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Implicative algebras and their relationship with triposes

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Introduction

The aim of this thesis is to present the notion of *implicative algebra* and to examine its connections with the concept of tripos.
Alexandre Miquel first introduced implicative algebras in his paper “Implicative algebras: a new foundation for realizability and forcing” [10] with the goal of creating an algebraic structure that could simultaneously factorize the model-theoretic constructions underlying both forcing and realizability.
Introduced by Paul Cohen in 1963 [2] [3], the main idea behind forcing is to interpret every formula φ of the considered theory as an element of a complete Boolean (or Heyting) algebra. On other hand, Kleene’s realizability, first introduced in 1945 [6], interprets each closed formula φ of the theory as the set of its *realizers*, i.e. a specific subset of a suitable algebra of programs. This method, originally restricted only to intuitionistic logic, was expanded by Krivine to classical logic [7]. In classical realizability, every closed formula is interpreted as the set of its *counter-realizers*, represented by a subset of the set of stacks associated to an algebra of classical programs.
Miquel’s work [10] demonstrates that implicative algebras can bring together these concepts thanks to the use of the same set to represent both realizers and truth values.
The thesis will proceed as follows. Firstly, we will review some preliminary notions about categories and triposes, with a focus on the category of Heyting algebras.
Subsequently, we will present the concept of implicative algebra and its key features, paying special attention to how this structure can interpret first-order logic. In particular, we will start by defining what an *implicative structure* is and showing how it can induce a *semantic type system* where the types correspond to its elements. Then, we will present the notion of *separator*, a particular type of subset of an implicative structure, that has a fundamental role in the definition of implicative algebra. After having defined this, we will focus on the study of the implicative algebras induced by particular types of structures (complete Heyting algebras cHAs, total

combinatory algebras CAs and abstract Krivine structures AKSs).
In chapter 3, we will start to examine the relationship between triposes and implicative algebras. Resuming Miquel's results [10], we will show how an implicative algebra can induce a specific type of tripos, called *implicative tripos*. As before, we will focus on analyzing the implicative triposes induced by cHAs, CAs and AKSs, showing how the concept of implicative tripos can simultaneously unify the notions of realizability and forcing triposes.
Afterwards, we will prove how, given a set-based tripos, it is possible to construct an implicative algebra that induces an implicative tripos isomorphic to the given one [9].
In the last two chapters, after presenting the notions of *geometric morphism* and *first-order logic morphism* between implicative triposes, we will analyze which types of functions between the corresponding implicative algebras can induce these morphisms.
These results will lead us to define new notions of morphisms between implicative algebras, and the consequent categories, that do not overlook but actually consider their relationship with triposes. Similarly to what Frey and Streicher have supposed in [4], these new categories allow us to shift our attention from the study of the categories of triposes to the study of the implicative algebras, much simpler algebraic structures, perhaps providing a new perspective on the former.

Chapter 1

Preliminaries

1.1 Some notions about categories

Definition 1.1. *A* **category** $\mathbb{C}$ *consists of*

- *a class* $\mathsf{Obj}(\mathbb{C})$ *of* **objects***;*
- *a class* $\mathsf{Hom}(\mathbb{C})$ *of* **morphisms***;*
- *two class functions* $\mathsf{dom}, \mathsf{cod} : \mathsf{Hom}(\mathbb{C}) \to \mathsf{Obj}(\mathbb{C})$ *called* **domain** *and* **codomain***;*
- *a class function* $\mathsf{id}_- : \mathsf{Obj}(\mathbb{C}) \to \mathsf{Hom}(\mathbb{C})$*;*
- *a class function*

$$\circ : \{(f,g) \in \mathsf{Hom}(\mathbb{C}) \times \mathsf{Hom}(\mathbb{C}) : \mathsf{cod}(f) = \mathsf{dom}(g)\} \to \mathsf{Hom}(\mathbb{C})$$
$$(f,g) \mapsto g \circ f$$

  such that:

  - $\mathsf{dom}(g \circ f) = \mathsf{dom}(f)$ *and* $\mathsf{cod}(g \circ f) = \mathsf{cod}(g)$ *for every* f, g *morphisms of* $\mathbb{C}$ *such that* $\mathsf{cod}(f) = \mathsf{dom}(g)$*;*
  - $\mathsf{dom}(\mathsf{id}_X) = \mathsf{cod}(\mathsf{id}_X) = X$ *for every* X *object of* $\mathbb{C}$*;*
  - $f \circ \mathsf{id}_{\mathsf{dom}(f)} = \mathsf{id}_{\mathsf{cod}(f)} \circ f = f$ *for every* f *morphism of* $\mathbb{C}$*;*
  - $h \circ (g \circ f) = (h \circ g) \circ f$ *for all* f, g, h *morphisms of* $\mathbb{C}$ *such that* $\mathsf{cod}(f) = \mathsf{dom}(g)$ *and* $\mathsf{cod}(g) = \mathsf{dom}(h)$.

If f *is a morphism such that* $\mathsf{dom}(f) = X$ *and* $\mathsf{cod}(f) = Y$*, we will denote it as* $f : X \to Y$*. We will denote as* $\mathsf{Hom}_{\mathbb{C}}(X,Y)$ *the class of morphisms from* X *to* Y*.*

Example. The category **Set** is defined as follows:

- the objects are sets;
- if X, Y are sets then $\mathsf{Hom}_{\mathbb{C}}(X, Y) = \{(X, f, Y) : f$ is a map from X to $Y\}$. We will write just f instead of (X, f, Y);
- the composition of $f : X \to Y$ and $g : Y \to Z$ is the usual composition $g \circ f : X \to Z$;
- the identity of X is determined by the usual identity map of X.

Example. Let $P = (P, \leq)$ be a preorder i.e. P is a set and $\leq$ is a binary operation on P that is reflexive and transitive. Then we can see P as a category in the following way:

- the objects of P are its elements, i.e. $\mathsf{Obj}(P) = P$;
- if $p, p' \in P$ then:

$$\mathsf{Hom}_P(p, p') = \begin{cases} \{(p, p')\} & \text{if } p \leq p' \\ \varnothing & \text{otherwise} \end{cases}$$

- if $p \in P$ then $\mathsf{id}_p = (p, p)$;
- $(q, r) \circ (p, q) = (p, r)$ for every $r, p, q \in P$ such that $p \leq q \leq r$.

Example. The category **PreOrd** is defined in the following way:

- the objects are preorders;
- a morphism from P to Q is a monotonic map between the corresponding sets;
- the composition of two morphisms is the usual composition of maps between sets;
- id_P is the usual identity map of the set P.

Definition 1.2. *Let* $\mathbb{C}$ *be a category. The* **opposite category** $\mathbb{C}^{op}$ *of* $\mathbb{C}$ *is defined as follows:*

- $\mathsf{Obj}(\mathbb{C}^{op}) = \mathsf{Obj}(\mathbb{C})$ *and* $\mathsf{Hom}(\mathbb{C}^{op}) = \mathsf{Hom}(\mathbb{C})$*;*

- $\mathsf{dom}^{\mathbb{C}^{op}} = \mathsf{cod}^{\mathbb{C}}$ *and* $\mathsf{cod}^{\mathbb{C}^{op}} = \mathsf{dom}^{\mathbb{C}}$*;*
- *if* f, g *are morphisms of* $\mathbb{C}$ *such that* $\mathsf{cod}^{\mathbb{C}^{op}}(f) = \mathsf{dom}^{\mathbb{C}^{op}}(g)$*, then* $g \circ^{\mathbb{C}^{op}} f = f \circ^{\mathbb{C}} g$

Definition 1.3. *Let* $f : X \to Y$ *be a morphism of a category* $\mathbb{C}$*. Then* f *is an* **isomorphism** *if there exists a morphism of* $\mathbb{C}$ $g : Y \to X$ *such that* $g \circ f = \mathsf{id}_X$ *and* $f \circ g = \mathsf{id}_Y$*.*

Now, let us recall the notion of pullback.

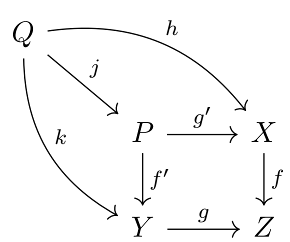
Definition 1.4. *Let* $f : X \to Z$ *and* $g : Y \to Z$ *be two morphisms of a category* $\mathbb{C}$*. A* **pullback of** f **along** g *is a tern* (P, g', f') *such that* P *is an object of* $\mathbb{C}$ *and* $f' : P \to Y$*,* $g' : P \to X$ *are two morphisms of* $\mathbb{C}$ *such that the following diagram commutes*

$$\begin{CD} P @>{g'}>> X \\ @V{f'}VV @VV{f}V \\ Y @>{g}>> Z \end{CD}$$

and such that if $Q \in \mathsf{Obj}(\mathbb{C})$ *and* $h : Q \to X$*,* $k : Q \to Y \in \mathsf{Hom}(\mathbb{C})$ *are such that the following diagram commutes*

$$\begin{CD} Q @>{h}>> X \\ @V{k}VV @VV{f}V \\ Y @>{g}>> Z \end{CD}$$

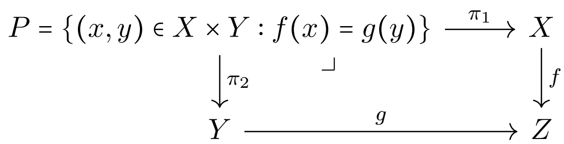
then there exists one and only one morphism $j : Q \to P$ *such that*



commutes, i.e. $h = g' \circ j$ *and* $k = f' \circ j$*. In such case, we will write:*

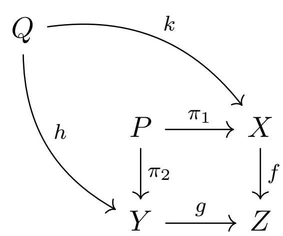
$$\begin{CD} P @>{g'}>> X \\ @V{f'}V{\lrcorner}V @VV{f}V \\ Y @>{g}>> Z \end{CD}$$

Lemma 1.1. *Let $f : X \to Y$ and $g : Y \to Z$ be two maps between sets. Then the following diagram is a pullback in* **Set***:*



where π_1 and π_2 are the projections of P. Furthermore, every pullback in **Set** *of f along g is isomorphic to (P, π_2, π_1).*

Proof. Clearly $f \circ \pi_1 = g \circ \pi_2$. If



commutes, then $f(k(q)) = g(h(q))$, hence $(k(q), h(q)) \in P$. Then, the unique map $j : Q \to P$ such that $\pi_1 \circ j = k$ and $\pi_2 \circ j = h$ is $j(q) = (k(q), h(q))$. Now, let (P', g', f') be another pullback of f along g. Since $f \circ g' = g \circ f'$ and $f \circ \pi_1 = g \circ \pi_2$, there exist unique maps $l : P' \to P$ and $l' : P \to P'$ such that:

$$\pi_1 \circ l = g' \qquad \pi_2 \circ l = f'$$
$$g' \circ l' = \pi_1 \qquad f' \circ l' = \pi_2$$

Then, clearly

$$\pi_1 \circ (l \circ l') = \pi_1 \qquad \pi_2 \circ (l \circ l') = \pi_2$$

By uniqueness, then $l \circ l' = \mathsf{id}_P$. Similarly, $l' \circ l = \mathsf{id}_{P'}$. Thus P' is isomorphic to P. □

1.1.1 Functors and natural transformations.

Definition 1.5. *Let $\mathbb{C}$ and $\mathbb{D}$ be two categories. A* **functor** *F from $\mathbb{C}$ to $\mathbb{D}$, denoted as $F : \mathbb{C} \to \mathbb{D}$, is a pair (F_0, F_1) of class functions where $F_0 : \mathsf{Obj}(\mathbb{C}) \to \mathsf{Obj}(\mathbb{D})$ and $F_1 : \mathsf{Hom}(\mathbb{C}) \to \mathsf{Hom}(\mathbb{D})$ such that:*

- *if $f : X \to Y$ is a morphism of $\mathbb{C}$ then $F_1(f)$ is a morphism of $\mathbb{D}$ from $F_0(X)$ to $F_0(Y)$;*

- $F_1(\mathsf{id}_X) = \mathsf{id}_{F_0(X)}$ *for every object* X *of* $\mathbb{C}$*;*
- *if* $f : X \to Y$ *and* $g : Y \to Z$ *are morphisms of* $\mathbb{C}$ *then* $F_1(g \circ f) = F_1(g) \circ F_1(f)$*.*

We will often write F instead of F_0 and F_1.

Example. Let $P = (P, \leq_P)$ and $Q = (Q, \leq_Q)$ be two preorders and $F : P \to Q$ be a map. Then

$$F \text{ is a functor} \iff F \text{ is monotonic}^1$$

Clearly if F is a functor and $p \leq_P p'$ then it must be $F(p) \leq_Q F(p')$. Conversely, if F is monotonic and $p \leq_P p'$ then $F((p,p')) = (F(p), F(p')) \in \mathsf{Hom}(\mathbb{D})$. Furthermore, $F((p,p)) = \mathsf{id}_{F(p)}$ for every $p \in P$ and if $p \leq_P q$ and $q \leq_P r$ then $F((q,r) \circ (p,q)) = (F(p), F(r)) = F((q,r)) \circ F((p,q))$. Thus F is a functor.

Definition 1.6. *Let* $\mathbb{C}$ *and* $\mathbb{D}$ *be two categories and* $F, G : \mathbb{C} \to \mathbb{D}$ *be two functors. A* **natural transformation** Φ *from* F *to* G *is a family of morphisms* Φ_X *of* $\mathbb{D}$ *for every* $X \in \mathsf{Obj}(\mathbb{C})$ *such that for every* $f \in \mathsf{Hom}_{\mathbb{C}}(X, Y)$ *the following diagram is commutative:*

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \downarrow{\scriptstyle \Phi_X} & & \downarrow{\scriptstyle \Phi_Y} \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

i.e. $\Phi_Y \circ F(f) = G(f) \circ \Phi_X$*.*
We will say that Φ *is a* **natural isomorphism** *if* Φ_X *is an isomorphism of* $\mathbb{D}$ *for every* $X \in \mathsf{Obj}(\mathbb{C})$*.*

Now, we can recall the notions of adjoints.

Definition 1.7. *Let* $\mathbb{C}$ *and* $\mathbb{D}$ *be two categories and* $F : \mathbb{C} \to \mathbb{D}$*,* $G : \mathbb{D} \to \mathbb{C}$ *be two functors.* F *is a* **left adjoint** *of* G *(or* G *is a* **right adjoint** *of* F*) if there exists an* **adjunction** *from* F *to* G*, i.e. there exists a family of bijections* $(\phi_{X,Y})_{X \in \mathsf{Obj}(\mathbb{C}), Y \in \mathsf{Obj}(\mathbb{D})}$ *such that*

$$\phi_{X,Y} : \mathsf{Hom}_{\mathbb{D}}(F(X), Y) \to \mathsf{Hom}_{\mathbb{C}}(X, G(Y))$$

is natural with respect to X *and* Y*, which means that for every* $f : X \to X' \in \mathsf{Hom}(\mathbb{C})$ *and* $g : Y \to Y' \in \mathsf{Hom}(\mathbb{D})$ *the following diagrams commute:*

[1] We have indicated with F the functor $(F, F \times F|_{\mathsf{Hom}(P)})$.

$$
\begin{array}{ccc}
\mathsf{Hom}_{\mathbb{D}}(F(X),Y) & \xleftarrow[-\circ F(f)]{} & \mathsf{Hom}_{\mathbb{D}}(F(X'),Y) \\
\big\downarrow \phi_{X,Y} & & \big\downarrow \phi_{X',Y} \\
\mathsf{Hom}_{\mathbb{C}}(X,G(Y)) & \xleftarrow[-\circ f]{} & \mathsf{Hom}_{\mathbb{C}}(X',G(Y))
\end{array}
$$

$$
\begin{array}{ccc}
\mathsf{Hom}_{\mathbb{D}}(F(X),Y) & \xrightarrow{g\circ -} & \mathsf{Hom}_{\mathbb{D}}(F(X),Y') \\
\big\downarrow \phi_{X,Y} & & \big\downarrow \phi_{X,Y'} \\
\mathsf{Hom}_{\mathbb{C}}(X,G(Y)) & \xrightarrow{G(g)\circ -} & \mathsf{Hom}_{\mathbb{C}}(X,G(Y'))
\end{array}
$$

In such case, we write $F \dashv G$.

Example. Let $P = (P, \leq_P)$, $Q = (Q, \leq_Q)$ be two preorders and $F : P \to Q$, $G : Q \to P$ be two functors, i.e. two monotonic maps.
Then

$$F \dashv G \quad \text{if and only if} \quad \forall p \in P, \forall q \in Q: \ F(p) \leq_Q q \text{ iff } p \leq_P G(q)$$

If ϕ is an adjunction from F to Q, then for every $p \in P$ and $q \in Q$ $\phi_{p,q}$: $\mathsf{Hom}_Q(F(p), q) \to \mathsf{Hom}_P(p, G(q))$ is a bijection. Thus, $F(p) \leq_Q q$ if and only if $p \leq_P G(q)$.
Conversely, if for every $p \in P, q \in Q$: $F(p) \leq_Q q$ if and only if $p \leq_P G(q)$ then $\phi_{p,q}$ is trivially defined.

1.2 Heyting algebras

Definition 1.8. *A* **partial order** *is a preorder $P = (P, \leq_P)$ such that $\leq_P$ is antisymmetric, i.e. for every $p, p' \in P$, if $p \leq_P p'$ and $p' \leq_P p$, then $p = p'$.*

We can define a category **Pos** in the following way:

- the objects are partial orders;
- a morphism from P to Q is a monotonic map from P to Q;
- the composition is the usual composition of maps;
- id_P is the usual identity map of the set P.

Definition 1.9. *A partial order $\mathbb{H} = (\mathbb{H}, \leq, \wedge, \vee, \to, \top, \bot)$ is a* **Heyting algebra** *if:*

1. *for every $a, b \in \mathbb{H}$ there exist a greatest lower bound and a least upper bound, denoted by $a \wedge b$ and $a \vee b$ respectively;*

2. $\top, \bot \in \mathbb{H}$ *such that* $\bot \leq c \leq \top$ *for all* $c \in \mathbb{H}$*;*

3. $\rightarrow : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H}$ *is an operation such that:*

$$c \wedge a \leq b \quad \textit{if and only if} \quad c \leq a \rightarrow b$$

Definition 1.10. *Let $\mathbb{H}$ and $\mathbb{K}$ be two Heyting algebras. A* **morphism of Heyting algebras** *is a monotonic map $\varphi : \mathbb{H} \rightarrow \mathbb{K}$ such that:*

1. $\varphi(h \wedge_{\mathbb{H}} h') = \varphi(h) \wedge_{\mathbb{K}} \varphi(h')$*;*

2. $\varphi(h \vee_{\mathbb{H}} h') = \varphi(h) \vee_{\mathbb{K}} \varphi(h')$*;*

3. $\varphi(h \rightarrow_{\mathbb{H}} h') = \varphi(h) \rightarrow_{\mathbb{K}} \varphi(h')$*;*

4. $\varphi(\bot_{\mathbb{H}}) = \bot_{\mathbb{K}}$

Let us observe that if $\varphi : \mathbb{H} \rightarrow \mathbb{K}$ is a morphism of Heyting algebras then $\varphi(\top_{\mathbb{H}}) = \top_{\mathbb{K}}$. Indeed, since $\top_{\mathbb{H}} = \bot_{\mathbb{H}} \rightarrow_{\mathbb{H}} \bot_{\mathbb{H}}$ then $\varphi(\top_{\mathbb{H}}) = \varphi(\bot_{\mathbb{H}}) \rightarrow_{\mathbb{K}} \varphi(\bot_{\mathbb{H}}) = \bot_{\mathbb{K}} \rightarrow_{\mathbb{K}} \bot_{\mathbb{K}} = \top_{\mathbb{K}}$.

We can now define the category **HA** in the following way:

- the objects are Heyting algebras;
- a morphism from $\mathbb{H}$ to $\mathbb{K}$ is a morphism of Heyting algebras from $\mathbb{H}$ to $\mathbb{K}$;
- the composition is the usual composition of maps;
- $\mathsf{id}_{\mathbb{H}}$ is the usual identity map of the set $\mathbb{H}$.

Definition 1.11. *Let $\mathcal{A}$ be a Heyting algebra. We say that $\mathcal{A}$ is a* **Boolean algebra** *if*

$$a \vee (a \rightarrow \bot) = \top$$

for every $a \in \mathcal{A}$.

Definition 1.12. *A Heyting algebra $\mathbb{H}$ is* **complete** *if every set-indexed family $(a_i)_{i \in I}$ of elements of $\mathbb{H}$ has both a greatest lower bound $\bigwedge_{i \in I} a_i \in \mathbb{H}$ and a least upper bound $\bigvee_{i \in I} a_i \in \mathbb{H}$.*

Definition 1.13. *Let $\mathbb{H}$ and $\mathbb{K}$ be two complete Heyting algebras. A* **morphism of complete Heyting algebras** *is a map $\varphi : \mathbb{H} \to \mathbb{K}$ that preserves arbitrary meets, arbitrary joins and the implication.*

Now, let us state a lemma that will be useful later.

Lemma 1.2. *Let $\mathbb{H}$, $\mathbb{K}$ be two Heyting algebras and $\varphi : \mathbb{H} \to \mathbb{K}$ be a bijective map between them. Then, φ is an isomorphism in* **Pos** *if and only if it is an isomorphism in* **HA**.

Proof. Clearly, if φ is an isomorphism in **HA** then φ is an isomorphism in **Pos**.
Conversely, let φ^{-1} be the inverse of φ in **Pos**. Let $x, y \in \mathbb{H}$:

$$x \leq y \implies \varphi(x) \leq \varphi(y)$$
$$\varphi(x) \leq \varphi(y) \implies \varphi^{-1}(\varphi(x)) \leq \varphi^{-1}(\varphi(y)) \implies x \leq y$$

i.e. $x \leq y$ if and only if $\varphi(x) \leq \varphi(y)$. Thus, clearly $\varphi(\top) = \top$ and $\varphi(\bot) = \bot$.
Since:

$$\varphi(x) \wedge \varphi(y) \leq \varphi(x) \implies \varphi^{-1}(\varphi(x) \wedge \varphi(y)) \leq x$$
$$\varphi(x) \wedge \varphi(y) \leq \varphi(y) \implies \varphi^{-1}(\varphi(x) \wedge \varphi(y)) \leq y$$

thus $\varphi^{-1}(\varphi(x) \wedge \varphi(y)) \leq x \wedge y$ and $\varphi(x) \wedge \varphi(y) \leq \varphi(x \wedge y)$. In addition,

$$x \wedge y \leq x \implies \varphi(x \wedge y) \leq \varphi(x)$$
$$x \wedge y \leq y \implies \varphi(x \wedge y) \leq \varphi(y)$$

then $\varphi(x \wedge y) \leq \varphi(x) \wedge \varphi(y)$. Analogously for $\vee$. Now, let us show that φ preserves $\to$: for every $z \in \mathbb{H}$ we have that

$$\begin{aligned} z \wedge \varphi(x) \leq \varphi(y) &\text{ iff } \varphi^{-1}(z \wedge \varphi(x)) \leq y \\ &\text{ iff } \varphi^{-1}(z) \wedge x \leq y \\ &\text{ iff } \varphi^{-1}(z) \leq x \to y \\ &\text{ iff } z \leq \varphi(x \to y) \end{aligned}$$

Thus φ is a isomorphism of **HA**. □

Definition 1.14. *Let $(P, \leq)$ be a poset and F a non-empty subset of P. F is a* **filter** *of P if:*

- *for every $x, y \in F$ there exists $z \in F$ such that $z \leq x$ and $z \leq y$;*

- *F is upwards closed.*

If $X \subseteq P$ we say that F is the filter **generated by** *X if F is the smallest filter containing X. If a filter is generated by a singleton then we say that it is* **principal**.

Definition 1.15. *If $\mathbb{H}$ is a Heyting algebra and F is a filter of $\mathbb{H}$, then $\mathbb{H}/F$ is the quotient set induced by the following relation:*

$$x \sim y \quad \Leftrightarrow \quad x \to y \in F \text{ and } y \to x \in F$$

As usual, if $\mathbb{H}$ is a complete Heyting algebra and I is a set, we can consider:

$$\mathbb{H}^I := \{\eta : I \to \mathbb{H} \text{ map}\}$$

Clearly, $\mathbb{H}^I$ is a complete Heyting algebra, where

$$\begin{aligned}
&(\eta \wedge^I \zeta)(i) = \eta(i) \wedge \zeta(i) \\
&(\eta \vee^I \zeta)(i) = \eta(i) \vee \zeta(i) \\
&(\eta \to^I \zeta)(i) = \eta(i) \to \zeta(i) \\
&\top^I(i) = \top \\
&\bot^I(i) = \bot
\end{aligned}$$

for every $i \in I$.

1.3 Frames and locales

Let us start by introducing the following categories:

Definition 1.16. *The category* **Frm** *of* **frames** *is defined as follows:*

- *the objects are complete lattices $H = (H, \leq)$ that satisfy the* infinite distributive law *i.e. $(H, \leq)$ is a complete lattice such that*

  $$a \wedge \bigvee B = \bigvee \{a \wedge b : b \in B\}$$

  for every $a \in H$ and $B \subseteq H$;

- *the morphisms from H to K are maps preserving finite meets and arbitrary joins;*

- *the composition is the usual composition of maps;*
- id_H *is the usual identity map of the set* H.

Definition 1.17. *The category* **Loc** *of* **locales** *is defined as the opposite category of* **Frm**. *The morphisms of* **Loc** *are called* **continuous maps**.

Let us observe that:

Theorem 1.1. *A complete lattices* $H = (H, \leq)$ *is a complete Heyting algebra if and only if* H *satisfies the infinite distributive law.*

Proof. ($\Rightarrow$) Let us suppose that H is a complete Heyting algebra. Let $a \in H$ and $B \subseteq H$. Clearly, $\bigvee\{a \wedge b : b \in B\} \leq a \wedge \bigvee B$. In addition,

$$a \wedge b \leq \bigvee\{a \wedge b : b \in B\} \quad \text{then} \quad b \leq a \to \bigvee\{a \wedge b : b \in B\}$$

for every $b \in B$. Thus:

$$\bigvee B \leq a \to \bigvee\{a \wedge b : b \in B\} \quad \text{then} \quad a \wedge \bigvee B \leq \bigvee\{a \wedge b : b \in B\}$$

Then H satisfies the infinite distributive law.
($\Leftarrow$) Let us suppose that H satisfies the infinite distributive law. Then, for every $a, b \in H$ we define

$$a \to b \ := \ \bigvee\{x \in H : x \wedge a \leq b\}$$

Then, for every $c \in H$

$$\begin{array}{llll} \text{if} & c \leq a \to b & \text{then} & c \wedge a \leq a \wedge (a \to b) \\ & & \text{then} & c \wedge a \leq a \wedge \bigvee\{x \in H : x \wedge a \leq b\} \\ & & \text{then} & c \wedge a \leq \bigvee\{a \wedge x : x \in H \text{ and } x \wedge a \leq b\} \\ & & \text{then} & c \wedge a \leq b \end{array}$$

Since if $c \wedge a \leq b$ then clearly $c \leq a \to b$, thus H is a complete Heyting algebra. ☐

Thus, the objects of **Frm, Loc** and **HA** are exactly the same, while the difference between these three categories is how the morphisms are defined.

Example. Let (X, τ) be a topological space. If we consider the lattice of its open subsets $(\tau, \subseteq)$ then, for every $a \in \tau$ and $S \subseteq \tau$:

$$a \cap \bigcup S = \bigcup\{a \cap s : s \in S\}$$

i.e. $(\tau, \subseteq)$ satisfies the infinite distributive law. Thus, it is a frame. Now, let (Y, σ) be another topological space and $f : X \to Y$ be a continuous map, i.e. a morphism of topological spaces. Then:

$$\begin{aligned} f^{-1} : \sigma &\to \tau \\ s &\mapsto f^{-1}(s) \end{aligned}$$

is clearly well defined and monotonic w.r.t $\subseteq$. Furthermore, f^{-1} preserves arbitrary unions and finite intersections. Then, $f^{-1} : (\sigma, \subseteq) \to (\tau, \subseteq)$ is a morphism of frames.

Lemma 1.3. *Let H and K be two frames and let $\varphi : H \to K$ be a finite meet- preserving map between them. Then φ is a morphism of frames if and only if there exists a map $\psi : K \to H$ such that $\varphi \dashv \psi$ where both φ and ψ are considered as functors between the categories induced by the preorders H and K. Furthermore, ψ is unique.*

Proof. $(\Rightarrow)$. Let φ be a morphism of frames. For every $a \in K$, we define

$$\psi(a) := \bigvee \{y \in H : \varphi(y) \leq a\}$$

It is obvious that ψ is monotone. Let $x \in H$. Clearly, $\varphi(x) \leq a$ implies that $x \leq \psi(a)$. Conversely,

$$\begin{aligned} \text{if } x \leq \psi(a) \quad & \text{then} \quad \varphi(x) \leq \varphi(\psi(a)) \\ & \text{then} \quad \varphi(x) \leq \bigvee \{\varphi(y) : y \in H \text{ and } \varphi(y) \leq a\} \\ & \text{then} \quad \varphi(x) \leq a \end{aligned}$$

Thus $\varphi \dashv \psi$.

$(\Leftarrow)$. Let $\psi : K \to H$ be such that $\varphi \dashv \psi$ and let $B \subseteq H$. Since φ preserves $\wedge$, it is monotone. Thus:

$$\bigvee_{b \in B} \varphi(b) \leq \varphi(\bigvee_{b \in B} b)$$

Conversely,

$$\begin{aligned} & \varphi(b) \leq \bigvee_{b \in B} \varphi(b) && \text{for every } b \in B \\ \text{then} \quad & b \leq \psi(\bigvee_{b \in B} \varphi(b)) && \text{for every } b \in B \\ \text{then} \quad & \bigvee_{b \in B} b \leq \psi(\bigvee_{b \in B} \varphi(b)) && \\ \text{then} \quad & \varphi(\bigvee_{b \in B} b) \leq \bigvee_{b \in B} \varphi(b) && \end{aligned}$$

Then φ preserves arbitrary joins and thus it is a morphism of frames.
We can observe that this property follows from a more general result, which is that every left adjoint preserves the colimits [8].
The uniqueness of ψ follows from the uniqueness of the adjoints between posets. □

1.4 Triposes

Definition 1.18. *A* **Set-based tripos** *is a functor* $\mathsf{P}\colon \mathbf{Set}^{op} \to \mathbf{HA}$ *such that:*

1. *for each* $X, Y \in \mathbf{Set}$ *and for each map* $f : X \to Y$*, the corresponding morphism of Heyting algebras* $\mathsf{P}f : \mathsf{P}Y \to \mathsf{P}X$ *has a left adjoint* $\exists f$ *and a right adjoint* $\forall f$ *when it is seen as a functor between posets, i.e.* $\exists f, \forall f : \mathsf{P}X \to \mathsf{P}Y$ *are monotonic maps such that for all* $p \in \mathsf{P}X$ *and* $q \in \mathsf{P}Y$

$$\exists f(p) \leq_{\mathsf{P}Y} q \Leftrightarrow p \leq_{\mathsf{P}X} \mathsf{P}f(q)$$
$$q \leq_{\mathsf{P}Y} \forall f(p) \Leftrightarrow \mathsf{P}f(q) \leq_{\mathsf{P}X} p$$

2. Beck-Chevalley condition. *For each pullback square in* **Set**

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & X_2 \\ \downarrow{\scriptstyle g_1} & \lrcorner & \downarrow{\scriptstyle g_2} \\ X_3 & \xrightarrow{f_2} & X_4 \end{array}$$

the following two diagrams commute:

$$\begin{array}{ccc} \mathsf{P}X_1 & \xrightarrow{\exists f_1} & \mathsf{P}X_2 \\ {\scriptstyle \mathsf{P}g_1}\uparrow & & {\scriptstyle \mathsf{P}g_2}\uparrow \\ \mathsf{P}X_3 & \xrightarrow{\exists f_2} & \mathsf{P}X_4 \end{array} \qquad \begin{array}{ccc} \mathsf{P}X_1 & \xrightarrow{\forall f_1} & \mathsf{P}X_2 \\ {\scriptstyle \mathsf{P}g_1}\uparrow & & {\scriptstyle \mathsf{P}g_2}\uparrow \\ \mathsf{P}X_3 & \xrightarrow{\forall f_2} & \mathsf{P}X_4 \end{array}$$

i.e. $\exists f_1 \circ \mathsf{P}g_1 = \mathsf{P}g_2 \circ \exists f_2$ *and* $\forall f_1 \circ \mathsf{P}g_1 = \mathsf{P}g_2 \circ \forall f_2$.

3. *there exists a* generic predicate*, i.e. there exists a set* Σ *and a predicate* $tr_\Sigma \in \mathsf{P}\Sigma$ *such that for all sets X, the decoding map*

$$\begin{aligned} [\![\ ]\!]_X : \Sigma^X &\to \mathsf{P}X \\ \sigma &\mapsto \mathsf{P}\sigma(tr_\Sigma) \end{aligned}$$

is surjective.

Remark. 1. Let P be a tripos and $f : X \to Y$ be a map between sets. If $\exists f$ and $\exists' f$ are both left adjoints for $\mathsf{P}f$ then:

$$\exists f(p) \le q \Leftrightarrow p \le \mathsf{P}f(q) \Leftrightarrow \exists' f(p) \le q$$

for every $p \in \mathsf{P}X$ and $q \in \mathsf{P}Y$. Thus, $\exists f = \exists' f$. Analogously we can prove that $\forall f$ is unique.
Let us observe that the notion of tripos does not imply that $\exists f$ and $\forall f$ are morphisms of Heyting algebras, but only monotonic maps. However, it is possible to define two functors:

$$\begin{aligned} \exists : \textbf{Set} &\to \textbf{Pos} \\ X &\mapsto \mathsf{P}X \\ f &\mapsto \exists f \end{aligned} \qquad\qquad \begin{aligned} \forall : \textbf{Set} &\to \textbf{Pos} \\ X &\mapsto \mathsf{P}X \\ f &\mapsto \forall f \end{aligned}$$

In fact, if $f : X \to Y$ and $g : Y \to Z$ are maps between sets then:

$$\begin{aligned} \exists(g \circ f) &= \exists g \circ \exists f \\ \exists(\mathsf{id}_X) &= \mathsf{id}_{\mathsf{P}X} \end{aligned} \qquad\qquad \begin{aligned} \forall(g \circ f) &= \forall g \circ \forall f \\ \forall(\mathsf{id}_X) &= \mathsf{id}_{\mathsf{P}X} \end{aligned}$$

2. Let

$$\begin{array}{ccc} I & \xrightarrow{f_1} & I_1 \\ \downarrow{\scriptstyle f_2} & \lrcorner & \downarrow{\scriptstyle g_1} \\ I_2 & \xrightarrow{g_2} & J \end{array}$$

be a pullback in **Set**; the Beck-Chevalley condition requires that the following two diagrams commute:

$$\begin{array}{ccc} \mathsf{P}I & \xrightarrow{\exists f_1} & \mathsf{P}I_1 \\ {\scriptstyle \mathsf{P}f_2}\uparrow & & \uparrow{\scriptstyle \mathsf{P}g_1} \\ \mathsf{P}I_2 & \xrightarrow{\exists g_2} & \mathsf{P}J \end{array} \qquad \begin{array}{ccc} \mathsf{P}I & \xrightarrow{\forall f_1} & \mathsf{P}I_1 \\ {\scriptstyle \mathsf{P}f_2}\uparrow & & \uparrow{\scriptstyle \mathsf{P}g_1} \\ \mathsf{P}I_2 & \xrightarrow{\forall g_2} & \mathsf{P}J \end{array}$$

But, we can show that it is not necessary to prove the commutativity of both. Indeed

$$\exists f_1 \circ \mathsf{P}f_2 = \mathsf{P}g_1 \circ \exists g_2 \Leftrightarrow \forall f_2 \circ \mathsf{P}f_1 = \mathsf{P}g_2 \circ \forall g_1$$

$\mathsf{P}g_1 \circ \exists g_2$ is a left adjoint of $\mathsf{P}g_2 \circ \forall g_1$, in fact if $p \in \mathsf{P}I_2$ and $p' \in \mathsf{P}I_1$:

$$\begin{aligned} \mathsf{P}g_1(\exists g_2(p)) \le p' &\Leftrightarrow \exists g_2(p) \le \forall g_1(p') \\ &\Leftrightarrow p \le \mathsf{P}g_2(\forall g_1(p')) \end{aligned}$$

Analogously, $\exists f_1 \circ \mathsf{P} f_2$ is a left adjoint of $\forall f_2 \circ \mathsf{P} f_1$:

$$\begin{aligned}\exists f_1(\mathsf{P} f_2(p)) \leq p' &\Leftrightarrow \mathsf{P} f_2(p) \leq \mathsf{P} f_1(p') \\ &\Leftrightarrow p \leq \forall f_2(\mathsf{P} f_1(p'))\end{aligned}$$

We can conclude by uniqueness of the left and the right adjoints.

3. The generic predicate is never unique. In particular, if $h : \Sigma' \to \Sigma$ has a right inverse, then $tr_{\Sigma'} = \mathsf{P}h(tr_\Sigma)$ is another generic predicate for P. Indeed, if $p \in \mathsf{P}X$, there exists $\sigma \in \Sigma^X$ such that $[\![\sigma]\!]_X = p$. Let $\bar{h}$ the right inverse of h, then

$$\mathsf{P}(\bar{h} \circ \sigma)(tr_{\Sigma'}) = \mathsf{P}(\bar{h} \circ \sigma)(\mathsf{P}h(tr_\Sigma)) = \mathsf{P}(h \circ \bar{h} \circ \sigma)(tr_\Sigma) = \mathsf{P}\sigma(tr_\Sigma) = p$$

Lemma 1.4. *Let $\mathsf{P} : \mathbf{Set}^{op} \to \mathbf{HA}$ be a tripos and let $f : X \to Y$ be a map between sets. Then if f has a right (or left) inverse then $\exists f$ and $\forall f$ are left (or right) inverses of $\mathsf{P}f$. Furthermore, if f has an inverse, $\exists f = \forall f$ is the inverse of $\mathsf{P}f$.*

Proof. Let $g : Y \to X$ be the right inverse of f, i.e. $f \circ g = \mathsf{id}_Y$. Then $\mathsf{P}g \circ \mathsf{P}f = \mathsf{P}(f \circ g) = \mathsf{P}\mathsf{id}_Y = \mathsf{id}_{\mathsf{P}Y}$. Let us observe that if $q, q' \in \mathsf{P}Y$ then $\mathsf{P}f(q) \leq \mathsf{P}f(q') \Rightarrow \mathsf{P}g(\mathsf{P}f(q)) \leq \mathsf{P}g(\mathsf{P}f(q'))$, i.e. $q \leq q'$. Hence:

$$\exists f(\mathsf{P}f(q)) \leq q' \Leftrightarrow \mathsf{P}f(q) \leq \mathsf{P}f(q') \Leftrightarrow q \leq q'$$
$$q' \leq \forall f(\mathsf{P}f(q)) \Leftrightarrow \mathsf{P}f(q') \leq \mathsf{P}f(q) \Leftrightarrow q' \leq q$$

Hence, if $q' = q$:

$$\exists f(\mathsf{P}f(q)) \leq q \qquad \text{and} \qquad q \leq \forall f(\mathsf{P}f(q))$$

Conversely, if we choose $q' = \exists f(\mathsf{P}f(q))$ and $q' = \forall f(\mathsf{P}f(q))$ we can prove:

$$q \leq \exists f(\mathsf{P}f(q)) \qquad \text{and} \qquad \forall f(\mathsf{P}f(q)) \leq q$$

Then, $\exists f \circ \mathsf{P}f = \forall f \circ \mathsf{P}f = \mathsf{id}_{\mathsf{P}Y}$.
The case where f has a left inverse is similar.
The case where f has an inverse is obvious from the previous two. ☐

Now, let us introduce a particular type of tripos.

