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WARPING
THE EFFECTIVE FIELD THEORY OF
STRING FLUX COMPACTIFICATIONS

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Alla mia famiglia

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1. Introduction

Although the Standard Model (SM) appears to correctly describe the behaviour of elementary particles and their interactions, it still presents some unsolved theoretical problems, like the so-called hierarchy problems, or the explanation of the mysterious dark matter, and moreover it is not known how to couple it to gravity in a consistent quantum theory of all known interactions. These facts suggest to consider the SM as an *effective theory*, valid only up to a certain energy scale Λ_{SM}, that is, as the low-energy manifestation of a more fundamental theory.

Several models have been developed which describe physics *beyond the Standard Model* (BSM). Starting from the Minimal Supersymmetric Standard Model, where supersymmetry is used as guiding principle for getting the natural extension of the SM [1–3], to String Theory, born to explain the strong interactions and grown up, once ''equipped'' with supersymmetry, as the best promising theory of everything, able to unify all kinds of elementary forces.

There exist five different types of critical [1] *Superstring Theories*: type I, type IIA, type IIB, heterotic $SO(32)$ and heterotic $E_8 \times E_8$. They live in a ten-dimensional spacetime and replace point-particles with several different oscillation modes of one-dimensional extended objects, the so-called ''strings''. All these different theories are linked among themselves by dualities, particular equivalences which relate different theories in different perturbative regimes (small–large or strong–weak dualities). These dualities suggested that these theories should be different manifestations of a unique theory, dubbed *M-theory*. It prescribes an eleven-dimensional spacetime and it is not a string theory. Nowadays, unfortunately, we know only the M-theory effective supergravity [4], ignoring its microscopical structure.

In most of the (semi-)realistic string/M-theory models, the unobserved 6/7 extra dimensions are assumed to be compactified on manifolds with an extremely small radius. The properties of the internal spaces manifest themselves in the physical properties of the corresponding effective four-dimensional theories. It is exactly the analysis of such physical properties one of the most intriguing and relevant aspects in string phenomenology.

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[1]That is, admitting a Poincaré invariant vacuum.

1. Introduction

This study is commonly carried out following the standard *Kaluza–Klein reduction* (KK) [5–8]. It consists in various steps, which can be summarised as follows. Firstly, one chooses a specific ten-dimensional vacuum field configuration, with a ten-dimensional manifold usually factorised as

$$M_{10} = M^{1,3} \times M_6 \quad , \tag{1.1}$$

with a Minkowskian $M^{1,3}$ and a compact M_6. Then, one expands all fields into modes on the internal M_6 and, finally, extracts the four-dimensional effective theory of light modes by integrating out the heavy ones. In fact, the latter have masses which are roughly inversely proportional to the compactification radius. Therefore, the corresponding modes are typically too massive to appear in the low-energy theory.

In the simplest cases, the reduction is carried out with a purely metric M_6. The ten-dimensional metric Ansatz is

$$ds_{10}^2 = ds_4^2 + ds_6^2 \quad , \tag{1.2}$$

and in the effective action there appear several massless scalar fields, which are not controlled by any potential. They are dubbed ''moduli'' and parametrise possible deformations of the internal metric ds^6 as well as of other ten-dimensional fields. They do not correspond to any particle in the SM and moreover their vevs define the coupling constants, making uncontrollable the effective theory!

To eliminate from the effective theory such problematic fields, one would like to find a mechanism to produce potentials giving sufficiently high masses to the moduli. This is referred to as the *moduli stabilisation problem*. The situation is ameliorated by performing the KK reduction starting from a non-purely metric background, in which non-trivial field-strengths are turned on. These kinds of backgrounds are more commonly called ''flux compactifications'', since field-strengths can be identified with their quantised flux along closed surfaces in M_6. Consistency requires fluxes to be accompanied by negatively charged sources, as *O-planes*. These are localised extended objects, which are non-dynamical at the perturbative level [9]. More generally, models can be enriched by the additional presence of *D-branes*, of positive charge. These are localised *dynamical* extended objects which generalise the concept of point-particles [10] and naturally lead to the appearance of non-abelian gauge groups and charged matter in the effective theory. One is naturally led to consider models maintaining a certain level of supersymmetry at high energy scales and this constrains the form and the number of local sources and fluxes. These models are under a much better control and supersymmetry can be spontaneously

broken at low energies, reproducing a more realistic non-supersymmetric effective theory.

Now, it is important to observe that the inclusion of fluxes, D-branes and O-planes modifies the spacetime geometry: a *warp factor* $e^{2A(y)}$ turns on, dressing the four-dimensional metric as follows

$$ds_{10}^2 = e^{2A(y)}ds_4^2 + ds_6^2 \quad .\tag{1.3}$$

This effect can have important physical implications. For instance, it is the basic ingredient in some ideas proposed to address the hierarchy problems [11, 12]. However, a non-trivial warping leads to significant complications in the reduction procedure, which are usually ignored by taking the so-called "large volume limit", neglecting the backreaction of fluxes and localised sources on the geometry and assuming the warping to be constant [13–15]. Although legitimate in some cases, this approximation is surely quite limiting and not always justified. Indeed, several attempts have been developed in order to compute the effective theory including a non-trivial warp factor [16–20].

Nonetheless, a systematic and simple way to include warping effects in the effective theory of flux compactifications is still missing. This thesis addresses this issue, presenting an approach, alternative to direct KK reduction, to compute the Kähler potential of the effective theory. Here is a summary of the thesis.

In Chapter 2 we introduce the ten-dimensional low-energy effective theory of Type IIB string theory. We first present the IIB supergravity, describing the light closed string states. Then, we include the open string sector, associated with Dp-branes, which can be regarded as non-perturbative objects on which open strings can end.

In Chapter 3 we review the basics of four-dimensional rigid supersymmetry and then of supergravity, in order to get the needed familiarity to recognise their structures in the four-dimensional effective theories obtained by dimensional reduction.

In Chapter 4 we deal with Type IIB compactifications. Firstly, in Section 4.1 we focus on compactifications on Calabi–Yau (CY) spaces, in which the only non-trivial field is the metric. We explain why moduli are present and where they come from. We show that the KK reduction (reviewed in Appendix B) leads to a four-dimensional $\mathcal{N} = 2$ supergravity, encoding information on the internal compact space.

Section 4.2, treats compactifications on orientifolds. The internal manifold M_6 corresponds to a CY modded out by a symmetry involution which projects out half of the starting background field content, reducing the effective theory to a more realistic $\mathcal{N} = 1$ supergravity. The orientifold projection introduces O-planes at involution fixed loci.

1. Introduction

In Section 4.3 we introduce fluxes, following [13]. Focusing on a background maintaining four-dimensional Poincaré invariance, we see how these fluxes can generate a four-dimensional scalar potential stabilising some moduli. Furthermore, such a potential gives a supersymmetry breaking mechanism to the effective $\mathcal{N} = 1$ supergravity. Then we include Dp-branes, with their own moduli, and see how their presence modifies the effective theory [21–24]. These results are obtained in the large volume limit, approximating the warping to a constant.

In Ch. 5 we present an alternative approach to dimensional reduction, in order to study the low-energy theory of IIB warped flux compactifications. This method, based on supersymmetry considerations [25–27], allows to take properly into account the non-trivial warp factor. We discuss how it allows in principle to compute the Kähler potential $K(\varphi, \bar{\varphi})$, which determines, in the effective action, the kinetic terms of a number of complex scalars φ^i parametrising the compactification moduli. We then explicitly apply this method to a simple class of warped compactifications on fluxed toroidal orientifolds $T^6/\mathbb{Z}_2$ [28], computing the explicit form of the associated low-energy Kähler potential.

2. IIB Supergravity

In string theory the string dynamics in the ten-dimensional spacetime is governed by the so-called *worldsheet theory*, which is a two-dimensional theory generalising the point-particle dynamics [29, 34]. In general, in a curved space, i.e. with non-flat background metric, the string worldsheet theory is interacting and usually not exactly solvable. However, it may be studied as an effective theory, using the so-called α' *expansion*, with the expansion parameter $\frac{\alpha'}{R^2} \sim \frac{\ell_s^2}{R^2} \sim \frac{E^2}{M_s^2}$. Here $\ell_s \equiv 2\pi\sqrt{\alpha'}$ and M_s are the length and energy string scales, while E and R are the length and energy scales at which the theory is studied. Because we do not observe strings, we should deal with energies well below M_s (which in many model is close to the four-dimensional Planck scale). In such a low-energy, or large radius, regime there is indeed not enough resolution to perceive the string spatial extent and its massive oscillation states, with masses quantized in terms of M_s. Only massless modes of oscillations are observable and thus strings behave as point-particles, with dynamics governed by an effective quantum field theory action, i.e. the supergravity action [1].

2.1. The closed string sector: the bulk

Type II supergravity theories are the maximally supersymmetric theories in ten-dimensional spacetime: they have 32 supercharges organized in two Majorana–Weyl spinors, thus $\mathcal{N} = 2$ [2]. There exist two types of such theories, differing for field content. Both type IIA and IIB are obtained naturally as the low-energy approximations of the type II superstring theories (closed string theories) and then their spectrum consists of the massless closed string modes, as illustrated in Tables 2.1 and 2.2.

The closed string sector organise in lef- and right- moving states. In each of these sectors NS or R refers to the two possible periodicity conditions on the worldsheet fermions. For both types II, the NSNS sector contains the dilaton scalar ϕ, the two-form B_2 and the metric g. The RR sector is different for the two types of supergravities. Type IIA

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[1] A full string theory description is only available in some special cases, like orbifolds or CFT constructions [29].

[2] Indeed ''II'' refers to the number of supersymmetries.

Sector	10d fields
NS-NS	ϕ, B_2, g
NS-R	λ^1, ψ_M^1
R-NS	λ^2, ψ_M^2
R-R	C_1, C_3

Table 2.1.: Massless spectrum of IIA supergravity.

Sector	10d fields
NS-NS	ϕ, B_2, g
NS-R	λ^1, ψ_M^1
R-NS	λ^2, ψ_M^2
R-R	C_0, C_2, C_4

Table 2.2.: Massless spectrum of IIB supergravity.

contains the odd p-forms C_1, C_3, while type IIB contains even p-forms: a scalar C_0, a two-form C_2 and a four-form C_4 with self-dual field-strength $\tilde{F}_5$. Fermions are in the NS-R and R-NS sectors: there are two Rarita–Schwinger fields, corresponding in the IIB case to two left-handed Majorana–Weyl gravitinos, and two spinors (right-handed Majorana–Weyl dilatinos). In type IIA the gravitinos and the dilatinos have opposite chirality [29].

The local symmetries of the type II theories are:

- ten-dimensional diffeomorphisms;

- gauge transformation of the B-field:

$$B_2 \rightarrow B_2 + d\lambda_1 \quad . \tag{2.1.1}$$

Differently from what happens for C_p fields, strings are charged under the B-field, as one can see from the two-dimensional string worldsheet action, see for instance [29];

- gauge transformations of p-form fields C_p:

$$C_p \rightarrow C_p + d\Lambda_{p-1} \quad . \tag{2.1.2}$$

There exist no states in the perturbative string spectrum charged under these fields, but branes, regarded as non-perturbative states in string theory, are charged under them [10];

- local superymmetry, with 32 supercharges arranged in two ten-dimensional Majorana–Weyl spinors (having 16 components each) of the same chirality for type IIB, opposite for type IIA. The spectrum is composed by a $\mathcal{N} = 2$ ten-dimensional gravity multiplet.

The great number of supersymmetries strongly constrains the low-energy actions for the massless states and uniquely fixes the type II supergravity actions up to two derivatives. In this thesis we are interested in the IIB type, whose massless spectrum is summarized in Table 2.2 [29].

We will study a IIB background, i.e. a field configuration that is a solution of the equations of motion, in which all fermionic fields vanish. Hence it is enough to consider just the bosonic sector of the tree-level IIB supergravity action [30] [3]:

$$
\begin{aligned}
S_{\text{IIB}}^{\text{sf}} &= S_{\text{NS}} + S_{\text{RR}} + S_{\text{CS}} \quad , \\
S_{\text{NS}} &= \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g_{\text{s}}}\, e^{-2\phi} \left(\mathcal{R}_{\text{s}} + 4(\nabla\phi)^2 - \frac{1}{2}|H_3|^2 \right) \quad , \\
S_{\text{R}} &= -\frac{1}{4\kappa_{10}^2} \int d^{10}x \sqrt{-g_{\text{s}}} \left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2 \right) \quad , \\
S_{\text{CS}} &= \frac{1}{4\kappa_{10}^2} \int C_4 \wedge H_3 \wedge F_3 \quad ,
\end{aligned}
$$

(2.1.3)

where $\frac{1}{2\kappa_{10}^2} = \frac{2\pi}{\ell_{\text{s}}^8} = \frac{2\pi}{(2\pi\sqrt{\alpha'})^8}$ gives the correct mass dimensions and it is often defined as $\kappa_{10}^2 \equiv \frac{(2\pi)^6}{M_{\text{P},10}^8}$. The field-strengths are defined as

$$
\begin{aligned}
F_p &= dC_{p-1} \quad \forall\, p = 1, 3, 5 \quad , \\
\tilde{F}_3 &= F_3 - C_0 H_3 \quad , \\
\tilde{F}_5 &= F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3 \quad .
\end{aligned}
$$

(2.1.4)

In many cases is more convenient to work in the *Einstein frame*, with the metric $g_{MN} = e^{-\phi/2}g_{\text{s}MN}$, defined in order to recover the usual Einstein–Hilbert term:

$$
\begin{aligned}
S_{\text{IIB}}^{\text{Ef}} &= \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g}\,\mathcal{R} - \frac{1}{4\kappa_{10}^2} \int d^{10}x \sqrt{-g} \bigg(e^{2\phi}|F_1|^2 + (\nabla\phi)^2 \\
&\quad + e^{\phi}|\tilde{F}_3|^2 + e^{-\phi}|H_3|^2 + \frac{1}{2}|\tilde{F}_5|^2 - C_4 \wedge H_3 \wedge F_3 \bigg) \quad .
\end{aligned}
$$

(2.1.5)

To be precise, the actions written above are not supersymmetric, because they possess more bosonic than fermionic degrees of freedom, due to the fact that they do not incorporate the self-duality condition

$$
\tilde{F}_5 = *\tilde{F}_5 \quad ,
$$

(2.1.6)

---

[3]Note that we use a different convention for the Hodge-$*$ with respect to [30], see Appendix A.

which has to be imposed as additional constraint at the level of equations of motion, reproducing the correct theory [4].

One can check that the equations of motion for the action (2.1.5), in absence of local sources, are [33]:

$$
\begin{aligned}
R_{MN} = {} & \frac{1}{2}\partial_M\phi\partial_N\phi + \frac{1}{2}e^{2\phi}\partial_M C_0\partial_N C_0 + \frac{1}{4\cdot 4!}\tilde{F}_{MABCD}\tilde{F}_N{}^{ABCD} \\
& + \frac{1}{4}\left(e^{\phi}\tilde{F}_{MAB}\tilde{F}_N{}^{AB} + e^{-\phi}H_{MAB}H_N{}^{AB}\right) \\
& - \frac{1}{48}g_{MN}\left(e^{\phi}\tilde{F}_{ABC}^2 + e^{-\phi}H_{ABC}^2\right) \quad,
\end{aligned}
$$
$$
d*(e^{\phi}\tilde{F}_3) = F_5\wedge H_3 \quad,
$$
$$
d*\tilde{F}_5 = d\tilde{F}_5 = H_3\wedge F_3 \quad,
$$
$$
d*(e^{-\phi}H_3 - e^{\phi}C_0\tilde{F}_3) = F_3\wedge F_5 \quad,
$$
$$
\nabla_M(e^{2\phi}\partial^M C_0) = -e^{\phi}*H_3\wedge\tilde{F}_3 \quad,
$$
$$
\nabla_M\nabla^M\phi = e^{2\phi}|F_1|^2 + \frac{1}{2}e^{\phi}|\tilde{F}_3|^2 - \frac{1}{2}e^{-\phi}|H_3|^2 \quad.
$$

(2.1.7)

We will see how the bulk action and therefore the equations of motion are modified when localized sources as D-branes (Section 2.2) or O-planes are present.

Later on we will need to know the supersymmetry transformations of the fermionic fields. Representing gravitinos, dilatinos and supersymmetry parameters by Weyl spinors $\psi_M, \lambda, \epsilon$, the supersymmetry variations are [34]:

$$
\delta\lambda = \frac{1}{2}(\slashed{\partial}\phi - ie^{\phi}\slashed{\partial}C_0)\epsilon + \frac{1}{4}\left(ie^{\phi}\tilde{\slashed{F}}_3 - \slashed{H}_3\right)\epsilon^* \quad,
$$
$$
\delta\psi_M = \left(\nabla_M + \frac{i}{8}e^{\phi}\slashed{F}_1\Gamma_M + \frac{i}{16}e^{\phi}\tilde{\slashed{F}}_5\Gamma_M\right)\epsilon - \frac{1}{8}\left(2\slashed{H}_M + ie^{\phi}\tilde{\slashed{F}}_3\Gamma_M\right)\epsilon^* \quad,
$$

(2.1.8)

where

$$
\slashed{F}_p \equiv \frac{1}{p!}F_{M_1...M_{\mathrm{P}}}\Gamma^{M_1...M_{\mathrm{P}}} \quad,
$$
$$
\slashed{F}_M \equiv \frac{1}{(p-1)!}F_{MN_1...N_{p-1}}\Gamma^{N_1...N_{p-1}} \quad.
$$

(2.1.9)

---

[4]The problem here is that one cannot write the action with a canonical kinetic term for a self-dual form, as $\tilde{F}_5\wedge*\tilde{F}_5$, because it vanishes. Neither a Lagrange multiplier field does help, ending up reintroducing the components it was intended to eliminate. An alternative approach to the one used here is to formulate a manifestly covariant action following Pasti, Sorokin and Tonin prescription [31, 32].

2.2. The open string sector: Dp-branes

Dp-branes are extended dynamical objects admitted by the string theory, which open new possibilities for the construction of realistic models in string theory [10]. As we will see below, they correspond to non-perturbative states of type II string theories [5].

At weak coupling, a Dp-brane is a $(p + 1)$-dimensional object, on which open strings can end. Let us explain how this happens. Open and closed strings share the same local worldsheet structure, which means that both have the same local dynamics, while differences arise as global boundary conditions [29]. In fact, to get the open strings e.o.m. from the variation of the Polyakov action, one has to require the vanishing of boundary terms (which do not arise in the case of closed strings). In order to make them vanish, one has basically two possibilities. To describe these, let us recall that the string worldsheet action, which is the two-dimensional generalisation of the particle worldline action, is described by the embedding $X^M(t, \sigma)$, $M = 0, ..., 9$. These can be thought as ten two-dimensional fields, depending on worldsheet parameters (t, σ). $\sigma \in [0, \ell]$, where ℓ is the string length. This means that the open string endpoints are at $\sigma = 0$ and ℓ [29].

In a ten-dimensional Poincaré invariant theory the variations of the worldsheet fields $\delta X^M(t, \sigma)$ are unconstrained and one finds that the vanishing of the boundary terms requires the so-called ''Neumann-Neumann boundary conditions'' (NN) on both endpoints:

$$\partial_\sigma X^M|_{\sigma=0,\ell} = 0 \quad , \tag{2.2.1}$$

corresponding to free endpoints moving through spacetime. Another way to make the boundary terms vanish is to impose $\delta X^M(t, \sigma)|_{\sigma=0,\ell} = 0$. It corresponds to pin the endpoints:

$$X^M|_{\sigma=0,\ell} = x_{0,\ell}^M \quad , \qquad x_0^i, x_\ell^i = \text{const} \quad . \tag{2.2.2}$$

These are the so-called ''Dirichlet-Dirichlet boundary conditions'' (DD). If one chooses a starting Minkowskian background, such boundary conditions will break the Poincaré invariance $ISO(1, 9)$ of the vacuum.

Generally, an open string can have different boundary conditions on each of its endpoints, i.e. not only NN or DD, but also ND or DN. Take, for instance, an open string which is described by the worldsheet theory of $p + 1$ fields $X^\mu(t, \sigma)$, $\mu = 0, ..., p$ satisfying the Neumann boundary condition and $9 - p$ fields satisfying the Dirichlet boundary condition $X^i(t, \sigma) = x_0^i$, $i = p + 1, ..., 9$. Focus on the endpoint at $\sigma = 0$. This string endpoint is confined to move within a p-dimensional hyperplane, termed a ''Dp-brane'',

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[5]Branes are non-perturbative states also of the type I theory, built by orientifolding type IIB [29].

with transverse position x_0^i [6].

The presence of Dp-branes breaks some of the symmetries of the vacuum. For instance, in a maximal symmetric background, the inclusion of a flat Dp-brane (neglecting its backreaction on the background, which modifies the geometry) breaks at least both the Lorentz invariance $SO(1,9) \to SO(1,p) \times SO(9-p)$, where $SO(1,p)$ is the residual symmetry on the brane, and the traslational invariance in the transverse directions. Moreover, branes break also some of all bulk supersymmetries.

Branes are *physical* objects, with a proper dynamics, described by the worldsheet theory of the open string sector. At low energies compared to the string scale, this dynamics is encoded in the massless open string modes and one can construct an effective action based only on them, exactly as in the supergravity case (see subsect. 2.2.1).

Analysing the open string oscillators, one finds that IIB/IIA theories are compatible only with "stable" branes with odd/even p respectively. The structure of the oscillators is similar to that of purely NN open superstrings, but the DD boundary conditions lead the open string center of mass to be localized on the brane. The corresponding particles propagate in its $(p+1)$-dimensional volume [29]. For a Dp-brane spanning the directions x^μ, $\mu = 0, ..., p$ and transversal to x^i, $i = p+1, ..., 9$, the massless spectrum is presented in Table 2.3. Counting of degrees of freedom shows that massless states fill in a $U(1)$ vector supermultiplet with respect to 16 supersymmetries in $p+1$ dimensions [7]. It can also be thought of as the $(p+1)$-dimensional reduction of a ten-dimensional $\mathcal{N} = 1$ vector multiplet, with the ten-dimensional vector field splitting into the $(p+1)$-dimensional gauge boson A_μ and $9-p$ real scalars ϕ^i, and with the ten-dimensional Majorana–Weyl spinor giving rise to fermions λ_α. Scalars ϕ^i can be regarded as the Goldstone bosons associated to the Minkowski bulk translational symmetries broken by the D-brane. Their vevs describe exactly the position of the brane in the transversal directions; their variations encode the brane transversal fluctuations (see subsection 2.2.1). The spinors λ_α can be regarded as goldstinos associated to the broken supersymmetries of Minkowski bulk, which allows 32 global supercharges.

The maximum number of the unbroken supersymmetries in a Minkowskian vacuum containing a Dp-brane is 16, that is half of the type II supersymmetries. This suggests to regard the brane as to a BPS state [29, 34]. This amount of supersymmetry prevents

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[6]Setting 10 Dirichlet conditions, one deals with a (-1)-brane, completely localized in space and time. It is interpreted as a "D-instanton", in the Euclideanized theory.

[7]Take, for example, a D3-brane. It presents a massless spectrum that is a $U(1)$ $\mathcal{N} = 4$ vector multiplet in four dimensions, composed by a gauge boson (2 d.o.f.), 6 real scalars and four Majorana spinors (2 d.o.f. each), for a total of 8 bosonic and 8 fermionic physical d.o.f..

the presence of tachyonic states and makes the brane stable.

Sector	$(p+1)$d fields
NS	$A_\mu(x^\mu)$
	$\phi^i(x^\mu)$
R	$\lambda_\alpha(x^\mu)$

Table 2.3.: Massless spectrum of a Dp-brane: fields live on the brane.

2.2.1. D-brane effective action

Here we present the D-brane (bosonic) action at energy well below the string scale, in the perturbative regime of string theory. The rigorous derivation follows from the worldsheet theory, including non-trivial background NSNS and RR fields ϕ, g, B_2, C_p. It yields to an action corresponding exactly to the *worldvolume action* for the brane, which is the higher-dimensional generalization of the string worldsheet action. This formulation is extremely useful to understand the physical meaning of the effective action. One introduces the brane worldvolume W, as the $p+1$-dimensional generalization of the string worldsheet, parametrized by $\sigma^\alpha = 0, ..., p$. To get the *spacetime* supersymmetric spectrum of Table 2.3, the supersymmetric action can be described by an embedding in superspace [35, 36]. We will only need the bosonic brane worldvolume action, which in the string frame is:

$$S^{\text{sf}}_{\text{D-brane}} = S_{\text{DBI}} + S_{\text{CS}} \quad , \tag{2.2.3}$$

$$S_{\text{DBI}} = -T_p \int_W d^{p+1}\sigma \ e^{-\phi} \sqrt{-\det(P[g - B_2] + \lambda F)} \quad , \tag{2.2.4}$$

$$S_{\text{CS}} = \mu_p \int_W \left[P\left(\sum_{\text{P}} C_p e^{-B_2} \right) e^{\lambda F} \right]_{p+1} \wedge \hat{A}(R) \quad , \tag{2.2.5}$$

where $\lambda = 2\pi\alpha'$ and $P[...]$ denotes the pullback onto the brane worldvolume W.
The term (2.2.4) is the Dirac–Born–Infeld action. It describes the coupling of the brane to the NSNS fields of the background. T_p is the brane tension and $F = dA$ is the field-strength of the $U(1)$ worldvolume gauge field A_α. This action comes from the combination of the Nambu–Goto action with the Born–Infeld action, both suitably generalized for a Dp-brane [8]. In fact, ignoring fields B_2 and A_α and also considering a constant dilaton, it reproduces exactly the Nambu–Goto action for an extended Dp-brane

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[8]The Nambu–Goto string action is the natural generalization of the free point-particle action:

2. IIB Supergravity

$$S_{\text{NG}}^{\text{Dp}-\text{brane}} = -T_p \int_W d^{p+1}\sigma \ \sqrt{-\det(P[g])} \quad . \tag{2.2.8}$$

The appearance of the combination

$$\mathcal{F} \equiv \lambda F - P[B_2] \tag{2.2.9}$$

is required by gauge invariance under (2.1.1). Indeed, open strings in a non-trivial background couple also to the gauge boson $A_\alpha(\sigma)$. The string worldsheet action, from which we extract the effective brane action, contains the term

$$S_\Sigma^{(A,B)} = -\frac{1}{2\pi\alpha'} \int_\Sigma B_2 + \int_{\partial\Sigma} A_1 \tag{2.2.10}$$

which is invariant if the gauge transformation of B_2 (2.1.1) is accompanied by the shift

$$A_1 \to A_1 + \frac{1}{2\pi\alpha'}\lambda_1 \quad . \tag{2.2.11}$$

Hence physical quantities can depend only on the gauge-invariant combination (2.2.9) [9]. The DBI action (2.2.4) encodes the dynamical nature of the brane embedding implicitly in the pull-back $P[...]$ of the bulk spacetime fields to the brane worldvolume. One can use the *static gauge*, i.e. that gauge choice for which the worldvolume diffeomorphisms symmetry is used to set the first $p+1$ components of the embedding fields equal to the worldvolume coordinates, i.e. $X^\alpha(\sigma) = \sigma^\alpha$. Then, brane transversal fluctuations around a fixed $X_o^i(\sigma)$ can be identified with the $9-p$ scalars as $X^i(\sigma) = X_o^i(\sigma) + \lambda\phi^i(\sigma)$,

---

$$S_{\text{NG}}^{\text{string}} = -T_s \int_\Sigma d^2\sigma \ \sqrt{-\det(P[g])} \quad , \tag{2.2.6}$$

where T_s string tension $T_s \equiv 1/(2\pi\alpha') \sim M_s^2$. $P[g]$ is the spacetime metric pulled back to the string worldsheet Σ.

The Born–Infeld action [37] was proposed as a non-linear generalization of the Maxwell theory, formulated in the attempt to eliminate the classical infinite self-energy of a charged point-particle (in a flat four-dimensional spacetime):

$$S_{\text{BI}} \sim \int d^4x \ \sqrt{-\det(\eta_{\mu\nu} + \lambda F_{\mu\nu})} \quad . \tag{2.2.7}$$

It possesses the crucial property to be generally covariant in a curved space, i.e. replacing $\eta_{\mu\nu} \to g_{\mu\nu}$. Here these actions are generalized in a higher dimensional structure and combined to give the brane action [34].

[9]To be more precise, in a supersymmetric worldsheet theory there are also spinors *on the worldsheet* combining in $\mathcal{F}$ to make it supersymmetrically invariant. Here we are studying just the bosonic sector.

$i = p + 1, ..., 9$ (λ is inserted to give the mass dimension one to the scalars ϕ^i). In this gauge one gets [10]:

$$S_{\text{DBI}} = -T_p \int_W d^{p+1}\sigma \, e^{-\phi} \big[- \det \big(g_{\alpha\beta} + \lambda^2 \partial_\alpha \phi^i \partial_\beta \phi^j g_{ij} + 2\lambda g_{i(\alpha}\partial_{\beta)}\phi^i - B_{\alpha\beta} + \lambda F_{\alpha\beta} \big) \big]^{\frac{1}{2}} \, ,$$

(2.2.12)

where the pullbacked fields must be regarded of as embedding-dependent. The complete fluctuations dependence is obtained expanding these fields around brane positions. Take, for instance, the $g_{\alpha\beta}$ term: it must be regarded of as

$$g_{\alpha\beta}(\sigma^\alpha, X_o^i + \lambda\phi^i) = g_{\alpha\beta}(\sigma^\alpha, X_o^i) + \partial_i g_{\alpha\beta}\lambda\phi^i + ... \quad . \tag{2.2.13}$$

Expanding (2.2.12) in powers of the field-strength F one finds that the term quadratic in λ is the $(p+1)$-dimensional generalization of the Maxwell action for $F_{\alpha\beta}$. Since this has a coupling like $1/g_{\text{YM}}^2 \sim 1/g_s$, one gets the relation $T_p \sim 1/g_s$, unveiling the non-perturbative nature of D-branes.

The Chern–Simons term (2.2.5) encodes the informations of the brane worldvolume couplings to the RR fields C_p. μ_p is the Dp-brane charge and it is equal to the tension, $\mu_p = T_p$ [11].

In the integral one has to pick up just the terms corresponding to $p + 1$ forms, which can be integrated on the worldvolume. There is not only the leading order contribution given by the electric coupling to the C_{p+1} field. Indeed, the CS action shows that when non-vanishing worldvolume field-strengths are present, they induce lower-dimensional D-brane charges, i.e. brane interacts also with lower degree RR forms. The pullback, makes manifest the interaction of the brane fluctuations ϕ^i with the RR fields. Note that, once again, B_2 and F enter the action through the gauge-invariant combination (2.2.9). Since they are two-forms, only $p = \text{odd/even}$ branes coupling RR fields are present in a type IIB/IIA theory. The last term in (2.2.5) is relevant only in the presence of spacetime curvature and it is called the *A-roof polynomial* $\hat{A}(R) = 1 + c \operatorname{Tr} R^2 + \cdots$, where c is a constant and R the curvature two-form [29].

The action (2.2.3) can be generalized to stacks of Dp-branes, giving rise to non-Abelian gauge theories, where scalars ϕ^i becomes $N \times N$ matrices. Both the DBI and the CS actions are consequently modified [10]. The importance of intersecting brane-world models has grown since it was discovered they can lead to SM-like gauge theories [29, 38, 39]. We

---

[10]The components of a spacetime tensor T_{MN} pulled-back to the brane worldvolume $(P[T])_{\alpha\beta} = T_{MN}\partial_\alpha X^M \partial_\beta X^M$.

[11]$\mu_p = T_p$ is typical of supersymmetric branes and it does not hold in general.

will see how Dp-branes can be consistently included in compactification backgrounds in the next Chapter.

3. $\mathcal{N} = 1$ Supergravity in $D = 4$

In this Section we present the main features of four-dimensional supergravity theories that we will encounter in the context of superstring compactifications. A complete dissertation is beyond the scope of the thesis and for more details we refer the reader to textbooks and exhaustive notes [40–42].

A supergravity theory is obtained by gauging a supersymmetric theory, i.e. promoting the global (also called *rigid*) supersymmetry to be *local*. Such a gauging leads to include general spacetime transformations (diffeomorphisms) among the symmetries and hence the theory corresponds to a supersymmetric version of gravity. In order to proceed to the description of the theory, it is advisable to refresh the fundamental characteristics of a rigid supersymmetric theory, whose supporting motivations we gave in the Introduction.

3.1. Basics of rigid supersymmetry in $D = 4$

Supersymmetry is a symmetry which relates bosons and fermions. Calling Q the generator, one has

$$
\begin{aligned}
Q(\text{fermion}) &= \text{boson} \quad , \\
Q(\text{boson}) &= \text{fermion} \quad .
\end{aligned}
\tag{3.1.1}
$$

Therefore, the theory has to contain an equal number of fermionic and bosonic degrees of freedom. The first and simpler supersymmetric theory in four dimensions was discovered by Wess and Zumino [2], which is the free theory of a Weyl fermion ψ [1] and a complex scalar ϕ [29]:

---

[1]Following the Bagger's and Wess' conventions [43], a Weyl spinor is a two-component spinor which can belong to the fundamental (lefthanded) $(1/2, 0)$ or to the antifundamental (righthanded) $(0, 1/2)$ representation of the Lorentz group. In the first case it is denoted by ψ_α (ψ_L), in the latter with $\bar{\psi}^{\dot\alpha}$ (ψ_R), where $\alpha, \dot\alpha = 1, 2$. In this notation a Dirac spinor is a reducible four-component spinor collecting two Weyl spinors as

$$
\Psi_D = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot\alpha} \end{pmatrix} \quad ,
\tag{3.1.2}
$$

3. $\mathcal{N} = 1$ Supergravity in $D = 4$

$$S = \int d^4x(-\partial_\mu\phi\partial^\mu\phi^* - i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi) \quad . \tag{3.1.3}$$

This action is invariant under the following supersymmetric variations

$$\delta_\epsilon\phi = \sqrt{2}\epsilon\psi \quad , \tag{3.1.4}$$

$$\delta_\epsilon\psi_\alpha = i\sqrt{2}(\partial_\mu\phi\sigma^\mu\bar{\epsilon})_\alpha \quad , \tag{3.1.5}$$

where ϵ_α is the infinitesimal transformation parameter and it is a fermionic (Grassmann odd) quantity of mass dimension $[\epsilon] = -1/2$.

