal distance to the surface of last scattering, $\tau_0 - \chi$ is the conformal time at which the photon is at the position $\chi\hat{\boldsymbol{n}}$. The deflection angle, i.e the angle formed by the deflected direction and the one that the photon would have in an unperturbed universe, is denoted as $\boldsymbol{\alpha}$ and it is related, in the first order approximation, with the lensing potential by $\boldsymbol{\alpha} = \nabla\psi$. Then, the lensed CMB temperature in a given direction $\hat{\boldsymbol{n}}$, is denoted with $\widetilde{T}(\hat{\boldsymbol{n}})$ and it is equal to the unlensed temperature, T, in the deflected direction $\hat{\boldsymbol{n}}' = \hat{\boldsymbol{n}} + \boldsymbol{\alpha}$, that is $\widetilde{T}(\hat{\boldsymbol{n}}) = T(\hat{\boldsymbol{n}} + \boldsymbol{\alpha})$.
Since we are considering a spatially-flat universe we can set in the above definition $f_K(\chi) = \chi$.

Then, one can expand the lensing potential in spherical harmonics $\psi = \sum_{lm} \psi_{lm} Y_{lm}$.
The power spectrum of the lensing potential in a given direction is

$$\langle \psi_{lm} \psi^*_{l'm'} \rangle = \delta_{ll'} \delta_{mm'} \mathcal{C}_l^{\psi} \tag{2.13}$$

$$\mathcal{C}_l^{\psi} = 16\pi \int \frac{dk}{k} \mathcal{P}_{\mathcal{R}}(k) \left[\int_0^{\chi_*} T_{\Phi_W}(k; \tau_0 - \chi) j_l(k_\chi) \frac{\chi_* - \chi}{\chi_* \chi} \right]^2 \tag{2.14}$$

here T_{Φ_W} is the transfer function and in a linear theory it relates the primordial comoving curvature perturbation to the Weyl potential t through $\Phi_W(\boldsymbol{k}, \tau) = T_{\Phi_W}(k; \tau)\mathcal{R}(\boldsymbol{k})$; while $j_\ell(r) = \sqrt{\pi/2r} J_{l+1/2}()r$ with $J_\ell(r)$ are the Bessel functions of first kind. The Weyl potential can be expressed in terms of the gravitational slip and of the Newtonian potential, therefore

$$\Phi_W = \frac{(1+\eta)}{2} \Psi \qquad \Longrightarrow \qquad \mathcal{P}_{\Phi_W} = \frac{(1+\eta)^2}{4} \mathcal{P}_\Psi \tag{2.15}$$

one finds that if $\eta \neq 1$ the power spectrum of the Weyl potential acquires a factor $(1+\eta)^2/2$ inside the integral of (2.14). Moreover, in [78], it is shown that the lensed B-mode power spectrum for $\ell << 1000$ and at lowest order can be approximated as follows

$$\widetilde{\mathcal{C}}_\ell^B \simeq \frac{1}{4} \int d\ell' \ell' \mathcal{C}_{\ell'}^{\psi} \mathcal{C}_{\ell'}^E \tag{2.16}$$

where $\mathcal{C}_\ell^E$ is the unlensed E-mode spectrum. Thus if $\eta \neq 1$, the presence of a gravitational slip affects the power spectrum of the lensing potential ($\mathcal{C}_\ell^{\psi}$) by a factor $(1+\eta)^2/4$ and since it enters linearly in (2.16), also the B-mode power spectrum gets enhanced, at large scales, by the same factor.

2.1.3 CMB and LSS cross-correlations

The accelerated expansion of the universe leads to the decay of the gravitational potentials and this decay in turn induces the ISW that, as we have just mentioned, is due to the time variation of the scalar potentials and it is responsible for secondary anisotropies of the CMB temperature fluctuations. A consequence of this fact is that, if there is a cluster of galaxies in a given direction of the sky, it is very likely to observe a correlation in the temperature anisotropies of the CMB in the same direction if the photons of the CMB have crossed that region during the accelerated expansion. The ΛCDM model predicts a positive correlation signal, but departures from this model can potentially change the sign of this correlation. In order to identify this signal, one can define the two-point angular cross correlation function between the surveys of the large scale structures and the CMB temperature anisotropy as follows [9]- [39]

$$C^X(\theta) = \langle \Delta_{ISW}(\hat{\boldsymbol{n}}_1) \delta_{LSS}(\hat{\boldsymbol{n}}_2) \rangle \tag{2.17}$$

where $\theta = |\hat{\boldsymbol{n}}_1 - \hat{\boldsymbol{n}}_2|$, $\Delta_{ISW}(\hat{\boldsymbol{n}})$ is the integral (2.11) and $\delta_{LSS}(\hat{\boldsymbol{n}})$ is the density constrast of a clump of luminous matter observed by a given survey in the direction $\hat{\boldsymbol{n}}_2$ and it is given by

$$\delta_{LSS}(\hat{\boldsymbol{n}}_2) = b \int dz \phi(z) \delta_m(\hat{\boldsymbol{n}}_2, z) \tag{2.18}$$

here b is the bias of the galaxy, $\phi(z)$ the selected function of the survey and δ_m is the matter density fluctuation.
Then one can expand $C^X(\theta)$ into a Legendre series

$$C^X(\theta) = \sum_{l=2}^{\infty} \frac{2l+1}{4\pi} C_l^X P_l(cos\theta) \tag{2.19}$$

where $P_l(cos\theta)$ are the Legendre polynomials and C_l^X is the power spectrum of the cross correlation which is given by the following expression

$$C_l^X = 4\pi \frac{9}{25} \int \frac{dk}{k} \Delta_{\mathcal{R}}^2 I_l^{ISW}(k) I_l^{LSS}(k) \tag{2.20}$$

where $\Delta_{\mathcal{R}}^2$ is the primordial power spectrum; $I_l^{ISW}(k)$ and $I_l^{LSS}(k)$ are given by

$$\begin{aligned} I_l^{ISW}(k) &= -\int dz e^{-\kappa(z)} \frac{d((1+\eta)\Psi_k)}{dz} j_l[kr(z)] \\ I_l^{LSS}(k) &= b \int dz \phi(z) \delta_k(z) j_l[kr(z)] \end{aligned} \tag{2.21}$$

where Ψ_k and δ_k are the Fourier components respectively of the Newtonian potential Ψ and matter perturbation; $j_l[kr(z)]$ are the spherical Bessel functions, $r(z)$ the comoving distance at "z" and $\kappa(\tau) = \int_\tau^{\tau_0} d\tau \dot{\kappa}(\tau)$ is the total optical depth.
Thus, from (2.20) and (2.21), one can see that a non-vanishing gravitational slip can affect not only the value of $I_l^{ISW}(k)$ but in principle it could affect also its sign with respect to $I_l^{LSS}(k)$.

2.2 Gravitational slip as signature of non-standard propagation of tensor modes

As like stated previously, the propagation of the scalar tensor modes are affected by modifications of gravity, in particular the presence of gravitational slip implies a non-standard propagation of these modes in a number of MG theories.
To understand what we mean by "non-standard" propagation let us consider the perturbed line element in a spatially flat FRW universe, neglecting the tensor and scalar perturbations, it can be written in the following way

$$ds^2 = -a(\tau)^2 \left[-d\tau^2 + (\delta_{ij} + h_{ij}) dx^i dx^j \right] \tag{2.22}$$

where h_{ij} is a symmetric, traceless and divergence-free tensor which represents the tensor perturbation of the metric, i.e. the GWs.
The equation of motion for the tensor modes can be obtained by linearising the Einstein's field equation. In GR one gets the following equation (in the Fourier space)

$$h_{ij}'' + 2\mathcal{H} h_{ij}' + c_T k^2 h_{ij} = a^2 \Pi_{ij}^T \tag{2.23}$$

where c_T is the tensor speed and, in GR, it is equal to the speed of light, i.e $c_t = 1$; the source term $\Pi_{ij}^T = (T_{ij} - T_k^k \delta_{ij}/3)^{(TT)}$ is the transverse-traceless projection of the anisotropic matter stress tensor that is zero for perfect fluids. The prime stands for the derivation with respect to conformal time. This equation describes the standard propagation of the GWs.
However, in a general theory of gravity, any additional degrees of freedom can alter this form, thus, in such a theory, the propagation equation for the tensor modes can be written as follows

$$h_{ij}'' + (2+\nu)\mathcal{H} h_{ij}' + c_T k^2 h_{ij} + a^2 m_g^2 h_{ij} = a^2 \Gamma \gamma_{ij} \tag{2.24}$$

where

- $\nu \equiv H^{-1} dM_*^2/dt$: this parameter characterize the running of the effective Planck mass, M_*, so that a theory of gravity that predicts an evolution in time of the Plank mass also predicts a non-standard propagation of the GWs;

- c_T: it is the tensor speed and, in general, it can differ from the speed of light[4], thus, if $c_T \neq 1$, the dispersion relation changes and also the propagation of the tensor modes;
- m_g^2: it is the squared mass of the graviton and it appears in massive bi-gravity;
- $\Gamma\gamma_{ij}$: here γ_{ij} is a transverse-traceless tensor which represent the source term; in particular, in the bimetric massive gravity, it is the tensor perturbation of the second metric; while if matter has anisotropic stress, γ_{ij} contains this contribution.

The first two quantities are defined in the matter Jordan frame. By comparing this equation with (2.23) one can see that departures from GR affect the homogeneous part of the propagation equation and thus, the non-standard behaviour of the tensor modes, is due to these kind of modifications. In particular, in scalar-tensor theories, where $m_g = 0$, one can have $\nu \neq 0$ and $c_T \neq 1$, so that the propagation of the GWs is modified; in what follows we will see that these parameters are strictly related to the gravitational slip and in Horndeski gravity it implies a non-standard propagation of GWs, while in beyond Horndeski theories it is not necessarily true. Also in Einstein-Aether models and in bimetric gravity a non-vanishing gravitational slip implies that the propagation of the GWs is modified.

[4]However, there are very stringent constraints on this parameter, in particular, the recent event GW100817, forces c_T to be very close to one at low redshifts.

Chapter 3

The Effective Field Theory approach: unifying single-field models of Dark Energy and Modified Gravity

In this chapter we introduce the Effective Field Theory (EFT) of DE which allows the unification of the description of single scalar-field models of dark energy and modified gravity. The advantage of this approach is that it is a very efficient way to describe a wide number of existing theories in the same language and in a formalism that has a clear connection to cosmological observations. In the following, we present two different EFT parametrizations : the first is the most general EFT approach to dark energy while, the second, the α-parametrization, is specifically designed to study the phenomenological aspects of Horndeski theories and, by adding new parameters, also of its extensions, among which we consider the "beyond Horndeski models". Using these formalisms, we write down the modified Poisson equation and the modified anisotropy constraint and we derive the analytic expressions for the phenomenological functions μ, η and Σ in the quasi-static approximation. In particular, using the second parametrization, we analyse for what values of the α-parameters there is a non-vanishing gravitational slip.

3.1 The most general EFT parametrization

The Effective Field Theory (EFT) approach was first applied in a cosmological context to study inflation [41]- [42], then, it was extended to the DE sector [40]- [43] with the aim of unifying the description of single-field models of DE and MG and also in order to have a theory that was readily testable by observations.
The idea is to start by a perturbed FRW universe in which there are gravity, a single scalar field and a matter sector obeying to the weak equivalence principle (WEP), i.e. it is assumed the existence of a conformal frame, the Jordan frame, in which all matter fields are minimally coupled to the metric. Then, it is possible to construct the most general action that is invariant under time-dependent spatial diffeomorphism by choosing the unitary gauge. In this gauge the dynamic of the scalar field perturbation is "eaten" by the metric, i.e. $\delta\phi$ vanishes and the time coordinate is function of the scalar field $t = t(\phi)$; thus the scalar field ϕ defines a preferred slicing of the spacetime in space-like hypersurfaces where $\phi = const$. Finally, the time diffeomorphism invariance, can be restored via "Stückelberg trick", which consists in performing an infinitesimal time diffeomorphism $t \to t + \pi(x^\mu), x^i \to x^i$ after which the scalar degree of freedom reappears as a new field ($\pi(x)$) in the action.

The EFT action is given by [43]

$$
\begin{aligned}
S = \int d^4x\sqrt{-g}\Big[&\frac{m_0^2}{2}\Omega(t)R + \Lambda(t) - c(t)\delta g^{00} \\
&+ \frac{M_2^4(t)}{2}(\delta g^{00})^2 + \frac{M_3^4(t)}{2}(\delta g^{00})^3 + \dots \\
&- \frac{\bar{M}_1^3(t)}{2}\delta g^{00}\delta K - \frac{\bar{M}_2^2(t)}{2}(\delta K)^2 - \frac{\bar{M}_3^2(t)}{2}\delta K^\mu_\nu \delta K^\nu_\mu + \dots \\
&+ \lambda_1(t)(\delta R)^2 + \lambda_2(t)\delta R_{\mu\nu}\delta K^{\mu\nu} + \gamma_1(t)C_{\mu\nu\sigma\lambda}C^{\mu\nu\sigma\lambda} + \gamma_2(t)\epsilon^{\mu\nu\sigma\lambda}C_{\mu\nu}{}^{\alpha\beta}C_{\sigma\lambda\alpha\beta} + \dots \\
&+ m_1^2(t)n^\mu n^\nu \partial_\mu g^{00}\partial_\nu g^{00} + m_2^2(t)(g^{\mu\nu} + n^\mu n^\nu)\partial_\mu g^{00}\partial_\nu g^{00} + \dots\Big] + S_m[g_{\mu\nu}, \psi_i]
\end{aligned}
\tag{3.1}
$$

where $h_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu$ is the induced spatial metric, $n_\mu \equiv -\partial_\mu\phi/\sqrt{-(\partial\phi^2)}$ is the normal to the hypersurfaces of constant time and $K_{\mu\nu} \equiv h^\rho_\mu \nabla_\rho n_\nu$ is the extrinsic curvature; then δg^{00}, δR, $\delta R^{\mu\nu}$, $\delta K^{\mu\nu} = K_{\mu\nu} - K^{(0)}_{\mu\nu} = K_{\mu\nu} + 3H(g^{\mu\nu} + n^\mu n^\nu)$ and δK are the perturbations respectively to the time-time component of the metric, the Ricci scalar, the Ricci tensor, the extrinsic curvature tensor and to its trace; $C^{\mu\nu\sigma\lambda}$ is the Weyl tensor.
Moreover, we have $m_0^2 = (8\pi G)^{-1}(= M_{pl}^2)$ (the bare squared Planck mass) and the operators Ω, Λ, c, M_i, λ_i, γ_i and m_i which are functions of time; the subset $\{M_i, \lambda_i, \gamma_i, m_i\}$ only affects the behaviour of the perturbations.
However, one finds that only a reduced set of these operators are required to fully characterize the linear perturbation theory. Thus, the EFT action describing the most general single-scalar field model of dark energy, up to quadratic order in perturbations, is

$$
\begin{aligned}
S = \int d^4x\sqrt{-g}\Big[&\frac{m_0^2}{2}\Omega(t)R + \Lambda(t) - c(t)\delta g^{00} \\
&+ \frac{M_2^4(t)}{2}(\delta g^{00})^2 - \frac{\bar{M}_1^3(t)}{2}\delta g^{00}\delta K - \frac{\bar{M}_2^2(t)}{2}(\delta K)^2 - \frac{\bar{M}_3^2(t)}{2}\delta K^i_j \delta K^j_i + \frac{\hat{M}^2(t)}{2}\delta g^{00}\delta R^{(3)} \\
&+ m_2^2(t)(g^{\mu\nu} + n^\mu n^\nu)\partial_\mu g^{00}\partial_\nu g^{00}\Big] + S_m[g_{\mu\nu}, \psi_i]
\end{aligned}
\tag{3.2}
$$

here the term $\delta R^{(3)}$ is the perturbation of the three dimensional spatial Ricci scalar of constant-time hypersurfaces.
This action encompasses a broad number of models, among which: DGP braneworld models, Galileons models, ghost condensate and in particular it contains the Horndeski gravity and beyond Horndeski theories.
From this action one can see that a theory belonging to this class can be specified by the following set of functions of time: $\{\Omega(t), M_1^3(t), M_2^4(t), M_3^2(t), M_2^4(t), \hat{M}^2(t), m_2^2(t)\}$.

3.1.1 Background evolution

The background evolution is described by

$$
S_0 = \int d^4x\sqrt{-g}\Big[\frac{m_0^2}{2}\Omega(t)R + \Lambda(t) - c(t)\delta g^{00}\Big] + S_m[g_{\mu\nu}, \psi_i] \tag{3.3}
$$

It is completely determined by the functions of time $\Omega(t)$, $\Lambda(t)$, $c(t)$ plus the scale factor $a(t)$, the average energy density $\bar{\rho}_m(t)$ and the average pressure $\bar{p}_m(t)$ of the matter fields that satisfy the standard continuity equation (in the Jordan frame). These last three, have to be chosen in such a way to mimic the ΛCDM expansion history.

By varying the action with respect to the metric and, assuming a spatially flat universe, one gets the background evolution equations

$$3m_0^2\Omega\left[H^2 + H\frac{\dot{\Omega}}{\Omega}\right] = \bar{\rho}_m - \Lambda + 2c \tag{3.4}$$

$$m_0^2\Omega\left[3H^2 + 2\dot{H} + \frac{\ddot{\Omega}}{\Omega} + 2H\frac{\dot{\Omega}}{\Omega}\right] = -\Lambda - \bar{p}_m \tag{3.5}$$

Then, using these expressions, one finds that two of the first three functions can be expressed in terms of the others parameters, so only one function of the set $\{\Omega(t), \Lambda(t), c(t)\}$ is needed to characterize the theory at background level.

$$c(t) = -\frac{\bar{\rho}_m + \bar{p}_m}{2} - m_0^2\Omega\left[\dot{H} + \frac{\ddot{\Omega}}{2\Omega} - \frac{H}{2}\frac{\dot{\Omega}}{\Omega}\right] \tag{3.6}$$

$$\Lambda(t) = -\bar{p}_m - m_0^2\Omega\left[3H^2 + 2\dot{H} + \frac{\ddot{\Omega}}{\Omega} + 2H\frac{\dot{\Omega}}{\Omega}\right] \tag{3.7}$$

Finally, one can recast the equations (3.4)-(3.5) introducing the DE energy density and the DE pressure.

$$H^2 = \frac{1}{3\Omega m_0^2}(\bar{\rho}_m + \rho_{DE}) \tag{3.8}$$

$$\dot{H} = -\frac{1}{2\Omega m_0^2}(\bar{p}_m + p_{DE} + \bar{\rho}_m + \rho_{DE}) \tag{3.9}$$

where

$$\rho_{DE} = -\Lambda - 2c - 3m_0^2 H\dot{\Omega}, \quad p_{DE} = \Lambda + m_0^2\ddot{\Omega} + 2m_0^2 H\dot{\Omega} \tag{3.10}$$

By combining the background equations and the matter continuity equation one obtains the continuity equation for dark energy

$$\dot{\rho}_{DE} + 3H(p_{DE} + \rho_{DE}) = 3m_0^2 H^2\dot{\Omega} \tag{3.11}$$

3.1.2 Perturbed equations: Poisson equation and anisotropy constraint

At this stage we can restore the time diffeomorphism invariance by using the above-mentioned "Stückelberg trick" so that the scalar degree of freedom explicitly appears in the action as π, which is the perturbed part of the scalar field.
The perturbed field equations can be obtained by varying the action (3.2) with respect to the metric and then by subtracting the background contribution (3.4)- (3.5).
The resulting equations can be written in this form

$$m_0^2\Omega\delta G^\mu_\nu = \delta T^{\mu(m)}_\nu + \delta T^{\mu(DE)}_\nu \tag{3.12}$$

From them, it is straightforward to obtain the Poisson equation and the anisotropy constraint

$$2m_0^2\Omega\frac{k^2}{a^2}\Phi = -\bar{\rho}_m\Delta + \Delta_P \tag{3.13}$$

$$m_0^2\Omega\frac{k^2}{a^2}(\Phi - \Psi) = \bar{p}_m\Pi + \Delta_S \tag{3.14}$$

with

$$\Delta_P = -2c(\dot{\pi}-\Psi) + m_0^2\dot{\Omega}\Big[-3\dot{H}\Phi + \frac{k^2}{a^2}\pi - 3(\dot{\Phi}+H\Psi)\Big] - 4M_2^4(\dot{\pi}-\Psi) - \bar{M}_1^3\Big[3\dot{\Phi}+3H\Psi+3\dot{H}\pi - \frac{k^2}{a^2}\pi\Big]$$
$$-2H\bar{M}_3^2\frac{k^2}{a^2}\pi - 4\hat{M}^2\frac{k^2}{a^2}(\Phi+H\pi) + 8m_2^2\frac{k^2}{a^2}(\Psi-\dot{\pi}) \tag{3.15}$$

$$\Delta_S = m_0^2\dot{\Omega}\frac{k^2}{a^2}\pi - \bar{M}_3^2\frac{k^2}{a^2}\Big[\dot{\pi} + \Big(H + \frac{2\dot{\bar{M}}_3}{\bar{M}_3}\Big)\pi\Big] + 2\hat{M}^2\frac{k^2}{a^2}(\Psi-\dot{\pi}) \tag{3.16}$$

These terms represent the DE contribution to the previous equations.

3.1.3 Phenomenological functions in QSA

In order to derive analytic expression for the functions $\eta(a,k)$, $\mu(a,k)$ and $\Sigma(a,k)$ one needs the equations (3.13)-(3.14) plus the equation of motion for π (obtained by varying the action with respect to this field) in the quasi-static limit.
The quasi-static approximation consists in neglecting the time variation of all parameters and perturbations of the metric which are assumed to be small with respect to the Hubble time

$$|\dot{X}| \lesssim H|X| \tag{3.17}$$

where X here stands for a generic perturbation of the metric or the scalar field. In particular, in [14], it is shown that this approximation breaks down outside of the sound horizon rather than outside the Hubble horizon.
In this limit the dominant contributions to the perturbation equations are those containing the terms δ_m and k^2/a^2 (i.e. the terms involving spatial derivative of the fields)

$$\frac{k^2}{a^2}|X| >> H^2|X| \tag{3.18}$$

This approximation works well for sub-horizon perturbation when, the mass of the scalar field, is at most of the order of the Hubble parameter (H) and thus, the oscillating mode of the scalar field perturbation, is suppressed relative to the matter-induced mode; while, when this mass is larger than H, the oscillating mode of the scalar field cannot be neglected and, this approximation, does not hold anymore.
Then, if this approximation can be applied, one obtains the following system of equations in the Fourier space

$$A_1\frac{k^2}{a^2}\Phi + A_2\frac{k^2}{a^2}\pi + A_3\frac{k^2}{a^2}\Psi \simeq -\bar{\rho}_m\Delta \tag{3.19}$$
$$B_1\Psi + B_2\Phi + B_3\pi \simeq 0 \tag{3.20}$$
$$C_1\frac{k^2}{a^2}\Phi + C_2\frac{k^2}{a^2}\Psi + \Big(C_\pi + C_3\frac{k^2}{a^2}\Big)\pi \simeq 0 \tag{3.21}$$

the expressions for the coefficients A_i, B_i, C_i ($i = 1,2,3$) and C_π are reported in appendix A (tab.A.1).
To obtain μ we have to put the equation (3.19) in the form (1.13) by using the other two equations in order to eliminate the fields π and Φ.
Finally, we get the following expression

$$\mu(a,k) = 2m_0^2\left[\frac{B_2C_3 - B_3C_1 + \frac{a^2}{k^2}B_2C_\pi}{A_1(B_3C_2 - B_1C_3) + A_2(B_1C_1 - B_2C_2) + A_3(B_2C_3 - B_3C_1) - \frac{a^2}{k^2}(A_1B_1 - A_3B_2)C_\pi}\right] \tag{3.22}$$

Analogously, one finds the analytic expression for the gravitational slip parameter

$$\eta(a,k) = \frac{B_3C_2 - B_1C_3 - \frac{a^2}{k^2}B_1C_\pi}{B_2C_3 - B_3C_1 + \frac{a^2}{k^2}B_2C_\pi} \tag{3.23}$$

Σ can be obtained by using the relation $\Sigma = \mu\frac{(1+\eta)}{2}$.

$$\Sigma(a,k) = m_0^2\left[\frac{B_3C_2 - B_1C_3 + B_2C_3 - B_3C_1 + \frac{a^2}{k^2}(B_2 - B_1)C_\pi}{A_1(B_3C_2 - B_1C_3) + A_2(B_1C_1 - B_2C_2) + A_3(B_2C_3 - B_3C_1) - \frac{a^2}{k^2}(A_1B_1 - A_3B_2)C_\pi}\right] \tag{3.24}$$

3.2 EFT of Horndeski gravity and beyond Horndeski

3.2.1 The α-parametrization

In [45]- [46] an alternative parametrization of the EFT theory of dark energy, specifically designed for Horndeski models, was introduced; it can be extended to beyond Horndeski models and others models by adding new parameters.

