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Cohomology of Coherent Sheaves on Projective Schemes

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Declaration of Authorship

I, Phrador Sainhery , declare that this thesis titled, 'Cohomology of Coherent Sheaves on Projective Schemes ' and the work presented in it is my own. I confirm that this work submitted for assessment is my own and is expressed in my own words. Any uses made within it of the works of other authors in any form (e.g., ideas, equations, figures, text, tables, programs) are properly acknowledged at any point of their use. A list of the references employed is included.

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Abstract

Let X be a projective scheme of dimension n over a an algebraically closed field k and let $\mathcal{O}_X$ denote its structure sheaf. Let $\mathcal{F}$ be any coherent sheaf of $\mathcal{O}_X$-modules. In this essay, we wish to compute the sheaf cohomology $H^i(X, \mathcal{F})$, for $i \in \mathbb{Z}$. We will see the Serre's duality which asserts that the k-finite dimensional vector space H^{n-i} is dual to $\mathsf{Ext}^i(\mathcal{F}, \omega_X)$, for $i \geq 0$, where ω_X is the canonical sheaf on X. Applying this result to a smooth projective curves of genus g, we will be able to prove the Riemann–Roch theorem.

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To my Dad, Mum and Sister

Chapter 1

Introduction

One of the strongest problem in Algebraic Geometry is to classify all algebraic varieties. Having a numerical invariant attached to a given space could solve a part of this problem. One modern technique to define this invariant is by cohomology. The aim of this essay is to initiate ourself in the utilization of cohomology in Algebraic Geometry . The first mathematician who brought the notion of cohomology of sheaves into Algebraic Geometry is J-P. Serre in his paper [Serre, 1955] , where he used Čech cohomology. Though, there are many ways of defining cohomology. Notably, in this Master thesis, we will see the approach of Grothendieck [Grothendieck, 1957]. He defined the cohomology as the right derived functors of the global section: starting from a short exact sequence of sheaves of abelian groups, $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ the global section functor $\Gamma(X, -)$ gives a left exact sequence

$$0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}''),$$

As the functor $\Gamma(X, -)$ is not right exact in general. One may ask, how to continue this sequence to the right. For that, Grothendieck used the language of homological algebra: fix a topological space X and consider the cohomology as a functor from $\mathcal{A}\mathrm{b}(X)$, the category of sheaves of abelian groups to $\mathcal{A}\mathrm{b}$, the category of abelian group. From a standard result in homological algebra, the definition of derived funtors uses injective resolution, that is, every sheaf of abelian groups is embedded into an injective sheaf $\mathcal{I}$, and there is an exact sequence

$$0 \to \mathcal{I} \to \mathcal{I}^1 \to \mathcal{I}^2 \to \cdots$$

where the $\mathcal{I}^i$ are injective sheaves. Despite this definition from derived functor is well suited for theoretical side, computing cohomology from resolutions is almost impossible in practice. We should go back to Čech cohomology introduced by Serre: take an open affine covering $\mathcal{U}$ of the topological X. Define the Čech cohomology as the cohomology group of the explicit complex of abelian groups with pth group

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \ldots < i_p} \mathcal{F}(U_{i_0} \cap \ldots \cap U_{i_p}).$$

We will see that under some hypotheses on X and on the sheaf $\mathcal{F}$, the resulting cohomology group from derived functor and from the Čech coincide.
We give a look carefully on the study of cohomology of particular sheaves, which are quasi-coherent and coherent sheaves of $\mathcal{O}_X$-modules. That is the reason why we will start our study in Chapter II with quasi-coherent and coherent sheaves on projective space.
Chapter III is devoted for the introduction of cohomology of sheaves. There we introduce the general notion, then right after, we study a vanishing theorem of Grothendiek. We proceed with the result about the cohomology of affine noetherian scheme. We end the chapter with some computation of cohomology of projective space.
In Chapter IV, discussed about the Serre duality theorem for cohomology of coherent sheaves and the at last but not least we study the famous Riemann-Roch theorem.
The main reference for this essay is the book of [Hartshorne, 1977].

Chapter 2

Sheaf of $\mathcal{O}_X$-modules

2.1 Sheaves of Modules

In this section we will see some basic properties of particular sheaves of modules, quasi-coherent and coherent sheaves.

Definition 2.1. Let $(X, \mathcal{O}_X)$ be a ringed space. A *sheaf of* $\mathcal{O}_X$*-modules* or simply an $\mathcal{O}_X$*-module* is a sheaf $\mathcal{F}$ on X, such that for each open set $U \subseteq X$, the group $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and if $V \subseteq U$, then $(a \cdot f)|_V = a|_V \cdot f|_V$ for every $a \in \mathcal{O}_X(U)$ and $f \in \mathcal{F}(U)$. A morphism $\mathcal{F} \to \mathcal{G}$ of sheaves of $\mathcal{O}_X$-modules is a morphism of sheaves such that for each open set $U \subseteq X$, the map $\mathcal{F}(U) \to \mathcal{G}(U)$ is a homomorphism of $\mathcal{O}_X(U)$-modules.

Definition 2.2. Let $\mathcal{F}$ and $\mathcal{G}$ be two $\mathcal{O}_X$-modules we denote the group of morphisms from $\mathcal{F}$ to $\mathcal{G}$ by $\mathsf{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, or sometimes $\mathsf{Hom}_X(\mathcal{F}, \mathcal{G})$. A sequence of $\mathcal{O}_X$-modules and morphisms is *exact* if it is exact as a sequence of sheaves of abelian groups.

If U is an open subset of X, and if $\mathcal{F}$ is an $\mathcal{O}_X$-module, then $\mathcal{F}|_U$ is an $\mathcal{O}_X|_U$-module. If $\mathcal{F}$ and $\mathcal{G}$ are two $\mathcal{O}_X$-modules, the presheaf $U \mapsto \mathsf{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ is indeed a sheaf which we call the sheaf $\mathcal{H}om$ and denote by $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. It is also an $\mathcal{O}_X$-module.

We define the tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ of two $\mathcal{O}_X$-modules to be the sheaf associated to the presheaf

$$U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U).$$

Definition 2.3. An $\mathcal{O}_X$-module $\mathcal{F}$ is *free* it is isomorphic to a direct sum of copies of $\mathcal{O}_X$. It is *locally free* if X can be covered by open sets U for which $\mathcal{F}|_U$ is a free $\mathcal{O}_X|_U$-module. In that case the *rank* of $\mathcal{F}$ on such an open is the number of copies of the structure sheaf needed (finite or infinite). If X is connected, the rank of a locally free sheaf is the same everywhere. A locally free sheaf of rank 1 is called *invertible sheaf.*

A sheaf of ideals on X is a sheaf of modules $\mathcal{I}$, which is a subsheaf of $\mathcal{O}_X$, that is $\mathcal{I}(U)$ is an ideal in $\mathcal{O}_X(U)$ for any open set U.

Example 2.1. *Let A be a ring, and $I \subset A$ be an ideal. Let $X = \text{Spec A}$. By definition, $V(I) \subset X$ is a closed subset. We identify $V(I)$ as an affine subscheme $Y = \text{Spec } A/I$, so that we have a short exact sequence of sheaves of modules*

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0,$$

which is the analogue to $0 \to I \to A \to A/I \to 0$.

Definition 2.4. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces, let $\mathcal{F}$ be an $\mathcal{O}_X$-module and $\mathcal{G}$ be an $\mathcal{O}_Y$-module. Note that the direct image $f_*\mathcal{F}$ is an $f_*\mathcal{O}_X$-module. Together with the morphism $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings on Y, we have a natural $\mathcal{O}_Y$-module structure on $f_*\mathcal{O}_X$. We call it the *direct image* of $\mathcal{F}$ or the *pushforward* of $\mathcal{F}$ by f.

Now let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules. Then $f^{-1}\mathcal{G}$ is an $f^{-1}\mathcal{O}_Y$-module. Because of the adjoint property of f^{-1} we have a morphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ of sheaves of rings on X. We define $f^*\mathcal{G}$ to be the tensor product

$$f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

We call it *inverse image* of $\mathcal{G}$ by the morphism f.

Remark 2.5. For any $\mathcal{O}_X$-module $\mathcal{F}$ and any $\mathcal{O}_Y$-module $\mathcal{G}$, there is a natural isomorphism of groups

$$\mathsf{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \cong \mathsf{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F})$$

Definition 2.6. Let $(X, \mathcal{O}_X)$ be a ringed space. We say that an $\mathcal{O}_X$-module $\mathcal{F}$ is *generated by its global sections at* $x \in X$ if the canonical homomorphism $\mathcal{F}(X) \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x} \to \mathcal{F}_x$ is surjective. We say that it is *generated by its global sections* if this is true at every point of X. Let S be a subset of $\mathcal{F}(X)$, we say that $\mathcal{F}$ is generated by S if $\{s_x\}_{s \in S}$ generates $\mathcal{F}_x$ for every $x \in X$.

Lemma 2.7. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then an $\mathcal{O}_X$-module $\mathcal{F}$ is a generated by its global sections if and only if there exist a finite set I and a surjective homomorphism of $\mathcal{O}_X$-modules $\mathcal{O}_X^{(I)} \to \mathcal{F}$, where $\mathcal{O}_X^{(I)}$ is the direct sum of $\mathcal{O}_X$ indexed by I. Moreover if $\mathcal{F}$ is generated by a set S of global section, then we can take $I = S$.*

Proof. If there exists a surjective homomorphism $\mathcal{O}_X^{(I)} \to \mathcal{F}$, immediately $\mathcal{F}$ is generated by its global sections since $\mathcal{O}_X^{(I)}$ is. Now we suppose that $\mathcal{F}$ is generated by its global sections. Let S be a subset of $\mathcal{F}(X)$ which generates $\mathcal{F}$. Let $\{\varepsilon_s\}_{s \in S}$ be the canonical basis of $\mathcal{O}_X^{(S)}$. For any open set U of X, the $\varepsilon_s|_U$ form a basis of $\mathcal{O}_X(U)^{(S)}$ over $\mathcal{O}_X(U)$. Let us consider the morphism $\psi : \mathcal{O}_X(U)^{(S)} \to \mathcal{F}$ which is defined by

$$\psi(U) : \sum_{s\in S} f_s \cdot \varepsilon_s|_U \mapsto \sum_{s\in S} f_s \cdot s|_U,$$

for any $f_s \in \mathcal{O}_X(U)$. Then ψ_x is surjective for every $x \in X$ and therefore ψ is surjective. □

2.2 Quasi-coherent Sheaves on an Affine Scheme

Definition 2.8. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is *quasi-coherent* if for every $x \in X$, there is an open neighborhood U of x and an exact sequence of $\mathcal{O}_X$-modules

$$\mathcal{O}_X^{(J)}|_U \to \mathcal{O}_X^{(I)}|_U \to \mathcal{F}|_U \to 0$$

Example 2.2. *On any scheme X, the structure sheaf $\mathcal{O}_X$ is quasi-coherent.*

We now want to classify quasi-coherent sheaves on an affine scheme $X = \text{Spec A}$.

Definition 2.9. Let A be a ring and let M be an A-module. We define the *sheaf associated* to M on Spec A, denoted $\widetilde{M}$, as follows. For each prime $\mathfrak{p} \subseteq A$, let $M_\mathfrak{p}$ be the localization of M at $\mathfrak{p}$. For any open set $U \subseteq X$, we define the group $\widetilde{M}(U)$ to be the set of functions $s : U \to \amalg_{\mathfrak{p}\in U} M_\mathfrak{p}$ such that for each $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in M_\mathfrak{p}$, and such that s is locally a fraction m/f, with $m \in M$ and $f \in A$. To be precise, we require that for each $\mathfrak{p} \in U$, there is a neighborhood V of $\mathfrak{p}$ in U, and there are elements $m \in M$ and $f \in A$, such that for each $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = m/f$ in $M_\mathfrak{q}$. We make $\widetilde{M}$ into a sheaf by using the obvious restriction map.

Proposition 2.10. *Let A be a ring, let M be an A-module, and let $\widetilde{M}$ be the sheaf on $X = \text{Spec A}$ associated to M. Then :*

(a) $\widetilde{M}$ is an $\mathcal{O}_X$-module;

(b) for each $\mathfrak{p} \in X$, the stalk $(\widetilde{M})_\mathfrak{p}$ of the sheaf $\widetilde{M}$ at $\mathfrak{p}$ is isomorphic to $M_\mathfrak{p}$

(c) for any $f \in A$, the A_f-module $\widetilde{M}(D(f))$ is isomorphic to the localized module M_f. In particular $\Gamma(\widetilde{M}, X) = M$

Proof. Hartshorne. □

Lemma 2.11. *Let M be an A-module. Then $M = 0$ if and only if $M_{\mathfrak{m}} = 0$ for every maximal ideal $\mathfrak{m} \subseteq A$.*

Proof. Let $x \in M$. Let us consider the ideal $J = \{a \in A | \ ax = 0\}$. Now, if $J \neq A$, then there is a maximal ideal $\mathfrak{m}$ of$A$ such that $J \subseteq A$. Since $M_{\mathfrak{m}} = 0$, there exist an $s \in A \setminus \mathfrak{m}$ such that $sx = 0$. Hence $s \in J$, which contradicts the assumption $J \subseteq \mathfrak{m}$. This implies that $J = A$, and $1 \in J$. Thus $x = 0$. $\square$

Proposition 2.12. *Let $X = \text{Spec A}$ be an affine scheme. Then the following properties are true.*

(a) Let $\{M_i\}$ be a family of A-modules. Then $(\oplus_i M_i)^{\sim} \cong \oplus_i \widetilde{M_i}$

(b) A sequence of A-module $L \to M \to N$ is exact if and only if the associated sequence of $\mathcal{O}_X$-modules $\widetilde{L} \to \widetilde{M} \to \widetilde{N}$ is exact.

(c) For any A-module M, the sheaf $\widetilde{M}$ is quasi-coherent.

(d) Let M, N be two A-modules. Then we have a canonical isomorphisms

$$(M \otimes_A N)^{\sim} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$$

Proof. (a) is true by definition. (b) Suppose the sequence $L \to M \to N$ is exact. Let $\mathfrak{p} \in \text{Spec A}$, the sequence $L_{\mathfrak{p}} \to M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is exact since $A_{\mathfrak{p}}$ is flat over A. The sequence is exact on stalks, hence $\widetilde{L} \to \widetilde{M} \to \widetilde{N}$ is exact. Conversely, let us suppose that we have an exact sequence.

$$\widetilde{L} \xrightarrow{\alpha} \widetilde{M} \xrightarrow{\beta} \widetilde{N}. \tag{2.1}$$

For any prime ideal $\mathfrak{p} \in \text{Spec A}$, we have the following commutative diagram

$$\begin{array}{ccccc} L & \xrightarrow{\alpha(X)} & M & \xrightarrow{\beta(X)} & N \\ \downarrow & & \downarrow & & \downarrow \\ L_{\mathfrak{p}} & \longrightarrow & M_{\mathfrak{p}} & \longrightarrow & N_{\mathfrak{p}} \end{array}$$

Where the vertical arrows are the localization maps. The bottom row is exact since the sequence of sheaves is exact. We deduce from this that $(\ker \beta(X)/\text{Im}\alpha(X))_{\mathfrak{p}} = 0$. In particular, the latter is true for any maximal ideal $\mathfrak{p}$, thus by Lemma $\ker \beta(X) = \text{Im}\alpha(X)$ and the sequence

(c) follows from (a) and (b).

(d) Let $L = M \otimes_A N$. For any principal open subset $D(f)$ of X, we have a canonical isomorphism of $\mathcal{O}_X(D(f))$-modules.

$$\begin{aligned}\widetilde{L}(D(f)) = (M \otimes_A N) \otimes_A A_f &\cong (M \otimes_A A_f) \otimes_{A_f} (N \otimes_A A_f) \\ &\cong \widetilde{M}(D(f)) \otimes_{\mathcal{O}_X(D(f))} \widetilde{N}(D(f)).\end{aligned}$$

This is isomorphism is compatible with the restriction homomorphisms. Therefore, it induces an isomorphism of $\mathcal{O}_X$-modules $\widetilde{L} \cong \widetilde{M} \otimes \widetilde{N}$, because the principal open subsets form a base for the topology of X. □

Proposition 2.13. *Let $\mathcal{F}$ be a quasi-coherent sheaf on a scheme X. We suppose X is noetherian. Then for any $f \in \mathcal{O}_X(X)$ the canonical homomorphism*

$$\mathcal{F}(X)_f = \mathcal{F}(X) \otimes_{\mathcal{O}_X(X)} \mathcal{O}_X(X_f) \to \mathcal{F}(X_f),$$

where $X_f := \{x \in X |\ f_x \in \mathcal{O}^*_{X,x}\}$, is an isomorphism.

Proof. We first show that every point $x \in X$ has an affine open neighborhood U such that the canonical homomorphism $\mathcal{F}(U)^\sim \to \mathcal{F}|_U$ is an isomorphism. By our assumption on X, there exist an open affine neighborhood U of x and an exact sequence of $\mathcal{O}_X$-modules

$$\mathcal{O}_X^{(J)}|_U \to \mathcal{O}_X^{(I)}|_U \to \mathcal{F}|_U \to 0.$$

Let $M = \mathrm{Im}(\alpha(U))$. By Proposition 2.12 we have an exact sequence

$$\mathcal{O}_X^{(J)}|_U \to \mathcal{O}_X^{(I)}|_U \to \widetilde{M} \to 0,$$

which implies that $\mathcal{F}|_U \cong \widetilde{M}$ and we have $M = \widetilde{M}(U) = \mathcal{F}(U)$. As X is noetherian, we can cover X with a finite number of affine open subsets U_i such that $\mathcal{F}|_{U_i} \cong \mathcal{F}(U_i)^\sim$. Let $V_i = U_i \cap X_f = D(f|_{U_i})$. Then X_f is the union of the $V_i := U_i \cap X_f = D(f|_{U_i})$. To ease notation we still denote by f its restriction to any open subset of X. With $\mathcal{O}_X(U_i)_f = \mathcal{O}_X(V_i)$ and the well known exact sequence which characterizes a sheaf, we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F}(X)_f & \longrightarrow & \oplus_i \mathcal{F}(U_i)_f & \longrightarrow & \oplus_{i,j}\mathcal{F}(U_i \cap U_j)_f \\ & & \downarrow & & \downarrow{\scriptstyle\gamma} & & \downarrow \\ 0 & \longrightarrow & \mathcal{F}(X_f) & \longrightarrow & \oplus \mathcal{F}(V_i) & \longrightarrow & \oplus_{i,j}\mathcal{F}(V_i \cap V_j) \end{array}$$

where the horizontals rows are exact. The homomorphism γ is an isomorphism because $\mathcal{F}|_{U_i} \cong \mathcal{F}(U_i)^\sim$. Again we may apply the same reasoning to $U_i \cap U_j$ since X is noetherian, and get $\oplus_{i,j}\mathcal{F}(U_i \cap U_j)_f \cong \oplus_{i,j}\mathcal{F}(V_i \cap V_j)$. Coming back to our diagram, we have $\mathcal{F}(X)_f \to \mathcal{F}(X_f)$ is an isomorphism. □

Theorem 2.14. *Let X be a scheme, and $\mathcal{F}$ an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is quasi-coherent if and only if for every open affine subset U of X, we have $\mathcal{F}|_U \cong \mathcal{F}(U)^\sim$.*

Proof. Suppose $\mathcal{F}$ is quasi-coherent and let U be an affine open subset of X. For any $f \in \mathcal{O}_X(U)$, we have $\mathcal{F}(U)_f \cong \mathcal{F}(D(f))$ by Proposition 2.13. Thus $\mathcal{F}|_U \cong \mathcal{F}(U)^\sim$. Conversely, let $X = \bigcup_i U_i$ be an affine open covering of X. By hypothesis, we have $\mathcal{F}(U_i)^\sim \cong \mathcal{F}|_{U_i}$, for each i, this is nothing else but Proposition 2.12 (c). □

Remark 2.15. In the language of category, we may rephrase the above theorem as follows. If $X = \text{Spec A}$, the functor $M \mapsto \widetilde{M}$ induces an equivalence of categories between the category of A-modules and the category of quasi-coherent $\mathcal{O}_X$-modules.

Example 2.3. *If $X = \text{Spec A}$ is an affine scheme, if $Y \subseteq X$ is the closed subscheme defined by an ideal $\mathfrak{a} \subseteq A$, and if $i : Y \to X$ is the inclusion morphism, then $i_*\mathcal{O}_Y$ is a quasi-coherent $\mathcal{O}_X$-module, since it is isomorphic to $(A/\mathfrak{a})^\sim$.*

Example 2.4. *If U is an open subscheme of a scheme X, with inclusion map $j : U \to X$, then the ideal sheaf $j_!(\mathcal{O}_U)$ by extending $\mathcal{O}_U$ by zero outside of U, is an $\mathcal{O}_X$-module, but it is not in general quasi-coherent. For example, suppose X is integral, and $V = \text{Spec A}$ is any open affine subset of X, not contained in U. Then $j_!(\mathcal{O}_U)|_V$ has no global section over V, and yet it is not a zero sheaf. Hence it cannot be of the form $\widetilde{M}$ for any A-module M.*

Proposition 2.16. *Let X be an affine scheme. Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be an exact sequence of $\mathcal{O}_X$-modules with $\mathcal{F}$ quasi-coherent. Then the sequence*

$$0 \to \mathcal{F}'(X) \to \mathcal{F}(X) \to \mathcal{F}''(X) \to 0,$$

is exact.

2.3 Coherent Sheaves

Definition 2.17. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is finitely generated if for every $x \in X$, there exist an open neighborhood U of x, an integer $n \geq 1$ and a surjective homomorphism $\mathcal{O}_X^n|_U \to \mathcal{F}|_U$. We say that $\mathcal{F}$ is coherent if it is finitely generated, and if for every every open subset U of X, and for every homomorphism $\alpha : \mathcal{O}_X^n|_U \to \mathcal{F}|_U$, the kernel Ker$\alpha$ is finitely generated.

For simplicity, we will not mention coherent sheaves unless the scheme is noetherian.

Proposition 2.18. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let us consider the following properties:*

(i) $\mathcal{F}$ is coherent

(ii) $\mathcal{F}$ is finitely generated

(iii) for every affine open subset U of X, $\mathcal{F}(U)$ is finitely generated over $\mathcal{O}_X(U)$.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Moreover, if X is locally noetherian, then these properties are equivalent.