The *supercurrent*

$$J_\alpha^\mu = (\partial_\nu\phi^*\sigma^\nu\bar{\sigma}^\mu\psi)_\alpha \tag{3.1.6}$$

is conserved $\partial_\mu J_\alpha^\mu = 0$ by using the equations of motion. Hence, it gives the following conserved *supercharges*

$$Q_\alpha = \int d^3x J_\alpha^0 \quad , \qquad \bar{Q}_{\dot{\alpha}} \equiv Q_\alpha^\dagger \quad . \tag{3.1.7}$$

These are the charge generators and they transform as Weyl spinors under the Lorentz group. One can check they satisfy the following *superalgebra*:

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}}P_\mu \quad , \tag{3.1.8}$$

where P_μ is the translation generator, while

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0 \quad , \tag{3.1.9}$$

$$[Q_\alpha, P_\mu] = [\bar{Q}_{\dot{\alpha}}, P_\mu] = 0 \quad . \tag{3.1.10}$$

Since it is constructed as a symmetry between fermions and bosons, which have different spins, the supersymmetric algebra can not be considered as an internal symmetry. This is reflected in the non-trivial commutation rules between supersymmetric and Lorentz generators $M_{\mu\nu}$:

$$[M_{\mu\nu}, Q_\alpha] = i(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta \quad , \tag{3.1.11}$$

$$[M_{\mu\nu}, \bar{Q}_{\dot{\alpha}}] = i(\bar{\sigma}_{\mu\nu})_{\dot{\alpha}}{}^{\dot{\beta}} \bar{Q}_{\dot{\beta}} \quad .$$

---

the Dirac mass term is hence $\Psi^\dagger\gamma^0\Psi = \chi_R^\dagger\psi_L + \psi_L^\dagger\chi_R = \chi^\alpha\psi_\alpha + \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}$, where we used the Weyl representation of gamma matrices $\gamma^0 = \sigma^2 \otimes \mathbb{1}$ and we called $\chi^\alpha = (\bar{\chi}^{\dot{\alpha}})^*$ and $\bar{\psi}_{\dot{\alpha}} = (\psi_a)^*$. Here $(\sigma^\mu)_{\alpha\dot{\alpha}} = (-\mathbb{1}, \sigma^i)$ and $(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = (-\mathbb{1}, -\sigma^i)$.

Notice that we presented the case of a $\mathcal{N} = 1$ supersymmetric theory. If $\mathcal{N} > 1$ one deals with an *extended* supersymmetry. In this case the supercharges are $Q_\alpha^I, \bar{Q}_{\dot\alpha}^I$ with $I = 1, ..., \mathcal{N}$ and the anticommutation rules become

$$\{Q_\alpha^I, \bar{Q}_{\dot\beta}^J\} = 2\sigma_{\alpha\dot\beta}^\mu P_\mu \delta^{IJ} \quad , \qquad \forall I, J = 1, ..., \mathcal{N} \ , \tag{3.1.12}$$

$$\{Q_\alpha^I, Q_\beta^J\} = \epsilon_{\alpha\beta} Z^{IJ} \quad , \qquad \{\bar{Q}_{\dot\alpha}^I, \bar{Q}_{\dot\beta}^J\} = \epsilon_{\dot\alpha\dot\beta}(Z^{IJ})^* \quad , \tag{3.1.13}$$

where Z^{IJ} are antisymmetric central charges, commuting with full algebra generators. The values of possible $\mathcal{N}$ is constrained by the theory of representations. In fact, $\mathcal{N}$ is directly linked to the maximum spin allowed for a particle in a given representation, which is one for a rigid supersymmetric theory and two for a supergravity theory. Consistency then requires $\mathcal{N} \leq 4$ and $\mathcal{N} \leq 8$ in the two cases respectively [40, 42].

From the rules governing the superalgebra one can study the theory of representations in terms of states [40–42]. Any representation of the full supersymmetric algebra is also a representation of the Poincaré algebra, in general reducible. Hence, an irriducible representation of the superalgebra corresponds to different Poincaré irreducible representations, i.e. to several particles. This is why such an irreducible representation is called *supermultiplet*.

Each supermultiplet contains bosonic and fermionic states, related by the action of generators $Q, \bar{Q}$ and having spins differing by units of half. Moreover, in the same supermultiplet particles have the same mass and bosonic degrees of freedom (d.o.f.) are equal to fermionic ones [40–42]. Since we deal with fields throughout the thesis, let us describe supermultiplets in terms of fields.

Fields representations. For us, the relevant supermultiplets are those of $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetries. For $\mathcal{N} = 1$ we present each supermultiplet distinguishing between the massless and the massive representation:

the chiral supermultiplet (matter multiplet): whether massless (called also *Wess–Zumino multiplet*) or massive, it contains a complex scalar and a Weyl fermion;

the vector supermultiplet: the massless one corresponds to a gauge boson and a Weyl fermion, both in the adjoint of the gauge group, while the massive one is composed by a vector, two Weyl spinors (of opposite chirality) and a real scalar. Notice that degrees of freedom are the same of those of a massless vector multiplet plus one massless matter multiplet. In fact, one can generates massive vector multiplets by a super-Higgs mechanism [29, 42];

the graviton supermultiplet: we are interested only in the massless representation, which contains the graviton and a gravitino. This kind of multiplet appears only in supergravity theories.

When $\mathcal{N} = 2$, supermultiplets we are interested in are only the massless ones:

the hypermultiplet (matter multiplet): it may contain a complex scalar and a Weyl fermion (*half-hypermultiplet*), or two complex scalars and two Weyl fermions (*hypermultiplet*);

the vector supermultiplet: it contains a gauge boson, a complex scalar and 2 Weyl fermions, all transforming in the adjoint of the gauge group;

the graviton supermultiplet: it corresponds to a graviton, 2 gravitinos and a vector (termed the *graviphoton*).

Superspace and superfields. A very useful mathematical instrument to deal with all those fields is the superspace formalism, since it allows to organise all components of a supermultiplet in a single *superfield*. This, in turn, allows to write general field theory actions. The superspace is the generalisation of the four-dimensional Minkowskian spacetime including two extra fermionic dimensions, described by two *anticommuting spinorial coordinates* $\theta_\alpha, \bar{\theta}_{\dot{\alpha}}$ [40, 42]. Superfields are fields defined in such a space, depending on all its coordinates $(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$. Due to the Grassmann nature of fermionic coordinates, it follows that superfields have a finite expansion in these coordinates and hence can encode a finite number of ordinary fields. So they can be naturally filled by the supermultiplet components. For a $\mathcal{N} = 1$ supersymmetric theory, the simplest type of superfield is the *chiral superfield* $\Phi(x, \theta, \bar{\theta})$, so named since it contains all the fields of the *off-shell* chiral supermultiplet

$$(\Phi, \psi, F) \quad . \tag{3.1.14}$$

Here Φ, ψ are the complex scalar (typically denoted as the superfield) and the Weyl fermion contained in the supermultiplet as described above. F is an auxiliary complex scalar field, introduced for convenience, in order to close the supersymmetry algebra off-shell. To be clearer, let us come back to the Wess–Zumino example (3.1.3). The model can be completely described by a chiral superfield with an action

$$S = \int d^4x(-\partial_\mu \Phi \partial^\mu \Phi^* - i\bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi + |F|^2) \quad , \tag{3.1.15}$$

invariant under

$$\delta_\epsilon \Phi = \sqrt{2}\epsilon\psi \quad , \tag{3.1.16}$$

$$\delta_\epsilon \psi_\alpha = i\sqrt{2}(\partial_\mu \Phi \sigma^\mu \bar{\epsilon})_\alpha + \sqrt{2}\epsilon_\alpha F \quad , \tag{3.1.17}$$

$$\delta_\epsilon F = i\sqrt{2}(\bar{\epsilon}\bar{\sigma}^\mu \partial_\mu \psi) \quad . \tag{3.1.18}$$

The additional degrees of freedom represented by F are unphysical. In fact they can be integrated out by applying the equations of motion. A chiral superfield has the following expansion in θ_α:

$$\Phi(x, \theta, \bar{\theta}) = \Phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) \quad , \tag{3.1.19}$$

where $y = x + i\theta\sigma^\mu\bar{\theta}$ and we adopt the same notation for the superfield and its complex scalar component. There exists also the *antichiral* superfield, which is simply the adjoint $\Phi^\dagger$ and contains the conjugate fields $(\Phi^*, \bar{\psi}_{\dot{\alpha}}, F^*)$.

Another kind of superfield is the *vector superfield* $V(x, \theta, \bar{\theta})$, fundamental for the introduction of gauge interactions in the theory. This superfield encodes all the information about the vector multiplet, described off-shell by (omitting gauge indices)

$$(A_\mu, \lambda_\alpha, \mathrm{D}, C, \chi_\alpha, N) \quad . \tag{3.1.20}$$

Here A_μ is the gauge boson with field-strength $F_{\mu\nu}$, λ_α is the *gaugino*, a Weyl spinor in the adjoint of the gauge group, while $\mathrm{D}, C, N, \chi_\alpha$ are auxiliary fields, three complex scalars and a Weyl spinor, equipping the multiplet with a generalised gauge invariance. Among these, the latter three are removed by a suitable gauge choice, such that V can be written as an expansion in the fermionic coordinates with field components A_μ, λ and D. The latter can then be integrated out, leaving A_μ and λ_α as dynamical fields. This superfield has the property to be self-adjoint $V^\dagger = V$.

$\mathcal{N} = 1$ *Supersymmetric actions.* At his point we possess all the basic instruments to understand the basic structure of a supersymmetric action, while its meticulous derivation is beyond the scope of this review.

Let us begin by presenting the most general renormalisable supersymmetric Lagrangian encoding interactions among a number of chiral superfields Φ_i:

$$\mathcal{L}_W = \int d^2\theta W(\Phi_i) + h.c. \quad , \tag{3.1.21}$$

where

$$W(\Phi_i) = \frac{1}{3}h^{ijk}\Phi_i\Phi_j\Phi_k + \frac{1}{2}m^{ij}\Phi_i\Phi_j + \lambda^i\Phi_i \qquad (3.1.22)$$

is the *superpotential*, a holomorphic function of chiral fields. It determines masses, Yukawa couplings and a scalar potential $V(\Phi_i)$. We shall stress that the holomorphicity of W is really crucial to supersymmetry. As one can show, the product of chiral superfields yields another chiral superfield. Now, since the integral in (3.1.21) selects the ''F-term'' of these chiral fields, i.e. the F auxiliary field which has a total derivative as supersymmetric variation (3.1.18), then the four-dimensional integral of (3.1.21) is invariant under supersymmetry. This means that the action derived from (3.1.21) is automatically invariant.

The kinetic terms for chiral multiplets are described by

$$\int d^2\theta d^2\bar{\theta}\Phi_i^\dagger\Phi_i \quad . \qquad (3.1.23)$$

The integral now selects the so-called ''D-term'', i.e. the $\theta^2\bar{\theta}^2$ component of the product $\Phi_i^\dagger\Phi_i$, which has again a total derivative as supersymmetric variation. Therefore also the kinetic action is manifestly supersymmetric invariant. It contains kinetic terms for the complex scalars Φ_i, for the Weyl spinors ψ_i and also for the auxiliary fields F_i. In the latter case the kinetic term is $\sum_i |F_i|^2$ which, by using the equations of motion $F_i^* = -\frac{\partial W}{\partial\Phi^i}$ (derived with the interaction (3.1.21)), gives rise to a contribution to the scalar potential of the form

$$V_F(\Phi_i) = \sum_i \left|\frac{\partial W}{\partial\Phi^i}\right|^2 \quad . \qquad (3.1.24)$$

Gauge bosons and gauge interactions are introduced with vector superfields. Given a vector multiplet V, one can include the gauge invariant field-strength $F_{\mu\nu}$, the gauginos λ_α and the auxiliary scalar D into a *spinorial* chiral superfield W_α, suitably defined in order to describe the kinetic terms by

$$\frac{1}{4}\text{Tr}\int d^2\theta W_\alpha W^\alpha + h.c. = \text{Tr}\left[-\frac{1}{4}F_{\mu\nu}^2 - i\lambda\sigma^\mu D_\mu\bar{\lambda} + \frac{1}{2}\text{D}^2\right] \quad , \qquad (3.1.25)$$

where D_μ is the gauge covariant derivative $D_\mu\bar{\lambda} = \partial_\mu\bar{\lambda} - \frac{i}{2}[A_\mu, \bar{\lambda}]$ [40, 42].

The interactions among the components of a vector superfield V and the components of a chiral superfield Φ are encoded by the covariantisation of (3.1.23):

$$\int d^2\theta d^2\bar{\theta}\Phi^\dagger e^V\Phi \quad . \qquad (3.1.26)$$

Other than being supersymmetric invariant, (3.1.25) and (3.1.26) are invariant under a generalised gauge symmetry, with chiral superfields Φ transforming in some representation of the gauge group. However, once gauged away C, χ_α, N, one is left with the physical fields $\psi, \Phi, A^\mu, \lambda$ transforming under ordinary gauge transformations [40, 42]. Note that (3.1.26) contains the gauge interactions between matter fields (Φ, ψ) and gauge bosons A^a_μ or gauginos λ^a. Moreover, it produces a linear term in D, which determines the equation of motion $\mu D^a = -g\Phi^\dagger T^a \Phi$, where g is the gauge coupling and T^a is a representation of the gauge generator, labelled by the gauge index a and some other representation indices [2] which are the same of Φ. Analogously to what happens for the kinetic terms of chiral fields, the kinetic term of D in (3.1.25) can be regarded as a contribution to the scalar potential $V(\Phi)$:

$$V_{\mathrm{D}} = \frac{1}{2}\sum_a |\mathrm{D}^a|^2 = \frac{g^2}{2}\sum_a |\Phi^\dagger T^a \Phi|^2 \quad . \tag{3.1.27}$$

We can generalise the above renormalisable supersymmetric actions for gauge and matter multiplets. It is exactly what happens in supergravity theories arising in string compactifications. There, as we will see in the next Chapters, We will deal with superfields which are singlets under the gauge interactions, therefore we will focus on this case in the following.

The generalisation of the chiral kinetic terms (3.1.23) is now:

$$\int d^2\theta d^2\bar{\theta} K(\Phi_i, \Phi^\dagger_j) \quad , \tag{3.1.28}$$

where $K(\Phi_i, \Phi^\dagger_j)$ is the *Kähler potential*, a real function of chiral fields. It leads to kinetic terms for the scalar components of the non-linear sigma model form, i.e. (here $\bar{\Phi}_{\bar{\imath}} \equiv \Phi^*_i$)

$$g_{i\bar{\jmath}}\partial_\mu \Phi^i \partial^\mu \bar{\Phi}^{\bar{\jmath}} \quad , \tag{3.1.29}$$

where

$$g_{i\bar{\jmath}} = \frac{\partial^2 K}{\partial\Phi^i \partial\bar{\Phi}^{\bar{\jmath}}} \tag{3.1.30}$$

is regarded as the Kähler metric of a Kähler manifold, parametrised by the complex scalar coordinates Φ^i.

It is also possible to give a field dependent generalisation of the gauge kinetic terms (3.1.25):

---

[2]We chose not to display these indices to simplify the notation.

$$\frac{1}{4} \int d^2\theta \; f(\Phi_i)\mathrm{Tr}[W^\alpha W_\alpha] + h.c. \sim (\mathrm{Re}\, f)_{ab} F^a_{\mu\nu} F^{b\;\mu\nu} + (\mathrm{Im}\, f)_{ab} F^a_{\mu\nu} \tilde{F}^{b\;\mu\nu} \quad , \qquad (3.1.31)$$

where $f(\Phi_i)$ is the holomorphic *gauge kinetic function* and $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$.

Notice that, in perturbation theory, loop corrections modify K, but, remarkably, the superpotential (3.1.22) turns out to be exact to all orders [3]. Indeed, one can show that, in the superspace formalism, such quantum corrections have a D-term structure. Furthermore, $W(\Phi)$ can be any holomorphic function and the scalar potential V_F (3.1.24) becomes

$$V_F = g^{i\bar{j}} \frac{\partial W}{\partial \Phi^i} \frac{\partial W}{\partial \bar{\Phi}^j} \quad , \qquad (3.1.32)$$

where $g^{i\bar{j}}$ is the inverse of the Kähler metric (3.1.30).

Supersymmetry breaking. We know that all fields in a supermultiplet have the same mass. However, at present day, there is no evidence of the existence of superpartners of the SM particles. This suggests that the supersymmetry must be broken at an energy scale at least above the electroweak one. One can think that, if realised in nature, supersymmetry is broken spontaneously, which means the Lagrangian to be supersymmetric invariant but the vacuum state to be not, i.e. $Q_\alpha|0\rangle \neq 0$. In global supersymmetric theories the order parameter is the groundstate energy $\langle 0|H|0\rangle$, which is shown to be positive semidefinite and vanishes only in case of unbroken supersymmetry (when $Q_\alpha|0\rangle = 0$) [40, 42]. Remembering that the vacuum state is by definition such that $\langle 0|H|0\rangle = \langle 0|V|0\rangle$ and that the scalar potential is in general the sum of $V_\mathrm{D} + V_F$, one finds that the supersymmetry breaking requires

$$\langle 0|F_i|0\rangle \neq 0 \quad \text{and/or} \quad \langle 0|\mathrm{D}^a|0\rangle \neq 0 \quad , \quad \text{for some } i, a. \qquad (3.1.33)$$

There exists also a supersymmetric Goldstone theorem, predicting the existence of a massless particles for each "broken degree of freedom". The broken generator Q_α is a spinor with 2 fermionic indices, hence the Goldstone particle is a Weyl spinor ψ_G, dubbed *goldstino*, and it is typically the partner of the non-vanishing F or D auxiliary term in (3.1.33).

3.2. Basics of $\mathcal{N} = 1$ supergravity in $D = 4$

We are now ready to introduce four-dimensional supergravity, in the same pragmatic approach followed in the previous Section.

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[3]W can get corrections by non-perturbative effects, e.g. instantons.

Promoting the supersymmetry to hold locally means allowing for parameters of the super-Poincaré group to vary in spacetime. Supersymmetry parameters become spacetime dependent spinors $\epsilon_\alpha(x), \bar{\epsilon}_{\dot{\alpha}}(x)$ and, as a consequence, the superalgebra involves local translation parameters which can be refarded of as diffeomorphisms. This in turn means that local supersymmetry requires gravity [42].

A supergravity theory is an interacting field theory containing the gravity multiplet and possible matter/vector multiplets of the underlying globally supersymmetric theory. In particular, in a four-dimensional $\mathcal{N} = 1$ supergravity the gravity multiplet consists in the graviton $g_{\mu\nu}$ and its superpartner, the gravitino ψ_α^μ. This is considered as the gauge particle of the theory, since it enters the action coupling to the conserved supercurrent, as (3.1.6) [40, 42]. Being non-renormalisable, a supergravity theory must be regarded as an *effective* theory, valid at energies well below the Planck energy scale M_{P}.

We are interested in the supergravity action describing gravity coupled to chiral and gauge multiplets, up to second order in derivatives. In the following $\partial_i \equiv \frac{\partial}{\partial \Phi^i}$. We start by giving a description of the supergravity coupled to chiral fields. The superspace sigma model Lagrangian is a particular generalisation of (3.1.28), that is [44]

$$\mathcal{L} = -3M_{\mathrm{P}}^2 \int d^2\theta d^2\bar{\theta} E \exp\left[-\frac{1}{3M_{\mathrm{P}}^2} K(\Phi_i, \Phi_j^\dagger)\right] \quad , \tag{3.2.1}$$

where $M_{\mathrm{P}}^2 = \kappa^{-2} = \frac{1}{8\pi G_N}$ is the reduced Planck mass and E is the superdeterminant of the superspace vielbeins (*supervielbeins*), encoding informations about the gravity multiplet. In the low-energy limit, $M_{\mathrm{P}} \to \infty$, gravity decouples, i.e. (3.2.1) is expanded as

$$-3M_{\mathrm{P}}^2 \int d^2\theta d^2\bar{\theta} E + \int d^2\theta d^2\bar{\theta} E K(\Phi_i, \Phi_j^\dagger) + \mathcal{O}(M_{\mathrm{P}}^{-2}) \tag{3.2.2}$$

One can check that the first term reproduces the Einstein and Rarita–Schwinger actions and the second term is the supergravity generalisation of the kinetic terms of chiral fields in a global supersymmetric theory in a flat spacetime (3.1.28). The third and higher order terms vanish in the low-energy limit. This analysis furnishes a proof of the validity of (3.2.1). Let us report, for future reference, the pure bosonic terms coming from the examination of (3.2.2):

$$\mathcal{L}_{\mathrm{Bos}} = \frac{M_{\mathrm{P}}^2}{2} \, eR - e \, g_{i\bar{\jmath}} \, g^{\mu\nu} \partial_\mu \Phi^i \partial_\nu \bar{\Phi}^{\bar{\jmath}} \quad , \tag{3.2.3}$$

where e is the determinant of the vierbeins e_μ^a, R is the Ricci scalar with mass dimension two, $g_{i\bar{\jmath}}$ is the Kähler metric (3.1.30) and it is dimensionless by definition since the Kähler potential has dimension two while Φ^i has mass dimension one. We are guaranteed on

the supersymmetric invariance of the Lagrangian since it has been derived from the superspace formalism [40, 42], as holds for the next term.

The supergravity generalisation of the superpotential (3.1.22) encoding the matter (chiral) couplings is:

$$\int d^2\theta \ \varepsilon W(\Phi^i) + h.c. \quad , \tag{3.2.4}$$

where ε is the chiral *superspace density* [40, 42]. Expanding (3.2.4) in components and eliminating the auxiliary fields by their equations of motion, one obtains some terms, among which the following pure bosonic contribution

$$e^{-1}\mathcal{L}_{\text{Bos}}^W = -e^{\frac{K}{M_{\text{P}}^2}} \left[g^{i\bar{j}} D_i W \overline{D_j W} - \frac{3}{M_{\text{P}}^2} |W|^2 \right] \equiv -V_F \tag{3.2.5}$$

to be added to (3.2.3). This new term has the form of a scalar potential and in fact it corresponds exactly to the supergravity generalisation of the V_F term (3.1.32). To be precise: the first term on the RHS comes from the F kinetic term, while the second term descends from the superpotential contribution. Notice that the superpotential must have mass dimension three. Here $g^{i\bar{j}}$ is the inverse of the Kähler metric and $D_i = \partial_i + \frac{1}{M_{\text{P}}^2}\partial_i K$ is the *Kähler derivative*, a covariant derivative with respect to the *Kähler invariance*

$$\begin{aligned} K(\Phi_i, \bar{\Phi}_j) &\to K(\Phi_i, \bar{\Phi}_j) + h(\Phi_i) + \bar{h}(\bar{\Phi}_j) \\ W &\to e^{-h(\Phi_i)} W \end{aligned} \tag{3.2.6}$$

enjoyed by the completely expanded supergravity action, where $h(\Phi_i)$ is a holomorphic function [40, 42]. Note that in case of $M_{\text{P}} \to \infty$, i.e. when gravity decouples, one recovers the V_F potential found for the global supersymmetry (3.1.32).

However, we prefer to deal with dimensionless quantities $\hat{K}, \hat{\Phi}^i, \hat{D}_i$, defined as:

$$K = M_{\text{P}}^2 \hat{K} \quad , \tag{3.2.7}$$

$$\Phi^i = M_{\text{P}} \hat{\Phi}^i \quad , \tag{3.2.8}$$

$$D_i = \frac{1}{M_{\text{P}}} \hat{D}_i = \frac{\partial}{\partial \hat{\Phi}^i} + \frac{\partial}{\partial \hat{\Phi}^i} \hat{K} \quad . \tag{3.2.9}$$

In terms of the hatted quantities the Lagrangian kinetic terms (3.2.3) and the scalar potential (3.2.5) become respectively (dropping hats):

$$\frac{M_{\text{P}}^2}{2} \ eR - e \ M_{\text{P}}^2 \ g_{i\bar{j}} \partial_\mu \Phi^i \partial^\mu \bar{\Phi}^{\bar{j}} \quad , \tag{3.2.10}$$

$$V_F = \frac{1}{M_{\text{P}}^2} \ e^K [|D_i W|^2 - 3|W|^2] \quad . \tag{3.2.11}$$

The introduction of gauge interactions, by the gauge superfield V, complicates the situation. A fortunate guess, as one can show, is to generalise the global supersymmetric lagragian (3.1.26) taking into account the sigma model type, as done for chiral fields only, that is [44]

$$\mathcal{L} = -3M_{\mathrm{P}}^2 \int d^2\theta d^2\bar{\theta} E \, \exp\left[-\frac{1}{3M_{\mathrm{P}}^2} K\left(\Phi_i, (\Phi^\dagger e^{2gV})_{\bar{j}}^\dagger\right)\right] \quad , \tag{3.2.12}$$

where we shall remember that we are assuming chiral fields to be singlets under the gauge group, therefore the Kähler potential is trivially gauge invariant. The supergravity action is complete once added to the chiral interactions encoded in the superpotential term (3.2.4) and the vector multiplet kinetic terms (3.1.25), generalised to the field dependent case in a way to lead to the gauge kinetic terms of (3.1.31). Eliminating the auxiliary fields one gets the final action, which has the following bosonic terms (using dimensionless quantities):

$$e^{-1}\mathcal{L}_{\mathrm{Bos}} = \frac{M_{\mathrm{P}}^2}{2} R - M_{\mathrm{P}}^2 g_{i\bar{j}} \partial_\mu \Phi^i \partial^\mu \bar{\Phi}^{\bar{j}} - \frac{1}{4}(\mathrm{Re}\,f)_{ab} F_{\mu\nu}^a F^{b\ \mu\nu} - \frac{1}{4}(\mathrm{Im}\,f)_{ab} F_{\mu\nu}^a \tilde{F}^{b\ \mu\nu} - V \quad . \tag{3.2.13}$$

Here V is the scalar potential, composed by F- and D-terms:

$$V = V_F + V_{\mathrm{D}} = \frac{e^K}{M_{\mathrm{P}}^2}\left[g^{i\bar{j}} D_i W \overline{D_j W} - 3|W|^2\right] + \frac{1}{2}(\mathrm{Re}\,f)_{ab}^{-1} \mathrm{D}^{*a}\mathrm{D}^b \quad , \tag{3.2.14}$$

where also the D^a auxiliary field has to be intended replaced by its equation of motion. The action can be recast in the language of forms as follows:

$$S_{\mathrm{Bos}} = \int \left(\frac{M_{\mathrm{P}}^2}{2} R * 1 + M_{\mathrm{P}}^2 g_{i\bar{j}} d\Phi^i \wedge *d\bar{\Phi}^{\bar{j}} - \frac{1}{2}(\mathrm{Re}\,f)_{ab} F^a \wedge *F^b - \frac{1}{2}(\mathrm{Im}\,f)_{ab} F^a \wedge F^b - V * 1\right) \quad . \tag{3.2.15}$$

In order to analyse the supersymmetry breaking in the contest of supergravity, it is useful to remember that typically one chooses a Minkowskian/AdS vacuum state, constraining gauge and fermion fields to vanish. Since supersymmetric variations of bosonic fields are proportional to fermionic fields (which vanish) and vice-versa, the only condition to preserve supersymmetry is the requirement of vanishing fermionic variations [40,42] [4]. In particular, given a graviton multiplet with a gravitino ψ_μ, a number of chiral multiplets Φ_i with Weyl spinors χ_i and a vector multiplet with gauginos λ^a, one finds that supersymmetry is preserved if [40, 42]

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[4]We are assuming constant scalars.

$$\delta\psi_\mu = 0 \quad \leftrightarrow \quad \text{condition on } \epsilon \quad , \tag{3.2.16}$$

$$\delta\chi_i = 0 \quad \leftrightarrow \quad D_i W = 0 \quad , \tag{3.2.17}$$

$$\delta\lambda_a = 0 \quad \leftrightarrow \quad \mathrm{D}^a = 0 \quad . \tag{3.2.18}$$

Supersymmetry is broken when instead

$$\begin{aligned} D_i W &\neq 0 \ , &&\text{for some } i \ , \quad \text{or} \\ \mathrm{D}^a &\neq 0 \ , &&\text{for some } a \ . \end{aligned} \tag{3.2.19}$$

The order parameters are now (up to positive factors) the vevs of auxiliary fields replaced by their equations of motion. Hence, in contrast to global supersymmetry, there is the possibility to have a vanishing cosmological constant $\langle V \rangle$ even if supersymmetry *is broken*: if

$$\langle W \rangle \neq 0 \tag{3.2.20}$$

then, by a suitable fine tuning, it can cancel the positive term given by the supersymmetry breaking F-term and/or D-term in (3.2.14). Note that unbroken supersymmetry requires a negative or vanishing cosmological constant, hence it can not take place in De Sitter vacua.

4. IIB Calabi–Yau compactifications

The mechanism of string compactification allows for the construction of models consistent with the perceived four-dimensional spacetime. In these models, string theory is defined on a ten-dimensional spacetime $M_{10} = M^{1,3} \times M_6$ [1], where M_6 is a compact manifold of typical size R. Then, at energies $E \ll 1/R$ the physics appears essentially four-dimensional. This observation is the starting point of most of the string phenomenological models, which assume that presently accessible energies cannot resolve the too small finite size of the extra dimensions, so that the low-energy physics is described by a four-dimensional *effective* theory [2].

The derivation of the four-dimensional effective theory associated with a given string compactification represents a crucial step in the study of these models. The derivation consists of various steps (see Appendix B). One first chooses a starting *background*, i.e. a field configuration solution of the ten-dimensional equations of motion, and studies *perturbatively* the theory of the dynamical fluctuations around it. The last step consists in integrating out extra dimensions to discover how the ten-dimensional theory manifests itself in our four-dimensional world. One finds that the effective field theory depends on the *structure* of M_6. Generically, except cases of exact solvable worldsheet theories, the system is analysed in the small curvature approximation ($R \gg \ell_s$) and high string scale ($E \ll M_s$). Since string oscillations modes have masses of order M_s, in this regime strings can be considered as massless particles whose physics is described by an effective field theory of supergravity (see Ch. 2). Furthermore, by performing a standard KK reduction, one gets a tower of Kaluza–Klein massive modes, with masses of order $1/R \ll M_s$. Therefore, at energies $E \ll 1/R$, one can include in the four-dimensional effective theory only the lightest KK modes.

In particular, we are interested in compactifications of type IIB supergravity, because they provide one of the richest frameworks for model building and includes the F-theory models as its non-perturbative completion, when also 7-branes are present [45, 46].

In the following Sections we firstly address the compactification on a Calabi–Yau of

---

[1]String quantization shows that superstring theories must have a *critical dimension* $D = 10$ to have a well behaviour at high energies. However, one can compactify also the bosonic string theory (which has crucial dimension $D = 26$) over a manifold with 22 dimensions M_{22}.

[2]Here on we will work in natural dimensions $\hbar = c = 1$.

type IIB supergravity, which leads to an effective $\mathcal{N} = 2$ four-dimensional supergravity. Orientifold projections, background fluxes and D-branes are subsequently introduced to reduce it to a more realistic four-dimensional $\mathcal{N} = 1$ supergravity, with a superpotential, non-trivial gauge couplings and chiral matter [3].

As anticipated in the Introduction, a general string compactification has a factorized background:

$$M_{10} = M^{1,3} \times M_6 \quad . \tag{4.0.1}$$

Typically one considers a compactification Ansatz in which $M^{1,3}$ is maximally symmetric, restricting it to a Minkowski, dS or AdS spacetime. The general metric Ansatz then takes the form:

$$ds^2 = e^{2A(y)}ds_4^2 + g_{mn}(y) \ dy^m dy^n \quad , \tag{4.0.2}$$

where e^{2A} is the so-called warp factor, or warping for short, and ds_4^2 is the $\mathrm{Mink}_4, \mathrm{AdS}_4$ or dS_4 metric, which preserves the Poincaré, $SO(2,3)$ or $SO(1,4)$ symmetry respectively. In order to preserve external maximal symmetry, the warp factor is allowed to depend on internal coordinates only. In the most general case, such a solution admits also non-trivial background fluxes and dilaton. In order to not spoil 4d maximal symmetry, the dilaton is allowed to vary only in the internal space $\phi(y)$, while background fluxes have a precise form as we will see in Section 4.3.

4.1. IIB compactification on Calabi–Yau threefolds

Now we specialise to *pure geometrical* IIB compactifications, where the only background field turned on is the metric g_{MN}, while the background fields H_3, F_1, F_3, F_5 are set to zero and the dilaton ϕ is constant.