In this approach, the evolution of linear perturbations, can be fully specified by the background evolution H(t), the constant ρ_{m0}, i.e. the matter density today, plus a set of four independent functions of time $\{\alpha_M(t),\ \alpha_B(t),\ \alpha_T(t),\ \alpha_K(t)\}$ (and $\alpha_H(t)$ for beyond Horndeski). The α-functions are arbitrary and independent of the first two, thus they characterize the physical properties of the dark energy model. This formulation completely separates the background description from the perturbations, contrary to the original EFT approach in which the operators $\{\Omega(t), \Lambda(t), c(t)\}$ enter both the background and the perturbations evolution.

The starting point is a general unitary gauge action expressed in the Arnowitt-Deser-Misner (ADM) coordinates. The ADM coordinate are particularly useful when the gradient of the scalar field is timelike and thus, one has a preferred slicing of the spacetime in spacelike hypersurfaces of constant ϕ which, in these coordinates, coincide with constant time hypersurfaces. The line element in this metric is the following

$$ds^2 = -N^2dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \tag{3.25}$$

here N_i is the shift function and N the lapse function; h_{ij} is the three-dimensional spatial metric.

In these coordinates one has that

$$X = g^{00} = -\frac{\dot\phi^2(t)}{N^2} \tag{3.26}$$

$$n_\mu = -\frac{\nabla_\mu\phi}{\sqrt{-X}} = (n_0 = -N, n_i = 0) \tag{3.27}$$

$$K_{\mu\nu} = (g^\sigma_\mu + n^\sigma n_\mu)\nabla_\sigma n_\nu \quad \Longrightarrow \quad K_{ij} = \frac{1}{2N}(\dot h_{ij} - D_iN_j - D_jN_i) \tag{3.28}$$

The most general unitary gauge action in the ADM formalism is [46]- [47]

$$S_g = \int d^4x\sqrt{-g}L(N, K_{ij}, R_{ij}, h_{ij}, D_i, t) \tag{3.29}$$

where D_i is the covariant derivative associated to the three-dimensional spatial metric (h_{ij}) and the Lagrangian for Horndeski and beyond Horndeski model is of the following form

$$\mathcal{L} = \sum_{i=2}^{5} \mathcal{L}_i \tag{3.30}$$

with

$$\mathcal{L}_2 = A_2(t, N), \tag{3.31}$$
$$\mathcal{L}_3 = A_3(t, N)K, \tag{3.32}$$
$$\mathcal{L}_4 = A_4(t, N)(K^2 - K_{ij}K^{ij}) + B_4(t, N)R, \tag{3.33}$$
$$\mathcal{L}_5 = A_5(t, N)(K^3 - 3KK_{ij}K^{ij} + 2K_{ij}K^{ik}K_k^j) + B_5(t, N)K^{ij}\left(R_{ij} - \frac{1}{2}h_{ij}R\right) \tag{3.34}$$

In particular, Horndeski theories can be obtained by imposing

$$A_4 = -B_4 + 2XB_{4,X}, \qquad A_5 = -\frac{X}{3}B_{5,X} \tag{3.35}$$

Then one can derive the evolution of the background by varying the action (3.29) with respect to the lapse function and the scale factor. The evolution of linear perturbations can be studied by expanding the Lagrangian in the action (3.29) up to quadratic order. In [46] it is shown that the EFT quadratic action for linear perturbations which give rise to propagation equation with no more than two space derivatives, can be written as follows

$$\begin{aligned} S^{(2)} = \int d^3x dt a^3 \frac{M_*^2}{2}\Big[&\delta K_{ij}\delta K^{ij} - \delta K^2 + (1+\alpha_T)\left(R\frac{\delta\sqrt{h}}{a^3} + \delta_2 R\right) \\ &+ \alpha_K H^2\delta N^2 - 2\alpha_B H\delta K\delta N + (1+\alpha_H)R\delta N\Big] + S_m^{(2)}[g_{\mu\nu}, \psi_i] \end{aligned} \tag{3.36}$$

this is equivalent to the standard EFT action in which $\bar{M}_2^2 = -\bar{M}_3^2$ and $m_2^2 = 0$. Here $S_m^{(2)}$ is the perturbed matter action in the Jordan frame and M_*, is the effective squared Planck mass, which, in general, is function of time. Thus, it is convenient to introduce the dimensionless parameter:

$$\alpha_M \equiv \frac{1}{H}\frac{d}{dt}lnM_*^2 \tag{3.37}$$

which characterizes the running of the Planck mass.
The relations among the α-parameters, the operators appearing in (3.70) and the Horndeski functions K and G_i (3.44), can be found in the table A.2.
This approach has the advantage of provide a clear connection between the parameters of the theory and the phenomenology of Horndeski gravity. On the other hand, the relation of these parameters with the original Lagrangian is much more confusing.
We can now briefly describe the physical meaning of the α-functions.

Planck mass run rate: α_M

It is defined by (3.37) and measures the running of the Planck mass. This parameter, along with α_T (and α_H in beyond Horndeski models), controls the existence of the gravitational slip and it is related with the non-standard propagation of tensor modes [32]- [33]. In general, when $\alpha_M \neq 0$ one also has $\alpha_B \neq 0$, this happens for all known models. Moreover, α_M modifies the evolution of the vector modes and affects the lensing potential as well as the amplitude of the primordial polarization peak in the B-mode power spectrum.

Tensor speed excess: α_T

It quantifies the deviation of the speed of gravitational waves from that of the light: $\alpha_T = c_T - 1$. A non-zero value implies higher order derivative coupling of the scalar field to the metric and this non-linearity gives rise to gravitational slip. Furthermore, this parameter affects the position of the primordial peak in the B-mode power spectrum.

Kinetic braiding: α_B

The non-vanishing of this function means a kinetic mixing between gravitational and scalar degrees of freedom, i.e. a non-zero contribution of the term $\delta K \delta N$ in the quadratic action that implies a coupling between the metric and the scalar field. Furthermore, if $\alpha_B \neq 0$ this parameter and α_K determine a transition scale in the dynamics of the gravitational potentials called "braiding scale", k_B. In particular, α_B controls whether dark energy clusters at all.

Kineticity: α_K

This function measures the independent kinetic energy of the scalar degree of freedom, that is the contribution deriving from the term $\delta N \delta N$. Thus, one has $\alpha_K \neq 0$ in minimal coupling models of dark energy. Large values of this parameter lead to a suppression of the sound speed of the scalar modes. As we just mentioned, it is related to the braiding scale and specifically it determines the scale at which dark energy begins to cluster.

Kinetic mixing with matter: α_H

This parameter has been introduced to extend the α-parametrization to beyond Horndeski models and thus, it is zero for Horndeski gravity. It characterizes theories in which there is a kinetic coupling between the matter fields and the additional scalar degree of freedom. Furthermore, it modifies the matter sound speed and contribute to gravitational slip. In [53], the authors show that this parameter leads to a damping of the matter power spectrum on both large and small scale. In the same paper, they show that it also affects the temperature CMB power spectrum at low multipoles leading to an enhancement of this latter as well as to a decreasing of the lensing potential.

In conclusion these property functions represent the maximum information about the nature of the dark energy that one can obtain from the evolution of linear cosmological perturbations. In particular, we are interesting in theories in which there is gravitational slip, i.e. a non-vanishing Δ_S in (3.14). We can rewrite the anisotropic constraint (3.14) with (3.16) in terms of the α functions

$$M_*^2 \frac{k^2}{a^2}[\Phi - \Psi] = \bar{p}_m \Pi + M_*^2 \frac{k^2}{a^2}\left[H\alpha_M \pi - \alpha_T(\Phi + H\pi) + \alpha_H(\Psi - \dot{\pi})\right] \tag{3.38}$$

It is easy to see that, since at late time the anisotropic stress sourced by matter is negligible, i.e $\Pi \sim 0$, the set $\{\alpha_T, \alpha_M, \alpha_H\}$ determines the relation between the two scalar potentials, thus it controls the existence of gravitational slip. In particular, in Horndeski gravity, it is different from zero if $M_*^2 \alpha_T = -\bar{M}_2^2 = \bar{M}_3^2 = 2X[2G_{4,X} - 2G_{5,\phi} - (\ddot{\phi} - \dot{\phi}H)G_{5,X}] \neq 0$ or if we have a theory with non-minimal coupling with gravity ($\dot{\Omega}(t) \neq 0$) (expression (3.16)), this means that to have gravitational slip at least one of the parameters α_T, α_M must be non-vanishing. Indeed, combining the expression in table A.2 of the appendix A, one finds $M_*^2(1 + \alpha_T) = m_0^2 \Omega$; this relation along with (3.37) and (3.14) shows that if $\alpha_T = 0$, the only way to have gravitational slip is by a non-zero value of α_M, i.e $dM_*^2/dt \neq 0$; this case is equivalent to have $\dot{\Omega} \neq 0$, that is a non-minimal coupling with gravity. In what follows we will see that there is a deep connection among non-standard propagation of GWs, non-minimal coupling with gravity and gravitational slip. In beyond Horndeski theories the situation is complicated by the presence of the property function α_H; this last gives rise to additional contribution to the gravitational slip in such a way that it is possible to have $\Phi \neq \Psi$ even if with $\alpha_M = 0 = \alpha_T$.

3.2.2 Stability conditions

There are some stability conditions that a theory of gravity must satisfy in order to be a viable model [18]- [19]:

a) no-ghost conditions: this kind of instabilities arises whenever in the high-k limit the kinetic matrix has negative eigenvalues that destabilise the high energy vacuum state;

b) positive squared speeds of propagation: it is required in order to avoid gradient instabilities at high values of k;

c) no-tachyonic instabilities: they are related with the presence of negative mass squared terms.

These conditions impose constraints on the free parameters of the theory. In particular, the first two stability conditions, in terms of the α-parametrization, are for tensor and scalar modes respectively [45]- [46]

$$Q_T = \frac{M_*^2}{8} > 0 \tag{3.39}$$

$$c_T^2 = 1 + \alpha_T > 0 \tag{3.40}$$

$$Q_S = \frac{2M_*^2\alpha}{(2-\alpha_B)^2} > 0 \tag{3.41}$$

$$\begin{aligned} c_S^2 = &-\frac{(2-\alpha_B)^2}{2\alpha}\left[1 + \alpha_T - 2\frac{1+\alpha_H}{2-\alpha_B}\left(1 + \alpha_M - \frac{\dot{H}}{H^2}\right) - \frac{2}{H}\frac{d}{dt}\left(\frac{1+\alpha_H}{2-\alpha_B}\right)\right] - \\ &-\frac{(1+\alpha_H)^2}{\alpha}\frac{(\bar{\rho}_m + \bar{p}_m)}{M_*^2 H^2} > 0 \end{aligned} \tag{3.42}$$

where we have defined $\alpha \equiv \alpha_K + \frac{3}{2}\alpha_B^2$
From the above expressions one can see that if $Q_T > 0$ then (3.41) is satisfied if $\alpha > 0$. In the following we assume that these stability conditions hold.
In particular, for Horndeski theories, the expression (3.42) reduces to

$$c_S^2 = -\frac{(2-\alpha_B)[\dot{H} - \frac{1}{2}H^2\alpha_B(1+\alpha_T) - H^2(\alpha_M - \alpha_T)] - H\dot{\alpha}_B + (\bar{\rho}_m + \bar{p}_m)/M_*^2}{H^2\alpha} > 0 \tag{3.43}$$

Thus one finds that the only difference in the stability conditions between Horndeski and beyond Horndeski model is in the gradient of the scalar sector.

3.2.3 Horndeski gravity

In 1974 Horndeski [65] derived the most general class of four-dimensional scalar-tensor theories whose Lagrangian leads to second-order equations of motion; this property ensures that this class of theories are free of Ostrogradski's instabilities[1]. In 2011, Deffayet et al. showed

[1] This kind of instabilities are those that arise when there are ghost-like degrees of freedom and are usually related to higher order time-derivative.

that the original Horndeski action is equivalent to the action of the so called generalized Galileon theories [66]- [67].
The Horndeski action can be cast in the following form

$$S_H[g_{\mu\nu},\phi] = \int d^4x\sqrt{-g}\sum_{i=2}^{5}\mathcal{L}_i \tag{3.44}$$

and the $\mathcal{L}_i$ are defined as follows

$$\mathcal{L}_2 = K(\phi,X), \tag{3.45}$$
$$\mathcal{L}_3 = -G_3(\phi,X)\Box\phi, \tag{3.46}$$
$$\mathcal{L}_4 = G_4(\phi,X)R + G_{4,X}(\phi,X)[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)], \tag{3.47}$$
$$\begin{aligned}\mathcal{L}_5 = G_5(\phi,X)G_{\mu\nu}(\nabla^\mu\nabla^\nu\phi) - \frac{1}{6}G_{5,X}(\phi,X)[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)+\\ + 2(\nabla^\mu\nabla_\rho\phi)(\nabla^\rho\nabla_\sigma\phi)(\nabla^\sigma\nabla_\mu\phi)]\end{aligned} \tag{3.48}$$

where ϕ is the scalar field and $X = -\frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi$ is its kinetic term; then $K(\phi,X)$ and $G_i(\phi,X)(i = 3,4,5)$ are functions of the first two and we use the notations $G_{i,X} \equiv \partial G_i/\partial X$, $G_{i,\phi} \equiv \partial G_i/\partial\phi$, $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$. Finally, g is the determinant of the metric, R the Ricci scalar and $G_{\mu\nu}$ the Einstein's tensor.
The Horndeski action includes a wide number of single scalar field models: to different choices of the functions K and G_i correspond different theories. In the table 3.1 the functions K and G_i are specified for some of the most popular theories belonging to this class.

	$K(\phi,x)$	$G_3(\phi,x)$	$G_4(\phi,x)$	$G_5(\phi,x)$
Quintessence	$X - V(\phi)$	0	$\frac{1}{2}M_{pl}^2$	0
K-essence	$K(\phi,X)$	0	$\frac{1}{2}M_{pl}^2$	0
Brans–Dicke	$\frac{\omega_{BD}}{\phi}M_{pl}X - V(\phi)$	0	$\frac{1}{2}M_{pl}$	0
Generalized Brans–Dicke	$\omega(\phi)X - V(\phi)$	0	$G_4(\phi)$	0
f(R) gravity	$-\frac{1}{2}M_{pl}^2(RF(R) - f(R))$	0	$\frac{1}{2}M_{pl}^2F(R)$	0
Covariant Galileons	c_2X	$2Xc_3/M^3$	$\frac{1}{2}M_{pl}^2 + X^2c_4/M^6$	X^2c_5/M^9
Derivative couplings	$X - V(\phi)$	0	$\frac{1}{2}M_{pl}^2$	$c\phi$
Gauss-Bonnet couplings	$X - V(\phi) + 8\xi^{(4)}(\phi)X^2(3 - lnX)$	$4\xi^{(3)}(\phi)X(7 - 3lnX)$	$\frac{1}{2}M_{pl}^2 + 4\xi^{(2)}(\phi)X(2 - lnX)$	$-4\xi^{(1)}(\phi)lnX$
Kinetic braidings	$K(\phi,X)$	$G_3(\phi,X)$	$\frac{1}{2}M_{pl}^2$	0

Table 3.1: Explicit expressions of the Horndeski functions K and $G_i(i = 3,4,5)$ for some of the well-known models belonging to this class. In this table we adopt the notations of [23] except that we use K instead of G_2 and $-G_3$ instead of G_3.
here $V(\phi)$ is the potential of the scalar field, M_{pl} is the bare Planck mass while m and c are constants; we consider covariant Galileon models without field potential where $c_i(i = 1,3,4,5)$ are dimensionless coefficients and M is a constant having the dimension of a mass.
ω_{BD} is the Brans-Dicke parameter; it can be shown that for $\omega_{BD} = 0$ this theory is equivalent to $f(R)$ gravity in the metric formalism with $V(\phi) = \frac{1}{2}M_{pl}^2(RF - f)$ [30]; while, for $\omega_{BD} \to \infty$, we recover GR with a quintessence scalar field.
$F(R) \equiv \partial f/\partial R$ corresponds to the propagating degree of freedom of the metric $f(R)$ gravity. $\xi^{(n)}(\phi) \equiv \partial\xi^n/\partial\phi^n$ where $\xi(\phi)$ is the coupling between the scalar field and the Gauss-Bonnet curvature invariant defined by $\mathcal{G} = R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$.

To complete our description we add to (3.44) the action for the matter fields assuming a matter perfect fluid minimally coupled to gravity. Then, the complete action is

$$S = \int d^4x\sqrt{-g}\Big[\sum_{i=2}^{5}\mathcal{L}_i + \mathcal{L}_m(g_{\mu\nu},\psi_i)\Big] \tag{3.49}$$

One can show that the EFT action (3.2) for the Horndeski theories corresponds to [44]

$$\begin{aligned} S = \int d^4x\sqrt{-g}\Big[&\frac{m_0^2}{2}\Omega(t)R + \Lambda(t) - c(t)\delta g^{00} + \frac{M_2^4(t)}{2}(\delta g^{00})^2 \\ &- \frac{\bar{M}_1^3(t)}{2}\delta g^{00}\delta K - \frac{\bar{M}_2^2(t)}{2}\Big(\delta K^2 - \delta K^i_j\delta K^j_i - \frac{1}{2}\delta g^{00}\delta R^{(3)}\Big)\Big] + S_m[g_{\mu\nu},\psi_i] \end{aligned} \tag{3.50}$$

thus we have $m_2^2 = 0$ and $2\hat{M}^2 = \bar{M}_2^2 = -\bar{M}_3^2$; in this case all the coefficients in table A.1 are k-independent.
In [44] Bloomfield finds via software analysis the correspondences among the EFT operators and the Horndeski free functions $K(\phi,X)$ and $G_i(\phi,X)$.

Phenomenological functions in Horndeski models (QSA)

After introducing the α-functions, we can rewrite the expressions (3.22)-(3.23)-(3.24) specializing to Horndeski gravity ($2\hat{M}^2 = \bar{M}_2^2 = -\bar{M}_3^2$ and $m_2^2 = 0$)

$$\mu(a,k) = \frac{m_0^2}{M_*^2}\left[1 + \frac{a^2}{k^2}M_C^2\right]\left[\frac{f_3}{2M_*^2 f_1} + \frac{a^2}{k^2}\frac{M_C^2}{1+\alpha_T}\right]^{-1} \tag{3.51}$$

$$\eta(a,k) = \left[\frac{f_5}{f_1} + \frac{a^2}{k^2}\frac{M_C^2}{1+\alpha_T}\right]\left[1 + \frac{a^2}{k^2}M_C^2\right]^{-1} \tag{3.52}$$

$$\Sigma(a,k) = \frac{m_0^2}{2M_*^2}\left[1 + \frac{f_5}{f_1} + \frac{a^2}{k^2}M_C^2\left(1 + \frac{1}{1+\alpha_T}\right)\right]\left[\frac{f_3}{2M_*^2 f_1} + \frac{a^2}{k^2}\frac{M_C^2}{1+\alpha_T}\right]^{-1} \tag{3.53}$$

here the quantities f_i are functions of time and their definitions can be found in table A.3; in particular, for the Horndeski gravity, their explicit expression are given by (A.4)-(A.6). The new term $M_C^2 \equiv \frac{C_\pi}{f_1}$ is a transition scale associated to the "Compton wavelength" of the scalar field ($\propto 1/M_C$) which is a fundamental quantity in the study of LSS. It is the scale inside which DE begins to cluster (if $\alpha_B \neq 0$) so that the growth of large-scale structures can differ from GR. We consider these functions in two limiting cases:

Super-Compton: $k << aM_C$

$$\mu_0 = \frac{m_0^2}{M_*^2}(1+\alpha_T) \tag{3.54}$$

$$\eta_0 = (1+\alpha_T)^{-1} = \frac{1}{c_T^2} \tag{3.55}$$

$$\Sigma_0 = \frac{m_0^2}{2M_*^2}(2+\alpha_T) \tag{3.56}$$

Sub-Compton: $k >> aM_C$

$$\mu_\infty = \frac{m_0^2}{M_*^2}\frac{(2M_*^2 f_1)}{f_3} = \frac{m_0^2}{M_*^2}\left[1+\alpha_T+\beta_\xi^2\right] \tag{3.57}$$

$$\eta_\infty = \frac{f_5}{f_1} = \left[1+\beta_B\beta_\xi/2\right]\left[1+\alpha_T+\beta_\xi^2\right]^{-1} \tag{3.58}$$

$$\Sigma_\infty = \frac{m_0^2}{2M_*^2}\left[2+\alpha_T+\beta_\xi^2+\beta_B\beta_\xi/2\right] \tag{3.59}$$

In the super-Compton limit it easy to see that any deviation from GR comes from the modification of the tensor sector as well as from the running of the Planck mass. In particular, as we will see in the next section, the value of the effective gravitational coupling in this limit at $a = 1$ must coincide with the value of the Newtonian constant measured in terrestrial experiments, where the fifth force is hidden by screening mechanisms. This fact implies $\mu_0(a = 1) = 1$ and

$\Sigma_0(a=1) = \frac{1}{2}(2+\alpha_T)(1+\alpha_T)^{-1}$, but allows variations in the past. Then if one observes $\mu_0 < 1$[2], in principle, if $\alpha_T \neq 0$, one can have $\Sigma_0 > 1$. However, it would require a fine-tuning of the functions involved in the above expressions, thus it is very likely that $\Sigma_0 < 1$.
In the sub-Compton limit the quantities β_ξ and β_B are defined in the appendix A (respectively (A.9) and (A.10)). They represent the contribution due to the propagation of the fifth force. Since $\beta_\xi^2 > 0$ it follows that, in general, $\mu_\infty > \mu_0$ and the condition $\mu_\infty \neq \mu_0$ or $\Sigma_\infty \neq \Sigma_0$ are signature of a fifth force. Thus, if a scale dependence is detected, so that one can distinguish between the two regimes, then a measurement of $\mu_\infty < \mu_0$ would disqualify Horndeski gravity. In particular the presence of the fifth force term makes possible to have $\mu_0 < 1$ and $\mu_\infty > 1$. As in the case $k << aM_C$, since Σ and μ are controlled by the same functions one should expect that if one measures $\mu_\infty > 1(< 1)$ then it is likely to have $\Sigma_\infty > 1(< 1)$.
These two cases are the most interesting because the observational window it is very likely to fall in one of these two regimes, indeed, for most of the well-known models, it is usually completely inside the Compton wavelength or completely outside it[3]. In particular if a scale dependence is observed, these kind of models would be ruled out[4]. Otherwise, if no scale-dependence is detected, one has to test independently these two regimes.

Screened gravitational coupling

Modified gravity can give rise to modifications of the effective gravitational couplings G_{matter}, G_{light} and thus, can affect both the growth of large-scale structures and weak lensing. In general, in accordance with what we have found in the previous section ($\mu_\infty > \mu_0$), this class of models predicts enhanced growth within the Compton wavelength; this enhancement is due to the fifth force induced by the scalar-matter interaction which is attractive for theoretically consistent Horndeski theories. However, in order to satisfy the constraints imposed by solar system tests of gravity and thus to suppress the propagation of the fifth force on small scale, screening mechanisms [75] are required and should be environmental dependent. Among these mechanisms, we find the chameleon mechanism [68]- [69] and the Vainshtein mechanism [74].
The former is based on the following idea: there is an effective scalar potential which is the sum of two terms, one of which is density-dependent; this potential gives rise to a mass term which depends on the local matter density in such a way that on cosmological scale the mass of the scalar field can be very small ($\sim H_0$) meanwhile, in high-density regions, the scalar field begins very massive and the the fifth force is hidden.
The Vainshtein mechanism is used in many alternative theories of gravity[5] to hide the propagation of the additional degrees of freedom via non-linear effects. In particular, in the context of Horndeski theories, this mechanism, also called "k-mouflage", is performed through the non-linear self-interaction terms[6] of the scalar field that lead to the decoupling of the field from matter within the Vainshtein radius which depends on the surrounding density.
Therefore, if such a screening mechanism is at work, the fifth force is suppressed around local source and the screened gravitational coupling is

$$G_{sc}(a) = \frac{1}{16\pi G_4(\phi(a))} = \frac{1}{8\pi M_*^2(a)} \tag{3.60}$$

[2]This can happens in self-accelerating models, in which the evolution of $M_*^2(a)$ can lead to a value of μ_0 less then one.

