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Flavour changing neutral currents and axions

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Introduction

The Standard Model of strong and electroweak interactions is the current theory of particle physics and its theoretical predictions have been in astonishing agreement with experimental data so far.
However, it is widely believed that the Standard Model (SM) needs an ultraviolet completion, namely an extension in order to describe physical interactions properly at high energies. The main reason suggesting an UV completion of the SM is the fact that we are still missing a quantum theory of gravity and the energy at which quantum gravity effects become manifest, namely the Planck scale ($M_P \sim 10^{19}$ GeV), constitutes the ultimate UV cutoff for the SM.
Nevertheless, there are some theoretical issues, which induce us to believe that the scale at which new physics (NP) appears should be much lower. The most compelling one is the so called hierarchy problem: the Higgs boson is a light particle compared to the Planck scale, but regarding the SM as an effective field theory with M_P as UV cutoff, no argument is able to justify naturally the huge gap between the electroweak and the fundamental mass scale.
Trusting the Naturalness principle, we would expect the existence of NP around the TeV scale, which we are probing these years through LHC experiments.
Furthermore, cosmological issues like Dark Matter lead us to think that the SM is presumably an incomplete theory and new elementary particles still have to be discovered.
Within the search of a more fundamental model which solves the problems presented, we focus in this thesis on an hypothetical class of particles known as *Axion-Like Particles* (ALPs). These particles inherit their name from the QCD Axion, postulated in 1977 by Peccei and Quinn to naturally solve the strong CP problem, namely the absence of CP violation in QCD. ALPs are the Pseudo-Nambu-Goldstone bosons (pNGb) of spontaneously broken $U(1)$ global symmetries, which appear in many extensions of the SM.
They can also acquire a mass m_a due to nonperturbative effects and in this case they consistute a promising candidate to explain DM nature.
While the theoretical difference between the axion and a generic ALP is that the latter does not need to solve the strong CP problem, the practical one is that for ALPs the symmetry breaking scale f_a and m_a can be treated as independent

parameters, while the two are instead related for the QCD axion. This fact, combined with cosmological observations, ultimately constrains the Axion mass to be set in the range between 10^{-5} and 10^{-2} eV. We actually focus on generic ALPs and thus consider f_a and m_a to be free parameters.
A powerful tool to describe ALPs contributions to physical observables is provided by the EFT approach. In this framework the SM Lagrangian is extended with non renormalizable operators. Even though the new Lagrangian is not renormalizable, it nevertheless provides definite predictions up to the UV cutoff f_a and in this energy range a perturbative expansion is well defined.
Within the EFT approach, NP contributions are included in a set of coefficients associated with higher dimensional operators, called Wilson coefficients. This affords to set up a model-independent analysis where NP contributions are parametrized by the Wilson coefficients. Then, the expression of such coefficients may be derived in specific NP models.
Among the most sensitive observables to NP we focus on those related to flavour physics. In the SM all the fermions carrying the same quantum numbers with respect to the SM gauge group come with three different replicas, known as flavours. The term flavour physics refers to interactions that distinguish among generations and in the SM all the source of flavour interactions is encoded in the Yukawa couplings of fermions with the Higgs field.
In particular we can make the distinction between flavour changing charged currents (FCCC) and neutral currents (FCNC): the former change the flavour of a fermion current altering its electric charge, while the latter instead conserve the electric charge. Although FCCC are allowed to occur in the SM at tree level, FCNC processes are possible only at the loop level and are furthermore suppressed by the GIM mechanism.
Consequently, the study of FCNC processes is a powerful tool to detect possible NP effects and enables us to put strong bounds on Beyond Standard Model (BSM) theories.
In this work we explore ALPs contributions to FCNC processes, formulating them in a model-independent approach via an ALP effective lagrangian. Our goal is to constrain ALP parameter space studying heavy meson FCNC decays. In fact, while astrophysics and cosmology impose severe constraints on ALP interactions in the sub-KeV mass range and TeV scale can be tested at LHC, the most efficient probes of ALPs couplings in the Mev-GeV region come from precision experiments performed at the charm/bottom quark scale.
In particular we analyze ALP couplings with electroweak bosons and fermions, assuming a flavour blind coupling. The comparison with data considers first each coupling separately, then the ensamble in combination and the resulting interference pattern is worked out in detail.
This thesis is developed in the following way: in chapter 1 we discuss the issues of the SM and we explain why we need an UV completion of the SM and at what energy we suppose it to occur. In chapter 2 we present briefly the con-

struction of the SM and we discuss in detail its flavour structure. In chapter 3 we describe the EFT approach and we present in detail an application of this method in flavour physics, within the SM framework. In chapter 4 we present the strong CP problem, we show how the axion arises as a natural solution and we shortly review the main features of axion models. In chapter 5 we introduce the ALP effective lagrangian and we put phenomenological bounds on its Wilson coefficients, analyzing FCNC processes. Incidentally, we discuss an example of a FCNC transition in the SM and we highlight the technical differences of the computations between an effective theory and a renormalizable theory.

Chapter 1

Why going beyond the Standard Model

1.1 Experimental evidences

The Standard Model of particle physics is the current theory describing the interactions of fundamental particles and has been successfully tested over the last fifty years.
However, there are various phenomena which induce us to believe that the Standard Model (SM) may not be our ultimate theory of particle physics. The majority of these issues comes from the comparison between the SM predictions with cosmological observations, the most remarkable of which are:

- *Dark Matter:* Today there is experimental evidence that the SM particles constitute just the 5% of the energy density of the Universe. About the 26% of the energy density should be given by Dark Matter (DM), namely matter made of electrically neutral particles not included in the SM and at most weakly interacting with the SM fields.
  Therefore the SM would presumably need a completion that introduces other particles in order to take into account for the presence of DM. There are various hypotheses about DM nature, between them for example supersymmetric particles, sterile neutrinos and the axion like particles (ALPs).

- *Dark Energy:* The remaining 69% of the energy density budget of the Universe consists of Dark Energy (DE), an unknown form of energy responsible for the acceleration of the expansion of the Universe.
  The simplest physical explanation for DE is that of an intrinsic, fundamental energy of space given by the presence of quantum fields. However, attempts to explain DE in terms of vacuum energy of SM fields lead to a mismatch of 120 orders of magnitude. If this interpretation of the nature

of DE is correct, then the SM needs to be extended with other particles, in order to match the experimental value of the DE energy density.

- *Matter-Antimatter asymmetry:* It is well established from cosmological observations that in the Universe there is exceedingly more matter than antimatter, this fact is known as baryon asymmetry. In order to produce such an asymmetry, some constraints, known as Sakharov conditions, need to be satisfied [34]. These conditions for the production of baryon asymmetry are B violation, C and CP violation and the presence of an out of equilibrium phase in the early Universe. While such conditions are all qualitatively satisfied by the SM, the presence of an out of equilibrium phase would require the Higgs boson mass to be less than 80 GeV, which is ruled out by experiment. In addition the CP violation amount provided by the SM is too small to account for the baryon asymmetry.
Extensions of the SM, like Grand Unified Theories (GUT) and Supersymmetry (SUSY), can solve this problem, for instance introducing new sources of CP violation.

- *Neutrino masses:* According to the SM, neutrinos are massless particles, however from neutrino oscillation experiments we know that they do indeed have a non vanishing mass.
Mass terms for the neutrinos can be added to the SM, but they require an extension of the SM particle content, namely the introduction of a right handed neutrino for a Dirac mass term or new heavy degrees of freedom for Majorana mass terms.

Even if we already know from the previous observations that the SM cannot be the ultimate theory of particle physics, since it needs to be extended, to date no experimental result is accepted as disproving the SM at the 5σ level, which is fixed to be the threshold of a New Physics (NP) discovery in particle physics.
However, there are some observables, like the muon $g-2$ factor, which show significant discrepancy from the SM predicted value and more sophisticated experimental tests are needed to shed light on the nature of these discrepancies.

1.2 Theoretical hints

Independently from the compelling experimental reasons, one can also theoretically argue that the SM cannot be a theory valid up to arbitrarily high energies. First of all the SM does not include gravitational interactions and moreover a consistent quantum theory of gravity is still missing.
Nevertheless, the energy scale at which quantum gravity effects eventually become manifest represents the ultimate UV cutoff for the SM. Even in the absence of any other kind of NP effects up to Planck scale ($M_{pl}{\sim}10^{19}$ GeV), the SM needs at least an UV completion to include gravitation.

However, there are some theoretical hints which make reasonable to think that new physics exists at lower energy, perhaps detectable by current collider experiments.

- One indication of NP below the Planck scale comes from the evolution of the three gauge couplings under the renormalization group. In the SM they almost merge at 10^{14}GeV, while their unification is accomplished exactly at $\sim 10^{16}$GeV in supersymmetric extensions of the SM. This means that at such energy scale there could be just one type of gauge interaction with a larger gauge group, containing the SM gauge group as a subgroup.
  This is very appealing from a theoretical point of view and supports the idea of GUT at such energy scale.

- Another theoretical clue comes from the smallness of neutrino masses, which would require an unnatural small Yukawa coupling with the Higgs field and the introduction of an unobserved right handed neutrino.
  Neutrino small masses may be more naturally provided by the following five dimensional operator, known as Weinberg operator, which is the only one compatible with SM symmetries:

  $$\mathcal{L}_5 = \frac{y}{\Lambda}(\tilde{\phi}^\dagger L)^T C(\tilde{\phi}^\dagger L) \tag{1.1}$$

  where $\tilde{\phi}$ is the charge conjugate of the Higgs field, L is the leptonic left-handed doublet, C is the charge conjugation operator, Λ is the UV scale which originates this effective operator and y is an $\mathcal{O}(1)$ coupling constant. Upon the Electroweak Symmetry Breakdown (EWSB), this operator produces the following mass for neutrino:

  $$m_\nu = \frac{yv^2}{\Lambda}, \tag{1.2}$$

  with $v = 246.2$GeV the VEV of the Higgs field and $m_\nu \sim 0.1$eV the neutrino mass. Inverting this relation we can get an estimate for the scale Λ, which yields a value $\mathcal{O}(10^{15})$ GeV.

- The Higgs boson mass is linked to the EW symmetry breaking scale, namely $v = 246.2$ GeV.
  However, one expects that large loop corrections would make the Higgs mass huge, comparable to the Planck scale, unless there is an incredible fine tuning cancellation between the radiative corrections and the bare mass.
  Trusting the principle of Naturalness, this fact suggests that new physics should appear at much lower scale, $\mathcal{O}$ (TeV).

- It's an experimental fact that QCD interactions do not manifest CP violation, this fact is known as the strong CP problem. The term $\theta_{QCD} G^a_{\mu\nu}\tilde{G}^{\mu\nu}_a$, which in principle appears in the SM Lagrangian and leads to CP violation in the strong sector, has an extremely small coefficient, $\theta_{QCD} \leq 10^{-10}$. The Peccei-Quinn mechanism can explain "naturally" this value and it also implies the existence of a new scalar particle, the QCD Axion.

1.3 Naturalness and the hierarchy problem

As we have seen in the previous section, most of the theoretical problems affecting the SM refer to the concept of "Naturalness". Therefore we need to define precisely what is the meaning of the Naturalness of a theory.
The definition of Naturalness given by 't Hooft is [26]:

At any energy scale μ, a physical parameter or set of physical parameters $\alpha_i(\mu)$ is allowed to be very small only if the replacement $\alpha_i(\mu) = 0$ would increase the symmetry of the system.

Apparently, this definition seems to come from nowhere, but it becomes clearer recalling that, if the classical action of a QFT has a certain symmetry, then this symmetry, if not anomalous, must be fulfilled by the quantum action as well. As a consequence, if the parameter α is zero, also its correction δ_α should vanish, in order to preserve the symmetry at the quantum level.
If on the other side the symmetry is slightly broken by the appearance of the parameter, then the parameter will receive quantum corrections proportional to itself ($\delta\alpha \sim \alpha$), because the symmetry should be restored in the $\alpha = 0$ limit. An example of the first case is provided by massless gauge bosons, whose mass is constrained to be zero by gauge invariance. The second case is instead represented by fermion masses in the SM: setting them to zero restores the chiral symmetry, which protects them against large loop corrections. In fact, fermion masses get corrections of the following form:

$$\delta_{m_f} \sim \frac{\alpha}{4\pi} m_f \log\frac{\Lambda}{m_f} \tag{1.3}$$

that remain acceptably small even if the theory has a UV cutoff at $M_{Pl} \sim 10^{19}$ GeV. Thus, as a consequence of the 't Hooft Naturalness principle, fermion masses are protected from planckian corrections and so a small value for m_f/M_{Pl} is natural.
Conversely, the scalar masses fail to break any symmetry of the action. This implies that considering a scalar field ϕ, loop corrections δm_ϕ^2 to m_ϕ^2 depend quadratically on the UV scale Λ, rather than logarithmically as in the case of fermion masses, making unnatural a physical mass m_ϕ far away from the fundamental scale of the Planck Mass.
This is indeed the case of the SM Higgs boson. Regarding the SM as an EFT

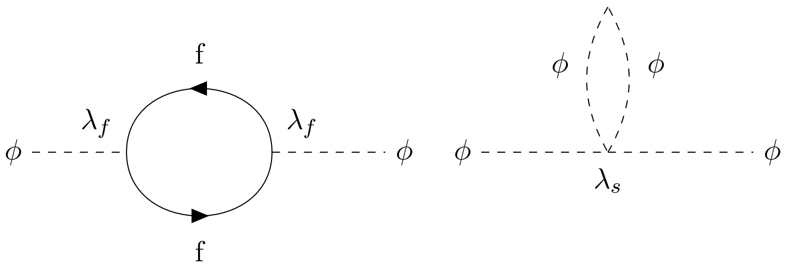


Figure 1.1: Two of the contributions to the Higgs mass in the Standard Model at one loop order: the diagram on the left involves a fermion loop, while the second is a self-correction of the Higgs propagator.

with UV cutoff at $\Lambda = M_{pl}$, since gravity has to couple with the Higgs field, the Higgs boson mass will receive a correction $\delta m_h^2 \sim M_{pl}^2$. One may define the amount of fine tuning between the Higgs mass and the radiative corrections as:

$$f \equiv \frac{m_h^2}{\delta m_h^2} \sim \frac{m_h^2}{M_{pl}^2} \sim 10^{-34} \tag{1.4}$$

An high fine tuning is regarded as unnatural in the absence of a mechanism to justify it. While there could be an ambiguity on the naturalness of a generic fine tuning parameter f, the fine tuning computed in Eq.(1.4) is unambiguously unnatural.
Thus, there is no way in the SM to naturally protect the Higgs mass from receiving these large corrections, which would arise also even if the bare mass of the Higgs were zero.
Thus the problem of the Higgs naturalness eventually turns into a hierarchy problem: is actually the SM valid up to the Planck scale? If this is the case, then the question becomes why the EW scale and the Planck scale are so far away from each other.
Viable solutions require either introducing some new symmetry that protects the Higgs mass, such as Supersymmetry, or stating that the UV cutoff of the SM as an effective field theory is much lower than the Planck scale.
For example, setting the fine tuning to be $f \leq 10^{-2}$, then the UV cutoff of the SM should be at $\mathcal{O}$ (TeV). This means that there would be some new physics residing in the "desert" between 10^3 and 10^{19} GeV, whose lower region up to 14 TeV is being explored in these years by LHC.

Chapter 2

Flavour physics

2.1 Construction of the Standard Model Lagrangian

In this section we briefly present the construction of the Standard Model of particle physics, formulated in its original version in the 60' by Glashow, Weinberg and Salam.
The SM is a renormalizable quantum field theory and resting on the experimental observations, we set up the following theoretical framework:

1. A gauge symmetry group G_{SM}, bearing the gauge boson fields

$$G_{SM} = SU(3)_C \otimes SU(2)_L \otimes U(1)_Y, \tag{2.1}$$

   where we refer to the first subgroup as Quantum Chromodynamics (QCD), while the latter constitute the electroweak sector of the SM.

2. In nature we observe 4 gauge bosons mediating electroweak forces: one of them, the photon, is massless, while Z and $W^\pm$ are massive instead. Hence $SU(2)_L$ gauge symmetry must be spontaneously broken, leaving us with a residual one-dimensional group corresponding to $U(1)_{EM}$, so

$$G_{SM} \to H = SU(3)_C \otimes U(1)_{EM}. \tag{2.2}$$

   A doublet of complex scalar fields, ϕ, denoted as Higgs field, is introduced to allow the spontaneous breaking of G_{SM} into the residual symmetry group H.

3. We introduce 3 generations of fermions, each of them consisting of 5 different representations of G_{SM}:

$$\begin{array}{ccccc} Q_{Li}(3,2)_{\frac{1}{6}} & , & u_{Ri}(3,1)_{\frac{2}{3}} & , & d_{Ri}(3,1)_{-\frac{1}{6}} \\ & & L_{Li}(1,2)_{\frac{1}{2}} & , & e_{Ri}(1,1)_{-1}. \end{array} \tag{2.3}$$

In this notation, for example, Q_{Li} are left-handed quarks (the index i=1,2,3 runs over families), which transform under the fundamental representation of $SU(3)_C$ and $SU(2)_L$ and carry weak hypercharge $Y = \frac{1}{6}$. We can decompose the two $SU(2)_L$ doublets into their components in this way:

$$Q_{Li} = \begin{pmatrix} u_{Li} \\ d_{Li} \end{pmatrix} \quad L_{Li} = \begin{pmatrix} \nu_{Li} \\ e_{Li} \end{pmatrix}. \tag{2.4}$$

Finally we have the Higgs field, which is a scalar transforming in the $(1,2)_{\frac{1}{2}}$ representation of G_{SM}.

Using the above considerations we can write down a renormalizable Lagrangian for the SM, which can be split into four sectors: gauge boson, Higgs, fermionic and Yukawian:

$$\mathcal{L}_{SM} = \mathcal{L}_B + \mathcal{L}_H + \mathcal{L}_F + \mathcal{L}_Y \tag{2.5}$$

where

$$\mathcal{L}_B = -\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu,a} - \frac{1}{4}W^b_{\mu\nu}W^{\mu\nu,b} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}, \tag{2.6}$$

$$\mathcal{L}_H = (D_\mu\phi)^\dagger(D^\mu\phi) - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2, \tag{2.7}$$

$$\mathcal{L}_F = \sum_i i\Big(\bar{Q}_{Li}\not{D}Q_{Li} + \bar{L}_{Li}\not{D}L_{Li} + \bar{u}_{Ri}\not{D}u_{Ri} + \bar{d}_{Ri}\not{D}d_{Ri} + \bar{e}_{Ri}\not{D}e_{Ri}\Big), \tag{2.8}$$

$$-\mathcal{L}_Y = \bar{Q}_{Li}Y^d_{ij}\phi d_{Rj} + \bar{Q}_{Li}Y^u_{ij}\tilde{\phi}u_{Rj} + \bar{L}_{Li}Y^e_{ij}\phi e_{Rj} + h.c., \tag{2.9}$$

where $G_{\mu\nu}$, $W_{\mu\nu}$, $B_{\mu\nu}$ are the field strength tensors associated to gluon fields (a=1...8), weak gauge bosons (b=1,2,3) and hypercharge boson B_μ, respectively. $\tilde{\phi} = i\sigma^2\phi^*$ is the charge conjugated of ϕ, μ^2 and λ are real parameters associated to the Higgs potential ($\mu^2 < 0$ and $\lambda > 0$) and $Y^{u,d,e}$ are 3×3 complex matrices known as Yukawa matrices.
The action of the gauge covariant derivative D_μ on matter fields is determined by the representation of G_{SM} under which they transform and takes the following form:

$$\begin{cases} D_\mu\phi = \big(\partial_\mu + igW_{b\mu}\frac{\sigma^b}{2} + ig'B_\mu\frac{1}{2}\big)\phi \\ D_\mu Q_L = \big(\partial_\mu + ig_sT^aG^a_\mu + igW_{b\mu}\frac{\sigma^b}{2} + ig'B_\mu\frac{1}{6}\big)Q_L \\ D_\mu L_L = \big(\partial_\mu + igW_{b\mu}\frac{\sigma^b}{2} - ig'B_\mu\frac{1}{2}\big)L_L \\ D_\mu u_R = \big(\partial_\mu + ig_sT^aG^a_\mu + ig'B_\mu\frac{2}{3}\big)u_R \\ D_\mu d_R = \big(\partial_\mu + ig_sT^aG^a_\mu - ig'B_\mu\frac{1}{6}\big)d_R \\ D_\mu e_R = \big(\partial_\mu - ig'B_\mu\big)e_R, \end{cases} \tag{2.10}$$

where T^a and σ^a are the generators of SU(3) and SU(2) respectively, in the fundamental representation.
$\mathcal{L}_B$ is the gauge boson part of the Lagrangian, containing boson kinetic terms and their cubic and quartic self interactions; this part of the Lagrangian contains three free parameters: g, g' and g_s, which are the gauge couplings associated to

the three simple subgroups of G_{SM}.
$\mathcal{L}_H$ is the Higgs doublet Lagrangian, containing its kinetic term, the interactions with the gauge fields and its potential. In order for ϕ to acquire a non vanishing VEV (v), we need to require $\mu^2 < 0$, while λ has to be positive in order to have a bounded potential. The relation between v and the free parameters of $\mathcal{L}_H$ is easily obtained minimizing the potential and reads

$$v = \sqrt{-\frac{\mu^2}{2\lambda}} \tag{2.11}$$

The mass of the Higgs boson is given in terms of v, which is experimentally known from the muon decay, and λ, which is instead a free parameter of the theory. At tree level the relation reads

$$m_h = \sqrt{2\lambda v^2} = (124.97 \pm 0.24)\text{GeV} \tag{2.12}$$

$\mathcal{L}_F$ contains fermion kinetic terms and their interactions with gauge fields, while $\mathcal{L}_Y$ contains the Yukawa interaction between fermions and the Higgs field. This interaction produces fermion masses after Electroweak Symmetry Breaking (EWSB). Due to Higgs mechanism, the electroweak gauge bosons W_1, W_2, W_3 and B mix to create the states which are physically observable. These physical states are:

$$W^{\pm}_{\mu} = \frac{W_{1\mu} \mp iW_{2\mu}}{\sqrt{2}}, \tag{2.13}$$

$$\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} B_\mu \\ W_{3\mu} \end{pmatrix}, \tag{2.14}$$

with the *Weinberg angle* $\theta_W = \arctan\frac{g'}{g} \sim 0.23$. $W^{\pm}$ and Z are the weak interaction mediator bosons, while A is the electromagnetic field. The former are massive, with mass linked to the VEV of the Higgs field in this way at tree level:

$$M_W = \frac{gv}{2} = 80.4\text{GeV} \tag{2.15}$$

$$M_Z = \frac{\sqrt{g^2 + g'^2}v}{2} = 91.2\text{GeV}. \tag{2.16}$$

We can rewrite $\mathcal{L}_F$ replacing the gauge fields $W_{a\mu}$ and B_μ with $W^{\pm}$ and Z. One then obtains:

$$\mathcal{L}_F = [\mathcal{L}_{Kin}] - \frac{g}{\sqrt{2}}(W^+_\mu J^{-\mu} + W^-_\mu J^{+\mu}) - eA_\mu J^\mu_{em} - \frac{g}{\cos\theta_W} Z_\mu J^\mu_Z \tag{2.17}$$

where

$$\begin{aligned} J^{-\mu} &= \bar{\nu}_L \gamma^\mu e_L + \bar{u}_L \gamma^\mu d_L \\ J^\mu_{em} &= -\bar{e}\gamma^\mu e + \frac{2}{3}\bar{u}\gamma^\mu u - \frac{1}{3}\bar{d}\gamma^\mu d \\ J^\mu_{3L} &= \bar{L}_L \gamma^\mu \frac{\sigma^3}{2} L_L + \bar{Q}_L \gamma^\mu \frac{\sigma^3}{2} Q_L \\ J^\mu_Z &= J^\mu_{3L} - \sin^2\theta_W J^\mu_{em}. \end{aligned} \tag{2.18}$$

We commonly refer to $J^{\pm}_{\mu}$ as charged currents terms (CC) and to J^{Z}_{μ} as neutral currents (NC) terms.