Definition 1.19. *Let $\mathbb{H}$ be a complete Heyting algebra. Then $\mathbb{H}$ induces the following* **Set***-based tripos, called* **Heyting tripos** *or* **forcing tripos***:*

$$\begin{aligned} P : \mathbf{Set}^{op} &\to \mathbf{HA} \\ X &\mapsto \mathbb{H}^X \\ f &\mapsto - \circ f \end{aligned}$$

1.4.1 Interpretation of triposes

Let us recall the main idea that connect triposes to logic, i.e. how a tripos $\mathsf{P} : \mathbf{Set}^{op} \to \mathbf{HA}$ can describe a type of intuitionistic higher-order logic.
We can think every set I as a " type" and the corresponding $\mathsf{P}I$ as the set of predicates over I. In this interpretation, if $p, q \in \mathsf{P}I$ then they can be seen as formulas $p(x), q(x)$ that depends on a variable x of type I.
Then, we can interpret the order of $\mathsf{P}I$ in the following way:

$$p \leq q \text{ means } (\forall x : I)(p(x) \Rightarrow q(x))$$
$$p = q \text{ means } (\forall x : I)(p(x) \Leftrightarrow q(x))$$

Furthermore, since $\mathsf{P}I$ is a Heyting algebra it is also possible to interpret $\wedge, \vee, \to$, true and false.
Now, let $f : I \to J$ be a map and let $q \in \mathsf{P}J$. Thus q can be interpreted as a predicate $q(y)$ depending on a variable y of type J. Then, $\mathsf{P}f : \mathsf{P}J \to \mathsf{P}I$ can have a role of " substitution map" in the sense that:

$$\mathsf{P}f(q) \text{ represents } q(f(x)) \text{ where } x : I$$

Since, $\mathsf{P}f$ is a morphism of $\mathbf{HA}$ the substitution commutes with $\wedge, \vee$ and $\to$ (as logical connectives).
Now, we can use $\exists f$ and $\forall f$ in order to express the existential and universal quantification along f. Indeed, if $p \in \mathsf{P}I$ then:

$$\exists f(p) \text{ means } (\exists x : I)(f(x) = y \wedge p(x))$$
$$\forall f(p) \text{ means } (\forall x : I)(f(x) = y \Rightarrow p(x))$$

Then:

$$\exists f(p) \leq q \text{ iff } p \leq \mathsf{P}f(q)$$
$$\mathsf{P}f(p) \leq q \text{ iff } p \leq \forall f(q)$$

represents:

$$(\forall y : J)\big((\exists x : I)(f(x) = y \wedge p(x)) \Rightarrow q(y)\big) \quad \text{iff} \quad (\forall x : I)\big(p(x) \Rightarrow q(f(x))\big)$$
$$(\forall x : I)\big(q(f(x)) \Rightarrow p(x)\big) \quad \text{iff} \quad (\forall y : J)\big(q(y) \Rightarrow (\forall x : I)(f(x) = y \Rightarrow p(x))\big)$$

$\exists f$ and $\forall f$ are not morphisms of $\mathbf{HA}$ then the existential and the universal quantification do not necessarily commute with all the connectives.

Let π, π' be the first projections of $I \times K$ and $J \times K$ respectively, then the following diagram

$$\begin{array}{ccc} I \times K & \xrightarrow{\pi} & I \\ \downarrow{\scriptstyle f \times \mathsf{id}_K} & \lrcorner & \downarrow{\scriptstyle f} \\ J \times K & \xrightarrow{\pi'} & J \end{array}$$

is clearly a pullback. The Beck-Chevalley condition ensures us that

$$\begin{array}{ccc} \mathsf{P}(I \times K) & \xrightarrow{\exists\pi} & \mathsf{P}I \\ {\scriptstyle \mathsf{P}(f \times \mathsf{id}_K)}\uparrow & \lrcorner & \uparrow{\scriptstyle \mathsf{P}f} \\ \mathsf{P}(J \times K) & \xrightarrow{\exists\pi'} & \mathsf{P}J \end{array} \qquad \begin{array}{ccc} \mathsf{P}(I \times K) & \xrightarrow{\forall\pi} & \mathsf{P}I \\ {\scriptstyle \mathsf{P}(f \times \mathsf{id}_K)}\uparrow & \lrcorner & \uparrow{\scriptstyle \mathsf{P}f} \\ \mathsf{P}(J \times K) & \xrightarrow{\forall\pi'} & \mathsf{P}J \end{array}$$

commute, thus:

$$(\forall x : I)[(\exists z : I \times K)(\pi(z) = x \wedge p((f \times \mathsf{id}_K)(z))) \\ = (\exists w : J \times K)(\pi'(w) = f(x) \wedge p(w))]$$
$$(\forall x : I)[(\forall z : I \times K)(\pi(z) = x \Rightarrow p((f \times \mathsf{id}_K)(z))) \\ = (\forall w : J \times K)(\pi'(w) = f(x) \Rightarrow p(w))]$$

i.e.

$$(\forall x : I)[(\exists z : K)p(y, z)(\{y := f(x), z := z\}) = (\exists z : K)(p(y, z)\{y := f(x)\})]$$
$$(\forall x : I)[(\forall z : K)(p(y, z)\{y := f(x), z := z\}) = (\forall z : K)(p(y, z)\{y := f(x)\})]$$

for every every predicate $p \in \mathsf{P}(J \times K)$. In addition, the role of Σ is to represent the " type of proposition", while

$$tr_\Sigma \quad \text{represents} \quad \text{"}\varphi\text{[2] is true" where } \varphi : \Sigma$$

i.e. the generic predicate expresses the formula asserting that a given proposition is true.

In this idea, the decoding map $[\![\ ]\!]_I$ allows us to turn any functional proposition into a predicate. If $f : I \to \Sigma$ then:

$$[\![f]\!]_I = \mathsf{P}f(tr_\Sigma) \quad \text{represents} \quad \text{"}f(x) \text{ is true" where } x : I$$

The surjectivity of the decoding map ensures us that every predicate of $\mathsf{P}I$ is represented by at least a functional proposition of Σ^I, which means that every predicate of $\mathsf{P}I$ is of the form "$f(x)$ is true", with f a functional proposition from I.

[2] We have used φ as a variable in order to highlight that the variable is a proposition.

Chapter 2

Implicative algebras

2.1 Implicative structures

Definition 2.1. *An* **implicative structure** *is a triple* $(\mathcal{A}, \leq, \to)$ *where* $(\mathcal{A}, \leq)$ *is a complete meet-semilattice, i.e. a poset where every set-indexed family* $(b_i)_{i\in I}$ *of elements of* $\mathcal{A}$ *has a greatest lower bound* $\bigwedge_{i\in I} b_i$*, and* $\to$ *is a binary operation called the* **implication of** $\mathcal{A}$ *such that if* $a, a', b, b' \in \mathcal{A}$ *and* $(b_i)_{i\in I}$ *is a family of elements of* $\mathcal{A}$*:*

- *if* $a' \leq a$ *and* $b \leq b'$ *then* $(a \to b) \leq (a' \to b')$
- $a \to \bigwedge_{i\in B} b_i = \bigwedge_{i\in I}(a \to b_i)$

We will denote $\bot = \bigwedge \mathcal{A}$ *and* $\top = \bigwedge \varnothing$*. Moreover, if* B *is a subset of* $\mathcal{A}$ *we will denote* $\bigwedge_{b\in B} b$ *as* $\bigwedge B$*.*

We write $a \to b \to c$ instead of $a \to (b \to c)$.

Let $\mathcal{A} = (\mathcal{A}, \leq, \to)$ be a fixed implicative structure, we can equip $\mathcal{A}$ with the following operators.

Definition 2.2. *Let* $a, b \in \mathcal{A}$*, the* **application of** a **to** b *is*

$$ab := \bigwedge\{c \in \mathcal{A} : a \leq (b \to c)\}$$

We write $a_1a_2a_3...a_n$ instead of $((a_1a_2)a_3)...a_n$ for all $a_1, a_2, ..., a_n \in \mathcal{A}$.

Lemma 2.1. *Let* $a, a', b, b' \in \mathcal{A}$*, then:*

1. (Monotonicity). *If* $a \leq a'$ *and* $b \leq b'$ *then* $ab \leq a'b'$*;*

2. (β-reduction). $(a \to b)a \leq b$;

3. (η-expansion). $a \leq (b \to ab)$;

4. (Minimum). $ab = \mathsf{min}\{c \in \mathcal{A} : a \leq (b \to c)\}$;

5. (Adjunction). $ab \leq c$ *if and only if* $a \leq (b \to c)$.

Proof. Let $a, b \in \mathcal{A}$, if we define $\mathsf{U}_{a,b} := \{c \in \mathcal{A} : a \leq (b \to c)\}$ then $ab = \bigwedge \mathsf{U}_{a,b}$.

1. *(Monotonicity).* Let $a \leq a'$ and $b \leq b'$, if $c \in \mathsf{U}_{a',b'}$ i.e. $a' \leq (b' \to c)$, then $a \leq a' \leq (b' \to c) \leq (b \to c)$, consequently $\mathsf{U}_{a',b'} \subseteq \mathsf{U}_{a,b}$. Hence, $ab = \bigwedge \mathsf{U}_{a,b} \leq \bigwedge \mathsf{U}_{a',b'} = a'b'$;

2. *(β-reduction).* Since $b \in \mathsf{U}_{a \to b, a}$ then $(a \to b)a = \bigwedge \mathsf{U}_{a \to b, a} \leq b$;

3. *(η-expansion).* $(b \to ab) = (b \to \bigwedge \mathsf{U}_{a,b}) = \bigwedge_{c \in \mathsf{U}_{a,b}} (b \to c) \geq a$;

4. *(Minimum).* By the previous point, $ab \in \mathsf{U}_{a,b}$ and $ab = \bigwedge \mathsf{U}_{a,b}$ then $ab = \mathsf{min} \mathsf{U}_{a,b}$;

5. *(Adjunction).* If $ab \leq c$ then $a \leq (b \to ab) \leq (b \to c)$. Conversely, if $a \leq (b \to c)$ then $c \in \mathsf{U}_{a,b}$, hence $ab = \bigwedge \mathsf{U}_{a,b} \leq c$.

□

Definition 2.3. *Let $f : \mathcal{A} \to \mathcal{A}$ be a map, then we can consider an associated element of $\mathcal{A}$, called the* **abstraction of** f*, in the following way:*

$$\boldsymbol{\lambda} f := \bigwedge_{a \in \mathcal{A}} (a \to f(a))$$

Lemma 2.2. *Let $f, g : \mathcal{A} \to \mathcal{A}$ and $a \in \mathcal{A}$:*

1. (Monotonicity). *If $f(a) \leq g(a)$ for all $a \in \mathcal{A}$ then $\boldsymbol{\lambda} f \leq \boldsymbol{\lambda} g$;*

2. (β-reduction). $(\boldsymbol{\lambda} f)a \leq f(a)$;

3. (η-expansion). $a \leq \boldsymbol{\lambda}(b \mapsto ab)$

Proof. Let $f, g : \mathcal{A} \to \mathcal{A}$ then:

1. *(Monotonicity).* Obvious from the first property in the definition of $\to$;

2. *(β-reduction).* By definition, $\boldsymbol{\lambda} f \leq (a \to f(a))$ hence $(\boldsymbol{\lambda} f)a \leq f(a)$ by Lemma 2.1;

3. *(η-expansion).* By Lemma 2.1, $a \preceq (b \to ab)$ then $\bigwedge_{b\in\mathcal{A}} a \preceq \bigwedge_{b\in\mathcal{A}}(b \to ab)$ i.e. $a \preceq \boldsymbol{\lambda}(b \mapsto ab)$.

□

2.1.1 Semantic typing

In this subsection we will study the *semantic type system* induced by an implicative structure $\mathcal{A}$, in which types correspond to the elements of $\mathcal{A}$.

Let us start by introducing terms. We call a *λ-term with parameters in* $\mathcal{A}$ any λ-term enriched with constants taken in $\mathcal{A}$. Given a closed λ-term t with parameters in $\mathcal{A}$, we can associate it with an element $t^{\mathcal{A}}$ of the implicative structure $\mathcal{A}$ defined inductively in the following way:

$$a^{\mathcal{A}} := a$$
$$(tu)^{\mathcal{A}} := (t^{\mathcal{A}})(u^{\mathcal{A}})$$
$$(\lambda x.t)^{\mathcal{A}} := \boldsymbol{\lambda}(a \mapsto (t\{x := a\})^{\mathcal{A}})$$ [1]

The next theorem states a fundamental property of the λ-term with parameters in $\mathcal{A}$.

Theorem 2.1. *Let t be a λ-term with parameters in $\mathcal{A}$ where $FV(t) = \{x_1, ..., x_n\}$ and $a_1 \preceq a_1', ..., a_n \preceq a_n'$ are parameters in $\mathcal{A}$ then:*

$$(t\{x_1 := a_1, ..., x_n := a_n\})^{\mathcal{A}} \preceq (t\{x_1 := a_1', ..., x_n := a_n'\})^{\mathcal{A}}$$

Proof. By induction on t.

- $t = a$: obvious;

[1] $t\{x := a\}$ denotes the λ-term obtained from t by replacing the variable x with a. In particular:

- if $t = \kappa$ then $t\{x := a\} = \kappa$ where κ is a parameter or a variable different from x;
- if $t = x$ then $t\{x := a\} = a$;
- if $t = us$ then $t\{x := a\} = (u\{x := a\})(s\{x := a\})$;
- if $t = \lambda y.u$ then $t\{x := a\} = \lambda y.(u\{x := a\})$;
- if $t = \lambda x.u$ then $t\{x := a\} = t$.

- $t = x$: obvious;
- $t = uz$: since $(u\{x_1 := a_1, ..., x_n := a_n\})^{\mathcal{A}} \leq (u\{x_1 := a'_1, ..., x_n := a'_n\})^{\mathcal{A}}$ and $(z\{x_1 := a_1, ..., x_n := a_n\})^{\mathcal{A}} \leq (z\{x_1 := a'_1, ..., x_n := a'_n\})^{\mathcal{A}}$ by inductive hypothesis, then $(t\{x_1 := a_1, ..., x_n := a_n\})^{\mathcal{A}} \leq (t\{x_1 := a'_1, ..., x_n := a'_n\})^{\mathcal{A}}$ by Lemma 2.1;
- $t = \lambda x.u$: since $(u\{x_1 := a_1, ..., x_n := a_n, x := a\})^{\mathcal{A}} \leq (u\{x_1 := a'_1, ..., x_n := a'_n, x := a\})^{\mathcal{A}}$ by inductive hypothesis, then

  $$\boldsymbol{\lambda}(a \mapsto (u\{x_1 := a_1, ..., x_n := a_n, x := a\})^{\mathcal{A}}) \leq \\ \leq \boldsymbol{\lambda}(a \mapsto (u\{x_1 := a'_1, ..., x_n := a'_n, x := a\})^{\mathcal{A}})$$

  by Lemma 2.2. Thus:

  $$(t\{x_1 := a_1, ..., x_n := a_n\})^{\mathcal{A}} \leq (t\{x_1 := a'_1, ..., x_n := a'_n\})^{\mathcal{A}}$$

  .

□

Definition 2.4. *A* **typing context** *is a finite (unordered) list* $\Gamma = x_1 : a_1, ..., x_n : a_n$ *where* $x_1, ..., x_n$ *are pairwise distinct* λ*-variables and* $a_1, ..., a_n \in \mathcal{A}$. *We write* $\mathsf{dom}(\Gamma) = \{x_1, ..., x_n\}$.
If Γ *and* Γ' *are typing contexts, we will write* $\Gamma' \leq \Gamma$ *if for every* $(x : a) \in \Gamma$ *there exists* $b \in \mathcal{A}$ *such that* $b \leq a$ *and* $(x : b) \in \Gamma'$.

Given a type context $\Gamma = x_1 : a_1, ..., x_n : a_n$, a λ-term t with parameters in $\mathcal{A}$ and an element $a \in \mathcal{A}$, we can define a typing judgment $\Gamma \vdash t : a$ in the following way:

$$\Gamma \vdash t : a \quad \text{if and only if} \quad FV(t) \subseteq \mathsf{dom}(\Gamma) \text{ and } (t[\Gamma])^{\mathcal{A}} \leq a$$

where, in the notation $t[\Gamma]$, Γ is interpreted as a list of variable assignments, i.e. $t[\Gamma]$ denotes the term $t\{x_1 := a_1, ..., x_n := a_n\}$.

Theorem 2.2. *Let* Γ, Γ' *be typing contexts,* t, u λ*-terms with parameters in* $\mathcal{A}$ *and* a, a', $b \in \mathcal{A}$ *then:*

- (Axiom). *If* $(x : a) \in \Gamma$ *then* $\Gamma \vdash x : a$;
- (Parameter). $\Gamma \vdash a : a$;
- (Subsumption). *if* $\Gamma \vdash t : a$ *and* $a \leq a'$ *then* $\Gamma \vdash t : a'$;

- (Context subsumption). *If* $\Gamma \leq \Gamma'$ *and* $\Gamma \vdash t : a$ *then* $\Gamma' \vdash t : a$*;*

- ($\top$-intro). *If* $FV(t) \subseteq \mathsf{dom}(\Gamma)$ *then* $\Gamma \vdash t : \top$*;*

- ($\to$-intro). *If* $\Gamma, x : a \vdash t : b$ *then* $\Gamma \vdash \lambda x.t : a \to b$*;*

- ($\to$-elim). *If* $\Gamma \vdash t : a \to b$ *and* $\Gamma \vdash u : a$ *then* $\Gamma \vdash tu : b$*;*

- (Generalization). *Let* $(a_i)_{i\in I}$ *be a set-indexed family of elements of* $\mathcal{A}$. *If* $\Gamma \vdash t : a_i$ *for all* $i \in I$, *then* $\Gamma \vdash t : \bigwedge_{i\in I} a_i$.

Proof. Axiom, Parameter, Subsumption, $\top$*-intro* and *Generalization* are obvious.
Context-subsumption follows from the monotonicity of substitution (Theorem 2.1).
In order to show *($\to$-intro)*, we assume $\Gamma, x : a \vdash t : b$ or equivalently $FV(t) \subseteq \mathsf{dom}(\Gamma, x : a)$ and $(t[\Gamma, x : a])^{\mathcal{A}} \leq b$; by definition of typing context it follows that $FV(\lambda x.t) \subseteq \mathsf{dom}(\Gamma)$ and

$$((\lambda x.t)[\Gamma])^{\mathcal{A}} = \bigwedge_{a_0\in\mathcal{A}} (a_0 \to (t[\Gamma, x := a_0])^{\mathcal{A}}) \leq a \to (t[\Gamma, x := a])^{\mathcal{A}} \leq a \to b.$$

Finally, in order to prove $\to$*-elim*, we suppose $FV(t), FV(u) \subseteq \mathsf{dom}(\Gamma)$, $(t[\Gamma])^{\mathcal{A}} \leq a \to b$ and $(u[\Gamma])^{\mathcal{A}} \leq a$, hence $FV(tu) \subseteq \mathsf{dom}(\Gamma)$ and by Lemma 2.1:

$$(tu[\Gamma])^{\mathcal{A}} = (t[\Gamma])^{\mathcal{A}}(u[\Gamma])^{\mathcal{A}} \leq (a \to b)a \leq b.$$

□

Lemma 2.3. *Let* t, u *be two closed* λ*-terms with parameters in* $\mathcal{A}$*. Then:*

- *if* $t \twoheadrightarrow_\beta u$ *then* $t^{\mathcal{A}} \leq u^{\mathcal{A}}$
- *if* $t \twoheadrightarrow_\eta u$ *then* $u^{\mathcal{A}} \leq t^{\mathcal{A}}$

Proof. Let us start by showing that if $t \to_{\beta,1} u$ then $t^{\mathcal{A}} \leq u^{\mathcal{A}}$.

1. if $t = (\lambda x.t_1)t_2$ and $u = t_1\{x := t_2\}$, then:

$$t^{\mathcal{A}} = (\lambda x.t_1)^{\mathcal{A}} t_2^{\mathcal{A}} = \boldsymbol{\lambda}(a \mapsto (t_1\{x := a\})^{\mathcal{A}})t_2^{\mathcal{A}} \leq (t_1\{x := t_2^{\mathcal{A}}\})^{\mathcal{A}} = u^{\mathcal{A}}$$

   by *3* of Lemma 2.2;

2. if $t = t's$ and $u = u's$ where $t' \to_{\beta,1} u'$, then:

$$t^{\mathcal{A}} = t'^{\mathcal{A}} s^{\mathcal{A}} \leq u'^{\mathcal{A}} s^{\mathcal{A}} = u^{\mathcal{A}}$$

by the monotonicity of the application (Lemma 2.1) and by inductive hypothesis. The case in which $t = st'$ and $u = su'$ is analogous;

3. if $t = \lambda x.t'$ and $u = \lambda x.u'$ where $t' \to_{\beta,1} u'$ then $(t'\{x := a\})^{\mathcal{A}} \leq (u'\{x := a\})^{\mathcal{A}}$ for all $a \in \mathcal{A}$ and hence:

$$t^{\mathcal{A}} = \boldsymbol{\lambda}(a \mapsto (t'\{x := a\})^{\mathcal{A}}) \leq \boldsymbol{\lambda}(a \mapsto (u'\{x := a\})^{\mathcal{A}}) = u^{\mathcal{A}}$$

by Lemma 2.2.

Clearly, $t \twoheadrightarrow_{\beta} u$ implies $t^{\mathcal{A}} \leq u^{\mathcal{A}}$ by transitivity of $\leq$.
If $t = \lambda x.ux$, hence $t \to_{\eta,1} u$ and

$$u^{\mathcal{A}} \leq \boldsymbol{\lambda}(a \mapsto u^{\mathcal{A}} a) = t^{\mathcal{A}}$$

by *3* of Lemma 2.2. Similarly to what we have done in the case of β-reduction above, we can conclude using Lemma 2.1, Lemma 2.2 and the transitivity of $\leq$. □

2.2 Implicative algebras

The most important feature of the implicative structures is that every element can represent at the same time a realizer and a truth value, i.e. a set of realizers satisfying some kind of closure property.
The idea is that we can associate every actual realizer t to a truth value $[t]$, called the *principal type of* t, defined as the meets of every truth value containing t.
Conversely, if a is a truth value we could also interpret it as a generalized realizer, in particular as the realizer whose principal type is a itself.
This point of view leads to an important problem: every truth value is realized at least by itself and $\perp$ realizes every truth value. This means we need to equip an implicative structure with a new kind of structure (*separator*) that should play the role of a sort of *criterion of consistency*.

In order to define it, we have to define before the following combinators:

$$\mathbf{K} := \lambda xy.x \qquad \mathbf{S} := \lambda xyz.xz(yz)$$

Lemma 2.4.

$$\mathbf{K}^{\mathcal{A}} = \bigwedge_{a,b\in\mathcal{A}} (a \to b \to a)$$

$$\mathbf{S}^{\mathcal{A}} = \bigwedge_{a,b,c\in\mathcal{A}} ((a \to b \to c) \to (a \to b) \to a \to c)$$

Proof. Clearly:

$$\mathbf{K}^{\mathcal{A}} = (\lambda x.(\lambda y.x))^{\mathcal{A}} = \bigwedge_{a\in\mathcal{A}} (a \to (\bigwedge_{b\in\mathcal{A}} (b \to a))) = \bigwedge_{a,b\in\mathcal{A}} (a \to b \to a)$$

Using semantic type rules, we can show

$$\mathbf{S}^{\mathcal{A}} \leq \bigwedge_{a,b,c\in\mathcal{A}} ((a \to b \to c) \to (a \to b) \to a \to c)$$

in the following way:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Axiom}}{\Gamma \vdash x : a \to b \to c} \quad \dfrac{\text{Axiom}}{\Gamma \vdash z : a}}{\Gamma \vdash xz : b \to c}\,\to\text{-elim.} \qquad \dfrac{\dfrac{\text{Axiom}}{\Gamma \vdash z : a} \quad \dfrac{\text{Axiom}}{\Gamma \vdash y : a \to b}}{\Gamma \vdash yz : b}\,\to\text{-elim.}}{\Gamma := x : a \to b \to c, y : a \to b, z : a \vdash xz(yz) : c}\,\to\text{-elim.}}{x : a \to b \to c, y : a \to b \vdash \lambda z.xz(yz) : a \to c}\,\to\text{-intro.}}{x : a \to b \to c \vdash \lambda yz.xz(yz) : (a \to b) \to a \to c}\,\to\text{-intro.}}{\vdash \lambda xyz.xz(yz) : (a \to b \to c) \to (a \to b) \to a \to c \text{ for all } a,b,c \in \mathcal{A}}\,\to\text{-intro.}}{\vdash \lambda xyz.xz(yz) : \bigwedge_{a,b,c\in\mathcal{A}}((a \to b \to c) \to (a \to b) \to a \to c)}\,\text{Gen.}$$

Conversely:

$$\bigwedge_{a,b\in\mathcal{A}} ((a \to b \to c) \to (a \to b) \to a \to c) \leq$$

$$\leq \bigwedge_{a,d,e\in\mathcal{A}} ((a \to ea \to da(ea)) \to (a \to ea) \to a \to da(ea))$$

so by item 3. of Lemma 2.1:

$$\begin{aligned}
&\bigwedge_{a,b\in\mathcal{A}} ((a \to b \to c) \to (a \to b) \to a \to c) \\
&\qquad \leq \bigwedge_{a,d,e\in\mathcal{A}} ((a \to da) \to (a \to ea) \to a \to da(ea)) \\
&\qquad \leq \bigwedge_{a,d,e\in\mathcal{A}} ((a \to da) \to e \to a \to da(ea)) \\
&\qquad \leq \bigwedge_{a,d,e\in\mathcal{A}} (d \to e \to a \to da(ea)) \\
&\qquad \leq \bigwedge_{d\in\mathcal{A}} \Big(d \to \bigwedge_{e\in\mathcal{A}} \Big(e \to \bigwedge_{a\in\mathcal{A}} (a \to da(ea))\Big)\Big) \\
&\qquad = (\lambda xyz.xz(yz))^{\mathcal{A}} = \mathbf{S}^{\mathcal{A}}
\end{aligned}$$

□

Now we can define:

Definition 2.5. *A* **separator** *of an implicative structure $\mathcal{A}$ is a subset $S \subseteq \mathcal{A}$ such that:*

1. (upwards closed.) *If $a \in S$ and $a \leq b$ then $b \in S$*
2. $\mathbf{K}^{\mathcal{A}}, \mathbf{S}^{\mathcal{A}} \in S$
3. (closed under modus ponens.) *If $(a \to b) \in S$ and $a \in S$ then $b \in S$*

Observation. Let S be an upwards closed subset of $\mathcal{A}$. Then:

S is closed under modus ponens $\Leftrightarrow$ S is closed under application.

Indeed, let us suppose S is closed under modus ponens and $a, b \in S$. By Lemma 2.1 $a \leq (b \to ab)$ and hence $b \to ab \in S$; since $b \in S$ and S is closed under modus ponens, $ab \in S$.
Conversely, if S is closed under application and $a, a \to b \in S$, then $(a \to b)a \in S$ and since $(a \to b)a \leq b$ by Lemma 2.1, then $b \in S$.

Definition 2.6. *Let $\mathcal{A}$ be an implicative structure. We define*

$$\mathsf{cc}^{\mathcal{A}} := \bigwedge_{a,b \in \mathcal{A}} \big(((a \to b) \to a) \to a\big)$$

We say that a separator S of $\mathcal{A}$ is **classical** *if $\mathsf{cc}^{\mathcal{A}} \in S$. While, S is* **consistent** *if $\bot \notin S$.*

Let us observe that:

$$\bigwedge_{b \in \mathcal{A}} (((a \to b) \to a) \to a) \leq (((a \to \bot) \to a) \to a)$$

Furthermore, for every $b \in \mathcal{A}$: $a \to \bot \leq a \to b$ thus $(a \to b) \to a \leq (a \to \bot) \to a$. Then $((a \to \bot) \to a) \to a \leq ((a \to b) \to a) \to a$, thus:

$$\mathsf{cc}^{\mathcal{A}} = \bigwedge_{a \in \mathcal{A}} \big(((a \to \bot) \to a) \to a\big)$$

Finally:

Definition 2.7. *An* **implicative algebra** *is a quadruple $(\mathcal{A}, \leq, \to, S)$ where $(\mathcal{A}, \leq, \to)$ is an implicative structure and S is a separator of $\mathcal{A}$.*

2.3 Interpreting first-order logic

Let $\mathcal{A}$ be an implicative structure and $a, b \in \mathcal{A}$, we will write:

$$
\begin{aligned}
a \times b &:= \bigwedge_{c \in \mathcal{A}} ((a \to b \to c) \to c) \\
a + b &:= \bigwedge_{c \in \mathcal{A}} ((a \to c) \to (b \to c) \to c) \\
\neg a &:= (a \to \bot)
\end{aligned}
$$

Theorem 2.3. Rules for $\times$.

1. $\dfrac{\Gamma \vdash t : a \qquad \Gamma \vdash u : b}{\Gamma \vdash \lambda z.ztu : a \times b}$

2. $\dfrac{\Gamma \vdash t : a \times b}{\Gamma \vdash t(\lambda xy.x) : a}$

3. $\dfrac{\Gamma \vdash t : a \times b}{\Gamma \vdash t(\lambda xy.y) : b}$

Proof. 1. Let $\Gamma' = \Gamma, z : a \to b \to c$ then:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Axiom}}{\Gamma' \vdash z : a \to b \to c} \qquad \dfrac{\Gamma \vdash t : a}{\Gamma' \vdash t : a}\,\text{C. subs.}}{\Gamma' \vdash zt : b \to c}\,\to\text{-elim.} \qquad \dfrac{\Gamma \vdash u : b}{\Gamma' \vdash u : b}\,\text{C. subs.}}{\Gamma' \vdash ztu : c}\,\to\text{-elim.}}{\Gamma \vdash \lambda z.ztu : (a \to b \to c) \to c \quad \text{for all } c \in \mathcal{A}}\,\to\text{-intro.}}{\Gamma \vdash \lambda z.ztu : a \times b}\,\text{Gen.}
$$

2.
$$
\dfrac{\dfrac{\Gamma \vdash t : a \times b}{\Gamma \vdash t : (a \to b \to a) \to a}\,\text{Subs.} \qquad \dfrac{\dfrac{\dfrac{\text{Axiom}}{\Gamma, x : a, y : b \vdash x : a}}{\Gamma, x : a \vdash \lambda y.x : b \to a}\,\to\text{-intro.}}{\Gamma \vdash \lambda xy.x : a \to b \to a}\,\to\text{-intro.}}{\Gamma \vdash t(\lambda xy.x) : a}\,\to\text{-elim.}
$$

3.
$$
\dfrac{\dfrac{\Gamma \vdash t : a \times b}{\Gamma \vdash t : (a \to b \to b) \to b}\,\text{Subs.} \qquad \dfrac{\dfrac{\dfrac{\text{Axiom}}{\Gamma, x : a, y : b \vdash y : b}}{\Gamma, x : a \vdash \lambda y.y : b \to b}\,\to\text{-intro.}}{\Gamma \vdash \lambda xy.y : a \to b \to b}\,\to\text{-intro.}}{\Gamma \vdash t(\lambda xy.y) : b}\,\to\text{-elim.}
$$

□

Theorem 2.4. Rules for $+$.

1. $$\frac{\Gamma \vdash t : a}{\Gamma \vdash \lambda zw.zt : a + b}$$

2. $$\frac{\Gamma \vdash t : b}{\Gamma \vdash \lambda zw.wt : a + b}$$

3. $$\frac{\Gamma \vdash t : a + b \qquad \Gamma, x : a \vdash u : c \qquad \Gamma, y : b \vdash v : c}{\Gamma \vdash t(\lambda x.u)(\lambda y.v) : c}$$

Proof. 1. Let $\Gamma' = \Gamma, z : a \to c, w : b \to c$ then:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Axiom}}{\Gamma' \vdash z : a \to c} \qquad \dfrac{\Gamma \vdash t : a}{\Gamma' \vdash t : a}\,\text{C. subs.}}{\Gamma' \vdash zt : c}\,\to\text{-elim.}}{\Gamma, z : a \to c \vdash \lambda w.zt : (b \to c) \to c}\,\to\text{-intro.}}{\Gamma \vdash \lambda zw.zt : (a \to c) \to (b \to c) \to c \quad \text{for all } c \in \mathcal{A}}\,\to\text{-intro.}}{\Gamma \vdash \lambda zw.zt : a + b}\,\text{Gen.}$$

2. Let $\Gamma' = \Gamma, z : a \to c, w : b \to c$ then:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Axiom}}{\Gamma' \vdash w : b \to c} \qquad \dfrac{\Gamma \vdash t : b}{\Gamma' \vdash t : b}\,\text{C. subs.}}{\Gamma' \vdash wt : c}\,\to\text{-elim.}}{\Gamma, z : a \to c \vdash \lambda w.wt : (b \to c) \to c}\,\to\text{-intro.}}{\Gamma \vdash \lambda zw.wt : (a \to c) \to (b \to c) \to c \quad \text{for all } c \in \mathcal{A}}\,\to\text{-intro.}}{\Gamma \vdash \lambda zw.wt : a + b}\,\text{Gen.}$$

3. Let $\alpha = (a \to c) \to (b \to c) \to c$

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash t : a + b}{\Gamma \vdash t : \alpha}\,\text{Subs.} \qquad \dfrac{\Gamma, x : a \vdash u : c}{\Gamma \vdash \lambda x.u : a \to c}\,\to\text{-intro.}}{\Gamma \vdash t(\lambda x.u) : (b \to c) \to c}\,\to\text{-elim.} \qquad \dfrac{\Gamma, y : b \vdash v : c}{\Gamma \vdash \lambda y.v : b \to c}\,\to\text{-intro.}}{\Gamma \vdash t(\lambda x.u)(\lambda y.v) : c}\,\to\text{-elim.}$$

□

We can also define the universal and the existential quantification of a family of truth values $(a_i)_{i \in I}$ in the following way:

$$\forall_{i \in I} a_i := \bigwedge_{i \in I} a_i \qquad \exists_{i \in I} a_i := \bigwedge_{c \in \mathcal{A}} \Big(\big(\bigwedge_{i \in I} (a_i \to c) \big) \to c \Big)$$

Theorem 2.5. Rules for $\forall$.

1. $$\frac{\Gamma \vdash t : a_i \qquad \textit{for all } i \in I}{\Gamma \vdash t : \forall_{i\in I} a_i}$$

2. $$\frac{\Gamma \vdash t : \forall_{i\in I} a_i \qquad i_0 \in I}{\Gamma \vdash t : a_{i_0}}$$

Proof. Obvious. □

Theorem 2.6. Rules for $\exists$.

1. $$\frac{\Gamma \vdash t : a_{i_0} \qquad i_0 \in I}{\Gamma \vdash \lambda z.zt : \exists_{i\in I} a_i}$$

2. $$\frac{\Gamma \vdash t : \exists_{i\in I} a_i \qquad \Gamma, x : a_i \vdash u : c \qquad \textit{for all } i \in I}{\Gamma \vdash t(\lambda x.u) : c}$$

Proof. 1. Let us consider:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, z : \bigwedge_{i\in I}(a_i \to c) \vdash z : \bigwedge_{i\in I}(a_i \to c)}\text{Axiom}}{\Gamma, z : \bigwedge_{i\in I}(a_i \to c) \vdash z : a_{i_0} \to c}\text{Subs.} \qquad \Gamma \vdash t : a_{i_0} \qquad i_0 \in I}{\Gamma, z : \bigwedge_{i\in I}(a_i \to c) \vdash zt : c}\to\text{-elim.}}{\Gamma \vdash \lambda z.zt : (\bigwedge_{i\in I}(a_i \to c)) \to c \ \text{ for all } c \in \mathcal{A}}\to\text{-intro.}}{\Gamma \vdash \lambda z.zt : \exists_{i\in I} a_i}\text{Gen.}$$

2. Since $\exists_{i\in I} a_i = \forall_{c\in\mathcal{A}}((\bigwedge_{i\in I}(a_i \to c)) \to c)$, we can prove:

$$\dfrac{\dfrac{\Gamma \vdash t : \exists_{i\in I} a_i}{\Gamma \vdash t : (\bigwedge_{i\in I}(a_i \to c)) \to c}\text{Subs.} \qquad \dfrac{\dfrac{\Gamma, x : a_i \vdash u : c \qquad \text{for all } i \in I}{\Gamma \vdash \lambda x.u : (a_i \to c) \qquad \text{for all } i \in I}\to\text{-intro.}}{\Gamma \vdash \lambda x.u : \bigwedge_{i\in I}(a_i \to c)}\text{Gen.}}{\Gamma \vdash t(\lambda x.u) : c}\to\text{-elim.}$$

□

Let α, β be two objects. Then, we define

$$\textbf{id}^{\mathcal{A}}(\alpha, \beta) = \begin{cases} \bigwedge_{a\in\mathcal{A}}(a \to a) & \text{if } \alpha = \beta \\ \top \to \bot & \text{otherwise} \end{cases}$$

Lemma 2.5. Rules for **id**

1. $$\dfrac{}{\Gamma \vdash \lambda x.x : \alpha = \alpha}$$

2. $$\dfrac{\Gamma \vdash t : \mathbf{id}^{\mathcal{A}}(\alpha, \beta) \qquad \Gamma \vdash u : p(\alpha)}{\Gamma \vdash tu : p(\beta)}$$ *where $p : M \to \mathcal{A}$ for some set M.*

Proof. Let us consider:

$$\dfrac{\dfrac{\dfrac{}{\Gamma, x : a \vdash x : a}\text{Axiom}}{\Gamma \vdash \lambda x.x : a \to a \text{ for all } a \in \mathcal{A}}\to\text{-intro.}}{\Gamma \vdash \lambda x.x : \mathbf{id}^{\mathcal{A}}(\alpha, \alpha)}\text{Gen.}$$

In order to prove the second rule, let us start by observing that if $a \in \mathcal{A}$ is such that

$$a \leq \mathbf{id}^{\mathcal{A}}(\alpha, \beta) \quad \text{then} \quad a \leq p(\alpha) \to p(\beta)$$

Indeed, if $\alpha = \beta$ then $a \leq \mathbf{id}^{\mathcal{A}}(\alpha, \beta)$ means that $a \leq \bigwedge_{b \in \mathcal{A}}(b \to b)$, thus $a \leq p(\alpha) \to p(\beta)$. While, if $\alpha \neq \beta$ then $a \leq \top \to \bot \leq p(\alpha) \to p(\beta)$, thus:

$$\dfrac{\dfrac{\Gamma \vdash t : \mathbf{id}^{\mathcal{A}}(\alpha, \beta)}{\Gamma \vdash t : p(\alpha) \to p(\beta)}\text{Subs.} \qquad \Gamma \vdash u : p(\alpha)}{\Gamma \vdash tu : p(\beta)}\to\text{-elim.}$$

□

2.3.1 $\mathcal{A}$-valued interpretations

Let $\mathcal{L}$ be a first-order language.

Definition 2.8. *An* **$\mathcal{A}$-valued interpretation** *of $\mathcal{L}$ is defined by:*

1. *a non-empty set M, called domain of interpretation;*
2. *a function $f^M : M^k \to M \in F$ for each k-ary function symbol f of $\mathcal{L}$;*
3. *a truth-value function $p^{\mathcal{A}} : M^k \to \mathcal{A}$ for each k-ary predicate symbol of $\mathcal{L}$.*

We can interpret every closed term of $\mathcal{L}$ with parameters in M in an element t^M of M , in the following way:

- if $t = m$ where $m \in M$ then $t^M = m$;
- if $t = f(t_1, ..., t_k)$ then $(f(t_1, ..., t_k))^M = f^M(t_1^M, ..., t_k^M)$

In addition, if ϕ is a closed $\mathcal{L}$-formula then we define $\phi^{\mathcal{A}}$ in the following way:

$$
\begin{array}{cc}
(t_1 = t_2)^{\mathcal{A}} := \mathbf{id}^{\mathcal{A}}(t_1^M, t_2^M) & (p(t_1, ..., t_k))^{\mathcal{A}} := p^{\mathcal{A}}(t_1^M, ..., t_k^M) \\
(\phi \Rightarrow \psi)^{\mathcal{A}} := \phi^{\mathcal{A}} \to \psi^{\mathcal{A}} & (\neg\phi)^{\mathcal{A}} := \phi^{\mathcal{A}} \to \bot \\
(\phi \wedge \psi)^{\mathcal{A}} := \phi^{\mathcal{A}} \times \psi^{\mathcal{A}} & (\phi \vee \psi)^{\mathcal{A}} := \phi^{\mathcal{A}} + \psi^{\mathcal{A}} \\
(\forall x \phi(x))^{\mathcal{A}} := \forall_{\alpha \in M}(\phi(\alpha))^M & (\exists x \phi(x))^{\mathcal{A}} := \exists_{\alpha \in M}(\phi(\alpha))^M
\end{array}
$$

Definition 2.9. *Le $\mathcal{A}$ be an implicative structure. The* **intuitionistic core of** $\mathcal{A}$ $\mathcal{S}_J^O(\mathcal{A})$ *is the smallest separator contained in $\mathcal{A}$.*
The **classical core of** $\mathcal{A}$ $\mathcal{S}_K^O(\mathcal{A})$ *is the smallest separator of $\mathcal{A}$, containing* $\mathsf{cc}^{\mathcal{A}}$.