[3]In general, for self-accelerating models it is of the order of the Hubble radius ($\lambda_C \sim H$), while in chameleon-type models it is very small ($\lambda_C < 1Mpc$)

[4]We may take as an example self-accelerating models where in general the mass of the scalar field is very small , then it follows that the observations fall in the limit $ka >> M_C$ where the scale dependence disappears.

[5]Fierz-Pauli massive gravity, Dvali–Gabadadze– Porrati (DGP), f($\mathcal{G}$) gravity

[6]A well-known example is the term $X\Box\phi$ appearing in the Lagrangian density of the cubic Galileon theories.

where we have assumed that $c_T = 1$; indeed such an assumption, after the aforementioned GW170817 event, has proved to be a very good assumption.
Moreover, this screened value of the gravitational coupling at $a = 1$ ($\equiv a_0$) and in the limit $k \rightarrow 0$, has to match the current value of the effective gravitational coupling measured in Cavendish-type experiments on Earth[7]

$$G_{eff}(a_0, k=0) = G\mu(a_0, k=0)) = \frac{1}{8\pi M_*^2(a_0)} \tag{3.61}$$

The relation (3.60) suggests the normalization $\mu_0(a_0) = 1$ so that $G_{sc}(a_0, k=0) = G$ and $M_*^2(a_0) = (8\pi G)^{-1}$. Thus one has that (in the limit $k \rightarrow 0$)

$$\mu_0(a) = \frac{M_*^2(a_0)}{M_*^2(a)} \qquad \Longrightarrow \qquad \begin{cases} M_*^2(a) < M_*^2(a_0) & \Longrightarrow \quad \mu_0(a) > 1 \\ M_*^2(a) > M_*^2(a_0) & \Longrightarrow \quad \mu_0(a) < 1 \end{cases} \tag{3.62}$$

So if in the past $M_*^2(a) < M_*^2(a_0)$, we can potentially have $G_{eff} < G$, but it is not a sufficient condition because in the full expression of μ , the effective Planck mass and the positive term related with the propagation of the fifth force (β_ξ^2) can combine in such a way that one can still have $G_{eff} > G$.
Moreover, the screened gravitational coupling must satisfies the constraints imposed by Lunar Laser Ranging (LLR) experiments [56]- [57]:

$$\left|\frac{\dot{G}_{sc}}{G_{sc}}\right| < 1.3 \cdot 10^{-12} yr^{-1} = 0.02 H_0 \quad \Longrightarrow \quad |\alpha_{0M}| < 0.02 \tag{3.63}$$

where $\dot{G}_{sc}(a_0) = -H_0\alpha_M/(8\pi M_*^2(a_0))$. Thus, the bound on the variation of the gravitational coupling implies a bound on the value of the running Planck mass parameter.
Additional bounds on the variation of the gravitational coupling come from observations of the light elements' abundances in the Universe [72], binary pulsars [73] and CMB. In particular, this variation, must be consistent with the standard primordial nucleosynthesis scenario (BBN) and this requires that the gravitational coupling at the BBN time should not differ by more than 10% from the value measured on Earth.

3.2.4 Beyond Horndeski models

As just mentioned previously, a viable theory of gravity must satisfies a set of stability conditions, among which there is the absence of ghost-instabilities, in particular according to the Ostrogradski's theorem, this kind of instability arises in theories with non-degenerate Lagrangian with higher time derivatives. For many years the Horndeski gravity has been considered the most general class of scalar-tensor theories which does not suffer from Ostrogradski's instabilities; recently, an extension of these theories has been introduced in [48]- [49]: they are a new class of scalar-tensor theories, dubbed "beyond Horndeski" models or GPLV (Gleyzes-Langlois-Piazza-Vernizzi) theories; they are the minimum extension of Horndeski gravity. Gleyzes et al. showed, via Hamiltonian analysis, that even if they have higher-order derivative equations of motion, the beyond Horndeski theories are free from Ostrogradski's instabilities, indeed the true propagating degrees of freedom, which are three as like as in Horndeski gravity, obey to second-order equations of motion.
The action of the beyond Horndeski models is that of the Horndeski theories plus two additional terms that modify $\mathcal{L}_4$ and $\mathcal{L}_5$

$$S_{BH}[g_{\mu\nu}, \phi] = \int d^4x\sqrt{-g}\sum_{i=2}^{5}\mathcal{L}_i^{BH} \tag{3.64}$$

[7] we live in a screened environment

and the $\mathcal{L}_i$ are defined as follows

$$\mathcal{L}_2^{BH} = K(\phi, X), \tag{3.65}$$
$$\mathcal{L}_3^{BH} = -G_3(\phi, X)\Box\phi, \tag{3.66}$$
$$\begin{aligned}\mathcal{L}_4^{BH} = &G_4(\phi, X)R + G_{4,X}(\phi, X)[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)]+ \\ &+ F_4(\phi, X)\epsilon^{\mu\nu\rho}{}_\sigma\epsilon^{\mu'\nu'\rho'\sigma}\nabla_\mu\phi\nabla_{\mu'}\phi\nabla_\nu\nabla_{\nu'}\phi\nabla_\rho\nabla_{\rho'}\phi\end{aligned} \tag{3.67}$$
$$\begin{aligned}\mathcal{L}_5^{BH} = &G_5(\phi, X)G_{\mu\nu}(\nabla^\mu\nabla^\nu\phi)- \\ &-\frac{1}{6}G_{5,X}(\phi, X)[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi) + 2(\nabla^\mu\nabla_\rho\phi)(\nabla^\rho\nabla_\sigma\phi)(\nabla^\sigma\nabla_\mu\phi)]+ \\ &+ F_5(\phi, X)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\mu'\nu'\rho'\sigma}\nabla_\mu\phi\nabla_{\mu'}\phi\nabla_\nu\nabla_{\nu'}\phi\nabla_\rho\nabla_{\rho'}\phi\nabla_\sigma\nabla_{\sigma'}\phi\end{aligned} \tag{3.68}$$

here the notation is the same we use in the previous section and $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric four-dimensional Levi-Civita tensor. This Lagrangian corresponds to the Horndeski Lagrangian (3.44)-(3.68) when the free functions $F_4(\phi, X)$ and $F_5(\phi, X)$ are setted to zero; indeed, from the table A.2, one can see that in this case $\alpha_H = 0$. In [48]- [49] it is also shown that one can use disformal transformations to relate the subclasses of these theories that one obtains by setting $\mathcal{L}_4 = 0$ or $\mathcal{L}_5 = 0$ (not at the same time) to theories with manifest second-order equations of motions, i.e. the Horndeski theories.
Here again one has to add the matter fields action under the same assumptions made previously. Then, the complete action is the following

$$S = \int d^4x\sqrt{-g}\Big[\sum_{i=2}^{5}\mathcal{L}_i^{BH} + \mathcal{L}_m(g_{\mu\nu}, \psi_i)\Big] \tag{3.69}$$

The standard EFT action (3.2) for beyond Horndeski theories corresponds to [44]

$$\begin{aligned}S = \int d^4x\sqrt{-g}\Big[&\frac{m_0^2}{2}\Omega(t)R + \Lambda(t) - c(t)\delta g^{00} + \frac{M_2^4(t)}{2}(\delta g^{00})^2 \\ &-\frac{\bar{M}_1^3(t)}{2}\delta g^{00}\delta K + \frac{\hat{M}^2(t)}{2}\delta g^{00}\delta R^{(3)} - \frac{\bar{M}_2^2}{2}\left(\delta K^2 - \delta K^i_j\delta K^j_i\right)\Big] + S_m[g_{\mu\nu}, \psi_i]\end{aligned} \tag{3.70}$$

thus we have $m_2^2 = 0$ and $\bar{M}_2^2 = -\bar{M}_3^2$.
While the correspondence between the free functions appearing in (3.64) and the operators appearing in the ADM action (3.31)-(3.34) can be found in appendix A.0.2.

Disformal transformations

A disformal transformation is a field redefinition of the metric tensor composed by a conformal transformation plus a transformation that modifies the lightcone structure

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} = \Omega^2(\phi, X)g_{\mu\nu} + \Gamma(\phi, X)\partial_\mu\phi\partial_\nu\phi \tag{3.71}$$

where Ω^2 and Γ are respectively the conformal and disformal factors; the WEP is preserved under disformal transformations if one assumes that Ω and Γ are the same for all matter species. In particular, the theories whose action is invariant under this kind of transformation can be subdivided in three classes:

a) *Horndeski theories* where the conformal and the disformal factors are independent by X: $\Omega(\phi)$, $\Gamma(\phi)$;

b) *Beyond Horndeski theories* where the disformal factor depends also by X: $\Omega(\phi)$, $\Gamma(\phi, X)$;

c) *Degenerate Higher Order Scalar-Tensor (DHOST)* which are the most general theories invariant under disformal transformations: $\Omega(\phi, X)$, $\Gamma(\phi, X)$.

When the dependence of the disformal factor by X is allowed, then the theory contains higher order derivatives; this is the case of b) and c).
If one chooses the scalar field as the time coordinate, $\partial_\mu \phi = \delta^0_\mu$, in the ADM formalism the components of the metric tensor transform as follows

$$\bar{N}^i = N^i, \qquad \bar{h}_{ij} = \Omega^2(\phi, X) h_{ij}, \qquad \bar{N}^2 = \Omega^2(\phi, X) N^2 - \Gamma(\phi, N) \tag{3.72}$$

while the determinant of the metric transforms accordingly

$$\sqrt{-\bar{g}} = \Omega^3 \sqrt{-g} \sqrt{\Omega^2 - \Gamma/N^2} \tag{3.73}$$

In [48], the authors showed that using this class of transformations, in particular the class a), one can map the subclasses of beyond Horndeski theories involving respectively $\mathcal{L}_4$ (with $\mathcal{L}_5 = 0$) and $\mathcal{L}_5$ (with $\mathcal{L}_4 = 0$) to a theory that has second-order equations of motions, i.e a theory that belongs to Horndeski models. In particular, in the former case, starting from a theory in which the coefficients $\bar{A}_4(\phi \bar{X})$ and $\bar{B}_4(\phi \bar{X})$ satisfy the first condition of (3.35), one obtains a theory in which the the new coefficients $A_4(\phi X)$, $B_4(\phi X)$ and Γ are related by

$$\Gamma_{4,X} = \frac{A_4 + B_4 - 2XB_{4,X}}{X^2 A_4} \tag{3.74}$$

in this way by choosing a Γ which is independent by X one obtains an Horndeski theory.
Analogously, in the second case, if the second condition of (3.35) is satisfied, the relation among $A_5(\phi X)$, $B_5(\phi X)$ and Γ is the following

$$\Gamma_{5,X} = \frac{3A_5 + XB_{5,X}}{3X^2 A_5} \tag{3.75}$$

so that, a Γ which is independent by X, gives a theory in which the new coefficients satisfy (3.35) (with $A_4 = B_4 = 0$).
In the same paper is pointed out that one cannot map an arbitrary theory with both $\mathcal{L}_4 \neq 0$ and $\mathcal{L}_5 \neq 0$ to an Horndeski theory because it is not possible to satisfy (3.74) and (3.75) at the same time.

Coupling to matter

In the previous section we said that the Lagrangian of beyond Horndeski models is preserved under disformal transformations whose disformal coefficient depends by the scalar field and its gradient; this last dependence leads to the so called Kinetic Matter Mixing (KMM). In [52] the authors define the KMM in a frame-independent way. They assume the existence of a frame in which the matter is minimally-coupled to gravity, i.e the Jordan frame, so that the matter action is, as usual,

$$S_m = \int d^4x \sqrt{-\bar{g}} \mathcal{L}_m(\bar{g}_{\mu\nu}, \psi_i) \tag{3.76}$$

where $\bar{g}_{\mu\nu}$ is the Jordan-frame metric which is disformally related to the metric $g_{\mu\nu}$ through (3.71). In the frame of $g_{\mu\nu}$, the KMM manifest itself with the appearing of a kinetic coupling between the matter fields and the new scalar degree of freedom which is parametrized by introducing the additional function of time

$$\alpha_{X,m} = \frac{X^2}{\Omega_m} \frac{\partial \Gamma_m}{\partial X} \tag{3.77}$$

which is valued on the background. In the Jordan frame $\alpha_{X,m} = 0$ and the information about the kinetic mixing are transferred to the beyond Horndeski function α_H, which now encodes the KMM. Thus, since it is a physical effect, the authors in order to characterize the kinetic matter mixing in a way which is independent from the reference frame, define a new parameter $\lambda^2 \propto (\alpha_H - \alpha_{X,m})^2$. It measures the degree of this mixing between matter and the scalar filed and, as it is shown in the reference, it does not depend on the frame. Furthermore, they show that if $\lambda \neq 0$, the KMM is present and the dispersion relation of the scalar modes is modified as follow

$$(\omega^2 - c_S^2 k^2)(\omega^2 - c_m^2 k) = \lambda^2 c_S^2 \omega^2 k^2 \tag{3.78}$$

thus, the propagation modes are mixed states of matter and scalar fields; this leads to several observational effects. In particular, in [52], the authors analyse the observational signatures of KMM finding that it affects both the matter power spectrum and the CMB.

Phenomenological functions in beyond Horndeski models (QSA)

As like as Horndeski gravity, assuming that the oscillating mode of the scalar filed fluctuation can be neglecting, we can rewrite the expressions (3.22)-(3.23)-(3.24) for beyond Horndeski models ($2\hat{M}^2 \neq \bar{M}_2^2 = -\bar{M}_3^2$ and $m_2^2 = 0$)

$$\mu(a,k) = \frac{m_0^2}{M_*^2}\left[1 + \frac{\Delta f_1}{f_1^H} + \frac{a^2}{k^2}M_C^2\right]\left[\frac{f_3^{BH}}{2M_*^2 f_1^H} + \frac{a^2}{k^2}M_C^2\frac{(1+\alpha_H)^2}{1+\alpha_T}\right]^{-1} \tag{3.79}$$

$$\eta(a,k) = \left[\frac{f_5^{BH}}{f_1^H} + \frac{a^2}{k^2}M_C^2\frac{1+\alpha_H}{1+\alpha_T}\right]\left[1 + \frac{\Delta f_1}{f_1^H} + \frac{a^2}{k^2}M_C^2\right]^{-1} \tag{3.80}$$

$$\Sigma(a,k) = \frac{m_0^2}{2M_*^2}\left[1 + \frac{\Delta f_1}{f_1^H} + \frac{f_5^{BH}}{f_1^H} + \frac{a^2}{k^2}M_C^2\left(1 + \frac{1+\alpha_H}{1+\alpha_T}\right)\right]\left[\frac{f_3^{BH}}{2M_*^2 f_1^H} + \frac{a^2}{k^2}M_C^2\frac{(1+\alpha_H)^2}{1+\alpha_T}\right]^{-1} \tag{3.81}$$

here the quantities f_i are defined in table A.3; $M_C^2 \equiv C_\pi / f_1^H$ where f_1^H is defined in (A.4). We write the functions f_i, for beyond Horndeski models as a sum of an Horndeski term (f_i^H) plus an additional term (Δf_i) arising from the contribution of the α_H: $f_i^{BH} \equiv f_i^H + \Delta f_i$. The explicit expressions of these functions can be found in A.0.2.
As for the Horndeski case, we consider these functions in the sub-Compton and super-Compton regimes:

Super-Compton: $k << aM_C$

$$\mu_0 = \frac{m_0^2}{M_*^2}\frac{1+\alpha_T}{(1+\alpha_H)^2} \tag{3.82}$$

$$\eta_0 = \frac{1+\alpha_H}{1+\alpha_T} \tag{3.83}$$

$$\Sigma_0 = \frac{m_0^2}{2M_*^2}\frac{(2+\alpha_H+\alpha_T)}{(1+\alpha_H)^2} \tag{3.84}$$

From the above expressions, one can see that in this case one can have gravitational slip even if the tensor speed excess is vanishing because of the presence of the property function α_H; moreover, also in Σ_0 and μ_0 there is more freedom than in the Horndeski's case and thus, one can have potentially $(\Sigma - 1)(\mu - 1) < 0$.

Sub-Compton: $k >> aM_C$

$$\mu_\infty = \frac{m_0^2}{M_*^2}\frac{2M_*^2 f_1^{BH}}{f_3^{BH}} = \frac{m_0^2}{M_*^2}\left[\frac{\alpha(c_S^H)^2(1+\alpha_T+\beta_\xi^2)+2\beta_{1H}}{\alpha(c_S^H)^2+\beta_{3H}}\right] \tag{3.85}$$

$$\eta_\infty = \frac{f_5^{BH}}{f_1^{BH}} = \frac{\alpha(c_S^H)^2\left(1+\beta_B\beta_\xi/2\right)+2\beta_{5H}}{\alpha(c_S^H)^2(1+\alpha_T+\beta_\xi^2)+2\beta_{1H}} \tag{3.86}$$

$$\Sigma_\infty = \frac{m_0^2}{2M_*^2}\left[\frac{\alpha(c_S^H)^2\left(2+\alpha_T+\beta_\xi^2+\beta_B\beta_\xi/2\right)+2(\beta_{1H}+\beta_{5H})}{\alpha(c_S^H)^2+\beta_{3H}}\right] \tag{3.87}$$

where the quantities β_{1H}, β_{3H} and β_{5H} are defined in A.0.2 ((A.34)-(A.37)).
The sub-Compton limit is much more complicated than the previous one because we have several additional terms, so it is not simple to analyse without any assumptions that simplify the expressions. The novelty here is that, thanks to the new terms β_{1H}, β_{3H} and β_{5H}, which can be negatives, in beyond Horndeski models the fifth force can be repulsive leading to a weakening gravity and a suppression of clustering, contrary to what happens in Horndeski gravity where it is always attractive; this effect is due to the presence of the kinetic matter mixing (KMM).

3.2.5 GWs propagation in scalar-tensor theories

Let us consider the perturbed line element for tensor modes in a spatially-flat FRW universe:

$$ds^2 = -dt^2 + a^2(t)(\delta_{ij}+h_{ij})dx^i dx^j \tag{3.88}$$

where h_{ij} is a symmetric, traceless and divergence-free tensor.
Then, the quadratic action for tensor perturbations can be written as

$$S_T^{(2)} = \int d^3x dt a^3 \frac{M_*^2}{8}\left[\dot{h}_{ij}^2 - \frac{c_T^2}{a^2}(\partial_k h_{ij})^2\right] + S_{m,T}^{(2)}[g_{\mu\nu},\psi_i] \tag{3.89}$$

By varying this action with respect to h_{ij} and using (3.37)-(3.42) one finds the evolution equation for tensor modes which, in the Fourier space, is

$$\ddot{h}_{ij} + (3+\alpha_M)H\dot{h}_{ij} + (1+\alpha_T)\frac{k^2}{a^2}h_{ij} = \frac{2}{M_*^2}\left(T_{ij} - \frac{\delta_{ij}}{3}T_k^k\right)^{(TT)} \tag{3.90}$$

here the term in brackets on the r.h.s is the transverse-traceless projection of the anisotropic matter stress tensor that vanishes for perfect fluids.
From this equation one immediately sees that if both α_M and α_T are zero, then the propagation of GWs is not affected by modifications of gravity. On the contrary, a non-zero value of α_M, introduces a modification in the friction term that affects the amplitude of GWs and moreover the same parameter, along with α_T, modifies the frequency of the observed GW. Thus, if at least one of these functions are different from zero, we have a non-standard propagation of tensor modes and since the same property functions (together with α_H) are responsible for $\eta \neq 1$, then, it becomes clear that there is a deep connection between these two effects.
In addition, as we mentioned in section 2.1.2, the tensor speed c_T determines the horizon crossing of the tensor modes and thus, a value of α_T different from zero, shifts the position of the first peak of the B-mode power spectrum. Furthermore, the theoretical BB-spectrum predicts another peak at $\ell \sim 5$ and one finds that also the position of this second peak depends on the tensor speed and thus it would be shifted if $c_T \neq 1$.
In [33]- [32] the authors pointed out that the existence of the gravitational slip is controlled by

the property functions α_M, α_T and α_H and that the first two are the same property functions responsable for the non-standard propagations of GWs. Thus, in many modified gravity theories, the presence of gravitational slip is related with the modifications of the tensor modes propagation. In Horndeski models, in principle, one can construct a theory in which there is no gravitational slip even if one of α_M, α_T are different from zero. However, this requires a very tuned choice of the α-functions, i.e. $\alpha_T = 0$ and $\alpha_M = -\alpha_B/2$, thus in general, in Horndeski gravity, the modification of the propagation of the GWs waves is associated to the presence of the gravitational slip. In beyond Horndeski models the situation is different and the presence of gravitational slip does not imply the non-standard propagation of the tensor modes, indeed one can have a theory with $\alpha_M = 0 = \alpha_T$ and thus a standard propagation of GWs but a non vanishing gravitational slip arising from the contribution of $\alpha_H \neq 0$.

Chapter 4

The day after GW170817: implications for scalar-tensor theories

On the 17th August of 2017 the LIGO-VIRGO collaborations detected the first binary neutron-star merger, the event GW170817; exactly $(1.74 \pm 0.05)s$ later, the Fermi and the International Gamma-Ray Astrophysics Laboratory detected its electromagnetic counterpart, the short gamma-ray burst GRB170817A [25]- [24].
This event puts a very strong constraint on the propagation speed of the tensor modes (GWs):

$$-3 \cdot 10^{-15} < c_T - 1 < 7 \cdot 10^{-16}, \qquad z \leq 0.009 \tag{4.1}$$

Before this event there were bounds on c_T coming from the non-observation of gravitational Cherenkov radiations and the variation of the orbital period of binaty pulsar. In particular, the former imposes a lower bound on c_T: for the cosmic rays which have a galactic origin this bound gives $1 - c_T < 2 \cdot 10^{-15}$, if the cosmic rays have an extragalactic origin the bound is of order $1 - c_T < 2 \cdot 10^{-19}$ [28].
These constraints in terms of the tensor speed excess translate in: $|\alpha_T| \leq 10^{-15}$.
As a result, many alternative theories of gravity have been ruled out. In the next sections we consider the consequences for two classes of theories: Horndeski and beyond Horndeski models.
However, one must take into account the fact that, in principle, one can have a $c_T \neq 1$ at hight redshift and, as pointed out in [24], its value may depend on the frequency at which it is measured, $c_T = c_T(k)$. In particular the event GW170817 has been detected at energy scale close to the cut-off scale ($\Lambda_3 = (M_{pl}^2 H_0^2)^{1/3}$) at which the EFT of dark energy breaks down. They also underline that an EFT may have a tensor speed different from the speed of light, but one must have $c_T = 1$ at high energy, specifically they show that it can happen for Horndeski theories.

4.1 Horndeski gravity after GW170817

In particular, focusing on Horndeski gravity, if we impose $c_T = 1$ then $\alpha_T = 0$ and this corresponds to the following constraint

$$\begin{aligned} M_*^2 \alpha_T = -\bar{M}_2^2 = \bar{M}_3^2 = \hat{M}^2 = 2X[2G_{4,X} - 2G_{5,\phi} - (\ddot{\phi} - \dot{\phi}H)G_{5,X}] = 0 \\ \Longrightarrow 2G_{4,X} - 2G_{5,\phi} - (\ddot{\phi} - \dot{\phi}H)G_{5,X} = 0 \end{aligned} \tag{4.2}$$

this relation must hold for any value of ϕ, its derivative and H, so if we exclude a cancellation among $G_{4,X}$, $G_{5,\phi}$, $G_{5,X}$ that requires a tuning among the functions appearing in the above relation [29], the condition (4.2) yields

$$G_{4,X} = G_{5,X} = G_{5,\phi} = 0 \quad \Longrightarrow \quad G_4(\phi), \quad G_5 = \text{const} \tag{4.3}$$

Then, thanks to the Bianchi identity the term $G_{\mu\nu}(\nabla^\mu\nabla^\nu\phi)$ in (3.68) vanishes and the Lagrangian density for Horndeski models is restricted to be of the following form

$$\mathcal{L}_H = K(\phi, X) - G_3(\phi, X)\Box\phi + G_4(\phi)R \tag{4.4}$$

This Lagrangian represents the subclass of the Horndeski models survived the event GW170817. In particular, it includes quintessence and k-essence models, BD theory and metric f(R) gravity. In these last models the evolution of the scalar field on cosmological scale is negligible and screening mechanisms are required in order to satisfy solar system constraints , thus, these theories, do not self-accelerate cosmological expansion. An other class of models which is consistent with the bound (4.1) includes kinetic braidings and their extensions; these theories, in general, exhibit self-acceleration.