2.1.1 Interlude on neutrino physics

Today three species of active neutrinos, one for each fermion generation, are thought to exist: this fact has been determined experimentally at LEP from the measurement of the invisible decay width of Z boson.
Neutrinos are different from other fermions because of some peculiar properties: they are purely left-handed, namely no neutrino of right chirality has ever been observed up to now, and they interact with other particles only through the weak interaction, since they don't carry neither electric nor colour charge.
Therefore, their observation is based on processes with very tiny cross sections, so that sensitive detectors and sophisticated techniques are required in their experimental hunt.
Even though in the original SM proposed by Glashow, Salam and Weinberg they were supposed to be massless particles, we know from the phenomenon of neutrino oscillation that they indeed possess a mass: actually, in neutrino oscillation experiments we are sensitive only to squared mass differences, and our best estimates on the sum of their masses is based on a combination of cosmological probes, which set the limit of $\sum_i m_{\nu_i} < 120$ meV$(95\% C.L.)$.
The fact of being electrically neutral opens the possibility that neutrinos are Majorana fermions, namely that they coincide with their own antiparticle. For neutrinos a Majorana mass term is allowed, which has the property of violating any $U(1)$ global symmetry, in particular lepton number.
The other possibility to give neutrino mass is through the Yukawa interaction, introducing a right-handed neutrino for each generation, inert under electroweak interaction, which would produce the canonical Dirac mass term.
For both kind of mass terms we would have to extend the original definition of the SM, since the Majorana mass term for the SM neutrino can be provided only by higher dimensional operators, implying the existence of high energy fields and the Dirac mass term requires the introduction of a right handed neutrino.
Up to now we do not have evidence to establish whether neutrinos are Dirac or Majorana particles, therefore in the following discussion, for simplicity, we will assume that their mass is generated by a Dirac mass term, keeping in mind that this is far from being the only possibility.

2.2 Flavour structure of the Standard Model

The term *Flavour physics* refers to interactions that distinguish among fermion flavours, where by flavour we mean one of the different generations of fermion fields carrying the same quantum numbers, namely belonging to the same representation of G_{SM}. In the SM all the source of flavour interactions is due to the

Yukawa matrices $Y^{e,u,d,\nu}$, as they are the only part of $\mathcal{L}_{SM}$ which distinguish between families. In the absence of Yukawa interaction or in the case where Yukawa matrices are proportional to the identity, besides the Gauge Symmetry, $\mathcal{L}_{SM}$, extended with right-handed neutrinos, possesses a global $U(3)^6$ symmetry, which can be split into its quark and lepton subgroup:

$$G_{Flavour} = U(3)^3_Q \otimes U(3)^3_L, \tag{2.19}$$

under which fermion fields mix between generations for each representation in this way:

$$\begin{aligned} Q_L &\to V_Q Q_L \qquad & L_L &\to V_L L_L \\ u_R &\to V_u u_R \qquad & d_R &\to V_d d_R \\ e_R &\to V_e e_R, \qquad & \nu_R &\to V_\nu \nu_R \end{aligned} \tag{2.20}$$

where V's are $U(3)$ matrices. When considering $Y^{e,u,d,\nu} \neq \{0, \mathbb{1}_3\}$, G_{SM} is now broken down to a smaller group, known as the global accidental symmetry group of the SM. We call this symmetry accidental since it's not a fundamental symmetry of our theory like the Gauge Symmetry, which was imposed a priori.

$$G_{Flavour} \to U(1)_B \otimes U(1)_L. \tag{2.21}$$

$U(1)_B$ is associated to baryon number conservation, while $U(1)_L$ is associated to lepton number conservation. [1] Let's now see how we can deduce the number of physical parameters of the SM looking in particular at $\mathcal{L}_Y$.
Going in the unitary gauge the Higgs doublet takes the simple form

$$\phi = \begin{bmatrix} 0 \\ \frac{h+v}{\sqrt{2}} \end{bmatrix}, \tag{2.22}$$

Since $Y^{e,\nu,u,d}$ are 3×3 complex matrices in flavour space they contain $4\times2\times3^2 = 72$ parameters. However, since in a local QFT physics shouldn't depend on the base used in field space, we are free to make a transformation in the fermion field space to diagonalize Yukawa matrices. As they are complex matrices, they are diagonalized by a bi-unitary transformation, with real and positive eigenvalues. From Eq (2.9) the Yukawian Lagrangian reads:

$$-\mathcal{L}_Y = \frac{h+v}{\sqrt{2}}\left[\bar{e}^i_R Y^{ij}_e e^j_L + \bar{\nu}^i_R Y^{ij}_\nu \nu^j_L + \bar{u}^i_R Y^{ij}_u u^j_L + \bar{d}^i_R Y^{ij}_d d^j_L\right] + h.c. \tag{2.23}$$

We now diagonalize Yukawa matrices making these unitary transformations on fermion fields:

$$f_L \to V_f f'_L \qquad f_R \to U_f f'_R \qquad (f = e, \nu, u, d) \tag{2.24}$$

[1] In the original version of SM, with massless neutrinos, individual lepton family numbers are conserved too at tree level. Taking into account for the presence of neutrinos, one $U(1)$ combination, namely $B+L$, is anomalous at one loop, so the only exactly conserved quantum number associated to the accidental symmetry group of the SM is $B-L$.

where primed indices are referred to fermion fields in the mass basis. We finally get this expression for $\mathcal{L}_Y$:

$$-\mathcal{L}_Y = \frac{h+v}{\sqrt{2}}\left[\bar{e'}_{Ri}(Y_e)^{ij}_{Diag}e'_{Lj}+\bar{\nu'}_{Ri}(Y_\nu)^{ij}_{Diag}\nu'_{Lj}+\bar{u'}_{Ri}(Y_u)^{ij}_{Diag}u'_{Lj}+\bar{d'}_{Ri}(Y_d)^{ij}_{Diag}d'_{Lj}\right]+h.c. \tag{2.25}$$

where $(Y_f)_{Diag} = U_f^\dagger Y_f V_f$ are the Yukawa matrices in the mass basis, which are related to quark masses by:

$$m_{fk} = (Y_f)^{kk}_{Diag}\frac{v}{\sqrt{2}}, \tag{2.26}$$

where the index k runs over fermion generations.
While diagonalizing Yukawa matrices, the change of basis obviously modifies $\mathcal{L}_F$ too. In particular NC and CC terms undergo these transformations:

$$\begin{array}{lcc} & NC & \bar{f_L}\gamma^\mu f_L \rightarrow \bar{f_L}V_f^\dagger\gamma^\mu V_f f_L = \bar{f_L}\gamma^\mu f_L \\ CC & & \bar{u_L}\gamma^\mu d_L \rightarrow \bar{u_L}V_u^\dagger\gamma^\mu V_d d_L = \bar{u_L}\gamma^\mu V_{CKM} d_L \\ & & \bar{\nu_L}\gamma^\mu e_L \rightarrow \bar{\nu_L}V_\nu^\dagger\gamma^\mu V_e e_L = \bar{\nu_L}\gamma^\mu U^\dagger_{PMNS} e_L \end{array} \tag{2.27}$$

The neutral currents remain unchanged, while charged currents are affected by the change of basis both in the quark and lepton sectors. In the quark sector the change of basis produces the *Cabibbo-Kobayashi-Maskawa* (CKM) unitary matrix, given by:

$$V_{CKM} = V_u^\dagger V_d = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}, \tag{2.28}$$

responsible for quark mixing, while in the leptor sector the change of basis produces the *Pontecorvo-Maki-Nakagawa-Sakata* (PMNS) unitary matrix, which is responsible for neutrino oscillations and is given by:

$$U_{PMNS} = V_e^\dagger V_\nu = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{pmatrix}. \tag{2.29}$$

Therefore we conclude that only CC interactions feel the change of basis, i.e. the misalignment between the mass basis and the flavour basis. As a consequence, only *Flavour Changing Charged Currents* (FCCC) are admitted in the SM at tree level, while *Flavour Changing Neutral Currents* (FCNC) can appear only at one-loop order, so they are suppressed with respect to the former. We will analyze FCNC processes in the next section, in particular the *Glashow-Iliopoulos-Maiani mechanism*(GIM), which explains the suppression of FCNC in the SM.
To stress the importance that the study of FCNC had in the development of the SM, we remind that in 1964, when only up, down and strange quarks were thought to exist, on the basis of the suppression of FCNC, the existence of a fourth quark, the charm, was predicted. Actually the charm quark would be discovered only 10 years later, in 1974.

2.2.1 CKM matrix and CP violation

We have seen that the CKM is a 3×3 unitary matrix in flavour space, which is responsible for flavour changing transitions in the quark sector. Let's consider now the general N generations case, where V_{CKM} is so of dimension N. A general $N \times N$ unitary matrix has N^2 parameters. $N(N-1)/2$ of these parameters may be taken as Euler angles which one introduces when dealing with rotations in a N dimensional space, while the remaining $N(N+1)/2$ are complex phases. We can ask ourselves whether these phases are all physical or not.

We know from quantum mechanics that the phase of a wavefunction is not a measurable quantity, namely a wavefunction ψ and $exp(i\phi)\psi$, where ϕ is a real number, are physically equivalent. The situation is the same in QFT: what matters is not the absolute phase but the relative phases of different fields. Therefore we should examine which phases in V_{CKM} are observable and which are not. The phases of the fields are arbitrary, so we can redefine them performing phase transformations in this way:

$$
\begin{aligned}
d_{Lk} &\to e^{i\phi_k} d_{Lk} \qquad && k = d, s, b \\
u_{Lj} &\to e^{i\phi_j} u_{Lj} \qquad && j = u, c, t
\end{aligned}
\tag{2.30}
$$

Under the above transformations, for any number of families, what happens is that:

$$
V_{kj} \to e^{i(\phi_k - \phi_j)} V_{kj} \tag{2.31}
$$

where j and k denote an up-kind and down-kind quark respectively. Considering the effect of this field redefinition on the other pieces of $\mathcal{L}_{SM}$, NC terms are manifestly invariant because they are flavour conserving; $\mathcal{L}_Y$ is affected because it connects L and R fields, however this can be remedied by rephasing any right-handed quark field with the same phase as corresponding left-handed one, such that $\mathcal{L}_Y$ remains unchanged. From the previous equation, we see that the number of phases which can be reabsorbed is given by the number of phase differences $\phi_j - \phi_k$. Since we have $2N$ of such phases, $2N-1$ of them will be unphysical.

Therefore, CKM has $N^2 - (2N-1) = (N-1)^2$ total parameters, of which $(N-1)(N-2)/2$ are complex phases. The same reasoning applies identically to the PMNS matrix, for the lepton sector. Thus, we have two physical phases for the 3 fermion generations case (one for each mixing matrix), which combined with the 12 Yukawa couplings and the 6 Euler angles of the CKM and PMNS matrix give the number of 20 independent parameters for $\mathcal{L}_Y$. Adding the gauge coupling constant g, g' and g_s, the free parameters of the Higgs potential λ and μ^2, and finally the QCD theta term, we are left with 26 free parameters for the SM, with Dirac mass term for the neutrino.[2]

On the other side, for the case of 2 families, the CKM matrix has one rotation

[2] If neutrinos are Majorana particles, than the PMNS would possess 3 physical phases and the SM would have 28 free parameters

angle and no phases. In this case the quark mixing matrix has the simple form:

$$V = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix}, \tag{2.32}$$

where θ_C is called *Cabibbo angle*. The importance of the irremovable phase in CKM matrix resides on the fact that it allows CP violation in the electroweak sector of the SM. To see this we recall the CP transformations for vector and axial fermion bilinears (V and A) and for the gauge fields (W):

$$\begin{aligned} V \quad & \bar{\psi}_1\gamma^\mu\psi_2 \xrightarrow{CP} -\bar{\psi}_2\gamma_\mu\psi_1 \\ A \quad & \bar{\psi}_1\gamma^\mu\gamma_5\psi_2 \xrightarrow{CP} -\bar{\psi}_2\gamma_\mu\gamma_5\psi_1 \\ W \quad & W^\pm_\mu \xrightarrow{CP} -W^{\mp\mu} \end{aligned} \tag{2.33}$$

Considering the quark part of the Charged Current Lagrangian in the mass basis, we have:

$$\mathcal{L}^q_{CC} = -\frac{g}{\sqrt{2}}\left[\bar{u}^i_L\gamma^\mu V^{CKM}_{ij} d^j_L W^+_\mu + \bar{d_L}^i\gamma^\mu V^{\dagger CKM}_{ij} u^j_L W^-_\mu\right], \tag{2.34}$$

Applying CP on $\mathcal{L}^q_{CC}$ we thus obtain the following Lagrangian:

$$\mathcal{L}'^q_{CC} = -\frac{g}{\sqrt{2}}\left[\bar{d}^j_L\gamma_\mu V^{CKM}_{ij} u^i_L W^{-\mu} + \bar{u}^j_L\gamma_\mu V^{CKM*}_{ji} d^i_L W^{+\mu}\right], \tag{2.35}$$

We note that CP operation interchanges the two terms of $\mathcal{L}^q_{CC}$ and $\mathcal{L}'^q_{CC}$ except for the fact that V_{ij} and V^*_{ij} are not interchanged. Thus CP is a good symmetry of the electroweak sector only if there is a choice of phase convention where all the couplings are real.
CP would not necessarily be violated in the three generations SM: for instance if two quarks of the same charge had the same mass, one mixing angle and the phase could be removed from CKM mixing matrix.
This happens because in this case we'd have an extra symmetry in the model, i.e. unitary transformations in the space spanned by the hypotetical degenerate mass quarks. Taking for example b and s quarks to be mass degenerate, we could build an s' quark proportional to the linear combination $V_{us}s + V_{ub}b$, whereby the up quark would couple only to two down-type quarks (d and s') and not to the orthogonal combination b'. Thus the element $V_{ub'}$ of our new CKM matrix would be zero and by use of unitarity we could show that in this case:

$$V = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta\cos\phi & \cos\theta\cos\phi & \sin\phi \\ \sin\theta\sin\phi & -\cos\theta\sin\phi & \cos\phi \end{pmatrix} \tag{2.36}$$

This matrix leads to a CP conserving theory because it's real and obviously this argument holds for any pair of up-type or down-type quarks. So we have six necessary but not sufficient conditions for CP violation in the EW sector:

$$\begin{aligned} &m_u \neq m_c && m_c \neq m_t && m_t \neq m_u \\ &m_d \neq m_s && m_s \neq m_b && m_b \neq m_d \end{aligned} \tag{2.37}$$

Likewise, if the value of any of the three mixing angle were 0 or $\pi/2$, then the phase could be removed. This can be checked writing the CKM matrix as product of three rotation matrices, one of them involving a phase: this phase could be absorbed by properly redefining the quark fields.
Therefore there are 8 necessary conditions on the angles and the phase of the CKM matrix, in order to have CP violation, i.e.

$$\theta_i \neq 0, \pi/2 \qquad \delta \neq 0, \pi \qquad j = 1, 2, 3 \tag{2.38}$$

Thus there are altogether 14 necessary conditions to have CP violation in the EW sector of the SM with three fermion generations.
These conditions can be incorporated into one parametrization independent constraint. To find this condition, one notes that as a consequence of the unitarity of CKM matrix, for any choice of *i,j,k,l*=1,2,3 we have:

$$Im[V_{ij} V_{kl} V^*_{il} V^*_{kj}] = J \sum_{m,n=1}^{3} \epsilon_{ikm} \epsilon_{jln} \tag{2.39}$$

where the quantity J is called *Jarkslog invariant* [31]. One actually observes that J is phase convention independent, since, rephasing one of the quarks and using the fact that each quark field appears twice, once in a complex conjugated matrix element, while the other one not, then we see that J does not change.
We now define the quark mass matrices m_u and m_d associated with up-type and down-type quarks respectively:

$$\begin{aligned} m_u &= -\frac{v}{\sqrt{2}} Y_u \\ m_d &= -\frac{v}{\sqrt{2}} Y_d \end{aligned} \tag{2.40}$$

Finally we can unify the 14 conditions for CP violation within the single relation

$$\text{Det } \mathbf{C} \neq 0 \tag{2.41}$$

where $\mathbf{C}$ is given by this commutator of the square of the quark mass matrices:

$$i\mathbf{C} = [m_u m_u^\dagger, m_d m_d^\dagger] \tag{2.42}$$

What is remarkable of this commutator is that its determinant is given by:

$$\text{Det } \mathbf{C} = -2J(m_t^2 - m_c^2)(m_c^2 - m_u^2)(m_u^2 - m_t^2)(m_b^2 - m_s^2)(m_s^2 - m_d^2)(m_d^2 - m_b^2) \tag{2.43}$$

Now we can easily see why this is a condition on CP violation in the EW sector of the SM: if the determinant is not zero then the quark mixing matrix cannot be made real by rephasing the quark fields, because J can't be zero, therefore CP is violated.

2.2.2 Unitarity of the CKM matrix

The unitarity of CKM matrix is manifest using an explicit parametrization. Obviously, as CKM can be obtained composing three rotation matrices, the parametrization is not unique, but the most used is the so called standard parametrization.
The standard parametrization of CKM matrix is given by [28]:

$$V_{CKM} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -s_{23}c_{12} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \qquad (2.44)$$

where $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$ (i,j=1,2,3) and δ the phase necessary for CP violation. Their values are approximately $s_{12} \sim 0.22$, $s_{23} \sim 0.042$, $s_{13} \sim 0.0041$ and $\delta \sim 69°$.
Consequently, to a good accuracy, $c_{13} = c_{23} = 1$ and the set of four independent parameters is given by:

$$s_{12} = |V_{us}|, \quad s_{13} = |V_{ub}|, \quad s_{23} = |V_{cb}|, \quad \delta \qquad (2.45)$$

The main phenomenological advantages of the standard parametrization over the other proposed in literature are basically these two:

- s_{12}, s_{13} and s_{23} are respectively related to $|V_{us}|$, $|V_{ub}|$ and $|V_{cb}|$, which can be measured independently in meson decays.
- The CP violating phase δ is multiplied by the very small angle s_{13}. This highlights the suppression of CP violation independently of the actual value of δ.

This parametrization is suitable for numerical calculations, however it is useful to make a change of parameters in order to see the hierarchical structure of CKM matrix more transparently. This scope is realized by the Wolfenstein parametrization [29], which is an approximated one, where each element is expanded as a power series in the small parameter $\lambda = |V_{us}| \sim 0.22$

$$V_{CKM} = \begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4) \qquad (2.46)$$

and so the four free parameters in this parametrization of CKM matrix are λ, A, ρ and η. Since λ is small, then it's sufficient to keep only the first few terms in this expansion.
In literature these redefinitions of the last two parameters are often used:

$$\bar{\rho} = \rho\big(1 - \frac{\lambda^2}{2}\big) \qquad \bar{\eta} = \eta\big(1 - \frac{\lambda^2}{2}\big) \qquad (2.47)$$

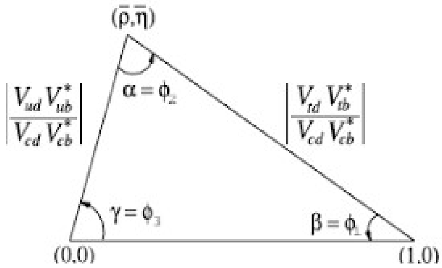


Figure 2.1: The rescaled Unitarity Triangle, all sides are divided by $V_{cd}V_{cb}^*$

the reason being convenience when extending the Wolfenstein parametrization to higher orders. The unitarity of CKM matrix implies various relations between its elements. In particular among these identities:

$$\sum_{k=1,2,3} V_{ki}V_{kj}^* = 0 \tag{2.48}$$

we focus on the one obtained for $i = 1$ and $j = 3$, namely

$$\begin{aligned} V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0 \\ \frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*} + \frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*} + 1 = 0 \end{aligned} \tag{2.49}$$

which is the so called "Unitarity Triangle", as it can be represented by a triangle in the complex plane.
This relation is very interesting, since it involves the sum of three terms which are all of $\mathcal{O}(\lambda^3)$, therefore the sides of the triangle are all of the same order of magnitude and so this relation is easier to test experimentally. In particular the sides as well the angles α, β and γ are accessible in many flavour changing observables involving CP asymmetries in various B meson decays.
In fact, it can be proved that the area of the unitarity triangle is related to CP violation via the Jarkslog invariant:

$$|J|/2 = A_\Delta, \tag{2.50}$$

where A_Δ denotes the area of the unitarity triangle. Moreover, since CKM matrix depends only on four parameters, we need four independent physical observables to completely determine it. After this procedure the theory becomes predictive and any other observable can be theoretically evaluated and compared with the experiments.
The numerical values of λ and A are known quite accurately from respectively $k \to \pi l \nu$ and $b \to c l \nu$ decays [30]:

$$\lambda = 0.22506 \pm 0.00050 \qquad A = 0.811 \pm 0.026 \tag{2.51}$$

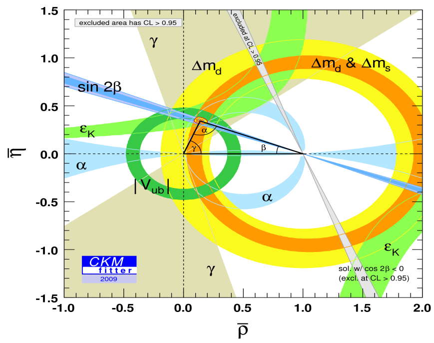


Figure 2.2: Constraints on the $\bar{\rho}$, $\bar{\eta}$. The shaded area have 95% CL. The individual constraints come from mass difference in B_d (Δm_d) and B_s (Δm_s), CP violation in neutral kaon system (ϵ_k) and in B_d system(sin 2β) , charmless semileptonic decays ($|V_{ub}|$) and from various B decays α and γ

We can then express all the relevant observables as a function of the two remaining parameters, namely ρ and η, and check if there is a region in the (ρ, η) plane which is consistent with all measurements.

The resulting constraints are shown in Figure 2.2. The agreement between theory and experiment is impressive, in particular this range of parameters can account for all measurements:

$$\bar{\rho} = 0.124^{+0.019}_{-0.018} \qquad \bar{\eta} = 0.356 \pm 0.011 \tag{2.52}$$

We can thus conclude that CKM matrix is the dominant source of CP violation in flavour changing process.
A fascinating question is whether there is room for New Physics contributions (NP) compatible with experimental bounds. As we will discuss in the following section the most promising sector is the one of FCNC, that, since are tree-level forbidden in the SM, can put strong constraints on any NP model.