Proof. By definition, (i) implies (ii). Let us suppose $\mathcal{F}$ is finitely generated. Let U be an affine open subset of X. Then U can be covered with a finite number of principal open subsets U_i such that there exists an exact sequence $\mathcal{O}_X^n|_{U_i} \to \mathcal{F}|_{U_i} \to 0$. It follows that the sequence of $\mathcal{O}_X(U_i)$-modules $\mathcal{O}_X^n(U_i) \to \mathcal{F}(U_i) \to 0$ is exact. In particular, $\mathcal{F}(U_i)$ is finitely generated over $\mathcal{O}_X(U_i)$. Since $\mathcal{F}(U_i) = \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(U_i)$, there exists a finitely generated sub-$\mathcal{O}_X$-module M of $\mathcal{F}(U)$ such that $M \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(Ui) = \mathcal{F}(U_i)$. Enlarging $\mathcal{F}$, if necessary, we may suppose that this equality holds for every i. Then the sequence $\widetilde{M} \to \mathcal{F}|_U \to 0$ is then exact because it is exact on every U_i, consequently, $M \to \mathcal{F}(U)$ is surjective and (ii) implies (iii).

We now suppose (iii) is true and X is locally noetherian. We want to show that $\mathcal{F}$ is coherent. Let V be an open subset of X and $\alpha : \mathcal{O}_X^n|_V \to \mathcal{F}|_V$ a homomorphism. We need to show that $\mathrm{Ker}(\alpha)$ is finitely generated. A this is a local property we may assume that V is affine. Then $\mathcal{F}|_V = \widetilde{N}$. Then $\mathrm{Ker}(\alpha) = (\mathrm{Ker}\alpha(V))^\sim$ by Proposition 2.12 (b). Now $\mathrm{Ker}\alpha(V)$ is finitely generated because $\mathcal{O}_X(V)$ is noetherian. Therefore $\mathrm{Ker}(\alpha)$ is finitely generated and $\mathcal{F}$ is coherent as desired. □

Proposition 2.19. *Let X be a scheme, we have the following properties of $\mathcal{O}_X$.*

(a) A direct sum of quasi-coherent sheaves is quasi-coherent ; a finite direct sum of finitely generated quasi-coherent sheaves is finitely generated.

(b) If $\mathcal{F}, \mathcal{G}$ is are quasi-coherent (resp. finitely generated quasi-coherent) sheaves, then so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$. Moreover for any affine open subsetof X, we have

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})(U) = \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U).$$

(c) The kernel, cokernel, and image of any morphisms of quasi-coherent sheaves are quasi-coherent.

(d) *Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be an exact sequence of $\mathcal{O}_X$-modules if two of them are quasi-coherent, so is the third.*

(e) *If X is locally noetherian, then the properties (c) and (d) are true for coherent sheaves.*

Proof. (a)–(c) The question is local, so we may assume X is affine and the results can be deduced from Remark 2.15.

(d) The only difficult case in is that where $\mathcal{F}'$ and $\mathcal{F}''$ are quasi-coherent. We might suppose that X is affine by Proposition 2.16, the sequence $0 \to \mathcal{F}'(X) \to \mathcal{F}(X) \to \mathcal{F}''(X) \to 0$. So we have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}(X)^{\sim} & \longrightarrow & \mathcal{F}(X)^{\sim} & \longrightarrow & \mathcal{F}''(X)^{\sim} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}'' & \longrightarrow & 0 \end{array}$$

where the first and the last vertical arrows are isomorphisms since $\mathcal{F}$ and $\mathcal{F}''$ were assumed to be quasi-coherent sheaves. Therefore, the five-lemma implies that $\mathcal{F}(X)^{\sim} \to \mathcal{F}$ is an isomorphism. (e) is trivially true. □

Proposition 2.20. *Let $X \to Y$ be a morphism of schemes.*

(a) *Let $\mathcal{G}$ be an $\mathcal{O}_Y$. Then for any $x \in X$, se have the canonical isomorphism*

$$(f^*\mathcal{G})_x \cong \mathcal{G}_{f(x)} \otimes \mathcal{O}_{X,x}.$$

(b) *Let us suppose $\mathcal{G}$ is quasi-coherent. Let U be a affine open subset of X such that $f(U)$ is contained in an affine open subset V of Y. Then*

$$f^*\mathcal{G}|_U \cong (\mathcal{G}(V) \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U)^{\sim}.$$

In paritcular, $f^*\mathcal{G}$ is quasi-coherent. It is finitely generated if $\mathcal{G}$ is finitely generated.

(c) *Let $\mathcal{F}$ be a quasi-coherent sheaf on X. If X is noetherian, or f is separated and quasi-compact then $f_*\mathcal{F}$ is quasi-coherent on Y.*

(d) *If f is finite and $\mathcal{F}$ quasi-coherent and finitely generated, then $f_*\mathcal{F}$ is quasi-coherent and finitely generated on Y.*

Proof. (a) We have that $f^{-1}\mathcal{G} = \mathcal{G}_{f(x)}$, so on stalks we have $(f^*\mathcal{G})_* \cong \mathcal{G}_{f(x)} \otimes \mathcal{O}_{X,x}$ as desired.

(b) Let $g : U \to V$ be the restriction of f to U. Then $f^*\mathcal{G}|_U = g^*\mathcal{G}|_V$. We may therefore assume $X = U = \text{Spec B}$ and $Y = V = \text{Spec A}$. We note that the property is obvious if $\mathcal{G} = \mathcal{O}_Y^{(I)}$ since we have $f^*\mathcal{O}_Y = \mathcal{O}_X$ and we f^* commutes direct sums. So we have $f^*\mathcal{O}_Y^{(I)} = (f^*\mathcal{O}_Y)^{(I)} = \mathcal{O}_X^{(I)} = (B^{(I)})^\sim$. For the general case, we have an exact sequence

$$K \to L \to \mathcal{G}(Y) \to 0,$$

with K, L are free modules over A. We then have exact sequences

$$\widetilde{K} \to \widetilde{L} \to \mathcal{G} \to 0, \text{ and } f^*\widetilde{K} \to f^*\widetilde{L} \to f^*\mathcal{G} \to 0.$$

Since β is associated to $\alpha_B : K \otimes_A B \to L \otimes B$, we have

$$f^*\mathcal{G} = \text{Coker } \beta = ((\text{Coker } \alpha) \otimes_A B)^\sim = (\mathcal{G}(Y)\otimes)B)^\sim,$$

and this proves (b).

(c) We may assume that Y is affine since the property wanted is of local nature on Y. For any $g \in \mathcal{O}_Y(Y)$, let g' denote its image in $\mathcal{O}_X(X)$. We have

$$f_*\mathcal{F}(D(g)) = \mathcal{F}(f^{-1}D(g)) = \mathcal{F}(D(g')) \overset{\text{Prop.2.13}}{\cong} \mathcal{F}(X)_{g'} = \mathcal{F}(X) \otimes_{\mathcal{O}_Y(Y)} \mathcal{O}_Y(D(g)).$$

(d) We may assume Y is affine. As f is affine, hence separated and quasi-compact, we just have seen that $f_*\mathcal{F} = \mathcal{F}(X)^\sim$. Since X is affine and finite over Y, $\mathcal{F}(X)$ is finitely generated over $\mathcal{O}_X(X)$ and consequently finitely generated over $\mathcal{O}_Y(Y)$. □

Definition 2.21. Let Y be a closed subscheme of X, and let $i : Y \to X$ be the inclusion morphism. We define the *ideal sheaf* of Y, denoted $\mathcal{I}_Y$, to be the kernel of the morphism $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Y$.

Proposition 2.22. *Let X be a scheme. For any closed subscheme Y of X, the corresponding ideal sheaf $\mathcal{I}_Y$ is quasi-coherent sheaf of ideals on X. Conversely, any quasi-coherent sheaf of ideals on X is the ideal sheaf of a uniquely determined closed subscheme of X.*

Proof. If Y is a closed subscheme of X, then the inclusion morphism $i : Y \to X$ is quasi-compact and separated so we apply Proposition 2.20 and thus $i^*\mathcal{O}_Y$ is quasi-coherent on X. Hence $\mathcal{I}_Y$, being the kernel of morphism of quasi-coherent sheaves, is quasi-coherent by Proposition 2.19. Idea of the converse: given a scheme X and a quasi-coherent sheaf of ideals $\mathcal{I}$, let Y be the support of the quotient sheaf $\mathcal{O}_X/\mathcal{I}$. Then Y is a subspace of X and $(Y, \mathcal{O}_X/\mathcal{I})$ is the unique closed subscheme of X with ideal sheaf $\mathcal{I}$. Need to check that Y is unique and $(Y, \mathcal{O}_X/\mathcal{I})$ is a closed subscheme with. This is a local question, then we may assume $X =$ Spec A. Since $\mathcal{I}$ is quasi-coherent, $\mathcal{I} = \tilde{\mathfrak{a}}$, for some ideal $\mathfrak{a}$. Then $(Y, \mathcal{O}_X/\mathcal{I})$ is just the closed subscheme of X determined by the ideal $\mathfrak{a}$. □

2.4 Quasi-coherent Sheaves on a Projective Scheme

Definition 2.23. Let S be a graded ring and let M be a graded S-module. We define the sheaf associated to M on Proj S, denoted by $\widetilde{M}$, as follows. For each $\mathfrak{p} \in$ Proj S, let $M_{(\mathfrak{p})}$ be the group of elements of degree 0 in localization $T^{-1}S$, where T is the multiplicative system of homogeneous element of S not in $\mathfrak{p}$. For any open subset $U \in$ Proj S, we define $\widetilde{M}(U)$ to be the set of functions $s : U \to \amalg_{\mathfrak{p}\in U} M_{(\mathfrak{p})}$ which are locally fractions. This means that for every $\mathfrak{p} \in U$, there is a neighborhood V of $\mathfrak{p}$ in U, and homogeneous elements $m \in M$ and $f \in S$ of the same degree, such that for every $\mathfrak{q} \in V$, we have $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = m/f$ in $M_{(\mathfrak{q})}$. We make $\widetilde{M}$ into a sheaf with the restriction maps.

Proposition 2.24. *Let S be a graded ring, and M a graded S-module. Let $X =$ Proj S.*

(a) For any $\mathfrak{p} \in X$, the stalk $\widetilde{M}_{(p)}$

(b) For any homogeneous $f \in S_+$, we have $\widetilde{M}|_{D_+(f)} \cong (M_{(\mathfrak{f})})^\sim$, via the isomorphism $D_+(f)$ with Spec $A_{(f)}$*, where $M_{(f)}$ denotes the group of elements of degree* 0 *in the localized module M_f.*

(c) $\widetilde{M}$ is a quasi-coherent $\mathcal{O}_X$-module. If S is noetherian and M is finitely generated , then $\widetilde{M}$ is coherent.

If $\varphi : M \to N$ is a morphism of graded S-modules then $m/s \mapsto \varphi(m)/s$ defines a morphism of $S_{(\mathfrak{p})}$-modules $\varphi_{(\mathfrak{p})} : M_{(\mathfrak{p})} \to N_{(\mathfrak{p})}$. We now give the following result on homogeneous localization that could be useful later.

Lemma 2.25. *Let S be a graded ring, and suppose we have an exact sequence of graded S-modules $M \to N \to L$. Then for any $\mathfrak{p} \in$ Proj S, the sequence $M_{(\mathfrak{p})} \to N_{(\mathfrak{p})} \to L_{(\mathfrak{p})}$ of $S_{(\mathfrak{p})}$-module is exact.*

Proof. Let $\varphi : M \to N$ and $\psi : N \to L$ be morphisms of S-modules forming the exact sequence. Clearly Im $\varphi_{(\mathfrak{p})} \subseteq \ker \psi_{(\mathfrak{p})}$. Now suppose n, t are homogeneous of the same degree k with $n \in N$ and $t \notin \mathfrak{p}$ such that $\psi(n)/t = \psi_{(\mathfrak{p})}(n/t) = 0$. This implies that $q\psi(n) = 0$ for some homogeneous $q \notin \mathfrak{p}$ say of degree j. Thus $\psi(qn) = 0$, $qn = \varphi(m)$ for some m of degree $j + k$. Then in $N_{(\mathfrak{p})}$, we have $n/t = qn/qt = \varphi(m)/qt = \varphi_{(\mathfrak{p})}(m/qt)$. Hence Im $\varphi_{(\mathfrak{p})} \supseteq \ker \psi_{(\mathfrak{p})}$ and the sequence of S-modules $M_{(\mathfrak{p})} \to N_{(\mathfrak{p})} \to L_{(\mathfrak{p})}$ is exact. □

Definition 2.26. Let S be a graded ring, and let $X = \text{Proj S}$. For any $n \in \mathbb{Z}$ we define the sheaf $\mathcal{O}_X(n)$ to be $S(n)^\sim$. We call $\mathcal{O}_X(1)$ the *twisting sheaf of Serre*. For any sheaf of $\mathcal{O}_X$-modules, we denote by $\mathcal{F}(n)$ the *twisted sheaf* $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

Proposition 2.27. *Let S be a graded ring and let $X = \text{Proj S}$. Assume that S is generated by S_1 as S_0-algebra.*

(a) The sheaf $\mathcal{O}_X(n)$ is an invertible sheaf on X.

(b) For any graded S-module M, $\widetilde{M}(n) \cong (M(n))^\sim$. In particular, $\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) \cong \mathcal{O}_X(n+m)$.

(c) Let T be another graded ring, generated by T_1 as T_0-algebra, let $\varphi : S \to T$ be a homomorphism preserving degrees and let $U \subseteq Y = \text{Proj T}$ and $f : U \to X$ be the morphism determined by φ. Then $f^*(\mathcal{O}_X(n)) = \mathcal{O}_X(n)|_U$.

Definition 2.28. Let S be graded ring, let $X = \text{Proj S}$ and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We define the graded S-module associated to $\mathcal{F}$ as a group, to be $\bigoplus_{n\in\mathbb{Z}} \Gamma(X, \mathcal{F}(n))$. We give it a structure of graded S-module as follows. If $s \in S_d$, then s determines in a natural way a global section $s \in \Gamma(X, \mathcal{O}_X(d))$. Then any $t \in \Gamma(X, \mathcal{F}(n))$ we define the product $s \cdot t$ in $\Gamma(X, \mathcal{F}(n+d))$ by taking the tensor product $s \otimes t$ and using the natural $\mathcal{F}(n) \otimes \mathcal{O}_X(d) \cong \mathcal{F}(n+d)$.

Proposition 2.29. *Let A be a ring, let $S = A[T_0, ..., T_r]$, $r \geq 1$, and let $X = \text{Proj S}$. Then $\bigoplus_{n\in\mathbb{Z}} \mathcal{O}_X(n) \cong S$.*

Lemma 2.30. *Let X be noetherian or separated and quasi-compact scheme, let $\mathcal{F}$ be a quasi-coherent sheaf on X, and let $\mathcal{L}$ an invertible sheaf on X. Let us fix a section $f \in \mathcal{F}(X)$ and $s \in \mathcal{L}(X)$.*

(a) If $f|_{X_s} = 0$, then there exists an $n \geq 1$ such that $f \otimes s^n = 0$ in $\mathcal{F} \otimes \mathcal{L}^n(X)$.

(b) Let $g \in \mathcal{F}(X_s)$. Then there exists an $n_0 \geq 1$ such that $g \otimes (s^n|_{Xs})$ lifts to a section of $\mathcal{F} \otimes \mathcal{L}^n(X)$

Proposition 2.31. *Let A be a ring $S = A[T_0, ..., T_r]$. Let $X = \text{Proj } S$. Then for any quasi-coherent sheaf $\mathcal{F}$ on X, there is a natural isomorphism of graded S-modules*

$$(\bigoplus_{n\in\mathbb{Z}} \Gamma(X, \mathcal{F}(n)))^\sim \cong \mathcal{F}.$$

Definition 2.32. For any scheme Y, we define the *twisting sheaf* $\mathcal{O}_{\mathbb{P}^r_Y}(1)$ on $\mathbb{P}^r_Y$ to be $g^*(\mathcal{O}(1))$, where $g : \mathbb{P}^r_Y \to \mathbb{P}^r_{\mathbb{Z}}$ is the natural map coming from the fiber product $\mathbb{P}^r_Y = \mathbb{P}^r_{\mathbb{Z}} \times_{\mathbb{Z}} Y$.

Definition 2.33. For any scheme Y, we define the *twisting sheaf* $\mathcal{O}_{\mathbb{P}^r_Y}(1)$ on $\mathbb{P}^r_Y$ to be $g^*(\mathcal{O}(1))$, where $g : \mathbb{P}^r_Y \to \mathbb{P}^r_{\mathbb{Z}}$ is the natural map coming from the fiber product $\mathbb{P}^r_Y = \mathbb{P}^r_{\mathbb{Z}} \times_{\mathbb{Z}} Y$.

Definition 2.34. If X is a scheme over Y, an invertible sheaf $\mathcal{L}$ is *very ample* relative to Y, if there is an immersion $i : X \to \mathbb{P}^r_Y$ for some r, such that $i^*(\mathcal{O}_{\mathbb{P}^r_Y}(1)) \cong \mathcal{L}$. A morphism $i : X \to Z$ is an *immersion* if it gives an isomorphism of X with an open subscheme of a closed subscheme of Z. We recall also that an invertible of $\mathcal{O}_X$-module is a locally free sheaf of rank one.

Definition 2.35. Let $\mathcal{E}$ be a locally free sheaf of a ringed space $(X, \mathcal{O}_X)$. The sheaf $\check{\mathcal{E}} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ is called the dual of $\mathcal{E}$.

Lemma 2.36. *Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module of finite rank. Let $\check{\mathcal{E}}$ denote the dual of $\mathcal{E}$. Then we have*

(a) An isomorphism $\check{\check{\mathcal{E}}} \cong \mathcal{E}$;

(b) For any $\mathcal{O}_X$-module $\mathcal{F}$, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \cong \check{\mathcal{E}} \otimes_{\mathcal{O}_X} \mathcal{F}$.

Proof. To prove (a), we define the map φ by

$$\varphi(U) : \mathcal{E}(U) \to \check{\check{\mathcal{E}}}(U) = \mathsf{Hom}(\mathcal{H}om(\mathcal{E}, \mathcal{O}_X)|_U, \mathcal{O}_X|_U)$$

which sends a section $e \in \mathcal{E}(U)$ the collection of maps

$$\{e(V)\}_V : \mathsf{Hom}(\mathcal{E}|_{U\cap V}, \mathcal{O}_X|_{U\cap V}) \to \mathcal{O}_X|_{U\cap V},$$

with $e(V)(\sigma) = \sigma(U \cap V)(e|_{U\cap V})$, where $\sigma : \mathcal{E}|_{U\cap V} \to \mathcal{O}_X|_{U\cap V}$. One checks on stalks, $\mathsf{Hom}(\mathcal{E}, \mathcal{O}_X)_P$ is $\mathsf{Hom}(\mathcal{E}_P, \mathcal{O}_{X,P})$, see for example [Serre, 1955]"Faisceaux algébriques cohérentes". Then the morphism φ induces the stalk morphism $\mathcal{E}_P \to \mathsf{Hom}(\mathsf{Hom}(\mathcal{E}_P, \mathcal{O}_{X,P}), \mathcal{O}_{X,P})$, which is given by $e_P \mapsto (\sigma_P \mapsto \sigma_P(e_P))$. Since $\mathcal{E}$ is locally free of finite rank, the stalk $\mathcal{E}_P$ is free of finite rank and the stalk map is the canonical isomorphism of free module of finite rank with its double dual. This shows the required isomorphism $\check{\check{\mathcal{E}}} \cong \mathcal{E}$.
For (b) we define the map α by $\alpha(U) : \mathcal{H}om(\mathcal{E}|_U, \mathcal{O}_X|_U) \otimes \mathcal{F}(U) \to \mathcal{H}om(\mathcal{E}|_U, \mathcal{F}|_U)$ such that $\psi \otimes f \mapsto (\psi \otimes f)(V)(e) = \psi(V)(e) \cdot f|_V \in \mathcal{F}(U \cap V)$, where $e \in \mathcal{E}(U \cap V)$. Now on stalks, φ_P is the map $\mathsf{Hom}(\mathcal{E}_P, \mathcal{O}_{X,P}) \otimes \mathcal{F}_P \to \mathsf{Hom}(\mathcal{E}_P, \mathcal{F}_P)$, given by $\psi \otimes f \mapsto \psi(e) \cdot f$ is an isomorphism of $\mathcal{O}_{X,P}$-modules, since $\mathcal{E}_P$ is a free $\mathcal{O}_{X,P}$-module. This shows (b). □

Lemma 2.37 (Projection formula)**.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module, $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module of finite rank. Then there* *is a natural isomorphism*

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}) \cong f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E}.$$

Proof. From part(a) of the previous lemma, we have the identification

$$f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E} \cong f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \check{\check{\mathcal{E}}}. \tag{2.2}$$

On the other hand, we have $\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F})$ Hartshorne [1977], for any $\mathcal{O}_Y$-module $\mathcal{G}$. We let $\mathcal{G} = \check{\mathcal{E}}$. By patching together over the opens of Y, we obtain an isomorphism of sheaves of $\mathcal{O}_Y$-modules.

$$\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}) \cong f_*\mathcal{H}om_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F})$$

Then we get,

$$\begin{aligned} f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E} &\cong \mathcal{H}om_{\mathcal{O}_Y}(\check{\mathcal{E}}, f_*\mathcal{F}) \\ &\cong f_*\mathcal{H}om_{\mathcal{O}_X}(f^*\check{\mathcal{E}}, \mathcal{F}) \\ &\cong f_*((f^*\check{\mathcal{E}})\check{} \otimes \mathcal{F}), \end{aligned}$$

where the first isomorphism follows from the previous Lemma part (b), the second one comes from relation (2.2) and the last uses again part (b) of the previous lemma. Now, one can check that $(f^*\check{\mathcal{E}})\check{} \cong f^*(\check{\check{\mathcal{E}}})$, and we have the result as desired. $\square$

Theorem 2.38 (Serre)**.** *Let X be a projective scheme over a noetherian ring A, let $\mathcal{O}_X(1)$ be an ample invertible sheaf on X and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then there is an integer n_0 such that for all $n \geq n_0$, the sheaf $\mathcal{F}(n)$ can be generated by a finite global sections.*

Proof. Since X is a projective scheme, we let $i : X \to \mathbb{P}^r_A$ be a closed immersion of X into a projective space over A, such that $i^*(\mathcal{O}_{\mathbb{P}^n}) = \mathcal{O}_X(1)$. Then $i_*\mathcal{F}$ is a coherent $\mathcal{O}_X$-module on $\mathbb{P}^r_A$, and moreover $i_*(\mathcal{F}(n)) = (i_*\mathcal{F})(n)$, and $\mathcal{F}(n)$ is generated by global sections if and only if $i_*(\mathcal{F}(n))$ is because $i_*\mathcal{F}(n)(\mathbb{P}^r_A) = \mathcal{F}(n)(X)$. So we reduce to the case $X = \mathbb{P}^r_A =$Proj$A[T_0, ..., T_r]$.