4.1.1 Classification of viable Horndeski models in terms of μ, η and Σ

In this subsection we consider the analytic expressions for the phenomenological functions (3.79)-(3.81) setting $c_T = 1$ and considering each subclass of surviving Horndeski theories. The general expressions, after imposing the bound (4.1), become

$$\mu(a,k) = \frac{m_0^2}{M_*^2}\left[1 + \frac{a^2}{k^2}M_C^2\right]\left[\frac{1}{1+\beta_\xi^2} + \frac{a^2}{k^2}M_C^2\right]^{-1} = \frac{m_0^2}{M_*^2}\left[1 + \frac{\Delta_1(a,k)}{1-\Delta_1(a,k)}\right] \tag{4.5}$$

$$\eta(a,k) = \left[\frac{1+\beta_B\beta_\xi/2}{1+\beta_\xi^2} + \frac{a^2}{k^2}M_C^2\right]\left[1 + \frac{a^2}{k^2}M_C^2\right]^{-1} = 1 + \Delta_2(a,k) \tag{4.6}$$

$$\Sigma(a,k) = \frac{m_0^2}{2M_*^2}\left[\frac{2+\beta_\xi^2+\beta_B\beta_\xi/2}{1+\beta_\xi^2} + \frac{a^2}{k^2}2M_C^2\right]\left[\frac{1}{1+\beta_\xi^2} + \frac{a^2}{k^2}M_C^2\right]^{-1} = \frac{m_0^2}{M_*^2}\left[1 + \Delta_3(a,k)\right] \tag{4.7}$$

with the parameters appearing in these expressions defined in (A.8)-(A.12) but now $\beta_\xi^2 = \frac{1}{2\alpha c_s^2}[\alpha_B + 2\alpha_M]^2$.
In particular, assuming that the stability conditions (3.39)-(3.42) hold, from the expression of Δ_2, Δ_1 and Δ_3[1] one can see that

- $\Delta_2 \neq 0$ if

$$\beta_B\beta_\xi/2 - \beta_\xi^2 = -\frac{2}{\alpha c_S^2}\alpha_M(2\alpha_M + \alpha_B) \neq 0 \quad \Longrightarrow \quad \eta \neq 1 \tag{4.8}$$

  This means that to have gravitational slip one must have $\alpha_M \neq 0$ and it must be $\alpha_M \neq -\alpha_B/2$.
  In terms of Horndeski free functions this implies one needs $dM_*^2/dt = dG_4/dt \neq 0 \Longrightarrow G_4 = G_4(\phi)$, i.e a conformal coupling to gravity.

- $\Delta_1 \neq 0$ if

$$\beta_\xi^2 = \frac{1}{2\alpha c_s^2}(\alpha_B + 2\alpha_M)^2 \neq 0 \quad \Longrightarrow \quad \mu > \frac{m_0^2}{M_*^2} \tag{4.9}$$

  Therefore, except for the cases $\alpha_M = 0 = \alpha_B$ and $\alpha_B = -2\alpha_M$, there is an enhancement of the gravitational interaction with matter.

[1] appendix A, (A.12)-(A.14)

- $\Delta_3 \neq 0$ if

$$\beta_B\beta_\xi/2+\beta_\xi^2=\frac{1}{\alpha c_S^2}\alpha_M(2\alpha_M+\alpha_B)(\alpha_M+\alpha_B)\neq 0 \quad\Longrightarrow\quad \Sigma\neq\frac{m_0^2}{M_*^2} \tag{4.10}$$

  Thus one finds $\Sigma = m_0^2/M_*^2$ for $\alpha_M = 0 = \alpha_B$, $\alpha_B = -2\alpha_M$ and $\alpha_B = -\alpha_M$, then, according to the sign of the product $(2\alpha_M + \alpha_B)(\alpha_M + \alpha_B)$, one has that $\Sigma \gtrless m_0^2/M_*^2$.

The two limiting cases, setting $\alpha_T = 0$, yield

Super-Compton: $k << aM_C$

$$\mu_0=\frac{m_0^2}{M_*^2} \tag{4.11}$$

$$\eta_0=1 \tag{4.12}$$

$$\Sigma_0=\frac{m_0^2}{M_*^2} \tag{4.13}$$

Sub-Compton: $k >> aM_C$

$$\mu_\infty=\frac{m_0^2}{M_*^2}\left[1+\beta_\xi^2\right] \tag{4.14}$$

$$\eta_\infty=\left[1+\beta_B\beta_\xi/2\right]\left[1+\beta_\xi^2\right]^{-1} \tag{4.15}$$

$$\Sigma_\infty=\frac{m_0^2}{2M_*^2}\left[2+\beta_\xi^2+\beta_B\beta_\xi/2\right] \tag{4.16}$$

One can see that if a k-dependence is detected, then, a measurement of $\Sigma_0 \neq \mu_0$ or $\eta_0 \neq 1$, would rule out the Horndeski theories; these stringent constraints derive from the fact that we have imposed $\alpha_T = 0$[2].
Furthermore a measurement of $\mu_\infty - \Sigma_\infty = \alpha_M(\alpha_B + 2\alpha_M)m_0^2/(2M_*^2) \neq 0$ would implies $\alpha_M \neq 0$ and $\alpha_B \neq -2\alpha_M$ that in turn, as we have seen, would mean a non-standard relation between the scalar potentials, i.e. $\eta_\infty \neq 1$.

Quintessence and K-essence: $\alpha_B = 0 = \alpha_M$

This subclass corresponds to $K = K(\phi, X)$, $G_3 = 0$, $G_4 = m_0^2/2$, thus the action is

$$\mathcal{S}=\int d^4x\sqrt{-g}\left[\frac{m_0^2}{2}R+K(\phi,X)\right]+\mathcal{S}_m[g_{\mu\nu},\psi_i] \tag{4.17}$$

In particular, the quintessence models, are characterized by a scalar field with canonical kinetic term and a slowly varying potential ($K(\phi, X) = X - V(\phi)$); while in K-essence models the scalar field has non-canonical kinetic term. K-essence models include: ghost condensate models, tachyon field and Dirac-Born-Infeld (DBI) theories. In this class of theories the DE equation of state is time-dependent and the phenomenological functions are trivial

$$\mu=1,\qquad \eta=1,\qquad \Sigma=1, \tag{4.18}$$

This result holds at any scale and any time, therefore the equations (1.13)-(1.16) are not modified and neither gravitational slip nor modification of G_{matter} G_{light} are expected for these theories. Thus, any deviation of these functions from the unity, as well as any detection of a scale-dependence or a time-dependence of the same, would rule out these models.

Kinetic braiding models and its extensions: $\alpha_B + \alpha_M = \frac{\dot{\phi}XG_{3,X}}{HG_4}$

In these models the Horndeski free functions are $K = K(\phi, X)$, $G_3 = G_3(\phi, X)$, $G_4 = G_4(\phi)$. This class includes two special cases: 1) $\alpha_B = -2\alpha_M$, and 2) $\alpha_M = 0$, $\alpha_B \neq 0$. In particular in the

[2]These are more stringent constraints with respect to those we found in section (3.2.3) where we did not impose $c_T = 1$.

former $\beta_\xi = 0$, in the latest $\beta_\xi = \beta_B/4$; the result, in both cases, is that $\Sigma = \mu$ and $\eta = 1$.
Kinetic braiding theories correspond to the particular choice

$$G_4 = m_0^2/2 \quad \Longrightarrow \quad \alpha_M = 0, \quad \alpha_B = 2\dot{\phi} X G_{3,X}/(H m_0^2)$$

This class is also a subclass of the covariant Galileons, which in [11] is dubbed H3 and which in particular contains the cubic Galileons[3]. The expressions (4.5)-(4.7) in this case are

$$\mu(a,k) = \Sigma(a,k) = \left[1 + \frac{a^2}{k^2} M_C^2\right]\left[\frac{2\alpha c_S^2}{2\alpha c_S^2 + \alpha_B^2} + \frac{a^2}{k^2} M_C^2\right]^{-1}, \qquad \eta = 1 \tag{4.19}$$

where we have used $\beta_\xi^2 = \frac{1}{2\alpha c_S^2}\alpha_B^2 = \beta_B^2/4$ and $M_*^2 = m_0^2$.
We have that for these models the two scalar potentials are equally enhanced by the cubic derivative coupling G_3 so that, also in this case, there is no gravitational slip.

Super-Compton: $k << aM_C$

$$\mu_0 = \Sigma_0 = 1 \tag{4.20}$$

$$\eta_0 = 1 \tag{4.21}$$

Sub-Compton: $k >> aM_C$

$$\mu_\infty = \Sigma_\infty = 1 + \frac{1}{2\alpha c_S^2}\alpha_B^2 \tag{4.22}$$

$$\eta_\infty = 1 \tag{4.23}$$

Since they are self-accelerating models the scalar field mass is small and $\lambda_C \sim 1/H$, consequently the observational window, in this case, falls in the small scale limit if the QSA holds; this means that if a scale dependence in μ and Σ were to prove to be true it would rule out this class of models. Moreover also an observation of $\Sigma \neq \mu$ or $\eta \neq 1$ would disqualify these theories.

Generalized Brans-Dicke theories: $\alpha_B = -\alpha_M$

Generalized BD theories are obtained by choosing: $K = K(\phi, X)$, $G_3 = 0$, $G_4 = G(\phi)$.
The GBD action, in the Jordan frame, can be cast in the following form[4]

$$\mathcal{S}_{GBD} = \int d^4x\sqrt{-g}\,\frac{m_0^2}{2}\left[F(\phi)R - Z(\phi)g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - 2U(\phi)\right] + \mathcal{S}_m[g_{\mu\nu}, \psi_i] \tag{4.24}$$

the variation of the action with respect to the metric gives the modified Einstein's equations

$$F G_{\mu\nu} - Z(\phi)\nabla_\mu\phi\nabla_\nu\phi + g_{\mu\nu}\left[Z\nabla^\rho\phi\nabla_\rho\phi + 2U\right] - \nabla_\mu\nabla_\nu F + g_{\mu\nu}\Box F = \frac{1}{m_0^2}T^{(m)}_{\mu\nu} \tag{4.25}$$

where we have used the notations of [105] [5].
Then, varying the action with respect to the scalar field, one obtains the Klein-Gordon type equation

$$(2ZF + 3F'^2)\Box\phi + \frac{1}{2}(2ZF + 3F'^2)\nabla^\mu\phi\nabla_\mu\phi = \frac{1}{m_0^2}F' T_k^{k(m)} - 4F'U + 2U'F \tag{4.26}$$

[3] Appendix C.2.

[4] an equivalent form of this action can be found in appendix D.

[5] in particular the choice $Z(\phi) = \frac{1}{2}(1 - 6Q^2)F(\phi)$, corresponds to Brans-Dicke theory with scalar potential if one redefine the scalar field in this way $F(\phi) = \phi' = e^{-2Q\phi}$ with $3 + 2\omega_{BD} = 1/(2Q^2)$; one recovers GR for $Q \longrightarrow 0$ ($\omega_{BD} \longrightarrow \infty$)

where the prime denotes the derivative with respect to the scalar field.
From the above equation we obtain the modified Friedmann equation

$$3FH^2 = \frac{1}{m_0^2}\bar{\rho}_m + 3\dot{F}H + \frac{1}{2}Z\dot{\phi}^2 + U \tag{4.27}$$

Finally, by perturbing at linear order the field equations and by subtracting the background equations, one gets the Poisson equation and the anisotropy constraint, that in QSA are

$$Fk^2\Psi = -4\pi Ga^2\Delta - \frac{1}{2}k^2\delta F \tag{4.28}$$

$$Fk^2(\Phi - \Psi) = k^2\delta F \tag{4.29}$$

It is straightforward to obtain the equation for the Weyl potential

$$k^2(\Phi + \Psi) = -\frac{8\pi Ga^{2\,2}}{F}\rho\Delta \tag{4.30}$$

Therefore, we can write the phenomenological parameter $\Sigma(a) = F(\phi)^{-1}$. This expression shows that Σ is inversely proportional to the background value of the conformal factor and it is k-independent. Then, one can use the relation (3.60) with the normalization $\mu(a = 1, k = 0) = 1$ so that $\Sigma(a = 1) = F(\phi(a = 1))^{-1} = 1$. Furthermore, the variation of the conformal factor is constrained by the presence of screening mechanisms in such a way that one must have $|F(z = 1) - F(z = 0)|/F(z = 0) \lesssim 10^{-6}$ [76][6].
In terms of the α-parametrization the phenomenological functions are

$$\begin{aligned} \mu(a,k) &= \frac{m_0^2}{M_*^2}\left[1 + \frac{a^2}{k^2}M_C^2\right]\left[\frac{2\alpha c_S^2}{2\alpha c_S^2 + \alpha_B^2} + \frac{a^2}{k^2}M_C^2\right]^{-1}, \quad \Sigma(a) = \frac{m_0^2}{M_*^2} \\ \eta(a,k) &= \left[\frac{2\alpha c_S^2 - \alpha_B^2}{2\alpha c_S^2 + \alpha_B^2} + \frac{a^2}{k^2}M_C^2\right]\left[1 + \frac{a^2}{k^2}M_C^2\right]^{-1} \end{aligned} \tag{4.31}$$

where we have used $\beta_\xi^2 = \frac{1}{2\alpha c_S^2}(-\alpha_B)^2 = \beta_B^2/4$.

Super-Compton: $k << aM_C$

$$\mu_0 = \Sigma_0 = \frac{m_0^2}{M_*^2} \tag{4.32}$$

$$\eta_0 = 1 \tag{4.33}$$

Sub-Compton: $k >> aM_C$

$$\mu_\infty = \frac{m_0^2}{M_*^2}\left[1 + \frac{1}{2\alpha c_S^2}\alpha_B^2\right], \quad \Sigma_\infty = \frac{m_0^2}{M_*^2} \tag{4.34}$$

$$\eta_\infty = \frac{2\alpha c_S^2 - \alpha_B^2}{2\alpha c_S^2 + \alpha_B^2} \tag{4.35}$$

This subclass of the survived Horndeski theories predicts gravitational slip. Furthermore, from the above expressions, one can see that the parameter Σ depends only by time and it is related with the effective value of the Planck mass. Additionally, since α and c_S^2 in a consistent theory of gravity are positive quantities, one has $\mu \geq m_0^2/M_*^2$, $\Sigma = m_0^2/M_*^2$ and $\eta \leq 1$. One can conclude that any measurement of these functions which are not consistent with these last conditions would rule out the GBD theories.

[6]For non-universally coupled models this constraints are less stringent and one can obtain value of Σ slightly different from one.

4.2 Beyond Horndeski models after GW170817

We said that the cut-off of the scalar-tensor EFT of dark energy, where the interactions between the scalar field fluctuations and the gravitons become important, lies near the scale at which the GW170817 has been detected. In [55] the authors show that, at this scale, the decay of gravitons becomes very efficient for theories with $\alpha_H \neq 0$, consequently, the observation of GWs would imply the ruling out of this kind of theories. However, they also show that one can avoid the decay of GWs by imposing that the propagation speed of scalar modes is equal to that of the light, i.e. $c_S^2 = 1$, it can be used like a constraint that allows the elimination of one of the α property functions.
If we impose the bound (4.1) and thus $c_T = 1$, i.e. $\alpha_T = 0$ we have

$$M_*^2 \alpha_T = 2X[2G_{4,X} - 2G_{5,\phi} - (\ddot{\phi} - \dot{\phi}H)G_{5,X} + 4X(F_4 - 3F_5 H\dot{\phi})] = 0 \tag{4.36}$$

this relation must hold for any value of ϕ, its derivative and H, thus one has to require that

$$G_{5,X} = 0, \qquad F_5 = 0, \qquad G_{4,X} - G_{5,\phi} + 2XF_4 = 0 \tag{4.37}$$

in [26] they show that the Lagrangian that satisfies these constraints is of the following form

$$\begin{aligned} \mathcal{L} = &K(\phi, X) - G_3(\phi, X) - B_4(\phi, X)R + \\ &+ \frac{2}{X} B_{4,X}(\phi, X)[\nabla^\mu \phi \nabla^\nu \phi \nabla_\mu \nabla_\nu \phi \Box \phi - \nabla_\mu \phi \nabla_\mu \nabla_\nu \phi \nabla_\lambda \phi \nabla^\lambda \nabla_\nu \phi] \end{aligned} \tag{4.38}$$

where $B_4(\phi, X) = (G_4 - XG_{5,\phi})/2$ and $B_{4,X} = -2XF_4$. In [26] the authors point out that the functions $G_{5,X}$ and F_5 must vanish separately in order to avoid the pathology $M_*^2 = 0$.

4.2.1 Phenomenological functions after GW170817

We can rewrite the phenomenological functions μ, η and Σ setting $\alpha_T = 0$

$$\mu(a,k) = \frac{m_0^2}{M_*^2}\left[\frac{f_1^{BH}}{f_1^H} + \frac{a^2}{k^2}M_C^2\right]\left[\frac{f_3^{BH}}{2M_*^2 f_1^H} + \frac{a^2}{k^2}M_C^2(1+\alpha_H)^2\right]^{-1} = \frac{m_0^2}{M_*^2}[1 + \Delta_1^{BH}(a,k)] \tag{4.39}$$

$$\eta(a,k) = \left[\frac{f_5^{BH}}{f_1^H} + \frac{a^2}{k^2}M_C^2(1+\alpha_H)\right]\left[\frac{f_1^{BH}}{f_1^H} + \frac{a^2}{k^2}M_C^2\right]^{-1} = 1 + \Delta_2^{BH}(a,k) \tag{4.40}$$

$$\begin{aligned} \Sigma(a,k) = &\frac{m_0^2}{2M_*^2}\left[\frac{f_1^{BH}}{f_1^H} + \frac{f_5^{BH}}{f_1^H} + \frac{a^2}{k^2}M_C^2(2+\alpha_H)\right]\left[\frac{f_3^{BH}}{2M_*^2 f_1^H} + \frac{a^2}{k^2}M_C^2(1+\alpha_H)^2\right]^{-1} = \\ &= \frac{m_0^2}{M_*^2}\left[1 + \Delta_3^{BH}\right] \end{aligned} \tag{4.41}$$

The quantities $\Delta_i^{BH}(a,k)$ $(i = 1, 2)$ are defined in A.0.2. From these expressions one can see that, in general, the new property function α_H gives rise to gravitational slip and modifies the gravitational interaction felt by both relativistic and non-relativistic particles. Indeed, unlike the Horndeski case, in which there are same combinations of the property functions α for which the additional contributions Δ_i^H (A.12)-(A.14) to the phenomenological functions cancel out for each scale and each time, here the expressions of Δ_i^{BH} (A.40)-(A.41) involve combinations of the α and background quantities, thus it is extremely unlikely to have a cancellation of these terms. In particular

- $\Delta_2^{BH} \neq 0$ (A.41)
  It is given by $\alpha_H(t)$ plus a contribution that depends on time and scale ; this means that in general, if $\alpha_H \neq 0$, one has gravitational slip, also when $\alpha_M = 0$ or $\alpha_M = -\alpha_B/2$, i.e. $\beta_\xi \beta_B/2 - \beta_\xi^2 = 0$, that correspond to the cases in which, in the Horndeski gravity, the gravitational slip disappears.

- $\Delta_1^{BH} \neq 0$ (A.40)
  It is given by $-(2\alpha_H + \alpha_H^2)/(1+\alpha_H)^2$ plus a contribution that depends on time and scale; thus, also in this case, in general $\Delta_2^{BH} \neq 0$ and the gravitational interaction with matter is modified with respect to GR.

- $\Delta_3^{BH} \neq 0$ (A.42)
  The phenomenological function Σ can be obtained by the first two through $\Sigma = \mu(1+\eta)/2$, thus one finds $\Delta_3 = \Delta_1 + \Delta_2/2 + \Delta_1\Delta_2/2$ and, in general, $\Delta_3^{BH} \neq 0$, because cancellations among these terms are unlikely. This means that also effective gravitational coupling $G_{light} = \Sigma G$ is modified in beyond Horndeski models.

The expressions of the three functions in the super-Compton and sub-Compton regimes are the following:

Super-Compton: $k << aM_C$

$$\mu_0 = \frac{m_0^2}{M_*^2}\frac{1}{(1+\alpha_H)^2} \tag{4.42}$$

$$\eta_0 = 1 + \alpha_H \tag{4.43}$$

$$\Sigma_0 = \frac{m_0^2}{2M_*^2}\frac{(2+\alpha_H)}{(1+\alpha_H)^2} \tag{4.44}$$

In this regime the phenomenological functions depends only by α_H and the effective squared Planck mass. We see that the situation is different with respect to the Horndeski theories; here there is always gravitational slip, i. e. $\eta_0 \neq 1$, because of the presence of the α_H functions. This means that if a scale dependence would be detected, then a measurement of $\eta_0 = 1$ would disqualify this class of theories. Furthermore an $\alpha_H \neq 0$ leads to a different gravitational interaction with matter with respect to GR, in particular there will be an enhancement $\mu_0 > m_0^2/M_*^2$ (or a decreasing $\mu_0 < m_0^2/M_*^2$) of this latter if $\alpha_H < 0$ ($\alpha_H > 0$) as one can see from (4.42). In this limit, also the effective gravitational interaction felt by relativistic particles, is affected by the presence of a non-zero α_H.

Sub-Compton: $k >> aM_C$

$$\mu_\infty = \frac{m_0^2}{M_*^2}\left[\frac{\alpha(c_S^H)^2(1+\beta_\xi^2) + 2\beta_{1H}}{\alpha(c_S^H)^2 + \beta_{3H}}\right] \tag{4.45}$$

$$\eta_\infty = \frac{\alpha(c_S^H)^2\left(1+\beta_B\beta_\xi/2\right) + 2\beta_{5H}}{\alpha(c_S^H)^2(1+\beta_\xi^2) + 2\beta_{1H}} \tag{4.46}$$

$$\Sigma_\infty = \frac{m_0^2}{2M_*^2}\left[\frac{\alpha(c_S^H)^2\left(2+\beta_\xi^2+\beta_B\beta_\xi/2\right) + 2(\beta_{1H}+\beta_{5H})}{\alpha(c_S^H)^2 + \beta_{3H}}\right] \tag{4.47}$$

The expressions of the phenomenological functions in the sub-Compton limit involves not only α_H but also the others property functions and the background functions, therefore, the discussion of this case, is not so simple as the super-Compton limit and, as just commented for the general case, one has modifications with respect to GR; indeed a cancellations among the additional contributions would require a tuning among the αs, H and $\dot{H}$.

Chapter 5

Constraining Horndeski gravity with μ, η and Σ

In this chapter we summarize what we have found in the previous one about Horndeski gravity and, in particular, we analyse the conjecture $(\mu-1)(\Sigma-1) \geq 0$. Finally, we briefly report some of the results obtained in the literature using the Boltzmann codes EFTCMB and EFTCosmoMC; in particular, we consider the following papers: Espejo (2018) [12], Peirone (2017) [13] and the Planck Collaboration 2015 [37].

5.1 Summary

Horndeski gravity, unless there is an highly unlikely fine-tuning among M_*^2, α_T, β_B and β_ξ^2, predicts

$$(\mu - 1)(\Sigma - 1) \geq 0 \tag{5.1}$$

in principle, a different sign in the two factors, is not impossible to obtain in Horndeski gravity, however, for anything which has been said, such a result would disfavour these models.
Moreover if one

- measures $\mu \leq m_0^2/M_*^2$ or $\Sigma \neq m_0^2/M_*^2 \implies$ GBD theories would be ruled out;
- measures $\eta \neq 1 \implies$ theories with $\alpha_M = 0$ or $\alpha_M = -\alpha_B/2$ would be ruled out;
- detects a k-dependence $\implies$ the Covariant Galileons would be ruled out and furthermore
  - i) $\mu_\infty < \mu_0$ would rule out all the Horndeski theories;
  - ii) $\mu_\infty \geq \mu_0$ Horndeski gravity would be not disqualified; in particular $\mu_\infty > \mu_0$ would imply a non vanishing β_ξ^2, i.e. it is evidence of a fifth force.

After GW170817

The bound (4.1) for $z \leq 0.009$ allows us to set $\alpha_T = 0$, this in turn implies $G_4 = G_4(\phi)$ and $G_5 = const$. The survived Horndeski theories can be classified in terms of the α parameters as follows:

A) Quintessence and K-essence: $\alpha_B = 0 = \alpha_M$;

B) Kinetic braiding and its extensions: $\alpha_B + \alpha_M = \frac{\dot{\phi} X G_{3,X}}{H G_4}$. In particular the case $\alpha_M = 0$ corresponds to kinetic braiding;

C) Generalized Brans-Dicke theories: $\alpha_B = -\alpha_M$.