In order to get a four-dimensional supersymmetric theory, one has to choose the compactification manifold M_6 *preserving* some amount of supersymmetry, i.e. a manifold admitting a certain number of Killing spinors. There are several reasons to search for a supersymmetric effective theory, mainly phenomenological, to construct appealing semi-realistic models of particle physics, in which to embed the SM gauge group and in which the supersymmetry is broken at energy scale lower than the string scale (typically at TeV scale). Compactifications preserving maximal supersymmetry are also simpler to study from a technical point of view, because one can show that in this case solutions to the

---

[3]By compactifying on more symmetric spaces, as six-tori or simple orientifolds thereof, one can get more supersymmetric effective theories.

supersymmetric conditions, first order differential equations, are automatically solutions of the supergravity equations of motion, which are of second order. Following considerations outlined below, one finds that, in purely geometric compactifications of type II theories, the manifolds preserving the minimal number of supersymmetries, which is 1/4 of the starting ones, are the Calabi–Yau manifolds.

First of all, in a starting background which preserves four-dimensional maximal symmetry, all vevs of fermionic fields are necessarily vanishing [4]. For this reason one restricts the analysis to purely bosonic solutions. Supersymmetries are preserved if the associated transformations leave the starting background invariant, i.e. if the supersymmetric variation of each field vanishes. For a generic bosonic O_{bos} or fermionic O_{ferm} field these variations have the following general structure:

$$\delta O_{\text{bos}} \sim [Q, O_{\text{bos}}] \sim O_{\text{ferm}} \quad ,$$
$$\delta O_{\text{ferm}} \sim [Q, O_{\text{ferm}}] \sim O_{\text{bos}} \quad ,$$

$$(4.1.1)$$

where Q is the supercharge. Thus, in a background with a maximal four-dimensional symmetry, $\delta O_{\text{bos}} = 0$ is trivially satisfied, since fermionic fields vanish, and one is left with the condition on fermionic variations. In the pure geometrical type IIB compactification at hand, by setting to zero (2.1.8) one gets:

$$\delta \lambda^{1,2} \sim \partial_M \phi \Gamma^M \epsilon_{1,2} = \partial_m \phi \gamma^m \epsilon_{1,2} = 0 \quad ,$$
$$\delta \psi_M^{1,2} \sim \nabla_M \epsilon_{1,2} = 0 \quad .$$

$$(4.1.2)$$

Equations for $\psi_M^{1,2}$ in (4.1.2) are the so-called *Killing spinor equations* for the two Majorana–Weyl spinors $\epsilon_{1,2}$ representing the infinitesimal supersymmetric parameters. These equations means that there should exist two covariantly constant spinors $\epsilon_{1,2}$. Choosing a real representation for gamma matrices (see Appendix A), $\epsilon_{1,2}$ are really real.

The factorized Ansatz (4.0.1) induces a decomposition of the ten-dimensional structure group $Spin(1,9) \rightarrow Spin(1,3) \times Spin(6) = SL(2, \mathbb{C}) \times SU(4)$. As a consequence, a ten-dimensional Weyl spinor $\mathbf{16}_{\mathbb{C}}$ can be decomposed with respect to that subgroup as

$$Spin(1,9) \rightarrow Spin(1,3) \times Spin(6)$$
$$\mathbf{16}_{\mathbb{C}} \quad \rightarrow \quad (\mathbf{2}, \mathbf{4}) \oplus (\bar{\mathbf{2}}, \bar{\mathbf{4}}) \quad ,$$

$$(4.1.3)$$

where $\mathbf{2}, \bar{\mathbf{2}}$ are four-dimensional Weyl spinors of opposite chirality (irrepses of $SL(2, \mathbb{C})$) and $\mathbf{4}, \bar{\mathbf{4}}$ are six-dimensional Weyl spinors of opposite chirality (irrepses of $SU(4)$). Chi-

---

[4]This is due to the structure decomposition (4.1.3). Such a decomposition shows that spinors transform non-trivially under $Spin(1,3)$ and then non-vanishing expectation values are not allowed.

ralities combine in order to reproduce the correct ten-dimensional chirality [5]. Hence, Killing spinors $\epsilon_{1,2}$ decompose as

$$
\begin{aligned}
\epsilon_1(x,y) &= \zeta_1(x) \otimes \eta_1(y) + c.c. \quad, \\
\epsilon_2(x,y) &= \zeta_2(x) \otimes \eta_2(y) + c.c. \quad.
\end{aligned}
\tag{4.1.4}
$$

Here $\zeta_{1,2} \equiv \zeta_+^{1,2}$ are two independent four-dimensional anti-commuting spinors of positive chirality and $\eta_{1,2} \equiv \eta_+^{1,2}$ are two six-dimensional commuting spinors of positive chirality, which can be independent or dependent, depending on the structure of M_6. Thus one has anti-commuting $\epsilon_{1,2} \equiv \epsilon_+^{1,2}$ of ten-dimensional positive chirality (lefthanded), as must be in a type IIB supergravity. Complex conjugation flips chirality: $\zeta_{1,2}^* \equiv \zeta_-^{1,2}$ and $\eta_{1,2}^* \equiv \eta_-^{1,2}$. We assume $\epsilon_{1,2}^\dagger \epsilon_{1,2} \neq 0$.

One can show that supersymmetry equations are just *necessary* conditions to get a supersymmetric vacuum, they solve the equations of motion [6]. Hence, one must impose the Bianchi identities, which are non-trivial constraints on the solutions. However, since these identities involve fluxes, they are trivially satisfied in case of pure geometric compactifications, and one is left with the supersymmetry conditions (4.1.2).

Using real ten-dimensional gamma matrices Γ_M, the dilatino variations, along with (4.1.4), gives

$$
\gamma^m \partial_m \phi (\zeta_{1,2} \otimes \eta_{1,2} + c.c.) = 0 \quad,
\tag{4.1.5}
$$

which are satisfied since we are assuming a constant dilaton ϕ.

The gravitino variations are more interesting. They constrain the properties of both the external and the internal spaces. Using the factorization of the gamma matrices [47], the covariant derivative of a spinor splits as

$$
\nabla_M = \partial_M + \frac{1}{4}\omega_M{}^{AB}\Gamma_{AB} \quad \longrightarrow \quad
\begin{cases}
\nabla_\mu \otimes \mathbb{1} + \dfrac{1}{2}e^A(\gamma_\mu\gamma_5 \otimes \gamma^m\partial_m A) \quad, \\[2mm]
\mathbb{1} \otimes \nabla_m \quad,
\end{cases}
\tag{4.1.6}
$$

where ∇_μ is the covariant derivative with respect to the unwarped external metric. Using (4.1.6) and (4.1.4), the equations (4.1.2) for the gravitinos become:

---

[5]Ten-dimensional chiral operator splits, under (4.0.1), as $\Gamma_{10} \equiv \gamma_5 \otimes \gamma_7$, where γ_5 and γ_7 are the four-dimensional (unwarped) and six-dimensional chiral operators respectively (see Appendix A for the conventions on gamma matrices).

[6]This fact can be elegantly described in the generalised geometry framework [25, 26, 47].

$$\nabla_\mu \zeta_{1,2} \otimes \eta_{1,2} - \frac{1}{2} e^A (\gamma_\mu \zeta_{1,2}^* \otimes \gamma^m \partial_m A \, \eta_{1,2}^*) + c.c. = 0 \quad , \qquad (4.1.7)$$

$$\zeta_{1,2} \otimes \nabla_m \eta_{1,2} + c.c. = 0 \quad . \qquad (4.1.8)$$

Equation (4.1.7) is solved only for a constant warping A [7] and therefore the spacetime is Minkowskian, as we are going to show. Indeed, with constant A, equation (4.1.7) reduces to

$$\nabla_\mu \zeta_{1,2} = 0 \quad . \qquad (4.1.9)$$

With a Christoffel connection [34]

$$[\nabla_\mu, \nabla_\nu] \zeta_{1,2} \sim R_{\mu\nu\rho\sigma} \gamma^{\rho\sigma} \zeta_{1,2} \sim \Lambda \gamma_{\mu\nu} \zeta_{1,2} \quad , \qquad (4.1.10)$$

where we used the expression for the Riemann tensor in the (unwarped) maximally symmetric 4d spacetime $R_{\mu\nu\rho\sigma} \sim \Lambda(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho})$, with cosmological constant Λ. Therefore, finally, we get:

$$\nabla_\mu \zeta_{1,2} = 0 \quad \Longrightarrow \quad \Lambda = 0 \quad \Longleftrightarrow \quad R_{\mu\nu\rho\sigma} = 0 \quad . \qquad (4.1.11)$$

Thus the external gravitino equations require a constant warp factor and a Minkowskian four-dimensional spacetime.

As a consequence the spin connection vanishes $\omega_M^{AB} = 0$ and $\zeta_{1,2}$ are constants, not just convariantly constant. They represent the infinitesimal parameters of the unbroken global four-dimensional supersymmetry. This is a first hint on the fact that the residual four-dimensional supersymmetry should have at least (i.e. for $\eta_1 \sim \eta_2$) $\mathcal{N} = 2$.

By repeating the above considerations for the gravitino internal equations (4.1.8) we arrive at the *Ricci flatness* condition for the internal metric [34]:

$$\nabla_m \eta_{1,2} = 0 \quad \Longrightarrow \quad R_{mn} = 0 \quad . \qquad (4.1.12)$$

Note that this does not mean that M_6 is flat, since the Riemann tensor can still be non-vanishing. Furthermore $\nabla_m(\eta_{1,2}^\dagger \eta_{1,2}) = 0$ which implies $\eta_{1,2}^\dagger \eta_{1,2} = \text{const}$.

In general, the existence of a covariantly constant spinor reduces the holonomy group of the Riemannian manifold M_6 [29,34,47]. Hence the supersymmetry conditions $\nabla_m \eta_{1,2} =$

---

[7]We wrote (4.1.7) in that form in order to compare spinors of the same chirality. This fact is useful to show that the equation is solved for constant A only. Indeed, in case of non-constant A, (4.1.7) is satisfied if $\eta_{1,2} \sim \gamma_m \eta_{1,2}^*$, which is impossible since $\eta_{1,2}^\dagger \gamma_m \eta_{1,2} = 0$ by chirality.

0 mean that M_6 must have at most holonomy $SU(3)$, i.e. it has to admit at least one covariantly constant spinor. We will see below that Ricci flat manifolds with $SU(3)$ holonomy are Calabi–Yau manifolds, but the discussion can be generalized to manifold of smaller holonomy.

In the first case, there exists only a covariantly constant spinor $\eta_1 = \eta_2 \equiv \eta$ and the compactification yields to a four-dimensional $\mathcal{N} = 2$ supersymmetry.

Otherwise, if the holonomy group is smaller, for instance is $SU(2)$ or trivial (in case of flat M_6, which is the particular case treated in this thesis, see Section 5.4), the equation $\nabla_m \eta = 0$ allows respectively two or four covariantly constant independent spinors η_α as solutions. Hence in general $\eta_+^{1,2}$ in (4.1.4) are a linear combination thereof:

$$\eta_+^{1,2} = \sum_\alpha c_\alpha^{1,2} \eta_\alpha \quad , \qquad \alpha = 1, 2 \ (1, ..., 4) \ . \tag{4.1.13}$$

Inserting (4.1.13) into (4.1.4), one gets four (eight) tensor products $\zeta_{1,2} \otimes \eta_\alpha$, which lead to 4 (8) independent four-dimensional Weyl spinors, giving 16 (32) independent real supercharges, i.e. a $\mathcal{N} = 4(8)$ residual supersymmetry in four dimensions.

Resuming, supersymmetry imposes two constraints on the Riemannian internal manifold M_6. A *topological constraint* on the (at most) $SU(3)$-structure of M_6, that is the existence of (at least) a globally defined non-vanishing invariant spinor η (see (4.1.4)), and a *differential constraint* on the covariantly constancy of η. Now we show how these constraints lead M_6 to be a Calabi–Yau three-fold CY_3.

4.1.1. The geometry of Calabi–Yau spaces

Let us take the case of strict $SU(3)$ holonomy, in which we can set $\eta \equiv \eta_1 = \eta_2$ and $\eta^\dagger \, \eta = 1$. We want to show that such a manifold is a Calabi–Yau, defined as follows:

Def: A *Calabi–Yau* manifold is a compact Kähler manifold M with a vanishing first Chern class.

Def: Given a hermitean manifold M, the *first Chern class* c_1 is defined as the cohomology class of the Ricci form $\mathfrak{R}$ divided by 2π:

$$c_1 \equiv \frac{[\mathfrak{R}]}{2\pi} \in H^{1,1}(M) \quad . \tag{4.1.14}$$

Def: On an hermitean manifold M, in terms of the complex local coordinates, the *Ricci form* $\mathfrak{R}$ is the (1,1)-form defined as

$$\mathfrak{R} \equiv i\mathfrak{R}_{i\bar{j}} dz^i \wedge dz^{\bar{j}} = i\partial\bar{\partial} \log \sqrt{g} \quad , \tag{4.1.15}$$

where

$$\mathfrak{R}_{i\bar{j}} \equiv R^k{}_{ki\bar{j}} = -\partial_i \partial_{\bar{j}} \log \sqrt{g} \quad , \tag{4.1.16}$$

as one can show for an hermitian manifold. $R^l{}_{li\bar{j}}$ is the only non-vanishing components of the curvature tensor and $g \equiv \det(g_{mn})$ [48], [49] [8].

Now we want to show that a six-dimensional Riemannian manifold M_6 with $SU(3)$ structure and holonomy is Kähler and has $c_1(M) = 0$ [9].

First of all let us see the implications of having a $SU(3)$-structure. The globally defined nowhere vanishing chiral spinor $\eta \equiv \eta_+$ (and metric g) can be used to build tensor as spinor bilinears. Remembering that six-dimensional gamma matrices are antisymmetric and that η is Grassmann even, we get:

$$\eta^T \eta = 0 \qquad \text{by chirality}^{10} \quad , \tag{4.1.17}$$

$$\eta^T \gamma_m \eta = 0 \qquad \text{by symmetry} \quad , \tag{4.1.18}$$

$$\eta^T \gamma_{mn} \eta = 0 \qquad \text{by symmetry (or chirality)} \quad , \tag{4.1.19}$$

$$I^m{}_n \equiv -i\eta^\dagger \gamma^m{}_n \eta \qquad \textit{glob. defined, nowhere vanishing} \quad , \tag{4.1.20}$$

$$\Omega_{mnp} \equiv \eta^T \gamma_{mnp} \eta \qquad \textit{glob. defined, nowhere vanishing} \quad . \tag{4.1.21}$$

The minus sign in (4.1.20) it is only matter of convention [11]. Fierz identities ensure that $I^m{}_n I^n{}_p = -\delta^m_p$ and therefore M_6 is at least almost complex, with the almost complex structure I [34]. As one can verify, the metric is hermitean with respect to the complex structure defined in (4.1.20), i.e. it satisfies

$$g_{mn} = I^p{}_m I^q{}_n g_{pq} \tag{4.1.22}$$

---

[8]Here are some notable properties of the Ricci form. Let us first define the action of *Dolbeault operators* $\partial, \bar{\partial}$. On a complex (p,q)-form ω: $\partial\omega = \partial_i \omega_{j_1\ldots j_p, \bar{k}_1\ldots\bar{k}_q} dz^i \wedge dz^{j_1} \wedge \cdots dz^{j_p} \wedge d\bar{z}^{k_1} \wedge \cdots d\bar{z}^{k_q}$ and $\bar{\partial}\omega = \partial_{\bar{\imath}} \omega_{j_1\ldots j_p, \bar{k}_1\ldots\bar{k}_q} dz^{\bar{\imath}} \wedge dz^{j_1} \wedge \cdots dz^{j_p} \wedge d\bar{z}^{k_1} \wedge \cdots d\bar{z}^{k_q}$. It is straightforward to show that $\mathfrak{R}$ is real, using $\bar{\partial}\partial = -\partial\bar{\partial}$, and that it is closed $d\mathfrak{R} = 0$, using $\partial\bar{\partial} = \frac{1}{2}d(\partial - \bar{\partial})$. However it is *not exact*, because $(\partial - \bar{\partial})\log\sqrt{g}$ is not globally defined (i.e. coordinate scalar [49]). Hence $\mathfrak{R}$ is used as the representative of a cohomology class in $H^{1,1}(M)$, termed the first Chern class. This class possesses the important property to be *analytic invariant*, that is, invariant under smooth changes of the metric $g_{mn} \to g_{mn} + \delta g_{mn}$ [49]. In fact, one can show that $\mathfrak{R} \to \mathfrak{R} + \delta\mathfrak{R}$ where $\delta\mathfrak{R}$ is exact [48], [49].

[9]The Riemannian structure, i.e. the existence of a globally defined metric tensor, is necessary also to define gamma matrices in the curved space as $\gamma_m = e^n_m \gamma_{\underline{n}}$, with the sechsbein e^n_m.

[10]We use an antisymmetric chiral operator γ_7 to build chiral projectors $P_{L,R}$, see Appendix A.

[11]Conventions are explained in Appendix A. We anticipate that in Chapter (5) it will turn out more convenient to change conventions adopted here.

and therefore M_6 is almost hermitean. An almost hermitean manifold has also an almost symplectic structure [47]. In fact, in this case it is possible to globally define a non-degenerate two-form

$$J \equiv \frac{1}{2}J_{mn}dy^m \wedge dy^n \qquad \text{with} \quad J_{mn} = g_{mp}I^p_{\ n} \quad . \qquad (4.1.23)$$

This is the almost symplectic structure.

Considering also the differential constraint furnished by the $SU(3)$ holonomy, for a metric-compatible connection ∇ [12], it happens that the Nijenhuis tensor $(N_I)^m_{\ np}$ vanishes [34], [47], [49]:

$$\begin{aligned} \nabla\eta = 0 \\ \nabla g = 0 \end{aligned} \quad \Longrightarrow \quad \nabla I = 0 \quad \Longrightarrow \quad (N_I)^m_{\ np} = 0 \quad . \qquad (4.1.24)$$

This means that the almost complex structure is integrable and hence M_6 is a complex manifold. To be precise, due to (4.1.22), M_6 is an hermitean manifold.

At this point, stated that M_6 has both a Riemannian and a complex structure, the Kähler structure is automatically implied. Let us see how.

On a complex manifold one can locally define a set of complex coordinates $z^i, \bar{z}^i$ with $i = 1, 2, 3$, defining an atlas of holomorphic coordinates. In these coordinates the components of the pre-symplectic structure J defined above are related to the (hermitean) metric as

$$J_{i\bar{j}} = ig_{i\bar{j}} \quad . \qquad (4.1.25)$$

Therefore

$$dJ = i\partial_i g_{j\bar{k}}dz^i \wedge dz^j \wedge d\bar{z}^{\bar{k}} + i\partial_{\bar{i}}g_{j\bar{k}}d\bar{z}^{\bar{i}} \wedge dz^j \wedge d\bar{z}^{\bar{k}} = 0 \quad , \qquad (4.1.26)$$

which follows from the metric compatibility condition $\nabla g = 0$, with a torsion-free connection [13] . We have just shown the integrability of the pre-symplectic structure: M_6 is thus a Kähler manifold, with Kähler form J.

---

[12]A connection ∇ is said to be *metric-compatible*, i.e. compatible with respect to the Riemannian structure with a metric g, if $\nabla g = 0$.

[13]On a hermitean manifold we can define an *hermitean connection* respecting the complex and Riemannian structures $\nabla g = \nabla I = 0$. The only non-vanishing components of the connection coefficients are the one pure in lower indices Γ^i_{jk} and its complex conjugate [48], [49]. Then $\nabla g = 0 \Rightarrow \partial_i g_{j\bar{k}} = g_{l\bar{k}}\Gamma^l_{ij}$ and analogously for the complex conjugate. Taking a torsion-free connection, Γ^i_{jk} are the Levi–Civita connection components, symmetric in j, k.

We have to demonstrate that the first Chern class vanishes. The demonstration follows directly from the Ricci flatness condition (4.1.12), once stated M_6 to be Kähler. In this case, in fact,

$$\mathfrak{R}_{i\bar{\jmath}} = R_{i\bar{\jmath}} \quad , \tag{4.1.27}$$

where $R_{i\bar{\jmath}} = R^k{}_{ik\bar{\jmath}}$ is the Ricci tensor in complex coordinates [48], [49]. Therefore, if a Kähler manifold M_6 admits a Ricci flat metric, the analytic invariance of c_1 ensures a vanishing first Chern class for all other metrics and hence $c_1(M_6) = 0$.

We have just demonstrated that M_6 is a Calabi–Yau manifold as previously defined. These manifolds are characterised also by a holomorphic $(3,0)$-form defined by the bilinear (4.1.21):

$$\Omega \equiv \frac{1}{3!}\Omega_{ijk}dz^i \wedge dz^j \wedge dz^k \quad . \tag{4.1.28}$$

Such form is associated to the almost complex structure of M_6 and it used, along with J, to define the $SU(3)$ structure in terms of forms (see [47]). Being Ω holomorphic [14], it is closed $d\Omega = 0$. It has the following components:

$$\Omega_{ijk}(z) = f(z)\epsilon_{ijk} \quad , \tag{4.1.29}$$

with $f(z)$ a no-where vanishing holomorphic function. Furthermore, Ω is not exact [15].

Calabi–Yau manifolds are really important because of the following general theorem, firstly conjectured by Calabi and then demonstrated by Yau:

Yau's thoerem: Let M be a compact, Kähler manifold, with Kähler metric g and Kähler form J. Suppose that ρ is a real, closed $(1,1)$-form on M with $[\rho] = 2\pi c_1(M)$. Then there *exists* a *unique* Kähler metric g' on M with Kähler form J' in the same class of J ($[J] = [J']$) and with Ricci form $\mathfrak{R}' = \rho$.

In other words, for a given compact Kähler manifold M with first Chern class $c_1(M)$, it

---

[14]$\bar{\partial}\Omega = 0$ is implied by $\nabla_{\bar{\imath}}\Omega_{jkl} = 0$, due to the metric compatibility and the $SU(3)$ holonomy ($\nabla g = \nabla\eta = 0$), and by the fact that M_6 is hermitean ($\Gamma^k_{\bar{\imath}j} = 0$).

[15]This is because by choosing (4.1.20), (4.1.21) one gets (see Appendix A)

$$i\Omega \wedge \bar{\Omega} \sim d\mathrm{V} \quad . \tag{4.1.30}$$

The integral of the volume form $d\mathrm{V}$ over M_6 is non-zero. It follows that $\Omega \wedge \bar{\Omega}$ is not exact. Hence Ω is not exact [49].

is always possible to pick up a metric g giving a particular Ricci form $\mathfrak{R} \in c_1(M)$, and that metric is *unique*. This means that any other metric g' corresponding to the same Ricci form $\mathfrak{R}$ is associated to a different Kähler manifold, differing in complex structure and/or Kähler structure.

We will use this fact to parametrize the space of the different supersymmetry preserving internal Calabi–Yau three-folds, that is the space of the different Kähler manifolds satisfying Ricci flatness (4.1.12).

A comment is due in order to proceed. We will often refer to "Calabi–Yau" (CY) as manifolds with strict $SU(3)$ holonomy, even if the definition is more general and valid also for manifolds with a smaller holonomy group, as one can check repeating the considerations above [47]. In particular, the original work in this thesis is based on a six-torus T^6, of local trivial holonomy, in which the reduction of supersymmetry is due to additional physical ingredients.

4.1.2. The Cohomology of a Calabi–Yau

In general, the Kaluza–Klein reduction (see Appendix B) on a ten-dimensional background $M^{1,3} \times M_6$ with a compact M_6, gives rise to massless four-dimensional fields as zero-modes of the internal Laplacian. Hence, in the case of a CY at hand (strict $SU(3)$ holonomy), zero-modes are in one-to-one correspondence with the harmonic forms on the Calabi–Yau threefold and thus their multiplicity is counted by the dimension of the non-trivial cohomology groups of the CY. These are decomposed in the following even/odd (p,q)-cohomology groups [16]:

---

[16]Let us briefly remember that on a complex manifold M one can define complex (p,q)-forms $\omega^{p,q}$ as having p holomorphic and q anti-holomorphic indices. A complex k-form ω is uniquely written in terms of such bidegree forms as $\omega_k = \sum_{p+q=k} \omega^{p,q}$.

The real exterior derivative on a (p,q)-form can be decomposed as $d = \partial + \bar{\partial}$. Each Dolbeault operator $\partial, \bar{\partial}$ defines its proper cohomology group $H^{p,q}_{\partial}(M), H^{p,q}_{\bar{\partial}}(M)$. The complex dimension $h^{p,q} \equiv \dim_{\mathbb{C}} H^{p,q}_{\bar{\partial}}(M)$ is called the *Hodge number*. In a Kähler manifold the Laplacians $\Delta_d = dd^{\dagger} + d^{\dagger}d, \Delta_{\partial} = \partial\partial^{\dagger} + \partial^{\dagger}\partial, \Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^{\dagger} + \bar{\partial}^{\dagger}\bar{\partial}$ are the same. It follows that $d, \partial, \bar{\partial}$-cohomology groups have the same harmonic representatives and then:

$$H^{p,q}(M) = H^{p,q}_{\partial}(M) = H^{p,q}_{\bar{\partial}}(M) \quad . \tag{4.1.31}$$

Using this and the more generic decomposition $H^k(M)^{\mathbb{C}} = \oplus_{p+q=k} H^{p,q}(M)$ one finds that in a Kähler manifold there is a relation between Hodge and Betti numbers b^k:

$$b^k \equiv \dim_{\mathbb{R}} H^k(M) = \dim_{\mathbb{C}} H^k(M)^{\mathbb{C}} = \sum_{k=p+q} h^{p,q} \quad . \tag{4.1.32}$$

$$H^{\text{even}} = H^{0,0} \oplus H^{1,1} \oplus H^{2,2} \oplus H^{3,3} \quad ,$$
$$H^{\text{odd}} = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3} \quad .$$

(4.1.33)

The Hodge numbers of a complex manifold are usually organised in the so-called *Hodge diamond*. The Hodge diamond of a CY three-fold takes the form:

$$
\begin{array}{ccccccc}
 & & & h^{3,3} & & & \\
 & & h^{3,2} & & h^{2,3} & & \\
 & h^{3,1} & & h^{2,2} & & h^{1,3} & \\
h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & & & h^{0,0} & & &
\end{array}
\xrightarrow{SU(3)\text{holonomy}}
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array} \quad .
$$

(4.1.34)

The Hodge diamond of a CY_3 has this particular form because the Hodge numbers of a generic CY n-fold satisfy $h^{p,0} = h^{n-p,0}$ [34,48]. Furthermore, in a Kähler manifold of complex dimension n, Hodge numbers are related also by complex conjugation and by Poicaré duality respectively as $h^{p,q} = h^{q,p}$ and $h^{p,q} = h^{n-p,n-q}$. Any compact connected Kähler manifold has $h^{0,0} = 1$, corresponding to constant functions. A simply-connected Kähler manifold has vanishing first homology group and then $b^1 = h^{1,0} = h^{0,1} = 0$ [48,49]. Hence a simply-connected CY three-fold is described in terms of $h^{1,1}$ and $h^{2,1}$ only. $h^{3,0} = 1$ refers to the unique (up to constant rescalings) holomorphic Ω and $b^1 = 0$ implies that on a CY threefold the metric components $g_{\mu n}$, leading to 4d massless vector fields, are absent [29], [34]. Note that the Hodge diamond does not furnish a complete characterisation of the CY, since inequivalent CY may have he same Hodge numbers.

It is useful to introduce a basis for the different cohomology groups by choosing the unique harmonic representative in each cohomology class, see Table 4.1. Here vol is the harmonic volume form of the CY, while $(\alpha_{\hat{K}}, \beta^{\hat{L}})$ is a real basis on $H^3(M_6, \mathbb{R})$ and $\tilde{\omega}^A$ is the real dual basis with respect to the real ω_A, which means that these have the non-vanishing intersection numbers

$$\int_{M_6} \omega_A \wedge \tilde{\omega}^B = \delta_A^B \quad , \quad \int_{M_6} \alpha_{\hat{K}} \wedge \beta^{\hat{L}} = \delta_{\hat{K}}^{\hat{L}} \quad .$$

(4.1.35)

As we will see below, these harmonic forms are used in the dimensional reduction of the ten-dimensional theory to four dimensions, and are also related to massless modes arising as metric deformations [50]. The latter ones deserve a more detailed analysis and we present them in the following subsection.

Cohomology group	dimension	basis
$H^{1,1}$	$h^{1,1}$	ω_A
$H^{2,2}$	$h^{1,1}$	$\tilde{\omega}^A$
H^3	$2h^{2,1}+2$	$(\alpha_{\hat{K}}, \beta^{\hat{L}})$
$H^{2,1}$	$h^{2,1}$	χ_K
$H^{1,2}$	$h^{2,1}$	$\bar{\chi}_K$
$H^{3,3}$	1	vol

Table 4.1.: Cohomology groups on a CY.

4.1.3. Calabi–Yau deformations

In this subsection we restrict to CY spaces in the stricter sense, i.e. Kähler manifolds of $SU(3)$ holonomy. In the following we will refer to the complex coordinates as $y^i, \bar{y}^{\bar{j}}$, to avoid confusion with complex structure moduli (z^k). Therefore in complex coordinates the Kähler form is

$$J = ig_{i\bar{j}}dy^i \wedge d\bar{y}^{\bar{j}} \quad . \tag{4.1.36}$$

Following Candelas and de la Ossa [50] we choose the pair (g, I) to describe the Calabi–Yau threefold ($SU(3)$ holonomy) and its deformations. Thus the manifold can be deformed in two ways. The first consists in directly deforming the complex structure, the other in deforming the components $g_{i\bar{j}}$ of the metric, in such a way to deform J [50].

The Yau's theorem comes here to the fore. It allows to parametrize the space of different Calabi–Yau manifolds with the space of the Ricci flat metrics. Let g_{mn} and $g_{mn} + \delta g_{mn}$ be two Ricci-flat metrics for the CY, referred each to a different Kähler class. Imposing a gauge fixing "coordinate condition", necessary to eliminate the metric deformations describing coordinate changes which are not of interest, the Ricci flatness $R_{mn}(g) = 0 = R_{mn}(g+\delta g)$ forces deformations δg to satisfy a differential equation, called the Lichnerowicz equation [50]. Zero-modes solutions to this equation separate in mixed type $\delta g_{i\bar{j}}$ and in pure type δg_{ij}, $\delta g_{\bar{i}\bar{j}}$ (omitting zero-mode index). They are in one-to-one correspondence with the elements of the cohomology groups $H^{1,1}$ and $H^{2,1}$. Indeed, $\delta g_{i\bar{j}}$ can be used to build the real harmonic $(1, 1)$-form

$$\delta J = i\delta g_{i\bar{j}} \, dy^i \wedge d\bar{y}^{\bar{j}} \quad , \tag{4.1.37}$$

while δg_{ij} can be used to build the following complex harmonic $(2, 1)$-form

$$\Omega_{ij}{}^{\bar{k}}\delta g_{\bar{k}\bar{l}} \, dy^i \wedge dy^j \wedge d\bar{y}^{\bar{l}} \quad . \tag{4.1.38}$$

From (4.1.37) it should be clear that variations of mixed type correspond to variations of the Kähler class. On the other hand, variations of mixed type correspond to variations of the complex structure. The proof follows remembering that $g_{mn} + \delta g_{mn}$ is still a Kähler metric. Hence, there must exist a coordinate system in which its pure parts vanish. However, only a non-holomorphic change of coordinates can remove these parts [50], that is, one must change complex structure.

The deformations of the Kähler form (4.1.37) can be expanded in a basis of harmonic $(1,1)$-forms and thus J can be described by $h^{1,1}$ real parameters v^A:

$$\delta g_{i\bar{j}} = -i\delta v^A (\omega_A)_{i\bar{j}} \,, \quad J_{i\bar{j}} = v^A (\omega_A)_{i\bar{j}} \,, \quad A = 1,...,h^{1,1} \quad . \tag{4.1.39}$$

To be precise, by integrating infinitesimal deformations δv^A, one can get finite deformations v^A. Among these, only those leading to positive definite metrics are allowed, or equivalently, those such that the volumes of complex two-cycles, complex four-cycles and M_6 are positive [17] [34]:

$$\int_{M_6} J \wedge J \wedge J > 0 \quad , \quad \int_{c_2} J \wedge J > 0 \quad , \quad \int_{c_4} J > 0 \quad . \tag{4.1.40}$$

These conditions are preserved under positive rescalings $J \to rJ$, for any $r > 0$. Hence, the subset of Kähler deformations v^A leading to a J satisfying (4.1.40) is called *Kähler cone*. Under these rescalings v^A span indeed a $h^{1,1}$-dimensional cone.

In both type II string theories there exists a real two-form field, the B_2 field. After the dimensional reduction on a Calabi–Yau threefold, for a pure geometrical compactification, internal components of such a field are components of an harmonic $(1,1)$-form $B_{i\bar{j}}$ (see Appendix B). This real form, closed on the CY, can be used to build the *complexified Kähler form*

$$\mathcal{J} \equiv B^{1,1} + iJ \quad . \tag{4.1.41}$$

The real scalars $b^A(x)$, which arise in the expansion of the internal part of the B_2 form, can be used to provide the imaginary parts of the complex moduli fields

$$t^A(x) = b^A(x) + iv^A(x) \tag{4.1.42}$$

parametrizing the $h^{1,1}$-dimensional complexified Kähler cone. By writing $v^A(x)$ we are anticipating that the reduction procedure will lead to promote parameters describing the

---

[17]Complex cycles C_{2k} ($k = 1, 2, 3$) are complex submanifold of M_6, defined by equations like $f(z) = 0$. Remarkably, they are volume-minimising in their homology classes [51]. Their volume is given by $\frac{1}{k!} \int_{C_{2k}} J^k$. Note that conditions (4.1.40) depend on conventions, as explained in Appendix A. Here we are adopting the most natural notation convention.

internal metric to four-dimensional scalar fields. As we will see, there exist also models in which b^A are not present. In these cases the complexification of Kähler moduli involves moduli coming from the expansion of other fields.