2.3 FCNC and GIM Mechanism in the SM

We saw in the previous section that the flavour diagonal structure of the basic vertices involving γ, Z and g forbids the appearance of FCNC processes at tree level. However, thanks to the flavour changing W vertex, we can construct one-loop and higher order diagrams mediating FCNC processes.
The fact that these processes take place at the loop level makes them very useful for testing the quantum structure of the theory and search physics beyond the SM.
One of the most studied processes involving FCNC processes is meson oscillation. It obviously involves only neutral mesons, which can mix with their antiparticles.

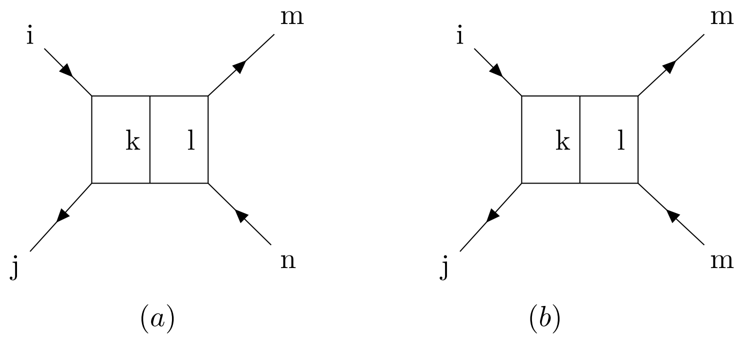


Figure 2.3: Box vertices

This can happen because neutral mesons are eigenstates of the strong and of the electromagnetic part of $\mathcal{L}_{SM}$ and are so characterized by well defined flavour and electric charge.
However $\mathcal{L}_{CC}$ contains vertices in which the flavour of the involved fermions is not conserved. Therefore mesons are not eigenstates of the SM Hamiltonian in general. In particular, for electrically neutral mesons, meson-antimeson transitions are allowed.
Lowest order Feynman graphs involved in these processes are the so called "box diagrams", which contain four powers of the weak gauge coupling constant g.
These diagrams involve in general both quarks and leptons as depicted in Figure 2.3. In vertex a the flavour violation takes place on both sides of the box, thus allowing $\Delta F = 2$ transitions, where F denote the flavour involved (+1 for particles, -1 for antiparticles). $\Delta F = 2$ sector involves neutral meson oscillation, namely the $K^0 - \bar{K}^0$,$D^0 - \bar{D}^0$, $B^0 - \bar{B}^0$.
Instead, the $\Delta F = 1$ sector spans a wide range of decays, such as the radiative $b \to s\gamma$, the semileptonic $b \to sl^+l^-$ and the purely leptonic $B_s \to \mu^+\mu^-$.
All these processes get contribution by a set of triple effective vertices, denoted in literature as "penguin diagrams", which have the topology depicted in Figure 2.4. In addition to the penguin vertex, the box vertex may also contribute to $\Delta F = 1$ processes, like in purely leptonic decay.
Both box and penguin effective vertices depend on the masses of the internal quarks or leptons and consequently are calculable functions of $x_i = m_i^2/M_W^2$, where i is the flavour index of the fermion running in the loop.
A set of basic universal functions can be found, which govern the physics of all FCNC processes. These functions were calculated by various authors, in particular by Inami and Lim [33].

We consider as example the $B_d - \bar{B}_d$ oscillation and try to understand how the amplitude of this process is suppressed by the GIM Mechanism.
Let's focus on the diagram a in Figure 2.5 and compute its amplitude, setting external quarks at zero momentum, such that all the particles running in the

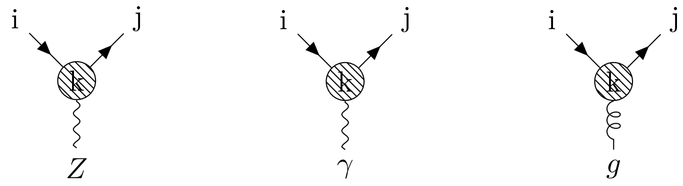


Figure 2.4: Penguin vertices

loop have the same momentum k.

$$A = \sum_{i,j=u,c,t} \frac{g^4}{4} \int \frac{d^4k}{(2\pi)^4} \frac{(-i)^2}{(k^2 - {M_W}^2)^2} \Big(g_{\mu\mu'} - \frac{k_\mu k_{\mu'}}{{M_W}^2}\Big)\Big(g_{\nu\nu'} - \frac{k_\nu k_{\nu'}}{{M_W}^2}\Big) \cdot \\ \big[\bar{u}_d \gamma^\mu P_L V^*_{id} \frac{-i}{\not{k} - m_i} V_{ib} \gamma^\nu P_L u_b\big] \big[\bar{v}_d \gamma^{\nu'} P_L V^*_{jd} \frac{-i}{\not{k} - m_j} V_{jb} \gamma^{\mu'} P_L v_b\big], \tag{2.53}$$

where $P_L = (1 - \gamma_5)/2$. Defining $\xi_i \equiv V^*_{id} V_{ib}$, as a consquence of CKM unitarity, we have $\xi_u = -(\xi_c + \xi_t)$.
From naive power counting we would expect A to be quadratically divergent, but this is not going to be the case, due to the unitarity of CKM matrix as we are going to see. Let's consider this combination in the first term between square brackets:

$$\sum_{i=u,c,t} \xi_i \gamma^\mu P_L \frac{1}{\not{k} - m_i} \gamma^\nu P_L = \sum_{i=u,c,t} \gamma^\mu \not{k} \frac{\xi_i}{k^2 - m_i^2} \gamma^\nu P_L \tag{2.54}$$

and using unitarity of the CKM we easiliy obtain:

$$\gamma^\mu \not{k} \Big[\xi_c \frac{m_c^2 - m_u^2}{(k^2 - m_c^2)(k^2 - m_u^2)} + \xi_t \frac{m_t^2 - m_u^2}{(k^2 - m_t^2)(k^2 - m_u^2)}\Big] \gamma^\nu P_L. \tag{2.55}$$

Therefore the first term between square brackets goes like $\frac{1}{k^3}$, for $k \to \infty$. An identical analysis holds also for the lower line of the diagram, so we clearly see from power counting that the amplitude is actually finite, as it behaves like:

$$\int \frac{d^4}{(2\pi)^4} \frac{1}{k^3 k^3} \to finite \tag{2.56}$$

Furthermore we note that that the amplitude would vanish if masses of up-type quarks were equal, as we see from Eq.(2.55). In this limit FCNC decays and transitions are absent. Thus, beyond tree level, the conditions for a complete GIM cancellation of FCNC processes are:

- Unitarity of CKM matrix
- Exact horizontal flavour symmetry which assures the equality of quark masses of a given charge.

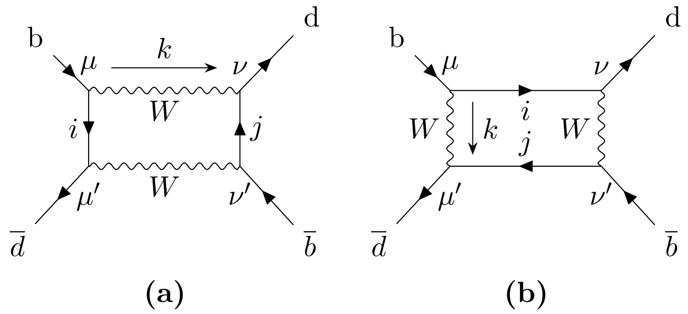


Figure 2.5: 1 loop box-diagrams for $B_d - \bar{B}_d$ oscillation

We emphasize that such a horizontal symmetry is very natural, as all the quantum numbers of fermions of a given charge are equal in the SM. However, such a horizontal symmetry is broken by the disparity of quark masses. This is in fact the origin of the breakdown of GIM Mechanism at the one-loop level and the appearance of FCNC transitions.

The size of this breakdown, and thus the strength of FCNC transitions, is controlled by the disparity of masses and can be affected by QCD corrections, which are the dominant ones at the $\mathcal{O}(m_b)$ scale.

Chapter 3

Effective field theory in flavour physics

In order to detect possible NP effects contributing to FCNC processes, we will focus mainly on meson decays, which take place at energy of $\mathcal{O}(1\text{Gev})$, namely the mass of the decaying particle.
The idea is to build a NP model, which generically will be defined by a Lagrangian Density of the form $\mathcal{L}_{NP} = \mathcal{L}_{SM} + \mathcal{L}_{BSM}$, where by BSM we mean any kind of physics beyond the SM. Then we should select a physical observable whose experimental knowledge and SM theoretical estimation allow us to put constraints on the BSM contribution to that observable.
We can argue why the study of SM suppressed processes, such as FCNC, is a very powerful tool to test BSM physics. NP models have to be compatible with observed physical quantities, in particular testing the SM suppressed ones can put strong constraints on the parameters of the BSM theory. Instead, testing a SM tree level observable is less useful, because the BSM contribution has to be subdominant and we should compute the SM value of that observable with many loops precision to put bounds on the BSM theory, which is a highly non trivial task.
We have at our disposal two main ways to describe NP effects in flavour physics: (i) to build an explicit UV completion of the SM; (ii) to analyze NP using an EFT approach. The former approach is more predictive, but also more model dependent, since we should completely specify the new fields at the UV scale besides SM ones. Instead, the EFT approach can provide a general description of NP at low energy, but has the drawback of shedding less light on NP scenarios that could arise at high energies, as it depends on a higher number of unknown parameters. We are going to describe EFT approach in detail in this section.

3.1 Effective field theories

The basic concept of EFT is to assume that heavy fields are no more dynamical degrees of freedom at low energy, so that we can remove them from the Lagrangian (we call this procedure ”integrating-out”) by using their equations of motion. The effect of this procedure is the appearance of new non renormalizable ”effective” vertices between the low energy fields. We can then parametrize NP contributions using the coefficients associated with these new Lagrangian terms:

$$\mathcal{L}_{EFT} = \mathcal{L}_{SM} + \sum_i C_i^{(d)} Q_i^{(d)}, \tag{3.1}$$

Since $\mathcal{L}_{SM}$ is a renormalizable Lagrangian, then the operators $Q_i^{(d)}$ should have $d > 4$ and so $\mathcal{L}_{EFT}$ is then not renormalizable.
We refer to the coefficients $C_i^{(d)}$ associated to the operators $Q_i^{(d)}$ as Wilson coefficients, whose mass dimension is then $4 - d$. The series of operators weighted by relative coefficients is known as Operator Product Expansion (OPE) and it can be interpreted as a series of effective vertices $Q_i^{(d)}$ multiplied by effective coupling constants $C_i^{(d)}$.
In general we will have an infinite number of these operators, but we can establish a hierarchy among them using dimensional analysis. Calling Λ the mass scale of NP generating the effective operator $Q_i^{(d)}$, we can rewrite its Wilson coefficient in this way:

$$C_i^{(d)} = \frac{c_i}{\Lambda^{d-4}} \tag{3.2}$$

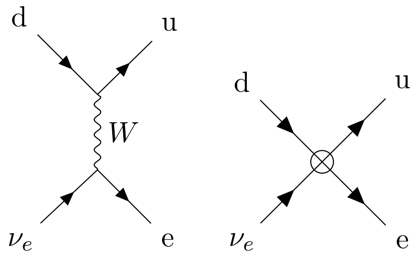
where c_i is a dimensionless coefficient. Therefore we can rewrite $\mathcal{L}_{EFT}$ as:

$$\mathcal{L}_{EFT} = \mathcal{L}_{SM} + \sum_i \frac{c_i}{\Lambda^{d-4}} Q_i^{(d)}. \tag{3.3}$$

The magnitude of each of operator in the expansion can be estimated by dimensional analysis as:

$$\int d^4x \frac{c_i}{\Lambda^{d-4}} Q_i^{(d)} \sim \left(\frac{E}{\Lambda}\right)^{d-4}, \tag{3.4}$$

so the higher the dimension of an effective operator, the more it is suppressed at low energies, where the EFT is valid. We commonly refer to these $d > 4$ operators as *irrelevant*, while $d = 4$ and $d < 4$ operators, which appear in $\mathcal{L}_{SM}$, are denoted respectively as *marginal* and *relevant*. Therefore we can restrict ourselves to the study of lowest dimensional operators contributing to the process of our interest, since they produce the largest corrections to the SM predictions.
Already in the SM framework, where we know the UV renormalizable theory, it can be useful to adopt an EFT approach. In this case the main advantage is a technical one, in the sense that computations are simpler with respect to the full theory. The most striking example of this feature is represented by weak decays



(a) **(b)**

Figure 3.1: β-decay at tree level in the SM (a) and in the Fermi (b) theory

of hadrons, such as the β decay, where we have two very separated energy scales at work: the W boson scale, $\mathcal{O}(80\text{ GeV})$, which mediates the process, and the mass scale of the decaying particle, $\mathcal{O}(1\text{ GeV})$. According to EFT, since we are considering a low energy process ($E << M_W$), we can integrate out $W^\pm$ bosons from the SM Lagrangian and compute observables within the new EFT, which is the Fermi theory of weak interactions, where the charged current interaction of fermions with W boson is now replaced by an effective four fermions interaction:

$$\mathcal{L}_{EFT} = -\frac{4G_F}{\sqrt{2}} J_\mu^+ J^{-\mu}, \tag{3.5}$$

where

$$J_\mu^+ = \bar{u}_{Li}\gamma_\mu V_{ij}^{CKM} d_{Lj} \tag{3.6}$$

and G_F is the Fermi constant, which has been measured by muon lifetime ($G_F = 1.16638 \cdot 10^{-5}\text{ GeV}^{-2}$). Thus, at tree level, EFT provides a simpler framework to perform computations. Things become more involved at the loop level, where we should take into account QCD corrections, which are the dominant ones at the hadron mass scales.
However, due to the asymptotic freedom of QCD, short distance corrections, namely contributions of hard gluons at the energy scales $\mathcal{O}(M_W)$, down to hadronic scales $\mathcal{O}(1\text{ GeV})$ can be treated in perturbation theory using renormalization group (RG) methods as we will discuss soon.
Our first task is the computation of the Wilson coefficients C_i in the ordinary perturbation theory. This can be achieved by the requirement that the amplitude A_{full} in the full theory should be reproduced in the effective theory. This procedure is called *matching* and consists generally of the following steps:

- Find all possible gauge-invariant operators of a given dimension allowed by symmetries and quantum numbers of a given physical process. We will consider up to $d = 6$ operators, since increasing their dimension calculations become more and more involved and less physically relevant.

- Write down the OPE with undetermined Wilson coefficients, namely:

$$\mathcal{L}_{EFT} = \sum_{i,d\leq 6} C_i Q_i^{(d\leq 6)} \tag{3.7}$$

- Determine the values of Wilson coefficients C_i such that:

$$A_{full} =< f|\mathcal{L}_{SM}|i >= A_{eff} = \sum_i C_i < f|Q_i|i > + \text{ higher order terms,} \tag{3.8}$$

by computing the amplitude for a given process both in the full and in the effective theory and keeping only leading order terms in the former, in order to perform a consistent matching.

3.1.1 Operator product expansion in the Standard Model

Let's discuss more in detail the structure of the OPE: actually we have been a little sloppy on the notation we used when we introduced it, to be more precise we should observe that both C_i and Q_i depend on a renormalization scale μ, such that the OPE assumes this form:

$$\sum_i C_i(\mu) Q_i(\mu). \tag{3.9}$$

The key point is that $C_i(\mu)$ summarize the physics contributions from scales higher than μ and, due to asymptotic freedom of QCD, they can be calculated perturbatively as long as μ is not to small. $C_i(\mu)$ will depend on high energy degrees of freedom, such as W,Z, Higgs boson and top quark. This dependence can be found evaluating loop diagrams with heavy particle exchanges and properly including short distance QCD effects.
We note that the value of μ can be chosen arbitrarily, as μ separates the contributions to a given amplitude into short-distance contributions at scales higher than μ and long-distance ones at scales lower than μ. For our scope we will choose μ to be of the scale of the decaying hadron mass, so $\mathcal{O}(m_b)$ in B meson decay for instance.
Indeed, very different energy scales are involved in the process, as $\mu << M_W, m_t$ and, as a consequence, large logarithms $\log M_W/\mu$ appear in the amplitude and they compensate the smallness of the QCD coupling constant α_s in the evaluation of the $C_i(\mu)$. Therefore the naive perturbation theory breaks down and terms $\alpha_s^n(\log M_W/\mu)^n$, $\alpha_s^n(\log M_W/\mu)^{n-1}$ etc. have to be resummed to all orders in α_s before a reliable result for C_i is obtained.
The solution to this problem is employing RG method. The renormalization group equations (RGEs) describe the change of renormalized quantities, such as Green functions and parameters, with the renormalization scale μ. Solving at leading order RGEs equations allows to sum up these large logarithms to all

orders in perturbation theory, and enables us to restore a perturbative approach, the so called RG improved perturbation theory.
Therefore our strategy in order to compute explicitly the Wilson coefficients is to perform the matching between the UV theory and the EFT at the cutoff scale, that is $\mathcal{O}(M_W)$. In this way we avoid the appearance of large logarithms and ordinary perturbation theory can be applied. Only after that the matching has been performed at high energy we'll use RGEs to run the C_i from high energy to low energy, when they can be used to compute physical observables of our interest. This last step amounts to sum up all the contributions of the kind $\alpha_s^i(\log M_W/\mu)^n$, $\alpha_s^n(\log M_W/\mu)^n$, to all orders in i and for $n \leq i$.
For simplicity we will compute only leading logarithms terms of the amplitudes and thus we will solve RGEs in the leading logarithmic approximation, which is equivalent to sum up $\alpha_s^n(\log M_W/\mu)^n$ terms.

3.2 Computation of Wilson Coefficients

To show explicitly how this procedure works, let's study in detail a simple example, namely the quark level decay $c \to s u \bar{d}$, neglecting higher order QCD effects for the moment. The tree-level amplitude is determined by the t-channel diagram with W boson exchange and is given by

$$\begin{aligned} iA &= -\frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud} \frac{M_W^2}{k^2 - M_W^2} (\bar{s}c)_{V-A} (\bar{u}d)_{V-A} \\ iA &= \frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud} (\bar{s}c)_{V-A} (\bar{u}d)_{V-A} + \mathcal{O}\Big(\frac{k^2}{{M_W}^2}\Big), \end{aligned} \tag{3.10}$$

where we defined $(\bar{s}c)_{V-A} = \bar{s}\gamma_\mu(1-\gamma_5)c$, and k is the transferred momentum through the W propagator. Since at low energy k^2 is very small as compared to M_W^2, then $\mathcal{O}(k^2/M_W^2)$ terms can be safely neglected and the full amplitude can be recast in this form:

$$iA = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud} (\bar{s}c)_{V-A} (\bar{u}d)_{V-A} + \text{Higher D Operators}, \tag{3.11}$$

where higher dimensional operators correspond to the terms of $\mathcal{O}(k^2/M_W^2)$. Neglecting these terms means neglecting higher dimensional operators, which is the core of EFT approach. In this simple example the amplitude in the effective theory is given by only one operator, namely $(\bar{s}c)_{V-A}(\bar{u}d)_{V-A}$ and its associated Wilson coefficient is simply 1, normalizing the amplitude as:

$$iA = \frac{G_F}{\sqrt{2}} \sum_i V_{CKM}^i C_i(\mu) Q_i(\mu) \tag{3.12}$$

Actually, in the EFT one can write another gauge invariant operator contributing to this process, with the same flavour but different colour structure. This

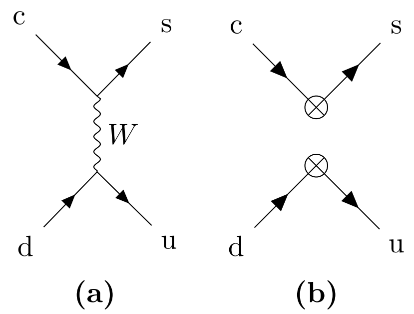


Figure 3.2: $c \to s u \bar{d}$ at the quark level in the SM (a) and in the effective (b) theory

operator is induced by higher order QCD effects and so it will have to be included in the low energy effective lagrangian, when considering also short-distance QCD effects.

3.2.1 OPE and short distance QCD corrections

We obtained for the $c \to s u \bar{d}$ transition, without QCD effects, the following amplitude in the EFT (Eq. 3.11):

$$iA^0_{eff} = \frac{G_F}{\sqrt{2}} V^*_{cs} V_{ud} (\bar{s}_\alpha c_\alpha)_{V-A} (\bar{u}_\beta d_\beta)_{V-A}, \tag{3.13}$$

where we are summing over colour indices.
Including QCD effects the effective amplitude is generalized to:

$$iA_{eff} = \frac{G_F}{\sqrt{2}} V^*_{cs} V_{ud} \big(C_1(\mu) Q_1(\mu) + C_2(\mu) Q_2(\mu) \big), \tag{3.14}$$

with

$$\begin{cases} Q_1 = (\bar{s}_\alpha c_\beta)_{V-A} (\bar{u}_\beta d_\alpha)_{V-A} \\ Q_2 = (\bar{s}_\alpha c_\alpha)_{V-A} (\bar{u}_\beta d_\beta)_{V-A} \end{cases} \tag{3.15}$$

The most striking feature of this amplitude is that in addition to the original operator Q_2, which was already present at tree level, a new operator Q_1 with same flavour, but different colour structure has been generated.
This happens, because, as we can see looking at the colour structure of diagrams (b) and (c) in Figure 3.3, due to the gluon exchange, these diagrams will contain the product of the colour charges $T^a_{\alpha\beta}$ and $T^a_{\gamma\sigma}$, which using colour algebra gives:

$$T^a_{\alpha\beta} T^a_{\gamma\sigma} = -\frac{1}{2N} \delta_{\alpha\beta} \delta_{\gamma\sigma} + \frac{1}{2} \delta_{\alpha\sigma} \delta_{\gamma\beta}. \tag{3.16}$$

The first term on the r.h.s of this equation gives a contribution to the operator Q_2, while the second term produces the new operator Q_1.

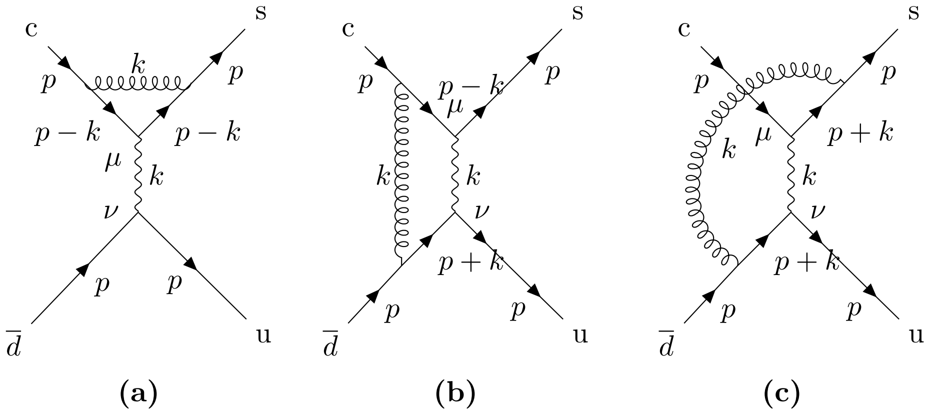


Figure 3.3: 1 loop current-current diagrams in the full theory

Including QCD corrections C_1 and C_2 become non trivial functions of the renormalization scale μ, α_s and M_W, while if QCD is neglected $C_1 = 0$, $C_2 = 1$ and we recover A^0_{eff}.