Now cover $X = \bigcup D_+(T_i)$. Since $\mathcal{F}$ is coherent, for each $i = 0, ..., r$ there is a finitely generated module M_i over $B_i = A[\frac{T_0}{T_i}, ..., \frac{T_r}{T_i}]$ such that $\mathcal{F}|_{D_+(T_i)} = \tilde{M}_i$. For each i, take a finite number of elements $s_{ij} \in M_i$ which generate this module. By a lemma we have seen before, there is an integer n such that $T_i^n \otimes s_{ij}$ extends to a global section t_{ij} of $\mathcal{F}(n)$. We can take one n to work for all i, j because they are finite. Now $\mathcal{F}(n)$ corresponds to a B_i-module M_i' on $D_+(T_i)$, and the map $T_i^n : \mathcal{F} \to \mathcal{F}(n)$ induces an isomorphism of M_i to M_i', so the sections $T_i^n \otimes s_{ij}$ generate M_i' and therefore the global sections $t_{ij} \in \mathcal{F}(n)(X)$ generate the sheaf $\mathcal{F}(n)$ everywhere. $\square$

Corollary 2.39. *Let X be projective over a noetherian ring A. Then any coherent sheaf $\mathcal{F}$ on X can be written as quotient of sheaf $\mathcal{E}$, where $\mathcal{E}$ is a finite direct sum of twisted structure sheaves $\mathcal{O}_X(n_i)$ for various integers n_i.*

Chapter 3

Cohomology of Coherent Sheaves

3.1 Derived functors and Cohomology

3.1.1 Derived Functors

We review some techniques of homological algebra in order to be able to define sheaf cohomology using derived functors of the global section functor.

Definition 3.1. An abelian category is a category $\mathcal{A}$, such that: for each $A, B \in \mathrm{Ob}\mathcal{A}$, $\mathsf{Hom}(A, B)$ has a structure of an abelian group, and the composition law is linear; finite direct sums exists; every monomorphisms is the kernel of its cokernel, every epimorphism is the cokernel of its kernel; and every morphism can be factored into an epimorphism followed by a monomorphism.

Example 3.1. *The following are all abelian categories.*

- $\mathcal{A}\mathrm{b}$*, the category of abelian groups.*
- *A-Mod, the category of modules over a ring A.*
- $\mathcal{A}\mathrm{b}(X)$*, the category of sheaves of abelian groups.*
- $\mathcal{M}\mathrm{od}(X)$*, the category of sheaves of $\mathcal{O}_X$-modules, on a ringed space $(X, \mathcal{O}_X)$.*

Definition 3.2. An object I of a category $\mathcal{A}$ is said to be *injective* if the functor $\mathsf{Hom}(-, I)$ is exact.

Definition 3.3. An abelian category $\mathcal{A}$ is said to *have enough injectives* if each object of $\mathcal{A}$ can be embedded in an injective object.

This is equivalent to saying that each object A of $\mathcal{A}$ admits an *injective resolution*, that is, a long exact sequence

$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots$$

where each I^i is injective. To see this, first embed A in an injective object I^0, then embed the cokernel of the inclusion $\varepsilon : A \to I^0$ in an injective I^1, and take for $I^0 \to I^1$ the composite $I^0 \to$ cokernel $\varepsilon \to I^1$, and so on.

Definition 3.4. A *complex* $A^\bullet$ in an abelian category $\mathcal{A}$ is a collection of objects A^i of $\mathcal{A}$, $i \in \mathbb{Z}$, together with morphisms $d^i : A^i \to A^{i+1}$ such that $d^{i+1} \circ d^i = 0$ for all i. The maps d^i are called the *differentials* of the complex $A^\bullet$.

The i-th *cohomology object* of the complex $A^\bullet$ is defined by

$$h^i(A^\bullet) = \ker d^i / \mathrm{Im}\ d^{i-1}$$

A *morphism* of complexes $f : A^\bullet \to B^\bullet$ is a collection of maps $f^i : A^\bullet \to B^\bullet$ which commutes with the differentials, i.e. that make the following diagram commutative.

$$\begin{tikzcd}
\cdots \arrow[r] & A^{i-1} \arrow[r, "d^{i-1}"] \arrow[d, "f^{i-1}"] & A^i \arrow[r, "d^i"] \arrow[d, "f^i"] & A^{i+1} \arrow[r] \arrow[d, "f^{i+1}"] & \cdots \\
\cdots \arrow[r] & B^{i-1} \arrow[r, "d^{i-1}"] & B^i \arrow[r, "d^i"] & B^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

Any such morphism induces a morphism

$$h^i(f) : h^i(A^\bullet) \to h^i(B^\bullet)$$

on the cohomology, defined by $h^i(f)(a + \mathrm{Im}\ d^i) := f^i(a) + \mathrm{Im}\ d^i$. Thus one may think of h^i as a functor on the category of complexes in $\mathcal{A}$.

Proposition 3.5. *Let $\mathcal{A}$ be an abelian category and $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ a short exact sequence of complexes in $\mathcal{A}$. Then there are natural maps $\delta^i : h^i(C^\bullet) \to h^{i+1}(A^\bullet)$ giving rise to a long exact sequence*

$$\cdots \to h^i(A^\bullet) \to h^i(B^\bullet) \to h^i(C^\bullet) \xrightarrow{\delta^i} h^{i+1}(A^\bullet) \to \cdots$$

Proof. [Weibel, 1995] (Th. 1.3.1) □

Definition 3.6. A covariant functor $F : \mathcal{A} \to \mathcal{B}$ from one abelian category to another is *additive* if for any two objects A, A' in $\mathcal{A}$, the induced map $\mathsf{Hom}(A, A') \to \mathsf{Hom}(FA, FA')$

is a homomorphism of abelian groups. F is left exact functor if it is additive and every short exact sequence

$$0 \to A' \to A \to A'' \to 0$$

in $\mathcal{A}$, the sequence

$$0 \to F(A') \to F(A) \to F(A'').$$

is exact in $\mathcal{B}$.

Definition 3.7. Let $\mathcal{A}$ be an abelian category with enough injectives, and let $F : \mathcal{A} \to \mathcal{B}$ be a covariant left exact functor. For each object A of $\mathcal{A}$, choose once and for all an injective resolution $I^\bullet$ of A. Then the *right derived functors* R^iF of F is defined by $R^iF(A) = h^i(FI^\bullet)$.

Theorem 3.8. *Let A be an abelian category with enough injectives, and let $F : \mathcal{A} \to \mathcal{B}$ be a covariant left exact functor to another abelian category $\mathcal{B}$. Then*

(a) For each $i \geq 0$, R^iF is an additive functor from $\mathcal{A}$ to $\mathcal{B}$. Furthermore, it is independent of the choices of injective resolutions made.

(b) There is a natural isomorphism $F \cong R^0F$

(c) For each short exact sequence $0 \to A' \to A \to A'' \to 0$ and for each $i \geq 0$ there is a natural morphism $\delta^i : R^iF(A'') \to R^{i+1}F(A')$, such that we obatain a long exact sequence

$$\cdots \to R^iF(A') \to R^iF(A) \to R^iF(A'') \xrightarrow{\delta^i} R^{i+1}F(A') \to R^{i+1}F(A) \to \cdots$$

(d) For each injective object I of $\mathcal{A}$, and for each $i > 0$, $R^iF(I) = 0$.

Definition 3.9. With $F : \mathcal{A} \to \mathcal{B}$ as in the theorem, an object J is acyclic for F if $R^iF(J) = 0$ for all $i > 0$.
In particular, if $0 \to A \to B \to C \to 0$ is exact and A is acyclic for F, then $0 \to FA \to FB \to FC \to 0$ is exact. The above theorem says that injectives are acyclic for any left exact functor.

Proposition 3.10. *If $0 \to A \to J^\bullet$ is an F-acyclic resolution of A, i.e., each J^i is acyclic for F, $i \geq 0$. Then there is a natural isomorphism $R^iF(A) \cong h^i(F(J^\bullet))$ for each $i \geq 0$.*

Proof. [Hilton and Stammbach, 1971](IV. Th. 4,4) □

3.1.2 Sheaf Cohomology as Derived Functors

Now we want to define the cohomology groups of a sheaf by taking the the right derived functors of the global section functor. Then, we must first ensure that the global section functor $\Gamma(X, -)$ is left exact and that the category $\mathcal{A}\mathrm{b}(X)$ has enough injectives.

Proposition 3.11. *The global section functor* $\Gamma(X, -)$ *is left exact.*

Proof. Let $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$ be a short sequence of $\mathcal{O}_X$-modules. We want to show that $0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X)$ is exact. At first place we have $\ker \beta(X) = (\ker \alpha)(X) = 0$ since $\ker \alpha = 0$. In the middle it is clear that $\mathrm{Im}\ \alpha(X) \subseteq \ker \beta(X)$. Let $s \in \mathcal{G}(X)$ which maps to 0 in $\mathcal{H}$, by definition s is local section of $\mathcal{F}$. But $\mathcal{F}$ is sheaf on X, it implies that s is a section of $\mathcal{F}(X)$. □

Proposition 3.12. *If A is a ring, then the category A-Mod of A-modules has enough injectives.*

Proof. [Hilton and Stammbach, 1971] (I.8.3) □

Proposition 3.13. *If $(X, \mathcal{O}_X)$ is a ringed space, then the category $\mathcal{M}\mathrm{od}(X)$ of $\mathcal{O}_X$-modules has enough injectives.*

Proof. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. For each point $x \in X$, let $\mathcal{F}_x$ denote the stalk of $\mathcal{F}$ at x. Fix an inclusion $\mathcal{F}_x \hookrightarrow I_x$ of $\mathcal{F}_x$ in an injective $\mathcal{O}_{X,x}$ module. For each point, let $j_x : \{x\} \to X$ denote the inclusion. Consider $\mathcal{I} = \prod_{x \in X}(j_x)_*(I_x)$, Here we consider I_x as a sheaf one the single point $\{x\}$. Then we have an inclusion $\mathcal{F} \hookrightarrow \mathcal{I}$. We have to show now that $\mathsf{Hom}_{\mathcal{O}_X}(-, \mathcal{I})$ is an exact functor. Let $\mathcal{G}$ be an $\mathcal{O}_X$-module. Using the adjointness property, we have

$$\mathsf{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{I}) = \prod_{x \in X} \mathsf{Hom}_{\mathcal{O}_X}(\mathcal{G}, (j_x)_* I_x) = \prod_{x \in X} \mathsf{Hom}_{\mathcal{O}_{X,x}}((j_x)^*(\mathcal{G}), I_x) = \prod_{x \in X} \mathsf{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, I_x)$$

This implies that $\mathsf{Hom}_{\mathcal{O}_X}(-, \mathcal{I})$ is exact, as a direct product of the stalk functor followed by $\mathsf{Hom}(-, I_x)$, which is exact because I_x is an injectives $\mathcal{O}_{X,x}$-module. This shows that $\mathcal{I}$ is an injective $\mathcal{O}_X$-module. □

Corollary 3.14. *The category $\mathcal{A}\mathrm{b}(X)$ of sheaves of abelian groups has enough injectives.*

Definition 3.15. Let X be a topological space. Let $\Gamma(X, -)$ be the global section functor from $\mathcal{A}\mathrm{b}(X)$ to $\mathcal{A}\mathrm{b}$. We define the *cohomology functors* $H^i(X, -)$ to be the right derived functors of $\Gamma(X, -)$. For any sheaf $\mathcal{F}$ on X, the groups $H^i(X, \mathcal{F})$ are the *cohomology groups* of $\mathcal{F}$.

Definition 3.16. A sheaf of abelian groups $\mathcal{F}$ on a topological space X is called *flasque* if for any inclusion $V \subseteq U$, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.

Proposition 3.17. *If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of sheaves and $\mathcal{F}'$ is flasque, then for each open $U \subseteq X$, the sequence $0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U) \to 0$ is exact.*

Lemma 3.18. *If $(X, \mathcal{O}_X)$ is a ringed space, any injective $\mathcal{O}_X$-module is flasque.*

Proposition 3.19. *If $\mathcal{F}$ is a flasque sheaf on a topological space X, then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.*

Proof. Since the category of sheaves of abelian groups $\mathcal{A}\mathrm{b}(X)$ has enough injectives, we embed $\mathcal{F}$ into an injective $\mathcal{I}$, and let $\mathcal{G}$ be the quotient:

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0.$$

Now $\mathcal{F}$ is flasque by hypothesis and $\mathcal{I}$ is flasque by Lemma 3.18, it follows from.. that $\mathcal{G}$ is also flasque. Then, from Proposition 3.17 we have an exact sequence

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{I}) \to \Gamma(X, \mathcal{G}) \to 0.$$

On the other hand, since $\mathcal{I}$ is injective, $H^i(X, \mathcal{I}) = 0$ for all $i > 0$ by Theorem 3.8.(e). Taking the long exact sequence of cohomology, we have $H^1(X, \mathcal{F}) = 0$ and $H^i(X, \mathcal{F}) \cong H^i(X, \mathcal{G})$ for each $i \geq 2$. But $\mathcal{G}$ is also flasque, we continue by induction on i and get the result. $\square$

Proposition 3.20. *Let $(X, \mathcal{O}_X)$ be a ringed space, then the derived functors of $\Gamma(X, -)$ coincide with the cohomology functors $H^i(X, -)$.*

Proof. We consider Γ as a functor from $\mathcal{A}\mathrm{b}(X)$ to $\mathcal{A}\mathrm{b}$. To calculate the derived functors of $\Gamma(X, -)$ on $\mathcal{M}\mathrm{od}(X)$, we use the injective resolution. But we saw in Lemma 3.18 that injective is flasque , and flasques are acyclic for the functor $\Gamma(X, -)$. So by Proposition 3.10, this resolution will give us the usual cohomology functors. In other words the following diagram commutes

$$\begin{array}{ccc}
\mathcal{M}\mathrm{od}(X) & \xrightarrow{R^i\Gamma} & \mathcal{M}\mathrm{od}(X) \\
\downarrow & & \downarrow \\
\mathcal{A}\mathrm{b} & \xrightarrow{\quad R^i\Gamma \quad} & \mathcal{A}\mathrm{b}
\end{array}$$

where the vertical arrows are the forgetful functors. $\square$

3.2 A Vanishing Theorem of Grothendieck

Theorem 3.21 (Grothendieck)**.** *Let X be a noetherian topological space of dimension n, then for all $i > n$ and all sheaves of abelian groups $\mathcal{F}$ on X, we have $H^i(X, \mathcal{F}) = 0$.*

The proof of this theorem needs some preliminary results, concerning direct limits of sheaves. We recall that the direct limit of the system $\{\mathcal{F}_i\}_i$ denoted $\varinjlim \mathcal{F}_i$, is the sheaf associated to the presheaf $U \mapsto \varinjlim \mathcal{F}_i(U)$. An exercise in [Hartshorne, 1977] (II, 1.11), shows that on a noetherian topological space, this presheaf is already a sheaf.

Lemma 3.22. *On a noetherian topological space, a direct limit of flasque sheaves is flasque.*

Proof. Let $\mathcal{F}_\alpha$ be a direct system of flasque sheaves. Then for any inclusion of open sets, $V \subseteq U$, and for each α, we have $\mathcal{F}_\alpha(U) \to \mathcal{F}_\alpha(V)$ is surjective. Since $\varinjlim$ is an exact functor, we get

$$\varinjlim \mathcal{F}_\alpha(U) \to \varinjlim \mathcal{F}_\alpha(V)$$

is also surjective. As we have discussed above, on a noetherian topological space, $\varinjlim \mathcal{F}_\alpha(U) = (\varinjlim \mathcal{F}_\alpha)(V)$ for any open set . So we have

$$(\varinjlim \mathcal{F}_\alpha)(U) \to (\varinjlim \mathcal{F}_\alpha)(V)$$

is surjective, and so $\varinjlim \mathcal{F}_\alpha$ is flasque. □

Proposition 3.23. *Let X be a noetherian topological space, and let $(\mathcal{F}_\alpha)_\alpha$ be a direct directed system of abelian sheaves. Then there are natural isomorphisms, for each $i \geq 0$*

$$\varinjlim H^i(X, \mathcal{F}_\alpha) \to H^i(X, \varinjlim \mathcal{F}_\alpha).$$

Lemma 3.24. *Let Z be a closed subset of X, let $\mathcal{F}$ be a sheaf of abelian groups on Z, and let $j : Z \to X$ be the inclusion. Then $H^i(Z, \mathcal{F}) = H^i(X, j_*\mathcal{F})$, where $j_*\mathcal{F}$ is the extension of $\mathcal{F}$ by zero outside Z.*

Proof. If $\mathcal{I}^\bullet$ is a flasque resolution of $\mathcal{F}$ on Y, then $j_*\mathcal{I}^\bullet$ is a flasque resolution of $j_*\mathcal{F}$ on X and for each $i, \Gamma(Z, \mathcal{F}) = \Gamma(X, j_*\mathcal{F})$, so we get the same cohomology groups. □

Remark 3.25. We sometime use an abuse of notation by writing $\mathcal{F}$ instead of $j_*\mathcal{F}$. This lemma shows that there will be no ambiguity about the cohomology groups.

Sketch of the proof of Theorem 3.21. The proof of the theorem is done by induction on $n = \dim X$, in several steps. We first fix some notation. If Z is a closed subset of X, let $U = X - Z$. For any sheaf $\mathcal{F}$ on X we let $\mathcal{F}_Z = j_*(\mathcal{F}|_Z)$, where $j : Z \to X$ is the inclusion. The sheaf $j_*(\mathcal{F}|_Z)$ is then obtained by extending $\mathcal{F}|_Z$ to zero outside Z. We let $\mathcal{F}_U = j_!(\mathcal{F}|_U)$, where $j_!(\mathcal{F}|_U)$ is the sheaf associated to the presheaf $V \mapsto \mathcal{F}(V)$ if $V \subseteq U$ and 0 otherwise. It easy to check that we have an exact sequence

$$0 \to \mathcal{F}_U \to \mathcal{F} \to \mathcal{F}_Z \to 0$$

Step 1. We reduce to the case X irreducible. If X is reducible, let Z be one of its irreducible components, and let $U = X - Z$. Then for any sheaf $\mathcal{F}$ on X, we have an exact sequence, as above. From the long exact sequence of cohomology, it will be sufficient to prove that $H^i(X, \mathcal{F}_Z) = 0$ and $H^i(X, \mathcal{F}_U) = 0$ for $i > n$. But Z is irreducible and beside, $\mathcal{F}_U$ can be regarded as as a sheaf on the closed subset $\overline{U}$, which has one fewer irreducible components, Lemma 3.24, and induction on the number of irreducible components, allow us to reduce to the case X irreducible.

Step 2. Suppose X is irreducible of dimension 0. Then the only open subset of X are X and the empty set. Otherwise, X would admit a proper irreducible closed subset, and $\dim X \geq 1$. Thus the functor $\Gamma(X, -)$ induces an equivalence of categories $\mathcal{A}\mathrm{b}(X) \to \mathcal{A}\mathrm{b}$. In particular $\Gamma(X, -)$ is exact so $H^i(X, \mathcal{F}) = R^i F(\Gamma(X, \mathcal{F})) = 0$.

Step 3. We now suppose X is irreducible of dimension $n \geq 1$. Let $\mathcal{F}$ be a sheaf on X. We want to reduce to the case $\mathcal{F}$ is generated by finite local sections. Let $S = \amalg_{U \subseteq X} \mathcal{F}(U)$, U runs over the quasi-compact open of X. Take any finite subset A of S, say $A = \{s_1, ..., s_d\}$. Let $\mathcal{F}_A$ be the subsheaf of $\mathcal{F}$ generated by all $s_i \in A$. Note that if $A' \subset A$ then $\mathcal{F}_{A'} \subset \mathcal{F}_A$. Thus $\{F_A\}$ forms a direct system over the set of all finite subset of S. One can check that $\varinjlim \mathcal{F}_A = \mathcal{F}$. It follows that $H^i(X, \mathcal{F}) = \varinjlim H^i(X, \mathcal{F}_A)$. Hence it suffices to prove the vanishing theorem for $\mathcal{F}_A$. Suppose that $\mathcal{F}$ is generated by the local sections $s_1, ..., s_d$. Let $\mathcal{F}' \subset \mathcal{F}$ be a subsheaf generated by $s_1, ..., s_{d-1}$. Then we have an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}/\mathcal{F}' \to 0.$$

From the long exact sequence of cohomology, and by induction on d, we reduce to the case that $\mathcal{F}$ is generated by at most a single section over some open U. In that case $\mathcal{F}$ is a quotient of the sheaf $\mathbb{Z}_U$, where the sheaf $\mathbb{Z}$ denotes the constant sheaf $\mathbb{Z}$ on X. We denote by $\mathcal{K}$ the kernel, it gives an exact sequence

$$0 \to \mathcal{K} \to \mathbb{Z}_U \to \mathcal{F} \to 0.$$

Again the long exact sequence of cohomology, will imply that it is sufficient to prove vanishing for $\mathcal{K}$ and for $\mathbb{Z}_U$.

Step 4. Let U be an open subset of X and let $\mathcal{R}$ be a subsheaf of $\mathbb{Z}_U$. For each $x \in U$, we look on the stalk $\mathcal{R}_x$, then either $\mathcal{R}_x = 0$ or $\mathcal{R}_x \cong d\mathbb{Z}$, d is a positive integer. If the former case happen, then skip to Step 5. If not there is a nonempty open subset $V \subseteq U$ such that $\mathcal{R}|_V = d \cdot \mathbb{Z}|_V$. Thus $\mathcal{R}_V \cong Z_V$, and we obtain an exact sequence

$$0 \to \mathbb{Z}_V \to \mathcal{R} \to \mathcal{R}/\mathbb{Z}_V \to 0.$$

Now the sheaf $\mathcal{R}/\mathbb{Z}_V$ is supported on the closed subset $\overline{U - V}$ of X which has dimension $< n$, since X is irreducible. So using Lemma 3.24 and the induction hypothesis, we have $H^i(X, \mathcal{R}/\mathbb{Z}_V) = 0$ for $i \geq 1$. Yet by the long exact sequence of cohomology, we need only to check vanishing for $\mathbb{Z}_V$.