Lemma 2.6. *Let ϕ be a closed formula of $\mathcal{L}$. Then:*

- *if ϕ is an intuitionistic tautology then $\phi^{\mathcal{A}} \in \mathcal{S}_J^O(\mathcal{A})$;*
- *if ϕ is an intuitionistic tautology then $\phi^{\mathcal{A}} \in \mathcal{S}_K^O(\mathcal{A})$.*

Proof. By induction on the derivation in natural deduction of the formula ϕ, we can use Lemmas 2.4, 2.3, 2.6, 2.5, 2.5 2.2 in order to find a closed λ-term t (if the derivation is classical, it can contains also $\mathsf{cc}^{\mathcal{A}}$) such that $\vdash t^{\mathcal{A}} : \phi^{\mathcal{A}}$. We can conclude by Lemma 2.7. □

2.3.2 Heyting algebras induced by implicative algebras

Let $\mathcal{A}$ be an implicative structure and $S \subseteq \mathcal{A}$ be a separator. We can consider a binary relation on $\mathcal{A}$ called **entailment**, induced by S, defined in the following way :

$$a \vdash_S b \Leftrightarrow (a \to b) \in S$$

for all $a, b \in \mathcal{A}$.

Lemma 2.7. *Let $S \subseteq \mathcal{A}$ be a separator, t be a λ-term without parameters in $\mathcal{A}$ such that $FV(t) = \{x_1, ..., x_n\}$ and $a_1, ..., a_n \in S$. Then:*

$$(t\{x_1 := a_1, ..., x_n := a_n\})^{\mathcal{A}} \in S.$$

Proof. If u is a λ-term, we define a term u_0 inductively on u, in the following way:

- if $u = x$ then $u_0 = x$

- if $u = ss'$ then $u_0 = s_0 s'_0$;
- if $u = \lambda x.s$ then $u_0 = \lambda^* x.s_0$

where λ^* is defined as:

- $\lambda^* x.x = \mathbf{SKK}$;
- $\lambda^* x.s = \mathbf{K}s$ if $s \in \{\mathbf{K},\mathbf{S}\}$ or if s is a variable different from x;
- $\lambda^* x.ss' = \mathbf{S}(\lambda^* x.s)(\lambda^* x.s')$;

It can be proved that $u_0 \twoheadrightarrow_\beta u$. We can also observe that if u is closed then u_0 is obtained only from $\mathbf{K}$ and $\mathbf{S}$ by application.
Let t and $a_1, ..., a_n$ be as in the statement, then we can consider the closed term $\tilde{t} := (\lambda x_1...x_n.t)_0$. Then clearly $\tilde{t}^{\mathcal{A}} a_1...a_n \in S$, since $\mathbf{K}^{\mathcal{A}}, \mathbf{S}^{\mathcal{A}} \in S$ and S is closed under application. Then:

$$\tilde{t}^{\mathcal{A}} a_1...a_n \leq (\lambda x_1...x_n.t)^{\mathcal{A}} a_1...a_n \leq (t\{x_1 := a_1, ..., x_n := a_n\})^{\mathcal{A}}$$

where we have used Lemma 2.3. Since S is upwards closed, we can conclude. □

Lemma 2.8. *The relation $\vdash_S$ is a preorder on $\mathcal{A}$.*

Proof. Let $a, b, c \in \mathcal{A}$.

- *Reflexivity.*

$$\dfrac{\dfrac{\text{Axiom}}{x : a \vdash x : a}}{\vdash \lambda x.x : a \to a}\ \to\text{-intro.}$$

  hence $(\lambda x.x)^{\mathcal{A}} \leq a \to a$. Since $(\lambda x.x)^{\mathcal{A}} \in S$ because of Lemma 2.7, then $a \to a \in S$, i.e. $a \vdash_S a$.

- *Transitivity.* Let us suppose $a \vdash_S b$ and $b \vdash_S c$, then:

$$\dfrac{\dfrac{\dfrac{\text{Axiom}}{\Gamma \vdash y : b \to c} \quad \dfrac{\dfrac{\text{Axiom}}{\Gamma \vdash x : a \to b} \quad \dfrac{\text{Axiom}}{\Gamma \vdash z : a}}{\Gamma \vdash xz : b}\ \to\text{-elim.}}{\Gamma \vdash y(xz) : c}\ \to\text{-elim.}}{x : a \to b, y : b \to c \vdash \lambda z.y(xz) : a \to c}\ \to\text{-intro.}$$

  where $\Gamma = x : a \to b, y : b \to c, z : a$. Hence $(\lambda z.y(xz)([x := a \to b, y := b \to c])^{\mathcal{A}} \leq a \to c$. So, $a \to c \in S$ by Lemma 2.7.

$\square$

We will denote with $\mathcal{A}/S = (\mathcal{A}/S, \leq_S)$ the poset induced by the relation of entailment $\vdash_S$, in particular:

- $\mathcal{A}/S = \{[a]_S : a \in \mathcal{A}\}$ is the quotient of $\mathcal{A}$ by the equivalence relation $\dashv\vdash_S$ where
$$a \dashv\vdash_S b \Leftrightarrow a \vdash_S b \text{ and } b \vdash_S a$$
- if $a, b \in \mathcal{A}$: $[a]_S \leq_S [b]_S \Leftrightarrow a \vdash_S b$

We will often use the notation $[a]$ instead of $[a]_S$.

Theorem 2.7. *Let $\mathcal{A}$ be an implicative structure and S be a separator of $\mathcal{A}$. Let us define $H = (\mathcal{A}/S, \leq_S)$ and, given $[a], [b] \in H$:*

$$\begin{aligned}
&[a] \wedge_H [b] := [a \times b]\\
&[a] \vee_H [b] := [a + b]\\
&[a] \to_H [b] := [a \to b]\\
&\top_H := [\top] = S\\
&\bot_H := [\bot] = \{c \in \mathcal{A} : \neg c \in S\}
\end{aligned}$$

then $H = (H, \wedge_H, \vee_H, \to_H, \bot_H, \top_H)$ is a Heyting algebra.

Proof. Let $a, b, c \in \mathcal{A}$.

- $\wedge_H$. Let $[c] \leq_S [a]$ and $[c] \leq_S [b]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{Axiom}}{\Gamma \vdash x : c \to a} \quad \dfrac{\text{Axiom}}{\Gamma \vdash z : c}}{\Gamma \vdash xz : a}\,\to\text{-elim.} \qquad \dfrac{\dfrac{\text{Axiom}}{\Gamma \vdash y : c \to b} \quad \dfrac{\text{Axiom}}{\Gamma \vdash z : c}}{\Gamma \vdash yz : b}\,\to\text{-elim.}}{\Gamma := x : c \to a, y : c \to b, z : c \vdash \lambda w.w(xz)(yz) : a \times b}\,\text{Th. 2.3}}{x : c \to a, y : c \to b \vdash \lambda zw.w(xz)(yz) : c \to a \times b}\,\to\text{-intro}$$

Since $c \to a,\ c \to b \in S$, we can conclude that $c \to a \times b \in S$, by Lemma 2.7, i.e. $[c] \leq_S [a \times b]$. Conversely:

$$\dfrac{\dfrac{\dfrac{\text{Axiom}}{z : a \times b \vdash z : a \times b}}{z : a \times b \vdash z\lambda xy.x : a}\,\text{Th. 2.3}}{\vdash \lambda z.z\lambda xy.x : a \times b \to a}\,\to\text{-intro.}$$

$$\dfrac{\dfrac{\dfrac{\text{Axiom}}{z : a \times b \vdash z : a \times b}\ \text{Th. 2.3}}{z : a \times b \vdash z\lambda xy.y : b}\ \to\text{-intro.}}{\vdash \lambda z.z\lambda xy.y : a \times b \to b}$$

so $[a\times b] \leq_S [a]$ and $[a\times b] \leq_S [b]$ by Lemma 2.3. Hence $\mathsf{inf}_H([a],[b]) = [a \times b]$;

- $\vee_H$. Let $[a] \leq_S [c]$ and $[b] \leq_S [c]$:

$$\dfrac{\dfrac{\dfrac{\text{Axiom}}{\Gamma \vdash z : a + b} \qquad \dfrac{\dfrac{\text{Axiom}}{\Gamma, w : a \vdash x : a \to c} \quad \dfrac{\text{Axiom}}{\Gamma, w : a \vdash w : a}}{\Gamma, w : a \vdash xw : c}\ \to\text{-elim.} \qquad \pi}{\Gamma := x : a \to c, y : b \to c, z : a + b \vdash z(\lambda w.xw)(\lambda u.yu) : c}\ \text{Th. 2.4}}{x : a \to c, y : b \to c \vdash \lambda z.z(\lambda w.xw)(\lambda u.yu) : a + b \to c}\ \to\text{-intro}$$

where π is:

$$\dfrac{\dfrac{\text{Axiom}}{\Gamma, u : b \vdash y : b \to c} \quad \dfrac{\text{Axiom}}{\Gamma, u : b \vdash u : b}}{\Gamma, u : b \vdash yu : c}\ \to\text{-elim.}$$

Then $a + b \to c \in S$, by Lemma 2.7. Furthermore,

$$\dfrac{\dfrac{\dfrac{\text{Axiom}}{x : a \vdash x : a}\ \text{Th. 2.4}}{x : a \vdash \lambda zw.zx : a + b}\ \to\text{-intro.}}{\vdash \lambda xzw.zx : a \to a + b}$$

$$\dfrac{\dfrac{\dfrac{\text{Axiom}}{x : b \vdash x : b}\ \text{Th. 2.4}}{x : b \vdash \lambda zw.wx : a + b}\ \to\text{-intro.}}{\vdash \lambda xzw.wx : b \to a + b}$$

Hence, $[a] \leq_S [a + b] \in S$ and $[b] \leq_S [a + b] \in S$. So $\mathsf{sup}_H([a],[b]) = [a + b]$;

- $\to_H$. Let $[c] \wedge_H [a] \leq_s [b]$, i.e. $(c \times a) \to b \in S$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{Axiom}}{\Gamma \vdash x : (c \times a) \to b} \qquad \dfrac{\dfrac{\text{Axiom}}{\Gamma \vdash y : c} \quad \dfrac{\text{Axiom}}{\Gamma \vdash z : a}}{\Gamma \vdash \lambda w.wyz : c \times a}\ \text{Th. 2.3}}{\Gamma := x : (c \times a) \to b, y : c, z : a \vdash x(\lambda w.wyz) : b}\ \to\text{-elim.}}{x : (c \times a) \to b, y : c \vdash \lambda z.x(\lambda w.wyz) : a \to b}\ \to\text{-intro.}}{x : (c \times a) \to b \vdash \lambda yz.x(\lambda w.wyz) : c \to a \to b}\ \to\text{-intro.}$$

Then $[c] \leq_S [a \to b]$. Conversely, if $[c] \leq_S [a \to b]$:

$$\dfrac{\dfrac{\dfrac{\text{Axiom}}{\Gamma \vdash x : c \to a \to b} \quad \dfrac{\dfrac{\text{Axiom}}{\Gamma \vdash y : c \times a}}{\Gamma \vdash y\lambda x'y'.x' : c}\text{Th. 2.3}}{\Gamma \vdash x(y\lambda x'y'.x') : a \to b}\to\text{-elim.} \quad \dfrac{\dfrac{\text{Axiom}}{\Gamma \vdash y : c \times a}}{\Gamma \vdash y\lambda x'y'.y' : a}\text{Th. 2.3}}{\dfrac{\Gamma := x : c \to a \to b, y : c \times a \vdash x(y\lambda x'y'.x')(y\lambda x'y'.y') : b}{x : c \to a \to b \vdash \lambda y.x(y\lambda x'y'.x')(y\lambda x'y'.y') : (c \times a) \to b}\to\text{-intro.}}\to\text{-elim.}$$

Then $(c \times a) \to b \in S$, i.e. $[c] \wedge_H [a] \leq_S [b]$.

- $\top_H$. If $s \in S$ then

$$\dfrac{\dfrac{\text{Param.}}{x : \top \vdash s : s}}{\vdash \lambda x.s : \top \to s}\to\text{-intro.}$$

thus $[\top] = [s]$. Furthermore, for every $c \in \mathcal{A}$:

$$\dfrac{\dfrac{}{x : c \vdash x : \top}\top\text{-intro.}}{\vdash \lambda x.x : c \to \top}\to\text{-intro.}$$

then $[c] \leq_S [\top]$. Hence, $\top_H = S$.

- $\bot_H$. For every $c \in \mathcal{A}$:

$$\dfrac{\dfrac{\dfrac{\text{Axiom}}{x : \bot \vdash x : \bot}}{x : \bot \vdash x : c}\text{Subs.}}{\vdash \lambda x.x : \bot \to c}\to\text{-intro.}$$

then $[\bot] \leq_S [c]$, i.e. $\bot_H = [\bot]$. Clearly, $[c] = [\bot]$ if and only if $c \to \bot \in S$; thus $\bot_{\mathbb{H}} = \{c \in \mathcal{A} : c \to \bot \in S\}$.

□

2.4 Examples

2.4.1 Complete Heyting algebras and implicative algebras

Let us fix a complete Heyting algebra $\mathbb{H} = (\mathbb{H}, \leq, \wedge, \vee, \to, \top, \bot)$.

Lemma 2.9. $(\mathbb{H}, \leq, \to)$ *is an implicative structure.*

Proof. Clearly $\mathbb{H}$ is a complete meet-semilattice. If $a' \leq a$ and $b \leq b'$ are elements of $\mathbb{H}$ then:

$$\begin{aligned} a \to b \leq a \to b \quad &\text{then} \quad a \to b \wedge a \leq b \\ &\text{then} \quad a \to b \wedge a' \leq b' \\ &\text{then} \quad a \to b \leq a' \to b' \end{aligned}$$

Furthermore, $\bigwedge_{i\in I}(a \to b_i) = a \to \bigwedge_{i\in I} b_i$. Indeed, since $a \to \bigwedge_{i\in I} b_i \leq a \to b_i$ for every $i \in I$, it is obvious that $a \to \bigwedge_{i\in I} b_i \leq \bigwedge_{i\in I}(a \to b_i)$ and since $\bigwedge_{i\in I}(a \to b_i) \leq a \to b_i$ for every $i \in I$

$$\begin{aligned} &\text{then} \quad \bigwedge_{i\in I}(a \to b_i) \wedge a \leq b_i \text{ for every } i \in I \\ &\text{then} \quad \bigwedge_{i\in I}(a \to b_i) \wedge a \leq \bigwedge_{i\in I} b_i \\ &\text{then} \quad \bigwedge_{i\in I}(a \to b_i) \leq a \to \bigwedge_{i\in I} b_i \end{aligned}$$

Hence, we have proved that $\mathbb{H}$ is an implicative structure. $\square$

Let $\mathcal{H} = (\mathbb{H}, \leq, \to)$ be the implicative structure induced by $\mathbb{H}$.
We can observe that the application in $\mathcal{H}$ coincides with the binary meet. Indeed, let $a, b, c \in \mathbb{H}$ then, by Lemma 2.1:

$$ab \leq c \quad \text{if and only} \quad a \leq b \to c \quad \text{if and only} \quad a \wedge b \leq c$$

thus $ab = a \wedge b$.
Furthermore,

$$a \times b = a \wedge b \qquad\qquad a + b = a \vee b$$

Indeed, for every $c \in \mathbb{H}$:

$$\begin{aligned} a \to b \to c \leq a \to b \to c \quad &\text{if and only if} \quad a \wedge b \wedge (a \to b \to c) \leq c \\ &\text{if and only if} \quad a \wedge b \leq (a \to b \to c) \to c \end{aligned}$$

thus $a \wedge b \leq a \times b$. Since $\top \wedge a \wedge b \leq a$ then $\top = a \to b \to a$. Thus:

$$a \times b = \bigwedge_{c\in\mathcal{A}} \big((a \to b \to c) \to c\big) \leq (a \to b \to a) \to a = \top \to a \leq a$$

where the last inequality follows from the fact that $\top \to a \leq \top \to a \Leftrightarrow \top \to a \leq a$. Analogously we can prove that $a \times b \leq b$, thus $a \times b \leq a \wedge b$.

Since $a \wedge (a \to c) = a(a \to c) \leq c$ and $b \wedge (b \to c) = b(b \to c) \leq c$ by Lemma 2.1 then:

$$\begin{aligned} & \big(a \wedge (a \to c)\big) \vee \big(b \wedge (b \to c)\big) \leq c \\ \text{then} \quad & \big(a \wedge (a \to c) \wedge (b \to c)\big) \vee \big(b \wedge (b \to c) \wedge (a \to c)\big) \leq c \\ \text{then} \quad & (a \vee b) \wedge (a \to c) \wedge (b \to c) \leq c \\ \text{then} \quad & a \vee b \leq \big((a \to c) \to (b \to c) \to c\big) \end{aligned}$$

thus $a \vee b \leq a + b$. While, let us observe that:

$$\top \to a = \bigvee\{c : c \wedge \top \leq a\} = a$$

Thus, let $c \in \mathcal{A}$ be such that $a \leq c$ and $b \leq c$, i.e. $a \to c = b \to c = \top$. Then

$$(a \to c) \to (b \to c) \to c = \top \to \top \to c = c$$

Thus $a + b \leq a \vee b$, because $a \vee b = \bigwedge\{c \in \mathbb{H} : a \leq c \text{ and } b \leq c\}$.

Lemma 2.10. *If t is a λ-term such that $FV(t) = \{x_1, ..., x_n\}$ and $a_1, ..., a_n \in \mathcal{H}$, then:*

$$a_1 \wedge ... \wedge a_n \leq (t\{x_1 := a_1, ..., x_n := a_n\})^{\mathcal{H}}$$

Furthermore, if t is closed then $t^{\mathcal{H}} = \top$.

Proof. By induction on t.

- if $t = x_1$ then $a \leq a = (x_1\{x_1 := a\})^{\mathcal{H}}$;
- if $t = u_1 u_2$ then

$$\begin{aligned} & (t\{x_1 := a_1, ..., x_n := a_n\})^{\mathcal{H}} \\ = \; & (u_1\{x_1 := a_1, ..., x_n := a_n\})^{\mathcal{H}} (u_2\{x_1 := a_1, ..., x_n := a_n\})^{\mathcal{H}} \\ = \; & (u_1\{x_1 := a_1, ..., x_n := a_n\})^{\mathcal{H}} \wedge (u_2\{x_1 := a_1, ..., x_n := a_n\})^{\mathcal{H}} \\ \geq \; & a_1 \wedge ... \wedge a_n \end{aligned}$$

  where the last inequality follows by the inductive hypothesis;

- if $t = \lambda y.u$ then

$$\begin{aligned} (t\{x_1 := a_1, ..., x_n := a_n\})^{\mathcal{H}} &= \bigwedge_{b \in \mathbb{H}} (b \to (u\{y := b, x_1 := a_1, ..., x_n := a_n\})^{\mathcal{H}}) \\ &\geq \bigwedge_{b \in \mathbb{H}} (b \to b \wedge a_1 \wedge ... \wedge a_n) \end{aligned}$$

by inductive hypothesis. Furthermore, since $a \leq b \to b \wedge a$, we can conclude that

$$(t\{x_1 := a_1, ..., x_n := a_n\})^{\mathcal{H}} \geq a_1 \wedge ... \wedge a_n$$

□

Now, we want to analyze the separators of an implicative algebra induced by a Heyting algebra.

Lemma 2.11. *Let $S \subseteq \mathbb{H}$. Then S is a separator for $\mathcal{H}$ if and only if it is a filter over $\mathbb{H}$.*

Proof. Let S be a separator. We have already proved that any separator is closed under application, thus $xy \in S$ for every $x, y \in S$. Since $xy = x \wedge y$ we can conclude that S is a filter.
Conversely, let S be a filter. By Lemma 2.10, $\mathbf{K}^{\mathbb{H}} = \mathbf{S}^{\mathbb{H}} = \top$ thus they are elements of S. If $x \to y, x \in S$, there exists $z \in S$ such that $z \leq x \to y$, hence $z \wedge x \leq y$, and $z \leq x$. Then, $z = z \wedge x \leq y$. Since S is upwards closed, then $y \in S$. Thus, S is closed under modus ponens. □

Lemma 2.12. *The following are equivalent:*

1. *$\mathbb{H}$ is a complete Boolean algebra;*
2. *$\mathsf{cc}^{\mathcal{H}} = \top$;*
3. *$t^{\mathcal{H}} = \top$ for all closed λ-terms with $\mathsf{cc}^{\mathcal{H}}$.*

Proof. (1) ⇒ (2). If $\mathbb{H}$ is Boolean, then clearly $((a \to \bot) \to a) \to a = \top$ for every $a \in \mathbb{H}$ thus $\mathsf{cc}^{\mathcal{A}} = \top$.
(2) ⇒ (3). If t is a closed λ-term with $\mathsf{cc}^{\mathcal{H}}$. We define a λ-term u such that $t = u\{x := \mathsf{cc}^{\mathcal{H}}\}$ and $FV(u) = \{x\}$. By Lemma 2.10, then $\top = \mathsf{cc}^{\mathcal{H}} \leq t^{\mathcal{H}}$.
(3) ⇒ (1). Since $\mathsf{cc}^{\mathcal{H}} = \top$, we have that $((a \to \bot) \to a) \to a = \top$ for every $a \in \mathbb{H}$ and since $((a \to \bot) \to a) \to a \leq ((a \to \bot) \to \bot) \to a$, then $\mathbb{H}$ is Boolean. □

2.4.2 Kleene's Realizability

In this subsection, we will study the relationship between implicative algebras and Kleene's realizability.
The main idea of Kleene's realizability is to identify every closed formula as the set of its realizers: fixed an algebra of programs P, every closed formula

φ is interpreted as a subset $[\![\varphi]\!]$ of P. Following this interpretation, for every closed formulas φ, ψ, we define:

$$[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \times [\![\psi]\!] \qquad\qquad [\![\varphi \vee \psi]\!] = [\![\varphi]\!] + [\![\psi]\!]$$

While, the existential and the universal quantification have the following expression:

$$[\![\forall x \phi(x)]\!] = \bigcap_{v \in \mathcal{M}} [\![\varphi(v)]\!] \qquad\qquad [\![\exists x \phi(x)]\!] = \bigcup_{v \in \mathcal{M}} [\![\varphi(v)]\!]$$

Our aim is to show that we can express Kleene's realizability in terms of implicative algebras. Let us start by defining:

Definition 2.10. *$(P, \cdot)$ is a* **partial applicative structure PAS** *if P is a non-empty set and $\cdot : P \times P \rightharpoonup P$ is a partial operation over P called* application. *If $x \cdot y$ is defined we write $x \cdot y \downarrow$ for every $x, y \in P$.*
If P is a PAS, we can define a binary operation on $\mathcal{P}(P)$, called **Kleene's implication**, *such that if $a, b \subseteq P$:*

$$a \to b := \{z \in P : \ \forall x \in a \ z \cdot x \downarrow\in b^2\}$$

Definition 2.11. *A* **partial combinatory algebra PCA** *is a PAS $(P, \cdot)$ such that there exist two elements $\mathsf{k}, \mathsf{s} \in P$ such that if $x, y, z \in P$:*

1. *$(\mathsf{k} \cdot x) \downarrow, (\mathsf{s} \cdot x) \downarrow$ and $((\mathsf{s} \cdot x) \cdot y) \downarrow$;*
2. *$(\mathsf{k} \cdot x) \cdot y \simeq x$;*
3. *$((\mathsf{s} \cdot x) \cdot y) \cdot z \simeq (x \cdot z) \cdot (y \cdot z)$*

where $\simeq$ indicates that either both sides of the equations are undefined or that they are both defined and equal.
A **combinatory algebra** *(***CA***) is a PCA such that the application $\cdot : P \times P \to P$ is total.*

Thus, if P is a non-empty set and $\cdot$ is a binary application on P, then $(P, \cdot)$ is a CA if there exist $\mathsf{k}, \mathsf{s} \in P$ such that, for all $x, y, z \in P$:

1. $(\mathsf{k} \cdot x) \cdot y = x$;
2. $((\mathsf{s} \cdot x) \cdot y) \cdot z = (x \cdot z) \cdot (y \cdot z)$.

[2]The notation $z \cdot x \downarrow\in b$ means that $z \cdot x \downarrow$ and $z \cdot x \in b$.

While, the corresponding Kleene's implication can be defined as:

$$a \to b = \{z \in P : \ \forall x \in a \ z \cdot x \in b\}.$$

Lemma 2.13. *If* $(P, \cdot)$ *is a CA, then* $\mathcal{A} = (\mathcal{P}(P), \subseteq, \to)$*, where* $\to$ *denote Kleene's implication, is an implicative structure.*

Proof. Clearly $(\mathcal{P}(P), \subseteq)$ is a complete meet-semilattice. Let $a, a', b, b' \subseteq P$ such that $a' \subseteq a$ and $b \subseteq b'$; we want to show that $a \to b \subseteq a' \to b'$. If $z \in a \to b$ then $z \cdot x \in b$ for every $x \in a$, thus $z \cdot x \in b'$ for every $x \in a'$, i.e. $a \to b \subseteq a' \to b'$. Now, let $(b_i)_{i \in I}$ be a set-indexed family of subsets of P. If $z \in P$ then:

$$\begin{aligned} z \in a \to \bigcap_{i \in I} b_i \quad &\text{iff} \quad \forall x \in a \ \ z \cdot x \in \bigcap_{i \in I} b_i \quad \text{iff} \quad \forall x \in a \ \ \forall i \in I \ \ z \cdot x \in b_i \ \ \forall x \in a \\ &\text{iff} \quad \forall i \in I \ \ z \in a \to b_i \quad \text{iff} \quad z \in \bigcap_{i \in I} (a \to b_i) \end{aligned}$$

Thus, $\mathcal{A}$ is an implicative structure. □

Lemma 2.14. *Let* $\mathcal{A} = (\mathcal{P}(P), \subseteq, \to)$ *be the implicative structure induced by a CA* $(P, \cdot)$*, then* $S = \mathcal{P}(P) \setminus \{\varnothing\}$ *is a separator of* $\mathcal{A}$*.*

Proof. Clearly, S is upwards closed. Now, let us prove that $\mathbf{K}^{\mathcal{A}}, \mathbf{S}^{\mathcal{A}} \in S$. Let $a, b, c \subseteq P$. Then:

$$\begin{aligned} a \to b \to a &= \{z \in P : \ \forall x \in a \ z \cdot x \in (b \to a)\} \\ &= \{z \in P : \ \forall x \in a, \ \forall y \in b \ (z \cdot x) \cdot y \in a\} \end{aligned}$$

thus, clearly $\mathsf{k} \in a \to b \to a$ and $\mathbf{K}^{\mathcal{A}} \in S$.
While

$$\begin{aligned} (a \to b \to c) \to (a \to b) \to a \to c &= \{z \in P : \forall x \in a \to b \to c \ z \cdot x \in (a \to b) \to a \to c\} \\ &= \{z \in P : \forall y \in a \to b, \forall w \in a, \forall x \in a \to b \to c \ (((z \cdot x) \cdot y) \cdot w) \in c\} \end{aligned}$$

Let us observe that $x \cdot w \in b \to c$ since $x \in a \to b \to c$ and $w \in a$, while $y \cdot w \in b$ because $y \in a \to b$ and $w \in a$. Then:

$$(((\mathsf{s} \cdot x) \cdot y) \cdot w) = (x \cdot w) \cdot (y \cdot w) \in c \quad \text{for every } x \in a \to b \to c, \ y \in a \to b, \ w \in a$$

thus $\mathbf{S}^{\mathcal{A}} \in S$.
Now, let $a \to b \in S$ and $a \in S$. Then, there exists $x, y \in P$ such that $x \in a \to b$ and $y \in a$, then clearly $x \cdot y \in b$, i.e. $b \neq \varnothing$. □

Let $(P, \cdot)$ be a CA and $\mathcal{A} = (P, \cdot, \rightarrow, \mathcal{P}(P) \setminus \varnothing)$ the implicative algebra induced. If $a, b \subseteq P$ then:

$$a \cdot b = \{x \cdot y : x \in a, y \in b\} = ab$$

Indeed, let $c \subseteq P$ then:

$$a \cdot b \subseteq c \quad \text{iff} \quad \forall x \in a, \forall y \in b \;\; x \cdot y \in c \quad \text{iff} \quad a \subseteq b \rightarrow c$$

thus, $a \cdot b = ab$ by Lemma 2.1.

2.4.3 Classical realizability

The main difference between classical and intuitionistic realizability is that in classical realizability every closed formula ϕ is not interpreted as the set of its realizers but as the set of its counter-realizers, i.e. $[\![\phi]\!] \in \mathcal{P}(\Pi)$ where Π is the set of stacks associated to an algebra of classical programs Λ. The set of its realizers are instead defined indirectly as the orthogonal set of $[\![\phi]\!] \in \mathcal{P}(\Pi)$ with respect to a particular relation $\perp\!\!\!\perp \subseteq \Lambda \times \Pi$.
As before, we will show how classical realizability can be expresses through implicative algebras.

Definition 2.12. *We say that* $\mathcal{K} = (\Lambda, \Pi, \oplus, \cdot, \mathsf{k}_{\_}, \mathsf{K}, \mathsf{S}, \mathsf{cc}, PL, \perp\!\!\!\perp)$ *is an* **abstract Krivine structure** *if:*

1. Λ *and* Π *are non empty-sets. We called their elements* $\mathcal{K}$*-terms and* $\mathcal{K}$*-stack respectively;*

2. $\oplus : \Lambda \times \Lambda \rightarrow \Lambda$ *is a map called application. We usually write tu instead of* $\oplus(t, u)$*;*

3. $\cdot : \Lambda \times \Pi \rightarrow \Pi$ *is a map called push;*

4. $\mathsf{k}_{\_} : \Pi \rightarrow \Lambda$ *is a map that associates every* $\pi \in \Pi$ *to a* $\mathcal{K}$*-term* k_π *called the continuation associated to* π*;*

5. K, S *and* cc *are three different elements of* Λ*;*

6. $PL \subseteq \Lambda$ *is closed under application and* $\mathsf{K}, \mathsf{S}, \mathsf{cc} \in PL$*.* PL *is called the set of proof-like* $\mathcal{K}$*-terms;*

7. $\perp\!\!\!\perp \subseteq \Lambda \times \Pi$ *is such that, for every* $t, u, v \in \Lambda$ *and* $\pi, \pi' \in \Pi$*:*

$$\begin{aligned} t \perp\!\!\!\perp u \cdot \pi &\implies tu \perp\!\!\!\perp \pi \\ t \perp\!\!\!\perp \pi &\implies \mathsf{K} \perp\!\!\!\perp t \cdot u \cdot \pi \\ t \perp\!\!\!\perp v \cdot uv \cdot \pi &\implies \mathsf{S} \perp\!\!\!\perp t \cdot u \cdot v \cdot \pi \\ t \perp\!\!\!\perp \mathsf{k}_\pi \cdot \pi &\implies \mathsf{cc} \perp\!\!\!\perp t \cdot \pi \\ t \perp\!\!\!\perp \pi &\implies \mathsf{k}_\pi \perp\!\!\!\perp t \cdot \pi' \end{aligned}$$

$\perp\!\!\!\perp$ *is called the pole of* $\mathcal{K}$*.*

If $a \subseteq \Pi$ we will denote:

$$a^{\perp\!\!\!\perp} := \{t \in \Lambda : \ \forall \pi \in a \ t \perp\!\!\!\perp \pi\}$$

Let us fix an AKS $\mathcal{K}$ and let $\mathcal{A} = (\mathcal{P}(\Pi), \supseteq, \to)$ where

$$a \to b := a^{\perp\!\!\!\perp} \cdot b = \{t \cdot \pi : t \in a^{\perp\!\!\!\perp}, \pi \in b\}$$

for every $a, b \subseteq \Pi$.

Lemma 2.15. $\mathcal{A} = (\mathcal{P}(\Pi), \supseteq, \to)$ *is an implicative structure.*

Proof. Clearly, $(\mathcal{P}(\Pi), \supseteq)$ is a complete meet-semilattice. Let $a, a', b, b' \subseteq \Pi$ be such that $a' \supseteq a$ and $b \supseteq b'$. If $z \in a' \to b'$ then $z = t \cdot \pi$ where $t \in a'^{\perp\!\!\!\perp}, \pi \in b$. Clearly, $a'^{\perp\!\!\!\perp} \subseteq a^{\perp\!\!\!\perp}$, thus $z \in a' \to b'$, i.e. $a \to b \supseteq a' \to b'$.
Now, let $(b_i)_{i \in I}$ a set-indexed family of subsets of Π.

$$\begin{aligned} a \to \bigcup_{i \in I} b_i &= \{t \cdot \pi : t \in a^{\perp\!\!\!\perp}, \pi \in b_i \text{ for some } i \in I\} = \bigcup_{i \in I} \{t \cdot \pi : t \in a^{\perp\!\!\!\perp}, \pi \in b_i\} \\ &= \bigcup_{i \in I} (a \to b_i) \end{aligned}$$

□

Theorem 2.8. *Let* $S = \{a \in \mathcal{A} : a^{\perp\!\!\!\perp} \cap PL \neq \emptyset\}$*. Then* S *is a classical separator of* $\mathcal{A}$*.*

Proof. Clearly, S is upwards closed: if $a, b \subseteq \Pi$ such that $a \in S$ and $a \supseteq b$ then $a^{\perp\!\!\!\perp} \subseteq b^{\perp\!\!\!\perp}$ thus $b^{\perp\!\!\!\perp} \cap PL \neq \emptyset$ and $b \in S$.
Let us observe that $\mathsf{K} \in (\mathbf{K}^{\mathcal{A}})^{\perp\!\!\!\perp}$. Indeed, let $\pi \in (a \to b \to a)$ for some $a, b \subseteq \Pi$ then

$$\pi = t \cdot u \cdot \pi' \ \text{ where } t \in a^{\perp\!\!\!\perp}, u \in b^{\perp\!\!\!\perp}, \pi' \in a$$

thus $t \cdot \pi'$ and $\mathsf{K} \perp\!\!\!\perp \pi$.
Now, let $\pi \in (a \to b \to c) \to (a \to b) \to a \to c$ for some $a, b, c \subseteq \Pi$. Then

$$\pi = t \cdot u \cdot v \cdot \pi' \quad \text{where} \quad t \in (a \to b \to c)^{\perp\!\!\!\perp}, u \in (a \to b)^{\perp\!\!\!\perp}, v \in a^{\perp\!\!\!\perp}, \pi' \in c$$

Clearly if $\tau \in b$ then $u \perp\!\!\!\perp v \cdot \tau$ and thus $uv \in b^{\perp\!\!\!\perp}$. Then:

$$v \cdot uv \cdot \pi' \in a \to b \to c \quad \text{then} \quad t \perp\!\!\!\perp v \cdot uv \cdot \pi' \quad \text{then} \quad \mathsf{S} \perp\!\!\!\perp \pi$$

hence $\mathsf{S} \in (\mathbf{S}^{\mathcal{A}})^{\perp\!\!\!\perp}$.
Let $a, b \subseteq \Pi$ and $\pi \in (((a \to b) \to a) \to a)$. Then $\pi = t \cdot \pi'$ where $t \in ((a \to b) \to a)^{\perp\!\!\!\perp}$ and $\pi' \in a$. Since $\pi' \in a$ then $\mathsf{k}_{\pi'} \perp\!\!\!\perp u \cdot \tau$ for every $u \in a^{\perp\!\!\!\perp}$ and $\tau \in b$. Since $t \in ((a \to b) \to a)^{\perp\!\!\!\perp}$, we have that $t \perp\!\!\!\perp \mathsf{k}_{\pi'} \cdot \pi'$ and consequently $\mathsf{cc} \perp\!\!\!\perp t \cdot \pi'$. Thus $\mathsf{cc}^{\mathcal{A}} \in S$.
If $a, a \to b \in S$ there exists $t \in a^{\perp\!\!\!\perp}$ and $u \in (a \to b)^{\perp\!\!\!\perp}$, thus $tu \in b^{\perp\!\!\!\perp}$. □

Chapter 3

Implicative triposes

Our aim in this chapter is to prove that every implicative algebra induces a **Set**-based tripos, called *implicative tripos*.

3.1 Defining $\mathcal{A}^I/S[I]$

Let us suppose that $\mathcal{A} = (\mathcal{A}, \leq, \to)$ is a fixed implicative and I a fixed set. Then we can define

$$\boldsymbol{\mathcal{A}^I} = (\mathcal{A}^I, \leq^I, \to^I)$$

where:

- $\mathcal{A}^I \; := \{\eta : I \to \mathcal{A} \text{ map}\}$
- $\eta \leq^I \zeta \Leftrightarrow \eta(i) \leq \zeta(i)$ for all $i \in I$
- $(\eta \to^I \zeta)(i) := \eta(i) \to \zeta(i)$ for all $i \in I$.

Lemma 3.1. *$\mathcal{A}^I = (\mathcal{A}^I, \leq^I, \to^I)$, defined above, is an implicative structure.*

Proof. $(\mathcal{A}^I, \leq^I)$ is a complete meet-semilattice: if $(\zeta_j)_{j\in J}$ is a set-indexed family of elements of $\mathcal{A}^I$ then we can define $\bigwedge_{j\in J} \zeta_j : I \to \mathcal{A}$, in the following way $(\bigwedge_{j\in J} \zeta_j)(i) := \bigwedge_{j\in J} \zeta_j(i)$. Clearly, $\bigwedge_{j\in J} \zeta_j$ is the greatest lower bound of $(\zeta_j)_{j\in J}$.
Given $\eta, \eta', \zeta, \zeta' \in \mathcal{A}^I$ such that $\eta' \leq^I \eta$ and $\zeta \leq^I \zeta'$, using the definition of $\leq^I$ and that $\mathcal{A}$ is an implicative structure, it is clear that $(\eta(i) \to \zeta(i)) \leq (\eta'(i) \to \zeta'(i))$ for every $i \in I$, and consequently that $(\eta \to \zeta) \leq^I (\zeta' \to \eta')$.
Furthermore:

$$\big(\eta \to \bigwedge_{j\in J} \zeta_j\big)(i) = \eta(i) \to \bigwedge_{j\in J} \zeta_j(i) = \bigwedge_{j\in J} (\eta(i) \to \zeta_j(i)) = \bigwedge_{j\in J} (\eta \to \zeta_j)$$

□

Now, our aim is to define a suitable separator for $\mathcal{A}^I$, in order to give it the structure of an implicative algebra.

Definition 3.1. *The* **uniform power separator** $S[I] \subseteq \mathcal{A}^I$ *is defined by*

$$\begin{aligned} S[I] &:= \{\eta \in \mathcal{A}^I : \exists s \in S, \forall i \in I, s \leq \eta(i)\} \\ &= \{\eta \in \mathcal{A}^I : \exists s \in S, s \leq \bigwedge_{i \in I} \eta(i)\} \\ &= \{\eta \in \mathcal{A}^I : \bigwedge_{i \in I} \eta(i) \in S\} \end{aligned}$$

The next lemma states that the notion of uniform power separator is well defined.

Lemma 3.2. *The power uniform separator $S[I]$ defined above is actually a separator.*

Proof. It is clear that $S[I]$ is upward closed: let $\eta \in S[I]$ and $\zeta \in \mathcal{A}^I$ such that $\eta \leq \zeta$, then there exists $s \in S$ such that $s \leq \eta(i)$ and consequently $s \leq \zeta(i)$ for all $i \in I$.
Furthermore,

$$\begin{aligned} \mathbf{K}^{\mathcal{A}^I}(i) = (\lambda xy.x)^{\mathcal{A}^I}(i) = \bigwedge_{\eta,\zeta \in \mathcal{A}^I} (\eta \to \zeta \to \eta)(i) &= \bigwedge_{\eta,\zeta \in \mathcal{A}^I} (\eta(i) \to \zeta(i) \to \eta(i)) \\ &= \bigwedge_{a,b \in \mathcal{A}} a \to b \to a = \mathbf{K}^{\mathcal{A}} \end{aligned}$$

then, since $\mathbf{K}^{\mathcal{A}} \in S$, we have that $\mathbf{K}^{\mathcal{A}^I} \in S[I]$. Analogously we can prove that $\mathbf{S}^{\mathcal{A}^I} \in S[I]$.
Now, we want to prove that S is closed under modus ponens. Let $(\eta \to \zeta), \eta \in S[I]$. So there exist $s, s' \in S$ such that $s \leq \eta(i) \to \zeta(i)$ and $s' \leq \eta(i)$ for every $i \in I$. By Lemma 2.1, $ss' \leq (\eta \to \zeta)(i)\eta(i) \leq \zeta(i)$ for every $i \in I$. Since S is closed under application, we have that $\zeta \in S[I]$. □

3.2 Implicative triposes

Theorem 3.1. *Let $\mathcal{A} = (\mathcal{A}, \leq, \to, S)$ be an implicative algebra. Then the correspondence:*

$$\begin{aligned} \mathsf{P} : \mathbf{Set}^{op} &\to \mathbf{HA} \\ \mathrm{I} &\mapsto \mathcal{A}^I / S[I] \\ f &\mapsto [- \circ f] \end{aligned}$$

defines a **Set***-based tripos, called the* **implicative tripos** *induced by* $\mathcal{A}$*.*

Proof. We already showed that $\mathsf{P}I$ is a Heyting algebra for every set I in Theorem 2.7.