3.2.2 Calculation of the amplitude in the SM

First order QCD corrections to $c \to su\bar{d}$ in the SM are given by the set of diagrams in Figure 3.3. Our goal is to calculate the amplitude for this process at first order both in the SM and in the EFT and then perform the matching in order to extract Wilson coefficients C_1 and C_2.
In the computation we will keep only $\sim \alpha_s{\cdot}\log$ terms and neglect constant contributions of $\mathcal{O}(\alpha_s)$, which amounts to the leading log approximation (LO). As a side remark, we will not consider QED corrections since $\alpha_{EM}/\alpha_s \sim 10^{-2}$. However, if our goal were performing the matching at NLO in QCD, then LO term in QED would be a competitive effect and it would not be negligible anymore. Let's start with the computation of diagram (a), which is the simplest one, since only operator Q_2 contributes to the amplitude and only three particles are running in the loop.

$$A = -\frac{g^2}{8}(ig_s)^2 V_{ud}V^*_{cs}\mu^{\epsilon}\int\frac{d^Dk}{(2\pi)^D}\left[\bar{s}_\beta\gamma^\rho T^a_{\beta\alpha} i\frac{\not{p}-\not{k}}{(p-k)^2}\gamma^\mu(1-\gamma_5) i\frac{\not{p}-\not{k}}{(p-k)^2}T^a_{\alpha\gamma}\gamma^\sigma c_\gamma\right] \\ \frac{-ig_{\rho\sigma}}{k^2}\frac{ig_{\mu\nu}}{M_W^2}\left[\bar{u}_\delta\gamma^\nu(1-\gamma_5)d_\delta\right] \tag{3.17}$$

In this computation, since we have to deal with a divergent diagram, we have to regularize it and we use the Dimensional Regularization (DR) approach. In order to have a dimensionless coupling constant in D dimension we performed the replacement $g_s \to g_s\mu^{\epsilon/2}$, where μ is a renormalization scale. Furthermore

we are considering an equal momentum p for all the external legs and we set all quark masses to zero.
Using the definition of G_F in terms of g and M_W and using colour algebra we get to

$$A = -\frac{G_F}{\sqrt{2}} V_{ud} V_{cs}^* g_s^2 \mu^\epsilon C_F \int \frac{d^D k}{(2\pi)^D} \frac{\left[\bar{s}_\beta \gamma^\rho (\not{p} - \not{k}) \gamma^\mu (\not{p} - \not{k}) \gamma_\rho (1-\gamma_5) c_\beta\right]}{k^2((p-k)^2)^2} \left[\bar{u}_\delta \gamma_\mu (1-\gamma_5) d_\delta\right] \tag{3.18}$$

From Dirac gamma matrices properties in D dimensions we have that:

$$\gamma^\rho (\not{p} - \not{k}) \gamma^\mu (\not{p} - \not{k}) \gamma_\rho \overset{A.13}{=} (-2+\epsilon)(\not{p} - \not{k}) \gamma^\mu (\not{p} - \not{k}), \tag{3.19}$$

therefore we have

$$A = -\frac{G_F}{\sqrt{2}} V_{ud} V_{cs}^* g_s^2 \mu^\epsilon C_F (-2+\epsilon) \int \frac{d^D k}{(2\pi)^D} \frac{\left[\bar{s}_\beta (\not{p} - \not{k}) \gamma^\mu (\not{p} - \not{k}) (1-\gamma_5) c_\beta\right]}{k^2((p-k)^2)^2} \left[\bar{u}_\delta \gamma_\mu (1-\gamma_5) d_\delta\right] \tag{3.20}$$

From Dirac equation for a massless particle it follows that:

$$\begin{cases} \bar{s}_\beta \not{p} = 0 \\ \not{p} c_\beta = 0 \end{cases} \tag{3.21}$$

$$A = -\frac{G_F}{\sqrt{2}} V_{ud} V_{cs}^* g_s^2 \mu^\epsilon C_F (-2+\epsilon) \int \frac{d^D k}{(2\pi)^D} \frac{\left[\bar{s}_\beta \not{k} \gamma^\mu \not{k} (1-\gamma_5) c_\beta\right] \left[\bar{u}_\delta \gamma_\mu (1-\gamma_5) d_\delta\right]}{k^2((p-k)^2)^2} \tag{3.22}$$

We can now recast the denominator using the Feynman parametrization:

$$\frac{1}{A^2 B} = 2 \int_0^1 dx \frac{x}{[B + (A-B)x]^3} \tag{3.23}$$

We can eliminate linear terms from the denominator D by the change of variable

$$k \to k - px \tag{3.24}$$

Performing the change of variables also at numerator N and using again Dirac equation we get this expression for N/D

$$\frac{N}{D} = \frac{\left[\bar{s}_\beta \not{k} \gamma^\mu \not{k} (1-\gamma_5) c_\beta\right] \left[\bar{u}_\delta \gamma_\mu (1-\gamma_5) d_\delta\right]}{(k^2 - c)^3}, \tag{3.25}$$

where $c = -p^2 x(1-x)$. After this change of variables the loop integral has become an even function of k. Since in the integration we can substitute $k_\alpha k_\beta \to$

$\frac{k^2}{4} g_{\alpha\beta}$, then we just have to evaluate this integral over k, which can be performed using (A.5):

$$K_1 = \int \frac{d^D k}{(2\pi)^D} \frac{k^2}{(k^2 - c)^3} = i \frac{(4\pi)^{\epsilon/2}}{(4\pi)^2} \frac{2}{\epsilon} \left(1 - \frac{\epsilon}{2} \log c\right) \tag{3.26}$$

We use $\overline{MS}$ renormalization scheme, so we can forget about the finite part that arises along with the divergences in Feynman diagrams. We thus get, using again D dimension algebra of gamma matrices:

$$A = -\frac{G_F}{\sqrt{2}} V_{ud} V_{cs}^* g_s^2 \mu^\epsilon C_F 2(1-\epsilon) S_2 \int_0^1 dx \; x K_1 \tag{3.27}$$

Where $S_2 = (\bar{s}_\alpha c_\alpha)_{V-A} (\bar{u}_\beta d_\beta)_{V-A}$ is just the tree level matrix element of the operator Q_2. Plugging in the explicit form of K_1 we obtain:

$$A = -i \frac{G_F}{\sqrt{2}} V_{ud} V_{cs}^* \frac{\alpha_s}{4\pi} (4\pi\mu^2)^{\epsilon/2} \frac{4}{\epsilon} C_F (1-\epsilon) S_2 \int_0^1 dx x \left(1 - \frac{\epsilon}{2} \log c\right) \tag{3.28}$$

Multiplying this expression by 2, namely the symmetry factor coming from the diagram in which the gluon is exchanged between the lower external legs, we obtain the final expression for the amplitude of diagram (a) [32]:

$$iA = \frac{G_F}{\sqrt{2}} V_{ud} V_{cs}^* \frac{\alpha_s}{4\pi} 2 C_F \left(\frac{2}{\epsilon} + \log \frac{\mu^2}{-p^2}\right) S_2 \tag{3.29}$$

We have still to calculate diagrams (b) and (c), but as they have a similar topology the calculation is almost identical, so we will compute here just diagram (b), while diagram (c) will be computed in the appendix.
Actually (b) and (c) are both logarithmically divergent diagrams but their sum is finite, as their divergences cancel due to the different signs of momenta k running in the loop. Let's now compute the amplitude of (b):

$$\begin{aligned} B = -\frac{g^2}{8} (i g_s)^2 V_{ud} V_{cs}^* \mu^\epsilon \int \frac{d^D k}{(2\pi)^D} & \left[\bar{s}_\beta \gamma^\mu (1-\gamma_5) i \frac{\not{p} - \not{k}}{(p-k)^2} \gamma_\rho T^a_{\beta\alpha} c_\alpha\right] \cdot \\ \frac{-i g_{\rho\sigma}}{k^2} \frac{-i g_{\mu\nu}}{k^2 - M_W^2} & \left[\bar{u}_\delta \gamma^\nu (1-\gamma_5) i \frac{\not{p} + \not{k}}{(p+k)^2} \gamma_\sigma T^a_{\delta\gamma} d_\gamma\right] \end{aligned} \tag{3.30}$$

In this calculation we actually didn't use the full W boson propagator, namely $-i(g_{\mu\nu} - k_\mu k_\nu / M_W^2)/(k^2 - M_W^2)$ in the unitary gauge, but we neglected $k_\mu k_\nu / M_W^2$ term because, since the divergent term coming from this piece cancels out with the corresponding one from diagram (c), then only finite terms survive. Moreover, these terms are suppressed by a p^2/M_W^2 factor with respect to the ones coming from $g_{\mu\nu}$ piece of the propagator, therefore they can be safely neglected. Another reason to neglect such terms is that our goal is to perform the matching between SM and EFT at LO approximation, so keeping only operators of dimension 6. If we keep also $\mathcal{O}(\frac{p^2}{M_W^2})$ terms in the SM calculation, then we would have to take into account also higher dimensional operators in the EFT, otherwise the matching would be inconsistent. From colour algebra we have that:

$$T^a_{\beta\alpha}T^a_{\delta\gamma} = \frac{1}{2}\big(\delta_{\beta\gamma}\delta_{\alpha\delta} - \frac{1}{3}\delta_{\alpha\beta}\delta_{\delta\gamma}\big) \tag{3.31}$$

The first factor gives contribution to Q_1, while the second one to Q_2. We will keep only the latter in the following, as they differ just by a numerical constant. Since we are computing the leading term of the finite part we can set directly $D = 4$ as will not have to handle with divergent integrals. Thus we obtain

$$B^{(S_2)} = -\frac{G_F M_W^2}{6\sqrt{2}}V_{ud}V^*_{cs}g_s^2 \int \frac{d^4k}{(2\pi)^4}\frac{\big[\bar{s}_\alpha\gamma^\mu(1-\gamma_5)(\not{p}-\not{k})\gamma_\rho c_\alpha\big]\big[\bar{u}_\beta\gamma_\mu(1-\gamma_5)(\not{p}+\not{k})\gamma_\rho d_\beta\big]}{(p-k)^2k^2(k^2-M_W^2)(p+k)^2} \tag{3.32}$$

Since we are interested in the LO term, we keep only the quadratic term in k at numerator, as linear terms in k and with no power of k will either vanish or be suppressed by a p^2/M_W^2 factor.
Writing M_W^2 at numerator as $k^2 - (k^2 - M_W^2)$ we can reduce the number of factors at denominator, by splitting the integral in this way:

$$\begin{aligned}
&\int \frac{d^4k}{(2\pi)^4}\frac{\big[\bar{s}_\alpha\gamma^\mu(1-\gamma_5)\not{k}\gamma_\rho c_\alpha\big]\big[\bar{u}_\beta\gamma_\mu(1-\gamma_5)\not{k}\gamma_\rho d_\beta\big]}{(p+k)^2(p-k)^2(k^2-M_W^2)} && (1)\\
-&\int \frac{d^4k}{(2\pi)^4}\frac{\big[\bar{s}_\alpha\gamma^\mu(1-\gamma_5)\not{k}\gamma_\rho c_\alpha\big]\big[\bar{u}_\beta\gamma_\mu(1-\gamma_5)\not{k}\gamma_\rho d_\beta\big]}{(p+k)^2(p-k)^2k^2} && (2)
\end{aligned} \tag{3.33}$$

Let's compute first (1), substituting $k_\nu k_\sigma \to \frac{k^2}{4}g_{\nu\sigma}$:

$$\frac{1}{4}\big[\bar{s}_\alpha\gamma^\mu\gamma^\nu\gamma^\rho(1-\gamma_5)c_\alpha\big]\big[\bar{u}_\beta\gamma_\mu\gamma_\nu\gamma_\rho(1-\gamma_5)d_\beta\big]\int\frac{d^4k}{(2\pi)^4}\frac{k^2}{(p+k)^2(p-k)^2(k^2-M_W^2)} \tag{3.34}$$

Let's try to simplify the structure of gamma matrices in this equation: since any 4×4 matrix can be expanded on the 16 matrices Γ^a, providing the basis of Clifford algebra, namely $\{I,\ \gamma_\mu,\ \sigma_{\mu\nu},\ \gamma_5\gamma_\mu,\ \mathrm{i}\gamma_5\}$, then we can rewrite

$$\gamma^\mu\gamma^\nu\gamma^\rho = S^{\mu\nu\rho\sigma}\gamma_\sigma + i\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma_5 \tag{3.35}$$

where $S^{\mu\nu\rho\sigma}$ is the symmetric tensor

$$S^{\mu\nu\rho\sigma} = g^{\mu\nu}g^{\mu\sigma} + g^{\nu\rho}g^{\mu\nu} - g^{\mu\rho}g^{\nu\sigma} \tag{3.36}$$

Using this decomposition, we can simplify gamma matrices contractions in this way:

$$\big[\bar{s}_\alpha\gamma^\mu\gamma^\nu\gamma^\rho(1-\gamma_5)c_\alpha\big]\big[\bar{u}_\beta\gamma_\mu\gamma_\nu\gamma_\rho(1-\gamma_5)d_\beta\big] = 16S_2, \tag{3.37}$$

where the factor 16 is produced by the sum of a factor 10 associated to the contraction of symmetric tensor indices and a factor of 6 coming from contracting antisymmetric Levi-Civita tensor indices. Therefore (3.34) becomes

$$4S_2 \int \frac{d^4k}{(2\pi)^4} \frac{k^2}{(p+k)^2(p-k)^2(k^2-M_W^2)} \tag{3.38}$$

Now we can approximate in the denominator $(p-k)^2 \rightarrow k^2$ and $(p-k)^2 \rightarrow k^2$. This approximation is allowed since, due to the presence of M_W^2 in the third factor, IR divergences will not appear. This approximation has the great advantage of reducing the denominator to 2 factors, enabling us to use only one Feynman parameter in the computation of the integral. We thus get for (1)

$$4S_2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2(k^2-M_W^2)}. \tag{3.39}$$

Instead, we cannot apply the same trick in integral (2) at Eq. (3.33), since it would be otherwise logarithmically IR divergent, so we have to keep both $(p-k)^2$ and $(p+k)^2$ terms. The amplitude so reads

$$B^{(S_2)} = \frac{4G_F}{6\sqrt{2}} V_{ud}V_{cs}^* g_s^2 \big(I_1 - I_2\big) S_2, \tag{3.40}$$

where

$$I_1 = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2(k^2-M_W^2)} \qquad I_2 = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(p-k)^2(p+k)^2} \tag{3.41}$$

These integrals can be easily evaluated using (A.4), after simplifying the denominator with the following Feynman parametrization formula:

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[B+(A-B)x]^2} \tag{3.42}$$

The computations of I_1 and I_2 yield the following results:

$$I_1 = \frac{i}{(4\pi)^2} \int_0^1 dx \big[\frac{2}{\epsilon} - \log(M_W^2 x)\big] \qquad I_2 = \frac{i}{(4\pi)^2} \int_0^1 dx \big[\frac{2}{\epsilon} - \log(-4p^2 x(1-x))\big] \tag{3.43}$$

When combining them, the divergent parts obviously cancel, since the original integral in Eq. (3.32) was itself convergent and we finally obtain this expression for the amplitude associated with the operator Q_2, where we have multiplied by the symmetry factor of 2:

$$iB^{(S_2)} = \frac{4G_F}{\sqrt{2}} V_{ud}V_{cs}^* \frac{\alpha_s}{3(4\pi)} \Big[\log\frac{M_W^2}{-p^2} - (\log 4 - 1)\Big] S_2 \tag{3.44}$$

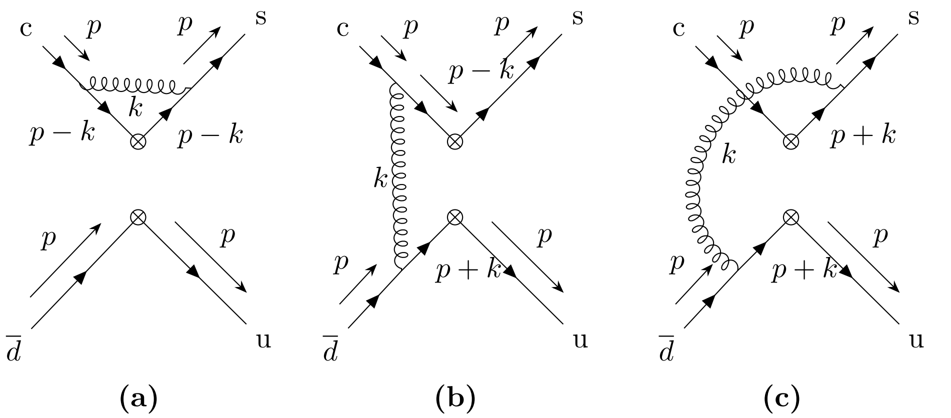


Figure 3.4: 1 loop current-current diagrams in the effective theory

Since our goal is to perform the matching in the LO approximation we keep only the dominant $\log\frac{M_W^2}{-p^2}$ term, neglecting constant contributions of $\mathcal{O}(\alpha_s)$.
The computation of diagram (c) is performed in the Appendix B. Here we only report the amplitude associated to the correction of S_2

$$iC^{(S_2)} = -\frac{G_F}{\sqrt{2}} V_{ud} V_{cs}^* \frac{\alpha_s}{3(4\pi)} \big(\log\frac{M_W^2}{-p^2} + 1\big) S_2 \tag{3.45}$$

In conclusion, summing up first order QCD corrections to the tree level result, we obtain the following expression for the SM amplitude at LO, in agreement with [32]:

$$iA_{full} = \frac{G_F}{\sqrt{2}} V_{ud} V_{cs}^* \Big[\big(1+2C_F\frac{\alpha_s}{4\pi}(\frac{2}{\epsilon}+\log\frac{\mu^2}{-p^2})\big)S_2+\frac{\alpha_s}{4\pi}\log\frac{M_W^2}{-p^2}S_2-3\frac{\alpha_s}{4\pi}\log\frac{M_W^2}{-p^2}S_1\Big] \tag{3.46}$$

3.2.3 Calculation of Q_1 and Q_2 in the EFT

The unrenormalized operators Q_1 and Q_2 are found at $\mathcal{O}(\alpha_s)$ by calculating the diagrams in Figure 3.4 and their symmetric counterparts. In this case the computations are much easier, therefore we will compute here only the contribution to Q_2 due to diagram (c), while other computations are really similar, therefore we omit them.

Let's now calculate the correction to the tree level operator ${Q_2}^{tree} = S_2$ given by diagram (c), where

$$S_2 = (\bar{s}_\alpha c_\alpha)_{V-A}(\bar{u}_\beta d_\beta)_{V-A}. \tag{3.47}$$

The amplitude of diagram (a) with the insertion of operator Q_2 reads:

$$Q_2^{(c)} = -\frac{g^2}{8}(ig_s)^2 V_{ud}V_{cs}^* \mu^\epsilon \int \frac{d^D k}{(2\pi)^D}\big[\bar{s}_\beta \gamma^\rho T^a_{\beta\alpha} i\frac{\not{p}+\not{k}}{(p+k)^2}\gamma^\mu(1-\gamma_5)c_\alpha\big]\cdot \\ \frac{-ig_{\rho\sigma}}{k^2}\frac{ig_{\mu\nu}}{M_W^2}\big[\bar{u}_\gamma \gamma^\nu (1-\gamma_5) i\frac{\not{p}+\not{k}}{(p+k)^2}T^a_{\gamma\delta}\gamma^\sigma d_\delta\big] \tag{3.48}$$

The product of colour charges gives:

$$T^a_{\beta\alpha}T^a_{\gamma\delta} = \frac{1}{2}\big(\delta_{\beta\delta}\delta_{\alpha\gamma} - \frac{1}{3}\delta_{\alpha\beta}\delta_{\delta\gamma}\big) \tag{3.49}$$

The first factor gives a correction to the tree-level operator S_1, while the second one to S_2. We carry only the latter in the following calculation. We thus obtain:

$$Q_2^{(c)} = \frac{G_F}{6\sqrt{2}} g_s^2 V_{ud}V_{cs}^* \mu^\epsilon \int \frac{d^D k}{(2\pi)^D}\frac{\big[\bar{s}_\beta \gamma^\rho (p \not{+} k)\gamma^\mu (1-\gamma_5)c_\beta\big]\big[\bar{u}_\alpha \gamma^\mu (p \not{+} k)\gamma^\rho (1-\gamma_5) d_\alpha\big]}{((p+k)^2)^2 k^2} \tag{3.50}$$

Making the shift $k \to (k+p)$ we can simplify the numerator, obtaining:

$$Q_2^{(c)} = \frac{G_F}{6\sqrt{2}} g_s^2 V_{ud}V_{cs}^* \mu^\epsilon \int \frac{d^D k}{(2\pi)^D}\frac{\big[\bar{s}_\beta \gamma^\rho \not{k}\gamma^\mu (1-\gamma_5)c_\beta\big]\big[\bar{u}_\alpha \gamma^\mu \not{k}\gamma^\rho (1-\gamma_5) d_\alpha\big]}{(k-p)^2 k^4} \tag{3.51}$$

Exploiting again the property $k_\nu k_\sigma \to \frac{k^2}{4} g_{\nu\sigma}$ under integration and projecting, as seen before, the structure of gamma matrices onto the basis of Clifford algebra we obtain:

$$Q_2^{(c)} = \frac{G_F}{6\sqrt{2}} g_s^2 V_{ud}V_{cs}^* \mu^\epsilon S_2 K_2 \quad , \tag{3.52}$$

where

$$K_2 = \int \frac{d^D k}{(2\pi)^D}\frac{1}{k^2 (k-p)^2}. \tag{3.53}$$

This integral can be performed easily using Feynman parametrization of (3.42) and (A.4). Reinserting tree level operator S_1, which we did not carry in this computation, we finally obtain the result for the correction to S_2 due to diagram (c):

$$Q_2^{(c)} = -\frac{G_F}{\sqrt{2}} V_{ud}V_{cs}^* \frac{\alpha_s}{3(4\pi)}\big[\frac{2}{\epsilon} + \log\frac{\mu^2}{-p^2}\big]\big(S_2 - 3S_1\big) \tag{3.54}$$

Computing the remaining diagrams, we get these expressions for the unrenormalized Q_1 and Q_2 at $\mathcal{O}(\alpha_s)$.