The second set of deformations are variations of the complex structure of the Calabi–Yau. They are in one-to-one correspondence with the harmonic $(2,1)$-forms (4.1.38). Expanding (4.1.38) in the harmonic basis of $H^{2,1}$ we get

$$
\begin{aligned}
\frac{1}{4}\Omega_{ij}{}^{\bar{k}}\delta g_{\bar{k}\bar{l}}\, dy^i \wedge dy^j \wedge d\bar{y}^{\bar{l}} &= \delta z^K \chi_K \\
&= \delta z^K \frac{1}{2}(\chi_K)_{ij\bar{k}}\, dy^i \wedge dy^j \wedge d\bar{y}^{\bar{k}} \quad,
\end{aligned}
\tag{4.1.43}
$$

where z^k are $h^{2,1}$ complex parameters describing the complex structure. They correspond to local coordinates for the complex structure moduli space [50]. The $\frac{1}{4}$ has been chosen for convenience. By inverting (4.1.43), one finds

$$
\delta g_{\bar{i}\bar{j}} = \frac{1}{||\Omega||^2}\delta z^K (\chi_K)_{ij\bar{j}}\,\bar{\Omega}^{ij}{}_{\bar{i}} \quad,
\tag{4.1.44}
$$

which shows how the metric changes under a deformation of the complex structure δz^k. Here $||\Omega||^2 \equiv \frac{1}{3!}\Omega_{ijk}\bar{\Omega}^{ijk}$.

Complex parameters t^A and z^K span the geometric Calabi–Yau moduli space. Candelas and de la Ossa showed that such a moduli space is at least locally a product

$$
\mathcal{M}_z \times \mathcal{M}_k \quad,
\tag{4.1.45}
$$

with $\mathcal{M}_z$ the manifold of complex structure of complex dimension $h^{2,1}$ and $\mathcal{M}_k$ the complexification of the space of parameters of the Kähler class, with complex dimension $h^{1,1}$. It is also shown they are special Kähler manifolds [50]. Here is a review of the main results.

The complex structure moduli space $\mathcal{M}_z$

This is the space parametrized by z^K. Its metric is given by

$$
G_{K\bar{L}} = -\frac{\int_{M_6} \chi_K \wedge \bar{\chi}_{\bar{L}}}{\int_{M_6} \Omega \wedge \bar{\Omega}} = \partial_K \partial_{\bar{L}} K^{\mathrm{cs}} \quad,
\tag{4.1.46}
$$

with the Kähler potential

$$
K^{\mathrm{cs}} = -\log\left(i\int_{M_6}\Omega \wedge \bar{\Omega}\right) \quad,
\tag{4.1.47}
$$

as one can verify by using

$$\frac{\partial}{\partial z^K}\Omega(z) = -\Omega(z)\frac{\partial}{\partial z^K}K^{\mathrm{cs}} + \chi_K(z,\bar{z}) \quad . \tag{4.1.48}$$

Notice that the three-form Ω is defined up to a complex rescaling by a holomorphic function [50]

$$\Omega \to e^{f(z)}\Omega \quad , \tag{4.1.49}$$

which changes also the Kähler potential (4.1.47) by a Kähler transformation

$$K^{\mathrm{cs}} \to K^{\mathrm{cs}} - f(z) - \bar{f}(\bar{z}) \quad . \tag{4.1.50}$$

Notable consequences are the possibility to construct a covariant derivative and the fact that the symmetry (4.1.49) allows to reduce the number of periods of Ω [18] and then to define $h^{2,1}$ special coordinates for $\mathcal{M}_z$ [50].

The complexified Kähler moduli space $\mathcal{M}_k$

$\mathcal{M}_k$ is spanned by the complex parameters t^A. The metric of this space is defined as

$$G_{A\bar{B}} = \frac{1}{2}G(\omega_A, \omega_B) \equiv \frac{1}{4V}\int_{M_6}\omega_A \wedge *_6\omega_B = \partial_A\partial_{\bar{B}}K^{\mathrm{k}} \quad , \tag{4.1.51}$$

where ω_A, ω_B are real $(1,1)$-forms and $*_6$ is the Hodge operator on the CY. The metric (4.1.51) is derivable from the Kähler potential

$$K^{\mathrm{k}} = -\log\left(\frac{4}{3}\int_{M_6}J \wedge J \wedge J\right) = -\log(8V) \quad , \tag{4.1.52}$$

where $J = v^A\omega_A$ is the Kähler form and V is the volume of the Calabi–Yau

$$V = \frac{1}{6}\int_{M_6}J \wedge J \wedge J \quad . \tag{4.1.53}$$

Its special character is well described in [50].

In the following we will abbreviate the intersection numbers as

---

[18]Periods of Ω are parameters used to describe the manifold to underline its special Kähler structure.

$$
\begin{aligned}
\mathcal{I}_{ABC} &= \int_{M_6} \omega_A \wedge \omega_B \wedge \omega_C \quad , \\
\mathcal{I}_{AB} &= \int_{M_6} \omega_A \wedge \omega_B \wedge J = \mathcal{I}_{ABC} v^C \quad , \\
\mathcal{I}_A &= \int_{M_6} \omega_A \wedge J \wedge J = \mathcal{I}_{ABC} v^B v^C \quad , \\
\mathcal{I} &= \int_{M_6} J \wedge J \wedge J = \mathcal{I}_{ABC} v^A v^B v^C = 6\mathrm{V} \quad .
\end{aligned}
\tag{4.1.54}
$$

Example: moduli of T^2

We have just seen that compactifications on CY threefolds are described by $h^{1,1}$ real parameters v^A and $h^{2,1}$ complex parameters z^K characterising completely the shape and the size of M_6. In order to better understand what does shape and size mean, let us see, for instance, the moduli of a CY onefold, a two-dimensional torus T^2.

Consider a two-torus $T^2 = S^1 \times S^1$ described by two real periodic coordinates $(x, y) \sim (x + 1, y + 1)$. We can introduce a complex coordinate $z = x + \lambda y$, where λ can be identified with the single complex structure. The torus is defined by the flat metric

$$
\begin{aligned}
ds^2 &= 2g_{z\bar{z}} dz d\bar{z} = 2|\alpha|^2 dz d\bar{z} \\
&= 2|\alpha|^2 (dx^2 + |\lambda|^2 dy^2 + 2\mathrm{Re}\,\lambda dx dy) \quad ,
\end{aligned}
\tag{4.1.55}
$$

where α is a generic complex constant. In this case the holomoprhic one-form is $d\Omega = dz$ and the Kähler form is $J = ig_{z\bar{z}} = 2\mathrm{Im}\,\lambda\, g_{z\bar{z}} dx \wedge dy = \sqrt{g} dx \wedge dy$. Hence J describes the volume, i.e. the *size*

$$
v \equiv \int J = \int dx dy \sqrt{g} = 2|\alpha|^2 \mathrm{Im}\,\lambda \quad .
\tag{4.1.56}
$$

The *shape*, on the other hand, is parametrised by the complex structure λ itself. If it is pure imaginary, one has a rectangular torus, described by

$$
\begin{aligned}
\mathrm{Re}\,z &\sim \mathrm{Re}\,z + 1 \quad , \\
\mathrm{Im}\,z &\sim \mathrm{Im}\,z + \mathrm{Im}\,\lambda \quad .
\end{aligned}
\tag{4.1.57}
$$

The torus starts to flex when the complex structure has also a real part, since the identification becomes

$$\mathrm{Re}\,z \sim \mathrm{Re}\,z + 1 + \mathrm{Re}\,\lambda \quad,$$
$$\mathrm{Im}\,z \sim \mathrm{Im}\,z + \mathrm{Im}\,\lambda \quad. \tag{4.1.58}$$

Notice that the metric can be rewritten explicitly in terms of parameters v, λ as

$$ds^2 = \frac{v}{\mathrm{Im}\,\lambda}(dx^2 + |\lambda|^2 dy^2 + 2\mathrm{Re}\,\lambda\, dx dy) \quad. \tag{4.1.59}$$

4.1.4. The effective theory

To obtain the effective four-dimensional theory for the compactification on a Calabi–Yau threefold M_6 one has to reduce the action (2.1.3) [19]. Massless metric fluctuations correspond to promoting geometric Kähler and complex structure parameters to 4d scalar fields $v^A(x)$ and $z^K(x)$. In addition, 4d massless fields come also from the expansion of a massless fluctuation of other IIB fields. Indeed, such fluctuations can be expanded in terms of the harmonic forms on M_6 presented in Table 4.1 as [14]:

$$B_2 = B_2(x) + b^A(x)\omega_A \quad, \qquad A = 1, ..., h^{1,1} \quad,$$
$$C_2 = C_2(x) + c^A(x)\omega_A \quad,$$
$$C_4 = D_2^A(x) \wedge \omega_A + V^{\hat{K}}(x) \wedge \alpha_{\hat{K}} + U_{\hat{K}}(x) \wedge \beta^{\hat{K}} + d^A(x)\tilde{\omega}^A \ , \quad \hat{K} = 0, ..., h^{2,1} \quad. \tag{4.1.60}$$

There are various types of four-dimensional fields appearing in these expansions. $b^A(x)$, $c^A(x)$ and $d^A(x)$ are scalars; $V^{\hat{K}}(x)$ and $U_{\hat{K}}(x)$ are one-forms [20]; $B_2(x)$, $C_2(x)$ and $D_2^A(x)$ are two-forms. The self-duality condition $*\tilde{F}_5 = \tilde{F}_5$ eliminates half of degrees of freedom of C_4. Following [14] we keep $d^A(x)$ and $V^{\hat{K}}(x)$.

For what concern the axion and the dilaton, their massless fluctuations appear as four-dimensional scalars $\phi(x)$ and $C_0(x)$ in the effective theory [21].

The low-energy action is computed inserting (4.1.39), (4.1.44), (4.1.60) in (2.1.5) and integrating over M_6. As expected, the effective action reproduces the (bosonic part of the) standard $\mathcal{N} = 2$ supergravity action, once moduli are organized in the (bosonic components of the) supermultiplets as presented in Table 4.2 [14,52–55]. To be precise, in order to recover the standard $\mathcal{N} = 2$ supergravity action one has to express also the so-called double-tensor multiplet in a hypermultiplet. This is achieved by dualizing

---

[19]For a review on the dimensional reduction see Appendix B.

[20]Turning on fluxes, $V^{\hat{K}}$ and $U_{\hat{K}}$ will correspond to magnetic and electric potentials.

[21]Taking the equations of motion (2.1.7) it is quite straightforward to check that the massless fluctuations of the dilaton and the axion correspond to zero-modes of the internal Laplacian and hence are constants in M_6.

$B_2(x)$ and $C_2(x)$ to two scalar fields [14]. Kinetic terms for geometric moduli $v^A(x)$ and $z^K(x)$ are of sigma-model type, determined respectively by the metrics on the space of geometric deformations G_{AB} (4.1.51) and G_{KL} (4.1.46).

In conclusion, one finds an effective $\mathcal{N} = 2$ supergravity action with only moduli and gauge kinetic terms, without a scalar potential V. This means that supersymmetry is unbroken.

supermultiplet	number	4d fields
gravity multiplet	1	$g_{\mu\nu}, V^0$
vector multiplets	$h^{2,1}$	V^K, z^K
hypermultiplets	$h^{1,1}$	v^A, b^A, c^A, d^A
double-tensor multiplet	1	B_2, C_2, ϕ, C_0

Table 4.2.: $\mathcal{N} = 2$ four-dimensional bosonic components of supermultiplets.

4.2. IIB compactification on Calabi–Yau orientifolds

Purely geometrical supersymmetric IIB CY compactifications lead to $\mathcal{N} = 2$ supergravity theory, as just seen. To reduce the supersymmetry one usually considers *orientifold projections*. For instance, CY orientifolds lead to an effective $\mathcal{N} = 1$ supergravity.

As we will see, the orientifold projection introduce new objects in the compactification background, the O-planes. Consistency implies the presence of D-branes and/or non-vanishing background fluxes. Fluxes enrich the effective supergravity with a non-trivial scalar potential, which can admit even non-supersymmetric (tree-level) vacua.

In general, the introduction of O-planes, branes and fluxes in the background modifies the CY geometry. As we will see in Sect. 4.3, these sources backreact and turn on a *non-trivial* warping $e^{2A(y)}$ in the metric (4.0.2). Such warp factor becomes very strong near local sources. We will often denote this Ansatz by

$$M^{1,3} \times M_6 \quad \xrightarrow{\text{sources + bg fluxes}} \quad M^{1,3} \times_{\text{w}} M_6 \quad . \tag{4.2.1}$$

Hence, in presence of fluxes, the compactification manifold is in general no more a CY and in order to give a proper characterisation to such manifold one has to work in the generalised geometry framework [47, 56].

Carrying out the effective theory by KK reduction in warped compactifications is really involved because of warping. Nevertheless, it is quite standard to by-pass the problem, by performing the dimensional reduction in the *large volume* limit. By taking the average compactification radius R to be large enough, one may assume that background fluxes are diluted and sources are smeared and then do not backreact on the geometry. Since the backreaction is encoded in the warp factor, as we will see, this limit corresponds to a constant warping and then the unwarped Ansatz $M^{1,3} \times M_6$ is restored [13, 17]. This simplifying assumption implies that the metrics on the moduli space of deformations agree with the CY case. Consequently, also the kinetic terms of the reduced action are equal to the ones obtained in CY compactifications, as we will see in Sect. 4.3. However, a non-trivial warp factor is a physically important feature in string compactifications and one should study how it contributes to the effective action.

To summarize, we proceed by steps. Firstly, after recalling the meaning of the orientifold projection in a type IIB theory, we specialise to the example of the O3/O7 CY orientifold. We study how the bulk massless spectrum of the Table 4.2 is reduced (4.2.1) and recall peculiarities of the effective action (focusing on the Kähler potential), in absence of background fluxes and neglecting the open string sector (4.2.2).

Then in Section 4.3, we turn on fluxes and study the associated compactification background. We specialise to a particular class of fluxed vacua, the "GKP Ansatz", and analyse features of the reduced effective action, still neglecting the open string sector.

Hence, we complete the analysis in subsection 4.3.1 allowing for the presence of D3-branes in the background. Since D-branes have their own dynamics, the effective action is modified. In particular, in the scalar sector additional moduli appear. We are interested in understanding how these brane moduli enter in the effective Kähler potential.

All these steps are based on the dimensional reduction carried out in the large volume limit. In the next Chapter we will present an elegant approach which allows to identify non-trivial warping contributions to the effective Kähler potential.

IIB orientifolds

Orientifold projections are a fundamental ingredient for model building, because they allow to include D-braes and fluxes and to reduce the supersymmetry of the effective action obtained from pure geometrical compactifications. In the following we will review the case of Calabi–Yau orientifold compactifications, but the same approach can be applied to compactifications on other other kinds of internal manifolds. The original work of this thesis lies in a toroidal orientifold compactification, which is rather special and so does not fit precisely in this more general class. Nevertheless, one can learn something

4. IIB Calabi–Yau compactifications

interesting from it (see Section 5.4).

    Orientifold theories are unoriented string theories constructed by starting from oriented ones as follows. Consider a string theory with isomorphic left-right moving worldsheet sectors, as the type II. The type I unoriented theory is built as a quotient by the *worldsheet parity* Ω_p, that exchanges left and right movers. States surviving the quotient, i.e. states which are considered equivalent under the left-right exchange, define the new theory. The quotient is called *orientifold*, since Ω_p flips the orientation of the string. see for instance [29].

    More in general, one can consider a type II theory modded out by a symmetry group $S\Omega_p$, where S is a symmetry composed by a group of target space symmetries σ and, eventually, by other operations to render $S\Omega_p$ a symmetry of the string theory. The symmetry group $S\Omega_p$ consists of perturbative symmetries of the string theory and then the orientifold projection can be imposed in the low-energy description of the string theory. Bosonic states transforms under Ω_p as in Table 4.3.

    Orientifold planes are the surfaces left invariant by the σ action. They are RR charged, they have negative tension and they are non-dynamical at lowest order in string theory, which will be always the case of interest. [14].

    In type IIB orientifold S depends on the model. As described below, in the simplest case it suffices S to be a symmetry σ of the ten-dimensional spacetime, but in other models $\sigma\Omega_p$ is no more a symmetry of the IIB string theory and S must be of the form $(-1)^{F_L}\sigma$, where $(-1)^{F_L}$ is another symmetry operation admitted by the IIB theory [22]. The IIB bosonic fields transformations under $(-1)^{F_L}$ and Ω_p are resumed in Table 4.3 [14].

10d fields	Ω_p	$(-1)^{F_L}$	$(-1)^{F_L}\Omega_p$
ϕ	$+$	$+$	$+$
g	$+$	$+$	$+$
B_2	$-$	$+$	$-$
C_0	$-$	$-$	$+$
C_2	$+$	$-$	$-$
C_4	$-$	$-$	$+$

Table 4.3.: IIB orientifolds: Ω_p and $(-1)^{F_L}$ action.

---

[22] F_L is the spacetime fermion number in the left-moving sector [29].

IIB Calabi–Yau orientifolds

In this particular case, consistency requires the internal discrete spacetime symmetry σ to be an isometric and holomorphic involution. Orientifold planes are the spacetime subspaces of fixed points of this involution. Op-planes wrap $(p-3)$-cycles and their negative RR charge $\mu_{Op-plane}$ must be cancelled by fluxes or by the charge of Dp-branes wrapped on similar cycles [15].

The possible Calabi–Yau orientifolds are the following:

- O3-D3 models, $S = \sigma(-1)^{F_L}$: locally, around an O3-plane located at $z^i = 0$, σ flips all complex coordinates $z^i \to -z^i$, for each $i = 1, 2, 3$. The allowed extended objects are space-filling O3-planes and D3-branes;

- O5-D5 models, $S = \sigma$: σ acts flipping locally two different coordinates z^i and z^j, leaving the other one z^k untouched ($i \neq j \neq k \neq i$). This model has O5-planes and D5 branes transverse to z^i and z^j and wrapped on 2-cycles;

- O7-D7 models, $S = \sigma(-1)^{F_L}$: σ acts locally as $z^i \to -z^i$ only in the i-th direction. The model contains O7-planes and D7-branes, both transverse to the i-th direction and wrapped on 4-cycles along the other directions;

- O9-D9 models, $S = \sigma = Id$: the trivial action of σ allows for the presence of ten-dimensional space-filling objects, as O9-planes and D9-branes.

Since consistency requires σ to be an isometry an holomorphic involution of the CY, it leaves invariant both the metric and the complex structure. It follows that the Kähler metric is left invariant too. In order to distinguish the different projections $\mathcal{O}$ above, one refers to the holomorphic three-form Ω transformation under σ:

$$O3/O7: \quad \mathcal{O}_1 = (-1)^{F_L}\Omega_{\mathrm{p}}\sigma \qquad \sigma\Omega = -\Omega \quad , \tag{4.2.2}$$

$$O5/O9: \quad \mathcal{O}_2 = \Omega_{\mathrm{p}}\sigma \qquad \sigma\Omega = \Omega \quad . \tag{4.2.3}$$

Given an orientifold action among those listed above, not all kinds of permitted D-branes and O-planes combine in a supersymmetric background. Including in the compactification different types of these objects can break the supersymmetry. It can be shown that only compactifications on orientifold O3/O7 models with D3/D7-branes and O5/O9 models with D5/D9-branes preserve a common four-dimensional $\mathcal{N} = 1$ supersymmetry. In other words, it can be shown that supersymmetry is preserved in presence of different types of D-branes with four non-common transversal directions, as D3/D7-branes *or* D5/D9-branes [29].

4.2.1. CY orientifolds with $O3/O7$-planes

Now we focus on the first type of orientifold projection $\mathcal{O}_1$ (4.2.2). The purpose is to present, following [14], the massless spectrum of the effective $\mathcal{N} = 1$ supergravity that arises in such compactifications.

To determine the effective theory massless spectrum, when the projection is taken into account, and to see how the massless fields assemble in $\mathcal{N} = 1$ supermultiplets, one has to look at the behaviour of fields under the projection $(-1)^{F_L}\Omega_{\mathrm{p}}\sigma$. In the four-dimensional reduced theory, in fact, only massless states which are invariant under orientifolding are selected. Remembering the action of Ω_{p} and $(-1)^{F_L}$ of Tab. 4.3, it is straightforward to check that the invariant states have to transform under σ as:

10d fields	σ
ϕ	$+$
g	$+$
B_2	$-$
C_0	$+$
C_2	$-$
C_4	$+$

Table 4.4.: IIB orientifolds: σ action.

Since σ is a holomorphic involution, the cohomology groups split in two eigenspaces under the σ action as:

$$H^{p,q} = H^{p,q}_+ \oplus H^{p,q}_- \tag{4.2.4}$$

and analogously every harmonic p,q-form. The Hodge numbers $h^{p,q}_+$ and $h^{p,q}_-$ are the dimensions of $H^{p,q}_+$ and $H^{p,q}_-$ respectively and satisfy $h^{p,q} = h^{p,q}_+ + h^{p,q}_-$.
Since the Hodge-$*$ operator commutes with σ [23], the Hodge numbers satisfy $h^{p,q}_\pm = h^{3-p,3-q}_\pm$. Recalling the action on the holomorphic three-form Ω (4.2.2) and that the volume form depends on the σ-invariant $\Omega \wedge \bar{\Omega}$ (A.8), one gets (see Table 4.5):

$$\begin{aligned}
h^{1,1}_\pm &= h^{2,2}_\pm \quad, & h^{1,2}_\pm &= h^{2,1}_\pm \quad, \\
h^{3,0}_+ &= h^{0,3}_+ = 0 \quad, & h^{3,0}_- &= h^{0,3}_- = 1 \quad, \\
h^{0,0}_+ &= h^{3,3}_+ = 1 \quad, & h^{0,0}_- &= h^{3,3}_- = 0 \quad.
\end{aligned} \tag{4.2.5}$$

---

[23]σ indeed preserves orientation and metric of the CY.

cohomology group		dimension		basis	
$H^{1,1}_+$	$H^{1,1}_-$	$h^{1,1}_+$	$h^{1,1}_-$	ω_α	ω_a
$H^{2,2}_+$	$H^{2,2}_-$	$h^{1,1}_+$	$h^{1,1}_-$	$\tilde{\omega}^\alpha$	$\tilde{\omega}^a$
$H^{2,1}_+$	$H^{2,1}_-$	$h^{2,1}_+$	$h^{2,1}_-$	χ_κ	χ_k
$H^{1,2}_+$	$H^{1,2}_-$	$h^{2,1}_+$	$h^{2,1}_-$	$\bar{\chi}_\kappa$	$\bar{\chi}_k$
H^3_+	H^3_-	$2h^{2,1}_+$	$2h^{2,1}_- + 2$	$(\alpha_\kappa, \beta^\lambda)$	$(\alpha_{\hat{k}}, \beta^{\hat{l}})$

Table 4.5.: Cohomology groups for a CY $O3/O7$ orientifold.

For the reduction procedure is fundamental to keep in the expansions (4.1.39), (4.1.44), (4.1.60) only those components which survive the projection, i.e. transforming under σ as in Tab. 4.4.

Four-dimensional scalars $\phi(x)$ and $C_0(x)$ survive. As for the scalars $v^A(x)$, only a subset remains in the spectrum:

$$J = v^\alpha \omega_\alpha \quad , \qquad \alpha = 1, ..., h^{1,1}_+ \quad . \tag{4.2.6}$$

From the invariance of g under σ along with (4.2.2), the surviving components of the complex structure deformations (4.1.44) are:

$$\delta g_{\bar{i}\bar{j}} = \frac{1}{||\Omega||^2} \delta z^k(x) (\chi_k)_{ij\bar{j}} \, \bar{\Omega}^{ij}{}_{\bar{i}} \quad , \qquad k = 1, ..., h^{2,1}_- \quad . \tag{4.2.7}$$

The expansions of form fields (4.1.60) are truncated as follows:

$$
\begin{aligned}
B_2 &= b^a(x)\omega_a \quad , & a &= 1, ..., h^{1,1}_- \quad , \\
C_2 &= c^a(x)\omega_a \quad , & & \\
C_4 &= D^\alpha_2(x)\omega_\alpha + V^\kappa(x) \wedge \alpha_\kappa + U_\kappa(x) \wedge \beta^\kappa + d_\alpha(x)\tilde{\omega}^\alpha \, , & \kappa &= 1, ..., h^{2,1}_+ \quad .
\end{aligned}
\tag{4.2.8}
$$

Imposing F_5 self-duality one reduces the degrees of freedom of C_4. It corresponds to two choices: one between electric and magnetic potentials $V^\kappa(x), U^\kappa(x)$, and the other between the two-form $D^\alpha_2(x)$ and the scalar $d^A(x)$. Although the first choice does not change the structure of the 4d spectrum (both are one-forms), the last one does determine the structure of some of the $\mathcal{N} = 1$ multiplets to be linear (choosing D^α_2) or chiral (choosing d^A) [24].

---

[24] A linear multiplet L has a real scalar L and the field stregth of a two-form D_2 as bosonic components.

supermultiplet	number	4d fields
gravity multiplet	1	$g_{\mu\nu}$
vector multiplets	$h_+^{2,1}$	V^k
chiral multiplets	$h_-^{2,1}$	z^k
	1	(ϕ, C_0)
	$h_-^{1,1}$	(b^a, c^a)
chiral/linear multiplets	$h_+^{1,1}$	$(v^\alpha, d^\alpha)/(v^\alpha, D_2^\alpha)$

Table 4.6.: $N = 1$ 4d bosonic spectrum of O3/O7-orientifold compactification.

These choices are however equivalent, as one should expect. The effective theory described in the linear multiplet formalism corresponds to the effective theory described in terms of chiral multiplets. Indeed, by dualising linear multiplets (i.e. by dualising D_2^α, or performing the dualisation in the superspace formalism), one recovers exactly the effective action in terms of chiral multiplets [14, 57].

Let us remark that components $c^a(x), b^a(x), V^\kappa(x), U_\kappa(x)$ appear in (4.2.8) only if the model allows for O7-planes. Indeed, in O3 models two/three-forms are locally even/odd under the σ action respectively. Hence, in such models $c^a(x), b^a(x), V^\kappa(x), U_\kappa(x)$ must vanish. Notice that, in the particular case of O3 models and flat space, all cohomology groups $H^{p,q}$ are even/odd under σ if $(p+q)$ is even/odd. This consideration will be used in Section 5.4, where we will deal with a $T^6/\mathbb{Z}_2$ orientifold compactification.

Fields describing the effective theory are organizable as the $\mathcal{N} = 1$ multiplets of Table (4.6). The spectrum is indeed a truncation of the $\mathcal{N} = 2$ spectrum of a Calabi–Yau compactification (Table (4.2)). The graviphoton V^0 disappeared; the $h^{2,1}$ vector multiplets decomposed into $h_+^{2,1}$ $\mathcal{N} = 1$ vector multiplets plus $h_-^{2,1}$ chiral multiplets; the $h^{1,1} + 1$ hypermultiplets are reduced into $h^{1,1} + 1$ chiral multiplets, losing half of their degrees of freedom [25].

4.2.2. $O3/O7$ **CY orientifolds without fluxes: the Kähler potential**

The four-dimensional effective action is computed by Kaluza–Klein reduction of the IIB action (2.1.5), for the orientifold-truncated expansions (4.2.6), (4.2.7), (4.2.8). We recall

---

[25]Totally, in the $h^{1,1} + 1$ hypermultiplets there are $4h^{1,1} + 4$ scalar fields: v^A, b^A, c^A, d^A(or D_2^A-dualization), ϕ, C_0 and other two scalars, dualizations of B_2 and C_2. Of these, only $v^\alpha, b^a, c^a, d^\alpha$(or D_2^α-dualization), ϕ, C_0 survive, for a total of $2h_-^{1,1} + 2h_+^{1,1} + 2 = 2h^{1,1} + 2$ degrees of freedom.

that there are two descriptions of the effective action, depending on the choice between d^α and D_2^α in (4.2.8). In the following we use chiral multiplets.

Once obtained the effective action by dimensional reduction, one has to rewrite it the standard $\mathcal{N} = 1$ supergravity form (3.2.15). This action would be characterised by a Kähler potential $K(\varphi^i, \bar\varphi^i)$, a holomorphic superpotential $W(\varphi^i)$ and holomorphic gauge-kinetic coupling functions $f(\varphi^i)$, all determined as functions of some chiral fields φ^i, called "the Kähler coordinates", to be identified in terms of moduli $z^k, \phi, C_0, b^a, c^a, v^\alpha, d^\alpha$. This identification is highly non-trivial.

Fortunately, complex structure deformations $z^k(x)$ are already good Kähler coordinates. The other fields combine in a more complicated way [14] [26]:

$$
\begin{aligned}
\tau &= C_0 + i e^{-\phi} \quad, \\
G^a &= c^a - \tau b^a \quad, \\
\rho_\alpha &= i d_\alpha + \frac{1}{2} \mathcal{I}_\alpha(v) - \zeta_\alpha(\tau, \bar\tau, G, \bar G) \quad,
\end{aligned}
\tag{4.2.9}
$$

where

$$
\zeta_\alpha = -\frac{i}{2\mathrm{Im}\,\tau} \mathcal{I}_{\alpha b c} \, G^b \, \mathrm{Im}\, G^c \quad.
\tag{4.2.10}
$$

In terms of coordinates $\varphi^i = (z^k, \tau, \rho_\alpha, G^a)$ the effective theory has a Kähler metric descending from the following Kähler potential

$$
\begin{aligned}
K &= K^{\mathrm{cs}} + K^{\mathrm{k}} + K^\tau \quad, \\
K^{\mathrm{cs}} &= -\log\left(i \int_{M_6} \Omega(z) \wedge \bar\Omega(\bar z) \right) \quad, \\
K^{\mathrm{k}} &= -2\log[\mathrm{V}(\tau, \rho, G)] \quad, \\
K^\tau &= -\log[-i(\tau - \bar\tau)] \quad,
\end{aligned}
\tag{4.2.11}
$$

where

$$
\mathrm{V}(v(\tau, \rho, G)) \equiv \frac{1}{6} \int_{M_6} J \wedge J \wedge J = \mathcal{I}_{\alpha\beta\gamma} v^\alpha v^\beta v^\gamma
\tag{4.2.12}
$$

has to be understood as a function of the Kähler coordinates τ, ρ, G which enter inverting $\rho_\alpha(v, \tau, G)$ in (4.2.9) to get $v(\tau, \rho, G)$. The inversion must be carried out case by case

---

[26] Note that only a subset of the intersection numbers (4.1.54) is invariant under the orientifold action. From Table 4.4 and the σ-invariance of J, one argues that $\mathcal{I}_{\alpha\beta c} = \mathcal{I}_{abc} = \mathcal{I}_{\alpha b} = \mathcal{I}_a = 0$, where to omit an index corresponds to a contraction with a v^α (for instance $\mathcal{I}_a = \mathcal{I}_{a\beta\gamma} v^\beta v^\gamma$). All the others can be non vanishing, in particular $\mathcal{I} = \mathcal{I}_{\alpha\beta\gamma} v^\alpha v^\beta v^\gamma = 6\mathrm{V}$.

and it is not possible to give a general explicit solution of $v(\tau, \rho, G)$. Hence the Kähler potential remains implicit. It enjoys the following features:

- it has a part describing the complex structure moduli which is exactly the orientifold projection of that found in the case of the compactification on a Calabi–Yau threefold (Section 4.1);

- it gives a metric which is diagonal in the complex structure moduli while it mixes τ, ρ, G, implying a moduli space of the form

$$\mathcal{M}_{\text{cs}}^{h_-^{2,1}} \times \mathcal{M}^{h^{1,1}+1} \quad , \tag{4.2.13}$$

where $\mathcal{M}^{h^{1,1}+1}$ is a Kähler manifold and $\mathcal{M}_{\text{cs}}^{h_-^{2,1}}$ is special Kähler [14];

- By considering only one Kähler modulus $v^\alpha \equiv v$ ($\rho_\alpha \equiv \rho$) parametrizing the internal volume, one can check that $\rho(v, \tau, G)$ in (4.2.9) can be easily inverted to find $v(\tau, \rho, G)$ and thus

$$K^{\text{k}} = -3 \log \left[\rho + \bar\rho + \frac{1}{\text{Im}\,\tau} \mathcal{I}_{1ab} \text{Im}\,G^a \text{Im}\,G^b \right] \quad . \tag{4.2.14}$$

One can proceed computing the gauge-kinetic coupling functions $f_{\kappa\lambda}$ and show that they are holomorphic in the complex structure moduli $f_{\kappa\lambda}(z^k)$ [14].

As expected, dealing simply with a truncation of the pure CY three-fold compactification, the reduced action presents no scalar potential V. Both $W = 0$ and $\text{D}_\kappa = 0$ and the supersymmetry remains unbroken.

Let us see now how background fluxes enter in the description.

4.3. Turning on background fluxes

We now generalise the discussion on compactifications to the case with a richer background [27]. The starting Ansatz is always (4.0.1), with the metric (4.0.2). The difference with respect to pure geometrical compactifications lies in the fact that now we allow for the presence of non-trivial background fluxes.