- P *is a functor.* Let I, J be sets and $f : I \to J$. Then f induces

  $$\begin{aligned} \mathcal{A}^f : \mathcal{A}^J &\to \mathcal{A}^I \\ \eta &\mapsto \eta \circ f \end{aligned}$$

  Let us suppose $\eta, \zeta \in \mathcal{A}^J$ are such that $\eta \dashv\vdash_{S[J]} \zeta$. This means that $\eta \to \zeta,\ \zeta \to \eta \in S[J]$, so there exist $s, s' \in S$ such that for all $j \in J$:

  $$s \leq \eta(j) \to \zeta(j) \qquad s' \leq \zeta(j) \to \eta(j)$$

  Then for all $i \in I$:

  $$s \leq (\eta \circ f)(i) \to (\zeta \circ f)(i) \qquad s' \leq (\zeta \circ f)(i) \to (\eta \circ f)(i)$$

  so $\mathcal{A}^f(\eta) \to \mathcal{A}^f(\zeta),\ \mathcal{A}^f(\zeta) \to \mathcal{A}^f(\eta) \in S[I]$, or equivalently $\mathcal{A}^f(\eta) \dashv\vdash_{S[I]} \mathcal{A}^f(\zeta)$.
  Therefore, the map $\mathcal{A}^f : \mathcal{A}^J \to \mathcal{A}^I$ factors into a map $\mathsf{P}f : \mathsf{P}J \to \mathsf{P}I$.
  We now have to verify that $\mathsf{P}f$ is a morphism of HAs.
  Let $p = [\eta], q = [\zeta] \in \mathsf{P}J$:

  $$\begin{aligned} \mathsf{P}f(p \wedge q) &= \mathsf{P}f([\eta] \wedge [\zeta]) = \mathsf{P}f([\eta \times \zeta]) = [(\eta \times \zeta) \circ f] \\ &= [i \mapsto (\eta \times \zeta)(f(i))] = [i \mapsto \eta(f(i)) \times \zeta(f(i))] \\ &= [(\eta \circ f) \times (\zeta \circ f)] = [\eta \circ f] \wedge [\zeta \circ f] \\ &= \mathsf{P}f(p) \wedge \mathsf{P}f(q) \end{aligned}$$

  Clearly, the proofs for the other connectives are similar. Then $\mathsf{P}f$ is a morphism of HAs.
  Furthermore, $\mathsf{P}(\mathsf{id}_J) = \mathsf{id}_{\mathsf{P}J}$: let $p = [\eta] \in \mathsf{P}J$ then

  $$\mathsf{P}(\mathsf{id}_J)(p) = \mathsf{P}(\mathsf{id}_J)([\eta]) = [\eta \circ \mathsf{id}_J] = [\eta] = p$$

  P preserves the composition of morphisms: if $f : I \to J$ and $g : K \to I$ then for every $p = [\eta] \in \mathsf{P}J$:

  $$\begin{aligned} \mathsf{P}(f \circ g)(p) &= \mathsf{P}(f \circ g)([\eta]) = [\eta \circ f \circ g] \\ &= \mathsf{P}g([\eta \circ f]) = \mathsf{P}g(\mathsf{P}f([\eta])) \\ &= (\mathsf{P}g \circ \mathsf{P}f)(p) \end{aligned}$$

- *Existence of right adjoints.* Let $f : I \to J$, if $\eta \in \mathcal{A}^I$, we can define

$$\begin{aligned} \forall^0 f(\eta) :\ & J \to \mathcal{A} \\ & j \mapsto \forall^0_f(\eta)(j) := \forall_{f(i)=j}\eta(i) \end{aligned}$$

If $\eta \vdash_{S[I]} \zeta$ then there is $s \in S$ such that:

$$s \leq \bigwedge_{i \in I} (\eta(i) \to \zeta(i))$$

Let $j \in J$ and $i \in I : f(i) = j$ then:

$$s \leq \eta(i) \to \zeta(i) \leq \Big(\bigwedge_{f(i')=j} \eta(i') \Big) \to \zeta(i)$$

so:

$$s \leq \bigwedge_{f(i)=j} \bigg(\Big(\bigwedge_{f(i')=j} \eta(i') \Big) \to \zeta(i) \bigg) = \Big(\bigwedge_{f(i')=j} \eta(i') \Big) \to \bigwedge_{f(i)=j} \zeta(i)$$

Then:

$$s \leq \bigwedge_{j \in J} (\forall^0_f(\eta)(j) \to \forall^0_f(\zeta)(j)) \qquad \text{i.e.} \qquad \forall^0_f(\eta) \vdash_{S[J]} \forall^0_f(\zeta)$$

This means that if $\eta \dashv\vdash_{S[I]} \zeta$ then $\forall^0_f(\eta) \dashv\vdash_{S[J]} \forall^0_f(\zeta)$. Hence, it is possible to define:

$$\begin{aligned} \forall f :\ & \mathsf{P}I \ \to \ \mathsf{P}J \\ & [\eta] \mapsto \big[\forall^0_f(\eta)\big] \end{aligned}$$

Given $p = [\eta] \in \mathsf{P}I$ and $q = [\zeta] \in \mathsf{P}J$, then:

$$\begin{aligned} \mathsf{P}f(q) \leq p &\Leftrightarrow [\zeta \circ f] \leq [\eta] \Leftrightarrow (\zeta \circ f) \to \eta \in S[I] \\ &\Leftrightarrow \bigwedge_{i \in I} \Big((\zeta \circ f)(i) \to \eta(i) \Big) \in S \Leftrightarrow \bigwedge_{j \in J} \bigwedge_{f(i)=j} \Big(\zeta(j) \to \eta(i) \Big) \in S \\ &\Leftrightarrow \bigwedge_{j \in J} \Big(\zeta(j) \to \bigwedge_{f(i)=j} \eta(i) \Big) \in S \Leftrightarrow \bigwedge_{j \in J} \Big(\zeta(j) \to \forall^0_f(\eta)(j) \Big) \in S \\ &\Leftrightarrow \zeta \to \forall^0_f(\eta) \in S[J] \Leftrightarrow q \leq \forall f(p) \end{aligned}$$

- *Existence of left adjoints.* Let $f : I \to J$, if $\eta \in \mathcal{A}^I$, we can define

$$\begin{aligned} \exists^0 f(\eta) : \; & J \to \mathcal{A} \\ & j \mapsto \exists^0_f(\eta)(j) := \exists_{f(i)=j}\eta(i) \end{aligned}$$

Let $\eta, \zeta \in \mathcal{A}^I$ such that $\eta \vdash_{S[I]} \zeta$, then there exists $s \in S$ such that:

$$s \leq \bigwedge_{i \in I} (\eta(i) \to \zeta(i))$$

We denote with $\alpha = \bigwedge_{i\in I}(\eta(i) \to \zeta(i))$. Then:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Axiom}}{\Gamma \vdash x : \exists^0_f(\eta)(j)} \qquad \dfrac{\pi}{\Gamma, z : \eta(i) \vdash y(sz) : c \text{ for all } i \in I : f(i) = j}}{\Gamma := s : \alpha, x : \exists^0_f(\eta)(j), y : \bigwedge_{f(i)=j} \zeta(i) \to c \vdash t := x\lambda z.y(sz) : c}\,\text{Th. 2.6}}{s : \alpha, x : \exists^0_f(\eta)(j) \vdash \lambda y.t : \left(\bigwedge_{f(i)=j} \zeta(i) \to c\right) \to c \text{ for all } c \in \mathcal{A}}\,\to\text{-intro.}}{s : \alpha, x : \exists^0_f(\eta)(j) \vdash \lambda y.t : \exists^0_f(\zeta)(j)}\,\text{Gen.}}{s : \alpha \vdash \lambda xy.t : \exists^0_f(\eta)(j) \to \exists^0_f(\zeta)(j) \text{ for all } j \in J}\,\to\text{-intro.}}{s : \alpha \vdash \lambda xy.t : \bigwedge_{j\in J}\left(\exists^0_f(\eta)(j) \to \exists^0_f(\zeta)(j)\right)}\,\text{Gen.}$$

where π is the following tree:

$$\dfrac{\dfrac{\dfrac{\text{Axiom}}{\Gamma' \vdash y : \bigwedge_{f(i)=j}(\zeta(i) \to c)}}{\Gamma' \vdash y : \zeta(i) \to c}\,\text{Subs.} \qquad \pi'}{\Gamma' := \Gamma, z : \eta(i) \vdash y(sz) : c}\,\to\text{-elim.}$$

and π' is:

$$\dfrac{\dfrac{\dfrac{\text{Axiom}}{\Gamma' \vdash s : \alpha}}{\Gamma' \vdash s : \eta(i) \to \zeta(i)}\,\text{Subs.} \qquad \dfrac{\text{Axiom}}{\Gamma' \vdash z : \eta(i)}}{\Gamma' \vdash sz : \zeta(i)}\,\to\text{-elim.}$$

i.e. we have proved that

$$(\lambda xy.x(\lambda z.y(sz)))^{\mathcal{A}} \leq \bigwedge_{j \in J} (\exists^0_f(\eta)(j) \to \exists^0_f(\zeta)(j))$$

hence:

$$\exists^0_f(\eta) \to \exists^0_f(\zeta) \in S[J] \qquad \text{i.e.} \qquad \exists^0_f \vdash_{S[J]} \exists^0_f(\zeta)$$

by Lemma 2.7. Thus, we can define:

$$\begin{aligned} \exists_f : \mathsf{P}I &\to \mathsf{P}J \\ [\eta] &\mapsto \left[\exists^0_f(\eta)\right] \end{aligned}$$

We want to prove that $\exists_f$ is actually the left adjoint of $\mathsf{P}f$.
Let us start by observing that if $p = [\eta] \in \mathsf{P}I$ and $q = [\zeta] \in \mathsf{P}J$, then:

$$p \le \mathsf{P}f(q) \Leftrightarrow \bigwedge_{j\in J} \bigwedge_{f(i)=j} (\eta(i) \to \zeta(j)) \in S$$

Since:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash x : \exists^0_f \eta(j)}\text{Axiom} \qquad \tau}{\Gamma := z : \curlywedge_{f(i)=j}(\eta(i) \to \zeta(j)), x : \exists^0_f \eta(j) \vdash t := x\lambda y.zy : \zeta(j)}\text{Th. 2.6}}{z : \curlywedge_{f(i)=j}(\eta(i) \to \zeta(j)) \vdash \lambda x.t : \exists^0_f \eta(j) \to \zeta(j) \text{ for all } j \in J}\to\text{-intro.}}{z : \curlywedge_{f(i)=j}(\eta(i) \to \zeta(j)) \vdash \lambda x.t : \curlywedge_{j\in J}(\exists^0_f \eta(j) \to \zeta(j))}\text{Gen.}$$

where τ is the following tree:

$$\dfrac{\dfrac{\dfrac{}{\Gamma, y : \eta(i) \vdash z : \curlywedge_{f(i)=j} \eta(i) \to \zeta(j)}\text{Axiom}}{\Gamma, y : \eta(i) \vdash z : \eta(i) \to \zeta(j)}\text{Subs.} \qquad \dfrac{}{\Gamma, y : \eta(i) \vdash y : \eta(i)}\text{Axiom}}{\Gamma, y : \eta(i) \vdash zy : \zeta(j) \text{ for all } i \in I : f(i) = j}\to\text{-elim.}$$

Then:

$$\bigwedge_{j\in J} \bigwedge_{f(i)=j} (\eta(i) \to \zeta(j)) \in S \Rightarrow \bigwedge_{j\in J} (\exists^0_f \eta(j) \to \zeta(j)) \in S$$

Conversely, let $\Gamma = x : \curlywedge_{j\in J}(\exists^0_f \eta(j) \to \zeta(j)), y : \eta(i)$ then:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash x : \bigwedge_{j\in J}(\exists^0_f \eta(j) \to \zeta(j))}\text{Axiom}}{\Gamma \vdash x : \exists^0_f \eta(j) \to \zeta(j)}\text{Subs.} \qquad \dfrac{\dfrac{}{\Gamma \vdash y : \eta(i)}\text{Axiom}}{\Gamma \vdash \lambda z.zy : \exists^0_f \eta(j)}\text{Th. 2.6}}{\Gamma \vdash x\lambda z.zy : \zeta(j)}\to\text{-elim.}}{x : \bigwedge_{j\in J}(\exists^0_f \eta(j) \to \zeta(j)) \vdash \lambda y.x\lambda z.zy : \eta(i) \to \zeta(j) \text{ for all } i \in I : f(i) = j}\to\text{-intro.}}{x : \bigwedge_{j\in J}(\exists^0_f \eta(j) \to \zeta(j)) \vdash \lambda y.x\lambda z.zy : \bigwedge_{f(i)=j}(\eta(i) \to \zeta(j)) \text{ for all } j \in J}\text{Gen.}}{x : \bigwedge_{j\in J}(\exists^0_f \eta(j) \to \zeta(j)) \vdash \lambda y.x\lambda z.zy : \bigwedge_{j\in J}\bigwedge_{f(i)=j}(\eta(i) \to \zeta(j))}\text{Gen.}}{\vdash \lambda xy.x\lambda z.zy : \bigwedge_{j\in J}(\exists^0_f \eta(j) \to \zeta(j)) \Rightarrow \bigwedge_{j\in J}\bigwedge_{f(i)=j}(\eta(i) \to \zeta(j))}\to\text{-intro.}
$$

hence:

$$\bigwedge_{j\in J}(\exists^0_f \eta(j) \to \zeta(j)) \in S \Rightarrow \bigwedge_{j\in J}\bigwedge_{f(i)=j}(\eta(i) \to \zeta(j)) \in S$$

Now, we can show that $\exists_f$ is the left adjoint of $\mathsf{P}f$:

$$\begin{aligned} p \le \mathsf{P}f(q) &\Leftrightarrow \bigwedge_{j\in J}\bigwedge_{f(i)=j}(\eta(i) \to \zeta(j)) \in S \\ &\Leftrightarrow \bigwedge_{j\in J}(\exists^0_f(\eta)(j) \to \zeta(j)) \in S \\ &\Leftrightarrow \exists f(p) \le q \end{aligned}$$

- *Beck-Chevalley condition.* Let us consider an arbitrary pullback in **Set**

$$\begin{array}{ccc} I & \xrightarrow{f_1} & I_1 \\ {\scriptstyle f_2}\downarrow & \lrcorner & \downarrow{\scriptstyle g_1} \\ I_2 & \xrightarrow{g_2} & J \end{array}$$

We have to prove that the following two diagrams commute:

$$\begin{array}{ccc} \mathsf{P}I & \xrightarrow{\forall f_1} & \mathsf{P}I_1 \\ {\scriptstyle \mathsf{P}f_2}\uparrow & & \uparrow{\scriptstyle \mathsf{P}g_1} \\ \mathsf{P}I_2 & \xrightarrow{\forall g_2} & \mathsf{P}J \end{array} \qquad \begin{array}{ccc} \mathsf{P}I & \xrightarrow{\exists f_1} & \mathsf{P}I_1 \\ {\scriptstyle \mathsf{P}f_2}\uparrow & & \uparrow{\scriptstyle \mathsf{P}g_1} \\ \mathsf{P}I_2 & \xrightarrow{\exists g_2} & \mathsf{P}J \end{array}$$

But, thanks to Remark 1.4, it is sufficient to prove commutativity of the first diagram. Furthermore, we can suppose:

- $I = \{(i_1, i_2) \in I_1 \times I_2 : g_1(i_1) = g_2(i_2)\}$
- $f_1(i_1, i_2) = i_1$ and $f_2(i_1, i_2) = i_2$ for all $(i_1, i_2) \in I$

since all pullbacks in **Set** are like this, up to isomorphism[8].
Let $p = [\eta] \in \mathsf{P}I_2$ then

$$\begin{aligned}(\forall f_1 \circ \mathsf{P}f_2)(p) &= \forall f_1([\eta \circ f_2]) = \forall f_1([(i_1, i_2) \mapsto \eta(i_2)]) \\ &= \left[i_1 \mapsto \bigwedge_{f_1(i_1', i_2') = i_1} \eta(i_2') \right] = \left[i_1 \mapsto \bigwedge_{g_2(i_2) = g_1(i_1)} \eta(i_2) \right] \\ &= \left[i_1 \mapsto \forall^0_{g_2}(\eta)(g_1(i_1)) \right] = \mathsf{P}g_1([\forall^0_{g_2}(\eta)]) \\ &= (\mathsf{P}g_1 \circ \forall g_2)(p)\end{aligned}$$

- *Generic predicate.* Let $\Sigma := \mathcal{A}$ and $tr_\Sigma := [\mathsf{id}_\mathcal{A}] \in \mathsf{P}\Sigma$. Then, we want to show that, if I is a set then the decoding map

$$\begin{aligned}[\![\;\;]\!]_I : \Sigma^I &\to \mathsf{P}I \\ f &\mapsto \mathsf{P}f(tr_\Sigma)\end{aligned}$$

is surjective. Let us suppose $p = [\eta] \in \mathsf{P}I$ then:

$$\mathsf{P}\eta(tr_\Sigma) = \mathsf{P}\eta([\mathsf{id}_\mathcal{A}]) = [\mathsf{id}_\mathcal{A} \circ \eta] = [\eta] = p$$

□

3.2.1 Implicative triposes and forcing triposes

In this section, we want to characterize the implicative triposes induced by complete Heyting algebras.
Let us start by fixing a complete Heyting algebra $\mathbb{H}$ and the subset $S = \{\top\} \subseteq \mathbb{H}$.
Clearly S is a filter, hence $\mathcal{H} = (\mathbb{H}, \leq, \to, S)$ is an implicative algebra.
If I is a set, then $S[I] = \{\top_{\mathcal{H}^I}\}$. Thus:

$$\mathcal{H}^I / S[I] \cong \mathcal{H}^I$$

Then, it is obvious that:

Lemma 3.3. *The implicative tripos induced by the implicative algebra $\mathcal{H} = (\mathbb{H}, \leq, \to, \{\top\})$ coincides with the forcing tripos induced by the complete Heyting algebra $\mathbb{H}$.*

In this case, we can observe that the adjoints have a particular easy definition. Indeed, if $f : I \to J$ and $\eta : I \to \mathbb{H}$ are two maps, then

$$\begin{aligned} \exists f(\eta) : J &\to \mathbb{H} \\ j &\mapsto \bigvee_{f(i)=j} \eta(i) \end{aligned} \qquad\qquad \begin{aligned} \forall f(\eta) : J &\to \mathbb{H} \\ j &\mapsto \bigwedge_{f(i)=j} \eta(i) \end{aligned}$$

Clearly, if $\eta \in \mathbb{H}^I$ and $\zeta \in \mathbb{H}^J$ then

$$\begin{aligned} \exists f(\eta) \le \zeta \quad &\text{if and only if} \quad \forall j \in J : \bigvee_{f(i)=j} \eta(i) \le \zeta(j) \\ &\text{if and only if} \quad \forall j \in J : \ \eta(i) \le \zeta(j) \quad \forall i \in I : f(i) = j \\ &\text{if and only if} \quad \forall i \in I : \ \eta(i) \le (\zeta \circ f)(i) \\ &\text{if and only if} \quad \eta \le \mathsf{P}f(\zeta) \end{aligned}$$

and

$$\begin{aligned} \zeta \le \forall f(\eta) \quad &\text{if and only if} \quad \forall j \in J : \ \zeta(j) \le \bigwedge_{f(i)=j} \eta(i) \\ &\text{if and only if} \quad \forall j \in J : \ \zeta(j) \le \eta(i) \quad \forall i \in I : f(i) = j \\ &\text{if and only if} \quad \forall i \in I : \ (\zeta \circ f)(i) \le \eta \\ &\text{if and only if} \quad \mathsf{P}f(\zeta) \le \eta \end{aligned}$$

Definition 3.2. *Let $\mathcal{A}$ be an implicative structure. Then*

$$⋔^{\mathcal{A}} := (\lambda xy.x)^{\mathcal{A}} \wedge (\lambda xy.y)^{\mathcal{A}} = \bigwedge_{a,b \in \mathcal{A}} (a \to b \to a \wedge b)$$

Let us observe that if $a, b \in \mathcal{A}$ then

$$⋔^{\mathcal{A}}\, ab \le a \qquad\qquad ⋔^{\mathcal{A}}\, ab \le b$$

This element has a fundamental role in defining which separators are filters and which are not. Indeed:

Lemma 3.4. *Let $\mathcal{A}$ be an implicative algebra and S be a separator of $\mathcal{A}$. The following are equivalent:*

1. $⋔^{\mathcal{A}} \in S$;
2. $[a \times b] = [a \wedge b] \in \mathcal{A}/S$ *for all* $a, b \in \mathcal{A}$;
3. S *is a filter w.r.t.* $\le$.

Proof. (1) ⇒ (2). If $a, b \in \mathcal{A}$ then:

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{Axiom}}{x : a \wedge b \vdash x : a \wedge b}}{x : a \wedge b \vdash x : a}\,\text{Subs.} \qquad \dfrac{\dfrac{\text{Axiom}}{x : a \wedge b \vdash x : a \wedge b}}{x : a \wedge b \vdash x : b}\,\text{Subs.}}{x : a \wedge b \vdash \lambda z.zxx : a \times b}\,\text{Th. 2.4}}{\vdash \lambda xz.zxx : a \wedge b \to a \times b}\,\to\text{-intro.}$$

thus $[a \wedge b] \vdash_S [a \times b]$. Conversely,

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Parameter}}{\Gamma \vdash \pitchfork^{\mathcal{A}} : \pitchfork^{\mathcal{A}}}}{\Gamma \vdash \pitchfork^{\mathcal{A}} : a \to b \to a \wedge b}\,\text{Subs.} \qquad \dfrac{\dfrac{\text{Axiom}}{\Gamma \vdash z : a \times b}}{\Gamma \vdash z\lambda xy.x : a}\,\text{Th. 2.4}}{\Gamma \vdash \pitchfork^{\mathcal{A}} (z\lambda xy.x) : b \to a \wedge b}\,\to\text{-elim.} \qquad \dfrac{\dfrac{\text{Axiom}}{\Gamma \vdash z : a \times b}}{\Gamma \vdash z\lambda xy.y : b}\,\text{Th. 2.4}}{\Gamma := z : a \times b \vdash \pitchfork^{\mathcal{A}} (z\lambda xy.x)(z\lambda xy.y) : a \wedge b}\,\to\text{-elim.}}{\vdash \lambda z.\, \pitchfork^{\mathcal{A}} (z\lambda xy.x)(z\lambda xy.y) : a \times b \to a \wedge b}\,\to\text{-intro.}$$

Then $[a \times b] = [a \wedge b]$.
(2) ⇒ (3). Let $a, b \in S$ then $[a] = [b] = [\top]$. Thus, $[a \wedge b] = [a \times b] = [\top \times \top] = [\top] \wedge [\top] = [\top]$, thus $a \wedge b \in S$.
(3) ⇒ (1). Let S be a filter. Since $(\lambda xy.x)^{\mathcal{A}}$ and $(\lambda xy.x)^{\mathcal{A}}$ are in S then also $\pitchfork^{\mathcal{A}} \in S$. □

Now, let us introduce two technical lemmas.

Lemma 3.5. *Let $S \subseteq \mathcal{A}$ be a separator. The following are equivalent:*

1. *S is finitely generated and $\pitchfork^{\mathcal{A}} \in S$;*
2. *S is a principal filter of $\mathcal{A}$;*
3. *$(\mathcal{A}/S, \leq_S)$ is complete and the quotient map from $\mathcal{A}$ to $\mathcal{A}/S$ commutes with arbitrary meets.*

Proof. (1) ⇒ (2). Let S be generated by $\{g_1, ..., g_n\}$. Let

$$\pitchfork_k^{\mathcal{A}} := \bigwedge_{i=1}^{k} (\lambda x_1 ... x_k . x_i)^{\mathcal{A}} = \bigwedge_{a_1, ..., a_k \in \mathcal{A}} (a_1 \to ... a_k \to a_1 \wedge ... \wedge a_k)$$

Let

$$\mathbf{Y} := (\lambda y f. f(yyf))(\lambda z g. g(zzg)) \qquad \Theta := (\mathbf{Y}(\lambda r.\, \pitchfork_{n+1}^{\mathcal{A}}\, g_1 ... g_n (rr)))^{\mathcal{A}}$$

Clearly, $\Theta \in S$. Furthermore, let us observe that for every λ-term with parameters:

$$\begin{aligned}(\mathbf{Y}(\lambda r.t))^{\mathcal{A}} &\leq \Big(\lambda f.f\big((\lambda zg.g(zzg))(\lambda zg.g(zzg))f\big)(\lambda r.t)\Big)^{\mathcal{A}} \\ &\leq \Big((\lambda r.t)\big((\lambda zg.g(zzg))(\lambda zg.g(zzg))(\lambda r.t)\big)\Big)^{\mathcal{A}} \\ &\leq (t\{r := \mathbf{Y}(\lambda r.t\}))^{\mathcal{A}}\end{aligned}$$

thus $\Theta \leq_{\pitchfork_{n+1}^{\mathcal{A}}} g_1 ... g_n(\Theta\Theta) \leq_{\pitchfork_{n+1}^{\mathcal{A}}} \wedge g_1 \wedge ... \wedge g_n \wedge \Theta\Theta$. Thus, if $a \in S0$ then $\Theta \leq a$, i.e. S is generated by $\{\Theta\}$.

(2) $\Rightarrow$ (3). Let S be generated by $\{\Theta\}$ and $(a_i)_{i\in I}$ be a set-indexed family of elements of $\mathcal{A}$. Since $\bigwedge_{i\in I} a_i \leq a_i$ for all $i \in I$, then:

$$[\bigwedge_{i\in I} a_i]_S \leq_S [a_i]_S \quad \forall i \in I$$

Thus, $[\bigwedge_{i\in I} a_i]_S$ is a lower bound of $([a_i]_S)_{i\in I}$. Now, let $\beta = [b]_S \in \mathcal{A}/S$ be another lower bound of $[a_i]_S$ for all $i \in I$,. Clearly, $b \to a_i \in S$ for all $i \in I$. By hypothesis, $\Theta \leq a$ for every $a \in S$ thus $\Theta \leq \bigwedge_{i\in I}(b \to a_i) = b \to \bigwedge_{i\in I} a_i$ i.e. $[b]_s \leq_S [\bigwedge_{i\in I} a_i]_S$. Hence, $[\bigwedge_{i\in I} a_i]_S$ is the greatest lower bound of $([a_i]_S)_{i\in I}$. Hence, we have showed that $(\mathcal{A}/S, \leq_S)$ is complete and that the quotient map $\mathcal{A} \to \mathcal{A}/S$ commutes with arbitrary meets.
(3) $\Rightarrow$ (2) $\Rightarrow$ (1). Let $(\mathcal{A}/S, \leq_S)$ be complete and $[\bigwedge_{i\in I} a_i]_S = \bigwedge_{i\in I}[a_i]_S$ for every set-indexed family of elements of $\mathcal{A}$. Let us observe that

$$[\bigwedge S]_S = \bigwedge_{s\in S}[s]_S = [\top]_S$$

thus $\bigwedge S \in S$. Clearly, then S is a principal filter generated by $\bigwedge S$ and by Lemma 3.4 $\pitchfork^{\mathcal{A}} \in S$. □

Lemma 3.6. *Let S be a separator of an implicative structure $\mathcal{A}$. The following are equivalent:*

1. $S[I] = S^I$*;*
2. *S is closed under all I-indexed meets.*

Proof. (1) $\Rightarrow$ (2). If $\eta : I \to \mathcal{A} \in S^I = S[I]$ then there exists $s \in S$ such that $s \leq \bigwedge_{i\in I} \eta(i)$, hence $\bigwedge_{i\in I} \eta(i) \in S$.

$(2) \Rightarrow (1)$. Clearly, $S[I] = \{\eta : I \to \mathcal{A} : \exists s \in S \text{ such that } s \leq \bigwedge_{i\in I} \eta(i)\} \subseteq S^I$ because S is upwards closed.
Let $\eta : I \to S$ and $s = \bigwedge_{i\in I} \eta(i)$. Since $s \leq \eta(i)$ for all $i \in I$ and $s \in S$ by hypothesis, then $\eta \in S[I]$. $\square$

Finally, we can characterize the forcing triposes.

Theorem 3.2. *Let $\mathsf{P} : \mathbf{Set}^{op} \to \mathbf{HA}$ be the implicative tripos induced by the implicative algebra $\mathcal{A} = (\mathcal{A}, \leq, \to, S)$. Then, the following are equivalent:*

1. *P is isomorphic to a forcing tripos;*
2. *S is a principal filter of $\mathcal{A}$;*
3. *S is finitely generated and $\pitchfork^{\mathcal{A}} \in S$.*

Proof. $(1) \Rightarrow (2)$. Let $\mathbb{H}$ be a complete Heyting algebra and ϕ be a natural isomorphism from P to $\mathsf{P}_{\mathbb{H}}$, where $\mathsf{P}_{\mathbb{H}}$ is the forcing tripos induced by $\mathbb{H}$. Clearly, if $1 = \{*\}$ is a fixed singleton, we have that $\phi_1 : \mathcal{A}/S \to \mathbb{H}$ is an isomorphism of $\mathbf{HA}$, thus $\mathcal{A}/S$ is a complete Heyting algebra. Now, let us fix a set I. For every $i \in I$ we define $\bar{i} : 1 \to I$ as $\bar{i}(*) = i$. Then:

$$\begin{aligned} \mathsf{P}\bar{i} : \mathcal{A}^I/S[I] &\to \mathcal{A}/S \\ [\eta]_{S[I]} &\mapsto [\eta(i)]_S \end{aligned} \qquad\qquad \begin{aligned} \mathsf{P}_{\mathbb{H}}\bar{i} : \mathbb{H}^I &\to \mathbb{H} \\ \zeta &\mapsto \zeta(i) \end{aligned}$$

Clearly, by naturality of ϕ, the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{A}/S & \xrightarrow{\phi_1} & \mathbb{H} \\ {\scriptstyle \mathsf{P}\bar{i}}\uparrow & & \uparrow{\scriptstyle \mathsf{P}_{\mathbb{H}}\bar{i}} \\ \mathcal{A}^I/S[I] & \xrightarrow{\phi_I} & \mathbb{H}^I \end{array}$$

Thus, for every $\eta : I \to \mathcal{A}$ and $i \in I$:

$$\begin{aligned} (\phi_1 \circ \mathsf{P}\bar{i})([\eta]_{S[I]}) &= (\mathsf{P}_{\mathbb{H}}\bar{i} \circ \phi_I)([\eta]_{S[I]}) \\ \phi_1([\eta(i)]_S) &= \phi_I([\eta]_{S[I]})(i) \end{aligned}$$

Now, let:

$$\begin{aligned} \rho_I : \mathcal{A}^I/S[I] &\to (\mathcal{A}/S)^I \\ [\eta]_{S[I]} &\mapsto (i \mapsto \mathsf{P}\bar{i}([\eta]_{S[I]})) = [\eta(i)]_S \end{aligned}$$

then the following diagram

$$
\begin{array}{ccc}
(\mathcal{A}/S)^I & \xrightarrow{\phi_1^I} & \mathbb{H}^I \\
\uparrow{\scriptstyle \rho_I} & & \uparrow{\scriptstyle \mathsf{id}_{\mathbb{H}^I}} \\
\mathcal{A}^I/S[I] & \xrightarrow{\phi^I} & \mathbb{H}^I
\end{array}
\qquad\qquad
\begin{array}{ccc}
[\eta(-)]_S & \overset{\phi_1^I}{\longmapsto} & \phi_1^I([\eta(-)]_S) \\
\uparrow{\scriptstyle \rho_I} & & \uparrow{\scriptstyle \mathsf{id}_{\mathbb{H}^I}} \\
[\eta(-)]_{S[I]} & \overset{\phi^I}{\longmapsto} & \phi^I([\eta(-)]_{S[I]})
\end{array}
$$

commute, since $\mathsf{id}_{\mathbb{H}^I}(\zeta) = i \mapsto \mathsf{P}_{\mathbb{H}}\bar{i}(\zeta)$.
Thus, ρ_I is an isomorphism because ϕ_I, ϕ_1^I and $\mathsf{id}_{\mathbb{H}^I}$ are isomorphism too.
Now, let us observe that for every $\eta, \zeta : I \to \mathcal{A}$:

$$
\begin{aligned}
[\eta]_{S^I} \leq_{S^I} [\zeta]_{S^I} &\Leftrightarrow \eta \to \zeta \in S^I \Leftrightarrow \eta(i) \to \zeta(i) \in S \quad \forall i \in I \\
&\Leftrightarrow [\eta(i)]_S \leq_S [\zeta(i)]_S \quad \forall i \in I
\end{aligned}
$$

then we can define $\alpha_I : \mathcal{A}^I/S^I \to (\mathcal{A}/S)^I$ such that $\alpha_I([\eta]_{S^I}) = i \mapsto [\eta(i)]_S$.
Clearly, it is an isomorphism.
If $\tilde{\mathsf{id}}$ is the is the inclusion of $\mathcal{A}^I/S[I]$ in $\mathcal{A}^I/S^I$ then:

$$
\begin{array}{ccc}
\mathcal{A}^I/S[I] & & \\
\downarrow{\scriptstyle \rho_I} & \searrow{\scriptstyle \tilde{\mathsf{id}}} & \\
\mathcal{A}^I/S^I & \xrightarrow{\alpha_I} & (\mathcal{A}/S)^I
\end{array}
$$

Thus, $\tilde{\mathsf{id}}$ is an isomorphism, thus $S[I] = S^I$. By Lemma 3.6, S is closed under all I-indexed meets. Thus S is a principal filter generated by $\bigwedge S$.
(2) ⇒ (1). Let S be a principal filter. By Lemma 3.5, then $\mathbb{H} := \mathsf{P}1 \cong \mathcal{A}/S$ is a complete Heyting algebra and S is closed under arbitrary meets. Thus by Lemma 3.6, $S[I] = S^I$. Then $\tilde{\mathsf{id}}$ and ρ_I -defined as above- are isomorphisms for all sets I. Thus, since ρ_I is clearly natural, we can conclude that P is isomorphic to $\mathsf{P}_{\mathbb{H}}$.
(2) ⇔ (3). By Lemma 3.5. □

3.2.2 Intuitionistic realizability triposes and quasi-implicative algebras

Definition 3.3. *Let $P = (P, \cdot, \mathsf{k}, \mathsf{s})$ be a PCA. The* **intuitionistic realizability tripos** *induced by P is defined as follows:*

$$
\begin{aligned}
P : \mathbf{Set}^{op} &\to \mathbf{HA} \\
I &\mapsto \mathcal{P}(P)^I / \lhd\rhd_I \\
f &\mapsto [- \circ f]
\end{aligned}
$$

where:

$$\eta \triangleright_I \zeta \quad \textit{if and only if} \quad \bigcap_{i\in I}(\eta(i) \to \zeta(i)) \neq \varnothing$$

for all $\eta, \zeta : I \to \mathcal{P}(P)$.

Theorem 3.3. *If* $P = (P, \cdot, \mathsf{k}, \mathsf{s})$ *is a CA then the intuitionistic realizability tripos induced by* P *coincides with the implicative tripos induced by the implicative algebra* $\mathcal{A} = (\mathcal{P}(P), \subseteq, \to, \mathcal{P}(P) \setminus \varnothing)$*, where* $\to$ *is the Kleene's implication induced by* P*.*

Proof. Let P be the intuitionistic realizability tripos induced by P and $\mathsf{P}^{\mathcal{A}}$ the implicative tripos induced by the implicative algebra $\mathcal{A} = (\mathcal{P}(P), \subseteq, \to, S)$ where $S = \mathcal{P}(P) \setminus \varnothing$. It is sufficient to just show that $\mathsf{P}I = \mathsf{P}^{\mathcal{A}}I$ for every set I. Thus, let I be a set and $\eta, \zeta : I \to \mathcal{P}(P)$, then:

$$\eta \vdash_{S[I]} \zeta \quad \text{iff} \quad \bigcap_{i\in I}(\eta(i) \to \zeta(i)) \in S \quad \text{iff} \quad \bigcap_{i\in I}(\eta(i) \to \zeta(i)) \neq \varnothing \quad \text{iff} \quad \eta \triangleright_I \zeta$$

□

Let P be a PCA. Similarly to the CA case, we can observe that $(\mathcal{P}(P), \subseteq)$ is a complete lattice and that Kleene's implication fulfills the first axiom of definition 2.1. Furthermore, if I is a not-empty set then Kleene's implication also satisfies the second axiom. Indeed, if $I \neq \varnothing$ and $a, b_i \subseteq P$ for all $i \in I$ then:

$$\begin{aligned} a \to \bigcap_{i\in I} b_i &= \{z \in P : \forall x \in a, z \cdot x \downarrow \in \bigcap_{i\in I} b_i\} = \{z \in P : \forall x \in a, z \cdot x \downarrow \in b_i \forall i \in I\} \\ &= \bigcap_{i\in I}(a \to b_i) \end{aligned}$$

While:

$$a \to \bigcap \varnothing = a \to P = \{z \in P : \forall x \in a, z \cdot x \downarrow\} \neq P = \bigcap \varnothing^1$$

This example leads us to define a new type of structure:

Definition 3.4. *A* **quasi-implicative structure** *is a triple* $(\mathcal{A}, \preceq, \to)$ *where* $(\mathcal{A}, \preceq)$ *is a complete meet-semilattice and* $\to$ *is a binary operation such that if* $a, a', b, b' \in \mathcal{A}$ *and* $(b_i)_{i\in I}$ *is a non-empty set-indexed family of elements of* $\mathcal{A}$*:*

[1] In the lattice $(\mathcal{P}(P), \subseteq)$, the intersection of a set-indexed family $(b_i)_{i\in I}$ of subsets of P is defined as $\bigcap_{i\in I} b_i := \{z \in P : z \in b_i \text{ for every } i \in I\}$, thus $\bigcap \varnothing = \{z \in P\} = P$.

- *if* $a' \leq a$ *and* $b \leq b'$ *then* $(a \to b) \leq (a' \to b')$
- $a \to \bigwedge_{i \in I} b_i = \bigwedge_{i \in I}(a \to b_i)$

Thus, the difference between a quasi-implicative and an implicative structures is that

$$a \to \top = \top \quad \text{for all } a \in \mathcal{A}$$

does not hold in the quasi-implicative structures.

Given a quasi-implicative structure $(\mathcal{A}, \leq, \to)$, we can define an associated implicative structure $(\mathcal{B}, \leq_{\mathcal{B}}, \to_{\mathcal{B}})$ called the ***completion of*** $\mathcal{A}$ in the following way:

1. $\mathcal{B} = \mathcal{A} \cup \{\top_{\mathcal{B}}\}$ where $\top_{\mathcal{B}}$ is a new element;
2. if $b, b' \in \mathcal{B}$ then: $b \leq_{\mathcal{B}} b'$ if and only if $b \leq b'$ or $b' = \top_{\mathcal{B}}$;
3. if $b, b' \in \mathcal{B}$ then:

$$b \to_{\mathcal{B}} b' = \begin{cases} b \to b' & \text{if } b, b' \in \mathcal{A} \\ \top_{\mathcal{A}} \to b' & \text{if } b = \top_{\mathcal{B}}, b' \in \mathcal{A} \\ \top_{\mathcal{B}} & \text{if } b' = \top_{\mathcal{B}} \end{cases}$$

Lemma 3.7. *The completion of a quasi-implicative structure is an implicative structure.*

Proof. Let $(\mathcal{A}, \leq, \to)$ be a quasi-implicative structure and $(\mathcal{B}, \leq_{\mathcal{B}}, \to_{\mathcal{B}})$ its completion. It is clear that $(\mathcal{B}, \leq_{\mathcal{B}}, \to)$ is a quasi-implicative structure. Let us show that it is actually complete: if $a \in \mathcal{B}$ then

$$a \to \bigwedge \varnothing = a \to \top_{\mathcal{B}} = \top_{\mathcal{B}} = \bigwedge \varnothing$$

□

Let $\mathcal{A}$ be a quasi-implicative structure. Similarly to what we have done for the implicative structures, we can equip $\mathcal{A}$ with two partial operations:

if $\{c \in \mathcal{A} : a \leq b \to c\} \neq \varnothing$	then	$ab = \bigwedge\{c \in \mathcal{A} : a \leq b \to c\}$
if f is a partial function from $\mathcal{A}$ to $\mathcal{A}$	then	$\boldsymbol{\lambda} f = \bigwedge_{a \in \mathsf{dom}(f)} (a \to f(a))$

We can also define a partial function $t \mapsto t^{\mathcal{A}}$ defined in the same way we did for the implicative structures.
If we define the judgment:

$$\Gamma \vdash t : a \Leftrightarrow F(t) \subseteq (\Gamma),\ (t[\Gamma])^{\mathcal{A}} \text{ is well defined},\ (t[\Gamma])^{\mathcal{A}} \leq a$$

then all the semantic rules we have proved in section 2.1.1 remain valid.
Furthermore, if we extend the notion of separator to the quasi-implicative structure we can also define:

Definition 3.5. *A* **quasi-implicative algebra** *is a quasi-implicative structure equipped with a separator.*

It is clear that every quasi-implicative algebra induces a tripos, called ***quasi-implicative tripos***, in the same way that every implicative algebra induces the implicative tripos.