After GW170817, in addition to the foregoing, we found that if one

- measures $\mu \neq \Sigma \implies$ k-essence and quintessence models along with kinetic braidings would be ruled out;
- detects a k-dependence then an observation of

  iii) $\mu_0 \neq \Sigma_0$ and $\eta_0 \neq 1$ would disqualify Horndeski gravity;

  iv) $\mu_\infty - \Sigma_\infty = \frac{m_0^2}{2M_*^2}\alpha_M(\alpha_B + 2\alpha_M) \begin{cases} \neq 0 & \implies \quad \alpha_M \neq 0 \text{ and } \alpha_M \neq -\alpha_B/2 \quad \implies \quad \eta_\infty \neq 1 \\ = 0 & \implies \quad \alpha_M = 0 \text{ o } \alpha_M = -\alpha_B/2 \quad \implies \quad \eta_\infty = 1 \end{cases}$

5.2 Analysing the conjecture $(\mu - 1)(\Sigma - 1) \geq 0$

Following [13], we consider the limit $k << aM_C$ and $k >> aM_C$ and analyse the conditions under which this constraint can be violated. It is clear, from what we have seen in section 4.1.1, that the theories belonging to class A) and the subclass of B) for which $\alpha_M = 0$ or $\alpha_M = -\alpha_B/2$ cannot violate this conjecture in any case since they predict $\Sigma = \mu$.

5.2.1 Super-Compton limit: $k << aM_C$

In this regime the phenomenological functions are given by (3.54)-(3.56), thus, in order to have gravitational slip ($\eta_0 \neq 1$), one has to require $\alpha_T \neq 0$. We know that at low redshifts this is impossible to achieve because of the bound (4.1); however, there are no constraints on c_T at high redshifts. A zero tensor speed excess, as we saw, implies also $\mu_0 = \Sigma_0$ and therefore, in this case, the constraint cannot be violated by Horndeski theories. If one allows for $\alpha_T \neq 0$ in principle there is any chance of breaking the conjecture:

a) $\mu_0 = \frac{1}{\Omega}(1 + \alpha_T)^2 > 1 \qquad \Sigma_0 = \frac{1}{\Omega}(1 + \alpha_T)(1 + \frac{\alpha_T}{2}) < 1$

$$\iff \quad (1 + \alpha_T)(1 + \frac{\alpha_T}{2}) < \Omega < (1 + \alpha_T)^2 \quad \implies \quad \alpha_T > 0 \quad \implies \quad \Omega > 1 \tag{5.2}$$

b) $\mu_0 = \frac{1}{\Omega}(1 + \alpha_T)^2 < 1 \qquad \Sigma_0 = \frac{1}{\Omega}(1 + \alpha_T)(1 + \frac{\alpha_T}{2}) > 1$

$$\iff \quad (1 + \alpha_T)^2 < \Omega < (1 + \alpha_T)(1 + \frac{\alpha_T}{2}) \quad \implies \quad \alpha_T < 0 \quad \implies \quad \Omega < 1 \tag{5.3}$$

where to obtain these inequalities we have used (3.54)-(3.56) and $M_*^2(1 + \alpha_T) = m_0^2\Omega$[1].
From the above expressions it is easy to see that a violation of (5.1), in this regime, would imply $\alpha_T \neq 0$ that, in turn, would constrain the conformal coupling to be $\Omega \neq 1$. But, as we mentioned before, these quantities are subjected to other strong constraints[2] and so the conclusion is that it would be very hard to obtain and also to observe a violation of the conjecture (5.1).

[1] Appendix A, table A.2

[2] In particular we have the following bounds: 1) $|\Omega(z = 0) - 1| < 0.1$ coming from the non-detection of the fifth force on Earth; 2)$|\Omega(z = 1100) - 1| < 0.1$ coming from the necessity to be consistent with BBN and CMB [13]- [76].

5.2.2 Sub-Compton limit: $k >> aM_C$

In this limit the phenomenological functions are given by (3.57)-(3.59). Unlike the previous case, one can have gravitational slip even if $\alpha_T = 0$, thanks to the presence of the fifth force term. Here again, we have two possibilities to violate the condition above:

a) $\mu_\infty = \frac{1}{\Omega}(1+\alpha_T)(1+\alpha_T+\beta_\xi^2) > 1 \qquad \Sigma_\infty = \frac{1}{\Omega}(1+\alpha_T)\left(1+\frac{\alpha_T}{2}+\frac{\beta_\xi^2+\beta_\xi\beta_B/2}{2}\right) < 1$

$$\iff \quad 1+\frac{\alpha_T}{2}+\frac{\beta_\xi^2+\beta_\xi\beta_B/2}{2} < \frac{\Omega}{1+\alpha_T} < 1+\alpha_T+\beta_\xi^2 \tag{5.4}$$

b) $\mu_\infty = \frac{1}{\Omega}(1+\alpha_T)(1+\alpha_T+\beta_\xi^2) < 1 \qquad \Sigma_\infty = \frac{1}{\Omega}(1+\alpha_T)\left(1+\frac{\alpha_T}{2}+\frac{\beta_\xi^2+\beta_\xi\beta_B/2}{2}\right) > 1$

$$\iff \quad 1+\alpha_T+\beta_\xi^2 < \frac{\Omega}{1+\alpha_T} < 1+\frac{\alpha_T}{2}+\frac{\beta_\xi^2+\beta_\xi\beta_B/2}{2} \tag{5.5}$$

Thus, also in this case, the breaking of the conjecture (5.1) would require very specific conditions. We can then consider the case in which $\alpha_T = 0$, so that the above disequalities simplify. In order to satisfy (5.4) and (5.5) one must require respectively [3]

$$\text{a)} \qquad \frac{\beta_\xi\beta_B}{2} < \beta_\xi^2 \quad \Longrightarrow \quad \alpha_M\left(\alpha_M+\frac{\alpha_B}{2}\right) > 0 \quad \Longrightarrow \quad \begin{cases} \alpha_M > 0, & \alpha_M > -\alpha_B/2 \\ \alpha_M < 0, & \alpha_M < -\alpha_B/2 \end{cases} \tag{5.6}$$

$$\text{b)} \qquad \frac{\beta_\xi\beta_B}{2} > \beta_\xi^2 \quad \Longrightarrow \quad \alpha_M\left(\alpha_M+\frac{\alpha_B}{2}\right) < 0 \quad \Longrightarrow \quad \begin{cases} \alpha_M > 0, & \alpha_M < -\alpha_B/2 \\ \alpha_M < 0, & \alpha_M > -\alpha_B/2 \end{cases} \tag{5.7}$$

where we used (A.9)-(A.10)[4].
Thus, one has that the condition b) cannot be achieved in GBD theories in which $\alpha_M = -\alpha_B$. Moreover, since $\alpha_M \propto \dot{\Omega}(a)$ and we have stringent bounds on this quantity as well as on $\Omega(a)$ and $H(a)$, the violation of the condition (5.1) is unlikely. Finally, the case b) is even harder to achieve: the fifth force is attractive and its presence increases the value of G_{eff}, this means that to have $\mu < 1$ one needs $\Omega > 1$, but as we have already said, it has to be close to one. The conclusion is that, in any case, to arrange the values of the different quantities in such a way to violate the conjecture (5.1), a fine-tuning among the parameters of the theory is required.

5.3 Horndeski gravity with EFTCAMB and EFTCosmoMC

5.3.1 Espejo(2018)-Peirone(2017)

In [12] and in [13] the authors perform numerical Monte Carlo simulations with the publicly available Boltzmann codes EFTCMB and EFTCosmoMC[5] in order to verify the consistency of this conjecture (5.1), to check if there is any correlation between the values of μ and Σ and also to test the validity of the QSA. They start to solve for the background solution requiring that it is consistent with the concordance model ΛCDM[6] and that the conformal coupling $\Omega(t)$ satisfies the constraints coming from CMB and BBN. Then, they check the stability conditions

[3]are the conditions (35) and (36) of [13]

[4]these expressions contains the pre factor $1/(\alpha c_S^2)$ that is positive if the stability conditions (3.39)-(3.42) hold.

[5]They are patches of the Einstein-Boltzmann solver CAMB.

[6]They impose theoretical prior in order to constrain the parameters' space in such a way to be consistent with observations; for example they impose a Gaussian prior on H(t).

of the model and, if these conditions hold, they evolve the liner perturbations finding the exact solution of Φ, Ψ and Δ; then, for a given model, they reconstruct the exact phenomenological functions. They make simulations by parametrizing the EFT functions using a Padé expansion and by considering three class of models: the full class of Horndeski gravity (Hor) with the bound (4.1), the class of models for which $c_T = 1$ (H_S) and GBD theories, which have a standard kinetic term.
In particular, in the former review, they sample the (a, k)-plane at the following values: $a \in \{0.25, 0.575, 0.9\}$, $k \in \{0.001, 0.05, 0.1\}$. The ensuing results are:

- The QSA breaks down for modes $k \lesssim 0.001$ h/Mpc even though they are inside the scalar field's sound horizon;

- The conjecture (5.1) is not violate for GBD models; it holds also for the other two class of models, except for about 10% of these theories that violate the conjecture at $k = 0.001h/Mpc$ giving $\mu < 1$ and $\Sigma > 1$. Furthermore, they show that the stability conditions (3.39)-(3.42) work against the breaking of this conjecture disfavouring solutions with $\mu < 1$ and $\Sigma > 1$.

- There are correlations among the background parameters and the phenomenological functions μ and Σ and these correlation should be taken into account when searching for signatures of MG.

In [13] the authors focusing on the reconstruction of w_{DE}, Σ and μ via PCA technique[7]. They derive many statistical properties of these functions among which the mean values and the distribution functions as well as their joint covariances and the functional forms of their correlations functions, independently for the three class of models. The results can be summarized as follows:

- The mean values of Σ and μ do not have significant variations with redshift, in particular for H_S and Hor these values remain very close to 1 ($\sim \sigma$); in GBD the means values are close to one within two standard deviations but tend to values below one because of the pre factor m_0^2/M_*^2. The mean value of w_{DE} is close to one at lower redshifts and tends to zero at higher redshifts[8].

- All classes of models show strong correlation between Σ and μ and it must be taken into account when these functions are constrained by data. The correlation between w_{DE} and Σ/μ decreases as the number of parameters used to characterize a given class of models increases[9]. Finally, the cross correlations among w_{DE}, Σ and μ vary significantly from one class of models to another: it is very strong for GBD, it is barely visible for H_S and negligible for Hor.

- They use the generalized CPZ parametrization and fit it to their numerical results to obtain the functional forms of the correlation functions. They found the same result for the three class of models: the correlation functions for μ and Σ scale with "a" while for w_{DE} scale with "lna".

[7]The Principal Component Analysis is a method that allows the reconstruction of these functions in a non-parametrically way by binning them in redshift and in scale. The advantage of this technique is that it compresses all information about modes coming from observations and uses them to derive constraints on the parameters of a specific model.

[8]This difference is due to the fact that at lower redshifts the mean values are strongly influenced by SN data while those at higher redshifts are determined by the dominant density component because of the tracking behaviour of the effective DE fluid which is induced by the non-minimal coupling of the scalar field.

[9]In particular they use two parameters to specify GBD models (Σ, Λ) and add two parameters for H_S and three for Hor

Parameter	TT+lowP+ +BSH	TT+lowP+ +WL	TT+lowP+ +BAO/RSD	TT+lowP+ +BAO/RSD+WL	TT,TE,EE	TT,TE,EE+ +BSH
Linear model						
α_{M0} (95%CL)	< 0.052	< 0.072	< 0.057	< 0.074	< 0.050	< 0.043
Exponential model						
α_{M0} (95%CL)	< 0.063	< 0.092	< 0.066	< 0.097	< 0.054	< 0.062
β	$0.87^{+0.57}_{-0.27}$	$0.91^{+0.54}_{-0.26}$	$0.88^{+0.56}_{-0.28}$	$0.92^{+0.53}_{-0.25}$	$0.90^{+0.55}_{-0.26}$	$0.92^{+0.53}_{-0.24}$

Table 5.1: In this table are shown the marginalized mean value at 68% confidence intervals for the EFT parameters obtained via MCMC (Monte Carlo Markov chain) methods using the Boltzmann code EFTCAMB.

5.3.2 Planck Collaboration 2015: dark energy and modified gravity

In [37] they use the data coming from the "Planck Collaboration I 2015" in combination with other data sets to analyse the implications for DE and MG models. Also in this analysis, it is used EFTCAMB to fit observational data sets via MC methods in order to constrain the parameters of the EFT of dark energy. In particular, they focusing on Horndeski models ($\alpha_H = 0$) and thus impose the conditions $m_2^2 = 0$ and $2\hat{M}^2 = \bar{M}_2^2 = -\bar{M}_3^2$ by setting these parameters to zero ($\bar{M}_2^2 = -\bar{M}_3^2 = 0$), consequently also α_T is setted to zero. Furthermore, they impose the condition $\alpha_M = -\alpha_B$ (GBD theories) and fix the background to be that of ΛCDM. With these choices the only free parameters of the theory is α_M and it is related with the conformal coupling of the EFT by[10]

$$\alpha_M = \frac{a}{\Omega}\frac{d\Omega}{da} \tag{5.8}$$

Constraints on α_M

The Planck Collaboration constrains the α_M parameter making the following ansatz: $\alpha_M = \alpha_{M0}a^{\beta}$, where α_{M0} is the actual value of the parameter and $\beta \in (0, 3]$ is a parameter that quantifies how fast GR is recovered in the past. The EFTCAMB allows us to choose among different models for the function $\Omega(a)$, Plank Collaboration fits data using the exponential and the linear models with $\Omega_0 = \alpha_{M0}/\beta \in [0, 1]$

$$\Omega(a) = exp\left(\frac{\alpha_{M0}}{\beta}a^{\beta}\right) - 1, \qquad \Omega(a) = \frac{\alpha_{M0}}{\beta}a \tag{5.9}$$

Therefore, for $\alpha_{M0} = 0$, one recovers ΛCDM and for small Ω_0 the exponential model reduces to the linear one. Moreover, as the scaling parameter goes to zero ($\beta \to 0$), then $\alpha_M = const$ and it cannot vanishes in the past. The results are shown in table 5.3.2 and in Fig.5.1. In particular, the viability conditions required by EFTCAMB lead to a sharp cut-off-off for $\beta \sim 1.5$ and strictly reduce the number of viable models; in [37] is pointed out that the filtering conditions implemented in EFTCAMP may be too restrictive and as a consequence may exclude more models than the necessary.

[10] table A.2.

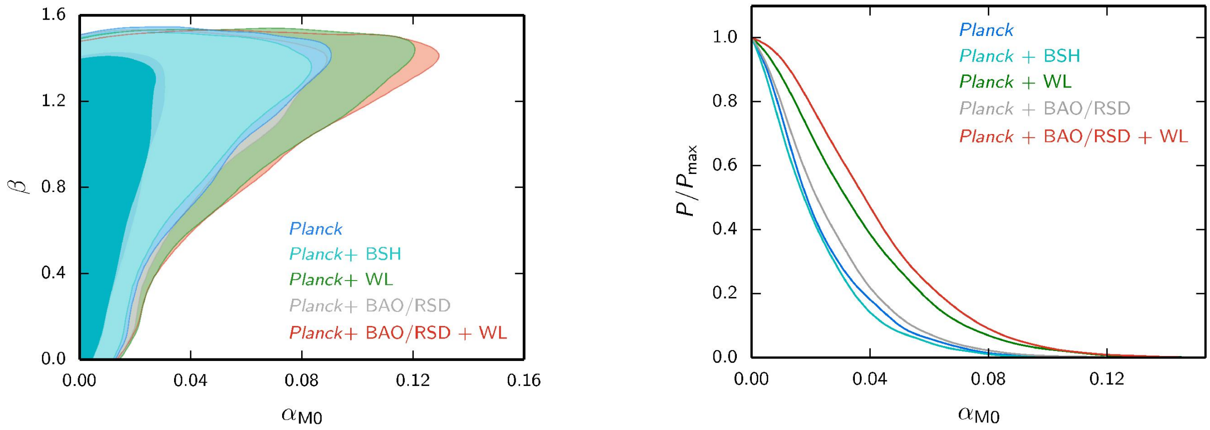

Figure 5.1: Marginalized posterior distributions for the parameters α_{M0} and β of the exponential model (at 68% and 95% C.L., on the left) and for the background parameter Ω of the linear model (on the right); here Planck refers to Planck TT+lowP. In particular, at perturbation level, in both models, for $\Omega_0 = 0$ the ΛCDM model is recovered. In the figure on the left one can see that the inclusion of WL data enlarges the contours of the marginalized posterior distribution; this fact is due to a moderately tension between WL data and the others.

Constraints on η, μ and Σ

In order to characterize deviations from GR, they choose the phenomenological functions η and μ parametrizing them in the following way

$$\mu(a,k) = 1 + f_1(a)\frac{1 + c_1(\lambda H/k)^2}{1 + (\lambda H/k)} \tag{5.10}$$

$$\eta(a,k) = 1 + f_2(a)\frac{1 + c_2(\lambda H/k)^2}{1 + (\lambda H/k)} \tag{5.11}$$

where the f_i are functions of time which quantify the extent of the deviations from GR, while the parameters c_i and λ are constants which give information about the scale dependence.
Then, they implement the model using two different parametrizations of the functions $f_i(a)$ $(i = 1, 2)$

$$\text{a) DE-related parametrization: } f_i(a) = E_{ii}\Omega_{DE}(a) \tag{5.12}$$

$$\text{b) time-related parametrization: } f_i(a) = E_{i1} + E_{i2}(1-a) \tag{5.13}$$

The two parametrizations are complementary, in particular in the former the time evolution is based on the effective dark energy density $\Omega_{DE}(a)$ and the deviations from GR are not allowed at early time; on the contrary, the last parametrization has two parameters: E_{i1} that characterizes deviations at low redshifts and E_{i2} that allows for deviations at high redshifts, i.e. at early time. The authors make clear that they are not using the QSA, but only a minimal parametrization that allow for a scale dependence of the phenomenological functions which is required because of their data cover a wide range of scales. Then, for each parametrization, they consider the scale-dependent case and the scale independent case by setting for this last case $c_1 = 1 = c_2$. The results of their analysis are reported in table 5.2 and can be summarized as follows:

- The scale dependence in any case does not play a significant role, indeed they find that the chi-quadro distributions are approximately the same in both scale dependent and independent case; thus they focusing on this last case, which is the simplest to analyse.

- The E_{ij} values are obtained by constraining data; from them the actual values of μ and η, denoted with the index zero[11], are reconstructed using the above parametrization; the today's value of Σ is obtained using (1.15). The plots of the marginalized distributions in the plane $(\mu_0 - 1, \eta_0 - 1)$ (for both DE-related and time-related parametrizations) and that in the plane $(\mu_0 - 1, \Sigma_0 - 1)$ (for the DE-related case) are shown respectively in Fig.5.3 and Fig.5.2.. In these plots the dashed lines represent the values of these functions predicted by the ΛCDM scenario ($\mu = \eta = \Sigma = 1$) and, in general, the results are compatible with this scenario, even if some tension is visible. In particular, they found that for the case a) the deviations, for each data set, is at least of 2σ (Plank TT+lowP); the tension increases when BAO+RSD are included reaching the maximal value of 3σ when they consider the full set of data (Plank TT+lowP+BAO+RSD+WL). Including also CMB lensing the tension decreases with a maximal value of 1.7σ, obtained when all data sets are combined. In the second parametrization (time-related) the tension is lower going from 1σ for PLanck TT+lowP to a maximum of 2.1σ for the combination Plank TT+lowP+BAO+RSD+WL. The higher tension observed in the DE-relates parametrization is probably due, as the authors comment in the paper, to the fact that the data set Planck TT+lowP , slightly prefers models which have a lower CMB temperature power spectra at low multipoles (ISW effect) and higher CMB lensing potential with respect to ΛCDM[12].

Parameter	Planck TT+ +lowP	Planck TT+ +lowP+BSH	Planck TT+ +lowP+WL	Planck TT+lowP+ +BAO/RSD	Planck TT+lowP +WL+BAO/RSD	Planck TT,TE,EE+ +lowP+BSH
E_{11}	$0.099^{+0.34}_{-0.73}$	$0.06^{+0.32}_{-0.69}$	$-0.20^{+0.19}_{-0.47}$	$-0.24^{+0.19}_{-0.33}$	$-0.30^{+0.18}_{-0.30}$	$0.08^{+0.33}_{-0.69}$
E_{22}	0.99 ± 1.3	1.03 ± 1.3	$1.92^{+1.4}_{-0.96}$	1.77 ± 0.88	2.07 ± 0.85	0.9 ± 1.2
$\mu_0 - 1$	$0.07^{+0.24}_{-0.51}$	$0.04^{+0.22}_{-0.48}$	$-0.14^{+0.13}_{-0.34}$	$-0.17^{+0.14}_{-0.23}$	$-0.21^{+0.12}_{-0.21}$	$0.06^{+0.23}_{-0.48}$
$\eta_0 - 1$	0.70 ± 0.94	0.72 ± 0.90	$1.36^{+1.0}_{-0.69}$	1.23 ± 0.62	1.45 ± 0.60	0.60 ± 0.86
$\Sigma_0 - 1$	0.28 ± 0.15	0.27 ± 0.14	$0.34^{+0.17}_{0.14}$	0.29 ± 0.13	0.31 ± 0.13	0.23 ± 0.13

Table 5.2: Marginalized mean values (68% C.L.) of the phenomenological parameters obtained by Planck Collaboration for the DE-related scale-independent case.

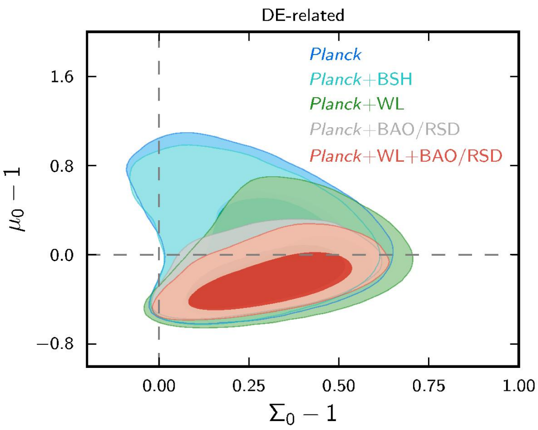

Figure 5.2: Contour plot for marginalized posterior distribution for 68% and 95% C.L. in the planes $(\mu_0 - 1, \Sigma_0 - 1)$ for the DE-related parametrization with no scale-dependence.

[11] In this paper μ_0, η_0 and Σ_0 refers to today's values of these functions, while in the previous sections we use the same notations for the super-Compton limit of the functions.

[12] Their best fit power spectrum Planck TT+lowP in the DE-related parametrization is obtained for the following values of the parameters: $E_{11} = -0.3, E_{22} = 2.2$. These values are close to those of the model with $E_{11} = 0, E_{22} = 1$ in Fig.5.4.

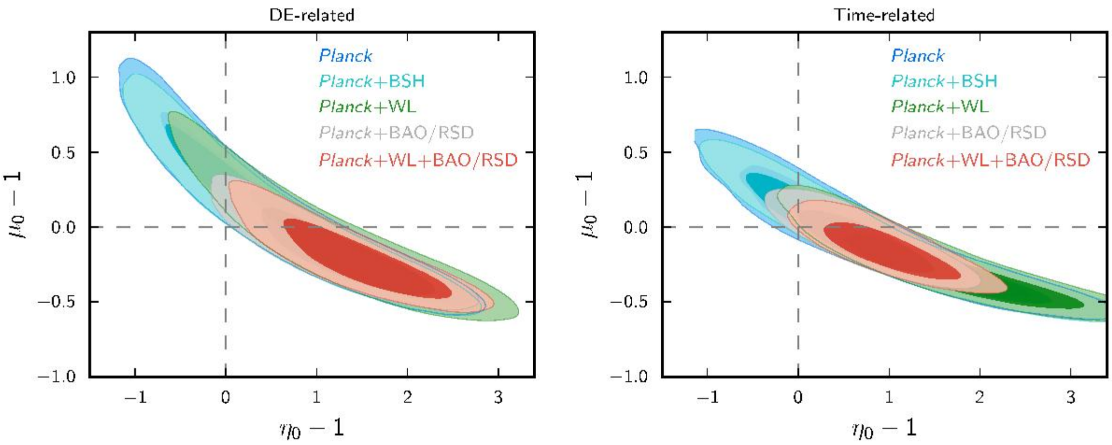

Figure 5.3: Contour plots for marginalized posterior distributions in the planes $(\mu_0 - 1, \eta_0 - 1)$ at 68% and 95% C.L. for the DE-related and time-related parametrizations with no scale-dependence. Here Plancks stands for PlanckTT+TEB. These plots show a similar behaviour for both parametrization.