Step 5. To complete the proof, we need to show that for any open subset $U \subset X$, we have $H^i(X, \mathbb{Z}_U) = 0$ for $i > n$. Let $Y = X - U$. Then we have an exact sequence

$$0 \to \mathbb{Z}_U \to \mathbb{Z} \to \mathbb{Z}_Y \to 0$$

Since X is irreducible $\dim Y < \dim X$, by induction hypothesis $H^i(X, \mathbb{Z}_Y) = 0$ for $i \geq n$. Also, the constant sheaf $\mathbb{Z}$ is flasque, thus $H^i(X, \mathbb{Z}) = 0$. So $H^i(X, \mathbb{Z}_U) = 0$ for $i > 0$, and this completes the proof. □

3.3 Cohomology of Noetherian Affine Schemes

This section will provide us a tool which can help to compute the cohomology of sheaves on a given space. We will prove here that if $X = \text{Spec A}$ is a noetherian affine scheme, then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$ and all quasi-coherent sheaves $\mathcal{F}$ of $\mathcal{O}_X$-modules.

We first fix some notations. For any ring A, and any ideal $\mathfrak{a} \subseteq A$, and any A-module M, we define the submodule $\Gamma_{\mathfrak{a}}(M)$ to be $\{m \in M |\ \mathfrak{a}^n m = 0,\ \text{for some } n > 0\}$. Let us also recall the definition of the support of a section of a sheaf.
Let $\mathcal{F}$ be a sheaf on X, and let $s \in \mathcal{F}(U)$ be a section over an open set U. Then the support of s, is the set

$$\text{Supp } s := \{x \in U |\ s_x \neq 0\},$$

where s_x denotes the germ of s in the stalk $\mathcal{F}_x$.

Lemma 3.26. *Let A be a ring, and M an A-module. Let $X = \operatorname{Spec} \mathrm{A}$, and let $\mathcal{F}$ be the sheaf $\widetilde{M}$. Then $\operatorname{Supp} m = V(\operatorname{Ann} m)$, for any $m \in \Gamma(X, \mathcal{F})$, where $\operatorname{Ann} m$ is the annihilator of m.*

Proof. Let $\mathfrak{p} \in V(\operatorname{Ann} m)$, then $\mathfrak{p} \supseteq V(\operatorname{Ann} m)$, so localizing at $\mathfrak{p}$ means everything in $\operatorname{Ann} m$ is localized as well. So $m_{\mathfrak{p}} \neq 0$. Conversely, let $\mathfrak{p} \in \operatorname{Supp} m$, then $m_{\mathfrak{p}} \neq 0$, which is equivalent to $am \neq 0$ for $a \notin \mathfrak{p}$, Then $a \notin \operatorname{Ann} m$. So $\operatorname{Ann} m \subseteq \mathfrak{p}$. This shows the lemma. □

Lemma 3.27. *Let A be a noetherian ring, let I be an injective A-module. Then the submodule $\Gamma_{\mathfrak{a}}(I)$ is also an injective A-module.*

Proof. [Hartshorne, 1977] (III, Lem. 3.2) □

Lemma 3.28. *Let A be a noetherian ring, let M be an A-module. Let $X = \operatorname{Spec} \mathrm{A}$ and let $\mathcal{F}$ be the sheaf $\widetilde{M}$. Let $Z = V(\mathfrak{a})$ for some ideal $\mathfrak{a} \subseteq A$. Then $\Gamma_{\mathfrak{a}}(M) = \Gamma_Z(X, \mathcal{F})$, where Γ_Z denotes sections with support in Z.*

Proof. We will use the fact that $\operatorname{Supp} m = V(\operatorname{Ann} m)$ from Lemma 3.26. Now by definition,

$$\begin{aligned} m \in \Gamma_Z(X, \mathcal{F}) &\Leftrightarrow \operatorname{Supp} m \subseteq Z \\ &\Leftrightarrow V(\operatorname{Ann} m) \subseteq V(\mathfrak{a}) \\ &\Leftrightarrow \mathfrak{a}^n \subseteq \operatorname{Ann} m \\ &\Leftrightarrow m \in \Gamma_{\mathfrak{a}}(M) \end{aligned}$$

□

Lemma 3.29. *Let I be an injective module over a noetherian ring A. Then for any $f \in A$, the natural map of I to its localization I_f is surjective.*

Proof. Let $\theta : I \to I_f$ be the natural map, and let $x \in I_f$ be any element. Then by definition of localization, there is a $y \in I$, and an $n \geq 0$ such that $x = \theta(y)/f^n$. For each $i > 0$, let $\mathfrak{b}_i$ be the annihilator of f^i in A. Then $\mathfrak{b}_1 \subseteq \mathfrak{b}_2 \subseteq ...$, and since A is noetherian, there is an r such that $\mathfrak{b}_r = \mathfrak{b}_{r+1} = ...$ We define a map φ from the ideal (f^{n+r}) of A to I by sending f^{n+r} to $f^r y$. This is possible, because if $f^{n+r}b = f^{n+r}c$, so $(b-c)f^{n+r} = 0$, then $b - c \in \mathfrak{b}_{n+r}$. As the annihilator of f^{n+r} is $\mathfrak{b}_{n+r} = \mathfrak{b}_r$ and $\mathfrak{b}_r$ annihilates $f^r y$, it follows that $(b-c)f^r y = 0$ and φ is well defined . Since I is injective, φ extends to a map $\psi : A \to I$. Let $\psi(1) = z$. Then $f^{n+r}z = f^{n+r}\psi(1) = \psi(f^{n+r}) = \varphi(f^{n+r}) = f^r y$. But this implies that $\theta(z) = \theta(y)/f^n$. Hence θ is surjective. □

Proposition 3.30. *Let I be an injective module over a noetherian ring A. Then the sheaf $\widetilde{I}$ on $X = \mathrm{Spec\ A}$ is flasque.*

Proof. We will use noetherian induction on $Y = \overline{\mathrm{Supp}\ \widetilde{I}}$. If Y consists of a single point of X. Then $\widetilde{I}$ is a skyscraper sheaf which is obviously flasque. In the general case, as X is affine, then to show that $\widetilde{I}$ is flasque, it will be sufficient to show for any open $U \subseteq X$, that $\Gamma(X, \widetilde{I}) \to \Gamma(U, \widetilde{I})$ is surjective. If $Y \cap U = \emptyset$, there is nothing to prove. If $Y \cap U \neq \emptyset$, we can find and open set $X_f := D(f)$ contained in U and $X_f \cap Y \neq \emptyset$. Let $Z = X - X_f$ and consider the following diagram:

$$\begin{tikzcd}
\Gamma(X, \widetilde{I}) \arrow[r] & \Gamma(U, \widetilde{I}) \arrow[r] & \Gamma(X_f, \widetilde{I}) \\
\Gamma_Z(X, \widetilde{I}) \arrow[r] \arrow[u] & \Gamma_Z(U, \widetilde{I}) \arrow[u] &
\end{tikzcd}$$

where Γ_Z denotes the sections with support in Z. Now given a section $s \in \Gamma(U, \widetilde{I})$, we consider its image s' in $\Gamma(X_f, \widetilde{I})$. But $\Gamma(X_f, \widetilde{I}) = I_f$, so using Lemma 3.29, there is a $t \in \Gamma(X, \widetilde{I})$ such that $t|_{X_f} = s'$. Then $s - t|_U$ goes to 0 in $\Gamma(X_f, \widetilde{I})$. So it has a support in Z. Hence, to complete the proof, it will be sufficient to show that $\Gamma_Z(X, \widetilde{I}) \to \Gamma_Z(U, \widetilde{I})$ is surjective.

Let $J = \Gamma_Z(X, \widetilde{I})$. If $\mathfrak{a} = (f) \subseteq A$, then $J = \Gamma_{\mathfrak{a}}(I)$, by Lemma 3.28, J is also injective A-module according to Lemma 3.27. Furthermore, the support of $\widetilde{I}$ is contained in $Y \cap Z$, which is strictly smaller than Y. Hence by our induction hypothesis, $\widetilde{J}$ is flasque. But $\Gamma_Z(U, \widetilde{I}) = \Gamma(U, \widetilde{J})$, we conclude that $\Gamma_Z(X, \widetilde{I}) \to \Gamma_Z(U, \widetilde{I})$ is surjective. $\square$

Theorem 3.31. *Let $X = \mathrm{Spec\ A}$, with A a noetherian ring. Then for all quasi-coherent sheaves $\mathcal{F}$ on X, and for all $i > 0$, we have, $H^i(X, \mathcal{F}) = 0$.*

Proof. Given a quasi-coherent sheaf $\mathcal{F}$, let $M = \Gamma(X, \mathcal{F})$, and take an injective resolution $0 \to M \to I^{\cdot}$ of M in the category of A-modules. Then we obtain an exact sequence of sheaves $0 \to \widetilde{M} \to \widetilde{I}^{\cdot}$ on X. Now $\mathcal{F} = \widetilde{M}$ and each $\widetilde{I^i}$ is flasque, we can use this resolution to compute the cohomology. Applying the functor $\Gamma(X, -)$, we recover the exact sequence of A-modules $0 \to M \to I^{\cdot}$. Hence, $H^0(X, \mathcal{F}) = 0$ and $H^i(X, \mathcal{F}) = 0$ for all $i > 0$. $\square$

Corollary 3.32. *Let X be a noetherian scheme, and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then $\mathcal{F}$ can be embedded in a flasque quasi-coherent sheaf $\mathcal{G}$.*

Proof. Since X is noetherian, we may cover it with a finite number open affines $U_i = \mathrm{Spec\ A_i}$. As $\mathcal{F}$ is quasi-coherent, we have $\mathcal{F}|_{U_i} = \widetilde{M_i}$ for each i. Let us embed M_i in an injective A_i-module I_i. For each, i let $f_i : U_i \to X$ be the inclusion map, and let $\mathcal{G} = \bigoplus (f_i)_* \widetilde{I}_i$. Hence we obtain a map $\mathcal{F} \to (f_i)_*(\widetilde{I}_i)$. Taking the direct sum over i gives a map $\mathcal{F} \to \mathcal{G}$ which is clearly injective. On the other hand $\widetilde{I}_i$ is flasque by Proposition and quasi-coherent on U_i, for each i. Hence $(f_i)_*$ is also quasi-coherent [Hartshorne, 1977](II, Ex. 1.16) and quasi-coherent. Taking the direct sum, we see that $\mathcal{G}$ is flasque and quasi-coherent. □

3.4 Čech Cohomology

We will study here the Čech cohomology groups for a sheaf of abelian groups on a topological space X. The aim of this section is to show that if X is a noetherian separated scheme, the sheaf is quasi-coherent, and the covering is an open affine covering, then these Čech cohomology groups coincide with the cohomology groups we have defined before. At first glance, one might be scared computing cohomology with resolutions, but thanks to this result, we could have a practical way of computing cohomology of quasi-coherent sheaves on a scheme.

Let X be topological space, and let $\mathcal{U} = (U_i)_{i \in I}$ be an open covering of X. Fix, once and for all, a well-ordering of the index set I. For any finite set of indices $i_1, ..., i_p \in I$ we let

$$U_{i_0 \dots i_p} = U_{i_0} \cap \dots \cap U_{i_p}.$$

Now let $\mathcal{F}$ be a sheaf of abelian groups on X. We define a complex of abelian groups. For each $p \geq 1$

$$C^\bullet(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \dots < i_p} \mathcal{F}(U_{i_0 \dots i_p}).$$

Thus an element $\alpha \in C^p(\mathcal{U}, \mathcal{F})$ is determined by giving an element $\alpha_{i_0, \dots, i_p} \in \mathcal{F}(U_{i_0, \dots, i_p})$, for each $(p+1)$-tuple $i_0 < \dots < i_p$. We define the differential map $d : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$, by setting

$$(d\alpha)_{i_0 \dots i_p} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0, \dots, \hat{i}_k, \dots, i_{p+1}}|_{U_{i_0, \dots, i_{p+1}}},$$

where $\hat{i}_k$ means we omit the index i_k. For example, if $p = 1$, then

$$(d\alpha)_{i_0, i_1, i_2} = \alpha_{i_1, i_2}|_{U_{i_0, i_1, i_2}} - \alpha_{i_0, i_2}|_{U_{i_0, i_1, i_2}} + \alpha_{i_0, i_1}|_{U_{i_0, i_1, i_2}}$$

A direct computation shows that $d^2 = 0$, so we have indeed a complex of abelian groups.

Definition 3.33. Let X be a topological space and let $\mathcal{U}$ be an open covering of X. For any sheaf of abelian groups $\mathcal{F}$ on X, we define the p-th Čech cohomology group of $\mathcal{F}$, with respect to the covering $\mathcal{U}$, to be

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = h^p(C^\bullet(\mathcal{U}, \mathcal{F})).$$

Remark 3.34. In general, Čech cohomology does *not* take short exact sequences of sheaves to long exact sequences of cohomology groups. For instance, if we take on X the open covering $\mathcal{U}$ containing only the open set X, then we will have $\check{H}^p(\mathcal{U}, \mathcal{F}) = \Gamma(X, \mathcal{F})$ if $p = 0$ and $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for $p > 0$. So the existence of long exact sequence in Čech cohomology would imply that the global section functor $\Gamma(X, -)$ is exact, which is not always the case.

Example 3.2. *Let $X = \mathbb{P}^1_k$. We consider the covering $\mathcal{U} = \{U_0, U_1\}$ of X with $U_0 = D_+(T_0) = \text{Spec } k[T_1/T_0]$, and $U_1 = D_+(T_1) = \text{Spec } k[T_0/T_1]$. Let us set $x = T_1/T_0$. We have*

$$C^0(\mathcal{U}, \mathcal{F}) = \mathcal{O}_X(U_0) \oplus \mathcal{O}_X(U_0) = k[x] \oplus k[1/x] \quad \textit{and} \quad C^1(\mathcal{U}, \mathcal{F}) = \mathcal{O}_X(U_{01}) = k[x, 1/x]$$

. We therefore get the differential

$$\begin{array}{rcl} k[x] \oplus k[1/x] & \to & k[x, 1/x] \\ (f, g) & \mapsto & f - g \end{array}$$

This is map is clearly surjective, so $\check{H}^1(\mathcal{U}, \mathcal{O}_X) = C^1(\mathcal{U}, \mathcal{F})/\text{Im } d = 0$. We conclude that $\check{H}^p(\mathcal{U}, \mathcal{O}_X) = 0$ for $p \geq 1$.

Lemma 3.35. *For any X,$\mathcal{U}$, and $\mathcal{F}$ as above, we have $\check{H}^0(\mathcal{U}, \mathcal{F}) = \Gamma(X, \mathcal{F})$.*

Proof. By definition, $\check{H}^0(\mathcal{U}, \mathcal{F})$ is the kernel of $d : C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F})$, which is the map

$$\begin{array}{rcl} \prod_{i \in I} \mathcal{F}(U_i) & \to & \prod_{i,j} \mathcal{F}(U_i \cap U_j) \\ (f)_{i \in I} & \mapsto & (f_j|_{U_i \cap U_j} - f_i|_{U_i \cap U_j})_{i,j}. \end{array}$$

So $df = 0$ says the sections f_i and f_j agree on $U_i \cap U_j$. Hence it follows from the definition of a sheaf that $\ker d = \mathcal{F}(X)$. □

Proposition 3.36. *Let X be topological space, let $\mathcal{U}$ be an open covering, and let $\mathcal{F}$ be a flasque sheaf of abelian groups on X. Then for all $p > 0$ we have $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$.*

Before proving this proposition, let us define a "sheafified " version of the Čech complex, and afterward we give a statement of a lemma using this sheafified version, which is the key point of the proof of our proposition.

For any open set $V \subseteq X$, let $f : V \to X$ denote the inclusion map. Now given $X, \mathcal{U}, \mathcal{F}$ as above we define a complex as follows. For each $p \geq 0$

$$\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \ldots < i_p} f_* \mathcal{F}|_{U_{i_0 \cdots i_p}},$$

we define as well a morphism of sheaves $d^p : \mathcal{C}^p(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^{p+1}(\mathcal{U}, \mathcal{F})$, by setting

$$(d^p(V)\alpha)_{i_0,\ldots,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0,\ldots,\hat{i}_k,\ldots,i_{p+1}}|_{V \cap U_{i_0,\ldots,i_p}}.$$

Lemma 3.37. *For any sheaf of abelian groups $\mathcal{F}$ on X, the complex $\mathcal{C}^p(\mathcal{U}, \mathcal{F})$ is a resolution of $\mathcal{F}$, i.e., there is a natural map $\varepsilon : \mathcal{F} \to \mathcal{C}^0(\mathcal{U}, \mathcal{F})$ such that the sequence of sheaves*

$$0 \to \mathcal{F} \xrightarrow{\varepsilon} \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^1(\mathcal{U}, \mathcal{F}) \to \cdots$$

is exact.

Proof of Proposition 3.36. Let us consider the resolution $0 \to \mathcal{F} \to \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$ given by Lemma 3.37. Since $\mathcal{F}$ is flasque, for any $i_0, \ldots, i_p$, $\mathcal{F}|_{U_{i_0,\ldots,i_p}}$ is flasque on $U_{i_0,\ldots,i_p}$ and f_* preserves flasque sheaves. The product of flasque sheaves is flasque. It follows that $\mathcal{C}^p(\mathcal{U}, \mathcal{F})$ is flasque for each $p \geq 0$. So we can use this flasque resolution to compute the usual cohomology groups of $\mathcal{F}$. But $\mathcal{F}$ is flasque, then $H^p(X, \mathcal{F}) = 0$ for $p > 0$. On the other hand, the answer given by this resolution is

$$h^p(\Gamma(X, \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}))) = h^p(C^\bullet(\mathcal{U}, \mathcal{F})) = \check{H}^p(\mathcal{U}, \mathcal{F}).$$

So we conclude that $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for $p > 0$. □

Lemma 3.38. *Let X be a topological space, and $\mathcal{U}$ an open covering. Then for each $p \geq 0$ there is a natural map, functorial in $\mathcal{F}$,*

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \to H^p(X, \mathcal{F}).$$

Proof. [Hilton and Stammbach, 1971](IV, Th. 4.4) □

Theorem 3.39. *Let X be a noetherian separated scheme, let $\mathcal{U}$ be an open affine covering of X, and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then for all $p \geq 0$, the natural* *maps of Lemma 3.38 give isomorphisms*

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \xrightarrow{\sim} H^p(X, \mathcal{F}).$$

Proof. For $p = 0$, we have by Lemma 3.35, $\check{H}^p(\mathcal{U}, \mathcal{F}) = \Gamma(X, \mathcal{F})$. On the other hand by definition we have $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$. Therefore, $\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F})$. For the general case, we embed $\mathcal{F}$ in a flasque, quasi-coherent sheaf $\mathcal{G}$, and let $\mathcal{R}$ be the quotient:

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{R} \to 0$$

For each $i_0 < ... < i_p$, the open set $U_{i_0 \dots i_p}$ is affine, since it is an intersection of affine open subsets of a separated scheme. Since $\mathcal{F}$ is quasi-coherent, the sequence of abelian groups

$$0 \to \mathcal{F}(U_{i_0 \dots i_p}) \to \mathcal{G}(U_{i_0 \dots i_p}) \to \mathcal{R}(U_{i_0 \dots i_p}) \to 0$$

is exact by Theorem 3.31. Taking the products, we obtain the corresponding sequence of Čech complexes

$$0 \to \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^\bullet(\mathcal{U}, \mathcal{G}) \to \mathcal{C}^\bullet(\mathcal{U}, \mathcal{R}) \to 0,$$

which is exact. From this we get a long exact sequence of Čech cohomology groups. Since $\mathcal{G}$ is flasque, its cohomology vanishes for $p \geq 1$, by Proposition 3.36, so we have an exact sequence

$$0 \to \check{H}^0(\mathcal{U}, \mathcal{F}) \to \check{H}^0(\mathcal{U}, \mathcal{G}) \to \check{H}^0(\mathcal{U}, \mathcal{R}) \to \check{H}^1(\mathcal{U}, \mathcal{F}) \to 0,$$

and isomorphisms

$$\check{H}^p(\mathcal{U}, \mathcal{R}) \cong \check{H}^{p+1}(\mathcal{U}, \mathcal{F}),$$

for each $p \geq 1$. Now we draw the following diagram,

$$\begin{tikzcd}
0 \arrow[r] & \check{H}^0(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[d, "\cong"] & \check{H}^0(\mathcal{U}, \mathcal{G}) \arrow[r] \arrow[d, "\cong"] & \check{H}^0(\mathcal{U}, \mathcal{R}) \arrow[r] \arrow[d, "\cong"] & \check{H}^1(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0(X, \mathcal{F}) \arrow[r] & H^0(X, \mathcal{G}) \arrow[r] & H^0(X, \mathcal{R}) \arrow[r] & H^1(X, \mathcal{F}) \arrow[r] & 0
\end{tikzcd}$$

and conclude that the map $\check{H}^1(\mathcal{U}, \mathcal{F}) \to H^1(X, \mathcal{F})$, is an isomorphism. But $\mathcal{R}$ is also quasi-coherent, since it is the kernel of a morphism of quasi-coherent sheaves, we obtain the result by induction on p. □

3.5 The Cohomology of Projective space

In this section we make explicit calculation of the cohomology of the sheaves $\mathcal{O}(n)$ on a projective space, by using Čech cohomology for a suitable open affine covering.

Let A be a noetherian ring, let $S = A[T_0, ..., T_r]$, and let $X = \text{Proj } S$ be the projective space $\mathbb{P}^r_A$ over A. Let $\mathcal{O}_X(1)$ be the twisting sheaf of Serre. For any $\mathcal{O}_X$-modules $\mathcal{F}$ we denote by $\Gamma_*(\mathcal{F})$ the graded S-module $\bigoplus_{n\in\mathbb{Z}} \Gamma(X, \mathcal{F}(n))$.

Theorem 3.40. *Let A be a noetherian ring, let $X = \mathbb{P}^r_A$, with $r \geq 1$. Then:*

(a) *The natural map $S \to \bigoplus_{n\in\mathbb{Z}} H^0(X, \mathcal{O}_X(n))$ is an isomorphism of graded S-modules;*

(b) *$H^i(X, \mathcal{O}_X(n)) = 0$ for $0 < i < r$ and all $n \in \mathbb{Z}$;*

(c) *$H^r(X, \mathcal{O}_X(-r-1)) \cong A$;*

(d) *The natural map $H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-r-1)) \to H^r(X, \mathcal{O}_X(-r-1)) \cong A$ is a perfect pairing of finitely generated free A-modules, for each $n \in \mathbb{Z}$.*

Proof. Let $\mathcal{F}$ be the quasi-coherent sheaf $\bigoplus_{n\in\mathbb{Z}} \mathcal{O}_X(n)$. Since cohomology commutes with arbitrary direct sums on a noetherian topological space, then the cohomology of $\mathcal{F}$ will be the direct sum of the cohomology of the sheaves $\mathcal{O}_X(n)$. We note also that all the cohomology groups in question have the natural structure of A-module.