$$Q_1^0 = \big[1 + 2C_F \frac{\alpha_s}{4\pi}(\frac{2}{\epsilon} + \log\frac{\mu^2}{-p^2})\big]S_1 + \frac{\alpha_s}{4\pi}\big[\frac{2}{\epsilon} + \log\frac{\mu^2}{-p^2}\big]S_1 - 3\frac{\alpha_s}{4\pi}\big[\frac{2}{\epsilon} + \log\frac{\mu^2}{-p^2}\big]S_2 \tag{3.55}$$

$$Q_2^0 = \big[1 + 2C_F \frac{\alpha_s}{4\pi}(\frac{2}{\epsilon} + \log\frac{\mu^2}{-p^2})\big]S_2 + \frac{\alpha_s}{4\pi}\big[\frac{2}{\epsilon} + \log\frac{\mu^2}{-p^2}\big]S_2 - 3\frac{\alpha_s}{4\pi}\big[\frac{2}{\epsilon} + \log\frac{\mu^2}{-p^2}\big]S_1 \tag{3.56}$$

where the coefficient $\frac{G_F}{\sqrt{2}} V_{ud} V_{cs}^*$ has been factorized from the definition of the operators.
It can be proved that divergences appearing in the first term of both equations can be removed through quark field renormalization, namely the bare field $q^{(0)}$ is renormalized according to:

$$q^{(0)} = Z_q^{1/2} q, \tag{3.57}$$

where $Z_q^{1/2}$ is the renormalization constant and q is the physical, i.e. renormalized field.
However, in contrast to the full amplitude, the resulting expressions are still divergent after quark field renormalization. This is because EFT is no more a renormalizable theory and therefore the standard renormalization procedure which works for the SM does not completely eliminate the divergences from the physical amplitude of the EFT.
Thus, to remove the additional divergences, a further renormalization, known as *operator renormalization* becomes necessary:

$$Q_i^{(0)} = Z_{ij} Q_j, \tag{3.58}$$

where the renormalization constant is the 2×2 matrix Z. This means that in general $Q_1^{(0)}$ and $Q_2^{(0)}$ are not renormalized independently from each other, namely they mix under renormalization.
We stress that, although operator renormalization seems a new concept, it can be described in terms of the completely equivalent renormalization of the Wilson coefficients C_i. This can be seen explicitly from the effective lagrangian, which can be written as:

$$C_i^{(0)} Q_i^{(0)} = Z_{ij} C_i Q_j, \tag{3.59}$$

and we can think of Z as the result of the renormalization procedure

$$C_i^{(0)} = Z_{ji} C_j \tag{3.60}$$

carried on the Wilson coefficients.
In our case we easily obtain from (3.55) and (3.56) the following renormalization matrix Z:

$$Z = 1 + \frac{2\alpha_s}{4\pi\epsilon} \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix} \tag{3.61}$$

Thus we finally obtain the renormalized operators Q_i, which are given by

$$\begin{cases} Q_1 = \left(1 + 2C_F \frac{\alpha_s}{4\pi} \log \frac{\mu^2}{-p^2}\right) S_1 + \frac{\alpha_s}{4\pi} \log \frac{\mu^2}{-p^2} S_1 - 3\frac{\alpha_s}{4\pi} \log \frac{\mu^2}{-p^2} S_2 \\ Q_2 = \left(1 + 2C_F \frac{\alpha_s}{4\pi} \log \frac{\mu^2}{-p^2}\right) S_2 + \frac{\alpha_s}{4\pi} \log \frac{\mu^2}{-p^2} S_2 - 3\frac{\alpha_s}{4\pi} \log \frac{\mu^2}{-p^2} S_1 \end{cases} \tag{3.62}$$

3.2.4 Extraction of the Wilson Coefficients

After performing quark field renormalization also in the full theory, we can finally match $A_{full} = A_{eff}$ where, according to our convention on the normalization of Wilson coefficients,

$$iA_{eff} = \frac{G_F}{\sqrt{2}} V_{ud} V_{cs}^* \sum_i C_i Q_i \tag{3.63}$$

The matching gives the following result for the Wilson coefficients:

$$C_1(\mu) = -3\frac{\alpha_s}{4\pi}\log\frac{M_W^2}{\mu^2}, \qquad C_2(\mu) = 1 + \frac{\alpha_s}{4\pi}\log\frac{M_W^2}{\mu^2} \tag{3.64}$$

However, as anticipated in the previous section, this result can only be used for $\mu = \mathcal{O}(M_W)$, while for $\mu \ll M_W$ we have to sum the large logarithms to all orders of perturbation theory before we can trust our result for C_i, using RGEs. For the study of RG properties of Q_1 and Q_2 it is useful to go to a different operators basis, defined by

$$Q_\pm = \frac{Q_2 \pm Q_1}{2} \qquad C_\pm = C_2 \pm C_1. \tag{3.65}$$

In the new basis, $Q_\pm$ are renormalized independently of each other, via:

$$Q_\pm^{(0)} = Z_\pm Q_\pm, \qquad Z_\pm = 1 + \frac{\alpha_s}{4\pi\epsilon}(\mp 3 \mp 1) \tag{3.66}$$

From (3.62) and (3.64) we read the following values respectively for $Q_\pm$ and $C_\pm$:

$$Q_\pm = \big(1 + 2C_F\frac{\alpha_s}{4\pi}\log\frac{\mu^2}{-p^2}\big)S_\pm + (1 \mp 3)\frac{\alpha_s}{4\pi}\log\frac{\mu^2}{-p^2}S_\pm \tag{3.67}$$

$$C_\pm = 1 + (1 \mp 3)\frac{\alpha_s}{4\pi}\log\frac{M_W^2}{\mu^2}, \tag{3.68}$$

where $S_\pm = (S_2 \pm S_1)/2$. From equations (3.67) and (3.68) we can see explicitly one of the most important features of OPE, namely the factorization of the effective amplitude in short-distance (Wilson coefficients) and long-distance (operators) contributions. This factorization is achieved by this splitting of the integration over virtual momenta:

$$\int_{-p^2}^{M_W^2} \frac{dk^2}{k^2} = \int_{\mu^2}^{M_W^2} \frac{dk^2}{k^2} + \int_{-p^2}^{\mu^2} \frac{dk^2}{k^2}. \tag{3.69}$$

This last equation manifests the fact that Wilson coefficients contain contributions above $\mu = \mathcal{O}(m_b)$, up to $\mathcal{O}(M_W)$, whereas low energy contributions are encoded into $Q_\pm$ operators.

RGEs for $C_\pm$ follow from the fact that the bare coefficients $C_\pm^{(0)}$ do not depend on the renormalization scale μ. Then, recasting Eq. (3.60) properly adapted to the diagonal basis we have that:

$$C_{\pm} = Z_{\pm} C_{\pm}^{(0)}, \tag{3.70}$$

and differentiating this equation we get

$$\frac{dC_{\pm}(\mu)}{d\log(\mu)} = \gamma_{\pm}(g_s) C_{\pm}(\mu). \tag{3.71}$$

Here $\gamma_{\pm}$ is the anomalous dimension of the operator $Q_{\pm}$ and is given by

$$\gamma_{\pm}(g_s) = \frac{1}{Z_{\pm}} \frac{dZ_{\pm}}{d\log(\mu)}. \tag{3.72}$$

We note that without QCD loop corrections, the coupling $C_{\pm}$ would be independent on μ; therefore the non trivial μ dependence of the Wilson coefficients is a pure quantum effect, which implies an anomalous scaling behaviour of $C_{\pm}$. For this reason we refer to $\gamma_{\pm}$ as the anomalous dimension of $Q_{\pm}$.
The solution of RGEs equations (3.71) is given by:

$$C_{\pm}(\mu) = U_{\pm}(\mu, \mu_W) C_{\pm}(\mu_W), \tag{3.73}$$

where $\mu_W = \mathcal{O}(M_W)$ and $U_{\pm}(\mu, \mu_W)$ is the evolution function:

$$U_{\pm}(\mu, \mu_W) = exp\Big[\int_{g_s(\mu_W)}^{g_s(\mu)} dg_s' \frac{\gamma_{\pm}(g_s')}{\beta(g_s')}\Big], \tag{3.74}$$

where $\beta(g_s)$ is the QCD Beta function, which is given at 1 loop by:

$$\beta(g_s) = -\beta_0 g^3/16\pi^2 \qquad \beta_0 = 11 - \frac{2}{3}f, \tag{3.75}$$

with f the number of effective flavours present at the scale μ. Using the value of $Z_{\pm}$ computed in (3.66), the one loop anomalous dimension of $Q_{\pm}$ is given by:

$$\gamma_{\pm}(\alpha_s) = \frac{\alpha_s}{4\pi}\gamma_{\pm}^{(0)} \qquad \gamma_{\pm}^{(0)} = \pm 2(3 \mp 1). \tag{3.76}$$

In order to complete our calculation we choose $\mu = M_W$ and so, from (3.68) $C_{\pm}(M_W) = 1$, and consequently we get for $\mu = \mathcal{O}(m_b)$:

$$C_{\pm}(\mu_b) = \Big[\frac{\alpha_s(M_W)}{\alpha_s(\mu)}\Big]^{\frac{\gamma_{\pm}^{(0)}}{2\beta_0}} \tag{3.77}$$

We have now summed up over all leading logarithms and this equation gives the Wilson coefficients $C_{\pm}$ in leading log approximation. For instance, taking $f = 5$ and plugging in the above equation the numerical values of $\gamma_{\pm}^{(0)}$ and β_0 we find

$C_+(\mu_b) \sim 0.85$ and $C_-(\mu_b) \sim 1.40$, therefore we have an enhancement of C_- and a suppression of C_+ relative to the case without QCD corrections, in which $C_\pm = 1$.
To summarize, we have seen explicitly for a typical weak decay process, in the leading log approximation, that RG improved perturbation theory enables us to implement a perturbative approach in the framework of low energy EFT, which is technically useful, as computations are simpler than in the full SM theory.

Chapter 4

Strong CP problem and axions

4.1 The strong CP problem

The strong CP problem is a naturalness issue dealing with QCD: from a theoretical point of view we expect such an SU(3) gauge theory coupled to massive quarks to be generically CP violating, but no experimental evidence of CP violation in the strong interactions has ever been observed. In this section we discuss why this experimental fact poses a naturalness problem in QCD and how axions can solve this issue. The QCD Lagrangian density for N quark flavours reads

$$\mathcal{L}_{QCD} = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu a} + \sum_{i=1}^{6} \bar{q}^i \left(i \not{D} - m_i \right) q^i, \tag{4.1}$$

where q_i are the quark fields and $G^a_{\mu\nu}$ is the gluon field-strength tensor.
We can easily see that in the limit $m_i \to 0$, the Lagrangian has a global $U(6)_V \times U(6)_A$ global symmetry.
However, since only up and down quark masses are smaller than the typical scale of the formation of hadrons, that is $\Lambda_{QCD} \sim 200$ MeV, then we should expect only $U(2)_V \times U(2)_A$ to be a good approximate global symmetry of the strong interactions. We can decompose this symmetry as

$$U(2)_V \times U(2)_A = SU(2)_V \times SU(2)_A \times U(1)_V \times U(1)_A \tag{4.2}$$

The $U(1)_V$ subgroup is associated to baryon number conservation and is broken by anomaly, while $SU(2)_V$ is an approximated symmetry, due to the quarks mass term. The axial symmetry group $U(2)_A$, instead, is spontaneously broken by the non-vanishing value of the QCD quark condensate, $< q\bar{q} > \neq 0$.
According to the Goldstone theorem, we should have four pNGB in the particle spectrum, with small mass compared to Λ_{QCD}. Although the three pions (π^0, $\pi^{\pm}$) forming an $SU(2)$ triplet are light, and therefore are suitable to be considered

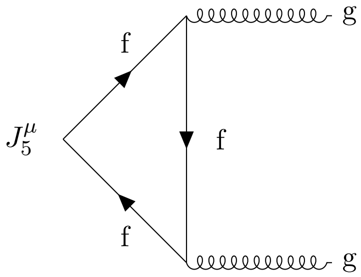


Figure 4.1: The triangle diagram responsible of the chiral anomaly. Here f is the fermion coupled to the gluons g. It can be shown that all the contribution to the anomaly is given by this diagram, so the anomaly is one loop exact, and thus is a non-perturbative effect

as the pNGBs of $SU(2)_A$ symmetry, the next particle in the hadron spectrum, namely the η meson, is much heavier, $m_\eta \sim$ 958 MeV, and so we cannot associate this particle to the $U(1)_A$ symmetry.
This is the $U(1)_A$ missing meson problem, namely the fact that we are still missing a pNGB in the particle spectrum.
The solution to this issue is that $U(1)_A$ is not a true symmetry at the quantum level, but is instead anomalous, so there is no approximate NGB associated with this chiral symmetry.
Although in the $m_{u,d} \to 0$ limit the Lagrangian is invariant under the $U(1)_A$ transformation

$$q_i \to e^{i\alpha\gamma_5/2} q_i \tag{4.3}$$

the associated current J_5^μ is not conserved even in the massless quark limit, but gets contribution from the triangle diagram and its divergence is given by

$$\partial_\mu J_5^\mu = \frac{g_s^2}{32\pi^2} G^a_{\mu\nu} \tilde{G}_a^{\mu\nu}, \tag{4.4}$$

where $\tilde{G}_{\mu\nu}$ is the dual field strength tensor.
Physical effects of the $G\tilde{G}$ term were neglected in early times because it turns out to be a total derivative,

$$G^a_{\mu\nu} \tilde{G}_a^{\mu\nu} = \partial_\mu K^\mu \tag{4.5}$$

where

$$K_\mu = \epsilon^{\mu\nu\rho\sigma} A^a_\nu \Big(G^a_{\rho\sigma} - \frac{2}{3} g_s f^{abc} A^b_\rho A^c_\sigma \Big) \tag{4.6}$$

By partial integration, the contribution to $G\tilde{G}$ is given by field configurations at infinity, which would not contribute to physical processes if the gauge fields went to zero at the boundary. However, Gerard 't Hooft realised that the QCD

vacuum is far from being trivial and there are actually topologically non-trivial field configurations, called instantons, that contribute to this operator, and thus it cannot be neglected [6].
These instantons are strongly related to the structure of the QCD vacuum, namely the different vacuum configurations of A^a_μ cannot be mapped to the trivial vacuum $A^a_\mu = 0$, but instead belong to different topological sectors. Since we have to take into account all the topologically different QCD vacua, in order to have a gauge invariant QCD vacuum, then we conclude that the total divergence is no more irrelevant and so the operator $G\tilde{G}$ cannot be neglected.
As a consequence, a $G\tilde{G}$ term must be admitted in the Lagrangian (4.1), because it is compatible with all symmetries of the SM gauge group and instanton configurations provide a non vanishing contribution to it. Thus, we are led to add the following term to the QCD Lagrangian, known as theta term:

$$\mathcal{L}_\theta = \frac{g_s^2}{32\pi^2} G^a_{\mu\nu} \tilde{G}^{\mu\nu}_a \theta \tag{4.7}$$

In addition to the QCD term, the θ parameter gets an additional contribution coming from the quark masses: in principle the Yukawa matrices are generic 3×3 complex matrices and, in order to go in the physical mass basis, we need to perform a $SU(N)$ chiral rotation of quarks. This step does not affect the theta term, since there is no anomaly associated to $SU(N)$. We are now left with diagonal but complex mass matrices M_u and M_d.

$$\mathcal{L}_{mass} = -m^i_u e^{-i\beta_i} \bar{u}^i_L u^i_R - m^j_d e^{-i\beta_j} \bar{d}^j_L d^i_R + h.c., \tag{4.8}$$

where $M^{ii}_u = -m^i_u e^{-i\beta_i}$ and $M^{jj}_d = -m^j_d e^{-i\beta_j}$. Their coefficients can be made real performing a $U(1)_A$ transformation (4.3) for each quark field, with angle $\alpha_i = \beta_i$, where β_i is the phase of the corresponding coefficient. As we have seen these transformations are anomalous and produce a contribution to the QCD theta parameter equal to $-\sum_i a_i = \sum_i \beta_i =$ Arg det$(M_u M_d)$. Therefore, the final coefficient of the $G\tilde{G}$ term is given by

$$\bar{\theta} = \theta + \text{Arg Det}(M_u M_d) \tag{4.9}$$

The theta term respects the symmetries of QCD, but violates time reversal and parity, while conserving charge conjugation, so it violates CP and induces a neutron electric dipole moment d_n whose upper limit is given by [5]:

$$d_n \sim \frac{e\theta m_q}{m_N^2} < 3 \cdot 10^{-26} e\text{cm} \tag{4.10}$$

The experimental bound on d_n implies a bound on the QCD θ parameter, $\theta < 10^{-9}$.
Here stands the strong CP problem, namely the fact that this parameter, in principle arbitrary, is unnaturally small. In fact, we notice that putting the θ

parameter to zero doesn't provide CP conservation in the SM, since it is already broken in the EW sector, so the symmetry of the Lagrangian does not increase and from the definition of Naturalness given by 't Hooft the theory is so not natural.

4.1.1 Peccei Quinn solution to strong CP problem

A natural solution to the strong CP problem is to enlarge the SM symmetry group with a global chiral U(1) symmetry, known as $U(1)_{PQ}$, for R. Peccei and H. Quinn [25].
This symmetry is spontaneously broken when one or more scalar fields take a vev and the associated GB is called Axion. We can ask ourselves how this could solve the strong CP problem. Basically, the idea is that by incorporating this symmetry in the theory one replaces the CP violating parameter θ with the dynamical CP conserving interactions of the axion field $a(x)$. Because the axion field is the NGB associated with the spontaneously broken PQ symmetry, it transforms as

$$a(x) \to a(x) + \alpha f_a, \tag{4.11}$$

where α is the parameter of the transformation and f_a the scale associated with symmetry breaking.
Therefore, if the effective Lagrangian $\mathcal{L}_{Eff}$ describing the full theory is $U(1)_{PQ}$ invariant, the axion field must only enter derivatively coupled. However, due to the chiral anomaly, $\mathcal{L}_{Eff}$ must also have a term in which the axion field couples directly to the gluon pseudoscalar density $G\tilde{G}$, to guarantee that current J^{μ}_{PQ} associated to $U(1)_{PQ}$ has a QCD chiral anomaly. The effective SM Lagrangian, endued with the extra $U(1)_{PQ}$ global symmetry is thus given by [15]

$$\mathcal{L}^{Eff} = \mathcal{L}_{SM} + \theta\frac{\alpha_s}{8\pi}G^a_{\mu\nu}\tilde{G}^{\mu\nu}_a + \frac{1}{2}\partial_\mu a\partial^\mu a + \mathcal{L}_{int}\left[\partial^\mu a/f_a, \psi\right] + \frac{a}{f_a}\xi\frac{\alpha_s}{8\pi}G^a_{\mu\nu}\tilde{G}^{\mu\nu}_a, \tag{4.12}$$

where ξ is a model dependent parameter associated with the anomaly of the $U(1)_{PQ}$ current

$$\partial_\mu J^\mu_{PQ} = \xi\frac{\alpha_s}{8\pi}G^a_{\mu\nu}\tilde{G}^{\mu\nu}_a \tag{4.13}$$

The presence of the last term in the above Lagrangian produces an effective potential for the axion field, thus its vev is no more arbitrary and is obtained by minimizing the effective potential.

$$\left\langle\frac{\partial V_{Eff}}{\partial a}\right\rangle = -\frac{\xi}{f_a}\frac{\alpha_s}{8\pi}\langle G^a_{\mu\nu}\tilde{G}^{\mu\nu}_a\rangle\big|_{<a>} = 0 \tag{4.14}$$

What Peccei and Quinn showed is that the periodicity of $\langle G\tilde{G}\rangle$ in the relevant theta parameter $(\theta + \xi\langle a\rangle/f_a)$ forces the axion to take the vev

$$\langle a\rangle = -\frac{f_a}{\xi}\theta \tag{4.15}$$

This solves the strong CP problem, since expressing $\mathcal{L}_{SM}^{Eff}$ in terms of the physical axion field, $a_{ph} = a - \langle a \rangle$, it does no longer contain the CP violating term. Furthermore, the effective potential, breaking explicitly $U(1)_{PQ}$ symmetry, provides the axion a mass term:

$$m_a^2 = \Big\langle \frac{\partial^2 V_{Eff}}{\partial a^2} \Big\rangle = -\frac{\xi}{f_a}\frac{\alpha_s}{8\pi}\frac{\partial}{\partial a}\langle G^a_{\mu\nu}\tilde{G}^{\mu\nu}_a \rangle|_{<a>} \tag{4.16}$$

Therefore, the SM with an additional $U(1)_{PQ}$ symmetry no longer has a QCD CP violating interaction. Instead, it contains additional interactions of a massive axion field both with matter and gluons characterized by a scale f_a:

$$\mathcal{L}^{Eff} = \mathcal{L}_{SM} + \frac{1}{2}\partial_\mu a_{ph}\partial^\mu a_{ph} - \frac{1}{2}m_a^2 a_{ph}^2 + \mathcal{L}_{int}\big[\partial^\mu a_{ph}/f_a, \psi\big] + \frac{a_{ph}}{f_a}\xi\frac{\alpha_s}{8\pi}G^a_{\mu\nu}\tilde{G}^{\mu\nu}_a \tag{4.17}$$

4.2 QCD Axion

We saw in the previous section that the PQ solution to the strong CP problem implies the existence of a pseudoscalar particle, the axion. The axion mixes with the π^0 and η mesons, since it has their same quantum numbers and takes a mass given by [38]:

$$m_a = \frac{\sqrt{m_u m_d}}{m_u + m_d}\frac{m_\pi f_\pi}{f_a} \tag{4.18}$$

Thus we see that the axion mass is inversely proportional to its decay constant f_a. Even if axions would not couple to photons directly, they inherit a coupling from their mixing with π^0 and η and the effective interaction can be written as

$$\mathcal{L}_{a\gamma\gamma} = -\frac{1}{4}g_{a\gamma\gamma}aF_{\mu\nu}F^{\mu\nu}, \tag{4.19}$$

where $g_{a\gamma\gamma}$ is a dimensionful coupling constant. Experimental constraints on $g_{a\gamma\gamma}$ can be inferred as a function of the axion mass from various astrophysical constraints and low energy data. For example, for $m_a \leq 1$ KeV solar neutrino data observations and Horizontal Branch stars data [4] result in $g_{a\gamma\gamma} \leq 10^{-10}\text{GeV}^{-1}$. Let's now review briefly the different ways proposed to implement the $U(1)_{PQ}$ symmetry, which lead to different axion models.

PQWW

The original model is known as the Peccei-Quinn-Weinberg-Wilczek (PQWW) axion model [7], [8], [9]. The PQ symmetry is realized with two Higgs doublets and the SM quarks:

$$\mathcal{L}_Y = -\bar{Q}_L Y_d \Phi_2 d_R - \bar{Q}_L Y_u \tilde{\Phi}_1 u_R + h.c. \tag{4.20}$$

where the Higgs fields Φ_i take vacuum expectation values v_i. The chiral PQ transformation takes the form

$$\Phi_i \to e^{i\alpha X_i}\Phi_i \qquad u_R \to e^{i\alpha X_u}u_R \qquad d_R \to e^{i\alpha X_d}d_R \tag{4.21}$$

where the PQ charges have to satisfy $X_u = X_1$ and $X_d = -X_2$ in order to have an invariant Lagrangian. The Higgs doublets take vev v_i after symmetry breaking and in particular the neutral part of the Higgs doublets can be parametrized as

$$\Phi_1^0 = \frac{1}{\sqrt{2}}(v_1 + h_1)e^{\frac{i\rho_1}{v_1}} \qquad \Phi_2^0 = \frac{1}{\sqrt{2}}(v_2 + h_2)e^{\frac{i\rho_2}{v_2}} \tag{4.22}$$

One linear combination of ρ_1 and ρ_2 provides the longitudinal component of the Z boson, the other one is the axion.
In this model the axion couples to the SM via the chiral rotations and the PQ charges of the SM fermions and expanding in powers of $1/v$, where $v = \sqrt{v_1^2 + v_2^2}$, the quark coupling is given at leading order in the expansion by $im_q(a/v)\bar{q}\gamma_5 q$.
The PQ chiral anomaly then induces couplings to gauge bosons via fermion loops $\propto aG\tilde{G}/v$ and $\propto aF\tilde{F}/v$.
In particular the gluon term is the term which leads to the PQ solution of the strong-CP problem.
One notes that all the axion couplings come suppressed by the scale v, which in the PQWW model is fixed to be the electroweak scale.
However, the PQWW axion is experimentally ruled out. The reason is that its physics is tied with the EWSB, and the experimental data do not agree with the predictions of the model. For instance, the following branching ratio can be estimated

$$Br(K^+ \to \pi^+ + a) \sim 10^{-5}\big(x + \frac{1}{x}\big)^2, \tag{4.23}$$

where $x = v_2/v_1$ and is many order of magnitude above the experimental bound

$$Br(K^+ \to \pi^+ + \text{invisible}) < 10^{-9} \tag{4.24}$$

Other models, with PQ symmetry breaking scale $f_a \gg v$, called "invisible" axion models, are still viable and we are going to present them below.