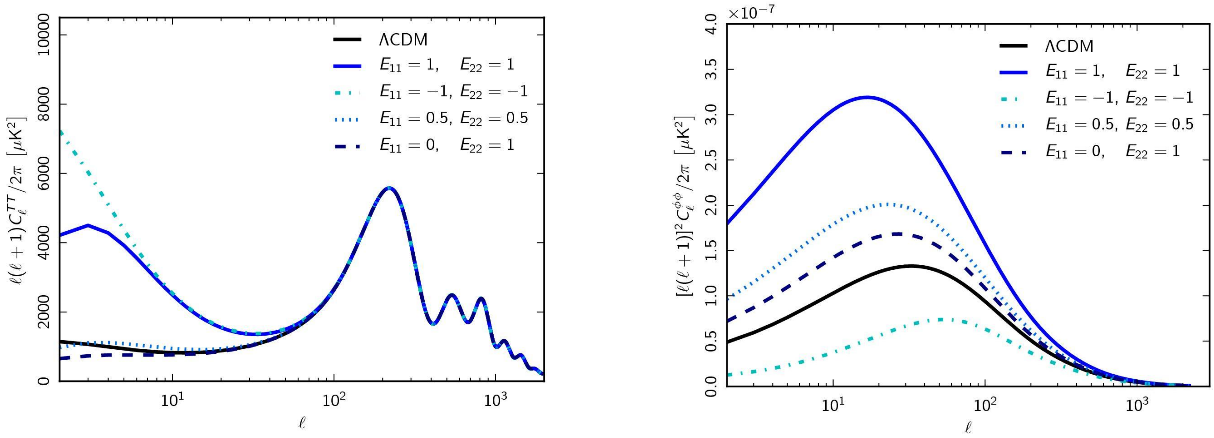

Figure 5.4: In these pictures it is shown how deviations from GR affect the power spectra of CMB temperature (left panel) and that of the lensing potential (right panel). From (5.10)-(5.11) one can see that these deviations, in the parametrization adopted in this paper, are characterized by the parameters E_{ii}; in particular for $E_{22} \neq 0$ gravitational slip appears and in any case the power spectra of CMB temperature is modifies at low multipoles, while the power spectra of the lensing potential increases if $E_{22} > 0$ and decreases for $E_{22} < 0$.

Chapter 6

Degeneracies with modifications to GR

Cosmological observations constrain the background evolution of a viable theory of gravity to be very close to that predicted by the ΛCDM scenario but it is well known that the evolution of cosmological perturbations of such a theory can still differ from it and thus, this theory, can have different observational signatures. However, from an observational point of view, it is not always so simple to distinguish among different theories of gravity and the reasons are the following

- different cosmological parameters can induce the same observational effects or moreover, the set of parameters which characterize a given theory of gravity can combine to give rise to the same observational signatures of another theory of gravity;
- systematic effects can reproduce physical effects on observables introducing bias in the cosmological parameters in such a way that they can be mistaken for modifications of gravity or, they can cancel out observational signatures of modified gravity. Furthermore, they can limit the precision under which one can constrain the MG parameters.

In this chapter we consider some of the possible degeneracies that one has to take into account when testing for departures from GR.

6.1 Cosmic degeneracy between modified gravity and massive neutrinos

The neutrino flavours oscillations is evidence of the fact that neutrinos are massive particles. Since they are the most abundant particles after photons, they play an important role in the history of the universe and in the structure formation and thus, the evidence of massive neutrinos, has important implications in cosmology. In particular, in [80], it is shown that in a cosmological model with massive neutrinos the free-streaming of these particles leads to a modification of the evolution of both matter and metric perturbations at scale below their free-streaming length which results into a scale-dependent suppression of the growth of structures at late time. It is due to the combination of two effects: 1) the matter-radiation equality take place a bit later than in a model with massless neutrinos; 2) the gravitational potentials slowly decays during the matter-dominated era on scales below the free-streaming scale. Thus, from observations of the CMB and LSS, one can obtain information about neutrinos, such as a bound on the sum of the neutrino masses.
However, it is also well known that many theories of modified gravity predict an enhancement of the growth of large-scale structures, due to the presence of an attractive fifth force, that could cancel out the effects of massive neutrinos and thus, from an observational point of view,

such a theory would be indistinguishable from GR with very light neutrinos. In the literature there are many studies on this topic and on the possibility of breaking such a degeneracy, called cosmic degeneracy, and in all these works, they use as representative models of modified gravity, the well known metric f(R) model.

In particular, in some of these works, the cosmic degeneracy is analysed using the DUSTGRAIN-pathfinder simulations[1], which is a suite of cosmological N-body simulations specifically designed to study cosmological models in which gravity is modified and neutrinos are massive particles. The modified gravity model implemented by the DUSTGRAIN project is the f(R) gravity model proposed by Hu and Sawicki in [86], for which it is possible to vary the values of the propagating degree of freedom $f_R(z=0) = df(R)/dR|_{z=0}$ and the mass of neutrinos in a certain range. Among these papers there are

Peel et al. (2018) [81]
In this paper the authors propose to use non-Gaussian information from weak-lensing statistic of order higher than second to break the cosmic degeneracy: the aperture mass skewness (third order), the kurtosis (fourth order) and the peak counts statistic as a function of map filtering scale and source galaxy redshift. They confirm the fact that at linear order and also at non-linear order, second-order statistics is not able to break the cosmic degeneracy; moreover, the main result they obtain is that the only higher-order observable, among those they have considered, which is potentially capable to distinguish efficiently between standard cosmology and modified gravity models with massive neutrinos, is the peak counts and its efficiency depends on redshift and angular scale.

Merten et al. (2018) [83] In this paper the authors explore the possibility to break cosmic degeneracy making use of three types of machine learning techniques. In particular, they use the DUSTGRAIN-pathfinder simulations to generate the mass map which constitute their main data set. Then, they apply to these data set different methods of characterization and classification. In order to characterize the mass maps they use, among others, a Convolutional Neural Network (CNN) to extract the characterising features directly from data. The conclusion is that these techniques are promising but further investigation is needed.

Hagstotz et al. (2019) [85]
It was already pointed out in previous papers the redshift dependence of this degeneracy; in this paper, the author suggest to break the cosmic degeneracy exploiting kinematic information related to either the growth rate on large-scales or the virial velocities inside of collapsed structures. They use the DUSTGRAIN-pathfinder simulations finding that the suppression of the growth due to neutrino's mass dominates in the past, while the enhancement due to the fifth force will dominate in the future evolution. They also study kinematic inside clusters, finding that the velocity dispersion is not affected by neutrinos and that, in general, kinematic information is a very powerful observable in order to detect a potential fifth force.

Finally we mention a recent works in which the authors propose to break this degeneracy with redshift space distortions (RSD).

Wright et al. (2019) [84]
In this analysis the authors use an extended version of the semi-analytical code COPTER which computes large-scale structure observables using perturbation theory, MGCOPTER. This version allows us to model the combined effects of modified gravity and massive neutrinos on real and redshift-space power spectra. The idea behind this study is that

[1] DUSTGRAIN stands for "Dark Universe Simulations to Test GRAvity In the presence of Neutrinos", more details can be found in [82].

the cosmological models that, at a given redshift have similar matter power spectrum, have different growth rate and thus, one can distinguish between them by exploiting the velocities information encoded in the RSD. They find that the information that one can obtain from the quadrupole of the redshift space power spectrum can in principle break the cosmic degeneracy, but further investigation is required. They also study the redshift evolution of the degeneracy finding that, in general, if it is present at a given redshift, it is likely that the same disappears at another redshift.

6.2 Systematic effects in cosmological tests and degeneracies with modifications of gravity

Systematic effects in cosmological tests can be degenerate with the modifications of gravity, therefore in order to distinguish between genuine signatures of modified gravity and to improve the precision in constraining cosmological parameters it is necessary to identify and to control them.

6.2.1 Weak gravitational lensing systematic effects

Weak gravitational lensing, in particular that due to LSS, i.e cosmic shear, represents a very powerful tool in testing gravity; but weak lensing measurements are subject to a number of systematic effects that can reduce the precision that one can reach in cosmological tests and thus, the capacity in distinguishing different cosmological models [87]- [89]. In what follows, we consider the three principal systematic effects that affect this kind of measurements: baryonic effects, photometric redshift uncertainties and galaxy intrinsic alignments.

Baryonic effects

It is well know that the physics of structure formation impacts on weak lensing observables and many studies demonstrate that on small scales the presence of baryonic matter cannot be ignored [88], otherwise one would introduce bias in the estimation of cosmological parameters; in particular, they affect the matter power spectrum. This is because baryons have different physical properties with respect to CDM and one of them is that baryons have non-zero pressure and this fact prevents clustering on small scales. Other effects are radiative cooling, star formation and supernova (SN) feedback [88]- [90]. In the literature there are many studies that investigate these systematic effect using hydrodynamical simulations.

Photometric redshift uncertainties

The spectroscopic distance measures of an object on cosmological scale, i.e the determination of the redshift, requires long time observations and a very sensitive instrumentation and thus, the estimation of the distances of all cosmological objects with high precision, is a very hard task to achieve in this way. For this reason, in order to determine redshift distances, alternative methods have been introduced, among them there are those based on photometric measurements[2]. This technique, also called photo-z's, is based on imaging data[3] which allows inferring

[2]The photometric redshift was first introduced by Baum in 1962, which used multi band photometry to estimate galaxy redshifts; then this technique was adopted by Pushell, Owen and Laing which in their paper used for the first time the term ”photometric redshift”.

[3]This means that the only information used in photometric redshift is the galaxy magnitude and colours through a few broad filters.

redshifts of a large number of objects; however, in photometric redshift techniques there are many factors which can limit the accuracy in redshift determination. In the literature can be found various papers on this topic and in particular, on the uncertainties associated with this kind of measurements, e.g. [91]- [92].

Intrinsic alignments (IA)

The intrinsic alignment of galaxies is one of the most relevant systematic effects in weak lensing measurements and thus, many studies has been made to understand and to mitigate this effect, e.g. [93]- [94]. For intrinsic alignment one means the fact that the shape and orientation of the galaxies are not random but are statistically determined by a number of factors (such as formation environment, history and galaxy type) that produce such an alignment that, when observed, can be mistaken for cosmic shear. It follows the necessity to identify these effects in order to isolate the weak lensing signals.

As an example of how these systematic effects can impact on the precision in constraining MG parameters we mention the paper [95] by Ferté et al. (2017). In this study, the authors use cosmic microwave background anisotropy measurements from the Planck satellite, cosmic shear from CFHTLenS and DES-SV, and redshift-space distortions from BOSS data release 12 and the 6dF galaxy survey to constrain the phenomenological functions μ and Σ. They also investigate the impact of systematics on the estimation of these functions by performing a forecast analysis through Fisher matrix technique[4]. Thus, they compare the constraints on μ and Σ obtained when the systematic effects are neglected, to the case in which they are not neglected, finding that: photometric redshift effects and the shear calibration do not affect the constraints on these functions; on the contrary, the intrinsic alignment, shifts the constraints on Σ to higher values. Therefore, they pay particular attention to this last effect showing that the presence of IA in the data leads to an underestimation of the uncertainties in the phenomenological functions.

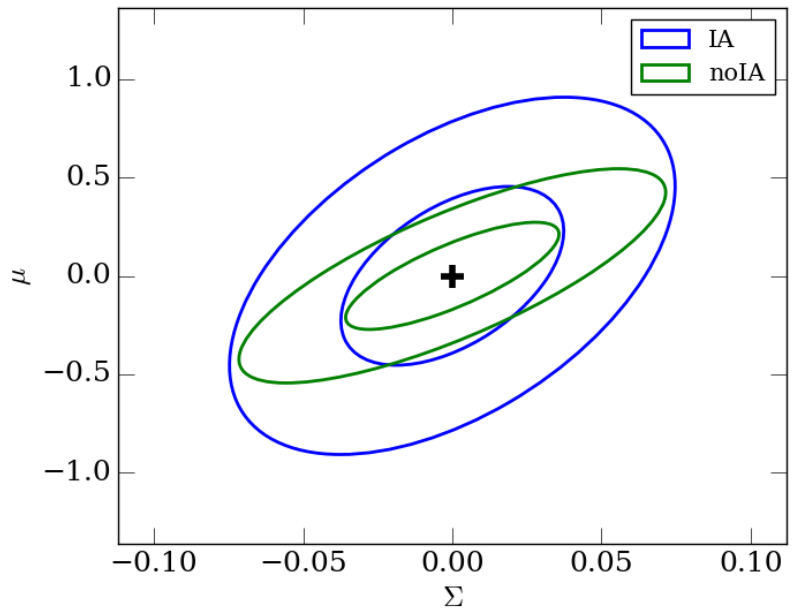

Figure 6.1: [95] This picture shows the forecasted 68% and 95% confidence contour plots on Σ_0 and μ_0 for a future DES-Y5 like survey when IA are considered (blue) and when it is neglected (green).

[4] Appendix D.

6.3 Degeneracies among MG parameters

The degeneracies among cosmological parameters and, in particular among those that characterize deviations from GR, arise because the observables can depend by combinations of them; however, one can try to break such degeneracies by combining different cosmological probes. One way to investigate degeneracies is based on the Fisher information formalism[5]. This approach allows us to forecast the precision under which one would be able to constrain the parameters of a given cosmological model and to check for correlations among these parameters.
In particular the phenomenological functions μ, η and Σ have been shown to be correlated each other and also with background parameters in many theories of MG gravity (e.g. Horndeski models, see section 5.3.1). To this respect, we mention two papers in which the authors study degeneracies in modified gravity theories.

A. Hojjati (2012) [96]
Once the background has been fixed, the author considers three class of models: 1) f(R) theory in which the phenomenological functions can be parametrized in terms of one parameter; 2)scalar-tensor theories which require three parameters; 3)general scalar-tensor theories for which he uses a BZ parametrization with five parameters. He also performs a pixelations of μ and η in order to study degeneracies at a given scale and redshift. Then, using the Fisher formalism, he derives the covariance matrix, $\boldsymbol{C}$, and thus the correlations among parameters C_{ij} (un-normalized correlations) and the normalized correlations given by (D.4). Then, he combines data sets coming from different observations[6] in order to understand the behaviour of the degeneracies. In particular, he finds degeneracies between μ, H and the galaxy bias[7] when he considers data sets from galaxy number counts (GC); while weak lensing (WL) observations measures the Weyl potential and thus may be used to constrain Σ, which is a combinations of μ and η, thus in these observations there is degeneracies among MG parameters and background quantities. In general, he finds that an higher number of parameters leads to less stringent constraints and that by combining WL and GC the degeneracies decrease; the residual degeneracies can be broken using the cross-correlation of WL and GC data at multiple tomographic redshift.

Casas et al. (2017) [101] This work is a forecasting analysis on modified gravity for forthcoming surveys in which authors choose to characterize deviations from GR with η and μ, considering both linear and non-linear scales. They use different time-dependent parametrizations of these functions neglecting the scale dependence and performing a redshift binning. The authors use the Fisher matrix approach for their analysis and in particular, they study the correlations between the two phenomenological functions using a prior covariance matrix derived from the CMB Planck observations obtained by performing a MCMC analysis. They compute the Fisher matrix on a set of 15 independent parameters $\{\Omega_m, \Omega_b, \ln 10^{10} A_s, n_s, \mu_i, \eta_i\}$, where η_i and μ_i are the values of the functions at five bin-intervals corresponding to the following redshift $z = \{0-0.5, 0.5-1.0, 1.0-1.5, 1.5-2.0, 2.0-2.5\}$. They also perform a Zerophase Component Analysis (ZCA) in order to further decorrelate MG parameters and to identify which combinations of the MG parameter amplitudes, in different redshift bins, are best constrained by future surveys. They find that, in general η and μ, for different redshift, are strongly correlated but, these correlations, are reduced or disappear when adding non-linear scales. They also

[5] Appendix D.

[6] CMB temperature and polarization (T and E), SNe observations, weak lensing (WL) shear of distant galaxies, galaxy number counts (GC) and their cross-correlation.

[7] Indeed these kind of observations measures $\delta_g = b\delta_m$ which can be combined with the equations (1.20)-(1.21) in the sub-horizon limit giving a second order equation in δ_g from which it is easy to see the origin of such degeneracy.

show that the higher are correlations the lower is the power in constraining MG parameters. Furthermore, their study shows that there is a strong anti-correlations between the primordial amplitude of the density fluctuation A_s and μ_i: this degeneracy can be understood by considering that a large initial fluctuation amplitude leads to the same physical effects of a larger growth of structures; but also this degeneracy is mitigate by including non-linear scales. Moreover, they quantify the effects of adding non-linear scales on the correlations comparing the FoC (D.5, appendix D) obtained using only linear scales and the one obtained by including non-linear scales: the former gives FoC$\simeq$ 62, the latest FoC$\simeq$ 35. Thus, this analysis, point out that by including non-linear scales one can significantly decrease the degree of degeneracy among MG parameters.

Chapter 7

Future cosmological constraints on GR and MG parameter forecasts

We are in the era of precision cosmology in which very significant steps forward have been made in the understanding of the universe thanks to a wide number of cosmological probes such as the CMB, galaxy clustering, weak lensing, and supernovae. The future cosmological surveys promise to further improve this understanding by providing an enormous amount of data with high precision. Among them we find: the Euclid mission, whose launch is planned for 2022, with the main aim of investigating the nature and the properties of dark energy, dark matter and gravity; DESI (Dark Energy Spectroscopic Instrument) which will start in the present year with the purpose of measuring the effect of dark energy on the expansion of the universe; the eBOSS (Extended Baryon Oscillation Spectroscopic Survey) which has the aims to improve constraints on the nature of dark energy and to measure with high precision most of the expansion history of the Universe.
In this chapter we briefly touch upon some of the recent forecasting analyses which allow an overview of how such future surveys will improve our ability in constraining MG parameters and thus in testing departures from GR.

Ferté et al. (2017) [95]
In this paper the authors use a wide range of observational data to constrain the phenomenological functions Σ_0 and μ_0[1] finding no evidence for modified gravity. Moreover, the authors, using the Fisher formalism, perform a forecasting analysis for the full five-years DES survey and an LSST-like survey showing that these forthcoming experiments will significantly improve the constraints on MG parameters Σ and μ. As we already mentioned in the previous chapter, they also investigate the impact of systematic effects on such constraints. In particular, the LSST survey, is expected to give standard deviations of an order of magnitude smaller than DES-Y5 survey. The projected uncertainties they obtained are the following

$$\text{DES-Y5} \qquad \sigma_{\mu_0} = 0.23, \qquad \sigma_{\Sigma_0} = 0.034 \tag{7.1}$$

$$\text{LSST} \qquad \sigma_{\mu_0} = 0.014, \qquad \sigma_{\Sigma_0} = 0.0027 \tag{7.2}$$

Casas et al. (2017) [101] We already mention this paper in regard of degeneracies among MG parameters. In this chapter we come back to it to give more details on their forecasts about the precision under which future surveys will be able to constrain MG parameters. Their forecasting analysis includes a wide number of forthcoming surveys (Euclid, DESI-ELG, SKA1 and SKA2) and make use of the Fisher information techniques which

[1] They use different definitions with respect those we have adopted in this dissertation: their $\{\Sigma, \mu\}$ = our $\{\Sigma, \mu\}-1$.

is applied to weak lensing and galaxy cluster using as prior the Planck data from CMB. They considers two regimes, the linear one and the midly non-liner one (up to $k \simeq 0.5$ h/Mpc), for which they obtain the power spectrum using respectively a Boltzmann code and two different methods for the latter (Halofit and a a semi-analytic prescription including screening mechanisms). As we said previously, they choose $\eta(a)$ and $\mu(a)$ to specify modifications of gravity and use two different approach for their analysis:

1) They bin the time dependence of the two phenomenological functions neglecting the scale dependence without specifying any parametrized evolution

$$p(z) = p(z_1) + \sum_{i=1}^{N-1} \frac{p(z_{i+1}) - p(z_i)}{2} \left[1 + \tanh\left(s \frac{z - z_{i+1}}{z_{i+1} - z_i} \right) \right] \tag{7.3}$$

where $z_1 = 0.5$ is the redshift value of the first bin, $p = \mu, \eta$, N is the number of binned value and $s = 10$ a smoothing parameter.

2) They use the same two parametrizations adopted by Planck Collaboration 2015 5.3.2 calling "late-time" parametrization and "early-time" parametrization what in [37] are called respectively DE-related (5.12) and time-related (5.13) parametrization.

We summarize their predictions on the accuracy that future survey will reach in constraining μ, η and Σ by reporting their results in the table 7.1.

	early-time parametrization				late-time parametrization			
	μ	η	Σ	MG FoM	μ	η	Σ	MG FoM
Fiducial	0.902	1.939	1.326	relative	1.042	1.719	1.416	relative
GC(nl-HS)								
Euclid	1.8%	7.9%	4.8%	6.6	1.7%	475%	291%	2.9
SKA1-SUR	12.6%	82.7%	52.6%	2.2	18.1%	165%	108%	1.7
SKA2	0.9%	3.4%	1.8%	8.3	0.7%	86.8%	53.2%	5.5
DES-ELG	8.2%	32%	28.6%	4.3	3.3%	899%	552%	1.8
WL(nl-HS)								
Euclid	2.8%	8.0%	3.4%	6.6	23.3%	40.9%	4.6%	4.5
SKA1	13.1%	37.1%	16.4%	3.4	220%	405%	36.8%	0.5
SKA2	2.4%	7.0%	2.9%	6.9	19%	33.2%	3.7%	4.9
GC+WL(lin)								
Euclid	3.0%	6.8%	3.4%	6.4	7.1%	10.6%	2.0%	6.6
SKA1	14.4%	29.6%	15.5%	3.3	26.4%	28.8%	13.6%	3.7
SKA2	2.5%	5.7%	2.7%	6.8	4.1%	5.5%	1.6%	7.5
GC+WL(lin)+Planck								
Euclid	2.4%	6.5%	2.8%	6.8	6.2%	9.8%	1.5%	6.9
SKA1	8.8%	22.2%	8.5%	4.5	12%	19.8%	3.8%	5.3
SKA2	2.1%	5.4%	2.3%	7.2	3.6%	5.2%	1.2%	7.8
GL+WL(nl-HS)								
Euclid	1.3%	4.4%	1.9%	8.1	1.6%	2.4%	1.0%	8.7
SKA1	8.2%	24.4%	10.5%	4.4	12.8%	11.0%	7.3%	5.5
SKA2	0.8%	2.7%	1.3%	8.8	0.7%	0.9%	0.6%	10.3
GL+WL(nl-HS)+Planck								
Euclid	1.3%	4.4%	1.9%	8.1	1.6%	2.4%	0.9%	8.9
SKA1	7.0%	20.8%	8.2%	4.9	3.5%	6.0%	2.7%	6.9
SKA2	0.8%	2.7%	1.3%	8.8	0.6%	0.9%	0.5%	10.3

Table 7.1: The table shows the forecasted results for the phenomenological functions from Casas et al. (2017) for the early-time and late-time parametrizations: 1σ fully marginalized errors on the cosmological parameters $\{\Omega_m, \Omega_b, \ln 10^{10} A_s, n_s, \mu_i, \eta_i\}$ comparing different surveys in the linear and non-linear case. For a detailed discussion of the results see [101].

We finally mention a recent paper in which the authors perform a forecasting study on the potential of future galaxy-cluster surveys in constraining the gravitational slip parameter η using mass profiles.

Pizzuti et al. (2019) [103]
They combine simulated information of galaxy cluster mass profiles inferred from galaxy cluster kinematics and lensing observations to reconstruct η in a model-independent way without make any assumptions on its functional form. In particular, the dynamical mass profile is obtained using the MAMPOSSt method, while the lensing one is obtained using previous information from the analysis of the cluster MACS 1206. They assume: 1) that the density distribution of the cluster is dominated by dark matter for which they assume a Navarro-Frenk-White (NFW) profile; 2)that clusters are in thermal and hydrostatic equilibrium (i.e are relaxed) and have a spherical symmetry; 3) flat ΛCDM background. By using these assumptions they arrive to an expression of the gravitational slip parameter in terms of the lensing and dynamical masses of the cluster, respectively M_{lens} and M_{dyn}

$$\eta(r) = \frac{\int^r \frac{ds}{s^2}[2M_{lens}(s) - M_{dyn}(s)]}{\int^r \frac{ds}{s^2} M_{lens}(s)} \tag{7.4}$$

in GR one has $M_{lens} = M_{dyn}$, thus $\eta = 1$.
They fix redshift at $z = 0$ and consider two cases: a) η scale-independent; b)η scale-dependent. Then, for each case, they reconstruct η from synthetic sample of 15 and 75 clusters taking as fiducial value $\eta = 1$. Their results can be summarized as follows:

- The forecasting constraining power of η, as was to be expected, is higher in the scale-independent case. The accuracies found in their analysis are reported in table 7.2;
- They investigate on the dependence of the constraints on cluster masses and number of galaxy members in the clusters. They find that the former dependence is moderate; while the number of tracers in the clusters has a negligible impact on the constraints. In order to deepen this effects, they consider increasing number of tracers (N_g) used in the dynamics fit for one single cluster finding that, for the scale-independent case , the constraining power does not change in an appreciable way passing from 27% uncertainties at 1σ for $N_g = 100$ to 21% for $N_g = 2000$ and for already $N_g \gtrsim 500$ the accuracy remains approximately the same. Thus, they conclude that, with a moderate number of clusters for which a hundred measured spectroscopic redshifts are available, combined with an honest lensing mass profile, one can reach an high precision in constraining the gravitational slip parameter.
- Finally, they point out that their analysis is purely statistical and several possible systematic effects have been neglected and thus, further investigation is required. This means that, once these effects are properly taken in account, future surveys can determine η down to the percent level of accuracy.