Lemma 3.8. *Let* $\mathcal{A} = (\mathcal{A}, \leq, \to)$ *be a a quasi-implicative structure and* $\mathcal{B} = (\mathcal{B}, \leq_{\mathcal{B}}, \to_{\mathcal{B}})$ *its completion. If* $\phi : \mathcal{A} \to \mathcal{B}$ *is the inclusion of* $\mathcal{A}$ *into* $\mathcal{B}$ *then* $\phi(\mathbf{K}^{\mathcal{A}}) = \mathbf{K}^{\mathcal{B}}$ *and* $\phi(\mathbf{S}^{\mathcal{A}}) = \mathbf{S}^{\mathcal{B}}$.

Proof.

$$\begin{aligned}
\mathbf{K}^{\mathcal{B}} &= \bigwedge_{a,b\in\mathcal{B}} (a \to_{\mathcal{B}} b \to_{\mathcal{B}} a) \\
&= \bigwedge_{a,b\in\mathcal{A}} (a \to_{\mathcal{B}} b \to_{\mathcal{B}} a) \wedge \bigwedge_{a\in\mathcal{A}} (a \to_{\mathcal{B}} \top_{\mathcal{B}} \to_{\mathcal{B}} a) \wedge \bigwedge_{b\in\mathcal{B}} (\top_{\mathcal{B}} \to_{\mathcal{B}} b \to_{\mathcal{B}} \top_{\mathcal{B}}) \\
&= \bigwedge_{a,b\in\mathcal{A}} (a \to b \to a) \wedge \bigwedge_{a\in\mathcal{A}} (a \to_{\mathcal{B}} \top_{\mathcal{A}} \to a) \wedge \bigwedge_{b\in\mathcal{B}} (\top_{\mathcal{B}} \to_{\mathcal{B}} \top_{\mathcal{B}}) \\
&= \bigwedge_{a,b\in\mathcal{A}} (a \to b \to a) \wedge \bigwedge_{a\in\mathcal{A}} (a \to \top_{\mathcal{A}} \to a) \wedge \top_{\mathcal{B}} \\
&= \bigwedge_{a,b\in\mathcal{A}} (a \to b \to a) \\
&= \bigwedge_{a,b\in\mathcal{A}} \phi(a \to b \to a) \\
&= \phi(\mathbf{K}^{\mathcal{A}})
\end{aligned}$$

Clearly $\mathbf{S}^{\mathcal{B}} \leq \phi(\mathbf{S}^{\mathcal{A}})$. Furthermore, we can observe that

$$\begin{aligned}
\text{if } a \in \mathcal{B}, b \in \mathcal{B}, c = \top_{\mathcal{B}} : & \ (a \to_{\mathcal{B}} b \to_{\mathcal{B}} \top_{\mathcal{B}}) \to_{\mathcal{B}} (a \to_{\mathcal{B}} b) \to_{\mathcal{B}} a \to_{\mathcal{B}} \top_{\mathcal{B}} \\
& = (a \to_{\mathcal{B}} b \to_{\mathcal{B}} \top_{\mathcal{B}}) \to_{\mathcal{B}} (a \to_{\mathcal{B}} b) \to_{\mathcal{B}} \top_{\mathcal{B}} \\
& = (a \to_{\mathcal{B}} b \to_{\mathcal{B}} \top_{\mathcal{B}}) \to_{\mathcal{B}} \top_{\mathcal{B}} \\
& = \top_{\mathcal{B}} \\
\text{if } a = \top_{\mathcal{B}}, b \in \mathcal{A}, c \in \mathcal{A} : & \ (\top_{\mathcal{B}} \to_{\mathcal{B}} b \to_{\mathcal{B}} c) \to_{\mathcal{B}} (\top_{\mathcal{B}} \to_{\mathcal{B}} b) \to_{\mathcal{B}} \top_{\mathcal{B}} \to_{\mathcal{B}} c \\
& = (\top_{\mathcal{A}} \to b \to c) \to (\top_{\mathcal{A}} \to b) \to \top_{\mathcal{A}} \to c \\
\text{if } a \in \mathcal{A}, b = \top_{\mathcal{B}}, c \in \mathcal{A} : & \ (a \to_{\mathcal{B}} \top_{\mathcal{B}} \to_{\mathcal{B}} c) \to_{\mathcal{B}} (a \to_{\mathcal{B}} \top_{\mathcal{B}}) \to_{\mathcal{B}} a \to_{\mathcal{B}} c \\
& = (a \to_{\mathcal{B}} \top_{\mathcal{A}} \to c) \to_{\mathcal{B}} \top_{\mathcal{B}} \to_{\mathcal{B}} a \to c \\
& = (a \to \top_{\mathcal{A}} \to c) \to_{\mathcal{B}} \top_{\mathcal{A}} \to a \to c \\
& = (a \to \top_{\mathcal{A}} \to c) \to \top_{\mathcal{A}} \to a \to c \\
\text{if } a = \top_{\mathcal{B}}, b = \top_{\mathcal{B}}, c \in \mathcal{A} : & \ (\top_{\mathcal{B}} \to_{\mathcal{B}} \top_{\mathcal{B}} \to_{\mathcal{B}} c) \to_{\mathcal{B}} (\top_{\mathcal{B}} \to_{\mathcal{B}} \top_{\mathcal{B}}) \to_{\mathcal{B}} \top_{\mathcal{B}} \to_{\mathcal{B}} c \\
& = (\top_{\mathcal{B}} \to_{\mathcal{B}} \top_{\mathcal{A}} \to c) \to_{\mathcal{B}} \top_{\mathcal{B}} \to_{\mathcal{B}} \top_{\mathcal{A}} \to c \\
& = (\top_{\mathcal{A}} \to \top_{\mathcal{A}} \to c) \to_{\mathcal{B}} \top_{\mathcal{A}} \to \top_{\mathcal{A}} \to c \\
& = (\top_{\mathcal{A}} \to \top_{\mathcal{A}} \to c) \to \top_{\mathcal{A}} \to \top_{\mathcal{A}} \to c
\end{aligned}$$

If $a, c \in \mathcal{A}$:

$$(a \to \top_{\mathcal{A}} a \to c) \to (a \to \top_{\mathcal{A}} a) \to a \to c \leq (a \to \top_{\mathcal{A}} \to c) \to \top_{\mathcal{A}} \to a \to c$$

Then $\mathbf{S}^{\mathcal{B}} \geq \phi(\mathbf{S}^{\mathcal{A}})$. □

If $\mathcal{A} = (\mathcal{A}, \leq, \to, S)$ is a quasi-implicative algebra, then it is obvious that

Definition 3.6. *Let $\mathcal{A} = (\mathcal{A}, \leq, \to, S)$ be a quasi-implicative algebra. The* **completion of** $\mathcal{A}$ *is $\mathcal{B} = (\mathcal{B}, \leq_{\mathcal{B}}, \to_{\mathcal{B}}, S_{\mathcal{B}} = S \cup \{\top_{\mathcal{B}}\})$.*

By Lemma 3.8, it is obvious that $S_{\mathcal{B}}$ is a separator.

Lemma 3.9. *Let $\mathcal{A} = (\mathcal{A}, \leq, \to, S)$ be a quasi-implicative algebra and $\mathcal{B} = (\mathcal{B}, \leq_{\mathcal{B}}, \to_{\mathcal{B}}, S_{\mathcal{B}} = S \cup \{\top_{\mathcal{B}}\})$ its completion. Then the quasi-implicative tripos induced by $\mathcal{A}$ is isomorphic to the implicative tripos induced by $\mathcal{B}$.*

Proof. Let ϕ be the inclusion map from $\mathcal{A}$ to $\mathcal{B}$. Let us start by observing that $S = S_{\mathcal{B}} \cap \mathcal{A} = \phi^{-1}(S_{\mathcal{B}})$, thus if I is a set and $\eta, \zeta \in \mathcal{A}^I$ then:

$$\begin{aligned} \eta \vdash_{S[I]} \zeta \quad & \text{iff} \quad \bigwedge_{i \in I} (\eta(i) \to \zeta(i)) \in S \\ & \text{iff} \quad \phi(\bigwedge_{i \in I} (\eta(i) \to \zeta(i))) \in S_{\mathcal{B}} \\ & \text{iff} \quad \bigwedge_{i \in I} (\phi(\eta(i)) \to_{\mathcal{B}} \phi(\zeta(i))) \in S_{\mathcal{B}} \\ & \text{iff} \quad \phi \circ \eta \vdash_{S[I]} \phi \circ \zeta \end{aligned}$$

Thus ϕ induces an injective map $\bar{\phi}_I : \mathcal{A}^I / S[I] \to \mathcal{B}^I / S_{\mathcal{B}}[I]$ for every set I. If $\eta \in \mathcal{B}^I$ we consider:

$$\tilde{\eta}(i) = \bigwedge_{c \in \mathcal{B}} ((\eta(i) \to_{\mathcal{B}} c) \to_{\mathcal{B}} c)$$

Let us observe:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash z : \eta(i) \to_{\mathcal{B}} c}\text{Axiom} \qquad \dfrac{}{\Gamma \vdash x : \eta(i)}\text{Axiom}}{\Gamma := x : \eta(i), z : \eta(i) \to_{\mathcal{B}} c \vdash zx : c}\to\text{-elim.}}{x : \eta(i) \vdash \lambda z.zx : (\eta(i) \to_{\mathcal{B}} c) \to_{\mathcal{B}} c \quad \text{for all } c \in \mathcal{B}}\to\text{-intro.}}{x : \eta(i) \vdash \lambda z.zx : \tilde{\eta}(i)}\text{Gen.}}{\vdash \lambda xz.zx : \eta(i) \to_{\mathcal{B}} \tilde{\eta}(i) \quad \text{for all } i \in I}\to\text{-intro.}}{\vdash \lambda xz.zx : \bigwedge_{i \in I}(\eta(i) \to_{\mathcal{B}} \tilde{\eta}(i))}\text{Gen.}$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash y : \tilde{\eta}(i)}\text{Axiom}}{\Gamma \vdash y : (\eta(i) \to_{\mathcal{B}} \eta(i)) \to_{\mathcal{B}} \eta(i)}\text{Subs.} \qquad \dfrac{\dfrac{}{\Gamma, x : \eta(i) \vdash x : \eta(i)}\text{Axiom}}{\Gamma \vdash \lambda x.x : \eta(i) \to_{\mathcal{B}} \eta(i)}\to\text{-in.}}{\Gamma := y : \tilde{\eta}(i) \vdash y\lambda x.x : \eta(i)}\to\text{-el.}}{\vdash \lambda y.y\lambda x.x : \tilde{\eta}(i) \to_{\mathcal{B}} \eta(i) \quad \text{for all } i \in I}\to\text{-intro.}}{\vdash \lambda y.y\lambda x.x : \bigwedge_{i \in I}(\tilde{\eta}(i) \to_{\mathcal{B}} \eta(i))}\text{Gen.}$$

Furthermore,

$$\tilde{\eta}(i) \leq (\eta(i) \to_{\mathcal{B}} \bot) \to_{\mathcal{B}} \bot \leq \bot \to_{\mathcal{B}} \bot = \bot \to \bot \leq \top_{\mathcal{A}}$$

thus we have showed $[\eta] = \bar{\phi}([\tilde{\eta}])$. Then $\bar{\phi}$ is an isomorphism of **Pos** and by Lemma 1.2 of **HA**. Since the naturality of $\bar{\phi}$ is obvious, we have showed that the quasi-implicative tripos induced by $\mathcal{A}$ is isomorphic to the implicative tripos induced by $\mathcal{B}$.

□

Thus:

Theorem 3.4. *If P is a PCA then the intuitionistic realizability tripos induced by P is isomorphic to the implicative tripos induced by the completion of P.*

3.2.3 Classical realizability triposes

Definition 3.7. *Let $\mathcal{A}$ be an implicative algebra induced by an AKS. Then the implicative tripos induced by $\mathcal{A}$ is called* **classical realizability tripos**.

Lemma 3.10. *Let $\mathcal{A} = (\mathcal{A}, \leq, \to, S)$ and $\mathcal{B} = (\mathcal{B}, \leq, \Rightarrow, U)$ be two implicative algebras. If there exists a surjective map $\psi : \mathcal{B} \to \mathcal{A}$ such that:*

1. *preserves arbitrary meets;*
2. *preserves implication;*
3. *$b \in U$ if and only if $\psi(b) \in S$*

then the corresponding triposes $\mathsf{P}^{\mathcal{A}}$ and $\mathsf{P}^{\mathcal{B}}$ are isomorphic.

Proof. Let $\eta, \zeta \in \mathcal{B}^I$:

$$\begin{aligned}
\eta \vdash_{U[I]} \zeta \quad & \text{if and only if} \quad \bigwedge_{i \in I} (\eta(i) \Rightarrow \zeta(i)) \in U \\
& \text{if and only if} \quad \psi(\bigwedge_{i \in I} (\eta(i) \Rightarrow \zeta(i))) \in S \\
& \text{if and only if} \quad \bigwedge_{i \in I} (\psi(\eta(i)) \to \psi(\zeta(i))) \in S \\
& \text{if and only if} \quad \psi \circ \eta \vdash_{S[I]} \psi \circ \zeta
\end{aligned}$$

Thus, we can define an injective function $\bar{\psi}_I : \mathcal{B}^I / U[I] \to \mathcal{A}^I / S[I]$ such that $\bar{\psi}_I([\eta]_{U[I]}) = [\psi \circ \zeta]$. Since ψ is surjective, $\bar{\psi}_I$ is a bijective monotonic map, thus it is an isomorphism of $\mathbf{HA}$, by Lemma 1.2. Since the naturality of $(\bar{\psi}_I)_{I \in \mathsf{Obj}(\mathbf{Set})}$ is obvious, we can conclude that $\mathsf{P}^{\mathcal{A}}$ and $\mathsf{P}^{\mathcal{B}}$ are isomorphic. □

Now, we can prove:

Theorem 3.5. *If $\mathcal{A}$ is a classical implicative algebra then there exists an AKS $\mathcal{K}$ such that the implicative tripos induced by $\mathcal{A}$ is isomorphic to the classical realizability tripos induced by $\mathcal{K}$.*

Proof. Let:

- $\Lambda = \Pi := \mathcal{A}$;
- $a \bigoplus b := ab$, $a \cdot b := a \to b$, $\mathsf{k}_a := a \to \bot$
- $\mathsf{K} := \mathbf{K}^{\mathcal{A}}$, $\mathsf{S} := \mathbf{S}^{\mathcal{A}}$, $\mathsf{cc} := \mathsf{cc}^{\mathcal{A}}$,
- $PL := S$ and $\perp\!\!\!\perp := \preceq$

Let us prove that $\perp\!\!\!\perp$ satisfies the axioms of the pole.

1.$t \preceq u \to \pi$ implies $tu \preceq \pi$
2. $t \preceq \pi$ implies $\mathbf{K}^{\mathcal{A}} \preceq t \to u \to t \preceq t \to u \to \pi$
3. $t \preceq v \to uv \to \pi$ implies $\mathbf{S}^{\mathcal{A}} \preceq (v \to uv \to \pi) \to (v \to uv) \to v \to \pi$
$\preceq (v \to uv \to \pi) \to u \to v \to \pi \preceq t \to u \to v \to \pi$
4. $t \preceq (\pi \to \bot) \to \pi$ implies $\mathsf{cc}^{\mathcal{A}} \preceq ((a \to \bot) \to \pi) \to \pi$ implies $t \to \pi$
5. $t \preceq \pi$ implies $\pi \to \bot \preceq t \to \pi'$

Thus, $\mathcal{K}$ is an AKS. Let us observe that if $\beta \subseteq \Pi$ then:

$$\beta^{\perp\!\!\!\perp} := \{a \in \mathcal{A} : a \preceq b \forall b \in \beta\} = \{a \in \mathcal{A} : a \preceq \bigwedge \beta\}$$

Let $\mathcal{B} = (\mathcal{P}(\mathcal{A}), \supseteq, \Rightarrow, U)$ the classical implicative algebra induced by $\mathcal{K}$. Let us observe that $U = \{\beta \subseteq \mathcal{A} : \bigwedge \beta \in S\}$.
Let $\psi : \mathcal{B} \to \mathcal{A}$ be such that $\psi(\beta) = \bigwedge \beta$. Let us show that ψ satisfies the conditions of Lemma 3.10. If $(\beta_i)_{i \in I}$ is a set-indexed family of elements of $\mathcal{B}$, then $\psi(\bigcup_{i \in I} \beta_i) = \bigwedge(\bigcup_{i \in I} \beta_i) = \bigwedge_{i \in I}(\bigwedge \beta_i) = \bigwedge_{i \in I} \psi(\beta_i)$. Let $\beta, \gamma \in \mathcal{B}$ then $\psi(\beta \Rightarrow \gamma) = \psi(\{b \to c : b \preceq \beta, c \in \gamma\}) = \bigwedge\{b \to c : b \preceq \bigwedge \beta, c \in \gamma\} = \bigwedge \beta \to \bigwedge \gamma = \psi(\beta) \to \psi(\gamma)$. Furthermore, $\beta \in U$ if and only if $\bigwedge \beta \in S$ if and only if $\psi(\beta) \in S$. Finally, we can apply Lemma 3.10. $\square$

Chapter 4

Every tripos is isomorphic to an implicative one

Let $\mathsf{P} : \mathbf{Set} \to \mathbf{HA}$ be a fixed $\mathbf{Set}$-based tripos. In this chapter we want to define an implicative algebra $\mathcal{A}$ such that its implicative tripos $\mathsf{P}^{\mathcal{A}}$ is isomorphic to P.

Let $tr_\Sigma \in \Sigma$ be a generic predicate of P and X be an arbitrary set. In the first part of this chapter, we will show how Σ^X can represent the set of *propositional functions over* X. In other words, if $\sigma \in \Sigma^X$ and $p \in \mathsf{P}X$ such that $[\![\sigma]\!]_X = p$, σ can be seen a sort of "code" for the predicate p. We will also show how the structure of Heyting algebra of $\mathsf{P}X$ can be derived from analogous operations on Σ.

Let us start by observing:

Lemma 4.1. *The decoding map $[\![\ ]\!]_X : \Sigma^X \to \mathsf{P}X$ is natural in X, which means that for each map $f : X \to Y$ the following diagram commutes:*

$$\begin{array}{ccc} \Sigma^X & \xrightarrow{[\![\ ]\!]_X} & \mathsf{P}X \\ {\scriptstyle -\circ f}\uparrow & & \uparrow{\scriptstyle \mathsf{P}f} \\ \Sigma^Y & \xrightarrow{[\![\ ]\!]_Y} & \mathsf{P}Y \end{array}$$

i.e. that $[\![\sigma \circ f]\!]_X = \mathsf{P}f([\![\sigma]\!]_Y)$

Proof. Let $\sigma \in \Sigma^Y$.

$$[\![\sigma \circ f]\!]_X = \mathsf{P}(\sigma \circ f)(tr_\Sigma) = \mathsf{P}f(\mathsf{P}\sigma(tr_\Sigma)) = \mathsf{P}f([\![\sigma]\!]_Y)$$

□

4.1 Defining $\wedge, \vee$ and $\to$

In this section, we aim to show how the connectives of $\mathsf{P}X$ descend from analogous operations on the set Σ.

Let $\pi_1, \pi_2 : \Sigma \times \Sigma \to \Sigma$ be the projections of $\Sigma \times \Sigma$. We can define:

$$\begin{array}{ll} \dot{\wedge} : \Sigma \times \Sigma \to \Sigma & [\![\dot{\wedge}]\!]_{\Sigma\times\Sigma} = [\![\pi_1]\!]_{\Sigma\times\Sigma} \wedge [\![\pi_2]\!]_{\Sigma\times\Sigma} \in \mathsf{P}(\Sigma\times\Sigma) \\ \dot{\vee} : \Sigma \times \Sigma \to \Sigma & [\![\dot{\vee}]\!]_{\Sigma\times\Sigma} = [\![\pi_1]\!]_{\Sigma\times\Sigma} \vee [\![\pi_2]\!]_{\Sigma\times\Sigma} \in \mathsf{P}(\Sigma\times\Sigma) \\ \dot{\to} : \Sigma \times \Sigma \to \Sigma & [\![\dot{\to}]\!]_{\Sigma\times\Sigma} = [\![\pi_1]\!]_{\Sigma\times\Sigma} \to [\![\pi_2]\!]_{\Sigma\times\Sigma} \in \mathsf{P}(\Sigma\times\Sigma) \end{array}$$

The existence of $\dot{\wedge}, \dot{\vee}$ and $\dot{\to}$ is ensured by the surjectivity of the decoding map and by the axiom of choice.
If $\sigma, \tau \in \Sigma^X$ then we will write $[\![\sigma(x) \mathbin{\dot{\wedge}} \tau(x)]\!]_{x\in X}$ instead of $[\![\dot{\wedge} \circ \langle\sigma,\tau\rangle]\!]_X$. We adopt analogous notation for $\dot{\vee}$ and $\dot{\to}$.

Theorem 4.1. *Let X be a set and $\sigma, \tau \in \Sigma^X$. Then:*

$$\begin{aligned} [\![\sigma(x) \mathbin{\dot{\to}} \tau(x)]\!]_{x\in X} &= [\![\sigma]\!]_X \to [\![\tau]\!]_X \\ [\![\sigma(x) \mathbin{\dot{\wedge}} \tau(x)]\!]_{x\in X} &= [\![\sigma]\!]_X \wedge [\![\tau]\!]_X \\ [\![\sigma(x) \mathbin{\dot{\vee}} \tau(x)]\!]_{x\in X} &= [\![\sigma]\!]_X \vee [\![\tau]\!]_X \end{aligned}$$

Proof.

$$\begin{aligned} [\![\sigma(x) \mathbin{\dot{\to}} \tau(x)]\!]_{x\in X} &= [\![\dot{\to} \circ \langle\sigma,\tau\rangle]\!]_{x\in X} = \mathsf{P}(\langle\sigma,\tau\rangle)([\![\dot{\to}]\!]_{\Sigma\times\Sigma}) \\ &= \mathsf{P}(\langle\sigma,\tau\rangle)([\![\pi_1]\!]_{\Sigma\times\Sigma} \to [\![\pi_2]\!]_{\Sigma\times\Sigma}) \\ &= \mathsf{P}(\langle\sigma,\tau\rangle)([\![\pi_1]\!]_{\Sigma\times\Sigma}) \to \mathsf{P}(\langle\sigma,\tau\rangle)([\![\pi_2]\!]_{\Sigma\times\Sigma}) \\ &= [\![\pi_1 \circ \langle\sigma,\tau\rangle]\!]_X \to [\![\pi_2 \circ \langle\sigma,\tau\rangle]\!]_X = [\![\sigma]\!]_X \to [\![\tau]\!]_X \end{aligned}$$

The other cases are similar. □

4.2 Defining $\bot$ and $\top$

Let us fix a terminal object $1 = \{*\} \in \mathbf{Set}$, i.e. a fixed singleton. We will indicate with $!_X : X \to 1$ the unique map from X to 1.
We choose $\dot{\bot}, \dot{\top} \in \Sigma$ such that

$$[\![\dot{\bot}]\!]_{*\in 1} = \bot_1 \in \mathsf{P}1 \qquad [\![\dot{\top}]\!]_{*\in 1} = \top_1 \in \mathsf{P}1$$

where we identify $\dot{\bot}$ and $\dot{\top}$ with the corresponding constant maps from 1 to Σ.
In the rest of the thesis, we will write $[\![\dot{\bot}]\!]_{x\in X}$ instead of $[\![\dot{\bot} \circ !_X]\!]_X$.

Theorem 4.2. *If X is a set then:*

$$[\![\dot{\bot}]\!]_{x\in X} = \bot_X \in \mathsf{P}X$$
$$[\![\dot{\top}]\!]_{x\in X} = \top_X \in \mathsf{P}X$$

Proof. By Lemma 4.1:

$$[\![\dot{\bot}]\!]_{x\in X} = [\![\dot{\bot}\circ !_X]\!]_X = \mathsf{P}!_X([\![\dot{\bot}]\!]_{*\in 1}) = \mathsf{P}!_X(\bot_1) = \bot_X$$
$$[\![\dot{\top}]\!]_{x\in X} = [\![\dot{\top}\circ !_X]\!]_X = \mathsf{P}!_X([\![\dot{\top}]\!]_{*\in 1}) = \mathsf{P}!_X(\top_1) = \top_X$$

where the last equalities of each row are due to the fact that $\mathsf{P}!_X$ is a morphism of Heyting algebras. □

4.3 Defining quantifiers

In this section we define the codes $\dot{\bigwedge}$ and $\dot{\bigvee}$ of $\forall$ and $\exists$.
Let us start by considering the following set:

$$E := \{(\xi, s) : \xi \in s\} \subseteq \Sigma \times \mathcal{P}(\Sigma)$$

and the corresponding projections

$$e_1 : E \to \Sigma \qquad e_2 : E \to \mathcal{P}(\Sigma)$$

The surjectivity of the decoding map allows us to pick two codes in $\Sigma^{\mathcal{P}(\Sigma)}$ in the following way:

$$\dot{\bigwedge} : \mathcal{P}(\Sigma) \to \Sigma \qquad [\![\dot{\bigwedge}]\!]_{\mathcal{P}(\Sigma)} = \forall e_2([\![e_1]\!]_E)$$
$$\dot{\bigvee} : \mathcal{P}(\Sigma) \to \Sigma \qquad [\![\dot{\bigvee}]\!]_{\mathcal{P}(\Sigma)} = \exists e_2([\![e_1]\!]_E)$$

If $h : Y \to \mathcal{P}(\Sigma)$ is such that $h(y) = Z$, then we will write $[\![\dot{\bigwedge} Z]\!]$ instead of $[\![\dot{\bigwedge} \circ h]\!]$.

Theorem 4.3. *Let X, Y be sets, $\sigma \in \Sigma^X$ and $f : X \to Y$, then:*

$$[\![\dot{\bigwedge}\{\sigma(x) : x \in f^{-1}(y)\}]\!]_{y\in Y} = \forall f([\![\sigma]\!]_X) \in \mathsf{P}Y$$
$$[\![\dot{\bigvee}\{\sigma(x) : x \in f^{-1}(y)\}]\!]_{y\in Y} = \exists f([\![\sigma]\!]_X) \in \mathsf{P}Y$$

Proof. If $h : Y \to \mathcal{P}(\Sigma)$ such that $h(y) = \{\sigma(x) : x \in f^{-1}(y)\}$ then:

$$[\![\dot{\bigwedge}\{\sigma(x) : x \in f^{-1}(y)\}]\!]_{y\in Y} = [\![\dot{\bigwedge} \circ h]\!]_Y = \mathsf{P}h([\![\dot{\bigwedge}]\!]_{\mathcal{P}(\Sigma)}) = \mathsf{P}h(\forall e_2([\![e_1]\!]_E))$$
$$[\![\dot{\bigvee}\{\sigma(x) : x \in f^{-1}(y)\}]\!]_{y\in Y} = [\![\dot{\bigvee} \circ h]\!]_Y = \mathsf{P}h([\![\dot{\bigvee}]\!]_{\mathcal{P}(\Sigma)}) = \mathsf{P}h(\exists e_2([\![e_1]\!]_E))$$

If we consider $G \coloneqq \{(\sigma(x), f(x)) : x \in X\} \subseteq \Sigma \times Y$ and the two following functions:

$$\begin{aligned} g : G &\to Y \\ (\xi, y) &\mapsto y \end{aligned} \qquad\qquad \begin{aligned} g' : G &\to E \\ (\xi, y) &\mapsto (\xi, h(y)) \end{aligned}$$

then the following diagram is a pullback in **Set**:

$$\begin{array}{ccc} G & \xrightarrow{g} & Y \\ {\scriptstyle g'}\downarrow & \lrcorner & \downarrow{\scriptstyle h} \\ E & \xrightarrow{e_2} & \mathcal{P}(\Sigma) \end{array} \qquad\qquad \begin{array}{ccc} (\xi, y) & \overset{g}{\longmapsto} & yh \\ {\scriptstyle g'}\downarrow & & \downarrow \\ (\xi, h(y)) & \overset{e_2}{\longmapsto} & h(y) \end{array}$$

In fact, let $l : I \to Y$ and $m = (m_1, m_2) : I \to E$ such that $e_2 \circ m = h \circ l$, i.e. $m_2(i) = h(l(i))$, then:

$$\begin{aligned} \phi : I &\to G \\ i &\mapsto (m_1(i), l(i)) \end{aligned}$$

is the only map such that $g \circ \phi = l$ and $g' \circ \phi = m$. Thus, by the Beck-Chevalley condition, the following diagrams commute:

$$\begin{array}{ccc} \mathsf{P}G & \xrightarrow{\forall g} & \mathsf{P}Y \\ {\scriptstyle \mathsf{P}g'}\uparrow & & \uparrow{\scriptstyle \mathsf{P}h} \\ \mathsf{P}E & \xrightarrow{\forall e_2} & \mathsf{P}\mathcal{P}(\Sigma) \end{array} \qquad \begin{array}{ccc} \mathsf{P}G & \xrightarrow{\exists g} & \mathsf{P}Y \\ {\scriptstyle \mathsf{P}g'}\uparrow & & \uparrow{\scriptstyle \mathsf{P}h} \\ \mathsf{P}E & \xrightarrow{\exists e_2} & \mathsf{P}\mathcal{P}(\Sigma) \end{array}$$

Hence:

$$\begin{aligned} [\![\dot{\bigwedge}\{\sigma(x) : x \in f^{-1}(y)\}]\!]_{y \in Y} &= (\mathsf{P}h \circ \forall e_2)([\![e_1]\!]_E) = (\forall g \circ \mathsf{P}g')([\![e_1]\!]_E) \\ [\![\dot{\bigvee}\{\sigma(x) : x \in f^{-1}(y)\}]\!]_{y \in Y} &= (\mathsf{P}h \circ \exists e_2)([\![e_1]\!]_E) = (\exists g \circ \mathsf{P}g')([\![e_1]\!]_E) \end{aligned}$$

Let $q : X \to G$ be defined by $q(x) = (\sigma(x), f(x))$. It is clear that q is surjective and consequently that it has a right inverse by (AC). Hence, $\exists q$ and $\forall q$ are left inverses of $\mathsf{P}q$, by Lemma 1.4. Then:

$$\begin{aligned} [\![\dot{\bigwedge}\{\sigma(x) : x \in f^{-1}(y)]\!]_{y \in Y} &= (\forall g \circ \mathsf{P}g')([\![e_1]\!]_E) = (\forall g \circ (\forall q \circ \mathsf{P}q) \circ \mathsf{P}g')([\![e_1]\!]_E) \\ [\![\dot{\bigvee}\{\sigma(x) : x \in f^{-1}(y)]\!]_{y \in Y} &= (\exists g \circ \mathsf{P}g')([\![e_1]\!]_E) = (\exists g \circ (\exists q \circ \mathsf{P}q) \circ \mathsf{P}g')([\![e_1]\!]_E) \end{aligned}$$

Since $\forall$ and $\exists$ are functors, $g \circ q = f$ and $e_1 \circ g' \circ q = \sigma$:

$$\begin{aligned} [\![\dot{\bigwedge}\{\sigma(x) : x \in f^{-1}(y)]\!]_{y \in Y} &= (\forall(g \circ q) \circ \mathsf{P}(g' \circ q))([\![e_1]\!]_E) = \forall f([\![e_1 \circ g' \circ q]\!]_X) \\ &= \forall f([\![\sigma]\!]_X) \\ [\![\dot{\bigvee}\{\sigma(x) : x \in f^{-1}(y)]\!]_{y \in Y} &= (\exists(g \circ q) \circ \mathsf{P}(g' \circ q))([\![e_1]\!]_E) = \exists f([\![e_1 \circ g' \circ q]\!]_X) \\ &= \exists f([\![\sigma]\!]_X) \end{aligned}$$

□

4.4 Defining the filter

We define

$$\Phi := \{\xi \in \Sigma : [\![\xi]\!]_{*\in 1} = \top_1\} \subseteq \Sigma$$

where we have identified ξ with the corresponding map from $\mathbf{1}$ to Σ.

Theorem 4.4. *Let X be a set and $\sigma, \tau \in \Sigma^X$ then:*

$$[\![\sigma]\!]_X \leq [\![\tau]\!]_X \Leftrightarrow \dot{\bigwedge}\{\sigma(x)\dot{\to}\tau(x) : x \in X\} \in \Phi$$

Proof.

$$\begin{aligned}
[\![\sigma]\!]_X \leq [\![\tau]\!]_X &\Leftrightarrow \top_X \leq [\![\sigma]\!]_X \to [\![\tau]\!]_X \Leftrightarrow \mathsf{P}!_X(\top_1) \leq [\![\sigma(x)\dot{\to}\tau(x)]\!]_{x\in X} \\
&\Leftrightarrow \top_1 \leq \forall !_X([\![\sigma(x)\dot{\to}\tau(x)]\!]_{x\in X}) \Leftrightarrow \top_1 \leq [\![\dot{\bigwedge}\{\sigma(x)\dot{\to}\tau(x) : x \in !_X^{-1}(1)\}]\!]_{*\in 1} \\
&\Leftrightarrow \top_1 = [\![\dot{\bigwedge}\{\sigma(x)\dot{\to}\tau(x) : x \in X\}]\!]_{*\in 1} \Leftrightarrow \dot{\bigwedge}\{\sigma(x)\dot{\to}\tau(x) : x \in X\} \in \Phi
\end{aligned}$$

□

Example. Let P be the implicative tripos induced by an implicative algebra $\mathcal{B} = (\mathcal{B}, \preceq, \to, U)$. Since its generic predicate is $tr_{\mathcal{B}} = [\mathsf{id}_{\mathcal{B}}] \in \mathsf{P}\mathcal{B}$, then $[\![\sigma]\!]_X = \mathsf{P}\sigma(\mathsf{id}_{\mathcal{B}}) = [\sigma] \in \mathsf{P}X$ for every set X and $\sigma \in \mathcal{B}^X$, i.e. the decoding map $[\![\ ]\!]_X$ coincides with the quotient map from $\mathcal{B}^X$ to $\mathsf{P}X$.
In such case, it is clear that

$$\dot{\wedge} = \times \qquad \dot{\vee} = + \qquad \dot{\to} = \to$$

since, for every $a, b \in \mathcal{B}$

$$\begin{aligned}
[a \times b] &= [a] \wedge [b] \in \mathcal{B}/U \\
[a + b] &= [a] \vee [b] \in \mathcal{B}/U \\
[a \to b] &= [a] \to [b] \in \mathcal{B}/U
\end{aligned}$$

Analogously, $\dot{\top} = \top_{\mathcal{B}}$ and $\dot{\bot} = \bot_{\mathcal{B}}$.
Furthermore, if e_1, e_2 are the projections of $E = \{(x, A) : x \in A \subseteq \mathcal{B}\}$ then

$$\begin{aligned}
\forall e_2([e_1]) &= [A \mapsto \bigwedge_{e_2(z)=A} e_1(z)] = [A \mapsto \bigwedge_{x\in A} x] \\
\exists e_2([e_1]) &= [A \mapsto \exists_{e_2(z)=A} e_1(z)] = [A \mapsto \exists_{x\in A} x]
\end{aligned}$$

thus $\dot{\bigwedge} = \curlywedge$ and $\dot{\bigvee} = \exists$.
Furthermore,

$$\Phi = \{x \in \mathcal{B} : [x] = [\top_{\mathcal{B}}]\} = U$$

4.5 Constructing the implicative algebra

4.5.1 Defining the set of atoms

Definition 4.1. *The* **set $\mathcal{A}_0$ of atoms** *is inductively defined as follows:*

1. *if $\xi \in \Sigma$ then $\dot{\xi} \in \mathcal{A}_0$*
2. *if $s \in \mathcal{P}(\Sigma)$ and $\alpha \in \mathcal{A}_0$ then $(s \mapsto \alpha) \in \mathcal{A}_0$.*

Basically, the elements of $\mathcal{A}_0$ are of the form: $s_1 \mapsto ... \mapsto s_n \mapsto \dot{\xi}$ where $s_1, ..., s_n$ are subsets of Σ and $\xi \in \Sigma$.
Now, we define a binary relation $\leq$ over $\mathcal{A}_0$ in the following way:

$$\frac{}{\dot{\xi} \leq \dot{\xi}} \qquad\qquad \frac{s \subseteq s' \quad \alpha \leq \alpha'}{s \mapsto \alpha \leq s' \mapsto \alpha'}$$

Lemma 4.2. *The relation $\leq$ is a preorder on $\mathcal{A}_0$.*

Proof. Let us prove that $\leq$ is a preorder.

- *Reflexivity.* Let $\alpha \in \mathcal{A}_0$, we prove by inductive hypothesis that $\alpha \leq \alpha$:
  1. if $\alpha = \dot{\xi}$ where $\xi \in \Sigma$, then $\dot{\xi} \leq \dot{\xi}$ by definition of $\leq$;
  2. let $\alpha = s \mapsto \alpha'$ where $s \subseteq \Sigma$ and $\alpha' \in \mathcal{A}_0$. Since $s \subseteq s$ and $\alpha' \leq \alpha'$ by induction, then $s \mapsto \alpha' \leq s \mapsto \alpha'$, i.e. $\alpha \leq \alpha$.
- *Transitivity.* Let $\alpha, \beta, \gamma \in \mathcal{A}_0$ such that $\alpha \leq \beta$ and $\beta \leq \gamma$. Then:
  1. if $\alpha = \dot{\xi}$, since $\alpha \leq \beta$ and $\beta \leq \gamma$ then $\beta = \gamma = \dot{\xi}$, hence clearly $\alpha \leq \gamma$;
  2. if $\alpha = s \mapsto \alpha'$ where $s \subseteq \Sigma$ and $\alpha' \in \mathcal{A}_0$, then β must be of the form $t \mapsto \beta'$ and consequently γ must be of the form $u \mapsto \gamma'$ too, where $s \subseteq t \subseteq u$, $\alpha' \leq \beta'$ and $\beta' \leq \gamma'$ so clearly $s \subseteq u$ and, by induction, $\alpha' \leq \gamma'$. Hence, $\alpha \leq \gamma$.

□

4.5.2 Defining $\mathcal{A}$

We consider a "conversion" function, defined by recursion as follows:

$$\begin{aligned} \phi_0 : \quad \mathcal{A}_0 &\to \Sigma \\ \dot{\xi} &\mapsto \xi \\ (s \mapsto \alpha) &\mapsto (\dot{\bigwedge} s) \dot{\to} \phi_0(\alpha) \end{aligned}$$

Definition 4.2. *Let*

- $\mathcal{A} := \mathcal{P}_{\uparrow}(\mathcal{A}_0) = \{s \subseteq \mathcal{A}_0 : s \text{ is upward closed}\}$*;*
- $\leq$ *be a binary relation on* $\mathcal{A}$ *defined as* $a \leq b \Leftrightarrow b \subseteq a$ *for all* $a, b \in \mathcal{A}$*;*
- $\to$ *be a binary function on* $\mathcal{A}$ *such that:*

$$a \to b := \{s \mapsto \beta : s \in \tilde{\phi}_0(a)^{\subseteq} \text{ and } \beta \in b\}$$

where

$$\begin{aligned} \tilde{\phi}_0 \ : \ & \mathcal{A} \to \mathcal{P}(\Sigma) \\ & a \mapsto \{\phi_0(\alpha) : \alpha \in a\} \end{aligned} \qquad s^{\subseteq} := \{s' \in \mathcal{P}(\Sigma) : s \subseteq s'\}$$

We can clarify the notion of $a \to b$:

$$\begin{aligned} a \to b &= \left\{s \mapsto \beta : s \in \{s' \in \mathcal{P}(\Sigma) : \tilde{\phi}_0(a) \subseteq s'\} \text{ and } \beta \in b\right\} \\ &= \{s \mapsto \beta : \tilde{\phi}_0(a) \subseteq s \text{ and } \beta \in b\} \\ &= \{s \mapsto \beta : s \in \mathcal{P}(\Sigma) \text{ such that } \phi_0(\alpha) \in s \text{ for all } \alpha \in a \text{ and } \beta \in b\} \end{aligned}$$

Lemma 4.3. $(\mathcal{A}, \leq, \to)$ *is an implicative structure.*

Proof.