4.2.1 Invisible axion models

KSVZ

One type of invisible axion model is the Kim-Shifman-Vainshtein-Zakharov (KSVZ) [10], [11]. In this model a new complex scalar ϕ, singlet under the SM gauge group, and a new heavy quark Q are introduced. The PQ field and the heavy quarks interact via the $U(1)_{PQ}$ invariant Yukawa term, which provides the heavy quark mass:

$$\mathcal{L}_Y = -\lambda_Q \phi \bar{Q}_L Q_R + h.c. \tag{4.25}$$

where the Yukawa coupling λ_Q is a free parameter. The PQ symmetry is spontaneously broken when the scalar ϕ acquires a vev due to a "Mexican-hat" type potential and the fermion Q obtains a large mass $m_Q \sim \lambda_Q f_a$.
At the classical level, the Lagrangian is unaffected by chiral rotations, and the field ϕ is not coupled to the SM. Nevertheless at the quantum level chiral rotations on Q affect the gluon pseudoscalar density via the chiral anomaly:

$$\mathcal{L} \to \mathcal{L} + \frac{\alpha}{32\pi^2} G\tilde{G} \tag{4.26}$$

The interactions of KSVZ axion with the gauge bosons arise at one loop as a result of the chiral anomaly. At the level of EFT, the induced topological term is the only modification to the SM lagrangian and, since the heavy fermion can be integrated out for $E \ll m_Q$, then the axion is directly coupled to the gauge bosons.
There is in particular an axion-photon coupling in this model that can be calculated via loops giving the EM anomaly and its value depends on the electromagnetic charges assigned to Q.
However the KSVZ axion has no tree-level couplings to the SM matter fields, while such terms are generated at one loop in the EFT.

DFSZ

The other popular invisible axion model is the Dine-Fischler-Srednicki-Zhitnisky (DFSZ) [12], [13], where the axion couples to the SM via the Higgs sector.
It contains two Higgs doublets, H_u, H_d, like in the PQWW model, however the complex scalar ϕ, which contains the axion as its angular degree of freedom, is introduced as a standard model singlet. The PQ and Higgs fields interact through the scalar potential

$$V = \lambda_H H_u H_d \phi^2 \tag{4.27}$$

This term is $U(1)_{PQ}$ invariant for ϕ with PQ charge +1, and the Higgs fields each with charge -1.
The Higgs also couples to all the SM fermions, providing them mass via the Yukawa interaction

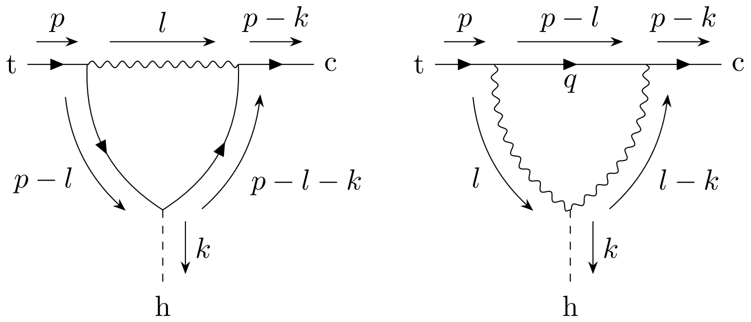
$$\mathcal{L}_Y \supset \bar{Q}_L y_u u_R H_u, \tag{4.28}$$

therefore in order to satisfy the symmetry, SM fermions have to be charged under $U(1)_{PQ}$.
After EW symmetry breaking, H is replaced by its vev, inducing axial current couplings between the axion and standard model fermions from the chiral term in the fermion mass matrix: $im_u(a/f_a)\bar{u}\gamma_5 u$. This axial current in turn induces the coupling between the axion and the QCD $G\tilde{G}$ via the colour anomaly.
Thus, the main phenomenological difference between this model and the KSVZ is that now there are tree level coupling between the axion and the SM fermions.

Chapter 5

ALPs Effective Field Theory

The Higgs boson discovery in 2012 has set spin zero particles at the center of searches for BSM physics. This may have been the first exploration of a new universe: scalar and pseudoscalar particles, elementary or composite, as portals to new physics.
Extra spin zero particles are in fact proposed as solutions to fundamentals and pressing problems in particle physics. A paradigmatic example is the strong CP problem of QCD, for which the solution relies on an anomalous global U(1) symmetry, the so called Peccei-Quinn symmetry, which is spontaneously broken. The associated (pseudo) Nambu-Goldstone boson (NGB), the axion, is in addition an excellent candidate to explain the Dark Matter content of the Universe. Many other extensions of the SM present one or several spontaneously broken global U(1) symmetries, thus predicting the appearance of Nambu-Goldstone bosons: we refer to these particles as axion-like particles (ALP). If ALPs get a tiny mass due to non-perturbative effect or explicit symmetry breaking, they are also good DM matter candidates. One important difference between a generic ALP and the QCD axion is that for the former the scale of symmetry breaking f_a and its mass m_a are treated as independent variables, while for the QCD axion the two parameters are inversely proportional. In full generality, the impact of ALPs depends on their nature and on the strength of their couplings; in practice the relevant characteristic of NGbs is that they only possess derivative couplings, because of the underlying shift symmetry.
We are going to explore in this chapter ALPs contributions to FCNC processes, formulating them via the linear realization of the ALP effective Lagrangian. We will consider the complete basis of bosonic ALP couplings to the electroweak sector, namely we will discuss the set of gauge invariant, independent leading order couplings to the W, Z, photon and Higgs doublet. In particular we will see that FCNC processes can get contributions from more couplings and we will have to consider them simultaneously in order to delimit the parameter space.



(a) **(b)**

Figure 5.1: Feynman diagrams for the decay $t \to cH$ in the unitary gauge.

5.1 FCNC top quark decay in the SM

In order to put bounds on ALP couplings through FCNC processes, we will have to compute Feynman amplitudes given by the penguin diagrams associated to the $d_i \to d_j a$ transition, where d is a generic down type quark and a is an ALP. In order to get some practice with quark FCNC decays involving a scalar or pseudoscalar particle, we focus now on the process $t \to cH$ in the SM, where H is the Higgs boson, with the goal of getting an estimate of the amplitude and of the branching ratio of this decay channel. This decay proceeds via one loop W boson exchanges and the corresponding diagrams are depicted in Figure 5.1.
Since the SM is a renormalizable theory, then we expect divergences appearing in the diagrams to cancel out, therefore the amplitude should be finite. The check of the finitness of the amplitude is left to the appendix, while here we will make an estimate of the decay width assuming the unrealistic parameters range [17].

$$M_W^2 \gg m_t^2 \gg m_b^2 \gg m_c^2 \tag{5.1}$$

This simplification neglects terms of $\mathcal{O}(m_t^2/M_W^2)$ but allows us to get without heavy computations an approximated amplitude, that is valid in the same parameter range, and to make comparison with the complete numerical computation. The amplitude of diagram (a) is given by:

$$\begin{aligned} A = -\frac{g^2}{8}(-\frac{ig}{2})\frac{1}{M_W}\sum_{q=d,s,b} V_{qc}V_{qt}^* m_q \int \frac{d^D l}{(2\pi)^D}\Big[\bar{u}_c\gamma^\mu(1-\gamma_5)i\cdot \\ \frac{\not{p}-\not{l}-\not{k}+m_q}{(p-l-k)^2-m_q^2} i\frac{\not{p}-\not{l}+m_q}{(p-l)^2-m_q^2}\gamma^\nu(1-\gamma_5)u_c\Big]\Big(\frac{-i}{l^2-M_W^2}\Big)\Big[g_{\mu\nu}-\frac{l_\mu l_\nu}{M_W^2}\Big] \end{aligned} \tag{5.2}$$

Using the orthogonality of L and R chiral projectors we obtain

$$A = -\frac{g^3}{4M_W}\sum_q V_{qc}V_{qt}^* m_q^2 \int \frac{d^D l}{(2\pi)^D} \frac{\Big[\bar{u}_c\gamma^\mu(2\not{p} - \not{k} - 2\not{l})\gamma^\nu P_L u_t\Big]\Big[g_{\mu\nu} - \frac{l_\mu l_\nu}{M_W^2}\Big]}{((p-l-k)^2 - m_q^2)((p-l)^2 - m_q^2)(l^2 - M_W^2)} \tag{5.3}$$

The dominant part of the amplitude comes from the contraction of the $g_{\mu\nu}$ piece of the W propagator, since the $l_\mu l_\nu$ part gives an $\mathcal{O}(m_t^2/M_W^2)$ correction, which is small in the limit $M_W^2 \gg m_t^2$. Applying gamma matrices algebra and putting $m_c = 0$ we get to

$$A = -\frac{g^3}{4M_W}\sum_q V_{qc}V_{qt}^* m_q^2 (I_1 + I_2), \tag{5.4}$$

where

$$\begin{cases} I_1 = 4\int \frac{d^D l}{(2\pi)^D} \frac{\big[\bar{u}_c \not{l} P_L u_t\big]}{((p-l-k)^2 - m_q^2)((p-l)^2 - m_q^2)(l^2 - M_W^2)} \\ I_2 = -2m_t \int \frac{d^D l}{(2\pi)^D} \frac{\big[\bar{u}_c P_R u_t\big]}{((p-l-k)^2 - m_q^2)((p-l)^2 - m_q^2)(l^2 - M_W^2)} \end{cases} \tag{5.5}$$

The integrals can be solved using the following Feynman parametrization formula:

$$\frac{1}{ABC} = 2\int_0^1 dx \int_0^{1-x} dy \frac{1}{\Big[A + (B-A)x + (C-A)y\Big]^3}, \tag{5.6}$$

where

$$\begin{cases} A = l^2 - M_W^2 \\ B = (p-l)^2 - m_q^2 \\ C = (p-l-k)^2 - m_q^2 \end{cases} \tag{5.7}$$

In order to get rid of linear terms at denominators, the shift $l \to l' = l - \big[p(x+y) - ky\big]$ is performed and we rearrange the integrals in the following form:

$$\begin{cases} I_1 = 8\int_0^1 dx \int_0^{1-x} dy \int \frac{d^D l}{(2\pi)^D} \frac{\big[\bar{u}_c(\not{l} + \not{p}(x+y) - \not{k}y) P_L u_t\big]}{(l^2 - a)^3} \\ I_2 = -4m_t \int_0^1 dx \int_0^{1-x} dy \int \frac{d^D l}{(2\pi)^D} \frac{\big[\bar{u}_c P_R u_t\big]}{(l^2 - a)^3}, \end{cases} \tag{5.8}$$

$$a = m_q^2(x+y) + M_W^2(1-x-y) - k^2 y - p^2(x+y) - p^2(x+y)^2 - k^2 y^2 + 2kpy \tag{5.9}$$

The linear term in l in I_1 vanishes by symmetry of the integration volume and using Dirac equation, recalling that we are putting $m_c = 0$, we get the following expression for $I_1 + I_2$:

$$I_1 + I_2 = -4m_t\big[\bar{u}_c P_R u_t\big]\int_0^1 dx \int_0^{1-x} dy (1-2x) \int \frac{d^D l}{(2\pi)^D} \frac{1}{(l^2 - a)^3} \tag{5.10}$$

From Eq. (A.4) and integrating over Feynman parameters in the limit $M_W^2 \gg m_q^2$ we thus get to the following value of iA

$$iA = \frac{g^3 m_t}{2M_W} \sum_q \frac{V_{qc}V_{qt}^* m_q^2}{16\pi^2} \big[\bar{u}_c P_R u_t\big] \int_0^1 dx \int_0^{1-x} dy \frac{1-2x}{a} \tag{5.11}$$

Finally, the integration over Feynman parameters in the range $M_W^2 \gg m_t^2$ produces the final amplitude for diagram (a), which reads

$$iA = \frac{g^3 m_t}{4M_W} \sum_q \frac{V_{qc}V_{qt}^* x_q}{16\pi^2} \big[\bar{u}_c P_R u_t\big] \tag{5.12}$$

where we have defined $x_q \equiv m_q^2/M_W^2$. Let's now proceed with the computation of diagram (b) in the same parameters range. The amplitude of the diagram reads

$$B = -\frac{g^2}{8} igM_W \sum_{q=d,s,b} V_{qc}V_{qt}^* m_q \int \frac{d^D l}{(2\pi)^D} \Big[\bar{u}_c \gamma^\mu (1-\gamma_5) i \frac{(\not{p} - \not{l} + m_q)}{(p-l)^2 - m_q^2} \gamma^\nu (1-\gamma_5) u_t\Big] \cdot \\ \Big(- \frac{i}{(l-k)^2 - M_W^2}\Big)\Big(g_{\mu\sigma} - \frac{(l-k)_\mu (l-k)_\sigma}{M_W^2}\Big)\Big(- \frac{i}{l^2 - M_W^2}\Big)\Big(g^\sigma_\nu - \frac{l_\nu l^\sigma}{M_W^2}\Big) \tag{5.13}$$

The dominant contribution comes from taking the contraction of the metric tensors inside W boson propagators, as other finite terms will be suppressed by powers of M_W.
Using orthogonality of chiral projectors and basic gamma matrices properties, the loop integral becomes

$$-2 \int \frac{d^D l}{(2\pi)^D} \frac{\big[\bar{u}_c (\not{p} - \not{l}) P_L u_t\big]}{((p-l)^2 - m_q^2)(l^2 - M_W^2)((l-k)^2 - M_W^2)} \tag{5.14}$$

Using the Feynman parametrization Eq. (5.6) and performing the loop momentum shift $l \to l' = l - (px + by)$ we are given with

$$-4 \int_0^1 dx \int_0^{1-x} dy \int \frac{d^D l}{(2\pi)^D} \frac{\big[\bar{u}_c \big(\not{p}(1-x) - \not{k}y - \not{l}\big) P_L u_t\big]}{(l^2 - b)^3}, \tag{5.15}$$

where $b = M_W^2(1-x) + m_q^2 x - p^2 x - k^2 y - (px + ky)^2$. Using Dirac equation and setting $m_c = 0$ yields the following amplitude:

$$B = 2g^3 M_W \sum_q V_{qc}V_{qt}^* m_t \big[\bar{u}_c P_R u_t\big] \int_0^1 dx \int_0^{1-x} dy (1-x-y) \int \frac{d^D l}{(2\pi)^D} \frac{1}{(l^2 - b)^3} \tag{5.16}$$

When we perform the integrals over Feynman parameters only terms proportional to quark masses survive, due to unitarity of CKM matrix and we finally obtain this expression for iB:

$$iB = -\frac{g^3 m_t}{M_W} \sum_q \frac{V_{qc} V_{qt}^* x_q}{16\pi^2} \left[\bar{u}_c P_R u_t\right] \tag{5.17}$$

The leading term of the sum is given by the bottom quark and thus we obtain the approximated amplitude for $t \to cH$:

$$M(t \to cH) \sim -\frac{3g^3}{64\pi^2} V_{cb} V_{tb}^* \frac{m_t m_b^2}{M_W^3} \left[\bar{u}_c P_R u_t\right] \tag{5.18}$$

This expression shows the basic dependence of the amplitude on both the top Yukawa coupling, through the factor m_t/M_W, and the remnants of the GIM cancellation, giving rise to the factor $V_{tb}^* V_{cb} \frac{m_b^2}{m_W^2}$.
Using some spinor algebra we get the following unpolarized squared amplitude:

$$|\bar{M}|^2 = \frac{1}{2}\left(\frac{3g^3}{64\pi^2}\right)^2 |V_{cb}|^2 |V_{tb}|^2 \frac{m_t^2 m_b^4}{M_W^6} \left(m_t^2 - m_H^2\right) \tag{5.19}$$

The differential decay width is given by:

$$d\Gamma = \frac{1}{2m_t} |\bar{M}|^2 d\phi, \tag{5.20}$$

where $d\phi$ is the differential phase space. Integrating over the full solid angle, the total decay width reads

$$\Gamma(t \to cH) = \frac{9}{512\pi^5} \frac{G_F^3}{\sqrt{2}} |V_{cb}|^2 |V_{tb}|^2 m_t^3 m_b^4 \left(1 - \frac{m_H^2}{m_t^2}\right)^2 \tag{5.21}$$

Extrapolating the formula Eq.(5.18) out of its validity region yields the following value for the branching ratio: $\mathcal{B}(t \to cH) \sim 7 \cdot 10^{-14}$, which differs by one order of magnitude from the numerical result reported in [36], namely $\mathcal{B}^{Num}(t \to cH) \sim 3 \cdot 10^{-15}$. The discrepancy is due to the fact that we neglected $\mathcal{O}(m_t^2/m_W^2)$ terms in the computation and we did not perform the running of the bottom quark mass to the top scale, which yields a factor $[m_b(m_t)/m_b(m_b)]^4 \sim 0.09$. However, the experimental bound on this decay from LHC is $\mathcal{B}^{Exp}(t \to cH) < 2.1 \cdot 10^{-3}$ [37], that is about twelve orders of magnitude less stringent than the SM prediction. Other top quark FCNC decay processes are characterized by similar SM rates:

$$\begin{cases} \mathcal{B}(t \to cZ) \sim 1 \cdot 10^{-14} \\ \mathcal{B}(t \to c\gamma) \sim 4.6 \cdot 10^{-14} \\ \mathcal{B}(t \to cg) \sim 4.6 \cdot 10^{-12} \end{cases} \tag{5.22}$$

These values are well beyond the detection capabilities of the LHC, which can optimistically probe a maximum of 10^{-5}. Consequently, measuring any excess in the branching ratios for top quark FCNC processes would be a clear evidence of physics beyond the SM.

5.2 Bosonic ALP Lagrangian

The most general effective Lagrangian describing ALP couplings to the SM bosons is given at LO in the linear expansion by [1]:

$$\mathcal{L}_{Eff} = \mathcal{L}_{SM} + \delta\mathcal{L}_{Eff}, \tag{5.23}$$

where

$$\mathcal{L}_{SM} \supset D_\mu \Phi^\dagger D^\mu \Phi - \Big(\bar{Q}_L Y^d \Phi d_{Rj} + \bar{Q}_L Y^u \tilde{\Phi} u_R + \bar{L}_L Y^e \Phi e_R + h.c.\Big) \tag{5.24}$$

$$\delta\mathcal{L}_{Eff} = \frac{1}{2}\partial_\mu a \partial^\mu a - \frac{1}{2}m_a^2 a^2 + c_{a\phi} O_{a\phi} + c_B O_B + c_W O_W, \tag{5.25}$$

with

$$\begin{aligned} O_{a\phi} &= i\frac{\partial^\mu a}{f_a}\Phi^\dagger \overleftrightarrow{D_\mu}\Phi \\ O_B &= -\frac{a}{f_a}B_{\mu\nu}\tilde{B}^{\mu\nu} \\ O_W &= -\frac{a}{f_a}W_{\mu\nu}\tilde{W}^{\mu\nu}, \end{aligned} \tag{5.26}$$

where c_i are real Wilson coefficients, Φ is the SM Higgs doublet, f_a is the ALP decay constant and $\tilde{X}^{\mu\nu}$ is the dual field strength of X boson.
We see that there is no explicit interaction term in the Lagrangian between the ALP and fermions, however fermion-axion interactions are there even if hidden. This can be seen by using the equation of motion of Φ and substituting the resulting expression into $O_{a\Phi}$, thus inducing a new Yukawa-axion coupling for which $O_{a\Phi}$ can be entirely traded up to $\mathcal{O}(a/f_a)$. Let's prove this statement explicitly now: taking the expression of $O_{a\Phi}$, we can get rid of the derivative acting on the ALP field integrating by parts and thus obtaining:

$$O_{a\Phi} = -\frac{ia}{f_a}\Big[\partial^\mu\big(\Phi^\dagger D_\mu \Phi\big) - \partial^\mu\big(D_\mu \Phi^\dagger\big)\Phi\Big] \tag{5.27}$$

The covariant derivative acts on the Higgs doublet as $D_\mu\Phi = (\partial_\mu + i\Gamma_\mu)\Phi$, where Γ_μ is the Yang-Mills connection. After some algebraic manipulations, the operator $O_{a\Phi}$ can be rewritten as:

$$O_{a\Phi} = \frac{ia}{f_a}\Box\Phi^\dagger\Phi + h.c. \qquad \Box = D_\mu D^\mu \tag{5.28}$$

The Euler-Lagrange equations for Φ and $\Phi^\dagger$ yield

$$\begin{cases} \Box\Phi^\dagger = X^\dagger\Phi^\dagger - \frac{\partial\mathcal{L}_{Yuk}}{\partial\Phi} \\ \Box\Phi = X\Phi - \frac{\partial L_{Yuk}}{\partial\Phi^\dagger}, \end{cases} \tag{5.29}$$

where $\mathcal{L}_{Yuk}$ is given by the second piece in Eq. (5.24) and

$$X = -ic_{a\Phi}\frac{\Box a}{f_a} - 2c_{a\Phi}\Gamma_\mu\frac{\partial^\mu a}{f_a} \tag{5.30}$$

Since X is made by operators suppressed by a power of the UV cutoff f_a, we can forget about it, as our goal is to obtain an equivalent operator $O_{a\Phi}$ at order $\mathcal{O}(a/f_a)$ and thus we plug in Eq. (5.28) only the terms given by $\frac{\partial L_{Yuk}}{\partial \Phi}$ and its hermitian conjugate.
The overall effect is that

$$O_{a\Phi} \to i\frac{a}{f_a}\Big[\bar{Q}_L Y_u \tilde{\Phi} u_R - \bar{Q}_L Y_d \Phi d_R - \bar{L}_L Y_e \Phi e_R\Big] + h.c. \tag{5.31}$$

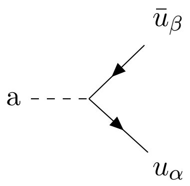
The Lagrangians containing the expressions given by Eqs. (5.26) and (5.31) respectively of $O_{a\Phi}$ give not the same Feynman rules, since the couplings between fields are different, but produce the same S matrix, as they are equivalent on-shell.
We note that the ALP-electroweak operators in Eq. (5.26) are flavour blind, nevertheless $O_{a\Phi}$ and O_W can take part in FCNC at one loop level through W boson exchange. At this order of perturbation theory, the parameter space of ALP-couplings in FCNC processes is thus bidimensional and is spanned by the Wilson coefficients c_W and $c_{a\Phi}$.