N_{clus}	**scale-independent**	**scale-dependent**
15	5%(1σ), 9%(2σ)	10%(1σ), 21%(2σ)
75	2%(1σ), 4%(2σ)	4%(1σ), 8%(2σ)

Table 7.2: In the table are shown the forecasting accuracy in constraining η for the scale-dependent (evaluated at the reference scale $r = 1.5$ Mpc) and for the scale-independent cases from Pizzuti et al. (2019)

.

The overall picture emerging from these forecasting analysis is encouraging: the huge amount of data coming from different cosmological probes, combined with the development of appropriate and differentiate analysis techniques to deal with them, lead us to believe that in the not too distant future, we would be able to reach unprecedented level of accuracy in testing gravity at cosmological scales.

Conclusions

In this thesis we studied how cosmological observations can be used to constrain modified gravity models reducing their parameters' space and eventually ruling out some of them. In this work we focus on the most general scalar-tensor theories with second order equations of motion, i.e. Horndeski gravity, and on its minimal extension, i.e beyond Horndeski models, paying particular attention to the subclasses of these theories which predict non-zero gravitational slip.

In the first chapter we wrote down the full set of the equations that describe the linear evolution of the cosmological perturbations, parametrizing the deviations from GR with $\mu(a,k)$, $\eta(a,k)$ and $\Sigma(a,k)$. The scale dependence of these functions allows for modifications of gravity on cosmological scales, while local gravity constraints, can still be satisfied. Following Bertschinger and Zukin [63], we distinguished between two classes of theories of gravity: scale-independent and scale-dependent models. We saw that on super-horizon scales the evolution of the perturbations, in a generic theory of gravity, can be completely specified by the gravitational slip parameter, $\eta(a)$; this is also true, on sub-horizon scales bigger than the Jeans wavelength, for scale-independent modified gravity models. On the contrary, the scale-dependent models, need two independent functions to fully characterize the departures from GR; in particular, we chose $\eta(a,k)$ and $\mu(a,k)$ and we found that, this last, is the only that affects the growth of matter perturbations on sub-horizon scale.

In the second chapter we clarified our definition of gravitational slip [33]: it is the mismatch between the two scalar potentials appearing in the perturbed line element of the FRW metric, in Newtonian gauge, due to modification of the geometrical part of the Einstein's field equations. Moreover, we saw that in Horndeski theories a non-standard propagation of GWs in general, implies gravitational slip, while in beyond Horndeski models this is no longer true. Then, we summarized very quickly the effects of a non-zero gravitational slip on LSS: it can induce changes in the matter power spectrum and it can also lead to distortions in the CMB temperature anisotropies and in the CMB polarization maps. Furthermore, when there is a non-vanishing gravitational slip, the lensing potential power spectrum acquires a factor $(1+\eta)^2/4$; as a consequence, the B-mode power spectrum, is affected by the same factor for $\ell << 1000$. Finally, the gravitational slip can also induce a change in the sign of CMB and LSS cross-correlations, which in the ΛCDM is predicted to be positive.

In the third chapter we introduced the EFT formalism and wrote down the modified Poisson equation and the modified anisotropy constraint in terms of the EFT operators. We obtained the analytic expressions for $\eta(a,k)$, $\mu(a,k)$ and $\Sigma(a,k)$ in the quasi-static approximation and we introduced the α- parametrization that allowed us to characterize very efficiently the phenomenology of the Horndeski and beyond Horndeski models. In particular, the modified anisotropic constraint, expressed in terms of this parametrization (3.38), showed that the existence of the gravitational slip is controlled by the functions α_M (running of the Planck mass), α_T (tensor speed excess) and α_H (kinetic matter mixing). We then considered the analytic expressions for the phenomenological functions in two limiting cases: super-Compton limit ($k << aM_C$) and sub-Compton limit ($k >> aM_C$) where M_C defines a transition scale between the two extreme cases and it is related with the Compton wavelength ($M_C \propto \lambda_C^{-1}$). These two limits are the most interesting cases because it is very probably that, the observational window

of a given theory, falls in one of these two regimes.
In the fourth chapter we accounted for the gravitational waves event GW170817 which constrained the GWs speed c_T to be very very close to the speed of light at low redshifts. We analysed the implications for scalar-tensor theories:
- Horndeski Lagrangian reduced to (4.4). Then we found that, if $\alpha_T = 0$, the conditions necessary to have gravitational slip depend only by the parameter α_M, which must be $\alpha_M \neq 0$ and $\alpha_M \neq -\alpha_B/2$. Following [23], we classified the survived Horndeski theories in terms of the α functions:
A) quintessence and K-essence: $\alpha_B = 0 = \alpha_M$;
B) kinetic braiding and its extensions: $\alpha_B + \alpha_M = (\dot{\phi} X G_{3,X})/(H G_4)$;
C) GBD theories: $\alpha_B = -\alpha_M$.
We saw that, only GBD theories and the extensions of the kinetic braiding with $\alpha_M \neq 0$ and $\alpha_M \neq -\alpha_B/2$[2], can exhibit gravitational slip, indeed, it is strictly related to the non-minimal coupling with gravity, i.e the Horndeski function $G_4(\phi)$, which cannot be $G_4 = const$ in order to have $\eta \neq 1$.
- Beyond Horndeski Lagrangian reduced to (4.38). In this case, we found that the analytic expressions for the phenomenological functions are much more cumbersome than the Horndeski case and, in general, one always has gravitational slip (because $\alpha_H \neq 0$). Moreover, in beyond Horndeski models, the fifth force can be repulsive leading to a weakening gravity and a suppression of clustering, contrary to what happens in Horndeski gravity where it is always attractive.
Furthermore, we analysed, for these two classes of models, how measurements of $\mu(a,k)$, $\eta(a,k)$ and $\Sigma(a,k)$ can be used to rule out subclasses of modified gravity theories.
In the fifth chapter we collected the result of the previous chapter about Horndeski gravity and, following [13], we verified that it is very likely that the functions μ and Σ satisfy the condition $(\mu-1)(\Sigma-1) \geq 0$, this means that a measurement of $(\mu-1)(\Sigma-1) < 0$ would disfavour this class of models. We also reported the results obtained in the following papers: [12], [13] and in [37].
In the sixth chapter we briefly considered the most important sources of degeneracies one must take into account in cosmological tests of modified gravity. In particular, we considered the cosmic degeneracies, i.e. the enhancement of the growth of structures predicted in some modified gravity theories can be cancelled out by the effect of massive neutrinos and thus, such a model, can be degenerate with GR with very light neutrinos. We also considered degeneracies arising from systematic effects (e.g. baryonic effect, photometric redshift uncertainties and intrinsic alignments) and degeneracies among MG parameters.
Finally, in the seventh chapter, we reported some recent forecasting analyses which allow an overview of how future surveys will improve our ability in constraining MG parameters and thus in discriminating between different theories of gravity.

[2] This special case was analysed in [36] and it was dubbed ”no slip gravity”.

Appendix A

Relations among EFT operators, α property functions and Horndeski-beyond Horndeski's free functions

The expression for the term C_π appearing in the equation of motion for the scalar field π, in the QSA, is

$$C_\pi = \frac{m_0^2}{4}\dot{\Omega}\dot{R}^{(0)} - 3\dot{H}c + \frac{3}{2}[\dot{H}(3H+\partial_t)+\ddot{H}]\bar{M}_1^3 + \frac{9}{2}\dot{H}^2\bar{M}_2^2 + \frac{3}{2}\dot{H}^2\bar{M}_3^2 \tag{A.1}$$

In the following table one can find the expressions for the coefficients A_i, B_i, C_i $(i = 1,2,3)$

Index	A_i	B_i	C_i
1	$2m_0^2\Omega + 4\hat{M}^2$	$-1 - \frac{2}{m_0^2\Omega}\hat{M}^2$	$m_0^2\dot{\Omega} + 2H\hat{M}^2 + 4\hat{M}\dot{\hat{M}}$
2	$-m_0^2\dot{\Omega} - \bar{M}_1^3 + 2H\bar{M}_3^2 + 4H\hat{M}^2$	1	$-\frac{m_0^2}{2}\dot{\Omega} - \frac{1}{2}\bar{M}_1^3 - \frac{3H}{2}\bar{M}_2^2 - \frac{H}{2}\bar{M}_3^2 + 2H\hat{M}^2$
3	$-8m_2^2$	$-\frac{\dot{\Omega}}{\Omega} + \frac{1}{m_0^2\Omega}\left(H + \frac{2\dot{\bar{M}}_3}{\bar{M}_3}\right)\bar{M}_3^2$	$c - \frac{1}{2}(H+\partial_t)\bar{M}_1^3 + \left(\frac{k^2}{2a^2} - 3\dot{H}\right)\bar{M}_2^2$ $+\left(\frac{k^2}{2a^2} - \dot{H}\right)\bar{M}_3^2 + 2(H^2 + \dot{H} + H\partial_t)\hat{M}^2$

Table A.1: In this table are shown the coefficients of the system of equations (3.19)-(3.21). One can notice that the only term that depends on the scale is C_3, in particular, in Horndeski gravity, this dependence vanishes.

In the following table can be found the correspondences among the Horndeski free functions $K(\phi,X)$,$G_i(\phi,X)$ (and $F_{4,5}(\phi,X)$ for beyond Horndeski models), the EFT operators and the α-functions. We adopt the notations of [45], here the definition of α_B differs from that in [46] and [23]; the relation between the two definitions is $\alpha_B \equiv \alpha_B^B = -2\alpha_B^G$

Parameters	Horndeski-Beyond Horndeski	EFT operators
$\boldsymbol{M_*^2}$	$2(G_4 - 2XG_{4,X} + XG_{5,\phi} - \dot\phi H X G_{5,X}) -$ $\boldsymbol{-8X^2(F_4 - 3H\dot\phi F_5)}$	$m_0^2\Omega + \bar{M}_2^2$
$\boldsymbol{M_*^2 H\alpha_M}$	$\frac{d}{dt}M_*^2$	$m_0^2\dot\Omega + \dot{\bar{M}}_2^2$
$\boldsymbol{M_*^2 H\alpha_B}$	$2\dot\phi(XG_{3,X} - G_{4,\phi} - 2XG_{4,\phi,X}) +$ $+8XH(G_{4,X} + 2XG_{4,X,X} - G_{5,\phi} - XG_{5,\phi,X}) +$ $\boldsymbol{+8XH(8HXF_4 + 4X^2F_{4,X}) +}$ $+2\dot\phi XH^2(3G_{5,X} + 2XG_{5,X,X}) -$ $\boldsymbol{-2\dot\phi XH^2(60HXF_5 + 24X^2F_{5,X})}$	$-\bar{M}_1^3 - m_0^2\dot\Omega$
$\boldsymbol{M_*^2\alpha_T}$	$2X(2G_{4,X} - 2G_{5,\phi} - (\ddot\phi - \dot\phi H)G_{5,X}) +$ $\boldsymbol{+8X^2(F_4 - 3F_5 H\dot\phi)}$	$-\bar{M}_2^2$
$\boldsymbol{M_*^2 H^2\alpha_K}$	$2X(K_{,X} + 2XK_{,X,X} - 2G_{3,\phi} - 2XG_{3,\phi,X}) +$ $+12\dot\phi XH(G_{3,X} + XG_{3,X,X} - 3G_{4,\phi,X} - 2XG_{4,\phi,X,X}) +$ $+12XH^2(G_{4,X} + 8XG_{4,X,X} + 4X^2G_{4,X,X,X}) -$ $-12XH^2(G_{5,\phi} + 5XG_{5,\phi,X} + 2X^2G_{5,\phi,X,X}) +$ $\boldsymbol{+12X(24XF_4 + 36X^2F_{4,X} + 8X^3F_{4,X,X}) +}$ $+4\dot\phi XH^3(3G_{5,X} + 7XG_{5,X,X} + 2X^2G_{5,X,X,X}) -$ $\boldsymbol{-4\dot\phi XH^3(120XF_5 + 132X^2F_{5,X} + 24X^3F_{5,X,X})}$	$2c + 4M_2^4$
$\boldsymbol{M_*^2\alpha_H}$	$\boldsymbol{8X^2(F_4 - 3F_5 H\dot\phi)}$	$2\hat{M}^2 - \bar{M}_2^2$

Table A.2: Here the relations among the operators of the EFT actions (3.70) and those appearing in the actions (3.44) and (3.64). The terms in bold in the middle column are those that one has to add with respect to the Horndeski case to extend this class to beyond Horndeski models; indeed, these terms contain the free function $F_4(\phi, X)$ and $F_5(\phi, X)$ that in Horndeski theory are zero.

f_1	f_2	f_3	f_4	f_5
$B_2C_3 - B_3C_1$	B_2C_π	$A_1(B_3C_2 - B_1C_3) + A_2(B_1C_1 - B_2C_2) + A_3(B_2C_3 - B_3C_1)$	$(A_3B_2 - A_1B_1)C_\pi$	$B_3C_2 - B_1C_3$

Table A.3: The table shows the f_i functions appearing in the quasi-static approximation of the phenomenological functions $\mu(a,k)$, $\eta(a,k)$ and $\Sigma(a,k)$

A.0.1 Horndeski gravity

In particular, for Horndeski gravity one finds, using tab.A.2, the following relations

$$2\hat{M}^2 = \bar{M}_2^2 = -\bar{M}_3^2, \quad m_2^2 = 0 \quad \Longrightarrow \quad M_*^2 = m_0^2\Omega + 2\hat{M}^2, \quad M_*^2\alpha_T = -2\hat{M}^2 \quad \Longrightarrow \quad M_*^2(1+\alpha_T) = m_0^2\Omega$$
$$\dot{\bar{M}}_2^2 = 4\dot{\hat{M}}\hat{M} = -2\bar{M}_3\dot{\bar{M}}_3, \quad -\dot{M}_1^3 = \dot{H}M_*^2\alpha_B + H^2M_*^2\alpha_M\alpha_B + HM_*^2\dot{\alpha}_B + m_0^2\ddot{\Omega}$$

Now using the tables A.1 and A.2, along with the expressions (3.6)-(3.37)-(3.42), one obtains the following coefficients

Index	A_i	B_i	C_i
1	$2M_*^2$	$-(1+\alpha_T)^{-1}$	$HM_*^2(\alpha_M - \alpha_T)$
2	$HM_*^2\alpha_B$	1	$\frac{HM_*^2}{2}\alpha_B$
3	0	$-H(\alpha_M - \alpha_T)(1+\alpha_T)^{-1}$	$\frac{H^2M_*^2}{2}\left[\alpha c_S^2 + \frac{\alpha_B^2}{2}(1+\alpha_T) + 2\alpha_B(\alpha_M - \alpha_T)\right]$

Table A.4: In this table are reported the coefficients of the system of equations (3.19)-(3.21) specialized to the case of Horndeski gravity and in terms of the α-parametrization

the coefficient C_3 can be also recast in this way

$$\begin{aligned} C_3 &= -\frac{\bar{\rho}_m + \bar{p}_m}{2} + \frac{M_*^2H^2}{2}[\alpha_B(\alpha_M+1) + 2\alpha_M - 2\alpha_T] + \frac{\dot{H}M_*^2}{2}[\alpha_B - 2(1+2\alpha_T)] + \frac{M_*^2H}{2}\dot{\alpha}_B \\ &= \frac{M_*^2H^2}{2}\alpha c_S^2\left[1 + \frac{\beta_B\beta_\xi}{2}\right] + \frac{M_*^2H^2}{2}\alpha_B[\alpha_M - \alpha_T] \end{aligned} \tag{A.2}$$

The term C_π in Horndeski gravity becomes

$$C_\pi = \frac{3}{2}M_*^2\dot{H}\left[\frac{\bar{\rho}_m + \bar{p}_m}{M_*^2} + (2-\alpha_B)\dot{H} - H\dot{\alpha}_B - \alpha_B\left(3H^2 + H^2\alpha_M + \frac{\ddot{H}H}{\dot{H}}\right)\right] \tag{A.3}$$

Finally one gets

$$f_1 = \frac{M_*^2H^2\alpha c_S^2}{2(1+\alpha_T)}\left[1 + \alpha_T + \beta_\xi^2\right] \tag{A.4}$$
$$f_3 = \frac{M_*^4H^2}{1+\alpha_T}\alpha c_S^2 \tag{A.5}$$
$$f_5 = \frac{M_*^2H^2\alpha c_S^2}{2(1+\alpha_T)}\left[1 + \beta_B\beta_\xi/2\right] \tag{A.6}$$

$$\Longrightarrow \quad \frac{2M_*^2 f_1}{f_3} = 1 + \alpha_T + \beta_\xi^2, \quad \frac{f_5}{f_1} = \frac{1 + \beta_B\beta_\xi/2}{1 + \alpha_T + \beta_\xi^2} \tag{A.7}$$

where we have defined the following quantities

$$\alpha \equiv \alpha_K + \frac{3}{2}\alpha_B \tag{A.8}$$

$$\beta_\xi^2 \equiv \frac{2}{\alpha c_S^2}\left[\frac{\alpha_B}{2}(1+\alpha_T) + (\alpha_M - \alpha_T)\right]^2 \tag{A.9}$$

$$\beta_B^2 \equiv \frac{2}{\alpha c_S^2}\alpha_B^2 \tag{A.10}$$

$$M_C^2 \equiv \frac{3\dot{H}(1+\alpha_T)}{\alpha c_S^2 H^2(1+\alpha_T+\beta_\xi^2)}\left[\frac{\bar{\rho}_m + \bar{p}_m}{M_*^2} + (2-\alpha_B)\dot{H} - H\dot{\alpha_B} - \alpha_B\left(3H^2 + H^2\alpha_M + \frac{\ddot{H}H}{\dot{H}}\right)\right] \tag{A.11}$$

$$\Delta_1(a,k) \equiv \frac{\beta_\xi^2}{1+\beta_\xi^2}\left[1 + \frac{a^2}{k^2}M_C^2\right]^{-1} \tag{A.12}$$

$$\Delta_2(a,k) \equiv \frac{\beta_B\beta_\xi/2 - \beta_\xi^2}{1+\beta_\xi^2}\left[1 + \frac{a^2}{k^2}M_C^2\right]^{-1} \tag{A.13}$$

$$\Delta_3(a,k) \equiv \frac{\beta_B\beta_\xi/2 + \beta_\xi^2}{2(1+\beta_\xi^2)}\left[\frac{1}{1+\beta_\xi^2} + \frac{a^2}{k^2}M_C^2\right]^{-1} \tag{A.14}$$

A.0.2 Beyond Horndeski gravity

The relations between the free functions appearing in (3.64) and those appearing in the ADM action (3.31)-(3.34) are the following

$$A_2 = K_2 + \sqrt{-X}\int dX \frac{G_{3,\phi}}{2\sqrt{-X}}, \tag{A.15}$$

$$A_3 = \int dX \sqrt{-X} G_{3,X} - 2\sqrt{-X} G_{4,\phi}, \tag{A.16}$$

$$A_4 = -G_4 + 2XG_{4,X} + \frac{X}{2}G_{5,\phi} - X^2 F_4, \tag{A.17}$$

$$B_4 = G_4 + \sqrt{-X}\int dX \frac{G_{5,\phi}}{4\sqrt{-X}}, \tag{A.18}$$

$$A_5 = -\frac{1}{3}\sqrt{-X^3}G_{5,X} + \sqrt{-X^5}F_5, \tag{A.19}$$

$$B_5 = -\int dX \sqrt{-X} G_{5,X} \tag{A.20}$$

Using the tables A.1 and A.2, along with the expressions (3.6)-(3.37)-(3.42), one finds the following coefficients

Index	A_i	B_i	C_i
1	$2M_*^2(1+\alpha_H)$	$-(1+\alpha_H)(1+\alpha_T)^{-1}$	$HM_*^2[\alpha_M-\alpha_T+\alpha_H(1+\alpha_M)]+M_*^2\dot{\alpha}_H$
2	$HM_*^2(\alpha_B+2\alpha_H)$	1	$\frac{HM_*^2}{2}(\alpha_B+2\alpha_H)$
3	0	$-H(\alpha_M-\alpha_T)(1+\alpha_T)^{-1}$	$\frac{M_*^2\dot{H}}{2}[\alpha_B+2\alpha_H-2(1+2\alpha_T)]+\frac{M_*^2H}{2}[\dot{\alpha}_B+2\dot{\alpha}_H]+$ $+\frac{M_*^2H^2}{2}[(\alpha_B+2\alpha_H)(1+\alpha_M)+2(\alpha_M-\alpha_T)]-\frac{(\bar{\rho}_m+\bar{p}_m)}{2}$

Table A.5: In this table are reported the coefficients of the system of equations (3.19)-(3.21) specialized to the case of beyond Horndeski theories and in terms of the α-parametrization

By comparing the table A.5 with the table A.4, one can see that, the coefficient A_i, B_i and C_i, for beyond Horndeski theories, can be written as a sum of the ones of the Horndeski gravity plus a new term which contains the parameter α_H

$$A_1^{BH} = A_1^H + 2M_*^2\alpha_H \tag{A.21}$$
$$A_2^{BH} = A_2^H + 2HM_*^2\alpha_H \tag{A.22}$$
$$A_3^{BH} = A_3^H = 0 \tag{A.23}$$
$$B_1^{BH} = B_1^H - \alpha_H(1+\alpha_T)^{-1} \tag{A.24}$$
$$B_2^{BH} = B_2^H = 1 \tag{A.25}$$
$$B_3^{BH} = B_3^H \tag{A.26}$$
$$C_1^{BH} = C_1^H + HM_*^2\alpha_H(1+\alpha_M) + M_*^2\dot{\alpha}_H \tag{A.27}$$
$$C_2^{BH} = C_2^H + HM_*^2\alpha_H \tag{A.28}$$
$$C_3^{BH} = C_3^H + \dot{H}M_*^2\alpha_H + H^2M_*^2\alpha_H(1+\alpha_M) + M_*^2H\dot{\alpha}_H \tag{A.29}$$

From the above expressions, one can obtain the f_i functions for beyond Horndeski model

$$f_1^{BH} = f_1^H + H^2M_*^2\Delta_C + \left[HM_*^2\alpha_H(1+\alpha_M) + M_*^2\dot{\alpha}_H\right]H\frac{\alpha_M-\alpha_T}{1+\alpha_T}$$
$$f_3^{BH} = f_3^H + \frac{M_*^4H^2}{(1+\alpha_T)}\Bigg\{(2-\alpha_B)(1+\alpha_H)\Delta_C + \alpha_H(\alpha_B+2\alpha_H)\left[\frac{\dot{H}}{H}(1+\alpha_H)-(1+\alpha_M)\right]$$
$$+\alpha_H\left[(2+\alpha_H)(1+\frac{\beta_\xi\beta_B}{2}) - \frac{\beta_\xi\beta_B}{\alpha_B}\right]\alpha(c_S^H)^2\Bigg\}$$
$$f_5^{BH} = f_5^H + \frac{M_*^2H^2}{(1+\alpha_T)}\left\{(1+\alpha_H)\Delta_C + \frac{\alpha_H}{2}\left[\frac{\dot{H}}{H^2}(\alpha_B-4\alpha_T-2) + \frac{1}{H}\dot{\alpha}_B + \alpha_B(1+\alpha_M) - \frac{(\bar{\rho}_m+\bar{p}_m)}{M_*^2H^2}\right]\right\}$$