1. Let us show that $(\mathcal{A}, \leq)$ is a complete lattice. Clearly $\leq$ is a partial order. Let $(b_i)_{i \in I}$ be a set-indexed family of elements of $\mathcal{A}$, then $\bigwedge_{i \in I} b_i = \bigcup_{i \in I} b_i$ and $\bigvee_{i \in I} b_i = \bigcap_{i \in I} b_i$. Obviously, $\top_{\mathcal{A}} = \varnothing$ and $\bot_{\mathcal{A}} = \mathcal{A}$.

2. Let $a, a', b, b' \in \mathcal{A}$ such that $a' \leq a$ and $b \leq b'$. We have to prove that $a \to b \leq a' \to b'$. Let $s \mapsto \beta \in a' \to b'$. Clearly, $\beta \in b$ and $\phi_0(\alpha) \in s$ for all $\alpha \in a$, because $a \subseteq a'$ and $b' \leq b$. Then $s \mapsto \beta \in a \to b$.

3. Let $a, b \in \mathcal{A}$, then:

$$\begin{aligned} a \to \bigwedge_{i \in I} b_i &= \{s \mapsto \beta : s \in \tilde{\phi}_0(a)^{\subseteq} \text{ and } \beta \in \bigcup_{i \in I} b_i\} \\ &= \bigcup_{i \in I} \{s \mapsto \beta : s \in \tilde{\phi}_0(a)^{\subseteq} \text{ and } \beta \in b_i\} \\ &= \bigwedge_{i \in I} (a \to b_i). \end{aligned}$$

□

4.5.3 Defining a new generic predicate of P

Let

$$\phi : \mathcal{A} \to \Sigma \qquad\qquad \psi : \Sigma \to \mathcal{A}$$
$$a \mapsto \dot{\bigwedge}\tilde{\phi}_0(a) \qquad\qquad \xi \mapsto \{\dot{\xi}\}$$

Let us consider $tr_{\mathcal{A}} \coloneqq [\![\phi]\!]_{\mathcal{A}} = \mathsf{P}\phi(tr_{\Sigma}) \in \mathsf{P}\mathcal{A}$.
We want to show that $tr_{\mathcal{A}}$ is a generic predicate for the tripos P.

Lemma 4.4. *$P\psi(tr_{\mathcal{A}}) = tr_{\Sigma}$.*

Proof.

$$\begin{aligned}\mathsf{P}\psi(tr_{\mathcal{A}}) &= \mathsf{P}\psi([\![\phi]\!]_{\mathcal{A}}) = [\![\phi \circ \psi]\!]_{\Sigma} = [\![\dot{\bigwedge}\tilde{\phi}_0(\{\dot{\xi}\})]\!]_{\xi \in \Sigma} \\ &= [\![\dot{\bigwedge}\{\xi\}]\!]_{\xi \in \Sigma} = [\![\dot{\bigwedge}\{\mathsf{id}_{\Sigma}(\xi') : \xi' \in \mathsf{id}_{\Sigma}^{-1}(\xi)\}]\!]_{\xi \in \Sigma} = \forall \mathsf{id}_{\Sigma}([\![\mathsf{id}_{\Sigma}]\!]_{\Sigma})\end{aligned}$$

By Lemma 1.4, $\forall\mathsf{id}_{\Sigma}$ is the inverse of $\mathsf{Pid}_{\Sigma} = \mathsf{id}_{\mathsf{P}\Sigma}$, then $\forall\mathsf{id}_{\Sigma} = \mathsf{id}_{\mathsf{P}\Sigma}$. Hence,

$$\mathsf{P}\psi(tr_{\mathcal{A}}) = \mathsf{Pid}_{\Sigma}([\![\mathsf{id}_{\Sigma}]\!]_{\Sigma}) = [\![\mathsf{id}_{\Sigma}]\!]_{\Sigma} = \mathsf{Pid}_{\Sigma}(tr_{\Sigma}) = tr_{\Sigma}$$

□

Lemma 4.5. *The predicate $tr_{\mathcal{A}} \in \mathsf{P}\mathcal{A}$ is a generic predicate for the tripos P.*

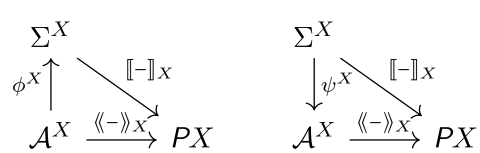
Proof. We want to show that

$$\langle\!\langle\ \rangle\!\rangle_X : \mathcal{A}^X \to \mathsf{P}X$$
$$\eta \mapsto \mathsf{P}\eta(tr_{\mathcal{A}})$$

is surjective. If $p \in \mathsf{P}X$ then there exists $\sigma \in \Sigma^X$ such that $\mathsf{P}\sigma(tr_{\Sigma}) = p$. Hence, $\mathsf{P}\sigma(\mathsf{P}\psi(tr_{\mathcal{A}})) = p$ by Lemma 4.4, that is $\mathsf{P}(\psi \circ \sigma)(tr_{\mathcal{A}}) = p$. Then, $\langle\!\langle \psi \circ \sigma \rangle\!\rangle_X = p$. □

Let X be a set. We will denote with $[\![-]\!]_X : \Sigma^X \to \mathsf{P}X$ and with $\langle\!\langle - \rangle\!\rangle_X : \mathcal{A}^X \to \mathsf{P}X$ the corresponding decoding maps, while we will use $\phi^X : \mathcal{A}^X \to \Sigma^X$ and $\psi^X : \Sigma^X \to \mathcal{A}^X$ to indicate the natural transformations induced by ϕ and ψ, i.e. $\phi^X(\eta) = \phi \circ \eta$ and $\psi^X(\sigma) = \psi \circ \sigma$.

Lemma 4.6. *Let X be a set. Then, the two following diagrams commute:*



i.e. $\langle\!\langle - \rangle\!\rangle_X = [\![-]\!]_X \circ \phi^X$ *and* $[\![-]\!]_X = \langle\!\langle - \rangle\!\rangle_X \circ \psi^X$.

Proof. Let $\eta \in \mathcal{A}^X$, then:

$$[\![\phi^X(\eta)]\!]_X = [\![\phi \circ \eta]\!]_X = \mathsf{P}\eta(\mathsf{P}\phi(tr_\Sigma)) = \mathsf{P}\eta(tr_{\mathcal{A}}) = \langle\!\langle \eta \rangle\!\rangle_X.$$

While, if $\sigma \in \Sigma^X$:

$$\langle\!\langle \phi^X(\sigma) \rangle\!\rangle_X = \langle\!\langle \phi \circ \sigma \rangle\!\rangle_X = \mathsf{P}\sigma(\mathsf{P}\psi(tr_{\mathcal{A}})) = \mathsf{P}\sigma(tr_\Sigma) = [\![\sigma]\!]_X.$$

□

4.5.4 Universal quantification in $\mathcal{A}$

As we did in subsection 4.3, we define:

$$E' := \{(a, A) : a \in A\} \subseteq \mathcal{A} \times \mathcal{P}(\mathcal{A}) \qquad e'_1 : E' \to \mathcal{A} \qquad e'_2 : E' \to \mathcal{P}(\mathcal{A})$$

where e'_1, e'_2 are the projections of E'.
We want to prove:

Theorem 4.5. $\langle\!\langle \curlywedge A \rangle\!\rangle_{A \in \mathcal{P}(\mathcal{A})} = \forall e'_2(\langle\!\langle e'_1 \rangle\!\rangle_{E'})$.

The meaning of this theorem is that the operator $\curlywedge : \mathcal{P}(\mathcal{A}) \to \mathcal{A}$ has the same role for the generic predicate $tr_{\mathcal{A}} \in \mathsf{P}\mathcal{A}$ that the operator $\dot{\bigwedge} \in \Sigma^{\mathcal{P}(\Sigma)}$ has for the generic predicate $tr_\Sigma \in \mathsf{P}\Sigma$.
In order to prove it, we first need to show the following property:

Lemma 4.7.

$$[\![\dot{\bigvee}\{\dot{\bigvee} s : s \in S\}]\!]_{S \in \mathcal{P}(\mathcal{P}(\Sigma))} = [\![\dot{\bigvee}(\bigcup S)]\!]_{S \in \mathcal{P}(\mathcal{P}(\Sigma))}$$
$$[\![\dot{\bigwedge}\{\dot{\bigwedge} s : s \in S\}]\!]_{S \in \mathcal{P}(\mathcal{P}(\Sigma))} = [\![\dot{\bigwedge}(\bigcup S)]\!]_{S \in \mathcal{P}(\mathcal{P}(\Sigma))}$$

Proof. Let consider the following sets and the corresponding projections:

$$\begin{array}{lll} E = \{(\xi, s) : \xi \in s\} \subseteq \Sigma \times \mathcal{P}(\Sigma) & e_1 : E \to \Sigma & e_2 : E \to \mathcal{P}(\Sigma) \\ F := \{(s, S) : s \in S\} \subseteq \mathcal{P}(\Sigma) \times \mathcal{P}(\mathcal{P}(\Sigma)) & f_1 : E \to \mathcal{P}(\Sigma) & f_2 : E \to \mathcal{P}(\mathcal{P}(\Sigma)) \\ G := \{(\xi, s, S) : \xi \in s \in S\} \subseteq \Sigma \times \mathcal{P}(\Sigma) \times \mathcal{P}(\mathcal{P}(\Sigma)) & g_1 : G \to E & g_2 : G \to F \end{array}$$

We can start by observing that:

$$\begin{aligned} [\![\dot{\bigvee}\{\dot{\bigvee} s : s \in S\}]\!]_{S \in \mathcal{P}(\mathcal{P}(\Sigma))} &= [\![\dot{\bigvee}\{\dot{\bigvee} f_1(z) : z \in f_2^{-1}(S)\}]\!]_{S \in \mathcal{P}(\mathcal{P}(\Sigma))} = \exists f_2([\![\dot{\bigvee} \circ f_1]\!]_F) \\ &= (\exists f_2 \circ \mathsf{P} f_1)([\![\dot{\bigvee}]\!]_{\mathcal{P}(\Sigma)}) = (\exists f_2 \circ \mathsf{P} f_1 \circ \exists e_2)([\![e_1]\!]_E) \end{aligned}$$

This is clearly a pullback:

$$
\begin{array}{ccc}
G & \xrightarrow{g_2} & F \\
{\scriptstyle g_1}\downarrow & \lrcorner & \downarrow{\scriptstyle f_1} \\
E & \xrightarrow{e_2} & \mathcal{P}(\Sigma)
\end{array}
\qquad\qquad
\begin{array}{ccc}
(\xi, s, S) & \xmapsto{g_2} & (s, S) \\
{\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle f_1} \\
(\xi, s) & \xmapsto{e_2} & s
\end{array}
$$

Then, by Beck-Chevalley, we have:

$$\exists g_2 \circ \mathsf{P} g_1 = \mathsf{P} f_1 \circ \exists e_2$$

Therefore, we obtain:

$$
\begin{aligned}
[\![\dot{\bigvee}\{\dot{\bigvee} s : s \in S\}]\!]_{S \in \mathcal{P}(\mathcal{P}(\Sigma))} &= (\exists f_2 \circ \exists g_2 \circ \mathsf{P} g_1)([\![e_1]\!]_E) = \exists (f_2 \circ g_2)([\![e_1 \circ g_1]\!]_G) \\
&= [\![\dot{\bigvee}\{(e_1 \circ g_1)(z) : z \in (f_2 \circ g_2)^{-1}(S)]\!]_{S \in \mathcal{P}(\mathcal{P}(\Sigma))} \\
&= [\![\dot{\bigvee}(\bigcup S)]\!]_{S \in \mathcal{P}(\mathcal{P}(\Sigma))}
\end{aligned}
$$

The other case is similar. □

Now we can prove Theorem 4.5:

Proof.

$$\langle\!\langle \curlywedge A \rangle\!\rangle_{A \in \mathcal{P}(\mathcal{A})} = \langle\!\langle \bigcup A \rangle\!\rangle_{A \in \mathcal{P}(\mathcal{A})} = [\![\phi(\bigcup A)]\!]_{A \in \mathcal{P}(\mathcal{A})} = [\![\dot{\bigwedge} \tilde{\phi}_0(\bigcup A)]\!]_{A \in \mathcal{P}(\mathcal{A})}$$

Let $\mathcal{P}\tilde{\phi}_0 : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{P}(\Sigma))$ such that $\mathcal{P}\tilde{\phi}_0(A) = \{\tilde{\phi}_0(a) : a \in A\}$. Then:

$$\langle\!\langle \curlywedge A \rangle\!\rangle_{A \in \mathcal{P}(\mathcal{A})} = [\![\dot{\bigwedge} \bigcup \mathcal{P}\tilde{\phi}_0(A)]\!]_{A \in \mathcal{P}(\mathcal{A})} = \mathsf{P}(\mathcal{P}\tilde{\phi}_0)([\![\dot{\bigwedge} \bigcup S]\!]_{A \in \mathcal{P}(\mathcal{P}(\Sigma))}$$

Thus, by Lemma 4.7:

$$
\begin{aligned}
\langle\!\langle \curlywedge A \rangle\!\rangle_{A \in \mathcal{P}(\mathcal{A})} &= \mathsf{P}(\mathcal{P}\tilde{\phi}_0)([\![\dot{\bigwedge}\{\dot{\bigwedge} s : s \in S\}]\!]_{S \in \mathcal{P}(\mathcal{P}(\Sigma))}) \\
&= [\![\dot{\bigwedge}\{\dot{\bigwedge}\tilde{\phi}_0(a) : a \in A\}]\!]_{A \in \mathcal{P}(\mathcal{A})} = [\![\dot{\bigwedge}\{\phi(a) : a \in A\}]\!]_{A \in \mathcal{P}(\mathcal{A})} \\
&= [\![\dot{\bigwedge}\{\phi(e_1'(p)) : p \in e_2'^{-1}(A)\}]\!]_{A \in \mathcal{P}(\mathcal{A})} = \forall e_2'([\![\phi \circ e_1']\!]_{E'}) \\
&= \forall e_2'(\langle\!\langle e_1' \rangle\!\rangle_{E'})
\end{aligned}
$$

□

Lemma 4.5 allows us to use the same argument of Theorem 4.3 in order to show:

Lemma 4.8. *Let $\eta \in \mathcal{A}^X$ and $f : X \to Y$ a map, then:*

$$\langle\!\langle \curlywedge \{\eta(x) : x \in f^{-1}(y)\} \rangle\!\rangle_{y \in Y} = \forall f(\langle\!\langle \eta \rangle\!\rangle_X) \in \mathsf{P}Y$$

4.5.5 Implication in $\mathcal{A}$

In this subsection, similarly to before, our aim is to show that the $\to : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ operator has the same role for the generic predicate $tr_{\mathcal{A}} \in \mathsf{P}\mathcal{A}$ that the operator $\dot{\to} \in \Sigma^{\Sigma\times\Sigma}$ has for the generic predicate $tr_{\Sigma} \in \mathsf{P}\Sigma$.
We need first to prove some technical lemmas.

Lemma 4.9. *Let $F := \{(s,s') : s \subseteq s'\} \subseteq \mathcal{P}(\Sigma) \times \mathcal{P}(\Sigma)$ and let $f_1, f_2 : F \to \mathcal{P}(\Sigma)$ be the corresponding projections. Then:*

$$[\![\dot{\bigvee} \circ f_1]\!]_F \leq [\![\dot{\bigvee} \circ f_2]\!]_F$$
$$[\![\dot{\bigwedge} \circ f_1]\!]_F \geq [\![\dot{\bigwedge} \circ f_2]\!]_F$$

Proof. Let us define the set $G := \{(\xi, \xi', (s,s')) : \xi \in s, \xi' \in s' \text{ and } (s,s') \in F\} \subseteq \Sigma \times \Sigma \times F$ and its projections $g_1, g_2 : G \to \Sigma$ and $g_3 : G \to F$.
We can observe that if $(s,s') \in F$ then $f_1(s,s') = \{g_1(\xi,\xi',(s,s')) : (\xi,\xi',(s,s')) \in G\} = \{g_1(z) : z \in g_3^{-1}((s,s'))\}$ and similarly $f_2(s,s') = \{g_2(z) : z \in g_3^{-1}((s,s'))\}$.
Then:

$$[\![\dot{\bigvee} \circ f_1]\!]_F = [\![\dot{\bigvee}\{g_1(z) : z \in g_3^{-1}(s,s')\}]\!]_{(s,s')\in F} = \exists g_3([\![g_1]\!]_G)$$
$$[\![\dot{\bigvee} \circ f_2]\!]_F = [\![\dot{\bigvee}\{g_2(z) : z \in g_3^{-1}(s,s')\}]\!]_{(s,s')\in F} = \exists g_3([\![g_2]\!]_G)$$
$$[\![\dot{\bigwedge} \circ f_1]\!]_F = [\![\dot{\bigwedge}\{g_1(z) : z \in g_3^{-1}(s,s')\}]\!]_{(s,s')\in F} = \forall g_3([\![g_1]\!]_G)$$
$$[\![\dot{\bigwedge} \circ f_2]\!]_F = [\![\dot{\bigwedge}\{g_2(z) : z \in g_3^{-1}(s,s')\}]\!]_{(s,s')\in F} = \forall g_3([\![g_2]\!]_G)$$

Let $g : G \to G$ be such that $g(\xi,\xi',(s,s')) = (\xi,\xi,(s,s'))$ then:

$$[\![g_1]\!]_G = [\![g_2 \circ g]\!]_G = \mathsf{P}g([\![g_2]\!]_G)$$

and , since $\exists g \dashv \mathsf{P}g \dashv \forall g$,

$$\exists g([\![g_1]\!]_G) \leq [\![g_2]\!]_G \leq \forall g([\![g_1]\!]_G)$$

Thus we can conclude:

$$[\![\dot{\bigvee} \circ f_1]\!]_F = \exists g_3([\![g_1]\!]_G) = \exists g_3(\exists g([\![g_1]\!]_G)) \leq \exists g_3([\![g_2]\!]_G) = [\![\dot{\bigvee} \circ f_2]\!]_F$$
$$[\![\dot{\bigwedge} \circ f_1]\!]_F = \forall g_3([\![g_1]\!]_G) = \forall g_3(\forall g([\![g_1]\!]_G)) \geq \forall g_3([\![g_2]\!]_G) = [\![\dot{\bigwedge} \circ f_2]\!]_F$$

where we have used that $g_3 = g_3 \circ g$. □

Corollary 4.1. *Let X be a set and $\eta, \zeta \in \mathcal{P}(\Sigma)^X$ such that $\eta(x) \subseteq \zeta(x)$ for every $x \in X$, then*

$$[\![\dot{\bigwedge} \circ \eta]\!]_X \geq [\![\dot{\bigwedge} \circ \zeta]\!]_X$$
$$[\![\dot{\bigvee} \circ \eta]\!]_X \leq [\![\dot{\bigvee} \circ \zeta]\!]_X.$$

Proof. Let $\mu \in (\mathcal{P}(\Sigma)\times\mathcal{P}(\Sigma))^X$ such that $\mu(x) = (\eta(x), \zeta(x))$ and F, f_1 and f_2 as in Lemma 4.9. Then:

$$\llbracket \dot{\bigwedge} \circ \eta \rrbracket_X = \llbracket \dot{\bigwedge} \circ f_1 \circ \mu \rrbracket_X = \mathsf{P}\mu(\llbracket \dot{\bigwedge} \circ f_1 \rrbracket_F)$$
$$\llbracket \dot{\bigvee} \circ \eta \rrbracket_X = \llbracket \dot{\bigvee} \circ f_1 \circ \mu \rrbracket_X = \mathsf{P}\mu(\llbracket \dot{\bigvee} \circ f_1 \rrbracket_F)$$

By Lemma 4.9:

$$\llbracket \dot{\bigwedge} \circ \eta \rrbracket_X \geq \mathsf{P}\mu(\llbracket \dot{\bigwedge} \circ f_2 \rrbracket_F) = \llbracket \dot{\bigwedge} \circ f_2 \circ \mu \rrbracket_X = \llbracket \dot{\bigwedge} \circ \zeta \rrbracket_X$$
$$\llbracket \dot{\bigvee} \circ \eta \rrbracket_X \leq \mathsf{P}\mu(\llbracket \dot{\bigvee} \circ f_2 \rrbracket_F) = \llbracket \dot{\bigvee} \circ f_2 \circ \mu \rrbracket_X = \llbracket \dot{\bigvee} \circ \zeta \rrbracket_X.$$

□

Now we can prove:

Lemma 4.10.

$$\llbracket \dot{\bigwedge} \{ \big(\dot{\bigwedge} s\big) \dot{\rightarrow} \xi : s \in u^{\subseteq},\ \xi \in t\} \rrbracket_{(u,t)\in\mathcal{P}(\Sigma)\times\mathcal{P}(\Sigma)} = \llbracket \dot{\bigwedge} \{ \big(\dot{\bigwedge} u\big) \dot{\rightarrow} \xi : \xi \in t\} \rrbracket_{(u,t)\in\mathcal{P}(\Sigma)\times\mathcal{P}(\Sigma)}$$

Proof. We start defining

$$G := \{(u, t, s, \xi) : u \subseteq s,\ \xi \in t\} \subseteq \mathcal{P}(\Sigma) \times \mathcal{P}(\Sigma) \times \mathcal{P}(\Sigma) \times \Sigma$$

$g_i : G \to \mathcal{P}(\Sigma)$ for $i = 1, 2, 3$ and $g_4 : G \to \Sigma$ the corresponding projections.
Let $g : G \to G$ such that $g(u, t, s, \xi) = (u, t, u, \xi)$.
Then:

$$\begin{aligned} &\llbracket \dot{\bigwedge} \{ \big(\dot{\bigwedge} s\big) \dot{\rightarrow} \xi : u \subseteq s, \xi \in t\} \rrbracket_{(u,t)\in\mathcal{P}(\Sigma)\times\mathcal{P}(\Sigma)} \\ =&\llbracket \dot{\bigwedge} \{ (\dot{\bigwedge} g_3(z)) \dot{\rightarrow} g_4(z) : z \in \langle g_1, g_2\rangle^{-1}(u,t)\} \rrbracket_{(u,t)\in\mathcal{P}(\Sigma)\times\mathcal{P}(\Sigma)} \\ =&\forall \langle g_1, g_2\rangle (\llbracket (\dot{\bigwedge} g_3(z)) \dot{\rightarrow} g_4(z) \rrbracket_{z\in G} = \forall \langle g_1, g_2\rangle (\llbracket \dot{\bigwedge} \circ g_3 \rrbracket_G \to \llbracket g_4 \rrbracket_G) \end{aligned}$$

Furthermore:

$$\begin{aligned} &\llbracket \dot{\bigwedge} \{ \big(\dot{\bigwedge} u\big) \mapsto \xi : \xi \in t\} \rrbracket_{(u,t)\in\mathcal{P}(\Sigma)\times\mathcal{P}(\Sigma)} = \\ =&\llbracket \dot{\bigwedge} \{ (\dot{\bigwedge} g_1(z)) \dot{\rightarrow} g_4(z) : z \in \langle g_1, g_2\rangle^{-1}(u,t) \rrbracket_{(u,t)\in\mathcal{P}(\Sigma)\times\mathcal{P}(\Sigma)} \\ =&\forall \langle g_1, g_2\rangle (\llbracket (\dot{\bigwedge} g_1(z)) \dot{\rightarrow} g_4(z) \rrbracket_{z\in G}) = \forall \langle g_1, g_2\rangle \big(\llbracket \dot{\bigwedge} \circ g_1 \rrbracket_G \to \llbracket g_4 \rrbracket_G\big) \end{aligned}$$

So we have to show that

$$\forall \langle g_1, g_2\rangle (\llbracket \dot{\bigwedge} \circ g_3 \rrbracket_G \to \llbracket g_4 \rrbracket_G) = \forall \langle g_1, g_2\rangle \big(\llbracket \dot{\bigwedge} \circ g_1 \rrbracket_G \to \llbracket g_4 \rrbracket_G\big)$$

(≤) Since $g_3 \circ g = g_1$ and $g_4 \circ g = g_4$:

$$\mathsf{P}g([\![\dot{\bigwedge} \circ g_3]\!]_G \to [\![g_4]\!]_G) = [\![\dot{\bigwedge} \circ g_3 \circ g]\!]_G \to [\![g_4 \circ g]\!]_G = [\![\dot{\bigwedge} \circ g_1]\!]_G \to [\![g_4]\!]_G$$

Then:

$$[\![\dot{\bigwedge} \circ g_3]\!]_G \to [\![g_4]\!]_G \le \forall g([\![\dot{\bigwedge} \circ g_1]\!]_G \to [\![g_4]\!]_G)$$

Thus:

$$\forall \langle g_1, g_2\rangle([\![\dot{\bigwedge} \circ g_3]\!]_G \to [\![g_4]\!]_G) \le \forall \langle g_1, g_2\rangle(\forall g([\![\dot{\bigwedge} \circ g_1]\!]_G \to [\![g_4]\!]_G))$$

Furthermore, $\langle g_1, g_2\rangle \circ g = \langle g_1 \circ g, g_2 \circ g\rangle = \langle g_1, g_2\rangle$ so

$$\forall \langle g_1, g_2\rangle([\![\dot{\bigwedge} \circ g_3]\!]_G \to [\![g_4]\!]_G) \le \forall \langle g_1, g_2\rangle([\![\dot{\bigwedge} \circ g_1]\!]_G \to [\![g_4]\!]_G)$$

(≥) Let F and $f_1, f_2 : F \to \mathcal{P}(\Sigma)$ defined as in Lemma 4.9.
If $z \in G$ we have that $g_1(z) \subseteq g_3(z)$, so by Corollary 4.1, $[\![\dot{\bigwedge} \circ g_3]\!]_G \to [\![g_4]\!]_G \ge [\![\dot{\bigwedge} \circ g_1]\!]_G \to [\![g_4]\!]_G$ and, consequently:

$$\forall \langle g_1, g_2\rangle([\![\dot{\bigwedge} \circ g_3]\!]_G \to [\![g_4]\!]_G) \ge \forall \langle g_1, g_2\rangle\big([\![\dot{\bigwedge} \circ g_1]\!]_G \to [\![g_4]\!]_G\big)$$

□

Lemma 4.11.

$$[\![\dot{\bigwedge}\{\theta \dot{\to} \xi : \xi \in s\}]\!]_{(\theta,s)\in\Sigma\times\mathcal{P}(\Sigma)} = [\![\theta \dot{\to} \dot{\bigwedge} s]\!]_{(\theta,s)\in\Sigma\times\mathcal{P}(\Sigma)}$$

Proof. Let $G := \{(\theta, \xi, s) : \xi \in s\} \subseteq \Sigma \times \Sigma \times \mathcal{P}(\Sigma)$ and $g_1, g_2 : G \to \Sigma$ and $g_3 : G \to \mathcal{P}(\Sigma)$ the corresponding projections, while π be the projection from $\Sigma \times \mathcal{P}(\Sigma)$ to Σ.

If $p \in \mathsf{P}(\Sigma \times \mathcal{P}(\Sigma))$ then:

$$\begin{aligned}
& p \leq [\![\dot{\bigwedge}\{\theta \dot{\to} \xi : \xi \in s\}]\!]_{(\theta,s)\in\Sigma\times\mathcal{P}(\Sigma)} \\
\Leftrightarrow\ & p \leq [\![\dot{\bigwedge}\{g_1(z) \dot{\to} g_2(z) : z \in \langle g_1, g_3\rangle^{-1}(\theta, s)\}]\!]_{(\theta,s)\in\Sigma\times\mathcal{P}(\Sigma)} \\
\Leftrightarrow\ & p \leq \forall\langle g_1, g_3\rangle([\![g_1]\!]_G \to [\![g_2]\!]_G) \\
\Leftrightarrow\ & \mathsf{P}\langle g_1, g_3\rangle(p) \leq [\![g_1]\!]_G \to [\![g_2]\!]_G \\
\Leftrightarrow\ & \mathsf{P}\langle g_1, g_3\rangle(p) \wedge [\![g_1]\!]_G \leq [\![g_2]\!]_G \\
\Leftrightarrow\ & \mathsf{P}\langle g_1, g_3\rangle(p) \wedge [\![\pi \circ \langle g_1, g_3\rangle]\!]_G \leq [\![g_2]\!]_G \\
\Leftrightarrow\ & \mathsf{P}\langle g_1, g_3\rangle(p \wedge [\![\pi]\!]_{\Sigma\times\mathcal{P}(\Sigma)}) \leq [\![g_2]\!]_G \\
\Leftrightarrow\ & p \wedge [\![\pi]\!]_{\Sigma\times\mathcal{P}(\Sigma)} \leq \forall\langle g_1, g_3\rangle([\![g_2]\!]_G) \\
\Leftrightarrow\ & p \leq [\![\pi]\!]_{\Sigma\times\mathcal{P}(\Sigma)} \to \forall\langle g_1, g_3\rangle([\![g_2]\!]_G) \\
\Leftrightarrow\ & p \leq [\![\theta]\!]_{(\theta,s)\in\Sigma\times\mathcal{P}(\Sigma)} \to [\![\dot{\bigwedge}\{g_2(z) : z \in \langle g_1, g_3\rangle^{-1}(\theta, s)\}]\!]_{(\theta,s)\in\Sigma\times\mathcal{P}(\Sigma)} \\
\Leftrightarrow\ & p \leq [\![\theta]\!]_{(\theta,s)\in\Sigma\times\mathcal{P}(\Sigma)} \to [\![\dot{\bigwedge}\{\xi : \xi \in s\}]\!]_{(\theta,s)\in\Sigma\times\mathcal{P}(\Sigma)} \\
\Leftrightarrow\ & p \leq [\![\theta \dot{\to} \dot{\bigwedge} s]\!]_{(\theta,s)\in\Sigma\times\mathcal{P}(\Sigma)}
\end{aligned}$$

Clearly, we can conclude that

$$[\![\dot{\bigwedge}\{\theta \dot{\to} \xi : \xi \in s\}]\!]_{(\theta,s)\in\Sigma\times\mathcal{P}(\Sigma)} = [\![\theta \dot{\to} \dot{\bigwedge} s]\!]_{(\theta,s)\in\Sigma\times\mathcal{P}(\Sigma)}$$

□

Corollary 4.2. *Let X be a set, $\sigma \in \Sigma^X$ and $t \in \mathcal{P}(\Sigma)^X$ then:*

$$[\![\dot{\bigwedge}\{\sigma(x) \dot{\to} \xi : \xi \in t(x)\}]\!]_{x\in X} = [\![\sigma(x) \dot{\to} \dot{\bigwedge} t(x)]\!]_{x\in X}$$

Proof.

$$[\![\dot{\bigwedge}\{\sigma(x) \dot{\to} \xi : \xi \in t(x)\}]\!]_{x\in X} = \mathsf{P}(\langle\sigma, t\rangle)([\![\dot{\bigwedge}\{\theta \dot{\to} \xi : \xi \in s\}]\!]_{(\theta,s)\in\Sigma\times\mathcal{P}(\Sigma)})$$

By Lemma 4.11:

$$\begin{aligned}
[\![\dot{\bigwedge}\{\sigma(x) \dot{\to} \xi : \xi \in t(x)\}]\!]_{x\in X} &= \mathsf{P}(\langle\sigma, t\rangle)([\![\theta \dot{\to} (\dot{\bigwedge} s)]\!]_{(\theta,s)\in\Sigma\times\mathcal{P}(\Sigma)}) \\
&= [\![\sigma(x) \dot{\to} \dot{\bigwedge} t(x)]\!]_{x\in X}
\end{aligned}$$

□

Now we can finally show:

Theorem 4.6. *Let π, π' the two projections from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$. Then*

$$\langle\!\langle a \to b \rangle\!\rangle_{(a,b)\in\mathcal{A}\times\mathcal{A}} = \langle\!\langle \pi \rangle\!\rangle_{\mathcal{A}\times\mathcal{A}} \to \langle\!\langle \pi' \rangle\!\rangle_{\mathcal{A}\times\mathcal{A}} \in \mathsf{P}(\mathcal{A} \times \mathcal{A})$$

Proof. Recall:

$$a \to b = \{s \mapsto \beta : s \in \tilde{\phi}_0(a)^{\subseteq} \text{ and } \beta \in b\}$$

where $s^{\subseteq} = \{s' \in \mathcal{P}(\Sigma) : s \subseteq s'\}$. Then:

$$\begin{aligned}
\langle\!\langle a \to b \rangle\!\rangle_{(a,b)\in\mathcal{A}\times\mathcal{A}} &= [\![\phi(a \to b)]\!]_{(a,b)\in\mathcal{A}\times\mathcal{A}} && \text{(by Lemma 4.6)}\\
&= [\![\dot{\bigwedge}\tilde{\phi}_0(a \to b)]\!]_{(a,b)\in\mathcal{A}\times\mathcal{A}}\\
&= [\![\dot{\bigwedge}\{\phi_0(s \mapsto \beta) : s \in \tilde{\phi}_0(a)^{\subseteq} \text{ and } \beta \in b\}]\!]_{(a,b)\in\mathcal{A}\times\mathcal{A}}\\
&= [\![\dot{\bigwedge}\{(\dot{\bigwedge} s)\dot{\to}\phi_0(\beta) : s \in \tilde{\phi}_0(a)^{\subseteq} \text{ and } \beta \in b\}]\!]_{(a,b)\in\mathcal{A}\times\mathcal{A}}\\
&= [\![\dot{\bigwedge}\{(\dot{\bigwedge} s)\dot{\to}\xi : s \in \tilde{\phi}_0(a)^{\subseteq} \text{ and } \xi \in \tilde{\phi}_0(b)\}]\!]_{(a,b)\in\mathcal{A}\times\mathcal{A}}
\end{aligned}$$

If we define $h : \mathcal{P}(\Sigma) \times \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ such that

$$h(u,t) \coloneqq \{(\dot{\bigwedge} s)\dot{\to}\xi : s \in u^{\subseteq} \text{ and } \xi \in t\}$$

then:

$$\begin{aligned}
\langle\!\langle a \to b \rangle\!\rangle_{(a,b)\in\mathcal{A}\times\mathcal{A}} &= [\![(\dot{\bigwedge} \circ h \circ \tilde{\phi}_0 \times \tilde{\phi}_0)(a,b)]\!]_{(a,b)\in\mathcal{A}\times\mathcal{A}}\\
&= \mathsf{P}(\tilde{\phi}_0 \times \tilde{\phi}_0)\big([\![\dot{\bigwedge} \circ h]\!]_{(u,t)\in\mathcal{P}(\Sigma)\times\mathcal{P}(\Sigma)}\big)\\
&= \mathsf{P}(\tilde{\phi}_0 \times \tilde{\phi}_0)\big([\![\dot{\bigwedge}\{(\dot{\bigwedge} s)\dot{\to}\xi : s \in u^{\subseteq} \text{ and } \xi \in t\}]\!]_{(u,t)\in\mathcal{P}(\Sigma)\times\mathcal{P}(\Sigma)}\big)
\end{aligned}$$

By Lemma 4.10:

$$\langle\!\langle a \to b \rangle\!\rangle_{(a,b)\in\mathcal{A}\times\mathcal{A}} = \mathsf{P}(\tilde{\phi}_0 \times \tilde{\phi}_0)\big([\![\dot{\bigwedge}\{(\dot{\bigwedge} u)\dot{\to}\xi : \xi \in t\}]\!]_{(u,t)\in\mathcal{P}(\Sigma)\times\mathcal{P}(\Sigma)}\big)$$

thus, by Corollary, 4.2

$$\begin{aligned}
\langle\!\langle a \to b \rangle\!\rangle_{(a,b)\in\mathcal{A}\times\mathcal{A}} &= \mathsf{P}(\tilde{\phi}_0 \times \tilde{\phi}_0)\big([\![(\dot{\bigwedge} u)\dot{\to}(\dot{\bigwedge} t)]\!]_{(u,t)\in\mathcal{P}(\Sigma)\times\mathcal{P}(\Sigma)}\big)\\
&= [\![(\dot{\bigwedge}\tilde{\phi}_0(a))\dot{\to}(\dot{\bigwedge}\tilde{\phi}_0(b))]\!]_{(a,b)\in\mathcal{A}\times\mathcal{A}}\\
&= [\![\phi(a)\dot{\to}\phi(b)]\!]_{(a,b)\in\mathcal{A}\times\mathcal{A}}\\
&= [\![\phi \circ a]\!]_{\mathcal{A}\times\mathcal{A}} \to [\![\phi \circ b]\!]_{\mathcal{A}\times\mathcal{A}}\\
&= \langle\!\langle \pi \rangle\!\rangle_{\mathcal{A}\times\mathcal{A}} \to \langle\!\langle \pi' \rangle\!\rangle_{\mathcal{A}\times\mathcal{A}}
\end{aligned}$$

□

Thus, we can use the same argument of Theorem 4.2 in order to prove:

Theorem 4.7. *Let X be a set and $\eta, \zeta \in \mathcal{A}^X$ then:*

$$\langle\!\langle \eta \to \zeta \rangle\!\rangle_X = \langle\!\langle \eta \rangle\!\rangle_X \to \langle\!\langle \zeta \rangle\!\rangle_X$$

4.5.6 Defining the separator

Now, we have to equip $\mathcal{A}$ with a separator to give it a structure of implicative algebra. The idea is to mimic what we did in section 4.4, where we have defined a sort of filter in the following way

$$\Phi := \{\xi \in \Sigma : [\![\xi]\!]_{*\in 1} = \top_1\} \subseteq \Sigma$$

Then, let us consider $S \subseteq \mathcal{A}$ such that:

$$S := \{a \in \mathcal{A} : \langle\!\langle a \rangle\!\rangle_{*\in 1} = \top_1\}$$

Thus:

$$S = \{a \in \mathcal{A} : \langle\!\langle a \rangle\!\rangle_{*\in 1} = \top_1\} = \{a \in \mathcal{A} : [\![\phi(a)]\!]_{*\in 1} = \top_1\} = \{a \in \mathcal{A} : \phi(a) \in \Phi\} = \phi^{-1}(\Phi)$$

Theorem 4.8. *The subset $S \subseteq \mathcal{A}$ is a separator of the implicative structure $\mathcal{A}$.*

Proof.

- *S is upward closed.* Let $a \in S$ and $b \in \mathcal{A}$ such that $a \leq b$, i.e. $b \subseteq a$. Thus $\tilde{\phi}_0(b) \subseteq \tilde{\phi}_0(a)$.
  Letφ and ψ be such that $* \in 1 \mapsto \tilde{\phi}_0(a)$ and $* \in 1 \mapsto \tilde{\phi}_0(b)$ respectively. Then by Corollary 4.1:

  $$\begin{aligned}[\![\phi(a)]\!]_{*\in 1} &= [\![\dot{\bigwedge}\tilde{\phi}_0(a)]\!]_{*\in 1} = [\![\dot{\bigwedge} \circ \varphi]\!]_{*\in 1} \leq [\![\dot{\bigwedge} \circ \psi]\!]_{*\in 1} = [\![\dot{\bigwedge}\tilde{\phi}_0(b)]\!]_{*\in 1} \\ &= [\![\phi(b)]\!]_{*\in 1}\end{aligned}$$

  Thus, we can conclude that $\langle\!\langle b \rangle\!\rangle_{*\in 1} = \top_{*\in 1}$.

- *S contains $\boldsymbol{K}^{\mathcal{A}}$ and $\boldsymbol{S}^{\mathcal{A}}$.* Let $\pi : 1 \times \mathcal{A} \to 1$ and $\pi' : (1 \times \mathcal{A}) \times \mathcal{A} \to 1 \times \mathcal{A}$ the first projections of $1 \times \mathcal{A}$ and $(1 \times \mathcal{A}) \times \mathcal{A}$ respectively.

  $$\begin{aligned}\langle\!\langle \mathbf{K} \rangle\!\rangle_{*\in 1} &= \Big\langle\!\!\Big\langle \bigwedge_{a,b\in\mathcal{A}} (a \to b \to a) \Big\rangle\!\!\Big\rangle_{*\in 1} \\ &= \langle\!\langle \bigwedge_{(-,a)} \{\bigwedge_{b\in\mathcal{A}} (a \to b \to a) : (-,a) \in \pi^{-1}(-)\} = \mathcal{A} \rangle\!\rangle_{*\in 1} \\ &= \forall\pi\Big(\Big\langle\!\!\Big\langle \bigwedge_{b\in\mathcal{A}} (a \to b \to a) \Big\rangle\!\!\Big\rangle_{(-,a)\in 1\times\mathcal{A}}\Big) \\ &= \forall\pi\Big(\Big\langle\!\!\Big\langle \bigwedge_{((-,a),b)} \{a \to b \to a : ((-,a),b) \in \pi'^{-1}((-,a))\} \Big\rangle\!\!\Big\rangle_{(-,a)\in 1\times\mathcal{A}}\Big) \\ &= \forall\pi(\forall\pi'(\langle\!\langle a \to b \to a \rangle\!\rangle_{((-,a),b)\in(1\times\mathcal{A})\times\mathcal{A}})) \\ &= \forall\pi(\forall\pi'(\top_{(1\times\mathcal{A})\times\mathcal{A}})) \\ &= \top_1\end{aligned}$$

  hence $\mathbf{K}^{\mathcal{A}} \in S$. Similarly, we can prove that $\mathbf{S}^{\mathcal{A}} \in S$.