Fluxes enter the equations of motion (2.1.7) and then play a fundamental role in determining solutions! In particular they *backreact* on the geometry, as one can see from

---

[27]Let us anticipate that the notation conventions for this Section and for next Chapter differ from the ones adopted so far, see Appendix A.

the Einstein equation (2.1.7), and this is enough to state that the internal manifold cannot be Ricci-flat any more. This backreaction is encoded in a non-trivial warp factor $e^{2A(y)}$ which dresses the four-dimensional metric in (4.0.2). Indeed, we will see that fluxes, along with localised objects like D-branes or O-planes, behave as sources for the warping.

Of course, non-vanishing background fluxes modify crucially supersymmetry equations too. Indeed, in presence of RR fluxes, supersymmetry conditions imply a relation between the ten-dimensional Majorana–Weyl parameters $\epsilon_{1,2}$ entering the dilatino and gravitino susy variations (2.1.8). As a consequence, four-dimensional spinors $\zeta_{1,2}$, components of $\epsilon_{1,2}$ as in 4.1, are related $\zeta_1 \sim \zeta_2 \equiv \zeta$. Since they can not be chosen independently any more, these backgrounds will generally preserve have an $\mathcal{N} = 1$ supersymmetry.

Supersymmetry conditions are found by setting fermionic variations (2.1.8) to vanish, along the lines of Section 4.1. In particular, from the analysis of supersymmetry conditions given by gravitinos, one finds that supersymmetric fluxed backgrounds are possible only if

$$M^{1,3} = \mathrm{Mink}_4, \mathrm{AdS} \qquad (4.3.1)$$

and that the internal manifold M_6 is no more Ricci flat [47].

To say something about the geometry of the internal manifold one has to work in the generalised geometry framework. In fact, in this framework one can split supersymmetry conditions in topological and differential conditions, as done for pure geometrical compactifications. In the particular case of Minkowskian flux compactification these conditions allow to identify the internal M_6 as a *generalised Calabi–Yau* [47,56]. This is a weaker condition with respect to the Calabi–Yau one, since it admits only half of the differentiable structures with respect to CY spaces [47]. For AdS compactification the geometric interpretation is not so clear.

A subclass of supersymmetric vacua: the GKP Ansatz

An important IIB flux background with four-dimensional Minkowski spacetime is the one proposed by Giddings, Kachru and Polchinski in [13] (see also [58]) and since the original work of this thesis is based on a background of this kind, it turns out appropriate to review that solution.

This vacuum turns out to be particularly fortunate since the internal manifold is a *warped* Calabi–Yau orientifold [28] (O-planes are necessary for consistency), or a non-Ricci-flat Kähler space in F-theory models. We are interested in the first class of models.

---

[28]Or, to be precise, a conformally CY [47].

4. IIB Calabi–Yau compactifications

In general, however, the ten-dimensional Einstein frame metric is:

$$ds^2 = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{-2A(y)}g_{mn}(y)dy^m dy^n \quad , \tag{4.3.2}$$

where g_{mn} (or simply g_6) is the *underlying* metric and it is Ricci-flat for the cases we are interested in, as we will see.

By taking M_6 to be a CY threefold ($SU(3)$ holonomy) one gets a model with $\mathcal{N} = 1$ (or $\mathcal{N} = 0$) supersymmetry. One can take also a M_6 of smaller holonomy, in which case the supersymmetry is enhanced. In Section 5.4, we will take the $T^6/\mathbb{Z}_2$ as the underlying orientifold, which can have at most a $\mathcal{N} = 4$ supersymmetry.

Generally, the only assumption of a starting Minkowskian $M^{1,3}$ allows for the presence of supersymmetry preserving space-filling D3/O3 and D7/O7, leading to F-theory compactifications on CY four-folds, so in the following we will maintain the presentation as more general as possible.

In order to easily describe this background solution, it is convenient to recast the IIB Einstein frame action (2.1.5) in terms of the three-form flux G_3 and the axio-dilaton τ, defined as:

$$\begin{aligned} \tau &= C_0 + ie^{-\phi} \quad , \\ G_3 &= F_3 - \tau H_3 \quad . \end{aligned} \tag{4.3.3}$$

Hence the IIB supergravity action becomes [29]:

$$\begin{aligned} S_{\text{IIB}} =& \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}\mathcal{R} - \frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-g}\left(\frac{\partial_M\tau\partial^M\tau}{(\text{Im}\,\tau)^2} + \frac{G_3\cdot\bar{G}_3}{3!\,\text{Im}\,\tau} + \frac{\tilde{F}_5^2}{2\cdot 5!}\right) \\ & - \frac{1}{8i\kappa_{10}^2}\int\frac{1}{\text{Im}\,\tau}C_4\wedge G_3\wedge\bar{G}_3 + S_{\text{loc}} \quad , \end{aligned} \tag{4.3.4}$$

where the S_{loc} includes local sources present in the supergravity background, as D-branes or orientifold O-planes.

Four-dimensional Poincaré symmetry restricts G_3 to have only internal legs

$$G_{MNP} \longrightarrow G_{mnp} \quad , \tag{4.3.5}$$

the self-dual $\tilde{F}_5$ to be of the form

$$\tilde{F}_5 = (1 + *)[d\alpha(y)\wedge dx^0\wedge dx^1\wedge dx^2\wedge dx^3] \tag{4.3.6}$$

---

[29]See Appendix A for conventions on the Hodge-$*$ used.

and the assio-dilaton to be [30]

$$\tau(y) \quad . \tag{4.3.7}$$

In F-theory models, i.e. type IIB solutions including D7-branes, $\tau(y)$ is related to an elliptically fibered CY four-fold [45, 46], while in CY orientifold models it is simply constant.

Three-form fluxes F_3, H_3 satisfy, in absence of sources, the Bianchi identities $dH_3 = dF_3 = 0$. Hence they are closed and belong to $H^3(M_6, \mathbb{Z})$. They are also quantized as

$$\begin{aligned} \frac{1}{2\pi\alpha'} \int_{\Sigma_3} F_3 = 2\pi\mathbb{Z} \quad , \\ \frac{1}{2\pi\alpha'} \int_{\Sigma_3} F_3 = 2\pi\mathbb{Z} \quad , \end{aligned} \tag{4.3.8}$$

where Σ_3 is a non-trivial 3-cycle supported on M_6. This flux quantization can be understood by considering, in the ten-dimensional IIB theory, the quantum amplitude of a process in which a Euclidean D1-brane wraps [31] on a trivial 2-cycle Π_2 in Σ_3. Π_2 splits Σ_3 in an internal part Σ_+ and an external part Σ_-, satisfying $\Sigma_+ - \Sigma_- = \Sigma_3$ where the minus sign is due to the orientation flip. The 2-cycle is the boundary of both these parts $\Pi_2 = \partial\Sigma_+ = \partial\Sigma_-$. Consider the CS contributions (2.2.5) to the path integral amplitude

$$\exp\left(i\mu_1 \int_{\Pi_2} C_2\right) = \exp\left(i\mu_1 \int_{\Sigma_\pm} F_3\right) \quad , \tag{4.3.9}$$

where we used the Stoke's theorem, since F_3 is globally well-defined while RR fields are not. Remember in fact that $F_3 = dC_2$ holds only *locally*, not globally since M_6 has in general a non-trivial cohomology, which means that F_3 is not exact. The two choices $\Sigma_\pm$ differ by a phase, but physically the integration has to lead to the same result, which is guaranteed if that phase is $2\pi\mathbb{Z}$:

$$\mu_1 \int_{\Sigma_+} F_3 - \mu_1 \int_{\Sigma_-} F_3 = \mu \int_{\Sigma_3} F_3 = 2\pi\mathbb{Z} \quad , \tag{4.3.10}$$

which is exactly the quantization condition of the RR F_3 (4.3.8).

The same considerations can be carried out for the amplitude of a process between fundamental strings, whose coupling to B_2 is given by the term of the worldsheet $\Sigma_{\mathrm{ws}} = \Pi_2$ action

---

[30]Poincaré invariance let F_1 to have only internal legs, hence $C_0(y)$. Furthermore, in a generic background $\phi(y)$.

[31]We are assuming a static gauge.

$$S_{B_2} = \frac{1}{2\pi\alpha'} \int_{\Pi_2} B_2 \quad . \tag{4.3.11}$$

No-go theorem

Before reviewing the GKP special solution, it is worth emphasising that localised sources are necessary in order to find solutions. Let us take the metric Ansatz (4.3.2). The Einstein equation of motion (2.1.7) for the non-compact components $R_{\mu\nu}$, in presence of localized sources with stress tensor T_{MN}^{loc} and in the metric Ansatz (4.3.2), can be recast as [32]

$$\nabla^2 e^{4A} = e^{8A} \frac{G_{mnp}\bar{G}^{mnp}}{12\text{Im}\,\tau} + e^{-4A}(\partial_m\alpha\partial^m\alpha + \partial_m e^{4A}\partial^m e^{4A}) + \frac{\kappa_{10}^2}{2}e^{2A}\mathcal{T}_{\text{loc}} \quad , \tag{4.3.12}$$

where

$$\mathcal{T}_{\text{loc}} \equiv \left(\sum_{M=5}^{9} T_M^M - \sum_{M=0}^{3} T_M^M \right)_{\text{loc}} \tag{4.3.13}$$

is the energy-momentum tensor contribution associated to localised sources. Contractions in (4.3.13) are intended with respect to the components of the ten-dimensional metric g_{MN}. Equation (4.3.12) leads to a no-go theorem in the absence of localized sources. In fact, integrating over the internal compact manifold M_6 the LHS vanishes while the RHS, composed by the flux and warping terms, is positive semidefinite. Therefore fluxes must vanish and warping must be constant.

The warping turns on only in the case of $\mathcal{T}_{\text{loc}} < 0$. In [13] it is shown that, to leading order in α', this condition can be satisfied in two cases: in F-theory models, with D7-branes, or in compactifications with negative tension objects, like anti-Dp-branes (branes of negative tension and charge) or O-planes. Anti-branes, however, do break supersymmetry and then one is led to considering CY orientifolds.

We obtained this no-go theorem using the Minkowskian compactification (4.3.2). However, similar considerations can be carried out for dS or AdS compactifications too, where the cosmological constant is $\Lambda > 0$, $\Lambda < 0$ respectively. In these cases one ends up with an equation analogous to (4.3.12), in which Λ contributes with an additional constant term on the RHS. Hence, for dS compactifications negative tension objects are needed too, while for AdS compactifications the cosmological constant provides a negative term itself and negative tension sources are no more necessary [13, 59, 60].

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[32]In (4.3.12) ∇^2 and all contractions are referred to the underlying (unwarped) metric g_6 and not to the internal (warped) one.

Another constraint comes from the equation of motion/Bianchi identity of $\tilde{F}_5$, which is [33]:

$$d\tilde{F}_5 = d * \tilde{F}_5 = H_3 \wedge F_3 + \ell_s^4 \hat{\delta}_6^{\text{loc}} \quad , \tag{4.3.14}$$

where $\ell_s^4 = 2\pi\kappa_{10}^2 T_3$ since the tension of a D3-brane is $T_3 = \mu_3 = (2\pi)^{-3}\alpha'^{-2}$. $\hat{\delta}_6^{\text{loc}}$ is the six-form local charge density encoding the distribution of D3-branes and O3-planes [34]:

$$\hat{\delta}_6^{\text{loc}} \equiv \sum_{I \in \text{D3's}} \hat{\delta}_6^I - \frac{1}{4} \sum_{J \in \text{O3's}} \hat{\delta}_6^J + ... \quad . \tag{4.3.16}$$

Here the factor $-\frac{1}{4}$ is because $T_{\text{O3}} = -\frac{1}{4}T_3$ [30]. The ellipses denote the other possible contributions, due to other kinds of localised sources (as exotic O-planes) or, in F-theory compactifications, to the α' contribution of the D7 CS action [13]. We will however treat only compactifications with constant τ and without other exotic sources.

The equation (4.3.14) tells us that fluxes behave as local sources as well, as anticipated. Integration over M_6 gives the no-tadpole cancellation condition

$$Q_3^{\text{loc}} + Q_3^{\text{flux}} = 0 \quad , \tag{4.3.17}$$

where

$$Q_3^{\text{flux}} \equiv \frac{1}{\ell_s^4} \int_{M_6} H_3 \wedge F_3 \tag{4.3.18}$$

and

$$Q_3^{\text{loc}} \equiv \int_{M_6} \hat{\delta}_6^{\text{loc}} \quad . \tag{4.3.19}$$

The equation (4.3.17) states that the *total* D3-charge from supergravity bulk fields and localized sources must vanish. From this condition one can realize that O-planes have negative tension. Let us take the O3-orientifold model as described in [30]: it is a compact Minkowski solution without background fluxes and with 16 D3-branes and 64 fixed O3-planes. Hence, from the no-tadpole cancellation one can understand that $T_{\text{O3}} = -\frac{1}{4}T_3$.

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[33]Notice that, in deriving (4.3.14) from the IIB action (4.3.4) one has to double the CS term. This is because any object with a D3 charge couples both electrically and magnetically to C_4 and the self duality of $\tilde{F}_5$ implies that this coupling are equal.

[34] A p-form density $\hat{\delta}_p$ is associated to a $6-p$-dimensional surface Σ in such a way that, given a $6-p$-form ω on M_6:

$$\int_{\Sigma} \alpha = \int_{M_6} \alpha \wedge \hat{\delta}_p \quad . \tag{4.3.15}$$

Hence, we can write a six-form density as $\hat{\delta}_6 = \delta^6(y)d^6y = \rho(y)d\text{V}$, with $\rho(y) = \delta^6(y)/\sqrt{g_6}$ and $d\text{V} \equiv \sqrt{g_6}d^6y$.

4. IIB Calabi–Yau compactifications

The GKP special solution

Giddings, Kachru and Polchinski showed that only in the presence of objects satisfying a *BPS-like condition* each, that is [13]

$$\frac{1}{4}\mathcal{T}_{\text{loc}} \geq T_3 \rho_3^{\text{loc}} \quad , \tag{4.3.20}$$

global constraints determine the solution completely. Here $\rho_3^{\text{loc}} \equiv \delta_6^{\text{loc}}/\sqrt{g_6}$, see (4.3.16) (see also footnote 34). By combining the Einstein equation (4.3.12) with the equation of motion for $\tilde{F}_5$ (4.3.14) one gets indeed:

$$\begin{aligned}
\nabla^2(e^{4A} - \alpha) &= \frac{e^{8A}}{6\,\text{Im}\,\tau}|iG_3 - *_6 G_3|^2 + e^{-4A}|\partial(e^{4A} - \alpha)|^2 \\
&\quad + 2\kappa_{10}^2 e^{2A}\left[\frac{1}{4}\mathcal{T}_{\text{loc}} - T_3\rho_3^{\text{loc}}\right] \quad ,
\end{aligned} \tag{4.3.21}$$

where $*_6$ denoted the Hodge star with respect to the underlying g_6. Integration over the compact internal space leads the LHS to vanish while assuming the BPS-like condition (4.3.20) the RHS vanishes if:

$$\star_6 G_3 = iG_3 \quad , \tag{4.3.22}$$

$$e^{4A} = \alpha \quad , \tag{4.3.23}$$

$$\frac{1}{4}\mathcal{T}_{\text{loc}} = T_3\rho_3^{\text{loc}} \quad . \tag{4.3.24}$$

Under these conditions, equations of motion and Bianchi identities give the solution. In particular, one finds that the Einstein equation for R_{mn} reduces to $R_{mn} \sim f(\partial_m\tau, T_{mn}^{\text{D7}})$ [13]. Furthermore, using (4.3.6) and under (4.3.23), equation (4.3.14) gives the following equation for the warp factor:

$$-\nabla^2 e^{-4A} = \frac{G_{mnp}\bar{G}^{mnp}}{12\text{Im}\,\tau} + \ell_{\text{s}}^4 \rho_3^{\text{loc}} \quad . \tag{4.3.25}$$

Henceforth set τ to be constant, i.e. we consider backgrounds without 7-branes. In this case, $R_{mn} = 0$ and M_6 is Ricci flat [13,58]. We are dealing with a O3 orientifold, with D3-branes and O3-planes, as anticipated. The local density ρ_3^{loc} is

$$\rho_3^{\text{loc}} = \sum_{I\in\text{D3's,O3's}} q_I \frac{\delta^6(y - Y_I)}{\sqrt{g_6}} \tag{4.3.26}$$

where $q_I = 1, -1/4$ for a D3-brane or O3-plane respectively.

Supersymmetry

Both supersymmetric and nonsupersymmetric solutions are possible, depending on the form of G_3. Since the underlying metric is Kähler, a generic solution to the ISD condition (4.3.22) is an harmonic form that can be Lefschetz decomposed as (see Appendix C):

$$G_3 = G^{0,3} + G^{2,1}_{\text{pr}} + G^{1,2}_{\text{np}} \quad , \tag{4.3.27}$$

where the last term is the non-primitive form $G^{1,2}_{\text{np}} = J \wedge \omega^{0,1}_{\text{pr}}$ of Lefschetz spin $l = 1$. If the compact manifold is a CY (in the stricter sense) $h^{0,1} = 0$ and this term is absent, while it is present in compactifications on manifolds of smaller holonomy, such as toroidal ones. It was shown that only the primitive $G^{2,1}_{\text{pr}}$ is allowed for supersymmetric ($\mathcal{N} = 1$) solutions [58]. As we will show below, we find that these supersymmetry conditions on the ten-dimensional background corresponds to supersymmetry conditions in the four-dimensional effective theory.

On the moduli of warped orientifold compactifications

These CY threefolds have in general $h^{2,1}$ geometric complex moduli and $h^{1,1}$ Kähler moduli, along with dilaton moduli and D3-brane positions moduli. Among these, the complex structure moduli and the assio-dilaton moduli are fixed by (4.3.22), which is indeed a condition on moduli and not on fluxes, fixed by quantization (4.3.8). In order to understand this fact, let us recall that the ISD condition implies the Lefschetz decomposition (4.3.27). Since we are dealing with a CY it reduces to $G_3 = G^{0,3} + G^{2,1}_{\text{pr}}$, where both components depend on complex structure and τ only.

This can be explained also in terms of the four-dimensional effective theory. As we will see below, Kähler deformations (along with the D3-brane moduli, when present) correspond to flat directions of the scalar potential arising in the effective action.

The universal modulus and the large volume limit.

Typically, in unwarped compactifications with ten-dimensional metric $ds^2_{10} = ds^2_4 + ds^2_6$, all equations specifying a solution are invariant under a constant rescaling of the internal metric:

$$g_6 \to a g_6 \quad , \tag{4.3.28}$$

where a is identified with a particular Kähler modulus, called "the universal modulus".

In presence of non-trivial warping in the metric (4.3.2), one has to define the universal modulus in another way. Indeed, under a rescaling of the underlying metric $g_6 \to a g_6$, the equation (4.3.25) would imply $e^{2A} \to a e^{2A}$, meaning that the internal metric $e^{-2A} g_6$ would remain unchanged.

4. IIB Calabi–Yau compactifications

In this case, the correct identification of the universal modulus follows from the observation that the solution of (4.3.25) determines the warp factor up to a constant a:

$$e^{-4A} = a + e^{-4A_0} \quad . \tag{4.3.29}$$

Here e^{-4A_0} is a particular solution of (4.3.25), associated to a fiducial metric $\hat{g}_{mn}$ which we can take to have unit volume $\hat{V}$. For instance, one can take e^{-4A_0} such that e^{-4A} behaves as a constant away from sources and diverges in their vicinity, i.e. such that in presence of a D3-brane located at y_0 we may locally write [17]:

$$e^{-4A} \approx a + \frac{4\pi\alpha'^2}{|y - y_0|^4} \quad . \tag{4.3.30}$$

Now, since the internal metric is $(a + e^{-4A_0})^{1/2}\hat{g}_6$, any changes of a modifies the internal volume. Therefore one is led to identify the universal modulus with the constant a. A further confirmation is given by the fact that in unwarped regions, away from sources, a corresponds to the definition of the universal modulus in the unwarped case (4.3.28).

To simplify the study of warped compactifications one usually adopts the *large volume/radius* limit, corresponding to take $a \to \infty$. This is justified by the fact that one can choose freely a, since is it an unfixed modulus. In this limit the internal volume of M_6 is

$$V_{\text{int}} \sim \int_{M_6} \sqrt{\det(a^{1/2}\hat{g}_6)} \sim a^{3/2}\hat{V} \sim a^{3/2} \quad , \tag{4.3.31}$$

since $\hat{V}$ is fixed, as discussed above. Hence from $V_{\text{int}} \sim R^6$ one gets the relation $a \sim R^4$, which explains the name of the approximation. In fact, taking this limit corresponds to taking a constant warp factor, which physically corresponds to smearing sources and diluting fluxes, neglecting their backreaction on the geometry. Although it is a useful tool to simplify the reduction and to get the effective action [35], it is still an approximation. Indeed, one would like to control warping effects, which may lead to relevant physical effects.

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[35] This limit allows to get the effective theory as done for a pure CY compactification in Section 4.1, eventually including flux effects. This procedure is however consistent only in a particular regime. As we will see, fluxes lead to stabilization of some moduli, giving them a mass of the order α'/R^3. On the other hand, KK modes have masses $m_{\text{KK}} \sim 1/R$. Hence, in the large volume limit, the mass induced by fluxes is smaller than the KK one. The regime allowed is between these two scales, where flux contributions are negligible and the effective theory corresponds to the one obtained from a pure geometrical compactification. However, fluxes become relevant at lower energies, near the flux scale and modify the effective action as we will describe.

The effective theory

Here we review the main features of the effective four-dimensional action, obtained by reducing the ten-dimensional theory compactified on a CY O3-orientifold and using the large volume limit. Here then the warp factor approaches a constant and $\tilde{F}_5 = 0$ (4.3.6).

In absence of fluxes H_3, F_3, the reduction leads to a supergravity theory with the massless spectrum of Table 4.6, without the vector multiplets V^κ and the chiral multiplets (b^a, c^a), cut by the O3-projection.

Assuming for simplicity the presence of a single Kähler modulus (i.e. $h^{1,1}_+ = 1$), the universal modulus a, the effective action has moduli kinetic terms determined by the Kähler potential [13]:

$$K = -\log[-i(\tau - \bar{\tau})] - 3\log(\rho + \bar{\rho}) - \log\left(-i\int_{M_6} \Omega \wedge \bar{\Omega}\right) \quad, \qquad (4.3.32)$$

where $\rho \equiv a + id$ is the scalar component of a chiral field ant it is composed by the universal modulus a and the C_4 modulus d. Notice that the last two terms of (4.3.32) give the Weyl–Petersson metric on the moduli space of a CY threefold, as seen in Sect. 4.1 [50].

Turning on three-form fluxes deforms the geometry of the compactification and generates a scalar potential for some moduli. This is obtained from the reduction of the kinetic $|G_3|^2$ term of the IIB action (4.3.4). In the large volume approximation and splitting the flux G_3 in its imaginary and anti-imaginary self-dual parts (ISD/AISD) defined as

$$\begin{aligned} G_3^\pm &\equiv \frac{1}{2}(G_3 \pm i *_6 G_3) \quad, \\ *_6\, G_3^\pm &= \mp i G_3^\pm \quad, \\ G_3 &= G_3^+ + G_3^- \quad, \end{aligned} \qquad (4.3.33)$$

one gets

$$S_{\text{flux}} = -V - \frac{i}{4\kappa_{10}^2 \text{Im}\,\tau}\int_{M_6} G_3 \wedge \bar{G}_3 \quad, \qquad (4.3.34)$$

The last term in the RHS is purely topological and does not contain moduli. It is proportional to the D3-charge given by three-form fluxes Q_3^{flux} (4.3.18) entering the no-tadpole cancellation (4.3.17). Thus the potential for moduli is determined only by the first term on the RHS of (4.3.34), which is

$$V \equiv -\frac{1}{2\kappa_{10}^2 \text{Im}\,\tau}\int_{M_6} G_3^+ \wedge *_6 \bar{G}_3^+ \quad. \qquad (4.3.35)$$

This potential depends on moduli defining the chiral field τ, which enter in V by the definition of G_3, and on geometric Kähler and complex structure moduli (a, z^k respec-

tively), which enter via the Hodge-$*_6$. The ISD condition for the special solution (4.3.22) corresponds in the four-dimensional theory to $V = 0$. This is a condition which can fix many moduli.

Expanding G_3 in the basis of 3-forms allowed by the orientifold projection, and using a little algebra, V can be recast in the following form:

$$V = \frac{1}{2\kappa_{10}^2}e^K(G^{i\bar{j}}D_iW\overline{D_jW}) \quad , \qquad \forall i, j = k, \tau \quad , \tag{4.3.36}$$

where $D_i \equiv \partial_i + \partial_i K$ and $G^{i\bar{j}} = \partial_i\partial_{\bar{j}}K$ are the Kähler covariant derivatives and the the Kähler metric respectively. W is the superpotential, which has a Gukov–Vafa–Witten form [61]

$$W_{\text{GVW}} = \int_{M_6} G_3 \wedge \Omega \tag{4.3.37}$$

and it is independent on ρ, since both G_3 and Ω are. Indices i, j in (4.3.36) run over k, τ only because of the particular structure of the effective theory obtained, which is a *no-scale* $\mathcal{N} = 1$ supergravity [62]. It is indeed straightforward to verify that (4.3.36) can be recast as the usual $\mathcal{N} = 1$ V_F:

$$V_F = \frac{1}{2\kappa_{10}^2}e^K(G^{i\bar{j}}D_iW\overline{D_jW} - 3|W|^2) \quad , \tag{4.3.38}$$

where now i, j run over *all* chiral indices k, τ, ρ. No-scale models are characterised by a Kähler potential such that the minimum of (4.3.38) possesses some flat directions (here parametrised by the chiral field ρ) and corresponds to a vanishing cosmological constant too. These models possess the interesting feature to allow for a vanishing cosmological constant even if the supersymmetry is broken. Indeed, from (4.3.36) it is immediate to check that $V = 0$ when

$$\begin{aligned} D_\tau W = 0 &\implies G^{3,0} = 0 \quad , \\ D_k W = 0 &\implies G^{1,2} = 0 \quad , \end{aligned} \tag{4.3.39}$$

while supersymmetry is preserved when $D_iW = 0$, for i labelling all chiral fields, without exceptions. Hence supersymmetry can be generally broken.

In order to keep supersymmetry unbroken (4.3.39) must be accompanied by the additional condition

$$D_\rho W = 0 \implies W = 0 \implies G^{0,3} = 0 \quad , \tag{4.3.40}$$

which further constrains $G_3 \in H^{2,1}(M_6)$. In the case of a compactification on a CY orientifold (strict $SU(3)$ holonomy), the three condition (4.3.39), (4.3.40) are enough to state that G_3 has to be primitive, in agreement with the ten-dimensional considerations. If

the CY has smaller holonomy, for instance taking a toroidal orientifold, those conditions are no more sufficient in order to get a primitive $G^{2,1}$ preserving supersymmetry. The additional condition $J \wedge G_3 = 0$ to be imposed arises from constraints on D-terms related to the superpotential by the enhanced supersymmetry [54].

Note that the no-scale structure holds also in the presence of more Kähler moduli, because it is determined by the ISD condition (4.3.22) which, as discussed before, leaves Kähler moduli unfixed. Since this condition corresponds to the vanishing of the scalar potential, it follows immediately that V has to be of the form (4.3.36), with indices running over all moduli except Kähler ones. Hence in these models the superpotential cannot depend on any Kähler modulus.

This structure, however, holds only in classical and large volume approximation. For instance, non-perturbative effects, like instantonic branes, furnish mechanism to stabilise the universal modulus and can generate a non-vanishing cosmological constant [63].

4.3.1. Introducing branes: the Kähler potential

Here we see how brane moduli fields, kept frozen so far, enter in the effective action. We work in a GKP background, compactified on a CY O3/O7 orientifold, with background three-form fluxes F_3 and H_3 turned on and including space-filling D3-branes [36]. The result is obtained in the large volume approximation.

Let us first recall that, in absence of branes, one obtains the effective action just plugging the orientifold truncated expansions (4.2.6), (4.2.7) and (4.2.8) including background fluxes (F_3, H_3), in the IIB action and by reducing it [14]. The final result is an effective action which differs from the one obtained in absence of fluxes only for the presence of a scalar potential term, as explained in Section 4.3. The Kähler potential is (4.2.11) and Kähler coordinates are (4.2.9).

In order to derive the effective action including branes, one strategy is to add bosonic and fermionic brane action terms to the IIB bulk action (2.1.5) and then to perform the reduction, in the large volume limit. This strategy, in which branes are considered as probes, has been followed in [21]. The results are reviewed here below. On the other hand, let us anticipate that the D3-moduli are automatically incorporated once the non-trivial warping is taken into account, see Chapter 5.

First of all one studies the bosonic sector, composed by the Dirac–Born–Infeld and by the Chern–Simons term of D3-brane action. One expands fields in fluctuations, includ-

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[36] D3-branes and O3-planes indeed satisfy the saturation condition (4.3.24) [13].

ing fluctuations of *transversal* positions in M_6, and computes the effective action. The fluctuations of D3-brane positions around given given points $Z_I^i \simeq Z_{(0)I}^i + \phi_I^i$ give rise to scalar moduli fields, collected in the complex scalars ϕ_I^i, with $i = 1, 2, 3$ and $I = 1, ..., N$ for N branes located apart from each other. These branes give rise to a gauge group $U(1)^N$. After that, one has to introduce the fermionic action, in order to satisfy supersymmtry [35, 36]. By reducing the action one finds that fermionic four-dimensional fields combine with bosonic ϕ_I^i into $\mathcal{N} = 1$ chiral multiplets. We refer the reader to [21] for the detailed discussion.

Obtained the low-energy four-dimensional action in presence of N D3-branes, one has to rewrite it in a standard supergravity form. However, the usual supergravity action (3.2.15) does not fit this situation and one finds that the effective action corresponds to a softly broken globally supersymmetric theory [21]. We are interested in the Kähler potential of the reduced theory. Let us remark that the analysis is carried out taking infinitesimal complex structure deformations δz^k. Remaining bulk moduli organise in the following scalar components of chiral fields (ρ_α, G^a, τ), which are defined as:

$$\tau = C_0 + ie^{-\phi} \quad , \qquad G^a = c^a - \tau b^a \quad ,$$

$$\rho_\alpha = id_\alpha + \frac{1}{2}\mathcal{I}_\alpha - \frac{i}{2\mathrm{Im}\,\tau}\mathcal{I}_{\alpha bc}\,G^b\,\mathrm{Im}G^c$$

$$+ i\mu_3\ell_s^2(\omega_\alpha)_{i\bar{j}}\sum_I \phi_I^i\left(\bar{\phi}_I^{\bar{j}} - \frac{i}{2}\delta\bar{z}^k(\chi_k)_l^{\bar{j}}\phi_I^l\right) \quad . \tag{4.3.41}$$

The Kähler potential $K(\tau, \rho, G, z, \phi)$ obtained differs from (4.2.11) only in the functional dependence of the second term (K^{k}), which is now

$$- 2\log\left[\mathrm{V}(v(\tau, \rho, G, z, \phi))\right] \quad . \tag{4.3.42}$$

This means that the presence of D3-branes mixes all Kähler coordinates τ, ρ, G, z, ϕ. As in the case of (4.2.11), in general, the dependence of K in terms of the Kähler coordinates can be stated only implicitly.

Let us take, for example, a single Kähler modulus v. It will correspond to a single chiral field ρ. In this case (4.3.41) can be inverted in terms of $v(\tau, \rho, G, z, \phi)$ and inserted in the (4.3.42), to get [21]:

$$-2\log\mathrm{V} = -3\log\left[\rho + \bar{\rho} - \frac{1}{\mathrm{Im}\,\tau}\mathcal{I}_{1ab}\,\mathrm{Im}G^a\mathrm{Im}G^b\right.$$

$$\left. + 2i\mu_3\ell_s^2(\omega_1)_{i\bar{j}}\sum_I(\phi_I^i\,\bar{\phi}_I^{\bar{j}}) + \frac{1}{2}\mu_3\ell_s^2\left((\omega_1)_{i\bar{j}}\,\delta\bar{z}^k\,(\bar{\chi}_k)_l^{\bar{j}}\sum_I(\phi_I^l\phi_I^i) + h.c.\right)\right] \quad . \tag{4.3.43}$$

The definition of the Kähler coordinates ρ_α in (4.3.41) and the particular form of the Kähler potential (4.3.43) confirm the non-trivial fibration of chiral fields ρ_α over the D3-brane moduli space discussed in [22, 23, 64] and typical of KK reductions [7].

5. An alternative to dimensional reduction

In this Chapter we will present an alternative strategy to compute the Kähler potential for a flux compactification, which does not require a systematic KK reduction. This strategy relies on the combination of supersymmetric considerations and some properties of instantonic branes, which can be used to ''probe'' these backgrounds, see [26], [25] [1]. This Chapter has been developed in parallel with the recent paper [27].

5.1. A look at the symmetry of the effective theory

The setup is the GKP background of Section 4.3 with ten-dimensional Einstein frame metric (4.3.2).

We saw that the warp factor must satisfy equation (4.3.25), where ρ_3^{loc} is given by (4.3.26), since we are interested in O3 CY orientifolds only.