For each $i = 1, ..., r$, let $U_i = D_+(T_i)$. Then each U_i is an open affine subset of X, and the U_i cover X, so we can compute the cohomology of $\mathcal{F}$ by using Čech cohomology for the covering $\mathcal{U} = (U_i)_i$. As we already saw that we have isomorphisms $\check{H}^p(\mathcal{U}, \mathcal{F}) \cong H^p(X, \mathcal{F})$ for $p \geq 0$. For any indices $i_0, ..., i_p$, the open set $U_{i_0,...,i_p}$ is just $D_+(T_{i_0} \cdots T_{i_p})$, so we have

$$\mathcal{F}(U_{i_0,...,i_p}) \cong \bigoplus_{n\in\mathbb{Z}} S(n)_{(T_{i_0}\cdots T_{i_p})} = S_{T_{i_0}\cdots T_{i_p}}.$$

Furthermore, the grading on $\mathcal{F}$ corresponds to the natural grading of $S_{T_{i_0}\cdots T_{i_p}}$ under this isomorphism. Thus the Čech complex of $\mathcal{F}$ is given by

$$C^{\cdot}(\mathcal{U}, \mathcal{F}) : \prod_{i=0}^{r} S_{T_i} \to \prod_{0\leq i<j\leq r} S_{T_iT_j} \to \cdots \to S_{T_0\cdots T_r},$$

and the modules all have a natural grading compatible with the grading on $\mathcal{F}$.

Since $H^0(X, \mathcal{F})$ is the kernel of the first map in the Čech complex, which is just S. This proves (a).

We now prove (c). We consider $H^r(X, \mathcal{F})$. It is the cokernel of the last map in the Čech complex, which is

$$d^{r-1} : \prod_{k=0}^{r} S_{T_0 \cdots \hat{T}_k \cdots T_r} \to S_{T_0 \cdots T_r}.$$

We think of $S_{T_0 \cdots T_r}$ as a free A-module with basis $T_0^{l_0} \cdots T_r^{l_r}$, with $l_i \in \mathbb{Z}$. The image of d^{r-1} is the free submodule generated by those elements for which at least one $l_i \geq 0$, i.e of the form

$$\sum_{i=0}^{r} (-1)^i \frac{P_i}{(T_0 \cdots \hat{T}_k \cdots T_r)^m} = \sum_{i=0}^{r} (-1)^i \frac{X_i^m P_i}{(T_0 \cdots T_r)^m}.$$

Thus $H^r(X, \mathcal{F})$ is a free A-module with basis consisting of the monomials

$$\{T_0^{l_0} \cdots T_r^{l_r} |\ l_i < 0 \text{ for each } i\}.$$

Furthermore the grading is given by $\sum l_i$. But there is only one such monomial of degree $-r-1$, namely $T_0^{-1} \cdots T_r^{-1}$, so we see that $H^r(X, \mathcal{O}_X(-r-1))$ is a free A-module of rank 1. This shows (c).

Now we prove (b), by induction on r. If $r = 1$, there is nothing to prove. So let $r > 1$. If we localize the complex $C^{\cdot}(\mathcal{U}, \mathcal{F})$ with respect to T_r, as graded S-modules, we get the Čech complex for the sheaf $\mathcal{F}|_{U_r}$ on the space U_r, with respect to the open affine covering $\{U_i \cap U_r |\ i = 0, ..., r\}$. This complex gives the cohomology of $\mathcal{F}|_{U_r}$ on U_r, which is 0 since U_r is affine. Since localization is an exact functor, we conclude that $H^i(X, \mathcal{F})_{T_r} = 0$ for $i > 0$. In other words, every element of $H^i(X, \mathcal{F})$ for $i > 0$ is annihilated by some power of T_r. To complete the proof of (b), we will show that for $0 < i < r$, multiplication by T_r induces a bijective map of $H^i(X, \mathcal{F})$ into itself. Then in this case, the module is 0. For this consider the exact sequence of graded S-modules

$$0 \to S(-1) \to S \to S/(T_r) \to 0.$$

This gives the exact sequence of sheaves.

$$0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to \mathcal{O}_H \to 0,$$

on X, where $H = V_+(T_r)$. Twisting by all $n \in \mathbb{Z}$ and taking the direct sum, we have

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{F}_H \to 0,$$

where $\mathcal{F}_H = \bigoplus_{n\in\mathbb{Z}} \mathcal{O}_H(n)$. The sequences remains exact since we tensor with invertible sheaf $\mathcal{O}_X(n)$. Taking the long exact sequence of cohomology, we get

$$\cdots \to H^{i-1}(X,\mathcal{F}_H) \to H^i(X,\mathcal{F}(-1)) \to H^i(X,\mathcal{F}) \to H^i(X,\mathcal{F}_H) \to \cdots$$

Considered as S-modules, $H^i(X,\mathcal{F}(-1))$ is just $H^i(X,\mathcal{F})$ shifted one place, and the map $H^i(X,\mathcal{F}(-1)) \to H^i(X,\mathcal{F})$ is the exact multiplication by T_r.

Now H is isomorphic to $\mathbb{P}^{r-1}_A$, we know also that $H^i(X,\mathcal{F}_H) = H^i(H, \bigoplus \mathcal{O}_H(n))$. So we apply our induction hypothesis to $\mathcal{F}|_H$, and find that $H^i(X,\mathcal{F}_H) = 0$ for $0 < i < r-1$. Furthermore, for $i = 0$ we have an exact sequence

$$0 \to H^0(X,\mathcal{F}(-1)) \to H^0(X,\mathcal{F}) \to H^0(X,\mathcal{F}_H) \to 0$$

by part (a), since $H^0(X,\mathcal{F}_H)$ is just $S/(T_r)$. At the other end of the exact sequence of cohomology, we have

$$0 \to H^{r-1}(X,\mathcal{F}_H) \to H^r(X,\mathcal{F}(-1)) \xrightarrow{\cdot T_r} H^r(X,\mathcal{F}) \to 0,$$

indeed, we have described $H^r(X,\mathcal{F})$ above as the free A-module with basis formed by negative monomials in $T_0, ..., T_r$. So we have $\cdot T_r$ is surjective. On the other hand, the kernel of $\cdot T_r$ is the free submodule generated by those negative monomials $T_0^{l_0}\cdots T_r^{l_r}$ with $l_r = -1$. Since $H^{r-1}(X,\mathcal{F}_H)$ is the free A-module with basis consisting of the negative monomials in $T_0, ..., T_{r-1}$, and δ is division by T_r, the sequence is exact. Putting these results all together, the long exact sequence of cohomology shows that the map multiplication by $T_r : H^i(X,\mathcal{F}(-1)) \to H^i(X,\mathcal{F})$ is bijective for $0 < i < r$, as required. $\square$

Theorem 3.41. *Let X be a projective scheme over a noetherian ring A, and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X over Spec A. Let $\mathcal{F}$ be a coherent sheaf on X. Then:*

(a) for each $i \geq 0$, $H^i(X,\mathcal{F})$ is a finitely generated A-module;

(b) there is an integer n_0, depending on $\mathcal{F}$, such that for each $i > 0$ and each $n \geq n_0$, $H^i(X,\mathcal{F}(n)) = 0$.

Proof. Since $\mathcal{O}_X(1)$ is a very ample sheaf on X over Spec$A$, there is a closed immersion $i : X \to \mathbb{P}^r_A$ of schemes over A, for some r, such that $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}^r}(1)$. If $\mathcal{F}$ is coherent on X then we saw in Proposition 2.20, that $i_*\mathcal{F}$ is coherent on $\mathbb{P}^r_A$, and the cohomology is the same, by Lemma 3.24. Thus we reduce to the case $X = \mathbb{P}^r_A$.

For $X = \mathbb{P}^r_A$, we observe that (a) and (b) are true for any sheaf of the form $\mathcal{O}_X(q),\ q \in \mathbb{Z}$. This follows straight away from Theorem 3.40. Hence the same is true for any finite direct sum of such sheaves.

To prove the theorem for arbitrary coherent sheaves, we use descending induction on i. For $i > r$, we have $H^i(X, \mathcal{F}) = 0$, since X can be covered by $r+1$ open affines, so using Čech cohomology, we get the result in this case.

Now, given a coherent sheaf $\mathcal{F}$ on X, we can write $\mathcal{F}$ as a quotient of a sheaf $\mathcal{E}$, which is a direct sum of sheaves $\mathcal{O}_X(q_i)$, for various integers q_i. Let $\mathcal{R}$ be the kernel,

$$0 \to \mathcal{R} \to \mathcal{E} \to \mathcal{F} \to 0 \tag{3.1}$$

Then $\mathcal{R}$ is also coherent. From the long exact sequence of cohomology, we get an exact sequence of A-modules

$$\cdots H^i(X, \mathcal{E}) \to H^i(X, \mathcal{F}) \to H^{i+1}(X, \mathcal{R}) \to \cdots$$

Now the module on left is finitely generated because $\mathcal{E}$ is a finite direct some of $\mathcal{O}_X(q_i)$, as we discussed above. The module on the right is finitely generated by our induction hypothesis. Since A is a noetherian ring, we conclude that the module on the middle is also finitely generated. This proves (a).
To prove part (b), we twist the short exact sequence (3.1). Since $\mathcal{O}_X(n)$ is invertible and therefore flat over $\mathcal{O}_X$, the sequence

$$0 \to \mathcal{R}(n) \to \mathcal{E}(n) \to \mathcal{F}(n) \to 0,$$

is exact for $n \in \mathbb{Z}$. We write down a piece of the long exact sequence

$$\cdots H^i(X, \mathcal{E}(n)) \to H^i(X, \mathcal{F}(n)) \to H^{i+1}(X, \mathcal{R}(n)) \to \cdots$$

Now for $n \gg 0$, the module on the left vanishes because $\mathcal{E}$ is a sum of $\mathcal{O}_X(q_i)$. The module on the right also vanishes for $n \gg 0$ by induction hypothesis. Hence $H^i(X, \mathcal{F}(n)) = 0$ for $n \gg 0$. Note that since there are only finitely many i involved in statement (b), namely $0 < i \leq r$, it is sufficient to determine n_0 separately for each i. This prove part (b). $\square$

Chapter 4

Serre's Duality and Riemann-Roch Theorem

This chapter is devoted for the study of Serre's duality and the Riemann-Roch theorem. The first section consists of gathering few algebraic results about differentials, defining the sheaf of differentials on a scheme X, which leads to the definition of a coherent sheaf, called canonical sheaf. The duality for the cohomology of this sheaf forms the main part of Serre's duality. The last section brings us to the Riemann-Roch theorem for projective curves. Its proof uses the result in the first section. We will see also the notion of divisors along the way. This algebraic concept holds an important role in the study of Riemann-Roch.

4.1 Serre's Duality

In this section we will study the Serre duality theorem for the cohomology of coherent sheaves on a projective scheme. We will pay more attention to the case of projective space $\mathbb{P}^n_k$ over an algebraically closed field k. We present its proof which is a direct the computations we have made in the next section. Then afterwards, we will give the statement for a projective scheme over k. We start by giving some algebraic results about Khäler differentials and then use them to define the canonical sheaf which plays an important role in the duality.

4.1.1 Khäler Differentials

Let A be a commutative ring with identity, let B be an A-algebra, and let M be an A-module. In this subsection we refer to [Matsumura and Algebra, 1980] for proofs.

Definition 4.1. An A-*derivation* of B into M is a map $d : B \to M$ such that

(1) d is additive,

(2) $d(bb') = bdb' + b'db$,

(3) $da = 0$ for all $a \in A$.

Definition 4.2. We define the *module of relative differential forms* of B over A to be a B-module $\Omega_{B/A}$, together with an A-derivation $d : B \to \Omega_{B/A}$ which satisfies the following universal property: for any B-module M and for any A-derivation $d' : B \to M$, there exists a unique B-module homomorphism $f : \Omega_{B/A} \to M$ such that $d' = f \circ d$.

Proposition 4.3. *Let B be an A-algebra. Let $f : B \otimes_A B \to B$ be the homomorphism defined by $f(b \otimes b') = bb'$, and let $I = \ker f$. Consider $B \otimes B$ as a B-module by multiplication on the left. Then I/I^2 inherits a structure of B-module. Define a map $d : B \to I/I^2$ by $db = 1 \otimes b - b \otimes 1$* (modulo I^2). *Then $\langle I/I^2, d\rangle$ is a module of relative differentials for B over A.*

Proposition 4.4. *If A' and B are A-algebras, let $B' = B \otimes A'$. Then $\Omega_{B'/A'} \cong \Omega_{B/A} \otimes_B B'$. Furthermore, if S is a multiplicative set in B, then $\Omega_{S^{-1}B/A} \cong S^{-1}\Omega_{B/A}$.*

Example 4.1. *The following example is fundamental for computation of modules of differentials. If $B = A[T_1, ..., T_n]$ is a polynomial ring over A, then $\Omega_{B/A}$ is free B-module of rank n generated by $dT_1, ..., dT_n$.*

Indeed, if $F \in B$ and let $d' : B \to M$ be an A-derivation into a B-module M. By the definition of derivation, we have that $d'F = \sum_i^n \frac{\partial F}{\partial T_i} d'T_i$, where $\frac{\partial F}{\partial T_i}$ is the partial derivative in the usual sense. Therefore d' is entirely defined by the images of the T_i. Now, let Ω be the free module generated by the symbols $dT_i, ..., dT_n$. We let $d : B \to \Omega$ be the map defined by $dF = \sum_i \dfrac{\partial F}{\partial T_i} dT_i$. It is easy to check that (Ω, d) fulfills the conditions of the universal property of the module $\Omega_{B/A}$. Therefore $\Omega_{B/A} \cong \Omega$.

Example 4.2. *Let B be a localization or a quotient of A, then $\Omega_{B/A} = 0$. Indeed, if $A \to B$ is surjective, $d(b) = ad(1) = 0$ for some preimage $a \in A$ of b. Let us suppose that $B = S^{-1}A$ is a localization of A. For any $b = B$, there exists an $s \in S$ such that $sb \in A$, and hence $sdb = d(sb) = 0$. But s is invertible in B, it follows that $db = 0$.*

Proposition 4.5. *(First Exact Sequence). Let $A \to B \to C$ be rings and homomorphisms. Then there is a natural exact sequence of C-modules*

$$\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0.$$

Proposition 4.6. *(Second Exact sequence). Let B an A-algebra, let I be an ideal of B, and let $C = B/I$. Then there is a natural exact sequence of C-modules*

$$I/I^2 \xrightarrow{\delta} \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0,$$

where for any $b \in I$, if $\bar{b}$ is its image in I/I^2, then $\delta\bar{b} = db \otimes 1$. Note in particular that I/I^2 has a natural structure of of C-module, and that δ is a C-linear map, even though it is defined via the derivation d.

Corollary 4.7. *If B is a finitely generated A-algebra, or if B is localization of finitely generated A-algebra, then $\Omega_{B/A}$ is a finitely generated B-module.*

Proof. If B is a finitely generated A-algebra, then it is quotient of a polynomial ring so the result follows from Proposition 4.6 and Example 4.2. □

Example 4.3. *Let $B = A[T_1, ..., T_n]$, and let $F \in B$. We let $C = B/(F)$, then by Example 4.1 and Proposition 4.6 we have*

$$\Omega_{C/A} = (\oplus_{i=0}^{n} CdT_i)/CdF,$$

with $dF = \sum_{i=0} \frac{\partial F}{\partial T_i} dT_i$.

Proposition 4.8. *Let $(B, \mathfrak{m})$ be a local ring which contains a field k isomorphic to its residue field $B/\mathfrak{m}$. Then the map $\delta : \mathfrak{m}/\mathfrak{m}^2 \to \Omega_{B/k} \otimes_B k$ of Proposition 4.6 is an isomorphism.*

Proof. From the Second Exact Sequence of Proposition 4.6, the cokernel of δ is $\Omega_{k/k} = 0$, so δ is surjective. To show that δ is injective, we show that the map

$$\delta' : \mathsf{Hom}_k(\Omega_{B/k} \otimes k, k) \to \mathsf{Hom}(\mathfrak{m}/\mathfrak{m}^2, k)$$

of dual vector space is surjective. The term on the left is isomorphic to $\mathsf{Hom}_B(\Omega_{B/k}, k)$, which by definition can be identified with the set $\mathrm{Der}_k(B, k)$ of k-derivation of B into k. If $d : B \to k$ is a derivation, then $\delta'(d)$ is obtained by restricting to $\mathfrak{m}$, and noting that $d(\mathfrak{m}^2) = 0$. To show that δ' is surjective, let $h \in \mathsf{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$. For any $b \in B$, we can write $b = \lambda + c$, with $\lambda \in k, c \in \mathfrak{m}$, in a unique way. Define $db = h(\bar{c})$, where $\bar{c} \in \mathfrak{m}/\mathfrak{m}^2$ is the image of c. One can check easily that d is a derivation of B to k, and that $\delta'(d) = h$. Thus δ' is surjective as desired. □

4.1.2 Sheaves of Differentials

We now carry the definition of the module of differentials over to schemes. Let $f : X \to Y$ be a morphism of schemes. We consider the diagonal morphism $\Delta : X \to X \times_Y X$. We know that Δ is locally closed immersion, and hence $\Delta(X)$ is closed in an open subset W of $X \times_Y X$.

Definition 4.9. Let $\mathcal{I}$ be the sheaf of ideals of $\Delta(X)$ in W. Then we define the *sheaf of relative differentials* of X over Y to be the sheaf $\Omega_{X/Y} = \Delta^*(\mathcal{I}/\mathcal{I}^2)$ on X.

Remark 4.10. We first note that $\mathcal{I}/\mathcal{I}^2$ has a natural structure of $\mathcal{O}_{\Delta(X)}$-module. Then since Δ induces an isomorphism of X to $\Delta(X)$, $\Omega_{X/Y}$ has a natural structure of $\mathcal{O}_X$-module. Furthermore, $\Omega_{X/Y}$ is a quasi-coherent since $\mathcal{I}$ is. If Y is a noetherian scheme and f is a morphism of finite type, then $X \times_Y X$ is also noetherian and so $\Omega_{X/Y}$ is coherent.

Remark 4.11. Now if $U = \text{Spec } A$ is an open affine subset of Y and $V = \text{Spec } B$ is an open affine of X, such that $f(V) \subseteq U$, then $V \otimes_U V$ is an open affine of $X \times_Y X$ isomorphic to Spec $(B \otimes_A B)$ and $\Delta(X) \cap (V \otimes_U V)$ is the closed subscheme defined by the kernel of the diagonal homomorphism $B \otimes_A B \to B$. Thus $\mathcal{I}/\mathcal{I}^2$ is the sheaf associated to the module I/I^2 of Proposition 4.3. It follows that $\Omega_{V/U} \cong (\Omega_{B/A})\tilde{}$. So our definition of the sheaf of differentials of X over Y is compatible, in the affine case, with the module of differentials defined above, via the functor $\sim$. This also tells us that we could have defined $\Omega_{X/Y}$ by covering X and Y with open affine subsets V and U as above, and glueing the corresponding sheaves $(\Omega_{B/A})\tilde{}$. The derivation $B \to \Omega_{B/A}$ glue together to give a map $d : \mathcal{O}_X \to \Omega_{X/Y}$ of sheaves of abelian groups on X, which is a derivation of the local ring at each point. Therefore we can carry our algebraic results to schemes, and obtain the following results.

Proposition 4.12. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. Then there is an exact sequence of sheaves on X,*

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

Proof. Follows from Propostion 4.5 and the definition of f^*. □

Proposition 4.13. *Let $f : X \to Y$ be a morphism, and let Z be a closed subscheme of X, with ideal sheaf $\mathcal{I}$. Then there is an exact sequence of sheaves on Z,*

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{\delta} \Omega_{X/Y} \otimes \mathcal{O}_Z \to \Omega_{Z/Y} \to 0.$$

Proof. Follows from Propostion 4.5. □

We will give an exact sequence relating the sheaf of differentials on a projective space to sheaves we already now. This results is fundamental, upon which we will have a tool for computations involving differentials on projective varieties.

Theorem 4.14. *Let A be a ring, let $Y = \text{Spec } A$, and let $X = \mathbb{P}^n_A$. Then there is an exact sequence of sheaves on X,*

$$0 \to \Omega_{X/Y} \to \mathcal{O}_X(-1)^{(n+1)} \to \mathcal{O}_X \to 0.$$

(The exponent $(n+1)$ in the middle means a direct sum of $n+1$ copies of $\mathcal{O}_X(-1)$.)

Before proving our theorem, let us give some few tools that will be used in the proof.

Definition 4.15. Let S be a graded ring and M a graded S-module. For $d \in \mathbb{Z}$, we define $M\{d\}$ to be the S-module $\bigoplus_{n \geq d} M$ with the grading $M\{d\}_l = M_{l+d}$ for $l \geq 0$ and $M\{d\}_l = 0$ for $l < 0$. This is a the graded S-submodule of $M(d)$ obtained by removing all negative grades.

If $\phi : M \to N$ is a morphism of graded S-modules then ϕ restricts to give a morphism of graded S-modules $\phi\{d\} : M\{d\} \to N\{d\}$.

Definition 4.16. We say that two graded S-modules M and N are quasi-isomorphic and write $M \sim N$ if there exists an integer $d \geq 0$ such that $M\{d\} \cong N\{d\}$. It is clear that if M and N are isomorphic as graded S-modules, they are quasi-isomorphic.

We say that $\phi : M \to N$ is a *quasi-epimorphism* if there exists $d \geq 0$ such that $M_l \to N_l$ is surjective for all $l \geq d$.

Lemma 4.17. *Let S be a graded ring and $\phi : M \to N$ a morphism of graded S-module. Then ϕ is a quasi-epimorphism if and only if* $\text{coker}\phi \sim 0$.