- *S is closed under modus ponens.* Suppose that $(a \to b) \in S$ and $a \in S$. Then
$$\top_1 = \langle\!\langle a \to b \rangle\!\rangle_{*\in 1} = \langle\!\langle a \rangle\!\rangle_{*\in 1} \to \langle\!\langle b \rangle\!\rangle_{*\in 1} = \top_1 \to \langle\!\langle b \rangle\!\rangle_{*\in 1}$$
thus $\langle\!\langle b \rangle\!\rangle_{*\in 1} = \top_1$.

□

Lemma 4.12. *Let X be a set and $\eta, \zeta \in \mathcal{A}^X$ then:*

$$\langle\!\langle \eta \rangle\!\rangle_X \leq \langle\!\langle \zeta \rangle\!\rangle_X \Leftrightarrow \bigwedge_{x \in X} (\eta(x) \to \zeta(x)) \in S$$

Proof.

$$\begin{aligned}
\langle\!\langle \eta \rangle\!\rangle_X \leq \langle\!\langle \zeta \rangle\!\rangle_X &\Leftrightarrow \top_X \leq \langle\!\langle \eta \rangle\!\rangle_X \to \langle\!\langle \zeta \rangle\!\rangle_X \\
&\Leftrightarrow \mathsf{P}!_X(\top_1) \leq \langle\!\langle \eta \to \zeta \rangle\!\rangle_X \\
&\Leftrightarrow \top_1 \leq \forall !_X(\langle\!\langle \eta \to \zeta \rangle\!\rangle_X \\
&\Leftrightarrow \top_1 \leq \left\langle\!\!\left\langle \bigwedge \{\eta(x) \to \zeta(x) : x \in !_X^{-1}(-) = X\} \right\rangle\!\!\right\rangle_{*\in 1} \\
&\Leftrightarrow \bigwedge_{x \in X} (\eta(x) \to \zeta(x)) \in S
\end{aligned}$$

□

4.6 Isomorphism

Let $\mathsf{P}^{\mathcal{A}}$: $\mathbf{Set}^{op} \to \mathbf{HA}$ be the implicative tripos induced by the implicative algebra $\mathcal{A}$ as we have described in chapter 3.
We can finally show:

Theorem 4.9. *The implicative tripos $\mathsf{P}^{\mathcal{A}}$ is isomorphic to the tripos P.*

Proof. For every set X, we consider $\rho_X := \langle\!\langle - \rangle\!\rangle_X : \mathcal{A}^X \to \mathsf{P}X$.
Let $\eta, \zeta \in \mathcal{A}^X$ then:

$$\eta \vdash_{S[X]} \zeta \Leftrightarrow \eta \to \zeta \in S[X] \Leftrightarrow \bigwedge_{x \in X} (\eta(x) \to \zeta(x)) \in S$$

then, by Lemma 4.12:

$$\eta \vdash_{S[X]} \zeta \Leftrightarrow \langle\!\langle \eta \rangle\!\rangle_X \leq \langle\!\langle \zeta \rangle\!\rangle_X \Leftrightarrow \rho_X(\eta) \leq \rho_X(\zeta)$$

and, consequently,
$$\eta \dashv\vdash_{S[X]} \zeta \Leftrightarrow \rho_X(\eta) = \rho_X(\zeta)$$

Hence, ρ_X induces a bijective map:

$$\tilde{\rho}_X : \mathsf{P}^{\mathcal{A}}X \to \mathsf{P}X$$

Furthermore, $\tilde{\rho}_X$ is an isomorphism of HA by Lemma 1.2.
We want to show that $\rho = \{\rho_X\}_{X \text{ set}}$ is a natural transformation. Let $f : X \to Y$ be a map between sets.

$$\begin{array}{ccc} \mathsf{P}^{\mathcal{A}}Y & \xrightarrow{\tilde{\rho}_Y} & \mathsf{P}Y \\ \downarrow{\scriptstyle \mathsf{P}^{\mathcal{A}}f} & & \downarrow{\scriptstyle \mathsf{P}f} \\ \mathsf{P}^{\mathcal{A}}X & \xrightarrow{\tilde{\rho}_X} & \mathsf{P}X \end{array}$$

Let $[\eta] \in \mathsf{P}^{\mathcal{A}}Y$, then:

$$\begin{aligned}(\mathsf{P}f \circ \tilde{\rho}_Y)([\eta]) &= \mathsf{P}f(\langle\!\langle \eta \rangle\!\rangle_Y) = \mathsf{P}f([\![\phi(\eta)]\!]_Y) = [\![\phi(\eta \circ f)]\!]_X \\ &= \langle\!\langle \eta \circ f \rangle\!\rangle_X = \tilde{\rho}_X([\eta \circ f]) = (\tilde{\rho}_X \circ \mathsf{P}^{\mathcal{A}}f)([\eta]).\end{aligned}$$

□

Example. Let P be the implicative tripos induced by $\mathcal{B} = (\mathcal{B}, \leq, \to, U)$. We have already observed that the decoding map corresponds to the quotient map and that $\dot{\to} = \to$, $\dot{\bigwedge} = \bigwedge$ and $\dot{\bigvee} = \exists$.
Then

$$tr_{\mathcal{A}} = \mathsf{P}\phi(tr_{\mathcal{B}}) = [\mathsf{id}_{\mathcal{B}} \circ \phi] = [\phi] \in \mathcal{B}^{\mathcal{A}}/U[\mathcal{A}]$$

and

$$\langle\!\langle \eta \rangle\!\rangle_X = [\phi \circ \eta] \in \mathcal{B}^X/U[X] \text{ for every } \eta \in \mathcal{A}^X$$

Thus Lemma 4.8 states that for every map $f : X \to Y$ and $\eta \in \mathcal{A}^X$

$$[y \mapsto \phi(\bigcup_{f(x)=y} \eta(x))] = [y \mapsto \bigwedge_{f(x)=y} \phi(\eta(x))] \in \mathcal{B}^Y/U[Y]$$

while Theorem 4.7 ensures that for every $\eta, \zeta \in \mathcal{A}^X$

$$[\phi \circ \eta \to_{\mathcal{A}} \zeta] = [\phi \circ \eta] \to_{\mathcal{B}} [\phi \circ \zeta] \in \mathcal{B}^X/U[X]$$

Then the natural isomorphism defined in Theorem 4.9 is:

$$\begin{aligned}\tilde{\rho}_X \ : \ & \mathsf{P}^{\mathcal{A}}X \to \mathsf{P}X \\ & [\eta]_{S[X]} \mapsto [\phi \circ \eta]_{U[X]}\end{aligned}$$

where $S = \phi^{-1}(U)$.

Chapter 5

Geometric morphisms

Let us start by introducing the notion of *geometric morphism*.

Definition 5.1. *Let* P *and* Q *be two* **Set***-based triposes. Let* $\Phi_+ : \mathsf{P} \to \mathsf{Q}$ *and* $\Phi^+ : \mathsf{Q} \to \mathsf{P}$ *be two natural transformations where both* P *and* Q *are considered as functors* **Set** $\to$ **PreOrd**. *If:*

1. $\Phi^+ \dashv \Phi_+$*, i.e. for every set* X*,* $\Phi^+_X \dashv \Phi_{+X}$ *where both* Φ^+_X *and* Φ_{+X} *are considered as functors between the categories induced by the preorders* $\mathsf{P}X$ *and* $\mathsf{Q}X$*;*

2. *for every set* X*,* $\Phi^+_X : \mathsf{Q}X \to \mathsf{P}X$ *preserves finite meets;*

then $\Phi = (\Phi_+, \Phi^+)$ *is a* **geometric morphism** *from* P *to* Q *[12].*

Let $\mathcal{A}$ and $\mathcal{B}$ be two implicative algebras and $\mathsf{P}^{\mathcal{A}}$ and $\mathsf{P}^{\mathcal{B}}$ the corresponding implicative triposes. In this chapter, we will prove that every pair of functions $\psi : \mathcal{A} \to \mathcal{B}$ and $\varphi : \mathcal{B} \to \mathcal{A}$ that satisfies some particular properties induces a geometric morphism from $\mathsf{P}^{\mathcal{A}}$ to $\mathsf{P}^{\mathcal{B}}$. Furthermore, we will also show that every geometric morphism between implicative triposes is of this type.

Theorem 5.1. *Let* $(\mathcal{A}, \leq, \to, S)$ *and* $(\mathcal{B}, \leq, \Rightarrow, U)$ *be two implicative algebras and* $\mathsf{P}^{\mathcal{A}}$ *and* $\mathsf{P}^{\mathcal{B}}$ *the two implicative triposes induced respectively by them. Let* $\psi : \mathcal{A} \to \mathcal{B}$ *and* $\varphi : \mathcal{B} \to \mathcal{A}$ *be two maps such that:*

1. *for every* $X \subseteq \mathcal{A} \times \mathcal{A}$ *and* $Y \subseteq \mathcal{B} \times \mathcal{B}$*:*

$$\text{if } \bigwedge_{(a,a')\in X} a \to a' \in S \text{ then } \bigwedge_{(a,a')\in X} \psi(a) \Rightarrow \psi(a') \in U$$
$$\text{if } \bigwedge_{(b,b')\in Y} b \Rightarrow b' \in U \text{ then } \bigwedge_{(b,b')\in Y} \varphi(b) \to \varphi(b) \in S$$

2. *for every* $X \subseteq \mathcal{A} \times \mathcal{B}$*:*

$$\bigwedge_{(a,b)\in X} \varphi(b) \to a \in S \text{ if and only if } \bigcurlywedge_{(a,b)\in X} b \Rightarrow \psi(a) \in U$$

3. *let* π_1, π_2 *be the projections of* $\mathcal{B} \times \mathcal{B}$ *then*

$$[\varphi \circ (\pi_1 \times_{\mathcal{B}} \pi_2)] = [(\varphi \circ \pi_1) \times_{\mathcal{A}} (\varphi \circ \pi_2)] \in \mathcal{A}^{\mathcal{B}\times\mathcal{B}}/S[\mathcal{B} \times \mathcal{B}]$$

Then ψ *and* φ *induce a geometric morphism between* $\mathsf{P}^{\mathcal{A}}$ *and* $\mathsf{P}^{\mathcal{B}}$*.*

Proof. If X is a set then we can define

$$\begin{aligned} \Phi_{+X} : \mathsf{P}^{\mathcal{A}}X &\to \mathsf{P}^{\mathcal{B}}X \\ [\eta] &\mapsto [\psi \circ \eta] \\ \Phi^{+}_{X} : \mathsf{P}^{\mathcal{B}}X &\to \mathsf{P}^{\mathcal{A}}X \\ [\beta] &\mapsto [\varphi \circ \beta] \end{aligned}$$

We want to show that $\Phi = (\Phi_+, \Phi^+)$ is a geometric morphism between $\mathsf{P}^{\mathcal{A}}$ and $\mathsf{P}^{\mathcal{B}}$.

- Φ_{+X} *and* Φ^{+}_{X} *are well defined.* Let $[\eta] = [\xi] \in \mathcal{A}^X/S[X]$, i.e.

  $$\bigwedge_{x\in X} \eta(x) \to \xi(x) \in S \quad \text{and} \quad \bigwedge_{x\in X} \xi(x) \to \eta(x) \in S.$$

  Clearly $\{(\eta(x), \xi(x)) : x \in X\}$ and $\{(\xi(x), \eta(x)) : x \in X\}$ are subsets of $\mathcal{A} \times \mathcal{A}$, then, by condition 1:

  $$\bigwedge_{x\in X} \psi(\eta(x)) \Rightarrow \psi(\xi(x)) \in U \quad \text{and} \quad \bigwedge_{x\in X} \psi(\xi(x)) \Rightarrow \psi(\eta(x)) \in U,$$

  hence $[\psi \circ \eta] = [\psi \circ \xi] \in \mathcal{B}^X/U[X]$. Analogously for φ.

- Φ_+ *and* Φ^+ *are natural transformations.* The first thing to show is that Φ_{+X} and Φ^{+}_{X} are monotone. Let $[\eta], [\xi] \in \mathcal{A}^X/S[X]$ such that $[\eta] \vdash [\xi]$, i.e. $\bigwedge_{x\in X} \eta(x) \to \xi(x) \in S$, then, we have already proved that:

  $$\bigcurlywedge_{x\in X} \psi(\eta(x)) \Rightarrow \psi(\xi(x)) \in U \text{ hence } \Phi_{+X}([\eta]) \vdash \Phi_{+X}([\xi])$$

  Analogously for Φ^{+}_{X}.

  Let $f : X \to Y$ be a map between sets, we have to show that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A}^X/S[X] & \xleftarrow[\mathsf{P}^{\mathcal{A}}(f)]{} & \mathcal{A}^Y/S[Y] \\ \downarrow{\scriptstyle \Phi_{+X}} & & \downarrow{\scriptstyle \Phi_{+Y}} \\ \mathcal{B}^X/U[X] & \xleftarrow[\mathsf{P}^{\mathcal{B}}(f)]{} & \mathcal{B}^Y/U[Y] \end{array}$$

Let $[\eta] \in \mathcal{A}^Y/S[Y]$ then:

$$\begin{aligned}(\Phi_{+X} \circ \mathsf{P}^{\mathcal{A}}(f))([\eta]) = \Phi_{+X}([\eta \circ f]) = [\psi \circ \eta \circ f] = \mathsf{P}^{\mathcal{B}}(f)([\psi \circ \eta])\\ = (\mathsf{P}^{\mathcal{B}}(f) \circ \Phi_{+Y})([\eta])\end{aligned}$$

Analogously for Φ^+.

- $\Phi^+ \dashv \Phi_+$. Let X be a set, $[\beta] \in \mathcal{B}^X/U[X]$ and $[\eta] \in \mathcal{A}^X/S[X]$ then:

$$\begin{aligned}\Phi^+_X([\beta]) \le [\eta] &\text{ if and only if } [\varphi \circ \beta] \vdash [\eta]\\ &\text{ if and only if } \bigwedge_{x \in X} \varphi(\beta(x)) \to \eta(x) \in S\end{aligned}$$

  Clearly $\{(\eta(x), \beta(x)) : x \in X\}$ is a subset of $\mathcal{A} \times \mathcal{B}$, hence, by condition 2.:

$$\begin{aligned}\Phi^+_X([\beta]) \le [\eta] &\text{ if and only if } \bigwedge_{x \in X} \beta(x) \Rightarrow \psi(\eta(x)) \in U\\ &\text{ if and only if } [\beta] \le \Phi_{+X}([\eta]).\end{aligned}$$

- Φ^+_X *preserves finite meets.* Let $[\beta], [\gamma] \in \mathcal{B}^X/U[X]$ then:

$$\begin{aligned}\Phi^+_X([\beta]) \wedge \Phi^+_X([\gamma]) = [\varphi \circ \beta] \wedge [\varphi \circ \gamma] = [(\varphi \circ \beta) \times_{\mathcal{A}} (\varphi \circ \gamma)]\\ \Phi^+_X([\beta] \wedge [\gamma]) = \Phi^+_X([\beta \times_{\mathcal{B}} \gamma]) = [\varphi \circ \beta \times_{\mathcal{B}} \gamma]\end{aligned}$$

  Clearly:

$$\bigwedge_{b,b \in \mathcal{B}'} (\varphi(b) \times_{\mathcal{A}} \varphi(b')) \to \varphi(b \times_{\mathcal{B}} b') \le \bigwedge_{x \in X} ((\varphi(\beta(x)) \times_{\mathcal{A}} \varphi(\gamma(x))) \to (\varphi(\beta(x) \times_{\mathcal{B}} \gamma(x)))$$

$$\bigwedge_{b,b \in \mathcal{B}'} \varphi(b \times_{\mathcal{B}} b') \to (\varphi(b) \times_{\mathcal{A}} \varphi(b')) \le \bigwedge_{x \in X} \varphi(\beta(x) \times_{\mathcal{B}} \gamma(x)) \to (\varphi(\beta(x)) \times_{\mathcal{A}} \varphi(\gamma(x)))$$

  Hence, by condition 3. and by the fact that S is upwards closed, we can conclude that:

$$\bigwedge_{x \in X} ((\varphi(\beta(x)) \times_{\mathcal{A}} \varphi(\gamma(x))) \to (\varphi(\beta(x) \times_{\mathcal{B}} \gamma(x))) \in S$$

$$\bigwedge_{x \in X} (\varphi(\beta(x) \times_{\mathcal{B}} \gamma(x)) \to (\varphi(\beta(x)) \times_{\mathcal{A}} \varphi(\gamma(x)))) \in S$$

  thus $[(\varphi \circ \beta) \times_{\mathcal{A}} (\varphi \circ \gamma)] = [\varphi \circ \beta \times_{\mathcal{B}} \gamma] \in \mathcal{A}^X/S[X]$, i.e. $\Phi^+_X([\beta]) \wedge \Phi^+_X([\gamma]) = \Phi^+_X([\beta \wedge \gamma])$.

□

Theorem 5.2. *Let $(\mathcal{A}, \leq, \to, S)$ and $(\mathcal{B}, \leq, \Rightarrow, U)$ be two implicative algebras and $\mathsf{P}^{\mathcal{A}}$ and $\mathsf{P}^{\mathcal{B}}$ the two implicative triposes induced respectively by them. Let Θ a geometric morphism from $\mathsf{P}^{\mathcal{A}}$ to $\mathsf{P}^{\mathcal{B}}$. Then Θ induces two maps $\psi : \mathcal{A} \to \mathcal{B}$ and $\varphi : \mathcal{B} \to \mathcal{A}$ that satisfy the conditions 1., 2. and 3. of the Theorem 5.1. Furthermore, the geometric morphism Φ induced by ψ and φ as described in Theorem 5.1 is Θ.*

Proof. Let $\Theta = (\Theta_+, \Theta^+)$ where $\Theta_+ : \mathsf{P}^{\mathcal{A}} \to \mathsf{P}^{\mathcal{B}}$ and $\Theta^+ : \mathsf{P}^{\mathcal{B}} \to \mathsf{P}^{\mathcal{A}}$.

- Θ *induces* ψ *and* φ. Let:

$$\Theta_{+\mathcal{A}}(tr_{\mathcal{A}}) = \Theta_{+\mathcal{A}}([\mathsf{id}_{\mathcal{A}}]) = [\bar{\Theta}_+(\mathsf{id}_{\mathcal{A}})] \in \mathcal{B}^{\mathcal{A}}/U[\mathcal{A}]$$
$$\Theta^+{}_{\mathcal{B}}(tr_{\mathcal{B}}) = \Theta^+{}_{\mathcal{B}}([\mathsf{id}_{\mathcal{B}}]) = [\bar{\Theta}^+(\mathsf{id}_{\mathcal{B}})] \in \mathcal{A}^{\mathcal{B}}/S[\mathcal{B}]$$

By axiom of choice, we can define:

$$\begin{aligned} \psi : \mathcal{A} &\to \mathcal{B} \\ a &\mapsto \bar{\Theta}_+(\mathsf{id}_{\mathcal{A}})(a) \end{aligned} \qquad\qquad \begin{aligned} \varphi : \mathcal{B} &\to \mathcal{A} \\ b &\mapsto \bar{\Theta}^+(\mathsf{id}_{\mathcal{B}})(b) \end{aligned}$$

- $\Theta = \Phi$. Let X be a set and $[\eta] \in \mathcal{A}^X/S[X]$. We can define

$$\begin{aligned} \{\eta\} : X &\to \mathcal{A} \\ x &\mapsto \eta(x) \end{aligned}$$

then $\mathsf{P}^{\mathcal{A}}\{\eta\}(tr_{\mathcal{A}}) = [\eta]$. Since Θ_+ is a natural transformation, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A}^{\mathcal{A}}/S[\mathcal{A}] & \xrightarrow{\mathsf{P}^{\mathcal{A}}\{\eta\}} & \mathcal{A}^X/S[X] \\ \downarrow{\scriptstyle \Theta_{+\mathcal{A}}} & & \downarrow{\scriptstyle \Theta_{+X}} \\ \mathcal{B}^{\mathcal{A}}/U[\mathcal{A}] & \xrightarrow{\mathsf{P}^{\mathcal{B}}\{\eta\}} & \mathcal{B}^X/U[X] \end{array}$$

then

$$\begin{aligned} \Theta_{+X}([\eta]) &= (\Theta_{+X} \circ \mathsf{P}^{\mathcal{A}}\{\eta\})(tr_{\mathcal{A}}) = (\mathsf{P}^{\mathcal{B}}\{\eta\} \circ \Theta_{+\mathcal{A}})(tr_{\mathcal{A}}) \\ &= \mathsf{P}^{\mathcal{B}}\{\eta\}([\psi]) = [\psi \circ \eta] \\ &= \Phi_{+X}([\eta]) \end{aligned}$$

Analogously, we can show $\Theta^+_X = \Phi^+_X$.

- *Condition 1.* Let $X \subseteq \mathcal{A} \times \mathcal{A}$ and $\bigwedge_{(a,a') \in X} a \to a' \in S$. Let us consider:

$$\begin{aligned} \eta : \ X &\to \mathcal{A} \\ (a, a') &\mapsto a \end{aligned} \qquad\qquad \begin{aligned} \zeta : \ X &\to \mathcal{A} \\ (a, a') &\mapsto a' \end{aligned}$$

  Then:

$$\bigwedge_{x \in X} \eta(x) \to \zeta(x) \in S$$

  i.e. $[\eta] \vdash [\zeta]$. Since Θ_{+X} is monotonous we have $\Theta_{+X}(\eta) \vdash \Theta_{+X}(\zeta)$, which means

$$\bigwedge_{x \in X} \psi(\eta(x)) \Rightarrow \psi(\zeta(x)) \in U.$$

  Analogously for φ.

- *Condition 2.* Let $X \subseteq \mathcal{A} \times \mathcal{B}$ and

$$\begin{aligned} \eta : \ X &\to \mathcal{A} \\ (a, b) &\mapsto a \end{aligned} \qquad\qquad \begin{aligned} \beta : \ X &\to \mathcal{B} \\ (a, b) &\mapsto b \end{aligned}$$

  Since $\Theta_X^{+} \dashv \Theta_{+X}$:

$$\Theta_X^{+}([\beta]) \vdash [\eta] \text{ if and only if } [\beta] \vdash \Theta_{+X}([\eta])$$

  so:

$$[\varphi \circ \beta] \vdash [\eta] \in S \text{ if and only if } [\beta] \vdash [\psi \circ \eta] \in U$$

  i.e.

$$\bigwedge_{x \in X} \varphi(\beta(x)) \to \eta(x) \in S \text{ if and only if } \bigwedge_{x \in X} \beta(x) \Rightarrow \psi(\eta(x)) \in U$$
$$\bigwedge_{(a,b) \in X} \varphi(b) \to a \in S \text{ if and only if } \bigwedge_{(a,b) \in X} b \Rightarrow \psi(a) \in U$$

- *Condition 3.* Let X be a set and $[\beta], [\gamma] \in \mathcal{B}^X / U[X]$, since $\Theta_X^{+}([\beta] \wedge [\gamma]) = \Theta_X^{+}([\beta]) \wedge \Theta_X^{+}([\gamma])$ we have that

$$[\varphi \circ (\beta \times_{\mathcal{B}} \gamma)] = [(\varphi \circ \beta) \times_{\mathcal{A}} (\varphi \circ \gamma)]$$

  then:

$$\bigwedge_{x \in X} (\varphi(\beta(x) \times_{\mathcal{B}} \gamma(x)) \to (\varphi(\beta(x)) \times_{\mathcal{A}} \varphi(\gamma(x)))) \in S$$
$$\bigwedge_{x \in X} ((\varphi(\beta(x)) \times_{\mathcal{A}} \varphi(\gamma(x))) \to \varphi(\beta(x) \times_{\mathcal{B}} \gamma(x))) \in S$$

□

Lemma 5.1. *Let* P *and* Q *be two* **Set***-based triposes and* $\Phi = (\Phi_+, \Phi^+)$ *a geometric morphism from* P *to* Q*. Then* Φ^+ *commutes with* $\exists$ *i.e. for every map between sets* $f : X \to Y$ *the following diagram*

$$\begin{array}{ccc} \mathsf{P}X & \xleftarrow[\Phi^+_X]{} & \mathsf{Q}X \\ \big\downarrow{\scriptstyle \exists^{\mathsf{P}} f} & & \big\downarrow{\scriptstyle \exists^{\mathsf{Q}} f} \\ \mathsf{P}Y & \xleftarrow[\Phi^+_Y]{} & \mathsf{Q}Y \end{array}$$

commutes.

Proof. Let us fix a map $f : X \to Y$ between sets and let $q \in \mathsf{Q}X$ and $p \in \mathsf{P}Y$. Then:

$$\begin{aligned} \exists^{\mathsf{P}} f(\Phi^+_X(q)) \leq p \quad &\text{if and only if} \quad \Phi^+_X(q) \leq \mathsf{P}f(p) \\ &\text{if and only if} \quad q \leq \Phi_{+X}(\mathsf{P}f(p)) \\ &\text{if and only if} \quad q \leq \mathsf{Q}f(\Phi_{+Y}(p)) \\ &\text{if and only if} \quad \exists^{\mathsf{Q}} f(q) \leq \Phi_{+Y}(p) \\ &\text{if and only if} \quad \Phi^+_Y(\exists^{\mathsf{Q}} f(q)) \leq p \end{aligned}$$

Thus $\exists^P f \circ \Phi^+_X = \Phi^+_Y \circ \exists^Q f$. □

Corollary 5.1. *Let* $(\mathcal{A}, \leq, \to, S)$ *and* $(\mathcal{B}, \leq, \Rightarrow, U)$ *be two implicative algebras and* $\mathsf{P}^{\mathcal{A}}$ *and* $\mathsf{P}^{\mathcal{B}}$ *the two implicative triposes induced respectively by them. Let* Θ *a geometric morphism from* $\mathsf{P}^{\mathcal{A}}$ *to* $\mathsf{P}^{\mathcal{B}}$ *and* $\varphi : \mathcal{B} \to \mathcal{A}$ *the map induced by* Θ *as described in Theorem 5.2. Then* φ *commutes with* $\exists$*, i.e. for every* $f : X \to Y$ *map between sets and* $\eta \in \mathcal{B}^X$*:*

$$[y \mapsto \exists_{f(x)=y} \varphi(\eta(x))] = [y \mapsto \varphi(\exists_{f(x)=y} \eta(x))] \in \mathcal{A}^Y / S[Y]$$

Proof. Obvious. □

Observation. Let $\mathcal{A}$ and $\mathcal{B}$ be the implicative algebras induced by two complete Heyting algebras $\mathbb{H}$ and $\mathbb{K}$ as described in chapter 2.
In such case, Theorem 5.1 and Theorem 5.2 imply the existence of the following one-to-one correspondence:

$$\left\{\text{Geometric morphisms from } \mathsf{P}^{\mathbb{H}} \text{ to } \mathsf{P}^{\mathbb{K}}\right\} \overset{1:1}{\longleftrightarrow} \left\{\text{Localic morphisms from } \mathbb{H} \text{ to } \mathbb{K}\right\}$$

$$\Phi = (\Phi_+, \Phi^+) \quad \longleftrightarrow \quad \varphi$$

Indeed, let Φ be a geometric morphism from $\mathsf{P}^{\mathbb{H}}$ to $\mathsf{P}^{\mathbb{K}}$ and let $\varphi : \mathbb{K} \to \mathbb{H}$ and $\psi : \mathbb{H} \to \mathbb{K}$ be the two maps induced by Φ as described in Theorem 5.2. Since φ preserves binary $\wedge$ and $\varphi \dashv \psi$ then φ is a morphism of frames by Lemma 1.3.
Conversely, if $\varphi : \mathbb{K} \to \mathbb{H}$ is a morphism of frames then let $\psi : \mathbb{H} \to \mathbb{K}$ be its unique right adjoint as defined in Lemma 1.3. Then, clearly, φ and ψ satisfy the conditions of Theorem 5.1 and thus they induce a geometric morphism Φ from $\mathsf{P}^{\mathbb{H}}$ to $\mathsf{P}^{\mathbb{K}}$.

It is clear that different pairs of functions can induce the same geometric morphism. Indeed, let (ψ_1, φ_1) and (ψ_2, φ_2) be two pairs of functions that satisfy the conditions of Theorem 5.1 and let Φ_1 and Φ_2 the two corresponding geometric morphisms induced. Then, it is obvious that:

$$\Phi_1 = \Phi_2 \quad \text{if and only if} \quad \begin{cases} [\psi_1] = [\psi_2] \in \mathcal{B}^{\mathcal{A}}/U[\mathcal{A}] \\ [\varphi_1] = [\varphi_2] \in \mathcal{A}^{\mathcal{B}}/S[\mathcal{B}] \end{cases}$$

In the last chapters, we have shown that there exists a correspondence between the geometric morphisms between **Set**-based triposes and a particular class of equivalence of functions between implicative algebras. This results lead us to define the following category:

- the objects are implicative algebras;
- for every implicative algebras $\mathcal{A} = (\mathcal{A}, \leq, \to, S)$ and $\mathcal{B} = (\mathcal{B}, \leq, \Rightarrow, U)$: $\mathsf{Hom}(\mathcal{A}, \mathcal{B}) = \{[(\psi, \varphi)] \in \mathcal{B}^{\mathcal{A}}/U[\mathcal{A}] \times \mathcal{A}^{\mathcal{B}}/S[\mathcal{B}] : (\psi, \varphi)$ satisfies the conditions of Theorem 5.1$\}$;
- $[(\theta, \xi)] \circ [(\psi, \varphi)] = [(\theta \circ \psi, \varphi \circ \xi)]$ for all morphisms $[(\psi, \varphi)], [(\theta, \xi)]$ such that $\mathsf{cod}(\psi) = \mathsf{dom}(\theta)$;
- for every implicative algebra $\mathcal{A}$: $\mathsf{id}_{\mathcal{A}} = [(\mathsf{id}_{\mathcal{A}}, \mathsf{id}_{\mathcal{A}})]$.

Introducing this new category allows us to have a new perspective on the study of the category of triposes and geometric morphisms, by changing the focus from triposes to the easier structures of implicative algebras.

Chapter 6

First-order logic morphisms

Similarly to what we have done in the last one, in this chapter, we will study which type of functions between implicative algebras induces and is induced by a *first-order logic morphism* between the two corresponding implicative triposes.

Definition 6.1. *Let* P *and* Q *be two* **Set***-based triposes. A* **first-order logic morphism** *from* P *to* Q *is a natural transformation* $\Phi : \mathsf{P} \Rightarrow \mathsf{Q}$ *such that* Φ *commutes with the left and and the right adjoints.*

Let $\mathcal{A} = (\mathcal{A}, \leq, \to)$ and $\mathcal{B} = (\mathcal{B}, \leq, \Rightarrow)$ be two implicative algebras. We will denote:

$$\begin{aligned} \bigwedge : \mathcal{P}(\mathcal{A}) &\to \mathcal{A} & \curlywedge : \mathcal{P}(\mathcal{B}) &\to \mathcal{B} \\ X &\mapsto \bigwedge_{x \in X} x & Y &\mapsto \curlywedge_{y \in Y} y \\ \exists : \mathcal{P}(\mathcal{A}) &\to \mathcal{A} & \exists : \mathcal{P}(\mathcal{B}) &\to \mathcal{B} \\ X &\mapsto \exists_{x \in X} x & Y &\mapsto \exists_{y \in Y} y \end{aligned}$$

Before we go any further, let us introduce a technical lemma that will be useful to us later.

Lemma 6.1. *Let* $\mathcal{A}$ *be an implicative algebra and* I *be a set. If* $a_i, b_i \in \mathcal{A}$ *for every* $i \in I$*, then:*

$$\curlywedge_{i \in I} (a_i \to b_i) \leq \curlywedge_{i \in I} a_i \to \curlywedge_{i \in I} b_i$$

Proof.

$$\curlywedge_{i \in I} (a_i \to b_i) \leq \curlywedge_{i \in I} (\curlywedge_{j \in I} a_j \to b_i) = \curlywedge_{j \in I} a_j \to \curlywedge_{i \in I} b_i$$

□

Theorem 6.1. *Let $\mathcal{A} = (\mathcal{A}, \leq, \rightarrow, S)$ and $\mathcal{B} = (\mathcal{B}, \leq, \Rightarrow, U)$ be two implicative algebras and $\mathsf{P}^{\mathcal{A}}$ and $\mathsf{P}^{\mathcal{B}}$ their implicative triposes.*
Let $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ be a map such that:

1. *$\varphi(S) \subseteq U$;*

2. *if π_1, π_2 are respectively the first and the second projections of $\mathcal{A} \times \mathcal{A}$ then:*

$$[\varphi \circ (\pi_1 \rightarrow \pi_2)] = [\varphi \circ \pi_1 \Rightarrow \varphi \circ \pi_2] \in \mathcal{B}^{\mathcal{A}\times\mathcal{A}}/U[\mathcal{A} \times \mathcal{A}]$$

3. $$[\varphi \circ \bigwedge] = [\curlywedge \circ \mathcal{P}\varphi] \in \mathcal{B}^{\mathcal{P}(\mathcal{A})}/U[\mathcal{P}(\mathcal{A})]$$

For every set X, let:

$$\begin{aligned} \Phi_X : \mathcal{A}^X/S[X] &\rightarrow \mathcal{B}^X/U[X] \\ [\eta] &\mapsto [\varphi \circ \eta] \end{aligned}$$

Then Φ is a natural transformation from $\mathsf{P}^{\mathcal{A}}$ to $\mathsf{P}^{\mathcal{B}}$, where they are both considered as functors from **Set** *to* **PreOrd**. *Furthermore, Φ preserves implication, $\top$ and $\wedge$ and commutes with the right adjoints.*

Proof. Let us start by observing that the second condition ensures that:

$$\curlywedge_{a,a' \in \mathcal{A}} \Big(\varphi(a \rightarrow a') \Rightarrow \varphi(a) \Rightarrow \varphi(a') \Big) \in U$$
$$\curlywedge_{a,a' \in \mathcal{A}} \Big((\varphi(a) \Rightarrow \varphi(a')) \Rightarrow \varphi(a \rightarrow a') \Big) \in U$$

Since U is upwards closed:

$$\curlywedge_{x \in X} \Big(\varphi(\eta(x) \rightarrow \zeta(x)) \Rightarrow \varphi(\eta(x)) \Rightarrow \varphi(\zeta(x)) \Big) \in U$$
$$\curlywedge_{x \in X} \Big((\varphi(\eta(x)) \Rightarrow \varphi(\zeta(x))) \Rightarrow \varphi(\eta(x) \rightarrow \zeta(x)) \Big) \in U$$

for every set X and $\eta, \zeta \in \mathcal{A}^X$. Now, we can prove the theorem.

- Φ_X *is well defined and monotonous.* Let X be a set and $\eta, \zeta \in \mathcal{A}^X$ such that $\eta \vdash_{S[X]} \zeta$, so:

$$\bigwedge_{x \in X} (\eta(x) \to \zeta(x)) \in S \qquad \text{(by definition)}$$

$$\varphi\big(\bigwedge_{x \in X} (\eta(x) \to \zeta(x))\big) \in U \qquad \text{(by condition \textit{1.})}$$

Furthermore, by condition *3.*:

$$[A \mapsto \varphi(\bigwedge_{a \in A} a)] = [A \mapsto \bigwedge_{a \in A} \varphi(a)]$$

i.e.

$$\bigwedge_{A \subseteq \mathcal{A}} \big(\varphi(\bigwedge_{a \in A} a) \Rightarrow \bigwedge_{a \in A} \varphi(a)\big) \in U \quad \text{and} \quad \bigwedge_{A \subseteq \mathcal{A}} \big(\bigwedge_{a \in A} \varphi(a) \Rightarrow \varphi(\bigwedge_{a \in A} a)\big) \in U$$

If we choose $A = \{\eta(x) \to \zeta(x) : x \in X\}$, then:

$$\varphi\big(\bigwedge_{x \in X} (\eta(x) \to \zeta(x))\big) \Rightarrow \bigwedge_{x \in X} \varphi(\eta(x) \to \zeta(x)) \in U$$

U is closed by modus ponens, so:

$$\bigwedge_{x \in X} \varphi(\eta(x) \to \zeta(x)) \in U$$

Furthermore, by condition *2.*:

$$\bigwedge_{x \in X} \big(\varphi(\eta(x) \to \zeta(x)) \Rightarrow \varphi(\eta(x)) \Rightarrow \varphi(\zeta(x))\big) \in U$$

thus, by Lemma 6.1:

$$\bigwedge_{x \in X} \varphi(\eta(x) \to \zeta(x)) \Rightarrow \bigwedge_{x \in X} \Big(\varphi(\eta(x)) \Rightarrow \varphi(\zeta(x))\Big) \in U$$

and

$$\bigwedge_{x \in X} \Big(\varphi(\eta(x)) \Rightarrow \varphi(\zeta(x))\Big) \in U$$

by modus ponens. Hence, we have shown that $\varphi \circ \eta \vdash_{U[x]} \varphi \circ \zeta$. Then, Φ_X is well defined and clearly monotonous.