The supersymmetry equations are solved by a ten-dimensional Weyl Killing spinor $\epsilon \equiv \epsilon_1 + i\epsilon_2$ of the form

$$\epsilon = e^{A/2}\zeta \otimes \eta \quad . \tag{5.1.1}$$

Here ζ and η are respectively a four-dimensional and a six-dimensional chiral spinors on the unwarped manifolds $M^{1,3}$ and M_6. As explained in Section 4.1, η (chosen such that $\eta^\dagger\eta = 1$) defines the Kähler form J and the holomorphic three-form Ω of the unwarped M_6 as follows [2]:

$$J_{mn} = i\eta^\dagger\gamma_{mn}\eta \quad , \qquad \Omega_{mnp} = e^{\phi/2}\eta^T\gamma_{mnp}\eta \quad . \tag{5.1.2}$$

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[1]The approach proposed in [26] is described within the framework of generalised geometry, which indeed furnishes an elegant and efficient way to deal with flux compactifications. The main idea of that paper, leading to the deduction of an implicit formulation of the Kähler potential, is however independent on generalised geometry and can be carried on in a simpler context, as we will discuss.

[2]The factor $e^{\phi/2}$ in (5.1.2) is in general needed to ensure holomorphicity. However this is irrelevant for us, since we deal with a constant ϕ. Here we are choose the (A.12) convention for J and the orientation defined by (A.11).

5. An alternative to dimensional reduction

J and Ω obey:

$$dV_6 = \frac{1}{3!} J \wedge J \wedge J = -\frac{i}{8} e^{-\phi} \Omega \wedge \bar{\Omega} \quad , \tag{5.1.3}$$

where

$$dV_6 = \sqrt{g_6} \, dy^1 \wedge \cdots \wedge dy^6 \tag{5.1.4}$$

is the volume form for the unwarped M_6.

A key point of the reasoning for the deduction of the Kähler potential lies in the observation of a symmetry of the dynamical field configuration generalising the vacuum Ansatz. In fact, in order to study the physics around a vacuum configuration, one has to consider the fluctuations of background fields around their vevs (Appendix B). For instance, at linear order, fluctuations of the ten-dimensional metric around the vacuum (4.3.2) are in general:

$$\begin{aligned}
\delta(e^{2A}g_{\mu\nu})(x,y) &= e^{2A(y)}(2\delta A(x,y)\eta_{\mu\nu} + \delta g_{\mu\nu}(x,y)) \quad , \\
\delta(e^{-2A}g_{mn})(x,y) &= e^{-2A(y)}(-2\delta A(x,y)g_{mn}(y) + \delta g_{mn}(x,y)) \quad .
\end{aligned} \tag{5.1.5}$$

Here $\delta g_{\mu\nu}(x,y)$ are the four-dimensional graviton Kaluza–Klein modes, whose zero-mode is the graviton; $\delta g_{mn}(x,y)$ are internal metric fluctuations, which depend on metric moduli $u^A(x)$ and on their respective KK modes. The warping depends on external coordinates x just through moduli and their KK tower of massive states. In fact, because of (4.3.25), a deformation δg_{mn} in turn implies a variation $\delta D \sim \frac{\partial D}{\partial u^A} u^A$.

Since we are interested only in the four-dimensional physics described by massless fields, we truncate the KK towers to zero-modes. The four-dimensional metric is truncated to the graviton $g_{\mu\nu}(x)$ and internal geometric massless zero-modes give rise to the appearance of moduli $u^A(x)$. One expects the ten-dimensional metric to be slightly modified by the presence of moduli smoothly varying in spacetime, in such a way that the equations of motion remain satisfied. However, the dynamical generalisation of the warped background Ansatz (4.3.2) can not include only fluctuations of the external and internal metrics (5.1.5). In fact, such a generalisation is not a KK consistent Ansatz, i.e. from which to start the reduction, because it does not satisfy the equations of motion. Extra off-diagonal terms are needed. This is the main difference with respect to the unwarped case: the correct dynamical metric Ansatz, describing fluctuations around the vacuum is [17, 19, 20, 65]:

$$\begin{aligned}
ds_{10}^2 &= e^{2A(y,u(x))} g_{\mu\nu}(x) dx^\mu dx^\nu + e^{-2A(y,u(x))} g_{mn}(y,u(x)) dy^m dy^n \\
&\quad + 2\partial_\mu \partial_\nu u^A(x) e^{2A(y)} K_A(y) dx^\mu dx^\nu + 2e^{2A(y)} B_{Am}(y)\partial_\mu u^A(x) dx^\mu dy^m \quad ,
\end{aligned} \tag{5.1.6}$$

where K_A and B_{Am} are called *compensators* for the metric [17]. This is the typical starting point for the reduction procedure, which is now much more involved, due to compensating terms. Here is why we would like to find a more general short-cut.

We start pointing out that the zero-modes Ansatz (5.1.6) possesses the following gauge symmetry:

$$g_{\mu\nu}(x) \to e^{-2\sigma(x)}g_{\mu\nu}(x) \quad , \tag{5.1.7}$$

$$g_{mn}(x,y) \to e^{2\sigma(x)}g_{mn}(x,y) \quad , \tag{5.1.8}$$

$$A(x,y) \to A(x,y) + \sigma(x) \quad . \tag{5.1.9}$$

Also the dynamical spinorial Ansatz, generalisation of the background Ansatz (5.1.1), has a similar gauge symmetry. Indeed, under

$$\zeta(x) \to e^{i/2\alpha(x)}\zeta(x) \quad , \tag{5.1.10}$$

$$\eta(x,y) \to e^{-i/2\alpha(x)}\eta(x,y) \tag{5.1.11}$$

the ten-dimensional dynamical Weyl spinor (5.1.1) is invariant.

Since the starting background is supersymmetric, then the four-dimensional effective theory must be supersymmetric. Such a theory will inherit these gauge symmetries. In particular, under these transformations, Ω (5.1.2) transforms as [3]

$$\Omega \to e^{3\sigma(x)-i\alpha(x)}\Omega \quad . \tag{5.1.12}$$

The invariance of the effective theory under this kind of transformation suggests to identify the effective theory as a *superconformal supergravity*. As we will see in the next Section, this theory is characterised by a Weyl–chiral symmetry, i.e. it is invariant under field transformations of the kind (5.2.1). This is enough to deduce the form of the Kähler potential as we will see in Section 5.3. Before entering the argument it is better to review the main aspects of such a theory. For a detailed discussion we refer the reader to [66].

5.2. Superconformal theory in a nutshell

Four-dimensional superconformal theory is typically used as an underlying theory, a tool for constructing standard supergravities. It possesses more symmetries than standard

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[3]γ_{mnp} in (5.1.2) in curved space are defined by sechsbeins, which have half Weyl weight with respect to g_6 (5.1.8).

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supersymmetric theories, based on the super-Poincaré group, and has the peculiarity of describing the gravitational coupling $M_{\rm P}^2$ as a the expectation value of a *function* of some scalar $\mathcal{N}(\Phi, \bar{\Phi})$. One recovers the Poincaré supergravity through an appropriate gauge-fixing, which breaks the local superconformal symmetries to the super-Poincaré subgroup.

A superconformal theory enjoys the super-Poincaré symmetry, which means invariance under general coordinate transformations, local Lorentz symmetry and local Q-supersymmetry. Additionally it possesses a chiral $U(1)$ symmetry, local dilatation invariance, special conformal symmetry and S-supersymmetry (see Table 5.1) [66].

Symmetry	g.c.t.	Lorentz	Q-susy	$U(1)$ chiral	dilatations	spec. conf.	S-susy
Generator	P_a	M_{ab}	Q	T	D	K_a	S
Gauge field	e_μ^a	ω_μ^{ab}	ψ_μ	A_μ	b_μ	f_μ^a	χ_μ

Table 5.1.: *Superconformal symmetries.*

Not all gauge fields in Table 5.1 are independent. In particular, ω_μ^{ab}, f_μ^a and χ_μ are composite fields and the only independent fields are the vierbeins e_μ^a, the gravitino ψ_μ, the dilatation field b_μ and the $U(1)$ gauge field A_μ. All these fields are collected in the *Weyl multiplet.*

The other superconformal multiplets in the theory are the chiral and the vector multiplets. A *chiral* multiplet Φ is defined as in rigid ordinary supersymmetry, i.e. by a complex scalar Φ (as usual, we identify the multiplet with its scalar component), a spinor of defined chirality ψ and an auxiliary complex scalar F. A Yang–Mills *vector* multiplet is defined by the gauge field A_μ, a gaugino λ and the auxiliary complex scalar D. The gaugino and the auxiliary field stay in the adjoint of the gauge group, labelled by a.

A generic field ϕ of a multiplet has its proper dilatation and chiral weighs (w, c) (called also Weyl–chiral weights), i.e. it transforms under a dilatation and a chiral transformation as [66]:

$$\phi \to \phi' = e^{w\sigma(x) + ic\alpha(x)}\phi \quad , \tag{5.2.1}$$

where $\sigma(x), \alpha(x)$ are the dilatation and chiral parameters respectively.

The superconformally invariant action of $\mathcal{N} = 1$ supergravity conformally coupled to $n + 1$ chiral multiplets Φ^I, $I = 0, ..., n$ of weights $(1, -1/3)$, and some gauge multiplets is determined by three functions: a real function $\mathcal{N}(\Phi, \bar{\Phi})$ and two holomorphic functions $\mathcal{W}(\Phi)$, $f_{\alpha\beta}(\Phi)$, respectively encoding the Kähler potential, the superpotential and the

gauge kinetic functions. The superconformal tensor calculus [67] leads to the complete component Lagrangian, whose bosonic part is [4]

$$
\begin{aligned}
(-\det g)^{-1/2}\,\mathcal{L} = {}&\frac{1}{2}\mathcal{N}R + 3\mathcal{N}_{I\bar{J}}\partial_{\mu}\Phi^{I}\partial^{\mu}\bar{\Phi}^{J} \\
&- \mathcal{W}_{I}(\mathcal{N}^{-1})^{I\bar{J}}\mathcal{W}_{\bar{J}} - \frac{1}{2}(\mathrm{Re}\,f_{ab})\mathrm{D}^{a}\mathrm{D}^{b} - \frac{1}{4}(\mathrm{Re}\,f_{ab})F_{\mu\nu}^{a}F^{\mu\nu b} + ... \quad ,
\end{aligned}
\tag{5.2.2}
$$

where R is the four-dimensional Ricci scalar, while $\mathcal{N}_I \equiv \frac{\partial \mathcal{N}}{\partial \Phi^I}$ and so on.

The first term in the superconformal Lagrangian $\mathcal{L}$ (5.2.2) is invariant under dilatations and chiral symmetry if $\mathcal{N}$ transforms as

$$
\mathcal{N} \to \mathcal{N}e^{2\sigma(x)} \quad ,
\tag{5.2.3}
$$

that is, if $\mathcal{N}$ transforms with opposite weights with respect to the metric g_4, which has weights $(w,c) = (-2,0)$. Choosing the chiral multiplets Φ to have weights $(1,0)$, one finds that $\mathcal{N}(\Phi,\bar{\Phi})$ is a homogeneous function of order two.

One may obtain a theory invariant invariant under dilatations and chiral transformations (5.2.1) by partially breaking the superconformal invariance, that is by breaking the special conformal symmetry and the S-supersymmetry. This is achieved by two suitable gauge choices, called "special conformal gauge" and "S-gauge" respectively [66].

Fixing also the gauges for dilatations and chiral transformations, one ends up with a theory invariant only under the super-Poincaré group. This is evident operating the change of variables $\Phi^I \to (Y, \varphi^i)$, $i = 1, ..., n$, defined by

$$
\Phi^I = Y f^I(\varphi^i) \quad ,
\tag{5.2.4}
$$

where $f^I(\varphi^i)$ are arbitrary holomorphic functions. Y has weights $(1, -1/3)$ while φ^i's have weights $(0,0)$. In this variables the scalar kinetic terms in (5.2.2) can be rewritten as [66]

$$
3\mathcal{N}_{I\bar{J}}\partial_{\mu}\Phi^{I}\partial^{\mu}\bar{\Phi}^{\bar{J}} = \frac{3}{4}\mathcal{N}^{-1}(\partial_{\mu}\mathcal{N})^{2} - \mathcal{N}\partial_{i}\partial_{\bar{j}}K\;\partial_{\mu}\varphi^{i}\partial^{\mu}\bar{\varphi}^{\bar{j}} \quad .
\tag{5.2.5}
$$

This is a Kählerian sigma model with a Kähler potential and metric [66]

$$
K(\varphi, \bar{\varphi}) = -3\log\left(\frac{\mathcal{N}(\varphi, \bar{\varphi})}{|Y|^2}\right) \quad ,
\tag{5.2.6}
$$

$$
g_{i\bar{j}} \equiv \partial_i \partial_{\bar{j}} K = \frac{\partial^2 K}{\partial \varphi^i \partial \bar{\varphi}^{\bar{j}}} \quad ,
\tag{5.2.7}
$$

where $\mathcal{N}(\varphi, \bar{\varphi})$ is a compact notation, which must be regarded as

---

[4]Our conventions differ from [66] by the redefinition $\mathcal{N} \to -3\mathcal{N}$.

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$$\frac{\mathcal{N}(\varphi, \bar{\varphi})}{|Y|^2} \equiv \mathcal{N}_{I\bar{J}} f^I(\varphi) \bar{f}^J(\bar{\varphi}) \quad . \tag{5.2.8}$$

The scalar fields φ^i are the n complex coordinates parametrizing the Kähler manifold, while Y is the so-called "conformon".

The transition to standard supergravity occurs by breaking dilatations, by fixing the "D-gauge"

$$D - \text{gauge} : \quad \mathcal{N} = M_{\mathrm{P}}^2 \quad , \tag{5.2.9}$$

i.e. when the modulus of the conformon is

$$|Y|^2 = e^{K/3} M_{\mathrm{P}}^2 \quad . \tag{5.2.10}$$

This means that there are only n physical degrees of freedom, corresponding to φ^i's, and that the conformal compensator Y is unphysical. With (5.2.10) the Lagrangian (5.2.2) becomes

$$\frac{1}{2} M_{\mathrm{P}}^2 R - M_{\mathrm{P}}^2 g_{i\bar{j}} \, \partial_\mu \varphi^i \partial^\mu \bar{\varphi}^{\bar{j}} + ... \quad . \tag{5.2.11}$$

The Kähler invariance follows from the non-uniqueness of the definition (5.2.4), which possesses the symmetry

$$\begin{aligned} Y' &= Y e^{g(\varphi)/3} \quad , \\ f'^I &= f^I e^{-g(\varphi)/3} \quad , \end{aligned} \tag{5.2.12}$$

for an arbitrary holomorphic function $g(\varphi)$, implying the invariance under

$$K \to K' = K + g(\varphi) + \bar{g}(\bar{\varphi}) \quad . \tag{5.2.13}$$

Since $\mathcal{W}$ has weights $(3, -1)$, it takes the form [66]:

$$\mathcal{W} = Y^3 M_{\mathrm{P}}^{-3} W(\varphi) \quad , \tag{5.2.14}$$

where $W(\varphi)$ is the superpotential of the supergravity theory. Since $\mathcal{W}$ depends on Φ^I (5.2.4), it is invariant under redefinitions (5.2.12). This in turn means that the superpotential $W(\varphi)$ enjoys the Kähler transformation

$$W \to W' = W e^{-g(\varphi)} \quad . \tag{5.2.15}$$

The correct standard $\mathcal{N} = 1$ supergravity in the Einstein frame is recovered finally gauging the $U(1)$ chiral symmetry by the

$$U(1) - \text{gauge}: \quad Y = \bar{Y} \quad . \tag{5.2.16}$$

Notice that this choice is broken by the transformations (5.2.12), which must be accompanied by a compensating chiral transformation [66].

5.3. The Kähler potential

From what just explained, it should be clear that the symmetry transformation of Ω (5.1.12) is a complexified Weyl symmetry of the same kind of (5.2.1). Hence, we can conclude that reducing the ten-dimensional action $S_{\text{IIB}}^{\text{Ef}}$ *without specifying a gauge* for the Weyl symmetry [5] and properly taking into account all compensator terms, we should get an effective action of the same type of (5.2.2). In particular, the scalars Φ^I should be defined in terms of the moduli u^A, organised in complex combinations.

Let us focus on the first two terms of that superconformal action (5.2.2). We know that $\mathcal{N}(\Phi, \bar{\Phi})$ defines the Kähler potential of the effective sigma model (5.2.6) and also enters in the superconformal Einstein term. Here is the crucial point. The Einstein term, in fact, does not depend on spacetime derivatives of chiral fields (since these are collected in the second term of (5.2.2)), hence it does not depend on $\partial_\mu u^A$. This means that the superconformal Einstein term can be obtained simply reducing the ten-dimensional Einstein term of $S_{\text{IIB}}^{\text{Ef}}$ neglecting the compensator terms in (5.1.6). This reduction gives

$$S_{\text{IIB}}^{\text{Ef}} = \frac{2\pi}{\ell_{\text{s}}^8} \int_{M^{1,3}} d^4x \sqrt{-\det g_4}\, R_4 \int_{M_6} d^6y \sqrt{\det g_6}\, e^{-4A(y,u)} \quad + ... \quad . \tag{5.3.1}$$

Matching this result with the superconformal action (5.2.2), the identification of $\mathcal{N}$ is immediate:

$$\mathcal{N} = \frac{4\pi}{\ell_{\text{s}}^8} \int_{M_6} d^6y \sqrt{\det g_6}\, e^{-4A(y,u)} \quad . \tag{5.3.2}$$

Hence the superconformal Kähler potential corresponds to the so-called *warped volume* $\mathcal{N} = \text{V}^{\text{w}}$, defined as the conversion factor between the four-dimensional and the ten-dimensional Planck mass [16, 17]. Under Weyl symmetry transformations (5.1.8), (5.1.9) $\text{V}^{\text{w}} \to e^{2\sigma(x)}\text{V}^{\text{w}}$ in agreement with (5.2.3).

One can then use the gauge-fixing procedure outlined above to obtain the standard supergravity formulation of the effective four-dimensional theory.

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[5]Typically one proceeds in the reduction passing to the Einstein frame, which corresponds to fix the gauge for Weyl symmetries.

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The conformal compensator Y

In order to compute the (implicit) Kähler potential $K(\varphi, \bar{\varphi})$ from the (implicit) $\mathcal{N}(\Phi, \bar{\Phi})$, we have firstly to identify a conformal compensator Y [6].

Let us work assuming the dilaton moduli and the complex structure moduli fixed, as it might happen in presence of background fluxes, see Section 4.3. The holomorphic three-form Ω, specifying the complex structure of M_6, is then fixed up to the normalisation. Indeed, because of the symmetry under (5.1.12), the normalisation cannot be fixed. However, by defining [7]

$$\Omega = \ell_{\rm s}^6 Y(x)^3 \Omega_0 \quad , \tag{5.3.3}$$

with a complex $Y(x)$ transforming as

$$Y(x) \to e^{\sigma(x) - \frac{i}{3}\alpha(x)} \quad , \tag{5.3.4}$$

one encodes the gauge symmetry in $Y(x)$ completely. Hence Ω_0 has a fixed normalisation, i.e. it satisfies:

$$dV_0 \equiv -\frac{i}{8} e^{-\phi} \Omega_0 \wedge \bar{\Omega}_0 = \frac{1}{3!} J_0 \wedge J_0 \wedge J_0 = \sqrt{\det g_0}\, d^6 y \quad , \tag{5.3.5}$$

with a dimensionless constant volume

$$V_0 = \int_{M_6} dV_0 = {\rm const} \quad . \tag{5.3.6}$$

In (5.3.5) J_0, g_0 are redefinitions of J, g. Indeed, since J, g, e^{2A} have Weyl–chiral weights $(2, 0)$ they can be redefined as done for Ω, in order to isolate this symmetry in $Y(x)$:

$$J = \ell_{\rm s}^4 |Y|^2 J_0 \quad , \tag{5.3.7}$$

$$g = \ell_{\rm s}^4 |Y|^2 g_0 \quad , \tag{5.3.8}$$

$$e^{2A} = \ell_{\rm s}^2 |Y|^2 e^{2D} \quad . \tag{5.3.9}$$

In terms of redefined quantities and Y, the ten-dimensional metric is (neglecting compensators):

$$ds_{10}^2 = \ell_{\rm s}^2 |Y|^2 e^{2D(y,u(x))} ds_4^2 + \ell_{\rm s}^2 e^{-2D(y,u(x))} ds_{6,0}^2 \quad . \tag{5.3.10}$$

---

[6] Henceforth let us denote g_6 by g.

[7] Powers of $\ell_{\rm s}$ appearing in redefinitions (5.3.3), (5.3.7), (5.3.8) and (5.3.9) are included for convenience.

At this point one should recognise that the redefinition (5.3.3) corresponds to a redefinition like (5.2.4) [8] [9]. $Y(x)$ is the conformal compensator which appears in the effective superconformal theory as an unphysical degree of freedom. Since it carries all informations on Weyl–chiral symmetry, it will be used to gauge-fix these symmetries as explained previously.

The implicit form of Kähler potential follows immediately from (5.2.6) along with (5.3.2) and (5.3.5):

$$
\begin{aligned}
K(\varphi, \bar{\varphi}) \ &= -3 \log \left(\frac{4\pi}{|Y|^2 \ell_s^8} \int_{M_6} d^6 y \sqrt{\det g_6} \ e^{-4A(y,u)} \right) \\
&= -3 \log \left(\frac{4\pi}{3!} \int_{M_6} e^{-4D(y,u)} \ J_0 \wedge J_0 \wedge J_0 \right) \quad .
\end{aligned}
\tag{5.3.11}
$$

Now, finally, we recover the effective supergravity theory by fixing the gauge of dilatations (5.2.10) and of $U(1)$ transformations (5.2.16).

Notice that, if we were dealing with an unwarped compactification, the fixed normalisation (5.3.6) would remove the overall rescaling from the set of the possible Kähler deformations of the internal metric (the universal modulus). However, in a warped compactification this is not the case. In fact, here the universal modulus is encoded in the warping itself (4.3.29) and no physical degrees of freedom are lost fixing V_0.

Now it is "only" matter of finding the correct definition of the Kähler coordinates φ^i in terms of the moduli, in order to make explicit the Kähler potential (5.3.11) in terms of them, i.e. as $K(\varphi, \bar{\varphi})$.

5.3.1. The holomorphic coordinates

Typically, as seen in Section 4.1, to find the correct definition of the holomorphic chiral fields φ^i in terms of the background moduli, one has to perform the dimensional reduction and to recast the four-dimensional effective in a supersymmetric form by a suitable definition of chiral fields. Here we follow a different and simpler strategy, which use probe Euclidean supersymmetric D3-branes. This is because we know that such instantonic

---

[8]On the other hand, redefinitions of J (or g) and e^{2A} do not correspond to redefinitions like (5.2.4), because they enter in the superconformal action via gauge invariant combinations. See for instance the first line of the Kähler potential (5.3.11) or the definition of chiral fields (5.3.18). Nothing changes passing to gauge fixed quantities J_0, e^{2D}.

[9]Let us remember that we are assuming complex structure moduli, introduced by Ω_0, to be fixed by fluxes, hence we denote by φ^i the chiral fields composed by Kähler and brane moduli only.

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corrections affect the effective action in a particular way. They can contribute to the superpotential by a term [29], [34]

$$W_{np} \sim e^{-S^{E3}} \quad , \tag{5.3.12}$$

where S^{E3} is the Euclidean D3-brane action

$$S^{E3} = S_{DBI}^{E3} - iS_{CS}^{E3} \quad , \tag{5.3.13}$$

derived in a Wick rotated vacuum. The relevant point is that four-dimensional super-symmetry requires this action to depend holomorphically on chiral fields φ^i. This fact allows for a definition of the same φ^i in terms of background moduli, as we are going to explain.

In warped compactifications the definition of the Chern–Simons term S_{CS} is problematic, being the RR potentials not globally defined [68], while the Dirac–Born–Infeld action S_{DBI} does not present such an issue and it can be computed in terms of background moduli [27]. Indeed, in order to find a correct holomorphic parametrization, the fact that $S_{DBI}^{E3} = \text{Re}\, S^{E3}(\varphi)$ suggests to impose that S_{DBI} must be the real part of chiral fields defined in terms of background moduli.

Let us specify to the simplified case of a O3 CY orientifold $M_6/\mathbb{Z}_2$, where C_2 and B_2 moduli are absent. We assume fixed complex structure moduli, due to background fluxes. We deal with $h_+^{1,1}$ Kähler moduli (one is the universal modulus and the other preserve the normalisation (5.3.6)) and $h_+^{2,2} = h_+^{1,1}$ moduli of C_4. When mobile space-filling D3-branes are present, one has to include the associated moduli. These are indeed the brane internal positions Y_I^m, with $I = 1, ..., N_{D3}$, which have a natural holomorphic parametrization in terms of complex coordinates of M_6: we denote chiral fields describing D3-moduli as Z_I^i, with $i = 1, 2, 3$. We are left with to identify the other chiral fields, denoted as ρ^A, which in general may depend on all other moduli, as we saw in subsection 4.3.1. $\text{Re}\,\rho^A$ are encoded in S_{DBI}^{E3}, while $\text{Im}\,\rho^A$ in S_{CS}^{E3}.

Introducing a basis of forms $\omega_A \in H_+^{1,1}(M_6, \mathbb{Z})$, we can expand J_0 as explained in Appendix A:

$$J_0 = -v^A \omega_A \quad . \tag{5.3.14}$$

The normalisation condition (5.3.6) becomes

$$V_0 = -\frac{1}{3!} v^A v^B v^C \mathcal{I}_{ABC} = \text{const} \quad . \tag{5.3.15}$$

Here $\mathcal{I}_{ABC}$ is the usual intersection number

$$\mathcal{I}_{ABC} = \int_{M_6} \omega_A \wedge \omega_B \wedge \omega_C \quad . \tag{5.3.16}$$

The constraint (5.3.15) implies that just $h^{1,1} - 1$ among the v^A are independent. The missing degree of freedom is encoded in the Kähler universal modulus a.

A Euclidean D3-brane wrapping a divisor $D^A \in M_6$, i.e. a linear combination of holomorphic four-cycles, has the following DBI action:

$$S_{\mathrm{DBI}}^{\mathrm{E3}} = 2\pi \int_{D^A} d^4x e^{-\phi} \sqrt{\det(g_{\mathrm{s}}|_{D^A} + \mathcal{F})} \quad , \tag{5.3.17}$$

where $\mathcal{F} = 2\pi\alpha' F_{\mathrm{D3}} - B_2|_{D^A}$ is the worldvolume flux on the brane. As already anticipated the instantonic Euclidean brane can preserve half of the background supersymmetry. The supersymmetry conditions for more general $\mathcal{N} = 1$ vacua have been derived in [26]. In the present case they read

$$\sqrt{\det(g_{\mathrm{s}}|_{D^A} + \mathcal{F})} = \frac{1}{2}\left(-e^{-4A+\phi}J \wedge J + \mathcal{F} \wedge \mathcal{F}\right) \quad , \tag{5.3.18}$$

where J refers to the internal (Einstein frame) metric g. In terms of J_0 (5.3.7) and e^{-4D} (5.3.9), (5.3.17) becomes

$$S_{\mathrm{DBI}}^{\mathrm{E3}} = \pi \int_{D^A} \left(-e^{-4D}J_0 \wedge J_0\right) + \frac{\pi}{\ell_{\mathrm{s}}^4} \int_{D^A} e^{-\phi}\mathcal{F} \wedge \mathcal{F} \quad . \tag{5.3.19}$$

Notice that the second term on the RHS is metric- (hence metric moduli-) independent, therefore it can be thought of as a constant and be ignored in what follows. Let us remark that, with our choice of conventions, the first term on the RHS is positive.

There are $h_+^{1,1}$ *independent* four-cycles of this kind which are Poincaré dual to the ω_A. Hence, all D3-branes wrapping a divisor D^A each (or a combinations thereof) furnish $S_{\mathrm{DBI}}^{\mathrm{E3}}$'s detecting all Kähler moduli. This is enough in order to compute the Kähler potential. Indeed, as we will see in the explicit example of subsection 5.4.1, it turns out that (5.3.11) depends just on geometric Kähler moduli and not on C_4 ones. Hence we do not need to compute $S_{\mathrm{CS}}^{\mathrm{E3}}$, which furnishes the ρ^A's dependence on C_4 moduli.

In other words, it is only matter of computing the following integrals

$$I^A \equiv -\frac{1}{2}\int_{D^A} e^{-4D}J_0 \wedge J_0 \quad . \tag{5.3.20}$$

For what just discussed, such integrals must correspond to

$$I^A = \mathrm{Re}\,\rho^A(v, a, Z_I, \bar{Z}_I) + (\mathrm{hol} + \overline{\mathrm{hol}}) \quad , \tag{5.3.21}$$

where $(\mathrm{hol} + \overline{\mathrm{hol}})$ depends on other chiral fields, as D3 positions. We shall stress that $\mathrm{Re}\,\rho^A(v, a, Z_I, \bar{Z}_I)$ depends on both the Kähler (v^A, a) and the brane (Z_I^i) moduli, which

enter in (5.3.20) through the warp factor. To compute the Kähler potential we will need to invert $\operatorname{Re}\rho^A(v, a, Z_I, \bar{Z}_I)$, expressing a in terms of complex coordinates ρ^A, Z_I^i. This means that, taking properly into account the backreaction, K will depend also on brane moduli, in agreement with what found by including branes as probes and carrying out the reduction (see subsect. 4.3.1).

In the following we present a simple explicit example.

5.4. A simple example

For this example we use one of the GKP backgrounds described in [28]. The internal manifold M_6 is a toroidal O3-orientifold $T^6/\mathbb{Z}_2$ with $T^6 = T^2 \times T^2 \times T^2$, parametrized by six real coordinates y^m, $m = 1, ..., 6$, with periodicity

$$y^m \sim y^m + 1 \quad . \tag{5.4.1}$$

These coordinates are conventionally split in x^i, y^i, $i = 1, 2, 3$ used to define the complex coordinates with the orientation convention of (A.11), i.e. with holomorphic one-forms

$$dz^i = dx^i + \lambda^{ij} dy^j \quad , \tag{5.4.2}$$

where the complex structure is defined by the matrix λ^{ij} and we have redefined

$$(y^1, y^2, y^3) \to (x^1, x^2, x^3) \quad , \quad (y^4, y^4, y^5) \to (y^1, y^2, y^3) \quad . \tag{5.4.3}$$

With this choice then:

$$\int_{T^6} dx^1 \wedge dx^2 \wedge dx^3 \wedge dy^1 \wedge dy^2 \wedge dy^3 = 1 \quad . \tag{5.4.4}$$

The normalisation is such that the holomorphic three-form is simply

$$\Omega_0 = dz^1 \wedge dz^2 \wedge dz^3 \quad . \tag{5.4.5}$$

The orientifold involution is given by $\mathcal{O}_1 = (-1)^{F_L}\Omega_{\mathrm{p}}\sigma$ (4.2.2) with σ acting on complex coordinates as

$$\sigma : \ z^i \to -z^i \ , \quad i = 1, 2, 3 \quad . \tag{5.4.6}$$

Such an involution gives rise to 64 O3-planes. In absence of O3-planes and fluxes, since the torus has trivial holonomy, the compactification would preserve maximal $\mathcal{N} = 8$ four-dimensional supersymmetry. As we saw in subsection 4.2.1, the orientifold action reduces the massless spectrum of the effective theory and in this case we are left with

10d component	#	4d field
$g_{\mu\nu}$	1	graviton
g_{mn}	21	scalars
$B_{m\nu}$	$b^1 = 6$	vectors
$C_{m\nu}$	$b^1 = 6$	vectors
C_{mnpq}	$b^4 = 15$	scalars
C_0	1	scalar
ϕ	1	scalar

Table 5.2.: Massless spectrum of closed string sector.

a $\mathcal{N} = 4$ supersymmetry. Fields describing the effective theory must then fill in $\mathcal{N} = 4$ supermultiplets. Moreover, in presence of O3-planes and vanishing background fluxes, the no-tadpole condition (4.3.17) implies the presence of D3-branes.

In order to find this spectrum one has to remember the transformation properties of the NSNS and RR fields under the worldsheet parity Ω_{p} and $(-1)^{F_L}$ given in Table 4.3. Moreover, in a O3 orientifold with flat metric cohomology groups are split under the σ action simply in H^{odd}_- and H^{even}_+. The massless bosonic spectrum is reported in Table 5.2. The components $B_{m\nu}, C_{m\nu}$ are now present since T^6 has a richer cohomology than a CY manifold, in particular $b^1 = 6$. Of the 21 metric moduli, 9 are Kähler moduli (related to $(\delta g_0)_{i\bar{j}}$) and 12 are complex structure moduli (linked to $(\delta g_0)_{ij}$).

However, differently from the compactification on a CY, where the Yau's theorem ensures that any complex and/or Kähler deformation implies a deformation of the Ricci-flat metric, in this case we have that three out of 12 complex structure moduli correspond to deformation of the complex structure at *fixed* metric. In fact, the complex structure of T^6 is fixed by 9 complex scalars encoded in the period matrix λ^{ij} of (5.4.2) [28], [69].

The graviton, six vector bosons, the axion and the dilaton are organised in the graviton multiplet with their fermionic partners, while the remaining six gauge fields and 36 real scalars fill into six gauge multiplets along with their fermionic partners [10].