Proposition 4.18. *Let S be a graded ring generated by S_1 as S_0-algebra. If M, N are quasi-isomorphic graded S-modules then there is a canonical isomorphism of sheaves of modules $\widetilde{M} \cong \widetilde{N}$. In particular if $M \sim 0$ then $\widetilde{M} = 0$.*

Proof of Theorem 4.14. Let $S = A[T_0, ..., T_n]$ be the homogeneous coordinate ring of X. Let E be the graded S-module $S(-1)^{(n+1)}$ with basis $e_0, ..., e_n$ in degree 1. Define a degree 0 homomorphism of graded S-modules $E \to S$ by sending $e_i \mapsto T_i$, and let M be the kernel. Then the exact sequence

$$0 \to M \to E \to S,$$

of graded S-modules gives rise to an exact sequence of sheaves on X,

$$0 \to \widetilde{M} \to \mathcal{O}_X(-1)^{(n+1)} \to \mathcal{O}_X \to 0.$$

Note that $\theta : E \to S$ is not surjective, but it is surjective in all degree ≥ 1 by Proposition 4.18, so the corresponding map of sheaves is surjective. We will now show that $\widetilde{M} \cong \Omega_{X/Y}$. Let $U_i = D_+(T_i)$ be the canonical affine open subset of X. Now $\theta_{(T_i)} : E_{(T_i)} \to S_{(T_i)}$ is a surjective homomorphism of S-modules and $E_{(T_i)}$ is a free $S_{(T_i)}$-module with basis $\{e_i/T_0, ..., e_i/T_n\}$.

Claim: the module $M_{(T_i)}$ is free with basis $\{\frac{1}{T^2}(T_ie_j - T_je_i)|\ i \neq j\}$. Indeed, let us take $t \in M_{(T_i)} = \ker \theta_{(T_i)}$, we may write

$$t = \frac{s_0}{T_i^m}\frac{e_0}{T_i} + \cdots + \frac{s_n}{T_i^m}\frac{e_n}{T_i},$$

Where each $s_k \in S_m, m \geq 1$. Since $\theta_{(T_i)}(t) = 0$, it follows that $s_0T_0 + \cdots s_nT_n = 0$ in S. Therefore,

$$\frac{s_0}{T_i^m}\left(\frac{e_0}{T_i} - \frac{T_0}{T_i}\frac{e_i}{T_i}\right) + \cdots + \frac{s_n}{T_i^m}\left(\frac{e_n}{T_i} - \frac{T_n}{T_i}\frac{e_i}{T_i}\right) = t - \frac{s_0T_0 + \cdots + s_nT_n}{T_i^{m+2}} = t.$$

So the elements $\frac{e_i}{T_i} - \frac{T_j}{T_i}\frac{e_i}{T_i}$ for $i \neq j$ at least generate $\ker \theta_{(T_i)}$ as $S_{(T_i)}$-module. One checks also that they are linearly independent. Thus $\widetilde{M}|_{U_i} \cong (M_{(T_i)})^\sim$ is a free $\mathcal{O}_{U_i}$-module generated by the sections $\frac{e_i}{T_i} - \frac{T_j}{T_i}\frac{e_i}{T_i}$ for $i \neq j$.

We define a map $\varphi_i : \Omega_{X/Y}|_{U_i} \to \widetilde{M}|_{U_i}$ as follows. As $U_i \cong \operatorname{Spec} A[\frac{T_0}{T_i}, ..., \frac{T_n}{T_i}]$, $\Omega_{X/Y}|_{U_i}$ is a free $\mathcal{O}_{U_i}$-module generated by $d\left(\frac{T_0}{T_i}\right), ..., d\left(\frac{T_n}{T_i}\right)$. So we define φ_i by sending $d\left(\frac{T_j}{T_i}\right) \mapsto \frac{1}{T_i^2}(T_ie_j - T_je_i)$. Thus φ_i is an isomorphism. It remains for us to check that φ_i glue together to give an isomorphism $\varphi : \Omega_{X/Y} \mapsto \widetilde{M}$ on all of X. Let us check the compatibility of these isomorphisms. On $U_i \cap U_j$, we have $\frac{T_k}{T_i} = \frac{T_k}{T_j}\frac{T_j}{T_i}$, for any k; hence we have, using properties of derivation,

$$\begin{aligned} z := d\left(\frac{T_k}{T_i}\right) &= \frac{T_k}{T_i}d\left(\frac{T_k}{T_i}\right) + \frac{T_k}{T_i}\left(\frac{T_k}{T_i}\right) \\ &= \frac{-T_jT_k}{T_i^2}d\left(\frac{T_i}{T_j}\right) + \frac{T_j}{T_i}d\left(\frac{T_k}{T_i}\right). \end{aligned}$$

Now $\varphi_i(z) = \frac{1}{T_i^2}(T_i e_k - T_k e_i)$, and on the other hand applying φ_j, we have

$$\begin{aligned}
\varphi_j(z) &= \frac{-T_j T_k}{T_i^2}\left[\frac{1}{T_j^2}(T_j e_i - T_i e_j)\right] + \frac{T_j}{T_i}\left[\frac{1}{T_j^2}(T_j e_k - T_k e_j)\right] \\
&= -\frac{T_k e_i}{T_i^2} + \frac{T_k e_j}{T_i T_j} + \frac{T_j e_k}{T_i T_j} - \frac{T_k e_j}{T_i T_j} \\
&= \frac{1}{T_i^2}(T_i e_k - T_k e_i) \\
&= \varphi_i(z).
\end{aligned}$$

Thus the isomorphisms φ_i glue and give the desired isomorphism $\varphi : \Omega_{X/Y} \to \widetilde{M}$. □

Corollary 4.19. *Let A be a ring, $Y = \operatorname{Spec} A$ and $X = \mathbb{P}^n_A$, then $\Omega_{X/Y}$ is locally free of rank n.*

Proof. We cover X with the open sets $U_i = D_+(T_i)$, and in the proof of Theorem 4.14, we showed that $\Omega_{X/Y} \cong \widetilde{M}|_{U_i}$ for each i. Since $\widetilde{M}|_{U_i} = (M_{(T_i)})^{\sim}$, this shows the result. □

We will study an application of the sheaf of differentials, to nonsingular varieties.

Definition 4.20. An (abstract) variety X over an algebraically closed field k is *non-singular* if all its local rings are regular local rings.

The following result gives a connection between non-singularity and differentials.

Theorem 4.21. *Let X be an irreducible separated scheme of finite type over an algebraically closed field k. Then $\Omega_{X/k}$ is a locally free sheaf of rank $n = \dim X$ if and only if X is a nonsingular variety.*

Definition 4.22. Let X be a nonsingular variety over k. We define the *tangent sheaf* of X to be $\mathcal{T}_X = \mathcal{H}om(\Omega_{X/k}, \mathcal{O}_X)$. It is locally free sheaf of rank $n = \dim X$. We define the *canonical sheaf* of X to be $\omega_X = \bigwedge^n \Omega_{X/k}$, the n-th exterior power (we refer to Exercise 5.16 of) of the sheaf of differentials.

Example 4.4. *Let $X = \mathbb{P}^n_k$. Taking the dual of the exact sequence of Theorem 4.14 gives us this exact sequence involving the tangent sheaf of $\mathbb{P}^n_k$:*

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1)^{(n+1)} \to \mathcal{T}_X \to 0.$$

To obtain the canonical sheaf of $\mathbb{P}^n_k$, we take the highest exterior powers of the exact sequence of Theorem 4.14, and find that $\omega_X \cong \mathcal{O}_X(-n-1)$.

$$\omega_X = \bigwedge^{n} \Omega_{X/Y} \cong \bigwedge^{n} \Omega_{X/Y} \otimes \mathcal{O}_X \cong \bigwedge^{n+1} \mathcal{O}_X(-1)^{(n+1)} \cong \mathcal{O}_X(-1)^{\otimes(n+1)} \cong \mathcal{O}_X(-n-1).$$

The second and third isomorphisms come from Exercise 5.16 in Hartshorne.

4.1.3 Ext Groups and Sheaves

We develop here some properties of Ext groups and sheaves, which we will need for the duality theorem. We work on a ringed space $(X, \mathcal{O}_X)$, and all sheaves will be sheaves of $\mathcal{O}_X$-modules. We recall that if $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules, $\mathsf{Hom}(\mathcal{F}, \mathcal{G})$ is the group of $\mathcal{O}_X$-module homomorphisms, and $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is the sheaf associated to the presheaf $U \mapsto \mathsf{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$.
For a fixed $\mathcal{F}$,

$$\mathsf{Hom}(\mathcal{F}, -) : \mathcal{M}\mathrm{od}(X) \to \mathcal{A}\mathrm{b} \text{ and } \mathcal{H}om(\mathcal{F}, -) : \mathcal{M}\mathrm{od}(X) \to \mathcal{M}\mathrm{od}(X)$$

are left exact covariant functors. Since $\mathcal{M}\mathrm{od}(X)$ has enough injectives, we can make the following definition.

Definition 4.23. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We define the functor $\mathsf{Ext}^i(\mathcal{F}, -)$ as the right derived functors of $\mathsf{Hom}(\mathcal{F}, -)$, and $\mathcal{E}xt^i(\mathcal{F}, -)$ as the right derived functor of $\mathcal{H}om(\mathcal{F}, -)$.

Consequently, according to the general properties of derived functors, we have $\mathsf{Ext}^0 = \mathsf{Hom}$, a long exact sequence for a short exact sequence in the second variable, $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) = 0$ for $i > 0$, $\mathcal{G}$ injective in $\mathcal{M}\mathrm{od}(X)$, and similarly for the $\mathcal{E}xt$ sheaves.

Lemma 4.24. *If $\mathcal{I}$ is an injective object of $\mathcal{M}\mathrm{od}(X)$, then for any open subset $U \subseteq X, \mathcal{I}|_U$ is an injective object of $\mathcal{M}\mathrm{od}(U)$.*

Proposition 4.25. *For any open subset $U \subseteq X$ we have*

$$\mathcal{E}xt^i_X(\mathcal{F}, \mathcal{G})|_U \cong \mathcal{E}xt^i_U(\mathcal{F}|_U, \mathcal{G}|_U).$$

Proof. Both sides give δ-functors in $\mathcal{G}$ from $\mathcal{M}\mathrm{od}(X)$ to $\mathcal{M}\mathrm{od}(U)$. They agree for $i = 0$, both side vanish for $i > 0$ and $\mathcal{G}$ injective, by previous lemma, so they are equal. $\square$

Proposition 4.26. *For any $\mathcal{G} \in \mathcal{M}\mathrm{od}(X)$, we have:*

(a) $\mathcal{E}xt^0(\mathcal{O}_X, \mathcal{G}) = \mathcal{G}$;

(b) $\mathcal{E}xt^i(\mathcal{O}_X, \mathcal{G}) = 0$ *for* $i > 0$;

(c) $Ext^i(\mathcal{O}_X, \mathcal{G}) = H^i(X, \mathcal{G})$ *for all* $i \geq 0$.

Proof. The functor $\mathcal{H}om(\mathcal{O}_X, -)$ is the identity functor so its derived functors are 0 for $i > 0$. This proves (a) and (b). The functors $\mathsf{Hom}(\mathcal{O}_X, -)$ and $\Gamma(X, -)$ are equal, so their derived functors (as functors from $\mathcal{M}\mathrm{od}(X)$ to $\mathcal{A}\mathrm{b}$) are the same. □

Proposition 4.27. *If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence in $\mathfrak{Mod}(X)$, then for any $\mathcal{G}$ we have a long exact sequence*

$$0 \to \mathsf{Hom}(\mathcal{F}'', \mathcal{G}) \to \mathsf{Hom}(\mathcal{F}, \mathcal{G}) \to \mathsf{Hom}(\mathcal{F}', \mathcal{G}) \to \cdots \to \mathsf{Ext}^1(\mathcal{F}'', \mathcal{G}) \to \mathsf{Ext}^1(\mathcal{F}, \mathcal{G}) \to \cdots,$$

and similarly for $\mathcal{E}xt$ sheaves.

Proof. Let $0 \to \mathcal{G} \to \mathcal{I}^\bullet)$ be an injective resolution of $\mathcal{G}$. For any injective sheaf $\mathcal{I}$, the functor $\mathsf{Hom}(-, \mathcal{I})$ is exact, so we get a short exact sequence of complexes

$$0 \to \mathsf{Hom}(\mathcal{F}'', \mathcal{I}^\bullet) \to \mathsf{Hom}(\mathcal{F}, \mathcal{I}^\bullet) \to \mathsf{Hom}(\mathcal{F}', \mathcal{I}^\bullet) \to 0$$

Taking the associated long exact sequence of cohomology groups h^i gives the sequence of Ext^i.

Similarly using Lemma 4.25, we see that $\mathcal{H}om(-, \mathcal{I})$ is an exact functor from $\mathcal{M}\mathrm{od}(X)$ to $\mathcal{M}\mathrm{od}(X)$. The same argument gives the exact sequence of $\mathcal{E}xt^i$. □

Proposition 4.28. *Suppose there is an exact sequence*

$$\cdots \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{F} \to 0$$

in $\mathfrak{Mod}(X)$, where the $\mathcal{L}_i$ are locally free sheaves of finite rank (in this case we say $\mathcal{L}_\bullet$ is a free resolution of $\mathcal{F}$). Then for any $\mathcal{G} \in \mathfrak{Mod}(X)$, we have

$$\mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) \cong h^i(\mathcal{H}om(\mathcal{L}_\bullet, \mathcal{G})).$$

Proof. Both sides are δ-functors in $\mathcal{G}$ from $\mathcal{M}\mathrm{od}(X)$ to $\mathcal{M}\mathrm{od}(X)$. For $i = 0$ they are equal, because $\mathcal{H}om(-, \mathcal{G})$ is contravariant and left exact. Both sides vanishes for $i > 0$ and $\mathcal{G}$ injective, because then $\mathcal{H}om(-, \mathcal{G})$ is exact so by universality of δ-functors, they are equal. □

Example 4.5. *If X is a scheme, which is quasi-projective over Spec A, where A is a noetherian ring, then any coherent sheaf $\mathcal{F}$ on X is quotient of a locally free sheaf. Thus any coherent sheaf on X has locally free resolution $\mathcal{L}_\bullet \to \mathcal{F} \to 0$. So the previous proposition tells us that we can calculate $\mathcal{E}xt$ by taking locally free resolutions in the first variable.*

Lemma 4.29. *If $\mathcal{L} \in \mathcal{M}\mathrm{od}(X)$ is locally free of finite rank, and $\mathcal{I}$ is injective, then $\mathcal{L} \otimes \mathcal{I}$ is also injective.*

Proof. We must show that the functor $\mathsf{Hom}(-, \mathcal{L} \otimes \mathcal{I})$ is exact. But it is the same functor as $\mathsf{Hom}(- \otimes \check{\mathcal{L}}, \mathcal{I})$, which is exact because $- \otimes \check{\mathcal{L}}$ is exact and $\mathcal{I}$ is injective. $\square$

Proposition 4.30. *Let $\mathcal{L}$ be a locally free sheaf of finite rank, and let $\check{\mathcal{L}} = \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$ be its dual. Then for any $\mathcal{F}, \mathcal{G} \in \mathcal{M}\mathrm{od}(X)$ we have*

$$\mathsf{Ext}^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \cong \mathsf{Ext}^i(\mathcal{F}, \check{\mathcal{L}} \otimes \mathcal{G}),$$

and for the sheaf $\mathcal{E}xt$ we have

$$\mathcal{E}xt^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \cong \mathcal{E}xt^i(\mathcal{F}, \check{\mathcal{L}} \otimes \mathcal{G}) \cong \mathcal{E}xt(\mathcal{F}, \mathcal{G}) \otimes \check{\mathcal{L}}.$$

Proof. The case $i = 0$, is already proved. For the general case, note that all of them are δ-functors in $\mathcal{G}$ from $\mathcal{M}\mathrm{od}(X)$ to $\mathcal{A}\mathrm{b}$ (respectively, $\mathcal{M}\mathrm{od}(X)$), since tensoring with $\check{\mathcal{L}}$ is an exact functor. For $i > 0$ and $\mathcal{G}$ injective they all vanish, by Lemma 4.29, so by universality they are equal. $\square$

We now give some properties which are more particular to the case of schemes.

Proposition 4.31. *Let X be a notherian scheme, let $\mathcal{F}$ be a coherent sheaf on X, let $\mathcal{G}$ be any $\mathcal{O}_X$-module, and let $x \in X$ be a point. Then we have*

$$\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})_x \cong \mathsf{Ext}^i_{\mathcal{O}_x}(\mathcal{F}_x, \mathcal{G}_x)$$

for any $i \geq 0$, where the right-hand side is Ext *over the local ring $\mathcal{O}_x$.*

Proof. The Ext over a ring A is defined as the right derived functor of $\mathsf{Hom}(M, -)$ for any A-module M, considered as a functor from $A\mathrm{Mod}$ to $A\mathrm{Mod}$.

Our question is local, by Proposition 4.25, we may assume that X is affine. Then $\mathcal{F}$ has a free resolution $\mathcal{L}_\bullet \to \mathcal{F} \to 0$ which on stalks at x gives a free resolution $(\mathcal{L}_\bullet)_x \to \mathcal{F}_x \to 0$. So by Proposition 4.28, we can calculate both sides by these resolutions. Since $\mathcal{H}om(\mathcal{L}, \mathcal{G})_x = \mathsf{Hom}_{\mathcal{O}_x}(\mathcal{L}_x, \mathcal{G}_x)$ for a locally free sheaf $\mathcal{L}$, and since the stalk functor is exact, we get the result.

Note that even the case $i = 0$ is not true without some special hypothesis on $\mathcal{F}$, such as $\mathcal{F}$ is coherent. $\square$

Lemma 4.32. *Let X be a projective scheme over a noetherian ring A and let $\mathcal{F} \to \mathcal{G}$ be a surjective morphism of coherent sheaves, then there exists $N > 0$ such that for all $n \geq N$ the morphism of A-modules $\Gamma(X, \mathcal{F}(n)) \to \Gamma(X, \mathcal{G}(n))$ is surjective.*

Proof. Let $\mathcal{K}$ be the kernel of $\mathcal{F} \to \mathcal{G}$. Then we have a short exact sequence

$$0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{G} \to 0.$$

Now tensoring with the flat sheaf $\mathcal{O}_X(n)$ gives us, $0 \to \mathcal{K}(n) \to \mathcal{F}(n) \to \mathcal{G}(n) \to 0$. The long exact sequence of cohomology,

$$0 \to \Gamma(X, \mathcal{K}(n)) \to \Gamma(X, \mathcal{F}(n)) \to \Gamma(X, \mathcal{G}(n)) \to H^1(X, \mathcal{K}(n)) \to \cdots,$$

By Serre's theorem 2.38, there is an $N > 0$ such that $H^1(X, \mathcal{K}(n)) = 0$ for all $n \geq N$. Therefore it follows that for all $n \geq N$, $\Gamma(X, \mathcal{F}(n)) \to \Gamma(X, \mathcal{G}(n))$ is surjective.

□

Lemma 4.33. *Let X be a projective scheme over a noetherian ring A and suppose we have an exact sequence of coherent sheaves for some $r \geq 3$*

$$\mathcal{F}^1 \to \mathcal{F}^2 \to \cdots \to \mathcal{F}^r.$$

Then there exists N such that for all $n \geq N$ the following sequence is exact.

$$\Gamma(X, \mathcal{F}^1(n)) \to \Gamma(X, \mathcal{F}^2(n)) \to \cdots \to \Gamma(X, \mathcal{F}^r(n))$$

Proof. Follows from the lemma above. □

Lemma 4.34. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be an isomorphism of ringed spaces. For $\mathcal{F}, \mathcal{G} \in \mathfrak{Mod}(X)$, there is a canonical isomorphism of sheaves of modules*

$$\theta : f_*(\mathcal{E}xt^i_X(\mathcal{F}, \mathcal{G})) \to \mathcal{E}xt^i_Y(f_*\mathcal{F}, f_*\mathcal{G})$$

Proof. (III, Ex. 6.10) □

Lemma 4.35. *Let $X = \operatorname{Spec} A$ be an affine scheme and let M, N be A-modules. Then for $i \geq 0$, there is a canonical morphism of sheaves of modules*

$$\lambda^i : \mathsf{Ext}^i_A(M, N)^\sim \to \mathcal{E}xt^i_X(\widetilde{M}, \widetilde{N})$$

Moreover, if A is a noetherian ring and M finitely generated, then this is an isomorphism.

Proof. (III, Ex. 6.7) □

Lemma 4.36. *Let X be a noetherian scheme, and let $\mathcal{F}, \mathcal{G} \in \mathcal{M}\mathrm{od}(X)$.*

(a) If $\mathcal{F}$ is coherent and $\mathcal{G}$ is quasi-coherent, then $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})$ is quasi-coherent for $i \geq 0$.

(b) If $\mathcal{F}, \mathcal{G}$ are both coherent then $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})$ is coherent for $i \geq 0$.

Proof. (a) Given a point $x \in X$, let U be an open affine neighborhood of x with the canonical isomorphism $f : U \to \mathrm{Spec}\ \mathcal{O}_X(U)$. Since $\mathcal{F}$ is coherent, $\mathcal{F}(U)$ is finitely generated module and we have the following canonical isomorphism for $i \geq 0$, by combining Proposition 4.25, Lemma 4.34 and Lemma 4.35 together

$$\begin{aligned} f_*\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})|_U &\cong f_*(\mathcal{E}xt^i(\mathcal{F}|_U, \mathcal{G}|_U)) \\ &\cong \mathcal{E}xt^i(f_*\mathcal{F}|_U, f_*\mathcal{G}|_U) \\ &\cong \mathcal{E}xt^i(\mathcal{F}(U)\tilde{}, \mathcal{G}(U)\tilde{}) \\ &\cong \mathsf{Ext}^i_{\mathcal{O}_X(U)}(\mathcal{F}(U), \mathcal{G}(U)\tilde{}. \end{aligned}$$

Which shows that $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})$ is a quasi-coherent sheaf of modules.
For (b), if $\mathcal{G}$ is coherent, $\mathcal{G}(U)$ is finitely generated, then the module $\mathsf{Ext}^i_{\mathcal{O}_X(U)}(\mathcal{F}(U), \mathcal{G}(U))$ is also finitely generated. Hence $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})$ is coherent. □

Proposition 4.37. *Let X be a projective scheme over a noetherian ring A, let $\mathcal{O}_X(1)$ be a very ample invertible sheaf, and let $\mathcal{F}, \mathcal{G}$ be coherent sheaves on X. Then there is an integer $n_0 > 0$, depending on $\mathcal{F}, \mathcal{G}$ and i, such that for every $n \geq n_0$ we have*

$$\mathsf{Ext}^i(\mathcal{F}, \mathcal{G}(n)) \cong \Gamma(X, \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}(n))).$$

Proof. The statement is true for $i = 0$ for any $\mathcal{F}, \mathcal{G}$ and n. If $\mathcal{F} = \mathcal{O}_X$, then the left hand side is $H^i(X, \mathcal{G}(n))$ by Proposition 4.26. So for $n \gg 0$ and $i > 0$ it is 0 by Serre's theorem 2.38. On the other hand, the right-hand side is always 0 for $i \gg 0$ by Proposition 4.26 (b), so we have the result for $\mathcal{F} = \mathcal{O}_X$.