Thus

$$f_1^{BH} = \frac{M_*^2H^2}{(1+\alpha_T)}\left[\frac{\alpha(c_S^H)^2}{2}(1+\alpha_T+\beta_\xi^2) + \beta_{1H}\right] \tag{A.30}$$
$$f_3^{BH} = \frac{M_*^4H^2}{(1+\alpha_T)}\left[\alpha(c_S^H)^2 + \beta_{3H}\right] \tag{A.31}$$
$$f_5^{BH} = \frac{M_*^2H^2}{(1+\alpha_T)}\left[\frac{\alpha(c_S^H)^2}{2}\left(1+\frac{\beta_B\beta_\xi}{2}\right) + \beta_{5H}\right] \tag{A.32}$$

where c_S^H is the sound speed of the scalar perturbations for the Horndeski gravity ($\alpha_H = 0$) which is given in (3.43). The other new parameters appearing in the above expressions are defined as follows

$$\Delta_C \equiv \frac{\dot{H}}{H^2}\alpha_H + (1+\alpha_M)\alpha_H + \frac{1}{H}\dot{\alpha}_H \tag{A.33}$$

$$\beta_{1H} \equiv \Delta_C(1+\alpha_M) - (\alpha_M - \alpha_T)\alpha_H \tag{A.34}$$

$$\beta_{3H} \equiv (2-\alpha_B)(1+\alpha_H)\Delta_C + \alpha_H(\alpha_B + 2\alpha_H)\left[\frac{\dot{H}}{H}(1+\alpha_H) - (1+\alpha_M)\right] \tag{A.35}$$

$$+ \alpha_H\left[(2+\alpha_H)(1+\frac{\beta_\xi\beta_B}{2}) - \frac{\beta_\xi\beta_B}{\alpha_B}\right]\alpha(c_S^H)^2 \tag{A.36}$$

$$\beta_{5H} \equiv \frac{\alpha(c_S^H)^2}{2}\left(1+\frac{\beta_B\beta_\xi}{2}\right)\alpha_H + \Delta_C(1+\alpha_H) - \frac{\alpha_H}{2}(2-\alpha_B)(\alpha_M-\alpha_T) \tag{A.37}$$

One can see that, in beyond Horndeski models, the expressions of the phenomenological functions are much more cumbersome because one has additional contributions coming from α_H; if one set this last parameter to zero then, the above expressions, reduce to the ones we found for the Horndeski gravity, indeed, in this case, $\Delta_C = 0$ and $\beta_{1H} = \beta_{3H} = \beta_{5H} = 0$.
We have thus

$$\frac{2M_*^2 f_1^{BH}}{f_3^{BH}} = \frac{\alpha(c_S^H)^2(1+\alpha_T+\beta_\xi^2) + 2\beta_{1H}}{\alpha(c_S^H)^2 + \beta_{3H}} \tag{A.38}$$

$$\frac{f_5^{BH}}{f_1^{BH}} = \frac{\alpha(c_S^H)^2\left(1+\beta_B\beta_\xi/2\right) + 2\beta_{5H}}{\alpha(c_S^H)^2(1+\alpha_T+\beta_\xi^2) + 2\beta_{1H}} \tag{A.39}$$

Finally, the quantities Δ_i^{BH} which appears in the expression of μ, η and Σ for the beyond Horndeski theories when $\alpha_T = 0$ are defined as follows

$$\begin{aligned}\Delta_1^{BH}(a,k) &= -\frac{(2\alpha_H+\alpha_H^2)}{(1+\alpha_H)^2} + \left[\frac{2M_*^2 f_1^{BH}(1+\alpha_H)^2 - f_3^{BH}}{f_1^H}\right]\left[\frac{f_3^{BH}}{2M_*^2 f_1^H} + \frac{a^2}{k^2}M_C^2(1+\alpha_H)^2\right]^{-1} \\ &= -\frac{(2\alpha_H+\alpha_H^2)}{(1+\alpha_H)^2} + M_*^2\left\{\frac{\alpha(c_S^H)^2[(1+\beta_\xi^2)(1+\alpha_H)^2-1] + \beta_{1H}(1+\alpha_H)^2 - \beta_{3H}}{\alpha(c_S^H)^2(1+\beta_\xi^2)}\right\}\cdot \\ &\cdot\left[\frac{\alpha(c_S^H)^2+\beta_{3H}}{\alpha(c_S^H)^2(1+\beta_\xi^2)} + \frac{a^2}{k^2}M_C^2(1+\alpha_H)^2\right]^{-1}\end{aligned} \tag{A.40}$$

$$\begin{aligned}\Delta_2^{BH}(a,k) &= \alpha_H + \left[\frac{f_5^{BH}-f_1^{BH}}{f_1^H}\right]\left[\frac{f_1^{BH}}{f_1^H} + \frac{a^2}{k^2}M_C^2\right]^{-1} \\ &= \alpha_H + \left[\frac{\alpha(c_S^H)^2\left(\beta_B\beta_\xi/2 - \beta_\xi^2\right) + 2(\beta_{5H}-\beta_{1H})}{\alpha(c_S^H)^2(1+\beta_\xi^2)}\right]\left[1 + \frac{2\beta_{1H}}{\alpha(c_S^H)^2(1+\beta_\xi^2)} + \frac{a^2}{k^2}M_C^2\right]^{-1}\end{aligned} \tag{A.41}$$

$$\Delta_3^{BH}(a,k) = \Delta_1^{BH}(a,k) + \frac{1}{2}\Delta_2^{BH}(a,k) + \frac{1}{2}\Delta_1^{BH}(a,k)\Delta_2^{BH}(a,k) \tag{A.42}$$

Appendix B

Modified field equations in the EFT of dark energy

In this appendix we report the Einstein's field equations, in the Newtonian gauge, obtained in [43] and where the background contribution has been subtracted, in the case of flat universe.

- 00-component: energy constraint

$$\begin{aligned} m_0^2\Omega\left[2\frac{k^2}{a^2}\Phi+6H(\dot{\Phi}+H\Psi)\right] &= -\delta\rho-\dot{\rho}_{DE}\pi-2c(\dot{\pi}-\Psi)+ \\ &+m_0^2\dot{\Omega}\left[3\pi\left(H^2-\dot{H}\right)+3H(\dot{\pi}-\Psi)+\frac{k^2}{a^2}\pi-3\left(\dot{\Phi}+H\Psi\right)\right] \end{aligned} \tag{B.1}$$

- 0i-component: momentum constraint

$$2m_0^2\Omega k^2\left[\dot{\Phi}+H\Psi\right]=(\bar{\rho}_m+\bar{p}_m)kv_m+(\rho_{DE}+p_{DE})k^2\pi+m_0^2\dot{\Omega}k^2(\dot{\pi}-\Psi) \tag{B.2}$$

- ij-traceless-component: anisotropy constraint

$$m_0^2\Omega\frac{k^2}{a^2}(\Phi-\Psi)=\bar{p}_m\Pi+m_0^2\dot{\Omega}\frac{k^2}{a^2}\pi-\bar{M}_3^2\frac{k^2}{a^2}\left[\dot{\pi}+\left(H+\frac{2\dot{\bar{M}}_3}{\bar{M}_3}\right)\pi\right]+2\hat{M}^2\frac{k^2}{a^2}(\Psi-\dot{\pi}) \tag{B.3}$$

- ij-trace-component: pressure equation

$$\begin{aligned} &m_0^2\Omega\left[2\ddot{\Phi}+2(3H^2+2\dot{H})\Psi+2H(\dot{\Psi}+3\dot{\Phi})+\frac{2k^2}{3a^2}(\Phi-\Psi)\right]=\delta p+\dot{p}_{DE}\pi+(\rho_{DE}+p_{DE})(\dot{\pi}-\Psi)+ \\ &+m_0^2\dot{\Omega}\left[\frac{\ddot{\Omega}}{\dot{\Omega}}(\dot{\pi}-\Psi)+\ddot{\pi}-\Psi-2\dot{\Phi}+3H\dot{\pi}-5H\Psi+3H^2\pi+\frac{2k^2}{3a^2}\pi\right] \end{aligned} \tag{B.4}$$

B.0.1 $\alpha-$parametrization

In [45]- [46] the authors find the following modified field equations in terms of the α property functions

- 00-component: energy constraint

$$\begin{aligned} &3(2-\alpha_B)H\dot{\Phi}+(6-\alpha_K-6\alpha_B)H^2\Psi+2(1+\alpha_H)\frac{k^2}{a^2}\Phi-(\alpha_K+3\alpha_B)H^2\dot{v}_X \\ &-\left[\frac{k^2}{a^2}(\alpha_B+2\alpha_H)-3\dot{H}\alpha_B+3\left(2\dot{H}+\frac{\bar{\rho}_m+\bar{p}_m}{M_*^2}\right)\right]Hv_X=-\frac{\bar{\rho}_m\delta}{M_*^2} \end{aligned} \tag{B.5}$$

- 0i-component: momentum constraint

$$2\dot{\Phi} + (2-\alpha_B)H\Psi - \alpha_B H\dot{v}_X - \left(2\dot{H} + \frac{\bar{\rho}_m + \bar{p}_m}{M_*^2}\right)v_X = -\frac{\bar{\rho}_m + \bar{p}_m}{M_*^2}v_m \tag{B.6}$$

- ij-traceless-component: anisotropy constraint

$$(1+\alpha_H)\Psi - (1+\alpha_T)\Phi - (\alpha_M - \alpha_T)Hv_X + \alpha_H\dot{v}_X = -\frac{3}{2}\frac{(\bar{\rho}_m + \bar{p}_m)}{M_*^2}\sigma_m \tag{B.7}$$

- ij-trace-component: pressure equation

$$\begin{aligned}
&2\ddot{\Phi} - \alpha_B H\ddot{v}_X + 2(3+\alpha_M)H\dot{\Phi} + (2-\alpha_B)H\dot{\Psi} \\
&+\left[H^2(2-\alpha_B)(3+\alpha_M) - (\dot{\alpha}_B H + \alpha_B\dot{H}) + 4\dot{H} - \left(2\dot{H} + \frac{\bar{\rho}_m + \bar{p}_m}{M_*^2}\right)\right]\Psi \\
&-\left[\left(2\dot{H} + \frac{\bar{\rho}_m + \bar{p}_m}{M_*^2}\right) + (\dot{\alpha}_B H + \alpha_B\dot{H}) + \alpha_B H^2(3+\alpha_M)\right]\dot{v}_X \\
&-\left[2\ddot{H} + 2\dot{H}H(3+\alpha_M) + \frac{\dot{\bar{p}}_m}{M_*^2} + \alpha_M H\frac{\bar{p}_m}{M_*^2}\right]v_X = \frac{\delta\bar{p}_m}{M_*^2} - \frac{k^2}{a^2}\frac{(\bar{\rho}_m + \bar{p}_m)}{M_*^2}\sigma_m
\end{aligned} \tag{B.8}$$

- equation of motion for the scalar velocity potential v_X

$$\begin{aligned}
&3\alpha_B H\ddot{\Phi} + \alpha_K H^2\ddot{v}_X - 3\left[\left(2\dot{H} + \frac{\bar{\rho}_m + \bar{p}_m}{M_*^2}\right) - (\dot{\alpha}_B H + \alpha_B\dot{H}) - \alpha_B H^2(3+\alpha_M)\right]\dot{\Phi}H^2M_\pi^2 v_X \\
&+(\alpha_K + 3\alpha_B)^2H^2\dot{\Psi} - 2\frac{k^2}{a^2}(\alpha_M - \alpha_T)H\Phi - \alpha_B H\frac{k^2}{a^2}\Psi + [2\alpha_K\dot{H} + \dot{\alpha_K}H + \alpha_K H^2(3+\alpha_M)]H\dot{v}_X - \\
&-\left[3\left(2\dot{H} + \frac{\bar{\rho}_m + \bar{p}_m}{M_*^2}\right) - (\dot{\alpha}_K + 3\dot{\alpha}_B)H - (3+\alpha_M)(\alpha_K + 3\alpha_B)H^2 - \dot{H}(2\alpha_K + 9\alpha_B)\right]H\Psi + \\
&+\left[-\left(2\dot{H} + \frac{\bar{\rho}_m + \bar{p}_m}{M_*^2}\right) + 2(\alpha_M - \alpha_T)H^2 + \alpha_B(1+\alpha_M)H^2 + (\dot{\alpha}_B H + \alpha_B\dot{H}) - \alpha_B H^2(3+\alpha_M)\right]\frac{k^2}{a^2}v_X + \\
&+2\frac{k^2}{a^2}\left\{\left[\alpha_H(1+\alpha_M) + (\dot{H}\alpha_H + \dot{\alpha_H}H)\right]v_X - \left[\alpha_H\dot{\Phi} + H\alpha_H(1+\alpha_M)\Phi - \dot{\alpha}_H\Phi + H\alpha_H\Psi\right]\right\} = 0
\end{aligned} \tag{B.9}$$

where the term M_π^2 is defined by

$$H^2M_\pi^2 = 3\dot{H}\left[(2-\alpha_B)\dot{H} + \frac{\bar{\rho}_m + \bar{p}_m}{M_*^2} - \dot{\alpha}_B H\right] - 3\alpha_B H\left[\ddot{H} + \dot{H}H(3+\alpha_M)\right] \tag{B.10}$$

the relation with the mass term defined in (A.3) is $H^2M_\pi^2 = 2C_\pi/M_*^2$

The scalar velocity potential is defined as $v_X \equiv -\delta\phi/\dot{\phi}$ and represents the perturbations of the scalar field. This quantity is introduced because the value of the scalar field, ϕ, is not an observable, then one can always redefine the field $\tilde{\phi} \equiv \tilde{\phi}(\phi)$ without changing observable quantities. The scalar field gradient can be thought of as a four velocity $u_\mu \equiv \partial_\mu/\sqrt{2X}$ if one requires that $\partial_\mu\phi$ is timelike. Consequently, ϕ, can be interpreted as a time variable and its gradient is the time direction for an observer which is at rest in this frame; thus v_x is the scalar velocity potential and it is invariant under redefinition of ϕ.

Appendix C

Some details on GBD and covariant Galileons theories

C.1 Generalized Fierz-Brans-Dicke theories

An equivalent form of the action (4.24) is

$$\mathcal{S}_{GBD} = \int d^4x\sqrt{-g}\frac{m_0^2}{2}\left[\phi R - \frac{\omega(\phi)}{\phi}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - 2\Lambda(\phi)\right] + \mathcal{S}_m[g_{\mu\nu},\psi_i] \tag{C.1}$$

the choice $\Lambda = 0$ and $\omega(\phi) = \omega_{BD}$ corresponds to Brans-Dicke theories, where ω_{BD} is the BD parameter. One can shows that a BD theory with $\omega_{BD} = 0$ is equivalent to metric f(R) gravity whose Lagrangian is $\mathcal{L}_{f(R)} = \frac{m_0^2}{2}f(R)$; while $\omega_{BD} = -3/2$ is equivalent to f(R) gravity in the Palatini formalism.
By varying the action (C.1) with respect to the metric, one obtains the field equations

$$\phi G_{\mu\nu} - \frac{\omega(\phi)}{\phi}\nabla_\mu\phi\nabla_\nu\phi + g_{\mu\nu}\left[\frac{\omega(\phi)}{2\phi}\nabla^\rho\phi\nabla_\rho\phi + \Lambda(\phi)\right] - \nabla_\mu\nabla_\nu\phi + g_{\mu\nu}\Box\phi = \frac{1}{m_0^2}T_{\mu\nu}^{(m)} \tag{C.2}$$

using the trace of (C.2) and varying with respect to ϕ one finds the Klein-Gordon equation for the scalar field

$$[3+2\omega(\phi)]\Box\phi + \frac{d\omega}{d\phi}\nabla^\mu\phi\nabla_\mu\phi + 4\Lambda(\phi) - 2\phi\frac{d\Lambda}{d\phi} = \frac{1}{m_0^2}T_k^{k(m)} \tag{C.3}$$

From (C.2)-(C.3) one obtains the modified Friedmann equations and the background equation for the scalar field

$$H^2 = \frac{1}{3m_0^2\phi}\bar{\rho}_m + \frac{1}{3\phi}\left[-3H\dot{\phi} + \frac{1}{2}\omega(\phi)\dot{\phi} + \Lambda\right] \tag{C.4}$$

$$\dot{H} = -\frac{2}{m_0^2\phi}(\bar{\rho}_m + \bar{p}_m) - \frac{1}{2\phi}\left[\ddot{\phi} + 5H\dot{\phi} + \omega(\phi)\dot{\phi}^2 - 2\Lambda\right] \tag{C.5}$$

$$\ddot{\phi} + \left[3H + \frac{\dot{\omega}}{2\omega+3}\right]\dot{\phi} = \frac{1}{2\omega+3}\left[\frac{1}{m_0^2}(\bar{\rho}_m + \bar{p}_m) - 2\phi\frac{d\Lambda}{d\phi} + 4\Lambda\right] \tag{C.6}$$

Then, by linearising the field equations and subtracting the background solutions, in QSA, one finds the Poisson equation and the anisotropy constraint

$$\phi_0 k^2\Psi = -4\pi G a^2\rho\Delta - \frac{1}{2}k^2\delta\phi \tag{C.7}$$

$$\phi_0 k^2(\Phi - \Psi) = k^2\delta\phi \tag{C.8}$$

where the scalar field can be written as $\phi(a,\boldsymbol{x}) = \phi_0(a) + \delta\phi(a,\boldsymbol{x})$, with ϕ_0 the background value of the scalar field.
The equation for the Weyl potential follows by the first two

$$k^2(\Phi + \Psi) = -\frac{8\pi G a^2}{\phi_0}\rho\Delta \tag{C.9}$$

In particular, in Brans-Dicke theories, i.e. $\omega(\phi) = \omega_{BD}$ and $\Lambda(\phi) = 0$, the parameter ω has been constrained by several solar system experiments. For instance, the Cassini mission [70] put strong constraints on the post-Newtonian deviation from the Einstein's gravity: $\omega > 4.3 \cdot 10^4$ at a 2σ level. The effective gravitational coupling measured by Cavendish-type experiments and the PPN (parametrized post-Newtonian) parameters γ, in this case, are given by

$$G_{eff} = \frac{G}{\phi_0}\frac{2\omega + 4}{2\omega + 3}, \qquad \gamma = \frac{1+\omega}{2+\omega} \tag{C.10}$$

C.2 Covariant Galileons

Galileon theories are a class of models characterized by non-linear derivative self-interactions whose Lagrangian is invariant under the Galileon shift symmetry in flat space-time $\partial_\mu\phi \to \partial_\mu\phi + b_\mu$. The covariant generalization is a special case of Horndeski gravity corresponding to the choice:

$$K(\phi, X) = c_2 X, \quad G_3(\phi, X) = 2\frac{c_3}{M^3}X, \quad G_4(\phi, X) = \frac{1}{2}m_0^2 + \frac{c_4}{M^6}X^2 \quad G_5(\phi, X) = \frac{c_5}{M^9}X^2 \tag{C.11}$$

where $M^3 = m_0 H_0^2$ so that M has dimension of a mass and makes the c_i coefficients dimensionless. They can be classified according to the highest power of ϕ appearing in their action: 1) cubic Galileons ($c_4 = 0 = c_5$), 2) quartic Galileons ($c_5 = 0$), 3) quintic Galileons ($c_i \neq 0$). These parameters are constrained by cosmological bounds coming from CMB and BAO and in [20] using Monte Carlo methods they find $c_4 = 0.008^{+0.11}_{-0.026}$, $c_5 = -0.013^{+0.023}_{-0.12}$ at 95% confidence level.
Covariant Galileons are self-accelerating models and in particular, in [27], it is shown that exists a tracker solution which asymptotically tends to a fixed the Sitter point; thus, the observational constraints coming from CMB can be satisfied. In this solution [1] the Hubble parameters and the time derivative of the scalar field are related by $\xi \equiv H(t)\dot{\phi}/(m_0 H_0^2) = const$. From the tables in appendix A one can derive the expressions for M_*^2, α_T and α_M

$$M_*^2 = m_0^2 - 6\frac{c_4}{M^6}X^2 - 4\dot{\phi}H\frac{c_5}{M^9}X^2 = m_0^2\left[1 - c_4\frac{3\xi^4}{2E^4} - c_5\frac{\xi^5}{E^4}\right] \tag{C.12}$$

$$M_*^2 H\alpha_M = 4\frac{1}{E^4}\frac{\dot{H}}{H}\left[\frac{3}{2}c_4\xi^4 + c_5\xi^5\right] \tag{C.13}$$

$$M_*^2\alpha_T = \frac{m_0^2}{E^4}\left[2c_4\xi^4 + c_5\xi^5\left(1 + \frac{\dot{H}}{H^2}\right)\right] \tag{C.14}$$

where $E \equiv H(t)/H_0$.
In the cubic Galileons $\alpha_T = 0 = \alpha_M$, therefore this theory predicts no gravitational slip. Moreover, the stringent bound on c_T (4.1) in turn implies strong constraints on c_4 and c_5 [25]:

$$|c_4| < \frac{\alpha_T}{2\xi^4} \sim 2.8 \cdot 10^{-17}\left(\frac{2}{\xi}\right)^4, \quad |c_5| < \frac{\alpha_T}{0.75\xi^5} \sim 3.8 \cdot 10^{-17}\left(\frac{2}{\xi}\right)^5 \tag{C.15}$$

[1] This solution must be reached before the scalar field gives a non negligible contribute to the energy density of the universe.

thus, if we exclude fine-tuning among the parameters of these models, the quintic and quartic covariant Galileons are strongly disfavoured after $GW170817$. On the other hand, the cubic Galileon models, even if satisfy the bound (4.1), are in tension with ISW data.

Appendix D

Fisher information

Fisher information formalism (Fisher, 1935) [97] is a statistical method widely used in cosmology to forecast the accuracy under which one is able to estimate the parameters of a models from a given data set in upcoming experiments and thus, the ability in constraining combined set of cosmological parameters. Indeed, the Fisher information is encoded in the Fisher matrix which, under some specific assumptions[1], turn out to be the inverse of of the covariance matrix which contains information about the uncertainties of the model's parameters and also about potential correlations (i.e. degeneracies) among them. Therefore, this approach allows to have an estimation of the parameter's errors before. It is a less expensive and quicker alternative to Markov Chain Monte Carlo (MCMC) [98].
Let us suppose to have a data set depending on n random real number $\{x_1, \ldots, x_n\}$ (that can be represented by the vector $\vec{x}$) whose probability distributions depends on a set of m model parameters $\{\theta_1, \ldots, \theta_m\}$ then the Fisher matrix is defined as follows [99]- [98]:

$$F_{ij} = -\left\langle \frac{\partial^2 \ln \mathcal{L}}{\partial\theta_i \partial\theta_j} \right\rangle \tag{D.1}$$

where $\partial^2 \ln \mathcal{L} / \partial\theta_i \partial\theta_j$ is the Hessian matrix that one obtains by expanding the log likelihood in Taylor series around its maximum, under the assumptions that it is locally a multi-variate Gaussian surface. Thus, the Fisher matrix, is the ensemble average of this Hessian matrix over observational data. A useful property of the Fisher matrix is that it is additive for independent data sets, this is consequence of the fact that the likelihood, for independent data sets, is the product of the likelihoods.
Assuming that the likelihood is a Gaussian around its maximum, Fisher matrix is related to the covariance matrix by

$$\boldsymbol{F} = \boldsymbol{C}^{-1} \tag{D.2}$$

where $\boldsymbol{C}$ is the covariance matrix.

D.0.1 Covariance matrix

The covariance matrix of $(\theta_1, \ldots, \theta_m) \equiv \theta$ is given by [101]

$$\boldsymbol{C} = \langle \Delta\theta \Delta\theta^T \rangle \tag{D.3}$$

where $\Delta\theta = \theta - \langle\theta\rangle$. The diagonal part of the matrix, contains the square of errors, i.e the variances, for each parameters; the off-diagonal elements represent the correlations among the

[1] In particular, one has to require that the likelihood assumes a multivariate Gaussian form in the model parameters, see for example [99], [100].

parameters, this means that if $\boldsymbol{C}$ is not diagonal there is degeneracy among the parameters which specify the cosmological model we are considering. Therefore, once $\boldsymbol{C}$ is known, one can compute the correlation matrix $\boldsymbol{P}$ as

$$P_{ij} = \frac{Cij}{\sqrt{C_{ii} C_{jj}}} \tag{D.4}$$

the correlations can be easily visualized using the marginalized ellipsoidal contour plot in the parameter hyperspace. The volume of the error ellipsoid plot is proportional to the square-root of the determinant of the covariance matrix, $\sqrt{\det \boldsymbol{C}}$ [2]. Then, it is possible to define respectively the "Figure of Merit (FoM)" and the "Figure of Correlations (FoC)" as follows

$$\text{FoM} = -\frac{1}{2}\ln[\det \boldsymbol{C}], \qquad \text{FoC} = -\frac{1}{2}\ln[\det \boldsymbol{P}] \tag{D.5}$$

The FoM quantifies how well model parameters are constrained by data sets, this means that more stringent constraints correspond to higher FoM; it is a very useful tool because can be used to compare how well the parameters are constrained in different experiments. The FoC measures the degree of correlation among parameters, the higher is the correlation (off-diagonal terms) the lower is the logarithm of the determinant[3], i.e. higher FoC. [101].

[2] If one diagonalize the covariance matrix then, it easy to see that, the determinant of this matrix, represents the logarithm of an error volume [101].

[3] With increasing correlations among parameters the determinant tend to zero and thus the logarithm tend to $-\infty$, the result is that the *FoC* tend to $+\infty$.

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