If $G_3 = 0$, the tadpole condition (4.3.17) imposes the presence of 16 spacetime filling D3-branes. Branes introduce new moduli. Each I-th D3 brane comes in fact with a worldvolume $\mathcal{N} = 4$ gauge supermultiplet, composed by a gauge boson and six real

---

[10]Possible $\mathcal{N} = 4$ massless supermultiplets are: the *gravity multiplet* encoding the following number of particles of given helicity $\mathbf{h}$ $(\mathbf{2}, 4 \times \mathbf{3/2}, 6 \times \mathbf{1}, 4 \times \mathbf{1/2}, 2 \times \mathbf{0})$, the *gravitino multiplet* $(\mathbf{3/2}, 4 \times \mathbf{1}, 7 \times \mathbf{1/2}, 8 \times \mathbf{0})$ and the *gauge multiplet* $(\mathbf{1}, 4 \times \mathbf{1/2}, 6 \times \mathbf{0})$. Purely matter multiplets are absent, since the number of supersymmetry generators is to high to avoid helicity one states. Hence there is no possibility to have a multiplet with fermions transforming in fundamental representations [41,42].

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scalars, with their fermionic partners. As we already explained, these scalars are the six transversal direction of the brane, i.e. the internal six real coordinates Y_I^m or the three complex Z_I^i.

Turning on internal components of background three-form fluxes F_3 and H_3, one see from their definitions that they survive the projection, while F_5 is completely determined by (4.3.14). F_3 and H_3 must be closed, as required by their Bianchi identities, and they have also to obey the quantisation conditions (4.3.8). One can expand fluxes in a basis of $H^3(M_6, \mathbb{Z})$ as

$$
\begin{aligned}
\frac{1}{\ell_s^2} F_3 &= a^0 \alpha_0 + a^{ij} \alpha_{ij} + b_{ij} \beta^{ij} + b_0 \beta^0 \quad, \\
\frac{1}{\ell_s^2} H_3 &= c^0 \alpha_0 + c^{ij} \alpha_{ij} + d_{ij} \beta^{ij} + d_0 \beta^0 \quad,
\end{aligned}
\tag{5.4.7}
$$

where $(a^0, a^{ij}, b_{ij}, b_0)$, $(c^0, c^{ij}, d_{ij}, d_0)$ are integers, taken *fixed* since we are interested in vacua with constant fluxes, and

$$
\begin{aligned}
\alpha_0 &= dx^1 \wedge dx^2 \wedge dx^3 \quad, \\
\alpha_{ij} &= \frac{1}{2} \epsilon_{ilm} dx^l \wedge dx^m \wedge dy^j \quad, \\
\beta^{ij} &= -\frac{1}{2} \epsilon_{jlm} dy^l \wedge dy^m \wedge dx^i \quad, \\
\beta_0 &= dy^1 \wedge dy^2 \wedge dy^3
\end{aligned}
\tag{5.4.8}
$$

is a basis of $H^3(M_6, \mathbb{Z})$.

By turning on background fluxes, one expects the supersymmetry to be reduced and some of the 38 moduli to be fixed. For instance, as we will see, one can get $\mathcal{N} = 1$ solutions. In these cases, generally, one can show that $12/15$ of C_4 scalars give mass to gauge fields by the Higgs mechanism [70, 71]. Six of them are partners of six Kähler moduli, fixed by supersymmetry conditions as explained below. The remaining 3 moduli of C_4 couple with the other three Kähler moduli in three massless matter multiplets.

The presence of H_3 and F_3 gives rise to the superpotential term $W_{\text{GVW}} = \int G_3 \wedge \Omega_0$ (4.3.37). Supersymmetry conditions impose G_3 to be of type $(2, 1)$ and primitive, i.e.

$$
(g_0)^{i\bar{j}} G_{i\bar{j}k} = 0 \quad \text{or} \quad J_0 \wedge G_3 = 0 \quad .
\tag{5.4.9}
$$

The requirement of G_3 to be $(2, 1)$ is equivalent to the vanishing of its $(3, 0)$, $(0, 3)$ and $(1, 2)$ components and can be translated into three conditions on W_{GVW}. In the present case, these these equations turn into 11 equations involving the complex axio-dilaton τ and the 9 complex components of λ^{ij} as variables, to be determined in function of integers

defining fluxes [28] [11]. Thus, in general, these equations can not be simultaneously satisfied and supersymmetry is broken. There are however cases in which, by a suitable choice of fluxes, they can be satisfied. We are interested in these classes of solutions.

At this point one is left with the imposition of the primitivity condition, which corresponds to 3 complex equations. It then fixes, in general, six Kähler moduli, as anticipated.

In the following discussion we specialise to the set-up given in the example in Sect. 4.1 of [28]. This is an $\mathcal{N} = 1$ supersymmetric vacuum with constant diagonal fluxes

$$(a^{ij}, b_{ij}, c^{ij}, d_{ij}) = (a, b, c, d,)\delta_{ij} \quad . \tag{5.4.10}$$

With this choice of fluxes the system of 11 equations for λ^{ij}, τ drastically simplifies and one can solve it, fixing both. One can check that the complex structure is diagonal

$$\lambda^{ij} = \lambda \delta^{ij} \tag{5.4.11}$$

and the value of λ depends on the choice of the integers that define the fluxes.

The flux contribution to the D3 charge in (4.3.17) is given by $Q_3^{\text{flux}} = \frac{1}{2} N_{\text{flux}}$, where

$$N_{\text{flux}} \equiv \frac{1}{\ell_{\text{s}}^4} \int_{T^6} H_3 \wedge F_3 = (b_0 c^0 - a^0 d_0) + 3(bc - ad) \in \mathbb{Z} \quad . \tag{5.4.12}$$

Since several combinations of these integers are possible, it means that there exist models with different N_{flux} and then with different numbers of D3-branes, fixed by the no-tadpole condition (4.3.17):

$$\frac{1}{2} N_{\text{flux}} + N_{\text{D3}} - \frac{1}{4} N_{\text{O3}} = 0 \quad . \tag{5.4.13}$$

Take, for instance:

$$(a_0, a, b, b^0) = (2, 0, 0, 2) \quad , \quad (c^0, c, d, d_0) = (2, -2, -2, -4) \quad . \tag{5.4.14}$$

With this choice equations for λ and τ give [28]

$$\lambda = \tau = e^{2\pi i/3} \tag{5.4.15}$$

and $N_{\text{flux}} = 12$ [12]. Then, (5.4.13) implies that in this model $N_{\text{D3}} = 10$.

Once chosen the values of $(a, b, c, d, a^0, b_0, c^0, d_0)$ and gotten λ, τ solving the 11 equations, we are guaranteed to have a G_3 of type $(2, 1)$ of the form:

---

[11]This is because also the holomorphic Ω_0 can be rewritten in terms of λ^{ij} in the basis (5.4.8) [28].

[12]Note that for odd values of N_{flux} consistency requires additional discrete NS/R fluxes to be switched on [28]. In this Section we avoid this issue, restricting to *even* N_{flux}.

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$$G_3 \sim dz^1 \wedge dz^2 \wedge d\bar{z}^3 + \text{cyc. perms. of } 123 \quad . \tag{5.4.16}$$

Now we must impose the primitivity condition $J_0 \wedge G_3 = 0$. As explained above, this condition furnishes six equations and one expects it generally fixes six out of the $h_+^{1,1} = 9$ Kähler moduli. However, this is not the case since the choice of three-form fluxes determining the G_3 in (5.4.16) is particular and the primitivity condition fixes just three Kähler moduli. It is indeed straightforward to check that the following Kähler form

$$J_0 = -v^A \omega_A \tag{5.4.17}$$

where ω^A is the basis of $H^{1,1}(M_6)$ with

$$\omega_1 = \frac{i}{\text{Im}\,\lambda} dz^1 \wedge d\bar{z}^1 \;, \quad \omega_2 = \frac{i}{\text{Im}\,\lambda} dz^2 \wedge d\bar{z}^2 \;, \quad \omega_3 = \frac{i}{\text{Im}\,\lambda} dz^3 \wedge d\bar{z}^3 \;,$$

$$\omega_4 = \frac{i}{2\text{Im}\,\lambda}(dz^2 \wedge d\bar{z}^3 + dz^3 \wedge d\bar{z}^2) \;, \quad \omega_5 = \frac{i}{2\text{Im}\,\lambda}(dz^3 \wedge d\bar{z}^1 + dz^1 \wedge d\bar{z}^3) \;, \tag{5.4.18}$$

$$\omega_6 = \frac{i}{2\text{Im}\,\lambda}(dz^1 \wedge d\bar{z}^2 + dz^2 \wedge d\bar{z}^1) \;,$$

satisfies $J_0 \wedge G_3$. Therefore the most general supersymmetry preserving metric takes the form

$$(g_0)_{i\bar{j}} = \frac{1}{\text{Im}\,\lambda} \underbrace{\begin{pmatrix} v_1 & v_6/2 & v_5/2 \\ v_6/2 & v_2 & v_4/2 \\ v_5/2 & v_4/2 & v_3 \end{pmatrix}}_{A_{i\bar{j}}} \rightarrow (g_0)_{mn} = \frac{2}{\text{Im}\,\lambda} \begin{pmatrix} A_{i\bar{j}} & \text{Re}\,\lambda\; A_{i\bar{j}} \\ \text{Re}\,\lambda\; A_{i\bar{j}} & |\lambda|^2\; A_{i\bar{j}} \end{pmatrix} \quad . \tag{5.4.19}$$

For future reference we furnish here the inverse metric:

$$(g_0)^{mn} = \frac{1}{2\text{Im}\,\lambda} \begin{pmatrix} |\lambda|^2\; A^{i\bar{j}} & -\text{Re}\,\lambda A^{i\bar{j}} \\ -\text{Re}\,\lambda\; A^{i\bar{j}} & A^{i\bar{j}} \end{pmatrix} \;, \quad A^{i\bar{j}} = \frac{1}{\det A} \begin{pmatrix} u_1 & u_6 & u_5 \\ u_6 & u_2 & u_4 \\ u_5 & u_4 & u_3 \end{pmatrix} \;. \tag{5.4.20}$$

where u_A are the following combinations of the constrained Kähler moduli v^A:

$$
\begin{aligned}
u_1 &\equiv v_2 v_3 - v_4^2/4 \quad, \\
u_2 &\equiv v_1 v_3 - v_5^2/4 \quad, \\
u_3 &\equiv v_1 v_2 - v_6^2/4 \quad, \\
u_4 &\equiv v_5 v_6/4 - v_1 v_4/2 \quad, \\
u_5 &\equiv v_4 v_6/4 - v_2 v_5/2 \quad, \\
u_6 &\equiv v_4 v_5/4 - v_3 v_6/2 \quad.
\end{aligned}
\tag{5.4.21}
$$

Hence

$$
g_0 \equiv \det(g_0)_{mn} = 2^6 (\det A)^2 \quad.
\tag{5.4.22}
$$

Parameters v^A's (> 0) are the six real Kähler moduli parametrising the flat directions of the effective $\mathcal{N} = 1$ supersymmetric theory.

Let us stop for a moment to comment the residual supersymmetry. To argue that it is minimal and not enhanced, one has to search for additional possible complex structures in which G_3 is $(2,1)$ and primitive. One can change the complex structure by taking $z^i \to \bar{z}^i$ for some or all $i = 1, 2, 3$. However no one of these additional possibilities preserve the required conditions and then this example has $\mathcal{N} = 1$ supersymmetry [28].

5.4.1. The Kähler potential and the universal modulus

We want to show now that the Kähler potential (5.3.11) can be recast only in terms of the universal modulus a.

First of all, one has to rewrite the equation for the warp factor (4.3.25) using the redefined quantities g_0 (5.3.10) and e^{-4D} (5.3.9). A little algebra gives

$$
d^6 y \sqrt{g} \frac{G_{mnp} \bar{G}^{mnp}}{12 \mathrm{Im}\,\tau} = \frac{*_6 G_3 \wedge \bar{G}_3}{2 \mathrm{Im}\,\tau} = \frac{i}{2 \mathrm{Im}\,\tau} G_3 \wedge \bar{G}_3 = H_3 \wedge F_3 = \ell_s^4 N_{\mathrm{flux}} d^6 y \quad,
\tag{5.4.23}
$$

where the third term follows by the ISD of G_3. Hence, defining

$$
\rho_{\mathrm{bg}} \equiv \frac{N_{\mathrm{flux}}}{\sqrt{g_0}} \quad,
\tag{5.4.24}
$$

the warping equation (4.3.25) becomes:

$$
-\nabla_0^2 e^{-4D} = \rho_{\mathrm{bg}} + \sum_{I \in \mathrm{D3's, O3's}} q_I \frac{\delta^6(y - Y_I)}{\sqrt{g_0}} \quad.
\tag{5.4.25}
$$

Integrating (5.4.25) over M_6 one gets the no-tadpole condition

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$$\int_{M_6} d^6 y \sqrt{g_0}\, \rho_{\mathrm{bg}} + Q_3^{\mathrm{loc}} = 0 \quad, \tag{5.4.26}$$

where $Q_3^{\mathrm{loc}} = \sum_{I \in \mathrm{D3's,O3's}} q_I = N_{\mathrm{D3}} - 16$ is the total D3 charge of localised D3's and O3's. The solution of (5.4.25) determines e^{-4D} up to a constant a, identifiable, as seen in Section 4.3, as the universal modulus:

$$e^{-4D} = a + e^{-4D_0} \quad. \tag{5.4.27}$$

Here e^{-4D_0} is a particular solution of the Poisson equation (5.4.25). This split possesses the redundancy

$$\begin{aligned} a &\to a + c \quad, \\ e^{-4D_0} &\to e^{-4D_0} - c \end{aligned} \tag{5.4.28}$$

which can be fixed simply choosing a particular value for the internal warped volume

$$V_0^{\mathrm{w}} \equiv \int_{M_6} e^{-4D_0} dV_0 \quad. \tag{5.4.29}$$

V_0^{w} is, in fact, a constant and its value is associated to the particular solution e^{-4D_0}. Take, for instance, the following split for (5.4.27)

$$e^{-4D} = \hat{a} + e^{-4\hat{D}_0} \quad, \tag{5.4.30}$$

where $e^{-4\hat{D}_0}$ is the particular solution

$$e^{-4\hat{D}_0} = \int_{M_6} d^6 y' \sqrt{g_0} G(y,y') \left[\sum_{I \in \mathrm{D3's,O3's}} q_I \frac{\delta^6(y' - Y_I)}{\sqrt{g_0}} + \rho_{\mathrm{bg}} \right] \quad, \tag{5.4.31}$$

where $G(y,y')$ is the Green's function, solution of

$$-\nabla^2_{0,(y)} G(y,y') = -\nabla^2_{0,(y')} G(y,y') = \frac{\delta^6(y - y')}{\sqrt{g_0}} - \frac{1}{2V_0} \quad, \tag{5.4.32}$$

with $V_0 = \int_{M_6} d^6 y \sqrt{g_0} = \frac{\sqrt{g_0}}{2}$. We can solve (5.4.32) passing in Fourier space, where

$$G(y,y') = \sum_{k \in \mathbb{Z}^6} A_k e^{2\pi i k \cdot (y - y')} \quad, \tag{5.4.33}$$

$$\delta(y - y') = \sum_{k \in \mathbb{Z}^6} e^{2\pi i k \cdot (y - y')} \quad. \tag{5.4.34}$$

By inserting these expansions in (5.4.32), one gets

$$A_k = \frac{1}{\sqrt{g_0}(2\pi)^2 k_0^2} \quad , \qquad k \in \mathbb{Z}^6 - \{0\} \quad . \tag{5.4.35}$$

where $k_0^2 = (g_0)^{mn} k_m k_n$. Thus, using the orthonormality condition in each single n-th direction on T^6

$$\int_0^1 dy^n e^{2\pi i k_n y^n} = \delta_{k_n,0} \quad , \tag{5.4.36}$$

it is straightforward to make (5.4.31) explicit:

$$e^{-4\hat{D}_0} = \sum_{I \in D3's,O3's} q_I \sum_{\mathbb{Z}^6 - \{0\}} A_k e^{2\pi i k \cdot (y - Y_I)} \quad . \tag{5.4.37}$$

Finally, by inserting (5.4.37) in (5.4.29) and by using again orthonormality, one gets:

$$\hat{V}_0^{\mathrm{w}} = \int_{M_6} d^6 y \, e^{-4\hat{D}_0} \sqrt{g_0} = 0 \quad . \tag{5.4.38}$$

More generally, since different particular solutions e^{-4D_0} differ by a constant shift (5.4.28), the warped volume (5.4.29) is

$$V_0^{\mathrm{w}} = \hat{V}_0^{\mathrm{w}} + \mathrm{const} \quad . \tag{5.4.39}$$

Being $\hat{V}_0^{\mathrm{w}} = 0$, this means that for every choice of the splitting (5.4.27), the associated warped volume is a moduli-independent constant [13]. The Kähler potential (5.3.11) can be written, in general, as

$$K = -3 \log(V_0 a + V_0^{\mathrm{w}}) - 3 \log 4\pi \quad , \tag{5.4.40}$$

or, more conveniently, using the particular split (5.4.30) as:

$$K = -3 \log(\hat{a} V_0) - 3 \log 4\pi \quad . \tag{5.4.41}$$

Thus the Kähler potential depends only on the universal modulus. In order to make (5.4.41) explicit in terms of chiral fields φ^i, we must identify the latter in terms of the moduli and then, if possible, invert these relations to get $\hat{a}(\varphi^i, \bar{\varphi}^i)$.

---

[13]The result (5.4.39) seems to depend on the constancy of $\sqrt{g_0}$, i.e. on the flatness of the metric, which makes $\hat{V}_0^{\mathrm{w}}$ vanish. However these same considerations can be easily generalised, as explained in [27].

5.4.2. Kähler potential and chiral fields

As we saw in subsection 5.3.1, possible chiral fields are the brane positions Z_I^i's and the closed string moduli ρ^A's. Kähler moduli enter only in the real part of the latter. Hence we are interested in computing only $\mathrm{Re}\,\rho^A(v, \hat{a}, Z, \bar{Z})$'s, following the parametrization (5.3.21).

In order to detect $\mathrm{Re}\,\rho^A$'s, we have to identify the six divisors D^A's over which to integrate to get integrals I^A's (5.3.20). Let us remember that a divisor takes the form $n_i z^i = \mathrm{const}$, where n_i are integers. Moreover, since we are dealing with a O3 orientifold, for each instantonic D3-brane wrapping a divisor defined by $n_i z^i = \mathrm{const}$, there is a corresponding brane-image localised at $n_i z^i = -\mathrm{const}$ in the recovering space T^6. The same holds for space-filling D3-branes. We have to take these images into account when integrating over T^6. In the following we identify divisors on $T^6/\mathbb{Z}_2$. The simplest ones are identified by:

$$
\begin{aligned}
z^1 = Z^1 &\quad \rightarrow \quad D^1 = T_2^2 \times T_3^2 \quad , \\
z^2 = Z^2 &\quad \rightarrow \quad D^2 = T_1^2 \times T_3^2 \quad , \\
z^3 = Z^3 &\quad \rightarrow \quad D^3 = T_1^2 \times T_2^2 \quad ,
\end{aligned}
\tag{5.4.42}
$$

where Z^1, Z^2, Z^3 are the transversal positions of D^1, D^2, D^3 respectively [14]. The other three divisors are less obvious. They are identified by:

$$
\begin{aligned}
z^2 - z^3 = Z^4 &\quad \rightarrow \quad D^4 = \frac{1}{2} T_1^2 \times \Sigma_1 \quad , \\
z^1 - z^3 = Z^5 &\quad \rightarrow \quad D^5 = \frac{1}{2} T_2^2 \times \Sigma_2 \quad , \\
z^1 - z^2 = Z^6 &\quad \rightarrow \quad D^6 = \frac{1}{2} T_3^2 \times \Sigma_3 \quad ,
\end{aligned}
\tag{5.4.43}
$$

where Z^4, Z^5, Z^6 denote transversal positions of D^4, D^5, D^6 respectively. In order to better understand the geometry of these cycles, let us take, for instance, D^4. It wraps T_1^2 and Σ_1, which is the two-torus of complex structure λ defined by the complex embedding

$$
\zeta = \sigma^1 + \lambda \sigma^2 \rightarrow
\begin{cases}
z^2 = \zeta + \mathrm{const} \\
z^3 = \zeta
\end{cases}
.
\tag{5.4.44}
$$

Σ_1 wraps the direction defined by $z^2 + z^3$, while it is transversal to T_1^2 and to the direction defined by $z^2 - z^3$. Along this direction it is located at $z^2 - z^3 = \mathrm{const} \equiv Z^4$. As a consequence, D^4 wraps directions z^1 and $z^2 + z^3$, while it is identified as a point of coordinate $z^2 - z^3 = Z^4$ in the transversal direction. The same holds for D^5, D^6. The

---

[14]Notice to not confuse complex coordinates z^i with D3 positions Z_I^i or transversal divisor positions $Z^1, ..., Z^6$ for each $D^1, ..., D^6$ respectively.

importance of having identified a good choice of divisors will be obvious below. Now we move on to compute the integrals I^A, working out I^1 and I^4 explicitly. Once understood how to obtain these two, the others are found by analogy.

$\underline{I^1}$.   Using the most convenient warp factor splitting (5.4.30) in (5.3.20):

$$
\begin{aligned}
I^1 &= -\frac{1}{2}\int_{D^1} e^{-4\hat{D}} J_0 \wedge J_0 \\
&= -\frac{1}{2}\int_{T_2^2 \times T_3^2} (\hat{a} + e^{-4\hat{D}_0}) J_0 \wedge J_0 \\
&\equiv I^1_{(1)} + I^1_{(2)} \quad .
\end{aligned}
\qquad (5.4.45)
$$

By inserting

$$
\begin{aligned}
J_0 \wedge J_0 = [&2u_3\,\omega_1 \wedge \omega_2 + 2u_2\,\omega_1 \wedge \omega_3 + 2u_1\,\omega_2 \wedge \omega_3 \\
&-4u_4\,\omega_1 \wedge \omega_4 - 4u_5\,\omega_2 \wedge \omega_5 - 4u_6\,\omega_3 \wedge \omega_6] \quad ,
\end{aligned}
\qquad (5.4.46)
$$

one gets

$$
I^1_{(1)} = 4u_1\hat{a} \quad , \qquad (5.4.47)
$$

$$
I^1_{(2)}(X^1, Y^1) = 4u_1 \int_{T_2^2 \times T_3^2} e^{-4\hat{D}_0} dx^2 \wedge dx^3 \wedge dy^2 \wedge dy^3 \quad , \qquad (5.4.48)
$$

where we emphasised that $I^1_{(2)}$ depends on the transversal positions (X^1, Y^1) of the divisor D^1. In order to obtain $I^1_{(2)}$ one should insert the warping solution (5.4.37), integrate and finally recast the result in complex variables, in such a way that to make possible the identification of $\mathrm{Re}\,\rho^1$ in the integral in I^1, as prescribed by (5.3.21).

As outlined in [24], there is a simpler way to proceed. It consists in transforming the six-dimensional Poisson equation for the warp factor (5.4.25) to a two-dimensional Poisson equation for $I^1_{(2)}$ and then to solve it in complex coordinates.

By multiplying both members of (5.4.25) by $J_0 \wedge J_0$ and by carrying out a cautious integration over the divisor D^1 [15] and using (5.4.26), one finds:

$$
-\nabla^2_{0,(X^1,Y^1)} I^1_{(2)} = \frac{8u_1}{\sqrt{g_0}}\left[\sum_{I\in \mathrm{D3's, O3's}} q_I \delta^2(X^1 - X^1_I; Y^1 - Y^1_I) - Q^{\mathrm{loc}}_3\right] \quad , \qquad (5.4.50)
$$

---

[15] Note that the integration over D^1 can be rewritten as:

$$
\int_{T_2^2 \times T_3^2} dx^2 \wedge dx^3 \wedge dy^2 \wedge dy^3 = 2\int_{T^6/\mathbb{Z}_2} d^6y\; \delta^2(x^1 - X^1; y^1 - Y^1) \quad , \qquad (5.4.49)
$$

where $\delta^2(x^1 - X^1; y^1 - Y^1) \equiv \delta(x^1 - X^1)\delta(y^1 - Y^1)$.

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where $\nabla^2_{0,(X^1,Y^1)} \equiv (g_0)^{mn} \frac{\partial^2}{\partial Y^m \partial Y^n}$ with $m,n = 1,4$, since these are the only derivative terms of ∇^2_0 surviving the integration. Equation (5.4.50) is a Poisson equation in the transversal positions of D^1. We are however interested in expliciting the dependence of $I^1_{(2)}$ on D3-positions. The Green's function method comes here to the fore. Indeed, using the Green's function $G(X^1, Y^1; x'^1, y'^1)$ satisfying

$$
-\nabla^2_{0,(X^1,Y^1)}G(X^1, Y^1; x'^1, y'^1) = -\nabla^2_{0,(x'^1,y'^1)}G(X^1, Y^1; x'^1, y'^1)
$$
$$
= \frac{\delta^2(X^1 - x'^1; Y^1 - y'^1)}{\sqrt{g_0}} - \frac{1}{\sqrt{g_0}} \quad , \tag{5.4.51}
$$

the solution of (5.4.50) is

$$
I^1_{(2)} = 8u_1 \int_{T_1^2} dx'^1 dy'^1 G(X^1, Y^1; x'^1, y'^1) \left[\sum_{I \in D3's, O3's} q_I \delta^2(X^1 - X_I^1; Y^1 - Y_I^1) - Q_3^{\text{loc}} \right] . \tag{5.4.52}
$$

At this point, it is crucial to notice that:

$$
-\nabla^2_{0,(X_I^1,Y_I^1)} I^1_{(2)} = \frac{8u_1 q_I}{\sqrt{g_0}} \left[\delta^2(X^1 - X_I^1; Y^1 - Y_I^1) - 1 \right] \quad , \tag{5.4.53}
$$

where (X_I^1, Y_I^1) are the transversal positions to D^1 of the I-th localised source (D3 or O3). We are interested in solving *this* equation. The complete solution for $I^1_{(2)}$ will be obtained by summing over all sources.

In complex variables [16], where

$$
\nabla^2_{0,(X_I^1,Y_I^1)} = \frac{2u_1 \text{Im}\,\lambda}{\det A} \frac{\partial}{\partial Z_I^1} \frac{\partial}{\partial \bar{Z}_I^1} \quad , \tag{5.4.56}
$$

equation (5.4.53) becomes:

$$
\frac{\partial}{\partial Z_I^1} \frac{\partial}{\partial \bar{Z}_I^1} I^1_{(2)} = -q_I \left(\delta^2(Z^1 - Z_I^1) - \frac{1}{2\text{Im}\,\lambda} \right) . \tag{5.4.57}
$$

---

[16]On a complex manifold M of $\dim_\mathbb{C} M = 1$ and with complex structure λ, we define the basis of the complex cotangent and tangent spaces as $(dz, d\bar{z})$ and $(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}})$ respectively, where

$$
dz = dx + \lambda dy \quad , \tag{5.4.54}
$$
$$
\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\text{Re}\,\lambda}{\text{Im}\,\lambda} \frac{\partial}{\partial x} - i \frac{1}{\text{Im}\,\lambda} \frac{\partial}{\partial y} \right) \quad , \tag{5.4.55}
$$

in a way that $\langle \frac{\partial}{\partial z}, dz \rangle = \langle \frac{\partial}{\partial \bar{z}}, d\bar{z} \rangle = 1$ and $\langle \frac{\partial}{\partial z}, d\bar{z} \rangle = \langle \frac{\partial}{\partial \bar{z}}, dz \rangle = 0$ [49].

The Dirac delta function in complex variables is defined, accordingly to [30], such that $\int d^2 z \delta^2(z, \bar{z}) = \int dx dy \delta(x)\delta(y) = 1$ where $d^2 z \equiv |Jac| dx dy = 2\text{Im}\,\lambda dx dy$ and thus $\delta(x)\delta(y) = 2\text{Im}\,\lambda \delta^2(z, \bar{z})$.

Notice that any dependence on Kähler moduli v^A has disappeared. Hence $I_{(2)}^1$ will depend only on the brane moduli. Equation (5.4.57) is solved for

$$-\frac{q_I}{2\pi}\log|\theta_1(Z^1 - Z_I^1|\lambda)|^2 + \frac{q_I}{\operatorname{Im}\lambda}(\operatorname{Im}(Z^1 - Z_I^1))^2 \quad , \tag{5.4.58}$$

where $\theta_1(z|\lambda)$ is a particular *theta function* defined on the two-torus [30] [17]. The complete solution for $I_{(2)}^1$ is then:

$$I_{(2)}^1 = \sum_{I\in\mathrm{D3's,O3's}} q_I\left[-\frac{1}{2\pi}\log|\theta_1(Z^1 - Z_I^1|\lambda)|^2 + \frac{1}{\operatorname{Im}\lambda}(\operatorname{Im}(Z^1 - Z_I^1))^2\right] \quad . \tag{5.4.59}$$

By re-summing (5.4.59) and (5.4.47), we get the integral I^1:

$$I^1 = 4u_1\hat{a} + \sum_{I\in\mathrm{D3's,O3's}} q_I\left[-\frac{1}{2\pi}\log|\theta_1(Z^1 - Z_I^1|\lambda)|^2 + \frac{1}{\operatorname{Im}\lambda}(\operatorname{Im}(Z^1 - Z_I^1))^2\right] \quad . \tag{5.4.60}$$

We can now rewrite I^1 as in (5.3.21) and identify $\operatorname{Re}\rho^1$. First of all notice that both the transversal position Z^1 of the divisor D^1 and O3-plane positions $(Z^1, Z^2, Z^3)_J$ ($J\in\mathrm{O3's}$) are *fixed*. Hence they contribute as constants in I^1. We can consistently restrict the summation index I to run over D3-branes only (and take $q_I = 1$). The $\log|\theta_1|^2$ term is already in the form $\mathrm{hol} + \overline{\mathrm{hol}}$. On the other hand, expanding

$$(\operatorname{Im}(Z^1 - Z_I^1))^2 = i(Z_I^1 - \bar{Z}_I^1)\operatorname{Im}Z^1 - \frac{1}{4}((Z_I^1)^2 + (\bar{Z}_I^1)^2) + \frac{1}{2}|Z_I^1|^2 + \mathrm{consts} \quad , \tag{5.4.61}$$

we see immediately that the term $|Z_I^1|^2$ obstructs to rewrite $(\operatorname{Im}(Z^1 - Z_I^1))^2$ as $\mathrm{hol} + \overline{\mathrm{hol}}$. Hence we are led to identify $\operatorname{Re}\rho^1$ as

$$\operatorname{Re}\rho^1 \equiv t_1 + \sum_{I\in\mathrm{D3's}} \frac{|Z_I^1|^2}{2\operatorname{Im}\lambda} \quad , \tag{5.4.62}$$

where

$$t_1 = 4u_1\hat{a} \tag{5.4.63}$$

must be regarded of as function of chiral fields $t_1(\operatorname{Re}\rho^1, Z_I^1, \bar{Z}_I^1)$.

---

[17]An important feature of this function is that it vanishes linearly when its first argument goes to zero $\theta_1(z|\lambda) \underset{z\to 0}{\sim} z$. It is hence straightforward to verify that (5.4.59) satisfies (5.4.57), just recalling that $\partial_z\partial_{\bar{z}}\log|z|^2 = 2\pi\delta^2(z,\bar{z})$. Notice that we are using a different periodicity condition with respect to [30], where $y \sim y + 2\pi$. This choice is reflected in the first argument of θ_1.

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In the same way we find also:

$$I^2 = 4u_2\hat{a} + \sum_{I\in\text{D3's}} \frac{|Z_I^2|^2}{2\text{Im}\,\lambda} + \text{hol}(Z_I^2) + \overline{\text{hol}}(\bar{Z}_I^2) + \text{consts} \quad, \tag{5.4.64}$$

$$I^3 = 4u_3\hat{a} + \sum_{I\in\text{D3's}} \frac{|Z_I^3|^2}{2\text{Im}\,\lambda} + \text{hol}(Z_I^3) + \overline{\text{hol}}(\bar{Z}_I^3) + \text{consts} \quad, \tag{5.4.65}$$

and so, accordingly to (5.3.21), we are led to choose:

$$\text{Re}\,\rho^2 \equiv t_2 + \sum_{I\in\text{D3's}} \frac{|Z_I^2|^2}{2\text{Im}\,\lambda} \quad, \tag{5.4.66}$$

$$\text{Re}\,\rho^3 \equiv t_3 + \sum_{I\in\text{D3's}} \frac{|Z_I^3|^2}{2\text{Im}\,\lambda} \quad, \tag{5.4.67}$$

with

$$\begin{aligned} t_2 &= 4u_2\hat{a} \quad, \\ t_3 &= 4u_3\hat{a} \quad. \end{aligned} \tag{5.4.68}$$

$\underline{I^4}$. The integral I^4 is computed by repeating the same steps followed for I^1, but its computation is a bit more involved, due to the definition of D^4.

First of all, let us work, for convenience, redefining coordinate spanning Σ_1 by $z^2+z^3 \equiv z^5$ and coordinates transversal to D^4 by $z^2 - z^3 \equiv z^4$. In real coordinates this means passing from (x^2, x^3, y^2, y^3) to (x^4, x^5, y^4, y^5) defined by

$$\begin{aligned} x^4 &\equiv x^2 - x^3 \quad, \\ x^5 &\equiv x^2 + x^3 \quad, \\ y^4 &\equiv y^2 - y^3 \quad, \\ y^5 &\equiv y^2 + y^3 \quad. \end{aligned} \tag{5.4.69}$$

Now, D3-brane positions are identified by $(Z_I^1, Z_I^4 \equiv Z_I^2 - Z_I^3, Z_I^5 \equiv Z_I^2 + Z_I^3)$ and the divisor D^4, located transversally in $z^4 = Z^4$, wraps directions (z^1, z^5). The integral we have to compute is:

$$\begin{aligned} I^4 &= -\frac{1}{2} \int_{D^4} e^{-4\hat{D}} J_0 \wedge J_0 \\ &= -\frac{1}{4} \int_{T_1^2 \times T_5^2} (\hat{a} + e^{-4\hat{D}_0}) J_0 \wedge J_0 \\ &\equiv I^4_{(1)} + I^4_{(2)} \quad. \end{aligned} \tag{5.4.70}$$

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A little algebra reveals that

$$I^4_{(1)} = 2u_3 + 2u_2 - 4u_4$$
$$I^4_{(2)}(X^4, Y^4) = \frac{u_3 + u_2 - 2u_4}{2} \int_{T^2_1 \times \Sigma_1} e^{-4\hat{D}_0} dx^1 \wedge dx^5 \wedge dy^1 \wedge dy^5 \quad , \tag{5.4.71}$$

where X^4, Y^4 are the real coordinates identifying D^4 transversal position.