If $\mathcal{F}$ is a locally free sheaf, we reduce to the case $\mathcal{F} = \mathcal{O}_X$ by Proposition 4.30. We have in the LHS, $\mathsf{Ext}^i(\mathcal{F}, \mathcal{G}(n)) = \mathsf{Ext}^i(\mathcal{O}_X, \mathcal{G}(n) \otimes \check{\mathcal{F}})$. On the RHS, $\Gamma(X, \mathcal{E}xt^i(\mathcal{O}_X, \mathcal{G}(n)) \otimes \check{\mathcal{F}})$.

Finally, if $\mathcal{F}$ is an arbitrary coherent sheaf, write it as a quotient of a locally free sheaf $\mathcal{E}$ (Corollary 2.39), and let $\mathcal{R}$ be the kernel:

$$0 \to \mathcal{R} \to \mathcal{E} \to \mathcal{F} \to 0.$$

Since $\mathcal{E}$ is locally free, by the early results, for $n \gg 0$, we have an exact sequence

$$0 \to \mathsf{Hom}(\mathcal{F}, \mathcal{G}(n)) \to \mathsf{Hom}(\mathcal{E}, \mathcal{G}(n)) \to \mathsf{Hom}(\mathcal{R}, \mathcal{G}(n)) \to \mathsf{Ext}^1(\mathcal{F}, \mathcal{G}(n)) \to 0$$

and isomorphisms, for all $i > 0$

$$\mathsf{Ext}^i(\mathcal{R}, \mathcal{G}(n)) \cong \mathsf{Ext}^{i+1}(\mathcal{F}, \mathcal{G}(n)).$$

In similar way, for the sheaves $\mathcal{H}om$ and $\mathcal{E}xt$, we use Lemma 4.36 and Proposition 4.30, then we obtain an exact sequence of coherent sheaves of modules

$$0 \to \mathcal{H}om(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om(\mathcal{E}, \mathcal{G}) \to \mathcal{H}om(\mathcal{R}, \mathcal{G}) \to \mathcal{E}xt^1(\mathcal{F}, \mathcal{G}) \to 0.$$

Replacing $\mathcal{G}$ by $\mathcal{G}(n)$, we get isomorphisms

$$\mathcal{E}xt^i(\mathcal{R}, \mathcal{G}(n)) \cong \mathcal{E}xt^{i+1}(\mathcal{F}, \mathcal{G}(n)).$$

Now Lemma 4.33 says that we can find an integer $N > 0$ such that for all $n \geq N$ the rows of the following diagram are exact,

$$\begin{tikzcd}
\mathsf{Hom}(\mathcal{E}, \mathcal{G}(n)) \arrow[r] \arrow[d, "\cong"] & \mathsf{Hom}(\mathcal{R}, \mathcal{G}(n)) \arrow[r] \arrow[d, "\cong"] & \mathsf{Hom}(\mathcal{F}, \mathcal{G}(n)) \arrow[r] \arrow[d] & 0 \\
\Gamma(X, \mathcal{H}om(\mathcal{E}, \mathcal{G}(n))) \arrow[r] & \Gamma(X, \mathcal{H}om(\mathcal{R}, \mathcal{G}(n))) \arrow[r] & \Gamma(X, \mathcal{H}om(\mathcal{F}, \mathcal{G}(n))) \arrow[r] & 0
\end{tikzcd}$$

Then there is an induced isomorphism of abelian groups $\mathsf{Hom}(\mathcal{F}, \mathcal{G}(n)) \to \Gamma(X, \mathcal{H}om(\mathcal{F}, \mathcal{G}(n)))$ for every $n \geq N$ and this prove the result for $i = 1$. But $\mathcal{R}$ is also coherent, so by induction we get the general result. $\square$

4.2 The Serre Duality Theorem

We present here the Serre duality theorem for the cohomology of coherent sheaves on a projective space and the proof. Next we will see that on a projective scheme over a field k there is for a will see that there is coherent sheaf ω_X°, the so-called dualizing sheaf. We end this section with the statement of Serre's duality for projective scheme over a field $k = \overline{k}$.

Theorem 4.38 (Duality for $\mathbb{P}_k^n$)**.** *Let k be a field, let $X = \mathbb{P}_k^n$ be the n-dimensional projective space over k, and let ω_X be the canonical sheaf on X.*

(a) $H^n(X, \omega_X) \cong k$. *Fix one isomorphism;*

(b) *for any coherent sheaf $\mathcal{F}$ on X, the natural pairing*

$$\mathsf{Hom}(\mathcal{F}, \omega_X) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X) \cong k,$$

is a perfect pairing of finite dimensional k-vector space;

(c) *for every $i \geq 0$ there is a natural functorial isomorphism*

$$\mathsf{Ext}(\mathcal{F}, \omega_X \to H^{n-i}(X, \mathcal{F})^\vee,$$

where $^\vee$ denotes the dual vector space, which for $i = 0$ is the one induced by the pairing of (b).

Proof. Part (a) follows from Example 4.4, which asserts that $\omega_X \cong \mathcal{O}_X(-n-1)$. Thus the result follows from a computation we have made before.

(b) We first note that the paring in (b) is natural, since any morphism $\mathcal{F} \to \omega_X$ induces a map of cohomology groups $H^i(X, \mathcal{F}) \to H^i(X, \omega_X)$ for each i. If $\mathcal{F} = \mathcal{O}_X(q)$ for some $q \in \mathbb{Z}$, then by the Ext property,

$$\begin{aligned} \mathsf{Hom}(\mathcal{F}, \omega_X) = \mathsf{Ext}^0(\mathcal{O}_X(q), \omega_X) &\cong \mathsf{Ext}^0(\mathcal{O}_X, \mathcal{O}_X(-q) \otimes \omega_X) \\ &\cong H^0(X, \omega_X(-q)). \end{aligned}$$

So the result follows from Theorem 2.38. Hence (b) holds also for a finite direct sum of sheaves of the form $\mathcal{O}_X(q_i)$. Now if $\mathcal{F}$ is an arbitrary coherent sheaf, we can write it as a cokernel $\mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$ of a map of sheaves $\mathcal{E}_1 \to \mathcal{E}_0$ where each $\mathcal{E}_i$ being a finite direct sum of sheaves of the form $\mathcal{O}_X(q_i)$ (Corollary 2.39). Now both $\mathsf{Hom}(-, \omega_X)$ and $H^n(X, -)^\vee$ are both left-exact contravariant functors, so we have the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathsf{Hom}(\mathcal{F}, \omega_X) \arrow[r] \arrow[d] & \mathsf{Hom}(\mathcal{E}_0, \omega_X) \arrow[r] \arrow[d, "\cong"] & \mathsf{Hom}(\mathcal{E}_1, \omega_X) \arrow[d, "\cong"] \\ 0 \arrow[r] & H^n(X, \mathcal{F})^\vee \arrow[r] & H^n(X, \mathcal{E}_0)^\vee \arrow[r] & H^n(X, \mathcal{E}_1)^\vee \end{tikzcd}$$

Using five-lemma, it implies that $\mathsf{Hom}(\mathcal{F}, \omega_X) \to H^n(X, \mathcal{F})^\vee$ is an isomorphism, and the statement (b) is proved.

(c) We sketch the main idea. One may check that both sides are contravariant δ-functors, for any $\mathcal{F}$ a coherent sheaf on X, indexed by $i \geq 0$. For $i = 0$ we have an isomorphism by (b). To show that they are isomorphic it will be sufficient to show that both sides are coeffaceable for $i > 0$, therefore the functors $\{\mathsf{Ext}^i(-, \omega_X\}_i$ and $\{H^n(X, -)^\vee\}_i$ are universal: given $\mathcal{F}$ coherent, it follows from Cor1.1 (Report3/19) that we can write $\mathcal{F}$

as a quotient of a sheaf $\mathcal{E} = \bigoplus_i^N \mathcal{O}_X(-q)$, with $q \gg 0$. Then

$$\mathsf{Ext}^i(\mathcal{E}, \omega_X) \cong \bigoplus_{i=1}^{N} \mathsf{Ext}^i(\mathcal{O}_X(-q), \omega_X) \cong \bigoplus_{i=1}^{N} \mathsf{Ext}^i(\mathcal{O}_X, \omega_X(q)) \cong \bigoplus H^i(X, \omega_X(q)) = 0,$$

the last isomorphism comes from (Report3/19 Th.2). on the other hand, for $0 < i \leq n$

$$H^{n-i}(X, \mathcal{E})^\vee = \bigoplus H^{n-i}(X, \mathcal{O}_X(-q))^\vee = 0,$$

again from Theorem 2.38. Thus both δ-functor are coeffaceable for $i > 0$ so they are universal. This shows (c). □

Definition 4.39. Let X be a proper scheme of dimension n over a field k. A *dualizing sheaf* for X is a coherent sheaf ω_X° on X, together with a *trace* morphism $t : H^n(X, \omega_X^\circ) \to k$, such that for all coherent sheaf $\mathcal{F}$ on X, the natural pairing

$$\mathsf{Hom}(\mathcal{F}, \omega_X^\circ) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X^\circ)$$

followed by t gives an isomorphism

$$\mathsf{Hom}(\mathcal{F}, \omega_X^\circ) \xrightarrow{\sim} H^n(X, \mathcal{F})^\vee$$

Proposition 4.40. *Let X be a proper scheme over a field k. Then a dualizing sheaf or X if it exits, is unique.*

Lemma 4.41. *Let X be a closed subscheme of codimension r of $Y = \mathbb{P}_k^N$. Then $\mathcal{E}xt_Y^i(\mathcal{O}_X, \omega_Y) = 0$ for all $i < r$.*

Lemma 4.42. *With the same hypotheses as Lemma 4.41, let $\omega_Y^\circ = \mathcal{E}xt_Y^r(\mathcal{O}_X, \omega_Y)$. Then for any $\mathcal{O}_X$-module $\mathcal{F}$, there is a functorial isomorphism*

$$\mathsf{Hom}_X(\mathcal{F}, \omega_X^\circ) \cong \mathsf{Ext}_Y^r(\mathcal{F}, \omega_Y).$$

Proposition 4.43. *Let X be a projective scheme over a field k. Then X has a dualizing sheaf.*

Proof. Embed X as a closed subscheme of $Y = \mathbb{P}_k^N$ for some N, let r be its codimension, and let $\omega_X^\circ = \mathcal{E}xt_Y^r(\mathcal{O}_X, \omega_Y)$. Then by Lemma 4.42, we have an isomorphism for any $\mathcal{O}_X$-module $\mathcal{F}$,

$$\mathsf{Hom}_X(\mathcal{F}, \omega_X^\circ) \cong \mathsf{Ext}_Y^r(\mathcal{F}, \omega_Y).$$

On the other hand, when $\mathcal{F}$ is coherent, the duality theorem for Y gives us an isomorphism,

$$\mathsf{Ext}_X^r(i_*\mathcal{F}, \omega_X^\circ) \cong H^{N-r}(\mathbb{P}_k^N, i_*\mathcal{F})^\vee.$$

But $N - r = n$, the dimension of X, so we obtain a functorial isomorphism, for any coherent sheaf $\mathcal{F}$ on X,

$$\mathsf{Hom}_X(\mathcal{F}, \omega_X^\circ) \cong H^n(\mathbb{P}_k^N, i_*\mathcal{F})^\vee \cong H^n(X, \mathcal{F})^\vee.$$

In particular, taking $\mathcal{F} = \omega_X^\circ$, the element $Id \in \mathsf{Hom}_X(\omega_X^\circ, \omega_X^\circ)$ gives us a homomorphism $t : H^n(X, \mathcal{F})^\vee) \to k$, which we take as our trace map. Then it is clear from the definition that the pair (ω_X°, t) is our dualizing sheaf for X. $\square$

Before stating the Serre duality for projective schemes, we will gather some results concerning Cohen-Macaulay rings, which are useful in the statement.

Definition 4.44. Let A be a ring, M an A-module and $a_1, ..., a_r$ a sequence of elements of A. We say that $a_1, ..., a_r$ is an M-regular sequence if a_1 is not a zero divisor in M, and for all $i = 2, ..., r$, a_i is not a zero divisor in $M/(a_1, ..., a_{r-1})M$.

If A is a local ring with maximal ideal $\mathfrak{m}$, we define the *depth* of M to be the maximum length of an M-regular sequence with all $a_i \in \mathfrak{m}$.

These definitions apply to the ring A itself, and we say that a local noetherian ring A is *Cohen-Macaulay* if depth $A = \dim A$

Now we enumerate some properties of Cohen-Macaulay rings.

Theorem 4.45. *Let A be a local noetherian ring with maximal ideal $\mathfrak{m}$.*

(a) If A is regular, then it is Cohen-Macaulay.

(b) If A is Cohen-Macaulay, then any localization of A at a prime ideal is also Cohen-Macaulay.

(c) If A is Cohen-Macaulay, then a set of elements $x_1, ..., x_r \in \mathfrak{m}$ forms an A-regular sequence if and only if $\dim A/(x_1, ..., x_r) = \dim A - r$

Definition 4.46. We say that a scheme is *Cohen-Macaulay* if all its local rings are Cohen-Macaulay.

Now we are ready to state the duality theorem for a projective scheme X.

Theorem 4.47 (Duality for a Projective Scheme)**.** *Let X be a projective scheme of dimension n over an algebraically closed field k. Let ω_X° be a dualizing sheaf for X, and let $\mathcal{O}_X(1)$ be a very ample sheaf on X. Then:*

(a) *for all $i \geq 0$ and $\mathcal{F}$ coherent on X, there are natural functorial maps*

$$\theta^i : \mathsf{Ext}^i(\mathcal{F}, \omega_X^\circ) \to H^{n-i}(X, \mathcal{F})^\vee,$$

such that θ^0 is the map given in the definition of dualizing sheaf;

(b) *the following conditions are equivalent:*

(i) *X is Cohen-Macaulay and equidimensional(i.e all irreducible components have the same dimension);*

(ii) *for any $\mathcal{F}$ locally fre on X, we have $H^i(X, \mathcal{F}(-q)) = 0$ for $i < n$ and $q \gg 0$;*

(iii) *the map θ^i of (a) are isomorphisms for all $i \geq 0$ and all $\mathcal{F}$ coherent on X.*

Proposition 4.48. *If X is a projective nonsingular variety over an algebraically closed field k, then the dualizing sheaf ω_X° is isomorphic to the canonical sheaf ω_X.*

Proof. [Hartshorne, 1977](III, Cor. 7.12) □

Remark 4.49. If X is a projective nonsingular curve, we find that $H^1(X, \mathcal{O}_X)$ and $H^0(X, \omega_X)$ are dual vector spaces. Indeed, X is a Cohen-Macaulay scheme by Theorem 4.45 (a), so we know from Serre duality in Theorem 4.47, that $\mathsf{Ext}^0(\mathcal{O}_X, \omega_X^\circ)$ and $H^1(X, \mathcal{O}_X)$ are dual vector spaces. On the other hand, using the above proposition, we have $\omega_X^\circ \cong \omega_X$. Thus the claim follows from the Ext property, and we come up with

$$\mathsf{Ext}^0(\mathcal{O}_X, \omega_X) \cong H^0(X, \omega_X) \cong H^1(X, \mathcal{O}_X)^\vee.$$

4.3 Riemann-Roch Theorem

In this section we will see an application of Serre duality in the proof of Riemann-Roch theorem for projective curves. We begin with an excursion on elementary results about divisors on a scheme, illustrating the notion to the case of smooth curves and we end up with the statement of the main theorem of this section and its proof.

4.3.1 Weil Divisors

Definition 4.50. We say a scheme X is regular in codimension one (or sometimes nonsingular in codimension one) if every local ring $\mathcal{O}_{X,x}$ of X dimension one is regular.

In this section we will consider schemes satisfying the following property: $(*)$ X is a noetherian integral separated scheme which is regular in codimension one.

Definition 4.51. Let X satisfy $(*)$. A *prime divisor* on X is a closed integral subscheme Y of codimension one. A *Weil divisor* is an element of the free abelian group $\mathrm{Div}(X)$ generated by the prime divisors. We write a divisor as $D = \sum n_i Y_i$, where the Y_i are prime divisors, the n_i are integers, and only finitely many are different from zero. If all the $n_i \geq 0$, we say that D is *effective*.

If Y is a prime divisor on X, let η be its generic point.Then the local ring $\mathcal{O}_{X,\eta}$ is a discrete valuation ring with quotient field $K(X)$ of X. We call the corresponding discrete valuation v_Y the *valuation* of Y. Now let $f \in K(X)^*$ be any nonzero rational function on X. Then v_Y is an integer. If it is positive, we say f has *zero* along Y, of that order; if it is negative, we say f has a *pole* along Y of order $-v_Y(f)$.

Lemma 4.52. *Let X satisfy $(*)$ and let $f \in K(X)^*$ be a nonzero function on X. Then $v_Y(f) = 0$ for all except finitely many prime divisors Y.*

Definition 4.53. Let X satisfy $(*)$ and let $f \in K(X)^*$. We define the divisor of f, denoted $\mathrm{div}(f)$, by

$$\mathrm{div}(f) = \sum v_Y(f) \cdot Y,$$

where the sum is taken over all prime divisor of X. By Lemma 4.52, this is a finite sum, hence it is a divisor. Any divisor which is equal to the divisor of a function is called *principal* divisor.

Remark 4.54. Because of the properties of valuations, we note that if $f, g \in K(X)^*$, then $\mathrm{div}(f/g) = \mathrm{div}(f) - \mathrm{div}(g)$. Therefore sending a function f to its divisor $\mathrm{div}(f)$ gives us a homomorphism of the multiplicative group $K(X)^*$ to the additive group $\mathrm{Div}(X)$, and the image, which consists of the principal divisors, is a subgroup of $\mathrm{Div}(X)$.

Definition 4.55. Let X satisfy $(*)$. Two divisors D and D' are said to be *linearly equivalent*, written $D \sim D'$, if $D - D' = \mathrm{div}(f)$, for some $f \in K(X)^*$. The group $\mathrm{Div}(X)$ of all divisors divided by the subgroup of principal divisors is called the *divisor class group* of X, and is denoted by $\mathrm{Cl}(X)$.

We will see a few special cases of calculation of divisor class group.

Proposition 4.56. *Let A be a noetherian domain. Then A is a unique factorization domain if and only if $X = \mathrm{Spec}\ A$ is normal and $\mathrm{Cl}(X) = 0$.*

Example 4.6. *If X is affine n-space $\mathbb{A}^n_k$ over a field k, then $\mathrm{Cl}(X) = 0$. Indeed, $X = \mathrm{Spec}\ k[T_1, ..., T_n]$, and the polynomial ring is a unique factorization domain.*

Proposition 4.57. *Let X be a projective space $\mathbb{P}^n_k$ over a field k. For any divisor $D = \sum n_i Y_i$, define the degree of D by $\deg D = \sum n_i \deg Y_i$, where $\deg Y_i$ is the degree of the hypersurface Y_i. Let H be the hyperplane $T_0 = 0$. Then:*

(a) *if D is any divisor of degree d, then $D \sim dH$*

(b) *for any $f \in K(X)^\times, \deg \operatorname{div}(f) = 0$;*

(c) *the degree function gives an isomorphism* $\deg : \mathrm{Cl}(X) \to \mathbb{Z}$

Proposition 4.58. *Let X satisfy $(*)$, let Z be a proper closed subset of X, and let $U = X \setminus Z$. Then:*

(a) *there is a surjective homomorphism* $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ *defined by*

$$D = \sum n_i Y_i \mapsto \sum n_i (Y_i \cap U),$$

where we ignore those $Y_i \cap U$ which are empty;

(b) *if* $\operatorname{codim}(Z, X) \geq 2$, *then* $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ *is an isomorphism;*

(c) *if Z is an irreducible subset of codimension 1, then there is an exact sequence*

$$\mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(U) \to 0,$$

where the first map is defined by $1 \mapsto 1 \cdot Z$.

Proof. (a) If Y is a prime divisor on X, then $Y \cap U$ is either empty or a prime divisor on U. If $[D_1] = [D_2]$ in $\mathrm{Cl}(X)$, then there exists an $f \in K(X)^\times$, such that $\operatorname{div}(f) = D_1 - D_2$, Considering f as a rational function on U, and if $\operatorname{div}(f) = \sum n_i Y_i$, with Y_i a prime divisor on X. We have $\operatorname{div}(f)|_U = \sum n_i (Y_i \cap U)$, so we obtain a homomorphism $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$. To show that it is surjective, let $D' = \sum n_i Y_i' \in \mathrm{Div}(U)$ with Y_i' a prime divisor on U. The closure $\overline{Y_i'}$ of Y_i' in X is a prime divisor on X, and $D = \sum n_i \overline{Y_i'}$ satisfies $D|_U = D'$. Hence $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ is surjective.

(b) The group $\mathrm{Div}(X)$ and $\mathrm{Cl}(X)$ depend only on subsets of codimension 1, so removing a closed subset of codimension ≥ 2 doesn't change anything.

(c) For the proof of exactness, suppose that $[D] \in \mathrm{Cl}(X)$ restricts to 0 in $\mathrm{Cl}(U)$. This means that D is a divisor of some $f \in K(U)^\times$. Since $K(U) = K(X)$ and the divisor of f in $\mathrm{Div}(X)$ restricts to the divisor of f in $\mathrm{Div}(U)$, it follows that we have $f \in K(X)^\times$ such that $D|_U = \operatorname{div}(f)|_U$. This implies that the difference $D|_U - \operatorname{div}(f)|_U$ is supported in $X \setminus U = Z$. So, if Z is irreducible $D - \operatorname{div}(f) \in Z$, and the kernel of $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ is just the subgroup of $\mathrm{Cl}(X)$ generated by $1 \cdot Z$. $\square$

Example 4.7. *Let Y be an irreducible curve of degree d in $X = \mathbb{P}^2_k$. For any divisor $[D] \in \mathrm{Cl}(X)$, write $[D] = \sum n_i Z_i$, with Z_i irreducible. Then $\mathrm{Cl}(X \setminus Y) = \mathbb{Z}/d\mathbb{Z}$. Let $U = X \setminus Y$. From Proposition 4.58, $[D]$ is in the kernel of $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ if and only if $\sum n_i(Z_i \cap U) = \mathrm{div}(f)|_U$ for some rational function f on $\mathbb{P}^2_k$. This means that on $\mathbb{P}^2_k$, we have $D - \mathrm{div}(f) = \sum n_i Z_i - \mathrm{div}(f) = k \cdot Y$, for some $k \in \mathbb{Z}$. It follows that the kernel is generated by Y. On the other hand the isomorphism of Proposition 4.57 $\deg : \mathbb{P}^2_k \to \mathbb{Z}$ gives the structure of $\mathrm{Cl}(U)$.*

4.3.2 Divisors on Curves

We will illustrate here the notion of divisors by paying special attention to the case of divisors on curves. We begin with some preliminaries about curves and morphisms of curves.