- $\Phi_X \circ \mathsf{P}^{\mathcal{A}} f = \mathsf{P}^{\mathcal{B}} f \circ \Phi_Y$. Let $f : X \to Y$ be a map between sets. We want to show that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A}^X/S[X] & \xleftarrow[\mathsf{P}^{\mathcal{A}}f]{} & \mathcal{A}^Y/S[Y] \\ \big\downarrow{\scriptstyle \Phi_X} & & \big\downarrow{\scriptstyle \Phi_Y} \\ \mathcal{B}^X/U[X] & \xleftarrow[\mathsf{P}^{\mathcal{B}}f]{} & \mathcal{B}^Y/U[Y] \end{array}$$

Let $[\eta] \in \mathcal{A}^Y/S[Y]$, then:

$$\begin{aligned}(\Phi_X \circ \mathsf{P}^{\mathcal{A}}f)([\eta]) = \Phi_X([\eta \circ f]) &= [\varphi \circ \eta \circ f] = \mathsf{P}^{\mathcal{B}}f([\varphi \circ \eta]) \\ &= (\mathsf{P}^{\mathcal{B}}f \circ \Phi_Y)([\eta])\end{aligned}$$

- Φ *preserves* $\to$. We have already observed that for every set X and $\eta, \zeta \in \mathcal{A}^X$:

$$\bigwedge_{x \in X} \Big((\varphi(\eta(x)) \Rightarrow \varphi(\zeta(x))) \Rightarrow \varphi(\eta(x) \to \zeta(x)) \Big) \in U$$

$$\bigwedge_{x \in X} \Big(\varphi(\eta(x) \to \zeta(x)) \Rightarrow \varphi(\eta(x)) \Rightarrow \varphi(\zeta(x)) \Big) \in U$$

which means:

$$\Phi_X([\eta] \to [\zeta]) = [\varphi \circ \eta \to \zeta] = [(\varphi \circ \eta) \Rightarrow (\varphi \circ \zeta)] = \Phi_X([\eta]) \Rightarrow \Phi_X([\zeta])$$

- Φ *commutes with* $\forall$. Let $f : X \to Y$ be a map between sets and $\eta \in \mathcal{A}^X$.

$$\begin{array}{ccc} \mathcal{A}^X/S[X] & \xrightarrow{\forall^{\mathcal{A}}f} & \mathcal{A}^Y/S[Y] \\ \big\downarrow{\scriptstyle \Phi_X} & & \big\downarrow{\scriptstyle \Phi_Y} \\ \mathcal{B}^X/U[X] & \xrightarrow{\forall^{\mathcal{B}}f} & \mathcal{B}^Y/U[Y] \end{array}$$

We have to show:

$$\begin{aligned}(\Phi_Y \circ \forall^{\mathcal{A}}f)([\eta]) &= (\forall^{\mathcal{B}}f \circ \Phi_X)([\eta]) \\ \Phi_Y([y \mapsto \bigwedge_{f(x)=y} \eta(x)]) &= \forall^{\mathcal{B}}f([\varphi \circ \eta]) \\ [y \mapsto \varphi(\bigwedge_{f(x)=y} \eta(x))] &= [y \mapsto \bigwedge_{f(x)=y} \varphi(\eta(x))]\end{aligned}$$

The third condition ensures:

$$\bigwedge_{X \subseteq \mathcal{A}} \big(\varphi \circ \bigwedge X \Rightarrow \bigwedge \mathcal{P}\varphi(X)\big) \in U$$

$$\bigwedge_{X \subseteq \mathcal{A}} \big(\bigwedge \mathcal{P}\varphi(X) \Rightarrow \varphi \circ \bigwedge X\big) \in U$$

For every $y \in Y$ let $f^{-1}(y) = \{x \in X : f(x) = y\}$. Since U is upwards closed:

$$\bigwedge_{y \in Y} \left(\varphi \circ \bigwedge \mathcal{P}\eta(f^{-1}(y)) \Rightarrow \bigwedge \mathcal{P}\varphi(\mathcal{P}\eta(f^{-1}(y)))\right) \in U$$
$$\bigwedge_{y \in Y} \left(\bigwedge \mathcal{P}\varphi(\mathcal{P}\eta(f^{-1}(y))) \Rightarrow \varphi \circ \bigwedge \mathcal{P}\eta(f^{-1}(y))\right) \in U$$

i.e.

$$\bigwedge_{y \in Y} \left(\varphi(\bigwedge_{f(x)=y} \eta(x)) \Rightarrow \bigwedge_{f(x)=y} \varphi(\eta(x))\right) \in U$$
$$\bigwedge_{y \in Y} \left(\bigwedge_{f(x)=y} \varphi(\eta(x)) \Rightarrow \varphi(\bigwedge_{f(x)=y} \eta(x))\right) \in U$$

- Φ_X *preserves* $\top$. Clearly $\top_{\mathsf{P}^{\mathcal{A}}X} = [x \mapsto \top_{\mathcal{A}}] \in \mathcal{A}^X / S[X]$. Let us observe that:

  $$\Phi_X(\top_{\mathsf{P}^{\mathcal{A}}X}) = \Phi_X([x \mapsto \top_{\mathcal{A}}]) = [\varphi \circ (x \mapsto \top_{\mathcal{A}})] = [x \mapsto \varphi(\top_{\mathcal{A}})]$$

  Since $\top_{\mathcal{A}} = \bigwedge \varnothing$, by condition *3.*:

  $$\varphi(\top_{\mathcal{A}}) \Rightarrow \bigwedge \varnothing \in U$$
  $$\bigwedge \varnothing \Rightarrow \varphi(\top_{\mathcal{A}}) \in U$$

  thus:

  $$\Phi_X(\top_{\mathsf{P}^{\mathcal{A}}X}) = [x \mapsto \bigwedge \varnothing] = [x \mapsto \top_{\mathcal{B}}] = \top_{\mathsf{P}^{\mathcal{B}}X}$$

- Φ *preserves* $\wedge$. Let $\eta, \zeta \in \mathcal{A}^X$. Φ_X is monotonous, so:

  $$[\eta] \wedge [\zeta] \vdash [\eta] \implies \Phi_X([\eta] \wedge [\zeta]) \vdash \Phi_X([\eta])$$
  $$[\eta] \wedge [\zeta] \vdash [\zeta] \implies \Phi_X([\eta] \wedge [\zeta]) \vdash \Phi_X([\zeta])$$

  Hence, $\Phi_X([\eta] \wedge [\zeta]) \vdash \Phi_X([\eta]) \wedge \Phi_X([\zeta])$.
  Now, we want to show the opposite inequality. $\mathsf{P}^{\mathcal{A}}X$ is a Heyting algebra, thus:

  $$\begin{aligned}[\eta] \wedge [\zeta] \vdash [\eta] \wedge [\zeta] \text{ then } & [\eta] \vdash [\zeta] \to ([\eta] \wedge [\zeta]) \\ \text{then } & [\eta] \vdash [\zeta] \to [\eta \times_{\mathcal{A}} \zeta] \\ \text{then } & [\eta] \vdash [\zeta \to \eta \times_{\mathcal{A}} \zeta]\end{aligned}$$

  Hence:

  $$\Phi_X([\eta]) \vdash \Phi_X([\zeta \to \eta \times_{\mathcal{A}} \zeta])$$

Since $\mathsf{P}^{\mathcal{B}}X$ is also a Heyting algebra and Φ_X preserves $\to$:

$$\begin{aligned}\Phi_X([\eta]) \vdash \Phi_X([\zeta \to \eta \times_{\mathcal{A}} \zeta]) &\text{ iff } \Phi_X([\eta]) \vdash \Phi_X([\zeta]) \Rightarrow \Phi_X([\eta \times_{\mathcal{A}} \zeta]) \\ &\text{ iff } \Phi_X([\eta]) \vdash \Phi_X([\zeta]) \Rightarrow \Phi_X([\eta] \wedge [\zeta]) \\ &\text{ iff } \Phi_X([\eta]) \wedge \Phi_X([\zeta]) \vdash \Phi_X([\eta] \wedge [\zeta])\end{aligned}$$

□

Theorem 6.2. *Let $\mathcal{A} = (\mathcal{A}, \leq, \to, S)$ and $\mathcal{B} = (\mathcal{B}, \leq, \Rightarrow, U)$ be two implicative algebras and $\mathsf{P}^{\mathcal{A}}$ and $\mathsf{P}^{\mathcal{B}}$ their implicative triposes. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a map such that:*

1. *$\varphi(S) \subseteq U$;*
2. *if π_1, π_2 are respectively the first and the second projections of $\mathcal{A} \times \mathcal{A}$ then:*

$$[\varphi \circ (\pi_1 \to \pi_2)] = [\varphi \circ \pi_1 \Rightarrow \varphi \circ \pi_2] \in \mathcal{B}^{\mathcal{A}\times\mathcal{A}}/U[\mathcal{A} \times \mathcal{A}]$$

3.

$$[\varphi \circ \bigwedge] = [\bigwedge \circ \mathcal{P}\varphi] \in \mathcal{B}^{\mathcal{P}(\mathcal{A})}/U[\mathcal{P}(\mathcal{A})]$$
$$[\varphi \circ \exists] = [\exists \circ \mathcal{P}\varphi] \in \mathcal{B}^{\mathcal{P}(\mathcal{A})}/U[\mathcal{P}(\mathcal{A})]$$

4.

$$\bigwedge_{a,b\in\mathcal{A}} \big(\varphi(a +_{\mathcal{A}} b) \Rightarrow (\varphi(a) +_{\mathcal{B}} \varphi(b))\big) \in U$$

For every set X, let:

$$\begin{aligned}\Phi_X : \mathcal{A}^X/S[X] &\to \mathcal{B}^X/U[X] \\ [\eta] &\mapsto [\varphi \circ \eta]\end{aligned}$$

Then Φ is a first-order logic morphism from $\mathsf{P}^{\mathcal{A}}$ to $\mathsf{P}^{\mathcal{B}}$.

Proof. By Theorem 6.1, we have just to prove that Φ preserves $\perp$ and $\vee$ and that commutes with $\exists$.

- Φ *preserves* $\perp$. Let us start by observing that $\perp = [x \mapsto \exists_\varnothing] \in \mathcal{A}^X/S[X]$ for every set X.

$$\exists_\varnothing = \bigwedge_{c\in\mathcal{A}} \big(\bigwedge \varnothing \to c\big) = \bigwedge_{c\in\mathcal{A}} \big(\top_{\mathcal{A}} \to c\big) = \top_{\mathcal{A}} \to \bigwedge_{c\in\mathcal{A}} c = \top_{\mathcal{A}} \to \perp_{\mathcal{A}}$$

  Since clearly $\perp = [x \mapsto \perp_{\mathcal{A}}]$:

$$\dfrac{\dfrac{\dfrac{\overset{\text{Axiom}}{\overline{x : \exists_\varnothing \vdash x : \top_{\mathcal{A}} \to \bot_{\mathcal{A}}}} \qquad \dfrac{}{x : \exists_\varnothing \vdash x : \top_{\mathcal{A}}}\,\top\text{-intro.}}{x : \exists_\varnothing \vdash xx : \bot_{\mathcal{A}}}\,\to\text{-elim.}}{\vdash \lambda x.xx : \exists_\varnothing \to \bot_{\mathcal{A}} \text{ for all } x \in X}\,\to\text{-intro.}}{\vdash \lambda x.xx : \bigwedge_{x \in X}(\exists_\varnothing \to \bot_{\mathcal{A}})}\,\text{Gen.}$$

Thus, $[x \mapsto \exists_\varnothing] = [x \mapsto \bot_{\mathcal{A}}] = \bot$. Then:

$$\Phi_X(\bot_{\mathsf{P}^{\mathcal{A}}X}) = \Phi_X([x \mapsto \exists_\varnothing]) = [\varphi \circ (x \mapsto \exists_\varnothing)] = [x \mapsto \varphi(\exists_\varnothing)]$$

By condition *3.*:

$$\varphi(\exists_\varnothing) \Rightarrow \exists_\varnothing \in U$$
$$\exists_\varnothing \Rightarrow \varphi(\exists_\varnothing) \in U$$

thus we can conclude:

$$\Phi_X(\bot_{\mathsf{P}^{\mathcal{A}}X}) = [x \mapsto \exists_\varnothing] = \bot_{\mathsf{P}^{\mathcal{B}}X}$$

- Φ *preserves* $\vee$. Let $\eta, \zeta \in \mathcal{A}^X$. Since Φ_X is monotonous:

$$[\eta] \vdash [\eta] \vee [\zeta] \implies \Phi_X([\eta]) \vdash \Phi_X([\eta] \vee [\zeta])$$
$$[\zeta] \vdash [\eta] \vee [\zeta] \implies \Phi_X([\zeta]) \vdash \Phi_X([\eta] \vee [\zeta])$$

hence $\Phi_X([\eta]) \vee \Phi_X([\zeta]) \vdash \Phi_X([\eta] \vee [\zeta])$.
Conversely, since:

$$\bigwedge_{a,b \in \mathcal{A}} \big(\varphi(a +_{\mathcal{A}} b) \Rightarrow \varphi(a) +_{\mathcal{B}} \varphi(b)\big) \leq$$
$$\leq \bigwedge_{x \in X} \big(\varphi(\eta(x) +_{\mathcal{A}} \zeta(x)) \Rightarrow \varphi(\eta(x)) +_{\mathcal{B}} \varphi(\zeta(x))\big)$$

we can conclude that $\Phi_X([\eta] \vee [\zeta]) \vdash \Phi_X([\eta]) \vee \Phi_X([\zeta])$ by condition *4.*

- Φ *commutes with* $\exists$. Similar to the commutativity with $\forall$ in Theorem 6.1.

□

Theorem 6.3. *Let* $\mathcal{A} = (\mathcal{A}, \leq, \to, S)$ *and* $\mathcal{B} = (\mathcal{B}, \leq, \Rightarrow, U)$ *be two implicative algebras and* $\mathsf{P}^{\mathcal{A}}$ *and* $\mathsf{P}^{\mathcal{B}}$ *the implicative triposes induced by them.*
Let Θ *be a first-order logic morphism from* $\mathsf{P}^{\mathcal{A}}$ *to* $\mathsf{P}^{\mathcal{B}}$*. Then* Θ *induces a map* $\varphi : \mathcal{A} \to \mathcal{B}$ *that satisfies the conditions* 1., 2., 3. *and* 4. *of Theorem 6.2. Furthermore, the first-order logic morphism* Φ *induced by* φ *as described in Theorem 6.2 is* Θ*.*

Proof. • *Θ induces φ and $\Theta = \Phi$.* Similarly to what we have done in Theorem 5.1, we can define:

$$\Theta_{\mathcal{A}}(tr_{\mathcal{A}}) = \Theta_{\mathcal{A}}([\mathsf{id}_{\mathcal{A}}]) = [\bar{\Theta}(\mathsf{id}_{\mathcal{A}})] \in \mathcal{B}^{\mathcal{A}}/U[\mathcal{A}]$$

and

$$\begin{aligned} \varphi : \mathcal{A} &\to \mathcal{B} \\ a &\mapsto \bar{\Theta}(\mathsf{id}_{\mathcal{A}})(a) \end{aligned}$$

by axiom of choice. Analogously to Theorem 5.1, we can prove that $\Theta_X([\eta]) = [\varphi \circ \eta] = \Phi_X([\eta])$, for every set X and for every $\eta \in \mathcal{A}^X$.

- *Condition 1.* Let $s \in S$. We want to show that $\varphi(s) \in U$. If $X = \{*\}$, we can consider:

$$\begin{aligned} \bar{s} : X &\to \mathcal{A} \\ * &\mapsto s \end{aligned}$$

Clearly $[\bar{s}] = \top_{\mathsf{P}^{\mathcal{A}}X}$. Since Θ is a first-order logic morphism, $\Theta_X(\top_{\mathsf{P}^{\mathcal{A}}X}) = \top_{\mathsf{P}^{\mathcal{B}}X}$. Hence $[\varphi \circ \bar{s}] = [x \mapsto \top_{\mathcal{B}}]$ and $\top_{\mathcal{B}} \Rightarrow \varphi(s) \in U$. Then $\varphi(s) \in U$ because $\top_{\mathcal{B}} \in U$.

- *Condition 2.* Let $X = \mathcal{A} \times \mathcal{A}$ and π_1, π_2 be respectively the first and the second projections of $\mathcal{A} \times \mathcal{A}$. By hypothesis, Θ_X is a morphism of **HA**, so it preserves the implication. Then:

$$\begin{aligned} \Theta_X([\pi_1] \to [\pi_2]) &= \Theta_X([\pi_1]) \Rightarrow \Theta_X([\pi_2]) \\ \Theta_X([\pi_1 \to \pi_2]) &= [\varphi \circ \pi_1] \Rightarrow [\varphi \circ \pi_2] \\ [\varphi \circ (\pi_1 \to \pi_2)] &= [\varphi \circ \pi_1 \Rightarrow \varphi \circ \pi_2] \end{aligned}$$

- *Condition 3.* Let $E = \{(a, A) : A \subseteq \mathcal{A} \text{ and } a \in A\} \subseteq \mathcal{A} \times \mathcal{P}(\mathcal{A})$ and π_1, π_2 the corresponding projections of E. By hypothesis, Θ commutes with the right adjoints, so the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A}^E/S[E] & \xrightarrow{\forall^{\mathcal{A}}\pi_2} & \mathcal{A}^{\mathcal{P}(\mathcal{A})}/S[\mathcal{P}(\mathcal{A})] \\ \downarrow{\Phi_E} & & \downarrow{\Phi_{\mathcal{P}(\mathcal{A})}} \\ \mathcal{B}^E/S[E] & \xrightarrow{\forall^{\mathcal{B}}\pi_2} & \mathcal{B}^{\mathcal{P}(\mathcal{A})}/U[\mathcal{P}(\mathcal{A})] \end{array}$$

Hence,

$$
\begin{aligned}
(\Phi_{\mathcal{P}(\mathcal{A})} \circ \forall^{\mathcal{A}}\pi_2)([\pi_1]) &= (\forall^{\mathcal{B}}\pi_2 \circ \Phi_E)([\pi_1]) \\
\Phi_{\mathcal{P}(\mathcal{A})}([A \mapsto \bigwedge_{\pi_2(z)=A} \pi_1(z)]) &= \forall^{\mathcal{B}}\pi_2([\varphi \circ \pi_1]) \\
[A \mapsto \varphi(\bigwedge_{\pi_2(z)=A} \pi_1(z))] &= [A \mapsto \curlywedge_{\pi_2(z)=A} \varphi(\pi_1(z))] \\
[A \mapsto \varphi(\bigwedge_{a\in A} a)] &= [A \mapsto \curlywedge_{a\in A} \varphi(a)] \\
[\varphi \circ \bigwedge] &= [\curlywedge \circ \mathcal{P}\varphi]
\end{aligned}
$$

Similar for $\exists$.

- *Condition 4.* Similarly to what we have done before, let $X = \mathcal{A}\times\mathcal{A}$ and π_1, π_2 be its projections. Since Θ_X is a morphism of **HA**, it preserves $\vee$:

$$
\begin{aligned}
\Theta_X([\pi_1] \vee [\pi_2]) &= \Theta_X([\pi_1]) \vee \Theta_X([\pi_2]) \\
\Theta_X([\pi_1 +_{\mathcal{A}} \pi_2]) &= [\varphi \circ \pi_1] \vee [\varphi \circ \pi_2] \\
[\varphi \circ (\pi_1 +_{\mathcal{A}} \pi_2)] &= [\varphi \circ \pi_1 +_{\mathcal{B}} \varphi \circ \pi_2]
\end{aligned}
$$

thus:

$$
\curlywedge_{a,b\in\mathcal{A}} \varphi(a +_{\mathcal{A}} b) \Rightarrow (\varphi(a) + \varphi(b)) \in U
$$

□

Observation. Let $\mathbb{H}$ and $\mathbb{K}$ be complete Heyting algebras. Let us show that $\varphi : \mathbb{H} \to \mathbb{K}$ is a map that satisfies the conditions expressed in Theorem 6.2 if and only if φ is a morphism of complete Heyting algebras, i.e. we want to show that there exists a one-to-one correspondence:

$$
\left\{\text{First-order logic morphisms } \mathsf{P}^{\mathbb{H}} \to \mathsf{P}^{\mathbb{K}}\right\} \overset{1:1}{\longleftrightarrow} \left\{\text{Morphisms of cHAs } \mathbb{H} \to \mathbb{K}\right\}
$$

Indeed, since we are working with implicative algebras induced by complete Heyting algebras, requiring that φ preserves $\forall, \exists$ and $\to$ - as expressed in Theorem 6.2 - is equivalent to require that φ preserves arbitrary meets, arbitrary joins and the implication, i.e. that φ is a morphism of complete Heyting algebras. Let us observe that this result follows from the fact that the separator of an implicative algebra induced by a complete Heyting algebra is defined as $\{\top\}$.

It is clear that the same first-order logic morphism can be induced by different functions: indeed, if φ_1 and φ_2 satisfy the conditions of Theorem 6.2 and Φ_1 and Φ_2 are the corresponding first-order logic morphisms induced, then:

$$\Phi_1 = \Phi_2 \qquad \text{if and only if} \qquad [\varphi_1] = [\varphi_2] \in \mathcal{B}^{\mathcal{A}}/U[\mathcal{A}]$$

Similarly to what we have done in chapter 5, we can now define a category such that:

- the objects are implicative algebras;
- for all implicative algebras $\mathcal{A} = (\mathcal{A}, \leq, \to, S)$ and $\mathcal{B} = (\mathcal{B}, \leq, \Rightarrow, U)$: $\mathsf{Hom}(\mathcal{A}, \mathcal{B}) = \{[\varphi] \in \mathcal{B}^{\mathcal{A}}/U[\mathcal{A}] : \varphi$ satisfies the conditions of Theorem 6.2$\}$;
- $[\psi] \circ [\varphi] = [\psi \circ \varphi]$ for all morphisms $[\psi], [\varphi]$ such that $\mathsf{cod}(\varphi) = \mathsf{dom}(\psi)$;
- for every implicative algebra $\mathcal{A}$: $\mathsf{id}_{\mathcal{A}} = [\mathsf{id}_{\mathcal{A}}]$.

6.1 Particular cases

In this section, we will describe some particular cases where the conditions of Theorem 6.2 on the map φ can be relaxed.
Let us fix two implicative algebras $\mathcal{A} = (\mathcal{A}, \leq, \to, S)$ and $\mathcal{B} = (\mathcal{B}, \leq, \Rightarrow, U)$ and their implicative triposes $\mathsf{P}^{\mathcal{A}}$ and $\mathsf{P}^{\mathcal{B}}$. Furthermore, let $\varphi : \mathcal{A} \to \mathcal{B}$ be a map such that:

1. $\varphi(S) \subseteq U$;
2. if π_1, π_2 are respectively the first and the second projections of $\mathcal{A} \times \mathcal{A}$ then:
$$[\varphi \circ (\pi_1 \to \pi_2)] = [\varphi \circ \pi_1 \Rightarrow \varphi \circ \pi_2] \in \mathcal{B}^{\mathcal{A} \times \mathcal{A}}/U[\mathcal{A} \times \mathcal{A}]$$
3. $$[\varphi \circ \bigwedge] = [\bigwedge \circ \mathcal{P}\varphi] \in \mathcal{B}^{\mathcal{P}(\mathcal{A})}/U[\mathcal{P}(\mathcal{A})]$$

As before, we will consider:

$$\begin{aligned} \Phi_X : \mathcal{A}^X/S[X] &\to \mathcal{B}^X/U[X] \\ [\eta] &\mapsto [\varphi \circ \eta] \end{aligned}$$

for every set X.

Lemma 6.2. *Let $\eta, \zeta \in \mathcal{A}^X$ for some set X. Then:*

$$[\varphi \circ (\eta +_{\mathcal{A}} \zeta)] = [\bigwedge_{\delta \in \mathcal{A}^X} \big((\varphi \circ \eta \Rightarrow \varphi \circ \delta) \Rightarrow (\varphi \circ \zeta \Rightarrow \varphi \circ \delta) \Rightarrow \varphi \circ \delta\big)]$$

Furthermore, :

$$[\varphi \circ \exists] = [X \mapsto \bigwedge_{a \in \mathcal{A}} \big(\bigwedge_{x \in X} (\varphi(x) \Rightarrow \varphi(a)) \Rightarrow \varphi(a)\big)]$$

Proof. Let us define $\bar{\eta}, \bar{\zeta} : \mathcal{A}^X \times X \to \mathcal{A}$ such that $\bar{\eta} = \eta \circ \pi_2$ and $\bar{\zeta} = \zeta \circ \pi_2$ where π_2 is the second projection of $\mathcal{A}^X \times X$. Let $\mathsf{ev} : \mathcal{A}^X \times X \to X$ be the evaluation map.
By Theorem 6.1, Φ preserves the implication so:

$$[\varphi \circ (\bar{\eta} \to \mathsf{ev}) \to (\bar{\zeta} \to \mathsf{ev}) \to \mathsf{ev}] = [(\varphi \circ \bar{\eta} \Rightarrow \varphi \circ \mathsf{ev}) \Rightarrow (\varphi \circ \bar{\zeta} \Rightarrow \varphi \circ \mathsf{ev}) \Rightarrow \varphi \circ \mathsf{ev}]$$

i.e.

$$\bigwedge_{z \in \mathcal{A}^X \times X} \big((\varphi \circ (\bar{\eta} \to \mathsf{ev}) \to (\bar{\zeta} \to \mathsf{ev}) \to \mathsf{ev})(z) \Rightarrow \\ \Rightarrow ((\varphi \circ \bar{\eta} \Rightarrow \varphi \circ \mathsf{ev}) \Rightarrow (\varphi \circ \bar{\zeta} \Rightarrow \varphi \circ \mathsf{ev}) \Rightarrow \varphi \circ \mathsf{ev})(z)\big) \in U$$

$$\bigwedge_{z \in \mathcal{A}^X \times X} \big(((\varphi \circ \bar{\eta} \Rightarrow \varphi \circ \mathsf{ev}) \Rightarrow (\varphi \circ \bar{\zeta} \Rightarrow \varphi \circ \mathsf{ev}) \Rightarrow \varphi \circ \mathsf{ev})(z) \Rightarrow \\ \Rightarrow (\varphi \circ (\bar{\eta} \to \mathsf{ev}) \to (\bar{\zeta} \to \mathsf{ev}) \to \mathsf{ev})(z)\big) \in U$$

By Lemma 6.1 and by the fact that U is upwards closed:

$$[\bigwedge_{\delta \in \mathcal{A}^X} (\varphi \circ (\bar{\eta} \to \mathsf{ev}) \to (\bar{\zeta} \to \mathsf{ev}) \to \mathsf{ev})(\delta, -)] = \\ = [\bigwedge_{\delta \in \mathcal{A}^X} \big((\varphi \circ \bar{\eta} \Rightarrow \varphi \circ \mathsf{ev}) \Rightarrow (\varphi \circ \bar{\zeta} \Rightarrow \varphi \circ \mathsf{ev}) \Rightarrow \varphi \circ \mathsf{ev}\big)(\delta, -)]$$

i.e.

$$[\bigwedge_{\delta \in \mathcal{A}^X} \big(\varphi \circ (\eta \to \delta) \to (\zeta \to \delta) \to \delta\big)] = \\ = [\bigwedge_{\delta \in \mathcal{A}^X} \big((\varphi \circ \eta \Rightarrow \varphi \circ \delta) \Rightarrow (\varphi \circ \zeta \Rightarrow \varphi \circ \delta) \Rightarrow \varphi \circ \delta\big)]$$

By condition *3.*:

$$[\varphi \circ (\eta +_{\mathcal{A}} \zeta)] = [\bigwedge_{\delta \in \mathcal{A}^X} \big((\varphi \circ \eta \Rightarrow \varphi \circ \delta) \Rightarrow (\varphi \circ \zeta \Rightarrow \varphi \circ \delta) \Rightarrow \varphi \circ \delta\big)]$$

Let $\eta : \mathcal{P}(\mathcal{A}) \times \mathcal{A} \to \mathcal{P}(\mathcal{A})$ such that $\eta(X, a) = \{x \to a : x \in X\}$ and let π_2 be the second the projection of $\mathcal{P}(\mathcal{A}) \times \mathcal{A}$. Then:

$$\begin{aligned}[\varphi \circ ((\bigwedge \circ \eta) \to \pi_2)] &= [\varphi \circ (\bigwedge \circ \eta)] \Rightarrow [\varphi \circ \pi_2] \\ &= [\curlywedge(\varphi \circ \eta)] \Rightarrow [\varphi \circ \pi_2]\end{aligned}$$

Furthermore, let $E = \{(x, X) : x \in X\}$ and $\pi_1', \pi_2' : E \times \mathcal{A} \to \mathcal{A}$ such that $\pi_1'((x, X), a) = x$ and $\pi_2'((x, X), a) = a$ then:

$$[\varphi \circ (\pi_1' \to \pi_2')] = [(\varphi \circ \pi_1') \Rightarrow (\varphi \circ \pi_2')] \in \mathcal{B}^{E \times \mathcal{A}}/U[E \times \mathcal{A}]$$

By Lemma 6.1:

$$[\curlywedge(\varphi \circ \eta)] = [(a, X) \mapsto \curlywedge\{\varphi(x) \to \varphi(a) : x \in X\}] \in \mathcal{B}^{\mathcal{P}(\mathcal{A}) \times \mathcal{A}}/U[\mathcal{P}(\mathcal{A}) \times \mathcal{A}]$$

Thus:

$$\begin{aligned}&\curlywedge_{a \in \mathcal{A}, X \subseteq \mathcal{A}} \Big(\varphi \circ (\bigwedge_{x \in X} (x \to a) \to a) \Rightarrow (\curlywedge_{x \in X} (\varphi(x) \to \varphi(a)) \to \varphi(a))\Big) \in U \\ &\curlywedge_{a \in \mathcal{A}, X \subseteq \mathcal{A}} \Big((\curlywedge_{x \in X} (\varphi(x) \to \varphi(a)) \to \varphi(a)) \Rightarrow \varphi \circ (\bigwedge_{x \in X} (x \to a) \to a)\Big) \in U\end{aligned}$$

by Lemma 6.1 and by the fact that U is upwards closed and that Φ commutes with right adjoints:

$$[\varphi \circ \exists] = [X \mapsto \curlywedge_{a \in \mathcal{A}} \Big(\curlywedge_{x \in X} (\varphi(x) \Rightarrow \varphi(a)) \Rightarrow \varphi(a)\Big)]$$

□

Proposition 6.1. *If there exists $\chi \in \mathcal{A}^{\mathcal{B}}$ such that:*

$$[\varphi \circ \chi] = [\mathsf{id}_{\mathcal{B}}] \in \mathcal{B}^{\mathcal{B}}/U[\mathcal{B}]$$

Then Φ is a first-order logic morphism from $\mathsf{P}^{\mathcal{A}}$ to $\mathsf{P}^{\mathcal{B}}$.

Proof. By Theorem 6.1, it is sufficient to show that Φ commutes with $\exists$ and that $[\varphi \circ (\eta +_{\mathcal{A}} \zeta)] \vdash [(\varphi \circ \eta) +_{\mathcal{B}} (\varphi \circ \zeta)]$ for $\eta, \zeta \in \mathcal{A}^X$.
Fixed $a, b \in \mathcal{A}$ we denote with $\beta_d := (\varphi(b) \Rightarrow d) \Rightarrow d$ and with $\alpha_d := (\varphi(a) \Rightarrow d) \Rightarrow \beta_d$ for every $d \in \mathcal{B}$. Let

$$u = \curlywedge_{d \in \mathcal{B}} ((\varphi \circ \chi)(d) \Rightarrow d) \in U \qquad u' = \curlywedge_{d \in \mathcal{B}} (d \Rightarrow (\varphi \circ \chi)(d)) \in U$$

Fixed $t = \lambda w.u'(yw)$ and $t' = \lambda w'.u'(zw')$, let us consider:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash u : u}\,\text{Param.}}{\Gamma \vdash u : \varphi(\chi(d)) \Rightarrow d}\;\text{Subs.} \qquad \pi}{\Gamma := x : \curlywedge_{c\in\mathcal{A}} \alpha_{\varphi(c)}, y : \varphi(a) \Rightarrow d, z : \varphi(b) \Rightarrow d \vdash u(xtt') : d}\;\to\text{-elim.}}{x : \curlywedge_{c\in\mathcal{A}} \alpha_{\varphi(c)}, y : \varphi(a) \Rightarrow d \vdash \lambda z.u(xtt') : \beta_d}\;\to\text{-intro.}}{x : \curlywedge_{c\in\mathcal{A}} \alpha_{\varphi(c)} \vdash \lambda yz.u(xtt') : \alpha_d}\;\to\text{-intro.}}{x : \curlywedge_{c\in\mathcal{A}} \alpha_{\varphi(c)} \vdash \lambda yz.u(xtt') : \varphi(a) +_{\mathcal{B}} \varphi(b)}\;\text{Gen.}}{\vdash \lambda xyz.u(xtt') : \curlywedge_{c\in\mathcal{A}} \alpha_{\varphi(c)} \Rightarrow \varphi(a) +_{\mathcal{B}} \varphi(b)}\;\to\text{-intro.}}{}
$$

where π is:

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, w : \varphi(a) \vdash x : \curlywedge_{c\in\mathcal{A}} \alpha_{\varphi(c)}}\,\text{Axiom}}{\Gamma, w : \varphi(a) \vdash x : \alpha_{\varphi(\chi(d))}}\;\text{Subs.} \qquad \pi'}{\Gamma \vdash xt : \beta_{\varphi(\chi(d))}}\;\to\text{-elim.} \qquad \dfrac{\text{Similar to } \pi'}{\Gamma \vdash t' : \varphi(b) \Rightarrow \varphi(\chi(d))}}{\Gamma \vdash xtt' : \varphi(\chi(d))}\;\to\text{-elim.}
$$

and π' is :

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma' \vdash u' : u'}\,\text{Param.}}{\Gamma' \vdash u' : d \Rightarrow \varphi(\chi(d))}\;\text{Subs.} \qquad \dfrac{\dfrac{}{\Gamma' \vdash y : \varphi(a) \Rightarrow d}\,\text{Axiom} \quad \dfrac{}{\Gamma' \vdash w : \varphi(a)}\,\text{Axiom}}{\Gamma' \vdash yw : d}\;\to\text{-elim.}}{\Gamma' := \Gamma, w : \varphi(a) \vdash u'(yw) : \varphi(\chi(d))}\;\to\text{-elim.}}{\Gamma \vdash t : \varphi(a) \Rightarrow \varphi(\chi(d))}\;\to\text{-intro.}
$$

Thus, by generalization and by Lemma 6.2:

$$[\varphi \circ (\eta +_{\mathcal{A}} \zeta)] \vdash [(\varphi \circ \eta) +_{\mathcal{B}} (\varphi \circ \zeta)]$$

By Lemma 6.2

$$[\varphi \circ \exists] = [X \mapsto \bigcurlywedge_{a\in\mathcal{A}} \big(\bigcurlywedge_{x\in X} (\varphi(x) \Rightarrow \varphi(a)) \Rightarrow \varphi(a)\big)]$$

Clearly

$$\bigcurlywedge_{b\in\mathcal{B}} \big(\bigcurlywedge_{x\in X} (\varphi(x) \Rightarrow b) \Rightarrow b\big) \leq \bigcurlywedge_{a\in\mathcal{A}} \big(\bigcurlywedge_{x\in X} (\varphi(x) \Rightarrow \varphi(a)) \Rightarrow \varphi(a)\big)$$

thus:

$$[\exists \circ \mathcal{P}\varphi] \vdash [\varphi \circ \exists]$$

since

$$\vdash \lambda y.y : \bigwedge_{X\subseteq\mathcal{A}}\left(\bigwedge_{b\in\mathcal{B}}\Big(\bigwedge_{x\in X}(\varphi(x)\Rightarrow b)\Rightarrow b\Big)\Rightarrow\bigwedge_{a\in\mathcal{A}}\Big(\bigwedge_{x\in X}(\varphi(x)\Rightarrow\varphi(a))\Rightarrow\varphi(a)\Big)\right)$$

Furthermore, let $\alpha_d := \bigwedge_{i\in I}(\varphi(a_i)\Rightarrow d)\Rightarrow d$ for every $a_i, d\in\mathcal{A}$ and for every set I:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma\vdash u:u}\,\text{Param.}}{\Gamma\vdash u:\varphi(\chi(d))\Rightarrow d}\,\text{Subs.}\qquad\dfrac{\dfrac{\dfrac{}{\Gamma\vdash x:\bigwedge_{c\in\mathcal{A}}\alpha_{\varphi(c)}}\,\text{Axiom}}{\Gamma\vdash x:\alpha_{\varphi(\chi(d))}}\,\text{Subs.}\qquad\tau}{\Gamma\vdash x\lambda z.u'(zy):\varphi(\chi(d))}\,\to\text{-elim.}}{\Gamma := x:\bigwedge_{c\in\mathcal{A}}\alpha_{\varphi(c)},\, y:\bigwedge_{i\in I}(\varphi(a_i)\Rightarrow d)\vdash u(x\lambda z.u'(zy)):d}\,\to\text{-elim.}}{x:\bigwedge_{c\in\mathcal{A}}\alpha_{\varphi(c)}\vdash\lambda y.u(x\lambda z.u'(zy)):\alpha_d\quad\text{for all } d\in\mathcal{B}}\,\to\text{-intro.}}{x:\bigwedge_{c\in\mathcal{A}}\alpha_{\varphi(c)}\vdash\lambda y.u(x\lambda z.u'(zy)):\bigwedge_{d\in\mathcal{B}}\alpha_d}\,\text{Gen.}}{\vdash\lambda xy.u(x\lambda z.u'(zy)).:\bigwedge_{c\in\mathcal{A}}\alpha_{\varphi(c)}\Rightarrow\bigwedge_{d\in\mathcal{B}}\alpha_d}\,\to\text{-intro.}$$

where τ is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma'\vdash u':u'}\,\text{Param.}}{\Gamma'\vdash u':d\Rightarrow\varphi(\chi(d))}\,\text{Subs.}\qquad\dfrac{\dfrac{}{\Gamma'\vdash z:\varphi(a_i)}\,\text{Axiom}\qquad\dfrac{\dfrac{}{\Gamma'\vdash y:\bigwedge_{i\in I}(\varphi(a_i)\Rightarrow d)}\,\text{Axiom}}{\Gamma'\vdash y:\varphi(a_i)\Rightarrow d}\,\text{Subs.}}{\Gamma'\vdash zy:d}\,\to\text{-elim.}}{\Gamma' := \Gamma, z:\varphi(a_i)\vdash u'(zy):\varphi(\chi(d))}\,\to\text{-elim.}}{\Gamma\vdash\lambda z.u'(zy):\varphi(a_i)\Rightarrow\varphi(\chi(d))\quad\text{for all } i\in I}\,\to\text{-intro.}}{\Gamma\vdash\lambda z.u'(zy):\bigwedge_{i\in I}(\varphi(a_i)\Rightarrow\varphi(\chi(d)))}\,\text{Gen.}$$

Thus, by Lemma 6.2:

$$[\exists\circ\mathcal{P}\varphi] = [\varphi\circ\exists]$$

□

Now, our aim is to show that condition *4.* of Theorem 6.2 is not necessary if the separator of $\mathcal{B}$ is a filter. Let us start by showing:

Lemma 6.3. *Let $\mathcal{A} = (\mathcal{A},\leq,\to,S)$ be an implicative algebra. If S is a filter and π_1,π_2 are the projections of $\mathcal{A}\times\mathcal{A}$ then*

$$[\pi_1 + \pi_2] = [\exists_{i=1,2}\pi_i]$$

Proof. We have to prove that:

$$\bigwedge_{a_1,a_2\in\mathcal{A}}(a_1+a_2\to\exists_{i=1,2}a_i)\in S$$
$$\bigwedge_{a,a_2\in\mathcal{A}}(\exists_{i=1,2}a_i\to a_1+a_2)\in S$$

Let consider:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma\vdash x:a_1+a_2}\text{Axiom}\qquad \dfrac{\dfrac{}{\Gamma,y:a_1\vdash y:a_1}\text{Axiom}}{\Gamma,y:a_1\vdash\lambda w.wy:\exists_{i=1,2}a_i}\text{Th. 2.6}\qquad \pi}{\Gamma:=x:a_1+a_2\vdash x(\lambda yw.wy)(\lambda zw.wz):\exists_{i=1,2}a_i}\text{Th. 2.4}}{\vdash\lambda x.x(\lambda yw.wy)(\lambda zw.wz):a_1+a_2\to\exists_{i=1,2}a_i\quad\text{for all } a_1,a_2\in\mathcal{A}}\to\text{-intro.}}{\vdash\lambda x.x(\lambda yw.wy)(\lambda zw.wz):\bigwedge_{a_1,a_2\in\mathcal{A}}(a_1+a_2\to\exists_{i=1,2}a_i)}\text{Gen.}$$

where π is:

$$\dfrac{\dfrac{}{\Gamma,z:a_2\vdash z:a_2}\text{Axiom}}{\Gamma,z:a_2\vdash\lambda w.wz:\exists_{i=1,2}a_i}\text{Th. 2.6}$$

Let us start by observing that if U is a filter then $\pitchfork^{\mathcal{B}}\in U$ by Lemma 3.4. Furthermore, let us recall that if $a_1,a_2\in\mathcal{A}$ then:

$$\pitchfork^{\mathcal{A}}a_1a_2\le a_1\qquad\pitchfork^{\mathcal{A}}a_1a_2\le a_2$$

Then:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma\vdash x:\bigwedge_{i=1,2}(a_i\to c)\to c}\text{Axiom}\qquad\dfrac{\text{Proved before}}{\Gamma\vdash\pitchfork^{\mathcal{A}}yz:\bigwedge_{i=1,2}(a_i\to c)}}{\Gamma:=x:\exists_{i=1,2}a_i,y:y:a_1\to c,z:a_2\to c\vdash x(\pitchfork^{\mathcal{A}}yz):c}\to\text{-elim.}}{x:\exists_{i=1,2}a_i,y:a_1\to c\vdash\lambda z.x(\pitchfork^{\mathcal{A}}yz):(a_2\to c)\to c}\to\text{-intro.}}{x:\exists_{i=1,2}a_i\vdash\lambda yz.x(\pitchfork^{\mathcal{A}}yz):(a_1\to c)\to(a_2\to c)\to c\quad\text{for all } c\in\mathcal{A}}\to\text{-intro.}}{x:\exists_{i=1,2}a_i\vdash\lambda yz.x(\pitchfork^{\mathcal{A}}yz):a_1+a_2}\text{Gen.}$$

$$\dfrac{\dfrac{x:\exists_{i=1,2}a_i\vdash\lambda yz.x(\pitchfork^{\mathcal{A}}yz):a_1+a_2}{\vdash\lambda xyz.x(\pitchfork^{\mathcal{A}}yz):\exists_{i=1,2}a_i\to a_1+a_2\quad\text{for all } a_1,a_2\in\mathcal{A}}\to\text{-intro.}}{\vdash\bigwedge_{a_1,a_2\in\mathcal{A}}(\exists_{i=1,2}a_i\to a_1+a_2)}\text{Gen.}$$

Thus $[\pi_1+\pi_2]=[\exists_{i=1,2}\pi_i]$ □

From the previous lemma it follows:

Corollary 6.1. *If U is a filter and*

$$[\varphi\circ\exists]=[\exists\circ\mathcal{P}\varphi]$$

then Φ is a first-order logic morphism.

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