5.3 ALP and FCNC processes

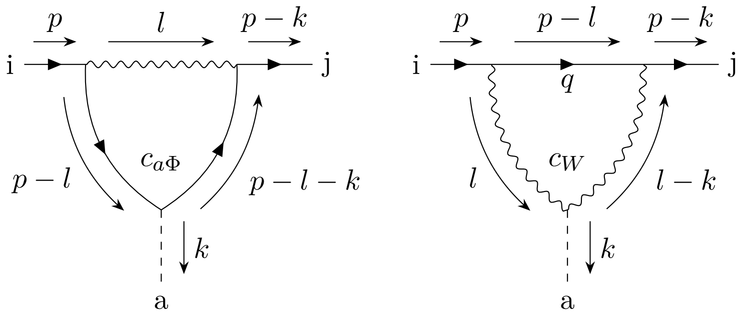
The effective interaction between the ALP and left-handed fermions can be expressed in full generality as:

$$\mathcal{L}_{eff}^{d_i \to d_j} = -g_{ij}^a(\partial_\mu a)\bar{d}_j\gamma^\mu P_L d_i + h.c, \tag{5.32}$$

where $P_L \equiv (1-\gamma_5)/2$, i and j denote flavour indices and g_{ij}^a is an effective coupling.
The operators $O_{a\Phi}$ and O_W give contributions to the flavour changing $d_i \to d_j a$ transition, with $i \neq j$, via one-loop $W^\pm$ exchange. The corresponding Feynman diagrams at the quark level are represented in Figure 5.2. From the expressions of O_W and $O_{a\Phi}$, we can show that the Feynman rules of ALP vertices with fermions and W bosons are the following:

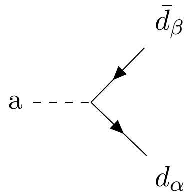


$$-\frac{(m_u)_\alpha}{f_a}\delta_{\alpha\beta}\gamma_5 c_{a\Phi} \tag{5.33}$$

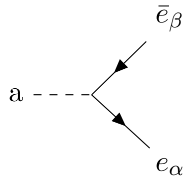


(a) **(b)**

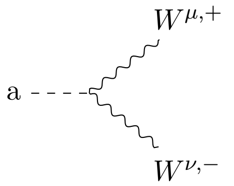
Figure 5.2: One-loop diagrams contributing to FCNC transitions, involving O_W and $O_{a\Phi}$



$$\frac{(m_d)_\alpha}{f_a}\delta_{\alpha\beta}\gamma_5 c_{a\Phi} \tag{5.34}$$



$$\frac{(m_e)_\alpha}{f_a}\delta_{\alpha\beta}\gamma_5 c_{a\Phi} \tag{5.35}$$



$$-\frac{4i}{f_a}c_W p^+_\alpha p^-_\beta \epsilon^{\mu\nu\alpha\beta} \tag{5.36}$$

Let's perform now the calculation of the amplitude of diagram (b), in order to find its contribution to the effective coupling g^a_{ij} :

$$B = -\frac{g^2}{8}\Big(\frac{-4i}{f_a}\Big)c_W \sum_{q=u,c,t} V_{qi}V^*_{qj}\int \frac{d^Dl}{(2\pi)^D}\Big[\bar{d}_j\gamma^\mu(1-\gamma_5)i\frac{\not{p}-\not{l}+m_q}{(p-l)^2-m_q^2}\gamma^\nu(1-\gamma_5)d_i\Big]\cdot$$
$$\Big(\frac{-i}{(l-k)^2-M_W^2}\Big)\Big[g_{\mu\sigma}-\frac{(l-k)_\mu(l-k)_\sigma}{M_W^2}\Big](k-l)_\alpha l_\beta \epsilon^{\sigma\delta\alpha\beta}\Big(\frac{-i}{l^2-M_W^2}\Big)\Big(g_{\nu\delta}-\frac{l_\nu l_\delta}{M_W^2}\Big) \tag{5.37}$$

$$B = \frac{g^2 c_W}{2 f_a} \sum_{q=u,c,t} V_{qi} V_{qj}^* \int \frac{d^D l}{(2\pi)^D} \Big[\bar{d}_j \gamma^\mu (1-\gamma_5) \frac{\not{p} - \not{l} + m_q}{(p-l)^2 - m_q^2} \gamma^\nu (1-\gamma_5) d_i\Big] \cdot$$
$$\Big(\frac{1}{(l-k)^2 - M_W^2}\Big) \Big[g_{\mu\sigma} - \frac{(l-k)_\mu (l-k)_\sigma}{M_W^2}\Big] (k-l)_\alpha l_\beta \epsilon^{\sigma\delta\alpha\beta} \Big(\frac{1}{l^2 - M_W^2}\Big) \Big(g_{\nu\delta} - \frac{l_\nu l_\delta}{M_W^2}\Big). \tag{5.38}$$

In these equations m_q is the mass of a given up-type quark q running in the loop, and in the approximation that m_{d_j}, $m_{d_j} \ll M_W$ we can set $p^2 \to 0$ and $k^2 \to 0$. In this limit we obtain:

$$B = -\frac{g^2 c_W}{2 f_a} \sum_{q=u,c,t} V_{qi} V_{qj}^* \int \frac{d^D l}{(2\pi)^D} \Big[\bar{d}_j \gamma^\mu (1-\gamma_5)(\not{p} - \not{l} + m_q) \gamma^\nu (1-\gamma_5) d_i\Big] \cdot \frac{\Big(g_{\mu\sigma} - \frac{(l-k)_\mu (l-k)_\sigma}{M_W^2}\Big)(k-l)_\alpha l_\beta \epsilon^{\sigma\delta\alpha\beta} \Big(g_{\nu\delta} - \frac{l_\nu l_\delta}{M_W^2}\Big)}{\Big(l^2 - m_q^2\Big)\big(l^2 - M_W^2\big)^2} \tag{5.39}$$

The factor m_q in the fermion propagator vanishes because of the orthogonality of left and right chiral projectors. Taking the product of the two W boson propagators we are left with 4 terms, where the only one non vanishing by symmetry is given by the product of the metric tensors $g_{\mu\sigma} g_{\nu\delta}$.
Thus we are left with this D dimensional integral to compute:

$$\int \frac{d^D l}{(2\pi)^D} \frac{l_\beta l_\sigma}{\Big(l^2 - m_q^2\Big)\big(l^2 - M_W^2\big)^2} \Big[\bar{d}_j \gamma_\mu \gamma^\sigma \gamma_\nu (1-\gamma_5) d_i\Big] k_\alpha \epsilon^{\mu\nu\alpha\beta} \tag{5.40}$$

Under integration over loop momenta we can substitute $l_\beta l_\sigma \to \frac{l^2}{4} g_{\beta\sigma}$, so that the integral simplifies to:

$$\frac{1}{4} \int \frac{d^D l}{(2\pi)^D} \frac{l^2}{\Big(l^2 - m_q^2\Big)\big(l^2 - M_W^2\big)^2} \Big[\bar{d}_j \gamma_\mu \gamma_\beta \gamma_\nu (1-\gamma_5) d_i\Big] k_\alpha \epsilon^{\mu\nu\alpha\beta}. \tag{5.41}$$

We can now decompose the product of gamma matrices into symmetric and antisymmetric parts, namely:

$$\gamma_\mu \gamma_\beta \gamma_\nu = S_{\mu\beta\nu\sigma} \gamma^\sigma - i \epsilon_{\sigma\mu\beta\nu} \gamma^\sigma \gamma_5, \tag{5.42}$$

where only the latter term survives the contraction with the epsilon tensor. The use of Levi-Civita contraction identity

$$\epsilon_{\mu\nu\beta\sigma} \epsilon^{\mu\nu\beta\alpha} = -6 \delta_\sigma^\alpha \tag{5.43}$$

yields the following amplitude:

$$B = -\frac{3 i g^2 c_W}{2 f_a} \Big[\bar{d}_j \gamma^\sigma (1-\gamma_5) d_i\Big] k_\sigma \sum_{q=u,c,t} V_{qi} V_{qj}^* \int \frac{d^D l}{(2\pi)^D} \frac{l^2}{\Big(l^2 - m_q^2\Big)\big(l^2 - M_W^2\big)^2} \tag{5.44}$$

Defining $\xi_q \equiv V_{qi}V_{qj}^*$ and setting $m_u = 0$, we can easily show exploiting unitarity of CKM matrix that:

$$\sum_{q=u,c,t} \frac{\xi_q}{l^2 - m_q^2} = \sum_{q=c,t} \frac{\xi_q m_q^2}{l^2(l^2 - m_q^2)}. \tag{5.45}$$

Since in our notation $P_L = (1 - \gamma_5)/2$ and defining $x_q \equiv m_q^2/M_W^2$, we thus get:

$$B = -\frac{3ig^2 c_W}{f_a}\Big[\bar{d}_j \gamma^\sigma P_L d_i\Big] k_\sigma \sum_{q=c,t} V_{qi}V_{qj}^* x_q \int \frac{d^D l}{(2\pi)^D} \frac{M_W^2}{\big(l^2 - m_q^2\big)\big(l^2 - M_W^2\big)^2} \tag{5.46}$$

Therefore we see GIM Mechanism at work, namely the logarithmic divergent loop integral is actually made to be finite as a consequence of the unitarity of CKM matrix. The integral can be easily computed with the following Feynman parametrization formula:

$$\frac{1}{A^2 B} = 2\int_0^1 dx \frac{x}{\big[B + (A - B)x\big]^3} \tag{5.47}$$

This formula yields for the loop integral:

$$2\int_0^1 dx x \int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2 - c)^3}, \qquad c = M_W^2 x + m_q^2(1 - x). \tag{5.48}$$

Computing the integral with (A.4) the amplitude reads now:

$$B = \frac{3g^2 c_W}{f_a}\Big[\bar{d}_j \gamma^\sigma P_L d_i\Big] k_\sigma \sum_{q=c,t} \frac{V_{qi}V_{qj}^*}{16\pi^2} x_q M_W^2 \int_0^1 dx \frac{x}{M_W^2 x + m_q^2(1 - x)} \tag{5.49}$$

Evaluating now the integral over x parameter, we get the final amplitude:

$$B = \frac{3g^2 c_W}{f_a}\Big[\bar{d}_j \gamma^\sigma P_L d_i\Big] k_\sigma \sum_{q=c,t} \frac{V_{qi}V_{qj}^*}{16\pi^2} g(x_q), \tag{5.50}$$

where the loop function is given by

$$g(x) = \frac{x\big[1 + x(\log x - 1)\big]}{(1 - x)^2} \tag{5.51}$$

By means of the expression of the effective Lagrangian (Eq. 5.32) we can extract the contribution to g_{ij}^a given by this loop diagram, which is given by:

$$g_{ij}^a(b) = \frac{3g^2 c_W}{f_a} \sum_{q=c,t} \frac{V_{qi}V_{qj}^*}{16\pi^2} g(x_q) \tag{5.52}$$

Let's now look at diagram (a), whose estimation, as we will see, involves some more technical aspects. From Eq. (5.33) the amplitude A of the diagram reads

$$A = -\frac{g^2}{8}\Big(-\frac{c_{a\Phi}}{f_a}\Big)\sum_{q=u,c,t} V_{qi}V_{qj}^* m_q \int \frac{d^D l}{(2\pi)^D}\Big[\bar{d}_j\gamma^\mu(1-\gamma_5)i\cdot$$
$$\frac{\not{p}-\not{l}-\not{k}+m_q}{(p-l-k)^2-m_q^2}\gamma_5 i\frac{\not{p}-\not{l}+m_q}{(p-l)^2-m_q^2}\gamma^\nu(1-\gamma_5)d_i\Big]\Big(\frac{-i}{l^2-M_W^2}\Big)\Big[g_{\mu\nu}-\frac{l_\mu l_\nu}{M_W^2}\Big] \tag{5.53}$$

Using the approximation of light external quarks at denominator and some elementary gamma matrices algebra at the numerator, we can recast the amplitude in the following form:

$$A = -\frac{ig^2 c_{a\Phi}}{4f_a}\sum_{q=u,c,t} V_{qi}V_{qj}^* m_q^2 \int \frac{d^D l}{(2\pi)^D}\frac{\Big[\bar{d}_j\gamma^\mu(1-\gamma_5)\not{k}\gamma^\nu d_i\Big]}{(l^2-m_q^2)^2(l^2-M_W^2)}\Big[g_{\mu\nu}-\frac{l_\mu l_\nu}{M_W^2}\Big] \tag{5.54}$$

The expression in square brackets is proportional to external quark masses when multiplied by the $g_{\mu\nu}$ piece of the W propagator, this can be seen writing $\not{k}$ as $\not{p}-(\not{p}-\not{k})$ and applying Dirac equation to the fermion fields. Therefore only the term proportional to $l_\mu l_\nu$ survives, and we are left with

$$A = \frac{ig^2 c_{a\Phi}}{4f_a}\sum_{q=u,c,t} V_{qi}V_{qj}^* m_q^2 \int \frac{d^D l}{(2\pi)^D}\frac{\Big[\bar{d}_j\not{l}\not{k}\not{l}(1-\gamma_5)d_i\Big]}{M_W^2(l^2-m_q^2)^2(l^2-M_W^2)} \tag{5.55}$$

Exploiting gamma matrices anticommutation relation yields:

$$\not{l}\not{k}\not{l}(1-\gamma_5) = \big[-\not{k}l^2+2(l\cdot k)\not{l}\big](1-\gamma_5), \tag{5.56}$$

so, introducing the chiral projector P_L and making the replacement $l_\mu l_\nu \to \frac{l^2}{4}g_{\mu\nu}$ in the integral we obtain

$$A = -\frac{ig^2 c_{a\Phi}}{4f_a}\sum_{q=u,c,t} V_{qi}V_{qj}^* x_q\Big[\bar{d}_j\gamma^\alpha P_L d_i\Big]k_\alpha \int \frac{d^D l}{(2\pi)^D}\frac{l^2}{(l^2-m_q^2)^2(l^2-M_W^2)} \tag{5.57}$$

Now, from power counting the integral is logarithmically divergent, and we do not get rid of this divergence by means of the unitarity of CKM matrix, because the sum over internal quarks does not cancel due to the factor x_q, therefore this amplitude is truly divergent.
Anyway, since we are using an EFT approach, we can put a UV cutoff over momentum integration and since the only UV scale at our disposal is the ALP scale f_a we fix $\Lambda \equiv f_a$.

Applying a Wick rotation in the loop integral via $l^0 \to il_E^0$ and keeping only the leading logarithm term we get:

$$\int_{m_q}^{f_a} \frac{d^D l}{(2\pi)^D} \frac{l^2}{(l^2 - m_q^2)^2 (l^2 - M_W^2)} = -\frac{i}{8\pi^2}\Big(\log\frac{f_a}{m_q} + \text{corrections}\Big) \tag{5.58}$$

Actually, there is some ambiguity in the definition of the IR cutoff m_q, since we should define at what scale μ the mass of the quarks is evaluated, whether at the scale of the experiment or at the scale m_q itself, where in the latter case m_q would correspond to the on-shell mass. However, this correction due to the running of the mass is a NLO correction, whose size is of the same order of the corrections that we neglected in the loop integral, so we forget about it in order to be consistent with our approximation.
Thus, we finally obtain the following amplitude for the diagram (a):

$$A = -\frac{ig^2 c_{a\Phi}}{4f_a} \sum_{q=u,c,t} \frac{V_{qi} V_{qj}^*}{16\pi^2} x_q \Big[\bar{d}_j \gamma^\alpha P_L d_i\Big] k_\alpha \log\frac{f_a^2}{m_q^2} \tag{5.59}$$

From Eq. (5.32) we extract the contribution to the effective coupling g_{ij}^a provided by the diagram (a):

$$g_{ij}^a(a) = -\frac{g^2 c_{a\Phi}}{4f_a} \sum_{q=u,c,t} \frac{V_{qi} V_{qj}^*}{16\pi^2} x_q \log\frac{f_a^2}{m_q^2} \tag{5.60}$$

Therefore, we are given with the following one loop result for the effective coupling g_{ij}^a, in accordance with [1]:

$$g_{ij}^a = g^2 \sum_{q=u,c,t} \frac{V_{qi} V_{qj}^*}{16\pi^2} \Big[\frac{3c_W}{f_a} g(x_q) - \frac{c_{a\Phi}}{4f_a} x_q \log\frac{f_a^2}{m_q^2}\Big] \tag{5.61}$$

We note that, since $g(x) \sim x + \mathcal{O}(x^2)$ for small x, up and charm quark contributions are subleading in both terms with respect to that of the top quark, which is the dominant one. We also observe that the logarithmic enhancement of the $c_{a\phi}$ term $\propto \log\,(f_a/m_q)$ should be particularly relevant for large values of f_a.
We saw in the previous section that the decay amplitude in the SM for a FCNC quark decay into the Higgs boson was finite, while in this computation a UV divergence appears in the diagram containing an ALP vertex with fermions (a), while the one where the ALP couples to W bosons (b) is indeed finite. While this is theoretically reasonable, since we are now working in an EFT, it may seem puzzling to understand how technically the amplitude results to be finite by replacing the ALP with the Higgs boson, but we should remember that, in addition to be a pseudoscalar, the ALP is also a NGB and so has an anomalous coupling with gauge bosons in order to preserve the shift symmetry.
Conversely, the Higgs coupling with the W boson is just proportional to the W

mass itself and the degree of divergence of the corresponding diagram is not lowered by vanishing contractions of loop momenta with the antisymmetric tensor.

5.3.1 Computation of $\Gamma\Big(K^+ \to \pi^+ a\Big)$

The differential decay rate for $K^+ \to \pi^+ a$ is given by:

$$d\Gamma = \frac{(2\pi)^4}{2M_K} \int \frac{d^3\bar{p}_1}{(2\pi)^3 2E_1} \frac{d^3\bar{p}_2}{(2\pi)^3 2E_2} \delta^{(4)}\big(k^\mu - p_1^\mu - p_2^\mu\big)|\bar{M}^2|, \tag{5.62}$$

where k^μ, p_1^μ and p_2^μ are the four momenta of K^+, π^+ and a respectively. Integrating over $\bar{p}_2$ and going to polar coordinates, where $d^3\bar{p}_1 = |\bar{p}_1^2| dp_1 d\Omega$ yields:

$$d\Gamma = \frac{1}{32\pi^2 M_K} \int \frac{dp_1}{E_1 E_2} \delta\Big(M_K - \sqrt{\bar{p}_1^2 + m_\pi^2} - \sqrt{\bar{p}_1^2 + m_a^2}\Big) |\bar{p}_1^2||\bar{M}^2| dp_1 d\Omega \tag{5.63}$$

The kinematic part can be easily solved using the well known property of the δ-function:

$$\delta(f(x)) = \sum_k \frac{\delta(x - x_k)}{|f'(x_k)|} \qquad f(x_k) = 0 \tag{5.64}$$

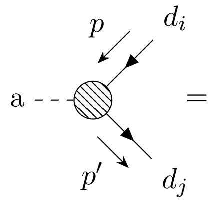
We thus get to the following expression:

$$d\Gamma = \frac{1}{64\pi^2 M_K} \lambda_{\pi a}^{1/2} \int dp_1 \delta\Big(|\bar{p}_1| - \frac{1}{2m_k}\big(M_K^2 - m_\pi^2 - m_a^2\big)\Big) |\bar{p}_1^2||\bar{M}^2| dp_1 d\Omega, \tag{5.65}$$

with

$$\lambda_{\pi a} = \Big[1 - \frac{(m_a + m_\pi)^2}{m_k^2}\Big]\Big[1 - \frac{(m_a - m_\pi)^2}{m_k^2}\Big] \tag{5.66}$$

We are now left with with the matrix element M to compute. The Feynman rule for the vertex defined by the Lagrangian in Eq.(5.32) is given by



$$= -ig_{ij}^a (p - p')_\mu \gamma^\mu (1 - \gamma_5) \tag{5.67}$$

Therefore the amplitude M may be written as

$$M = -\frac{i}{2} g_{ij}^a (p - p')_\mu < k^+ |\bar{d}_j \gamma^\mu (1 - \gamma_5) d_i| \pi^+ > \tag{5.68}$$

Since initial and final states have the same parity, only vector current contribution is not vanishing. Its matrix element between kaon and pion states decomposes into two form factors, f_+ and f_- as

$$< k^+ |\bar{d}_j \gamma_\mu d_i| \pi^+ >= (p_k + p_\pi)_\mu f_+(q^2) + (p_k - p_\pi)_\mu f_-(q^2), \tag{5.69}$$

where $q_\mu = (p_k - p_\pi)_\mu$ is the 4-momentum transferred between the kaon and the pion. We can define from f_+ and f_- the following scalar form factor f_0:

$$f_0(q^2) = f_+(q^2) + \frac{q^2}{M_K^2 - M_\pi^2} f_-(q^2), \tag{5.70}$$

which satisfies by definition the relation $f_+(0) = f_0(0)$. We can so rewrite M in terms of f_0 as

$$M = -\frac{i}{2} g_{ij}^a (m_k^2 - m_\pi^2) f_0(q^2) \tag{5.71}$$

Taking the square modulus of M and integrating over the solid angle $d\Omega$ we finally obtain this expression for the decay width:

$$\Gamma\Big(K^+ \to \pi^+ a\Big) = \frac{M_K^3 |g_{\pi a}|^2}{64\pi} f_0^2(m_a^2) \lambda_{\pi a}^{1/2} \Big(1 - \frac{m_\pi^2}{m_k^2}\Big)^2 \tag{5.72}$$

An analogous expression can be obtained for the $B \to Ka$ decay performing the proper replacements. Finally we read the relevant scalar form factors for Kaon and B-meson decays respectively from Refs. [2], [3].
We are now ready to start our phenomenological analysis, in order to put constraints on the parameter space spanned by c_W and $c_{a\Phi}$, comparing the experimental values with the theoretical ones, computed considering both the SM contribution and the ALPs one.

5.4 Phenomenological bounds on ALPs couplings

We perform our analysis in the scenario of an ALP which does not decay into visible particles in the detector. This situation can occur if a has sufficiently large coupling to a stable dark sector, as motivated by several DM models. The experimental constraints relevant for different m_a ranges are given by:

- $m_a \in (0, m_K - m_\pi)$.
  The most constraining experimental limits on $\mathcal{B}(K \to \pi\nu\bar{\nu})$ come from the E787 and E949 experiments, performed at Brookhaven National Laboratory. The value reported in these searches is $\mathcal{B}(K^+ \to \pi^+\nu\bar{\nu})^{Exp} = (1.73^{1.15}_{-1.05}) \times 10^{-10}$ [19], slightly above the SM prediction, $\mathcal{B}(K^+ \to \pi^+\nu\bar{\nu})^{SM} = (9.11 \pm 0.72) \times 10^{-11}$ [18], where the SM uncertainty is dominated by the current precision on CKM matrix parameters.
  The NA62 experiment at the CERN SPS aims to measure $\mathcal{B}(K^+ \to \pi^+\nu\bar{\nu})$ attaining at the SM rates in the very near future [22].