We start rewriting (5.4.25) in terms of new coordinates (5.4.69) and multiplying both its members by $J_0 \wedge J_0$. Then, we integrate over D^4, getting [18]:

$$-\nabla^2_{0,(X^4,Y^4)} I^4_{(2)} = \frac{4u_3 + 4u_2 - 8u_4}{\sqrt{g_0}} \left[\sum_{I \in D3's, O3's} q_I \delta^2(X^4 - X^4_I; Y^4 - Y^4_I) - Q^{\text{loc}}_3 \right] \quad , \tag{5.4.73}$$

where $\nabla^2_{0,(X^4,Y^4)}$ is the 2d Laplacian surviving the integration. Equation (5.4.73) for $I^4_{(2)}$ corresponds to (5.4.50) for $I^1_{(2)}$. Therefore, repeating above arguments based on Green's function (just replace (X^1, Y^1) with (X^4, Y^4)), we arrive at the following equation, which corresponds to (5.4.53):

$$-\nabla^2_{0,(X^4_I,Y^4_I)} I^4_{(2)} = \frac{(4u_3 + 4u_2 - 8u_4)q_I}{\sqrt{g_0}} \left[\delta^2(X^4 - X^4_I; Y^4 - Y^4_I) - Q^{\text{loc}}_3 \right] \quad . \tag{5.4.74}$$

This equation encodes the dependence of $I^4_{(2)}$ on (X^4_I, Y^4_I), i.e. from the transversal positions to D^4 of the I-th source. In complex variables [19], we have

$$\nabla^2_{0,(X^4_I,Y^4_I)} \equiv \frac{\text{Im}\,\lambda}{\det A}(2u_3 + 2u_2 - 4u_4)\frac{\partial}{\partial Z^4}\frac{\partial}{\partial \bar{Z}^4} I^4_{(2)} \quad , \tag{5.4.76}$$

and equation (5.4.74) can be rewritten as:

---

[18]The integration over $\int_{T^2_1 \times \Sigma_1}$ can be rewritten using the redefined coordinates as:

$$\int_{T^2_1 \times \Sigma_1} dx^1 \wedge dx^5 \wedge dy^1 \wedge dy^5 = 4 \int_{T^6} d^6y \, \delta^2(x^4 - X^4; y^4 - Y^4)$$
$$= 8 \int_{T^6/\mathbb{Z}_2} d^6y \, d^6y \, \delta^2(x^4 - X^4; y^4 - Y^4) \quad . \tag{5.4.72}$$

For notation see footnote 15.

[19]Since $dz^4 = dz^2 - dz^3$ and $dz^5 = dz^2 + dz^3$, then:

$$\frac{\partial}{\partial z^4} = \frac{1}{2}\left(\frac{\partial}{\partial z^2} - \frac{\partial}{\partial z^3}\right) , \qquad \frac{\partial}{\partial z^5} = \frac{1}{2}\left(\frac{\partial}{\partial z^2} + \frac{\partial}{\partial z^3}\right) \quad . \tag{5.4.75}$$

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$$\frac{\partial}{\partial Z_I^4}\frac{\partial}{\partial \bar{Z}_I^4} I_{(2)}^4 = -q_I \left(\frac{1}{2}\delta^2(Z^4 - Z_I^4) - \frac{1}{4\mathrm{Im}\,\lambda} \right) \quad . \tag{5.4.77}$$

Hence, once again, we get an equation free of Kähler moduli. The general solution for $I_{(2)}^4$ is:

$$I_{(2)}^4 = \sum_{I \in \mathrm{D3's,O3's}} q_I \left[-\frac{1}{4\pi}\log|\theta_1(Z^4 - Z_I^4|\lambda)|^2 + \frac{1}{2\mathrm{Im}\,\lambda}(\mathrm{Im}\,(Z^4 - Z_I^4))^2 \right] \quad . \tag{5.4.78}$$

Thus, the integral I^4 (5.4.70) has the following form:

$$I^4 = (2u_3 + 2u_2 - 4u_4)\hat{a} + \sum_{I \in \mathrm{D3's}} \left(\frac{|Z_I^4|^2}{4\mathrm{Im}\,\lambda} + \mathrm{hol}(Z_I^4) + \overline{\mathrm{hol}}(\bar{Z}_I^4) \right) + \mathrm{consts} \quad . \tag{5.4.79}$$

By expressing it in terms of (Z_I^2, Z_I^3), i.e. as

$$
\begin{aligned}
I^4 &= (2u_3 + 2u_2 - 4u_4)\hat{a} \\
&\quad + \sum_{I \in \mathrm{D3's}} \left(\frac{|Z_I^2|^2 + |Z_I^3|^2 - 2\mathrm{Re}\,(Z_I^2\bar{Z}_I^3)}{4\mathrm{Im}\,\lambda} + \mathrm{hol}(Z_I^2 - Z_I^3) + \overline{\mathrm{hol}}(\bar{Z}_I^2 - \bar{Z}_I^3) \right) + \mathrm{consts} \\
&= \frac{\mathrm{Re}\,\rho^2}{2} + \frac{\mathrm{Re}\,\rho^3}{2} - 4u_4\hat{a} \\
&\quad + \sum_{I \in \mathrm{D3's}} \left(-\frac{\mathrm{Re}\,(Z_I^2\bar{Z}_I^3)}{2\mathrm{Im}\,\lambda} + \mathrm{hol}(Z_I^2 - Z_I^3) + \overline{\mathrm{hol}}(\bar{Z}_I^2 - \bar{Z}_I^3) \right) + \mathrm{consts} \quad ,
\end{aligned}
\tag{5.4.80}
$$

the identification of $\mathrm{Re}\,\rho^4$ follows immediately:

$$\mathrm{Re}\,\rho^4 \equiv t_4 + \sum_{I \in \mathrm{D3's}} \frac{\mathrm{Re}\,(Z_I^2\bar{Z}_I^3)}{2\mathrm{Im}\,\lambda} \quad , \tag{5.4.81}$$

where

$$t_4 = 4u_4\hat{a} \quad . \tag{5.4.82}$$

Analogously:

$$
\begin{aligned}
I^5 &= \frac{\mathrm{Re}\,\rho^1}{2} + \frac{\mathrm{Re}\,\rho^3}{2} - 4u_5\hat{a} \\
&\quad + \sum_{I \in \mathrm{D3's}} \left(-\frac{\mathrm{Re}\,(Z_I^3\bar{Z}_I^1)}{2\mathrm{Im}\,\lambda} + \mathrm{hol}(Z_I^3 - Z_I^1) + \overline{\mathrm{hol}}(\bar{Z}_I^3 - \bar{Z}_I^1) \right) + \mathrm{consts}
\end{aligned}
\tag{5.4.83}
$$

and

$$
\begin{aligned}
I^6 = {} & \frac{\mathrm{Re}\,\rho^1}{2} + \frac{\mathrm{Re}\,\rho^2}{2} - 4u_6\hat{a} \\
& + \sum_{I\in\mathrm{D3's}} \left(-\frac{\mathrm{Re}(Z_I^1\bar{Z}_I^2)}{2\mathrm{Im}\,\lambda} + \mathrm{hol}(Z_I^1 - Z_I^2) + \overline{\mathrm{hol}}(\bar{Z}_I^1 - \bar{Z}_I^2) \right) + \mathrm{consts} \quad .
\end{aligned}
\tag{5.4.84}
$$

Therefore we identify remaining chiral fields by:

$$
\mathrm{Re}\,\rho^5 \equiv t_5 + \sum_{I\in\mathrm{D3's}} \frac{\mathrm{Re}(Z_I^3\bar{Z}_I^1)}{2\mathrm{Im}\,\lambda} \quad,
\tag{5.4.85}
$$

$$
\mathrm{Re}\,\rho^6 \equiv t_6 + \sum_{I\in\mathrm{D3's}} \frac{\mathrm{Re}(Z_I^1\bar{Z}_I^2)}{2\mathrm{Im}\,\lambda} \quad,
\tag{5.4.86}
$$

where

$$
\begin{aligned}
t_5 &= 4u_5\hat{a} \quad, \\
t_6 &= 4u_6\hat{a} \quad.
\end{aligned}
\tag{5.4.87}
$$

We have just found that Kähler moduli (v^A, a) can be expressed in terms of chiral fields ρ^A, Z_I^i as:

$$
\begin{aligned}
t_1 &= \hat{a}(4v_2v_3 - v_4^2) = \mathrm{Re}\,\rho^1 - \sum_{I\in\mathrm{D3's}} \frac{|Z_I^1|^2}{2\mathrm{Im}\,\lambda} \\
t_2 &= \hat{a}(4v_1v_3 - v_5^2) = \mathrm{Re}\,\rho^2 - \sum_{I\in\mathrm{D3's}} \frac{|Z_I^2|^2}{2\mathrm{Im}\,\lambda} \\
t_3 &= \hat{a}(4v_1v_2 - v_6^2) = \mathrm{Re}\,\rho^3 - \sum_{I\in\mathrm{D3's}} \frac{|Z_I^3|^2}{2\mathrm{Im}\,\lambda} \\
t_4 &= \hat{a}(v_5v_6 - 2v_1v_4) = \mathrm{Re}\,\rho^4 - \sum_{I\in\mathrm{D3's}} \frac{\mathrm{Re}(Z_I^2\bar{Z}_I^3)}{2\mathrm{Im}\,\lambda} \\
t_5 &= \hat{a}(v_4v_6 - 2v_2v_5) = \mathrm{Re}\,\rho^5 - \sum_{I\in\mathrm{D3's}} \frac{\mathrm{Re}(Z_I^3\bar{Z}_I^1)}{2\mathrm{Im}\,\lambda} \\
t_6 &= \hat{a}(v_4v_5 - 2v_3v_6) = \mathrm{Re}\,\rho^6 - \sum_{I\in\mathrm{D3's}} \frac{\mathrm{Re}(Z_I^1\bar{Z}_I^2)}{2\mathrm{Im}\,\lambda}
\end{aligned}
\tag{5.4.88}
$$

.

We can finally explicit the the Kähler potential (5.4.41). From the observation that

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$$\begin{aligned}
\hat{a}^3 V_0^2 &= \hat{a}^3 \frac{V_{T^6}^2}{4} \\
&= 4^2 \hat{a}^3 (\det A)^2 \\
&= \hat{a}^3 (4v_1 v_2 v_3 - v_4^2 v_1 - v_5^2 v_2 - v_6^2 v_3 + v_4 v_5 v_6)^2 \\
&= \frac{1}{4} \left(t_1 t_2 t_3 + 2 t_4 t_5 t_6 - t_1 t_4^2 - t_2 t_5^2 - t_3 t_6^2 \right) \quad ,
\end{aligned} \tag{5.4.89}$$

we get:

$$\begin{aligned}
K &= -\log(\hat{a}^3 V_0^2) - \log((4\pi)^3 V_0) \\
&= -\log \left(t_1 t_2 t_3 + 2 t_4 t_5 t_6 - t_1 t_4^2 - t_2 t_5^2 - t_3 t_6^2 \right) - \log(16\pi^3 V_0) \quad .
\end{aligned} \tag{5.4.90}$$

We remark that all t^A's must be thought of as functions of chiral fields ρ^A's, which enter through their real part, and Z_I^i, as in (5.4.88). The Kähler potential we have just found then involves a non-trivial combination of these fields and, fortunately in this model, we were able to find its explicit form. In fact, in general, the inversion $\hat{a}(\text{Re}\,\rho, Z, \bar{Z})$ is not always possible, see Sect. 4.1.

The last term in (5.4.90) is a constant and it does not matter for the computation of the effective kinetic terms for chiral fields. However, it can play a non-trivial role in the determination of the four-dimensional scalar potential V_F, arising from W_{GVW}, see Sect. 4.3.

We note that each $\text{Re}\,\rho^A$ has the structure

$$\text{Re}\,\rho^A = t_A + \frac{1}{2} \sum_{I \in \text{D3's}} k_A(Z_I, \bar{Z}_I) \quad , \tag{5.4.91}$$

where $k_A(Z_I, \bar{Z}_I)$ can be identified as a set of potentials, as explained in [27]. They are not globally defined and under a holomorphic transformation they transform as

$$k^A(Z_I, \bar{Z}_I) \to k^A(Z_I, \bar{Z}_I) + f^A(Z_I) + \bar{f}^A(\bar{Z}_I) \tag{5.4.92}$$

for some holomorphic $f^A(Z_I)$.

The structure (5.4.91), confirms the non-trivial fibration of ρ^A over the D3-brane moduli space found in subsection 4.3.1. Indeed, to get a well-behaving Kähler potential K (5.4.90) under transformations (5.4.92), ρ^A must transform as

$$\rho^A \to \rho^A + \sum_{I \in \text{D3's}} f^A(Z_I) \quad , \tag{5.4.93}$$

in order to leave each $t^A(\text{Re}\,\rho^A, Z_I, \bar{Z}_I)$ invariant.

Moreover, the form of $\text{Re}\,\rho^A$ agrees with (and completes) the one obtained in [21] for small fluctuations of D3-brane positions, see (4.3.43). Indeed, by setting $\delta z^k = G^a = 0$

(these moduli are absent in our model) and by expanding brane positions around given points $Z_I^i \simeq Z_{(0)I}^i + \phi_I^i$ we recover (4.3.43). Indeed, in this expansion, $k_A(\phi_I^i, \phi_I^j) \sim -i(\omega_A)_{i\bar{j}}(Z_{(0)I}, \bar{Z}_{(0)I})\phi_I^i\bar{\phi}_I^j$.

Finally, note that by setting $v_4 = v_5 = v_6 = 0$ one gets:

$$K = -\sum_{i=1}^{3} \log\left(\mathrm{Re}\,\rho^i - \sum_{I \in \mathrm{D3's}} \frac{|Z_I^i|^2}{2\mathrm{Im}\,\lambda}\right) - \log(16\pi^3 \mathrm{V}_0) \quad . \tag{5.4.94}$$

This expression coincides with the Kähler potential proposed in [72], where it is argued that (5.4.94) can be interpreted as the Kähler potential of the coset

$$\frac{U(1, N_{\mathrm{D3}}+1)}{U(1) \times U(1+N_{\mathrm{D3}})} \times \frac{U(1, N_{\mathrm{D3}}+1)}{U(1) \times U(1+N_{\mathrm{D3}})} \times \frac{U(1, N_{\mathrm{D3}}+1)}{U(1) \times U(1+N_{\mathrm{D3}})} \quad . \tag{5.4.95}$$

However, the discussion of [72] assumes the presence of a generic background flux G_3, which leads to fix six of the nine Kähler moduli as discussed above (it corresponds to taking a pure diagonal metric (5.4.19)) and then it should be generalised. Indeed, our result shows that there exist $\mathcal{N} = 1$ models compactified on a fluxed T^6 orientifold which have a more extended space of Kähler and D3-brane moduli.

6. Conclusions

In this thesis we reviewed the main aspects of IIB compactifications, focusing on the Kähler potential they give rise in the four-dimensional effective theory.

We started from pure geometric supersymmetric compactifications, which are CY compactifications. Although such models have remarkably physical and geometrical properties, they give $\mathcal{N} = 2$ supersymmetric effective theories with no chiral matter and no supersymmetry breaking mechanism. In order to include these ingredients one has to introduce fluxes and localised sources in the background.

Hence we studied the more phenomenologically attractive fluxed compactifications. We specialised to a particular class of compactifications to Mink$_4$, the GKP backgrounds, in which the compact manifold is still Kähler, and, in absence of 7-branes, even a CY orientifold. We saw that fluxes and localised sources turn on a warp factor. Such a non-trivial warping makes particularly cumbersome the reduction procedure and for this reason it is commonly approximated to a constant. We reviewed the structure of the corresponding (unwarped) effective theory. However, a non-trivial warp factor is a physically important feature in flux compactifications and one would like to understand how it affects the effective four-dimensional theory.

In order to address this problem we followed an alternative approach to dimensional reduction, based on four-dimensional local super-conformal symmetry and holomorphicity of brane instantons contributions, as presented in [25–27]. The original work of this thesis was to apply such a method to a simple class of models, in order to explicitly check how the warping contributes to the Kähler potential. Results agree with previous works [16, 17, 21, 22, 24]. Remarkably, we showed that brane moduli enter in the Kähler potential via the warp factor. The contribution found is compatible with the approximate corresponding contribution obtained in [21], where D3-branes are included as probes and their contribution is obtained by dimensional reduction in the large volume limit, in the approximation of small fluctuations of brane positions. In the particular case in which Kähler moduli $v_4 = v_5 = v_6 = 0$, our result agrees with [72] and then shows that the discussion presented there is not completely exhaustive. Notice that, in this simple model, fluxes do not modify the structure of the Kähler potential, in agreement with what found in [20] by dimensional reduction. However, it has been recently shown that more in general fluxes do change this structure, see [27].

6. Conclusions

In this work we detected the dependence of the Kähler potential on the geometric Kähler moduli and on the brane moduli, freezing out complex structure moduli, assumed fixed by fluxes. We leave to future work the inclusion of these moduli, in order to check if they enter in the Kähler potential mixing chiral fields, as found in [21] (see (4.3.42) and (4.3.43)).

Ringraziamenti

Desidero ricordare tutti coloro che hanno reso possibile la stesura di questa tesi e coloro che mi hanno accompagnato in questi anni di formazione universitaria.

Ringrazio, innanzitutto, il Prof. Luca Martucci, Relatore, sempre disponibile e prodigo di spiegazioni e consigli. Senza il suo aiuto e l'entusiasmo trasmesso per la Teoria di Stringa, in tutti i suoi mirabili aspetti, questa tesi non esisterebbe. Desidero ringraziare il Prof. Gianguido Dall'Agata, Controrelatore nell'occasione, per i suggerimenti e i chiarimenti. Colgo l'occasione anche per ringraziarlo di avermi seguito, in ruolo di Relatore, nel lavoro di tesi di laurea triennale, avendo così accresciuto il mio interesse verso la Fisica oltre il Modello Standard.

Vorrei, quindi, ringraziare tutta la mia famiglia, per il sostegno quotidiano e i miei zii, sempre vicini.

Ricordo con immenso piacere le innumerevoli, divertenti e proficue (a volte interminabili) discussioni di Fisica avute in qualunque luogo, a qualsiasi ora con Carlo, Andrea e Martino (e/o occasionalmente interessati sconosciuti). Spero non finiscano ora. Un sentito ringraziamento va a Gianluca, sempre cordiale ed ospitale. Come dimenticare tutti i pomeriggi di Laboratorio trascorsi insieme, tra una risata e l'altra, altrimenti infiniti e noiosi?

Lonigo, 2 Dicembre 2014

Appendices

A. Conventions

- The generic Ansatz $M_{10} = M^{1,3} \times M_6$ has ten-dimensional coordinates denoted by x^M, with $M = 0, ..., 9$. These split in x^μ, $\mu = 0, ...3$, and in y^m, $m = 1, ..., 6$, which are coordinates on $M^{1,3}$ and M_6 respectively. Local flat frame coordinates are identified with underlined indices $(x^{\underline{\mu}}, y^{\underline{m}})$ and vielbeins are $e_M^{\underline{N}}$.

- The Hodge-$*$ operator adopted throughout the thesis acts one the basis of a p-form α_p as:

$$*dx^{M_1} \wedge \cdots \wedge dx^{M_P} \equiv \frac{1}{(D-p)!} \sqrt{-g}\, g^{M_1 N_1} \cdots g^{M_P N_p} \epsilon_{L_1 \cdots L_{D-p} N_1 \cdots N_p}\, dx^{L_1} \wedge \cdots \wedge dx^{L_{D-p}} \quad ,$$

(A.1)

Therefore:

$$
\begin{aligned}
*\alpha_p \wedge \alpha_p &= |\alpha_p|^2 \sqrt{g} d^D x \\
&= \frac{1}{p!} \alpha_p^2 \, d\text{V} \\
&= \frac{1}{p!} \alpha_{M_1 \dots M_P} \alpha^{M_1 \dots M_P} \, d\text{V} > 0 \quad .
\end{aligned}
$$

(A.2)

This choice in particular implies a change of the Chern–Simons action sign in the IIB action (2.1.3), (2.1.5) with respect to [30].

- For the most general warped Ansatz $M_{10} = M^{1,3} \times_{\mathrm{w}} M_6$ with metric

$$ds^2 = e^{2A} ds_4^2 + ds_6^2 \quad , \tag{A.3}$$

we can define ten-dimensional gamma matrices in the local flat frame. Calling $\gamma_{\underline{\mu}}$ four-dimensional gamma matrices associated to the unwarped $M^{1,3}$ and $\gamma_{\underline{m}}$ six-dimensional gamma matrices related to M_6, ten-dimensional $\Gamma_{\underline{M}}$ split as:

A. Conventions

$$\begin{aligned}
\Gamma_{\underline{\mu}} &= e^{-A}\gamma_{\underline{\mu}} \otimes \mathbb{1} \quad, \\
\Gamma_{\underline{m}} &= \gamma_5 \otimes \gamma_{\underline{m}} \quad.
\end{aligned} \tag{A.4}$$

$\gamma_5 = i\gamma_{\underline{0123}}$ and $\gamma_7 = -i\gamma_{\underline{123456}}$ are the four-dimensional and six-dimensional chiral operators, while the ten-dimensional one is

$$\Gamma_{10} = \Gamma_{\underline{0\ldots9}} = \gamma_5 \otimes \gamma_7 \quad. \tag{A.5}$$

It results convenient to use a real representation for ten-dimensional gammas, since than a generic IIA/IIB ten-dimensional Majorana–Weyl spinor ϵ can be decomposed as $\epsilon = \zeta \otimes \eta + c.c.$, with ζ, η Weyl spinors in unwarped $M^{1,3}$ and M_6 respectively. Hence, as suggested by (A.4), one has to choose a real representation for $\gamma_{\underline{\mu}}$ and a pure imaginary representation for $\gamma_{\underline{m}}$.

- There are two main ways to deal with the internal manifold in string compactifications. In Sect. 4.1 we saw that for a six-dimensional CY, a Riemannian manifold of strict $SU(3)$ holonomy, there exists a single covariantly constant chiral spinor η which can be used to build the Kähler form J and the holomorphic three-form Ω by defining their components as:

$$J_{mn} \equiv -i\eta^\dagger \gamma_{mn}\eta \quad \rightarrow \quad J = ig_{i\bar{j}}dz^i \wedge d\bar{z}^{\bar{j}} \quad, \tag{A.6}$$

$$\Omega_{mnp} \equiv \eta^T \gamma_{mnp}\eta \quad \rightarrow \quad \Omega = \frac{1}{3!}\Omega_{ijk}dz^i \wedge dz^j \wedge dz^k \quad. \tag{A.7}$$

With these definitions, remembering that $\Omega_{ijk} = f(z)\epsilon_{ijk}$ with a holomorphic f and that $\sqrt{g} = 2^3 \det g_{i\bar{j}}$, one can verify:

$$i\frac{\Omega \wedge \bar{\Omega}}{||\Omega||^2} = \frac{1}{3!}J \wedge J \wedge J = \pm d\mathrm{V} \quad, \tag{A.8}$$

where

$$\begin{aligned}
||\Omega||^2 &= \frac{1}{3!}\Omega_{ijk}\bar{\Omega}^{ijk} \\
&= \frac{1}{3!}g^{i\bar{i}}g^{j\bar{j}}g^{k\bar{k}}\Omega_{ijk}\bar{\Omega}_{\bar{i}\bar{j}\bar{k}} \\
&= \frac{2^3}{\sqrt{g}}|f|^2 \quad.
\end{aligned} \tag{A.9}$$

A typical choice is $f = 1$, i.e. $\Omega = dz^1 \wedge dz^2 \wedge dz^3$.

The the sign in front of the volume form in (A.8) depends on the orientation choice. The more natural orientation is

$$
\begin{aligned}
dz^1 &= dy^1 + idy^2 \quad, \\
dz^2 &= dy^3 + idy^4 \quad, \\
dz^3 &= dy^5 + idy^6 \quad,
\end{aligned}
\tag{A.10}
$$

since it gives $\frac{1}{3!} J \wedge J \wedge J = dV$. With this orientation also the volume form of complex one- and two- cycles, given by the pull-back on the cycle of J and $\frac{1}{2} J \wedge J$ respectively, are positive definite. A complex r-cycle volume form is said to be *positive definite* if its wedge product with the remaining complex forms, defining volumes in transversal directions, reproduces the total volume form dV. Otherwise, it is said to be *negative definite*. In particular, chosen the orientation

$$
\begin{aligned}
dz^1 &= dy^1 + idy^4 \quad, \\
dz^2 &= dy^2 + idy^5 \quad, \\
dz^3 &= dy^3 + idy^6 \quad,
\end{aligned}
\tag{A.11}
$$

all volume forms of complex r-cycles, with $r = 1, 2, 3$, are negative definite. In order to ameliorate the situation one can change the sign in the definition of J (A.6)

$$
J_{mn} \equiv i\eta^\dagger \gamma_{mn} \eta \quad \rightarrow \quad J = -ig_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}} \quad \rightarrow \quad J = -v^A \omega_A \quad, \tag{A.12}
$$

with positive Kähler moduli v^A, as usual. Now (A.8) becomes

$$
-i\frac{\Omega \wedge \bar{\Omega}}{||\Omega||^2} = \frac{1}{3!} J \wedge J \wedge J = dV \quad. \tag{A.13}
$$

Therefore volume forms of complex 1-cycles and 3-cycles are positive definite, while volume forms of complex 2-cycles remain negative definite. This implies that volumes of holomorphic four-cycles c_4 are given by

$$
V_{c_4} = -\frac{1}{2} \int_{c_4} J \wedge J \quad. \tag{A.14}
$$

This convention it adopted from Section 4.3 on. In fact, using the Hodge-$*$ (A.1), this convention is compatible with the ISD condition of G_3 (4.3.22).

B. Dimesional reduction

Given a ten-dimensional action and a starting background, to determine the resulting theory in four dimensions one has to perform the KK reduction [8].

Let us review how it works in the case of a generic field $\Phi_{MN...}$. To study the dynamics of fluctuations around the background, one begins expanding each ten-dimensional field component, conventionally denoted $\Phi_{\mu\nu...}^{mn...}(x,y)$, around its vacuum expectation value:

$$\Phi_{\mu\nu...}^{mn...}(x,y) = \langle \Phi_{\mu\nu...}^{mn...}(x,y) \rangle + \delta\Phi_{\mu\nu...}^{mn...}(x,y) \quad . \tag{B.1}$$

Then, substituting (B.1) in the corresponding ten-dimensional equation of motion and keeping only linear terms (and eventually fixing the gauge to eliminate redundant degrees of freedom), one gets typically an equation of the form

$$(\mathcal{O}_{\text{ext}} + \mathcal{O}_{\text{int}})\delta\Phi_{\mu\nu...}^{mn...}(x,y) = 0 \quad , \tag{B.2}$$

where $\mathcal{O}_{\text{ext}}$, $\mathcal{O}_{\text{int}}$ are n-order differential operators ($n = 1$ for fermions, $n = 2$ for bosons), which depend on the specific field.

Being the internal manifold compact, one can expand fluctuations in series, in a basis of eigenfunction for the respective $\mathcal{O}_{\text{int}}$:

$$\delta\Phi_{\mu\nu...}^{mn...}(x,y) = \sum_i \phi_{\mu\nu...}^i(x) Y_i^{mn...}(y) \quad , \tag{B.3}$$

with

$$\mathcal{O}_{\text{int}} Y_i^{mn...}(y) = \lambda^i Y_i^{mn...}(y) \quad . \tag{B.4}$$

Replacing (B.1) into (B.2) one finds that the mass of each four-dimensional mode $\phi_{\mu\nu...}^i(x)$ is quantized in terms of λ^i. This is the so-called "Kaluza–Klein tower of states". Eigenvalues λ^i are typically proportional to some power p of the internal momentum, i.e. inversely proportional to the same power of the average compactification radius, $\lambda^i \sim 1/R^p$ ($p > 0$). For a small R, KK states can be very massive with respect to the energy scale of the effective theory and then they can be integrated out. We say that the KK tower is truncated to the massless zero-modes $\phi_{\mu\nu...}^0(x)$.

The effective four-dimensional action is obtained replacing in the ten-dimensional action the expansions (B.1), truncating to zero-modes and finally integrating over the

internal manifold, up to second order in fluctuations.

It is worth noting, however, that this truncation is not consistent in many cases: the heavier fields might induce interactions of the zero-mode that are not suppressed by inverse powers of the heavy mass. Sometimes (this is typical of warped compactifications) even the lighter massive modes should be considered in the reduction, since their contribution can in principle modify the effective theory [18].

B.1. An example: B_2

Let us work out, for instance, the reduction of a p-form, the B_2 field in the simpler case of the pure geometrical IIB compactification explained in Section 4.1, i.e. with vanishing background fluxes and constant background dilaton. In this situation the equation of motion (2.1.7) for $B_2 = B_2^{\mathrm{bg}} + \delta B_2$, with $B_2^{\mathrm{bg}} = \mathrm{const}$, at first order in the variations becomes:

$$\Delta \delta B_2 = d * d \delta B_2 = 0 \quad . \tag{B.1.1}$$

Remembering that in a factorised background $M^{1,3} \times M_6$ the Laplacian splits as $\Delta = \Delta_4 + \Delta_6$, one gets:

$$(\Delta_4 + \Delta_6)\delta B_2 = 0 \quad . \tag{B.1.2}$$

Hence the number of four-dimensional massless fields is given by the number of the zero-modes of the internal Laplacian, which are the Betti number b^p of M^6. For instance, taking M_6 to be a CY in the stricter sense, the four-dimensional massless spectrum generated by the B-field is presented in Table B.1.

δB_{MN}	$\delta B_{\mu\nu}$	$\delta B_{\mu n}$	δB_{mn}
type of field in 4d	2-form	1-forms	scalars
type of field in M_6	scalar	1-forms	2−forms
# of fields in 4d	$b^0 = 1$	$b^1 = 0$	$b^2 = h^{1,1}$

Table B.1.: B_2 zero modes.

Hence, in the effective action will appear also b^2 four-dimensional scalars. These are the moduli fields of B_2. More generally, if we had a M_6 such that $b^1 \neq 0$, also four-dimensional vectors would appear in the effective four-dimensional action as additional

fields. Hence one finds that the four-dimensional theory is determined by geometrical and topological details of the internal space.

C. Lefschetz decomposition

On a compact Kähler manifold M of complex dimension d and with a Kähler form J one can define a $SU(2)$ algebra, called *Lefschetz algebra*, by defining three operators L_+, L_-, L_3 which act on a generic harmonic n-forms as:

$$L_3 G \equiv \frac{d-n}{2} G \ ,$$

$$L_+ G \equiv J \lrcorner G = \frac{1}{2(n-2)!} J^{m_1 m_2} G_{m_1 m_2 m_3 \ldots m_n} dx^{m_3} \wedge \cdots \wedge dx^{m_n} \ , \tag{C.0.1}$$

$$L_- G \equiv J \wedge G \ .$$

One can check that $L_\pm$ raises/lowers the L_3-eigenvalue by one. "States", i.e. harmonic forms, are classified by L^2 and L_3 eigenvalues as spin states $|l, m\rangle$. A *primitive n-form* G_{pr} is a state of highest weight $|l, +l\rangle$ $(l = \frac{d-n}{2})$, i.e. such that

$$L_+ G_{\mathrm{pr}} = 0 \ , \qquad L_-^{2l+1} G_{\mathrm{pr}} = 0 \quad . \tag{C.0.2}$$

Hence primitive forms in the *middle cohomology*, i.e. with $n = d$, are singlets $|0, 0\rangle$ annihilated by both $L_\pm$. A general harmonic form can be obtained acting with L_- on a primitive form:

$$\mathrm{Harm}^n(M) = \oplus_k L_-^k \mathrm{Harm}_{\mathrm{pr}}^{n-2k}(M) \quad . \tag{C.0.3}$$

This is the Lefschetz decomposition. A crucial property of this decomposition is that it is compatible with the Hodge decomposition. Indeed, one can show that $[L^2, *] = 0$, which means that one can simultaneously diagonalise both the Lefschetz spin and the Hodge-$*$. For a spin l harmonic $(d-q, q)$-form ω:

$$* \omega = (-1)^{l+q} \omega \quad d \text{ even} \ , \qquad * \omega = (-1)^{l+q}(-i)\omega \quad d \text{ odd} \ . \tag{C.0.4}$$

A generic harmonic (p, q)-form can thus be decomposed as (remember footnote 16):

$$\mathrm{Harm}^{p,q}(M) = \oplus_k L_-^k \mathrm{Harm}_{\mathrm{pr}}^{p-k, q-k}(M) \quad . \tag{C.0.5}$$

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