Definition 4.59. Let k be an algebraically closed field. A *curve* is a nonsingular integral scheme X of dimension 1, proper over k.

Lemma 4.60. *Let $f : X \to Y$ be a morphism of separated schemes over a noetherian schemes S. Let Z be a closed subscheme of X which is proper over S. Then $f(Z)$ is closed in Y and proper over S with its image subscheme structure.*

Proof. Ex 4.4 Hartshorne. □

Proposition 4.61. *Let X be a proper nonsingular curve over k, let Y be any curve over k, and let $f : X \to Y$ be a morphism. Then either (1) $f(X)$ is a point, or (2) $f(X) = Y$. In case (2), $K(X)$ is a finite extension field of $K(Y)$, f is a finite morphism, and Y is also proper.*

Proof. Since X is proper over k, $f(X)$ must be closed in Y, and proper over k, by Lemma 4.60. On the other hand, $f(X)$ is irreducible. Thus either $f(X) = pt$ or $f(X) = Y$, and in case Y is also proper. In case (2), f is dominant, it induces an inclusion $K(Y) \subseteq K(X)$ of function fields. Since both fields are finitely generated extension fields transcendence degree 1 of k, $K(X)$ must be a finite algebraic extension of $K(Y)$. It remains for us to show that f is a finite morphism, to do so, let $V = \mathrm{Spec}\, B$ be any open affine subset of Y. Let A be the integral closure of B in $K(X)$. Then A is a finite B-module (see for e.g [Samuel and Zariski, 1975] V, Th. 9), and $\mathrm{Spec}\, A$ is isomorphic to an open subset U of X. One can check that $U = f^{-1}V$, and this shows that f is finite morphism. □

The above proposition allows us to define the following.

Definition 4.62. Let $f : X \to Y$ be a finite morphism of curves, we define the degree of f to be the degree of the field extension $[K(X) : K(Y)]$.

Now we come to study of divisors on curves. If X is a curve, then X satisfies the property $(*)$ used above, so it makes sense to talk about divisors on X. A prime divisor is just a closed point, so an arbitrary divisor can be written $D = \sum n_i P_i$, where P_i are closed points, and $n_i \in \mathbb{Z}$. We define the *degree* of D to be $\sum n_i$.

Definition 4.63. Let $\phi : X \to Y$ be a morphism of singular curves and $\phi^* : K(Y) \to K(X)$ be the corresponding morphism of function fields. The *ramification index* of ϕ at a closed point P of X is

$$e_\phi(P) = v_P(\phi^* t_Q),$$

where $t_Q \in \mathcal{O}_{Y,Q}$ is a uniformizer at $Q = \phi(P)$.

Remark 4.64. Note that $\phi^* Q$ is independent of the choice of the uniformizer t_Q. Indeed if t'_Q is another uniformizer at Q, then $t_Q = u t'_Q$, where u is a unit in $\mathcal{O}_{Y,Q}$. For any point $P \in X$ with $\phi(P) = Q$, $\phi^* u$ will be a unit in $\mathcal{O}_{X,P}$, so $v_P(\phi^* t_Q) = v_P(\phi^* t'_Q) = e_\phi(P)$.

Definition 4.65. Let $\phi : X \to Y$ be a morphism of nonsingular curves, we define the pullback map ϕ^* on divisors, to be the homomorphism $\phi^* : \mathrm{Div}Y \to \mathrm{Div}X$ defined by

$$\phi^*(Q) = \sum_{P \in \phi^{-1}(Q)} e_\phi(P) \cdot P$$

We extend the definition by linearity to all divisors on Y.

Remark 4.66. Using ϕ^* to denote both the pullback map $\mathrm{Div}(Y) \to \mathrm{Div}(X)$ and the dual morphism $K(Y) \to K(X)$ might seem like an unfortunate collision of notation, but it is standard an intentional. Noting that the kernel of the map $k(X)^\times \to \mathrm{Div}(X)$ is just $k^\times$, so up to scalars we can identify a function $f \in K(X)$ with the corresponding divisor $\mathrm{div}(f) \in \mathrm{Div}(X)$.

Proposition 4.67. *Let $f : X \to Y$ be a finite morphism of nonsingular curves. Then for any divisor D on Y we have* $\deg f^* D = \deg f . \deg D$

Corollary 4.68. *A principal divisor on a complete nonsingular curve X has degree 0. Consequently the degree map induces a surjective homomorphism* $\deg : \mathrm{Cl}X \to \mathbb{Z}$

Proof. Let $f \in K(X)$. If $f \in k$ then $\mathrm{div}(f) = 0$, so there is nothing to prove. Assume $f \notin k$. Then the inclusion of fields $k(f) \subseteq K(X)$ induces a finite morphism $\varphi : X \to \mathbb{P}^1_k$ as follows: if $\varphi = g/h$, with $g, h \in K[X]$, represented by homogeneous functions of the same degree, and h nonzero, then φ is given by $(g : h)$.

Now let $(x : y)$ be a homogeneous coordinate for $\mathbb{P}^1_k$ and define $0 = (0 : 1)$ and $\infty = (1 : 0)$. Note that 0 and ∞ are both rational point on $\mathbb{P}^1_k$, then we may identify them with the closed the corresponding closed points. Let $t_0 = \frac{x}{y}$ and $t_\infty = \frac{y}{x}$ be the respective uniformizers at 0 and ∞. The images of these uniformizers under the field embedding $\varphi^* : K(\mathbb{P}^1_k) \to K(X)$, induced by ϕ are by definition

$$\varphi^* t_0 = t \circ \varphi = \frac{g}{h} = f \quad \text{and} \quad \varphi^* t_\infty = t_\infty \circ \varphi = \frac{h}{g} = 1/f.$$

Let us consider a closed point $P \in X$ for which $\varphi(P) = 0$. The ramification index of φ at P is by definition

$$e_\varphi(P) = v_P(\varphi^* t_0) = v_P(f).$$

The same way if we consider a closed point $P \in X$ for which $\varphi(P) = \infty$, we have

$$e_\varphi(P) = v_P(\varphi^* t_\infty) = v_P(1/f) = -v_P(f).$$

Applying the pullback map $\varphi^* : \mathrm{Div}(\mathbb{P}^1_k) \to \mathrm{Div}(X)$ to the divisor $[0]$ yields

$$\varphi^*[0] = \sum_{P \in \varphi^{-1}(0)} v_P(f) P.$$

Similarly,

$$\varphi^*[\infty] = \sum_{P \in \varphi^{-1}(\infty)} -v_P(f) P.$$

It follows that

$$\varphi^*([0] - [\infty]) = \varphi^*[0] - \varphi^*[\infty] = \sum_{P \in \varphi^{-1}(0)} v_P(f) P - \sum_{P \in \varphi^{-1}(\infty)} v_P(f) P = \mathrm{div}(f).$$

Using Proposition 4.67, $\deg \varphi^*([0] - [\infty]) = 0$ since $[0] - [\infty]$ has degree 0. Hence $\deg \mathrm{div}(f) = 0$ on X.

Thus the degree of a divisor on X depends only on its linear equivalence class, and we obtain a homomorphism as stated. It is surjective because the degree of a single point is 1. □

4.3.3 Cartier Divisors

Definition 4.69. Let X be a scheme. For each open affine subset $U = \mathrm{Spec}\, A$, let S be the set of element of A which are not zero divisors, and let $K(U)$ be the localization of A by the multiplicative closed set S. We call $K(U)$ the total quotient ring of A. For each open set U, let $S(U)$ denote the set of elements of $\Gamma(U, \mathcal{O}_X)$ which are not zero

divisors in each local ring $\mathcal{O}_{X,x}$ for each $x \in U$. Then the rings $S(U)^{-1}\Gamma(U, \mathcal{O}_X)$ form a presheaf, whose associated sheaf of rings $\mathcal{K}$ we call the *sheaf of total quotient rings* of $\mathcal{O}$.
We denote by $\mathcal{K}^{\times}$ the sheaf (of multiplicative groups) of invertible elements in $\mathcal{K}$, similarly we denote by $\mathcal{O}^{\times}$ the sheaf of invertible element in $\mathcal{O}_X$.

Since all regular functions are rational, we have the following short exact sequence

$$0 \to \mathcal{O}^{\times} \to \mathcal{K}^{\times} \to \mathcal{K}^{\times}/\mathcal{O}^{\times} \to 0.$$

Definition 4.70. A *Cartier divisor* on a scheme X is a global section of the sheaf $\mathcal{K}^{\times}/\mathcal{O}^{\times}$. Hence a Cartier divisor can be represented by a collection of pair (U_i, f_i), where $\{U_i\}$ is an open cover of X and $f_i \in \Gamma(U_i, \mathcal{K}^{\times})$. In other words, it is "locally" defined by a single rational function, and the ratio of such rational functions on the intersection is regular.
A Cartier divisor is *principal* if it is in the image of $\Gamma(X, \mathcal{K}^{\times})$, in other words, it is globally defined by a single rational function. Two Cartier divisors are *equivalent* if their ratio is principal.

Proposition 4.71. *Let X be an integral, separated noetherian scheme, all of whose local rings are unique factorization domains. Then the group* $\mathrm{Div}(X)$ *of Weil divisors on X is isomorphic to the group of Cartier divisors* $\Gamma(X, \mathcal{K}^{\times}/\mathcal{O}^{\times})$*, and furthermore, the principal Weil divisors correspond to the principal Cartier divisors under this isomorphism.*

We will see now that invertible sheaves on a scheme are closely related to divisor classes modulo linear equivalence.

Proposition 4.72. *If $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on a ringed space X, so is $\mathcal{L} \otimes \mathcal{M}$. If $\mathcal{L}$ is any invertible sheaf on X, then there exists an invertible sheaf $\mathcal{L}^{-1}$ on X such that $\mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathcal{O}_X$.*

Proof. If $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves, then both are locally free of rank 1. And using $\mathcal{O}_X \otimes \mathcal{O}_X \cong \mathcal{O}_X$, the first statement is true. For the second statement, let $\mathcal{L}$ be any invertible sheaf, and take $\mathcal{L}^{-1} = \mathcal{L}\check{} = \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$. Then $\mathcal{L}\check{} \otimes \mathcal{L} \cong \mathcal{H}om(\mathcal{L}, \mathcal{L}) \cong \mathcal{O}_X$. □

Definition 4.73. For any ringed space X, we define the *Picard group* of X denoted by $\mathrm{Pic}(X)$, to be the group of invertible sheaves on X, under the operation $\otimes$. The previous proposition shows that in fact it is a group.

Definition 4.74. Let D be a Cartier divisor on a scheme X, represented by $\{(U_i, f_i)\}$ as above. We define a subsheaf $\mathcal{L}(D)$ of the sheaf of total quotient rings $\mathcal{K}$ by taking

$\mathcal{L}(D)$ to be the sub-$\mathcal{O}_X$-module of $\mathcal{K}$ generated by f_i^{-1} on U_i. This is well-defined, since f_i/f_j is invertible on $U_i \cap U_j$, so f_i^{-1} and f_j^{-1} generate the same $\mathcal{O}_X$-module. We call $\mathcal{L}(D)$ the sheaf associated to D.

Proposition 4.75. *Let X be a scheme. Then:*

(a) for any Cartier divisor D, $\mathcal{L}(D)$ is an invertible sheaf on X. The map $D \mapsto \mathcal{L}(D)$ gives a 1-1 correspondence between Cartier divisors on X and invertible subsheaves of $\mathcal{K}$;

(b) $\mathcal{L}(D_1 - D_2) \cong \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}$

(c) $D_1 \sim D_2$ if and only if $\mathcal{L}(D_1) \cong \mathcal{L}(D_2)$ as abstract invertible sheaves (i.e., disregarding the embedding in $\mathcal{K}$).

Corollary 4.76. *On any scheme X, the map $D \mapsto \mathcal{L}(D)$ gives an injective homomorphism of the group* $\mathrm{CaCl}(X)$ *of Cartier divisors modulo linear equivalence to* $\mathrm{Pic}(X)$.

Proposition 4.77. *If X is an integral scheme, the homomorphism* $\mathrm{CaCl}(X) \to \mathrm{Pic}(X)$ *of Corollary 4.76 is an isomorphism.*

Corollary 4.78. *If X is a noetherian, integral, separated scheme, all of whose local rings are unique fatorization domains, then there is a natural isomorphism* $\mathrm{CaCl}(X) \cong \mathrm{Pic}(X)$.

Definition 4.79. A Cartier divisor on a scheme is effective if it can be represented by $\{(U_i, f_i)\}$, where all the $f_i \in \Gamma(U_i, \mathcal{O}_{U_i})$.In that case we define the associated subscheme of codimension 1, Y, to be the closed subscheme defined by the sheaf of ideal $\mathcal{I}$ which is locally generated by f_i.

Remark 4.80. This definition gives a 1-1 correspondence between effective Cartier divisors on X and *locally principal closed* subschemes Y, i.e., subschemes whose sheaf of ideals is locally generated by a single element.

Proposition 4.81. *Let D be an effective Cartier divisor on a scheme X, and let Y be associated locally principal closed subscheme. Then $\mathcal{I}_Y \cong \mathcal{L}(-D)$.*

4.4 Riemann-Roch Theorem

Definition 4.82. Let X be a projective scheme over a field k, and let $\mathcal{F}$ be a coherent sheaf on X. We define the *Euler Characteristic* of $\mathcal{F}$ by

$$\chi(\mathcal{F}) = \sum_i (-1)^i \dim_k H^i(X, \mathcal{F}).$$

Definition 4.83. Let X be a projective scheme of dimension r over a field k. We define the *arithmetic genus* p_a of X by

$$p_a(X) = -(1)^r(\chi(\mathcal{O}_X - 1)).$$

We define the geometric genus of X to be $p_g(X) := \dim_k H^0(X, \omega_X)$. We note that if X is a curve, by Remark 4.49, we have $p_a(X) = p_g(X)$, so we may call it simply the *genus* of X, and denote it by g.

Proposition 4.84. *Let X be an integral scheme of dimension r over an algebraically closed field k. If X is integral then*

$$p_a(X) = \sum_{i=0}^{r-1}(-1)^i \dim_k H^{r-i}(X, \mathcal{O}_X).$$

Proof. By definition of the arithmetic genus, and noting that $H^0(X, \mathcal{O}_X) \cong k$ if X is integral, we obtain

$$\begin{aligned} p_a(X) &= (-1)^r(\chi(\mathcal{O}_X - 1)) \\ &= -(-1)^r + \sum_{j=0}^{r}(-1)^{j+r} \dim_k H^j(X, \mathcal{O}_X) \\ &= \sum_{i=0}^{r-1}(-1)^i \dim_k H^i(X, \mathcal{O}_X) \end{aligned}$$

□

Remark 4.85. Let X be a curve over k. Let $D = \sum_{P \in X} n_p \cdot P$.

$$L(D) := H^0(X, \mathcal{L}(D)) \text{ and } l(D) := \dim L(D)$$

From [Liu et al., 2002], $L(D)$ could be described as

$$L(D) = \{f \in K(X)^\times | \ v_P(f) + n_P \geq 0, \forall P\} \cup \{0\}.$$

Lemma 4.86. *Let D be a divisor on a curve X. If $\deg D = 0$, then $l(D) \neq 0$ if and only if $D \sim 0$. If $\deg D < 0$, then $l(D) = 0$.*

Proof. Suppose $l(D) \neq 0$. Take $f \in L(D) \setminus \{0\}$. Then $\mathrm{div}(f) + D \geq 0$. As is D linearly equivalent to $\mathrm{div}(f) + D$. We conclude that $\deg D \geq 0$. If $\deg D = 0$, then D is linearly equivalent to an effective divisor of degree 0. But there is only one such, namely the zero divisor. □

Lemma 4.87. *Let k be a field. Let X be a proper scheme over k. Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of coherent sheaves on X. Then*

$$\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'').$$

Proof. From the short exact sequence of the statement we obtain the associated long exact sequence of cohomology

$$0 \to H^0(X, \mathcal{F}') \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}'') \to H^1(X, \mathcal{F}') \to \cdots .$$

The rank-nullity theorem says that

$$0 = \dim_k H^0(X, \mathcal{F}') - \dim_k H^0(X, \mathcal{F}) + \dim_k H^0(X, \mathcal{F}'') - \dim_k H^1(X, \mathcal{F}') + \cdots$$

Pulling all the terms with $\mathcal{F}$ to the left hand side gives us the result as required. □

Since X has dimension 1, the sheaf of relative differential $\Omega_{X/k}$ of X is an invertible sheaf on X, and so is equal to the canonical sheaf ω_X on X. We call any divisor in the corresponding linear equivalence class a *canonical divisor* and denote it by K_X. We are ready to state the Riemann-Roch.

Theorem 4.88 (Riemann-Roch)**.** *Let D be a divisor on a curve X of genus g. Then*

$$l(D) - l(K_X - D) = \deg D + 1 - g.$$

Proof. We have the correspondence between the divisor $K_X - D$ and the invertible sheaf $\omega_X \otimes \mathcal{L}(D)^{\check{}}$, by Proposition 4.75. Since X is projective, we may apply Serre duality and, Ext properties, and find that

$$\begin{aligned} H^1(X, \mathcal{L}(D))^\vee &\cong \mathsf{Ext}^0(\mathcal{L}(D), \omega_X) \\ &\cong \mathsf{Ext}^0(\mathcal{O}_X, \mathcal{L}(D)^{\check{}} \otimes \omega_X) \\ &\cong H^0(X, \mathcal{L}(D)^{\check{}} \otimes \omega_X) \end{aligned}$$

This computation tells us that $H^0(X, \omega_X \otimes \mathcal{L}(D)^{\check{}})$ is dual to $H^1(X, \mathcal{L}(D))$. By definition, $l(D) - l(K_X - D) = \dim_k H^0(X, \mathcal{L}(D)) - \dim_k H^0(X, \mathcal{L}(K_X - D))$. Thus we came up with

$$l(D) - l(K_X - D) = \dim_k H^0(X, \mathcal{L}(D)) - \dim_k H^1(X, \mathcal{L}(D)) = \chi(\mathcal{L}(D)).$$

Hence showing the theorem amounts to proving that for any divisor D

$$\chi(\mathcal{L}(D)) = \deg D + 1 - g.$$

First we consider the case $D = 0$. Then $\mathcal{L}(D) \cong \mathcal{O}_X$, and our formula says

$$\dim_k H^0(X, \mathcal{O}_X) - \dim_k H^1(X, \mathcal{O}_X) = 1 - g = 0 + 1 - g.$$

Since $H^0(X, \mathcal{O}_X) = k$ and $\dim_k H^1(X, \mathcal{O}_X) = g$, by the definition of genus, the formula holds.

Now, let D be any divisor, and let P be any point. Since any divisor can be reached from 0 in a finite number steps by adding or subtracting a point each time, it will be sufficient to show that the formula is true for D if and only if it is true for $D + P$.

We can view P as a closed subscheme of X, and let $\kappa(P)$ denote the skyscraper sheaf $\mathcal{O}_X/\mathcal{I}_P$, where $\mathcal{I}_P$ is the ideal sheaf of the closed subset $\{P\}$. From Proposition 4.81, $\mathcal{I}_P$ is isomorphic to $\mathcal{L}(-P)$. Therefore we have an exact sequence

$$0 \to \mathcal{L}(-P) \to \mathcal{O}_X \to \kappa(P) \to 0.$$

Tensoring with the locally free sheaf or rank one $\mathcal{L}(D + P)$, we get

$$0 \to \mathcal{L}(D) \to \mathcal{L}(D + P) \to \kappa(P) \to 0.$$

Now, taking the Euler characteristic on this short exact sequence, and noting $\chi(\kappa(P)) = 1$, we have

$$\chi(\mathcal{L}(D + P)) = \chi(\mathcal{L}(D)) + 1.$$

On the other hand, $\deg(D + P) = \deg(D) + 1$, so our formula is true for D if and only if it is true for $D + P$, as required. And the theorem is proved. □

Example 4.8. *On a curve X of genus g, the canonical divisor K_X has degree $2g - 2$. Indeed, applying Riemann-Roch theorem to $D = K_X$ yields,*

$$l(K) - l(0) = \deg K_X + 1 - g.$$

Since $l(K_X) = \dim_k H^0(X, \omega_X) = g$ and $l(0) = 1$, it implies that $\deg K_X = 2g - 2$.

Example 4.9. *An curve X is* elliptic *if its genus is 1. On an elliptic curve, the canonical divisor K_X has degree 0, by the previous example. Moreover, $l(K_X) = g = 1$, then from Lemma 4.86 we conclude that $K_X \sim 0$.*

Example 4.10. *Let E be an elliptic curve, let O_E be a point of E, and let $\mathrm{Pic}^{\circ}(E)$ denote the subgroup of $\mathrm{Pic}(E)$ corresponding to divisors of degree 0. We show here that the map $P \mapsto \mathcal{L}(P - O_E)$ gives the a one-to-one correspondence between the set of points of E and the element of $\mathrm{Pic}^{\circ}(E)$. Therefore we get a group structure with O_E as identity, on the set of points of E.*

To do so, we will show that if D is any divisor of degree 0 on E, then there exists a unique point $P \in E$ such that $D \sim P - O_E$. We apply Riemann-Roch to $D + O_E$,and obtain

$$\begin{aligned} l(D+O_E) - l(K_E - D - O_E) &= \deg(D+O_E) + 1 - g \\ &= 1 + 1 - 1. \end{aligned}$$

By the previous example, $\deg K_E = 0$, so $\deg(K_E - D - O_E) = -1$, and Lemma 4.86 implies that $l(K_E - D - O_E) = 0$. Therefore $l(D + O_E) = 1$. This means there is a unique effective divisor linearly equivalent to $D + O_E$. As the degree of $D + O_E$ is 1, it must be a single point P. Thus we have shown that there is a unique point $P \sim D + O_E$.

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