- $m_a \in (0, m_B - m_K)$
  The study of the decay $B \to K\nu\bar{\nu}$ was performed by the Belle collaboration. The experimental limit on the Branching ratio is given by $\mathcal{B}(B \to K\nu\bar{\nu}) < 2.7 \times 10^{-5}$ [20], which lays a factor 2.7 above the SM prediction [21]. In

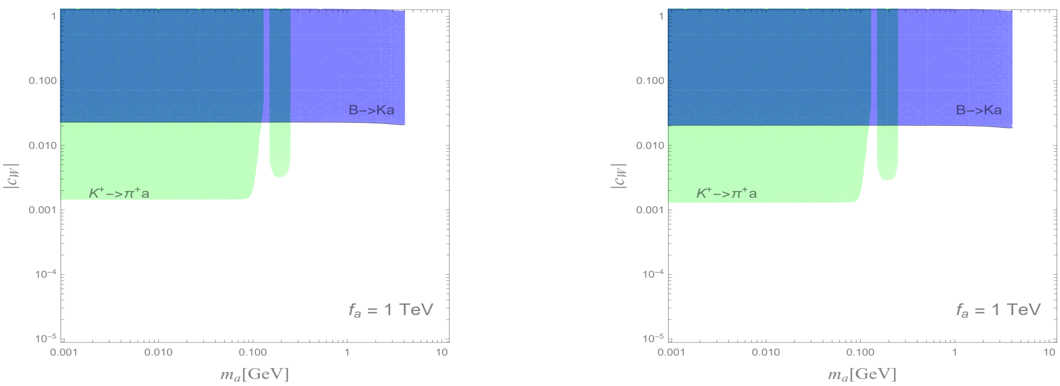


Figure 5.3: Invisible Alp: limits on the absolute values of c_W (left panel) and $c_{a\Phi}$ (right panel) as a function of the ALP mass. The exclusion contours were derived from the experimental bounds on $\mathcal{B}(K^+ \to \pi^+ + \text{inv})$ (green) and $\mathcal{B}(B \to K + \text{inv})$ (blue), by fixing $f_a = 1$ TeV and by setting the other coupling to zero.

the near future, Belle-II experiment aims at measuring the SM value with a $\mathcal{O}(10\%)$ precision [23].

We will infer the constraints on ALPs-electroweak couplings given by data in two different setups: firstly within the one coupling at a time approach, where either c_W or $c_{a\Phi}$ are switched on, then we will consider the simultaneous presence of both couplings.
In Figure 5.3 the allowed values of c_W and $c_{a\Phi}$ are depicted as a function of the ALP mass, where only the corresponding coupling is added to the SM.
The constraints derived on the parameter spaces $\{m_a, c_W\}$ and $\{m_a, c_{a\Phi}\}$ coincide with those derived in [1].
The case illustrated corresponds to $f_a = 1$ TeV and we observe that the quantitative similarity of the contour regions of the two couplings is just accidental. It can be checked that the limits on $c_{a\Phi}$ become stronger than those for c_W for large values of f_a, as a consequence of the logarithmic dependence on f_a of its contribution.
We can note that K-meson constraints are about one order of magnitude stronger than those derived for B-meson decays, although limited to a more restricted parameter range. However, such constraints become weaker in the ALP mass range 150 MeV $\leq m_a \leq$ 260 MeV, since the search for $K^+ \to \pi^+ \nu\bar{\nu}$ is background limited in this region and therefore the branching ratio limit is bigger.
When both $c_{a\Phi}$ and c_W are simultaneously considered, an interesting interference pattern can take place.
In Figure 5.4 the results of combining the experimental constraints for $f_a = 1$ TeV and $m_a \leq 0.1$ GeV are represented.
We can observe that when the relative sign between the couplings is positive, then a blind direction in the parameter space appears, due to the cancellation of the

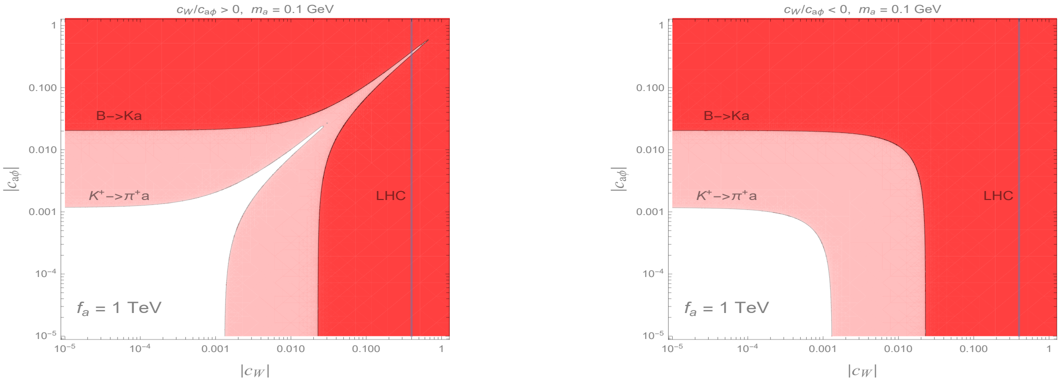


Figure 5.4: Invisible Alp: allowed $\{c_W, c_{a\Phi}\}$ parameter space when the two couplings are simultaneously present. The superposition of the constraints from $K^+ \to \pi^+ + \text{inv}$ (pink) and $B^+ \to K^+ + \text{inv}$ (red) is shown for the illustrative case with $f_a = 1$ TeV and $m_a \leq 0.1$ GeV. The left (right) panel shows the destructive (constructive) interference of the two couplings for $c_W/c_{a\Phi} > 0$ ($c_W/c_{a\Phi} < 0$). The blue solid line corresponds to the current limit from mono-W searches at the LHC with 3.2 fb^{-1} of data [24].

two contributions. This unconstrained direction is exactly aligned for Kaon and B-meson decays, therefore additional experimental information is then needed to strengthen the constraint along this direction.
One possibility would be to consider the D-meson decay $D \to \pi\nu\bar{\nu}$, which is sensitive to a different combination of $c_{a\Phi}$ and c_W, as up and down-type quark contributions to the term proportional to $c_{a\Phi}$ have opposite sign (Eqs. 5.33 and 5.34).
Nevertheless, these decays suffer from heavy GIM suppression, since the contribution of the loop is proportional to $(m_b/m_W)^2$ and no such experiments have been performed to date. A more promising possibility is to consider LHC constraints which are sensitive to a specific ALP coupling. For example, LHC searches for mono-W final states are only sensitive to the coefficient c_W. The authors of Ref [24] derived the following bound from 3.2 fb^{-1} luminosity of LHC data:

$$\frac{|c_W|}{f_a} \leq 0.41\ \text{TeV}^{-1}, \tag{5.73}$$

these constraints have been superimposed in Figure 5.4.
Similarly, the study of $pp \to t\bar{t} + \text{MET}$ at LHC would constrain only $c_{a\Phi}$, but such an analysis goes beyond the goal of our work.
Finally, from these plots we can observe that typically LHC constraints are weaker than flavour ones, except in the region of parameter space where the FCNC processes are suppressed due to the destructive interference between c_W and $c_{a\Phi}$ contributions. In this situation, the combination of low and high-energy bounds becomes an important tool to shed light on new physics.

Conclusions

Axions and ALPs phenomenology is a very promising research field, with important implications both in particle physics and in cosmology. In this context, FCNC processes provide a unique framework for testing the SM, as they enclose a huge variety of suppressed decays, through which one can look for deviations of the experimental results from the SM predictions and put bounds on ALP parameter space. However, so far no NP signals have been detected in flavour physics and this demands for more experimental advances to attain SM rates, which are going to be pursued in the very near future.
In the absence of data supporting a specific model of physics BSM, effective lagrangians provide a powerful model-independent tool based on the SM gauge symmetry.
In this work we considered the complete basis of bosonic electroweak ALP effective operators, at leading order of the expansion in powers of the inverse of the symmetry breaking scale, i.e. dimension 5 operators. Since this basis is flavour blind, its impact on flavour changing transitions starts at the loop level. Indeed, the experimental accuracy achieved in rare-decay physics is so good that loop-induced contributions by ALPs may provide the best bounds in a large fraction of the parameter space.
We derived constraints on ALP couplings in the scenario of invisible ALP: this assumption is physically motivated, as it can arise if the ALP has a sufficiently large coupling to a dark sector, as motivated by several DM models, making ($a \rightarrow$inv) the dominant decay channel of the ALP.
We first performed the analysis considering just one operator at a time and we found out that searches for K meson decays provide more stringent constraints than for B decay, even if limited to a more restricted ALP mass range. Then we took into account simultaneously ALP couplings to fermions and W boson and we studied the two dimensional parameter space spanned by the Wilson coefficients $c_{a\Phi}$ and c_W, reproducing the costraints derived in [1].
An interesting pattern of constructive/destructive interference has been revealed, depending on the relative sign of the couplings and on the mass range considered. In this way, the previous bounds stemming from kaon and B-decay data, obtained in the one coupling at a time approach, are alleviated by the appearence

of an unconstrained direction in the parameter space.
Moreover, we outlined how LHC constraints sensitive to a single ALP coupling, while generally considerably weaker than flavour bounds, can actually provide complementary information to low-energy probes, exploring otherwise inaccessible directions in the ALP parameter space.
These searches will be improved in the next years thanks to the experimental data collected at Na62, LHCb and Belle-II, providing hopeful opportunites to detect NP.
This thesis can be extended in several ways as we performed our analysis setting up many assumptions. For instance, we could perform the analysis also in the case of visible ALP decays, as presented in Reference [1]. Moreover, we could relax the hypothesis of flavour blind coupling of the ALP to fermions, introducing flavour diagonal couplings c_{aff}. This theoretical setup may be useful to improve the current experimental bounds of ALP couplings to fermions and may be the topic of a future work.

Appendix A

Loop integrals in D dimensions

When performing loop computations, one has to deal with the propagators of the particles running in the loops, which can be combined in a more suitable way by the integration over Feynman parameters:

$$\frac{1}{A_1 A_2 \dots A_n} = \int_0^1 dx_1 \dots dx_n \delta(1 - \sum_i x_i) \frac{(n-1)!}{[x_1 + A_1 + x_2 A_2 + \dots + x_n A_n]^n} \tag{A.1}$$

Once we have done this step, the expression between square brackets in the denominator will be a quadratic function of the integration momentum k^μ, for a one loop integral. In general the denominator will contain both quadratic and linear terms in k. We can eliminate the latter shifting the integration momentum $k \to l$. After this shift the denominator takes the form $(l^2 - \Delta)^n$.
Obviously the same shift has to be performed at numerator. Here, terms with an odd number of powers of l vanish by symmetric integration and we can replace, still due to symmetry:

$$l^\mu l^\nu \to \frac{l^2}{D} g^{\mu\nu} \tag{A.2}$$

$$l^\mu l^\nu l^\rho l^\sigma \to \frac{1}{D(D+2)} (l^2)^2 (g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}). \tag{A.3}$$

These integrals can be evaluated either in Euclidean space after Wick-rotating, or, as we have done in our computations, using the following D-dimensional integrals in Minkowski spacetime:

$$\int \frac{d^D l}{(2\pi)^D} \frac{1}{(l^2 - \Delta)^n} = \frac{(-1)^n i}{(4\pi)^{D/2}} \frac{\Gamma(n - \frac{D}{2})}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{D}{2}} \tag{A.4}$$

$$\int \frac{d^D l}{(2\pi)^D} \frac{l^2}{(l^2 - \Delta)^n} = \frac{(-1)^{n-1} i}{(4\pi)^{D/2}} \frac{D}{2} \frac{\Gamma(n - \frac{D}{2} - 1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{D}{2} - 1} \tag{A.5}$$

$$\int \frac{d^D l}{(2\pi)^D} \frac{l^\mu l^\nu}{(l^2-\Delta)^n} = \frac{(-1)^{n-1} i}{(4\pi)^{D/2}} \frac{g^{\mu\nu}}{2} \frac{\Gamma(n-\frac{D}{2}-1)}{\Gamma(n)} \Big(\frac{1}{\Delta}\Big)^{n-\frac{D}{2}-1} \tag{A.6}$$

$$\int \frac{d^D l}{(2\pi)^D} \frac{(l^2)^2}{(l^2-\Delta)^n} = \frac{(-1)^n i}{(4\pi)^{D/2}} \frac{D(D+2)}{4} \frac{\Gamma(n-\frac{D}{2}-2)}{\Gamma(n)} \Big(\frac{1}{\Delta}\Big)^{n-\frac{D}{2}-2} \tag{A.7}$$

If the integral is divergent, we can extract the behaviour near $D = 4$ expanding:

$$\Big(\frac{1}{\Delta}\Big)^{n-\frac{D}{2}} = 1 - \frac{\epsilon}{2} \log(\Delta) + \mathcal{O}(\epsilon^2) \tag{A.8}$$

where we have defined $\epsilon \equiv 4 - D$. We also need Euler Gamma function expansion around its poles, which is the following for $n = 0, 1 \dots$

$$\Gamma(-n + \frac{\epsilon}{2}) = \frac{(-1)^n}{n!} \big[\frac{2}{\epsilon} + \psi(n+1) + \mathcal{O}(\epsilon)\big], \tag{A.9}$$

being $\psi(1) = \gamma_E$ the Euler-Mascheroni constant

$$\gamma_E = -0.5772 \dots \tag{A.10}$$

We finally write down some gamma matrices contraction identities in D dimension which we used in our computations:

$$\gamma^\mu \gamma_\mu = D \tag{A.11}$$

$$\gamma^\mu \gamma^\nu \gamma_\mu = -(D-2)\gamma^\nu \tag{A.12}$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4 g^{\nu\rho} - (4-D)\gamma^\nu \gamma^\rho \tag{A.13}$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\gamma^\sigma \gamma^\rho \gamma^\nu + (4-D)\gamma^\nu \gamma^\rho \gamma^\sigma \tag{A.14}$$

Appendix B

Computation of diagram c in Fig. 3.3

Here we report the computation for the amplitude C of Figure 3.3 (c) in section 3.2. Since it's topologically similar to diagram (b), we will use the same techniques in the calculation. The amplitude C is given by:

$$\begin{aligned} C = -\frac{g^2}{8}(ig_s)^2 V_{ud}V_{cs}^*\mu^\epsilon \int \frac{d^D k}{(2\pi)^D}\left[\bar{s}_\beta \gamma^\rho T^a_{\beta\alpha} i\frac{\not{p}+\not{k}}{(p+k)^2}\gamma^\mu(1-\gamma_5)c_\alpha\right]\cdot \\ \frac{-ig_{\rho\sigma}}{k^2}\frac{-ig_{\mu\nu}}{k^2-M_W^2}\left[\bar{u}_\delta\gamma^\nu(1-\gamma_5)i\frac{\not{p}+\not{k}}{(p+k)^2}\gamma_\sigma T^a_{\delta\gamma}d_\gamma\right] \end{aligned} \tag{B.1}$$

Using colour algebra identity (3.49) and keeping only the piece contributing to the coefficient of S_2 we obtain

$$C^{(S_2)} = \frac{-G_F}{6\sqrt{2}}M_W^2 V_{ud}V_{cs}^* g_s^2 \int \frac{d^4k}{(2\pi)^4}\frac{\left[\bar{s}_\alpha\gamma^\rho(\not{p}+\not{k})\gamma^\mu(1-\gamma_5)c_\alpha\right]\left[\bar{u}_\beta\gamma_\mu(\not{p}+\not{k})\gamma_\rho(1-\gamma_5)d_\beta\right]}{k^2(k^2-M_W^2)((p+k)^2)^2} \tag{B.2}$$

We can now make a shift of integration momentum k, namely $k \to k+p$, in order to eliminate the external momentum p from the numerator; this yields the following expression:

$$C^{(S_2)} = \frac{-G_F}{6\sqrt{2}}M_W^2 V_{ud}V_{cs}^* g_s^2 \int \frac{d^4k}{(2\pi)^4}\frac{\left[\bar{s}_\alpha\gamma^\rho\not{k}\gamma_\mu(1-\gamma_5)c_\alpha\right]\left[\bar{u}_\beta\gamma_\mu\not{k}\gamma_\rho(1-\gamma_5)d_\beta\right]}{k^4((k-p)^2-M_W^2)(k-p)^2} \tag{B.3}$$

Using decomposition of gamma matrices, we can simplify the contractions at numerator as:

$$\left[\bar{s}_\alpha\gamma^\rho\gamma^\nu\gamma^\mu(1-\gamma_5)c_\alpha\right]\left[\bar{u}_\beta\gamma_\mu\gamma_\nu\gamma_\rho(1-\gamma_5)d_\beta\right] = 4S_2 \tag{B.4}$$

We note that this relation has a factor 4 of difference with respect to the corresponding one in diagram (b), this is because, due to the exchange of the order of

gamma matrices in the contractions, symmetric and antisymmetric parts of the decomposition now give opposite contribution.
Rewriting at numerator $M_W^2 = (k-p)^2 - \big[(k-p)^2 - M_W^2\big]$, and substituting in the integration $k_\nu k_\sigma \to \frac{g_{\nu\sigma}}{4} k^2$ the amplitude considerably simplifies:

$$C^{(S_2)} = \frac{-G_F}{6\sqrt{2}} M_W^2 V_{ud} V_{cs}^* g_s^2 \big(I - J\big) S_2, \tag{B.5}$$

where

$$I = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2\big(k^2 - M_W^2\big)} \qquad J = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2(k-p)^2} \tag{B.6}$$

Using Feynman parametrization formula (3.43) we obtain these results for I and J:

$$I = \frac{i}{(4\pi)^2} \int_0^1 dx \big[\frac{2}{\epsilon} - \log(M_W^2 x)\big] \qquad J = \frac{i}{(4\pi)^2} \int_0^1 dx \big[\frac{2}{\epsilon} - \log(-p^2 x(1-x))\big] \tag{B.7}$$

Combining and reinserting S_1 tree-level, which we did not carry in the computation, we get the final expression for the amplitude of diagram (c):

$$C = -\frac{G_F}{\sqrt{2}} V_{ud} V_{cs}^* \frac{\alpha_s}{3(4\pi)} \Big[\log\Big(\frac{M_W^2}{-p^2}\Big) + 1\Big] (S_2 - 3S_1) \tag{B.8}$$

Since we are computing the amplitude in the LO approximation, the finite $\mathcal{O}(\alpha_s)$ constant term can be neglected, so we have reproduced the result quoted in (3.45).

Appendix C

Computation of the divergent parts of diagrams in Fig. 5.1

Let's start with the computation of the divergent part of the amplitude associated to diagram (a).
Taking Eq. (5.3) and performing the shift over loop momentum $l \to l' = l - \big[p(x+y) - ky\big]$ we obtain

$$A = -\frac{g^3}{2M_W}\sum_{q=d,s,b} V_{qc}V_{qt}^* m_q^2 \int_0^1 dx \int_0^{1-x} dy \int \frac{d^D l}{(2\pi)^D} \frac{\Big[\bar{u}_c\gamma^\mu(2\not{p}(1-x-y) - \not{k}(1-2y) - 2\not{l})\gamma^\nu P_L u_t\Big]}{(l^2-a)^3} \cdot \Big[g_{\mu\nu} - \frac{(l+p(x+y)-ky)_\mu(l+p(x+y)-ky)_\nu}{M_W^2}\Big] \tag{C.1}$$

In order to select the divergent terms, we need to pick up quadratic terms in l; there are three of such contributions, namely the one taking the $l_\mu l_\nu$ piece of W boson propagator (I_1), and two taking one power of l from the W propagator and the other one from the fermion propagator (I_2 and I_3).
The divergent part of the loop integrals reads:

$$\begin{cases} I_1 = \frac{1}{2}\Big[\bar{u}_c\big(2\not{p}(1-x-y) - \not{k}(1-2y)\big)P_L u_t\Big] \int \frac{d^D l}{(2\pi)^D}\frac{l^2}{(l^2-a)^3} \\ I_2 = I_3 = 2\Big[\bar{u}_c\big(\not{p}(x+y) - \not{k}y\big)P_L u_t\Big] \int \frac{d^D l}{(2\pi)^D}\frac{l^2}{(l^2-a)^3} \end{cases} \tag{C.2}$$

Applying Eq.(A.5) and summing up the three integrals we end up with the following expression for the divergent part associated to diagram (a):

$$A_{Div} = -\frac{ig^3}{2M_W}\sum_q \frac{V_{qc}V_{qt}^*}{16\pi^2} m_q^2 [\bar{u}_c\gamma^\alpha P_L u_t] \int_0^1 dx \int_0^{1-x} dy \Big[\big(2+6(x+y)\big)p_\alpha - (1+6y)k_\alpha\Big] \tag{C.3}$$

Performing the integration over Feynman parameters and using Dirac equation we get the final result, which reads

$$A_{Div} = -\frac{3ig^3}{2M_W} \sum_q \frac{V_{qc}V_{qt}^*}{16\pi^2} m_q^2 \Big(m_b \big[\bar{u}_c P_R u_t\big] + m_c \big[\bar{u}_c P_L u_t\big] \Big) \tag{C.4}$$

Let's move now to diagram (b). Starting from Eq.(5.13), after some algebraic manipulation and the introduction of Feynman parameters, we get to the following amplitude:

$$B = -g^3 M_W \sum_{q=d,s,b} V_{qc}V_{qt}^* \int_0^1 dx \int_0^{1-x} dy \int \frac{d^D l}{(2\pi)^D} \frac{\Big[\bar{u}_c \gamma^\mu \big(\not{p}(1-x) - \not{k}y - \not{l}\big)\gamma^\nu P_L u_t\Big]}{(l^2-b)^3} \cdot$$
$$\Big[g_\mu^\delta - \frac{(l+px-k(1-y))^\delta (l+px-k(1-y))_\mu}{M_W^2}\Big]\Big[g_{\delta\nu} - \frac{(l+px+ky)_\delta (l+px+ky)_\nu}{M_W^2}\Big] \tag{C.5}$$

where

$$b = M_W^2(1-x) + m_q^2 x - p^2 x - k^2 y - (px+ky)^2 \tag{C.6}$$

We have to select quartic terms in l, since the quadratic ones produce a logarithmic divergence, which is canceled by unitarity.
There are 5 of such terms, one is obtained picking up all the four powers of l from the W boson propagators and the other four picking up three powers of l from the W propagators and the other power from the fermion propagator.
Summing up all these contributions we obtained the following expression for the divergent part:

$$B_{Div} = \frac{g^3}{4M_W^3} \sum_{q=d,s,b} V_{qc}V_{qt}^* \big[\bar{u}_c \gamma^\alpha P_L u_t\big] \int_0^1 dx \int_0^{1-x} dy \Big[(2+8x)p_\alpha - (5-8y)k_\alpha\Big] \int \frac{d^D l}{(2\pi)^D} \frac{l^4}{(l^2-b)^3} \tag{C.7}$$

The loop integral can be computed using Eq.(A.7) and yields

$$B_{Div} = \frac{ig^3}{2M_W^3} \sum_{q=d,s,b} \frac{V_{qc}V_{qt}^*}{16\pi^2} \big[\bar{u}_c \gamma^\alpha P_L u_t\big] 6 \int_0^1 dx \int_0^{1-x} dy \Big[(2+8x)p_\alpha - (5-8y)k_\alpha\Big] \cdot$$
$$\big(M_W^2(1-x) + m_q^2 x - p^2 x - k^2 y - (px+ky)^2\big) \tag{C.8}$$

Applying unitarity, only the term proportional to m_q^2 survives and using Dirac equation we obtain

$$B_{Div} = \frac{3ig^3}{2M_W} \sum_q \frac{V_{qc}V_{qt}^*}{16\pi^2} m_q^2 \Big(m_b \big[\bar{u}_c P_R u_t\big] + m_c \big[\bar{u}_c P_L u_t\big] \Big) \tag{C.9}$$

Comparing Eq.(C.4) and Eq.(C.9) we finally see that divergences in the amplitude of the process $t \to cH$ cancel out, in agreement with the fact that the SM is a renormalizable theory.

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